Dual pairs for matrix groups

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In honour of Darryl Holm's 70th birthday.

Abstract

In this paper we present two dual pairs that can be seen as the linear analogues of the following two dual pairs related to fluids: the EPDiff dual pair due to Holm and Marsden, and the ideal fluid dual pair due to Marsden and Weinstein.

1 Introduction

1.1 Definitions and new results

Let M be a symplectic manifold, P_1, P_2 Poisson manifolds, and let $\mathrm{J}_1, \mathrm{J}_2$ be a pair of Poisson surjective submersions

$$P_1 \overset{\mathrm{J}_1}{\leftarrow} M \overset{\mathrm{J}_2}{\rightarrow} P_2,$$

The maps $\mathrm{J}_1, \mathrm{J}_2$ are said to form a *Lie-Weinstein dual pair* [18, 16] if the tangent distributions to the fibres of $\mathrm{J}_1, \mathrm{J}_2$ are symplectically orthogonal, i.e.,

$$(\ker T\,\mathrm{J}_1)^\omega = \ker T\,\mathrm{J}_2\,.$$

They are said to form a *Howe dual pair* [16] if the Poisson subalgebras $\mathrm{J}_1^*(C^\infty(P_1))$ and $\mathrm{J}_2^*(C^\infty(P_2))$ of $(C^\infty(M), \{\cdot,\cdot\})$ centralise one another. Under mild conditions [16, Proposition 11.1.3], the definitions of Lie-Weinstein and Howe dual pairs may be shown to be equivalent. When such dual pairs exist, the Poisson structures in P_1 and P_2 are closely related. In particular, it can be shown (see for example [3, Theorem E.13]) that there is a one-to-one correspondence between symplectic leaves of P_1 and P_2.

Several authors have considered the analogue of dual pairs in situations where $\mathrm{J}_1, \mathrm{J}_2$ are not necessarily submersions, and have suggested conditions under which the above-mentioned one-to-one correspondence between symplectic leaves in P_1 and P_2 still holds. We mention in particular the work of Ortega [15, 16] on singular dual pairs, and that of Balleier and Wurzbacher [2] on the centralising Howe condition without the submersion property. In this paper, we will use the term "dual pair" to mean in this generalised sense, satisfying many of the desired properties of dual pairs, but not necessarily Lie-Weinstein. Specifically, we are concerned with the situation where a dual pair on the symplectic manifold M arises from a pair of Hamiltonian actions of finite-dimensional Lie groups G_1, G_2, with corresponding equivariant momentum maps $\mathrm{J}_1 : M \to \mathfrak{g}_1^*$ and $\mathrm{J}_2 : M \to \mathfrak{g}_2^*$.

We first discuss a criterion we call *mutual transitivity*, meaning the fibres of J_1 are G_2-orbits and the fibres of J_2 are G_1-orbits. We show that when this criterion is satisfied, the

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coadjoint orbits in $\mathrm{J}_1(M) \subset \mathfrak{g}_1^*$ and $\mathrm{J}_2(M) \subset \mathfrak{g}_2^*$ have a one-to-one correspondence. Additionally, any reduced space for the G_1-action is symplectomorphic to a coadjoint orbit in $\mathrm{J}_2(M)$, and similarly with 1 and 2 switched.

We then give several examples of mutually transitive dual pairs. We initially discuss a dual pair first considered by Balleier and Wurzbacher [2]. We then construct two Lie-Weinstein dual pairs, inspired by dual pairs related to fluid mechanics, and describe explicitly the (co)adjoint orbit correspondence between their images. The first of these was considered in [7, pp.502-506], although the full adjoint orbit correspondence was not provided there. The second is to our knowledge novel. We also point out some interesting relations between the momentum maps of the three examples considered, reminiscent of the seesaw pairs of dual pairs [8].

For other discussions of the relationship between reduced spaces and coadjoint orbits in matrix group dual pairs, see [2, 7, 10]. In [2, Proposition 2.6], the authors show that mutual transitivity is a consequence of the *symplectic Howe condition* (stating $\mathrm{J}_1^*(C^\infty(\mathfrak{g}_1^*))$ and $\mathrm{J}_1^*(C^\infty(\mathfrak{g}_2^*))$ centralise one another), plus properness of both actions. Since in both of our main examples the action is not proper (see Sections 4.3 and 5.2), we focus instead on mutual transitivity.

1.2 Motivation for Weinstein's definition of dual pairs

Inspired by the work of Lie [12], Kazdhan, Kostant, and Sternberg [7], and Howe [6], dual pairs were introduced by Weinstein [18] in the context of the following problem. Suppose P is a Poisson manifold, and we wish to find a canonical form for the Hamiltonian flow generated by some $h \in C^\infty(P)$. One approach to doing so is to introduce a so-called *symplectic realisation* of P, which is a surjective Poisson map $\mathrm{J} : M \to P$, where M is some symplectic manifold. Since J is Poisson, it respects dynamics in the sense that it intertwines the Hamiltonian flow in M generated by $h \circ \mathrm{J}$ and the Hamiltonian flow in P generated by h. Since Hamiltonian flows in a symplectic manifold always have a local canonical form, expressed in Darboux coordinates, this gives a local canonical form for the Hamiltonian flow in P. This is the essential idea behind the introduction of *Clebsch variables* in continuum problems [14, 5].

Suppose further that $\mathrm{J} : M \to P$ is a submersion with connected fibres, and let $\mathcal{D}$ denote the (regular) foliation defined by these fibres. Let $D := T\mathcal{D}$ be the corresponding involutive distribution, and D^ω the symplectically orthogonal distribution. Using the submersion and Poisson properties of J, it may be shown that D^ω is also involutive, and so integrates to a regular foliation $\mathcal{D}^\omega$. If we further suppose $M/\mathcal{D}^\omega$ has a smooth structure making the projection $\pi : M \to M/\mathcal{D}^\omega$ a submersion, then $M/\mathcal{D}^\omega$ may be given a Poisson structure with respect to which π is Poisson, and we end up with a with a pair of Poisson maps $P \overset{\mathrm{J}}{\leftarrow} M \overset{\pi}{\rightarrow} M/\mathcal{D}^\omega$ with symplectically orthogonal fibres. Then the Hamiltonian flow in M generated by $h \circ \mathrm{J}$ preserves the leaves of $\mathcal{D}^\omega$, and so trajectories of this flow are mapped by π to points of $M/\mathcal{D}^\omega$. Hence in some sense P describes the dynamics of the problem, while $M/\mathcal{D}^\omega$ describes the dynamical invariants of its symplectic realisation.

The above construction motivates the general definition of a Lie-Weinstein dual pair $P_1 \overset{\mathrm{J}_1}{\leftarrow} M \overset{\mathrm{J}_2}{\rightarrow} P_2$. In general the fibres of $\mathrm{J}_1, \mathrm{J}_2$ are orbits (at least formally) of infinite-dimensional Lie groups, with Lie algebras isomorphic to the Poisson subalgebras $\mathrm{J}_2^*(C^\infty(P_2))$ and $\mathrm{J}_1^*(C^\infty(P_1))$ of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$. In this paper we are interested in the case where the dual pair arises from a pair of Hamiltonian Lie group actions of groups G_1, G_2, with corresponding equivariant momentum maps $\mathrm{J}_i : M \to \mathrm{J}_i(M) \subset \mathfrak{g}_i^*$.

1.3 Motivation for the particular dual pairs considered in this paper

The original motivation for this paper was to find two dual pairs that can be seen as the linear analogues of the following two dual pairs related to fluids: the EPDiff dual pair introduced by Holm and Marsden in [5] and the ideal fluid dual pair introduced by Marsden and Weinstein in [14] (proven to be indeed dual pairs in [4], where the additional technicalities in defining infinite-dimensional dual pairs are discussed). EPDiff stands here for Euler-Poincaré equation on the diffeomorphism group.

The EPDiff dual pair involves the manifold of embeddings $\operatorname{Emb}(S,N)$ of a compact manifold S into a manifold N, acted on by the diffeomorphism groups $\operatorname{Diff}(N)$ from the left and by $\operatorname{Diff}(S)$ from the right. The momentum maps for the lifted cotangent actions, restricted to the open subset $T^*\operatorname{Emb}(S,N)^\times$ of $T^*\operatorname{Emb}(S,N)$ that consists of nowhere zero 1-form densities, define a dual pair[1]

$$\mathfrak{X}(N)^* \xleftarrow{\mathrm{J}_L} T^*\operatorname{Emb}(S,N)^\times \xrightarrow{\mathrm{J}_R} \mathfrak{X}(S)^*.$$

The linear analogues of these actions are the left $\mathrm{GL}(n,\mathbb{R})$-action and the right $\mathrm{GL}(m,\mathbb{R})$-action on the manifold of rank m matrices $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ (identified with linear injective maps $:\mathbb{R}^m\to\mathbb{R}^n$). We show in Section 5 that the lifted cotangent momentum maps, restricted to an open subset of $T^*\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$, define a dual pair

$$\mathfrak{gl}(n,\mathbb{R})^*_{\mathrm{J}_L} \xleftarrow{\mathrm{J}_L} \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})\times\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \xrightarrow{\mathrm{J}_R} \mathfrak{gl}(m,\mathbb{R})^*_{\mathrm{J}_R},$$

where $\mathfrak{gl}(n,\mathbb{R})^*_{\mathrm{J}_L}$ denotes the image of J_L in the dual Lie algebra $\mathfrak{gl}(n,\mathbb{R})^*$, and similarly for $\mathfrak{gl}(m,\mathbb{R})^*_{\mathrm{J}_R}$.

For the ideal fluid dual pair one notices that, given a compact manifold S endowed with a volume form μ, and a manifold M endowed with an exact symplectic form ω, the group $\operatorname{Diff}_{\mathrm{vol}}(S)$ of volume preserving diffeomorphisms and the group $\operatorname{Diff}_{\mathrm{ham}}(M)$ of Hamiltonian diffeomorphisms act in a Hamiltonian way on $\operatorname{Emb}(S,M)$. The symplectic form considered on $\operatorname{Emb}(S,M)$ is built in a natural way with the differential forms μ and ω. The momentum map for the right $\operatorname{Diff}_{\mathrm{vol}}(S)$-action together with the momentum map for the left action of the quantomorphism group, a one dimensional central extension of $\operatorname{Diff}_{\mathrm{ham}}(M)$ that integrates $C^\infty(M)$, define a dual pair

$$C^\infty(M)^* \xleftarrow{\mathrm{J}_L} \operatorname{Emb}(S,M)^\times \xrightarrow{\mathrm{J}_R} \mathfrak{X}_{\mathrm{vol}}(S)^*.$$

In the special case $M=T^*N$ the inclusion of $\operatorname{Emb}(S,M)$ in $T^*\operatorname{Emb}(S,N)$ naturally induced by the volume form μ is symplectic. Thus the linear analogue of the symplectic manifold $\operatorname{Emb}(S,M)$ is the manifold of rank m matrices $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$ (identified with linear injective maps $:\mathbb{R}^m\to\mathbb{R}^{2n}$) and endowed with symplectic form induced by the cotangent symplectic form on $T^*\mathrm{M}_{n\times m}(\mathbb{R})$, namely the linear symplectic form on the vector space $\mathrm{M}_{2n\times m}(\mathbb{R})$

$$\Omega(E,F) := \operatorname{Tr}(E^\top \mathbb{J} F), \qquad \text{where } \mathbb{J} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}.$$

It is not difficult to see that the maximal Lie subgroups of $\mathrm{GL}(2n,\mathbb{R})$ and $\mathrm{GL}(m,\mathbb{R})$ that preserve this symplectic form are the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ and the orthogonal group $\mathrm{O}(m)$. We show in Section 4 that the momentum maps for these two actions define a dual pair

$$\mathfrak{sp}(2n,\mathbb{R})^*_{\mathrm{J}_L} \xleftarrow{\mathrm{J}_L} \mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R}) \xrightarrow{\mathrm{J}_R} \mathfrak{o}(m)^*_{\mathrm{J}_R}.$$

[1] See [4] for the precise definition of dual pair in the infinite-dimensional context.

