Relative Motion of Free and Tethered Satellites

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Abstract

This research introduces a novel approach to the analysis of relative motion modelling of free and tethered satellites. For relative motion of free satellites the variational Kepler problem is considered and a geometrical method is developed in which the general solution of the variational Kepler problem is solved in the physically relevant relative position coordinates and expressed only in terms of the constants of motions. The method also allows for variations of parameters which is relevant in dust comet tail modelling where the different sizes of the dust particles relative to the nucleus in turn gives rise to a difference in mass parameter.

In the part of the research devoted to tethered satellites, different conservative models of tethered satellites are related mathematically to provide a unified framework and it is established in what limit they may provide useful insight into the underlying dynamics. First, the infinite dimensional model is regularised through the resistance against bending and then linked to a finite dimensional model, the slack-spring model, through a conjecture on the singular perturbation of tether thickness. Using a developed variational, symplectic integrator of the regularised system, numerical evidence is provided for the validity of the conjecture. Moreover, numerical computations of an orbiting tether system document that bending may be significant in regions of phase space. The slack-spring model is then naturally related to a billiard model in the limit of an inextensible spring. Next, the motion of a dumbbell model, which is lowest in the hierarchy of models, is identified within the motion of the billiard model through a theorem on the existence of invariant curves by exploiting Moser’s twist map theorem. Numerical computations provide insight into the dynamics of the billiard model.

To investigate the slack-spring limit further, a Galerkin approximation of the full massive tether model is considered. Here it is shown that the the slack-spring dynam-
ics can be identified with the slow dynamics on a normally elliptic slow manifold with bifurcations. Using averaging and a blow-up near the bifurcation it is in this thesis proven that the slow manifold persists adiabatically. It is believed that extending and generalising this result to more degrees of freedoms would attract considerable interest within both academia and industry.

The research also focuses attention on optimal attitude control. The research in this direction develops a novel geometrical and coordinate-independent approach to variational attitude dynamics and obtains within the linear approximation explicit, analytic expressions for the constrained $L^2$-optimal torque. The optimal torque is applied to two different formation flying missions scenarios where the range of validity of the linear approximation was also quantified. The results demonstrate an error of $\approx 1^\circ$ for a net rotation of $25^\circ$. A feedback law is also suggested in which the optimal control is updated via measurements of the instantaneous attitude and angular velocities.

Finally, this research considers a gravitational two-body problem where one of the bodies is modelled as a pseudo-rigid body. The other body is assumed to be a rigid sphere. Due to the rotational and "re-labelling" symmetries, the system is shown to possess conservation of angular momentum and "circulation". By following the classical reduction procedure undertaken in the study of the two-body problem of a general rigid body and a rigid sphere, a similar reduced non-canonical Hamiltonian system is computed. The classical two-body problem then becomes a natural subsystem. Then relative equilibria of the system are considered and it is shown that the notions of locally central and planar equilibria coincide. Finally, it is shown that Riemann's theorem on pseudo-rigid bodies has an extension to planar relative equilibrium of this system.
The research has resulted in the following publications/submissions:


- based on Chap. 4 and Chap. 5: K. U. Kristiansen, P. Palmer and M. Roberts. A unification of models of tethered satellites, accepted for publication in SIAM Journal of Applied Dynamical System subject to revision of minor corrections, 2010;


- Y. H. Shang, K. U. Kristiansen and P. Palmer. On the descent of a lander, submitted for publication in Journal of Guidance, Control and Dynamics, 2010 (this work has not been included in the thesis);


and the following papers are in preparation:
• based on Chap. 7: K. U. Kristiansen, P. Palmer and M. Roberts. The persistence of a slow manifold with bifurcation in tether modelling;

• based on Chap. 8: K. U. Kristiansen, N. Horri, P. Palmer and M. Roberts. Optimal control of relative attitude.

The research has also been communicated through the following talks and poster presentations:

• 2010 Aug.: Talk at Nonlinear Science and Complexity conference in Ankara;

• 2010 May: Poster at London Dynamical System Group meeting at Imperial College;

• 2010 May: Seminar speaker at University of Surrey, Department of Mathematics;

• 2009 Oct.: Seminar speaker at Technical University of Denmark, Department of Mathematics;

• 2009 Sep.: Poster at CELMEC conference on Celestial Mechanics. Acknowledged as being among the best three posters;

• 2008-10: Talks at AstroNet meetings in Ankara (2008), Toruń (2009) and Turku (2010);

• 2009 May: Talk at SIAM conference in Snowbird on Applications of Dynamical Systems;

• 2008 Sep.: Talk at AstroNet meeting during the International Astronautical Congress in Glasgow;

• 2008 Jun.: Poster presentation at EuroScience Open Forum in Barcelona;

• 2008 Apr.: Talk at AstroNet meeting in Surrey.

In an application for a EPSRC Post-Doctoral Research Fellowship Grant titled “Singular Perturbed Hamiltonian Systems with Slow Manifold Bifurcations” some of the interesting mathematical problems highlighted in the thesis has been suggested for further investigation in a three year research position.
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Dedicated to the memory of my father Niels Kristian Kristiansen: du som lænte mig ikke at give op.
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Introduction

The thesis focuses on relative motion of satellites. In the following section we shall give a short overview of the topic. At the end of the chapter the scope of the research and its aims and objectives will be described along with the research’s novelties.

1.1 Overview and motivation

There is a general trend in satellite mission designs to replace larger satellites with smaller satellites in formation. The coordination of smaller satellites has several benefits over single larger satellites. For one thing, the mission risk is decreased by allowing for cheaper replacements and therefore higher redundancy. Furthermore, with the development of small satellite technology and corresponding increase in their capabilities, formations have become feasible financially. Finally, and perhaps most importantly, a formation provides a more flexible mission platform by allowing reconfiguration for adaptive mission objectives so that for example research targets can be viewed from multiple angles at the same time providing unprecedent high measurement resolutions [60].

Several missions are already flying in formation. In the CLUSTER II space weather and environment mission [38], for example, now flying successfully in its ninth year, four satellites are placed in a tetrahedral formation. This allows for an
optimal resolution of the three dimensional gradients of the space measurements. Other existing successful formation flying missions include the NASA EO-1 remote sensing mission [48, 57] and the GRACE mission mapping the Earth’s gravity field [140]. Xiang and Jørgensen recent survey [157] also highlights a significant interest in this area by listing other planned and existing missions.

A particular way of keeping satellites in close proximity is to tether them together. The potential of such a system has recently been highlighted by the NASA’s proposed SPECS (Submillimeter Probe of the Evolution of Cosmic Structure) interferometer mission [95, 40, 28, 24] using tether formation flight to detect submillimeter-wavelength light from the early universe. Here the tethered formation would make up a very large scale, yet relatively light-weight, almost rigid structure of apertures in a controlled formation. This structure meets the requirements of a large interferometric baseline with adaptive baseline changes with minimal fuel consumption. As for the CLUSTER II mission, the formation again allow for a flexible platform in which a finer resolution may be attained compared to a large monolithic satellite system.

The initial interest in tethered satellites was probably due to the discovery of an orbitally stable relative equilibrium. In such a relative equilibrium or steady motion the system is normal to a circular orbit on which the centre of mass moves. A rigid satellite with a boom has a similar equilibrium and this is in general referred to as the effect of gravity gradient stabilisation [9]. The stability holds true for sup-kilometer tether lengths and a tether can therefore provide a stable, large yet relatively light-weight space-structure. This is an obvious advantage compared with the free orbiting satellites as the shape of any free satellite formation is neither rigid nor stable to perturbations. On the other hand, a major issue with tethered satellites is the extreme difficulty with bringing such a system into this stable configuration. There are no natural gateways to reach this equilibrium. As part of this issue, is the problem of how to safely release and separate a tethered system from a launcher without tether entanglement. These issues have been highlighted through several space experiments by the various space agencies. While these have verified the tether concept in general, they have also through several mission failures (TSS-IR and ESA mission YES [15]) documented the extreme difficulty with controlling and modelling this highly non-linear system. The reference [26] provides a recent detailed survey of suggested space tether applications and missions.

There are other immediate questions and problems that need to be dealt with for the further development of formation flying of free and tethered satellites. For example, perhaps the most obvious problem is how to avoid collisions between the satellites. In a realistic formation flying mission, satellites will have to be reconfigured by the use of thrusters and how to do so in a fuel-optimal way while avoiding collisions is not obvious. See [51] for a recent approach. Furthermore, the dynamics of relative motion of satellites is complex and yet an understanding of it is vital in planning a formation flying mission. In principle, numerical integrators for free satellite system
1.1 Overview and motivation

can now propagate initial conditions with very high accuracy. However, numerical computations often provide little insight into the causes and form of the dynamics. Analytical models, on the other hand, may provide a more detailed understanding of the dynamics and properties of the dynamics can be attributed to certain causes. Due to the complexity and the infinite dimensionality of a tethered satellite system, the reliability of numerical computations are here more questionable. Therefore, analytical models for the relative motion of tethered systems are perhaps even more important than for free relative motion of satellite systems. The main drive of this thesis is to rigorously develop analytical models further, and unify existing models, of both free and tethered satellites.

The control of the attitude of the spacecrafts orbiting in formation is essential in fulfilling any mission objectives. The attitude has to be properly aligned to obtain and maintain desired orientations for on-board instruments like solar panels, antennas or telescopes. In particular, in a formation flying mission the desired attitude of one satellite will in general depend upon the attitude but also the position of the other satellites in the formation. For inspection missions, for example, the desired attitude of a chaser satellite will depend upon the position and attitude of the inspected target. This problem therefore requires modelling of the attitude dynamics but also of the relative orbital dynamics.

On the other hand, to stretch the lifetime of a mission as far as possible, it is important to control the attitude using the least fuel possible within the given limitations. Nowadays, however, the most common attitude control software of small satellites generally includes a simple proportional-derivative (PD) quaternion feedback controller as proposed in [152]. This in spite of the fact that this controller is not optimal. Moreover, such controllers only address asymptotic convergence and are therefore in some sense not well-suited for the problem of achieving a certain attitude at a given time. This is becoming increasingly more important with the enhanced requirements for formation flying missions. To solve this problem optimally and analytically is therefore highly relevant. In particular, analytical expressions are computationally feasible and they may also provide further insight into the full nonlinear control problem. Another drive of the thesis is to develop explicit expressions for optimal attitude control within a linear theory and apply it to a formation flying missions scenario.

One may think of a tethered satellite system as a very flexible structure. A very classical way of modelling a flexible structure is as a pseudo-rigid body. As opposed to a rigid body whose configurations are given by orientation preserving isometries, the configurations of a pseudo-rigid body are described by invertible affine transformations. Such bodies were first considered by Newton in *Principia*. Since then several prominent authors including Dedekind, Dirichlet, Jacobi and Riemann [31, 36, 68, 124] have considered these bodies and in particular their relative equilibria of ellipsoidal configurations. Roche in 1847 [126] and Darwin in 1906 [30] also
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considered similar, but more restrictive, bodies for modelling tidal forces. It was here Darwin defined and calculated the Roche limit. More recently, the work on spinning pseudo-rigid bodies has been put into a modern language by Chandrasekhar [19] and Roberts and Sousa Dias [125] and been applied to problems in elasticity by Lewis and Simó [91].

With the recent progress of observational techniques and the increased interest in binary asteroids the full Newtonian two-body problem with non-spherical bodies have attracted significant attention [150, 96, 132, 11]. In fact studies have indicated that about 16% of the near-Earth Asteroids may be systems of relative orbiting asteroid pairs [99]. The formulation of this problem has been posed and studied in many references [150, 96, 132, 11, 149]. More recently, the question has been addressed whether the binary systems have random shapes or instead obey some general results of rotating deformable bodies, see e.g. [32]. To approach this problem, the models have to be extended to account for the deformation of the body. This can be attained, at least to a first approximation, by replacing the rigid bodies with pseudo-rigid bodies. The main drive of the thesis in this regard is to develop these models further while considering the possibility of generalising some of the results due to Riemann et al.

1.2 Scope of research

The research focuses on relative motion of satellites through four different sub-topics: Free relative motion modelling, modelling of relative motion of tethered satellites, optimal relative attitude control and finally a gravitational two-body problem of a pseudo-rigid body and a rigid sphere.

Free relative motion modelling

In this thesis the linearisation about an arbitrary Keplerian elliptic, parabolic or hyperbolic reference orbit is considered while particularly addressing the symmetries and the conserved quantities of the system. The methods considered by previous authors do not explicitly address the issue of constants of motion. This also manifests itself in complicated expressions and a lack of geometrical interpretation. The research will also consider variations in the mass parameter making the solutions applicable to modelling the dynamics of dust particles in comet tails. Indeed, dust particles leave a comet nucleus due to variations in orbital parameters but the variation in particles sizes will also give rise to variations in the solar radiation pressure which effects the motion of the particles relative to the nucleus. Since the solar radi-
1.2 Scope of research

Atmospheric pressure has the same form as the gravitational term, variations in particle sizes can be accounted for by variations in the mass parameter.

Tethered Satellites

The tether literature presents a number of different tether models of varying complexity. These models have been studied and used in many references on orbiting tethered satellites. However, the relationships between them have never been explored in detail. Furthermore, the effects of bending resistance have not been studied. In this thesis the task of unifying the models mathematically is begun by showing how simpler models can be derived rigorously from more complicated models, and how solutions of the former perturb to solutions of the latter. The work mainly focuses attention on the conservative models. Such a unification would justify the use of simpler models and therefore allow for the development of simpler control algorithms. These simpler control algorithms could then be validated on/extended onto the full system using a rigorous numerical scheme. The conclusions are supported by numerical computations via a developed numerical integrator.

Relative attitude control

Recently Palmer [115] considered $L^2$-optimal, or minimum energy [114], control of relocation of satellites in near-circular formations. The advantage of this approach, rather than traditional linear quadratic control approaches, is that it allows for closed-form solutions. Since then it has also been applied to elliptic orbits and other linear non-autonomous systems in [22, 21]. In this thesis this control approach is applied to the linearised attitude dynamics controlled with reaction wheels. External torques such as gravity gradient will be neglected.

The linearisation of the attitude dynamics often provides the basis for control algorithms on the full nonlinear system, see e.g [135] p. 113. In [135], however, the linearised attitude dynamics are derived using Euler angles as coordinates for the rotation matrix. In this research it will also be the aim to develop a coordinate independent and, similarly to the relative motion modelling above, a more geometrical approach to the linearised attitude dynamics.
Pseudo-rigid bodies

In [113, 134] the modelling of the gravitational two-body problem was extended by letting one of the bodies be a self-gravitating pseudo-rigid body. In this joint work with Mikhail Vereshchagin, the work of [113, 134] is put into the language of geometric theory of Hamiltonian systems while streamlining the approach and notation with the now standard reduction procedure for the two-body problem of a rigid body and a rigid sphere. Not only, will this procedure reduce the necessary equations of motion, but furthermore, as opposed to the method undertaking in [113, 134], the results will go through essentially unchanged if the pseudo-rigid body is assumed incompressible. The work will also investigate the possibility of obtaining a more general version of Riemann’s theorem on the classification of relative equilibria of pseudo-rigid bodies.

1.3 Aims and objectives

1.3.1 Aims

The aims of the research are listed here:

Free satellite motion

- Develop a geometrical framework for relative satellite motion;
- Obtain solutions of the variational Kepler problem with possible variations of mass parameters about any reference solution;
- Classify the solutions and apply the results to design of formation flying mission.

Tether satellite motion

- Unify and extend existing models of tethered satellites;
- Establish how solutions of models perturb into solutions of other models;
- Classify the dynamics of the models lowest in the hierarchy of models;
- Develop a numerical integrator of conservative massive tether models.
1.3 Aims and objectives

Relative attitude control

- Develop a simple geometrical coordinate-independent framework for variational attitude dynamics that is appropriate for optimal control;
- Determine analytical expression for the $L^2$-optimal control torque;
- Investigate the range of validity of the optimal linear control torque on the full nonlinear system.

Pseudo-rigid bodies

- Model the two-body problem of a pseudo-rigid body and a rigid sphere using appropriate coordinates also used in the reduction of the classical, rigid, full two-body problem;
- Obtain an extension of Riemann's theorem to the two-body problem.

1.3.2 Objectives

The objectives supporting the aims are listed here:

Free satellite motion

- Analyse symmetries and conserved quantities of variational equations;
- Integrate the variational equations and express the solutions in terms of the conserved quantities.

Tether satellite motion

- Analyse the effects of bending in elastic space tether modelling;
- Investigate the effects of vanishing thickness for stiff tethers;
- Analyse the singular perturbation of tether thickness on stiff tethers numerically;
- Analyse the singular perturbation of tether thickness on stiff tethers through a Galerkin approximation to the massive tether model.
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Relative attitude control

- Develop a coordinate independent and geometrical approach to the variational attitude dynamics;
- Set up constrained $L^2$-minimising control problem on the variational attitude dynamics;
- Apply the $L^2$-optimal control torque on realistic formation flying mission scenarios.

Pseudo-rigid bodies

- Investigate symmetries and conserved quantities of the two-body problem;
- Derive reduced equations of motions and reduced bracket structure;
- Investigate relative equilibria of the system while leaving the symmetric potential unspecified.

1.4 Novelty

The research contains the following novelty:

- Free relative motion modelling:
  - Obtaining a global solution expressed in terms of the relevant conserved quantities of the variational Kepler problem;
  - A classification of the solutions of the variational Kepler problem;
- Relative motion of tethered satellites:
  - A unification of well-posed models of tethered satellites;
  - The formulation of a conjecture on the slack-spring system as a limit system of the full massive tether model;
  - A theorem on an almost invariant slow manifold of a Galerkin approximation to the massive tether model on which the slow dynamics is described by the slack-spring model;
  - A theorem that shows the persistence of dumbbell dynamics within the billiard model;
1.5 Thesis structure

- Relative attitude:
  - Application of minimum energy control to attitude control;
  - Obtaining analytical expressions for $L^2$-optimal torque in variational attitude dynamics and applying these in a realistic formation flying mission.

- Pseudo-rigid bodies:
  - The reduction of the two-body problem of a pseudo-rigid body and rigid sphere;
  - An extension of Riemann's theorem to a two-body problem.

1.5 Thesis structure

In the following chapter the relevant literature is reviewed. This is followed by a chapter on the general variational Kepler problem. The main result of this chapter is stated as Theorem 1 on the general solution of the variational Kepler expressed only in terms of the variations of the conserved quantities. In Chap. 4 the different conservative models of tethered satellites are presented. Chap. 5 then seeks to unify the different conservative tether models. One of the key points in this chapter is the introduction of the regularising bending resistance and the formulation of Conjecture 1 on the relationship between the infinite dimensional massive tether model and the finite dimensional slack-spring model. Moreover, rigorous justification is provided for the validity of the dumbbell model in Theorem 3. The chapter concludes with numerical computations of the billiard model and a discussion of perturbations. Discussions are also provided of the effects of damping on the different tether models. In Chap. 6 a variational and symplectic numerical integrator is developed for the integration of the massive tether model. Numerical computations of an orbiting tether system are provided and the integrator is also used to provide further justification of the Conjecture 1. Chap. 7 investigates the slack-spring model in a Galerkin approximation. The main result of this chapter is Theorem 4. This provides the persistence of a slow manifold with bifurcation on which the slow dynamics is given by the slack-spring model. Finally, Chap. 8 and Chap. 9 consider the $L^2$-optimal attitude control and the two-body problem of a pseudo-rigid body and a rigid sphere, respectively. The main results of these two chapters are the development of explicit expressions of the optimal torque and Theorem 8 and Theorem 9 providing extensions of Riemann's classical theorem on ellipsoidal figures of relative equilibria. The thesis is concluded in Chap. 10 where open problems are also discussed.
Introduction
Chapter 2

Literature review

2.1 Introduction

In order to understand how the research contributes to today's needs, the relevant literature is reviewed. This is broken into four different categories corresponding to the different sub-topics of the thesis: free relative motion, tether relative motion, relative attitude dynamics and pseudo-rigid bodies.

2.2 Free relative motion

Perhaps the best known relative motion model is Hill’s equations [62], adapted to the problem of relative satellite navigation by Clohessy and Wiltshire [25] in 1960s. This is the linearisation of Keplerian relative dynamics around a circular reference orbit. However, six years prior to Clohessy and Wiltshire, Lawden derived the basic equations of relative motion for the more general case of eccentric orbits [86] (c.f. [14]). Tschauner and Hempel [144] independently formulated similar solutions to Lawden’s around the same time. The approach is in fact a generalisation of the Clohessy-Wiltshire equations, solving the same problem but linearising around an eccentric orbit rather than a circular one. While they still use Hill’s frame, they employ the true anomaly as the independent variable.
As formation flying missions are becoming a reality, there has been a renewed interest in relative motion modelling within the last decade. Important references include [14, 103] for the linear theory, [37, 54, 130, 133, 74, 112, 75, 153] for inclusion of higher order geopotential terms and [72, 147, 55] for nonlinearity effects. One of the few common threads within the existing literature on relative motion is that virtually all of the methods use a rotating and accelerating local coordinate frame. This approach makes analysis and visualisation of the motion rather straightforward. However, the perturbations to the Keplerian potential are usually defined in the Earth centered inertial frame. This is one of the primary reasons why the addition of the simple $J_2$ perturbation term greatly complicates the equations. The modelling of the motion is therefore actually hampered by the employment of this accelerating rotating frame.

Perhaps more importantly, these methods do not explicitly address the issue of constants of motion. For the motion of a satellite under a Keplerian potential, the energy, the angular momentum and the eccentricity are all conserved. For the case of the two satellites; the 'relative energy', the 'relative angular momentum', and the 'relative eccentricity' are also conserved. If these quantities are not conserved, e.g. in numerical integration, the relative orbits will get distorted over time. For example, any deviation from the relative energy will manifest itself as an alongtrack drift.

Palmer and Imre [117, 66] incorporated conservation of relative energy and relative angular momentum. In [117] they solved the linearised relative motion in the inertial coordinate frame exploiting these quantities, and in [66] they developed a symplectic numerical relative orbit propagator imposing the conservation laws and incorporating high order geopotentials. The results demonstrated a 40% improvement in computational time compared with propagating separate Keplerian orbits with a similar level of accuracy. Nevertheless, Palmer and Imre [117, 66] did not address the conservation of relative eccentricity. Furthermore, there does not seem to be any generalised approach to the general variational Kepler problem. Neither do the approaches previously considered in the literature consider the possibilities of variations in mass parameters. The latter could be of interest in comet tail modelling. In the tails of comets moving on highly eccentric orbits around the sun dust particles separate from the comet nucleus not only with a difference in the orbital parameters, but also due to the difference in the particle sizes relative to the size of the nucleus. This difference gives rise to an enhanced solar radiation pressure. Since the solar radiation pressure has the same form as the gravitational term, only opposite in sign, this variation can be accounted for by a variation in the mass parameter. See also [47]. To the author's knowledge there does not exist any analytical model of relative motion of the dust particles in the comet tails.
2.3 Tether relative motion

The available tether models in the literature can basically be divided into two groups: massive and massless tethers. In the following we shall present the ideas behind these widely used models. We shall also discuss some references where the models are analysed and applied.

Models and applications

Gravity and elasticity are important effects on the dynamics of tethered satellites. The effects of gravity are incorporated through the usual $1/r$-potential, neglecting perturbations due to the non-spherical earth. However, as it is custom in the modelling of free relative satellite motion, there is also often within the tether modelling literature made use of the fact that realistic tether lengths are much shorter than the distance to the attracting body which justifies a linearisation of the gravitational field about the system's centre of mass. On the other hand, there are different models in the literature for the elasticity of the tether. For a massive tether, linear elasticity is usually adapted, see e.g. [9, 78, 8]. Visco-elasticity to account for the dissipation in the tether is also incorporated via the Kelvin-Voigt model in some work, see e.g. [9, 78] and outside the space science research in [53]. Nonlinear elasticity is considered in [9] where the authors look at stability of certain equilibria and the effect of different elasticity laws. Furthermore, since tethers are usually very stiff, models for the case of inextensible tethers have also been proposed [119, 63, 69]. Though this is a very interesting limit to study, the models are very complicated and involve a free boundary value problem and an algebraic equation for the in-extensibility constraint which makes the equations very difficult to solve numerically and study analytically. The models are also in some sense singular. As an example of this singularity, one could think of the two end masses, the satellites, moving beyond the natural length of the tether, which would result in a $\delta$-distribution, changing the motion of the satellites abruptly to enforce the in-extensibility constraint. In [63], a free inextensible tether is considered in a quasi-stationary Stokes flow. Due to the absence of masses at the end-points, the author can enforce the inextensibility constraints directly hence easing the numerical and analytical study. This, however, is not possible in the space tether dynamics. Johansen et. al. in [69] analyse a discrete version to study small oscillations of a hanging rope with one free end. The free end again changes the nature of the problem and the singularity in tethered satellites models, as described above, is not present, again simplifying numerical and analytical analysis.

Any resistance against bending is neglected in the studies above. In [9] it is even claimed that due to the vanishing bending stiffness parameters of relevant thin tether materials, bending has little influence on the system dynamics compared to
other perturbation effects, see p. 59 in [9]. Since bending is a regularising term (biharmonic), such an argument would from a mathematical point of view have to be justified more rigorously. A mathematical justification is, however, apparently not available in the literature. In [98, 83, 89] discontinuities in tether tension are even observed in their numerical computations, yet they are not commented as a source of error.

Another model for tethered satellites is the dumbbell model, see e.g. [9, 128] with other references available in [15]. Here the tether is replaced by a rigid rod. Clearly, such a model can only capture motion in which the tether is taut and is not adequate to describe the motion when the tether folds and bends. Nevertheless, for the motion near the stable relative equilibria, librational motion, and for fast rotational motion, where the centrifugal forces keep the tether taut, this model is widely considered to be adequate. This has, however, not been justified mathematically. This is among the aims of the research.

Electrodynamic tethers have recently received a significant attention, see e.g. [120, 67, 155, 156, 27, 24] for some very recent references. For an electrodynamic tether a current running through the orbiting tether interacts with a magnetic field so that the system generates power. The power generated could be used on-board or for stabilisation [9]. The electrodynamical forces on the tether depend upon the attitude of the system relative to the magnetic field. In the dumbbell model the system is assumed rigid and the addition of the electrodynamical forces are therefore straightforward and widely used. For example, Peláez and Scheeres in [120] use a dumbbell model to study electrodynamical tethers as a method of orbit control near Jupiter, which is known to have a large magnetic field. It is not the aim to analyse electrodynamical tethers in this research and we therefore refer to [15] for a more detailed review on the subject.

Dumbbell models have also been used in three-body dynamics, see e.g. Farquhar [41] and more recently by Sanjurjo-Rivo et al. [128], where the dynamics of a tethered system near collinear libration points are studied. The idea proposed by the two papers is to adjust the tether length in order to stabilise the motion near the unstable equilibria. This application is up-to-date with the amount of research that studies the dynamics near libration points and with the future missions planned by the various space agencies. Sanjurjo-Rivo et al. extended the initial work by Farquhar to account for tether mass and developing a control scheme. Both papers acknowledge that the models cannot capture any dynamics where tether is in compression.

The reference [17] also considers the dumbbell model with a linearised gravitational field as a model of a tethered satellite system. Through an analytical study of the periodic orbits emanating from the stable relative equilibrium, where the system is aligned along the local radial direction, the authors investigate the effect of eccentricities on a tethered system. By using normal forms and averaging it is shown that
2.3 Tether relative motion

the family is stable to planar perturbations but unstable to non-planar perturbations for small but non-zero eccentricities. This is due to a parametric resonance with the orbital frequency. The instability was also known from numerical computations done in [9].

There is a finite dimensional model for massless tethers available in the literature that also account for tether slackness, referred to as slack-spring model in [9], p. 63. It is argued that the massless tether does not affect the motion when the satellites are closer together than the natural length of tether, whereas it acts like a perfect spring when they move apart beyond the natural length. It has not been analysed if this model can be obtained as a limit of the massive tether model. In Beletsky and Levin, [9] p. 63, a large and small vehicle is considered for the slack-spring model and the gravitational field is linearised about the larger vehicle’s Keplerian orbit. In [9], the slack-spring model is used to study the planar stability of deployment of small vehicle. Out-of-plane perturbations are not considered and the system is aligned with local vertical by controlling the tether length. A spring model also received attention recently in [18]. Here the slackness of the tether is however neglected and the authors extend their study from [17] on the dumbbell model to study the periodic orbits emanating from the stable relative equilibria. Their results are interesting, in the sense that contrary to the results for the rigid dumbbell model [17], it is shown that this family is stable for sufficiently small eccentricities and sufficiently large stiffnesses, both to planar and non-planar perturbations. Using analytical tools, normal forms and averaging, and numerical computations the authors also carry out a comprehensive bifurcation analysis of the orbiting spring model.

From the slack-spring model, Beletsky and Pankova, [19], suggested a model for inextensible tethers as a dynamical billiard problem. They do not present any formal derivation of the model as a limit system of other models, but they present a Poincaré-mapping to study the dynamics. It is not mentioned in the text, but from Fig 1 [19] it is obvious that one satellite is assumed to move on a circular orbit. In [134] the billiard model is used to study a visco-elastic tether and the transient chaotic oscillations of a tethered spaceship-satellite system. The visco-elasticity, that is the dissipation due to the tether jerks, is taken into account by restitution factors. It shows numerically that, as might be expected for a nonlinear, almost Hamiltonian system, transient chaos occurs before the system converges to the stable equilibria.

These models where the complexity is reduced dramatically are very interesting in terms of capturing approximations to the real dynamics. However, they have not been justified mathematically anywhere in the literature and it is not obvious in what limit these models are valid. Furthermore, the overall dynamics does not seem to have been analysed, since the research only focuses on part of the dynamics.

The massive tether model is singularly perturbed via the small tether thickness and the large stiffness. The literature on Hamiltonian singular perturbation theory
is therefore reviewed in the following section.

## 2.4 Singular perturbation theory

Systems involving different time and/or space scales described through a small positive parameter, say $\epsilon$, arise in a wide variety of scientific problems. Important examples include: molecular physics and the Born-Oppenheimer approximation [102], chemical enzyme kinetics and the Michaelis-Menten mechanism [104], predator-prey and reaction-diffusion models [107], the evolution and stability of the solar system [85] and tethered satellites. The main advantage of identifying slow and fast variables is dimension reduction by which all the fast variables, say $v$, are "slaved" to the slow ones, say $u$, through the "slow manifold" $M_0$. Dimension reduction is one of the main aims and tools for a dynamicist and the elimination of fast variables is very useful in for example numerical computations. In fact, since fast variables require more computational effort and evaluations, this reduction often bridges the gap between tractable and intractable computations. An example of this is the long time (Gyears) integration of the solar system, see [85], or perhaps the reduction from slow-fast tether system to slack-spring model.

The singular perturbed systems have the following general form:

$$\begin{align*}
\frac{du}{dt} &= f(u, v, \epsilon), \\
\epsilon \frac{dv}{dt} &= g(u, v, \epsilon),
\end{align*}$$

where $\epsilon$ is the small parameter. If we apply the time scaling $t \mapsto \epsilon^{-1}t$ then we transform (2.1) into:

$$\begin{align*}
\dot{u} &= \epsilon f(u, v, \epsilon), \\
\dot{v} &= g(u, v, \epsilon).
\end{align*}$$

(2.2)

Obviously, the two systems are only equivalent for $\epsilon \neq 0$. Different terminology is used, but we shall by the fast system refer to the fast part of (2.2) and by frozen refer to its limiting system. If we naively equate $\epsilon = 0$ in (2.1) then we obtain a differential equation for the slow variable $u$ and an algebraic equation $0 = g(u, v, 0)$. The set of points $\{(u, v)|g(u, v, 0) = 0\}$ satisfying this equations is "the slow manifold" $M_0$. On the other hand, equating $\epsilon = 0$ in (2.2), yields a differential equation for the fast variables $v$ depending upon the slow variables $u$ as parameters. In this limiting system the slow manifold is the set of equilibria.

The name slow manifold is quite unfortunate as we shall not require the slow manifold to be a manifold. In fact, a main focus in the research will be on slow manifolds where $\partial_v g(u, v, 0)|_{M_0}$ vanishes. The name is, nevertheless, widely used and we shall therefore also continue with this abuse of notation.
In the author's opinion the main problem of singular perturbation theory is to determine the fate of the slow manifold for $\epsilon \neq 0$, but small, and thus connect the apparent two different limit systems for $\epsilon \neq 0$. Fenichel [15, 11] was probably the first to address this rigorously. Fenichel proved the persistence of slow manifolds in finite dimensional systems where the dynamics normal to the slow manifold, described by the frozen system, is normally hyperbolic. Fenichel proved his results by first showing the persistence of the stable and unstable manifolds of $M_0$. Due to their transverse intersection, Fenichel then from this, concluded the existence of a perturbed invariant slow manifold $M_\epsilon$, $\epsilon$-close to $M_0$, for $\epsilon \neq 0$ but sufficiently small.

**Normally elliptic/parabolic slow manifold**

On the other hand, slow manifolds that are not normally hyperbolic are in general not expected to persist, because typical perturbations are believed to destroy it [117]. Non-hyperbolic slow manifolds, however, do occur in many interesting applications of Hamiltonian systems where any invariant set cannot be contracting and generic stability is therefore associated with oscillatory normal behaviour. We call the corresponding slow manifolds normally elliptic.

Since persistence of normally elliptic manifolds cannot be expected in general, one usually aims for something less. The general principle in physics for slow-fast systems is that the slow system is well-approximated (usually something like: $O(\epsilon^p)$-close, $p > 0$, over time-scales of order $O(\epsilon^{-1})$) by averaging the full system over the fast variables. If such an estimate can be established for a function on phase space then it is said to be adiabatically invariant. This principle fails to be true in general. The procedure of averaging can, however, be made rigorous for systems with only one fast phase or one fast degree of freedom in the Hamiltonian setting, see [3] and [52], respectively. For one fast degree of freedom systems the corresponding frozen system is an integrable one degree of freedom system depending on the slow variables as parameters. Since the slow manifold is normally elliptic, action angle coordinates can be introduced in a neighbourhood of the corresponding elliptic equilibrium in the fast space such that the frozen Hamiltonian only depends on the action variable. The key observation in the paper by Gelfreich and Lerman [52] is then that this transformation into action angle coordinates on the fast space can be lifted to a transformation on the full space for $\epsilon \neq 0$ but small that preserves the Hamiltonian structure and is $\epsilon$-close in the slow variables. The authors then generalise the classical KAM averaging iteration scheme [1, 5] that via averaging of the remainder over the fast angle moves the angle dependent terms to higher order in $\epsilon$ and this way obtaining an improved slow manifold. Such a procedure generally diverge. Nevertheless, in the case of an analytic Hamiltonian the authors use Neishtadt-type estimates [118, 119] to obtain

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2Formal definition of adiabatic invariance: $I$ is an adiabatic invariant if for any $\rho > 0$ the measure of the set of initial conditions $\{I(0)||I(t) - I(0)| > \rho\}$ vanishes for $\epsilon \to 0$ [15]. Chapter 5 §4.2.
an optimal number of iterations \( n = n(\epsilon) = \mathcal{O}(\epsilon^{-1}) \) moving the angle dependency to exponential small order in \( \epsilon \). This provides the persistence of an almost invariant slow manifold with only exponentially small error.\(^3\) But not only that, and perhaps more importantly, it also follows that the dynamics in a neighbourhood of the slow manifold can be reduced by an excellent approximation.

In general this method of improving the adiabatic invariance of a slow manifold via averaging fails for two or more fast degrees of freedoms. In particular, when there are more than one fast degree of freedom then the frozen system is in general not integrable. Furthermore, when the frozen system is integrable, or near-integrable, the procedure in general also fails, due to resonances between the fast variables \([5]\). In fact, the averaging procedure is in general limited to systems in which the frequencies can be controlled. This is, however, in general not the case near slow manifolds, where the frequencies in the frozen system depends on the slow variables and the required control can therefore only be attained through very restricted assumptions.

**Dynamic bifurcation or changing stability**

Averaging is only valid away from bifurcations of the slow manifold. However, since such a slow manifold bifurcation is directly associated with a bifurcation in the frozen system which depends on the slow variables as parameters, this is a general phenomenon. As opposed to traditional bifurcation theory, it is the slow variables (internal parameters) that unfold the bifurcation rather than external parameters. In \([3]\) this situation is referred to as dynamic bifurcation whereas \([90]\) uses changing stability. This is for example the case of the infinite dimensional tether system. Other examples include moving interfaces between ordered and disordered crystalline states \([131, 46, 137]\). The research will aim to investigate the dynamic bifurcation in a Galerkin approximation to the massive tether model and how the slack-spring model can be understood as an appropriate limit system.

In the following the relevant literature on attitude control is reviewed.

**2.5 Relative attitude control**

For a formation of satellites to perform a unified space mission the attitudes of the satellites have to be properly synchronised. To achieve this, a centralised control

\(^3\) Similar estimates are in fact obtained in \([5]\), Proposition 3 p. 208. Furthermore, these Neishtadt-type estimates have been used in generic integrable Hamiltonian systems by Nekhoroshev \([111]\), see also \([122]\), in estimating the rate of Arnold diffusion (exponentially slow).
strategy has several advantages over a *decentralised* attitude control scheme [13, 71, 151]. In a centralised control strategy a leader satellite is considered with a leading, reference attitude and the attitude of the following satellites are then viewed relative to this reference. For a decentralised control scheme, for example, the number of communication links grow quadratically with the number of satellites as opposed to linearly for a centralised scheme. Therefore a decentralised control scheme may place unnecessary computational burden on the satellites. Moreover, for the decentralised scheme a malfunction of a single satellite will in general affect the attitude of the whole formation. On the other hand, a decentralised attitude control scheme may actually benefit from reduced absolute and relative attitude errors [118, 20].

Despite the obvious benefits of optimal control, attitude control of satellites is still predominantly based on standard and non-optimal control laws, such as proportional-derivative (PD) quaternion feedback controller as proposed in [152]. The lack of commercial success of optimal control laws is partially due to their implementation complexity and that they are often considered numerically infeasible for on-board software on small satellites. Nonlinear optimal controllers that perhaps circumvent these issues, such as Lyapunov based inverse optimal control techniques [18] which are increasingly considered for certain control system applications do, in general, only address asymptotic convergence. This is also the case for PD or infinite horizon Linear Quadratic Regulator (LQR) controllers. While this is often adequate, it is nevertheless, at least to some extent, not well-suited for the general problem of achieving a certain attitude at a given time. This type of objective is becoming increasingly more important with the enhanced requirements for formation flying missions. For a PD controller, for example, one will have to choose the appropriate gain matrices to achieve a certain settling time by trial and error [152]. On the other hand, a fixed time optimal controller addresses this issue directly. Such controllers have been considered elsewhere, but due to the general nonlinearity of the system and torque limitations, the approaches are usually purely numerical. See [158, 20] and reference therein. In [158], for example, a numerical scheme was proposed to solve the minimum energy problem for rest to rest manoeuvre via a discretisation of the full nonlinear system. Again the implementation complexity associated with such an approach may be numerically infeasible for on-board software on small satellites. There do not seem to be any references that investigate the capabilities of a fixed time, minimum energy controller based on a linearisation of the attitude dynamics on the full nonlinear system. As in [158, 20], where optimal orbit reconfiguration was considered, this would allow for the development of analytic solutions of the optimal control. This is among the aims of the research. Explicit expressions of the control would circumvent the issues associated with complex numerical algorithms. In principle, torque constraints and reachability issues could then also be addressed directly through the analytic expressions [116].

An inspection mission provides an example of a problem of relative attitude in a formation flying mission. Here the attitude of a *chaser* satellite has to be prop-
erly aligned relative to the position and attitude of the inspected target satellite. A number of inspection experiments have been conducted recently on small satellite missions such as BX-1, inspection of Shenzhou VII spacecraft [94], and Snap-1, inspection of Tsinghua-1 [84, 146]. These were, however, time limited experiments with little focus on the relative attitude control as this was not required to ensure that the target would appear in the field of view of the chaser for the limited time of consideration. The problem was also considered in [143] in the case of a specified bounded and co-planar periodic relative orbit trajectory. The chaser is assumed to be underactuated so that there is no thrust available for rotation about the principle axis which is assumed perpendicular to the orbital plane. The reference then determines an underactuated finite horizon $L^2$-optimal attitude control numerically through a discretisation of the full nonlinear system, which is described using quaternions, so that the satellite at the completion of the manoeuvre is properly aligned with the relative position vector. It is not clear what is actually gained compared to a linearisation about a reference solution. Through a linearisation explicit, analytic solutions could have been obtained. These analytic solutions would certainly ease the numerical burden associated with the optimisation algorithm.

The reference [20] considers a decentralised control strategy and investigates the feasibility of a controller based upon a state-dependent Ricatti equation technique on a formation flying mission scenario. The controller is also compared with standard PD and LQR controllers. While the state-dependent Ricatti equation technique was demonstrated to possess some advantages in terms of convergence, stability, large angle manoeuvres and for small actuators with small torque constraints, it does not directly address the problem of achieving a certain attitude at a given time. Moreover, there does not seem to be a general analytic approach to this problem.

A tethered system is a very flexible structure. A very classical and simple way of modelling a flexible system is as a pseudo-rigid body. In the following the literature on this topic is reviewed. In particular, as the research on this topic aims to study the two-body problem of a pseudo-rigid body and a rigid body, the review will primarily focus attention on two recent papers on this problem.

2.6 Pseudo-rigid bodies

The study of the Newtonian many-body problem has a long history and still to this day attracts the interest of many mathematicians, physicists and engineers. Often in the study, however, the interacting bodies are assumed to be spherical or point masses. More recently, with the progress of observational techniques and the increased interest in binary asteroids [150, 96, 132, 11], models with non-spherical bodies have been developed and studied. The main effect is the coupling between
the orbital and rotational motion. Moreover, as the configuration space \( SO(3) = \{\text{group of orientation preserving isometries in } \mathbb{R}^3\} \) of a rigid body is a non-linear space the introduction of local coordinates leads to singularities and very complicated equations of motion. There is therefore a significant gain by exploiting the symmetries underpinning the system to reduce and simplify the equations of motion. The reference [159] was probably the first attempt in a systematic way to make use of the rotational symmetry of the Newtonian system of a sphere and a non-spherical rigid body to reduce the equations in a coordinate-free way. Since then this reduction procedure has been used and extended by many authors, see for example [90, 132, 11, 129].

In [113, 131] the modelling of the two-body problem was taken one step further by letting one of the bodies be self-gravitating and deformable via the action of general orientation-preserving, invertible matrix. Such matrices make up the Lie-group \( GL^+(3) = \{Q \in \mathbb{R}^{3 \times 3} | \det Q > 0\} \). The corresponding body is called a pseudo-rigid body or an affine rigid body. A similar but more restricted system for modelling tidal forces was considered by Roche in 1847 [126] and extended by Darwin in 1906 [30]. Both Roche and Darwin accounted for the presence of energy dissipation due to the non-conservative tides which eventually led to synchronized rotation and relative circular orbit motion. See also the review paper [15] for a recent exposition. It was here Roche defined and calculated the “Roche limit”. This limit is defined as the distance within which a celestial body, held together only by its own gravity, will disintegrate due to a second celestial body’s tidal forces.

Without the gravitational interaction from another body, such pseudo-rigid bodies have received attention in many references [31, 36, 68, 121, 19, 91, 125, 130]. The interest was initiated by Newton in *Principia*, where he showed that a spinning axisymmetric self-gravitating body of fluid that is rotating slowly about the symmetry axis will be oblate. Jacobi in 1834 [68] extended the work of Newton, but also work of Maclaurin, to show that a self-gravitating fluid also can take on ellipsoidal shapes. The solutions of Jacobi, Maclaurin and Newton were, however, still all rigid. In a frame rotating with the body the fluid is stationary. Dirichlet and Dedekind, [30] and [31], respectively, opened a new direction when they found a symmetry that applied to Jacobi’s solution generated a new solution in which the body is stationary in shape but the fluid particles follow elliptical paths in planes orthogonal to a principle axis of the ellipsoid. Dirichlet’s paper inspired Riemann to turn his attention to the problem. In [124] he gave a classification of the solutions of Dirichlet’s equations for which the ellipsoidal shape of the body remains constant. At the heart of this classification lies what is now known as Riemann’s theorem: the angular velocity and circulation (i) lie in the same principle plane and (ii) if the angular velocity is parallel to a principle axis then the vorticity must also lie along that same principle axis. By now this work has been unified and extended by for example Chandrasekhar [19] and Roberts and Sousa Dias [125]. Pseudo rigid bodies have also been applied to elasticity, spinning gas clouds, atomic nuclei etc. (see [125] and references therein).
The reference [113] models the dynamics of the two-body problem of a pseudo-rigid body and a rigid sphere. For the configurations of the pseudo-rigid body a polar decomposition is used:

$$\forall Q \in GL^+(3) \exists R \in SO(3), U \in \text{symm}^+(3) : Q = RU,$$

where $\text{symm}^+(3)$ is the space of symmetric positive definite matrices. While this decomposition circumvent singularity issues associated with the singular decomposition:

$$\forall Q \in GL^+(3) \exists R, S \in SO(3), \hat{A} \in \text{diag}^+(3) : Q = R\hat{A}S^T,$$

for spheroidal configurations, see e.g. [64, 125], this method is not well-suited for a complete reduction and for addressing all possible symmetries and the geometry underpinning the system. Indeed, the reference [113] does for example not mention the possible conservation of fluid circulation. Furthermore, the equations of motion are formulated as differential equations of matrices, which is not very convenient for either analytical or numerical investigations. Particularly, the addition of constraints such as incompressibility greatly complicates the analysis. Finally, since the paper in the end restricts attention to relative equilibria little is actually gained by avoiding the singularities of the singular decomposition.

The reference [134] considers the two-body problem of a pseudo-rigid body and a rigid body as a model of rubble-pile satellites. The equations for steady motions are derived but with several approximations. First of all, the problem is restricted in the sense that pseudo-rigid body does not affect the motion of the rigid body. Furthermore, it is assumed that the ratio of the inter-particle distances of the pseudo-rigid body to the distance to the rigid body is small, justifying an expansion in which only linear terms are retained. The setting was used to derive a generalised Roche limit which was applied to Mars's moons.

The review of the literature highlights that the extended two-body problem of a pseudo-rigid body and a sphere has not been considered within the framework now standard in the consideration of the classical full two-body problem, cf. e.g. [150, 96, 132, 11]. By considering this problem within this setting the rigid two-body problem will become a natural subsystem. Furthermore, as opposed to the approach undertaken in [113, 134], the analysis will go through essentially unchanged if the pseudo-rigid body is assumed to be incompressible.

### 2.7 Discussion and conclusion

The literature review has highlighted a need for considering several different problems on relative motion modelling. First of all the existing literature does not provide a
2.7 Discussion and conclusion

A general unified approach to linearised relative motion models. Furthermore, the existing approaches do not effectively address the symmetries of the system. Addressing symmetries is not only of academic interest, it also provides a geometrical interpretation of the problem.

The literature review on tethered satellites primarily highlighted the use of a range of different models with varying complexity. In particular, relatively simple models, such as the dumbbell model where the flexible tether system is modelled as a rigid body, are widely used. The validity of these simpler models does, however, not seem to have been exploited rigorously. It would be very relevant to unify these existing models and establish in what limit they may provide insight into the true dynamics.

As the massive tether model is singularly perturbed the literature on singular perturbed Hamiltonian systems was also reviewed. The review highlighted that slow manifolds with bifurcation had received relatively little attention. As bifurcations is a general phenomenon it would be interesting and relevant to obtain some results in this regard. In fact it will be made clear later on that slow manifold bifurcation is an essential phenomenon in the justification of finite dimensional tether models.

