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EXTENDING THE KNOPS-STUART-TAHERI TECHNIQUE TO C^1 WEAK LOCAL MINIMIZERS IN NONLINEAR ELASTICITY

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ABSTRACT. We prove that any C^1 weak local minimizer of a certain class of elastic stored-energy functionals $I(u) = \int_\Omega f(\nabla u)\,dx$ subject to a linear boundary displacement $u_0(x) = \xi x$ on a star-shaped domain Ω with C^1 boundary is necessarily affine provided f is strictly quasiconvex at ξ. This is done without assuming that the local minimizer satisfies the Euler-Lagrange equations, and therefore extends in a certain sense the results of Knops and Stuart, and those of Taheri, to a class of functionals whose integrands take the value $+\infty$ in an essential way.

1. INTRODUCTION

This short paper advances arguments to be found in [22] concerning the relative energies of C^1 weak local minimizers of energy functionals of the form

$$I(u) = \int_\Omega f(\nabla u(x))\,dx. \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a star-shaped domain with a C^1 boundary, $u : \Omega \to \mathbb{R}^m$ belongs to an appropriate Sobolev space, and $f : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$ belongs to a particular class of quasiconvex functions that are sufficiently smooth where finite. Previous works on this topic, most notably [13] and [22], established the uniqueness of sufficiently smooth solutions of the Euler-Lagrange equations associated with the functional (1.1) and subject to a linear boundary displacement. Formally, these are solutions of the system

$$\operatorname{div} Df(\nabla u) = 0, \tag{1.2}$$

where as usual $Df(A)$ is the $m \times n$ matrix whose (i,j) entry is $\frac{\partial f(A)}{\partial A_{ij}}$.

The technique referred to in the title, first used by Knops and Stuart in nonlinear elastostatics [13] and later developed by Taheri in [22], can be distilled into two steps, the ultimate goal of which is to compare two energies $I(u)$ and $I(v)$, say, where u and v agree on $\partial\Omega$ and at least one of them is a stationary point in some appropriate sense. The first step is to write the energies as integrals over the boundary $\partial\Omega$. The second hinges on the observation that if u and v agree on $\partial\Omega$ and are sufficiently smooth, then $\nabla u(x) - \nabla v(x)$ is a matrix of rank one provided

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$x \in \partial\Omega$. Thus one can use rank-one convexity of f to order $\int_{\partial\Omega} f(\nabla u(x))$ and $\int_{\partial\Omega} f(\nabla v(x))$, and hence, by step 1, to order $I(u)$ and $I(v)$. (See (2.1) and (1.5) below for the definition of rank-one convexity and quasiconvexity, respectively.)

In the intervening period the results contained in [13] applying to nonlinear elasticity were rederived by Sivaloganathan [21] using an interesting invariant integral method. Both [21] and [13] rely crucially on the smoothness of the solution to (1.2) to circumvent potential difficulties associated with the so-called stored-energy functions commonly used in nonlinear elasticity theory. In the case $m = n = 3$, for example, the corresponding f are polyconvex and take the form

$$f(A) = g(A, \operatorname{cof} A, \det A), \tag{1.3}$$

where g is convex on $\mathbb{R}^{3\times 3}_+ \times \mathbb{R}^{3\times 3}_+ \times \mathbb{R}_+$, and $f(A) = +\infty$ if $\det A \le 0$. This class of functions was introduced and subsequently developed by Ball in [1], [2], and studied by others, including but not limited to [20], [6], [7], and [18]. See [3] for an overview.