In the special case $m = 2n$ this dual pair appears in [17] (in the context of semiclassical quantum mechanics).

1.4 Outline of paper

In Section 2, we define the notion of mutually transitive actions, describe the coadjoint orbit and coadjoint orbit-reduced space correspondences, and discuss the relation between mutually transitivity and Lie-Weinstein dual pairs. In Section 3, we describe the $(\mathrm{U}(n), \mathrm{U}(m))$ dual pair, first considered by Balleier and Wurzbacher [2], and prove it satisfies mutual transitivity. In Section 4, we construct the $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(m))$ dual pair, which is the analogue of the ideal fluid dual pair, prove it satisfies mutual transitivity, explicitly describe the (co)adjoint orbit correspondence, and point out some connections with the $(\mathrm{U}(n), \mathrm{U}(m))$ dual pair. In Section 5, we construct the $(\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(m, \mathbb{R}))$ dual pair, which is the analogue of the EPDiff dual pair, prove it satisfies mutual transitivity, explicitly describe the (co)adjoint orbit correspondence, and point out some connections with the $(\mathrm{U}(n), \mathrm{U}(m))$ dual pair.

2 Mutually transitive actions and dual pairs

In this section, we first introduce the notion of *mutually transitive actions*, and indicate the resulting coadjoint orbit correspondence. We then describe how mutual transitivity allows us to view reduced spaces of one action as coadjoint orbits of the other. We emphasise here that by contrast with other treatments in the literature, this correspondence invokes only smoothness of the actions, and does not require properness. Finally we outline the relationship between mutual transitivity and Lie-Weinstein dual pairs.

Propositions 2.2, Lemma 2.7, and Proposition 2.8 are also proved in [2, Theorem 2.9]. We choose to include them here for completeness, since the first two are short, while our treatment of the last differs somewhat from that in [2].

A fuller treatment of dual pairs and related concepts can be found in [11, Section IV.7], [16, Chapter 11], and [2].

2.1 Mutually transitive actions

Let (M, ω) be a symplectic manifold, and let $\Phi_1 : G_1 \times M \to M$ and $\Phi_2 : G_2 \times M \to M$ be symplectic actions. We assume M, G_1, and G_2 are all finite-dimensional (dual pairs in infinite dimensions are discussed in [4]).

Definition 2.1. We say the actions Φ_1, Φ_2 are *mutually transitive* if the following three properties hold:

- Φ_1 and Φ_2 commute,
- Φ_1 and Φ_2 are Hamiltonian actions, with corresponding equivariant momentum maps $\mathrm{J}_1 : M \to \mathfrak{g}_1^*$ and $\mathrm{J}_2 : M \to \mathfrak{g}_2^*$,
- each level set of J_1 is a G_2-orbit and vice versa, i.e., for any $x \in M$,

$$\mathrm{J}_1^{-1}(\mathrm{J}_1(x)) = G_2 \cdot x \qquad \text{and} \qquad \mathrm{J}_2^{-1}(\mathrm{J}_2(x)) = G_1 \cdot x.$$

Denoting the coadjoint orbit in $\mathfrak{g}_i^*$ through μ_i by $\mathcal{O}_{\mu_i}$, we then have

Proposition 2.2. *Let Φ_1, Φ_2 be mutually transitive actions, with equivariant momentum maps* $\mathrm{J}_1, \mathrm{J}_2$*. Then for all $x \in M$,*

$$\mathrm{J}_1^{-1}(\mathcal{O}_{\mathrm{J}_1(x)}) = \mathrm{J}_2^{-1}(\mathcal{O}_{\mathrm{J}_2(x)}).$$

Proof.

$$\begin{aligned}
\mathrm{J}_1^{-1}(\mathcal{O}_{\mathrm{J}_1(x)}) &= G_1 \cdot \mathrm{J}_1^{-1}(\mathrm{J}_1(x)) && \text{since } \mathrm{J}_1 \text{ is } G_1\text{-equivariant} \\
&= G_1 \cdot (G_2 \cdot x) && \text{since } \Phi_2 \text{ is transitive on the fibres of } \mathrm{J}_1 \\
&= G_2 \cdot (G_1 \cdot x) && \text{since the actions } \Phi_1 \text{ and } \Phi_2 \text{ commute} \\
&= G_2 \cdot \mathrm{J}_2^{-1}(\mathrm{J}_2(x)) && \text{since } \Phi_1 \text{ is transitive on the fibres of } \mathrm{J}_2 \\
&= \mathrm{J}_2^{-1}(\mathcal{O}_{\mathrm{J}_2(x)}) && \text{since } \mathrm{J}_2 \text{ is } G_2\text{-equivariant.}
\end{aligned}$$

□

Corollary 2.3 ([2, Theorem 2.9(i)])**.** *In this situation, there exists a one-to-one correspondence between coadjoint orbits in $\mathrm{J}_1(M)$ and $\mathrm{J}_2(M)$ given by*

$$\mathcal{O}_{\mu_1} \mapsto \mathrm{J}_2(\mathrm{J}_1^{-1}(\mathcal{O}_{\mu_1}))= \mathrm{J}_2(\mathrm{J}_1^{-1}(\mu_1))$$

or equivalently

$$\mathcal{O}_{\mathrm{J}_1(x)} \mapsto \mathcal{O}_{\mathrm{J}_2(x)} \qquad \textit{for } x \in M.$$

Proof. Proposition 2.2 shows that the defined map between coadjoint orbits is invertible. □

2.2 The relation between coadjoint orbits and reduced spaces

Let Φ be any Hamiltonian action of G on M with equivariant momentum map J, and let $\pi : M \to M/G$ denote the quotient map. For $\mu \in \mathrm{J}(M)$, let G_μ denote the coadjoint stabiliser subgroup of G at μ, let M_μ be the set $\mathrm{J}^{-1}(\mu)/G_\mu \simeq \mathrm{J}^{-1}(\mathcal{O}_\mu)/G$, and let $\pi^\mu : \mathrm{J}^{-1}(\mu) \to M_\mu \subset M/G$ denote the restriction of π to $\mathrm{J}^{-1}(\mu)$. In favourable situations (for example, if the group action Φ is free and proper), M_μ can be given a differentiable structure with respect to which π^μ is a submersion, and a symplectic structure ω_{M_μ} satisfying $(\pi^\mu)^*\omega_{M_\mu} = (i^\mu)^*\omega$, where $i^\mu : \mathrm{J}^{-1}(\mu) \hookrightarrow M$ is the inclusion. The resulting symplectic manifold (M_μ, ω_{M_μ}) is called the *reduced space* or the *Marsden-Weinstein-Meyer quotient* at $\mu \in \mathfrak{g}^*$.

In this subsection, we demonstrate that as a consequence of mutual transitivity, such a differentiable and symplectic structure can always be defined on the reduced spaces corresponding to each action. Moreover, the reduced space M_{μ_1} of the G_1-action is symplectomorphic to a coadjoint orbit $\mathcal{O}_{\mu_2} \subset \mathfrak{g}_2^*$ (for some related μ_2).

We first recall the concept of an *initial submanifold*, and its relationship to group orbits.

Definition 2.4. [16, Section 1.1.8] Let M be a manifold, and N a subset of M endowed with its own manifold structure, such that the inclusion $i : N \hookrightarrow M$ is an immersion (i.e., N is an immersed submanifold of M). We say N is an *initial submanifold* of M if for any manifold P, a map $g : P \to N$ is smooth iff $i \circ g : P \to M$ is smooth.

Remark 2.5. Stated differently, Definition 2.4 says that if $h : P \to M$ is a smooth map with image contained in N, then h corestricts to a smooth map $h' : P \to N$.

Lemma 2.6. [16, Proposition 2.3.12 (i)] If $\Phi : G \times M \to M$ is a smooth G-action, then the orbit $G \cdot x$ through $x \in M$ is an initial submanifold of M.

Before proving our main result Proposition 2.8, we first give a supplementary lemma.

Lemma 2.7. *Let Φ_1, Φ_2 be mutually transitive actions, with equivariant momentum maps $\mathrm{J}_1, \mathrm{J}_2$. Choose $x \in M$, and let $\mu_i = \mathrm{J}_i(x),\ i = 1, 2$. Then the smooth map $\mathrm{J}_2 : M \to \mathfrak{g}_2^*$ restricts to a surjective submersion $\mathrm{J}_2^{\mu_1} : \mathrm{J}_1^{-1}(\mu_1) \to \mathcal{O}_{\mu_2}$ satisfying*

$$(i^{\mu_1})^* \omega = (\mathrm{J}_2^{\mu_1})^* \omega^+_{\mathcal{O}_{\mu_2}}, \tag{1}$$

where $\omega^+_{\mathcal{O}_{\mu_2}}$ is the positive Kostant-Kirillov-Souriau form on the coadjoint orbit $\mathcal{O}_{\mu_2} \subset \mathfrak{g}_2^*$.

Proof. Since $\mathrm{J}_1^{-1}(\mu_1) = G_2 \cdot x$ is a G_2-orbit, and J_2 is G_2-equivariant, the image $\mathrm{J}_2(\mathrm{J}_1^{-1}(\mu_1))$ equals the coadjoint orbit $\mathcal{O}_{\mu_2}$. Since $\mathrm{J}_1^{-1}(\mu_1)$ is a submanifold and $\mathcal{O}_{\mu_2}$ is an initial submanifold (by Lemma 2.6), the restriction $\mathrm{J}_2^{\mu_1} : \mathrm{J}_1^{-1}(\mu_1) \to \mathcal{O}_{\mu_2}$ is smooth. By equivariance of J_2, it follows that $\mathrm{J}_2^{\mu_1}$ is a submersion.