The literature review also underlined a need for relative attitude control to accommodate the enhanced requirements for formation flying missions. A recent approach to linear quadratic optimal control theory considered in [115] is also believed to be applicable to relative attitude control. Obtaining analytic expressions for optimal control is also practically very useful as they in principle circumvent the usual issues associated with the computation intensity of other approaches to optimal control theory.

Finally, the literature review emphasized that the two-body problem of a pseudo-rigid body and a sphere has not been viewed properly within the context of geometrical Hamiltonian theory.
Literature review
3.1 Introduction

In this chapter, the general variational Kepler problem is considered. The solutions of this problem is perhaps the most widely used models for relative motion of free satellites. Here the work by Palmer and Imre in [117] is extended by providing a generalised mathematical foundation and using the conservation of relative energy, angular momentum and eccentricity to obtain solutions of the general variational Kepler problem, expressed only in terms of these conserved quantities and a time delay. Variations in mass parameter is also accounted for. The Keplerian orbits are easily interpreted geometrically through conic sections and the conserved quantities and this framework provides the variational solutions with a natural, similar geometry. This geometry is useful in applications of formation flying mission design which is demonstrated by the construction of a tetrahedron formation. Furthermore, a complete solution of the variational equations is presented for all elliptic, hyperbolic and parabolic reference trajectories, including the zero angular momentum case. First, however, the Kepler problem is revisited.
3.2 The Kepler problem

The two-body problem is given by the Hamiltonian
\[ H(q, p) = \frac{1}{2} |p|^2 - \frac{\mu}{|q|}, \]
with \((q, p) \in \mathbb{R}^3\), and Hamilton’s equations:
\[
\dot{q} = \partial_p H(q, p) = p,
\]
\[
\dot{p} = -\partial_q H(q, p) = -\frac{\mu}{|q|^2} q,
\]
where \(| \cdot |\) is the Euclidean norm. By the rotational invariance, \(H(Rq, Rp) = H(q, p)\) for every rotation matrix \(R\), it follows that
\[ L(q, p) = q \wedge p, \]
is a conserved quantity. From the conservation of \(L\) follows that the motion takes place in a plane, which is called the orbital plane:
\[ P = \{ x \in \mathbb{R}^3 | (x, k) = 0 \}, \tag{3.1} \]
where \(k\) is a unit vector in the direction of \(L\). If \(L(q, p) = 0\) then \(q \parallel p\) and the motion is radial and may, depending on the energy, only exist for a finite time.

Due to a more hidden symmetry [29], the eccentricity vector
\[ e(q, p) = -\frac{q}{|q|} + \frac{1}{\mu} p \wedge L(q, p), \]
is also conserved. The seven scalar conserved quantities \(H(q, p) = h, L(q, p) = 1\) and \(e(q, p) = \epsilon\), are related through two equations:
\[
(l, \epsilon) = 0, \tag{3.2}
\]
\[
\epsilon^2 - 1 = \frac{2l^2}{\mu^2 h}, \tag{3.3}
\]
where \(\epsilon = |\epsilon|\) and \(l = |l|\). These equations define a smooth, five dimensional submanifold \(\mathcal{M}\) in \((l, \epsilon, h)\)-space. In other words, there exists five independent conserved quantities.

It follows from (3.2) and (3.3) that the eccentricity vector is in the orbital plane and its magnitude can be determined from \(L\) and \(H\), hence only the direction of the eccentricity vector is conserved independently. Moreover, by taking the dot product
of e with q one finds that \( t \rightarrow q(t) \) traces out a conic section determined by the norm of the eccentricity: elliptic \( e < 1 \), parabolic \( e = 1 \) or hyperbolic if \( e > 1 \).

For \( e < 1 \) a qualitative description of the relative motion of satellites on Keplerian orbits can be obtained from Kepler's third law. This says that the period of the periodic orbit only depends upon the semi-major axis \( a \), and since the energy constant, \( h \), is directly related to the semi-major axis:

\[
    h = -\frac{\mu}{2a}, \tag{3.4}
\]

it follows that initially neighbouring satellites with different energies or different mass-parameters will evidently drift apart. For \( e < 1 \) variations in the other constants of motion only give rise to bounded relative motion. In the following section a general Hamiltonian setting to variational equations with parameters is introduced.

### 3.3 General Hamiltonian setting of variational equations

We consider a Hamiltonian \( H : M \times I \rightarrow \mathbb{R} \) where \((M, \omega)\) is a symplectic manifold and \( I \subset \mathbb{R} \) is an open interval of definition for a parameter \( \mu \). Recall that a symplectic form \( \omega \) on \( M \) is a bilinear form \( \omega|_z : T_z M \times T_z M \rightarrow \mathbb{R} \) for every \( z \in M \) satisfying:

\[
    \omega|_z (v, w) = -\omega|_z (w, v) \quad \text{(skew-symmetry)}
\]

\[
    \omega|_z (v, w) = 0 \quad \forall w \in T_z M \Rightarrow v = 0 \quad \text{(non-degeneracy)}.
\]

Here \( T_z M \) is the vector space of tangent vectors to \( M \) at \( z \) and \( TM = \bigcup_{z \in M} \{z\} \times T_z M \) is the tangent bundle. Through the symplectic form, \( H \) defines a unique vector field \( X_h \) satisfying:

\[
    \omega(X_h, w) = dH(z, \mu)(w), \quad \forall w \in T_z M,
\]

where we by \( d \) interpret the differential of a function on \( M \). Similarly, the variational vector field \( X_{\delta h} : TM \rightarrow T(TM) \) is the unique vector field satisfying

\[
    \delta \omega(X_{\delta h}, (w, \delta w)) = d((\delta H((z, \mu), (\delta z, \delta \mu)))(w, \delta w), \forall (w, \delta w) \in T(z, \delta z)(TM). \tag{3.5}
\]

Here \( d \) is the differential of a function on \( TM \) and \( \delta \omega \) is a symplectic form on \( TM \), see e.g. [99]. The variational Hamiltonian \( \delta H \) is conserved: \( d\delta H(X_{\delta h}) = \delta \omega(X_{\delta h}, X_{\delta h}) \equiv 0 \). Let \( G \) be a Lie-group acting smoothly upon \( M \) that leaves \( H \) invariant:

\[
    H(gz, \mu) = H(z, \mu), \quad \forall g \in G, \forall z \in M. \tag{3.6}
\]
By Noether's theorem [64] there exists an associated conserved quantity $C = C(z, \mu)$ of dimension equal to the dimension of $G$. The $G$-action lifts naturally to an action on $TM$, and from (3.6) it directly follows that

$$\delta H(gz, \mu)(g\delta z, \delta \mu) = \delta H(z, \mu)(\delta z, \delta \mu),$$

and hence any $G$-invariance of $H$ translates to $G$-invariance of $\delta H$. As a corollary any $(G$-equivariant) conserved quantity, $C(z, \mu)$, of the original system gives rise to a $(G$-equivariant) conserved quantity, $\delta C$, of the Hamiltonian system $(TM, \delta H, \delta \omega)$. In coordinates:

$$\delta C(z, \mu)(\delta z, \delta \mu) = (\nabla C(z, \mu), (\delta z, \delta \mu)).$$

For the particular case of the Kepler Hamiltonian:

$$H(q, p, \mu) = \frac{1}{2}|p|^2 - \frac{\mu}{|q|},$$

the variational Hamiltonian is

$$\delta H(q, p, \mu, \delta q, \delta p, \delta \mu) = (p, \delta p) + \mu \frac{q}{|q|^3} (q, \delta q) - \frac{\delta \mu}{|q|},$$

with $\delta \omega = d\delta q \wedge dp + dq \wedge d\delta p$ and

$$X_{\delta H} = \begin{pmatrix}
\frac{p}{|q|^3}q \\
\frac{\mu}{|q|^3} \\
- \frac{\mu}{|q|^5} \left(1 - 3\frac{q^2}{|q|^2}\right) \delta q - \frac{\delta \mu}{|q|^3} q
\end{pmatrix}.$$

Moreover, by (3.8)

$$\delta L(q, p, \mu, \delta q, \delta p, \delta \mu) = \delta q \wedge p + q \wedge \delta p,$$

$$\delta e(q, p, \mu, \delta q, \delta p, \delta \mu) = - \frac{\delta q}{|q|} + \frac{\langle q, \delta q \rangle q}{|q|^3} + \frac{1}{\mu} \delta p \wedge L(q, p) + \frac{1}{\mu^2} p \wedge \delta L(q, p, \delta q, \delta p) - \frac{\delta \mu}{\mu^2} p \wedge L(q, p),$$

are also conserved. We call the quantities (3.9), (3.10) and (3.11) the relative Hamiltonian, the relative angular momentum and relative eccentricity, respectively. The variables $(\delta q, \delta p)$ are first order approximations to the relative motion of satellites near a reference orbit described by $(q, p)$.

Setting $\delta H(z, \delta z) = \delta h$, $\delta L(z, \delta z) = \delta l$ and $\delta e(z, \delta z) = \delta e$ we then obtain from (3.2) and (3.3) the following relations:

$$\langle \delta l, e \rangle + \langle l, \delta e \rangle = 0,$$

$$\langle e, \delta e \rangle = \frac{2l}{\mu^2} h + \frac{l^2}{\mu^2} \delta h - \frac{2l^2}{\mu^2} \frac{\delta \mu}{\mu}.$$
3.4 Solutions of Kepler’s variational equations

The main tool used in the construction of the variational solutions will rely on
the following. Since \(2n - 1\) (\(n = \) the degrees of freedom) independent quantities are
conserved, the Kepler problem falls into the category of maximally super-integrable
systems \([142]\), and its solution may therefore be expressed in terms of \(h, l, e, \mu\) and
time in the form \(t - t_0\), so that

\[
z \equiv (q, p) = z(h, l, e, \mu, t - t_0).
\]

As a consequence of the chain rule and the rule of mixed partials, solutions of the
variational equations can be obtained by taking independent variations with respect
to \(h, l, e\) and \(t_0\), see also Wiesel and Pohlen \([154]\). Indeed, let \(c \in \mathbb{R}^6\) be coordinates
on the six dimensional sub-manifold, \(\mathcal{M}\), embedded within \((h, l, e, \mu)\)-space and given
by the equations (3.2) and (3.3), and view \(z(t - t_0)\) as a vector-valued function on
\(\mathcal{M}\). Then

\[
\delta z(t - t_0)(c) (\delta c) = \sum_{i=1}^{6} \partial_c z(t - t_0)(c) \delta c_i, \quad \delta c \in T_c \mathcal{M} \cong \mathbb{R}^6.
\]

Differentiation with respect to \(t\) gives

\[
d_t (\partial_c z(t - t_0)(c) \delta c_i) = \partial_t z(t - t_0) \delta c_i
\]

\[
= \partial_t J \nabla_z H(z(t - t_0)(c)) \delta c_i
\]

\[
= J \nabla_z H(z(t - t_0)(c)) (\partial_c z(t - t_0)(c)) \delta c_i,
\]

showing that \(\partial_c z(t - t_0)(c) \delta c_i\) for \(i = 1, \ldots, 6\) solve the variational equations. Fixing
\(c\) and instead viewing \(z(t - t_0)\) as a vector-valued function of \(t_0\), which enters in the
form \(t - t_0\), we obtain

\[
\delta z(t - t_0)(\delta t_0) = \dot{z} \delta t_0 = -\dot{z} \delta t_0,
\]

and therefore differentiating this with respect to \(t\) shows that the vector field along
the reference orbit is a solution of the variational problem. This is a well-known
fact. In the following section it is the aim to determine the solution of the variational
equations of Kepler’s problem by exploiting this geometrical setting.

3.4 Solutions of Kepler’s variational equations

If, for \(c \neq 0,1\),

\[
\delta L(z, \delta z) = \delta l = \delta l_1 i + \delta l_2 j + \delta l_3 k,
\]

\[
\delta e(z, \delta z) = \delta e = \delta c_1 i + \delta c_2 j + \delta c_3 k,
\]
where \( i = \frac{e}{e} \), \( j = \frac{1 + e}{1 - e^2} \) and \( k = \frac{1}{l} \), see Fig. 3.1, then (3.12) and (3.13) may be written in coordinates as:

\[
\begin{align*}
\delta e_0 &= -\frac{e}{l} \delta l_1, \\
\frac{e}{1 - e^2} \delta e_1 + \frac{\delta l_3}{l} &= -\frac{\delta h}{2h} + \frac{\delta \mu}{\mu}.
\end{align*}
\]

The case when \( e = 0,1 \) will be returned to later.

For \( e \neq 1 \) (3.15) and (3.16) show that every variation can be described using \( \delta l, \delta e_1, \delta e_2 \) and \( \delta \mu \). Writing

\[ q(t; l, e) = |q| (i \cos \nu + j \sin \nu), \]

where everything on the right hand side is a function of \( t - t_0, l, \) and \( \mu \), we obtain

\[
\begin{align*}
\delta q &= \delta q_i i + \delta q_j j + q_1 \left( \frac{\delta e}{e} - \frac{e}{e^2} (e, \delta e) \right) \\
&\quad + q_2 \left( \frac{\delta l \wedge e + 1 \wedge \delta e}{l \wedge e} - \frac{1 \wedge \delta e}{(l \wedge e)} (l \wedge e, \delta l \wedge e + 1 \wedge \delta e) \right), \quad \epsilon \neq 0,1, \quad (3.17)
\end{align*}
\]

where \( q_1 \equiv |q| \cos \nu \) and \( q_2 \equiv |q| \sin \nu \) with \( |q| = l^2/(\mu(1 + e \cos \nu)) \).

Note that \( q_1 \) and \( q_2 \) only depend upon the norm of \( e \) and \( l \). Since \( e = e_1 \) and
3.4 Solutions of Kepler's variational equations

Let \( l = l_3 \) it follows that

\[
\begin{align*}
\delta q_1 &= \partial q_1 \delta l_3 + \partial_\ell q_1 \delta \ell_1, \\
\delta q_2 &= \partial q_2 \delta l_3 + \partial_\ell q_2 \delta \ell_1.
\end{align*}
\]

By linearity the variations can be considered independently. Therefore, first let \( \delta l_3 = 0 \) and \( \delta \ell_1 = 0 \). Then (3.17) gives

\[
\delta q = -\frac{\epsilon}{\epsilon} \delta \epsilon_2 \delta \ell_1 + \frac{q_1}{\epsilon} \delta \epsilon_2 \delta \ell_1 - \left( q_1 \delta l_1 + q_2 \frac{\epsilon}{l(l+\epsilon)} \delta l_2 \right) k, \quad \epsilon \neq 0, 1,
\]

or compactly:

\[
\delta q = \frac{1^\wedge \epsilon}{l(l+\epsilon)} \delta \epsilon_2 + \frac{\epsilon}{l(l+\epsilon)} \delta l_1 + \frac{\epsilon}{l(l+\epsilon)} \delta l_2, \quad \epsilon \neq 0, 1. \tag{3.18}
\]

Consider \( \delta l_1 = 0 = \delta l_2 \) and \( \delta \epsilon_2 = 0 \), so that \( \delta i = 0 = \delta j \). Then (3.17) becomes:

\[
\delta q = \delta q_1 + \delta q_2
\]

\[
= (\partial q_1 i + \partial q_2 j) \delta l_3 + (\partial q_1 i + \partial q_2 j) \delta \ell_1 + (\partial q_1 k + \partial q_2 l) \delta \mu.
\]

The partial derivatives of \( q_1 \) and \( q_2 \) are:

\[
\partial q_1 = \frac{2l}{\mu} \frac{\cos \nu}{1+\epsilon \cos \nu} + \frac{\mu}{\epsilon} \frac{-\sin \nu}{(1+\epsilon \cos \nu)^2} \partial \nu
\]

\[
\partial q_2 = \frac{2l}{\mu} \frac{\sin \nu}{1+\epsilon \cos \nu} + \frac{\mu}{\epsilon} \frac{\epsilon + \cos \nu}{(1+\epsilon \cos \nu)^2} \partial \nu
\]

\[
\partial q_1 = \frac{-l^2}{\mu} \frac{\cos \nu}{1+\epsilon \cos \nu} + \frac{\epsilon + \cos \nu}{(1+\epsilon \cos \nu)^2} \partial \nu
\]

\[
\partial q_2 = -\frac{l^2}{\mu} \frac{\cos \nu}{(1+\epsilon \cos \nu)^2} \sin \nu + \frac{\mu}{\epsilon} \frac{-\sin \nu}{(1+\epsilon \cos \nu)^2} \partial \nu
\]

\[
\partial q_1 = -\frac{l^2}{\mu} \frac{\cos \nu}{1+\epsilon \cos \nu} + \frac{l^2}{\mu} \frac{1}{1+\epsilon \cos \nu} \sin \nu \partial \nu
\]

\[
\partial q_2 = -\frac{l^2}{\mu} \frac{\sin \nu}{1+\epsilon \cos \nu} + \frac{l^2}{\mu} \frac{1}{1+\epsilon \cos \nu} \sin \nu \partial \nu
\]

To obtain the partial derivatives of \( \nu \) the variables are separated of the differential equation of \( \nu = \nu(t) \):

\[
\nu = \frac{\mu^2}{l^3} (1 + \epsilon \cos \nu)^2,
\]
which follows from conservation of angular momentum: \( q^2 \nu = l \), to obtain

\[
\frac{2}{(1 - e^2)^{3/2}} \arctan \left( \frac{(1 - e)(1 - \cos \nu)}{\sqrt{1 - e^2} \sin \nu} \right) = \frac{\mu^2}{l^3} (t - t_0) + \frac{\epsilon}{(1 - e^2)(1 + e \cos \nu)} \sin \nu.
\]

This equality extends to \( e > 1 \) upon replacing \( \arctan \) with \( \arctanh \). Differentiation of (3.25) with respect to \( l \) gives:

\[
\partial_l \nu = -\frac{3}{l^3} \left( 1 + e \cos \nu \right)^2 (t - t_0),
\]

which, by continuity of \( \nu \) as a function of \( e \) and \( l \), extends to \( e = 1, l \neq 0 \). Next, for \( e \neq 1 \) differentiation with respect to \( \epsilon \) gives, after some manipulations:

\[
\partial_\epsilon \nu = -\frac{3 \epsilon}{1 - e^2} \frac{\mu^2}{l^3} (1 + e \cos \nu)^2 (t - t_0) + \frac{\sin \nu (2 + e \cos \nu)}{1 - e^2}, \quad e \neq 1. \]

Notice the singularities for \( e = 1 \) and \( l = 0 \). These cases will be returned to in section 3.4.2. Similarly, differentiation of (3.25) with respect to \( \mu \) gives:

\[
\partial_\mu \nu = \frac{2 \mu^2}{l^3} (1 + e \cos \nu)^2 (t - t_0). \]

Using (3.19) and (3.20) it follows that

\[
(\partial q_1 + \partial q_2) \frac{\delta l_3}{l} = 2 \left( q - \frac{3}{2} p(t - t_0) \right) \frac{\delta l_3}{l},
\]

\[
(\partial q_1 + \partial q_2) \frac{\delta \epsilon_1}{l} = -\frac{3 \epsilon}{1 - e^2} p(t - t_0) \delta \epsilon_1
\]

\[
+ \frac{l^3}{(1 + e \cos \nu)^2} \left( e^2 \cos^2 \nu - 1 - \sin^2 \nu (1 + e \cos \nu) \right) \frac{1}{1 - e^2} \delta \epsilon_1
\]

\[
+ \frac{l^3}{(1 + e \cos \nu)^2} \sin \nu \left( e + \cos \nu \right) (2 + e \cos \nu) - \cos \nu (1 - e^2) \right) \frac{1}{1 - e^2} \delta \epsilon_1,
\]

\[
(\partial q_1 + \partial q_2) \frac{\delta \mu}{\mu} = (q - 2p(t - t_0)) \frac{-\delta \mu}{\mu}. \]

By (3.16):

\[
-3p(t - t_0) \frac{\delta l_3}{l} = \frac{3 \epsilon}{1 - e^2} p(t - t_0) \delta \epsilon_1 = \frac{3}{2} p(t - t_0) \frac{\delta h}{h} - 3p(t - t_0) \frac{\delta \mu}{\mu},
\]

\[
2q \frac{\delta l_3}{l} = q \frac{-\delta h}{h} - 2q \frac{\epsilon \delta \epsilon_1}{1 - e^2} + 2q \frac{\delta \mu}{\mu}. \]
so that, in terms of $\delta h$ and $\delta \epsilon_1$, the sum of (3.30), (3.29) and (3.31) become:

$$
\left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta i_3 + \left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta \epsilon_1 + \left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta \mu \\
= \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} - \frac{2}{1 - \epsilon^2} q \delta \epsilon_1 + \left( -2q + 3p(t - t_0) \right) \frac{-\delta \mu}{\mu} \\
+ \frac{l^2}{1 + \epsilon \cos \nu} \left( \epsilon^2 \cos^2 \nu - 1 - \sin^2 \nu(1 + \epsilon \cos \nu) \right) \frac{\delta \epsilon_1}{1 - \epsilon^2} \\
+ \frac{l^2}{1 + \epsilon \cos \nu} \sin \nu \left( (\epsilon + \cos \nu)(2 + \epsilon \cos \nu) - \cos \nu(1 - \epsilon^2) \right) \frac{\delta \epsilon_1}{1 - \epsilon^2} \\
+ (q - 2p(t - t_0)) \frac{-\delta \mu}{\mu} \\
= \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} - \frac{l^2}{\mu} \left( 1 + \frac{\sin^2 \nu}{1 + \epsilon \cos \nu} \right) \frac{\delta \epsilon_1}{1 - \epsilon^2} \\
+ \frac{l^2}{1 + \epsilon \cos \nu} \sin \nu \cos \nu \frac{\delta \epsilon_1}{1 - \epsilon^2} + (q + p(t - t_0)) \frac{-\delta \mu}{\mu},
$$

or compactly:

$$
\left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta i_3 + \left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta \epsilon_1 + \left( \frac{d}{dt} q_1 + \frac{d}{dt} q_2 \right) \delta \mu \\
= \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} - \frac{l^2}{\mu} \left( 1 + \frac{\sin^2 \nu}{1 + \epsilon \cos \nu} \right) \frac{\delta \epsilon_1}{1 - \epsilon^2} \\
+ (q + p(t - t_0)) \frac{-\delta \mu}{\mu}.
$$

For $\epsilon \neq 0, 1$ the variational solution has therefore been obtained:

$$
\delta q(t; \delta t_0, \delta \epsilon, \delta l, \delta h) = -p \delta t_0 + k \wedge q \frac{\delta \epsilon_2}{\epsilon} + j \wedge q \frac{\delta l_1}{l} + i \wedge q \frac{\delta l_2}{l} \\
+ \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} - \left( \frac{l^2}{\mu} i + \sin \nu q \wedge k \right) \frac{1}{1 - \epsilon^2} \delta \epsilon_1 \\
+ (q + p(t - t_0)) \frac{-\delta \mu}{\mu}.
$$

### 3.4.1 Circular reference orbits: $\epsilon = 0$

When $\epsilon = 0$ then $i$ and $j$ are not well-defined as the circular motion is completely isotropic, and hence variations $\delta i$ and $\delta j$ are singular in $\epsilon$. As a consequence, it may also be noted that for $\epsilon = 0$, $l \wedge q = \frac{l}{\mu} p$ and the solution obtained by differentiating the vector field along the reference orbit, (3.14), is therefore proportional to the solution corresponding to variations in $\epsilon_2$ which appears as the second term on the
right hand side of (3.32). However, $k$ is still well-defined and defines the orbital plane in which any orthonormal basis $\{i, \lambda\} \subset P$ (3.1) may be chosen. To account for variations in $\epsilon$ at $0$ two independent variations may be taken e.g. along the direction of $i$ and $\lambda$. In detail, two eccentric orbits with eccentricity vector pointing in the direction of $i$ and $\lambda$:

\[
q = \frac{l^2/\mu}{1 + \epsilon \cos \nu} (\cos \nu_i + \sin \nu \lambda),
\]

respectively

\[
q = \frac{l^2/\mu}{1 + \epsilon \sin \nu} (\cos \nu_i + \sin \nu \lambda),
\]

are considered, see also Fig. 3.2, and by taking linear variations with respect to $\epsilon$ at $\epsilon = 0$ it follows that

\[
\delta q = \frac{l^2}{\mu} \left( (1 + \sin^2 \nu) \delta \epsilon_1 + \cos \nu \sin \nu \delta \epsilon_2 \right) i
\]

\[
+ \frac{l^2}{\mu} (\cos \nu \sin \nu \delta \epsilon_1 - (1 + \cos^2 \nu) \delta \epsilon_2) \lambda.
\]

Figure 3.2: Two elliptic sections, whose eccentricity vectors are mutually orthogonal in the orbital plane, near a circular orbit.

The terms in the complete solution (3.32) describing the variation of $\delta l_0$, $\delta l_1$, $\delta l_2$, $\delta h$ and $\delta \mu$ are still valid for $\epsilon = 0$ when replacing $i$ and $j$ with $i$ and $\lambda$ respectively.
Therefore for \( e = 0 \):

\[
\delta q(t; \delta t_0, \delta e, \delta l, \delta h) = -p \delta t_0 + \frac{\lambda \wedge q}{l} \delta l_0 + \frac{\nu \wedge q}{l} \delta l_2 + \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} + \frac{l^2}{\mu} (- (1 + \sin^2 \nu) \delta e_1 + \cos \nu \sin \nu \delta e_2) \nu \\
+ \frac{l^2}{\mu} (\cos \nu \sin \nu \delta e_1 - (1 + \cos^2 \nu) \delta e_2) \lambda \\
+ (-q + p(t - t_0)) \frac{-\delta \mu}{\mu}.
\]

By moving into a rotating frame and setting \( \delta \mu = 0 \) it is straightforward to show that the solutions coincide with the solutions of the Hill-Clohessy-Wiltshire equations [62].

### 3.4.2 Parabolic reference orbits: \( e = 1 \)

When \( e = 1 \) then by (3.3) either \( l = 0 \) or \( h = 0 \). If \( l = 0 \) then the motion is purely radial, \( \dot{\nu} = 0 \). Let us initially assume that \( l \neq 0 \) and therefore \( h = 0 \). The solution in (3.32) does not extend trivially to \( e = 1 \) due to the apparent singularity in \( \partial_\nu \) (3.27). However, the singularity is removable by continuity of \( \nu \) as a function of \( l \) and \( e \). By successive application of L'Hôpital's rule to (3.25), we arrive at:

\[
\partial_\nu |_{e=1} = \frac{2 \sin \nu (1 + 3 \cos \nu + \cos^2 \nu)}{5 (1 + \cos \nu)^2}.
\]

Inserting this into (3.21) and (3.22) the variations in \( q \) due to variations in \( e_1 \) is then obtained:

\[
(\partial_\nu q_1 + \partial_\nu q_2)|_{e=1} \delta e_1 = \frac{l^2}{\mu} \left( 2 + 4 \cos \nu + \cos^2 \nu - 2 \cos^3 \nu \right) \nu \delta e_1 \\
+ \frac{l^2}{\mu} \left( \sin \nu (\cos \nu + 2(1 + \cos^2 \nu)) \right) \nu \delta e_1.
\]

On the other hand, \( \partial_\nu \) (3.26) extends trivially:

\[
\partial_\nu |_{e=1} = \frac{3 \mu^2}{l^3} (1 + \cos \nu)^2 (t - t_0),
\]

and so does the variations with respect \( I_3 \) (3.29):

\[
(\partial_\nu q_1 + \partial_\nu q_2)|_{e=1} \delta I_3 = 2 \left( q - \frac{3}{2} p(t - t_0) \right) \frac{\delta I_3}{l}.
\]
For $\epsilon = 1$, $h = 0$, $l \neq 0$ the following has therefore been constructed:
\[
\delta \mathbf{q}(t; \delta t_0, \delta \epsilon, \delta l, \delta h) = -\mathbf{p} \delta t_0 + k \wedge q \delta \epsilon_2 + j \wedge q \frac{\delta l_1}{l} + i \wedge q \frac{\delta l_2}{l} \\
+ 2 \left(q - \frac{3}{2}p(t - t_0)\right) \frac{\delta l_3}{l} \\
- \frac{t^4/\mu^3}{5(1 + \cos \nu)^2} (2 + 4 \cos \nu + \cos^2 \nu - 2 \cos \nu) i \delta h \\
+ \frac{t^4/\mu^3}{5(1 + \cos \nu)^2} \sin \nu (\cos \nu + 2(1 + \cos^2 \nu)) j \delta h \\
+ (q - 2p(t - t_0)) \frac{-\delta \mu}{\mu}.
\]
(3.34)

where it has been used that for $\epsilon = 1$ and $h = 0$ we have $\delta \epsilon_1 = \frac{t^2}{\mu}\delta h$.

Next, let $l = 0$ and $h \neq 0$. Then $\mathbf{q}$ is purely radial, having only a component in the direction of the eccentricity vector, say $\mathbf{q} = q_1 \epsilon = q_1 l$, $q_1 > 0$, and may, due to the possibility of collision with the central body only exist for finite time. Letting $(j, k)$ be an orthonormal basis for the plane perpendicular to the $l = \epsilon$ direction it again can be seen that $j \wedge \mathbf{q}$ and $k \wedge \mathbf{q}$ are independent solutions corresponding to variations in $\delta \epsilon_2$ and $\delta \epsilon_3$ respectively. By insertion one may also verify that $(q - \frac{3}{2}p(t - t_0)) \frac{-\delta h}{\mu}$ and $(q - 2p(t - t_0)) \frac{-\delta \mu}{\mu}$ are solutions. Therefore, for $l = 0$ and $h \neq 0$, five linearly independent solutions of the variational problem have been obtained:
\[
\delta \mathbf{q} = -\mathbf{p} \delta t_0 - k \wedge q \delta \epsilon_2 + j \wedge q \delta \epsilon_3 + (q - 3 \frac{3}{2}p(t - t_0)) \frac{-\delta h}{h} \\
+ (q - 2p(t - t_0)) \frac{-\delta \mu}{\mu}.
\]
(3.35)

To obtain the full solution of the linear problem:
\[
\delta \mathbf{q} = \delta \mathbf{p},
\]
\[
\delta \mathbf{p} = \frac{\mu}{q_1^3} \text{diag}(2, -1, -1) \delta \mathbf{q} + \frac{-\delta \mu}{q_1^3},
\]
(3.36)

only two additional, linearly independent solutions are required. For these let us assume $\delta \mathbf{q} = (0, \delta q_2, 0)$, $\delta \mathbf{p} = (0, \delta p_2, 0)$. Then $\delta \mathbf{L}$ explicitly reads:
\[
\delta \mathbf{L} = (0, 0, q_1 \delta p_2 - p_1 \delta q_2).
\]

From the conservation of $\delta \mathbf{L} = (0, 0, \delta l_3)$, using $\delta p_2 = \delta q_2$, a first order, ordinary differential equation for $\delta q_2$ is obtained:
\[
\dot{\delta q}_2 = \frac{p_1}{q_1} \delta q_2 + \frac{\delta l_3}{q_1} \\
= d_i (\log q_1) \delta q_2 + \frac{\delta l_3}{q_1}.
\]
3.4 Solutions of Kepler’s variational equations

Particular and homogeneous solutions are given by \( \delta q_1 = \int q_1^{-2} dt \) and \( q_1 \) respectively. The homogeneous solution, \( \delta q = (0, q_1, 0) \), already enters (3.35) in the form \( k \land q \) and the particular solution will therefore suffice. By assuming \( \delta q = (0, 0, \delta q_3) \) the solution \( \delta q_3 = \delta l_2 q_1 / q_1^{-2} dt \) is similarly obtained. It follows:

\[
\delta q = -p \delta t_0 - k \land q \delta c_2 + j \land q \delta c_3 + (q - 3/2 p (t - t_0)) - \frac{\delta h}{h} \\
+ \delta l_3 (0, q_1 \int q_1^{-3} dt, 0) + \delta l_2 (0, 0, q_1 \int q_1^{-3} dt). \tag{3.37}
\]

For \( h = 0 \) the reduced equations for \( q = (q_1, 0, 0), p = (q_1, 0, 0) \):

\[
\frac{q_1^2}{q_1} = -2 \frac{\mu}{q_1} = 0,
\]

may be explicitly solved by separation of variable to obtain:

\[
q_1(t) = \left( q_1(t_0) \pm \frac{3}{2} \sqrt{2 \mu (t - t_0)} \right)^{2/3}. \tag{3.38}
\]

This only exists for \( t - t_0 \in (-\frac{2}{3 \sqrt{2 \mu}} q_1(t_0), \infty) \) and \( t - t_0 \in (-\infty, \frac{2}{3 \sqrt{2 \mu}} q_1(t_0)) \). For radially expanding and contracting orbits take + and − in (3.38), respectively. From this it follows that the two variational solutions:

\[
(q - 3/2 p (t - t_0)) = \frac{2 q_1(t_0)}{(\pm 12 \sqrt{2 \mu (t - t_0) + 8 q_1(t_0)})^{1/3}}, \tag{3.39}
\]

and

\[
P = \frac{2 \sqrt{2 \mu}}{(\pm 12 \sqrt{2 \mu (t - t_0) + 8 q_1(t_0)})^{1/3}},
\]

are proportional to each other. To obtain a seventh linearly independent solution (3.9) is solved for the conservation of \( \delta h \):

\[
\delta h = p_1 \delta p_1 + \frac{\mu}{q_1} \delta q_1,
\]

\[
\delta q_1 = -\frac{\mu}{p_1 q_1^3} \delta q_1 + \frac{\delta h}{p_1} = \frac{p_1 \delta q_1 + \delta h}{p_1} = \delta_1 (p_1) \delta q_1 + \frac{\delta h}{p_1}.
\]

The homogeneous solution space is spanned by \( \delta q_1 = p_1 \) and by separation an inhomogeneous solution is obtained:

\[
\delta q_1 = p_1 \int p_1^{-2} dt \delta h = \frac{1}{80 \mu} \left( 12 \sqrt{2 \mu (t - t_0) + 8 q_1(t_0)} \right) 4/3 \delta h.
\]
3.4.3 Summary of results

The conclusions can now be collected in the following theorem:

**Theorem 1** The complete solution of the variational equations of Kepler's problem for $\epsilon \neq 0,1$ with

$$q = \frac{l^2/\mu}{1 + \epsilon \cos \nu} \cos i + \frac{l^2/\mu}{1 + \epsilon \cos \nu} \sin j, \tag{3.40}$$
$$p = -\frac{\mu}{l} \sin \nu i + \frac{\mu}{l}(\epsilon + \cos \nu)j, \tag{3.41}$$

is

$$\delta q(t; \delta t_0, \delta \epsilon, \delta I, \delta h) = -p \delta t_0 + k \wedge \frac{\delta c_2}{\epsilon} + j \wedge \frac{\delta l_1}{l} + i \wedge \frac{\delta l_2}{l}$$
$$+ \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} - \left( \frac{l^2}{\mu} i + \sin \nu j \wedge k \right) \frac{1}{1 - \epsilon^2} \delta \epsilon_1$$
$$+ (-q + p(t - t_0)) \frac{-\delta \mu}{\mu}, \tag{3.42}$$

which lifts to

$$\delta p(t; \delta t_0, \delta \epsilon, \delta I, \delta h) = \frac{\mu}{|q|^3} q(t - t_0) \wedge k + \frac{\delta c_2}{\epsilon} + j \wedge \frac{\delta l_1}{l} + i \wedge \frac{\delta l_2}{l}$$
$$- \frac{1}{2} \left( p - 3 \frac{\mu}{|q|^3} q(t - t_0) \right) \frac{-\delta h}{h} - \left( \sin \nu p \wedge k + \mu^2/|l^3(1 + \epsilon \cos \nu)^2 \cos \nu j \wedge k \right) \frac{1}{1 - \epsilon^2} \delta \epsilon_1$$
$$- \frac{\mu}{|q|^3} q(t - t_0) \frac{-\delta \mu}{\mu}, \tag{3.43}$$

where the subscripts 1, 2 and 3 refer to the coordinates of $i = \frac{\xi}{\epsilon}$, $j = \frac{\xi \omega}{\epsilon \Omega}$ and $k = \frac{I}{l}$ respectively.
For a circular reference orbit ($e = 0$):

$$
\delta q = -p \delta t_0 + \frac{\lambda}{l} q \delta l_1 + \frac{t}{l} q \delta l_2 + \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} + \frac{l^2}{\mu} \left[ \left( \cos \nu \sin \nu \delta \epsilon_1 \right) \right] \frac{\delta \mu}{\mu} \\
+ \frac{l^2}{\mu} \left( \cos \nu \sin \nu \delta \epsilon_1 - (1 + \cos^2 \nu) \delta \epsilon_2 \right) \lambda \\
+ \left[ \left( -q + p(t - t_0) \right) \right] \frac{-\delta \mu}{\mu},
$$

(3.44)

$$
\delta p = \frac{\mu}{|q|^3} q \delta t_0 + \frac{\lambda}{l} p \delta l_1 + \frac{t}{l} p \delta l_2 - \frac{1}{2} \left( p - \frac{3}{2} \mu q(t - t_0) \right) \frac{-\delta h}{h} + \frac{l^2}{\mu} \left[ \left( \cos \nu \sin \nu \delta \epsilon_1 + \left( \cos^2 \nu - \sin^2 \nu \right) \delta \epsilon_2 \right) \right] \lambda \\
+ \frac{l^2}{\mu} \left[ \left( \cos \nu \sin \nu \delta \epsilon_1 + 2 \cos \nu \sin \nu \delta \epsilon_2 \right) \lambda \right] \\
- \frac{\mu}{|q|^3} \frac{-\delta \mu}{\mu}.
$$

(3.45)

where $(\epsilon, \lambda) \subset \mathcal{P}$ is any orthonormal basis in the orbital plane and subscripts 1 and 2 now refer to coordinates with respect to $\epsilon$ and $\lambda$ respectively.

For a parabolic reference orbit where $e = 1$, $l \neq 0$ and $h = 0$:

$$
\delta q = -p \delta t_0 + k q \delta \epsilon_2 + j \frac{\delta l_1}{l} + i \frac{\delta l_2}{l} \\
+ \frac{2}{l} \left( q - \frac{3}{2} p(t - t_0) \right) \frac{\delta l_3}{l} + \frac{l^4}{\mu^3} \left( \frac{2 - 2 \cos \nu + \cos^2 \nu - 2 \cos^3 \nu}{2(1 + \cos \nu)} \right) \frac{i \delta h}{h} \\
+ \frac{l^4}{\mu^3} \left( \frac{2 - 2 \cos \nu + \cos^2 \nu}{2(1 + \cos \nu)} \right) \frac{j \delta h}{h} \\
+ \left( p - 2 \mu q(t - t_0) \right) \frac{-\delta \mu}{\mu},
$$

(3.46)

$$
\delta p = \frac{\mu}{|q|^3} q \delta t_0 + k \frac{p \delta \epsilon_2}{l} + j \frac{p \delta l_1}{l} + i \frac{p \delta l_2}{l} \\
- \left( \frac{p - 3 \mu q(t - t_0)}{|q|^3} \right) \frac{\delta l_3}{l} + \frac{l^4}{\mu^3} \left( \frac{2 \sin \nu \cos \nu (1 + 3 \cos \nu + \cos^2 \nu)}{2 \sin \nu \cos \nu (1 + \cos \nu)} \right) \frac{i \delta h}{h} \\
+ \frac{1}{\mu} \left( 3 - 2 \cos \nu \left( 2 - 2 \cos \nu - \cos^2 \nu \right) \right) \frac{j \delta h}{h} \\
- \left( \frac{\mu q(t - t_0)}{|q|^3} \right) \frac{-\delta \mu}{\mu}.
$$

(3.47)
For a purely radial reference orbit where \( e = 1, l = 0 \) and \( h \neq 0 \):

\[
\begin{align*}
\delta q &= -p\delta t_0 - k\wedge q\delta c_2 + j\wedge q\delta c_3 + \left( q - \frac{3}{2} p(t - t_0) \right) \frac{-\delta h}{h} \\
&+ \delta l_3(0, q_1 \int q_1^{-2} dt, 0) + \delta l_2(0, 0, q_1 \int q_1^{-2} dt) \\
&+ (q - 2p(t - t_0)) \frac{-\delta \mu}{\mu} \\
\delta p &= \left[ \frac{\mu}{q^2} q\delta t_0 - k\wedge p\delta c_2 + j\wedge p\delta c_3 - \frac{1}{2} \left( p - 3 \frac{\mu}{q^2} q(t - t_0) \right) \frac{-\delta h}{h} \right] \\
&+ \delta l_3(0, p_1 \int p_1^{-2} dt, 0, 0) + \delta l_2(0, 0, p_1 \int p_1^{-2} dt, 0) \\
&- \left( p - 2\frac{\mu}{q^2} q(t - t_0) \right) \frac{-\delta \mu}{\mu}.
\end{align*}
\]

(3.48)

When \( e = 1, l = 0 \) and \( h = 0 \) then \( q - \frac{3}{2}p(t - t_0) \frac{-\delta h}{h} \) in (3.48) is replaced with \( \delta h(p_1 \int p_1^{-2} dt, 0, 0), \int p_1^{-2} dt = \frac{1}{8\mu} \left( \pm 12\sqrt{2\mu(t - t_0)} + 8q(t_0) \right)^{4/3} \) where + and - refer to radially expanding and contracting reference solutions respectively.

\( \square \)

For \( 0 < e < 1 \) and \( \delta \mu = 0 \) the solutions are in agreement with the solutions obtained by Palmer and Imre in [117] (see equations (41)-(44) and note that \( A \) and \( B \) are related to the conserved quantities through: \( A = -\mu^2 \delta t_0/\mu \) and \( B = \delta c_2/\epsilon \)).

As expected, and discussed in section 3.2, only the variations in \( h \) and \( \mu \) give rise to unbounded relative motion for \( e < 1 \). The unboundedness is not physical and is only a result of our linearisation. An illustration of the effects of the variations in the six quantities \( t_0, t_1, t_2, c_2, c_1 \) and \( h \) in terms of conic sections is seen in Fig. 3.3 for \( e < 1 \). For \( e \geq 1 \) the reference trajectory \((q, p)\) is no longer periodic, in fact \(|q(t)| \to \infty\) as \( t \to \infty \) for \( e > 1 \) and \( e = 1 \) with \( h \geq 0 \). Therefore, the variational solutions around hyperbolic and parabolic orbits may grow unbounded even if \( \delta h = 0 \) and \( \delta \mu = 0 \). Moreover, the implication of infinitesimal difference in energies for the case of a parabolic and hyperbolic reference is no longer straightforward. For parabolic and hyperbolic orbits the velocities converge to \( 0 \) or a constant \( \pm v_\infty \) respectively as \( t \to \pm \infty \), cf. e.g. (3.41). Hence \( q - \frac{3}{2}p(t - t_0) \), the variational solution corresponding to variations in \( h \), can either approach \( \pm \infty \) or \( 0 \) when \( t \to \pm \infty \).

For sufficiently large \( t \) the azimuthal variation may be neglected since \( \nu \to 0 \) for \( \nu \to \nu^* \), where \( \nu^* \) is such that \( 1 + \epsilon \cos \nu^* = 0 \), and the drift to infinity is asymptotically radial. Similarly for \( t \to -\infty \). For \( h = 0 \) it then directly follows from (3.39) that

\[
|q - \frac{3}{2}p(t - t_0)| \to 0, \quad \text{for} \quad t \to \infty, \ h = 0,
\]
3.5 Synchrone and syndyne dust comet tails

for \( t \to \pm \infty \). For \( h > 0 \), \( \lim_{t \to -\infty} \psi = v_\infty = \sqrt{2h} < \infty \) and hence \( |q| - \sqrt{2h}t \to \text{const.} \) for \( t \to \infty \), so that

\[
\left| q - \frac{3}{2} p(t - t_0) \right| - \frac{\delta h}{h} - \frac{|\delta h|}{\sqrt{2h}} t \to \text{const.} \quad \text{for} \quad t \to \infty, \quad h > 0, \quad \delta h \neq 0.
\]

It follows that only for parabolic reference orbits will neighbouring satellites, whose orbits are only separated by an infinitesimal difference in energy, catch up at infinity.

In the following section, it is shown how these solutions can be used to obtain analytical solution of synchrone and syndyne dust tails [47]. Synchrone and syndyne dust tails are the respective tails due to variations only in particle sizes and in the time of which the particles left the nucleus.

3.5 Synchrone and syndyne dust comet tails

Many comets move on almost parabolic orbits [47], so we therefore restrict attention to \( \varepsilon = 1 \). Furthermore, comet tails are relatively flat compared to their sizes in the orbital plane of the comet. It is therefore reasonable to approximate the synchrone and syndyne tails as planar so that \( \delta l_1 = 0 = \delta l_2 \).

Since the dust particles all are assumed to originate from the nucleus we have to add the initial condition

\[
\delta q|_{v=v_0} = 0.
\]

(3.50)

Then if \( \delta q|_{v=v_0} = 0 \) with \( \delta p|_{v=v_0} = \delta \mathbf{p}_0 \) for \( v_0 \in (-\pi, \pi) \) (3.9), (3.10) and (3.11) give:

\[
\delta l = (k, q_0 \wedge \delta \mathbf{p}_0), \tag{3.51}
\]

\[
\delta h = (p_0, \delta \mathbf{p}_0) - \frac{\delta \mu}{\theta_0}, \tag{3.52}
\]

\[
\delta \epsilon_2 = -\frac{1}{\mu} (\delta \mathbf{p}_0 + \left( \frac{\delta l}{l} - \frac{\delta \mu}{\mu} \right) p_0, l), \tag{3.53}
\]
such that (3.50) together with (3.34) after some manipulations yield:

\[
\frac{\mu^2}{l^3} \delta t_0 = A(\nu_0) \delta \epsilon_2 + B(\nu_0) \frac{\delta l_3}{l} + C(\nu_0) \frac{-\delta \mu}{\mu}, \tag{3.54}
\]

\[
A(\nu_0) = -\frac{2 - \cos \nu_0 - 4 \cos^2 \nu_0 - 2 \cos^3 \nu_0}{5(1 + \cos \nu_0)^2 \cos \nu_0},
\]

\[
B(\nu_0) = \frac{4}{5} \tan \nu_0 + \frac{4}{5} \frac{\sin \nu_0}{(1 + \cos \nu_0)^2},
\]

\[
C(\nu_0) = \frac{2}{5} \tan \nu_0 + \frac{2}{5} \frac{\sin \nu_0}{(1 + \cos \nu_0)^2} = \frac{1}{2} B(\nu_0),
\]

valid for \(\nu_0 \neq \pm \pi/2\). In obtaining this expression we have inverted \(\delta \epsilon_2\) (3.53) for the component of \(\delta p_0\) parallel to \(q_0\) which is not possible for \(\nu_0 = \pm \pi/2\). For \(\nu_0 = \pm \pi/2\) one may instead invert \(\delta h\) (3.52) for this component. After some manipulations one then obtains:

\[
\frac{\mu^2}{l^3} \delta t_0 = D(\nu_0) \frac{l^2}{\mu^2} \delta h + E(\nu_0) \frac{\delta l}{l} + F(\nu_0) \frac{-\delta \mu}{\mu},
\]

\[
D(\nu_0) = \frac{2 - \cos \nu_0 - 4 \cos^2 \nu_0 - 2 \cos^3 \nu_0}{5(1 + \cos \nu_0)^2 \sin \nu_0},
\]

\[
E(\nu_0) = \frac{2l^2}{(1 + \cos \nu_0)^2 \sin \nu_0},
\]

\[
F(\nu_0) = \frac{l^3}{(1 + \cos \nu_0)^2 \sin \nu_0} = \frac{1}{2} E(\nu_0),
\]

now valid for \(\nu_0 \neq 0\). Upon inserting (3.54) into (3.34) the solution satisfying (3.50) with \(\nu_0 \neq \pm \pi/2\) we obtain

\[
\delta q = \left( \kappa \times q - \frac{l^3}{\mu^2} A(\nu_0) p \right) \delta \epsilon_2
\]

\[
+ \left( 2 \left( q - \frac{3}{2} p(t - t_0) - \frac{l^3}{\mu^2} B(\nu_0) p \right) \delta l_3 \right)
\]

\[
- \frac{l^4/\mu^5}{5(1 + \cos \nu)^2} \left( 2 + 4 \cos \nu + \cos^2 \nu - 2 \cos^3 \nu \right) j \delta h
\]

\[
+ \frac{l^4/\mu^5}{5(1 + \cos \nu)^2} \sin \nu \left( \cos \nu + 2(1 + \cos^2 \nu) \right) j \delta h
\]

\[
+ \left( (q - 2p(t - t_0)) - \frac{l^3}{\mu^2} C(\nu_0) p \right) \frac{-\delta \mu}{\mu}.
\]

Particularly for syndyne and synchronc tails \(\delta p_0 = 0\) so that by (3.51), (3.52)
3.6 Design of formation flying trajectories

and (3.53) it follows:

\[
\delta q = \left( q \wedge k + \frac{r^3}{\mu^2} A(v_0)p \right) \frac{l}{\mu}(p_0, i)
\]

\[- \frac{l^2/\mu}{5(1 + \cos \nu)} (2 + 4 \cos \nu + \cos^2 \nu - 2 \cos^3 \nu) i
\]

\[+ \frac{l^2/\mu}{5(1 + \cos \nu)} \sin \nu (\cos \nu + 2(1 + \cos^2 \nu)) j
\]

\[+ \left( (q - 2p(t - t_0)) - \frac{r^3}{\mu^2} C(v_0)p \right) \frac{-\delta \mu}{\mu}
\]  

(3.55)

If we view \(\delta q\) in (3.55) as a function of \(\delta \mu\) only then we obtain an analytical expression for the synchrone tail. By virtue of the linearity in \(\delta \mu\) this tail is straight and it is in the direction of positive radial and negative in-track, see also Fig. 3.4. This in good agreement with computations of the full nonlinear system, see e.g. [47]. If we instead view \(\delta q\) as a function of \(v_0\) we obtain an analytic formula for the syndyne tail. Due to the non-linearity in \(v_0\) this tail curves. Highly exaggerated examples are shown in Fig. 3.4 for two different values of \(v_0\).