The results of this paper apply to stored-energy functions for which additional regularity results, such as those of [6], are available. Introduced by Ball in [1], these f take the special form

$$f(A) = F(A) + h(\det A), \tag{1.4}$$

where $h : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ is convex and satisfies $h(s) = +\infty$ for all $s \le 0$, and where $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is C^1, quasiconvex and satisfies for some $q \ge n$ and all $n \times n$ matrices A the inequality

$$c|A|^q \le F(A) \le C(1 + |A|^q)$$

with constants $c, C > 0$. We recall that a function $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if

$$\int_\Omega f(A + \nabla\varphi)\, dx \ge \int_\Omega f(A)\, dx \tag{1.5}$$

for all $m \times n$ matrices A and all Lipschitz functions φ vanishing on $\partial\Omega$, and strictly quasiconvex if (1.5) holds with strict inequality whenever $\varphi \ne 0$. See [9] for further details.

Taheri's approach [22] applies to C^1 integrands f satisfying a p-growth condition

$$|f(A)| \le c(1 + |A|^p), \tag{1.6}$$

where $1 \le p < \infty$, c is a constant and A is any $m\times n$ real matrix. Although condition (1.6) is clearly not satisfied by integrands such as (1.4), [22] nevertheless contains an innovation which can be exploited in the context of stored-energy functions. Taheri observes that the conservation law [13, Proposition 2.1] relied on by Knops and Stuart can be replaced by a weaker conservation law, the so-called energy-momentum equations:

$$\operatorname{div}\left(f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)\right) = 0. \tag{1.7}$$

Here, $f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)$ is Eshelby's energy-momentum tensor; it is classically derived by applying Noether's theorem to the variational symmetry $x \mapsto x + a$, $a \in \mathbb{R}^n$. It is well-known that (1.7) can be derived rigorously not only for weak local minimizers of functionals whose integrands f satisfy (1.6) but also for stored-energy functions such as (1.4). See [4] or [6] for details.

The Euler-Lagrange equation (1.2), however, may not automatically hold for general forms of the stored energy including functions of the form (1.4), even while

(1.7) holds. See [8] for an example; see also [12], [19] and [11]. Indeed, it forms part of the hypotheses of the main results in [13], [21] and [22]. But in this paper we note that the full Euler-Lagrange equations are not needed in order to apply Taheri's argument [22]. In fact, it is sufficient that the weak local minimizer is only a 'subsolution' of the Euler-Lagrange equations in a small neighbourhood of the boundary. This point is clarified in Section 3.2 below, but to give an initial idea let us suppose for now that u is a smooth solution of the Euler-Lagrange equation (1.2). A straightforward approximation argument can be used to check that

$$\int_{\Omega} Df(\nabla u)\cdot\nabla u\,dx = \int_{\partial\Omega} Df(\nabla u(y))\cdot u(y)\otimes\nu(y)\,d\mathcal{H}^{n-1}(y),$$

where ν is the outward pointing normal to $\partial\Omega$. By 'subsolution' we mean, roughly speaking, that

$$\int_{\Omega} Df(\nabla u)\cdot\nabla u\,dx \leq \int_{\partial\Omega} Df(\nabla u(y))\cdot u(y)\otimes\nu(y)\,d\mathcal{H}^{n-1}(y). \tag{1.8}$$

We therefore introduce in Section 3 a functional $K(u)$ with the property that $K(u) < \infty$ implies that a suitable version of (1.8) holds. In particular, we do not assume that u is a solution of the Euler-Lagrange system (1.2). $K(u)$ is effectively a limiting measure of the 'twist' of the function u near the boundary of the domain: we return to this point below. To conclude the summary, inequality (1.8) then allows us to compare the bulk energies

$$I(u^{\text{hom}}) \geq I(u),$$

where u^{hom} is the one-homogeneous extension of $u|_{\partial\Omega}$ and u is the C^1 weak local minimizer. For less regular u a weaker statement can be deduced; its limitations can most profitably be viewed in the context of [14].

The paper is organized as follows. In Section 3 we motivate and discuss the functional K referred to above. The main result of Section 3 is Lemma 3.3, yielding an inequality such as (1.8) subsequently used in Section 4 to compare the energies $I(u^{\text{hom}})$ and $I(u)$. The results apply to general boundary data up to the end of Section 4.1; in Section 4.2 the boundary data is assumed to be linear and admissible in the sense outlined in Section 2 below. The paper concludes with a brief discussion of how these methods might be adapted to weak local minimizers that are not necessarily C^1.