Now taking $\xi, \zeta \in \mathfrak{g}_2$, and using the equivariance of J_2, we have

$$\begin{aligned}\omega_x(\xi \cdot x, \zeta \cdot x) &= d_x\langle \mathrm{J}_2, \xi\rangle(\zeta \cdot x) = \zeta \cdot x \,\langle \mathrm{J}_2, \xi\rangle = \langle -\operatorname{ad}^*_\zeta \mathrm{J}_2(x), \xi\rangle \\ &= \langle \mathrm{J}_2(x), [\xi, \zeta]\rangle = (\omega^+_{\mathcal{O}_{\mu_2}})_{\mathrm{J}_2(x)}(-\operatorname{ad}^*_\xi \mathrm{J}_2(x), -\operatorname{ad}^*_\zeta \mathrm{J}_2(x)).\end{aligned}$$

Again using equivariance of J_2, plus the fact that $\mathrm{J}_1^{-1}(\mu_1)$ is a G_2-orbit, gives (1). □

Proposition 2.8. *Let Φ_1, Φ_2 be mutually transitive actions, with equivariant momentum maps $\mathrm{J}_1, \mathrm{J}_2$. Then any reduced space under the G_1-action is symplectomorphic to a coadjoint orbit in $\mathrm{J}_2(M) \subset \mathfrak{g}_2^*$, and similarly with 1 and 2 switched. Explicitly, for $x \in M$,*

$$M_{\mathrm{J}_1(x)} \simeq \mathcal{O}_{\mathrm{J}_2(x)}, \qquad M_{\mathrm{J}_2(x)} \simeq \mathcal{O}_{\mathrm{J}_1(x)},$$

via a resp. G_2- and G_1-equivariant symplectomorphism.

Proof. Again choose $x \in M$, and let $\mu_i = \mathrm{J}_i(x),\ i = 1, 2$. By equivariance of J_1 and J_2 and the mutual transitivity property, it is not difficult to show that for any $y \in \mathrm{J}_1^{-1}(\mu_1)$,

$$(G_1)_{\mathrm{J}_1(y)} \cdot y = (G_2)_{\mathrm{J}_2(y)} \cdot y = G_1 \cdot y \cap G_2 \cdot y.$$

Hence the fibres of the restrictions $\pi_1^{\mu_1} : \mathrm{J}_1^{-1}(\mu_1) \to M_{\mu_1}$ and $\mathrm{J}_2^{\mu_1} : \mathrm{J}_1^{-1}(\mu_1) \to \mathcal{O}_{\mu_2}$ agree, and we get a bijection $\chi : M_{\mu_1} \to \mathcal{O}_{\mu_2}$ making the following diagram commute.

$$\begin{array}{ccc} & \mathrm{J}_1^{-1}(\mu_1) & \\ {\scriptstyle \pi_1^{\mu_1}} \swarrow & & \searrow {\scriptstyle \mathrm{J}_2^{\mu_1}} \\ M_{\mu_1} & \xrightarrow{\quad \chi \quad} & \mathcal{O}_{\mu_2} \end{array} \tag{2}$$

Pulling the smooth structure on $\mathcal{O}_{\mu_2}$ back to M_{μ_1} via χ implies that $\pi_1^{\mu_1}$ is also a smooth submersion. The fibres $(G_1)_{\mu_1} \cdot y$ of $\pi_1^{\mu_1}$ are integral manifolds of the degeneracy directions $(\mathfrak{g}_1)_{\mu_1} \cdot y$ of the restriction $(i^{\mu_1})^*\omega$. Then the usual Marsden-Weinstein-Meyer construction implies the existence of a reduced symplectic structure $\omega_{M_{\mu_1}}$ on M_{μ_1} satisfying

$$(i^{\mu_1})^* \omega = (\pi_1^{\mu_1})^* \omega_{M_{\mu_1}}. \tag{3}$$

By (1) and commutativity of diagram (2), we have

$$(i^{\mu_1})^*\omega = (\pi_1^{\mu_1})^*(\chi)^*\omega^+_{\mathcal{O}_{\mu_2}}. \tag{4}$$

Then combining (3) and (4),

$$(\pi_1^{\mu_1})^*\omega_{M_{\mu_1}} = (\pi_1^{\mu_1})^*(\chi)^*\omega^+_{\mathcal{O}_{\mu_2}},$$

and since $\pi_1^{\mu_1}$ is a surjective submersion,

$$\omega_{M_{\mu_1}} = (\chi)^*\omega^+_{\mathcal{O}_{\mu_2}},$$

i.e., χ is a symplectomorphism. Since the G_1- and G_2-actions on M commute, the G_2-action drops to M_{μ_1}. Then commutativity of the diagram (2) and G_2-equivariance of $\mathrm{J}_2^{\mu_1}$ implies that χ is G_2-equivariant.

A similar argument shows that $M_{\mu_2} \subset M/G_2$ is symplectomorphic to $\mathcal{O}_{\mu_1} \subset \mathfrak{g}_1^*$. □

The map χ in the above proof has a natural interpretation: using the identity $(i^{\mu_1})^*\omega = (\pi_1^{\mu_1})^*\omega_{M_{\mu_1}}$ it is easily shown that χ is the momentum map of the induced G_2-action on M_{μ_1}.

2.3 The relation to Lie-Weinstein dual pairs

In this subsection we make contact with the notion of dual pair in the Weinstein's original sense [18].

Definition 2.9. [16, Definition 11.1.1] Let M be a symplectic manifold, and P_1, P_2 Poisson manifolds. A pair of Poisson maps

$$P_1 \xleftarrow{\mathrm{J}_1} M \xrightarrow{\mathrm{J}_2} P_2,$$

is called a *Lie-Weinstein dual pair* if $\mathrm{J}_1, \mathrm{J}_2$ are surjective submersions satisfying

$$(\ker T\mathrm{J}_1)^\omega = \ker T\mathrm{J}_2.$$

Proposition 2.10. *Let Φ_1, Φ_2 be mutually transitive actions on M, and suppose the momentum maps $\mathrm{J}_1, \mathrm{J}_2$ have constant rank. Then $\mathrm{J}_1(M), \mathrm{J}_2(M)$ can be given smooth structures such that $\mathrm{J}_1(M) \xleftarrow{\mathrm{J}_1} M \xrightarrow{\mathrm{J}_2} \mathrm{J}_2(M)$ is a Lie-Weinstein dual pair.*

Proof. Since the maps $\mathrm{J}_i : M \to \mathfrak{g}_i^*$ are equivariant, they are Poisson ([13, Proposition 12.4.1]), and since $\mathrm{J}_i(M)$ is a union of symplectic leaves, this property still holds when the J_i are corestricted to their images.

Since $\mathrm{J}_1^{-1}(\mathrm{J}_1(x)) = G_2 \cdot x$, we have that

$$\ker T_x \mathrm{J}_1 = T_x(\mathrm{J}_1^{-1}(\mathrm{J}_1(x))) = \mathfrak{g}_2 \cdot x,$$

the first equality being a consequence of the constant rank property (see for example the discussion on page 8 of [16]). Then

$$(\ker T_x \mathrm{J}_1)^\omega = (\mathfrak{g}_2 \cdot x)^\omega = \ker T_x \mathrm{J}_2,$$

where the second equality is a standard result. So the dual pair condition holds.

We define a smooth structure on $\mathrm{J}_1(M)$ as follows: let $y \in \mathrm{J}_1(M)$, and $x \in \mathrm{J}_1^{-1}(y)$. Since J_1 has constant rank, there exist local charts (U_x, ϕ_x) about x and (V_y, ψ_y) about y with respect to which J_1 takes the form of a projection, i.e.,

$$\psi_y \circ \mathrm{J}_1 \circ \phi_x^{-1}(a_1, \dots, a_m) = (a_1, \dots, a_k, 0, \dots, 0), \tag{5}$$

with k independent of x, y. The first k components of ψ_y, restricted to $W_y = \mathrm{J}_1(U_x)$, defines a local coordinate chart $\eta_y : W_y \to \mathbb{R}^k$ about y. To show any two such charts are compatible, consider charts $(W_y, \eta_y), (W_{y'}, \eta_{y'})$, with $W_y \cap W_{y'} = \mathrm{J}_1(U_x) \cap \mathrm{J}_1(U_{x'}) \neq \varnothing$. By constructing a shifted chart $((\Phi_2)_{g_2}(U_x), \phi_x \circ (\Phi_2)_{g_2^{-1}})$ if necessary, we can without loss of generality assume $U_x \cap U_{x'} \neq \varnothing$. From (5), the level sets of $\mathrm{J}_1|_{U_x \cap U_{x'}}$ are expressed as $(a_1, \dots, a_k) = \text{const.}$ and $(a_1', \dots, a_k') = \text{const.}$ in respective local coordinates, and so the first k components of the (smooth) transition function $\phi_{x'} \circ \phi_x^{-1}$ only depend on the coordinates $(a_1, \dots, a_k)$. From

$$\eta_y \circ \mathrm{J}_1 \circ \phi_x^{-1}(a_1, \dots, a_m) = (a_1, \dots, a_k), \qquad \eta_{y'} \circ \mathrm{J}_1 \circ \phi_{x'}^{-1}(a_1', \dots, a_m') = (a_1', \dots, a_k'), \tag{6}$$

we deduce that

$$\eta_{y'} \circ \eta_y^{-1}(a_1, \dots, a_k) = (\eta_{y'} \circ \mathrm{J}_1 \circ \phi_{x'}^{-1})(\phi_{x'} \circ \phi_x^{-1}(a_1, \dots, a_k)) = (f_1(a_1, \dots, a_k), \dots, f_k(a_1, \dots, a_k))$$

for smooth functions $f_1, \dots, f_k$. From either of equations (6), $\mathrm{J}_1 : M \to \mathrm{J}_1(M)$ is a surjective submersion with respect to this smooth structure. A similar argument holds for $\mathrm{J}_2 : M \to \mathrm{J}_2(M)$. □

Remark 2.11. In general, the image of a constant rank map can exhibit so-called *multiple points*, i.e., points where the tangent space to the image cannot be defined consistently—see [11, Appendix 1, Section 1.8] for a discussion. The latter part of the proof of Proposition 2.10 essentially shows that as a consequence of the fact that level sets of J_1 are G_2-orbits, such multiple points do not exist for $\mathrm{J}_1(M)$.

Remark 2.12. The smooth structures on $\mathrm{J}_i(M)$ in Proposition 2.10 are necessarily unique [9, Theorem 4.31], and the $\mathrm{J}_i(M)$ are immersed submanifolds of $\mathfrak{g}_i^*$.

Remark 2.13. Careful examination of the proof of Proposition 2.10 shows that it is sufficient to know that *one* of the momentum maps has constant rank. From this, the Lie-Weinstein condition $(\ker T\,\mathrm{J}_1)^\omega = \ker T\,\mathrm{J}_2$ can be deduced, from which it follows that the other momentum map also has constant rank.

Example 2.14. The constant rank condition on $\mathrm{J}_1, \mathrm{J}_2$ is necessary for proving that mutual transitivity implies the Lie-Weinstein condition. For an example where the Lie-Weinstein condition fails to hold, consider $\mathbb{R}^2$ with its usual symplectic structure, and $G_1 = G_2 = \mathrm{SO}(2)$ with its usual action on $\mathbb{R}^2$. This action is Hamiltonian, with momentum map $\mathrm{J}(x, y) = \frac{1}{2}(x^2 + y^2)$ (on identifying $\mathfrak{so}(2)$ with $\mathbb{R}$). The action trivially commutes with itself, and the $\mathrm{SO}(2)$-orbits agree with the level sets of the momentum map. However

$$\ker T_{(x,y)}\,\mathrm{J} = \begin{cases} \mathbb{R}(y, -x) & (x, y) \neq (0, 0) \\ \mathbb{R}^2 & (x, y) = (0, 0) \end{cases},$$

and so

$$\left(\ker T_{(0,0)}\,\mathrm{J}\right)^\omega = \{(0, 0)\} \neq \mathbb{R}^2 = \ker T_{(0,0)}\,\mathrm{J}\,.$$

So the Lie-Weinstein condition fails to hold at the origin.

