Semigeostrophic time evolution

of velocity gradient tensor invariants

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ABSTRACT

The behaviour of quadratic invariants of the velocity gradient tensor is explored when the time evolution is governed by semigeostrophic forms of the shallow water equations. The evolution equation of a certain Jacobian involving the geostrophic flow is formally similar to its counterpart under the primitive shallow water equations. The resultant deformation and the Frobenius norm do not behave in this symmetrical way. A product of the study is a straightforward derivation of the semigeostrophic potential vorticity conservation property. Results are extended to 3D baroclinic flow by using isentropic coordinates.

Key Words: shallow water equations; isentropic coordinates; potential vorticity; resultant deformation; Jacobian; Frobenius norm

Running head: Velocity gradient tensor invariants

1. Introduction

A geometric invariant is a quantity whose mathematical form is unchanged under rotation of the coordinate axes. Horizontal divergence (δ) and the vertical component (ζ) of vorticity are two familiar meteorological quantities of this type, and both are simple linear functions of the elements of the 2×2 horizontal velocity gradient tensor (**A**). Certain quadratic functions of the elements of **A** are also geometric invariants, but are less frequently discussed. Examples are the resultant deformation (D_R) and the Jacobian (J) of the Cartesian flow components with respect to the Cartesian coordinates. Another is the Frobenius norm (Q) of **A**, which may be expressed algebraically in terms of D_R and J .

Roulstone, White and Clough (2014) – here denoted RWC – discussed these quadratic invariants and studied their behaviour under shallow water dynamics. As expected, the time evolution equations turned out to be more complicated than that of the potential vorticity (PV – also a geometric invariant) but they involve only familiar quantities and operators, and in essence they are no more complicated than the time evolution equation of the divergence δ .

RWC was motivated in part by the work of Cantwell (1992), Martin *et al.* (1998) and others on the time evolution of the geometric invariants of the 3×3 velocity gradient tensor for incompressible flow governed by the 3D Navier-Stokes equations.

RWC also studied the behaviour when the time evolution is governed by quasi-geostrophic (QG) forms of the shallow water equations, the geometric invariants being defined in terms of the geostrophic flow rather than the total flow. The results were in many respects formally similar to the shallow water primitive equation results, but a systematic difference was the absence of certain terms involving the divergence δ . This difference was traceable to the non-divergence of the geostrophic flow and its use as the advecting velocity (as well as the *advected* velocity) in the QG model. The occurrence was noted of a simple explicit QG time-evolution equation for the ageostrophic vorticity (in terms of the current geostrophic and ageostrophic flows).

This note investigates the behaviour when semigeostrophic (SG) dynamics governs the time evolution. An important feature of SG dynamics is the use of the full flow to advect the geostrophic flow in the momentum equation. Such hybrid treatment of the flow is not found in the Navier-Stokes equations, or in the complete shallow water equations, or in QG approximations to them. The time evolution of geometric invariants under SG dynamics is thus of particular interest.

Having first summarised relevant equations and notation (section 2), the SG case is explored in section 3, and results are compared with those found in RWC for the complete and QG cases. The study suggests a relatively straightforward derivation of the PV conservation law of the SG model in shallow water, as is discussed in section 4. In section 5 it is noted that the results on time evolution and PV conservation may be readily extended to the 3D SG equations (on an *f*-plane) by using isentropic coordinates. Concluding remarks are contained in section 6.

2. Basic equations and notation

2.1 Kinematics

RWC give background to the following minimal outline.

The elements of the 2D velocity gradient tensor, **A**, are the first partial derivatives of the flow velocity components u and v with respect to the corresponding Cartesian coordinates x and y:

$$\mathbf{A} \equiv \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \tag{1}$$

Divergence δ and vorticity ζ are given in terms of the diagonal and off-diagonal elements of **A** as:

$$\delta \equiv u_x + v_y \ , \tag{2}$$

$$\zeta \equiv v_x - u_y \ . \tag{3}$$

The Jacobian $J(u,v)$ of u and v with respect to x and y is the determinant of **A**:

$$J(u,v) \equiv u_x v_y - v_x u_y \, . \tag{4}$$

The quantities δ, ζ and $J(u,v)$ are geometric invariants: (2), (3) and (4) are formally unchanged under rotation of the coordinate axes.

The resultant deformation D_R is given by

$$(D_R)^2 \equiv (D_1)^2 + (D_2)^2 \, . \tag{5}$$

Here

$$D_1 \equiv u_x - v_y \, , \tag{6}$$

$$D_2 \equiv v_x + u_y \, , \tag{7}$$

are the deformation components. D_R is a geometric invariant, but D_1 and D_2 individually are not.