2. Notation and preliminaries

We denote the $m \times n$ real matrices by $\mathbb{R}^{m\times n}$, and unless stated otherwise we sum over repeated indices. We denote those $n \times n$ real matrices with positive determinant by $\mathbb{R}^{n\times n}_{+}$, and the identity matrix by $\mathbf{1}$. Throughout B is the unit ball in $\mathbb{R}^2$, and B_t the ball centred at 0 with radius t. We say that a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}\cup\{\infty\}$ is rank-one convex if

$$f(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda f(\xi_1) + (1-\lambda)f(\xi_2) \tag{2.1}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(\xi_1 - \xi_2) = 1$ and all $\lambda \in [0, 1]$. When f is everywhere real-valued this condition is implied by quasiconvexity; for extended real-valued f the implication need not hold. See [9, Chapter 5] for a proof of the former, and [5] for an example of the latter.

Other standard notation includes $||\cdot||_{k,p;\Omega}$ for the norm on the Sobolev space $W^{k,p}(\Omega)$, $||\cdot||_{p;\Omega}$ for the norm on $L^p(\Omega)$, and $\rightharpoonup$ to represent weak convergence in both of these spaces. $\mathcal{H}^k$ represents k-dimensional Hausdorff measure. The tensor product of two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ is written $a \otimes b$; it is the $m \times n$ matrix whose (i, j) entry is $a_i b_j$. The inner product of two matrices $X, Y \in \mathbb{R}^{m\times n}$ is $X \cdot Y = \operatorname{tr}(X^T Y)$. This obviously holds for vectors too.

The functional I will henceforth be

$$I(u) = \int_\Omega f(\nabla u)\,dx,$$

where f is defined in (1.4). In addition, we assume that there are constants t_0, $s > 0$, $c_2 > c_1 > 0$ such that

$$c_1 t^{-s-j} \leq (-1)^j \frac{d^j h(t)}{dt^j} \leq c_2 t^{-s-j} \tag{2.2}$$

for $j = 0, 1, 2$ and all $t \in (0, t_0)$. This assumption allows us to apply the results of [6] later in the paper.

Since the set Ω is assumed to be star-shaped with a C^1 boundary we can write

$$\Omega = \{x \in \mathbb{R}^n :\ |x| < d(\theta(x))\},$$

where $\theta(x) = \frac{x}{|x|}$ for nonzero x, and $d : \mathbb{S}^{n-1} \to \mathbb{R}$ is C^1. In this notation the normal $N(\theta(x))$ to $\partial\Omega$ at $x \in \partial\Omega$ is

$$N(\theta(x)) = \frac{1}{\alpha(\theta)}\left(\theta - (\mathbf{1} - \theta \otimes \theta)\frac{\nabla d}{d}\right),$$

where α is chosen so that $|N| = 1$.

Let

$$\mathcal{A}_{u_0} = \{v \in W^{1,n}(\Omega, \mathbb{R}^n) :\ I(v) < \infty,\ \operatorname{tr} v = \operatorname{tr} u_0\},$$

where $\operatorname{tr} u_0$ is the trace of a fixed function for which $I(u_0) < \infty$.

Definition 2.1. We shall say that $u \in \mathcal{A}_{u_0}$ is a weak local minimizer of I in $\mathcal{A}_{u_0}$ if there exists $\delta > 0$ such that any $v \in \mathcal{A}_{u_0}$ satisfying $||v - u||_{1,\infty;\Omega} \leq \delta$ necessarily satisfies $I(v) \geq I(u)$.