For pairs of group actions, the Lie-Weinstein condition is closely related with the notion of *mutually completely orthogonal* actions—see [11] for further details.

We conclude with a standard useful criterion for deducing that a momentum map has constant rank.

Lemma 2.15. [16, Corollary 4.5.13] *If Φ is a (locally) free Hamiltonian G-action, then J is a submersion, and in particular has constant rank.*

3 The $(\mathrm{U}(n), \mathrm{U}(m))$ actions on $\mathrm{M}_{n\times m}(\mathbb{C})$

Following [2], in this section we consider the natural Hamiltonian actions of $\mathrm{U}(n)$ and $\mathrm{U}(m)$ on $\mathrm{M}_{n\times m}(\mathbb{C})$. We show that these actions are mutually transitive, and consequently deduce the coadjoint orbit and reduced space correspondences (Corollary 2.3 and Proposition 2.8). We note that Balleier and Wurzbacher instead derive these properties as a consequence of the *symplectic Howe condition* [2, Definition 2.4] on the actions—see [2, Section 5.1] for details.

In what follows, we view elements of $\mathrm{M}_{n\times m}(\mathbb{C})$ either as matrices or as linear maps from $\mathbb{C}^m$ to $\mathbb{C}^n$, depending on context.

3.1 Commuting Hamiltonian actions

First, note that $\mathrm{M}_{n\times m}(\mathbb{C})$ is a complex inner product space, with Hermitian inner product

$$(E, F) = \mathrm{Tr}(E^\dagger F).$$

The imaginary part of this inner product defines a linear symplectic form

$$\Omega(E, F) := \mathrm{Im}\,\mathrm{Tr}(E^\dagger F) = \frac{1}{2i}\,\mathrm{Tr}(E^\dagger F - F^\dagger E),$$

and $\mathrm{M}_{n\times m}(\mathbb{C})$ is a linear Kähler space, with obvious complex structure. It is straightforward to show that the natural left $\mathrm{U}(n)$- and right $\mathrm{U}(m)$-actions act symplectically on $\mathrm{M}_{n\times m}(\mathbb{C})$, considered as a symplectic manifold. In fact, these actions are Hamiltonian, and we can easily compute corresponding momentum maps.

Proposition 3.1. (i) *A momentum map $\mathrm{J}_L : \mathrm{M}_{n\times m}(\mathbb{C}) \to \mathfrak{u}(n)^*$ corresponding to the left $\mathrm{U}(n)$-action is given by*

$$\langle \mathrm{J}_L(E), \zeta\rangle = \frac{1}{2}\Omega(\zeta E, E).$$

(ii) *A momentum map $\mathrm{J}_R : \mathrm{M}_{n\times m}(\mathbb{C}) \to \mathfrak{u}(m)^*$ corresponding to the right $\mathrm{U}(m)$-action is given by*

$$\langle \mathrm{J}_R(E), \xi\rangle = \frac{1}{2}\Omega(E\xi, E).$$

Proof. Both results follow from the general expression for the momentum map of a linear symplectic action on a symplectic vector space—see for example [13, Section 12.4, Example (a)]. □

Remark 3.2. Both momentum maps are easily seen to be equivariant,

$$\mathrm{J}_L(UE) = \mathrm{Ad}^*_{U^{-1}}(\mathrm{J}_L(E)) \qquad \text{and} \qquad \mathrm{J}_R(EV) = \mathrm{Ad}^*_V(\mathrm{J}_R(E)),$$

hence Poisson with respect to the (+) Lie-Poisson structure on $\mathfrak{u}(n)^*$, respectively (−) Lie-Poisson structure on $\mathfrak{u}(m)^*$.

3.2 Lie algebra-valued momentum maps

Given a (real) Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(N,\mathbb{C})$, we define the trace form $\langle\!\langle \cdot,\cdot \rangle\!\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$\langle\!\langle \xi,\zeta \rangle\!\rangle = \operatorname{Re}\operatorname{Tr}(\xi\zeta). \tag{7}$$

If $\mathfrak{g}$ is invariant under conjugate transpose, then $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ is non-degenerate, since

$$\langle\!\langle \xi,\xi^\dagger \rangle\!\rangle = \text{Re Tr}(\xi\xi^\dagger) > 0 \text{ for } \xi \neq 0.$$

We can use the non-degeneracy of $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ to translate the momentum maps J_L and J_R from Proposition 3.1 into Lie algebra-valued momentum maps. We note in particular that if $\mathfrak{g} \subset \mathfrak{gl}(N,\mathbb{C})$ integrates to $G \subset \mathrm{GL}(n,\mathbb{C})$, then the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ provided by the trace form is G-equivariant.

Proposition 3.3. (i) *The Lie algebra-valued momentum map* $\mathrm{j}_L : \mathrm{M}_{n\times m}(\mathbb{C}) \to \mathfrak{u}(n)$ *is*

$$\mathrm{j}_L(E) = \frac{i}{2}EE^\dagger.$$

(ii) *The Lie algebra-valued momentum map* $\mathrm{j}_R : \mathrm{M}_{n\times m}(\mathbb{C}) \to \mathfrak{u}(m)$ *is*

$$\mathrm{j}_R(E) = \frac{i}{2}E^\dagger E.$$

Proof. (i) For $\zeta \in \mathfrak{u}(n)$, $E \in \mathrm{M}_{n\times m}(\mathbb{C})$,

$$\begin{aligned}\langle \mathrm{J}_L(E),\zeta\rangle &= \frac{1}{2}\Omega(\zeta E, E) = \frac{1}{2}\operatorname{Im}\operatorname{Tr}(E^\dagger\zeta^\dagger E) = -\frac{1}{2}\operatorname{Im}\operatorname{Tr}(EE^\dagger\zeta) \qquad \text{using } \zeta^\dagger = -\zeta \\ &= \frac{1}{2}\operatorname{Re}\operatorname{Tr}(iEE^\dagger\zeta) = \left\langle\!\!\left\langle \frac{i}{2}EE^\dagger,\zeta \right\rangle\!\!\right\rangle.\end{aligned}$$

Since $\frac{i}{2}EE^\dagger \in \mathfrak{u}(n)$, the result follows.
(ii) Similar. □

3.3 The mutually transitive property

Proposition 3.4. (i) $\mathrm{U}(n)$ *acts transitively on the level sets of* j_R.

(ii) $\mathrm{U}(m)$ *acts transitively on the level sets of* j_L.

Proof. (i) From $\mathrm{j}_R(E) = \frac{i}{2}E^\dagger E$, it is clear that the level sets of j_R are invariant under the left $\mathrm{U}(n)$-action.

Now suppose $\mathrm{j}_R(E) = \mathrm{j}_R(E')$, implying $E^\dagger E = (E')^\dagger E'$. Let E_a denote the ath column of E, considered as a vector in $\mathbb{C}^n$. So we have the m^2 conditions

$$E_a^\dagger E_b = (E_a')^\dagger E_b' \qquad a,b = 1,\dots,m. \tag{8}$$

The set $\{E_1,E_2,\dots,E_m\} \subset \mathbb{R}^n$ has a maximal linearly independent subset $\{E_{a_1},\dots,E_{a_k}\}$ for some $k \leq m$, and such a subset constitutes a basis for the subspace $\operatorname{im} E \subset \mathbb{C}^n$. We claim that $\{E'_{a_1},\dots,E'_{a_k}\}$ is a basis for $\operatorname{im} E'$.

Firstly, suppose $\sum_{i=1}^k \alpha_i E'_{a_i} = 0$ for some $\alpha_i \in \mathbb{C}$. Then for any $c = 1,\dots,m$,

$$0 = (E_c')^\dagger\left(\sum_{i=1}^k \alpha_i E'_{a_i}\right) = E_c^\dagger\left(\sum_{i=1}^k \alpha_i E_{a_i}\right),$$

using the conditions (8). It follows that $\sum_{i=1}^k \alpha_i E_{a_i} \in \operatorname{im} E \cap (\operatorname{im} E)^\perp = \{0\}$ (where $^\perp$ denotes orthogonality with respect to the usual inner product in $\mathbb{C}^n$). Hence $\sum_{i=1}^k \alpha_i E_{a_i} = 0$, and so linear independence of the E_{a_i} guarantees that $\alpha_i = 0$ for all $i = 1, \ldots, k$, proving linear independence of $\{E'_{a_1}, \ldots, E'_{a_k}\}$.

Also, for any $c = 1, \ldots, m$, there exist $\beta_i \in \mathbb{C}$ such that $E_c = \sum_{i=1}^k \beta_i E_{a_i}$. Then

$$(E'_d)^\dagger \left(E'_c - \sum_{i=1}^k \beta_i E'_{a_i} \right) = E_d^\dagger \left(E_c - \sum_{i=1}^k \beta_i E_{a_i} \right) = 0 \qquad d = 1, \ldots, m,$$

implying that $E'_c - \sum_{i=1}^k \beta_i E'_{a_i} \in \operatorname{im} E' \cap (\operatorname{im} E')^\perp = \{0\}$, i.e., $E'_c = \sum_{i=1}^k \beta_i E'_{a_i}$. Hence $\{E'_{a_1}, \ldots, E'_{a_k}\}$ span $\operatorname{im} E'$.

Now define $U : \operatorname{im} E \to \operatorname{im} E'$ by $U(E_{a_i}) \coloneqq E'_{a_i}$. From (8) we see U is an isometry. It can be extended to the entire space $\mathbb{C}^n$ by picking an arbitrary isometry $(\operatorname{im} E)^\perp \to (\operatorname{im} E')^\perp$, giving $U \in \mathrm{U}(n)$.

From the discussion above, we see that if $E_c = \sum_{i=1}^k \beta_i E_{a_i}$, then $E'_c = \sum_{i=1}^k \beta_i E'_{a_i}$. It follows that

$$U(E_c) = \sum_{i=1}^k \beta_i U(E_{a_i}) = \sum_{i=1}^k \beta_i E'_{a_i} = E'_c$$

for all $c = 1, \ldots, m$, and so $E' = UE$. Hence E and E' lie in the same $\mathrm{U}(n)$-orbit.

(ii) Same method as part (i), except applied to rows of E instead of columns. □

We have proved mutual transitivity of the $(\mathrm{U}(n), \mathrm{U}(m))$ actions on $\mathrm{M}_{n\times m}(\mathbb{C})$. Thus we get a (generalised) dual pair of momentum maps

$$\begin{array}{ccccc} & & \mathrm{M}_{n\times m}(\mathbb{C}) & & \\ & \swarrow^{\mathrm{j}_L} & & \searrow^{\mathrm{j}_R} & \\ \mathfrak{u}(n)_{\mathrm{j}_L} & & & & \mathfrak{u}(m)_{\mathrm{j}_R} \end{array} \tag{9}$$

where $\mathfrak{u}(n)_{\mathrm{j}_L}$ and $\mathfrak{u}(m)_{\mathrm{j}_R}$ are the images of the left and right momentum maps respectively.

