THE CAHN-HILLIARD EQUATION WITH SINGULAR POTENTIALS AND DYNAMIC BOUNDARY CONDITIONS

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Abstract. Our aim in this paper is to study the Cahn-Hilliard equation with singular potentials and dynamic boundary conditions. In particular, we prove, owing to proper approximations of the singular potential and a suitable notion of variational solutions, the existence and uniqueness of solutions. We also discuss the separation of the solutions from the singularities of the potential. Finally, we prove the existence of global and exponential attractors.

1. Introduction

The Cahn-Hilliard system

$$\begin{cases} \partial_t u = \kappa\Delta_x\mu, \quad \kappa > 0, \\ \mu = -\alpha\Delta_x u + f(u), \quad \alpha > 0, \end{cases} \tag{1.1}$$

plays an essential role in materials science as it describes important qualitative features of two-phase systems related with phase separation processes. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. We refer the reader to, e.g., [5], [6], [28], [29], [31], [32], [40] and [41] for more details. Here, u is the order parameter (it corresponds to a (rescaled) density of atoms) and μ is the chemical potential. Furthermore, f is a double-well potential whose wells correspond to the phases of the material. A thermodynamically relevant potential is the following logarithmic (singular) potential:

$$f(s) = -2\kappa_0 s + \kappa_1 \ln\frac{1+s}{1-s}, \quad s\in(-1,1), \quad 0<\kappa_0<\kappa_1, \tag{1.2}$$

although such a potential is very often approximated by regular ones (typically, $f(s) = s^3 - s$). Finally, κ is the mobility and α is related to the surface tension at the interface.

This system, endowed with Neumann boundary conditions for both u and μ (meaning that the interface is orthogonal to the boundary and that there is no mass flux at the

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boundary) or with periodic boundary conditions, has been extensively studied and one now has a rather complete picture as far as the existence, uniqueness and regularity of solutions and the asymptotic behavior of the solutions are concerned. We refer the reader, among a vast literature, to, e.g., [1], [10], [17], [18], [19], [20], [26], [30], [34], [37], [38], [39], [40], [41], [46], [50] and [51].

Now, the question of how the process of phase separation (that is, the spinodal decomposition) is influenced by the presence of walls has gained much attention recently (see [21], [22], [27] and the references therein). This problem has mainly been studied for polymer mixtures (although it should also be important in other systems, such as binary metallic alloys): from a technological point of view, binary polymer mixtures are particularly interesting, since the occurring structures during the phase separation process may be frozen by a rapid quench into the glassy state; micro-structures at surfaces on very small length scales can be produced in this way.

In that case, we again write that there is no mass flux at the boundary. Then, in order to obtain the second boundary condition, following the phenomenological derivation of the Cahn-Hilliard system, we consider, in addition to the usual Ginzburg-Landau free energy

$$\Psi_{GL}(u,\nabla u)=\int_\Omega(\frac{\alpha}{2}|\nabla_x u|^2+F(u))\,dx, \tag{1.3}$$

where $F'=f$ and Ω is the domain occupied by the material (the chemical potential μ is defined as a variational derivative of Ψ_{GL} with respect to u), and assuming that the interactions with the walls are short-ranged, a surface free energy of the form

$$\Psi_\Gamma(u,\nabla_\Gamma u)=\int_\Gamma(\frac{\alpha_\Gamma}{2}|\nabla_\Gamma u|^2+G(u))\,dS,\quad \alpha_\Gamma>0 \tag{1.4}$$

(thus, $\Psi=\Psi_{GL}+\Psi_\Gamma$ is the total free energy of the system), where Γ is the boundary of Ω and ∇_Γ is the surface gradient. Writing finally that the system tends to minimize the excess surface energy, we end up with the following boundary condition:

$$\frac{1}{d}\partial_t u-\alpha_\Gamma\Delta_\Gamma u+g(u)+\alpha\partial_n u=0,\quad \text{on } \Gamma, \tag{1.5}$$

where Δ_Γ is the Laplace-Beltrami operator, ∂_n is the normal derivative, $g=G'$ and $d>0$ is some relaxation parameter, which is usually referred to as dynamic boundary condition, in the sense that the kinetics, i.e., $\partial_t u$, appears explicitly. Furthermore, in the original derivation, one has $G(u)=\frac{1}{2}a_\Gamma u^2-b_\Gamma u$, where $a_\Gamma>0$ accounts for a modification of the effective interaction between the components at the walls and b_Γ characterizes the possible preferential attraction (or repulsion) of one of the components by the walls (when b_Γ vanishes, there is no preferential attraction). We also refer the reader to [3] and [23] for other physical derivations of such dynamic boundary conditions, obtained by taking the continuum limit of lattice models within a direct mean-field approximation and by applying a density functional theory, to [44] for the derivation of dynamic boundary conditions in the context of two-phase fluids flows and to [47] and [48] for an approach based on concentrated capacity.

The Cahn-Hilliard system, endowed with dynamic boundary conditions, has been studied in [9], [24], [35], [42], [45] and [50] for regular potentials f and g. In particular, one now has satisfactory results on the existence, uniqueness and regularity of solutions and on the asymptotic behavior of the solutions.

The case of nonregular potentials and dynamic boundary conditions is essentially more complicated and less understood. Indeed, to the best of our knowledge, even the existence of weak energy solutions has only recently been established in that case, under the additional restriction that the boundary nonlinearity g has the right sign at the singular points ± 1, namely,

$$\pm\, g(\pm 1) > 0 \tag{1.6}$$

(see [25]; see also [7] where sign conditions are considered in the context of the Caginalp phase-field system). Furthermore, the questions related with the longtime behavior of the solutions (e.g., in terms of global attractors or/and exponential attractors) have not been considered in the literature.

The aim of the present paper is to give a thorough study of the singular Cahn-Hilliard problem endowed with dynamic boundary conditions. As we will see below, the main difficulty here lies in the fact that the combination of dynamics boundary conditions and of singular potentials can produce additional strong singularities on the corresponding solutions close to the boundary (especially in the case where the sign condition (1.6) is violated). In that case, even the simplest 1D stationary problems may not have solutions in a usual (or distribution) sense (due to the jumps of the normal derivatives close to the boundary produced by the singularities, see Example 6.2).

Nevertheless, we can construct a sequence of solutions of regular approximations of our singular problem which converges to a unique trajectory which is then naturally identified with the "solution" of the limit singular problem. As already pointed out, this trajectory may not be a solution of our equations in the usual (distribution) sense, so that the notion of a solution must be properly modified. To do so, we consider, in the spirit of [4] (see also [12]), the variational *inequality* associated with the problem and define a (variational) solution in terms of this variational inequality, see Section 3 for details.

Of course, important questions are when the solution thus defined is a usual distribution solution of the equations and which additional regularity one can expect from such a variational solution. Actually, we prove that the variational solutions are always Hölder continuous in space and are solutions in the usual sense if they do not reach the pure states on the boundary, namely, if

$$|u(t,x)| < 1 \tag{1.7}$$

for almost all $(t,x) \in \mathbb{R}^+ \times \Gamma$. One possible condition which guarantees that condition (1.7) holds is exactly the aforementioned sign condition (1.6) (see Proposition 4.5). Alternatively, this condition is always satisfied if the singularities of the nonlinearity f are strong enough, namely, if

$$\lim_{u\to\pm 1} F(u) = \infty, \quad F(u) := \int_0^u f(s)\, ds,$$

see Section 4. Furthermore, using some proper modification of the Moser iteration scheme, we can also show that any trajectory $u(t)$ is separated from the singularities ± 1 if, in addition,

$$\frac{f(u)}{u} \geq \frac{C}{(1-u^2)^p}, \quad p > 1$$

(see Remark 4.9; see also [8] for a similar condition for the Caginalp system). In that case, we have $|u(t,x)| \leq 1-\delta$ for some $\delta > 0$ and, consequently, the problem becomes factually nonsingular and can be further investigated by using the techniques devised for the Cahn-Hilliard equation with regular potentials. Unfortunately, this last condition is not satisfied by the physically relevant logarithmic potentials and we indeed need the variational inequalities (and solutions) in order to deal with such a potential.

The next, natural, step is to study the asymptotic behavior of the system. In particular, we are interested here in the study of finite-dimensional global attractors. We recall that the global attractor is the smallest compact set of the phase space which is fully invariant by the flow and attracts the bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the longtime behavior of the problem. Furthermore, when the global attractor has finite dimension (in the sense of covering dimensions such as the fractal and the Hausdorff dimensions), then, even though the initial phase space is infinite-dimensional, the dynamics of the system is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. We refer the reader to, e.g., [2], [36], [49] and the references therein for extensive reviews and discussions on this subject. One powerful method, in order to prove the existence of the finite-dimensional global attractor, is to prove the existence of a so-called exponential attractor (in particular, this approach does not necessitate, contrary to the usual one, based on the Lyapunov exponents, the differentiability of the underlying semigroup). An exponential attractor is a compact and semiinvariant set which contains the global attractor, has finite fractal dimension and attracts all bounded sets of initial data at an exponential rate. We refer the reader to, e.g., [11], [13], [14] and [36] for more details and discussions on exponential attractors.

We thus prove the existence of global and exponential attractors for our problem. We emphasize that this result is obtained under general assumptions (without any sign assumption or any assumption of the form (1.7)) and is thus valid for the variational solutions (which may not be solutions in the usual sense). In particular, such solutions may reach the singularities ± 1 on sets of positive measure on the boundary $\mathbb{R}^+ \times \Gamma$ or even on the whole boundary $\mathbb{R}^+ \times \Gamma$. This fact does not allow us to use the techniques devised in [34] to establish the existence of finite-dimensional attractors for the singular Cahn-Hilliard system with usual boundary conditions (these techniques are strongly based on the fact that $u(t)$ is separated from the singularities for almost all $t \geq 0$, which is not true in our case in general). Instead, we prove the finite-dimensionality of the global attractor by using a proper modification of the techniques developed in [15] for porous media equations.

This paper is organized as follows. In Section 2, we define proper (regular) approximations of the singular potential and derive uniform (with respect to these approximations) a priori estimates which allow us, in Section 3, to formulate the variational inequality associated with the singular Cahn-Hilliard system with dynamic boundary conditions and verify the existence and uniqueness of a solution for this inequality. We also study the further regularity of the solutions. Then, in Section 4, we give sufficient conditions which ensure that the solutions are separated from the singularities of f and, thus, satisfy the equations in the usual (distribution) sense. Section 5 is devoted to the asymptotic behavior of the system. Finally, we give, in Appendix 1, several auxiliary results. We also construct a simple example which shows that the solutions may not satisfy the dynamic boundary conditions in the usual sense for logarithmic potentials.

2. Approximations and uniform a priori estimates

We consider the following equations (for simplicity, we set all constants equal to 1):

$$\begin{cases} \partial_t u = \Delta_x \mu, \quad \partial_n \mu\big|_\Gamma = 0, \\ \mu = -\Delta_x u + \tilde{f}(u) + h_1, \quad u\big|_{t=0} = u_0, \end{cases} \tag{2.1}$$

in a bounded smooth domain Ω of $\mathbb{R}^3$, endowed with dynamic boundary conditions on $\Gamma := \partial\Omega$,

$$\partial_t \psi - \Delta_\Gamma \psi + g(\psi) + \partial_n u = h_2, \quad \psi := u\big|_\Gamma. \tag{2.2}$$

Here, u and μ are unknown functions, Δ_x and Δ_Γ are the Laplace and Laplace-Beltrami operators on Ω and Γ, respectively, $\tilde{f}$ and g are known nonlinearities, $h_1 \in L^2(\Omega)$ and $h_2 \in L^2(\Gamma)$ are given external forces and ∂_n stands for the normal derivative, n being the unit outer normal to Γ.

We assume that the nonlinearity $\tilde{f}$ has the form

$$\tilde{f}(z) := f(z) - \lambda z, \tag{2.3}$$

where $\lambda \in \mathbb{R}$ is a given constant and the singular function f satisfies

$$\begin{cases} 1. \quad f \in C^2((-1,1)), \\ 2. \quad f(0) = 0, \quad \lim_{u\to\pm 1} f(u) = \pm\infty, \\ 3. \quad f'(u) \geq 0, \quad \lim_{u\to\pm 1} f'(u) = +\infty, \\ 4. \quad \operatorname{sgn} u \cdot f''(u) \geq 0. \end{cases} \tag{2.4}$$

Since the function f is defined on the interval $(-1,1)$ only and has singularities at ± 1, we a priori assume that

$$|u(t,x)| < 1 \text{ almost everywhere in } \mathbb{R}^+ \times \Omega. \tag{2.5}$$

We finally assume that the second nonlinearity g is regular on the segment $[-1,1]$,

$$g \in C^2([-1,1]). \tag{2.6}$$

Then, we can assume, without loss of generality, that g is smoothly extended to the whole line, $g \in C^2(\mathbb{R})$, and $g(z) = z + g_0(z)$ with $\|g_0\|_{C^2(\mathbb{R})} \leq C$ for some positive constant C.

In order to solve the singular problem (2.1), we approximate the nonlinearity f by the following family of smooth functions:

$$f_N(u) := \begin{cases} f(u), \quad |u| \leq 1 - 1/N, \\ f(1-1/N) + f'(1-1/N)(u-1+1/N), \quad u > 1-1/N, \\ f(-1+1/N) + f'(-1+1/N)(u+1-1/N), \quad u < -1+1/N, \end{cases} \tag{2.7}$$

and we set $\tilde{f}_N(u) := f_N(u) - \lambda u$. We then consider the approximate problems

$$\begin{cases} \partial_t u = \Delta_x \mu, \quad \partial_n \mu\big|_\Gamma = 0, \\ \mu = -\Delta_x u + \tilde{f}_N(u) + h_1, \quad u\big|_{t=0} = u_0, \end{cases} \tag{2.8}$$

endowed with the same dynamic boundary conditions (2.2).

The main aim of the present section is to derive several uniform (with respect to $N \to \infty$) a priori estimates for the solutions $(u, \mu) = (u_N, \mu_N)$ of problems (2.8), (2.2) which will allow us (in the next section) to pass to the limit $N \to \infty$ and establish the existence of a solution for the singular problem (the existence, uniqueness and regularity of solutions for the regular case, such as in the approximate problems (2.8), (2.2), are now well-understood and will not be considered in the present paper, see [24], [25], [35], [42] and [45] for detailed expositions and related problems).