3. Weak local minimizers with positive twist near the boundary

It is clear from the definition of the functional I that any admissible function u necessarily satisfies $\det \nabla u > 0$ almost everywhere. Our strategy, by analogy with [22], will be to compare $I(u^{\text{hom}})$ with $I(u)$, where u is a C^1 weak local minimizer of I and u^{hom} is the one-homogeneous extension of the restriction of u to $\partial\Omega$. (See below for details.) In particular, were $\det \nabla u^{\text{hom}} > 0$ to fail on a set of positive Lebesgue measure, then the desired inequality

$$I(u^{\text{hom}}) \geq I(u)$$

would be trivial. Using the functional K described below we are able to restrict attention to those admissible u for which $\det \nabla u^{\text{hom}} > 0$ holds $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$; properties of one-homogeneous functions then imply that $\det \nabla u^{\text{hom}} > 0$ holds $\mathcal{L}^n$-almost everywhere in Ω.

3.1. **One-homogeneous extensions and the functional K.** Let $u \in \mathcal{A}_{u_0}$, let $t \in (0,1]$ and define $u_t(x) = u(t\theta d(\theta))$ for $x \in \Omega$ such that $|x| = td(\theta(x))$. Thus u_t is the restriction of u to the boundary of the set

$$\Omega_t = \{x \in \Omega : \; |x| < td(\theta)\}.$$

We define the one-homogeneous extension u_t^{hom} of u_t by

$$u_t^{\text{hom}}(x) = \frac{|x|}{td(\theta)} u(t\theta d(\theta))$$

for each $x \in \Omega$. Then $\nabla u_t(x)$ exists for almost every $x \in \Omega_t$, and in this case it follows that

$$\nabla u_t^{\text{hom}}(x) = \nabla u(t\theta d(\theta)) + \left(\frac{u(t\theta d(\theta))}{td(\theta)} - \nabla u(t\theta d(\theta))\theta\right) \otimes \alpha N.$$

Hence

$$\det \nabla u^{\text{hom}}(x) = \text{cof}\, \nabla u(\theta d(\theta)) \cdot \left(u(t\theta d(\theta)) \otimes \frac{\alpha N}{td(\theta)}\right). \tag{3.1}$$

Since $\det \nabla_t^{\text{hom}}$ clearly depends only on $\theta(x)$, it follows that $\det \nabla u_t^{\text{hom}} > 0$ $\mathcal{L}^n$-almost everywhere if and only if

$$\text{cof}\, \nabla u(t\theta d(\theta)) \cdot \left(u(t\theta d(\theta)) \otimes \frac{\alpha N}{td(\theta)}\right) > 0 \quad \mathcal{H}^{n-1}\text{-a.e.} \tag{3.2}$$

Remark 3.1. When Ω is the unit ball B in $\mathbb{R}^2$ and when u is sufficiently smooth, condition (3.2) with $t = 1$ is equivalent to the condition that $u^{\text{hom}}(\partial B)$ is the boundary of a star-shaped region. The definition of u^{hom} then implies that $u^{\text{hom}}(B)$ is star-shaped. Alternatively, maps u with $\det \nabla u^{\text{hom}} > 0$ $\mathcal{H}^1$-a.e. may be interpreted as having a 'positive twist' at the boundary ∂B. To see this we appeal to a result of Littlewood [15, Theorem 253]. Indeed, setting

$$w(e^{i\alpha}) = u_1(\cos\alpha, \sin\alpha) + iu_2(\cos\alpha, \sin\alpha),$$

writing $w = R(\alpha)e^{i\Phi(\alpha)}$, and using $N(\theta(x)) = \theta(x) = x$ when $x \in \partial B$, $d(\theta(x)) = 1$ for all $x \in B$, it follows from

$$\text{cof}\, \nabla u(\theta) \cdot (u(\theta) \otimes \theta) = \text{Re}\, (\overline{iw}\partial_\alpha w)$$