Remark 3.5. The momentum maps $\mathrm{j}_L, \mathrm{j}_R$ in fact define a singular dual pair, in the sense of Ortega [15, 16].

3.4 Adjoint orbit correspondence

We briefly recall the description of the adjoint orbit correspondence from [2]. Assuming for concreteness that $n \geq m$, any $E \in \mathrm{M}_{n\times m}(\mathbb{C})$ has a unique singular-value decomposition $E = U\Sigma V^\dagger$, where $U \in \mathrm{U}(n), V \in \mathrm{U}(m)$, and

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_m \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$. The expressions for the momentum maps (Proposition 3.3) imply that $\mathrm{j}_L(E)$ is in the adjoint orbit of the diagonal matrix $\operatorname{diag}[\frac{i}{2}\sigma_1^2, \frac{i}{2}\sigma_2^2, \ldots, \frac{i}{2}\sigma_m^2, 0, \ldots, 0] \in \mathfrak{u}(n)$, while $\mathrm{j}_R(E)$ is in the adjoint orbit of $\mathrm{diag}[\frac{i}{2}\sigma_1^2, \frac{i}{2}\sigma_2^2, \ldots, \frac{i}{2}\sigma_m^2] \in \mathfrak{u}(m)$. The correspondence between such orbits, for all $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$, is one-to-one (note our conventions for j_R introduce a minus sign relative to [2]).

3.5 Restriction to a Lie-Weinstein dual pair

For completeness, we now characterise the subset of $\mathrm{M}_{n\times m}(\mathbb{C})$ where the generalised dual pair (9) becomes a Lie-Weinstein dual pair. As before, assume $n \geq m$ for concreteness.

Proposition 3.6. *The momentum maps $\mathrm{j}_L, \mathrm{j}_R$ define a Lie-Weinstein dual pair on the (open) subset $\mathrm{M}_{n\times m}^{rk\, m}(\mathbb{C})$ of full rank matrices in $\mathrm{M}_{n\times m}(\mathbb{C})$.*

Proof. The right $\mathrm{U}(m)$-action is free on $\mathrm{M}_{n\times m}^{\mathrm{rk}\, m}(\mathbb{C})$. Then using Lemma 2.15, j_R has constant rank there, and then Remark 2.13 implies the result. □

Proposition 3.7. *The set $\mathrm{M}_{n\times m}^{rk\, m}(\mathbb{C})$ is the largest subset of $\mathrm{M}_{n\times m}(\mathbb{C})$ on which $\mathrm{j}_L, \mathrm{j}_R$ define a Lie-Weinstein dual pair.*

Proof. Let $E \in \mathrm{M}_{n\times m}(\mathbb{C})$ have singular-value decomposition $U\Sigma V^\dagger$, where Σ is as described in the previous section. Suppose σ_{m-k} is the last non-zero σ_i (implying $\sigma_{m-k+1} = \ldots = \sigma_m = 0$). Note $k = 0$ is possible. From

$$T_E\, \mathrm{j}_R(X_E) = \frac{i}{2}(X^\dagger E + E^\dagger X),$$

we see that $\ker T_E\, \mathrm{j}_R$ consists of matrices $X = U\widetilde{X}$, where $\widetilde{X} \in \mathrm{M}_{n\times m}(\mathbb{C})$ is non-zero only in the lower $m \times (n-m+k)$ block. Hence $\mathrm{im}\, T_E\, \mathrm{j}_R = nm - m(n-m+k) = m(m-k)$. This equals $\dim \mathfrak{u}(m) = m^2$ iff all of the σ_i are non-zero, which occurs iff E has full rank m. □

4 Matrix analogue of ideal fluid dual pair

In this section, we describe a symplectic structure on $\mathrm{M}_{2n\times m}(\mathbb{R})$ and demonstrate that the left (resp. right) action of $\mathrm{Sp}(2n,\mathbb{R})$ (resp. $\mathrm{O}(m)$) is Hamiltonian. We then show that on a suitable subset of $\mathrm{M}_{2n\times m}(\mathbb{R})$, the $\mathrm{Sp}(2n,\mathbb{R})$- and $\mathrm{O}(m)$-actions are mutually transitive, and deduce that they define a Lie-Weinstein dual pair. Finally, we describe explicitly the correspondence between adjoint orbits in the images of the respective momentum maps.

This dual pair was originally discussed in [7, pp.502-506].

4.1 Commuting Hamiltonian actions

The vector space $\mathrm{M}_{2n\times m}(\mathbb{R})$ has a symplectic form

$$\Omega(E,F) \coloneqq \mathrm{Tr}(E^\top \mathbb{J} F),$$

where $\mathbb{J} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$. As before, we think of the pair $(\mathrm{M}_{2n\times m}(\mathbb{R}), \Omega)$ as a symplectic manifold by using the canonical isomorphism $T_E\, \mathrm{M}_{2n\times m}(\mathbb{R}) \simeq \mathrm{M}_{2n\times m}(\mathbb{R})$.

The natural left $\mathrm{Sp}(2n,\mathbb{R})$- and right $\mathrm{O}(m)$-actions act symplectically on $\mathrm{M}_{2n\times m}(\mathbb{R})$, considered as a symplectic manifold. These actions are Hamiltonian, with momentum maps $\mathrm{J}_L : \mathrm{M}_{2n\times m}(\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R})^*$ and $\mathrm{J}_R : \mathrm{M}_{2n\times m}(\mathbb{R}) \to \mathfrak{o}(m)^*$ given by

$$\langle \mathrm{J}_L(E), \zeta\rangle = \frac{1}{2}\Omega(\zeta E, E) \qquad \text{and} \qquad \langle \mathrm{J}_R(E), \xi\rangle = \frac{1}{2}\Omega(E\xi, E). \tag{10}$$

Again, both momentum maps are equivariant,

$$\mathrm{J}_L(SE) = \mathrm{Ad}^*_{S^{-1}}(\mathrm{J}_L(E)) \qquad \text{and} \qquad \mathrm{J}_R(EO) = \mathrm{Ad}^*_O(\mathrm{J}_R(E)).$$

4.2 Lie algebra-valued momentum maps

We recall the trace form (7)

$$\langle\!\langle \xi, \zeta \rangle\!\rangle = \mathrm{Re}\,\mathrm{Tr}(\xi\zeta) = \mathrm{Tr}(\xi\zeta),$$

now defined on the Lie algebras $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{o}(m)$. Using $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ to identify Lie algebras with their duals, we can again define Lie algebra-valued momentum maps.

Proposition 4.1. *(i) The Lie algebra-valued momentum map* $\mathrm{j}_L : \mathrm{M}_{2n\times m}(\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R})$ *is*

$$\mathrm{j}_L(E) = -\frac{1}{2}EE^\top \mathbb{J}.$$

(ii) The Lie algebra-valued momentum map $\mathrm{j}_R : \mathrm{M}_{2n\times m}(\mathbb{R}) \to \mathfrak{o}(m)$ *is*

$$\mathrm{j}_R(E) = -\frac{1}{2}E^\top \mathbb{J} E.$$

Proof. (i) For $\zeta \in \mathfrak{sp}(2n,\mathbb{R})$, $E \in \mathrm{M}_{2n\times m}(\mathbb{R})$,

$$\begin{aligned}\langle \mathrm{J}_L(E), \zeta\rangle &= \frac{1}{2}\Omega(\zeta E, E) = \frac{1}{2}\mathrm{Tr}(E^\top \zeta^\top \mathbb{J} E) && \text{by (10)}\\ &= \frac{1}{2}\mathrm{Tr}(EE^\top\zeta^\top\mathbb{J}) = \frac{1}{2}\mathrm{Tr}(-EE^\top\mathbb{J}\zeta) && \text{since } \zeta \in \mathfrak{sp}(2n,\mathbb{R})\\ &= \left\langle\!\!\left\langle -\frac{1}{2}EE^\top\mathbb{J}, \zeta \right\rangle\!\!\right\rangle.\end{aligned}$$

Since $-\frac{1}{2}EE^\top\mathbb{J} \in \mathfrak{sp}(2n,\mathbb{R})$, the result follows.
(ii) Similar. □

4.3 The mutually transitive property on full rank matrices

In contrast with the case of the $\mathrm{U}(n)$- and $\mathrm{U}(m)$-actions on $\mathrm{M}_{n\times m}(\mathbb{C})$, demonstration of the mutually transitive property of the $\mathrm{Sp}(2n,\mathbb{R})$- and $\mathrm{O}(m)$-actions requires restriction to a subset of $\mathrm{M}_{2n\times m}(\mathbb{R})$. To this end, let $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R}) \subset \mathrm{M}_{2n\times m}(\mathbb{R})$ denote the matrices of rank m. In order for $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$ to be nonempty, we require $m \le 2n$. Defining $f : \mathrm{M}_{2n\times m}(\mathbb{R}) \to \mathbb{R}$ by $f(E) = \det(E^\top E)$, we see that $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R}) = f^{-1}((0,\infty))$, and so $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$ is an open subset of $\mathrm{M}_{2n\times m}(\mathbb{R})$. It follows that Ω remains non-degenerate when restricted to $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$. Additionally, since elements of $\mathrm{O}(m)$ and $\mathrm{Sp}(2n,\mathbb{R})$ have full rank, their group actions preserve $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$. We denote restrictions of Ω, j_L, j_R to $\mathrm{M}^{\mathrm{rk}\,m}_{2n\times m}(\mathbb{R})$ by the same symbols for convenience.

Proposition 4.2. (i) $\mathrm{Sp}(2n,\mathbb{R})$ *acts transitively on the level sets of* $\mathrm{j}_R : \mathrm{M}^{rk\,m}_{2n\times m}(\mathbb{R}) \to \mathfrak{o}(m)$.

(ii) $\mathrm{O}(m)$ *acts transitively on the level sets of* $\mathrm{j}_L : \mathrm{M}^{rk\,m}_{2n\times m}(\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R})$.

Before proving Proposition (4.2), we need the following standard result.

Proposition 4.3 (Witt's theorem)**.** [1, Theorem 3.9] *Let V be a finite-dimensional vector space, over a field $\mathbb{F}$ of characteristic different from 2, and $q : V \times V \to \mathbb{F}$ a symmetric or anti-symmetric nondegenerate bilinear form on V. If $f : U \to U'$ is a (linear) isometry between two subspaces of V, then f extends to an isometry of V.*

Proof of Proposition (4.2). (i) Since $\mathrm{j}_R(E) = -\frac{1}{2}E^\top \mathbb{J} E$, clearly the left $\mathrm{Sp}(2n,\mathbb{R})$-action preserves the level sets of j_R.

Now suppose $\mathrm{j}_R(E) = \mathrm{j}_R(E')$, implying $E^\top \mathbb{J} E = (E')^\top \mathbb{J} E'$. Letting E_a denote the ath column of E, considered as a vector in $\mathbb{R}^{2n}$, this gives the m^2 conditions

$$E_a^\top \mathbb{J} E_b = (E'_a)^\top \mathbb{J} E'_b \qquad \text{for } a, b = 1, \ldots, m.$$

Define $S : \operatorname{im} E \to \operatorname{im} E'$ by $S(E_a) = E'_a$ (this is well-defined, since the columns E_a are linearly independent). So the above condition becomes

$$\omega(E_a, E_b) = \omega(SE_a, SE_b),$$

where $\omega(X, Y) := X^\top \mathbb{J} Y$ denotes the standard symplectic form on $\mathbb{R}^{2n}$. By Witt's theorem, there exists a linear extension $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserving ω. Then $S \in \mathrm{Sp}(2n,\mathbb{R})$, and $E' = SE$. So E' and E lie in the same $\mathrm{Sp}(2n,\mathbb{R})$-orbit.