As usual, it is convenient to rewrite problem (2.8) in an equivalent form by using the inverse Laplacian $A := (-\Delta_x)^{-1}$ (endowed with Neumann boundary conditions). To be more precise, since the first eigenvalue of the Laplacian with Neumann boundary conditions vanishes, we assume that the operator A is defined on the functions with zero mean value only and maps them onto the functions with zero mean value as well. Then, applying this operator to both sides of (2.8), we have

$$A\partial_t u := (-\Delta_x)^{-1}\partial_t u = \Delta_x u - \tilde{f}_N(u) - h_1 + \langle \mu \rangle\,, \tag{2.9}$$

where $\langle v \rangle$ stands for the mean value of the function v over Ω. Furthermore, taking into account (2.2), we see that

$$\langle \mu \rangle = -\langle \Delta_x u \rangle + \left\langle \tilde{f}_N(u) \right\rangle + \langle h_1 \rangle = \partial_t \langle u \rangle_\Gamma + \langle g(u) \rangle_\Gamma - \langle h_2 \rangle_\Gamma - \left\langle \tilde{f}_N(u) \right\rangle + \langle h_1 \rangle\,, \tag{2.10}$$

where $\langle v \rangle_\Gamma := \frac{1}{|\Omega|}\int_\Gamma v(x)dS$. We also mention that problem (2.8) possesses the mass conservation law

$$\langle u(t) \rangle \equiv \langle u(0) \rangle = c \tag{2.11}$$

and, thus, $\langle \partial_t u \rangle = 0$ and the left-hand side of (2.9) is well-defined. Finally, keeping in mind the singular limit $N \to \infty$, we only consider the initial data u_0 for which $c \in (-1, 1)$.

We start with the usual energy equality.

Lemma 2.1. *Let the above assumptions hold and let u be a sufficiently regular solution of* (2.9)*. Then, the following identity holds:*

$$\begin{aligned} \frac{d}{dt}\Big(\frac{1}{2}\|\nabla_x u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_\Gamma u(t)\|^2_{L^2(\Gamma)} + (\tilde{F}_N(u(t)), 1)_\Omega + (h_1, u(t))_\Omega + \\ + (G(u(t)), 1)_\Gamma - (h_2, u(t))_\Gamma\Big) + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)} = 0, \end{aligned} \tag{2.12}$$

where $\tilde{F}_N(z) := \int_0^z \tilde{f}_N(s)\,ds$, $G(z) := \int_0^z g(s)\,ds$, $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ stand for the inner products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively, and $\|z\|^2_{H^{-1}(\Omega)} := (Az, z)_\Omega$.

Indeed, multiplying (2.9) by $\partial_t u$, integrating over Ω and by parts and taking into account (2.2), together with the identity $\langle \partial_t u \rangle = 0$, we deduce (2.12).

Corollary 2.2. *Let the above assumptions hold and let, in addition, N be large enough. Then, any (sufficiently regular) solution u of problem* (2.9) *satisfies:*

$$\begin{aligned}(2.13)\quad &\|u(t)\|^2_{H^1(\Omega)}+\|u(t)\|^2_{H^1(\Gamma)}+(F_N(u(t)),1)_\Omega+\\ &\qquad+\int_0^t(\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}+\|\partial_t u(s)\|^2_{L^2(\Gamma)})\,ds\le\\ &\le C(\|u(0)\|^2_{H^1(\Omega)}+\|u(0)\|^2_{H^1(\Gamma)}+(F_N(u(0)),1)_\Omega+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}),\end{aligned}$$

where $F_N(z):=\int_0^z f_N(s)\,ds$ and the constant C is independent of t and $u(0)$.

Indeed, owing to our assumptions on f and the explicit form of the approximations f_N, see (2.7), we can easily show that

$$2F_N(z)+C\ge\tilde F_N(z)\ge\frac12F_N(z)-C \tag{2.14}$$

if $N\ge N_0(\lambda)$ is large enough, where the constant C only depends on λ. Integrating now (2.12) with respect to t and using (2.14), the fact that $g_0(u)$ is globally bounded and obvious estimates, we end up with (2.13).

As a next step, we obtain the *dissipative* analogue of estimate (2.13).

Lemma 2.3. *Let the assumptions of Lemma 2.1 hold, u be a sufficiently regular solution of* (2.9) *and N be large enough (depending on λ and $c=\langle u_0\rangle$). Then, the following estimate holds:*

$$\begin{aligned}(2.15)\quad &\|u(t)\|^2_{H^1(\Omega)}+\|u(t)\|^2_{H^1(\Gamma)}+(F_N(u(t)),1)_\Omega+\\ &\qquad+\int_t^{t+1}(\|\partial_t u(s)\|^2_{H^{-1}(\Omega)}+\|\partial_t u(s)\|^2_{L^2(\Gamma)}+\|f_N(u(s))\|_{L^1(\Omega)})\,ds\le\\ &\le C(\|u(0)\|^2_{H^1(\Omega)}+\|u(0)\|^2_{H^1(\Gamma)}+(F_N(u(0)),1)_\Omega)e^{-\alpha t}+C(1+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}),\end{aligned}$$

where the positive constants C and α are independent of N and u, but can depend on the value c in the mass conservation (2.11). *In addition, the following smoothing property holds:*

$$\begin{aligned}(2.16)\quad &\|u(t)\|^2_{H^1(\Omega)}+\|u(t)\|^2_{H^1(\Gamma)}+(F_N(u(t)),1)_\Omega\le\\ &\le Ct^{-1}(\|u(0)-c\|^2_{H^{-1}(\Omega)}+\|u(0)\|^2_{L^2(\Gamma)}+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}+1),\quad t\in(0,1],\end{aligned}$$

where the constant C is independent of N.

Proof. We have, owing to assumptions (2.4) on the nonlinearity f and the fact that $c\in(-1,1)$,

$$\tilde f_N(z).(z-c)\ge\alpha f_N(z)z-C\ge\alpha/2|f_N(z)|-C_1,\quad z\in\mathbb{R}, \tag{2.17}$$

where N is large enough and the positive constants α and C_i depend on c and λ, but are independent of N (see [35]). Multiplying now equation (2.9) by $\bar u(t):=u(t)-c$ and using the above inequality, we find

$$
\frac{1}{2}\frac{d}{dt}(\|\bar u(t)\|^2_{H^{-1}(\Omega)}+\|\bar u(t)\|^2_{L^2(\Omega)})+\alpha((f_N(u(t)),u(t))_\Omega+ \\ +\|\bar u(t)\|^2_{H^1(\Omega)}+\|\bar u(t)\|^2_{H^1(\Gamma)})\le C(1+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}), \tag{2.18}
$$

for some positive constants α and C. Applying the Gronwall inequality to this relation, we obtain

$$
\|\bar u(t)\|^2_{H^{-1}(\Omega)}+\|\bar u(t)\|^2_{L^2(\Gamma)}+ \\ +\int_t^{t+1}(\|\bar u(s)\|^2_{H^1(\Omega)}+\|\bar u(s)\|^2_{H^1(\Gamma)}+(f_N(u(s),u(s))_\Omega)\,ds\le \\ \le C(\|\bar u(0)\|^2_{H^{-1}(\Omega)}+\|\bar u(0)\|^2_{L^2(\Gamma)})e^{-\alpha t}+C(1+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}) \tag{2.19}
$$

for some positive constants C and α. In order to finish the proof of the lemma, there only remains to note that, owing to the monotonicity of the function f_N,

$$
F_N(z)\le f_N(z).z,\quad z\in\mathbb{R}. \tag{2.20}
$$

Then, the smoothing property (2.16) follows in a standard way from (2.13), (2.19) and (2.20) and the dissipative estimate (2.15) is an immediate consequence of the dissipative estimate (2.19) (in a weaker norm) and the smoothing property (2.16), together with (2.13). This finishes the proof of Lemma 2.3. □

We are now ready to obtain additional regularity on $\partial_t u(t)$. To this end, we differentiate equation (2.9) with respect to t and set $\theta(t):=\partial_t u(t)$. Then, this function solves

$$
(-\Delta_x)^{-1}\partial_t\theta=\Delta_x\theta-\tilde f'_N(u)\theta+\langle\partial_t\mu\rangle\,,\quad \theta\big|_{t=0}=\theta_0, \tag{2.21}
$$

where $\theta_0:=-\Delta_x(\Delta_x u_0-\tilde f_N(u_0)-h_1)$, and

$$
\partial_t\theta-\Delta_\Gamma\theta+\partial_n\theta+g'(u)\theta=0,\quad \text{on }\Gamma.
$$

Lemma 2.4. *Let the assumptions of Lemma 2.1 hold. Then, the following estimate is valid for the derivative $\theta(t):=\partial_t u(t)$:*

$$
\|\theta(t)\|^2_{H^{-1}(\Omega)}+\|\theta(t)\|^2_{L^2(\Gamma)}+\int_t^{t+1}(\|\theta(s)\|^2_{H^1(\Omega)}+\|\theta(s)\|^2_{H^1(\Gamma)})\,ds\le \\ \le C(\|u(0)\|^2_{H^1(\Omega)}+\|u(0)\|^2_{H^1(\Gamma)}+\|\theta(0)\|^2_{H^{-1}(\Omega)}+\|\theta(0)\|^2_{L^2(\Gamma)})e^{-\alpha t}+ \\ +C(1+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}), \tag{2.22}
$$

where the positive constants C and α can depend on the total mass c, but are independent of N. In addition, the following smoothing property holds:

$$
\|\theta(t)\|^2_{H^{-1}(\Omega)}+\|\theta(t)\|^2_{L^2(\Gamma)}\le \\ \le Ct^{-2}(\|u(0)-c\|^2_{H^{-1}(\Omega)}+\|u(0)\|^2_{L^2(\Gamma)}+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)}+1),\quad t\in(0,1], \tag{2.23}
$$

where the constant C is independent of N.

Proof. We multiply equation (2.21) by $\theta(t)$, integrate over Ω and use the fact that $\tilde{f}_N'(u) \geq -\lambda$. Then, using also the boundary conditions and the fact that $g'(u)$ is uniformly bounded, we find

$$\begin{aligned}(2.24)\quad \frac{d}{dt}(\|\theta(t)\|^2_{H^{-1}(\Omega)} + \|\theta(t)\|^2_{L^2(\Gamma)}) + \alpha(\|\theta(t)\|^2_{H^1(\Omega)} + \|\theta(t)\|^2_{H^1(\Gamma)}) \leq \\ \leq C(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)}),\end{aligned}$$

for some positive constants α and C which are independent of N. Interpolating between H^{-1} and H^1 and applying the Gronwall inequality to this relation, we obtain the desired estimate (2.22). Combining this estimate with (2.15) and (2.16) and arguing in a standard way, we end up with (2.23) and finish the proof of the lemma. □

The next lemma gives H^1-estimates on the solutions for every fixed time $t \geq 0$.

Lemma 2.5. *Let the above assumptions hold. Then, for every fixed $t \geq 0$, the following estimate holds:*

$$\begin{aligned}(2.25)\quad \|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{H^1(\Gamma)} + \|f_N(u(t))\|_{L^1(\Omega)} \leq \\ \leq C(1 + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)} + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}),\end{aligned}$$

where the constant C depends on c, but is independent of t and N.

Indeed, multiplying equation (2.9) by $\bar{u}(t) := u(t) - c$ and arguing as in the derivation of (2.18) (but now without integrating with respect to t), we deduce the desired estimate (2.25). Here, we have used the inequality (2.17) again.

Furthermore, using (2.25) and expression (2.10) for the mean value of μ, we have

$$(2.26)\qquad |\langle \mu(t)\rangle| \leq C(1 + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)} + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}).$$

We finally rewrite equation (2.9) in the form of a nonlinear elliptic problem,

$$(2.27)\qquad \begin{cases} \Delta_x u(t) - f_N(u(t)) - u(t) = \tilde{h}_1(t) := h_1 - \\ \qquad\qquad - u(t) - \lambda u(t) + (-\Delta_x)^{-1}\partial_t u(t) - \langle \mu(t)\rangle, \quad \text{in } \Omega, \\ \Delta_\Gamma u(t) - u(t) - \partial_n u(t) = \tilde{h}_2(t) := -h_2 + g_0(u(t)) + \partial_t u(t), \quad \text{on } \Gamma, \end{cases}$$

for every fixed t and note that the estimates derived above yield the following control of the right-hand sides in (2.27):

$$\begin{aligned}(2.28)\quad \|\tilde{h}_1(t)\|_{L^2(\Omega)} + \|\tilde{h}_2(t)\|_{L^2(\Gamma)} \leq \\ \leq C(1 + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)} + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)})\end{aligned}$$

for some positive constant C which is independent of N.

Therefore, additional smoothness on the solution $u := u_N$ can be obtained by a proper elliptic regularity theorem (see [35]). Unfortunately, in contrast to the case of regular potentials, this problem does not satisfy the maximal regularity estimate in L^2 for singular potentials f, see Appendix 1. Nevertheless, the partial regularity formulated below is crucial for what follows.

Lemma 2.6. *Let the above assumptions hold and set $\Omega_\varepsilon := \{x \in \Omega,\ \ d(x,\Gamma) > \varepsilon\}$. Denote by $n = n(x)$ some smooth extension of the unit normal vector field at the boundary inside the domain Ω. Let also $D_\tau u := \nabla_x u - (\partial_n u)n$ be the tangential part of the gradient $\nabla_x u$. Then, for every $\varepsilon > 0$, the following estimate holds:*

$$\|u(t)\|_{C^\alpha(\Omega)} + \|\nabla_x D_\tau u(t)\|_{L^2(\Omega)} + \|u(t)\|_{H^2(\Omega_\varepsilon)} + \|u(t)\|_{H^2(\Gamma)} \leq \\ \leq C_\varepsilon(\|\tilde{h}_1(t)\|_{L^2(\Omega)} + \|\tilde{h}_2(t)\|_{L^2(\Gamma)}) \tag{2.29}$$

for some positive constants α and C_ε which are independent of N.

The proof of this estimate is based on some variant of the nonlinear localization technique and is given in Appendix 1 (see Theorem 6.1).

We summarize the a priori estimates obtained so far in the following theorem which is the main result of this section.

Theorem 2.7. *Let the above assumptions hold and let u be a sufficiently regular solution of problem (2.9) with a sufficiently large N (depending on the constant λ and the total mass $c \in (-1, 1)$). Then, the following estimate is valid for every $\varepsilon > 0$:*

$$\|u(t)\|^2_{C^\alpha(\Omega)} + \|u(t)\|^2_{H^2(\Gamma)} + \|u(t)\|^2_{H^2(\Omega_\varepsilon)} + \|u(t)\|^2_{H^1(\Omega)} + \\ + \|\partial_t u(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Gamma)} + \\ + \|\nabla_x D_\tau u(t)\|^2_{L^2(\Omega)} + \|f_N(u(t))\|_{L^1(\Omega)} + \int_t^{t+1} (\|\partial_t u(s)\|^2_{H^1(\Omega)} + \|\partial_t u(s)\|^2_{H^1(\Gamma)})\, ds \leq \\ \leq C(1 + \|u(0)\|^2_{H^1(\Omega)} + \|u(0)\|^2_{H^1(\Gamma)} + \|\partial_t u(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Gamma)})^2 e^{-\beta t} + \\ + C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)})^2, \tag{2.30}$$

where the positive constants α, β and C (which can depend on ε) are independent of $N \to \infty$. In addition, the following smoothing property holds:

$$\|\partial_t u(t)\|_{H^{-1}(\Omega)} + \|\partial_t u(t)\|_{L^2(\Gamma)} \leq Ct^{-1}(\|u(0) - c\|_{H^{-1}(\Omega)} + \|u(0)\|_{L^2(\Gamma)} + \\ + \|h_1\|_{L^2(\Omega)} + \|h_2\|_{L^2(\Gamma)} + 1), \quad t \in (0, 1], \tag{2.31}$$

where the constant C is also uniform with respect to $N \to \infty$.