In the following section, the usefulness of having solutions that are easily interpreted geometrically is demonstrated in the design of formation flying trajectories.

### 3.6 Design of formation flying trajectories

Here trajectories of four satellites are constructed so they in position space form a regular tetrahedral formation. Any inter-satellite formation will, at least in general, be distorted over the period of the orbit. The aim is therefore to obtain the tetrahedral formation at a certain stage of the orbit and, in the theme of the paper, it will be constructed by determining the required values of the conserved quantities. An application might be a mission orbiting the Sun to visit the Kuiper belt.

The Cluster mission II is currently flying and operating successfully in its ninth year exploiting the benefits of a tetrahedral formation. Furthermore, several future missions are planning to use such formations. This interest is due to the fact that the tetrahedral formations allow for resolving three dimensional gradients, for example in the context of space weather and environment. This mission scenario therefore provides a realistic example where the use of our solutions can be demonstrated.

The formation shall be chosen so that the four satellites are all at aphelion when at the Kuiper belt and, perhaps most importantly, all reach their aphelion at the same
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time. The former will maximise the duration of the formation, whereas the latter is enforced by \( \delta t_0 = 0 \) and fixing the energy \( h \) or, equivalently, the semi-major axis \( a \) cf. (3.4). Note, however, that \( \delta t_0 \neq 0, \delta h \neq 0 \) would allow for different injection times which in certain applications might be useful.

Changing \( l_2 \), within the first order approximation, does not change anything at aphelion and perihelion, see e.g. Fig. 3.3, and therefore, at least initially, \( \delta l_2 \) is equated to 0. Therefore only the three variations \( \delta e_1, \delta e_2, \) and \( \delta l_1 \) are considered, which respectively have the effect of changing the magnitude of the eccentricity, changing the orientation of the conic section within the orbital plane, and rotating the orbital plane about \( \mathbf{j} \) the direction of semilatus rectum. For \( \delta l_0 = 0, \delta h = 0 \) and \( \delta l_2 = 0 \) (3.42) reads

\[
\delta q = k \wedge q \frac{\delta e_2}{\varepsilon} + j \wedge q \frac{\delta l_1}{l} - \left( \frac{l^2}{\mu} i + \sin \nu \mathbf{q} \wedge \mathbf{k} \right) \frac{1}{1 - \varepsilon^2} \delta e_1.
\]

In particular at aphelion:

\[
\delta q|_{\nu=\pi} = -ae_{1}i - a(1+\varepsilon)\frac{\delta e_2}{\varepsilon}j + a(1+\varepsilon)\frac{\delta l_1}{l}k,
\]

where \( l^2/\mu = a(1-\varepsilon^2) \) has been used.

The configuration of the regular tetrahedron at aphelion is given by a centre point and a rotation matrix \( \mathbf{R}_{\alpha,\beta,\gamma} \in SO(3) \) mapping the inertial frame to a body fixed frame: \( \mathbf{R}_{\alpha,\beta,\gamma} : (i,j,k) \rightarrow (i',j',k') \). Let the reference orbit pass through the centre point and let \( \mathbf{R}_{\alpha,\beta,\gamma} \) depend on three Euler angles \( \alpha, \beta \) and \( \gamma \). As illustrated in Fig. 3.5, \( \alpha \) is the angle between \( i \) and the line of nodes \( n \), \( \beta \) is the angle between \( k \) and \( k' \) and finally \( \gamma \) is the angle between the line of nodes and \( \mathbf{j} \). With these definitions \( \mathbf{R}_{\alpha,\beta,\gamma} \) reads:

\[
\mathbf{R}_{\alpha,\beta,\gamma} = \begin{pmatrix}
c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & -c_\alpha s_\gamma - s_\alpha c_\beta c_\gamma & s_\beta s_\alpha \\
c_\alpha c_\gamma + s_\alpha c_\beta s_\gamma & -s_\alpha s_\gamma + c_\alpha c_\beta c_\gamma & -s_\beta c_\alpha \\
 s_\beta c_\gamma & s_\beta s_\gamma & c_\beta 
\end{pmatrix},
\]

where the usual compact notation has been used: \( s_\nu = \sin \nu \) and \( c_\nu = \cos \nu \) for every \( \nu \).

In Fig. 3.6 the tetrahedron is shown in the body frame \( (i',j',k') \). Let \( k' \) be directed through one of the vertices, which is called 1. The axes \( i' \) and \( j' \) are defined so that \( 2's \) projection onto the plane

\[\{x \in \mathbb{R}^3 | \langle x, k' \rangle = 0\},\]

\(^1\text{Aphelion and perihelion are the points on the orbit of greatest respectively least distance to the Sun.}\)
equals $\alpha^\prime$ for $c = \cos(\pi/2 - \theta^*)$, where $\theta^* = \arccos(-1/3) \approx 109.5^\circ$ is the angle between the segments joining the centre with the vertices. Here 3 is the vertex located a positive $2\pi/3$, i.e. $120^\circ$, rotation about $k'$ from 2. The four satellites are coloured respectively black, green, red and cyan in Fig. 3.6. Straightforward geometry shows that the distance from the tetrahedron centre to the vertices is $R = \sqrt{3}/2\ell$, where $\ell$ is the side-length.

Let the satellites, 1, 2, 3 and 4, be positioned at $\delta q^{(i)}$, $i = 1, 2, 3$ and 4 from the centre point. Then in the body frame it follows that

$$\delta q^{(1)} = Rk',
\delta q^{(2)} = R(\sin \theta^* Y + \cos \theta^* k'),
\delta q^{(3)} = R\left(-\frac{1}{2}\sin \theta^* i' + \frac{\sqrt{3}}{2}\sin \theta^* j' + \cos \theta^* k'\right),
\delta q^{(4)} = R\left(-\frac{1}{2}\sin \theta^* i' - \frac{\sqrt{3}}{2}\sin \theta^* j' + \cos \theta^* k'\right),$$

(3.58)

which may be related to the inertial frame through $R_{\alpha,\beta,\gamma}$.

Using (3.56), the following equations are obtained for the formation to have the configuration specified by $R_{\alpha,\beta,\gamma}$ at aphelion:

$$\langle \delta q^{(i)}, i \rangle = -a\delta e_1^{(i)},$$
$$\langle \delta q^{(i)}, j \rangle = -a(1 + \epsilon)\delta e_2^{(i)},$$
$$\langle \delta q^{(i)}, k \rangle = a(1 + \epsilon)\delta l_1^{(i)},$$

for $i = 1, 2, 3$ and 4, where $\delta l_1^{(i)}, \delta e_1^{(i)}, \delta e_2^{(i)}$ are the variations in the three constants of motion for satellite $i$. Solving the linear equations gives

$$\delta e_1^{(i)} = -\frac{1}{a}\langle \delta q^{(i)}, i \rangle,$$
$$\delta e_2^{(i)} = -\frac{\epsilon}{a(1 + \epsilon)}\langle \delta q^{(i)}, j \rangle,$$
$$\delta l_1^{(i)} = \frac{l}{a(1 + \epsilon)}\langle \delta q^{(i)}, k \rangle.$$

In Fig. 3.7, the evolution of a formation is visualised. Here it has for simplicity
been chosen $R_{\alpha, \beta, \gamma} = I$, and:

\[
\begin{align*}
\delta \epsilon_1^{(1)} &= 0, & \delta \epsilon_2^{(1)} &= 0, & \delta \ell_1^{(1)} &= \frac{Rl}{a(1 + \epsilon)}, \\
\delta \epsilon_1^{(2)} &= \frac{R \sin \theta^*}{a}, & \delta \epsilon_2^{(2)} &= 0, & \delta \ell_1^{(2)} &= \frac{R \cos \theta^*}{a(1 + \epsilon)}, \\
\delta \epsilon_1^{(3)} &= \frac{R \sin \theta^*}{2a}, & \delta \epsilon_2^{(3)} &= -\frac{\sqrt{3} \epsilon R \sin \theta^*}{2a(1 + \epsilon)}, & \delta \ell_1^{(3)} &= \frac{R \cos \theta^*}{a(1 + \epsilon)}, \\
\delta \epsilon_1^{(4)} &= \frac{R \sin \theta^*}{2a}, & \delta \epsilon_2^{(4)} &= \frac{\sqrt{3} \epsilon R \sin \theta^*}{2a(1 + \epsilon)}, & \delta \ell_1^{(4)} &= \frac{R \cos \theta^*}{a(1 + \epsilon)}.
\end{align*}
\]

The units on the axes in Fig. 3.7 are such that 1 equals the semi-major axis. Furthermore, $\epsilon = \frac{1}{2}$ and the side length of the tetrahedron is $\ell = \frac{1}{2}$ which corresponds to half the semi-major axis. This is physically unreasonable, but it is convenient in terms of the visualisation.

As expected, the formation is distorted as the satellites move away from aphelion. At perihelion the formation has, when compared to aphelion, been slightly distorted and reflected about the orbital plane. At semilatus rectum, the satellites are all in the same orbital plane, which is also in agreement with Fig. 3.3. By varying $l_2$ one may move the satellites out of this common plane, see e.g. Fig. 3.3. This variation will not change the configuration at aphelion and perihelion.

### 3.7 Discussion on perturbations

The approach developed rely on the super-integrability of the Kepler problem. Any perturbation will in general destroy the symmetries and the integrability, let alone the super-integrability. Particularly, the formation dynamics predicted by the solution above will eventually differ significantly from the true formation. However, the net effect of perturbations depends on the size of the perturbation and also on time scales. Therefore, the solutions may in fact be a good approximation to the true formation dynamics near certain reference trajectories over a large period of time. For example, the effect of the oblateness of the Earth decreases as $q$ increases as $q^{-4}$ and is therefore primarily significant for near Earth orbiting formations [66]. Moreover, for highly eccentric orbits, parabolic or hyperbolic orbits this effect is only important through the short duration flight through perigee such that a significant effect may not even be expected on a time scale of many orbital periods. In this latter scenario the third body effects might contribute with more important perturbations.

Several references account for the oblateness of the Earth, see [42, 59, 66, 130, 133, 153]. For example in [66] this perturbation was included in a numerical formation.
3.8 Conclusion

Analytical solutions of the variational equation of the Kepler problem including variations in the mass parameter about any reference orbit was obtained in a compact form by relying on the super-integrability of the Kepler problem. The solutions were written in terms of the relevant conserved quantities: relative energy, relative angular momentum and relative eccentricity vector and the geometrical setting, in which the solutions were derived, allowed for a straightforward design of tetrahedral formations on highly eccentric orbits. The inclusion of possible variations in mass parameter is thought to be very powerful in comet dust tail modelling. In the inversion of light intensity measurements for data related to comet origin, numerical propagation of millions of test particles are usually performed [50, 73, 105]. For such a large number of particles, the analytical solutions certainly relax the numerical effort significantly.
Figure 3.3: An illustration of the effects of the six independent variations in terms of conic sections for $\epsilon < 1$. The first five result in bounded relative motion whereas variation in energy results in unbounded solutions of variational equations in accordance to Kepler's third law.
3.8 Conclusion

Figure 3.4: Comet tails: a synchrone tail (---) and a syndyne tail (—). The nucleus orbit is shown via dash-dot —. The synchrone tail is straight due to the linearity of the variational solutions in $\delta \mu$. On the other hand the syndyne tail is curved due to the nonlinearity in $\nu_0$. 
Figure 3.5: Euler angles.

Figure 3.6: The tetrahedron in the body frame. The four satellites, 1, 2, 3 and 4, are coloured respectively black, green, red and cyan.
3.8 Conclusion

(a) Satellite formation at aphelion. (b) Satellite formation at semilatus rectum. (c) Satellite formation at perihelion.

Figure 3.7: A visualisation of a formation near an elliptic orbit. The formation is designed to be a regular tetrahedron at aphelion. The formation is distorted as it moves around the Sun and they will all be in the same orbital plane at semilatus rectum.
Chapter 4

Models of Tethered Satellites

4.1 Introduction

In this chapter several mathematical models of tether dynamics are presented. We consider two tethered satellites orbiting a spherical Earth, see Fig. 4.1. The tether is modelled using linear elasticity and we neglect non-conservative forces such as drag and visco-elasticity. Apart from the inclusion of bending resistance the equations obtained below are well-known in the tether literature, see e.g. [9].

4.2 Massive tether

As indicated in Fig. 4.1 the satellites are modelled as point masses positioned at $x$ and $y$ with masses $m_x$ and $m_y$, respectively. The tether is parametrised by

$$ r : [0, T] \times [0, l] \ni (t, s) \mapsto r(t, s) \in \mathbb{R}^3, $$

$l$ being the natural length of the tether. Letting $\rho_t$ denote the line density, $E$ Young's modulus and $A$ the cross sectional area the equations of the tethered system can be derived using a Lagrangian approach. Assuming linear elasticity the Lagrangian of
the usual massive tether system considered in the literature is

\[ L(w, \partial_t w) = K(\partial_t w) - P(w), \]

where

\[ w = (x, y, r), \]

and \( K \) and \( P \) are the kinetic and the potential energies of the system:

\[ K(\partial_t w) = \frac{1}{2} m_x |dx|^2 + \frac{1}{2} m_y |dy|^2 + \frac{1}{2} \rho_i \int_0^t |\partial_r r|^2 \, ds, \quad (4.1) \]

\[ P(w) = -\mu \frac{m_x}{|x|} - \mu \frac{m_y}{|y|} - \mu \rho_i \int_0^t |r|^{-1} \, ds + \frac{E A}{2} \int_0^t (|\partial_r r| - 1)^2 \, ds, \quad (4.2) \]

\( \mu \) being the Earth’s gravitational constant, \([79]\). The last term on the right of (4.2) gives Hooke’s law, see e.g. \([9]\). We furthermore impose the boundary conditions

\[ r(t, 0) = x(t) \quad \text{and} \quad r(t, 1) = y(t). \quad (4.3) \]

Hamilton’s principle states that the solution, satisfying

\[ w|_{t=0} = w_0 \quad \text{and} \quad w|_{t=T} = w_T, \quad (4.4) \]

is a critical path of the action \( S \) given by

\[ S(w) = \int_0^T L(w, \partial_t w) \, dt. \quad (4.5) \]

\(^1\) \( d_t = \frac{d}{dt} \) and \( \partial_x = \frac{\partial}{\partial x} . \)

\(^2\) \( |\cdot| \) and \( \|\cdot\| \) denote Euclidean norm and norms in infinite dimensional spaces, e.g. \( L^2 \) or Sobolev norms, respectively.
If \( w \) is classical, i.e. continuously differentiable with \( r \in C^2([0,T] \times [0,l]) \), then by Hamilton’s principle \( w \) satisfies the Euler-Lagrange equations:

\[
\begin{align*}
mx \ddot{x} &= -\mu \frac{m_x}{|x|^3} x + EA \left( a_1(|\partial_x r|)\partial_x r \right)_{s=0}, \\
my \ddot{y} &= -\mu \frac{m_y}{|y|^3} y - EA \left( a_1(|\partial_y r|)\partial_y r \right)_{s=1}, \\
\rho_1 \partial_t^2 r &= -\mu \frac{\rho_1}{|r|^3} r + EA \partial_s \left( a_1(|\partial_s r|)\partial_s r \right), \\
x &= r|_{s=0}, \\
y &= r|_{s=1},
\end{align*}
\]

where \( a_1 \) is defined by

\[
a_\zeta : \mathbb{R} \setminus \{0\} \ni x \mapsto a_\zeta(x) = \frac{x - \zeta}{x}, \quad \text{for every } \zeta > 0,
\]

with \( \zeta = 1 \). The more general \( a_\zeta, \zeta > 0 \), will later appear in the discussion and analysis of the slack-spring model. We shall discuss and analyse the assumption on \( w \) being classical in Chap. 5.

As we shall argue later, the equations are ill-posed. To regularise them we add the term

\[
B(r) = \frac{EI}{2} ||K[r]||^2,
\]

to \( P(r) \), to account for resistance against bending. Here \( I \) is the moment area of inertia, which for a circular cross-section is proportional to the fourth power of the diameter \( h \):

\[
I = \mathcal{O}(h^4),
\]

and

\[
K(r) = \frac{\partial^2 r \times \partial_s r}{||\partial_s r||^3}
\]

is the geometrical curvature. Whenever \( r \) is unit-speed parametrised then \( K(r) = ||\partial_s^2 r|| \), see e.g. [123] pp. 24-25, and therefore, by virtue of the linear elasticity assumption: \( ||\partial_s r|| \lesssim 1 \), we arrive at the approximation:

\[
B(r) \approx \frac{EI}{2} ||\partial_s^2 r||^2.
\]

From this approximation the inclusion of \( \frac{EI}{2} ||\partial_s^2 r||^2 \) in the potential gives rise to a
linear highest order differential operator in the Euler-Lagrange equations:

\[ m_x \ddot{x} = -\mu \frac{m_x}{|x|^3} x + EA \left( a_1 (|\partial_x r|) \partial_x r \right) \big|_{s=0} - EI \partial_x^2 r \big|_{s=0}, \tag{4.14} \]

\[ m_y \ddot{y} = -\mu \frac{m_y}{|y|^3} y - EA \left( a_1 (|\partial_y r|) \partial_y r \right) \big|_{s=1} + EI \partial_y^2 r \big|_{s=1}, \tag{4.15} \]

\[ \rho \ddot{r}_s = -\mu \frac{\rho}{|r|^3} r + EA \partial_s \left( a_1 (|\partial_s r|) \partial_s r \right) - EI \partial_s^2 r, \tag{4.16} \]

now equipped with the natural boundary conditions

\[ \partial_s^2 r = 0 \quad \text{for} \quad s = 0, 1. \tag{4.17} \]

The natural boundary conditions correspond to the inability of the tether to transfer bending to the hinged end-points.

The equations (4.14), (4.15), (4.16) together with (4.3), (4.17) and initial conditions

\[ w(0) = w_0, \quad \dot{w}(0) = \dot{w}_0, \]

establish the initial boundary value problem with dynamical boundaries. We shall leave the introduction of appropriate sets of initial conditions to Chap. 5. The relative equilibria of these models are not well-studied, although, a related problem is studied in [80].

To account for dissipation due to tether oscillations the Kelvin-Voigt force [78, 9] can be added to the equations. This term is simply included by replacing \( a_1 \) by

\[ \tilde{a}_1 = a_1 + \alpha |\partial_s r|^{-1} \partial_s |\partial_s r| = a_1 + \alpha |\partial_s r|^{-2} \partial_s^2 r, \tag{4.18} \]

where \( \alpha \geq 0 \) is a dissipation constant.

### 4.3 Massless tethers and the slack-spring model

In the slack-spring model the tether inertia is neglected and the tether only affects the motion when it is taut and the distance between the satellites is greater than the natural length \( l \). The direction of the tether force is directed along the relative position vector as an ideal spring with stiffness \( k = \frac{E l}{A} \). The equations are:

\[ m_x \ddot{x} = -\mu \frac{m_x}{|x|^3} x + k \tilde{a}_1 (|y - x|) (y - x), \tag{4.19} \]

\[ m_y \ddot{y} = -\mu \frac{m_y}{|y|^3} y + k \tilde{a}_1 (|y - x|) (x - y), \tag{4.20} \]
where

$$\delta_t(p) = \mathbf{1}_{\{p \geq 0\}} \delta_t(p), \text{ for every } p \geq 0,$$  \hspace{1cm} (4.21)

and $\mathbf{1}_{\{p \geq 0\}}$ is the Heaviside-function. Let $M = m_x + m_y$ be the total mass and $\mu_x = \frac{m_x}{M}$ and $\mu_y = \frac{m_y}{M} = 1 - \mu_x$ the mass ratios. Writing the Lagrangian in terms of the centre of mass and relative position coordinates:

$$q = \mu_y y + \mu_x x,$$
$$\delta q = y - x,$$

see Fig. 4.2, and applying the Legendre transformation, we end up with the Hamiltonian:

$$H_{ST}(q, \delta q, p, \delta p) = \frac{1}{2\xi} |p|^2 + \frac{1}{2} |\delta p|^2 - \frac{\mu}{\mu_x |q + \mu_x \delta q|} - \frac{\mu}{\mu_y |q - \mu_y \delta q|} + \kappa \mathbf{1}_{|\delta q| \leq l} \left( |\delta q| - l \right)^2,$$  \hspace{1cm} (4.22)

endowed with the symplectic form:

$$\omega = dq \wedge dp + d\delta q \wedge d\delta p.$$

Here ST in $H_{ST}$ stands for slack tether, $\xi = 1/(\mu_x \mu_y)$ and $\kappa = km_y m_x / 2M$.

The Hamiltonian is $SO(3)$-invariant and therefore conserves angular momentum:

$$J = q \wedge p + \delta q \wedge \delta p.$$ 

Relative equilibria of the slack-spring system are critical points of the Hamiltonian restricted to the level sets of the momentum map. To study the planar equilibria it
is beneficial to introduce the true anomaly \( \nu \) and the shape coordinate \( \theta \), which is invariant under the action of \( S^1 \), together with the two radii \( r \) and \( \delta r \), see Fig. 4.2. In particular we apply the following symplectomorphism to symplectic polar coordinates:

\[
\begin{align*}
\delta q &= r \begin{pmatrix} \cos \nu \\ \sin \nu \end{pmatrix}, \\
\delta \theta &= \delta r \begin{pmatrix} \cos(\nu + \theta) \\ \sin(\nu + \theta) \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
p_\nu &= rp \cdot (-\sin \nu, \cos \nu) + \delta r \delta p \cdot (-\sin(\nu + \theta), \cos(\nu + \theta)), \\
p_\theta &= \delta r \delta p \cdot (-\sin(\nu + \theta), \cos(\nu + \theta)), \\
p_r &= p \cdot (\cos \nu, \sin \nu), \\
p_{\delta r} &= \delta p \cdot (\cos(\nu + \theta), \sin(\nu + \theta)).
\end{align*}
\]

Then \( \nu \) becomes cyclic in the Hamiltonian:

\[
H_{ST}(r, \delta r, \nu, \theta, p_\nu, p_{\delta r}, p_r, p_\theta) = \frac{1}{2}p_\nu^2 + \frac{1}{2}p_{\delta r}^2 + \frac{1}{2}r^2(p_\nu - p_\theta)^2 + \frac{1}{2}\delta r^2p_\theta^2 \\
- \frac{1}{\mu} \sqrt{r^2 + \mu_\nu^2 \delta r^2} \cos \theta - \frac{1}{\mu_\nu} \sqrt{r^2 + \mu_\nu^2 \delta r^2} - 2\mu_\nu r \delta r \cos \theta \\
+ \kappa \delta r \delta t (\delta r - \delta t)^2,
\]

and \( J = p_\nu \in \mathbb{R} \). The study of relative equilibria and their stability becomes a straightforward, though tedious, computation. This shows that there exist two different types of relative equilibria: the tether can be either tangent or normal to the circular orbit on which the centre of mass moves, see Fig. 4.3. Due to the inability of the slack-spring to be in compression there do not exist any relative equilibria where the relative position between the satellites \( \delta q \) is perpendicular to the plane in which the centre of mass moves. Upon introducing elevation coordinates \( z \) and \( \delta z \) to the direction of \( J = (0, 0, p_\nu) \), the energy-momentum method, [99], can for realistic tether lengths, \( l \ll r \), be used to show that the relative equilibria with \( z = \delta z = 0 \) for which the system is aligned normal to a circular orbit are orbitally stable. On the other hand, the tangential relative equilibria are unstable. For more details on the stability and bifurcations when \( l = \mathcal{O}(r) \) see [82, 11].

To account for dissipation in the slack-spring model we can replace \( a \) by \( \tilde{a} = a_1 + a_2|\delta q|^{-1}d_1|\delta q| \). This is a slack-spring version of the Kelvin-Voigt model. This assumes that the system does not dissipate energy when the spring is slack. Let us consider the effect of this dissipation on an orbitally stable relative equilibrium where \( H_{ST}(J^{-1}(e), e > 0) \) is positive definite and where the system is normal to the circular orbiting centre of mass. Then the tether is stretched \( \delta r > 1 \). We may therefore consider a neighborhood of the equilibrium in which \( \delta r > 1 \). Then in the coordinates introduced above, the equations of motions with dissipation therefore coincide with Hamilton's equations except for the equation for \( p_{\delta r} \), which now reads:

\[
p_{\delta r} = -\delta \delta r H_{ST} - 2\kappa \delta p_{\delta r}.
\]
4.4 The dumbbell model

In the dumbbell model the tether is replaced by a rigid rod. The system is again Hamiltonian, now on $T^*Q$, where $Q = (\mathbb{R}^3 \times S^2_p) \setminus C$, $S^2_p = \{ q \in \mathbb{R}^3 | ||q|| = 1 \}$ and $C$ being the closed collision set. The symplectic polar coordinates introduced above for the slack-spring model are, upon fixing $\delta r = l$, also appropriate in the study of relative equilibria of the dumbbell dynamics, cf. [81, 82]. There exist three relative equilibria: tangent and normal to the circular orbiting centre of mass, as seen in Fig. 4.3, and finally an equilibrium for which the dumbbell attitude is normal to the plane defined by the $SO(3)$-orbit. The dumbbell model therefore has an additional relative equilibrium compared to the slack-spring model. A comprehensive stability analysis is provided in [81, 82].
4.5 Conclusions

In this chapter the different tether models have been presented. The models were divided into two groups: the massive tether models and the massless tether models. As opposed to previous studies the resistance against bending was taken into account. For the slack-spring model a reduction by angular momentum was presented. This system has two relative equilibria: the tether can be either tangent or normal to the circular orbit on which the centre of mass moves. For realistic tether lengths and tether stiffnesses the former is orbitally stable whereas the latter is unstable. In the following chapter it is the aim to unify these different models.


A Unification of Non-Dissipative Models of Tethered Satellites

5.1 Introduction

In this chapter, it is aimed to unify the different models presented in Chap. 4. The chapter is organised as follows: In section 5.2 the well-posedness of the massive tether models is investigated. We show that the inclusion of resistance to bending regularises the problem so that the system admits a unique strong, local solution. We also show that non-collision and non-singular parametrised solutions exist for all time. Following in section 5.3 we present a conjecture which states that in the limit of vanishing tether thickness, and for a large set of initial conditions, the solutions of the massive tether model converge to solutions of the slack-spring model. In section 5.4.2 the billiard model is then derived as the inextensible limit of the slack-spring model. This billiard model is then studied in section 5.5 using a Poincaré map. In the case that the centre of mass of the system is moving on a circular orbit we reduce this to a two dimensional symplectic map. The dumbbell dynamics is finally identified within the billiard model as persistent invariant curves.
5.2 Well-posedness of the massive tether models

In Chap. 4 it was assumed that the critical point of the action, (4.5), was classical. More often than not to establish existence in variational problems and partial differential equations it is necessary to enlarge the set of the admissible functions. With a bit of extra care the derivation of the Euler-Lagrange equations can be extended to these enlarged spaces.

In the following we shall investigate the well-posedness of the tether modelling including and neglecting the resistance against bending, $EI \neq 0$ respectively $EI = 0$. Recall that $E$ denotes Young's modulus while $I$ denotes the moment area of inertia (4.12). Despite the complete neglect of this term in the engineering literature we shall see that, at least from a mathematically point of view, its inclusion is essential.

5.2.1 $EI \neq 0$

For simplicity we set all constants to 1 and introduce $u = r - s y - (1 - s)x$ so the boundary conditions become homogeneous. Let $f(x) = -|x|^{-3}$, $x \in \mathbb{R}^3 \backslash \{0\}$. The equations (4.14), (4.15) and (4.16) then take the following form

$$
\begin{align*}
\partial_t^2 u &= -\partial_s^2 u + \partial_s \left( a_1(|\partial_r r|)\partial_r r \right) + h(u, x, y)(s), \\
\partial_t^2 x &= f(x) + a_1(|\partial_r r|)\partial_r r|_{s=0} - \partial_s^2 u|_{s=0}, \\
\partial_t^2 y &= f(y) - a_1(|\partial_r r|)\partial_r r|_{s=1} + \partial_s^2 u|_{s=1}, \\
u &= 0 = \partial_s^2 u, \quad \text{for } s = 0, 1,
\end{align*}
$$

with

$$
\begin{align*}
h(u, x, y)(s) &= f(r) - s(\partial_t^2 y + (1 - s)\partial_t^2 x) \\
&= f(r) - s(f(y) - a_1(|\partial_r r|)\partial_r r|_{s=1} + \partial_s^2 u|_{s=1}) \\
&\quad - (1 - s)(f(x) + a_1(|\partial_r r|)\partial_r r|_{s=0} - \partial_s^2 u|_{s=0}),
\end{align*}
$$

together with a set of initial conditions:

$$
\begin{align*}
u|_{t=0} = u_0 \in U, & \quad \partial_t u|_{t=0} = u_0 \in V, \\
x(0) = x_0 \in \mathbb{R}^3 \backslash \{0\}, & \quad \dot{x}(0) = \dot{x}_0 \in \mathbb{R}^3, \\
y(0) = y_0 \in \mathbb{R}^3 \backslash \{0\}, & \quad \dot{y}(0) = \dot{y}_0 \in \mathbb{R}^3.
\end{align*}
$$

Here we have introduced the following spaces

$$
\begin{align*}
U &= \{ u \in W^2((0, 1); \mathbb{R}^3) \mid u = 0 = \partial_s^2 u \text{ for } s = 0, 1 \}, \\
V &= W^2((0, 1); \mathbb{R}^3) \cap W^1_0((0, 1); \mathbb{R}^3).
\end{align*}
$$
5.2 Well-posedness of the massive tether models

Here $W^n$ is the $n$'th Sobolev space. In particular, $W^n_0$ is the completion of $C_0^\infty$ in the $W^n$-norm whose elements' first $n - 1$ (weak) derivatives all leave zero trace on the boundary, [118] Theorem 9.16 p. 123. For a comprehensive and very rigorous introduction to Sobolev spaces see [1, 56]. For a less formal description see [39]. We equip $U \times V$ with a $W^4 \times W^2$-type norm:

$$
\|(u, v)\|_{U \times V}^2 = \|\delta_x^2 v\|_{L^2((0, 1); \mathbb{R}^3)}^2 + \|\delta_x^4 u\|_{L^2((0, 1); \mathbb{R}^3)}^2.
$$

That this defines a norm on $U \times V$ follows from Poincaré's inequality [39]. Let

$$
\mathcal{S}_T = C^\infty_{L^\infty}([0, T); \mathbb{R}^3) \times C^\infty_{L^\infty}([0, T); U) \times C^\infty_{L^\infty}([0, T); \mathbb{R}^3) \times C^1_{L^\infty}([0, T); \mathbb{R}^3) \times C^1_{L^\infty}([0, T); \mathbb{R}^3)
$$

and

$$
\mathcal{X} = \left\{ (x_0, u_0, y_0, x_0, v_0, y_0)| |x_0|, |y_0|, |r_0| > 0, |\partial_x r_0| > 0, u_0 \in U, v_0 \in V \right\},
$$

where $r_0 = u_0 + sy_0 + (1-s)x_0$. Local existence, uniqueness and continuous dependence on initial conditions may then be proved:

**Theorem 2** The system of (5.1), (5.2) and (5.3) with initial conditions (5.5) admits a unique strong solution in $\mathcal{S}_T$ (5.6), for some $T = T(w_0) > 0$, that depends continuously on the initial conditions: $w_0 \in \mathcal{X}$ (5.7) within its interval of existence. If the solution satisfy

$$
|\partial_x r, |r| \geq \delta,
$$

for some $\delta > 0$, then the solution exists globally so that $T = \infty$. Finally, the solution preserves energy.

**Proof** The techniques involved are standard, see e.g. [145, 141], and we therefore only aim to give a proof of the existence. The uniqueness, continuous dependence on initial conditions and energy preservation will follow from similar estimates to those obtained below.

For this we will assume that (5.8) holds true for some (small) $\delta > 0$ and that it holds true with strict inequality at $t = 0$. In the following let $C_i$, $i \in \mathbb{N}$, be constants that only depend upon initial conditions and $\delta$. We will prove the existence by a Galerkin approximation. For this we will need to obtain a priori estimates. First, we note that from the energy conservation it follows by (5.8) that

$$
\|\delta_x^2 u\|_{L^2} \leq C_1.
$$

Next, we shall then show that this allows us to obtain a higher order a priori estimate of $(u, \partial_t u)$ in $L^\infty([0, T]; U \times V)$. Here $T > 0$ is some fixed constant. Upon dotting
the equation for \( \mathbf{r} \) by \( \partial_t \partial_x^4 \mathbf{u} \) and integrating by parts, we arrive at:

\[
\frac{1}{2} \partial_t \| \partial_r^2 \mathbf{v} \|_{L^2}^2 + \frac{1}{2} \partial_t \| \partial_r^4 \mathbf{u} \|_{L^2}^2 + (\partial_t^2 \mathbf{y} - f(\mathbf{y}), \partial_t \partial_r^2 \mathbf{u})|_{t=1} - (d_t^2 \mathbf{x} - f(\mathbf{x}), \partial_t \partial_r^4 \mathbf{u})|_{t=0} = (\partial_r^3(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}, \partial_t \partial_r^3 \mathbf{u})) + (\partial_r^3 f(\mathbf{r}), \partial_t \partial_r^3 \mathbf{u}),
\]

where \((\cdot, \cdot)\) and \((\langle \cdot, \cdot \rangle)\) are the Euclidean and \(L^2((0, 1); \mathbb{R}^3)\) inner products respectively, or simply by (5.2) and (5.3):

\[
\frac{1}{2} \partial_t \| \partial_r^2 \mathbf{v} \|_{L^2}^2 + \frac{1}{2} \partial_t \| \partial_r^4 \mathbf{u} \|_{L^2}^2 + \frac{1}{2} \partial_t |d_t^2 \mathbf{y} - f(\mathbf{y})|^2 + \frac{1}{2} \partial_t |d_t^2 \mathbf{x} - f(\mathbf{x})|^2 + (\partial_r^3(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}, \partial_t \partial_r^3 \mathbf{u})) + (\partial_r^3 f(\mathbf{r}), \partial_t \partial_r^3 \mathbf{u})
\]

\[
+ (d_t^2 \mathbf{y} - f(\mathbf{y}), \partial_t(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}|_{t=1} + (d_t^2 \mathbf{x} - f(\mathbf{x}), \partial_t(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}|_{t=0}). \tag{5.10}
\]

The equations (5.2) and (5.3) also give:

\[
\frac{1}{2} d_t |x|^2 + \frac{1}{2} d_t |x|^2 = (x, d_t x) + (f(x), d_t x) + (a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}|_{t=0} - \partial_r^2 \mathbf{u}|_{t=0}, d_t x),
\]

\[
\frac{1}{2} d_t |y|^2 + \frac{1}{2} d_t |y|^2 = (y, d_t y) + (f(y), d_t y) + (-a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}|_{t=1} + \partial_r^2 \mathbf{u}|_{t=1}, d_t y),
\]

which together with (5.10) upon consecutive applications of standard functional analytic inequalities guarantees the existence of \( C_{10} \) and \( C_{11} \) such that

\[
\frac{1}{2} \partial_t \left( \| \partial_r^2 \mathbf{v} \|_{L^2}^2 + \| \partial_r^4 \mathbf{u} \|_{L^2}^2 + |x|^2 + |d_t x|^2 + |y|^2 + |d_t y|^2 + |d_t^2 x - f(\mathbf{x})|^2 + |d_t^2 y - f(\mathbf{y})|^2 \right)
\]

\[
\leq C_{10} + C_{11} \left( \| \partial_r^2 \mathbf{v} \|_{L^2}^2 + \| \partial_r^4 \mathbf{u} \|_{L^2}^2 + |x|^2 + |d_t x|^2 + |y|^2 + |d_t y|^2 + |d_t^2 x - f(\mathbf{x})|^2 + |d_t^2 y - f(\mathbf{y})|^2 \right). \tag{5.11}
\]

The main difficulty here is to obtain the required control of the term

\[
(\partial_r^3(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r}), \partial_t \partial_r^3 \mathbf{u}).
\]

However, by (5.8) and (5.9) it follows upon applying the H"older inequality that

\[
|\partial_r^3(a_1(\partial_x \mathbf{r}) \partial_x \mathbf{r})| \leq C_2 |\partial_r^2 \mathbf{r}|^3 + C_3 |\partial_x \mathbf{r}|^2 |\partial_x^2 \mathbf{r}| + C_4 |\partial_x^3 \mathbf{r}|
\]

\[
= C_2 |\partial_r^2 \mathbf{u}|^3 + C_3 |\partial_x \mathbf{r}|^2 |\partial_x^2 \mathbf{u}| + C_4 |\partial_x^3 \mathbf{u}|, \tag{5.12}
\]

\(^{3}\text{Strictly upon extension by continuity.}\)
5.2 Well-posedness of the massive tether models

and therefore
\[
\langle (\partial_t^2(a_1(|\partial_x^r|)\partial_x^r), \partial_t \partial_x^3 u) \rangle \leq (\text{using Cauchy-Schwarz inequality in } \mathbb{R}^3)
\leq \| \partial_t^2(a_1(|\partial_x^r|)\partial_x^r) \|_{L^1} \| \partial_t \partial_x^3 u \|_{L^2}
\leq (\text{using (5.12) and Young's inequality})
\leq \frac{1}{2} C_2 \| \partial_x^2 u \|_{L^2}^2 + \frac{1}{2} C_3 \| \partial_x^2 u \|_{L^2}^2 + \frac{1}{2} C_4 \| \partial_t^3 u \|_{L^2}^2
\]

To estimate the first term on the right hand side of this inequality we use the Gagliardo-Nirenberg inequality [92] to interpolate \( L^6 \) between \( L^2 \) and \( W^2 \cap W_0^2 \):
\[
\| \partial_x^2 u \|_{L^2}^2 \leq C_5 \| \partial_x^2 u \|_{L^6}^2 \| \partial_x^3 u \|_{L^2} \leq (\text{using (5.9)}) \leq C_6 C_7 \| \partial_x^3 u \|_{L^2}.
\]

For the second term we use the embedding \( W^4 \hookrightarrow L^\infty \):
\[
\| \partial_x^2 u \|_{L^2}^2 \leq \| \partial_x^2 u \|_{L^6}^2 \| \partial_x^3 u \|_{L^6} \leq (\text{using (5.9)}) \leq C_7 \| \partial_x^3 u \|_{L^6}^2 \leq C_8 C_9 \| \partial_x^3 u \|_{L^6}^2.
\]

It therefore follows that
\[
\langle (\partial_t^2(a_1(|\partial_x^r|)\partial_x^r), \partial_t \partial_x^3 u) \rangle \leq C_7 + C_8 \| \partial_x^3 u \|_{L^2} + C_9 \| \partial_t^3 u \|_{L^2},
\]

Through Gronwall's inequality, (5.11) gives:
\[
\left( \| \partial_x^2 v \|_{L^2}^2 + \| \partial_x^3 u \|_{L^2}^2 + |x|^2 + |d_t x|^2 + |y|^2 + |d_t y|^2 + |d^2 x - f(x)|^2 + |d^2 y - f(y)|^2 \right)
\leq \left( 2C_{10} \left( \| \partial_x^2 v \|_{L^2}^2 + \| \partial_x^3 u \|_{L^2}^2 + |x|^2 + |d_t x|^2 + |y|^2 + |d_t y|^2 + |d^2 x - f(x)|^2 + |d^2 y - f(y)|^2 \right) \right) \exp(2C_{11} t).
\]

Finally, from (5.1) it follows that \( \partial_t^2 u \in L^\infty ([0,T]; L^2) \).

We are now ready to prove the existence of the solution. To do so we let \( \{ e_i \}_{i=1}^\infty \) be the orthonormal basis in \( L^2 \) generated by the eigenvectors of the self-adjoint operator \( \partial_x^2 \) defined on the space \( U \). Furthermore, we let \( \Pi_N \) be the orthoprojector to the first \( N \) eigenvectors in \( L^2 \), \( L^2_N = \Pi_N L^2 \). We write \( u_N = \Pi_N u \) and \( r_N = u_N + (1-s)x + sy \) and consider the approximation
\[
\partial_t^2 \Pi_N r_N = -\partial_t^2 r_N + \Pi_N \partial_x (a_1(|\partial_x^r|)\partial_x^r r_N) + \Pi_N f(r_N),
\]
\[
\partial_t^2 x = f(x) + a_1(|\partial_x^r|)\partial_x^r r_N \big|_{s=0} - \partial_t^3 u_N \big|_{s=0},
\]
\[
\partial_t^2 y = f(y) - a_1(|\partial_x^r|)\partial_x^r r_N \big|_{s=1} + \partial_t^3 u_N \big|_{s=1}.
\]
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This is now a finite dimensional system with smooth right hand side and the existence of the solution of the approximation therefore follows. We recall the property

$$\langle \Pi_N F, v_N \rangle = \langle F, v_N \rangle,$$

(5.16)

for every $v_N \in L^2_{\text{N}}$ and $F \in L^2$. Furthermore, if $z_N = v_N + f(s)w$, $v_N \in L^2_{\text{N}}$, $w \in \mathbb{R}^3$ and $f \in L^2((0,1);\mathbb{R})$ then

$$\langle \Pi_N F, z_N \rangle = \langle F, \Pi_N z_N \rangle$$

$$= \langle F, z_N - w\Pi_N f(s) \rangle$$

$$= \langle F, z_N \rangle - \langle F, w\Pi_N f(s) \rangle,$$

and

$$\langle F, w\Pi_N f(s) \rangle \leq \|F\|_{L^2}\|w\|\|\Pi_N f(s)\|_{L^2}$$

$$\leq \frac{1}{2} \|\Pi_N f(s)\|_{L^2} (\|F\|_{L^2}^2 + |w|^2).$$

Here the right hand side approaches 0 uniformly for $N \to \infty$. The estimates above can then with little effort be repeated to conclude that

$$\|\partial_s \partial_s^2 u_N\|_{L^2}, \|\partial_t^4 u_N\|_{L^2}, \|\partial_t^2 u_N\|_{L^2} \leq C,$$

with $C$ independent on $N$. In fact (5.11) extends identically due to (5.16). We can then pass to the limit $N \to \infty$ to conclude weak-* convergence to a $\xi = (u, \partial_t u)$ in $L^\infty([0,T];U \times V) \cap \{\partial_t^2 u \in L^\infty([0,T];L^2)\}$. However, by the compactness of the embedding:

$$L^\infty([0,T];U \times V) \cap \{\partial_t^2 u \in L^\infty([0,T];L^2)\} \subset C^0_{\infty}([0,T];V \times L^2),$$

see e.g. [141], we may actually conclude strong converge to $\xi$ in $C^0_{\infty}([0,T];V \times L^2)$. To show that this limit solves the equation we have to pass to the limit in the nonlinear term:

$$\partial_s (a_1(|\partial_s r_N|) \partial_s r_N) = M(\partial_s r_N) \partial_s^2 u_N \to M(\partial_s r) \partial_s^2 u,$$

(5.17)

where

$$M(p) = a_1(|p|)I + \frac{pp^T}{|p|^2} \in \mathbb{R}^{3 \times 3}, \quad p \in \mathbb{R}^3.$$  

(5.18)

To do so we first recall that $W^2((0,1);\mathbb{R}^3) \subset C^0_{\infty}([0,1];\mathbb{R}^3)$ and therefore $M_N = M(\partial_t r N) \to M$ by (5.8) in $C^0_{\infty}([0,1];\mathbb{R}^3)$. We write $M_N = M + \epsilon_N$ with $\epsilon_N \to 0$ in $C^0_{\infty}([0,1];\mathbb{R}^3)$ so that

$$\|M_N \partial_s^2 u_N - M \partial_s^2 u\|_{L^2} \leq \|M(\partial_s^2 u_N - \partial_s^2 u)\|_{L^2} + \|\epsilon_N\|_{L^2}\|\partial_s^2 u_N\|_{L^2} \to 0,$$
5.2 Well-posedness of the massive tether models

for $N \to \infty$. Therefore it has been shown that $M_N \partial^2_t u_N \to M \partial^2_t u$ in $L^2$ and $\xi$ is therefore a solution. By repeating the arguments in [145] it can actually be established that $\xi \in C^2_L([0,T];U \times V)$.

Now, recall that (5.8) was assumed to hold true with strict inequality at $t = 0$. Then by the continuity of $r$ and $\partial_r r$ it follows that (5.8) still holds true for $T$ sufficiently small. This completes the proof of local existence and also global existence if singularities are not encountered. $\blacksquare$

5.2.2 Well-posedness with Kelvin-Voigt dissipation.

The addition of the dissipative Kelvin-Voigt term (replacing $a_1$ by $\tilde{a}_1$ (4.18)) complicates this analysis. The Galerkin method relies on an $a$ priori $U \times V$-estimate similar to the one established above. Upon multiplying the equations by $\partial_t \partial^2_t u_T$ and integrating by parts we end up with (5.10) but with $a_1$ replaced by $\tilde{a}_1$ (4.18). However, the term $\langle \partial^2_u (\tilde{a}_1(|r_t|) \partial_t r), \partial_t \partial^2_t u \rangle$, cannot be controlled in $U \times V$ as a term including $\partial^2_t v$ appears. We need $U \times V$-estimates to control the traces appearing in the boundary equations (5.2) and (5.3). There is a lack of two weak derivatives. These issues could certainly be circumvented by the addition of a term $\partial_t \partial^3_t r$ due to bending dissipation. As we have mainly restricted attention to conservative models, this shall not be pursued further in this research.