(ii) Since $\mathrm{j}_L(E) = -\frac{1}{2}EE^\top \mathbb{J}$, clearly the right $\mathrm{O}(m)$-action preserves the level sets of j_L.

Now suppose $\mathrm{j}_L(E) = \mathrm{j}_L(E')$, implying $EE^\top = E'(E')^\top$. This can be put into a form similar to Proposition (3.4)(i) by letting $F = E^\top, F' = (E')^\top$. Following a similar argument as there, we obtain an isometry $O : \mathbb{R}^m \to \mathbb{R}^m$ with $F' = OF$, i.e., $E' = EO^\top$. Since $O^\top \in \mathrm{O}(m)$, we see that E and E' are related by the right $\mathrm{O}(m)$ action.

□

We have proved mutual transitivity of the $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(m))$ actions. Since $\mathrm{O}(m)$ acts freely on $\mathrm{M}_{2n\times m}^{\mathrm{rk}\,m}(\mathbb{R})$, we conclude by Lemma 2.15 that j_R, and so also j_L (Remark 2.13), has constant rank on $\mathrm{M}_{2n\times m}^{\mathrm{rk}\,m}(\mathbb{R})$, and so the momentum maps define a Lie-Weinstein dual pair

$$\begin{array}{ccc} & \mathrm{M}_{2n\times m}^{\mathrm{rk}\,m}(\mathbb{R}) & \\ {}^{\mathrm{j}_L}\swarrow & & \searrow^{\mathrm{j}_R} \\ \mathfrak{sp}(2n,\mathbb{R})_{\mathrm{j}_L} & & \mathfrak{o}(m)_{\mathrm{j}_R} \end{array}$$

where $\mathfrak{sp}(2n,\mathbb{R})_{\mathrm{j}_L}$ and $\mathfrak{o}(m)_{\mathrm{j}_R}$ are the images of the left and right momentum maps respectively.

Remark 4.4. For $m < 2n$, the $\mathrm{Sp}(2n,\mathbb{R})$-action has non-compact isotropy group at points of $\mathrm{M}_{2n\times m}^{\mathrm{rk}\,m}(\mathbb{R})$. Hence it cannot be proper [16, Proposition 2.3.8 (i)].

4.4 Adjoint orbit correspondence

By Corollary 2.3 there is a one-to-one correspondence between coadjoint orbits in the images $\mathfrak{sp}(2n,\mathbb{R})^*_{\mathrm{J}_L}$ and $\mathfrak{o}(m)^*_{\mathrm{J}_R}$. Equivalently, since $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is Ad-invariant, we have a correspondence between adjoint orbits in $\mathfrak{sp}(2n,\mathbb{R})_{\mathrm{j}_L}$ and $\mathfrak{o}(m)_{\mathrm{j}_R}$.

From [19] we know that every matrix $E \in \mathrm{M}_{2n\times m}(\mathbb{R})$ of rank m has an singular-value-decomposition-like representation as $E = SDO$ with $S \in \mathrm{Sp}(2n,\mathbb{R})$, $O \in \mathrm{O}(m)$, and D given by

$$D = \begin{array}{c} \begin{matrix} \scriptstyle p & \scriptstyle q & \scriptstyle p \end{matrix} \\ \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \scriptstyle p \\ \scriptstyle q \\ \scriptstyle r \\ \scriptstyle p \\ \scriptstyle q \\ \scriptstyle r \end{matrix} ,$$

where Σ is a diagonal block with positive entries $\sigma_1, \dots, \sigma_p$. Here $q = m - 2p$ is imposed by the rank condition, and $r = n - p - q = n - m + p$. Since $\mathrm{j}_R(E)$ is $\mathrm{O}(m)$-conjugate to

$$\mathrm{j}_R(D) = -\frac{1}{2} D^\top \mathbb{J} D = -\frac{1}{2} \begin{array}{c} \begin{matrix} \scriptstyle p & \scriptstyle q & \scriptstyle p \end{matrix} \\ \begin{bmatrix} 0 & 0 & \Sigma^2 \\ 0 & 0 & 0 \\ -\Sigma^2 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \scriptstyle p \\ \scriptstyle q \\ \scriptstyle p \end{matrix} \in \mathfrak{o}(m),$$

we conclude that the image of j_R consists of the adjoint orbits of $\mathrm{O}(m)$ that correspond to normal forms that are block diagonal, with entries $-\frac{1}{2}\begin{bmatrix} 0 & \sigma_1^2 \\ -\sigma_1^2 & 0 \end{bmatrix}, \dots, -\frac{1}{2}\begin{bmatrix} 0 & \sigma_p^2 \\ -\sigma_p^2 & 0 \end{bmatrix}$, and a $q \times q$ zero block. On the other hand $\mathrm{j}_L(E)$ is $\mathrm{Sp}(2n, \mathbb{R})$-conjugate to

$$\mathrm{j}_L(D) = -\frac{1}{2} D D^\top \mathbb{J} = -\frac{1}{2} \begin{array}{c} \begin{matrix} \scriptstyle p & \scriptstyle q & \scriptstyle r & \scriptstyle p & \scriptstyle q & \scriptstyle r \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & \Sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\Sigma^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \scriptstyle p \\ \scriptstyle q \\ \scriptstyle r \\ \scriptstyle p \\ \scriptstyle q \\ \scriptstyle r \end{matrix} \in \mathfrak{sp}(2n, \mathbb{R}),$$

hence the image of j_L consists of the adjoint orbits of $\mathrm{Sp}(2n, \mathbb{R})$ that correspond to normal forms that are block diagonal, with entries $-\frac{1}{2}\begin{bmatrix} 0 & \sigma_1^2 \\ -\sigma_1^2 & 0 \end{bmatrix}, \dots, -\frac{1}{2}\begin{bmatrix} 0 & \sigma_p^2 \\ -\sigma_p^2 & 0 \end{bmatrix}$, q blocks of type $-\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and an $r \times r$ zero block.

We conclude that the adjoint orbit correspondence is between the two above mentioned orbits, characterized by the integer p and the positive values $\sigma_1, \dots, \sigma_p$.

Remark 4.5. The adjoint orbit correspondence was described in [7, p.505] in the special case $q = 0$.

4.5 Relations between the $(\mathrm{U}(n), \mathrm{U}(m))$ and $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(m))$ momentum maps

As symplectic manifolds, $(\mathrm{M}_{n\times m}(\mathbb{C}), \Omega_{\mathbb{C}})$ and $(\mathrm{M}_{2n\times m}(\mathbb{R}), \Omega_{\mathbb{R}})$ are isomorphic, where we now use obvious notation to distinguish between symplectic forms. In fact, under the identification

$$E_{\mathbb{C}} = E_1 + iE_2 \in \mathrm{M}_{n\times m}(\mathbb{C}) \longleftrightarrow E_{\mathbb{R}} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \in \mathrm{M}_{2n\times m}(\mathbb{R})$$

we see that

$$\Omega_{\mathbb{C}}(E_{\mathbb{C}}, F_{\mathbb{C}}) = \operatorname{Im}\operatorname{Tr}(E_{\mathbb{C}}^{\dagger} F_{\mathbb{C}}) = \operatorname{Tr}(E_1^{\top} F_2 - E_2^{\top} F_1) = \Omega_{\mathbb{R}}(E_{\mathbb{R}}, F_{\mathbb{R}}).$$

We can realize $\mathfrak{u}(n)$ as a Lie subalgebra of $\mathfrak{sp}(2n,\mathbb{R})$ with the map

$$\ell : \zeta_1 + i\zeta_2 \in \mathfrak{u}(n) \mapsto \begin{bmatrix} \zeta_1 & -\zeta_2 \\ \zeta_2 & \zeta_1 \end{bmatrix} \in \mathfrak{sp}(2n,\mathbb{R})$$

(noting that $\zeta_1^{\top} = -\zeta_1, \zeta_2^{\top} = \zeta_2$). Denoting by i the inclusion of $\mathfrak{o}(m)$ into $\mathfrak{u}(m)$, we obtain:

Proposition 4.6. *The diagram*

$$\begin{array}{ccccc} \mathfrak{sp}(2n,\mathbb{R})^* & \xleftarrow{\mathrm{J}_{\mathrm{Sp}(2n,\mathbb{R})}} & \mathrm{M}_{2n\times m}(\mathbb{R}) & \xrightarrow{\mathrm{J}_{\mathrm{O}(m)}} & \mathfrak{o}(m)^* \\ \ell^* \downarrow & & \downarrow = & & \uparrow i^* \\ \mathfrak{u}(n)^* & \xleftarrow{\mathrm{J}_{\mathrm{U}(n)}} & \mathrm{M}_{n\times m}(\mathbb{C}) & \xrightarrow{\mathrm{J}_{\mathrm{U}(m)}} & \mathfrak{u}(m)^* \end{array}$$

commutes, where here momentum maps are labelled by their corresponding groups.

Proof. Let $E_{\mathbb{R}} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \in \mathrm{M}_{2n\times m}(\mathbb{R})$ be arbitrary. For all $\zeta = \zeta_1 + i\zeta_2 \in \mathfrak{u}(n)$ we have

$$\ell(\zeta)E_{\mathbb{R}} = \begin{bmatrix} \zeta_1 & -\zeta_2 \\ \zeta_2 & \zeta_1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \zeta_1 E_1 - \zeta_2 E_2 \\ \zeta_2 E_1 + \zeta_1 E_2 \end{bmatrix} \longleftrightarrow (\zeta_1 + i\zeta_2)(E_1 + iE_2) = \zeta E_{\mathbb{C}} \in \mathrm{M}_{n\times m}(\mathbb{C}),$$

and so

$$\langle \ell^*(\mathrm{J}_{\mathrm{Sp}(2n,\mathbb{R})}(E_{\mathbb{R}})), \zeta \rangle = \frac{1}{2}\Omega_{\mathbb{R}}(\ell(\zeta)E_{\mathbb{R}}, E_{\mathbb{R}}) = \frac{1}{2}\Omega_{\mathbb{C}}(\zeta E_{\mathbb{C}}, E_{\mathbb{C}}) = \langle \mathrm{J}_{\mathrm{U}(n)}(E_{\mathbb{C}}), \zeta \rangle.$$

The identity $i^* \circ \mathrm{J}_{\mathrm{U}(m)} = \mathrm{J}_{\mathrm{O}(m)}$ is proved similarly. □

5 Matrix analogue of the EPDiff dual pair

In this section, upon identifying $T^* \mathrm{M}_{n\times m}(\mathbb{R})$ with $\mathrm{M}_{2n\times m}(\mathbb{R})$, we describe the lifted cotangent action of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(m,\mathbb{R})$ on $\mathrm{M}_{2n\times m}(\mathbb{R})$, and demonstrate that these actions are mutually transitive, and deduce that they define a Lie-Weinstein dual pair on a suitable subset of $\mathrm{M}_{2n\times m}(\mathbb{R})$. We then give an explicit description of the adjoint orbit correspondence. Finally, we outline the relationship between $(\mathrm{GL}(n,\mathbb{R}), \mathrm{GL}(m,\mathbb{R}))$ momentum maps and the $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(m))$ momentum maps of the previous section.