Remark 2.8. We thus have a uniform H^2-estimate on the solution u *inside* the domain and an L^2-estimate on the gradient of the tangential derivatives $\nabla_x D_\tau u$. In contrast to this, we do not have a uniform control of the second normal derivative $\partial_n^2 u$ close to the boundary. Nevertheless, since the L^1-norm of the nonlinearity f_N is controlled, (2.27), together with the control of the tangential derivatives, allow us to estimate the L^1-norm of $\partial_n^2 u$. We thus have the control

$$\|u(t)\|_{W^{2,1}(\Omega)} \leq C(1 + \|\tilde{h}_1\|_{L^2(\Omega)} + \|\tilde{h}_2\|_{L^2(\Gamma)}). \tag{2.32}$$

This, in turn, gives a control of the L^1-norm of the normal derivative $\partial_n u$ at the boundary (owing to a proper trace theorem),

$$\|\partial_n u(t)\|_{L^1(\Gamma)} \leq C\|u(t)\|_{W^{2,1}(\Omega)}. \tag{2.33}$$

As we will see in the next section, estimates (2.32) and (2.33) remain true for the limit (as $N \to \infty$) solution u of the singular problem as well and, consequently, the trace of $\partial_n u(t)$ at the boundary is well-defined. However, owing to the nonreflexivity of L^1-spaces, this trace *may not coincide* with the limit of $\partial_n u_N(t)\big|_\Gamma$ computed on the boundary by using the dynamic boundary condition (2.2). Therefore, the boundary condition (2.2) may be violated for the limit singular solution. As we will see below, this indeed happens, even in the 1D case with smooth data. We overcome this difficulty by using monotonicity arguments and a proper variational formulation of problem (2.9), see Section 3.

We conclude this section by establishing the standard uniform Lipschitz continuity of the solution u of problem (2.9) with respect to the initial data.

Proposition 2.9. *Let the above assumptions hold and let $u_1(t)$ and $u_2(t)$ be two (sufficiently regular) solutions of problem* (2.9) *such that*

$$\langle u_1(0)\rangle = \langle u_2(0)\rangle = c.$$

Then, the following estimate holds:

$$\|u_1(t)-u_2(t)\|_{H^{-1}(\Omega)} + \|u_1(t)-u_2(t)\|_{L^2(\Gamma)} \le \\ \le C(\|u_1(0)-u_2(0)\|_{H^{-1}(\Omega)} + \|u_1(0)-u_2(0)\|_{L^2(\Gamma)})e^{Kt}, \tag{2.34}$$

where the constants C and K are independent of t, N, u_1 and u_2.

Proof. Let $v(t) = u_1(t) - u_2(t)$. Then, this function solves

$$\begin{cases} (-\Delta_x)^{-1}\partial_t v - \Delta_x v + [\tilde f_N(u_1) - \tilde f_N(u_2)] = \langle \mu_1 - \mu_2\rangle, & \text{in } \Omega,\\ \partial_t v - \Delta_\Gamma v + \partial_n v + [g(u_1) - g(u_2)] = 0, & \text{on } \Gamma. \end{cases} \tag{2.35}$$

Taking the scalar product of the first equation with $v(t)$, integrating by parts and using the facts that $\langle v(t)\rangle = 0$, $\tilde f_N'(z) \ge -\lambda$ and the nonlinearity g' is globally bounded, we obtain

$$\frac{d}{dt}(\|v(t)\|^2_{H^{-1}(\Omega)} + \|v(t)\|^2_{L^2(\Gamma)}) + \\ + \alpha(\|v(t)\|^2_{H^1(\Omega)} + \|v(t)\|^2_{H^1(\Gamma)}) \le |\lambda|\|v(t)\|^2_{L^2(\Omega)} + C\|v(t)\|^2_{L^2(\Gamma)} \tag{2.36}$$

for some positive constants α and C which are independent of N. Interpolating between H^{-1} and H^1 in order to estimate the L^2-norm of v in Ω and applying the Gronwall inequality, we find (2.34) and finish the proof of the proposition. □

3. The singular problem: variational formulation and well-posedness

The aim of this section is to pass to the limit $N \to \infty$ in (2.9) and prove the existence and uniqueness of solutions of the limit singular problem (2.1). As already mentioned, this limit solution is not necessarily a usual distribution solution of the equations and we need to define it in a proper way. To this end, we first fix a constant $L > 0$ such that

$$\|\nabla_x u\|^2_{L^2(\Omega)} - \lambda\|u\|^2_{L^2(\Omega)} + L\|u\|^2_{H^{-1}(\Omega)} \ge 1/2\|u\|^2_{H^1(\Omega)} \tag{3.1}$$

for all $u \in H^1(\Omega)$ with $\langle u\rangle = 0$ and introduce the quadratic form

$$B(u,v) := (\nabla_x u, \nabla_x v)_\Omega - \lambda(u,v)_\Omega + L((-\Delta_x)^{-1}\bar u, \bar v)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma, \tag{3.2}$$

where $u, v \in H^1(\Omega) \cap H^1(\Gamma) := \{u,\ u \in H^1(\Omega),\ u|_\Gamma \in H^1(\Gamma)\}$ (and a similar definition holds for similar spaces) and $\bar u := u - \langle u\rangle$, $\bar v = v - \langle \bar v\rangle$. Then, obviously,

$$B(u,\bar u) = B(\bar u,\bar u) \geq 0 \tag{3.3}$$

for all $u \in H^1(\Omega) \cap H^1(\Gamma)$.

The limit problem (2.9), corresponding to $N = \infty$, reads, *formally*,

$$\begin{cases} (-\Delta_x)^{-1}\partial_t u = \Delta_x u - f(u) + \lambda u + \langle\mu\rangle - h_1, \quad \text{in } \Omega, \\ \mu := -\Delta_x u + f(u) - \lambda u + h_1, \quad u\big|_\Gamma = \psi, \\ \partial_t\psi - \Delta_\Gamma\psi + g(\psi) + \partial_n u = h_2, \quad \text{on } \Gamma, \\ u\big|_{t=0} = u_0, \quad \psi\big|_{t=0} = \psi_0. \end{cases} \tag{3.4}$$

We test the first equation (again formally) with the function $u - v$, where $v = v(t,x)$ is smooth and satisfies

$$\langle u(t) - v(t)\rangle \equiv 0.$$

Then, after an integration by parts, we have

$$\begin{aligned}(A\partial_t u, u-v)_\Omega + (\partial_t u, u-v)_\Gamma + B(u,u-v) + (f(u),u-v)_\Omega = \\ = L(Au,u-v)_\Omega - (g(u),u-v)_\Gamma - (h_1,u-v)_\Omega + (h_2,u-v)_\Gamma.\end{aligned}$$

Finally, since B is positive and f is monotone, we have

$$B(u,u-v) \geq B(v,u-v), \quad (f(u),u-v)_\Omega \geq (f(v),u-v)_\Omega,$$

which yields

$$\begin{aligned}(A\partial_t u, u-v)_\Omega + (\partial_t u, u-v)_\Gamma + B(v,u-v) + (f(v),u-v)_\Omega \leq \\ \leq L(Au,u-v)_\Omega - (g(u),u-v)_\Gamma - (h_1,u-v)_\Omega + (h_2,u-v)_\Gamma.\end{aligned}$$

We recall that this inequality holds (again formally) for any properly chosen test function v such that $\langle v(\tau)\rangle \equiv c$. We are now ready to define a variational solution of the limit problem (3.4).

Definition 3.1. Let

$$(u_0,\psi_0) \in \Phi := \{(u,\psi) \in L^\infty(\Omega) \cap L^\infty(\Gamma), \quad \|u\|_{L^\infty(\Omega)} \leq 1, \quad \|\psi\|_{L^\infty(\Gamma)} \leq 1\}. \tag{3.5}$$

A pair of functions (u,ψ), $u = u(t,x)$, $x \in \Omega$, $\psi = \psi(t,x)$, $x \in \Gamma$, is a variational solution of problem (3.4) if

$$u(t)\big|_\Gamma = \psi(t) \ \text{ for almost all } t > 0, \ u(0) = u_0, \quad \psi(0) = \psi_0, \tag{3.6}$$

$$\begin{aligned} &1) \quad -1 < u(t,x) < 1 \ \text{ for almost all } (t,x) \in \mathbb{R}^+ \times \Omega, \\ &2) \ (u,\psi) \in C([0,\infty), H^{-1}(\Omega) \times L^2(\Gamma)) \cap L^2([0,T], H^1(\Omega) \times H^1(\Gamma)), \ \forall T > 0, \\ &3) \ f(u) \in L^1([0,T]\times\Omega), \ (\partial_t u, \partial_t\psi) \in L^2([\tau,T], H^{-1}(\Omega) \times L^2(\Gamma)), \ \forall T > \tau > 0, \end{aligned} \tag{3.7}$$

$\langle u(t)\rangle \equiv \langle u(0)\rangle$ and the variational inequality

$$\begin{aligned}(A\partial_t u(t), u(t)-w)_\Omega + (\partial_t u(t), u(t)-w)_\Gamma + B(v,u(t)-w) + (f(w),u(t)-w)_\Omega \leq \\ \leq L(Au(t),u(t)-w)_\Omega - (g(u(t)),u(t)-w)_\Gamma - (h_1,u(t)-w)_\Omega + (h_2,u(t)-w)_\Gamma\end{aligned} \tag{3.8}$$

is satisfied for almost every $t > 0$ and every test function $w = w(x)$ such that

$$w \in H^1(\Omega) \cap H^1(\Gamma), \quad f(w) \in L^1(\Omega)$$

and $\langle w\rangle = \langle u(0)\rangle$. Note that the relation $u(t)\big|_\Gamma = \psi(t)$ is assumed to hold *only* for $t > 0$. At the initial time $t = 0$, no relation between u_0 and ψ_0 is assumed. However, for $t > 0$ the function ψ can be found if u is known. Therefore, we can indeed write the variational inequality (3.8) in terms of the function u only.

Before studying the existence and uniqueness of variational solutions, it is convenient to rewrite the variational inequality in terms of test functions $v = v(t, x)$ depending on t and x. More precisely, let the test function v satisfy the regularity assumptions (3.7) and $\langle v(t)\rangle \equiv \langle u_0\rangle = c$ (we will call this class of functions *admissible test functions* below). Then, we can write inequality (3.8) with $w = v(t)$ for almost all $t > 0$. Moreover, due to the regularity assumptions (3.7) on u and v, we see that all terms obtained are in L^1 with respect to t. Thus, we can integrate this inequality with respect to t, which gives

$$\begin{aligned}
(3.9)\quad & \int_s^t [(A\partial_t u, u - v)_\Omega + (\partial_t u, u - v)_\Gamma]d\,\tau + \\
& \qquad + \int_s^t [B(v, u - v) + (f(v), u - v)_\Omega]\, d\tau \leq \\
& \leq \int_s^t [L(u, A(u - v))_\Omega - (g(u), u - v)_\Gamma - (h_1, u - v)_\Omega + (h_2, u - v)_\Gamma]\, d\tau
\end{aligned}$$

for all $t > s > 0$.

The next theorem gives the uniqueness of such variational solutions.

Theorem 3.2. *Let the nonlinearities f and g and the external forces h_1 and h_2 satisfy the assumptions of Section 2. Then, the variational solution of problem* (3.4) *(in the sense of Definition 3.1) is unique and is independent of the choice of L satisfying* (3.1). *Furthermore, for every two variational solutions u_1 and u_2 such that $\langle u_1(0)\rangle = \langle u_2(0)\rangle$, the following estimate holds:*

$$\begin{aligned}
(3.10)\quad & \|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} + \|\psi_1(t) - \psi_2(t)\|_{L^2(\Gamma)} \leq \\
& \qquad \leq Ce^{Kt}(\|u_1(0) - u_2(0)\|_{H^{-1}(\Omega)} + \|\psi_1(0) - \psi_2(0)\|_{L^2(\Gamma)}),
\end{aligned}$$

where the constants C and K are independent of t, u_1 and u_2.

Proof. We first need to deduce one more variational inequality for a solution u. Let w be a test function satisfying the assumptions of Definition 3.1 and set

$$v_\alpha := (1 - \alpha)u + \alpha w, \quad \alpha \in [0, 1].$$

Then, owing to assumption (2.4)(4), the function $|f(u)|$ is convex and, therefore,

$$|f(v_\alpha)| \leq |f(u)| + |f(w)|.$$

Consequently, v_α is an admissible test function for every $\alpha \in [0, 1]$. Inserting $v = v_\alpha$ in the variational inequality (3.9), dividing it by α and using the fact that $u \in AC([s, t], H^{-1}(\Omega)\cap$

$L^2(\Omega)$) (here, AC stands for absolutely continuous), we see that

$$\begin{aligned}(3.11)\quad \int_s^t[(A\partial_t u, u-w)_\Omega+(\partial_t u, u-w)_\Gamma]\,d\tau+\\ +\int_s^t[B(v_\alpha, u-w)+(f(v_\alpha), u-w)_\Omega]\,d\tau\le\\ \le\int_s^t[L(u, A(u-w))_\Omega-(g(u), u-w)_\Gamma-(h_1, u-w)_\Omega+(h_2, u-w)_\Gamma]\,d\tau.\end{aligned}$$

Passing to the limit $\alpha\to 0$ in (3.11) and using the Lebesgue dominated convergence theorem for the nonlinear term, we end up with the desired additional vartiational inequality, namely,

$$\begin{aligned}(3.12)\quad \int_s^t[(A\partial_t u, u-w)_\Omega+(\partial_t u, u-w)_\Gamma]\,d\tau+\\ +\int_s^t[B(u, u-w)+(f(u), u-w)_\Omega]\,d\tau\le\\ \le\int_s^t[L(u, A(u-w))_\Omega-(g(u), u-w)_\Gamma-(h_1, u-w)_\Omega+(h_2, u-w)_\Gamma]\,d\tau,\end{aligned}$$

where $w=w(t,x)$ is an arbitrary admissible test function.

We are now ready to prove the uniqueness. Let u_1 and u_2 be two variational solutions of problem (3.4). We consider the variational inequality (3.9) with $u=u_1$ and $v=u_2$, together with the additional variational inequality (3.12) with $u=u_2$ and $w=u_1$ (this makes sense, since u_1 and u_2 are admissible test functions), and sum the two resulting inequalities. Then, the terms containing B, f, h_1 and h_2 vanish and, using, in addition, the fact that $u_i\in AC([s,t], H^{-1}(\Omega)\cap L^2(\Gamma))$, $i=1,2$, we end up with the following inequality:

$$\begin{aligned}(3.13)\quad \frac12(\|u_1(t)-u_2(t)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}-\|u_1(s)-u_2(s)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)})\le\\ \le\int_s^t[L\|u_1(\tau)-u_2(\tau)\|^2_{H^{-1}(\Omega)}-(g(u_1(\tau))-g(u_2(\tau)), u_1(\tau)-u_2(\tau))_\Gamma]\,d\tau,\end{aligned}$$

where $\|u\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}=\|u\|^2_{H^{-1}(\Omega)}+\|u\|^2_{L^2(\Gamma)}$. Using now the fact that $g\in C^1([-1,1])$ and applying the Gronwall inequality to (3.13), we see that

$$\|u_1(t)-u_2(t)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}\le Ce^{K(t-s)}\|u_1(s)-u_2(s)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}$$

for some positive constants C and K which are independent of $t>s>0$ and u_i, $i=1,2$. Passing to the limit $s\to 0$ in this estimate and using the continuity (3.7)(2) of u_1 and u_2, we deduce the desired estimate (3.10) which, in particular, gives the uniqueness.