The Frobenius norm Q is the sum of the squares of the elements of **A**:

$$Q^2 \equiv (u_x)^2 + (u_y)^2 + (v_x)^2 + (v_y)^2 \, . \tag{8}$$

Q is a quadratic invariant, and is related to $J(u,v)$ and D_R by

$$Q^2 = (D_R)^2 + 2J(u,v) \tag{9}$$

(RWC, Eq (24)). Relationships involving the linear invariants δ and ζ as well as $J(u,v)$,

D_R and/or Q also hold. An example is

$$2Q^2 = D_R^2 + \delta^2 + \zeta^2 \tag{10}$$

(RWC, Eq (22).)

Subscript notation has been used in (1) – (4) and (6) – (8) to denote partial differentiation. The explicit notation $\partial/\partial x$, $\partial/\partial y$ will also be used (especially when superscripts or other subscripts occur).

In an established mathematical terminology, $\mathbf{A}$ could be called the Jacobian tensor of (u,v) with respect to (x,y), and $J(u,v)$ the Jacobian determinant. We will, however, continue to call $\mathbf{A}$ the horizontal velocity gradient tensor, and $J(u,v)$ the Jacobian.

Jacobians other than $J(u,v)$ occur in later sections, and their arguments will be explicitly indicated. $J(u,v)$ itself will be simply denoted J unless confusion seems likely.

2.2 Semigeostrophic (SG) shallow-water dynamics

The inviscid, *f*-plane, SG shallow water model consists of the horizontal momentum equations

$$\frac{Du_G}{Dt} - fv_A = 0, \qquad \frac{Dv_G}{Dt} + fu_A = 0, \tag{11,12}$$

and the continuity equation:

$$\frac{Dh}{Dt} + h\delta = 0. \tag{13}$$

Here *h* is the depth of the fluid (relative to a flat bed) and the material derivative is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad . \tag{14}$$

In (11, 12) the Coriolis parameter, f , is a constant, and the ageostrophic flow components u_A and v_A are defined as

$$u_A \equiv u - u_G, \qquad v_A \equiv v - v_G, \qquad (15, 16)$$

where

$$u_G \equiv -\frac{g}{f}\frac{\partial h}{\partial y}, \qquad v_G \equiv \frac{g}{f}\frac{\partial h}{\partial x} \qquad (17, 18)$$

are the geostrophic flow components.

The SG momentum equations (11) – (12) retain the full flow in the material derivative (see (14)), but the advected quantities are the components of the geostrophic flow. This is widely known as the geostrophic momentum approximation (see Hoskins (1975), for example). The continuity equation (13) is in its complete (shallow water) form.

Replacing u_G and v_G respectively with u and v in (11, 12) gives the usual 'primitive equation' shallow water model.

Replacing u and v respectively with u_G and v_G in (14), and the term $h\delta$ in (13) with $h_0\delta$ (h_0 being a constant value), gives the QG shallow water model studied in section 7 of RWC.

Having stated (11) and (12), many SG studies then make a transformation to 'geostrophic coordinates' $X = x + (v_G / f)$, $Y = y - (u_G / f)$ (see Hoskins 1975) and this is widely regarded as a key feature of SG modelling. However, the coordinate transformation is sometimes followed by the imposition of approximations which, strictly, vitiate the PV conservation property that may be demonstrated for (11) – (13) (see section 4, below). For discussion, see McWilliams and Gent (1980) – who note two variants of the SG model – and Craig (1993), p 3354. In this study we do not apply the geostrophic coordinate transformation, but consider (11), (12) and (13) as they stand. We regard the geostrophic momentum approximation embodied in (11) and (12) as the essence of SG dynamics because it assures retention of the PV conservation property.

3. Time evolution under SG shallow-water dynamics

3.1 Core equations

Straightforward differentiation of (11) and (12) – noting (14) – gives time-evolution equations for $\partial u_G/\partial x$, $\partial u_G/\partial y$, $\partial v_G/\partial x$, $\partial v_G/\partial y$:

$$\frac{D}{Dt}\left(\frac{\partial u_G}{\partial x}\right)+\frac{\partial u}{\partial x}\frac{\partial u_G}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u_G}{\partial y}-f\frac{\partial v_A}{\partial x}=0, \tag{19}$$

$$\frac{D}{Dt}\left(\frac{\partial u_G}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial u_G}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u_G}{\partial y}-f\frac{\partial v_A}{\partial y}=0, \tag{20}$$

$$\frac{D}{Dt}\left(\frac{\partial v_G}{\partial x}\right)+\frac{\partial u}{\partial x}\frac{\partial v_G}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v_G}{\partial y}+f\frac{\partial u_A}{\partial x}=0, \tag{21}$$

$$\frac{D}{Dt}\left(\frac{\partial v_G}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial v_G}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial v_G}{\partial y}+f\frac{\partial u_A}{\partial y}=0. \tag{22}$$