5.2.3 $EI = 0$

For simplicity we set $\mu = 0$ and all other constants to 1. The equations (4.6), (4.7) and (4.8) then become:

$$
\begin{align*}
\partial^2_t x &= (a_1(|r|) \partial_r r)|_{s=0}, \\
\partial^2_t y &= - (a_1(|r|) \partial_r r)|_{s=1}, \\
\partial^2_t r &= \partial_r (a_1(|r|) \partial_r r), \quad x = r|_{s=0}, \quad y = r|_{s=1}.
\end{align*}
$$

In the following we demonstrate that in this case the hope of obtaining existence of a strong solution is futile. The problem is quasi-linear, which follows from the computation in (5.17). The matrix $M(p)$ (5.18) is symmetric and it is therefore diagonalisable for every $p \neq 0$ with real eigenvalues and orthogonal eigenspaces. We furthermore notice that $b(p) = \frac{p \gamma}{|p|^2}$ is singular with $\ker b(p) = p^\perp$. Let $v \in \ker b(p)$ then

$$
M(p)v = a_1(|p|)v,
$$
showing that $v$ is an eigenvector of $M(p)$ with eigenvalue $a_1(|p|) = \frac{|p|-1}{|p|}$. It follows that $\text{span } p$ is an eigenspace and we easily show that

$$M(p)p = p,$$

and $1$ is the corresponding eigenvalue. We have shown that $\lambda$ is an eigenvalue of $M(p)$ if and only if

$$\lambda = \begin{cases} 1, & \text{algebraic multiplicity } = 1, \\ \frac{|p|-1}{|p|}, & \text{algebraic multiplicity } = 2 \end{cases}$$

(5.19)

with corresponding eigenspaces:

$$E(1) = \text{span } (p),$$

$$E \left( \frac{|p|-1}{|p|} \right) = p^\perp = \{ v \in \mathbb{R}^3 | v \cdot p = 0 \}. \quad (5.20)$$

The matrix $M(p)$ is therefore positive definite if and only if $|p| > 1$ and in particular the system of equations changes type when $|\partial_r r| = 1$. It is hyperbolic when $|\partial_r r| > 1$ whereas it will have components that are elliptic when $|\partial_r r| < 1$. The Euler-Tricomi equation [121] is a linear system that exhibits similar change of type in part of the phase space and in general one expects loss of regularity, a shock, when $|\partial_r r|$ moves through the unit circle.

This qualitative analysis suggests the ill-posedness of the classical tether equations. This ill-posedness will particularly hamper numerical integration. One can, for example, not expect conservation of energy through a shock. A drift in energy is indeed observed in the numerical computations in [83, 89] along with apparent tether discontinuities. In Chap. 6 the effect of the regularisation in numerical integration for similar parameter values shall be considered.

In the following section we conjecture that the slack-spring model is a limit of the massive tether model as the diameter of a stiff tether goes to 0. This will also force $ET \to 0$ (see (4.12) and (5.21)).

### 5.3 The vanishing thickness limit

Tethers are thin and longitudinally stiff. It therefore seems relevant to study the limit of vanishing thickness together with an assumption on the stiffness. Let $h$ denote the diameter of a tether with constant circular cross-section such that $A = \frac{\pi}{4} h^2$, $\rho_i = \frac{\pi}{2} h^2 \rho$ and $I = \frac{\pi}{64} h^4$ (4.12). Then (4.16) may be written as

$$\frac{\pi}{4} h^2 \rho_i \partial_t^2 r = -\mu \frac{\pi}{4} h^2 \rho |r|^{-3} r + \frac{\pi}{4} E h^2 \partial_r^2 (a_1(|\partial_t r|) \partial_t r) - \frac{\pi}{64} E h^4 \partial_t^4 r.$$
Now, if we assume
\[ E = \hat{E}h^{-2}, \quad \text{(5.21)} \]
and normalise appropriately, we have:
\[ h^2 (\partial_t^2 r + \partial_x^2 r) = \partial_x (a_1(\partial_x r)|\partial_x r) + h^2 f(r). \quad \text{(5.22)} \]
together with
\[ \begin{align*}
\partial_t^2 x + f(x) &= (a_1(\partial_x r)|\partial_x r)|_{s=0} - h^2 \partial_x^2 r|_{s=0} = 0, \\
\partial_t^2 y + f(y) &= -(a_1(\partial_x r)|\partial_x r)|_{s=1} + h^2 \partial_x^2 r|_{s=0} = 0, \\
\partial_x r|_{s=0} &= x, \quad r|_{s=1} = y, \\
\partial_x r|_{s=0,1} &= 0. 
\end{align*} \quad \text{(5.23)} \]
The assumption that \( E = \mathcal{O}(h^{-2}) \) is appropriate since the boundary terms
\[ (a_1(\partial_x r)|\partial_x r)|_{s=0,1} \]
are explicitly independent of \( h \). For any other polynomial relation these terms would either vanish or diverge upon equating \( h = 0 \). By Theorem 2, this system admits a unique local solution for every \( h > 0 \). As mentioned we can only guarantee global existence if singularities are not encountered. To avoid having to deal with the possibility that the solution in general only exists locally, we shall in the following replace \( f \) and \( a_1 \) by smooth mollifications: \( f^\text{mol}(z) = \chi_\delta(z)f(z) \) and \( a_1^\text{mol}(z) = \chi_\delta(z)a_1(z) \) respectively, where \( \chi_\delta : \mathbb{R} \to \mathbb{R} \) is a smooth function satisfying
\[ \begin{align*}
\chi_\delta(z) &= 1, \quad \text{whenever } z \geq \delta, \\
\chi_\delta(z) &\leq 1 \quad \text{whenever } \delta/2 \leq z \leq \delta, \\
\chi_\delta(z) &= 0 \quad \text{whenever } 0 \leq z \leq \delta/2, 
\end{align*} \]
for some (small) \( \delta > 0 \).

The limit \( h \to 0 \) is singular. Our hope is that as \( h \to 0 \) the solution of (5.22) and (5.23) will converge to some sort of weak solution. The full weak solution will not be well-defined, indeed we lose all possible \( W^4 \)-estimates on \( r \) as \( h \to 0 \). Nonetheless, we conjecture that the behaviour of the boundaries is well-defined, and in particular that for certain initial conditions it converges as \( h \to 0 \) to that of the solution of the slack-spring problem.

**Conjecture 1** For \( h > 0 \) let \( x^h \) and \( y^h \) solve the boundary equations of (5.22) and (5.23) with initial conditions:
\[ \begin{align*}
(x(0), y(0)) &= (x_0, y_0) \in \mathbb{R}^6 \setminus \{0\}, \\
(x(0), y(0)) &= (\hat{x}_0, \hat{y}_0) \in \mathbb{R}^6, \\
(u(0), v(0)) &= (u_0, v_0), \quad \text{(5.26)}
\end{align*} \]
Let \( x, y \) be the solutions of the slack-spring model:

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= f^\text{mol}(x) + \bar{u}_1(|y - x|(y - x)), \\
\frac{d^2 y}{dt^2} &= f^\text{mol}(y) + \bar{u}_1(|y - x|(x - y)),
\end{align*}
\]

with initial condition (5.24) and (5.25). Then for almost all initial conditions:

\[
|(x^h, y^h)(t) - (x, y)(t)|_{\mathbb{R}^2} = O(h) \quad \text{for} \quad 0 < t < O(h^{-p}), \quad \text{for some} \quad p > 0.
\]

We aim to give a rigorous proof of this in future work. Here we argue from a qualitative perspective that the assertion seems reasonable. Equating \( h = 0 \) in (5.22) we obtain an ordinary differential equation:

\[
\frac{ds}{dt}(a_1(|r|)r) = 0,
\]

implying

\[
a_1(|r|) \frac{\partial r}{\partial r} = \text{const.} \in \mathbb{R}^3,
\]

and

\[
\frac{\partial r}{\partial r} = \text{const.} + 1,
\]

with const. = |const.|. We obtain, by assuming |\( \partial r \)| \( \neq 0 \), that

\[
\begin{align*}
|\partial r| &= 1 \quad \text{for const.} = 0, \\
0 &= 0 \quad \text{for const.} \neq 0.
\end{align*}
\]

The former is not possible when the satellites are separated by a distance greater than \( l = 1 \), while the latter is not stable in the sense of Euler buckling when \(|y - x| < l|\), \([6]\).

To demonstrate Euler buckling we imagine \( x \) and \( y \) are fixed along the first inertial axis in free space, i.e. \( f = 0 \), in the plane with \( x = (0, 0) \) and \( y = (1 - d, 0) \), \( d < 1 \).

We are left with

\[
h^2 (\partial^2_r + \partial^4_r) = a_1(|r|) \frac{\partial r}{\partial r},
\]

\[
r|_{s=0} = 0, \quad r|_{s=1} = (1 - d, 0) \quad \text{and} \quad \partial^2_r r|_{s=0,1} = 0.
\]

Linearisation about the compressed equilibrium \( r = ((1 - d)s, 0) \) gives:

\[
h^2 (\partial^2_r + \partial^4_r) = \text{diag}(1, -d/(1 - d)) \partial^2_r r.
\]

(5.31)
Through the ansatz \((r_1^{(n)}, r_2^{(n)}), r_i^{(n)} = \exp(i\omega_i^{(n)} t) \sin(n\pi s), i = 1, 2\) we obtain:

\[
\left(\omega_2^{(n)}\right)^2 = -d/(1-d)h^{-2} + (\pi n)^2.
\]

Solving \(\omega_2^{(n)} = 0\) for \(h = h(d, n)\) gives a critical thickness:

\[
h_c = \frac{1}{\pi n} \sqrt{\frac{d}{1-d}}.
\]

in the sense that \(h < h_c\) implies that the \(n^{th}\) eigenmode is unstable. Finally, notice that \(h_{crit} \to \infty\) for \(d \to 1\) for fixed \(n\).

If \(|\partial_x r| = 1\) then the tether does not affect the motion of \(x\) and \(y\). This follows from the definition of \(\omega_1\) and by differentiating \(|\partial_x r| = 1\) twice and using the boundary conditions \(\partial_x^2 r|_{s=0,1} = 0\). On the other hand, when \(u = 0\), or \(r = sy + (1-s)x\), the boundary terms, entering the equations for \(x\) and \(y\), equal the effect of a spring with stiffness 1 connecting the two satellites.

The buckling result does not imply the non-existence of compressed tether motion. Certainly, zero angular momentum solutions provide a counter-example. However, we believe that the buckling result will imply that the set of initial conditions for which the result is not true is small, in some sense. This is the reason for the phrase: for almost initial conditions. In the construction of a rigorous proof this phrase and proper estimates on the convergence rate \(p\) will have to be made precise.

We shall return to the validity of Conjecture 1 later in the thesis. In Chap. 6, for example, we will provide further evidence of the conjecture through numerical computations and in Chap. 7 we will prove a similar result for a Galerkin approximation. In the concluding chapter when discussing open problems we will also suggest a list of projects aiming to bring to the mathematical theory on singular perturbation theory to a level in which a proof of the tether conjecture can be given. We will now revisit the slack-spring model and introduce the billiard model as the limit of an inextensible spring.
5.4 The slack-spring model

5.4.1 Linearisation of the gravitational field

The slack-spring model with Hamiltonian (4.22), repeated here for convenience:

$$H_{ST}(q, \delta q, p, \delta p) = \frac{1}{2\xi} |p|^2 + \frac{1}{2} |\delta p|^2 - \frac{1}{\mu_x} \frac{\mu}{|q + \mu_x \delta q|} - \frac{1}{\mu_y} \frac{\mu}{|q - \mu_y \delta q|}$$

$$+ \alpha_1 |\delta q_{\perp l}| (|\delta q| - l)^2,$$

(5.32)

is 12-dimensional. In section 4.3 restricting to planar dynamics and introducing appropriate polar coordinates, we were able to reduce to 3 degrees of freedom, see (4.23). However, even 6 dimensions are too many to easily visualise the dynamics. To overcome this problem we may make use of the fact that in practise \( l \ll r \). We therefore replace the gravitational term in Hamilton's equations with its linearised versions about \( \delta q = 0 \). We obtain

$$\dot{q} = \frac{\mu}{|q|^3} p,$$

$$\dot{p} = -\frac{1}{\xi} \frac{\mu}{q} q,$$

$$\dot{\delta q} = \delta p,$$

$$\dot{\delta p} = -\frac{\mu}{|q|^3} \left( I - 3 \frac{qq^T}{|q|^2} \right) \delta q - 2\alpha_1 (|\delta q|) \delta q.$$

Within this approximation the centre of mass is independent of the relative motion and moves on a Keplerian orbit. The Keplerian motion conserves eccentricity \( e \) and for \( 0 \leq e < 1 \) the motion is bounded and periodic. We therefore replace the original Hamiltonian system with a family of time-periodic Hamiltonians parametrised by \( e \in [0, 1) \). If we introduce the true anomaly \( \nu \) Fig. 4.2 as an independent variable and normalisations such that \( \nu = (1 + e \cos \nu)^2 \) and \( l = 1 \), then the Hamiltonian takes the following form:

$$H_{ST}(\delta q, \delta p, \nu; e) = \frac{1}{2\kappa} |\delta p|^2 - \frac{1}{2} (1 + e \cos \nu) \langle \delta q, A(\nu) \delta q \rangle + \frac{K}{\kappa} \alpha_1 |\delta q_{\perp l}| (|\delta q| - l)^2,$$

(5.33)

where

$$A(\nu) = I - 3 \begin{pmatrix} \cos^2 \nu & \sin \nu \cos \nu & 0 \\ \sin \nu \cos \nu & \sin^2 \nu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Finally, by moving into a rotating frame:

\[ \delta q = R(\nu)\delta q', \]
\[ \delta p = R(\nu)\delta p', \]

where \( R(\nu) \in \text{SO}(3) \) for every \( \nu \):

\[ R(\nu) = \begin{pmatrix} \cos \nu & -\sin \nu & 0 \\ \sin \nu & \cos \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

we obtain a new Hamiltonian

\[ H_{ST}(\delta q', \delta p', \nu; e) = \frac{1}{2\nu} |\delta p'|^2 + \langle \delta q', \Omega \wedge \delta p' \rangle 
- (1 + e \cos \nu) \left( (\delta q'_1)^2 - \frac{1}{2} (\delta q'_2)^2 - \frac{1}{2} (\delta q'_3)^2 \right) 
+ \frac{\kappa}{\nu} \delta q' \cdot (|\delta q'| - 1)^2, \]

(5.34)

where \( \delta q' = (\delta q'_1, \delta q'_2, \delta q'_3) \), \( \delta p' = (\delta p'_1, \delta p'_2, \delta p'_3) \) and

\[ \Omega = (0, 0, 1). \]

(5.35)

The Hamiltonian \( H_{ST}(\delta q', \delta p', \nu; e) \), corresponding to a circular orbiting centre of mass, is independent of \( \nu \) and \( H_{ST} \) is conserved. The six dimensional phase space is therefore foliated by five dimensional sub-manifolds, or three dimensional sub-manifolds if we restrict to planar motion. In the latter case visualisations are possible with 2-dimensional Poincaré maps.

In the dumbbell model the distance between the spacecraft is assumed constant and equal to \( l = 1 \). Therefore, by replacing the Euclidean configuration space above with \( S^2 \) for the attitude of the dumbbell we obtain the dumbbell model with linearised gravity, see e.g. [17]. For \( e = 0 \) and restricting to planar motion we obtain a time-independent one degree of freedom integrable Hamiltonian system. We shall return to this "underlying" integrable system when we later identify the dumbbell dynamics within the billiard dynamics. We mention that for the dumbbell model with small \( e \) most of the invariant curves of the planar dumbbell dynamics will persist by considering the stroboscopic, symplectic map and using Kolmogorov-Arnol’d-Moser (henceforth abbreviated KAM) theory.

### 5.4.2 The inextensible limit of the slack-spring model

Our aim in this subsection is to study the inextensible limit of the slack-spring model. We will show that the impact of \( \delta q \) with \( |\delta q| = 1 \) approximates a \( \delta \)-distribution as
\( \kappa \to \infty \), the effect of which is to reverse the direction of the radial momentum, \( p_\theta \mapsto -p_\theta \), leaving the remaining variables continuous in time.

The slack-spring problem can be viewed as a hybrid system: an integrable Hamiltonian flow within \( |\delta q'| < 1 \) and a different flow beyond where the spring affects the motion. Since the flow within \( |\delta q'| < 1 \) is not affected by the spring, or the value of \( \kappa \), for the purpose of our study it suffices to study the region: \( |\delta q'| \geq 1 \). To do so we consider the related spring-system, i.e. replacing \( a_1 \) (4.21) by \( a_1 \) (4.10). We assume \( \epsilon = 0 \) and restrict to planar dynamics for simplicity. The arguments can easily be extended for \( 0 < \epsilon < 1 \) and the non-planar case.

We introduce the polar coordinates: \( q' = \delta r \cos \theta, \sin \theta \). Upon replacing \( a_1 \) by \( a_1 \), Hamilton's equations, with Hamiltonian (5.34), become:

\[
\begin{align*}
\dot{\delta r} &= \delta r \dot{\theta}^2 + 3 \delta r \cos(\theta)^2 + 2 \kappa (1 - \delta r) + 2 \delta r \dot{\theta}, \\
\dot{\delta r} &= -3 \delta r \dot{\theta} \sin(\theta) - 2 \delta r \dot{\delta r},
\end{align*}
\]

We introduce the slow time \( \tau = \epsilon^{-1} t \) and set \( \delta r(\tau) = 1 + \epsilon \delta r_1(\tau) \) with \( \epsilon^2 = \kappa^{-1} \) to obtain:

\[
\begin{align*}
\delta r_1'' &= -2 \delta r_1 + \mathcal{O}(\epsilon), \\
\dot{\theta} &= -3 \cos(\theta) \sin(\theta) + \mathcal{O}(\epsilon).
\end{align*}
\]

Therefore, if \( \delta r_1(0) = 0 \) with \( \delta r_1(0) = B \), then after truncating terms of order \( \epsilon \),

\[
\delta r_1(t) = B \sin(\sqrt{2} \kappa t)
\]

and it follows that the effect of moving beyond \( \delta r = 1 \) is approximated by the bounce map: \( \delta r \mapsto -\delta r \), leaving the other variables, \( \delta r, \theta \) and \( \dot{\theta} \), continuous. Together the bounce map and the Keplerian flow between bounces define the billiard model.

The Kelvin-Voigt dissipation enters on the right hand side of (5.36) via the term \( -2 \kappa a \delta r \). If we assume that the damping factor is small and in particular satisfy \( \alpha = \tilde{\alpha} \epsilon \) for some \( \tilde{\alpha} \in [0,2) \), then the calculations made above can be repeated to show that the truncation satisfies:

\[
\delta r_1'' = -2 \delta r_1 - 2 \tilde{\alpha} \delta r_1'.
\]

Therefore

\[
\delta r_1 = \frac{B}{\sqrt{2 - \tilde{\alpha}}} \exp \left( -\tilde{\alpha} \sqrt{\kappa} t \right) \sin \left( \sqrt{2 - \tilde{\alpha}^2} \sqrt{\kappa} t \right),
\]

so that in the limit of \( \epsilon = 0 \):

\[
\delta r \mapsto -\delta r \exp \left( \frac{-\tilde{\alpha} \pi}{\sqrt{2 - \tilde{\alpha}}} \right) = -\delta r \left( 1 - \frac{\pi}{\sqrt{2}} \tilde{\alpha} + \mathcal{O}(\tilde{\alpha}^2) \right).
\]
The dissipation can therefore be accounted for within the billiard model via the restitution factors $\exp\left(-\frac{\sqrt{\epsilon}}{\sqrt{2}a}\right)$. This is done in [138]. This reference considers a fixed circular orbiting centre of mass and shows numerically that, as might be expected for a nonlinear, almost Hamiltonian system, transient chaos before the system converges to the stable equilibria.

In the following section the billiard model is studied further. The overall aim shall be to identify the dumbbell dynamics within the dynamics of the billiard model.

5.5 The billiard model

Between collisions the flow is given by the Hamiltonian:

$$Q(\delta q',\nu, \delta p', \epsilon) = \mathcal{E} + \frac{1}{2\nu} \langle \delta p', \delta p' \rangle - (1 + \epsilon \cos \nu) \left( \langle \delta q'_1 \rangle^2 - \frac{1}{2} \langle \delta q'_2 \rangle^2 - \frac{1}{2} \langle \delta q'_3 \rangle^2 \right),$$

(5.37)

with canonical symplectic structure: $\omega = d\delta q' \wedge dp' + d\nu \wedge d\mathcal{E}$. Recall that $\mathbf{q} = (0, 0, 1)$ (5.35). This is just (5.34) without the slack-spring term and where we have introduced the negative energy $\mathcal{E}$ as the canonical conjugate of $\nu$. This Hamiltonian is integrable as it is obtained from the variations of the integrable Kepler problem. In fact, for $0 \leq \epsilon < 1$ there is a five dimensional family of periodic solutions of the variational equations cf. Theorem 1. By linearity these solutions can be scaled such that they never intersect $|\delta q'| = 1$. This set of solutions is an integrable periodic subset of the billiard dynamics. The sixth remaining solution of the variational equations is a linear drift due to variations in energy, see Chap. 3. We consider initial conditions for the billiard map on $|\delta q'| = 1$. If these initial conditions correspond to a periodic solution of the variational equations, then the relative position vector certainly returns to $|\delta q'| = 1$. Otherwise, by the linear drift, the relative position is radially expanding. It therefore follows that every point on the section $|\delta q'| = 1$ for which $d_0|\delta q'| < 0$ is mapped through the flow of (5.37) to a point on $|\delta q'| = 1$ with $d_0|\delta q'| > 0$. This defines a map $B_\epsilon$, parametrised by the eccentricity $\epsilon$, mapping wall-collisions to wall-collisions. Since $p_{T\nu} \mapsto -p_{T\nu}$ leaves $Q$ invariant and $Q$, as a time-independent Hamiltonian, is conserved on the integral curves between collisions, $B_\epsilon$ maps the level-sets of $Q$, $Q = \xi$, into themselves. Therefore

$$B_\epsilon(z_0, \xi) = (z_1, \xi), \quad z_0, z_1 = z_1(z_0, \xi) \in T^*(S^2 \times S^1).$$

Here $S^2$ is for $|\delta q'| = 1$ measuring the collision attitude while $S^1$ is for the true anomaly $\nu$. As is usual for Hamiltonian Poincaré maps, the mapping

$$P_\epsilon : z_0 \mapsto z_1,$$

(5.38)
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is smooth and symplectic on $T^*(S^2 \times S^1)$.

For $\epsilon = 0$ further reduction is possible. Indeed, in this case $\nu$ is cyclic in the Hamiltonian (5.37), and $\mathcal{E}$ is therefore conserved, say $\mathcal{E} = c$. Hence

$$P_0(x_0, v_0, c) = (x_1, v_0 + \Delta \nu(x_0, c), c), \quad x_0, x_1 = x_1(x_0, c) \in T^* S^2, \quad (5.39)$$

and we define $P^{\text{red}}_0 : x_0 \mapsto x_1$ with $\mathcal{E} = c$. This is a family of four dimensional smooth symplectic maps parametrised by $c$. The planar restriction defines a family of 2 dimensional symplectic maps on the cylinder $T^* S^1$.

5.5.1 The dumbbell motion

The dumbbell motion is embedded within the billiard model as trajectories grazing along the boundary. As already mentioned in the last paragraph of section 5.4.1, this dumbbell motion is integrable when restricted to planar motion and $\epsilon = 0$. The interesting question is when this specific dynamics persists for slack tethers or, in other words, when is it obtainable as a limit within the billiard map. The existence of these regions of persistence will provide subsets of phase space in which the dumbbell model will be a valid approximation to the full dynamics.

For $\epsilon = 0$ the $\nu$-independence of $Q$ allows us to identify $\mathcal{E}$ with the associated energy function. In polar coordinates we may then write $E = -\mathcal{E}$ as

$$E = \frac{1}{2} \dot{r}^2 + \frac{1}{2} \dot{\theta}^2 - \frac{3}{2} \delta^2 \cos^2 \theta. \quad (5.40)$$

We say that a frequency $\omega$ satisfies the Diophantine condition if there exists $\tau \geq 1$ and $C > 0$ such that $|n\omega - m| > Cn^{-\tau}$, $C > 0$ for any $n, m \in \mathbb{N}$. We then obtain the following theorem:

**Theorem 3** Any invariant curve of the dumbbell model with $\epsilon = 0$, $\lambda = \delta^2 + 2\dot{\theta} + 3 \cos^2 \theta > 0$ and an induced frequency $\omega$ satisfying the Diophantine condition, persists within the reduced billiard map.

**Proof** The proof is inspired by a proof of the existence of invariant curves in magnetic billiards in [12]. It goes as follows:

1° Obtain an approximation to the billiard map using the escape velocity as a small parameter;

2° Derive an area-preserving, twist-mapping approximation using canonical transformations;

3° Use Moser’s twist theorem, [106] Theorem 2.11, to conclude the existence of invariant curves near the boundary.
In polar coordinates the Hamiltonian (5.37) with $\epsilon = 0$ reads:

$$Q = \frac{1}{2} p_r^2 + \frac{p_\theta^2}{2\delta r^2} - p_\theta + \frac{1}{2} \delta r^2 - \frac{3}{2} \delta r^2 \cos^2 \theta,$$

equipped with $\omega = d\delta r \wedge dp_r + d\theta \wedge dp_\theta$. As we will be considering trajectories grazing along the boundary $\delta r = 1$, we introduce $\epsilon$ and $\delta r$ so that $\delta r = 1 + \epsilon \delta \hat{r}$. It is moreover appropriate to introduce $p_{\delta r} = e^{1/2} p_\delta$ and $t \mapsto e^{-1/2} t$. Upon forgetting the tildes the Hamiltonian is transformed into:

$$Q = \frac{1}{2} \epsilon p_{\delta r}^2 + \frac{p_\theta^2}{2(1 + \epsilon \delta r)^2} - p_\theta + \frac{1}{2} (1 + \epsilon \delta r)^2 - \frac{3}{2} (1 + \epsilon \delta r)^2 \cos^2 \theta$$

$$= \frac{1}{2} \epsilon p_{\delta r}^2 + h(\theta, p_\theta) - \epsilon \lambda(\theta, p_\theta) \delta r + O(\epsilon^2),$$

equipped with $\omega = e d\delta r \wedge dp_r + e^{-1/2} d\theta \wedge dp_\theta$. Here

$$h(\theta, p_\theta) = \frac{1}{2} p_\theta^2 - p_\theta - \frac{3}{2} \cos^2 \theta,$$

is the Hamiltonian of the associated dumbbell model with $\epsilon = 0$, and

$$\lambda(\theta, p_\theta) = p_\theta^2 - 1 + 3 \cos^2 \theta,$$

which satisfies

$$-\epsilon^{-1} \partial_r Q \rightarrow \lambda$$
as $\epsilon \rightarrow 0$.

**Remark 1** The quantity $\lambda = \dot{\theta}^2 + 2 \dot{\theta} + 3 \cos^2 \theta + O(\epsilon)$ is the acceleration of $\delta r$. Physically, see e.g. [9], $\lambda$ with $\epsilon = 0$ is the tension in the associated dumbbell required to keep it unit-speed parametrized. If $\lambda$ is $O(e^{1/2})$, or even $\lambda < 0$, then the estimates below are not valid. In particular, Hamilton’s equations for $\delta r$ and $p_{\delta r}$ are

$$\epsilon \delta r = c p_{\delta r},$$

$$c p_{\delta r} = O(\epsilon^2).$$

Therefore if a trajectory is initiated on the boundary with $\delta r(0) = 0$ and $\dot{\delta} r(0) < 0$ then, within the truncation of these equations, $\delta r$ will only return to 0 if $\lambda > 0$. See also Fig. 5.1. However, we may notice that $\lambda \leq 0$ gives $-1 - \sqrt{1 - 3 \cos^2 \theta} \leq \dot{\theta} \leq \sqrt{1 - 3 \cos^2 \theta} - 1 \leq 0$, and therefore the negativity of $\lambda$ is only an issue when the tethered system’s rotation opposes the direction of rotation of the centre of mass (retrograde orbits). □
Figure 5.1: If (a) \( \lambda > 0 \) then the trajectory remains close to the boundary provided the radial escape velocity \( p_{\theta r} \) is sufficiently small. This is in general not the case for (b) \( \lambda < 0 \).

We will assume for the moment that
\[
\lambda \geq \delta > 0. \tag{5.43}
\]
We will return to this in 2°. As the billiard map is symplectic on energy level sets, we will reduce by energy. To return to the boundary \( p_{\theta r} \) will obviously have to change sign, so we will eliminate \( \delta r \) rather than \( p_{\theta r} \) via the conservation of energy:
\[
Q(\delta r, p_{\theta r}, \theta, p_{\theta}) = c. \tag{5.44}
\]
We have:
\[
\partial_{\theta} Q = -\epsilon \lambda + O(\epsilon^2).
\]
Therefore, for \( \epsilon \) small enough, using the implicit function theorem and assumption (5.43), we can solve (5.44) for \( \delta r \). Notice that \( 0 > h - c = O(\epsilon) \). Let us therefore set
\[
h - c = \epsilon \tilde{h}(\theta, p_{\theta}, c) < 0.
\]
Moreover let
\[
\delta r = M_0(p_{\theta r}, \theta, p_{\theta}, c) = M_0(p_{\theta r}, \theta, p_{\theta}, c) + \epsilon M_1(p_{\theta r}, \theta, p_{\theta}, c) + O(\epsilon^2).
\]
By insertion we obtain:
\[
M_0 = \frac{\tilde{h}}{\lambda} \frac{p_{\theta r}^2}{2\lambda}.
\]
Next, we eliminate $p_s$ which is conjugate to $\delta r$ by replacing time with $p_s$. We have:

\[
\begin{align*}
\frac{d\theta}{dp_s} &= -\epsilon^{3/2}\frac{\partial_{p_s} Q}{\partial r Q}, \\
\frac{dp_s}{dp_s} &= \epsilon^{3/2} \frac{\partial Q}{\partial r Q}.
\end{align*}
\]

But from (5.44) it follows upon using the chain rule that:

\[
\begin{align*}
\partial_{p_s} Q + \partial_{qs} Q \partial_{q} M_e &= 0, \\
\partial_{qs} Q + \partial_{q} Q \partial_{q_s} M_e &= 0,
\end{align*}
\]

and therefore:

\[
\begin{align*}
\frac{d\theta}{dp_s} &= \partial_{p_s} \left(\epsilon^{3/2} M_e\right), \\
\frac{dp_s}{dp_s} &= -\partial_{q} \left(\epsilon^{3/2} M_e\right).
\end{align*}
\]

The reduced system is therefore Hamiltonian with Hamiltonian function $\epsilon^{3/2} M_e$ and symplectic form $d\theta \wedge dp_s$. Here

\[
\epsilon^{3/2} \partial_{z} M_0 = \epsilon^{1/2} \frac{\partial_{s} h}{\lambda} - \epsilon^{3/2} \frac{\partial_{s} \lambda}{\lambda} \left(\bar{p_s} + \tilde{h}\right), \quad z = q \text{ or } p_s.
\]

Therefore:

\[
\begin{align*}
\frac{d\theta}{dp_s} &= \epsilon^{1/2} \frac{\partial_{s} h}{\lambda} + \cdots, \\
\frac{dp_s}{dp_s} &= -\epsilon^{1/2} \frac{\partial_{s} \lambda}{\lambda} + \cdots.
\end{align*}
\]

To approximate the billiard map the truncation of this system has to be integrated up until the trajectory returns to the boundary corresponding to $\delta r = 0$. In the following we approximate the required integration time. First we notice that from (5.44) it follows that on the boundary, given by $\delta r = 0$, we have $p_s = -\frac{1}{2} N_0 + O(\epsilon)$, $N_0 = 2\sqrt{-2h}$. Hence by (5.41) and (5.42), or

\[
\delta r = \lambda + O(\epsilon),
\]

we obtain

\[
\delta r = \frac{\lambda^2}{2} - \frac{1}{2} N_0 t + O(\epsilon).
\]

The equation $\delta r = 0$ to be solved for the return time $\Delta t > 0$, therefore solves to

\[
\Delta t = \frac{N_0}{\lambda} + O(\epsilon),
\]
which is positive for sufficiently small $\epsilon$ by assumption (5.43). In terms of $p_{\sigma \tau}$ the return time becomes $\Delta p_{\sigma \tau} = N_0 + O(\epsilon)$. If $\tau = \frac{M^2}{N^2}$ is a new time then the return time becomes

$$\Delta \tau = 1 + O(\epsilon), \tag{5.45}$$

and the equations read:

$$\frac{d\theta}{d\tau} = \epsilon^{1/2} \frac{\partial_{\theta} h}{N_0 \lambda} + \cdots,$$

$$\frac{dp_\theta}{d\tau} = -\epsilon^{1/2} \frac{\partial_{p_\theta} h}{N_0 \lambda} + \cdots. \tag{5.46}$$

The truncation of (5.46) is a time re-parametrisation of the dumbbell model with Hamiltonian $h$ and according to (5.45) its time-one map approximates the billiard map. Notice also that the truncation preserves $N_0$ since it conserves $h$. Therefore in terms of the action-angle variables $(\phi, J)$ of the dumbbell the truncation of (5.46) reads:

$$\dot{\phi} = \frac{\epsilon^{1/2}}{N_0 \lambda} \omega(J),$$

$$\dot{J} = 0.$$

Now, introduce $\phi \mapsto \psi$ where

$$\psi = \frac{\int_0^\phi \lambda(\tau, J) d\tau}{\lambda}, \quad \bar{\lambda} = \frac{1}{2\pi} \int_0^{2\pi} \lambda(\tau, J) d\tau,$$

and $\epsilon \mapsto \tilde{\epsilon} = \epsilon N_0^2$. The new $\tilde{\epsilon}$ is still small since $N_0 = O(1)$ and the truncation preserves it. Then the equations are transformed into:

$$\dot{\psi} = \tilde{\epsilon}^{1/2} \omega(J),$$

$$\dot{J} = 0.$$

Since the dumbbell model with $\epsilon = 0$ is just a pendulum equation, the time-one map of the dumbbell obviously satisfies the twist condition $\partial_\phi \omega(J) \neq 0$ [106] away from the separatrices. It is therefore only left to be shown that (5.43) holds in parts of the phase space. We may write

$$\lambda = 2\eta + 6 \cos^2 \theta + 2\dot{\theta}(\theta, \eta),$$

$$= 2\eta + 6 \cos^2 \theta \pm 2\sqrt{2\eta + 3 \cos^2 \theta},$$

where $2\eta = \dot{\theta}^2 - 3 \cos^2 \theta$ is the energy function related to $h$ and therefore conserved. At the $\theta = 0, \pi$ equilibria $\eta = -3/2$ and therefore $\lambda = 3$. Moreover, $\lambda > 0$ for sufficiently large $\eta$. 

2
For $\lambda > 0$ we have an area-preserving approximation of the billiard map through the time-one map of the truncation of the map (5.46). Moser's twist map theorem then guarantees the persistence of Diophantine tori.

\textbf{Remark 2} The arguments can also be repeated for a more general class of linear time-independent Hamiltonian vector-fields describing the flow between collisions. Another example could be the variational equations about the collinear Lagrange points in the circular restricted three body problem. Moreover, similar techniques have been used in magnetic billiards, see e.g. [12].

The tori which do not persist the perturbation, in particular tori with rational frequencies, break up into island chains and chaos [4]. In the following section we show some diagrams of numerical computations of the billiard map, particularly bringing to attention the dynamics away from the KAM tori.

5.5.2 Numerical computations of the billiard map for $e = 0$

We focus our attention on $e = 0$ and the family of 2-dimensional billiard maps describing the planar billiard dynamics. Again we use $\theta$ and $\dot{\theta}$ as coordinates on the cylinder $TS^1$, see Fig. 4.2 for the definition of $\theta$.

The invariant sets defined by $E = c$ (5.40) are disconnected for $E < 0$. For $E < 0$ the dynamics are confined to two regions of configuration space: $|\delta r \cos \theta| \geq \sqrt{-\frac{E}{2}}$, see Fig. 5.2. For $E \geq 0$ any point of configuration space, $\delta r \leq 1$, can be visited by the dynamics. In particular, collisions between the satellites can occur if and only if $E > 0$. The topology of the sets $E^{-1}(c)$, with $c \geq -\frac{3}{2}$, obviously implies that the billiard mapping is only defined on a proper subset of $(-\pi, \pi) \ni \theta$.

Fig. 5.3 shows four examples of the billiard map restricted to the level sets of $E$. In Fig. 5.3 (a), (b), (c) and (d) $E$ is fixed at $-0.7$, $0.1$, $1$ and $5$. On the boundary curves $\delta r = 0$, i.e. the dumbbell limit. Due to reflectional symmetries about the lines $\theta = 0$ and $\theta = \pi/2$ the sections with $\theta$ in only one of the regions $(0, \pi/2)$, $(\pi/2, \pi)$, $(-\pi, -\pi/2)$ and $(-\pi/2, 0)$ uniquely define the billiard map.

There is an obvious difference in the dynamics of direct and retrograde orbits, i.e. $\dot{\theta} > 0$ and $\dot{\theta} < 0$ respectively. Similar differences can be observed in the circular restricted three body problem in rotating coordinates, or in magnetic billiards [12]. In general, retrograde orbits have more energy as they need to be faster to reach the next collision.
In (a), $E = -0.7$, there are two obvious dominating regular regions: invariant curves near the boundary and an elliptic island. Between these regions we see both chaotic regions and additional smaller regular islands. The large elliptic island surrounds a nonlinear normal mode emerging from the stable fixed point. As $E$ is increased the qualitative picture in Fig. 5.3 (a) persist until $E = 0$ where the two white regions in Fig. 5.2 collide to enable transfer between the two half discs. Immediately after $E = 0$ the dynamics is predominantly chaotic, see Fig. 5.3 (b). Increasing the energy to $E = 1$ regularises the dynamics near the top boundary and resonance islands appear, see Fig. 5.3 (c). Increasing the energy even further to $E = 5$, Fig. 5.3 (d), regularises the dynamics near the lower boundary, again creating resonance islands. The behaviour of the invariant curves near the boundary is in agreement with Theorem 3 and Remark 1.

Projections of the five periodic orbits identified with periodic and fixed points for the billiard map, see Fig. 5.3, are visualised in Fig. 5.4. By the implicit function theorem, periodic and fixed points can be continued onto neighbouring energy surfaces provided $\lambda$ is not an eigenvalue of the linearised map. In Fig. 5.4 (a), the stable fixed point visible in Fig. 5.3 (a) has been continued for $E$ near $-0.7$.

The invariant curves near the boundaries are co-dimension 1 and they therefore act as absolute barriers to the motion. In particular, for these reasons, trajectories emanating from $\delta r = 0$ cannot, regardless of their initial energy $E$, reach these curves and regions of phase space without a control mechanism.
5.5 The billiard model

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{billiard_models.png}
\caption{Visualisation of the billiard map for $E$-values equal to -0.7, 0.1, 1 and 5. On the boundary curves $\delta r = 0$, i.e. the dumbbell limit. The periodic points pointed out by the arrows in (a), (c) and (d) are visualised as projections of periodic orbits in Fig. 5.4.}
\end{figure}

5.5.3 3D billiard dynamics

The spatial motion is described through a 4 dimensional map and is therefore harder, if not impossible, to visualise. We can, however, ask for conditions for which the planar fix points of the reduced billiard map are stable to out-of-plane disturbances. Any equilibrium $z_0 \in T^*S^1$ of the planar dynamics satisfy:

$$P_0^{\text{red}}(z_0, 0, 0) = (z_0, 0, 0).$$

From the diagrams above we can read off two eigenvalues $\lambda_1$ and $\lambda_2$ of $dP_0^{\text{red}}(z_0, 0, 0)$. Since $dP_0^{\text{red}}$ is symplectic three situations can occur: elliptic, hyperbolic or parabolic. Similar three situations can occur for the remaining two eigenvalues $\lambda_3$ and $\lambda_4$. By Lyapunov’s center theorem periodic orbits can be continued if $\lambda_3$ and $\lambda_4$ are complex.
conjugated on the unit circle in the complex plane and $i\lambda_2/\lambda_3$ is never an integer (no resonance). On the other hand if $\lambda_3 > 1 > \lambda_4 = \lambda_3^{-1}$ then there will exist invariant stable and unstable manifolds emanating from the planar fix point and the fix point will be unstable. We can repeat the arguments for periodic points of $P^{\text{red}}$ by considering appropriate compositions.

Numerically, the stability of the planar motion to out-of-plane disturbances can be studied and visualised through Lyapunov exponents. In Fig. 5.5 the largest Lyapunov exponents associated with out-of-plane perturbations are visualised on the cylinder for $E = 1$ and $E = 5$ in resp. (a) and (b). Regions of positive Lyapunov exponents are visible as red regions. Initiating the tethered system in these regions will, in general, result in growing out-of-plane oscillations. Notice, that the invariant curves are in the region where the largest Lyapunov exponent is almost 0 and hence the invariant curves are, at least formally, stable to out-of-plane perturbations. The white curves in Fig. 5.5 show the initial collisions of trajectories emanating from $5r_0 = 0$. It is seen that these trajectories fall within the unstable region. That pattern persists for larger values $E$.

In the following section we shall aim to give a short qualitative description of the dynamics when $e \neq 0$.

### 5.5.4 Perturbations: $e \neq 0$

To understand what happens to the invariant structure as we move away from $e = 0$ we have to go from the reduced map and back to the original billiard map $P_0$, see (5.39). Every fix point $z_0$ of the reduced map then induces a circle map of the original map:

$$P_0(z_0, v_0, c) = (z_0, v_0 + \Delta v_0(z_0, c), c),$$

through the mapping $v_0 \mapsto v_0 + \Delta v_0(z_0, c)$. Again restricting to the planar case, then every generic fix point $z_0$ of the reduced map for $E = c$ can by the implicit function theorem be continued, defining a curve $z_0 = z_0(E)$ of fix points. Through $P_0$ and the circle map we obtain an invariant object homeomorphic to a cylinder. The question is whether these invariant cylinders persist the perturbation of eccentricity. The problem is again a problem of small divisors and we have to rely on KAM theory.

One would only expect the persistence of circles for which the frequencies of the circle map $\nu \mapsto \nu + \Delta \nu$ are sufficiently irrational.

Similarly, for any invariant curve, $P_0$ defines an invariant object homeomorphic to a 2-torus. We would expect the persistence of invariant tori for which the induced symplectic mapping is sufficiently irrational. The remaining interesting question is of course: what is the measure of the invariant curves that persist? This is not obvious.
5.6 Conclusions

We shall not make any rigorous attempt on stating and proving the conjectures above within this research. For other possible future work, one could perhaps also continue with numerical computations similar to those done in [18] on the orbiting spring problem and investigate the effects of eccentricity on the stability of symmetric periodic orbits within the billiard model. A rigorous investigation of the effects of dissipation is, however, believed to be more capable of attracting significant interest from the space community.

5.6 Conclusions

In this chapter the different tether models have been related mathematically and it has been established in what limits they may provide useful models of tether dynamics. Firstly, the massive tether model was linked to the slack-spring model through a conjecture on the limit of vanishing thickness. Then the slack-spring model was related to the billiard model in the limit of an inextensible tether. Next, the motion of the dumbbell model was identified within the dynamics of the billiard model through a theorem on the existence of invariant curves. Finally, numerical computations provided some insights into the dynamics of the billiard map for the case of an underlying circular orbiting centre of mass.

In the following chapter we will develop a numerical integrator of the conservative massive tether model.
Figure 5.4: The periodic orbits corresponding to the periodic points of the billiard map indicated in Fig. 5.3. Unstable and stable periodic orbits are visualised. By the implicit function theorem, the periodic points can be continued onto neighbouring energy surfaces.
Figure 5.5: Largest Lyapunov exponents associated with out-of-plane perturbations for $E = 1$ resp. $E = 5$. 
A Unification of Non-Dissipative Models of Tethered Satellites
6.1 Introduction

In this chapter a symplectic, variational integrator of the massive tether model is developed which is used to investigate the necessity of the inclusion of resistance against bending in an example of an orbiting rubber tether connecting point masses. Furthermore, numerical evidence for the validity of the conjecture is provided.

6.2 A variational integrator for space tether dynamics

In the numerical integration of a hyperbolic partial differential equation where the phase space is infinite dimensional, not only time has to be discretised, it is also necessary to replace the phase space with an appropriate finite dimensional projection. Several choices can be made. For example the Euler-Lagrange equations may
be discretised directly using finite differences. This projection will in general not be Hamiltonian. Moreover, the differential operators appearing in the Euler-Lagrange equations are of higher order than those appearing in the variational form. A discretisation of the higher order operators requires additional smoothness which may in turn decrease the rate of convergence [139]. In particular, it is often the case that the existence of a solution is only guaranteed in the weak- or variational setting. Variational integrators circumvent these issues by discretising the variational form directly. Since this method ensures that the discretised system of equations is Hamiltonian, one can conserve geometrical structure through standard symplectic integrators [129, 88, 58, 101].

6.2.1 Space discretised Lagrangian

The Lagrangian of the massive tether model is ((4.1) and (4.2) with the addition of bending (4.13), repeated here for convenience):

\[
L = K - P, \\
K = \frac{1}{2}m_x |dx|^2 + \frac{1}{2}m_y |dy|^2 + \frac{1}{2}\rho \int_0^l |\partial_s r|^2 ds,
\]

\[
P = -\mu \frac{m_x}{|x|} - m_y |y| - \mu \rho \int_0^l |r|^{-1} ds + \frac{EA}{2} \int_0^l (|\partial_s r| - 1)^2 ds \\
+ \frac{EI}{2} \int_0^l |\partial_s^2 r|^2 ds.
\]

This is discretised spatially using a sequence of local Hermite elements [139]. The Lagrangian is defined on a set of \( r \)-functions whose second order spatial derivatives exist almost everywhere and the functions satisfy the boundary conditions \( r|_{s=0} = \mathbf{x} \) and \( r|_{s=l} = \mathbf{y} \). The Hermite elements form a non-orthogonal basis of the corresponding Sobolev space.

The sequence of Hermite elements is defined as follows. For given \( N \), the interval \([0, l]\) is divided into \( N \) equally sized intervals of length \( h = l/N \) with \( N + 1 \) nodal points. For each nodal point \( k \in \{1, \ldots, N + 1\} \) two functions \( \phi_k^1 \) and \( \phi_k^2 \) are defined as shown in Fig. 6.1. The functions \( \phi_k^1 \) and \( \phi_k^2 \) both have their support on the interval \( ((k - 1)h, (k + 1)h) \) and \( \phi_k^1 (kh) = 1 \) with \( \frac{d}{dh} \phi_k^1 (kh) = 0 \), while \( \phi_k^2 (kh) = 0 \) with \( \frac{d}{dh} \phi_k^2 (kh) = 1 \). The first and last elements \( \phi_1^1, \phi_1^2, \phi_{N+1}^1, \phi_{N+1}^2 \) have support on \( (0, h) \) and \( (Nh, (N + 1)h) \), respectively.