Thus, we only need to prove that the above definition of a solution is independent of the choice of L. To this end, we assume that u_1 is a variational solution for $L=L_1$ and u_2 is a variational solution for $L=L_2$. Let also $u_1(0)=u_2(0)$. Using then the obvious relation

$$B_{L_1}(v,u_1-v)-L_1(u_1,A(u_1-v))_\Omega=\\ =B_{L_2}(v,u_1-v)-L_2(u_1,A(u_1-v))_\Omega-(L_1-L_2)\|u_1-v\|^2_{H^{-1}(\Omega)}$$

and arguing exactly as in the proof of (3.10), we have

$$\frac{1}{2}(\|u_1(t)-u_2(t)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}-\|u_1(s)-u_2(s)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)})\le\\ \le\int_s^t[L_1\|u_1(\tau)-u_2(\tau)\|^2_{H^{-1}(\Omega)}-(g(u_1(\tau))-g(u_2(\tau)),u_1(\tau)-u_2(\tau))_\Gamma]\,d\tau, \tag{3.14}$$

which coincides with (3.13) and, therefore, also leads to estimate (3.10). Thus, $u_1\equiv u_2$ and Theorem 3.2 is proved. □

We are now able to prove the existence of a variational solution u of problem (3.4) by passing to the limit $N\to\infty$ in equations (2.9).

Theorem 3.3. *Let the assumptions of the previous theorem hold. Then, for every pair $(u_0,\psi_0)\in\Phi$, problem* (3.4) *possesses a unique variational solution (u,ψ) in the sense of Definition 3.1. Furthermore, this solution regularizes as $t>0$ and all the uniform estimates obtained in Section 2 hold for the solutions of the the limit singular equation* (3.4)*. In particular, the following estimate is valid for every $\varepsilon>0$:*

$$\|u(t)\|^2_{C^\alpha(\Omega)}+\|u(t)\|^2_{H^2(\Gamma)}+\\ +\|u(t)\|^2_{H^2(\Omega_\varepsilon)}+\|u(t)\|^2_{H^1(\Omega)}+\|\partial_t u(t)\|^2_{H^{-1}(\Omega)}+\|\partial_t u(t)\|^2_{L^2(\Gamma)}+\\ +\|\nabla_x D_\tau u(t)\|^2_{L^2(\Omega)}+\|f(u(t))\|_{L^1(\Omega)}+\int_t^{t+1}(\|\partial_t u(s)\|^2_{H^1(\Omega)}+\|\partial_t u(s)\|^2_{H^1(\Gamma)})\,ds\le\\ \le C\frac{t^4+1}{t^4}(1+\|h_1\|^2_{L^2(\Omega)}+\|h_2\|^2_{L^2(\Gamma)})^2,\quad t>0, \tag{3.15}$$

for some positive constants α and C which are independent of t and u (we recall that $\Omega_\varepsilon:=\{x\in\Omega,\ d(x,\Gamma)>\varepsilon\}$), where $D_\tau u$ denotes the tangential part of $\nabla_x u$ (see Lemma 2.6); in addition, all norms in the left-hand side of (3.15) *make sense for any variational solution u.*

Proof. Let u_N be the solution of the approximate problem (2.9). Then, repeating the derivation of the variational inequality (3.9), we see that

$$\int_s^t[(A\partial_t u_N,u_N-v)_\Omega+(\partial_t u_N,u_N-v)_\Gamma]d\,\tau+\\ +\int_s^t(B(v,u_N-v)+(f_N(v),u_N-v)_\Omega)\,d\tau\le\\ \le\int_s^t[L(u_N,A(u_N-v))_\Omega-(g(u_N),u_N-v)_\Gamma-(h_1,u_N-v)_\Omega+(h_2,u_N-v)_\Gamma]\,d\tau \tag{3.16}$$

for every admissible test function v and every $t > s > 0$ (we recall that the solution u_N of the regularized problem (2.9) is smooth and all the formal calculations performed in the derivation of (3.8) can be easily justified in that case).

Our aim is to pass to the limit $N \to \infty$ in (3.16). We start with the case where the initial datum u_0 is smooth and satisfies the additional conditions

$$|u_0(x)| \le 1 - \delta, \quad \delta > 0, \quad \psi_0 := u_0\big|_{\Gamma}. \tag{3.17}$$

Then, according to Theorem 2.7, the sequence u_N satisfies the uniform estimate (2.30) and, therefore, we can assume, without loss of generality, that u_N converges to some limit function u in the following sense:

$$\begin{aligned} &1) \quad u_N \to u \ \text{ weakly-}* \text{ in } L^\infty([0,T],(H^1(\Omega)\cap H^2(\Gamma))\cap H^2(\Omega_\varepsilon)),\\ &2) \quad \partial_t u_N \to \partial_t u \ \text{ weakly-}* \text{ in } \ L^\infty([0,T],H^{-1}(\Omega)\cap L^2(\Gamma))\\ &\qquad\qquad\qquad\qquad \text{and weakly in } L^2([0,T],H^1(\Omega)\cap H^1(\Gamma)),\\ &3) \quad D^2_\tau u_N \to D^2_\tau u \ \text{ weakly-}* \text{ in } L^\infty([0,T],L^2(\Omega)),\\ &4) \quad u_N \to u \ \text{ strongly in } C^\gamma([0,T]\times\Omega) \text{ for some } \gamma > 0. \end{aligned} \tag{3.18}$$

Indeed, the last strong convergence follows from the facts that u_N is uniformly bounded in $L^\infty([0,T],C^\alpha(\Omega))$, $\alpha > 0$, and $\partial_t u_N$ is uniformly bounded in $L^\infty([0,T],H^{-1}(\Omega))$ (owing to Theorem 2.7 and the assumption that the initial datum u_0 is smooth and is separated from the singularities ± 1).

These convergence results allow us to pass to the limit $N \to \infty$ in (3.16) and prove that the limit function satisfies (3.9) for any admissible function v. The only nontrivial term containing the nonlinearity f_N can be treated by using the inequality

$$|f_N(v)| \le |f(v)|,$$

the fact that $f(v) \in L^1([0,T]\times\Omega)$ and the Lebesgue dominated convergence theorem.

Thus, we only need to show that the function u thus constructed satisfies the regularity assumptions (3.7). The only nontrivial statements that we need to prove are that (3.7)(1) holds and $f(u) \in L^1([0,T]\times\Omega)$ (the other ones are immediate consequences of (3.18)). Let us check the first one. Since the L^1-norm of $f_N(u_N)$ is uniformly bounded, we conclude from the expression of the function f_N that

$$\operatorname{meas}\{(t,x) \in [T,T+1]\times\Omega,\ |u_M(t,x)| > 1 - \frac{1}{N}\} \le \varphi(\frac{1}{N}), \quad M \ge N, \tag{3.19}$$

where

$$\varphi(x) := \frac{C}{\max\{|f(1-x)|,|f(x-1)|\}} \tag{3.20}$$

for some constant C which is independent of $T \in \mathbb{R}^+$, $M \ge N$ and $N \in \mathbb{N}$. Thus, passing to the limit $M, N \to \infty$ in (3.19) and using the fact that $\varphi(x) \to 0$ as $x \to 0$, we conclude that

$$\operatorname{meas}\{(t,x) \in [T,T+1]\times\Omega,\ |u(t,x)| = 1\} = 0$$

and (3.7)(1) is verified. In order to prove that $f(u)$ is integrable, there only remains to note that the already proved statement (3.7)(1), together with the convergence (3.18)(4), imply the almost everywhere convergence $f_N(u_N) \to f(u)$ and, therefore, owing to the Fatou lemma,

$$\|f(u)\|_{L^1([T,T+1]\times\Omega)} \le \liminf_{N\to\infty} \|f_N(u_N)\|_{L^1([T,T+1]\times\Omega)} < \infty. \tag{3.21}$$

Thus, u is indeed the desired variational solution of (3.4) and estimate (3.15) immediately follows from (3.18) and Theorem 2.7 (in order to deduce the L^1-estimate on $f(u)$, one needs to use, in addition, inequality (3.21)).

Finally, we are now able to remove assumption (3.17). To this end, we approximate the initial datum $(u_0, \psi_0) \in \Phi$ by a sequence (u_0^k, ψ_0^k) of smooth functions satisfying (3.17) (of course, with $\delta = \delta_k$ which can tend to zero as $k \to \infty$) in such a way that

$$\|u_0 - u_0^k\|_{L^2(\Omega)} \to 0, \quad \|u_0^k\big|_\Gamma - \psi_0\|_{L^2(\Gamma)} \to 0, \quad \langle u_0^k \rangle \equiv \langle u_0 \rangle . \tag{3.22}$$

Let $(u_k(t), \psi_k(t))$ (where $\psi_k = u_k\big|_\Gamma$) be a sequence of variational solutions of problem (3.4) satisfying $(u_k(0), \psi_k(0)) = (u_0^k, \psi_0^k)$ (whose existence is proved above). Then, owing to the uniform Lipschitz continuity estimate (3.10) and assumption (3.22), (u_k, ψ_k) is a Cauchy sequence in $C([0,T], H^{-1}(\Omega) \times L^2(\Gamma))$ and, therefore, the limit function

$$(u, \psi) := \lim_{k\to\infty} (u_k, \psi_k)$$

exists and also belongs to $C([0,T], H^{-1}(\Gamma) \times L^2(\Omega))$. The fact that u is a variational solution of (3.4), as well as estimate (3.15), can be verified, based on the uniform estimates derived in Section 2, exactly as was done above for smooth initial data. This finishes the proof of Theorem 3.3. □

Corollary 3.4. *Under the assumptions of Theorem 3.3, equation* (3.4) *generates a solution semigroup $S(t)$ in the phase space Φ,*

$$S(t)(u_0, \psi_0) := (u(t), \psi(t)), \quad S(t) : \Phi \to \Phi, \quad t \ge 0, \tag{3.23}$$

where $(u(t), \psi(t))$ is the unique variational solution of problem (3.4) *with initial datum (u_0, ψ_0). Furthermore, this semigroup is globally Lipschitz continuous,*

$$\|S(t)(u_0^1, \psi_0^1) - S(t)(u_0^2, \psi_0^2)\|_{H^{-1}(\Omega)\times L^2(\Gamma)} \le Ce^{Kt} \|(u_0^1 - u_0^2, \psi_0^1 - \psi_0^2)\|_{H^{-1}(\Omega)\times L^2(\Gamma)}, \tag{3.24}$$

in the metric of the space $\Phi^w := H^{-1}(\Omega) \times L^2(\Omega)$.

We now start investigating the analytic structure of a solution $(u(t), \psi(t))$ of problem (3.4) (this will be continued in Section 4).

Proposition 3.5. *Let $(u(t), \psi(t))$ be a variational solution of problem* (3.4) *constructed in Theorem 3.3. Then, $\psi(t) = u(t)\big|_\Gamma$ for $t > 0$ and, for every $\varphi \in C_0^\infty((0,T)\times\Omega)$ such that $\langle \varphi(t) \rangle \equiv 0$, there holds*

$$\begin{aligned} &\int_{\mathbb{R}^+} ((-\Delta_x)^{-1}\partial_t u(t), \varphi(t))_\Omega \, dt = \\ &= \int_{\mathbb{R}^+} ((\Delta_x u(t), \varphi(t))_\Omega - (f(u(t)), \varphi(t))_\Omega + \lambda(u(t), \varphi(t))_\Omega - (h_1, \varphi(t))_\Omega) \, dt. \end{aligned} \tag{3.25}$$

Furthermore,

$$u \in L^\infty([\tau, T], W^{2,1}(\Omega)), \quad T > \tau > 0, \tag{3.26}$$

and the trace of the normal derivative on the boundary,

$$[\partial_n u]_{int} := \partial_n u\big|_\Gamma \in L^\infty([\tau,T],L^1(\Gamma)),\quad T>\tau>0, \tag{3.27}$$

exists.

Proof. Since, according to Theorem 2.7, the approximating sequence u_N is uniformly bounded in $L^\infty([\tau,T],H^2(\Omega_\varepsilon))$, for any $\varepsilon>0$, the sequence $f_N(u_N)$ is also uniformly bounded in $L^\infty([\tau,T],L^2(\Omega_\varepsilon))$. This fact, together with the almost everywhere convergence established in the proof of Theorem 3.3, guarantee that $f_N(u_N)\to f(u)$ weakly in $L^2([\tau,T]\times\Omega_\varepsilon)$ for all $\varepsilon>0$ and this, in turn, allows us to verify identity (3.25) by passing to the limit in the analogous identity for the approximate solutions u_N.

In order to check the remaining statements of the proposition, we first deduce from (3.25) that

$$(-\Delta_x)^{-1}\partial_t u(t) = \Delta_x u(t) - f(u(t)) + \lambda u(t) - h_1 + c(t) \tag{3.28}$$

for some function $c\in L^\infty([\tau,T])$, $T>\tau>0$. At this point, equality (3.28) is understood as an equality in $L^2_{\rm loc}([\tau,T]\times\Omega)$. However, owing to estimate (3.15), $f(u)\in L^\infty([\tau,T],L^1(\Omega))$ and the term $(-\Delta_x)^{-1}\partial_t u$ also belongs at least to this space. Thus, we see that

$$\Delta_x u\in L^\infty([\tau,T],L^1(\Omega))$$

and, consequently, since $\nabla_x D_\tau u$ is controlled by (3.15), we can finally conclude that (3.26) holds, which gives the existence of the trace (3.27) and finishes the proof of Proposition 3.5. $\square$

Note that, using the obvious fact that $\langle(-\Delta_x)^{-1}\partial_t u\rangle=0$, we can find the explicit formula for the function $c(t)$ in (3.28), namely,

$$c(t) = \langle\Delta_x u(t) - f(u(t)) + \lambda u(t) - h_1\rangle = -\langle\mu(t)\rangle \tag{3.29}$$

and, therefore, the first equation of (3.4) is satisfied in a usual sense (say, as an equality in $L^2_{loc}([\tau,T]\times\Omega)$ or/and almost everywhere).

We now investigate the third equation of (3.4) (the equation on the boundary). According to Theorem 2.7, we see that the approximating sequence $(u_N(t),\psi_N(t))$ satisfies

$$\|\partial_t\psi_N(t)\|_{L^\infty([\tau,T],L^2(\Gamma))} + \|\psi_N(t)\|_{L^2([\tau,T],H^2(\Gamma))}\le C$$

and, therefore, using the fact that the approximate solutions satisfy the second equation of (3.4), we can assume, without loss of generality, that we have the convergence

$$[\partial_n u]_{ext} := \lim_{N\to\infty}\partial_n u_N\big|_\Gamma \in L^\infty([\tau,T],L^2(\Gamma)),\quad T>\tau>0, \tag{3.30}$$

where the limit is understood as a weak-star limit in $L^\infty([\tau,T],L^2(\Gamma))$. Then, obviously,

$$\partial_t\psi - \Delta_\Gamma\psi + g(\psi) + [\partial_n u]_{ext} = h_2,\quad \text{on } \Gamma, \tag{3.31}$$

and, in order to verify that the variational solution (u,ψ) satisfies equations (3.4) in the usual sense, there only remains to check that

$$[\partial_n u]_{int} = [\partial_n u]_{ext}\ \text{ for almost every } (t,x)\in\mathbb{R}^+\times\Gamma. \tag{3.32}$$

However, as the example in Appendix 1 shows, this identity can be violated even in the simplest 1D stationary case. In the next section, we formulate several sufficient conditions which ensure that (3.32) holds for every (variational) solution of (3.4).