Eqs (19)-(22) describe the time evolution of the four elements of $\mathbf{A}$ evaluated for the *geostrophic* flow (u_G, v_G). This reflects the fact that the SG momentum equations (11, 12) describe the time evolution of u_G and v_G. Time evolution equations are therefore sought for the quadratic invariants evaluated for the geostrophic flow, not for the total flow. This approach is typical of analyses of conservation properties of approximate dynamical models in many meteorological contexts, and is justified by results. It enables the rationale of Cantwell's (1992) Navier-Stokes study to be applied to the SG case.

3.2 SG time evolution equation for J_G

The appropriate geostrophic Jacobian is J_G defined as

$$J_G \equiv J(u_G, v_G) = \frac{\partial u_G}{\partial x}\frac{\partial v_G}{\partial y} - \frac{\partial v_G}{\partial x}\frac{\partial u_G}{\partial y}\,. \tag{23}$$

From Eqs (19) – (22) a time evolution equation for J_G may be obtained as

$$\frac{DJ_G}{Dt} + J_G\delta + f\left[J(u_G, u_A) + J(v_G, v_A)\right] = 0. \tag{24}$$

Eq. (24) is a reasonably compact form; moreover, it is remarkably

similar to the *f*-plane time evolution equation found in the primitive equation case. For the case $\beta = 0$, RWC's (49) becomes:

$$\frac{DJ}{Dt} + J\delta + f\left[J(u_G, u_A) + J(v_G, v_A)\right] = 0. \tag{25}$$

The only modification of (25) seen in (24) is that J_G appears instead of J in the first two terms. As in (25), the material derivative in (24) is the unapproximated version (14); the divergence δ involves the total horizontal flow, and the two Jacobian terms that involve the ageostrophic flow components are unchanged.

The QG counterpart of (24) (see (90) of RWC) departs much more from (25). As well as featuring J_G rather than J, it has D/Dt replaced with the QG form D/Dt_G (see (75) of RWC); and the term $J_G\delta$ vanishes because δ appears as $\delta = \delta_G = 0$.

Both (24) and (25) may be condensed further by using (13) to combine the material derivative term and the term involving δ. See RWC's Eq (50).

3.3 SG time evolution equation for D_R^G

The appropriate geostrophic deformation is D_R^G given by

$$\left(D_R^G\right)^2 \equiv \left(D_1^G\right)^2 + \left(D_2^G\right)^2 = \left(\frac{\partial u_G}{\partial x} - \frac{\partial v_G}{\partial y}\right)^2 + \left(\frac{\partial v_G}{\partial x} + \frac{\partial u_G}{\partial y}\right)^2. \tag{26}$$

From (19) – (22) and (26), a lengthy calculation (outlined in Appendix A) leads to the following time-evolution equation for D_R^G:

$$\frac{D}{Dt}\left(D_R^G\right)^2 + \left(D_R^G\right)^2 \delta + \left(2f + \zeta_G\right)\left[D_2^G D_1^A - D_1^G D_2^A\right] = 0. \tag{27}$$

The primitive equation version (obtained from RWC's (60) with $\beta = 0$) is

$$\frac{D}{Dt}\left(D_R\right)^2 + 2\left(D_R\right)^2 \delta + 2f\left[D_2^G D_1^A - D_1^G D_2^A\right] = 0. \tag{28}$$

The factor of 2 that occurs in the second term in (28) is unity in (27), and the factor of $2f$ in the third term in (28) is $2f + \zeta_G$ in (27). The SG form (27) is thus slightly more complicated than the PE form (28); it is not obtained by simply replacing D_R with D_R^G in the latter.

The QG version of (28) (see (93) of RWC) involves D_R^G instead of D_R, and has D/Dt_G instead of D/Dt. Also, the term $\left(D_R^G\right)^2 \delta$ vanishes because δ appears as $\delta = \delta_G = 0$. Nevertheless, the QG version captures the form of the third term in (28).