With these basis functions there exists an expression for \( r \) in each element. In
6.2 A variational integrator for space tether dynamics

Figure 6.1: Hermite (cubic) elements

particular for \( s \in [(k-1)h, kh] \) this reads:

\[
\begin{aligned}
    r_i(t, s) &= \alpha_{i,k-1}(t) + \alpha'_{i,k-1}(t)(s - (k-1)h) \\
    &+ (3\alpha_{i,k}(t) - 3\alpha_{i,k-1}(t) - \alpha'_{i,k}(t)h - 2\alpha'_{i,k-1}(t)h)(s - (k-1)h)^2/h^2 \\
    &+ (2\alpha_{i,k-1}(t) - 2\alpha_{i,k}(t) + \alpha'_{i,k}(t)h + \alpha'_{i,k-1}(t)h)(s - (k-1)h)^3/h^3,
\end{aligned}
\]

for \( i = 1, 2, 3 \), where:

\[
\begin{aligned}
    r_i(t, (k-1)h) &= \alpha_{i,k-1}(t), \\
    r_i(t, kh) &= \alpha_{i,k}(t), \\
    \partial_s r_i(t, (k-1)h) &= \alpha'_{i,k-1}(t), \\
    \partial_s r_i(t, kh) &= \alpha'_{i,k}(t).
\end{aligned}
\]

To ensure that \( r \) satisfies the boundary conditions, we set \( \alpha_{i,0} = x_i \) and \( \alpha_{i,N+1} = y_i \).

See [139] for further details on the finite element method.

The integrals

\[
\int_{(k-1)h}^{kh} (|\partial_s r| - 1)^2 ds,
\]

with \( r \) as in (6.1), cannot be evaluated analytically. Instead the Simpson's 3/8 quadrature rule is applied to approximate these integrals:

\[
\int_a^b f(x)dx = \frac{3}{8}(b-a) \left( f(a) + 3f \left( \frac{2a+b}{3} \right) + 3f \left( \frac{a+2b}{3} \right) + f(b) \right) + \frac{(b-a)^3}{6480} f^{(4)}(\xi),
\]

for some \( \xi \in (a, b) \). Notice, that the order is consistent with the order of the Hermite discretisation.

From this spatial discretisation a finite-dimensional Hamiltonian system is obtained which may be integrated in time by applying a standard symplectic method.
For the parameter values considered henceforth, the problem has different time-scales, particularly when $E$ is large. In the examples that follow an implicit method is therefore used, namely the 3-stage Gauss-Legendre symplectic scheme, see e.g. [129] p. 32.

### 6.3 Orbiting tether system

Here an orbiting tether system is considered. The gravitational field near a circular orbit is linearised and the equations are written in the rotating frame. This is an approximation used throughout the literature, see e.g. [9]. The equations then take the following form:

\[
\begin{align*}
\rho \ddot{r} + \frac{EI}{I^2 \omega^2} \dot{\theta} \dot{r} - \frac{EA}{I^2 \omega^2} \partial_s (a_1(\theta, r)) \partial_r r &= \rho g r + 2 \rho l \omega^{-1} \omega \wedge \partial_r r, \\
m_x \ddot{x} &= -\frac{EI}{I^2 \omega^2} \dot{\theta} \dot{r} |_{\theta=0} + \frac{EA}{I^2 \omega^2} a_1(\theta, r) \partial_r r |_{\theta=0} + m_x g x + 2 m_x \omega^{-1} \omega \wedge \partial_r x, \\
m_y \ddot{y} &= \frac{EI}{I^2 \omega^2} \dot{\theta} \dot{r} |_{\theta=1} - \frac{EA}{I^2 \omega^2} a_1(\theta, r) \partial_r r |_{\theta=1} + m_y g y + 2 m_y \omega^{-1} \omega \wedge \partial_r y,
\end{align*}
\]}

(6.2)

where

\[ g = \text{diag}(3, 0, -1), \]

and $\omega = (0, 0, \omega)$ is the orbital angular velocity. By Kepler's third law $\omega = \sqrt{\mu/R^3}$.

The first two components are the local vertical and local horizontal. The local horizontal is tangent to the circular orbit and the local vertical is perpendicular to this in the orbital plane, see Fig. 6.2. The third coordinate is along the direction of $\omega$. Furthermore, for simplicity a new time-variable $t = \omega t$ has been introduced and the spatial variable has been scaled so that $s \in (0, 1)$. Finally, $r, x, y$ have been non-dimensionalised with respect to $l$. It is natural to define the following non-dimensional numbers:

\[
\begin{align*}
\kappa_t &= \frac{EA}{\rho l^2 \omega^2}, & \kappa_s &= \frac{EA}{ml \omega^2}, \\
\rho_t &= \frac{EI}{l^4 \omega^2}, & \rho_s &= \frac{EI}{ml^3 \omega^2}.
\end{align*}
\]
For a circular tether cross-section: 
\[ A = \pi \frac{5}{4} h^2, \quad \rho_l = \frac{5}{64} h^2, \quad \text{and} \quad I = \frac{5}{64} h^4. \]

The equations (6.2) are the Euler-Lagrange equations of the Lagrangian:

\[
L = \frac{1}{2} m_x \dot{x}^2 + \frac{1}{2} m_y \dot{y}^2 + \frac{1}{2} \rho_l \int_0^l |\dot{\rho}_s r|^2 ds \\
+ \frac{1}{2} m_x (x, \dot{x}) + \frac{1}{2} m_y (y, \dot{y}) + \frac{1}{2} \rho_l \int_0^1 (r, \dot{r} v) \\
+ m_x (x, \dot{x} \wedge \omega) + m_y (y, \dot{y} \wedge \omega) + \rho_l \int_0^1 (r, \partial_r r \wedge \omega) ds \\
+ \frac{\rho_l \mu_1}{2} \int_0^l (|\partial_s r| - 1)^2 ds + \frac{\rho_l \mu_2}{2} \int_0^l |\partial_s x|^2 ds,
\]

where \( \dot{} = \frac{d}{dt} \).

Figure 6.2: A tethered satellite system near a circular orbit. The invariance of the symmetric configuration about the circular orbit is exploited in the numerical example. Here 1 and 2 show the local vertical and horizontal directions, respectively.

Attention is restricted to the dynamics in the orbital plane writing \( r = (r_1, r_2) \), \( x = (x_1, x_2) \), \( y = (y_1, y_2) \), and \( m_x = m_y = m \). Then the following equations:

\[
\begin{align*}
    r_1(s) + r_1(1-s) &= 0, \\
    \dot{r}_1(s) + \dot{r}_1(1-s) &= 0, \\
    r_2(s) + r_2(1-s) &= 0, \\
    \dot{r}_2(s) + \dot{r}_2(1-s) &= 0,
\end{align*}
\]

\( s \in [0, 1] \),

define the fixed point set of a discrete symmetry operation and hence an invariant set. An example of the configuration on this invariant set can be seen in Fig. 6.2. In
particular, one may notice that on this set \( x + y = 0 \) and \( \dot{x} + \dot{y} = 0 \), and the system centre of mass

\[
e = \chi x + \chi y + (1 - 2\chi) \int_0^l r ds,
\]

where \( \chi = \frac{m}{2m + \rho} \), satisfies \( e = 0 \). This symmetry is numerically very useful since it allows us to only evaluate half of the vector field, e.g. \( s \in (0, 1/2) \).

Similar parameters to those in [83, 89] are considered: \( E = 0.01 \) GPa, \( m_x = m_y = 0.1 \) kg, \( \rho_l = 970 \) kg/m³, a thickness of \( h = 5.5 \) mm, and a length of \( l = 1 \) m so that

\[
\kappa_l = 1.03 \times 10^4 \text{s}^{-2} \times \omega^{-2}, \quad \mu_l = 1.95 \times 10^{-2} \text{s}^{-2} \times \omega^{-2}.
\]

For comparison with the results in [83, 89] let \( \omega = 1 \) s⁻¹. A more realistic value \( \omega \approx 10^{-2} \) s⁻¹ for a low Earth orbit would require higher precision numerics. Nevertheless, it is anticipated that these values would lead to similar conclusions.

The effect of the bending term is more severe in some regions compared to others. For example the two systems with and without bending both share the two equilibria where the system is either along the local vertical or local horizontal. In fact, since the local vertical equilibrium is stable, the difference between the two models is not expected to be dramatic near this configuration. Two different initial conditions are therefore considered, one in a region where the bending effects are small and another in a region where they cannot be neglected.

First an unstrained initial condition is considered near the stable relative equilibrium pointing along local vertical. The initial linear velocity distribution is shown in Fig. 6.3 configuration (a). This initial condition corresponds to a stable and regular initial condition for the billiard map. For comparison a fixed time step of \( 10^{-4} \) for both \( EI = 0 \) and \( EI \neq 0 \) was used together with 41 grid points. A tolerance of \( 10^{-12} \) was set for the Newton iteration performed within the implicit integrator.

In Fig. 6.4 (a) the tether motion of the half of the system furthest from the Earth is shown. Results with and without bending are shown via full and dotted lines, respectively. Fig. 6.4 (b) shows the difference in Euclidean norm between the endpoints of the two systems. As can be seen from this figure, the difference remains small. However, a slight secular drift is observed towards the end of the integration. Nevertheless, it is concluded that the two systems evolve similarly, suggesting that the effect of bending remains small. This is confirmed in (c) where the evolution of the ratio of bending energy, see (4.11), to energy is shown. Though increasing from an initially zero value the bending energy only contributes about 0.25% of the total energy. In (d) the change in energy is shown for both \( EI = 0 \) and \( EI \neq 0 \). When \( EI = 0 \) a similar behaviour to that observed in [83, 89] is seen: the energy is secular drifting. With bending \( (EI \neq 0) \), however, no secular change in energy is observed.
6.3 Orbiting tether system

The velocity distribution is linear between the two end point velocities. The dynamics of the initial configurations along local vertical (a) and along local horizontal (b) are seen in Fig. 6.4 and Fig. 6.5, respectively.

To provide a theoretical explanation of this, we shall, as in section 5.3, imagine \( x \) and \( y \) to be fixed along the first inertial axis in free space in the plane with \( x = (0, 0) \) and \( y = (1 - d, 0) \), \( d < 1 \). This gives

\[
\rho \partial_t^2 \mathbf{r} + EI \partial_s^4 \mathbf{r} = EA \partial_s (a_1 (|\partial_s \mathbf{r}|) \partial_s \mathbf{r}),
\]

\( \mathbf{r}|_{s=0} = 0, \mathbf{r}|_{s=1} = (1 - d, 0), \) and \( \partial_s^2 \mathbf{r}|_{s=0,1} = 0 \). Linearisation about the compressed equilibrium \( \mathbf{r} = ((1 - d)s, 0) \) gives (see also (5.31)):

\[
\rho \partial_t^2 \mathbf{r} + EI \partial_s^4 \mathbf{r} = EA \text{diag}(1, -d/(1 - d)) \partial_s^2 \mathbf{r}.
\]

Through the ansatz \( \mathbf{r} = (r_1^{(n)}, r_2^{(n)}), r_j^{(n)} = \exp(i \omega_j^{(n)} \tau) \sin(n \pi s), j = 1, 2 \) it is observed that if \( EI = 0 \) then all modes \( \sin(n \pi s) \) are unstable. The numerical truncation will therefore continuously neglect unstable terms. On the other hand, for the regularised system \( EI \neq 0 \) only a finite number of eigenmodes will be unstable.

Next, an initial configuration along the local horizontal is considered as shown in Fig. 6.3 (b). The velocity is directed so that the satellites are initially approaching each other and rotating anti-clockwise (retrogade). Again for comparison between the two systems with \( EI = 0 \) and \( EI \neq 0 \) a fixed time step of \( 10^{-5} \) for both systems was used together with 41 grid points. In Fig. 6.5 (a) the motion of the two systems is shown. In (b) the distance between the satellites is shown. From both (a) and (b) it is obvious that the two systems quickly diverge. For slightly perturbed initial conditions a similar behaviour is observed. In (c) the ratio of bending energy to energy is shown for \( EI \neq 0 \). This example demonstrates that the resistance against bending may be extremely significant and in practice helps keep satellites apart.
Figure 6.4: In (a) the tether motion of the half of system furthest from the Earth is shown. Again results with and without bending are shown via full and dotted lines, respectively. Figure (b) shows the difference in the distance between the end-points of the two systems. This difference remains small, though a slight secular drift is observed. In (c) the ratio of the bending energy, see (4.11), to energy is shown. Figure (d) show the change in energy. For $EI = 0$ a secular drift in energy is visible.
6.4 Numerical evidence of conjecture

To provide numerical evidence of Conjecture 1 a unit length tether system in free space is considered. From (5.22) and (5.23) with \( \mu = 0 \):

\[
\begin{align*}
\hbar^2 (\partial_t^2 r + \partial_s^4 r) &= \partial_s (a_1 (|\partial_s r|) \partial_s r), \\
\partial_t^2 x &= (a_1 (|\partial_s r|_{s=0}) \partial_s r) |_{s=0} - \hbar^2 \partial_s^2 r |_{s=0}, \\
\partial_t^2 y &= -(a_1 (|\partial_s r|_{s=1}) |\partial_s r| |_{s=1}) + \hbar^2 \partial_s^2 r |_{s=1}, \\
r |_{s=0} = x, \quad r |_{s=1} = y, \\
\partial_s^2 r |_{s=0,1} = 0.
\end{align*}
\] (6.3)

Figure 6.5: In (a) and (b) the tether motions with and without resistance against bending, respectively, are shown. For identical initial conditions the two systems diverge very rapidly. In (c) the distance between the satellites is shown. In (d) the ratio of bending energy to energy is shown for \( EI = 0 \) and \( EI \neq 0 \). The bending is significant.
As before, attention is restricted to the planar motion for which an invariant set is given by:

\[
\begin{aligned}
    r_1(s) + r_1(1-s) &= 0, & \dot{r}_1(s) + \dot{r}_1(1-s) &= 0, \\
    r_2(s) - r_2(1-s) &= 0, & \dot{r}_2(s) - \dot{r}_2(1-s) &= 0, \\
    r_2(s) - r_1(s) &= 0, & \dot{r}_2(s) - \dot{r}_1(s) &= 0, \\
    \cos^2(5s) - \sin^2(1 - s) &= 0, & \cos^2(5s) - \sin^2(1 - s) &= 0.
\end{aligned}
\]

In particular \(x_1 + y_1 = 0, \dot{x}_1 + \dot{y}_1 = 0\) while \(x_2 - y_2 = 0, \dot{x}_2 - \dot{y}_2 = 0\).

This problem should have behaviour similar to that predicted by Conjecture 1. However, the straight line configuration is invariant in free space and in particular, by continuous dependency on initial conditions, given any \(T > 0\) and \(\delta > 0\) there exist initial conditions sufficiently small such that after time \(T\) the solution is \(O(\delta)\)-close to this invariant line. These initial conditions, which are not covered by Conjecture 1, may, however, be exponentially small in \(h\) and \(\delta\). Attention is therefore restricted to small, but not too small, perturbations \((\sim 10^{-9})\) about the invariant line for \(|x - y| \geq 1\).

At \(t = 0\), let \(x_0 = (1/2,0)\) with \(\dot{x}_0 = (-0.1,0)\), \(r_0 = ((1-s)x_0 + sy_0, 10^{-9} \times \sin \pi s)\), and \(r_0 = (y_0 - x_0, 0) = (0.2,0)\), so that the end-points are initially approaching each other. In Fig. 6.6 (a), (b), and (c) the dynamics of the tethered system is seen for one unit of time and \(h^2 = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\), respectively.

In the Newton scheme used to solve for the solution of the implicit scheme a tolerance was set of \(10^{-12}\). For the four scenarios \(h^2 = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\) fixed time steps of \(10^{-4}, 10^{-5}, 2 \times 10^{-6}, 10^{-6}\), respectively, were used. The changes in energy are seen in Fig. 6.7. They are bounded and conserved to the order of \(10^{-13}\).
6.4 Numerical evidence of conjecture

Figure 6.6: The dynamics of the tethered system for $h^2 = 10^{-3}, 10^{-4}, 10^{-5}$, and $10^{-6}$, respectively. The initial conditions are: $x_0 = (1/2, 0)$ with $\dot{x}_0 = (-0.1, 0)$, $r_0 = (1 - s)x_0 + sy_0 + (0, 10^{-9} \times \sin \pi s)$, and $\dot{r}_0 = y_0 - x_0 = (0.2, 0)$. 
According to the conjecture $\dot{x} \approx \dot{x}_0 = \text{const.}$, should be expected, within this period of time so that $x \approx x_{\text{slack}} = \ell \dot{x}_0 + x_0$. In Fig. 6.9 (a) $\Delta x_1(t) = |x_1(t) - x_{\text{slack}}|$ is plotted for $h^2 = 10^{-3}$, $10^{-4}$, $10^{-5}$, and $10^{-6}$. The difference $\Delta x_1$ decays with decreasing values of the thickness $h^2$ in accordance to the conjecture. The difference $\Delta x_1(1)$ as a function of $h^2$ is seen in Fig. 6.10 in a log-log scale. The straight lines connecting the point corresponding to $h = 10^{-3}$ with $h = 10^{-6}$, $10^{-4}$, and $10^{-5}$ have slopes of values $\approx 0.96$, $\approx 0.91$, and $\approx 0.89$, respectively. It is therefore estimated that $p/2 \approx 1$ in Conjecture 1.

On the other hand, by integrating backwards the satellites will initially enter the region $|x - y| > 1$ in which the predicted effect of the tether on the boundaries equals that of a spring. By integrating backwards to $t = -3$ the satellites will also eventually enter the slack region, cf. Fig. 6.8 which shows the evolution of the norm $|x - y|$ for $h^2 = 10^{-3}$. Particularly, Fig. 6.8 shows that $|x - y| = 1$ for $t \approx -2.2$. In Fig. 6.9 (b)
6.4 Numerical evidence of conjecture

Δx₁, now defined by:

\[
\Delta x_1(t) = \begin{cases} 
  |x_1(t) - x_1^{\text{spring}}| & \text{for } |x - y| \geq 1, \\
  |x_1(t) - x_1^{\text{slack}}| & \text{for } |x - y| \leq 1,
\end{cases}
\]

is visualised for \( h^2 = 10^{-3}, 10^{-4}, 10^{-5}, \) and \( 10^{-6} \) over the period of 3 units of time. Again, the difference \( \Delta x_1(t) \) is seen to remain small within the spring region and in accordance to the conjecture the difference decays with the decreasing thickness \( h^2 \).

Figure 6.8: The evolution of the norm \( |x - y| \) during the backward integration for \( h^2 = 10^{-3} \).

![Figure 6.8](image)

Figure 6.9: The difference \( \Delta x_1(t) = |x_1(t) - x_1^{\text{spring}}| \) for different values of the thickness \( h^2 \). It is seen that \( \Delta x_1 \) decays with decreasing values of the thickness \( h^2 \) in accordance to the conjecture.

The difference \( \Delta x_1(-3) \), as a function of \( h^2 \), is shown in Fig. 6.10. Now, the straight lines connecting the point corresponding to \( h = 10^{-6} \) with \( h = 10^{-5}, 10^{-4}, \) and \( 10^{-3} \) have slopes of values \( \approx 0.96, \approx 0.89, \) and \( \approx 0.75 \), respectively. This is again consistent with an estimate of \( p/2 \approx 1 \).
Figure 6.10: The difference $\Delta x_1(1)$ and $\Delta x_1(-3)$, respectively, for different values of the thickness $h^2$. The points are in both cases approximately on a straight line with slope 1.

6.5 Conclusion

A symplectic integrator was developed for the regularised, massive tether model by using a spatial discretisation by Hermite elements of the variational form. Using this integrator the motion of an orbiting tether system with and without the inclusion of the regularizing bending term was compared. The examples showed that the effects of bending may be severe even on very short time scales, at least when the tether is in compression. For an initial condition near the stable relative equilibrium it was shown that the two systems with and without bending evolved similarly with the bending energy remaining small. However, even for this example the system without bending was more sensitive to numerical instabilities and the energy difference grew because of the development of shock fronts. Finally, numerical evidence was provided for the validity of a conjecture on the relationship between the massive tether model and the slack spring model: there exists a set of initial conditions for which the flow of the boundaries within the regularised massive tether model converge to the slack spring model in the limit of vanishing thickness of the stiff tether. The rate of this convergence was through the numerical experiments estimated to be of order $O(h^2)$ where $h$ is the thickness of the tether.

In the following chapter we will investigate the slack-spring model further via a Galerkin approximation of the full massive tether model.
7.1 Introduction

In Chap. 5 it was from an infinite dimensional model conjectured that in the limit of vanishing thickness of a stiff tether part of the dynamics was close to a finite dimensional model: the slack-spring model. This was further supported via numerical computations in Chap. 6. In this chapter these arguments are made rigorous on a Galerkin approximation of the tether system. In particular, the following Hamiltonian system arises through a two-element Galerkin approximation of the tether dynamics and by restricting to the plane and an invariant fix-point set of a discrete symmetry.
The Hamiltonian function is equipped with the non-canonical symplectic form \( \omega = du \wedge dU + dv \wedge dV \). Here \( k > 0 \) is a stiffness constant, \( \Gamma \geq 0 \) is the conserved angular momentum and \( \epsilon > 0 \) is related to the thickness of the tether which again is considered small. Due to the non-canonical symplectic form, Hamilton's equations become singularly perturbed or slow-fast:

\[
\begin{align*}
\dot{u} &= U, \\
\dot{v} &= V, \\
\dot{U} &= -\partial_u P_A, \\
\dot{V} &= -\partial_v P_A.
\end{align*}
\]

Within singular perturbation theory \((u, U)\) are called slow variables while \((v, V)\) are called fast variables: their velocities are of order \(O(1)\) and \(O(1/\epsilon)\), respectively. For \(\epsilon > 0\), it is convenient to introduce the slow time \(t \mapsto t/\epsilon\) such that (7.2) becomes:

\[
\begin{align*}
\dot{u} &= \epsilon U, \\
\dot{v} &= V, \\
\dot{U} &= -\partial_u P_A, \\
\dot{V} &= -\partial_v P_A,
\end{align*}
\]

which are still Hamiltonian with (7.1) but now equipped with \(\omega = \epsilon^{-1} du \wedge dU + dv \wedge dV\). If we formally set \(\epsilon = 0\) in (7.2) and (7.3), then two limit systems are obtained. In (7.2), the formal limit leads to algebraic equations: \(V = 0\) and \(\partial_u P = 0\). The set of points:

\[
M = M_1 \cup M_2,
\]

\[
M_1 = \{V = 0, u^2 + v^2 = 1, |u| < 1\},
\]

\[
M_2 = \{V = 0, v = 0\},
\]

satisfying these equations, see Fig. 7.1, is called the slow manifold and the corresponding system: \(\dot{u} = U, \dot{U} = -\partial_u P(u, v^*; 0), (u, v^*(u), U, 0) \in W\) is called the slow system. On the other hand, in (7.3), the formal limit leads to the fast, or frozen, system: \(\dot{v} = V, \dot{V} = -\partial_v P(u, v)\) with \(u\) now considered as a parameter.

One of the main tasks in singular perturbation theory is to determine the destiny of the slow manifold \(M\) for \(\epsilon > 0\) but small. Unless \(M\) is hyperbolic, then it is
unlikely that there is an invariant manifold nearby, because typical perturbations are believed to destroy it [97]. Therefore one usually aims for something less: almost invariance. For singular perturbed Hamiltonian systems with only one fast degree of freedom and an analytic Hamiltonian, [52] showed the existence of an almost invariant slow manifold nearby. By “almost” it is understood that the error field\(^1\) is of order \(O(e^{-C/\varepsilon})\), \(C > 0\) and therefore trajectories remain close to the slow manifold for exponentially long time. These arguments fail to be true in general for systems with multiple fast variables. Furthermore, the arguments rely on a lower bound of the frequency associated with the fast motion. This is violated near slow manifold bifurcations where the time scales become comparable.

In the Galerkin model, as it is also visualised in Fig. 7.1, the slow manifold \(M\) (7.4) bifurcates at \(u = \pm 1\) corresponding to a super-critical, \(Z_2\)-symmetric pitchfork bifurcation in the frozen system. The case of sub-critical pitchfork bifurcations in two degrees of freedom slow-fast systems has received attention elsewhere [46, 131, 137]. Particularly, the references show the persistence of a singular heteroclinic solution connecting equilibria on the slow manifold before and after the bifurcation. It is also shown that the heteroclinic connection remains close to the union of the normally hyperbolic branches of the slow manifolds before and after the perturbation. The situation we are addressing is similar in the sense that we prove the existence of a set that remains close to the union of the normally elliptic branches of the slow manifold. We will do so by showing that a large set of trajectories crosses the unperturbed

\(^1\)The error field is the normal component of the vector field restricted to the slow manifold, [97]
Separatrix lobes after the dynamic pitchfork bifurcation of $M$.

Separatrix crossing for $1 \frac{1}{2}$ degrees of freedom Hamiltonian systems with slowly varying parameters have been studied by a number of researchers, [16, 23, 61, 110, 33, 34, 35]. Only [23] provides rigorous results, while the others present results as careful estimates. Chow and Todd [23] investigate separatrix crossing in one degree of freedom Hamiltonian systems with a slowly varying parameter. The authors present relatively simple geometrically motivated proofs. However, to the author's knowledge, this type of behavior has not yet been investigated for two degrees of freedom system. Our results will rely on the results of [23] and an appropriate blow-up transformation.

We define the (continuous) action $J = J(u, U; c)$ as $1/2\pi$ times the area at $\epsilon = 0$ of the region $H \leq \epsilon$ in the $(v, V)$-plane for the given value of $(u, U)$:

$$J = \frac{1}{\pi} \int_{v_m}^{v_M} \sqrt{2\epsilon - U^2 - 2P(u, v)} dv,$$

where $v_m$ and $v_M$ are such that $J$ is continuous across the separatrix set. See figure Fig. 7.2. We shall show that the union of the normally elliptic branches of the slow manifolds persists in an adiabatic sense. More precisely, we show the following:

**Theorem 4** Fix $u_0$, an integer $k > 0$ and an energy constant $c > 0$. Then for sufficiently small $\rho$ and $\tilde{\epsilon} = \epsilon \rho^{-3}$ consider the set $N_c$ of initial conditions $(u_0, v_0, U_0, V_0) \in H^{-1}(c)$ with $I = O(\rho^3)$. Let $N_{c; V}$ denote the corresponding projection onto the $(u, V)$-plane with measure $O(\rho^3)$. Then we have:

(i) There exists $\tilde{N}_{c; V} \subset N_{c; V}$ with $\text{measure}(N_{c; V} \setminus \tilde{N}_{c; V}) = O(\rho^3 \tilde{\epsilon}^2 e^{-C_1/\tilde{\epsilon}})$, $C_1 > 0$, so that the change in action for initial conditions $(u_0, v_0, U_0, V_0) \in H^{-1}(c)$ that project to $(v_0, V_0) \in \tilde{N}_{c; V}$ is bounded by:

$$\Delta I = O(\rho^3),$$

for $k$ returns to $|u| = 1$ with $d_1|u| \neq 0$, and in particular the projection of the solution $(u(t), v(t), U(t), V(t))$ with initial conditions $(u_0, v_0, U_0, V_0) \in H^{-1}(c)$ with $(v_0, V_0) \in N_{c; V}$ onto the $(v, V)$-plane remains within a ball of radius $O(\rho^3)$ centered about $(v, V) = (\pm \sqrt{1 - u^2}, 0)$ while $|u| < 1$, and $(v, V) = (0, 0)$ while $|u| \geq 1$;

(ii) If the solution $(u(t), v(t), U(t), V(t))$ with initial conditions $(u_0, v_0, U_0, V_0) \in H^{-1}(c)$ with $(v_0, V_0) \in N_{c; V}$ does not return to $|u| = 1$, then $\Gamma^2 \geq 2c$, $|u| \geq 1$ and the projection onto the $(v, V)$-plane remains within a ball of radius $O(\rho^3)$ centered about $(0, 0)$ for an exponentially long time $O(e^{-C_2/\tilde{\epsilon}})$, $C_2 > 0$. □

The chapter is organised as follows: The first section describes the model and how we arrive at (7.1). The next section is devoted to the proof of Theorem 4.
7.2 The toy-model

Let $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$ be the positions of the end-points and, as in the previous two chapters, let $r = r(s)$ be the position of the tether described through the arc-length parameter $s$. The Lagrangian of a unit-density tether connecting point masses of unit-mass in free space and without bending resistance is then given by (see also (4.1) and (4.2)):

$$L = \frac{1}{2} |x|^2 + \frac{1}{2} |y|^2 + \epsilon^2 \int_0^1 |\dot{r}|^2 ds - \frac{k\epsilon^2}{2} \int_0^1 (|\partial_s r| - 1)^2 ds,$$

(7.5)

where $\epsilon$, rather than the $h$ previously used, represents the small perturbation parameter measuring the tether thickness. The scenario of interest here, cf. Conjecture 1, is when the stiffness satisfies $k = k\epsilon^{-2}$. The Galerkin approximation then consists of replacing $r$ with the approximation

$$r \approx \begin{cases} 
2(z - x)s + x & \text{for } s \in (0, 1/2], \\
2(z - y)(1 - s) + y & \text{for } s \in (1/2, 1),
\end{cases}$$

(7.6)

where $z = z(t)$ is the position of the “centre point” of the tether at $s = 1/2$. The system is seen in Fig. 7.3. Inserting (7.6) into (7.5) an “approximative” Lagrangian is obtained:

$$L_{app} = \frac{1}{2} \left( 1 + \frac{\epsilon^2}{6} \right) (|\dot{x}|^2 + |\dot{y}|^2) + \frac{\epsilon^2}{12} (z, \dot{x} + \dot{y}) + \frac{\epsilon^2}{6} |\dot{z}|^2
- \frac{k}{4} \left( (2(z - x)| - 1)^2 + (2(z - y)| - 1)^2 \right).$$
The system possesses rotational and translational symmetry. We first reduce by the translational symmetry by introducing centre of mass and relative coordinates:

\[ c = f_1 x + f_2 y + f_3 z, \]
\[ q = \frac{1}{2}(x - y), \]
\[ Q = z - \frac{1}{2}(x + y), \]

where \( f_1 = f_2 = \frac{1}{4}\sqrt{2\epsilon^2} \) and \( f_3 = \frac{1}{2}\sqrt{2\epsilon^2} \). Here \( q \) and \( Q \) are illustrated in Fig. 7.3.

![Figure 7.3: Toy model.](image)

Then the Lagrangian reads:

\[ L_{app} = \frac{1}{2} c^2 + \frac{1}{2} \alpha_c |q|^2 + \frac{1}{2} \epsilon^2 \beta_c |Q|^2 - \frac{k}{4} \left( (|Q - q| - 1)^2 + (|Q + q| - 1)^2 \right), \]

where

\[ \alpha_c = 2 \left( 1 + \frac{\epsilon^2}{6} \right), \quad \beta_c = \frac{1}{12} \frac{8 + \epsilon^2}{2 + \epsilon^2}. \]

By the symmetry, \( c \) is cyclic and a reduced Lagrangian, say \( l_{app} \), may therefore be introduced by:

\[ l_{app} = \frac{1}{2} \alpha_c |q|^2 + \frac{1}{2} \epsilon^2 \beta_c |Q|^2 - \frac{k}{4} \left( (|Q - q| - 1)^2 + (|Q + q| - 1)^2 \right). \]
Next, to reduce by the rotational symmetry we introduce a rotating orthonormal coordinate system with origin at \( \frac{1}{2}(x+y) \), first axis along direction of \( q \) and third axis perpendicular to the plane containing \( x, y \) and \( z \). In particular, we let 
\[
q = R \sigma_1, \quad Q = R \sigma_2,
\]
with \( R \in SO(3) \) defining the rotating coordinate system and where 
\[
\sigma_1 = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix},
\]
are the coordinates of \( q \) and \( Q \), respectively, in the rotating coordinate frame. Therefore, we obtain:
\[
R^T q = \hat{\Omega} \sigma_1 + \sigma_1, \quad (7.8)
\]
\[
R^T Q = \hat{\Omega} \sigma_2 + \sigma_2, \quad (7.9)
\]
where \( \hat{\Omega} = R^T \hat{R} \in so(3) \) is a skew-symmetric matrix. The hat-map which is defined by
\[
\hat{\sigma} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \in so(3), \quad (7.10)
\]
for every \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3 \), defines an isomorphism between the Lie-algebra \( so(3) \) and \((\mathbb{R}^3, \cdot \wedge \cdot)\). We shall also make use of the hat-map isomorphism in the following two chapters. Inserting (7.8) and (7.9) into (7.7) therefore gives
\[
J_{app} = \frac{1}{2} \langle \hat{\sigma}, M \hat{\sigma} \rangle + \frac{1}{2} \langle \Omega, \hat{\Omega} \rangle + \langle \Omega, C^{(1)} \rangle + \langle \Omega, C^{(2)} \rangle + \langle \Omega, I \rangle
\]
\[
- \frac{k}{4} \left( r_1 - 1 \right)^2 - \frac{k}{4} \left( r_2 - 1 \right)^2, \quad (7.11)
\]
where \( \sigma = (u, v, w), r_1 = \sqrt{\left( u + w \right)^2 + v^2}, r_2 = \sqrt{\left( u - w \right)^2 + v^2} \) and \( M, C \) and \( I \) are given by:
\[
M = \text{diag} \left( \alpha_1 \epsilon_2 \beta_1, \epsilon^2 \beta_2, \epsilon \beta_3 \right),
\]
\[
C = \epsilon^2 \beta_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w & -v \end{pmatrix},
\]
\[
I = \alpha_1 \hat{\alpha}_1 \hat{\sigma}_1 + \epsilon^2 \beta_2 \hat{\alpha}_2 \hat{\sigma}_2,
\]
and in particular \( I_{33} = \alpha_1 v^2 + \epsilon^2 \beta_2 (v^2 + w^2) \). By introducing:
\[
A = I^{-1} C = I_{33}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w & -v \end{pmatrix}, \quad B^{-1} = M - A^T I A,
\]
we re-write (7.11) as

\[ l_{app} = \frac{1}{2}((\Omega + A\dot{\sigma}), I(\Omega + A\dot{\sigma})) + \frac{1}{2}(\dot{\sigma}, B^{-1}\dot{\sigma}) - \frac{k}{4}(r_1 - 1)^2 + (r_2 - 1)^2. \]

This is motivated by the gauge invariant field theory in [93]. See also [76]. The Legendre transformation is then easily computed for \( \epsilon \neq 0 \):

\[ G = \partial_l l_{app} = I(\Omega + A\dot{\sigma}), \]
\[ \pi = \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \partial_\sigma l_{app} = A^T G + B^{-1}\dot{\sigma}, \]

so that the Hamiltonian becomes

\[ H = (G, \Omega) + (\pi, \dot{\sigma}) - l_{app} \]
\[ = \frac{1}{2}(\pi - A^T G, B(\pi - A^T G)) + \frac{1}{2}(G, I^{-1}G) + \frac{k}{4}(r_1 - 1)^2 + (r_2 - 1)^2. \]

By Lie-Poisson reduction [64, 100] this Hamiltonian is equipped with a reduced Poisson structure:

\[ \{ f, g \}(z) = \partial_x f^T A(x) \partial_x g, \]

where \( z = (G, \sigma, \pi) \) and

\[ A(x) = \begin{pmatrix} \hat{G} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \]

Hamilton's equations therefore become:

\[ \dot{G} = G \wedge \partial_G H = G \wedge \left(I^{-1}G - AB\left(\pi - A^T G\right)\right), \]
\[ \dot{\sigma} = \partial_\pi H = B\left(\pi - A^T G\right) \]
\[ = \left( \begin{array}{cc} \frac{1}{\alpha_x} & 0_{1 \times 2} \\ 0_{2 \times 1} & \epsilon^{-2} b \end{array} \right) \left( \begin{array}{c} U \\ V \\ W \end{array} \right) + \frac{G_3}{\alpha_x u^2} \left( \begin{array}{c} 0 \\ -v \\ v \end{array} \right), \]
\[ \dot{\pi} = -\partial_\sigma H, \]

where we have introduced the \( 2 \times 2 \) matrix:

\[ b = \frac{1}{\alpha_x \beta_w u^2} \left( \begin{array}{cc} I_{33} - \epsilon^2 \beta_v v^2 & -\epsilon^2 \beta_v w v \\ -\epsilon^2 \beta_w v w & I_{33} - \epsilon^2 \beta_w w^2 \end{array} \right). \]
Now, if $\overline{V} = V/\varepsilon$ and $\overline{W} = W/\varepsilon$ then the equations:

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} = \mathcal{B} \begin{pmatrix}
U \\
V \\
W
\end{pmatrix} + \frac{\varepsilon G_3}{\alpha \varepsilon \bar{u}^2} \begin{pmatrix}
0 \\
0 \\
-\bar{v}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\dot{U} \\
\dot{V} \\
\dot{W}
\end{pmatrix} = -\partial_{\sigma} H,
\]

where

\[
\mathcal{B} = \begin{pmatrix}
\frac{1}{\alpha \varepsilon} & 0_{1 \times 2} \\
0_{2 \times 1} & b
\end{pmatrix},
\]

have the classical form of a singular perturbed Hamiltonian system. We omit the bar henceforth. These equations are still Hamiltonian with the Hamiltonian function:

\[
H = \frac{1}{2} (\pi - A^T G, B (\pi - A^T G)) + \frac{1}{2} (G, G^{-1} G) + \frac{k}{4} (r_1 - 1)^2 + (r_2 - 1)^2,
\]

and the Poisson matrix:

\[
\Lambda(z) = \begin{pmatrix}
\tilde{G} & 0 & 0 \\
0 & 0 & \text{diag}(1, \varepsilon, \varepsilon) \\
0 & -\text{diag}(1, \varepsilon, \varepsilon) & 0
\end{pmatrix}.
\]

Here $(u, V)$ and $(w, W)$ are the fast variables and $(u, U)$ are the slow variables. By the reduction procedure it also follows that the total angular momentum $|G|$ is conserved. In fact, it is a Casimir function:

\[
\frac{1}{2} \frac{d}{dt} |G|^2 = \langle \dot{G}, G \rangle = \langle G \wedge \partial_{G} H, G \rangle = 0. \quad (7.12)
\]

### 7.2.1 Planar system with $\{w = 0, \dot{w} = 0\}$

The fix point set $\{w = 0, \dot{w} = 0\}$ of the tangent lift of the reflectional symmetry operation: $(u, v, w) \mapsto (u, v, -w)$ defines an invariant symplectic manifold on which the physical system remains an isosceles triangle with $r_1 = r_2$. We shall in the following restrict attention to this invariant manifold and planar motion where $G = (0, 0, \Gamma)$, $\Gamma = \text{conserved}$. The Hamiltonian then simplifies to

\[
H(u, v, U, V) = \frac{1}{2} \alpha^{-1} U^2 + \frac{1}{2} \beta^{-1} V^2 + P_A(u, v; \varepsilon),
\]

\[
P_A(u, v; \varepsilon) = \frac{\Gamma^2}{2} \alpha^{-1} u^2 + \frac{\varepsilon^2 \beta^{-1} v^2}{2} + P(u, v),
\]

\[
P(u, v) = \frac{k}{2} \left( \sqrt{u^2 + v^2} - 1 \right)^2,
\]
with symplectic form $\omega = du \wedge dU + edv \wedge dV$. Here $u$ and $v$ now measure the half-length between the end-points and the height of the isosceles triangle, respectively. Let $U \mapsto U = \alpha U, V \mapsto V = \xi V$ where $\xi = \sqrt{\lambda} \alpha$, and $e \mapsto \ell = \frac{\theta}{\xi}, \Gamma \mapsto \Gamma = \sqrt{\Gamma T}$, $k \mapsto \hat{k} = \alpha k$. Lets drop the primes. Finally, $t \mapsto t/\epsilon$ transforms the Hamiltonian system into (7.1), repeated here for convenience:

$$H(u, v, U, V) = \frac{1}{2} U^2 + \frac{1}{2} V^2 + P_A(u, v; \epsilon),$$

$$P_A(u, v; \epsilon) = \frac{1}{2} \frac{\Gamma^2}{2 u^2 + \epsilon^2 u^2} + P(u, v),$$

$$P(u, v) = \frac{k}{2} \left( \sqrt{u^2 + v^2} - 1 \right)^2,$$

$$w = \epsilon^{-1} du \wedge dU + dv \wedge dV.$$

By restricting attention to the set $\{w = 0, \dot{w} = 0\}$ we have obtained a system with only one fast degree of freedom and we therefore avoid having to deal with resonances. We will briefly consider the possibility of extending the result to the full two-element Galerkin approximation in section 7.4 below.

It is straightforward to show that the slow manifold $M_1 = \{u^2 + v^2 = 1, |u| < 1\}$ is normally elliptic, while $M_2 = \{v = 0, U = 0\}$ is normally hyperbolic for $|u| < 1$ and normally elliptic $|u| > 1$. See Fig. 7.1. For $|u| = 1$ the linearisation has two zero eigenvalues. The origin $(u, V) = 0$ of the frozen system therefore undergoes a pitchfork bifurcation for $|u| = 1$ such that for $|u| < 1$ there exist three equilibria, two stable and one unstable with a homoclinic orbit. The phase portrait of the frozen system is sketched in Fig. 7.4.

![Figure 7.4: Frozen system.](image)

The slow system is defined by the Hamiltonian

$$H^{(i)}_{\text{slow}} = H_0|M_i, \quad i = 1, 2.$$
In particular
\[
H^{(1)}_{\text{slow}}(u, U) = \frac{1}{2} U^2 + \frac{1}{2} \Gamma^2, \quad (7.14)
\]
\[
H^{(2)}_{\text{slow}}(u, U) = \frac{1}{2} U^2 + \frac{1}{2} \Gamma^2 + \frac{k}{2} (|u| - 1)^2,
\]
and the slow dynamics on $M_1$ and $M_2$ are equivalent to a system of free end-points and a system of end-points connected by a spring, respectively. Theorem 4 implies that a neighbourhood of the union of the normally elliptic part of $M$ is adiabatically invariant. The slow dynamics on $M$ and its neighbourhood are therefore close to the slack-spring. The theorem therefore proves a similar result to that conjectured in Conjecture 1 for a Galerkin approximation of the massive tether model.

Notice that $|u| \geq 1$ in (ii) in Theorem 4 since $M_1$ can only be locally invariant as trajectories eventually pass through $|u| = 1$. Furthermore, due to the symmetry $(v, V) \rightarrow (-v, -V)$, $M_2$ is invariant for any $\epsilon$. Through (7.14) this therefore implies that $\Gamma^2 \geq 2c$ in (ii) in Theorem 4. Also, by the invariance of $M_2$ the following follows from continuous dependency on initial conditions: for a given $T > 0$ and $\rho > 0$ there exists a $\delta = \delta(T, \rho) > 0$ such that for any $(v_0, V_0) \in B_{\delta}(0, 0)$ the solution satisfies $(v, V)(t) \in B_{\rho}(0, 0)$ for $0 \leq t \leq T$. This set of initial conditions does not behave like that of the slack-spring. This is partially the reason for the exclusion of an exponential small set in Theorem 4. The other contributing factor to the exclusion of an exponential small set is due to the presence of separatrices in the fast system for $|u| < 1$, see Fig. 7.4.

### 7.3 Proof of Theorem 4

The proof of Theorem 4 will be obtained through the following three steps:

1° Using averaging we show that a neighbourhood of $M_2$ in $(v, V)$-plane is almost invariant for $|u| \geq 1$.

2° We combine a blow-up transformation and results from Chow and Young (2004) to show that a neighbourhood of the section $|u| = 1$ with $d_t|u| < 0$ is mapped into the lobes of the separatrices at $|u| = 1 - a$, $a = a(\epsilon) > 0$, but small.