4. Additional regularity and separation from the singularities

The main aim of this section is to study the analytic properties of the variational solutions u of problem (3.4), especially close to the singular points ± 1. We start with the following result which gives an additional regularity on $u(t,x)$ close to the points where $|u(t,x)|<1$.

Proposition 4.1. *Let the assumptions of Theorem 3.2 hold and let u be a variational solution of problem* (3.4)*. Let also $\delta>0$, $T>0$ be given and set*

$$\Omega_\delta(T):=\{x\in\Omega,\quad |u(T,x)|<1-\delta\}. \tag{4.1}$$

Then, $u\in W^{2,2}(\Omega_\delta(T))$ and the following estimate holds:

$$\|u\|_{H^2(\Omega_\delta(T))}\le Q_{\delta,T}, \tag{4.2}$$

where the constant $Q_{\delta,T}$ only depends on T and δ, but is independent of the concrete choice of the solution u.

Proof. Since the solution $u(T,x)$ is Hölder continuous with respect to x (see (3.15)), there exists a smooth nonnegative cut-off function $\theta(x)$ such that

$$\begin{cases} 1)\ \ \theta(x)\equiv 1, \quad x\in\Omega_\delta(T),\\ 2)\ \ \theta(x)\equiv 0, \quad x\in\Omega\backslash\Omega_{\delta/2}(T),\\ 3)\ \ \|\theta\|_{C^2(\mathbb{R}^3)}\le K_{\delta,T}, \end{cases} \tag{4.3}$$

where $K_{\delta,T}$ depends on the constants in (3.15), but is independent of the concrete choice of the solution u.

Furthermore, let $u_N(t,x)$ be a sequence of approximate solutions of problems (2.9) which converges to the variational solution $u(t,x)$ as $N\to\infty$. Then, since this convergence holds in the space $C^\gamma([t,T]\times\Omega)$ for some $\gamma>0$,

$$|u_N(T,x)|<1-\delta/4, \quad x\in\Omega_{\delta/2}(T) \tag{4.4}$$

if N is large enough. Set now $v_N(x):=\theta(x)u_N(T,x)$. Then, this function obviously solves the following elliptic boundary value problem (compare with (2.27)):

$$\begin{cases} \Delta_x v_N-v_N=h_1(u_N):=\theta f_N(u_N(T))+\theta\tilde h_1(T)+2\nabla_x\theta.\nabla_x u_N(T)+u_N(T)\Delta_x\theta,\\ v_N\big|_\Gamma=w_N,\\ \Delta_\Gamma w_N-w_N-\partial_n v_N=h_2(u_N):=\\ \qquad\qquad :=\theta\tilde h_2(T)+2\nabla_\Gamma\theta.\nabla_\Gamma u_N(T)+u_N(T)\Delta_\Gamma\theta-u_N(T)\partial_n\theta, \end{cases} \tag{4.5}$$

where the functions $\tilde h_i$, $i=1,2$, are the same as in (2.27). In addition, owing to estimates (2.23), (2.25), (2.28), (4.3) and (4.4), we see that

$$\|h_1(u_N)\|_{L^2(\Omega)}+\|h_2(u_N)\|_{L^2(\Gamma)}\le Q_{\delta,T}, \tag{4.6}$$

where the constant $Q_{\delta,T}$ is independent of N and of the concrete choice of the solution u. Applying the H^2-regularity theorem to the linear elliptic problem (4.5) (see [35]) and recalling (4.3), we deduce that

$$\|u_N(T)\|_{H^2(\Omega_\delta(T))} \le Q_{\delta,T} \tag{4.7}$$

and, consequently, by passing to the limit $N \to \infty$, we see that $u(T) \in H^2(\Omega_\delta(T))$ and (4.2) holds. This finishes the proof of the proposition. □

Remark 4.2. Applying the L^p-regularity theorem to the elliptic boundary value problem (4.5), together with a proper interpolation inequality, we have

$$\|u(T)\|_{W^{2,p}(\Omega_\delta(T))} \le Q_{\delta,p,T}$$

for any $p < \infty$. However, it seems difficult to obtain further regularity results on u in $\Omega_\delta(T)$ by directly using equations (2.9) or (4.5), owing to the presence of the nonlocal term $\langle \mu(t)\rangle$ (which is only L^∞ with respect to t). Alternatively, one can use the standard interior estimates for the initial fourth-order problem (2.1). Then, it is not difficult to see that the factual regularity of the solution u in $\Omega_\delta(T)$ is only restricted by the regularity of the data f, g, h_i, $i = 1, 2$, and Ω (and, if these data are of class C^∞, the solution u is of class C^∞ in $\Omega_\delta(T)$ as well).

Corollary 4.3. *Let the assumptions of Theorem 3.2 hold and let u be a variational solution of problem* (3.4). *Assume, in addition, that*

$$|u(t_0, x_0)| < 1$$

for some $(t_0, x_0) \in \mathbb{R}^+ \times \Gamma$, with $t_0 > 0$. Then, there exists a neighborhood $(t_0-\delta, t_0+\delta)\times V$ of (t_0, x_0) in $\mathbb{R} \times \Gamma$ such that

$$[\partial_n u]_{int}(t, x) = [\partial_n u]_{ext}(t, x), \quad \forall (t, x) \in (t_0 - \delta, t_0 + \delta) \times V. \tag{4.8}$$

In particular, if

$$|u(t, x)| < 1 \quad \textit{for almost all } (t, x) \in \mathbb{R}^+ \times \Gamma, \tag{4.9}$$

then the equality $[\partial_n u]_{ext} = [\partial_n u]_{int}$ holds almost everywhere in $\mathbb{R}^+ \times \Gamma$ and, therefore, the variational solution u solves equations (3.4) *in the usual sense.*

Proof. Since the solution u is Hölder continuous with respect to t and x, there exists $\delta > 0$ such that the inequality

$$|u(t, x)| \le 1 - \delta$$

holds for all (t, x) belonging to some neighborhood $(t_0 - \delta, t_0 + \delta) \times V_\delta$ of (t_0, x_0) in $\mathbb{R} \times \Omega$. According to Proposition 4.1, the sequence u_N of approximate solutions (converging to the variational solution u) satisfies

$$\|u_N\|_{L^\infty([t_0-\delta,t_0+\delta],H^2(V_\delta))} \le C,$$

where the constant C is independent of N. Consequently, we can assume, without loss of generality, that $u_N \to u$ weakly-star in this space. Thus,

$$\partial_n u_N\big|_\Gamma \to \partial_n u\big|_\Gamma$$

weakly in $L^2([t_0 - \delta, t_0 + \delta] \times V)$ (for a proper choice of the small neighborhood V of x_0). This convergence, together with the definition (3.30) of the function $[\partial_n u]_{ext}$, give the desired equality (4.8). Thus, the first part of the statement is proved and the second one is an immediate consequence of the first one, which finishes the proof of Corollary 4.3. □

Thus, in order to prove that any variational solution u of problem (3.4) satisfies the equations in the usual sense, it is sufficient to check (4.9). The next corollary shows that this will be the case if the nonlinearity $f(u)$ has sufficiently strong singularities at ± 1.

Corollary 4.4. *Let the assumptions of Theorem 3.2 hold and let, in addition, the potential $F(u)$ be such that*

$$\lim_{u\to\pm 1} F(u)=\infty. \tag{4.10}$$

Then, for every variational solution u of problem (3.4),

$$F(u(t))\in L^1(\Gamma) \quad \text{and} \quad \|F(u(t))\|_{L^1(\Gamma)}\le C_T$$

for almost all $t\ge T>0$ and condition (4.9) *holds.*

Proof. Let u_N be a sequence of approximate solutions converging to the variational solution u. Applying estimate (6.4) in Appendix 1 to the elliptic problem (2.27), we infer that

$$\|F_N(u_N(t))\|_{L^1(\Gamma)}\le C_T, \quad t\ge T, \tag{4.11}$$

where the constant C_T is independent of N. Using assumption (4.10) and arguing as in the proof of Theorem 3.3, we see that condition (4.9) indeed holds. Then, owing to the convergence $u_N\to u$ in $C^\gamma([0,T]\times\Omega)$, $\gamma>0$, we conclude that $F_N(u_N)\to F(u)$ almost everywhere in $\mathbb{R}^+\times\Gamma$. The Fatou lemma finally yields that $F(u(t))\in L^1(\Gamma)$, which finishes the proof of the corollary. □

In particular, condition (4.9) is satisfied if the nonlinearity f is of the form

$$f(u)\sim\frac{u}{(1-u^2)^p} \tag{4.12}$$

with $p>1$. Unfortunately, the assumptions of Corollary 4.4 are violated in the physically most relevant case of a logarithmic potential,

$$f(u)=\ln\frac{1+u}{1-u}. \tag{4.13}$$

Furthermore, as explained in Appendix 1, in that case, the variational solution u may indeed not be a solution in the usual sense (even in the 1D stationary case). However, the next proposition gives another type of sufficient condition (in terms of the nonlinearity g and the boundary external forces h_2) which guarantees the equality $[\partial_n u]_{ext}=[\partial_n u]_{int}$ and holds for the logarithmic potential (4.13).

Proposition 4.5. *Let the assumptions of Theorem 3.2 hold and let, in addition, the following inequalities hold:*

$$g(-1)+\varepsilon\le h_2(x)\le g(1)-\varepsilon, \quad x\in\Gamma, \tag{4.14}$$

for some $\varepsilon>0$. Then, condition (4.9) *holds and*

$$\|f(u)\|_{L^1([t,t+1]\times\Gamma)}\le C_{\varepsilon,T}, \quad t\ge T>0, \tag{4.15}$$

where the constant $C_{\varepsilon,T}$ is independent of the concrete choice of the variational solution u. In particular, every variational solution of (3.4) *solves this system in the usual sense.*

Proof. As above, it is sufficient to derive the uniform (with respect to $N \to \infty$) estimate (4.15) for the approximate solution u_N of (2.9). In order to do so, we rewrite the system in the elliptic-parabolic form

$$\begin{cases} \Delta_x u_N(t) - f_N(u_N(t)) - u_N(t) = \tilde{h}_1(t), \\ u_N\big|_\Gamma = \psi_N, \\ \partial_t \psi_N - \Delta_\Gamma \psi_N + \partial_n u_N + g(\psi_N) = h_2. \end{cases} \tag{4.16}$$

Furthermore, since only the values of g on the segment $[-1, 1]$ are important for the limit problem, we can assume, without loss of generality, that

$$g(-u) + \varepsilon \le h_2(x) \le g(u) - \varepsilon, \quad u \in \mathbb{R}, \quad |u| \ge 1, \quad x \in \Gamma.$$

It follows from these inequalities and the continuity of g that

$$(g(z) - h_2(x)).f_N(z) \ge \frac{\varepsilon}{2}|f_N(z)| + C_\varepsilon, \quad z \in \mathbb{R}, \quad x \in \Gamma, \tag{4.17}$$

where the constant C_ε depends on ε and g, but is independent of N.

We now multiply the first equation of (4.16) by $f_N(u_N)$ and integrate with respect to x. Then, integrating by parts and using estimate (4.17), we find

$$\begin{aligned} \frac{d}{dt}\int_\Gamma F_N(u_N(t))\, dS + (f_N'(u_N(t))\nabla_x u_N(t), \nabla_x u_N(t))_\Omega + \\ + (f_N'(u_N(t))\nabla_\Gamma u_N(t), \nabla_\Gamma u_N(t))_\Gamma + \\ + 1/2\|f_N(u_N(t))\|^2_{L^2(\Omega)} + \varepsilon/2\|f_N(u_N(t))\|_{L^1(\Gamma)} \le C(1 + \|\tilde{h}_1(t)\|^2_{L^2(\Omega)}). \end{aligned} \tag{4.18}$$

Integrating this inequality with respect to time and using the facts that $f_N' \ge 0$ and the L^2-norm of $\tilde{h}_1(t)$ is controlled (see (2.22) and (2.28)), we obtain

$$\|f_N(u_N)\|_{L^1([t,t+1]\times\Gamma)} \le \frac{2}{\varepsilon}(\|F_N(u_N(t))\|_{L^1(\Gamma)} + \|F_N(u_N(t+1))\|_{L^1(\Gamma)}) + C_{\varepsilon,T}. \tag{4.19}$$

There only remains to note that the right-hand side of (4.19) is controlled, owing to estimate (6.4) (exactly as in the proof of (4.11)). Therefore, (4.19) gives uniform bounds on the L^1-norm of $f_N(u_N)$ on the boundary. Passing to the limit $N \to \infty$ now gives the statement of the proposition (exactly as in the proof of Corollary 4.4). □

Remark 4.6. As already mentioned, the variational solution u may not solve equations (3.4) in the usual sense if conditions (4.10) and (4.14) are violated (see Example 6.2 for details). Furthermore, arguing as in this example, it is not difficult to show that, for any singular nonlinearity f which does not satisfy (4.10), there exist "nonusual" variational solutions of problem (3.4) if the external forces h_2 are large enough.

We conclude this section by establishing that every solution u of problem (3.4) is separated from the singularities ± 1 if the nonlinearity f is singular enough. To this end, we need to require *at least* condition (4.10) to be satisfied (see again Example 6.2). Actually, we will require slightly more, namely, that the nonlinearity f satisfies the following inequalities:

$$\frac{\kappa_1}{(1-u^2)^{p-1}} \le \frac{f(u)}{u} \le \frac{\kappa_2}{(1-u^2)^M} \tag{4.20}$$

for some positive constants κ_i, $i = 1, 2$, and M and where $p > 2$ (recall that condition (4.10) is violated if $p < 2$, so that the sufficient condition (4.20) is close to the necessary one). In addition, we assume more regularity on the external forces h_1 and h_2, namely,

$$h_1 \in L^3(\Omega), \quad h_2 \in L^\infty(\Gamma). \tag{4.21}$$

Theorem 4.7. *Let the assumptions of Theorem 3.2 hold and let, in addition,* (4.20) *and* (4.21) *be satisfied. Then, every variational solution u of problem* (3.4) *is separated from the singularities ± 1, namely, the following estimate holds:*

$$|u(t,x)| \le 1 - \delta_T, \quad t \ge T > 0, \quad x \in \Omega, \tag{4.22}$$

where the constant δ_T depends on T, but is independent of u, t and x.

Proof. We only give below the formal derivation of estimate (4.22) which can be justified as above by approximating the solution u by a sequence u_N of solutions of the regularized problem (2.9). Our proof is based on the following lemma.