3.4 SG time evolution equation for Q_G

The appropriate geostrophic Frobenius norm Q_G is given by

$$\left(Q_G\right)^2 \equiv \left(\frac{\partial u_G}{\partial x}\right)^2 + \left(\frac{\partial u_G}{\partial y}\right)^2 + \left(\frac{\partial v_G}{\partial x}\right)^2 + \left(\frac{\partial v_G}{\partial y}\right)^2 \quad . \tag{29}$$

From (19) – (22) and (29), the time-evolution equation obeyed by Q_G is found to be

$$\frac{D}{Dt}\left(Q_G\right)^2 + \left(Q_G\right)^2 \delta = \left(f + \zeta_G\right)\left[D_1^G D_2^A - D_2^G D_1^A\right] - f\zeta_G \delta. \tag{30}$$

(See Appendix A for a sketch of the calculation.)

The corresponding primitive equation form (see RWC's (67), noting that $\delta_G = 0$ in the current *f*-plane context) is

$$\frac{D}{Dt}Q^2 + \left(Q^2 + D_R^2\right)\delta = f\left[D_1^G D_2^A - D_2^G D_1^A\right] - f\zeta_G\delta. \qquad (31)$$

The SG form (30) differs from the PE form (31) not only in the expected appearance of Q_G rather than Q: the second l.h.s. term in (30) lacks the contribution of D_R^2 that is seen in (31); and the first r.h.s. term in (30) has a factor of $\left(f + \zeta_G\right)$ rather than f.

The QG counterpart of (31) (see (94) of RWC) involves Q_G instead of Q, and has D/Dt_G instead of D/Dt. Also, it lacks the term in δ on the left-hand side. Its right-hand side can be shown to have the same form as that of (31), however.

Given the relationship (9) (applied to the geostrophic flow), subtracting (30) from (27) should give (25). That this is indeed so may be shown by applying identity (63) of RWC (with $\delta_G = 0$) after the subtraction. (Had (9) been used to derive one of the time-evolution equations from the other two, it could not have been used as a check on the analysis.)

3.5 SG divergence equation

A divergence equation is readily formed by adding (19) and (22). After a little manipulation it can be written as

$$2J_G + f\zeta_A + J\left(u_G, v_A\right) - J\left(v_G, u_A\right) = 0. \qquad (32)$$

The two Jacobian terms in (32) that involve the ageostrophic flow components are not present in the QG version ((80) of RWC). Neither do they occur in the primitive equation form ((34) of RWC). In the QG case, use of the appropriate version of (25) enables a simple expression for $D\zeta_A/Dt_G$ to be obtained; see (95) of RWC. This manoeuvre is not useful in the SG case because appropriate expressions for $D/Dt[J(u_G, v_A)]$ and $D/Dt[J(v_G, u_A)]$ are not available.

If the ageostrophic flow (u_A, v_A) were known, one might consider solving (32) for the geostrophic flow represented by an appropriate streamfunction. The problem would then be to solve an equation of Monge-Ampère type, under an ellipticity condition. See Larchevêque (1993), for example. However, standard procedures for time integration of the SG equations use the PV equation to evolve the geostrophic flow, and a need to calculate it from the ageostrophic flow does not arise.

4. Shallow water SGPV conservation

It is well known that the shallow water SG equations (1) – (3) imply the Lagrangian conservation law

$$\frac{D}{Dt} q_{SG} = 0, \tag{33}$$

in which the SG potential vorticity q_{SG} is given by

$$hq_{SG} = f + \frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} + \frac{1}{f}\left(\frac{\partial u_G}{\partial x}\frac{\partial v_G}{\partial y} - \frac{\partial v_G}{\partial x}\frac{\partial u_G}{\partial y}\right) = f + \zeta_G + \left(\frac{J_G}{f}\right). \tag{34}$$

The quantity q_{SG} is formally different from the QGPV (see RWC's (78)). It is also formally different from the primitive equation form $q = (f + \zeta)/h$ as regards the occurrence of the term in J_G in (34).

Although SGPV conservation as expressed by (33) and (34) is a familiar property, and is a key aspect of the SG model, it is not a transparent result, and derivations are typically not straightforward. It may be demonstrated by Hamiltonian methods (Salmon 1983), while Shutts and Cullen (1987) and Chynoweth and Sewell (1991) give algebraic proofs for the 3D, baroclinic case. An algebraic proof given by Allen et al. (1990) deftly exploits the fact that the SG momentum equations are linear in the velocity

components. See also White (2002) and Ehrendorfer (2004).