3° Again using averaging we show that a neighbourhood of $M_1$ is almost invariant for $|u| \leq 1 - a$. 
Averaging near $M_2$ for $|u| \geq 1$

We shall exploit the symmetry $(u, U) \mapsto (-u, -U)$ so that attention may be restricted to the bifurcation value $u = 1$. In particular, we will centre the slow coordinate about $u = 1$ through $u \mapsto \bar{u} = 1 - u$, $U \mapsto \bar{U} = -U$. We also define $\bar{P}_A(\bar{u}, \bar{v}; \varepsilon) = P_A(u, v; \varepsilon)$, $\bar{P}(\bar{u}, \bar{v}) = P(u, v)$. We shall henceforth again drop the primes. The frozen system is described by the Hamiltonian:

$$h(v, V; u) = \frac{1}{2} v^2 + \frac{k}{2} \left( \sqrt{v^2 + (1-u)^2} - 1 \right)^2 + O(\varepsilon^3),$$

and symplectic form $\omega = dv \wedge dV$ which is a one degree freedom integrable system. By the smallness of $\varepsilon$ the terms of order $O(\varepsilon^3)$ may be neglected. Then for $u \leq 0$ every point $(v, V)$ is on a periodic orbit and action angle coordinates may be introduced. The action $I$ is $1/2\pi$ times the area enclosed by the level set $h = \varepsilon$:

$$I = \frac{1}{2\pi} \int_{v_m}^{v_M} \sqrt{2c - 2P(u, v)} dv,$$

$$v_M(c, u) = -v_m(c, u) = \sqrt{\left(1 + \sqrt{2c}\right)^2 - (1-u)^2}.$$

In the following it is the aim to use averaging near $I = 0$ to move the angle dependency in the Hamiltonian to higher orders in $\varepsilon$. The averaging procedure relies on the frequency being bounded away from zero. Therefore extra care is required near the bifurcation point $u = 0$ where the linear frequency vanishes. Let $\mathcal{I}(c, u, v) = \sqrt{2c - 2P(u, v)} \geq 0$ be the integrand in (7.16). For $u \leq 0$ we obtain the following inequalities:

$$\mathcal{I}(c, u, v) \leq \mathcal{I}(c + \delta c, u, v) \leq \mathcal{I}(c + \delta c, 0, v),$$

$$v_M(c, u) - v_m(c, u) = 2v_M(c, u) \leq 2v_M(c + \delta c, u) \leq 2v_M(c + \delta c, 0),$$

for any $\delta c > 0$. Hence $\partial_h I|_{u=0} \geq \partial_h h$ and therefore:

$$\partial_h h|_{u=0} \leq \partial_I h.$$

The frequency $\partial_I h$ is therefore bounded from below by the frequency of the system with $u = 0$. Notice that this frequency only vanishes for $I = 0$. Moreover, for $u = 0$ an expansion of the frozen Hamiltonian about $(v, V) = (0, 0)$ gives:

$$h = \frac{1}{2} v^2 + \frac{k}{2} \left( \sqrt{1+v^2} - 1 \right)^2 + O(\varepsilon^3)$$

$$= \frac{1}{2} v^2 + \frac{k}{2} \left( \frac{v^4}{4} + O(\varepsilon^5) \right) + O(\varepsilon^5).$$
7.3 Proof of Theorem 4

Therefore consider \( p \) small and let \( v \mapsto \dot{v} = \rho^{-1}v, \ V \mapsto \dot{V} = \rho^{-2}V \) such that the truncation gives:\(^2\)

\[
\rho \dot{\rho} = \rho^2 \dot{V}, \quad \rho^2 \dot{V} = -\rho^3 \frac{k}{2} \rho^3 + \rho^5 \mathcal{O}(\rho^5).
\]

Now, by \( t \mapsto \rho t \) the Hamiltonian is transformed into:

\[
\dot{h} = \rho^{-4} h = \frac{1}{2} \dot{V}^2 + \frac{k}{2} \left( \frac{\dot{\rho}^4}{4} + \rho^2 \mathcal{O}(\rho^4) \right) + \mathcal{O}(\epsilon^2 \rho^{-2}).
\]

The constant \( \rho \) is small, but otherwise arbitrary, and it may therefore be assumed that \( \epsilon \ll \rho \). After some calculations it can be shown that the truncation in action-angle coordinates is given by

\[
\dot{h}(I) = \frac{c}{k^{3/3}} I^{2/3}, \quad c \approx 0.150.
\] (7.17)

The equations on the full space after applying \( v \mapsto \dot{v} = \rho^{-1}v, \ V \mapsto \dot{V} = \rho^{-2}V, \ u \mapsto \dot{u} = u, \ U \mapsto \dot{U} \) and \( t \mapsto \rho t \) are Hamiltonian with:

\[
\dot{H} = \rho^{-4} H = \frac{1}{2} \rho^{-4} \dot{I}^2 + \frac{1}{2} \dot{V}^2 + \rho^{-4} P(\dot{u}, \rho \dot{u}), \quad \dot{\omega} = \epsilon^{-1} \rho^{-3} \dot{d} \dot{u} + \epsilon^2 \dot{d} \dot{V}.
\]

Now, there exists, see e.g. [52], a generating function \( s(\dot{\phi}, \dot{V}; u) \) that via the equations:

\[
\dot{v} = \partial_{\dot{\phi}} s, \quad \dot{V} = \partial_u s,
\]

generates a transformation \( (\dot{v}, \dot{V}) \mapsto (\ddot{v}, \ddot{V}) \) with \( d\dot{v} \wedge d\dot{V} = d\ddot{v} \wedge d\ddot{V} \) that puts the frozen system given by \( \dot{h} \) into action-angle coordinates:

\[
\dot{h}(\ddot{v}, \ddot{V}; u) = \dot{h}(I; u),
\] (7.18)

where \( \ddot{v} = \sqrt{2I} \cos \dot{\phi}, \ \ddot{V} = \sqrt{2I} \sin \dot{\phi} \). As in [52] this transformation in \( (\dot{v}, \dot{V}) \)-plane can be lifted to a transformation on the entire phase space preserving \( \frac{1}{\epsilon^2} d\dot{u} \wedge d\dot{U} + d\dot{v} \wedge d\dot{V} \) via the function:

\[
S(\dot{u}, \dot{U}, \dot{v}, \dot{V}) = \frac{\dot{u} \dot{U}}{\epsilon \rho^3} + \mathcal{s}(\dot{u}, \dot{v}, \dot{V}),
\]

with \( \mathcal{s}(\dot{u}, \dot{v}, \dot{V}) = s(\dot{v}, \dot{V}; u) \), and the equations:

\[
\dot{u} = \epsilon \rho^3 \partial_u S = \dot{u}, \quad \dot{U} = \epsilon \rho^3 \partial_u S = \dot{U} + \epsilon \rho^3 \partial_u \mathcal{s}.
\]

\(^2\)We shall throughout use \( \mathcal{z}, \mathcal{H} \) or \( \mathcal{h} \) to denote blow-up variables and Hamiltonians, respectively. Moreover, \( \mathcal{z}, \mathcal{z}, \mathcal{z} \) will denote variables which are close to \( z \) and used successively in the averaging procedure to move angle dependency to higher order in \( \epsilon \).
The Slack-Spring Model as a Limit System of a Galerkin Approximation to the

Massive Tether Model

This transformation is identical in \( \dot{u} \) and \( \epsilon \rho^3 \)-close in \( \dot{U} \). The existence and regularity of this transformation for \( \epsilon \) sufficiently small follows from the implicit function theorem, see also [52]. The new Hamiltonian reads:

\[
\hat{H}(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \frac{1}{2} \rho^{-4} \left( \ddot{u}^2 + \frac{2r^2}{(1 - \ddot{u})^2} \right) + \dot{h}(\ddot{u}, \dot{I}) + \epsilon \rho^{-3} R_1(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}; \epsilon),
\]

where \( R_1(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}; \epsilon) = \epsilon^{-1} \rho \left( \hat{H} - \hat{H}^{(0)} \right) \). By (7.17) it follows that outside the cylinder \( \tilde{I} = 1 \) we have \( \omega \equiv \partial_\phi \tilde{h} \geq \frac{1}{27 \pi^2} > 0 \). Moreover, \( \hat{H} \) is real analytic outside \( \tilde{I} \geq 1 \) with \( \ddot{u} \leq 0 \). Through the generating function:

\[
S(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \frac{\dot{u} \dot{V}}{\epsilon \rho^{3}} + \epsilon \rho^{-1} s(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}),
\]

and the equations:

\[
\ddot{u} = \epsilon \rho^3 \partial_\phi S = \ddot{u} + \epsilon \rho^3 \partial_\phi s, \quad \ddot{v} = \partial_\phi S = \ddot{v} + \epsilon \rho^{-1} \partial_\phi s,
\]

we then transform the Hamiltonian into:

\[
\hat{H}(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \hat{H}^{(0)}(\ddot{u}, \ddot{U}, \frac{1}{2} \dot{v}^2 + \frac{1}{2} \ddot{V}^2)
+ \epsilon \rho^{-1} \left( \omega \partial_\phi s - R_1(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}; h) \right) + O(\epsilon^2 \rho^{-2}),
\]

where \( \dot{v} = \sqrt{2 \tilde{I}} \cos \phi \), \( \ddot{v} = \sqrt{2 \tilde{I}} \sin \phi \). To remove terms of \( O(\epsilon^2 \rho^{-1}) \) we choose \( s \) as the unique solution with zero average, [52] Lemma 4, satisfying:

\[
\partial_\phi s = \frac{R_1 - \ddot{R}_1}{\omega}, \quad \ddot{R}_1 = -\frac{1}{2\pi} \int_0^{2\pi} R_1(\ddot{u}, \sqrt{2 \tilde{I}} \cos \phi, \dot{U}, \sqrt{2 \tilde{I}} \sin \phi; \epsilon) d\phi.
\]

Then:

\[
\hat{H}(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \hat{H}^{(1)}(\ddot{u}, \dot{v}, \dot{I}) + O(\epsilon^2 \rho^{-2}),
\]

where \( \hat{H}^{(1)} = \hat{H}^{(0)} + \epsilon R_1 \). Since \( \hat{H} \) is real analytic outside \( \tilde{I} \geq 1 \) with \( \ddot{u} \leq 0 \) this procedure can be iterated \( n \) times to obtain, see also [52]:

\[
\hat{H}(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \hat{H}^{(n)}(\ddot{u}, \dot{v}, \dot{I}) + O(\epsilon^n \rho^{-n}),
\]

where \( \dot{I} = \dot{I} + O(\epsilon \rho^{-2}) \), or by un-doing the time scaling:

\[
\hat{H}(\ddot{u}, \ddot{v}, \dot{U}, \dot{V}) = \rho \hat{H}^{(n)}(\ddot{u}, \dot{v}, \dot{I}) + O(\epsilon^n \rho^{-n+1}).
\]
In fact, in [52] it is shown that the remainder can be made exponentially small using estimates by [108, 109] on the optimal \( n = n(\epsilon) = O(\epsilon^{-1}) \). If \( \Gamma^2 \geq 2c \) this therefore shows (ii) in Theorem 4. Henceforth we shall therefore assume \( \Gamma^3 < 2c \). Then on the level set \( H^{-1}(c), c > 0 \) it only takes \( O(1/\epsilon) \) time before \( u = 0 \) again, provided \( \rho \) is sufficiently small. Therefore:

**Lemma 1** While \( u \leq 0 \) and \( \tilde{I}(0) \geq 1 \):

\[
|\tilde{I}(t) - \tilde{I}(0)| \leq C\epsilon^{-1}, \quad C > 0,
\]

or since \( \tilde{I} = \tilde{I} + O(\epsilon\rho^{-1}) \):

\[
|\tilde{I}(t) - \tilde{I}(0)| \leq \tilde{C}\epsilon^{-1}, \quad \tilde{C} > 0,
\]

for \( \epsilon \) sufficiently small.

In terms of the original action \( I = \rho^3 \tilde{I} \) we have:

\[
|I(t) - I(0)| = O(\epsilon^2).
\]

while \( u \leq 0 \) and \( I \geq \rho^3 \). If \( I(0) < \rho^3 \) then either \( I(1/\epsilon) < \rho^3 \) or \( I(t_0) = \rho^3 \) for some \( t = t_0 \). For \( 1/\epsilon \geq t \geq t_0 \) Lemma 1 applies. Since \( \epsilon\rho^{-1} \ll 1 \) then \( \epsilon\rho^2 \ll \rho^3 \) and therefore as a corollary:

**Corollary 1** While \( u \leq 0 \) we have:

\[
|I(t) - I(0)| = O(\rho^3),
\]

for any \( I(0) = O(\rho^3) \).

In the following we show that a set of trajectories cross the (ghost) separatrices near the bifurcation and henceforth remain captured inside the lobes. We do so by applying a blow up transformation near the bifurcation \( u = 0 \) but with \( u > 0 \).

**2° Capture**

If \( u > 0 \) then \((v, V) = (0, 0)\) is an unstable equilibrium of the frozen system with homoclinic connections. From (7.15) it follows that the separatrix lobe is \( O(u^{1/2}) \) in \( v \)-direction and \( O(u) \) in \( V \)-direction. In particular, the area of the lobes satisfy:

\[
\text{Area of lobes} = O(u^{3/2}). \tag{7.19}
\]

Furthermore, since the quadratic part, in terms of \( v \) and \( V \), of the frozen system (7.15) is \( \frac{1}{2}V^2 - \frac{b}{2}uv^2 \), it follows that the growth rate of the unstable equilibrium is
Now again consider $p > 0$ small. Then if $u = p^2 \dot{u}$ and $U = \dot{U}$, then $v = p \dot{v}$, $V = p^2 \dot{V}$ blow up the separatrix lobes to $O(1)$ in both $v$ and $V$-direction. Similarly, the transformation of time $t \mapsto \tilde{t} = pt$ blow up the growth rate. Finally, we obtain a new Hamiltonian system:

$$\dot{H} = \rho^{-4} H = \frac{1}{2} \rho^{-4} \dot{U}^2 + \frac{1}{2} \dot{V}^2 + \dot{P}_A(\dot{u}, \dot{v}; c),$$

$$\dot{\omega} = \epsilon^{-1} \rho^{-1} d\dot{u} \wedge d\dot{U} + d\dot{h} \wedge d\dot{V},$$

where

$$\dot{P}_A(\dot{u}, \dot{v}; \rho) = \rho^{-4} P_A(p^2 \dot{u}, p \dot{v})$$

$$= k \left( \dot{u}^2 - \dot{u} \dot{v}^2 + \frac{1}{4} \dot{v}^4 + O(p^2) \right) + \frac{1}{2} \left( \rho^{-4} (1 - \rho^4 \dot{u}^2) + O(\epsilon^2 \rho^{-2}) \right).$$

Hamilton's equations now read:

$$\frac{d\dot{u}}{dt} = \epsilon p^{-3} \dot{U}, \quad \frac{d\dot{v}}{dt} = \dot{V}$$

$$\frac{d\dot{U}}{dt} = -\epsilon p \partial_u \dot{P}_A, \quad \frac{d\dot{V}}{dt} = -\partial_v \dot{P}_A.$$

For $\dot{u}$ to be slow we assume $\epsilon \ll \rho^3$. We say that the separatrices are not fully developed when this inequality is violated by for example $u = O(\epsilon^2/3)$. Now, if $\tilde{\epsilon} = \epsilon \rho^{-3}$ is sufficiently small, then $\tilde{\epsilon}^{-1} d\tilde{u} = \tilde{U}$ does not change sign on long time scales. Through the transformation $\tilde{t} \mapsto \tilde{t} = \frac{1}{\sqrt{2c - \Gamma^2}} \epsilon^{-1} \tilde{u}$ we therefore obtain the equations:

$$\frac{\dot{U}}{\sqrt{2c - \Gamma^2}} \frac{d\dot{u}}{dt} = \dot{V}, \quad \frac{\dot{U}}{\sqrt{2c - \Gamma^2}} \frac{d\dot{V}}{dt} = -\partial_u \dot{P}_A,$$

which are Hamiltonian with:

$$K_{\epsilon}(\dot{u}, \dot{V}; \dot{u}, h, c) = -\rho^{-4} \sqrt{2c - \Gamma^2} \dot{U}(\dot{u}, \dot{V}; \dot{u}, h, c)$$

$$= -\rho^{-4} \sqrt{2c - \Gamma^2} \left( 2c - \rho^4 (V^2 + 2 \dot{P}_A(\dot{u}, \dot{v})) \right)$$

$$= -\rho^{-4} (2c - \Gamma^2) + \frac{1}{2} \left( k + \Gamma^2 \left( 3 + \frac{1}{2c - \Gamma^2} \right) \right) \dot{u}^2$$

$$+ \frac{1}{2} \dot{V}^2 + \frac{k}{2} \left( -\dot{u} \dot{v}^2 + \frac{1}{4} \dot{v}^4 \right) + O(\epsilon^2, \rho^2),$$

$$\omega = d\dot{h} \wedge d\dot{V},$$

with $\dot{u}$ as a slowly varying parameter, $\frac{d\dot{u}}{dt} = \sqrt{2c - \Gamma^2} \dot{u}$, and where $\dot{U} = \dot{U}(\dot{u}, \dot{V}; \dot{u}, h, c)$ solves $\dot{H}(\dot{u}, \dot{U}, \dot{v}, V; h) = c$. Now, $K$ satisfies the hypothesis of Theorem 1 of [23] with $\epsilon$ replaced with $\tilde{\epsilon} = \epsilon \rho^{-3}$.
Theorem 5 For $\varepsilon$ sufficiently small the measure of the set of initial conditions $(v_0, V_0)$ of (7.20) that is not captured is of order:

$$O\left(\frac{e^{-1/\varepsilon}}{\varepsilon^2}\right).$$

The change in action is:

$$\Delta I = O(\varepsilon \ln \varepsilon^{-1}),$$
or in terms of the original action:

$$\Delta I = O(\varepsilon \ln \varepsilon^{-1}).$$

As a corollary we obtain the following result:

Corollary 2 For every $c > 0$ and $\rho$, $\varepsilon = \varepsilon \rho^{-3}$ sufficiently small consider the set $N_c$ of initial conditions $(u, v, U, V)(0) = (0, v_0, U_0, V_0) \in H^{-1}(c)$ satisfying $U_0 > 0$ and $I = O(\rho^3)$. Then there exists a subset of initial conditions $N_c \subset N_c$ with measure $m(N_c \setminus N_c) = O(\rho^3 \varepsilon^{-2} e^{-1/\varepsilon})$ and a $\nu^* = O(\rho^3)$ so that the following holds true: The projection of the set:

$$N_{c, \nu^*} = \{\text{set of first return of } N_c \text{ to the section } u = \nu^* \text{ with } U > 0\},$$
on to the $(v, V)$-plane will be contained within and be bounded away from the unperturbed separatrices. The change in action is $\Delta I|_{N_{c, \nu^*}} = O(\rho^3)$.

Remark 3 We call the projection of $N_{c, \nu^*}$ onto the $(v, V)$-plane for $N_{c, \nu^*}$.

Proof For $u = O(\rho^3)$ the measure of the lobes is $O(\rho^3)$ cf. (7.19). Therefore, since $I$ is adiabatically invariant away from the separatrices [10, 23, 110], it follows that the trajectory of any initial condition in $N_c$ satisfying, say, $\frac{B}{2} \rho^3 \leq I(v_0, V_0) \leq B \rho^3$, for some $B > 0$, remains bounded away from the separatrices for $u \leq \tilde{u}$, $\tilde{u} = \tilde{u}(B) = O(\rho^3)$. These initial conditions apply to Theorem 5. In particular, only an exponential small set does not cross the separatrices and the change in action through the crossing is $O(\varepsilon \ln \varepsilon^{-1})$. On the other hand, Theorem 5 does not apply to the set of trajectories that crosses the separatrices before they were fully developed. The change in the action for this set of trajectories may be as large as the area of the lobes at $u = \tilde{u}$: $\Delta I = O(\rho^3)$. Finally, since the lobes are expanding it follows that if $(v, V)$ at some $t = t_0$ is within one of separatrix lobes then $(v, V)(t \geq t_0)$ remains within that lobe at least while $\rho^{-2}u = O(1)$. See also the proof of Theorem 1 in [23]. The existence of $N_c$ and $\nu^* > \tilde{u}$ with the given properties therefore follows.

We shall in the following introduce $A$ such that $u^* = A \rho^2$. In the next section we investigate the further destiny of the set $N_{e, \nu^*}$ by averaging near $M_1$. 


3° Averaging near $M_1$

Consider $M^\pm = M_1 \cap \{ \pm \nu > 0 \} = \{(u,v,U,V) | u = f_{\pm}(u) \equiv \pm \sqrt{u(2-u)} \}, \nu = 0, u \in (0,2)$ and $2 - u^* \geq u \geq u^*, \ u^* = A\rho^2 > 0$, with $\rho$ sufficiently small so that Corollary 2 applies. As a first step, we straighten out $M^\pm_1$, see also [52], of the Hamiltonian (7.13) through the generating function:

$$s(u,v,U,V) = vV + \frac{u\dot{U}}{\epsilon} - \dot{V} f_{\pm}(u),$$

and the equations:

$$\dot{u} = \epsilon \partial_u s = u,$$

$$\dot{v} = \partial_v s = v - \dot{f}_\pm(u),$$

$$U = \epsilon \partial_u s = \dot{U} - \epsilon \dot{V} \partial_u f_{\pm}(u),$$

$$V = \partial_v s = \dot{V}.$$

In the new coordinates $(\bar{u}, \bar{v}, \bar{U}, \bar{V})$ we have $M^\pm_1 = \{\dot{v} = 0, \dot{V} = 0\}$ and the Hamiltonian reads:

$$H = \frac{1}{2} \bar{U}^2 + \frac{1}{2} \bar{V}^2 + \frac{1}{2} \frac{\Gamma^2}{(1 - \bar{u}^2)^2 + \epsilon^2 (\bar{u} \pm \sqrt{\bar{u}(2 - \bar{u})})^2} + \frac{k}{2} \left( \sqrt{\bar{u}^2 + 2\bar{v} \sqrt{\bar{u}(2 - \bar{u})} + 1 - 1} \right)^2 + \epsilon \frac{\bar{U} \dot{V}(1 - \bar{u})}{\sqrt{\bar{u}(2 - \bar{u})}} + \frac{1}{4} \epsilon^2 \frac{\bar{V}^2(1 - \bar{u})^2}{\bar{u}(2 - \bar{u})}. $$

By the discrete symmetry: $(u,V) \mapsto (-v,-V)$ we may restrict attention to $M^+_1$.

We now introduce the proper scaling $\bar{u} = \rho^2 \bar{u}, \bar{v} = \rho \bar{v}, \bar{U} = \bar{U}, \bar{V} = \rho \bar{V}, t \mapsto \rho t$ such that:

$$\bar{H} = \rho^{-4} H = \frac{1}{2} \rho^{-4} \bar{U}^2 + \frac{1}{2} \bar{V}^2 + \frac{1}{2} \frac{\rho^{-4} \Gamma^2}{(1 - \rho^4 \bar{u}^2)^2 + \epsilon^2 (\bar{u} \pm \sqrt{u(2 - \bar{u})})^2} + \rho^{-4} \frac{k}{2} \left( \sqrt{\rho^4 \bar{u}^2 + 2\rho^2 \bar{v} \sqrt{\bar{u}(2 - \rho^2 \bar{u})} + 1 - 1} \right)^2 - \epsilon \rho^{-3} \frac{\bar{U} \dot{V}(1 - \rho^2 \bar{u})}{\sqrt{\bar{u}(2 - \rho^2 \bar{u})}} - \frac{1}{4} \epsilon^2 \rho^{-2} \frac{\bar{V}^2(1 - \rho^2 \bar{u})^2}{\bar{u}(2 - \rho^2 \bar{u})},$$

$$\omega = \epsilon^{-1} \rho^{-1} d\bar{u} \wedge d\bar{U} + d\rho \wedge d\bar{V}.$$

The corresponding frozen system is given by:

$$\bar{h} = \frac{1}{2} \bar{V}^2 + \rho^{-4} \frac{k}{2} \left( \sqrt{\rho^4 \bar{u}^2 + 2\rho^2 \bar{v} \sqrt{\bar{u}(2 - \rho^2 \bar{u})} + 1 - 1} \right)^2 + O(\epsilon). \quad (7.21)$$
By assumption $\varepsilon = \varepsilon \rho^{-3} \ll 1$ so we may initially ignore the perturbation. A Taylor expansion about $\theta = 0$ then gives:

$$h = \frac{1}{2} \dot{\theta}^2 + \frac{k}{2} \dot{\theta}^2 (2 - \rho^2 \theta^2) + O(\theta^3).$$

From the quadratic part of the Hamiltonian we obtain the linear frequency:

$$\sqrt{k \theta^2 (2 - \rho^2 \theta^2)}$$

which is $\geq \sqrt{k A (2 - A \rho^2)}$. In particular, we conclude, using Corollary 2 and the fact that the flow map is continuous, that, provided $A$ is sufficiently large and $\varepsilon$ is sufficiently small, there exists an $N_\varepsilon$ such that $N_\varepsilon^{\psi, \nu}$ is contained in a neighbourhood of $M_\varepsilon^{\psi, \nu}$ with a corresponding frequency that is bounded from below by 1.

We are then in a position to apply the averaging procedure from [52] also used above in $1^\circ$. First, we transform the frozen system (7.21) into action angle variables:

$$h(\theta, V; \theta) = h(I; \dot{\theta}),$$

via a generating function $s(\theta, V; \theta)$ and the equations:

$$\dot{\theta} = \partial_{\theta} s, \quad \dot{V} = \partial_{\theta} s,$$

where $\theta = \sqrt{2I \cos \phi}, \quad V = \sqrt{2I \sin \phi}$. Secondly, this transformation is lifted to a transformation on the entire phase space preserving $\frac{1}{\rho} d\theta \wedge dV + \dot{\theta} \wedge d\theta$ via the function:

$$s(\theta, V, \phi, V) = \frac{\dot{V}}{\rho} + \delta(\theta, \dot{\theta}, V),$$

with $\delta(\theta, \dot{\theta}, V) = s(\theta, V; \theta)$, and the equations:

$$\dot{\theta} = \epsilon \rho \partial_{\theta} s = \dot{\theta},$$

$$\dot{V} = \epsilon \rho \partial_{\theta} s = \dot{V} + \epsilon \rho \partial_{\theta} \delta.$$

This transformation is identical in $\theta$ and $\epsilon \rho$-close in $\dot{\theta}$. The new Hamiltonian reads:

$$\hat{H}(\theta, \dot{\theta}, \dot{V}, \dot{V}) = \frac{1}{2\rho} \dot{\theta}^2 + \frac{k}{2} \dot{\theta}^2 (2 - \rho^2 \theta^2) + \epsilon \rho^{-3} R(\theta, \dot{\theta}, \dot{V}, \dot{V}; \epsilon),$$

where $R(\theta, \dot{\theta}, \dot{V}, \dot{V}; \epsilon) = \epsilon^{-1}\rho \left( \hat{H} - \hat{H}^{(0)} \right)$. Through the generating function:

$$s(u, \dot{u}, \dot{U}, \dot{V}) = v \dot{V} + \frac{\dot{u} \dot{U}}{\epsilon \rho} + \epsilon \rho^{-3} s(\theta, \dot{\theta}, V, V),$$
and the equations:

\[ \ddot{u} = \epsilon \rho \partial_\rho S = \ddot{u} + \epsilon^2 \rho^2 \partial_\rho s, \quad \ddot{U} = \epsilon \rho \partial_\rho S = \ddot{U} + \epsilon^2 \rho^2 \partial_\rho s, \]

\[ \ddot{v} = \partial_\varphi S = \ddot{v} + \epsilon \rho^{-3} \partial_\varphi s, \quad \ddot{V} = \partial_\varphi S = \ddot{V} + \epsilon \rho^{-3} \partial_\varphi s, \]

we transform the Hamiltonian into:

\[ \dot{H}(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}) = \dot{H}^{(0)}(\ddot{u}, \ddot{U}, \frac{1}{2} \dot{u}^2 + \frac{1}{2} \dot{V}^2) \]

\[ + \epsilon \rho^{-3} \left( \varpi \partial_\varphi s - R_1(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}; \epsilon) \right) + O(\epsilon^2 \rho^{-6}), \]

where \( \ddot{u} = \sqrt{2I} \cos \phi, \ddot{v} = \sqrt{2I} \sin \phi \) and \( \varpi = \partial_\varphi \dot{h} \). To remove terms of \( O(\epsilon \rho^{-3}) \) we choose \( s \) as the unique solution with zero average, [52] Lemma 4, satisfying:

\[ R_i = -\frac{1}{2\pi} \int_0^{2\pi} R_i(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}; \epsilon) \, d\phi. \]

Then:

\[ \dot{H}(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}) = \dot{H}^{(1)}(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}) + O(\epsilon^2 \rho^{-6}). \]

Again the averaging procedure can be iterated to obtain:

\[ \dot{H}(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}) = \dot{H}^{(n)}(\ddot{u}, \ddot{v}, \ddot{U}, \ddot{V}) + O(\epsilon^{n+2}). \]

For \( \rho \) sufficiently small it only takes \( O(1/\epsilon) \) time before \( |1 - u| = 1 \) again. Therefore:

**Lemma 2** There exists a set \( N_c \) with the given properties from Corollary 2 such that for initial conditions \( (u_0, V_0) \in N_{c_0}^*, u_0 = u^* = A \rho^3, U_0 < 0 \), with \( \rho, \epsilon \rho^{-3} \ll 1 \) sufficiently small and \( A \) sufficiently large, \((u, V)\) remains bounded away from the separatrices and particularly:

\[ |I(t) - I(0)| = O(\epsilon), \quad I(0) = O(\rho^3), \]

while \( 2 - u^* \geq u \geq u^* \).

This finishes the proof of Theorem 4. In the following section we will discuss a generalisation to the full two-element Galerkin approximation.
7.4 Discussion on general planar system

The general planar system can be written as

\[ H = \frac{\Gamma^2}{u^2} + \frac{1}{2} U^2 + \frac{1}{2} V^2 + \frac{1}{2} W^2 + \frac{k}{4} \left( (r_1 - 1)^2 + (r_2 - 1)^2 \right) + O(\epsilon), \quad (7.22) \]

where \( r_1 = \sqrt{(u + w)^2 + v^2}, \ r_2 = \sqrt{(u - w)^2 + v^2} \) and the symplectic form \( \omega = e^{-1} du \wedge dU + dv \wedge dV + dw \wedge dW. \) To extend the result of Theorem 4 to this system is not straightforward. The problem is now complicated by the presence of resonances near the slow manifold \( M. \) The averaging procedure used in 1° and 3° in the proof of Theorem 4 does therefore not generalise directly.

On the other hand, the procedure in 2° can be repeated in conjunction with partially averaging of the fast variables \((u, W).\) We show that this can done without addressing resonances since near the bifurcation the time-scales associated with \((w, W)\) are well-separated from the time-scales associated with \((v, V)\) and \((u, U).\) First, we note that if we expand the frozen system ((7.22) with \( u \) and \( U \) fixed and \( \epsilon = 0 \)) about \( M_1 \) and \( M_2 \) we obtain the quadratic parts:

\[ H_2^{(1)} = \frac{1}{2} V^2 + \frac{1}{2} W^2 + \frac{k}{2} (1 - u^2) v^2 + \frac{k}{2} u^2 w^2, \]

respectively

\[ H_2^{(2)} = \frac{1}{2} V^2 + \frac{1}{2} W^2 + \frac{k}{2} \left\{ \frac{|u| - 1}{u} v^2 + \frac{k}{2} w^2 \right\}. \]

In \( H_2^{(1)} \) we have introduced \( \hat{v} = v \mp \sqrt{1 - u^2} \) and \( \hat{V} = V \) and for convenience again dropped the tildes. The quadratic parts show that the linear frequency associated with \((w, W)\) only vanishes at \( u = 0 \) near \( M_1. \) If \( \Gamma \neq 0, \) however, then from (7.22) it follows that \( |u| \geq \delta \) for some \( \delta > 0 \) depending upon the angular momentum \( \Gamma \) and the energy \( H = c. \) The bifurcation at \( |u| = 1 \) is therefore only associated with the \((u, U)-variables. \) Furthermore, if we extend the blow-up transformation from 2° near the bifurcation value \( u = 1 \) in the following way:

\[
\begin{align*}
u &= 1 - \rho^2 \hat{u}, \quad U = \hat{U}, \\
v &= \rho \hat{v}, \quad V = \rho^2 \hat{V}, \\
w &= \rho^2 \hat{w}, \quad W = \rho^2 \hat{W},
\end{align*}
\]

and \( t \mapsto \rho^2 t, \) with \( \rho \) small but \( \epsilon \ll \rho \) (\( \rho = O(\epsilon^{1/3}) \) for example will do cf. (7.23)), then we transform (7.22) into:

\[ \dot{H} = \rho^{-4} H = \frac{1}{2} \rho^{-4} \dot{u}^2 + \frac{1}{2} \rho^{-2} \dot{v}^2 + \frac{1}{2} \dot{w}^2 + \frac{k}{2} \rho^{-2} \dot{u}^2 + \frac{k}{2} \rho^{-2} \dot{u}^2 + \frac{k}{2} \rho^{-2} \left( -\dot{u} v^2 + \frac{1}{4} \dot{v}^2 \right) + O(\rho^2), \]
The \textbf{Slack-Spring Model as a Limit System of a Galerkin Approximation to the Massive Tether Model}

now equipped with the symplectic form \( \omega = e^{-1} \rho^{-2} d\dot{u} \wedge d\dot{\theta} + e^{-1} d\dot{v} \wedge d\dot{v} + d\dot{w} \wedge d\dot{\theta} \).

Finally, \( \dot{u} = \sqrt{e} \dot{p} \) and \( \dot{\theta} = \sqrt{e} \dot{p} \) transform the system into a singular perturbed (now through \( \rho \)) Hamiltonian system with one fast degree of freedom \( \dot{v} \):

\[
\begin{align*}
\ddot{u} &= \rho \partial_\theta \dot{H} = \rho \left( e^{\rho^{-2} \dot{U}} \right), \\
\ddot{\theta} &= -\rho \partial_\theta \dot{H}, \\
\dot{v} &= \rho \partial_v \dot{H}, \\
\dot{w} &= \partial_w \dot{H}, \\
\dot{\theta} &= -\partial_\theta \dot{H}.
\end{align*}
\]

We can then for \( \rho \) sufficiently small with exponential good agreement: \( O(e^{-c/\rho}) \), restrict to the level set of an action \( I = \frac{1}{2} \left( \dot{w}^2 + k\dot{u}^2 \right) + O(\rho) \) obtained through the averaging iteration \([52]\). On these level sets we can then repeat the arguments from \( 2^0 \) to conclude that every trajectory, except for an exponentially small measure set, crosses the unperturbed separatrices with a small change in the continuous action.

The only obstacle to a generalisation of the result in Theorem 4 to the full two-element Galerkin approximation is therefore the averaging procedure in \( 1^0 \) and \( 3^0 \). Perhaps a combination of the method of MacKay \([97]\) with the energy method could be of use here. We shall not pursue this further within this research, but will discuss this possible extension and the energy method further in the concluding chapter when addressing the open problems for future work.

\section{7.5 Conclusion}

In this chapter a Galerkin approximation of the full massive tether model was analysed in the singular limit of vanishing thickness for a stiff tether. The singular perturbed, truncated system was shown to possess a slow manifold with bifurcation. This problem is not well-studied within the Hamiltonian setting where the interesting lower dimensional slow manifolds often are normally elliptic. It was shown that the slow dynamics on the normally elliptic branches of the slow manifold coincide with the slack-spring approximation and a theorem on the adiabatic invariance of these branches of the slow manifold was proven. This analysis provides further insight into the slack-spring approximation.

We believe that extending and generalising this result to more degrees of freedoms would attract considerable interest from mathematicians, physicists, astronomers and chemists. Indeed, it is very common for mathematical models from the various fields
7.5 Conclusion

to have a separation of time-scales giving rise to slow and fast variables. The slow manifold bifurcations are traditionally thought of as limitations of the slow-fast theory. In this research, however, we have both obtained further analytical and numerical results, supporting previous results for slowly varying one degree of freedom systems, that a reduction principle may still be valid near these points. We shall in the concluding chapter, Chap. 10, suggest a detailed direction for future work on this topic.

In the following chapter we consider geometric attitude control and provide analytical expressions for $L^2$-optimal control. The optimal control is shown to be possibly quite powerful in applications to formation flying.
The Slack-Spring Model as a Limit System of a Galerkin Approximation to the Massive Tether Model
8.1 Introduction

The control of the attitude of a satellite is crucial in meeting any mission requirements. In this chapter we develop an appropriate geometrical setting for optimal attitude control based on a linearisation about a reference attitude solution. Within this setting we obtain explicit expressions for the constrained $L^2$-optimal torque. The method is at the end of the chapter applied to two different realistic formation flying mission scenarios. For simplicity we shall neglect any external torques such as gravity gradient. First, however, we will revisit the attitude dynamics of a rigid body.

8.2 Attitude Dynamics

The attitude of a rigid body $B$ is described through a rotation matrix $R(t) \in SO(3)$ so that every point of the rigid body $X$ is mapped to a new point in space $x(t) = R(t)X$. In the absence of external forces the energy of the system coincides with the kinetic
Optimal Relative Attitude Control

Energy which is given by:

\[ K(R, \dot{R}) = \frac{1}{2} \int_B \rho(X)|\dot{x}(t)|^2 dX \]
\[ = \frac{1}{2} \int_B \rho(X)|\dot{R}(t)X|^2 dX \]
\[ = \frac{1}{2} \int_B \rho(X)|R^{-1}\dot{R}(t)X|^2 dX \]
\[ = \frac{1}{2} \int_B \rho(X)|R^{-1}\tilde{\Omega}X|^2 dX \]
\[ = \langle (\tilde{\Omega}, \tilde{\Omega}^T) \rangle, \]

where \( \langle (A, B) \rangle = \frac{1}{2} \text{tr}(AB^T) \) and \( \tilde{\Omega} = R^{-1}\dot{R} \in so(3) \) is a skew-symmetric matrix.

Physically, \( \Omega \) is the angular velocity of the rigid body in body coordinates. We have here also defined the coefficient of inertia matrix \( J = \int_B \rho(X)XX^T dX \).

We have in fact through the hat-map isomorphism (7.10) reduced the kinetic energy to a function on \( \mathbb{R}^3 \):

\[ k(\Omega) = K(I, \tilde{\Omega}) = \frac{1}{2} \langle \Omega, \Omega \rangle, \]

where \( I \) is the moment of inertia satisfying:

\[ I = \text{tr} J I - J. \]

There is no loss in generality in assuming that \( I \) is diagonal. The angular momentum of the system in body coordinates is \( \mathbf{G} = \tilde{\Omega}I \) and by Lie-Poisson reduction theory [64] it follows that \( \mathbf{G} \) satisfies the reduced equation:

\[ \dot{\mathbf{G}} = \mathbf{G} \wedge I^{-1}\mathbf{G}. \]

(8.1)

The Euler equation is Hamiltonian with the reduced Hamiltonian \( (H = k \) expressed in terms of angular momentum):

\[ H(G) = \frac{1}{2} \langle G, I^{-1}G \rangle, \]

and the Poisson brackets:

\[ \{ f, g \} = \langle \partial_G f, G \wedge \partial_G g \rangle. \]
This equation is known as the Euler equation for a rigid body. The angular momentum $G$ conserves the Hamiltonian and the magnitude of the angular momentum vector (see also (7.12)) so that

$$C(G) = \frac{1}{2} |G|^2,$$

is also conserved. In $G$-space the dynamics is therefore restricted to the intersections of the level sets: $C(G) = \frac{1}{2} c^2$ (note: $|G| = c$), $c > 0$ (sphere) and $H(G) = h$ (ellipsoid). The attitude dynamics is therefore integrable and can be solved using elliptic functions. The attitude of the system is finally recovered by the definition of $\Omega$:

$$\dot{R} = R \hat{\Omega}.$$

Since the Euler equations have been reduced by the $SO(3)$-symmetry of the system, the relative equilibria coincide with equilibria of the reduced system given by the Euler equations (8.1). If the rigid body does not possess any symmetry axis, then the rigid body has only three types of relative equilibria. The three types of relative equilibria correspond to the rigid body spinning around each of its three principle axes, i.e. only one component of $G$ is non-zero. The equilibria in which the body is spinning around its largest and smallest principle axes are both stable, whereas the third type of equilibrium is unstable [64].

If the spacecraft is equipped with a reaction wheel then the angular momentum of the wheel $h$ enters Euler’s equation in the following way:

$$\dot{G} + \dot{h} = (G + h) \wedge I^{-1}G,$$

see e.g. [152, 135]. Here $G + h$ is now the total angular momentum of the system, whose magnitude is still conserved, and in equilibrium the total angular momentum is now parallel to the angular velocity of the spacecraft: $G + h \parallel I^{-1}G$. In the following section we shall consider the variational attitude problem.

### 8.3 Variational attitude dynamics

According to Chap. 3 section 3.3 the variations $\delta G$ conserve:

$$\delta H(\delta G, G) = (\delta G, I^{-1}G),$$

and

$$\delta C(\delta G, G) = (\delta G, G),$$

and satisfy:
\[
\delta G = G \wedge \partial_G \delta H + \delta G \wedge \partial_G \delta H = G \wedge \mathbb{I}^{-1} \delta G + \delta G \wedge \mathbb{I}^{-1} G.
\] (8.6)

The Euler equation (8.1) and its variational equation (8.6) can also be recovered by the variational Hamiltonian \( \delta H \) and the Poisson bracket:
\[
\{f,g\}_\delta = (\partial_G f, G \wedge \partial_G g) + (\partial_G f, G \wedge \partial_G g + \delta G \wedge \partial_G g).
\]
Both (8.2) and (8.5) are Casimir functions of this bracket.

With an internal angular momentum (8.3) the variations take the following form:
\[
\delta G + \delta h = (G + h) \wedge \mathbb{I}^{-1} \delta G + (\delta G + \delta h) \wedge \mathbb{I}^{-1} G,
\] (8.7)
and conserve \((\delta G + \delta h, G + h)\). To obtain the attitude we solve for the rotation matrix \( R = R(t) \in SO(3) \):
\[
\dot{R} = R \dot{\Omega}, \quad \dot{\Omega} = \mathbb{I}^{-1} \dot{G} \in so(3).
\]

By variations we obtain:
\[
\frac{d\delta R}{dt} = R \delta \Omega + \delta R \dot{\Omega}.
\] (8.8)

but how do we interpret \( \delta R \in T_R SO(3) \), let alone solve it using coordinates? We shall in the following make use of the group structure of \( SO(3) \) and the fact that \( T_I SO(3) = so(3) \). See also Fig. 8.1 and [64]. First we rewrite (8.8) as:
\[
\delta \dot{\Omega} = -(R^{-1} \delta R) \dot{\Omega} + R^{-1} \delta \dot{R},
\] (8.9)
and set \( \check{\Sigma} = R^{-1} \delta R \in so(3) \). Here \( \check{\Sigma} \) is easily interpreted, we can particularly identify it with an element \( \Sigma \) in \( \mathbb{R}^3 \) using the hat-map. We therefore aim to obtain an equation for \( \check{\Sigma} \), in turn \( \Sigma \), and then recover \( \delta R \) by the definition. From \( R^{-1} R = I \) follows that
\[
\frac{d}{dt} R^{-1} = -R^{-1} \dot{R} R^{-1}.
\] (8.10)
Therefore, differentiating \( \check{\Sigma} \) gives
\[
\frac{d\check{\Sigma}}{dt} = -R^{-1} \dot{R} R^{-1} \delta R + R^{-1} \delta \dot{R}.
\]
Inserting this into (8.9) gives:
\[
\delta \dot{\Omega} = -\check{\Sigma} \dot{\Omega} + \frac{d\check{\Sigma}}{dt} + \dot{\Omega} \check{\Sigma}
\]
\[
= \frac{d\check{\Sigma}}{dt} + [\dot{\Omega}, \check{\Sigma}],
\]
or undoing the hat-map:

\[ \dot{\Sigma} = \delta \Omega + \Sigma \wedge \Omega \]
\[ = I^{-1} \delta G + \Sigma \wedge I^{-1} G. \]  

(8.11)

We recover \( \delta R \) through the definition of \( \Sigma \):

\[ \delta R = R \dot{\Sigma}. \]  

(8.12)

The linear approximation of the attitude is therefore given by:

\[ R + \delta R = R \left( I + \dot{\Sigma} \right). \]  

(8.13)

This approximation is almost orthogonal in the following sense:

\[ (R + \delta R)^{-1} - (R + \delta R)^T = O(|\Sigma|^2). \]  

(8.14)

A skew-symmetric matrix also appears implicitly in a linearisation of an attitude matrix in Euler angle coordinates. Consider for example the rotation matrix:

\[
R_{\alpha,\beta,\gamma} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c\gamma & -s\gamma \\
0 & s\gamma & c\gamma
\end{pmatrix}
\begin{pmatrix}
c\beta & 0 & -s\beta \\
0 & 1 & 0 \\
s\beta & 0 & c\beta
\end{pmatrix}
\begin{pmatrix}
ca & -sa & 0 \\
sa & ca & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( c = \cos \theta, s = \sin \theta \).
where $\alpha$, $\beta$ and $\gamma$ are roll, pitch and yaw, respectively, $s_v = \sin\nu$, $c_v = \cos\nu$, $\forall \nu \in S^1$ and expand about $\alpha = 0$, $\beta = 0$ and $\gamma = 0$:

$$R_{\alpha, \beta, \gamma} = \begin{pmatrix} 1 & -\alpha & -\beta \\ \alpha & 1 & -\gamma \\ \beta & \gamma & 1 \end{pmatrix} + \mathcal{O}(\alpha^2, \beta^2, \gamma^2)$$

$$= I + \begin{pmatrix} \gamma \\ \beta \\ \alpha \end{pmatrix} + \mathcal{O}(\alpha^2, \beta^2, \gamma^2).$$

The calculation also highlights that we may think of the components of $\Sigma$ as small angles. This viewpoint shall be useful later on when we shall apply the theory to examples and quantify the error introduced by replacing the nonlinear system with a linear approximation.

As an introduction to the method we later apply to attitude control, we will in the following section compute an explicit expression for the $L^2$-optimal control with boundary constraints for any general non-autonomous mechanical linear system.

### 8.4 Linear quadratic optimal control problem

We shall as in [115, 22, 21, 114] consider a linear quadratic optimal control problem with finite horizon. Let $T$ be the period of time considered. We shall throughout abbreviate $L^2((0,\bar{T}),\mathbb{R}^\ell)$ by $L^2_T$, but let $L^2_T = L^2$, and use the following notation:

$$\langle (f,g) \rangle_t = \int_0^T (f(t),g(t))_\mathbb{R}^\ell dt,$$

for the inner product.

We consider a linear mechanical system $x \in \mathbb{R}^{2m}$ with $m$ degrees of freedom:

$$\dot{x} = \Lambda(t)x + \Gamma u, \quad (8.15)$$

with boundary conditions $x(0) = x_0$ and $x(T) = x_T$ where $\Gamma^T = (0_{m \times m} I_{m \times m})$. We assume that a fundamental matrix $\Phi(t)$ is known. We also define $\Psi(t) = (\Psi^{(1)}(t) \; \Phi^{(2)}(t)) = \Phi(T) \Phi^{-1}$, $\Psi^{(1)}(t) \in \mathbb{R}^{2m \times m}$, $l = 1,2$, which by repeating the argument that lead to (8.10) can be shown to satisfy:

$$\dot{\Psi} = -\Psi \Lambda.$$

We then show the following theorem using standard variational calculus:
8.4 Linear quadratic optimal control problem

Proposition 1 Assume that the "controllability Gramian":
\[
\left( \int_0^T \Psi^{(2)}(s)\Psi^{(2)}(s)^T ds \right)
\]
is invertible. Then the control \( u \) that minimises the \( L^2 \)-norm subject to (8.15) and boundary conditions \( x(0) = x_0 \) and \( x(T) = x_T \) is given by:
\[
u(t)^T = \lambda^T \Psi^{(2)}(t),
\]
where the Lagrange multiplier \( \lambda \) satisfy
\[
\lambda = \left( \int_0^T \Psi^{(2)}(s)\Psi^{(2)}(s)^T ds \right)^{-1} (x_T - \Psi(0)x_0).
\]

PROOF We consider the linear quadratic optimal control problem through the Lagrangian [159]:
\[
L(u) = \frac{1}{2} \|u\|_{L^2_m}^2 - \langle \lambda, x(T) - x_T \rangle_{\mathbb{R}^{2m}},
\]
with \( \lambda \in \mathbb{R}^{2m} \), and variations of constants:
\[
x(t) = \Phi(t)\Phi(0)^{-1}x_0 + \int_0^t \Phi(t)\Phi(s)^{-1}\Gamma u(s)ds.
\]
The control \( u \) is optimal if the variation of (8.18) vanishes at \( u \). Through (8.19) (with \( t = T \)) the variation of \( L \) becomes
\[
\delta L = \langle (u, \phi) \rangle_m - \langle \lambda, \int_0^T \Psi(s)\Gamma \phi(s)ds \rangle_{\mathbb{R}^{2m}},
\]
for \( \phi \in L_m^2 \). We re-write (8.20) as
\[
\delta L = \langle (u, \phi) \rangle_m - \langle (\lambda, \Psi \Gamma \phi) \rangle_{2m}
\]
\[
= \langle (u, \phi) \rangle_m - \langle (\Psi^{(2)} \lambda, \phi) \rangle_m
\]
\[
= \langle (u - (\Psi^{(2)} \lambda), \phi) \rangle_m.
\]
It now follows that for the variation to vanish for every \( \phi \in L_m^2 \) the control \( u \) must satisfy
\[
u = (\Psi^{(2)})^T \lambda.
Inserting this into (8.19) then gives:

\[ x_T = \Psi(0)x_0 + \int_0^T \Psi^{(2)}(s)\Psi^{(2)}(s)^T ds \lambda, \]

or by assumption:

\[ \lambda = \left( \int_0^T \Psi^{(2)}(s)\Psi^{(2)}(s)^T ds \right)^{-1} (x_T - \Psi(0)x_0). \]

This completes the proof. ■

When \( A \) is independent of time then we may set \( \Phi = \exp(At) \) so that \( \Psi(s) = \Phi(T-s) \). Therefore as a corollary:

**Corollary 3** Assume that the “controllability Gramian”:

\[ \left( \int_0^T \Phi^{(2)}(T-s)\Phi^{(2)}(T-s)^T ds \right), \]

is invertible. Then the control \( u \) that minimises the \( L^2 \)-norm subject to (8.15) and boundary conditions \( x(0) = x_0 \) and \( x(T) = x_T \) is given by:

\[ u(t)^T = \lambda^T \Phi^{(2)}(T-t), \]

where the Lagrange multiplier \( \lambda \) satisfy

\[ \lambda = \left( \int_0^T \Phi^{(2)}(T-s)\Phi^{(2)}(T-s)^T ds \right)^{-1} (x_T - \Phi(T)x_0). \]

This result is known, see e.g. [114]. The matrix inverse appearing in (8.17) only needs to be computed once as it is independent of the boundary conditions. For the satellite applications considered in [115, 22], for example, it could easily be computed off-line and stored on-board within the satellite software.