Lemma 4.8. *Let the assumptions of the theorem hold and let $u(t)$ be a (variational) solution of problem* (3.4). *Then, for every $q > 0$, $f(u) \in L^q([t, t+1] \times \Omega)$ for all $t > 0$ and the following estimate holds:*

$$\|f(u)\|_{L^q([t,t+1]\times\Omega)} \le C_{T,q}, \quad t \ge T > 0, \tag{4.23}$$

where the constant $C_{T,q}$ is independent of t and u.

Proof. We rewrite system (3.4) in the form of a coupled elliptic-parabolic problem,

$$\begin{cases} \Delta_x u - f(u) - u = \tilde{h}_1(t), \quad u\big|_\Gamma = \psi, \\ \partial_t \psi - \Delta_\Gamma \psi + \partial_n u = \tilde{h}_2(t). \end{cases} \tag{4.24}$$

Then, owing to the regularity estimate (3.15) on the solution u and conditions (4.21), we have

$$\|\tilde{h}_1(t)\|_{L^3(\Omega)} + \|\tilde{h}_2(t)\|_{L^\infty(\Gamma)} \le C_T, \quad t \ge T. \tag{4.25}$$

We introduce the function $\varphi(u) := \frac{1}{1-u^2}$. Then,

$$\varphi'(u) = \frac{2u}{(1-u^2)^2} = 2u\varphi(u)^2.$$

We multiply the first equation of (4.24) by $u\varphi(u)^{n+1}$, where $n > 1$ is an arbitrary fixed exponent, and integrate with respect to $x \in \Omega$. Then, integrating by parts and using the obvious transformations

$$(\nabla_x u, u\varphi'(u)|\varphi(u)|^n \nabla_x u)_\Omega = \frac{1}{2}(|\nabla_x u|^2, |\varphi'(u)|^2 |\varphi(u)|^{n-2})_\Omega \ge C_n \|\nabla_x(|\varphi(u)|^{n/2})\|^2_{L^2(\Omega)},$$

$$\int_0^u v\varphi(v)^{n+1}\,dv = \frac{1}{2}\int_0^u \varphi'(v)\varphi(v)^{n-1}\,dv = \frac{1}{2n}(|\varphi(v)|^n - 1)$$

and

$$f(u)u\varphi(u)^{n+1} \geq \frac{\kappa_1}{2}\varphi(u)^{n+p} - C_n$$

(owing to the first inequality in (4.20)), we have

$$\begin{aligned}(4.26)\quad \frac{d}{dt}\|\varphi(u)\|^n_{L^n(\Gamma)} + 2\kappa(\||\varphi(u)|^{n/2}\|^2_{H^1(\Omega)} + \|\varphi(u)\|^{n+p}_{L^{n+p}(\Omega)}) \leq \\ \leq C\|\tilde h_1(t)\|_{L^3(\Omega)}\|\varphi(u)\|^{n+1}_{L^{3/2(n+1)}(\Omega)} + C\|\tilde h_2(t)\|_{L^\infty(\Gamma)}\|\varphi(u)\|^{n+1}_{L^{n+1}(\Gamma)} + C \leq \\ \leq C_T(\|\varphi(u)\|^{n+1}_{L^{3/2(n+1)}(\Omega)} + \|\varphi(u)\|^{n+1}_{L^{n+1}(\Gamma)} + 1),\end{aligned}$$

where $\kappa > 0$ and the constant C_T depends on n, but is independent of $t \geq T > 0$ and the concrete choice of the solution u. Let us estimate the right-hand side of (4.26). In order to estimate the boundary term, we use the following trace inequality

$$\|V\|^{s+1}_{L^{s+1}(\Gamma)} \leq C(\|V\|^2_{H^1(\Omega)} + \|V\|^{2s}_{L^{2s}(\Omega)})$$

(which can be easily obtained by using a proper interpolation inequality and Sobolev's embedding theorem). Using this inequality with $V = |\varphi|^{n/2}$ and $s = 1 + p/n$, we find

$$\|\varphi(u)\|^{n+p/2}_{L^{n+p/2}(\Gamma)} \leq C(\||\varphi(u)|^{n/2}\|^2_{H^1(\Omega)} + \|\varphi(u)\|^{n+p}_{L^{n+p}(\Omega)})$$

and, therefore, since $p > 2$, we can rewrite (4.26) without any boundary term in the right-hand side,

$$\begin{aligned}(4.27)\quad \frac{d}{dt}\|\varphi(u)\|^n_{L^n(\Gamma)} + 2\kappa'(\|\varphi(u)\|^n_{L^{3n}(\Omega)} + \|\varphi(u)\|^{n+p}_{L^{n+p}(\Omega)}) + \\ + \kappa\|\varphi(u)\|^{n+1}_{L^{n+1}(\Gamma)} \leq C_T(\|\varphi(u)\|^{n+1}_{L^{3/2(n+1)}(\Omega)} + 1)\end{aligned}$$

for some positive constants κ' and C_T which depend on n and T, but are independent of t and u (here, we have implicitly used the embedding $H^1 \subset L^6$ and replaced the exponent $n + p/2$ by $n + 1$ in the boundary term). In order to estimate the right-hand side of this inequality, we use one more interpolation inequality,

$$\|\varphi(u)\|^s_{L^r(\Omega)} \leq \|\varphi(u)\|^{\theta s}_{L^{3n}(\Omega)}\|\varphi(u)\|^{(1-\theta)s}_{L^{n+p}(\Omega)} \leq C(\|\varphi(u)\|^n_{L^{3n}(\Omega)} + \|\varphi(u)\|^{n+p}_{L^{n+p}(\Omega)}),$$

where $s \in [0,1]$ and q are such that

$$\frac{1}{r} = \frac{\theta}{3n} + \frac{1-\theta}{n+p}, \quad \frac{1}{s} = \frac{\theta}{n} + \frac{1-\theta}{n+p}.$$

Solving these equations for $r = 3/2(n+1)$ and $n > p - 1$, we have

$$s = (n+1)\frac{2n-p}{2n-p-(p-2)} > n+1$$

(since $p > 2$). Thus,

$$C_T\|\varphi(u)\|^{n+1}_{L^{3/2(n+1)}(\Omega)} \leq \kappa'(\|\varphi(u)\|^n_{L^{3n}(\Omega)} + \|\varphi(u)\|^{n+p}_{L^{n+p}(\Omega)}) + C'_T$$

and (4.27) yields, noting that $p>2$ and $L^{n+1}\subset L^n$,

$$\frac{d}{dt}\|\varphi(u(t))\|^n_{L^n(\Gamma)}+\kappa'\|\varphi(u(t))\|^{n+2}_{L^{n+2}(\Omega)}+\kappa\|\varphi(u(t))\|^{n+1}_{L^n(\Gamma)}\le C_T. \tag{4.28}$$

Multiplying this inequality by $(t-T)^{n+1}$, we obtain

$$\begin{aligned}\frac{d}{dt}[(t-T)^{n+1}\|\varphi(u(t))\|^n_{L^n(\Gamma)}]+\kappa'(t-T)^{n+1}\|\varphi(u(t))\|^{n+2}_{L^{n+2}(\Omega)}\le\\ \le -\kappa[(t-T)\|\varphi(u(t))\|_{L^n(\Gamma)}]^{n+1}+C(n+1)[(t-T)\|\varphi(u(t))\|_{L^n(\Gamma)}]^n+\\ +C_T(t-T)^{n+1}\le C_{n,T}.\end{aligned} \tag{4.29}$$

Integrating (4.29) with respect to $t\in[T,T+2]$, we finally end up with

$$\int_T^{T+2}(t-T)^{n+1}\|\varphi(u(t))\|^{n+2}_{L^{n+2}(\Omega)}\,dt\le C'_{n,T}.$$

Since n is arbitrary, this last inequality, together with the second inequality in (4.20), finish the proof of the lemma. □

It is now not difficult to finish the proof of the theorem. To this end, we note that, owing to estimate (3.15) and Sobolev's embedding theorem,

$$\|u(t)\|_{W^{2-1/3,3}(\Gamma)}\le C\|u(t)\|_{H^2(\Gamma)}\le C_T,\quad t\ge T. \tag{4.30}$$

On the other hand, owing to the lemma and the first equation of (3.4), there holds

$$\|\Delta_x u(t)\|_{L^q([t,t+1],L^3(\Omega))}\le C_T.$$

Thus, owing to the maximal regularity for the Laplacian in L^3 and Sobolev's embedding theorem,

$$\|\nabla_x u(t)\|_{L^q([t,t+1]\times\Omega)}\le C_{T,q},\quad t\ge T, \tag{4.31}$$

for any $q\ge1$. Furthermore, it follows from (3.15), (4.23) and (4.31) that

$$\begin{aligned}&\|\varphi(u)\|_{L^r([t,t+1],W^{1,r}(\Omega))}\le\\ &\le C(\|\varphi(u)\|_{L^{2r}([t,t+1]\times\Omega)}+\|\varphi'(u)\|_{L^{2r}([t,t+1]\times\Omega)})(1+\|\nabla_x u\|_{L^{2r}([t,t+1]\times\Omega)})\le C_{r,T},\\ &\|\partial_t\varphi(u)\|_{L^{2-\varepsilon}([t,t+1],L^{6-\varepsilon}(\Omega))}\le C\|\varphi(u)\|_{L^{r_\varepsilon}([t,t+1]\times\Omega)}\|\partial_t u\|_{L^2([t,t+1]\times H^1(\Omega))}\le C_{\varepsilon,T},\end{aligned} \tag{4.32}$$

where $\varepsilon>0$ and $r>0$ are arbitrary and the constants $C_{r,T}$ and $C_{\varepsilon,T}$ are independent of u and $t\ge T$. Fixing finally $r\gg1$ and $\varepsilon\ll1$ in such a way that

$$W^{1-\varepsilon}([t,t+1]\times\Omega)\cap L^r([t,t+1],W^{1,r}(\Omega))\subset C([t,t+1]\times\Omega),$$

we deduce from (4.32) that

$$\sup_{(s,x)\in[t,t+1]\times\Omega}\Big|\frac{1}{1-u^2(s,x)}\Big|=\|\varphi(u)\|_{C([t,t+1]\times\Omega)}\le C_T,\quad t\ge T,$$

which gives (4.22) and finishes the proof of Theorem 4.7. □

Remark 4.9. It is not difficult to see that assumption (4.21) can be slightly relaxed, namely,

$$h_1 \in L^{r_1(p)}(\Omega), \quad h_2 \in L^{r_2(p)}(\Gamma), \quad r_1(p) < 3, \quad r_2(p) < \infty$$

and $r_1(p) \to 3$, $r_2(p) \to \infty$ as $p \to 2$ (where $p > 2$ is the exponent in inequalities (4.20)). It is also worth noting that the proof of Lemma 4.8 does not involve the Laplace-Beltrami operator in the boundary conditions (and we have used it *only* in the derivation of estimate (4.30)). Consequently, arguing in a slightly more accurate way (e.g., by obtaining the L^∞-estimate on $\varphi(u)$ directly from the estimates of Lemma 4.8 by a Moser iteration technique), one can extend the theorem to less regular boundary conditions,

$$\partial_t u + \partial_n u + g(u) = h_2, \quad x \in \Gamma.$$

Finally, the aforementioned Moser scheme also allows to remove the second inequality in (4.20) and to obtain the separation from the singularities by only using the first inequality in (4.20) for the function f. We will come back to these questions elsewhere.

5. Long-time behavior: attractors and exponential attractors

In this concluding section, we study the asymptotic behavior of the trajectories of the solution semigroup (3.23) acting on the phase space Φ, endowed with the metric of Φ^w. We first recall that problem (2.1) enjoys the mass conservation

$$\langle u(t)\rangle \equiv \langle u(0)\rangle := c. \tag{5.1}$$

Therefore, it is natural to consider the restrictions of our semigroup to the hyperplanes

$$\Phi_c := \{(u,\psi) \in \Phi, \quad \langle u\rangle = c\}, \quad c \in (-1,1), \quad S(t) : \Phi_c \to \Phi_c. \tag{5.2}$$

The following proposition gives the existence of the global attractor $\mathcal{A}_c$ for this semigroup. We recall that, by definition, a set $\mathcal{A}_c \subset \Phi_c$ is the global attractor for the semigroup (5.2) if

1) It is compact in Φ_c.
2) It is strictly invariant, i.e., $S(t)\mathcal{A}_c = \mathcal{A}_c$, $t \geq 0$.
3) It attracts Φ_c as $t \to \infty$, i.e., for every neighborhood $\mathcal{O}(\mathcal{A}_c)$ of $\mathcal{A}_c$ in Φ_c, there exists $T = T(\mathcal{O})$ such that

$$S(t)\Phi_c \subset \mathcal{O}(\mathcal{A}_c), \quad t \geq T.$$

We refer the reader to, e.g., [2], [36] and [49] for details (we note that, in our situation, the phase space Φ_c is, by definition, bounded and, therefore, we need not involve bounded sets in the definition of the global attractor).

Proposition 5.1. *Let the assumptions of Theorem 3.2 hold. Then, for every $c \in (-1,1)$, the semigroup $S(t)$ associated with the variational solutions of problem* (2.1) *acting on the hyperplane* (5.2) *(endowed with the metric of Φ^w) possesses the global attractor $\mathcal{A}_c$. Furthermore, this attractor is bounded in the space $C^\alpha(\Omega) \times C^\alpha(\Gamma)$, $\alpha < 1/4$, and is generated by all complete trajectories of the semigroup $S(t)$ (i.e., by all variational solutions $(u(t), v(t))$ which are defined for all $t \in \mathbb{R}$).*

Indeed, owing to estimate (3.24), the semigroup $S(t)$ has a closed graph in Φ_c. On the other hand, owing to estimate (3.15), this semigroup possesses an absorbing set which is compact in Φ_c (endowed with the metric of Φ^w) and bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma)$. Thus, the existence of $\mathcal{A}_c$, together with all properties stated in the proposition, follow from a proper abstract attractor's existence theorem (see, e.g., [2], [36] and [49]).

Our next task is to prove the finite-dimensionality of the global attractor $\mathcal{A}_c$ constructed above and the existence of a so-called exponential attractor. We recall that, by definition, a set $\mathcal{M}(c) \subset \Phi_c$ is an exponential attractor for the semigroup $S(t)$ if

1) It is compact in Φ_c.
2) It is semiinvariant, i.e., $S(t)\mathcal{M}(c) \subset \mathcal{M}(c)$, $t \geq 0$.
3) It has finite fractal dimension in Φ_c.
4) It attracts Φ_c exponentially fast as $t \to \infty$, i.e.,

$$\operatorname{dist}_{\Phi_c}(S(t)\Phi_c, \mathcal{M}(c)) \leq Ce^{-\gamma t}, \quad t \geq 0,$$

for some positive constants γ and C. Here and below, $\operatorname{dist}_V(X,Y)$ stands for the non-symmetric Hausdorff distance between sets in V.