The expression (24) for DJ_G/Dt enables the SGPV conservation law (33) (with (34)) to be obtained rapidly. From (3), (20) and (21) the vorticity equation follows in the form

$$\frac{D\zeta_G}{Dt} + (f + \zeta_G)\delta - J(u_G, u_A) - J(v_G, v_A) = 0. \tag{35}$$

The two Jacobian terms in (35) do not have counterparts in the primitive equation form ((32) of RWC) but – apart from a constant factor – they are precisely equal to two of the Jacobian terms that appear in (24). Eliminating them between (24) and (35) gives

$$\frac{D}{Dt}\left[f + \zeta_G + \left(\frac{J_G}{f}\right)\right] + \left[f + \zeta_G + \left(\frac{J_G}{f}\right)\right]\delta = 0. \tag{36}$$

The SGPV conservation law (33) (with (34)) follows from (36) upon use of the continuity equation (13).

This proof depends for its brevity on the prior derivation of expression (24) for DJ_G/Dt; it demonstrates an advantage of investigating the behaviour of quadratic geometric invariants when considering the Lagrangian conservation properties of a set of equations.

5. Extension to 3D SG dynamics

In order to elucidate the properties, utility and applicability of the 3D baroclinic SG equations, isentropic coordinate forms have been derived and examined by Hoskins and Draghici (1977), Craig (1993) and others, typically

with the use of geostrophic coordinates in the horizontal and neglect of small terms. By employing isentropic coordinates in the vertical, but retaining the usual Cartesian coordinates in the horizontal, the results obtained in previous sections for the shallow-water

SG model may be readily extended to 3D baroclinic flow. For convenience, the case of a perfect gas is considered.

The 3D motion (assumed adiabatic) is 2-dimensional on isentropic surfaces, and potential temperature (θ) may be used as a vertical coordinate so long as the stratification is stable (i.e. so long as $\partial\theta/\partial z > 0$). The SG momentum equations (11,12) are unchanged, but the differentiations in the form (14) of the material derivative are now taken at constant θ:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}\Big|_\theta + u\frac{\partial}{\partial x}\Big|_\theta + v\frac{\partial}{\partial y}\Big|_\theta \quad . \tag{37}$$

Rather than via depth *h*, the geostrophic flow is defined in terms of the Montgomery potential $M \equiv gz + c_p T$, z being height, T temperature and c_p specific heat at constant pressure. See Holton (1992), section 4.6, and Hoskins *et al*. (1985).

The continuity equation (13) takes the form

$$\frac{D}{Dt}\left(\frac{\partial p}{\partial\theta}\right) + \left(\frac{\partial p}{\partial\theta}\right)\delta^\theta = 0. \tag{38}$$

Here p is pressure and δ^θ is the horizontal divergence on isentropic surfaces:

$$\delta^\theta \equiv \frac{\partial u}{\partial x}\Big|_\theta + \frac{\partial v}{\partial y}\Big|_\theta \,. \tag{39}$$

Re-tracing the analysis of sections 2-5, one obtains a vorticity equation in the form

$$\frac{D}{Dt}\left[f + \zeta_G^\theta + \left(\frac{J_G^\theta}{f}\right)\right] + \left[f + \zeta_G^\theta + \left(\frac{J_G^\theta}{f}\right)\right]\delta^\theta = 0\,. \tag{40}$$

Here

$$\zeta_G^\theta \equiv \frac{\partial v}{\partial x}\Big|_\theta - \frac{\partial u}{\partial y}\Big|_\theta \tag{41}$$

$$J_G^\theta \equiv \frac{\partial u_G}{\partial x}\Big|_\theta \frac{\partial v_G}{\partial y}\Big|_\theta - \frac{\partial v_G}{\partial x}\Big|_\theta \frac{\partial u_G}{\partial y}\Big|_\theta \,. \tag{42}$$

and D/Dt is given by (37).

Applying the continuity equation (38) in (40) leads immediately to

$$\frac{D}{Dt} q_{SG}^{\theta} = 0. \tag{43}$$

The SGPV, q_{SG}^{θ} , in (43) is given by the simple relation

$$\left(\frac{\partial p}{\partial \theta}\right) q_{SG}^{\theta} = f + \zeta_G^{\theta} + \left(\frac{J_G^{\theta}}{f}\right). \tag{44}$$

Using a Legendre transform approach, Chynoweth and Sewell (1991) considered the SG model in several coordinate systems, without making additional approximations. In terms of an appropriate potential function, they obtained the conservation law (43) with SGPV in a determinant form that is algebraically equivalent to (44).

In pressure coordinates, (44) assumes the significantly more complicated (though familiar) form given in Appendix B. The most straightforward way to establish SGPV conservation in pressure coordinates is evidently to derive the relevant result in isentropic coordinates, as above, and then to transform to the pressure system.

6. Concluding remarks

A number of results have emerged from this study of
the time evolution of quadratic geometric invariants under semigeostrophic (SG) dynamics.