5.1 Commuting Hamiltonian actions

We identify $T^* \mathrm{M}_{n\times m}(\mathbb{R})$ with $\mathrm{M}_{n\times m}(\mathbb{R}) \times \mathrm{M}_{n\times m}(\mathbb{R}) \simeq \mathrm{M}_{2n\times m}(\mathbb{R})$ using the non-degenerate pairing on $\mathrm{M}_{n\times m}(\mathbb{R})$ given by $(X,Y) \mapsto \operatorname{Tr}(X^{\top}Y)$. More precisely,

$$(Q,P) : X \in T_Q \mathrm{M}_{n\times m}(\mathbb{R}) \simeq \mathrm{M}_{n\times m}(\mathbb{R}) \mapsto \operatorname{Tr}(P^{\top}X). \tag{11}$$

Thus the canonical symplectic form on the cotangent bundle is the same as that induced by the constant symplectic form on the linear space $\mathrm{M}_{2n\times m}(\mathbb{R})$:

$$\Omega(X,Y) = \operatorname{Tr}(X^{\top}\mathbb{J}Y), \quad X, Y \in \mathrm{M}_{2n\times m}(\mathbb{R}).$$

On $\mathrm{M}_{n\times m}(\mathbb{R})$ we consider the left $\mathrm{GL}(n,\mathbb{R})$-action and the right $\mathrm{GL}(m,\mathbb{R})$-action, together with their cotangent lifted action on $T^*\mathrm{M}_{n\times m}(\mathbb{R})$:

$$A\cdot(Q,P)=(AQ,(A^\top)^{-1}P),\quad A\in\mathrm{GL}(n,\mathbb{R}) \tag{12}$$

and

$$(Q,P)\cdot B=(QB,P(B^\top)^{-1}),\quad B\in\mathrm{GL}(m,\mathbb{R}). \tag{13}$$

Proposition 5.1. *The left* $\mathrm{GL}(n,\mathbb{R})$*-action and right* $\mathrm{GL}(m,\mathbb{R})$*-action on* $T^*\mathrm{M}_{n\times m}(\mathbb{R})$ *are Hamiltonian with cotangent momentum maps*

$$\langle \mathrm{J}_L(Q,P),\xi\rangle=\mathrm{Tr}(QP^\top\xi),\quad \xi\in\mathfrak{gl}(n,\mathbb{R})$$

and

$$\langle \mathrm{J}_R(Q,P),\eta\rangle=\mathrm{Tr}(P^\top Q\eta),\quad \eta\in\mathfrak{gl}(m,\mathbb{R}).$$

Proof. Every cotangent lifted action is Hamiltonian and has an equivariant momentum map. The left action momentum map is

$$\langle \mathrm{J}_L(Q,P),\xi\rangle=(Q,P)(\xi Q)\overset{(11)}{=}\mathrm{Tr}(P^\top\xi Q)=\mathrm{Tr}(QP^\top\xi),\quad \xi\in\mathfrak{gl}(n,\mathbb{R}).$$

Similarly,

$$\langle \mathrm{J}_R(Q,P),\eta\rangle=(Q,P)(Q\eta)\overset{(11)}{=}\mathrm{Tr}(P^\top Q\eta),\quad \eta\in\mathfrak{gl}(m,\mathbb{R})$$

is the cotangent bundle momentum map for the right action. □

When using the trace form $\langle\langle X,Y\rangle\rangle=\mathrm{Tr}(XY)$ to identify $\mathfrak{gl}(n,\mathbb{R})^*$ with $\mathfrak{gl}(n,\mathbb{R})$, the momentum maps above take the concise expressions

$$\mathrm{j}_L(Q,P)=QP^\top\in\mathfrak{gl}(n,\mathbb{R}),\quad \mathrm{j}_R(Q,P)=P^\top Q\in\mathfrak{gl}(m,\mathbb{R}). \tag{14}$$

5.2 The mutually transitivity property on full rank matrices

Let $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})\subset\mathrm{M}_{n\times m}(\mathbb{R})$ denote the subset of rank m matrices. In the sequel we will at times identify $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ with linear injective maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. For $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ to be non-empty, we require $m\le n$. $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ is an open subset of $\mathrm{M}_{n\times m}(\mathbb{R})$, and so $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})\times\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ is an open subset of $T^*\mathrm{M}_{n\times m}(\mathbb{R})\simeq\mathrm{M}_{n\times m}(\mathbb{R})\times\mathrm{M}_{n\times m}(\mathbb{R})$. Hence the symplectic form on $T^*\mathrm{M}_{n\times m}(\mathbb{R})$ restricts to a symplectic form on $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})\times\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$. It is clear that $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})\times\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ is preserved by the cotangent lifted actions of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(m,\mathbb{R})$.

Proposition 5.2. *The group* $\mathrm{GL}(m,\mathbb{R})$ *acts transitively on level sets of the left momentum map* j_L *restricted to* $\mathrm{M}^{rk\,m}_{n\times m}(\mathbb{R})\times\mathrm{M}^{rk\,m}_{n\times m}(\mathbb{R})$.

Proof. The cotangent $\mathrm{GL}(m,\mathbb{R})$-action (13) preserves the fibers of the momentum map j_L in (14):

$$\mathrm{j}_L((Q,P)\cdot B)=\mathrm{j}_L(QB,P(B^\top)^{-1})=QB(P(B^\top)^{-1})^\top=QP^\top=\mathrm{j}_L(Q,P).$$

Suppose now that $\mathrm{j}_L(Q,P)=\mathrm{j}_L(Q',P')$. From $QP^\top=Q'P'^\top$ we deduce that the linear injective mappings corresponding to Q and Q' have the same range, since both $P^\top$ and $P'^\top$ correspond to linear surjective maps $\mathbb{R}^n\to\mathbb{R}^m$. Thus there exists $B\in\mathrm{GL}(m,\mathbb{R})$ with $Q'=QB$, and by inserting in the above identity we get $QP^\top=QBP'^\top$. By the injectivity of Q follows $P^\top=BP'^\top$. This ensures that $(Q',P')=(Q,P)\cdot B$. □

To prove the transitivity of the $\mathrm{GL}(n,\mathbb{R})$-action on level sets of the right momentum map j_R, we will need the fact that any two matrices in $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \subset \mathrm{M}_{n\times m}(\mathbb{R})$ can be completed to invertible $n\times n$ matrices by using the same matrix.

Lemma 5.3. *Assume $m<n$. Given matrices $Q_1, Q_2 \in \mathrm{M}^{rk\,m}_{n\times m}(\mathbb{R})$, there exists $X \in \mathrm{M}_{n\times(n-m)}(\mathbb{R})$ such that the order n square matrices $[Q_1\ X]$ and $[Q_2\ X]$ are invertible.*

Proof. An easy induction argument on m ensures that there exists a subspace V of $\mathbb{R}^n$ that is simultaneously a complement to both m-dimensional subspaces $\operatorname{im} Q_1$ and $\operatorname{im} Q_2$. Then we choose a basis of V and we build the matrix X whose columns are these basis vectors. (The induction argument is based on the fact that there exists $v\in\mathbb{R}^n$ that doesn't belong to these two m-dimensional subspaces, so the subspaces $\mathbb{R}v + \operatorname{im} Q_1$ and $\mathbb{R}v+\operatorname{im} Q_2$ both have dimension $m+1$.) □

Proposition 5.4. *The group $\mathrm{GL}(n,\mathbb{R})$ acts transitively on level sets of the right momentum map j_R restricted to $\mathrm{M}^{rk\,m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{rk\,m}_{n\times m}(\mathbb{R})$.*

Proof. The cotangent $\mathrm{GL}(n,\mathbb{R})$-action (12) preserves the fibers of the momentum map j_R in (14):

$$\mathrm{j}_R(A\cdot(Q,P)) = \mathrm{j}_R(AQ,(A^\top)^{-1}P) = Q^\top A^\top (A^\top)^{-1}P = Q^\top P = \mathrm{j}_R(Q,P).$$

Suppose now that $\mathrm{j}_R(Q,P) = \mathrm{j}_R(Q',P')$, i.e.,

$$(Q')^\top P' = Q^\top P. \tag{15}$$

We are looking for $A\in \mathrm{GL}(n,\mathbb{R})$ with properties $Q' = AQ$ and $A^\top P' = P$. The special case $m=n$ is easy, because in this case all $Q,Q',P,P'\in\mathrm{GL}(n,\mathbb{R})$, so we can put $A = Q'Q^{-1}\in\mathrm{GL}(n,\mathbb{R})$ which gives us $A^\top P' = (Q^{-1})^\top (Q')^\top P' = P$.

Next we consider the general case $m<n$. Since both P and P' are injective, there exists a matrix $C\in\mathrm{GL}(n,\mathbb{R})$ such that $P = C^\top P'$. Since $C^{-1}Q' \in \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$, by Lemma 5.3 there exists a matrix $X\in \mathrm{M}_{n\times(n-m)}(\mathbb{R})$ such that the two order n matrices $D = [Q\ X]$ and $[(C^{-1}Q')\ X]$ are both invertible. Putting $X' = CX$, the order n matrix $D' = [Q'\ X'] = C[(C^{-1}Q')\ X]$ is invertible too.

Now the matrix $A = D'D^{-1}\in\mathrm{GL}(n,\mathbb{R})$ satisfies $A\cdot(Q,P) = (Q',P')$. Indeed, because

$$(X')^\top P' = X^\top C^\top P' = X^\top P,$$

we have that

$$A^\top P' = (D^{-1})^\top (D')^\top P' = (D^{-1})^\top \begin{bmatrix}(Q')^\top P'\\ (X')^\top P'\end{bmatrix} \overset{(15)}{=} (D^{-1})^\top \begin{bmatrix} Q^\top P\\ X^\top P\end{bmatrix} = (D^{-1})^\top D^\top P = P.$$

On the other hand

$$AQ = D'D^{-1}Q = [Q'\ X']\begin{bmatrix} I\\ 0\end{bmatrix} = Q',$$

thus getting the required transitivity conditions. □

We have proved mutual transitivity of the $(\mathrm{GL}(n,\mathbb{R}),\mathrm{GL}(m,\mathbb{R}))$ actions. By injectivity of elements of $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$, it is straightforward to see that the right action (13) of $\mathrm{GL}(m,\mathbb{R})$

on $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ is free. So by Lemma 2.15, j_R and j_L (compare Remark 2.13) have constant rank, and thus the momentum maps $\mathrm{j}_L, \mathrm{j}_R$ define a Lie-Weinstein dual pair

$$\begin{array}{ccc} & \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) & \\ {}^{\mathrm{j}_L}\swarrow & & \searrow^{\mathrm{j}_R} \\ \mathfrak{gl}(n,\mathbb{R})_{\mathrm{j}_L} & & \mathfrak{gl}(m,\mathbb{R})_{\mathrm{j}_R} \end{array}$$

where $\mathfrak{gl}(n,\mathbb{R})_{\mathrm{j}_L}$ and $\mathfrak{gl}(m,\mathbb{R})_{\mathrm{j}_R}$ are the images of the left and right momentum maps respectively.