We also recall that the usual construction of an exponential attractor is based on the so-called squeezing (or smoothing) property for the difference of solutions (or their proper modifications, see [11], [13], [14] and [36] for details). The main difficulty here lies in the singular nature of the equations at ± 1. In particular, the difference between two singular solutions u_1 and u_2 does not possess any regularization. Nevertheless, as we will see below, our nonlinearity is strictly monotone near the singularities ± 1 and, far from these singularities, the problem still possesses the usual parabolic smoothing property. This fact, together with the Hölder continuity of the solutions and some localization technique, allow to construct an exponential attractor by using a proper modification of the techniques developed in [13] and [15]. However, some additional difficulties arise here, due to the fact that the H^{-1}-norm is not local.

Theorem 5.2. *Let the assumptions of Theorem 3.2 hold. Then, the semigroup $S(t)$ acting on the phase space Φ_c (endowed with the metric of Φ) possesses an exponential attractor $\mathcal{M}(c)$ which is bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma)$, $\alpha < 1/4$.*

Proof. We first note that, owing to Theorem 3.3, there exists a compact (for the metric of Φ^w) absorbing set

$$\mathbb{B}_c := S(1)\Phi_c \tag{5.3}$$

such that

$$S(t)\mathbb{B}_c \subset \mathbb{B}_c \tag{5.4}$$

and

$$\begin{aligned} \|u\|_{C^\alpha([t,t+1]\times\Omega)} + \|u(t)\|_{H^2(\Gamma)} + \|\partial_t u(t)\|_{H^{-1}(\Omega)} + \|\partial_t u(t)\|_{L^2(\Gamma)} + \\ + \|f(u(t))\|_{L^1(\Omega)} + \|\partial_t u\|_{L^2([t,t+1],H^1(\Omega))} + \|\partial_t u\|_{L^2([t,t+1],H^1(\Gamma))} \leq R \end{aligned} \tag{5.5}$$

for some fixed constant R which depends on c, but is independent of $(u(0),\psi(0)) \in \mathbb{B}_c$. In particular, for every point $(u,\psi) \in \mathbb{B}_c$, we have

$$\psi = u\big|_\Gamma$$

and, consequently, we generally write u instead of ψ in the boundary norms.

Thus, we only need to construct an exponential attractor $\mathcal{M}(c)$ for the semigroup $S(t)$ restricted to the semiinvariant absorbing set $\mathbb{B}_c$. As usual, to do so, we need to obtain proper estimates on the difference of two solutions $u_1(t)$ and $u_2(t)$ starting from the set $\mathbb{B}_c$. Furthermore, we use the following natural norm on the phase space $\mathbb{B}_c$:

$$\|u_1-u_2\|^2_{\Phi^w} := \|u_1-u_2\|^2_{H^{-1}(\Omega)} + \|u_1-u_2\|^2_{L^2(\Gamma)} = \\ = \|(-\Delta_x)^{-1/2}(u_1-u_2)\|^2_{L^2(\Omega)} + \|u_1-u_2\|^2_{L^2(\Gamma)}. \tag{5.6}$$

A crucial point in the proof is the global Lipschitz continuity in this norm (see Theorem 2.2),

$$\|u_1(t)-u_2(t)\|^2_{\Phi^w} + \\ + \int_t^{t+1}(\|u_1(s)-u_2(s)\|^2_{H^1(\Omega)} + \|u_1(s)-u_2(s)\|^2_{H^1(\Gamma)})\,ds \le Ce^{Kt}\|u_1(0)-u_2(0)\|^2_{\Phi^w}, \tag{5.7}$$

where the positive constants C and K are independent of $u_1(0), u_2(0) \in \mathbb{B}_c$.

We now consider an arbitrary small ε-ball $B(\varepsilon, u_0, \Phi^w)$ in the space $\mathbb{B}_c$ (endowed with the metric of Φ^w) and centered at u_0, where $0 < \varepsilon \le \varepsilon_0 \ll 1$ (and the parameter ε_0 will be fixed below). Let also $u^0(t)$, $t \ge 0$, be the solution of problem (3.4) starting from u_0.

As in (4.1), we introduce the sets

$$\Omega_\delta(u_0) := \{x \in \Omega,\ |u_0(x)| < 1-\delta\},\quad \overline{\Omega}_\delta(u_0) := \{x \in \Omega,\ \ |u_0(x)| > 1-\delta\}, \tag{5.8}$$

where δ is a sufficiently small positive number. Then, since the function $u_0(x)$ is uniformly Hölder continuous in Ω, there holds

$$d(\partial\Omega_{\delta_1}(u_0), \partial\Omega_{\delta_2}(u_0)) \ge C_{\delta_1,\delta_2} > 0,\quad \delta_1 \ne \delta_2, \tag{5.9}$$

where the constant C_{δ_1,δ_2} depends on δ_i, $i = 1,2$, but is independent of the concrete choice of $u_0 \in \mathbb{B}_c$.

As a next step, we note that, owing to the uniform Hölder continuity of the trajectory $u^0(t)$ (in space and time), there exists $T = T(\delta)$ such that

$$|u^0(t)| \le 1-\frac{\delta}{2},\ x \in \Omega_\delta(u_0),\ t \in [0,T], \\ |u^0(t)| \ge 1-2\delta,\ x \in \overline{\Omega}_{2\delta}(u_0),\ t \in [0,T], \tag{5.10}$$

and, furthermore, owing again to the uniform Hölder continuity,

$$\|u_1(t)-u_2(t)\|_{C(\Omega)} \le C\|u_1(t)-u_2(t)\|^\kappa_{\Phi^w}\|u_1(t)-u_2(t)\|^{1-\kappa}_{C^\alpha(\Omega)} \le C_T\varepsilon^\kappa,$$

for all $u_1(0), u_2(0) \in B(\varepsilon, u_0, \Phi^w)$. We can thus fix $\varepsilon_0 = \varepsilon_0(\delta)$ in such a way that

$$|u(t)| \le 1-\frac{\delta}{4},\quad x \in \Omega_\delta(u_0),\quad t \in [0,T], \\ |u(t)| \ge 1-4\delta,\quad x \in \overline{\Omega}_{2\delta}(u_0),\quad t \in [0,T], \tag{5.11}$$

for all trajectories $u(t)$ starting from the ball $B(\varepsilon, u_0, \Phi^w)$ with $\varepsilon \le \varepsilon_0$.

We also introduce the cut-off function $\theta \in C^\infty(\mathbb{R}^3, [0,1])$ such that

$$\theta(x) \equiv 0,\quad x \in \overline{\Omega}_\delta(u_0),\quad \theta(x) \equiv 1,\quad x \in \Omega_{2\delta}(u_0). \tag{5.12}$$

Such a function exists, owing to condition (5.9). Furthermore, it follows from this condition that this function can be chosen in such a way that it satisfies the additional assumption

$$\|\theta\|_{C^k(\mathbb{R}^3)} \le C_k, \tag{5.13}$$

where $k \in \mathbb{N}$ is arbitrary and the constant C_k depends on δ, but is independent of the choice of $u_0 \in \mathbb{B}_c$, see [15] for details.

Finally the second estimate of (5.11) yields

$$f'(u(t,x)) \ge \Lambda(\delta), \quad x \in \overline{\Omega}_{2\delta}(u_0), \quad t \in [0,T], \tag{5.14}$$

for all trajectories $u(t)$ starting from the ball $B(\varepsilon, u_0, \Phi^w)$, where

$$\Lambda(\delta) := \min\{f'(1-4\delta), f'(-1+4\delta)\}.$$

Since $f'(u) \to \infty$ as $u \to \pm 1$, then $\Lambda(\delta) \to \infty$ as $\delta \to 0$ and we can fix $\delta > 0$ in such a way that $\Lambda(\delta)$ is arbitrarily large. This will be essentially used in the next lemma which gives some kind of smoothing property for the difference of two solutions u_1 and u_2 and is crucial for our construction.

Lemma 5.3. *Let the above assumptions hold. Then, there exists $\delta > 0$ such that the following estimate holds:*

$$\|u_1(T) - u_2(T)\|^2_{\Phi^w} \le e^{-\beta T}\|u_1(0) - u_2(0)\|^2_{\Phi^w} + \\ + C\int_0^T \|\theta(u_1(s) - u_2(s))\|^2_{L^2(\Omega)}\, ds, \tag{5.15}$$

where the positive constants δ, β and C are independent of $u_1, u_2 \in B(\varepsilon, u_0, \Phi^w)$, s and $u_0 \in \mathbb{B}_c$.

Proof. As usual, we only give the formal derivation of this estimate which can be justified by approximating the variational solutions $u_1(t)$ and $u_2(t)$ by appropriate solutions of the regular equation (2.9). Set $v(t) := u_1(t) - u_2(t)$. Then, this function (formally) solves

$$\begin{cases} \partial_t v = -\Delta_x(\Delta_x v - l(t)v + \lambda v), \quad \partial_n(\Delta_x v - l(t)v + \lambda v)\big|_\Gamma = 0, \\ \partial_t v - \Delta_\Gamma v + \partial_n v + m(t)v = 0, \quad \text{on } \Gamma, \end{cases} \tag{5.16}$$

where

$$l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\, ds, \quad m(t) := \int_0^1 g'(su_1(t) + (1-s)u_2(t))\, ds.$$

Multiplying this equation by $(-\Delta_x)^{-1}v(t)$, integrating over Ω and using the fact that $\langle v(t)\rangle \equiv 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{\Phi^w} + \|\nabla_x v(t)\|^2_{L^2(\Omega)} + (l(t)v(t), v(t))_\Omega \le \lambda\|v(t)\|^2_{L^2(\Omega)} + K\|v(t)\|^2_{L^2(\Gamma)}, \tag{5.17}$$

where $K = \|g'\|_{C([-1,1])}$. We estimate the most complicated term $(l(t)v, v)$ as follows:

$$\int_\Omega l(t,x)|v(x)|^2\, dx \ge \int_{\overline{\Omega}_{2\delta}(u_0)} l(t,x)|v(x)|^2\, dx \ge \\ \ge \Lambda\|v\|^2_{L^2(\Omega)} - \Lambda\|v\|^2_{L^2(\Omega_{2\delta}(u_0))} \ge \Lambda\|v\|^2_{L^2(\Omega)} - \Lambda\|\theta v\|^2_{L^2(\Omega)}. \tag{5.18}$$

Thus, inequality (5.17) reads

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{\Phi^w}+\|\nabla_x v(t)\|^2_{L^2(\Omega)}+(\Lambda-\lambda)\|v(t)\|^2_{L^2(\Omega)}\le K\|v(t)\|^2_{L^2(\Gamma)}+\Lambda\|\theta v(t)\|^2_{L^2(\Omega)}. \tag{5.19}$$

Furthermore, using the trace-interpolation estimate

$$\|v\|^2_{L^2(\Gamma)}\le C\|v\|_{H^1(\Omega)}\|v\|_{L^2(\Omega)}\le C(\Lambda-\lambda)^{-1/2}(\|v\|^2_{H^1(\Omega)}+(\Lambda-\lambda)\|v\|^2_{L^2(\Omega)})$$

and fixing δ in such a way that $KC\le 1/2(\Lambda(\delta)-\lambda)^{1/2}$, we finally end up with

$$\frac{d}{dt}\|v(t)\|^2_{\Phi^w}+\|\nabla_x v(t)\|^2_{L^2(\Omega)}+\beta(\|v(t)\|^2_{L^2(\Omega)}+\|v(t)\|^2_{L^2(\Gamma)})\le 2\Lambda\|\theta v(t)\|^2_{L^2(\Omega)}, \tag{5.20}$$

where $\beta>0$. Using the Poincaré inequality $\|v\|_{H^{-1}(\Omega)}\le C\|v\|_{L^2(\Omega)}$, together with the Gronwall inequality, we deduce (5.15) and finish the proof of the lemma. □

Thus, owing to Lemma 5.3, the semigroup $S(t)$ is a contraction, up to the term $\|\theta(u_1-u_2)\|_{L^2([0,T]\times\Omega)}$. The next lemma gives some kind of compactness for this term.

Lemma 5.4. *Let the above assumptions hold. Then, the following estimate holds:*

$$\begin{aligned}\|\partial_t(\theta(u_1-u_2))\|_{L^2([0,T],H^{-3}(\Omega))}+&\\ +\|\theta(u_1-u_2)\|_{L^2([0,T],H^1(\Omega))}&\le Ce^{KT}\|u_1(0)-u_2(0)\|_{\Phi^w},\end{aligned} \tag{5.21}$$

where the constants C and K are independent of $u_i(0)\in B(\varepsilon,u_0,\Phi^w)$, $i=1,2$, and $u_0\in\mathbb{B}_c$.

Proof. The second term in the left-hand side of (5.21) can be easily estimated by (5.7) (and the fact that $\nabla_x\theta$ is uniformly bounded). So, we only need to estimate the time derivative. To this end, we recall that $\partial_t v$ ($v=u_1-u_2$) satisfies

$$\partial_t v=-\Delta_x(\Delta_x v-l(t)v)$$

in the sense of distributions. Therefore, for any test function $\varphi\in C_0^\infty(\Omega)$, there holds

$$\begin{aligned}\langle\partial_t(\theta v(t)),\varphi\rangle_\Omega=-\langle\Delta_x v(t)-l(t)v(t),\Delta_x(\theta\varphi)\rangle_\Omega=&\\ =\langle\nabla_x v(t),\nabla_x\Delta_x(\theta\varphi)\rangle_\Omega+\langle l(t)v(t),\Delta_x(\theta\varphi)\rangle_\Omega&.\end{aligned}$$

Since $\operatorname{supp}\theta\subset\Omega_\delta(u_0)$, (5.11) yields

$$|\langle l(t)v,\Delta_x(\theta\varphi)\rangle_\Omega|\le C\|v\|_{L^2(\Omega)}\|\varphi\|_{H^2(\Omega)}$$

and, thus,

$$|\langle\partial_t(\theta v(t)),\varphi\rangle_\Omega|\le C_1\|v(t)\|_{H^1(\Omega)}\|\varphi\|_{H^3(\Omega)}.$$

This estimate, together with (5.7), give the desired estimate (5.21) on the time derivative and finish the proof of the lemma. □

It is now not difficult to finish the proof of the theorem. We introduce the functional spaces

$$\begin{aligned}&\mathbb{H}_1:=L^2([0,T],H^1(\Omega))\cap H^1([0,T],H^{-3}(\Omega)),\\ &\mathbb{H}:=L^2([0,T],L^2(\Omega)).\end{aligned} \tag{5.22}$$

Then, obviously, $\mathbb{H}_1$ is compactly embedded into $\mathbb{H}$. We also introduce, for any $u_0 \in \mathbb{B}_c$, the linear operator

$$\mathbb{K}_{u_0} : B(\varepsilon, u_0, \Phi^w) \to \mathbb{H}_1$$

by

$$\mathbb{K}_{u_0} u(0) := \theta u(\cdot), \ \ u(t) \text{ solves } (3.4)$$

(where the constants δ, T and the cut-off function θ are such that Lemmas 5.3 and 5.4 hold). Then, on the one hand, owing to Lemma 5.4, the map $\mathbb{K}_{u_0}$ is uniformly Lipschitz continuous,

$$\|\mathbb{K}_{u_0}(u_1 - u_2)\|_{\mathbb{H}_1} \leq L\|u_1 - u_2\|_{\Phi^w}, \quad u_1, u_2 \in B(\varepsilon, u_0, \Phi^w), \quad \varepsilon \leq \varepsilon_0, \tag{5.23}$$

where the Lipschitz constant L is independent of the choice of $u_0 \in \mathbb{B}_c$ and $\varepsilon \leq \varepsilon_0$. On the other hand, it follows from Lemma 5.3 that

$$\|S(T)u_1 - S(T)u_2\|_{\Phi^w} \leq (1-\gamma)\|u_1 - u_2\|_{\Phi^w} + C\|\mathbb{K}(u_1 - u_2)\|_{\mathbb{H}}, \tag{5.24}$$

where $\gamma > 0$ and $C > 0$ are also independent of $u_0 \in \mathbb{B}_c$, $\varepsilon \leq \varepsilon_0$ and $u_1, u_2 \in B(\varepsilon, u_0, \Phi^w)$.