Remarkably, the Jacobian of the geostrophic flow with respect to the horizontal Cartesian coordinates behaves in a formally similar way to its counterpart in the complete shallow water equations: the former simply replaces the latter in its time evolution equation, and the the material derivative remains unchanged. Less surprisingly, divergence terms that vanish in the quasi-geostrophic (QG) case (Roulstone *et al.* 2014) are much better represented because the contribution of the ageostrophic flow – the sole contribution on an *f*-plane – is included.

Such divergence terms are also better represented in the SG time evolution of the total deformation and Frobenius norm than in the QG case. However, the SG time evolution equations for these two invariants are not formally similar to their counterparts in the primitive shallow water equations: neither equation results simply from replacement of the complete invariant by the geostrophically evaluated quantity. The symmetrical behaviour of the Jacobian's time evolution equation is all the more striking because neither the vorticity equation nor the divergence equation exhibits it (see (32) and (35)). Our discussion of these properties has been mainly descriptive; a deeper theoretical narrative should be sought in future work.

The detailed analytical results pass a necessary test that stems from a known algebraic relationship involving the Jacobian, the resultant deformation and the Frobenius norm.

It has been found that a compact prognostic equation for the ageostrophic vorticity that occurs in the QG model does not readily extend to the SG case.

A product of the study is a straightforward derivation of the potential vorticity conservation law of the SG shallow water model. This derivation hinges on the availability of the time evolution equation for the Jacobian of the geostrophic flow with respect to the horizontal coordinates, and on a cancellation with terms in the vorticity equation. The desirability is indicated of exploring the behaviour of quadratic geometric invariants when examining the conservation properties of a set of equations.

Results have been extended to the 3D baroclinic SG model by the use of isentropic coordinates, and it has been suggested that the easiest way to establish the PV conservation property of the 3D SG system in non-isentropic coordinates (such as pressure) is first to obtain the isentropic property and then to transform it.

The time evolution of quadratic geometric invariants under other approximate specifications of the dynamics is a promising subject for future study. The Green-Naghdi equations (see Miles and Salmon 1985) have good conservation properties and Hamiltonian structure, and thus have a pedigree comparable to that of the SG equations. Does the behaviour found here for SG dynamics occur also in the Green-Naghdi case? Another interesting candidate for further study is the class of 3D QG models. These have good conservation properties, and Hamiltonian structures, but (unlike SG) are not precisely transformable between different vertical coordinate systems (see Charney and Stern 1962, p163, and Berrisford et al. 1993, p780). The technique of transformation from isentropic coordinates used in the present study would therefore have to be applied with particular caution.

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Appendix A

Outline derivations of Eqs (27) and (30) from Eqs (19) – (22)

The geostrophic deformation components are defined in terms of u_G and v_G as

$$D_1^G \equiv \frac{\partial u_G}{\partial x} - \frac{\partial v_G}{\partial y}, \qquad D_2^G \equiv \frac{\partial v_G}{\partial x} + \frac{\partial u_G}{\partial y}, \tag{A1, A2}$$

and the ageostrophic components similarly, but in terms of u_A and v_A.

From (19) – (22) it follows that

$$\frac{D}{Dt}\left(D_1^G\right) + D_1^G\delta + J\left(v_A, u_G\right) + J\left(u_A, v_G\right) - fD_2^A = 0, \tag{A3}$$

$$\frac{D}{Dt}\left(D_2^G\right) + D_2^G\delta + J\left(v_A, v_G\right) - J\left(u_A, u_G\right) + fD_1^A = 0. \tag{A4}$$

The Jacobian terms in (A3) and (A4) may be re-expressed by using

$$2\frac{\partial u_G}{\partial x} = \delta_G + D_1^G, \qquad 2\frac{\partial v_G}{\partial y} = \delta_G - D_1^G, \tag{A5}$$

$$2\frac{\partial v_G}{\partial x} = D_2^G + \zeta_G, \qquad 2\frac{\partial u_G}{\partial y} = D_2^G - \zeta_G, \tag{A6}$$

and similar relations for the ageostrophic flow. After some labour, one finds from (A5) and (A6) that (for the case $\delta_G = 0$ that is of current interest):

$$2\left[J\left(v_A, u_G\right) + J\left(u_A, v_G\right)\right] = D_2^G\zeta_A - D_2^A\zeta_G - D_1^G\delta_A, \tag{A7}$$

$$2\left[J\left(v_A, v_G\right) - J\left(u_A, u_G\right)\right] = D_1^A\zeta_G - D_1^G\zeta_A - D_2^G\delta_A. \tag{A8}$$

A time-evolution equation for $\left(D_R^G\right)^2 = \left(D_1^G\right)^2 + \left(D_2^G\right)^2$ may be obtained by multiplying (A3) and (A4) respectively by $2D_1^G$ and $2D_2^G$, and adding the results. Carrying out this procedure, having applied (A7) and (A8), and finally noting $\delta_A = \delta$ (since $\delta_G = 0$), one obtains

$$\frac{D}{Dt}\left(D_R^G\right)^2 + \left(D_R^G\right)^2\delta + \left(2f + \zeta_G\right)\left[D_2^G D_1^A - D_1^G D_2^A\right] = 0, \tag{A9}$$

which is (27).