Remark 5.5. For $m < n$, the $\mathrm{GL}(n,\mathbb{R})$-action has non-compact isotropy group at points of $\mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$. Hence it cannot be proper [16, Proposition 2.3.8 (i)].

5.3 Adjoint orbit correspondence

As in the discussion of subsection 4.4, we have a one-to-one correspondence between $\mathrm{GL}(n,\mathbb{R})$-orbits in the image of j_L and $\mathrm{GL}(m,\mathbb{R})$-orbits in the image of j_R.

We now characterise the images of j_L and j_R, and the adjoint orbit correspondence between these images. Define the sets

$$S_L := \{\zeta \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{rank}\zeta = m\}$$
$$S_R := \{\xi \in \mathfrak{gl}(m,\mathbb{R}) \mid \operatorname{rank}\xi \geq 2m - n\}.$$

Lemma 5.6. *(i)* $\mathfrak{gl}(n,\mathbb{R})_{\mathrm{j}_L} \subset S_L$.

(ii) $\mathfrak{gl}(m,\mathbb{R})_{\mathrm{j}_R} \subset S_R$.

Proof. Thinking of $(Q,P) \in \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\,m}_{n\times m}(\mathbb{R})$ as linear maps, we have that $Q : \mathbb{R}^m \to \mathbb{R}^n$ is injective, while $P^\top : \mathbb{R}^n \to \mathbb{R}^m$ is surjective. Then

(i) $\operatorname{rank}\mathrm{j}_L(Q,P) = \dim\operatorname{im} QP^\top = \dim\operatorname{im} P^\top = m$, the second equality following from the injectivity of Q.

(ii) $\operatorname{rank}\mathrm{j}_R(Q,P) = \dim\operatorname{im} P^\top Q = m - \dim\ker P^\top Q \geq m - \dim\ker P^\top = 2m - n$, where we use that $\dim\ker P^\top = n - m$ by the rank-nullity theorem.

□

In fact, the momentum maps $\mathrm{j}_L, \mathrm{j}_R$ are surjective onto S_L, S_R:

Proposition 5.7. *(i)* $\mathfrak{gl}(n,\mathbb{R})_{\mathrm{j}_L} = S_L$.

(ii) $\mathfrak{gl}(m,\mathbb{R})_{\mathrm{j}_R} = S_R$.

Proof. (i) Let $\zeta \in S_L$. We wish to show ζ is in the image of j_L. By left $\mathrm{GL}(n,\mathbb{R})$-equivariance of j_L, we may assume ζ is in (real) Jordan canonical form

$$\zeta = \begin{bmatrix} J_{c_1}(\lambda_1) & & & & & & \\ & \ddots & & & & & \\ & & J_{c_p}(\lambda_p) & & & & \\ & & & J_{d_1}(0) & & & \\ & & & & \ddots & & \\ & & & & & J_{d_q}(0) & \\ & & & & & & 0_{(n-m-q)\times(n-m-q)} \end{bmatrix} \in \mathfrak{gl}(n,\mathbb{R}), \tag{16}$$

where $J_{c_i}(\lambda_i) \in \mathrm{M}_{c_i \times c_i}(\mathbb{R})$ denotes the (real) ith Jordan block corresponding to *non-zero* generalised eigenvalue λ_i, and $J_{d_j}(0) \in \mathrm{M}_{d_j \times d_j}(\mathbb{R})$ is the jth *non-trivial* (i.e., with $d_j \geq 2$) Jordan block corresponding to *zero* generalised eigenvalues. The dimension of the zero block follows from the condition

$$m = \operatorname{rank} \zeta = \sum_{i=1}^{p} c_i + \sum_{j=1}^{q} (d_j - 1), \tag{17}$$

which implies $n - \sum_{i=1}^{p} c_i - \sum_{j=1}^{q} d_j = n - m - q$.

To construct a suitable $(Q,P) \in \mathrm{M}^{\mathrm{rk}\, m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\, m}_{n\times m}(\mathbb{R})$ mapping to ζ under j_L: for $d \geq 2$, let $I_{d-1} \in \mathrm{M}_{(d-1)\times(d-1)}(\mathbb{R})$ denote the identity matrix, $\check{I}_{d-1} = \begin{bmatrix} I_{d-1} \\ 0_{1\times(d-1)} \end{bmatrix} \in \mathrm{M}_{d\times(d-1)}(\mathbb{R})$, and $\hat{I}_{d-1} = \begin{bmatrix} 0_{1\times(d-1)} \\ I_{d-1} \end{bmatrix} \in \mathrm{M}_{d\times(d-1)}(\mathbb{R})$. It is straightforward to check that

$$\check{I}_{d-1} \hat{I}_{d-1}^{\top} = J_d(0). \tag{18}$$

Take

$$Q = \begin{bmatrix} J_{c_1}(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J_{c_p}(\lambda_p) & & & \\ & & & \check{I}_{d_1-1} & & \\ & & & & \ddots & \\ & & & & & \check{I}_{d_q-1} \\ & & 0_{(n-m-q)\times m} & & & \end{bmatrix}, \qquad P = \begin{bmatrix} I_{c_1} & & & & & \\ & \ddots & & & & \\ & & I_{c_p} & & & \\ & & & \hat{I}_{d_1-1} & & \\ & & & & \ddots & \\ & & & & & \hat{I}_{d_q-1} \\ & & 0_{(n-m-q)\times m} & & & \end{bmatrix}. \tag{19}$$

By (17), $(Q,P) \in \mathrm{M}^{\mathrm{rk}\, m}_{n\times m}(\mathbb{R}) \times \mathrm{M}^{\mathrm{rk}\, m}_{n\times m}(\mathbb{R})$. Using (19), (18), and (16), it may be checked that $\mathrm{j}_L(Q,P) = QP^{\top} = \zeta$.

(ii) Let $\xi \in S_R$. Again, by right $\mathrm{GL}(m,\mathbb{R})$-equivariance of j_R we may assume that ξ is in (real) Jordan canonical form

$$\xi = \begin{bmatrix} J_{c_1}(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J_{c_p}(\lambda_p) & & & \\ & & & J_{d_1-1}(0) & & \\ & & & & \ddots & \\ & & & & & J_{d_q-1}(0) \end{bmatrix} \in \mathfrak{gl}(m,\mathbb{R}), \tag{20}$$

where c_i, d_j, and λ_i obey the same conventions as in part (i) (in particular, $d_j \geq 2$), but where now we let $J_1(0)$ denote the 1×1 zero matrix instead of explicitly writing a zero block. Defining $\check{I}_{d-1}, \hat{I}_{d-1}$ as before, we have

$$\hat{I}_{d-1}^{\top} \check{I}_{d-1} = J_{d-1}(0). \tag{21}$$

From the condition $2m - n \leq \operatorname{rank} \xi = m - q$, implying $n - m - q \geq 0$, we see that the matrices (19) are well-defined, with $(Q, P) \in \mathrm{M}_{n\times m}^{\mathrm{rk}\, m}(\mathbb{R}) \times \mathrm{M}_{n\times m}^{\mathrm{rk}\, m}(\mathbb{R})$. Using (19), (21), and (20), it may be checked that $\mathrm{j}_R(Q, P) = P^{\top} Q = \xi$.

□

Examining the proof of Proposition 5.7 gives an explicit characterisation of the adjoint orbit correspondence for our dual pair:

Corollary 5.8. *Given*

- *integers p, q with $0 \leq p \leq m$ and $0 \leq q \leq \min\{m, n - m\}$;*
- *complex numbers $\lambda_1, \ldots, \lambda_p$, with $\operatorname{Im} \lambda_i \geq 0$;*
- *integers $c_1, \ldots, c_p, d_1, \ldots, d_q$ satisfying $c_i \geq 1$ and c_i even if $\operatorname{Im} \lambda_i > 0$, $d_j \geq 2$, and $\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} (d_j - 1) = m$.*

Then there is a one-to-one correspondence between the adjoint orbits through elements (16) in $\mathfrak{gl}(n, \mathbb{R})_{\mathrm{j}_L}$ and (20) in $\mathfrak{gl}(m, \mathbb{R})_{\mathrm{j}_R}$.

5.4 Relations between the $(\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(m, \mathbb{R}))$ and $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(m))$ momentum maps

We can realize $\mathfrak{gl}(n, \mathbb{R})$ as a Lie subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$ with the map

$$\ell : \zeta \in \mathfrak{gl}(n, \mathbb{R}) \mapsto \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta^{\top} \end{bmatrix} \in \mathfrak{sp}(2n, \mathbb{R}).$$

Denoting by i the inclusion of $\mathfrak{o}(m)$ into $\mathfrak{gl}(m, \mathbb{R})$, we obtain:

Proposition 5.9. *The diagram*

$$\begin{array}{ccccc} \mathfrak{sp}(2n, \mathbb{R})^* & \xleftarrow{\mathrm{J}_{\mathrm{Sp}(2n,\mathbb{R})}} & \mathrm{M}_{2n\times m}(\mathbb{R}) & \xrightarrow{\mathrm{J}_{\mathrm{O}(m)}} & \mathfrak{o}(m)^* \\ {\scriptstyle \ell^*}\downarrow & & \downarrow{\scriptstyle =} & & \uparrow{\scriptstyle i^*} \\ \mathfrak{gl}(n, \mathbb{R})^* & \xleftarrow{\mathrm{J}_{\mathrm{GL}(n,\mathbb{R})}} & T^* \mathrm{M}_{n\times m}(\mathbb{R}) & \xrightarrow{\mathrm{J}_{\mathrm{GL}(m,\mathbb{R})}} & \mathfrak{gl}(m, \mathbb{R})^* \end{array}$$

commutes, where here momentum maps are labelled by their corresponding groups.

Proof. Let $E = \begin{bmatrix} Q \\ P \end{bmatrix} \in \mathrm{M}_{2n\times m}(\mathbb{R}) = T^* \mathrm{M}_{n\times m}(\mathbb{R})$ be arbitrary. For all $\zeta \in \mathfrak{gl}(n, \mathbb{R})$ we have

$$\begin{aligned} \langle \ell^*(\mathrm{J}_{\mathrm{Sp}(2n,\mathbb{R})}(E)), \zeta \rangle &= \frac{1}{2} \Omega(\ell(\zeta) E, E) = \frac{1}{2} \operatorname{Tr}(E^{\top} \ell(\zeta)^{\top} \mathbb{J} E) = \frac{1}{2} \operatorname{Tr}(Q^{\top} \zeta^{\top} P + P^{\top} \zeta Q) \\ &= \operatorname{Tr}(Q P^{\top} \zeta) = \langle \mathrm{J}_{\mathrm{GL}(n,\mathbb{R})}(E), \zeta \rangle. \end{aligned}$$

The identity $i^* \circ \mathrm{J}_{\mathrm{GL}(n,\mathbb{R})} = \mathrm{J}_{\mathrm{O}(m)}$ is proved similarly. □

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