In the following section we will apply these ideas to the computation of the optimal control of the attitude dynamics with reaction wheels. Here, however, as opposed to the “classical” form considered in Proposition 1, the torque \( \delta h \), that is sought minimised, enters along with \( \delta h \) (cf. (8.3)).
8.5 Optimal control of variational attitude dynamics

Consider any reference solution of Euler’s equations and set $I^{-1}G(t) = c(t)$ and $C(t) = \tilde{c}(t) - \left(\tilde{G}(t) + \tilde{h}(t)\right)I^{-1}$. Then:

\[ \delta G = -C\delta G - \tilde{c}\delta h - \delta h, \]  
(8.21)

and

\[ \Sigma = -\Sigma + I^{-1}\delta G. \]  
(8.22)

Let $\Phi(t)$ be the fundamental matrix associated with (8.21) and (8.22) with $\delta h = 0$ and $\delta G = 0$, respectively. Notice that $\Phi$ has the following form:

\[ \Phi(t) = \begin{pmatrix} \Phi^{(1,1)}(t) & 0 \\ \Phi^{(2,1)}(t) & \Phi^{(2,2)}(t) \end{pmatrix}, \]  
(8.23)

where

\[ \Phi^{(2,1)}(t) = \Phi^{2,2}(t) \int_0^t \Phi^{(2,2)}(s) \Phi^{(2,2)}(s)^{-1} I^{-1}\Phi^{(1,1)}(s)ds, \]

with inverse:

\[ \Phi(t)^{-1} = \begin{pmatrix} \left(\Phi^{(1,1)}(t)\right)^{-1} & 0 \\ -\left(\Phi^{(2,2)}(t)\right)^{-1}\Phi^{(2,1)}(t)\left(\Phi^{(1,1)}(t)\right)^{-1} \left(\Phi^{(2,2)}(t)\right)^{-1} \end{pmatrix}. \]  
(8.24)

The function

\[ \Psi(t) = \begin{pmatrix} \Psi^{(1)}(t) & \Psi^{(2)}(t) \end{pmatrix} = \Phi(T)\Phi(t)^{-1}, \]  
(8.25)

is also defined. From (8.23) and (8.24) it follows that the upper right block of $\Psi$ also vanishes so that

\[ \Psi = \begin{pmatrix} \Psi^{(1,1)} & 0 \\ \Psi^{(2,1)} & \Psi^{(2,2)} \end{pmatrix}, \]  
(8.26)

and

\[ \Psi = -\Sigma \begin{pmatrix} -C & 0 \\ I^{-1} & -\tilde{c} \end{pmatrix}. \]  
(8.27)
Furthermore, let

\[ x = \left( \begin{array} {c} \delta G \\ \Sigma \end{array} \right), \tag{8.28} \]

so that by variations of constants (8.21) and (8.22) read:

\[ x(t) = \Phi(t)\Phi(0)^{-1}x_0 - \Phi(t)\int_0^t \Phi(s)^{-1} \left( \delta \dot{h}(s) + \dot{\delta h}(s) \right) ds. \tag{8.29} \]

The following \( L^2 \)-norm of the torque:

\[ J = \frac{1}{2} \int_0^T |\dot{h} + \dot{\delta h}|^2 ds, \tag{8.30} \]

is now sought minimised subject to (8.21) and (8.22) and boundary conditions \( x(0) = x_0 \) and \( x(T) = x_T \). Initially, both \( \delta h(0) = \delta h_0 \) and \( \delta h(T) = \delta h_T \) are assumed fixed. Saturation levels are neglected. This constrained minimisation is again studied through variations of constants and by considering the Lagrangian:

\[ L = \int - (\lambda, x(T) - x_T), \]

with \( \lambda \in \mathbb{R}^3 \). Through the variations of constants ((8.29) with \( t = T \)) the variation of the Lagrangian gives:

\[ \delta L(\delta h, \delta \dot{h})(\phi, \dot{\phi}) = \langle \delta \dot{h} + \dot{h}, \phi \rangle_3 + \langle \lambda, \dot{\psi}^{(1)} \phi + \psi^{(1)} \dot{\phi} \rangle_3, \]

for every \( \phi \in W = \{ f \in C^\infty([0,T];\mathbb{R}^3) | f(0) = 0 \} \). By integration by parts and using that \( \phi \) vanishes at \( t = 0 \) it follows that

\[ \delta L = -\langle \delta \dot{h} + \dot{h}, \phi \rangle_3 + \langle \left( \Psi^{(1)} \right)^T \lambda, \phi \rangle_3 - \langle \left( \dot{\psi}^{(1)} \right)^T \lambda, \phi \rangle_3 + \langle \delta h(T), \phi(T) \rangle_3 + \langle \psi^{(1)} \phi \rangle_3 \]

\[ = \langle -\delta \dot{h} - \dot{h} + \left( \Psi^{(1)} \right)^T \lambda - \left( \dot{\psi}^{(1)} \right)^T \lambda, \phi \rangle_3 + \langle \delta h(T) + \dot{h}(T) + \lambda_3, \phi(T) \rangle_3, \tag{8.31} \]

where \( \lambda_1 = (\lambda_1, \lambda_2, \lambda_3) \) denotes the first three components of \( \lambda \). Therefore \( \delta L = 0 \), for every \( \phi \in W \), implies the natural boundary conditions:

\[ \delta h(T) + \dot{h}(T) + \lambda_1 = 0 \tag{8.32} \]

and \(-\dot{\delta h} - \dot{h} + \left( \Psi^{(1)} \right)^T \lambda - \left( \dot{\psi}^{(1)} \right)^T \lambda \in W^\perp = \{0\} \). The last equality follows from the denseness of \( W \) in \( L^2 \). The optimal torque therefore satisfy the differential equation:

\[ \dot{\delta h} + \dot{h} = \left( \Psi^{(1)} e^{(1)} \right)^T \lambda. \]
But by (8.27) this may be re-written as

$$\delta h + \dot{h} = I^{-1} \left( \Psi^{(1)}(\hat{G} + \hat{h}) - \Psi^{(2)} \right)^T \lambda.$$  \hfill (8.33)

Integrating this equation once from $T$ to $t \leq T$ and using (8.32) give:

$$\delta h + \dot{h} = -\lambda_t - \int_t^T I^{-1} \left( \Psi^{(1)}(\hat{G} + \hat{h}) - \Psi^{(2)} \right)^T ds \lambda.$$  

From (8.23) and (8.24) it follows that

$$\delta h + \dot{h} = -\lambda_t - \int_t^T I^{-1} \left( \Psi^{(1)}(\hat{G} + \hat{h}) - \Psi^{(2)} \right)^T ds \lambda.$$  

This could also be obtained through (8.27). Therefore:

$$\delta h + \dot{h} = \left( \int_t^T (\hat{G} + \hat{h}) \left( \Psi^{(1)}(s) \right)^T I^{-1} ds + \left( \begin{array}{c} -I \\ 0 \end{array} \right) \Psi^{(2,1)} \right) \lambda.$$  \hfill (8.34)

We collect the conclusions in the following general result:

**Proposition 2** Assume that the controllability Gramian, which is defined below, is invertible. Then the constrained torque minimizing (8.42) is given by (8.34) where the Lagrange multiplier satisfy (8.29) with $t = T$. This linear equation in $\lambda$ defines the controllability Gramian.  

**Zero angular momentum case**

If $|G + h| = 0$ then (8.33) does not depend upon the first three components of $\lambda$. It follows from (8.21) that in this case $|\delta G + \delta h|$ is conserved. Therefore when restricting attention to variations compatible with the zero angular momentum condition: $\delta G + \delta h = 0$, (8.21) becomes:

$$\delta G = -\delta h.$$  \hfill (8.35)

This reduced equation can also be obtained directly from taking the variations of (8.3) with $G + h = 0$. In particular, (8.3) is linear when restricting to $G + h = 0$. 
From Proposition 1 with \( u = -\delta h \) it then follows that the optimal torque is given by

\[
\dot{\delta h} + \delta h = \left( I - \left( \Psi^{(2,1)} \right)^T \right) \mu,
\]

with \( \mu \) as Lagrange multipliers. This is also consistent with (8.34):

\[
\dot{\delta h} + \delta h = -\lambda_1 + \left( \Psi^{(2,1)} \right)^T \lambda_2.
\]

Here \( \lambda_2 = (\lambda_4, \lambda_5, \lambda_6) \) denotes the last three components of \( \lambda \). Finally, (8.36) is recovered by identifying \((-\lambda_1, \lambda_2)\) with \( \mu \) as Lagrange multipliers.

These calculations are in the following made a bit more explicit by considering the particular class of zero-angular momentum reference solutions where \( I^{-1}G = (0,0,\omega^2) \), with \( \omega \geq 0 \) and \( h = -G \). These reference solutions are particularly relevant as three-axis stabilised satellites operate in such configurations. Then it follows from (8.11) that

\[
\Phi^{(2,2)}(t) = \begin{pmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

so that by (8.23):

\[
\Psi(t) = \begin{pmatrix}
I \\
\int_0^t \Phi^{(2,2)}(t-s)\Omega^{-1}ds \\
0 & -\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where for \( \omega \neq 0 \):

\[
\int_0^t \Phi^{(2,2)}(t-s)\Omega^{-1}ds = \omega^{-1}
\begin{pmatrix}
\Omega^{-1} \sin (\omega t) & \Omega^{-1} (1 - \cos (\omega t)) & 0 \\
-\Omega^{-1} (1 - \cos (\omega t)) & \Omega^{-1} \sin (\omega t) & 0 \\
0 & 0 & \frac{\pi}{4}
\end{pmatrix}.
\]

Otherwise if \( \omega = 0 \) then

\[
\int_0^t \Phi^{(2,2)}(t-s)\Omega^{-1}ds = t\Omega^{-1}.
\]

For \( \omega = 0 \) one therefore obtains:

\[
\dot{\delta h} = \begin{pmatrix}
-\lambda_1 \\
-\lambda_2 \\
-\lambda_3
\end{pmatrix} + \omega^{-1}
\begin{pmatrix}
\lambda_4 \Omega^{-1} \sin (\omega (T - t)) + \lambda_5 \Omega^{-1} (1 - \cos (\omega (T - t))) \\
-\lambda_4 \Omega^{-1} (1 - \cos (\omega (T - t))) + \lambda_5 \Omega^{-1} \sin (\omega (T - t)) \\
\lambda_6 \Omega^{-1} \sin (\omega (T - t))
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\lambda_1 + \omega^{-1} \lambda_4 \Omega^{-1} \\
-\lambda_2 + \omega^{-1} \lambda_5 \Omega^{-1} \\
-\lambda_3
\end{pmatrix} + \begin{pmatrix}
r \sin (\omega (T - t) + \phi_{10}) \\
r \sin (\omega (T - t) + \phi_{20}) \\
\frac{\pi}{4}
\end{pmatrix}.
\]
Here the last equality has introduced $r, \phi_{10}$ and $\phi_{20}$ given by the equations:

$$r = \omega^{-1} \sqrt{\lambda_{2} \lambda_{1}^{-2} + \lambda_{3} \lambda_{2}^{-2}},$$
$$\sin \phi_{10} = -r^{-1} \omega^{-1} \lambda_{3} \lambda_{2}^{-1},$$
$$\sin \phi_{20} = r^{-1} \omega^{-1} \lambda_{4} \lambda_{1}^{-1}.$$

Therefore:

$$\left| \delta h_{1} \right| \leq \left| -\lambda_{1} + \omega \lambda_{3} \lambda_{2}^{-1} \right| + r \max_{t \in [0,T]} \sin \left( \omega (T - t) + \phi_{10} \right) \leq \left| -\lambda_{1} + \omega \lambda_{3} \lambda_{2}^{-1} \right| + r,$$
$$\left| \delta h_{2} \right| \leq \left| -\lambda_{2} - \omega \lambda_{4} \lambda_{1}^{-1} \right| + r \max_{t \in [0,T]} \sin \left( \omega (T - t) + \phi_{20} \right) \leq \left| -\lambda_{2} - \omega \lambda_{4} \lambda_{1}^{-1} \right| + r,$$
$$\max \left| \delta h_{3} \right| = \max \left( \left| \lambda_{3} \right|, \left| -\lambda_{3} + \lambda_{5} \lambda_{3}^{-1} T \right| \right).$$

(8.37)

As the instantaneous torque is bounded in practice, these inequalities are very useful as they provide sufficient conditions for whether the given manoeuvre is feasible within the given limitations. For $\omega = 0$ it follows that

$$\delta h = \begin{pmatrix} -\lambda_{1} \\ -\lambda_{2} \\ -\lambda_{3} \end{pmatrix} + \begin{pmatrix} \lambda_{4} T^{-1} \\ \lambda_{5} T^{-1} \\ \lambda_{6} T^{-1} \end{pmatrix}.$$

This is in agreement with known results for the double integrator see e.g [114] p. 232.

In a formation flying mission the satellites are to be synchronised in order to perform a unified mission objective. For example for an inspection mission or building a telescope in space, the attitude of one satellite is often only a small variation with respect to the attitude of the neighbouring satellites. This is why it is believed that the derived solutions could be very useful in formation flying missions. Indeed, given one reference manoeuvre of, say, a leading satellite, then the solutions derived in this section provide the $L^{2}$-optimal relative manoeuvre. For clarity the simple algorithm is described in the following list:

1. Identify a reference solution $(G_{\text{ref}}(t), R_{\text{ref}}(t));$

2. From the prescribed initial conditions: $(G_{0}, R_{0}),$ manoeuvre time $T$ and desired final solution at $t = T$: $(G_{T}, R_{T})$ compute the relative quantities $\delta G_{0}, \delta G_{T}$ and $\Sigma_{0}, \Sigma_{T}$ through the equations:

$$G_{0} = G_{\text{ref}}(0) + \delta G_{0},$$
$$G_{T} = G_{\text{ref}}(T) + \delta G_{T},$$
$$\Sigma_{0} = (R_{\text{ref}}(0)^{T} R_{0} - I)^{A},$$
$$\Sigma_{T} = (R_{\text{ref}}(T)^{T} R_{T} - I)^{A}.$$
Here (8.13) has been used for the last two equalities and \((A)^{A} = \frac{1}{2} (A - A^{T})\) denotes the skew-symmetric part of the matrix \(A\);

3. From \(\delta G_{0}, \delta G_{T}\) and \(\Sigma_{0}, \Sigma_{T}\) compute the \(L^{2}\)-optimal relative torque \(\delta \dot{h}\) through Proposition 2.

In the following two sections the possible potential of these results is demonstrated through two realistic mission scenarios.

8.6 Application to an inspection mission

In this section we apply the \(L^{2}\)-optimal torque to a mission scenario where a chaser satellite is inspecting a target satellite. We will assume that the target satellite is on a slightly eccentric orbit and that the chaser is controlled on a circular orbit with the same semi-major axis so that the relative dynamics is approximated by (Theorem 1 (3.45) with \(\delta \mu = 0, \delta h = 0, \delta \epsilon_{2} = 0\):

\[
\delta \mathbf{q} = -\mathbf{p} \delta t_{0} + \frac{i \wedge \mathbf{q}}{l} \delta l_{1} + \frac{i \wedge \mathbf{q}}{l} \delta l_{2}
- \frac{l^{2}}{\mu} (1 + \sin^{2} \nu) \delta \epsilon_{1} \mathbf{i} + \frac{l^{2}}{\mu} \cos \nu \sin \nu \delta \epsilon_{j} \mathbf{j}.
\]

(8.38)

For simplicity we shall also set \(\delta l_{2} = 0\). The rotating Hill frame is described by the orthogonal basis \(\{i', j', k\}\) which is related to the inertial frame \(\{i, j, k\}\), \(i = (0, 0, 1), j = (0, 1, 0), k = (0, 0, 1)\), through the rotation matrix \(R_{\text{ref}}(\nu)\):

\[
i' = R_{\text{ref}}(\nu)^{T} i, \quad j' = R_{\text{ref}}(\nu)^{T} j,
R_{\text{ref}}(\nu) = \begin{pmatrix}
\cos \nu & -\sin \nu & 0 \\
\sin \nu & \cos \nu & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(8.39)

The satellite is assumed to be spinning around its third principal axis with a reference angular momentum vector \(G_{\text{ref}} = (0, 0, I_{g} \omega)\), where \(\omega = \frac{I_{g} \omega}{I_{g}}\) is the orbital frequency, so that the body principal axis coincide with the Hill frame \(\{i', j', k\}\). It is also assumed that the satellite is three-axis stabilised with zero total angular momentum so that \(G + h = 0\). Half of the angle of the sector in the \(\{i', j'\}\)-plane indicated by the dotted lines in Fig. 8.2 (a) is given by \(\arctan \left( \frac{\omega^{-1} \delta t_{1}}{\delta t_{0}} \right)\). The corresponding angle in the \(\{j', k\}\)-plane in Fig. 8.2 (b) is given by \(\arctan \left( \frac{\omega^{-1} \delta t_{1}^{-1}}{\delta t_{0}} \right)\). Therefore let us imagine that the chaser’s camera is mounted along the second principal axis, which by assumption corresponds to \(j'\). Then the target satellite will be within the chaser’s camera’s field of view provided \(\arctan \left( \frac{\omega^{-1} \delta t_{1}}{\delta t_{0}} \right)\) is small enough. To obtain
higher precision pointing the analytic solutions for the control developed above is applied to the full non-linear system. In particular, the aim is to apply the control so that the second principal axis of the chaser satellite coincides with the relative position vector at perigee \( \nu = 0 \), apogee \( \nu = \pi \) and semilatus rectum \( \nu = \pi/2, 3\pi/2 \) as indicated in Fig. 8.2. The range of validity of the linear approximation can be quantified by varying the ratio \( \delta \varepsilon_1/\omega \delta \varepsilon_0 \) assuming for simplicity that \( \delta \varepsilon_1 t_{-1} = \delta \varepsilon_1 \). Let \( \xi \) denote the corresponding required net rotation.

![Figure 8.2: An example of the orbit of the target relative to the chaser in a rotating Hill's frame. The target is contained within a sector emanating from the chaser. Half of the angle of the sector indicated by the dotted lines in (a) and (b) is given by \( \arctan \left( \sqrt{\frac{1}{\sin^2 \varepsilon_1 \cos \varepsilon_1}} \right) \) and \( \arctan \left( \sqrt{\frac{1}{\sin^2 \varepsilon_r \cos \varepsilon_r}} \right) \) respectively. The perigee \( \nu = 0 \), apogee \( \nu = \pi \) and semilatus rectum \( \nu = \pi/2, 3\pi/2 \) are indicated on the relative orbit.](image)

The relative position (8.38) is in Hill's frame given by \( \delta \mathbf{q}' = \mathbf{R}_{\text{ref}}(\nu)t^T \delta \mathbf{q} \). Similarly, in the body frame of the chaser, described by the perturbed rotation matrix \( \mathbf{R} = \mathbf{R}_{\text{ref}} + \delta \mathbf{R} + \mathcal{O}(|\Sigma|^2) = \mathbf{R}_{\text{ref}}(1 + \hat{\Sigma} + \mathcal{O}(|\Sigma|^2)) \), the relative position is given by

\[
\delta \mathbf{q}'' = \left( \mathbf{I} - \hat{\Sigma} + \mathcal{O}(|\Sigma|^2) \right) \mathbf{R}_{\text{ref}}(\nu)t^T \delta \mathbf{q}.
\]

Therefore

\[
\delta \mathbf{q}'' = \left( \mathbf{I} - \hat{\Sigma} + \mathcal{O}(|\Sigma|^2) \right) \delta \mathbf{q}',
\]

everything evaluated at \( t = T \), and to obtain the desired pointing it is the aim to bring both \( \delta \mathbf{q}'(T) \) and \( \delta \mathbf{q}''(T) \) to zero. From this it is obtained that:

\[
\Sigma_1(T) = \frac{\delta \mathbf{q}_1}{\delta \mathbf{q}_2},
\]

\[
\Sigma_2(T) = \frac{\delta \mathbf{q}_1}{\delta \mathbf{q}_2},
\]
and for simplicity $\Sigma_2(T) = 0$ so that the satellite has no net rotation about the second principal axis.

To quantify the difference between the full nonlinear solution and the linear approximation the following skew-symmetric matrix is defined:

$$\hat{\Sigma}_{\text{exact}} \equiv \left(R_{\text{ref}}^T R_{\text{exact}} - I\right)^A.$$

Here $R_{\text{exact}}$ is the attitude matrix from the full nonlinear system. If $R_{\text{exact}}$ is replaced with the linear approximation $R_{\text{ref}} + \delta R$ (8.13) then $\Sigma_{\text{exact}} = \Sigma$. Therefore if $\Sigma_{\text{exact}}$ is close to $\Sigma$ then the nonlinear system is said to be close to the linear system. Recall that the Euler equation is linear for zero angular momentum reference solution, see (8.3), so the linear approximation only introduces errors in $R$.

The satellite is initially assumed to be in its reference configuration: $\Sigma_0 = 0$, $\delta G_0 = 0$ and $\nu = \pi/2$. The target satellite is then regardless of $\delta G_1$ and $\delta G_1$ aligned along the $j'$-axis. It is also assumed that the chaser satellite has the following moment area of inertia values:

$$I_1 = 10 \text{ kgm}^2,$$
$$I_2 = 12 \text{ kgm}^2,$$
$$I_3 = 14 \text{ kgm}^2.$$

This corresponds to realistic values of a micro-satellite (10 - 100 kg). Then the necessary torque for changing the attitude of the satellite is computed analytically so that the target at apogee $\nu = \pi$, within the linear approximation, is aligned with the second principal axis and $\delta G(T) = 0$. The computation is continued in the same manner to maneuver the satellite at increments of $\Delta \nu = \pi/2$ until $\nu$ again equals $\pi/2$, but at each step $\Sigma_{\text{exact}}$ is used as new initial conditions for $\Sigma$. In Fig. 8.3 the orbit of the target relative to the chaser's circular orbit is shown in three different frames and for two different values of $\xi = 7^\circ$ and $14^\circ$ with $\delta \zeta_1 = \delta I_1 t^{-1}$. The blue curve shows the orbit in the Hill frame described through $R_{\text{ref}}$, the black dotted line shows the orbit in the frame described by the linear approximation $R_{\text{ref}}(I + \hat{\Sigma})$ and finally the red line shows the orbit in the frame described by $R_{\text{exact}}$. Recall that the frame described by the linear approximation $R_{\text{ref}}(I + \hat{\Sigma})$ is only orthonormal to order $O(\left|\Sigma\right|^2)$ (8.14). It is seen that the full nonlinear system tracks the linear approximation. In Fig. 8.4 the corresponding values of $\Sigma_{\text{exact}}$ and $\Sigma$ are shown for $\xi = 14^\circ$ and $\delta \zeta_1 = \delta I_1 t^{-1}$. The time evolution shows good agreement between the linear approximation and the nonlinear system. In Fig. 8.5 the time evolutions of the optimal angular momentum $\delta h$ and torque $\delta h$ are shown. Since it is preferred to leave the orbital period unspecified the units of the components of the angular momentum are kgm$^2$rad$^{-1}$. Moreover, the rotation about the third axis effectively decouples into a double integrator and the torque $\delta h_3$ is therefore a straight line in
8.6 Application to an inspection mission

The two remaining components are coupled and the torque profile curves in time. The relative difference at the end of the manoeuvre when $\nu$ again equals $\pi/2$, is plotted for different values of $\xi$. The last equality follows from the fact that the control is chosen precisely so that $\Sigma$ vanishes at the end of the manoeuvre. Another way of quantifying the error is via the angle:

$$\Delta \phi \equiv \arctan \frac{\sqrt{\delta q_1''^2 + \delta q_3''^2}}{|\delta q_2''|},$$

where $(\delta q_1'', \delta q_2'', \delta q_3'')$ are the coordinates of the target in the body coordinates described through $R_{\text{exact}}$. The angle $\Delta \phi$ would vanish if the target satellite was aligned with the second principal axis of the satellite. The angle is shown in Fig. 8.6 (b) for different values of $\xi$. Again there is a good agreement with the full nonlinear system and the linear system. For $\xi = 20^\circ$, for example, there is a relative error of 2.85% in $\Sigma$. This relative error corresponds to an absolute error of approximately 0.57° cf. (8.40). This is also in good agreement with the value 0.56° of the angle $\Delta \phi$ for $\xi = 20^\circ$ (see Fig. 8.6 (b)). Fig. 8.6 (c) finally shows the cost of the complete manoeuvre composed of the four increments.

8.6.1 Comparison with a PD-controller

The satellite manoeuvre considered can also be performed using a standard PD quaternion feedback controller [152]:

$$\delta \dot{q} = K_p q_e + K_d q_e \Gamma^{-1} G,$$

where $K_p$ and $K_d$ are positive definite gain matrices and $q_e$ is the vector part of the error quaternion [135]. However, as this type of controller only addresses asymptotically convergence it is not possible to complete each increment of $\Delta \nu = \pi/2$ of the manoeuvre while satisfying both the specified final attitude requirement and non-zero angular velocity. Instead one can use this type of controller to stabilise ($\Omega = 0$) the system about the desired direction and “hope to choose” the appropriate gains so
that the target satellite appears in the desired direction at the right time. This is not expected to be very torque efficient. In particular, torque is used to slow down the satellite at the completion of each increment of the manoeuvre. However, to match the following desired attitude, the satellite angular velocity will have to increase again. In Fig. 8.7 the target’s relative orbit for $\xi = 14^\circ$ and $\delta e_1 = \delta \theta I^{-1}$ is visualised in the Hill frame, the frame obtained by solving the full nonlinear system with the optimal control and finally the frame obtained by using the PD-controller. For a fair comparison the gains for the PD controller have been chosen to

$$K_p = 1.0 \text{ rad}^{-2} I, \quad K_d = 0.87 \text{ rad}^{-1} I,$$

by trial and error, so that the used $L^2$-cost (8.42) equals the optimal cost: 31.5 kg² m⁻¹ rad⁻¹ used for the optimal controller. The trajectory has been projected onto the two different planes in (a) and (b). The objective was, as before, to control the attitude so that the relative position vector at the end of the manoeuvre was along the second principal axis (the $j'$ axis in the figure). It is made obvious that the optimal controller performs better for the given cost. In fact the error in terms of $\Delta \phi$ (8.41) is
8.7 Application to a tetrahedral formation

In Chap. 3 section 3.6 it was shown how one could construct a tetrahedral formation at aphelion by choosing appropriate initial conditions at perihelion. To further...
demonstrate the usefulness of having analytical solutions of the optimal control, we show how to change the attitude of the four satellites from given initial conditions at perihelion so that each satellite has a principal axis coinciding with the corresponding relative position to the centre point of the tetrahedron. We will again assume that the satellites are three-axis stabilised so that $\mathbf{G} + \mathbf{h} = 0$.

In Fig. 8.9 (Fig. 3.7 (a) and (c) repeated) the four satellites 1, 2, 3 and 4 are shown at aphelion and perihelion. For simplicity it is assumed that satellite 1's attitude is at rest with a principal axis perpendicular to the reference orbital plane. Within the approximation of neglecting disturbances it will therefore also be at rest at perihelion and in particular the relative position from 1 to the centre of the tetrahedron will coincide with a principal axis. For the remaining three satellites: 2, 3 and 4 a reference solution is assumed with $\mathbf{G}_{\text{ref}} = (0, 0, \bar{\omega})$ and that the body frame, described by the reference attitude matrix $\mathbf{R}_{\text{ref}}$, initially coincide with the frame given by $\{i', j', k\}$. Here $i'$ and $j'$ describe the local vertical and local horizontal and where $\bar{\omega} = \frac{2\pi}{T}$ is the mean orbital frequency. In the absence of any control the satellites will at aphelion just have rotated $180^\circ$ about the inertial $k$-axis relative to their initial configuration.
at perihelion. The satellites 2, 3 and 4 are at aphelion positioned at (see also [77])

\[
\begin{align*}
\delta q^{(2)} &= R(-\sin \theta^*i + \cos \theta^*k), \\
\delta q^{(3)} &= R \left( \frac{1}{2} \sin \theta^*i - \frac{\sqrt{3}}{2} \sin \theta^*j + \cos \theta^*k \right), \\
\delta q^{(4)} &= R \left( \frac{1}{2} \sin \theta^*i + \frac{\sqrt{3}}{2} \sin \theta^*j + \cos \theta^*k \right),
\end{align*}
\]  

(8.43)
Figure 8.7: The target's relative orbit for $\xi = 10^\circ$ and $\delta e_i = \delta l_i l_i^{-1}$ in the Hill frame (blue), the frame obtained through solving the full nonlinear system with the optimal control (black dotted) and finally the frame obtained by using the PD-controller (red). The optimal controller performs better for the given cost. The errors in terms of $\Delta \phi$ (8.41) are $0.33^\circ$ and $12^\circ$.

Figure 8.8: In (a) the three components of the angular momenta are shown in the body frames given through the optimal controller (dotted curves) and the PD controller. It is seen that the angular momentum for the PD controller decreases. In (b) the components of the torque arc shown (dotted again corresponds to the optimal controller). The two controller have very distinct torque profiles.
8.7 Application to a tetrahedral formation

relative to the tetrahedron’s centre point. Recall that \( R = \sqrt{3} S \ell, \theta^* \approx 109.3^\circ \) and that \( \ell \) is the side length of the tetrahedron. In this example the optimal control is constructed so that within the linear theory the following net rotations is obtained relative to the reference rotation \( \mathbf{R}_\text{ref} \):

- For satellite 2: a rotation of \( \theta^* - \pi/2 \approx 19.3^\circ \) about the \( j \)-axis;
- For satellite 3: a rotation of arctan \( \left( \frac{1/2 \sin \theta^*}{\sqrt{3/2 \sin \theta^*}} \right) = -30^\circ \) about the \( k \)-axis and a rotation of \( \theta^* - \pi/2 \approx 19.3^\circ \) about the \( j \)-axis;
- For satellite 4: a rotation of arctan \( \left( \frac{1/2 \sin \theta^*}{\sqrt{3/2 \sin \theta^*}} \right) = 30^\circ \) about the \( k \)-axis and a rotation of \( \theta^* - \pi/2 \approx 19.3^\circ \) about the \( j \)-axis.

These rotations will imply that the first principal axis of the satellite 2 and the second principle axes of the satellites 3 and 4 coincide with the corresponding relative position vector to the centre of the tetrahedron.

Within the linear approximation the coordinates in the body frame of each satellite is given by:

\[
\delta \mathbf{q}^{(i),\nu} = \mathbf{R}_\text{ref}^T \left( \mathbf{I} - \Sigma^{(i)} \right) \delta \mathbf{q}^{(i)},
\]

all evaluated at aphelion, for \( i = 2, 3 \) and 4. Here \( \mathbf{R}_\text{ref}^T = \text{diag}(-1, -1, 1) \) (cf. (8.39) with \( \nu = \pi \) at aphelion). The aim is now to drive \( (\delta \mathbf{q}_2^{(2),\nu}, \delta \mathbf{q}_3^{(2),\nu}) \) and \( (\delta \mathbf{q}_1^{(3),\nu}, \delta \mathbf{q}_2^{(3),\nu}), \delta \mathbf{q}_4^{(3),\nu}) \), \( j = 3, 4 \) to zero. It follows that

\[
\Sigma_1^{(2)}(T) = 0, \\
\Sigma_2^{(2)}(T) = -\frac{\delta \mathbf{q}_2^{(2)}}{\delta \mathbf{q}_1^{(2)}}, \\
\Sigma_3^{(2)}(T) = \frac{\delta \mathbf{q}_2^{(2)}}{\delta \mathbf{q}_1^{(2)}},
\]

and

\[
\Sigma_1^{(3)}(T) = \frac{\delta \mathbf{q}_1^{(3)}}{\delta \mathbf{q}_2^{(3)}}, \\
\Sigma_2^{(3)}(T) = 0, \\
\Sigma_3^{(3)}(T) = -\frac{\delta \mathbf{q}_1^{(3)}}{\delta \mathbf{q}_2^{(3)}},
\]
for \( j = 3, 4 \) and where \( \delta \mathbf{q}^{(i)}, i = 2, 3, 4, \) are given by (8.43).

In Fig. 8.10 the results are presented via a projection of the relative position between the satellites and the centre point onto each of the planes orthogonal to the corresponding camera-axis. The axes have been made non-dimensional with respect to \( \ell \). The \( \circ \)'s are measured in the body frame when \( \delta \mathbf{h} = 0 \) so that \( \delta \mathbf{R} = 0 \), whereas the \( * \)'s are measured in the body frame obtained from solving the full nonlinear attitude problem with the analytic optimal control \( \delta \mathbf{h} \). The \( \triangle \)'s are measured in the frame described by the linear approximation \( \mathbf{R}_{\text{ref}} + \delta \mathbf{R} = \mathbf{R}_{\text{ref}} \left( \mathbf{I} + \hat{\mathbf{h}} \right) \). By construction the \( \square \)'s are all positioned at (0, 0) (for visualisation purposes they have, however, been slightly separated). The net rotation is smallest for satellite 2 and the error is therefore also smallest in this case: \( \approx 1.1^\circ \) compared to an error of \( \approx 5.4^\circ \) for satellites 3 and 4. The errors can be improved if it is assumed that the true attitude is measured during the manoeuvre. Then the analytic optimal torque can be re-computed from the measurements. In Fig. 8.10 the \( \circ \)'s are obtained by re-evaluating the optimal control half-way through the manoeuvre. From a single re-computation of the analytic torque an improvement of \( \approx 55\% \) is observed in the attitude of satellites 3 and 4. The improvement is \( \approx 20\% \) for satellite 1.

### 8.8 Discussion on perturbations

In this paper attention is focused upon the variations of Euler equations and the attitude dynamics using a geometrical and analytic approach to obtain optimal attitude control. Perturbations such as structural deformations and vibrations, sloshing effects and the gravity gradient are neglected. In practice these effects and uncertain-
Figure 8.10: The figure shows the projections of the relative positions between the satellites and the centre point onto each of the planes orthogonal to the corresponding camera-axis. The axis have been made non-dimensional with respect to \( \ell \). The \( \circ \)'s are measured in the body frame when \( \delta \mathbf{R} = 0 \) whereas the \( * \)'s are measured in the body frame obtained from solving the full nonlinear attitude problem with the optimal control \( \delta \mathbf{h} \). The \( \square \)'s are measured in the frame described by the linear approximation \( \mathbf{R}_{\text{ref}} + \delta \mathbf{R} = \mathbf{R}_{\text{ref}} (\mathbf{I} + \delta \mathbf{\hat{R}}) \). By construction the \( \square \)'s are all positioned at \((0,0)\). Finally, the \( \diamond \)'s are obtained by re-evaluating the optimal control half-way through the manoeuvre.

Ties cannot be neglected and will somehow have to be accounted for via feedback laws. A simple feedback law can be added on top of the open-loop solutions by re-evaluating the analytic optimal control using updated measurements. This was done in both examples above. One could also use the open-loop solutions in conjunction with a PD law to track the linearly optimal trajectories. However, the net effect of perturbations depends on the size of the perturbation and also on time scales. Therefore, the solutions may in fact be a good approximation to the true attitude dynamics near certain reference solutions over a large period of time. For example, it is expected that the gravity gradient is only significant in low Earth orbits. Indeed, it decreases as \( 1/|q|^3 \). Moreover, the effect is also negligible for almost spherical satellites. The approach could nevertheless easily be extended to account for gravity gradient. In fact, the calculations would go through almost unchanged. Then the orbit dynamics would affect the attitude dynamics and in particular it would practically limit the relevant reference solutions to relative equilibria with principal axes.
aligned with local horizontal and local vertical.

8.9 Conclusion

A general geometrical setting to variational attitude dynamics was developed for the purpose of obtaining analytic solutions of $L^2$-optimal control subject to boundary constraints. First, through variational calculus we obtained an analytic expression for the optimal control for a general linear mechanical non-autonomous system. Next, this method was applied to attitude control with reaction wheels. Simple explicit expressions was provided and applied to two formation flying mission scenario. The effect of torque saturation levels and reachability can easily be addressed through the explicit expressions. See also (8.37). The first application was to an inspection mission scenario. Here the range of validity of the linear approximation was also quantified. For a net rotation of 25° an error of $\approx 1°$ was observed. A comparison also showed that for this example the optimal controller out-performed a standard PD controller. The second application built on top of a previous application considered in Chap. 3. The attitude of a tetrahedral formation of satellites was controlled so that each satellite in the formation had a principal axis aligned with the relative position vector to the centre of the tetrahedron. With the recent interest in tetrahedral formation flying this mission scenario provides a realistic example where the use of our solutions has been demonstrated.
The Two-Body Problem of a Pseudo-Rigid Body and a Rigid Sphere

9.1 Introduction

In this chapter, which is based on joint work with Mikhail Vereshchagin, the gravitational two-body problem of a pseudo-rigid body and a rigid sphere is considered. In the first section this model will be discussed and the unreduced Hamiltonian system will be presented. This is followed by a section on the symmetries of the system, the related conserved quantities and the appropriate reduction procedure undertaken. In the final section the relative equilibria are studied. Here it is first shown that the notions of locally central equilibria and planar equilibria coincide. The problem of a rigid body and a sphere is a natural subsystem of our equations, and this result therefore also extends to this case. Another advantage of the approach undertaken here compared to [113, 134] is that the analysis goes through essentially unchanged if the pseudo-rigid body is assumed to be incompressible. Finally, it is proven that Riemann's classical theorem for pseudo-rigid bodies has a natural extension to planar relative equilibria of the considered two-body problem.
9.2 The model

We consider a sphere and a deformable body with masses \( m_1 \) and \( m_2 \), respectively, interacting through Newtonian gravitation. See Fig. 9.1. We assume that the configuration space of this system is \( \mathbb{R}^3 \times \mathbb{R}^3 \times GL^+(3) \). The former two spaces describe the centres of masses of the sphere and the pseudo-rigid body while the latter describes the deformation of the pseudo-rigid body \( B \) with respect to its centre of mass. Applying a \( GL^+(3) \) matrix to the pseudo-rigid body preserves the centre of mass. The self-gravitating potential and potential interaction between the two bodies only depend on the relative position and the configuration of the body described by \( GL^+(3) \). The system therefore possesses translational symmetry, the centre of mass moves with constant velocity, and we can reduce the system by introducing a centre of mass of the system and relative coordinates. Let \( x \) be the relative position of the two centres of masses and let \( Q \in GL^+(3) \). Then upon proper scaling, see e.g. [90], the kinetic energy of the system is:

\[
K(x, \dot{Q}) = \frac{1}{2} |x|^2 + \langle \dot{Q}, \dot{Q} J \rangle,
\]

where we by

\[
\langle (V, W) \rangle = \frac{1}{2} \text{tr}(V W^T)
\]

again denote the Riemannian inner product\(^1\) on the tangent spaces of \( GL^+(3) \), and \( J \in \text{diag}^+(3) \) is the moment coefficient of inertia of the reference configuration, see e.g. [64]. We assume that the reference configuration \( B_0 \) is spherical. It is then without loss of generality to assume that \( J = I \). Indeed, we can just replace \( Q \) by \( QJ^{-1/2} \) to achieve this. By the singular value decomposition:

\[
Q = R \hat{A} S^T,
\]

where \( R, S \in SO(3) \), and \( \hat{A} \in \text{diag}^+(3) \), it therefore follows that the configuration of the pseudo-rigid body at any time is ellipsoidal with principal axis half-lengths equal to the diagonal entries of \( \hat{A} \).

The potential of the system naturally splits into three parts \( U = U_{\text{grav}} + U_{\text{elit}} + U_{\text{class}} \). The first part \( U_{\text{grav}} \) is due to the gravitational interaction between the sphere and the pseudo-rigid body. This potential is simply the Newtonian inter-particle gravitational interaction integrated up over the pseudo-rigid body \( B \):

\[
U_{\text{grav}} = -\int_B \frac{\mu}{|x + z|} dz,
\]

\(^1\)The reason for including the factor of \( \frac{1}{2} \) will become apparent later.
9.2 The model

Figure 9.1: The two-body problem of a sphere and a pseudo-rigid body. Here $x$ denotes the relative position. The matrix $Q \in GL^+(3)$ describes the configuration of the pseudo-rigid body so that any point, say, $w$ of the pseudo-rigid body in its reference coordinate is mapped to a new point $Qw$ in the deformed pseudo-rigid body. We assume that the reference configuration is spherical and it therefore follows that the configuration of the pseudo-rigid body is ellipsoidal at all times.

Figure 9.1: The two-body problem of a sphere and a pseudo-rigid body. Here $x$ denotes the relative position. The matrix $Q \in GL^+(3)$ describes the configuration of the pseudo-rigid body so that any point, say, $w$ of the pseudo-rigid body in its reference coordinate is mapped to a new point $Qw$ in the deformed pseudo-rigid body. We assume that the reference configuration is spherical and it therefore follows that the configuration of the pseudo-rigid body is ellipsoidal at all times.

where $\mu$ is the universal gravitational constant. There are simplifications available for ellipsoids, see [11], but we do not need them. The reason we are concerned with such configurations will be explained later.

The second part of the potential $U_{\text{self}}$ is due to self-gravitating forces on the pseudo-rigid body. The expression for that for a homogeneous ellipsoid with unit-
density is given by Dirichlet's formula:

$$U_{\text{self}} = \frac{1}{2} \int_S \mu \left( \int_0^\infty \Phi(u, z) du \right) dz,$$

$$\Phi(u, z) = \frac{1}{\sqrt{(d_1^2 + u)(d_2^2 + u)(d_3^2 + u)}} \left( \sum_{i=1}^3 \frac{z_i^2}{d_i^2 + u} - 1 \right),$$

where $d_i$ are the half-lengths of the principle axis, see [113]. The final term in the potential is due to possible elastic forces on the body and its surface. Such potentials are considered and described in [91]. Compared to [113] we do not require the body to be homogeneous. Instead we restrict attention to the larger class of spherically symmetric pseudo-rigid bodies:

**Definition 1** We call a pseudo-rigid body spherically symmetric if in its reference spherical configuration the potential $U$ is rotational invariant. \square

A pseudo-rigid body is only spherically symmetric if material parameters, such as density and elasticity, in its reference spherical configuration only depend upon the distance from the centre of body. We collect the hypotheses:

**H1** The rigid sphere is external to the pseudo-rigid body.

**H2** The reference configuration is spherically symmetric.

From the kinetic energy we define the following Legendre transformations:

$$\langle F_l(x), v \rangle + \langle F_l(q), V \rangle = dk(x, q)(v, V) = (x, v) + 2\langle (q, V) \rangle,$$

for every $v \in \mathbb{R}^3$, $V \in T_q GL^+(3)$, so that $y = x$ and $P = 2Q$ are the momenta canonically associated with $x$ and $Q$, respectively [100]. The Hamiltonian is the function on the phase space $\mathcal{P} \equiv T^* (\mathbb{R}^3 \times GL^+(3))$ defined by:

$$H(x, y, Q, P) = \langle y, \dot{x} \rangle + \langle (P, Q) \rangle - K(x, Q) + U(x, Q)$$

$$= \frac{1}{2} \langle (x, y) + \frac{1}{4} \langle (P, P) \rangle + U(x, Q), (9.3)$$

equipped with canonical symplectic structure associated with the Poisson bracket:

$$\{f, g\}(x, y, Q, P) = \langle \delta_x f, \delta_y g \rangle - \langle \delta_x g, \delta_y f \rangle + \langle \frac{\delta f}{\delta Q}, \frac{\delta g}{\delta P} \rangle - \langle \frac{\delta g}{\delta Q}, \frac{\delta f}{\delta P} \rangle,$$

---

\^We identify the dual $T_q GL^+(3)$ with $T_q GL^+(3)$ via the inner product (9.1).
for \( f, g \in C^\infty(\mathcal{P}) \). This system is not very convenient to work with. First of all Hamilton's equations will include matrix equations. Furthermore, it is not straightforward to account for incompressibility within these equations. This would have to be done through Lagrange multipliers. We can, however, circumvent these issues by choosing appropriate coordinates that allow for reduction of the system. In these coordinates constraints such as incompressibility are also easily accounted for.

### 9.3 Symmetry and reduction

In this section we shall make use of symmetries to reduce the system. We shall throughout make use of the hat-map (7.10). This map defines an isomorphism between the Lie-algebra \( so(3) \) and \( (\mathbb{R}^3, \cdot \wedge \cdot) \) but also between \( (so(3), \langle \cdot, \cdot \rangle) \) and \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle) \) as inner-product spaces. The latter property is the reason for the factor of \( \frac{1}{2} \) introduced in the definition of the trace inner product. Finally, it also has the following properties \[ [125, 96] \):

\[
\begin{align*}
T & = T, \\
-2 & = T - \frac{1}{2} T, \\
zw & = zw, \\
g(z \wedge w) & = gzgw = (gz) \wedge (gw),
\end{align*}
\]

and

\[
\begin{align*}
\hat{z} & = z, \\
\hat{z}w & = zw, \\
gz & = gzg^T, \\
gz & = gzg^T.
\end{align*}
\]

for any \( z \in \mathbb{R}^3, w \in \mathbb{R}^3 \) and \( g \in SO(3) \). We now define two actions of \( SO(3) \) on \( \mathcal{P} \) by:

\[
\begin{align*}
l_g : \mathcal{P} \ni (x, y, Q, P) & \mapsto (gx, gy, gQ, gP) \in \mathcal{P}, \\
r_g : \mathcal{P} \ni (x, y, Q, P) & \mapsto (x, y, Qg, Pg) \in \mathcal{P},
\end{align*}
\]

for \( g \in SO(3) \). We then have the following consequence of the hypotheses:

**Proposition 3** The Hamiltonian system (9.3) is subject to the hypotheses is invariant under \( l_g \) and \( r_g \), i.e. \( H(r_g \circ l_g(z)) = H(z), \forall z \in \mathcal{P} \) and \( \forall (g, h) \in SO(3)^2 \). □

The first part regarding \( l_g \) is precisely what is exploited in the work in the two-body problem of a rigid body and a sphere. See for example \[150, 96, 132, 11, 149\]. By Noether's theorem \[64\] the symmetries \( l_g \) and \( r_g \) generate conserved quantities \( J_I \) and \( J_r \), respectively. Since the symmetries are due to the left and right actions of \( SO(3) \) the conserved quantities are maps from \( \mathcal{P} \) to the dual \( so(3)^* \) defined by:

\[
\begin{align*}
\langle \hat{J}_I(x, y, Q, P), \hat{\Sigma} \rangle & = \langle y, \Sigma \wedge x \rangle + \langle P, \hat{\Sigma} Q \rangle, \\
\langle \hat{J}_r(Q, P), \hat{\Sigma} \rangle & = \langle [P, Q] \hat{\Sigma} \rangle,
\end{align*}
\]
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for every $\mathbf{\hat{\Sigma}} \in so(3)$, see for example [64] Chapter 8. Therefore we have:

$$
\langle \mathbf{\hat{J}}_t(Q, P), \mathbf{\hat{\Sigma}} \rangle = \langle \mathbf{x} \wedge \mathbf{y}, \mathbf{\Sigma} \rangle + \langle (PQ^T, \mathbf{\hat{\Sigma}}) \rangle = \langle \mathbf{x} \wedge \mathbf{y} + PQ^T, \mathbf{\hat{\Sigma}} \rangle,
$$

$$
\langle \mathbf{\hat{J}}_r(Q, P), \mathbf{\hat{\Sigma}} \rangle = \langle Q^T P, \mathbf{\hat{\Sigma}} \rangle,
$$

so that upon identifying $so(3)^* \leftrightarrow so(3)$ through the inner product and taking the skew-symmetric part to ensure that $\mathbf{\hat{J}}_t, \mathbf{\hat{J}}_r \in so(3)$:

$$
\mathbf{\hat{J}}_t(x, y, Q, P) = \mathbf{x} \wedge \mathbf{y} + \frac{1}{2} (PQ^T - QP^T),
$$

$$
\mathbf{\hat{J}}_r(Q, P) = \frac{1}{2} (Q^T P - P^T Q).
$$

Here $\mathbf{\hat{J}}_t$ and $\mathbf{\hat{J}}_r$ are the total angular momentum and the circulation, respectively. See also [125, 64]. The momentum maps are left and right equivariant to the action of $SO(3)$ in the following sense:

$$
J_t(gx, gy, gQ, gP) = gJ_t(Q, P)g^T = g\mathbf{\hat{J}}_t,
$$

$$
J_r(gQ, gP) = h^T J_r(Q, P)h = h^T \mathbf{\hat{J}}_r,
$$

for every $(g, h) \in SO(3)^2$. We have here used (9.5) in the last equality. Furthermore, the momentum maps are right and left invariant in the following sense:

$$
\mathbf{\hat{J}}_t(x, y, Qh, Ph) = \mathbf{\hat{J}}_t(x, y, Q, P),
$$

$$
\mathbf{\hat{J}}_r(gQ, gP) = \mathbf{\hat{J}}_r(Q, P).
$$

We shall make use of these facts in the following lemma which will be useful later on when proving the extension of Riemann’s theorem. Here we will also make explicit use of the singular value decomposition (9.2) and define the skew-symmetric matrices:

$$
\mathbf{\hat{\Omega}} = R^T \tilde{R}, \quad \mathbf{\hat{\Lambda}} = S^T \tilde{S} \in so(3).
$$

At a relative equilibrium the trajectory follows an orbit of a one-parameter subgroup of the symmetry group $SO(3)^2$:

$$
x(t) = \exp(\Omega t)x_0, \quad Q(t) = R(t)\tilde{A}_0 S(t)^T,
$$

where $\tilde{A}_0$ is a constant diagonal matrix and

$$
R(t) = \exp(\tilde{\Omega} t), \quad S(t) = \exp(\tilde{\Lambda} t).
$$

Here $\Omega$ and $\Lambda$ are the angular velocities associated with the rigid body and orbital rotation and the internal rotation of particles of the body, respectively. We then have the following property:
Lemma 3 At relative equilibria:

\[ J_I \land \Omega = 0, \quad J_r \land \Lambda = 0. \]

PROOF The conservation of \( J_I \) and \( J_r \) imply that:

\[
\begin{align*}
J_I(x(t), y(t), Q(t), P(t)) &= J_I(x(0), y(0), Q(0), P(0)), \\
J_r(Q(t), P(t)) &= J_r(Q(0), P(0)).
\end{align*}
\]  

By setting \( g = \exp(\tilde{\Omega} t) \) and \( h = \exp(-\tilde{\Lambda} t) \) in (9.8) and (9.9) we have:

\[
\begin{align*}
J_I(x(0), y(0), Q(0), P(0)) &= \exp(\tilde{\Omega} t) J_I(x(0), y(0), Q(0), P(0)), \\
J_r(Q(0), P(0)) &= \exp(\tilde{\Lambda} t) J_r(Q(0), P(0)).
\end{align*}
\]  

Differentiating with respect to \( t \) at \( t = 0 \) gives:

\[ \tilde{\Omega} J_I = 0, \quad \tilde{\Lambda} J_r = 0, \]

or simply by (9.4):

\[ \Omega \land J_I = 0, \quad \Lambda \land J_r = 0. \]

An almost identical calculation is made in [125].