Multiplication of (19) – (22) by (2×) the respective advected quantities soon leads to

$$\frac{D}{Dt}(Q_G)^2 + 2(Q_G)^2\delta - 2I = 2f\left[(\nabla u_G)\cdot(\nabla v_A) - (\nabla v_G)\cdot(\nabla u_A)\right]. \qquad \text{(A10)}$$

In (A10),

$$I \equiv \left[\frac{\partial u_G}{\partial x} J(u_G, v_A) + \frac{\partial v_G}{\partial x} J(v_G, v_A)\right] - \left[\frac{\partial u_G}{\partial y} J(u_G, u_A) + \frac{\partial v_G}{\partial y} J(v_G, u_A)\right]. \qquad \text{(A11)}$$

By applying (A5) and (A6) and their ageostrophic counterparts, (A11) may be re-written (for the case $\delta_G = 0$) as

$$2I = \frac{1}{2}\left[(D_R^G)^2 + (\zeta_G)^2\right]\delta_A + \zeta_G\left(D_1^G D_2^A - D_2^G D_1^A\right). \qquad \text{(A12)}$$

By use of (10) for the geostrophic flow (noting $\delta_G = 0$) (A12) reduces to

$$2I = (Q_G)^2\delta_A + \zeta_G\left(D_1^G D_2^A - D_2^G D_1^A\right). \qquad \text{(A13)}$$

Further, the r.h.s. of (A10) may be re-written using RWC's (66) for the case $\delta_G = 0$:

$$2\left[(\nabla u_G)\cdot(\nabla v_A) - (\nabla v_G)\cdot(\nabla u_A)\right] = \left(D_1^G D_2^A - D_2^G D_1^A\right) - \zeta_G\delta_A . \qquad \text{(A14)}$$

Upon noting that $\delta = \delta_A$ (because $\delta_G = 0$), use of (A14) and (A15) in (A10) gives

$$\frac{D}{Dt}(Q_G)^2 + (Q_G)^2\delta = (f + \zeta_G)\left[D_1^G D_2^A - D_2^G D_1^A\right] - f\zeta_G\delta, \qquad \text{(A15)}$$

which is (30).

Expressions (A5), (A6) and their ageostrophic counterparts may be used to demonstrate identities (63) and (66) of RWC (which allow $\delta_G \neq 0$).

Appendix B

Transforming SGPV from isentropic to pressure coordinates

For any appropriately smooth function F, the x derivatives at constant θ and p are related by

$$\left.\frac{\partial F}{\partial x}\right|_{\theta} = \left.\frac{\partial F}{\partial x}\right|_{p} + \frac{\partial F}{\partial p}\left.\frac{\partial p}{\partial x}\right|_{\theta} . \tag{B1}$$

The y derivatives at constant θ and p obey a similar relation. The case $F = \theta$ shows that

$$\left.\frac{\partial \theta}{\partial x}\right|_{p} + \frac{\partial \theta}{\partial p}\left.\frac{\partial p}{\partial x}\right|_{\theta} = 0 , \tag{B2}$$

and a similar relation involving y derivatives.

Repeated application of (B1) and (B2) to the right-hand side of (44) – noting (41) and (42) – gives the lengthy result

$$q_{SG}^{\theta} = \left(f + \zeta_G^{p}\right)\frac{\partial \theta}{\partial p} - \frac{\partial v_G}{\partial p}\left.\frac{\partial \theta}{\partial x}\right|_{p} + \frac{\partial u_G}{\partial p}\left.\frac{\partial \theta}{\partial y}\right|_{p} + \frac{1}{f}\left(K_1 + K_2 + K_3\right). \tag{B3}$$