It is known (see, e.g., [16]; see also [33]) that inequalities (5.23) and (5.24), together with the compactness of the embedding $\mathbb{H}_1 \subset \mathbb{H}$, guarantee the existence of an exponential attractor $\mathcal{M}_d(c) \subset \mathbb{B}_c$ for the discrete semigroup $S(nT)$ acting on the phase space $\mathbb{B}_c$ (endowed with the topology of Φ^w). Furthermore, (5.7), together with the control (5.5) of the time derivative, yield that the semigroup $S(t)$ is uniformly Hölder continuous with respect to time and space in $[0, T] \times \mathbb{B}_c$. Thus, the desired exponential attractor $\mathcal{M}(c)$ for the continuous semigroup $S(t)$ on $\mathbb{B}_c$ can be obtained by the standard formula

$$\mathcal{M}(c) := \cup_{t \in [0,T]} \mathcal{M}_d(c).$$

Finally, although we have formally constructed the exponential attractor $\mathcal{M}(c) \subset \mathbb{B}_c \subset C^\alpha(\Omega) \times C^\alpha(\Gamma)$ in the topology of Φ^w only, the control of the C^α-norm of $\mathbb{B}_c$, together with a proper interpolation inequality, give the finite-dimensionality and the exponential attraction in the initial topology of Φ_c as well. This finishes the proof of Theorem 5.2. $\square$

6. Appendix 1. Some auxiliary results

In this section, we establish several estimates which are used in the paper. We start with regularity results for the following singular elliptic boundary value problem:

$$\begin{cases} \Delta_x u - u - f(u) = \tilde{h}_1, & \text{in } \Omega, \\ \partial_n u + u - \Delta_\Gamma u = \tilde{h}_2, & \text{on } \Gamma, \end{cases} \tag{6.1}$$

where $\tilde{h}_1 \in L^2(\Omega)$, $\tilde{h}_2 \in L^2(\Gamma)$ and the nonlinearity f satisfies conditions (2.4). As above, a solution u of this problem should be understood as a variational solution, analogously to Definition 3.1. Therefore, in order to justify the estimates given below, we factually need to deduce the corresponding uniform estimates for regularized problems of the form (6.1), where f is replaced by its approximations f_N (defined by (2.7)), and then pass to the limit $N \to \infty$. Since this passage to the limit is explained in details in Section 3, we give below the formal derivation of these estimates directly for the limit singular problem (6.1), leaving the justifications to the reader.

Theorem 6.1. *Let the above assumptions hold. Then, the following estimate holds for the solution u of problem* (6.1)*:*

$$\|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Gamma)} + \|f(u)\|_{L^1(\Omega)} \le C(1 + \|\tilde h_1\|^2_{L^2(\Omega)} + \|\tilde h_2\|^2_{L^2(\Gamma)}), \tag{6.2}$$

where the constant C is independent of $\tilde h_1$ and $\tilde h_2$. Furthermore, $u \in C^\alpha(\Omega) \cap H^2(\Gamma)$ with $\alpha < 1/4$ and the following estimate holds:

$$\|u\|^2_{C^\alpha(\Omega)} + \|u\|^2_{H^2(\Gamma)} \le C(1 + \|\tilde h_1\|^2_{L^2(\Omega)} + \|\tilde h_2\|^2_{L^2(\Gamma)}). \tag{6.3}$$

Finally, $F(u) \in L^1(\Gamma)$, where $F(z) := \int_0^z f(s)\,ds$, $\nabla_x D_\tau u \in L^2(\Omega)$, $u \in H^2(\Omega_\varepsilon)$, for every $\varepsilon > 0$, where $\Omega_\varepsilon := \{x \in \Omega,\ d(x,\Omega) > \varepsilon\}$, and the following estimate holds:

$$\|F(u)\|_{L^1(\Gamma)} + \|u\|^2_{H^2(\Omega_\varepsilon)} + \|\nabla_x D_\tau u\|^2_{L^2(\Omega)} \le C_\varepsilon(\|\tilde h_1\|^2_{L^2(\Omega)} + \|\tilde h_2\|^2_{L^2(\Gamma)}). \tag{6.4}$$

Proof. Estimate (6.2) can be obtained by multiplying the first equation by u and integrating over Ω (note that the existence and uniqueness of the solution u can be obtained exactly as in Section 3). So, we only need to give a formal derivation of estimates (6.3) and (6.4).

The derivation of these estimates is based on a standard localization technique. Thus, we only give below a sketch of the proof, leaving the details to the reader. Let θ be a smooth nonnegative cut-off function such that $\theta(x) = 1$ if $d(x,\Gamma) \ge \varepsilon$ and $\theta(x) = 0$ if $d(x,\Gamma) \le \varepsilon/2$ which satisfies, in addition, the inequality

$$|\nabla_x \theta(x)| \le C\theta^{1/2}(x). \tag{6.5}$$

Then, multiplying equation (6.1) by

$$\sum_{i=1}^3 \partial_{x_i}(\theta(x)\partial_{x_i} u),$$

integrating by parts and using estimate (6.2) (in order to estimate the lower-order terms) and the fact that $f' \ge 0$, we deduce that

$$\|u\|^2_{H^2(\Omega_\varepsilon)} \le C(1 + \|\tilde h_1\|^2_{L^2(\Omega)} + \|\tilde h_2\|^2_{L^2(\Gamma)}), \tag{6.6}$$

where the constant $C = C_\varepsilon$ depends on $\varepsilon > 0$, but is independent of u, $\tilde h_1$ and $\tilde h_2$.

Since $H^2 \subset C^\alpha$, $\alpha < 1/2$, there only remains, in order to finish the proof of the theorem, to study the function u in a small ε-neighborhood of the boundary Γ.

Let $x_0 \in \Gamma$ and $y = y(x)$ be local coordinates in the neighborhood of x_0 such that $y(x_0) = 0$ and Ω is defined in these coordinates by the condition $y_1 > 0$. Then, in the variable y, problem (6.1) reads

$$\begin{cases} \sum_{i,j=1}^3 \partial_{y_i}(a_{ij}(y)\partial_{y_j} u) + \sum_{i=1}^3 b_i(y)\partial_{y_i} u + c(y)u - f(u) = \tilde h_1, & y_1 > 0, \\ \sum_{i,j=2}^3 \partial_{y_i}(d_{ij}(y)\partial_{y_j} u) + \sum_{i=2}^3 e_i(y)\partial_{y_i} u + g(y)u + \tilde h_2 = \partial_{y_1} u, & y_1 = 0, \end{cases} \tag{6.7}$$

where a_{ij}, b_i, c, d_{ij}, e_i and g are smooth functions which satisfy uniform ellipticity assumptions.

Differentiating the first equation of (6.7) with respect to y_k, $k = 2, 3$, multiplying the resulting equation by ϕv_k, where $v_k := \partial_{y_k} u$ and ϕ is a smooth nonnegative cut-off function which is equal to one in the ball $|y| \le \varepsilon$ and zero outside the ball $|y| \ge 2\varepsilon$ and satisfies

(6.5), using again the fact that $f' \geq 0$ and the ellipticity assumption on the a_{ij}, we find, after standard transformations,

$$\gamma(\phi|\nabla_x v_k|, |\nabla_x v_k|)_\Omega + (\phi v_k, a_{11}(y)\partial_{y_1} v_k)_\Gamma + (\phi v_k, v_k)_\Omega \leq C(\|u\|^2_{H^1(\Omega)} + \|\tilde{h}_1\|^2_{L^2(\Omega)}), \tag{6.8}$$

where the positive constants C and γ are independent of u. Differentiating then the second equation of (6.7) with respect to y_k, inserting the expression for $\partial_{y_1} v_k$ thus obtained into (6.8) and arguing analogously, we have

$$\begin{aligned}\gamma(\phi|\nabla_x v_k|, |\nabla_x v_k|)_\Omega + \gamma(\phi|\nabla_{y_2,y_3} v_k|, |\nabla_{y_2,y_3} v_k|)_\Gamma + (\phi v_k, v_k)_\Omega + (\phi v_k, v_k)_\Gamma \leq \\ \leq C(\|u\|^2_{H^1(\Omega)} + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Gamma)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}).\end{aligned} \tag{6.9}$$

Combining this estimate with (6.2), we finally end up with

$$\begin{aligned}\|\phi u\|^2_{L^2(\mathbb{R}^+_{y_1}, H^2(\mathbb{R}^2_{y_2,y_3}))} + \|\phi u\|^2_{H^1(\mathbb{R}^+_{y_1}, H^1(\mathbb{R}^2_{y_2,y_3}))} + \|\phi u\big|_{y_1=0}\|^2_{H^2(\mathbb{R}^2_{y_2,y_3})} \leq \\ \leq C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}),\end{aligned} \tag{6.10}$$

where the constant C is independent of x_0 and u.

Returning to the variable x and using the fact that the boundary point x_0 is arbitrary (and that Γ is smooth), we infer from (6.10) that

$$\|\nabla_x D_\tau u\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\Gamma)} \leq C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}). \tag{6.11}$$

In addition, owing to the embedding

$$L^2(\mathbb{R}, H^2(\mathbb{R}^2)) \cap H^1(\mathbb{R}, H^1(\mathbb{R}^2)) \subset C^\alpha(\mathbb{R}^3), \quad \alpha < 1/4,$$

estimate (6.10), together with (6.6), also imply the estimate

$$\|u\|^2_{C^\alpha(\Omega)} \leq C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}), \quad \alpha < 1/4.$$

Thus, in order to finish the proof of the theorem, we only need to estimate the L^1-norm of $F(u)$ on the boundary. To this end, we also use the localized equations (6.7), but now multiply the first one by $\phi\partial_{y_1} u$. Then, after obvious transformations, we have

$$\begin{aligned}&\partial_{y_1}(\frac{1}{2}\phi(y)a_{11}(y)|\partial_{y_1} u|^2 + \phi(y)F(u)) \geq \\ &\geq -C(\phi + |\nabla_y \phi| + |D^2_y \phi|)(|\nabla_y u|^2 + |\partial^2_{y_1,y_2} u|^2 + |\partial^2_{y_1,y_3} u|^2 + |D^2_{y_2,y_3} u|^2 + F(u) + |\tilde{h}_1|^2),\end{aligned} \tag{6.12}$$

where the constant C is independent of $x_0 \in \Gamma$ and u. Integrating this estimate with respect to $y \in \mathbb{R}^+ \times \mathbb{R}^2$ and using (6.2) and (6.10), together with the fact that $\phi \neq 0$ only in a small neighborhood of the boundary, we see that

$$\begin{aligned}\int_{\mathbb{R}^2} (\phi(0, y_1, y_2)F(u(0, y_1, y_2)) - \phi(0, y_1, y_2)a_{11}(0, y_1, y_2)|\partial_{y_1} u(0, y_1, y_2)|^2)\, dy_1\, dy_2 \leq \\ \leq C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}).\end{aligned} \tag{6.13}$$

Thus, keeping in mind the fact that $\phi(y)$ is nonnegative and is equal to one close to the given point $x_0 \in \Gamma$, we conclude, returning to the variable x, that

$$\|F(u)\|_{L^1(\Gamma)} \leq C\|\partial_n u\|^2_{L^2(\Gamma)} + C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}).$$

There now only remains to note that, owing to estimate (6.11) and the second equation of (6.1), we can control the L^2-norm of $\partial_n u$ on the boundary,

$$\|\partial_n u\|^2_{L^2(\Gamma)} \le C(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}).$$

The control of the L^1-norm of $F(u)$ on the boundary thus follows and Theorem 6.1 is proved. $\square$

Our next task is to give an example of equations (6.1) for which the solution u does not satisfy the equations in the usual sense, but only in the variational sense described in Section 3. We recall that such an example cannot be found if the potential $F(u)$ is singular at ± 1. However, the situation is essentially different if the potential $F(u)$ has finite limits as $u \to \pm 1$. Indeed, in that case, the control of the L^1-norm of $F(u)$ is of no use and the singular part of the boundary (where $|u(x)| = 1$) may now have positive measure and may even coincide with the whole boundary. As we can see from the following example, the equality $[\partial_n u]_{int} = [\partial_n u]_{ext}$ can be violated at such singular points.

Example 6.2. We consider the following example of a one dimensional boundary value problem of the form (6.1):

$$y'' - f(y) = 0, \quad y'(\pm 1) = K \ge 0, \quad x \in [-1, 1], \tag{6.14}$$

where the function f satisfies assumptions (2.4) and, in addition, $F(1) = F_1 < \infty$ and $f(-y) = -f(y)$, which is of course a particular case of our general theory. Then, an analysis of the above ODE shows that, for relatively small values of K, this problem has a regular usual solution $y_K(x)$ which is odd,

$$y_K(-x) = -y_K(x)$$

(owing to the symmetry and the uniqueness), and is separated from the singularities of f. However, there exists a critical value K_+ such that, for $K > K_+$, y_K coincides with the singular solution y_+ of the problem

$$y_+'' - f(y_+) = 0, \quad y_+(1) = 1, \quad y_+(-1) = -1.$$

Thus, the usual solution of (6.14) does not exist for $K > K_+$. However, for these values of K, it can be uniquely defined as a variational solution. For the reader's convenience, we also give below a simple alternative proof of the above nonexistence fact which can be partially extended to the multi-dimensional case. Since $\|y_K\|_{L^\infty([-1,1])} \le 1$, the usual interior regularity techniques (see the interior regularity estimate in Theorem 6.1) show that

$$|y_K'(x)| \le C, \quad |y_K(x)| \le 1 - \delta, \quad x \in (-1/2, 1/2), \tag{6.15}$$

where the positive constants C and δ are *independent* of K. Multiplying now equation (6.14) by y', integrating over $[0, 1]$ and using (6.15), we obtain

$$|\frac{1}{2}|y_K'(1)|^2 - F(y_K(1))| \le C, \tag{6.16}$$

where the constant C is again independent of K. Thus, y_K cannot satisfy the boundary condition $y_K'(1) = K$ if K is large enough and $F(1)$ is finite.

Remark 6.3. Let $y_K(x)$ be a variational solution of problem (6.14) as constructed in the previous section. Then, since this solution is odd, it automatically satisfies the equation

$$y'' - f(y) = \langle y'' - f(y) \rangle_{[-1,1]}$$

and, therefore, it is a (variational) equilibrium for the corresponding 1D Cahn-Hilliard problem of the form (2.1). Thus, even in the 1D case, problem (2.1) can have variational solutions which do not satisfy the boundary conditions in the usual sense.

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