9.3.1 Decomposition

In our reduction procedure we will make use of the singular value decomposition \( Q = RAS^T \) (9.2). This decomposition is not unique. However if \( Q(t) \) is an analytic path in \( GL^+(3) \) then there exists analytic paths \( R(t) \), \( A(t) \) and \( S(t) \) such that \( Q(t) = R(t)A(t)S(t)^T \) \[64\]. This will, in particular, hold at relative equilibria. For a more detailed discussion see \[125, 64\]. The reference \[64\] also gives the following easy interpretation:

- \( S^T \) rotates the coordinates in the reference frame.
- \( \tilde{A} \) stretches the body along the instantaneous principle axis of \( S^T(B_0) \).
- \( R \) rotates the deformed body.

See also Fig. 9.2. Upon replacing \( GL^+(3) \) as configuration manifold by the product
Figure 9.2: The action of $Q$ can through singular value decomposition be decomposed into the following steps: 1) a rotation $S^2$ of the reference sphere; 2) a deformation $\hat{A}$ along the instantaneous principle axis; 3) a rotation $R$ of the ellipsoid.

$SO(3) \times \text{diag}^+(3) \times SO(3)$, we obtain a new expression for the kinetic energy:

$$K(\dot{x}, (R\hat{A}S)^T) = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + \langle (R, R\hat{A}^2) \rangle + \langle (\hat{A}, \hat{A}) \rangle$$

$$+ 2\langle (R\hat{A}S^T, R\hat{A}S^T) \rangle + \langle (S, S\hat{A}^2) \rangle,$$

and, through straightforward calculations, momenta:

$$R\hat{M} = RA^2 - RA^2(R^T R)^T + 2R\hat{A}S^T S\hat{A},$$

$$\hat{B} = \hat{A},$$

$$S\hat{N} = S\hat{A}^2 - S\hat{A}^2(S^T S)^T + 2S\hat{A}R^T R\hat{A}.$$

(9.13) (9.14)

Here we have again taken skew-symmetric parts to ensure that $\hat{M}$ and $\hat{N}$ belong to $so(3)$. The Hamiltonian then takes the following form:

$$H(x) = \langle y, x \rangle + \langle (R\hat{M}, R) \rangle + \langle (B, \hat{A}) \rangle + \langle (S\hat{N}, S) \rangle - K(\dot{x}, (R\hat{A}S)^T)$$

$$+ U(x, R\hat{A}S^T)$$

(9.15)

equipped with the canonical symplectic structure associated with the Poisson bracket:

$$\{f, g\}(x) = \langle \partial_x f, \partial_y g \rangle - \langle \partial_x g, \partial_y f \rangle + \langle \frac{\delta f}{\delta R^T M}, \frac{\delta g}{\delta R M} \rangle - \langle \frac{\delta g}{\delta R^T M}, \frac{\delta f}{\delta R M} \rangle$$

$$+ \langle \frac{\delta f}{\delta A}, \frac{\delta g}{\delta B} \rangle - \langle \frac{\delta g}{\delta A}, \frac{\delta f}{\delta B} \rangle + \langle \frac{\delta f}{\delta S}, \frac{\delta g}{\delta S} \rangle - \langle \frac{\delta g}{\delta S}, \frac{\delta f}{\delta S} \rangle, \quad f, g \in C^\infty(P),$$
for \( f, g \in C^\infty(\mathcal{P}) \) and where \( z = (x, y, R, R\tilde{M}, \tilde{A}, \tilde{B}, S, S\tilde{N}) \). The action \( \tau_h \circ \iota_g \) is through the decomposition mapped to the action

\[
\varphi_{gh}(z) = (gx, gy, gR, gR\tilde{M}, \tilde{A}, \tilde{B}, hS, hS\tilde{N}), \quad (g, h) \in SO(3)^2,
\]

which leaves the Hamiltonian (9.15) invariant. The Hamiltonian therefore descends to a Hamiltonian function \( h \) on the quotient space \( \mathcal{P}/SO(3)^2 \). We may define a model for \( \mathcal{P}/SO(3)^2 \) by taking \( (g, h) = (R^T, S^T) \) in (9.16) so that

\[
\hat{z} = (R^Tx, R^Ty, I, \hat{M}, \hat{A}, \hat{B}, I, \hat{N}) \in \mathcal{P}.
\]

Let \( \lambda = R^Tx, \mu = R^Ty \) and

\[
\hat{R} = R\hat{\Omega}, \quad \hat{S} = S\hat{\Lambda}.
\]

In particular, (9.13) and (9.14) then simplify to:

\[
\hat{M} = \hat{I}_d\hat{A} - \hat{I}_c\hat{A},
\]

\[
\hat{N} = \hat{I}_d\hat{A} - \hat{I}_c\hat{A},
\]

with inverse

\[
\hat{\Omega} = \hat{I}_d\hat{M} + \hat{I}_c\hat{N},
\]

\[
\hat{\Lambda} = \hat{I}_d\hat{N} + \hat{I}_c\hat{M},
\]

where

\[
I_d = \text{tr} \hat{A}^2 - \hat{A}^2 = \text{diag} (d_1^2 + d_2^2, d_1^2 + d_3^2, d_2^2 + d_3^2),
\]

\[
I_c = \text{diag} (2d_1d_3, 2d_1d_3, 2d_1d_3),
\]

\[
I^d = (I^2_d - I^2_c)^{-1}I_d,
\]

\[
I^c = (I^2_d - I^2_c)^{-1}I_c,
\]

\[
\text{and } \hat{\Lambda} = \text{diag}(d_1, d_2, d_3).
\]

Remark 4 (Singularities): Notice from (9.24) and (9.25) that \( I^c \) and \( I^d \) do not exist if \( d_i = d_j, \) for some \( i, j \in \{1, 2, 3\} \) with \( i \neq j, \) i.e. if the body is spheroidal. This is a consequence of our decomposition. The spheroidal configurations are, however, exceptional. There is therefore little loss of generality by restricting attention to ellipsoidal configurations, as in Riemann’s classical theorem.

Upon identifying \( (\hat{A}, \hat{B}) \) with \( (A = (d_1, d_2, d_3), B = (b_1, b_2, b_3)) \in T^*R^3 \) a straightforward calculation shows that:

\[
h(w) \equiv H(\bar{z}) = \frac{1}{2}(B, B) + \frac{1}{2}(\mu, \mu) + \frac{1}{2}(M, I^dM) + \frac{1}{2}(N, I^dN) + (N, I^cM)
\]

\[
+ u(\lambda, A),
\]

\[
w = (M, \lambda, \mu, A, B, N),
\]

(9.26)
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where \( u(\lambda, A) = U(\lambda, \tilde{A}) \).

The Poisson structure also descends to a non-canonical Poisson structure on the reduced space. In [150] it is shown how one obtains the reduced brackets for the system of a rigid body and a sphere. One can repeat the exact same calculations to obtain the part of the reduced bracket related to the left invariance. Similarly, it can be shown that the reduced bracket related to the right invariance is just the standard reduced rigid body bracket, see for example [64]. We therefore have:

**Theorem 6** The reduced system on \( \mathcal{P}/SO(3)^2 \) is described by the Hamiltonian (9.26) equipped with the Poisson structure matrix:

\[
J = \begin{pmatrix}
\dot{\lambda} & \mu & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
\tilde{\mu} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & N & 0 \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}
\]

Since \( \partial_M h = \mathbb{I}^d M + \mathbb{I}^c N = \Omega \) and \( \partial_N h = \mathbb{I}^d N + \mathbb{I}^c M = \Lambda \), Hamilton’s equations read:

\[
\begin{align*}
\dot{M} &= M \wedge \Omega + \lambda \wedge \partial_\lambda u, \\
\dot{\lambda} &= \lambda \wedge \Omega + \mu, \\
\dot{\mu} &= \mu \wedge \Omega - \partial_\mu u, \\
\dot{N} &= N \wedge \Lambda, \\
\dot{\Lambda} &= B, \\
\dot{B} &= -\partial_\Lambda h.
\end{align*}
\] (9.27)

The initial Hamiltonian system has now been reduced by its symmetries and relative equilibria therefore coincide with equilibria of the reduced system described by Theorem 6 and the equations in (9.27).

Now, \( L = M + \lambda \wedge \mu \) and \( N \) are the body angular momentum and body circulation, respectively, so that \( J_1 = RL \) and \( J_r = SN \) (see (9.6) and (9.7)). By virtue of the reduction we have:

**Proposition 4** The functions \( C = C(\|N\|, \|L\|) \), are Casimir functions of the system. In particular, \( |N| \) and \( |L| \) are conserved.

Besides decreasing the necessary degrees of freedom, the introduced reduction also allows us to decouple the dependency of the effective rotations \( R \) and \( S \) (9.17).
In particular, in the rotational frame the mutual attraction between the bodies is independent of the attitude of the pseudo-rigid body. Instead, the rotation affects the orbital motion, and vice versa, via the dependency of the angular momenta $M$ and $N$.

### 9.4 Relative equilibria

By the reduction, the relative equilibria of the system are solutions of the following system:

$$
0 = M \wedge \Omega + \lambda \wedge \partial_{\lambda} u, \quad (9.28)
$$

$$
0 = \lambda \wedge \Omega + \mu, \quad (9.29)
$$

$$
0 = \mu \wedge \Omega - \partial_{\lambda} u, \quad (9.30)
$$

$$
0 = N \wedge \Lambda, \quad (9.31)
$$

$$
0 = B, \quad (9.32)
$$

$$
0 = \partial_{\lambda} h, \quad (9.33)
$$

We shall assume throughout that the pseudo-rigid body is not spheroidal. For further simplifications it is advantageous to eliminate $\mu$ from (9.29) so that (9.30) and the total angular momentum read:

$$
0 = \lambda |\Omega|^2 - \Omega(\lambda, \Omega) - \partial_{\lambda} u. \quad (9.34)
$$

and

$$
L = M + \Omega|\lambda|^2 - \lambda(\lambda, \Omega), \quad (9.35)
$$

respectively. At relative equilibria (9.8) and (9.9) give:

$$
L = J_{\lambda}(x_0, y_0, \hat{A}_0, P_0), \quad N = J_r(\hat{A}_0, P_0).
$$

Lemma 3 may therefore be restated as:

**Corollary 4** At relative equilibria:

$$
L \wedge \Omega = 0, \quad N \wedge \Lambda = 0.
$$

**Proof** Here we show that this results can also be deduced directly from the reduced equations. Indeed, the latter condition coincides with equation (9.31). For the former condition, take the right outer product of $L$ expressed by (9.35) with $\Omega$, so that

$$
M \wedge \Omega - \lambda \wedge \Omega(\lambda, \Omega) = 0.
$$
The first item of this equation, using equation (9.28), is equal to $-\lambda \wedge \partial_{\lambda} u$. In turn, $\partial_{\lambda} u$ can be eliminated from (9.30). After these substitutions, the first item is equal to the negative of the second one, and hence $L \wedge \Omega = 0$. The corollary is therefore completed.

As for the rigid body case [132], the relative equilibria can be divided into two types: *locally central* and *non-locally central*. We also define planar equilibria in the following definition:

**Definition 2.1.** A relative equilibrium is said to be locally central if the mutual attraction and relative position vectors are parallel, i.e. $\lambda \wedge \partial_{\lambda} u = 0$.

2. A relative equilibrium is said to be planar if the total angular momentum vector $L$ is perpendicular to relative position vector $\lambda$, i.e. $(L, \lambda) = 0$.

However, the following theorem implies that the two notions of locally central and planar equilibria actually coincide:

**Theorem 7** Assume that the pseudo-rigid body satisfies the hypotheses. Then a relative equilibrium of the system is planar if and only if it is locally central:

$$(L, \lambda) = 0 \iff \lambda \wedge \partial_{\lambda} u = 0.$$  

**Proof** First notice that by using (9.35) that $(L, \lambda) = 0$ is equivalent to $(M, \lambda) = 0$.

Let $\lambda \wedge \partial_{\lambda} u = 0$. Then from equation (9.28) it follows that $M \wedge \Omega = 0$. Taking the left outer product of the last equation with $\lambda$, we obtain:

$$\Omega(M, \lambda) = M(\lambda, \Omega).$$  

Then, taking the left outer product of equation (9.30) with $\lambda$, and applying the assumption $\lambda \wedge \partial_{\lambda} u = 0$, we obtain $\lambda \wedge \Omega(\lambda, \Omega) = 0$. Thus, there are two feasible cases. If $\lambda \wedge \Omega = 0$ then from equation (9.29) it follows that $\mu$ vanishes and from equation (9.30) it then follows that the gradient $\partial_{\lambda} u$ vanishes. This can only be true if the sphere is internal to the pseudo rigid body. This contradicts hypothesis H1. Thus, $(\lambda, \Omega)$ vanishes, and then, since $\Omega$ does not vanish, from equation (9.36) it follows that $(M, \lambda)$ vanishes. Thus, the sufficient condition has been proved.

Let $(L, \lambda) = 0$. By Corollary 4 it follows that $(\Omega, \lambda) = 0$. Therefore, by eliminating $M$ in (9.35) and inserting this into (9.28), we have:

$$\lambda \wedge \partial_{\lambda} u = 0.$$  

Thus, the necessary condition has been proved.
We note that the two-body problem of a rigid body and a sphere is a natural subsystem:

\[
\begin{align*}
\dot{M} &= M \wedge \Omega + \lambda \wedge \partial \lambda u, \\
\dot{\lambda} &= \lambda \wedge \Omega + \mu, \\
\dot{\mu} &= \mu \wedge \Omega - \partial \lambda u,
\end{align*}
\]

with \( A \) fixed, and hence the Theorem also holds true for this case.

Riemann's theorem describes geometrical properties of the angular velocity and the circulation of a pseudo-rigid body in a relative equilibrium. In the following, we show that Riemann's theorem extends to the relative equilibria of the two-body systems of this paper whenever \( A \) is aligned with one of the principle axes of the ellipsoid. First, however, we note that from Corollary 4 it follows that \( L \) and \( \Omega, N \) and \( A \) are parallel pairwise. Here \( \Omega = 0 \) would imply that \( \mu = 0 \) and \( \partial \lambda u = 0 \) and therefore contradicts that the system is in equilibrium. If \( A \neq 0 \) we therefore introduce \( k_{\Omega} \) and \( k_{\lambda} \) so that \( L = k_{\Omega} \Omega \) and \( N = k_{\lambda} A \).

**Theorem 8** Assume that the system is in a relative equilibrium where the pseudo-rigid body is not spheroidal and where \( A \) is aligned with the \( l \)th principle axis of the pseudo-rigid body. Denote by the integers \( m \) and \( n, m \neq n, m, n \neq l \), the two remaining principle axes. Then:

1° If \( A \neq 0 \) and either of the following equations hold true:

\[
k_{\lambda} = d_{n}^{2} + d_{m}^{2} - 3 \left( d_{l}^{2} - d_{m}^{2} \right) \left( d_{l}^{2} - d_{n}^{2} \right) \lambda_{l}^{-2}
\]

or

\[
d_{l}^{2} = \frac{1}{6} \lambda_{l}^{2} + \frac{1}{2} k_{\lambda}, \quad d_{m}^{2} = -\frac{1}{6} \lambda_{l}^{2} + \frac{1}{2} k_{\lambda},
\]

and \( \frac{1}{2} \lambda_{l}^{2} < k_{\lambda} < \frac{5}{2} \lambda_{l}^{2} \) and \( d_{n} \) arbitrary so that \( d_{l}^{2} - d_{m}^{2} = \frac{1}{3} \lambda_{l}^{2} \), or finally Riemann's theorem hold true: (i) the angular velocity of the pseudo-rigid body vector \( \Omega \) and the internal rotation velocity vector \( A \) lie in the same principal plane of the body and (ii) if one of the vectors is aligned with a principal axis, then the other vector is aligned along the same axis.

2° If \( A = 0 \) and either

\[
d_{l}^{2} = \frac{1}{2} k_{\Omega} - \lambda_{l}^{2}, \quad d_{m}^{2} = \frac{1}{2} k_{\Omega},
\]

and \( 2 \lambda_{l}^{2} < k_{\Omega} < 4 \lambda_{l}^{2} \) or \( \Omega \) is in a principle plane of the body.
Proof For 1° assume that $\mathbf{A} \neq 0$. Then from (9.35), (9.18) and (9.19) it follows that

$$
\left( \begin{array}{cc}
L_d - (k_\Omega - |\lambda|^2) I - \lambda \lambda^T & -I_e \\
-I_e & L_d - k_A I
\end{array} \right) \left( \begin{array}{c}
\Omega \\
\Lambda
\end{array} \right) = 0, \quad (9.37)
$$

Since $\lambda$ is assumed to be aligned with one principle axis, it follows that $\lambda \lambda^T = \text{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ with only one diagonal element non-zero. Therefore there exists a non-zero solution for $(\Omega_i, \Lambda_i)$ if and only if the determinant of the linear system

$$
\left( (L_d)_{ii} - k_\Omega + |\lambda|^2 - \lambda_i^2 \right) \Omega_i - (L_e)_{ii} \Lambda_i = 0,
\left( (L_d)_{ii} - k_A \right) \Lambda_i - (L_e)_{ii} \Omega_i = 0
$$

vanishes. We compute:

$$
\left( (L_d)_{ii} - k_\Omega + |\lambda|^2 - \lambda_i^2 \right) k_A - (L_d)_{ii} (k_\Omega + k_A - |\lambda|^2 + \lambda_i^2) + (L_e)^2_{ii} - (L_e)_{ii}^2 = 0. \quad (9.38)
$$

From the definitions of $L_d$ (9.22) and $L_e$ (9.23) we then obtain the following lemma:

Lemma 4 If $(\Omega_i, \Lambda_i)$ and $(\Omega_j, \Lambda_j)$ are non-zero solutions of (9.37) with $i \neq j$, then either $\lambda_i = \lambda_j = 0$ and $d_i = d_j$ or the following two equations hold true

$$
k_{\Omega} + k_A = d_i^2 + d_j^2 - 2d_k^2 + |\lambda|^2
- \lambda_1^2 (d_i^2 + d_k^2 - k_A) (d_i - d_k)^{-1} + \lambda_2^2 (d_i^2 + d_k^2 - k_A) (d_i - d_k)^{-1}, \quad (9.39)
$$

where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$. $\Box$

**Proof** We solve (9.38) for $k_{\Omega} + k_A$ and $k_A k_A$ for $i = i$ and $i = j$.

If Riemann's theorem hold true then one of the pairs $(\Omega_i, \Lambda_i)$ vanishes. Now assume otherwise. Since the numbering is arbitrary, we may without loss of generality assume that $\lambda_1 \neq 0$ and $\lambda_2 = 0 = \lambda_3$. Since the pseudo-rigid body is assumed to be ellipsoidal, it follows from Lemma 4, and in particular (9.39), that:

$$
d_i^2 + d_k^2 - 2d_j^2 + \lambda_1^2 (d_i^2 + d_k^2 - k_A) (d_i^2 - d_k^2)^{-1}
= d_i^2 + d_k^2 - 2d_ j^2 + \lambda_1^2 (d_i^2 + d_k^2 - k_A) (d_i^2 - d_k^2)^{-1}
= d_i^2 + d_k^2 - 2d_j^2 + \lambda_1^2.
$$

After some straightforward manipulations this can be shown to imply that either

$$
k_A = d_i^2 + d_k^2 - 3 \left( d_i^2 - d_k^2 \right) \left( d_i^2 - d_k^2 \right) \lambda_1^2.
$$
or
\[ d_i^2 = \frac{1}{6} |\lambda|^2 + \frac{1}{2} k_n, \quad d_j^2 = -\frac{1}{6} |\lambda|^2 + \frac{1}{2} k_n, \]
and \( d_k \) arbitrary for \( j \neq k, j, k \in \{2, 3\} \). The conditions \( d_i < \lambda_i \) and \( d_j > 0 \) yield the inequalities: \( \frac{1}{3} \lambda_i^2 < k_n < \frac{2}{3} \lambda_i^2 \). Moreover, by solving for \( k_n \) it also follows from the latter equation in (9.40) that
\[ d_i^2 - d_j^2 = \frac{1}{3} \lambda_i^2. \]

In the following we will show the last part of Riemann's theorem. Since \( L_c \) is invertible it follows from (9.37) that
\[ \Omega = L_c^{-1} (L_d - k_n I) \lambda. \]

Assume then that \( \lambda_i = \lambda_j = 0, i \neq j \). Then by (9.41) it follows that either \( \Omega_i = \Omega_j = 0 \) or \( d_i = d_j \). The latter contradicts the body being ellipsoidal. We can repeat the same arguments for \( \Omega_i = \Omega_j = 0 \). The last part of Riemann's theorem has therefore been shown.

For \( 2^o \) let \( \lambda = 0 \). Then the first rows of (9.37) give
\[ \left( L_d - (k_n - |\lambda|^2) I - \lambda \lambda^T \right) \Omega = 0. \]

All component of \( \Omega \) may only be non-zero if the matrix appearing in (9.42) have zero rank. This implies that
\[ d_i^2 = \frac{1}{2} k_n - \lambda_i^2, \quad d_j^2 = \frac{1}{2} k_n. \]
From \( 0 < d_i^2 < \lambda_i^2 \) it follows that \( 2 \lambda_i^2 < k_n < 4 \lambda_i^2 \). The proof is completed.

The proof of the theorem did only rely on Corollary 4 and the properties of the conserved quantities and the angular velocities \( \Omega \) and \( \lambda \). Therefore the result applies to other similar two-body problems, for example in molecular dynamics. By making use of other properties in a relative equilibrium of the considered gravitational two-body problem we can show the following:

**Proposition 5** The second property of Riemann's theorem, see (ii) in Theorem 8, can only hold true in a locally central equilibrium.

**Proof** Assume otherwise so that \( \Omega_i = 0 = \Omega_j, i \neq j \) and therefore \( \lambda_i = 0 = \lambda_j \). Then from (9.37) we have:
\[ \lambda_i(\lambda, \Omega) = 0 = \lambda_j(\lambda, \Omega), \]
so that \((\lambda, \Omega) = 0\) or \(\lambda = 0 = \lambda_f\). The former implies through (9.34) that the equilibrium is planar. By assumption the latter must therefore hold true. But then \(\lambda \parallel \Omega\) and through (9.29) and (9.30) it follows that \(\mu = 0\) and \(\partial_{\lambda} u = 0\), respectively. This is absurd in equilibria. This completes the proof. ■

It can also be shown that the exceptions to the validity of Riemann's theorem in Theorem 8 cannot hold true in a locally central relative equilibrium. This follows from Theorem 7 and the following lemma:

**Lemma 5** In a locally central equilibrium the point mass is located along a principal axis of the body.

**Proof** This follows directly from the fact that the pseudo-rigid body is ellipsoidal. See also [132]. ■

Indeed, we have the following:

**Theorem 9** If the system is in a locally central equilibrium, then Riemann’s theorem hold true: (i) the angular velocity of the pseudo-rigid body vector \(\Omega\) and the internal rotation velocity vector \(\lambda\) lie in the same principal plane of the body and (ii) if one of the vectors is aligned with a principal axis, then the other vector is aligned along the same axis.

**Proof** By the assumptions of the theorem and Lemma 5 it follows that: (a) \(\lambda\) is aligned with a principle axis, say \(i\), and (b) \((\Omega, \lambda) = 0\) so that \(\Omega\) is contained in a principle plane with \(\Omega_i = 0\). Therefore by (9.41) it follows that either \(k_\lambda\) is such that \(\bar{e}_i^{-1}(\bar{I}_d - k_\lambda I)\) is in a principle plane with \(\bar{I}_d = 0\). By inserting (9.41) into (9.37) we obtain the following equation:

\[
0 = \left((\bar{I}_d - (k_\lambda + |\lambda|^2) I - \lambda \lambda^T) \bar{e}_i^{-1}(\bar{I}_d - k_\lambda I) - \bar{e}_i\right) \Lambda.
\]

From this equation it follows that if \(\left(\bar{e}_i^{-1}(\bar{I}_d - k_\lambda I)\right)_i = 0\) then \(\Lambda_i = 0\). The first part of Riemann’s theorem has been completed. The last part is proved by repeating the arguments in Theorem 8.

It is now natural to ask what happens when the equilibrium is non-planar. We do not expect a generalisation of Riemann’s theorem beyond Theorem 8. In the following we shall instead investigate the non-planar equilibria with the particular aim of diminishing the necessary equations while gaining further insight into the underlying geometry. Although, we pretty much follow the method proposed by Scheeres in [11], we find our approach clearer and simpler as unlike [132] we present explicit formulas for obtaining all the variables describing the relative equilibrium once \(\lambda\) is found.
9.4 Relative equilibria

9.4.1 Non-locally central relative equilibrium

Let $\lambda \wedge \partial_\lambda u \neq 0$. Then if we take the inner product of equation (9.28) with $\Omega$, we obtain that the vectors $\Omega$, $\lambda$ and $\partial_\lambda u$ all lie in the same plane. Furthermore, by taking the inner product of equation (9.30) with $\Omega$, we obtain that $(\Omega, \partial_\lambda u) = 0$. Hence, the vectors $\Omega$ and $\partial_\lambda u$ are perpendicular. The vectors $\lambda$, $\partial_\lambda u \wedge \lambda$ and $\partial_\lambda u \wedge (\partial_\lambda u \wedge \lambda)$ therefore form an orthogonal basis in $\mathbb{R}^3$. Let us denote this basis by $F_\lambda$. In this basis the vector $\Omega$ has only one non-zero component as it is parallel to $\partial_\lambda u \wedge (\partial_\lambda u \wedge \lambda)$. This allows us to write $\Omega$ in the following way:

$$\Omega = \pm |\Omega| \frac{\partial_\lambda u \wedge (\partial_\lambda u \wedge \lambda)}{|\partial_\lambda u \wedge (\partial_\lambda u \wedge \lambda)|},$$

(9.43)

Note that since the system is symmetric with respect to the reflection $(\Omega, \lambda, \mu) \mapsto -(\Omega, \lambda, \mu)$ the choice of a sign in (9.43) is not important. The magnitude of $\Omega$ can be found by taking the inner product of equation (9.34) with $\partial_\lambda u$:

$$|\Omega|^2 = \frac{|\partial_\lambda u|^2}{(\lambda, \partial_\lambda u)},$$

(9.44)

Here $(\lambda, \partial_\lambda u)$ does not vanish. Indeed this would imply $\partial_\lambda u = 0$ and therefore $|\lambda| = \infty$. Moreover, it is strictly positive. Finally, after some simplifications, the vector $\Omega$ can be rewritten in terms of $\lambda$ and the potential in the following way:

$$\Omega = \pm \frac{\partial_\lambda u \wedge (\partial_\lambda u \wedge \lambda)}{\sqrt{(\lambda, \partial_\lambda u) \left(\lambda^2|\partial_\lambda u|^2 - (\lambda, \partial_\lambda u)^2\right)}},$$

(9.45)

Let us now first assume that $\lambda \neq 0$. Then from (9.41) it follows that

$$(I_d - k_\lambda I)\Lambda = I_\epsilon \Omega,$$

(9.46)

where $k_\lambda$ is a parameter. The matrix $(I_d - k_\lambda I)$ is diagonal and may only have one zero non-diagonal component, as otherwise it would imply that the pseudo-rigid body was spheroidal. We therefore consider two different scenarios. In the first scenario the matrix is invertible so that:

$$\Lambda = (I_d - k_\lambda I)^{-1}I_\epsilon \Omega,$$

(9.47)

Next, let the matrix have a zero component so that $\Omega_i = 0$. From this $k_\lambda$ can be determined along with the other components of $\Lambda$ by inverting corresponding diagonal elements of the matrix. We then leave the remaining component $\Lambda_i$, rather than $k_\lambda$ as above, as a parameter. We have:

$$\Lambda_i = \Lambda_{i0}, \quad \Lambda_j = \frac{2djd_k}{d^2 - d^2_j} \Omega_i, \quad \Lambda_k = \frac{2djd_j}{d^2_i - d^2_k} \Omega_k,$$

(9.48)
where \((i, j, k)\) are in a cyclic permutation. After the introduced eliminations of \(\mu, \Omega\) and \(\Lambda\) equations (9.28) and (9.33) form a closed subsystem, which should be solved for the position \(\lambda\) of the relative equilibria. These equations are vectorial. It means that there are six scalar equations. However, the amount of equations determining the relative equilibrium position can be diminished. Each of the equations can be treated as a vector which should vanish. Any vector can be resolved in a unique way along an orthogonal basis, and the condition that a vector vanishes is equivalent to the condition that all its components in an orthogonal basis vanish. We shall use the basis \(\mathcal{F}_\lambda\) for this purpose. In this basis equation (9.28) has a zero component along the vector \(\partial_{\lambda} u \wedge (\partial_{\lambda} u \wedge \lambda)\). This allows us to reduce the equation (9.28) to two scalar equations, and together with equations (9.33) they give us the minimum number of equations for finding the relative equilibrium position:

\[
\langle M^*, \partial_{\lambda} u \rangle + \langle \lambda, \partial_{\lambda} u \rangle \left( \lambda^2 \partial_{\lambda} u^2 - \langle \lambda, \partial_{\lambda} \lambda \rangle \right) = 0, \tag{9.49}
\]

\[
(M^*, \lambda \wedge \partial_{\lambda} u) = 0 \tag{9.50}
\]

\[
\partial_{\lambda} h = 0. \tag{9.51}
\]

Here

\[
M^* = \pm (I_d - I_c - k_\lambda I_d^{-1} I_c) \partial_{\lambda} u \wedge (\partial_{\lambda} u \wedge \lambda), \tag{9.52}
\]

is a re-normalised angular momentum of the pseudo-rigid body valid only when \((I_c - k_\lambda I)\) is invertible. Otherwise we replace this vector by:

\[
M^* = \pm I_c \partial_{\lambda} u \wedge (\partial_{\lambda} u \wedge \lambda) - I_c \lambda, \tag{9.53}
\]

where \(\Lambda\) is given by (9.48).

We now consider the case when \(\Lambda\) vanishes. Then \(M = I_d \Omega\) and condition (9.31) is identically satisfied. We therefore have the same three equations as above but with \(M^*\) given by (9.53) evaluated at \(\Lambda = 0\).

Compared to the equations found by [132], we have obtained five equations as opposed to two due to the extra degrees of freedom in our system. The angular momentum vector also has a more complicated form. Again the rigid body case can be considered as a subsystem considering \(\Lambda = 0\) and \(d_1, d_2\) and \(d_3\) as constants.

### 9.5 Conclusion

By extending the reduction procedure of the two body problem of a rigid body and a sphere we reduced the system of the two body problem of a spherical symmetric pseudo-rigid body and a sphere so that the corresponding rigid body problem was a
9.5 Conclusion

natural subsystem. We showed that the pseudo-rigid body problem possesses similar properties and structure to the corresponding rigid body problem. In particular, we showed that the notions of locally central and planar relative equilibria coincide. This result also includes the rigid body problem. We also showed that Riemann's theorem of pseudo rigid bodies had a natural extension for planar relative equilibria.
The Two-Body Problem of a Pseudo-Rigid Body and a Rigid Sphere
In this final chapter the thesis is concluded. First we summarise the findings and conclusions. We finish the chapter by first listing the main research achievements that contribute to the state of the art and follow this by discussing the open problems for future work.

10.1 Summary of conclusions

The research has addressed several topics related with satellite relative motion and has primarily with mathematical rigor sought to analyse, develop and unify relative motion models of free and tethered satellites. First, analytical solutions of the variational equation of the Kepler problem with variational mass parameter were obtained about any reference orbit in a compact form by relying on the super-integrability of the Kepler problem. The solutions provide approximations to the relative motion of satellites near any Keplerian reference orbit and were written in terms of the relevant conserved quantities: relative energy, relative angular momentum and relative eccentricity vector. The geometrical setting, in which the solutions were derived, also allowed for a straightforward design of tetrahedral formations. The inclusion of possible variations in mass parameter is thought to be very powerful in comet dust tail modelling. In the inversion of light intensity measurements for comet related data, millions of test particles are usually propagated numerically [50, 73, 105]. For
such a large number of particles, the analytical solutions certainly relax the required numerical effort significantly.

Next, several different tether models were related mathematically to provide a unified framework and it was established in what limits they could provide insight into the full tether dynamics. Firstly, the massive tether model was linked to the slack-spring model through a conjecture on the limit of vanishing thickness. Then the slack-spring model was related to the billiard model in the limit of an inextensible tether. Next, the motion of the dumbbell model was identified within the dynamics of the billiard model through a theorem on the existence of invariant curves. The existence of the invariant curves within the planar billiard model with an underlying circular orbiting centre of mass imply that the tethered system cannot reach these practically relevant, stable regions of phase space without control. Numerical computations provided some insights into the dynamics of the billiard map for the case of an underlying circular orbiting centre of mass. It was also shown that the classical tether model was ill-posed due to the development of shock waves when in compression. The equations were regularised by the addition of bending resistance. A symplectic integrator was then developed for the regularised, massive tether model by using a spatial discretisation by Hermite elements of the variational form. Using this integrator the motion of an orbiting tether system with and without the inclusion of the regularising bending term was compared. The examples showed that the effects of bending may be severe even on very short timescales, at least when the tether is in compression. For an initial condition near the stable relative equilibrium it was shown that the two systems with and without bending evolved similarly with the bending energy remaining small. However, even for this example the system without bending was more sensitive to numerical instabilities and the energy difference grew because of the development of shock fronts. The developed integrator was also used to provide numerical evidence for the validity of the conjecture on the relationship between the massive tether model and the slack spring model. The rate of this convergence was through the numerical experiments estimated to be of order $\mathcal{O}(h^2)$ where $h$ is the thickness of the tether.

Further insight into the slack-spring approximation was also provided through a Galerkin approximation of the full massive tether model. The singular perturbed, truncated system was shown to possess a slow manifold with bifurcation. This type of problem is not well-studied within the Hamiltonian setting where the interesting lower dimensional slow manifolds often are normally elliptic. This is quite a surprise as the slow manifold bifurcation is a general phenomenon. A theorem on the adiabatic invariance of the slow manifold was proven using rigorous averaging and a blow up near the bifurcation. It was finally shown that the slow dynamics on the slow manifold coincided with the slack-spring approximation.

Next, a general geometrical setting to variational attitude dynamics was developed. The geometrical framework avoided the use of coordinates to describe the
10.2 Research achievements and contributions to the state of the art

relative and reference attitude. Constrained $L^2$-optimal torque was then obtained for the attitude dynamics controlled with reaction wheels. This result does not extend trivially from previous studies as the torque that is sought minimised generally enters the equations along with its anti-derivative. Explicit expressions were nevertheless provided and applied to two realistic formation flying mission scenarios. The first application was to an inspection mission scenario. Here the range of validity of the linear approximation was also quantified. For a net rotation of $25^\circ$ an error of $\approx 1^\circ$ was observed. In the second application the attitude of a tetrahedral formation of satellites was controlled so that each satellite in the formation had a principle axis aligned with the relative position vector to the centre of the tetrahedron. With the recent interest in tetrahedral formation flying this mission scenario provided a realistic example where the use of the analytic solutions were demonstrated.

In joint work with Mikhail Vereshchagin, the thesis finally considered a gravitational two-body problem of a pseudo-rigid body and a rigid sphere. For the case of a spherical symmetric pseudo-rigid body we reduced the system by its symmetries through an extension of the standard reduction procedure of the two-body problem of a rigid body and a sphere. This way the corresponding rigid body problem became a natural subsystem. We then showed that the pseudo-rigid body problem possessed similar properties and structure to the corresponding rigid body problem. In particular, we showed that the notions of locally central and planar relative equilibria coincided. We also showed that Riemann's theorem of pseudo rigid bodies had a natural extension for planar relative equilibria.

10.2 Research achievements and contributions to the state of the art

For clarity we list the main research achievements that contribute to the state of the art:

- Complete, geometrical interpretable solution of the variational Kepler problem expressed in the physically relevant relative position coordinates;

- A unification of conservative tether models including:
  - A theorem on the well-posedness of the bending regularised massive tether model;
  - The formulation of a conjecture on the relationship between full massive tether model and finite dimensional slack-spring supported by numerical evidence;
The identification of the dumbbell dynamics within the billiard model through a theorem on invariant curves;

- The identification of an interesting, both mathematically and practically, and rich process of reduction near slow manifold bifurcations;

- A theorem on the slack-spring model as a limit system of a Galerkin approximation of the massive tether model;

- The development and application to realistic mission scenarios of constrained $L^2$-optimal torque of attitude control;

- Reduction of the gravitational two-body problem of a pseudo-rigid body and rigid sphere;

- A theorem on an extension of Riemann's theorem to the two-body problem.

10.3 Open problems

The research has highlighted many interesting routes for possible further research exploration. We will in the following presents some of these possibilities for future work.

Comet tail modelling

First of all, it would be interesting to investigate the usefulness of having analytical expressions for comet dust tail modelling. In comet dust tail modelling it is of interest to invert light intensity measurements in the tail to obtain the distribution of dust particles sizes, ejection velocities and the distribution of the true anomalies when the particles were ejected from the nucleus [47]. Usually the dust tail is assumed to be collision-less [47, 50, 73, 105]. Without further modelling this problem is in fact in general ill-posed as the objective is to invert a map from a four dimensional space $(\delta \mu, \delta \eta, \delta \nu)$, assuming that the tail is co-planar with the nucleus orbit, to the plane in which the light intensity measurements are made. However, with further modelling on the ejection velocities the problem may admit a general solution. In particular, if all particles are assumed to leave the nucleus with zero velocity then through the analytical solutions obtained, the tail can be equipped with synchrone/syndyne-coordinates and the inversion can in fact be done analytically. For more complicated models of the ejection velocities numerical methods are required. These are very computational intensive and typically involve propagation of millions of test particles [50, 73, 105]. For such a large number of particles, the analytical solutions may
10.3 Open problems

relax the numerical effort considerably. Future research may also show that they can provide further insight into the tail formation.

Slow manifolds with bifurcations

The research on tethered satellites begun the unification of the different conservative models. The research in this direction highlighted several interesting topics for possible further exploration. The massive tether model was, for example, conjectured to be related to a slack-spring model in the limit of vanishing tether thickness. We did not manage to prove this conjecture within the three years of research. However, it is believed that the general theory of multi-dimensional, singular perturbed Hamiltonian systems is not yet at a stage that allows for rigorous investigation of this problem. The problem is related to the persistence of an almost everywhere normally elliptic, symplectic, lower dimensional slow manifold with bifurcations embedded within an infinite dimensional symplectic phase space. In fact, we saw through a discretisation of the full system that even the finite dimensional analogue had received little attention. In a post-doctoral EPSRC research proposal we have proposed to use this real-life application as a guide into the development of a new mathematical theory of slow manifolds with bifurcations in Hamiltonian systems with several fast degrees of freedom. We will in the following present a list of projects that aim to enhance the understanding of these systems and this type of behaviour and ultimately to bring the theory to a level in which a proof of the tether conjecture, Conjecture 1, can be given.

Project A: Dynamic bifurcation in a Galerkin approximation of tether system

To encapsulate and demonstrate the dynamic bifurcation of the tether system further within a finite dimensional analogue, it may useful to consider higher order Galerkin approximations of the full planar system. We saw in section 7 that the Galerkin approximation with two elements was equivalent to a system of two end point masses, the satellites, connected to a third particle with small mass $e$ via unit natural length springs. On an invariant, two-degree of freedom symplectic subspace, we showed that a relatively large set of trajectories behaved like that of the slack-spring model. The difficulties with extending this result to the higher order Galerkin approximations are due to resonances, but also due to the fact that more fast degrees of freedom bifurcates when the distance between the end-points equal the natural length. By considering higher order Galerkin approximations we may gain a further insight into the bifurcating modes and what is needed to provide a general proof of the conjecture. This system may also provide a relatively simple example where the proposed method
of combining MacKay's method with an appropriate energy method, see Project C, can be applied.

**Project B: Dynamic transcritical and centre-saddle bifurcations in slow-fast Hamiltonian systems**

We believe it is possible to apply the geometric approach of Chow and Young [23] on dynamic bifurcations to other types of Hamiltonian bifurcations. The transcritical bifurcation, for example, was considered in [87] for a one degree of freedom Hamiltonian systems with a slowly varying parameter. Contrary to the pitchfork bifurcation their results show that it is an exception for trajectories to remain close to the stable branches as the slowly varying parameter drifts through the bifurcation value. This is due to the presence of a (collapsing) separatrix in the frozen system which separates bounded motion from unbounded motion. Trajectories which are initially bounded eventually cross this unperturbed separatrix and moves into the region of unbounded motion. On the other hand, which the authors do not note, this implies that trajectories coming from infinity may be trapped within the separatrices bounding the stable equilibrium after bifurcation. The results are not presented as rigorous results, and it therefore seems natural to try to obtain a geometrically motivated proof for the crossing along the lines of Chow and Young's proof for the symmetric pitchfork [23].

In [33] the centre-saddle bifurcation is considered. The authors also consider slowly varying one degree of freedom Hamiltonian systems. The corresponding frozen system has two centres and one saddle before the bifurcation. At the bifurcation value one of the centres coalesce with the saddle to form a nonlinear saddle in a centre-saddle bifurcation. After the bifurcation there is therefore only one center. The authors show, primarily using asymptotic expansions and results of [110], that trajectories initially inside the collapsing unperturbed separatrices eventually cross the separatrices and continue after the bifurcation to oscillate about the remaining centre, for which the averaging is again valid, with a small change in the action. Again we believe that these results could be obtained rigorously using more geometrical motivated ideas and methods.

**Project C: Almost invariance of slow manifolds for analytic slow-fast Hamiltonian systems using the geometrical method of MacKay**

The classical averaging procedure in general fails for several fast degrees freedoms. However, in my opinion this may just be a consequence of the method. In fact, the geometrical method of MacKay presented in [97] is an alternative iterative procedure that improves on a slow manifold without noticing the presence of resonances. This method, valid for any numbers of degrees of freedom, simply makes use of the implicit function theorem to obtain an improved slow manifold as critical points of the
Hamiltonian restricted to the fast normal space. For $C^r$ systems this procedure may be iterated $r$ times. In fact, Mackay conjectures that this procedure could be iterated in the analytical case using the Neishstadt estimates [108, 109], also used by Gelfreich and Lerman in [52], to obtain an almost invariant slow manifold with exponentially small error. One of the benefits of Mackay's method is that the slow manifold at each step includes all nearby equilibria of the real system. This cannot be guaranteed in general by the averaging method. Furthermore, Mackay's error-estimates are pointwise in the slow vector field and therefore improve near equilibria. Another advantage is that it only relies on the implicit function theorem and is therefore applicable on any Banach space, making the method suitable for PDEs. The third project of the research will be to prove the conjecture of MacKay [97] and obtain slow manifolds with exponentially small error vector field. It will be the aim in the research to do so in general Banach spaces so that the results also apply to PDEs.

The geometrical approach of MacKay benefits from the fact that it does not address resonances. However, this is also why extra work is required to obtain powerful results on the long-term stability near the slow manifold. This is of great importance in the problems of dynamic bifurcation. Indeed, this will be crucial in Project A and ultimately for the validity of the tether conjecture: Project D. For this maybe one could develop the energy method further by providing sufficient conditions, perhaps something like: definiteness of the Hamiltonian restricted to the fast space, and a procedure for identifying proper adiabatic Lyapunov functions for slow manifolds.

Project D: Proof of tether conjecture

The previous projects all natural lead up to a proof of the tether conjecture of Conjecture 1. The success of this project depends on the success of the previous projects. To prove the conjecture, we will need an understanding of slow manifold bifurcations (Project A and B) but also invariance and stability of slow manifolds for PDEs (Project C). A natural starting point would be to identify dynamic bifurcation phenomena in simpler, scalar PDEs. An example could be a perturbed, scalar, nonlinear wave equation with homogeneous, Neumann boundary conditions and a slowly varying parameter that unfolds a pitchfork bifurcation scenario of homogeneous steady states:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \lambda u - u^3, \quad \partial_x u|_{x=0,1} = 0, \quad \lambda = \epsilon \ll 1, \epsilon > 0.$$ 

We believe this could also be of interest in the theory of nonlinear wave equations. The dissipative analogue, including a term proportional to $\partial_t u$ or $-\partial_t \partial_x^2 u$ on the left hand side, might also be a good, tractable starting point. In general, dissipative infinite dimensional systems are thought to be easier than their conservative counterparts so we believe initial progress could be made this way.
Effect of dissipation on tethered satellite systems

In this research, attention has been focused on conservative models of tethered satellites. It would be interesting and relevant in future work on tethered satellites to address the effect of energy dissipation in more details. In particular, it would be interesting to obtain results on the massive tether system’s global attractor. Here one might speculate or conjecture that the global attractor is the set of relative equilibria. It is, however, believed that this problem is very difficult to address rigorously. Perhaps the methods used by Ball in [7] to show a similar statement for the damped semi-linear wave equations could be of use in this direction.
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