Here

$$K_1 = \left(\frac{\partial v_G}{\partial p}\left.\frac{\partial u_G}{\partial y}\right|_{p} - \frac{\partial u_G}{\partial p}\left.\frac{\partial v_G}{\partial y}\right|_{p}\right)\left.\frac{\partial \theta}{\partial x}\right|_{p} , \tag{B4}$$

$$K_2 = \left(\frac{\partial u_G}{\partial p}\left.\frac{\partial v_G}{\partial x}\right|_{p} - \frac{\partial v_G}{\partial p}\left.\frac{\partial u_G}{\partial x}\right|_{p}\right)\left.\frac{\partial \theta}{\partial y}\right|_{p} , \tag{B5}$$

$$K_3 = \left(\left.\frac{\partial u_G}{\partial x}\right|_{p}\left.\frac{\partial v_G}{\partial y}\right|_{p} - \left.\frac{\partial u_G}{\partial y}\right|_{p}\left.\frac{\partial v_G}{\partial x}\right|_{p}\right)\frac{\partial \theta}{\partial p} . \tag{B6}$$

K_1, K_2 and K_3 all originate from the Jacobian term J_G^{θ} in (44), and together they have a clear scalar product form in the x, y, p coordinate system. The other terms in (B3) constitute a geostrophic approximation to the usual p-coordinate PV – see, for example, Hoskins, McIntyre & Robertson (1985).

Each term in (B3) has a counterpart in the expression for SGPV given by Hoskins (1975). [Rather than pressure itself, the vertical coordinate in Hoskins (1975) is a "pseudo-height" proportional to p^{κ}, where $\kappa \equiv R/c_p$. This difference is not crucial. For a discussion of pressure-based pseudo-heights, see White and Beare (2005).]

References

Allen JS, Barth JA, Newberger PA. 1990. On intermediate models for barotropic continental shelf and slope flow fields. Part I: formulation and comparison of exact solutions. *J. Phys. Oceanog.* **20:** 1017-1042.

Berrisford P, Marshall JC, White AA. 1993. Quasigeostrophic potential vorticity in isentropic coordinates. *J. Atmos. Sci.* **50:** 778-782.

Cantwell BJ. 1992. Exact solution of a restricted Euler equation for the velocity gradient tensor. *Phys. Fluids A***4**: 782-793.

Charney JG, Stern ME. 1962. On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.* **19:** 159-172.

Chynoweth S, Sewell MJ. 1991. A concise derivation of the semi-geostrophic equations. *Q. J. R. Meteorol. Soc.* **117:** 1109-1128.

Craig GC. 1993. A scaling for the three-dimensional semigeostrophic approximation. *J. Atmos. Sci.* **50:** 3350-3355.

Ehrendorfer M. 2004. A vector derivation of the semigeostrophic potential vorticity equation. *J. Atmos. Sci.* **61:** 1461-1466.

Holton JR. 1992. *An Introduction to Dynamic Meteorology.* Academic Press: New York.

Hoskins BJ. 1975. The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmos. Sci.* **32:** 233-242.

Hoskins BJ, Draghici I. 1977. The forcing of ageostrophic motion according to the semigeostrophic equations and in an isentropic coordinate model. *J. Atmos. Sci.* **34:** 1859-1867.

Hoskins BJ, McIntyre ME, Robertson AW. 1985. On the use and significance of isentropic potential vorticity maps. *Q. J. R. Meteorol. Soc.* **111:** 877-946.

Larchevêque M. 1993. Pressure field, vorticity field, and coherent structures in two-dimensional incompressible turbulent flows. *Theoret. Comput. Fluid Dynamics* **5:** 215-222.

Martin J, Dopazo C, Valiñ o L. 1998. Dynamics of velocity gradient invariants in turbulence: Restricted Euler and linear diffusion models. *Phys. Fluids* **10**: 2012-2025.

McWilliams JC, Gent, PR. 1980. Intermediate models of planetary circulations in the atmosphere and ocean. *J. Atmos. Sci.* **37:** 1657-1678.

Miles J, Salmon R. 1985. Weakly dispersive nonlinear gravity waves. *J. Fluid Mech.* **157:** 519-531.

Roulstone I, White AA, Clough SA. 2014. Geometrical invariants of the horizontal velocity gradient tensor and their dynamics in shallow water flow. *Q. J. R. Meteorol. Soc.* **140:** 2527-2534.

Salmon R. 1983. Practical use of Hamilton's principle. *J. Fluid Mech.* **132:** 431-444.

Shutts GJ, Cullen MJP. Parcel stability and its relation semigeostrophic theory. *J. Atmos. Sci.* **44:** 1318-1330.

White AA. 2002. A view of the equations of meteorological dynamics and various approximations. In *Large-scale Atmosphere-Ocean Dynamics I,* Norbury J, Roulstone I. (eds.): 1-100. Cambridge University Press, Cambridge, UK.

White AA, Beare RJ. 2005. Flavours of pseudo-height. *Q. J. R. Meteorol. Soc.* **131:** 759-764.