that

$$\det \nabla u^{\text{hom}} = R^2 \partial_\alpha \Phi. \tag{3.3}$$

Now, [15, Theorem 253] states that the positivity $\mathcal{H}^1$-a.e. of

$$\text{Re}\left(\frac{zw'(z)}{w(z)}\right)$$

with $z = e^{i\alpha}$ is necessary and sufficient for

$$\{w(e^{i\alpha}) : \alpha \in [0, 2\pi]\}$$

to be star-shaped. A short calculation shows that

$$\text{Re}\left(\frac{zw'(z)}{w(z)}\right) = \partial_\alpha \Phi,$$

which has the same sign as the term $R^2\partial_\alpha\Phi$ appearing in (3.3). Therefore (3.2) holds if and only if $u^{\text{hom}}(B)$ is star-shaped.

Remark 3.2. Littlewood's proof can be adapted to show that general two-dimensional star-shaped domains for which (3.2) holds are such that $u^{\text{hom}}(\Omega)$ is also star-shaped. Whether the same is true for star-shaped Ω and sufficiently smooth maps $u : \Omega \to \mathbb{R}^n$, $n \geq 3$, is an interesting question. We note that u may be required to satisfy certain smoothness and invertibility hypotheses in order to infer $u(B) = u^{\text{hom}}(B)$ from the fact that $u^{\text{hom}} = u$ on ∂B. See [16] for results of this kind.

Now for smooth enough u the assumption of (3.2) at the boundary $\partial\Omega$ would suffice for our purposes; but for less regular competitors we need to strengthen (3.2) to hold 'asymptotically close to $\partial\Omega$'. To make this precise, let $s \geq 3$ be an integer, let $t \in [\frac{1}{2}, 1]$ and define

$$e_t^{(s)}(x) = \chi_{B_t \setminus B_{t-\frac{1}{s}}}(x) \frac{\alpha N}{d}.$$

Let $\sigma : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ be smooth, convex and such that

$$\lim_{y \to 0+, \infty} \sigma(y) = +\infty.$$

Definition 3.1. Let $v \in \mathcal{A}_{u_0}$ and define

$$K(v) = \operatorname{ess} \liminf_{t \to 1} \liminf_{s \to \infty} \int_\Omega \sigma(\operatorname{cof} \nabla v(x) \cdot v(x) \otimes e_t^{(s)}(x))\, dx. \tag{3.4}$$

3.2. **Consequences of $K(v) < \infty$.** The goal of this section is to derive a version of inequality (1.8) for a sequence of sets Ω_{t_n} where $t_n \to 1$. Thus we aim to prove that

$$\int_{\Omega_{t_n}} Df(\nabla u) \cdot \nabla u \, dx \leq \int_{\partial\Omega_{t_n}} Df(\nabla u) \cdot u \otimes \frac{\alpha N}{d}\, d\mathcal{H}^{n-1} \tag{3.5}$$

for a sequence $t_n \to 1^-$. First we note that the weak energy-momentum equations associated with the functional I still have a key role to play.

Proposition 3.1. *Let u be a weak local minimizer of I in $\mathcal{A}$. Then the weak energy-momentum equations hold:*

$$\int_\Omega (f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)) \cdot \nabla\varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^n). \tag{3.6}$$

If u is in addition $C^1(\Omega)$, then

$$\frac{1}{\det \nabla u} \in L^p(\Omega') \quad \forall\, p \in (1, \infty) \tag{3.7}$$

and each $\Omega' \Subset \Omega$.

Proof. The energy-momentum equations are usually derived by considering so-called inner variations of the form

$$u_\delta(x) := u(x + \delta\varphi(x)),$$

where φ is a fixed but arbitrary test function. Provided δ is sufficiently small, it is easily checked that u_δ is both admissible and $W^{1,\infty}$-close to u. Consequently the limit

$$\lim_{\delta \to 0} \frac{I(u_\delta) - I(u)}{\delta}$$

is zero whenever it exists. One can now follow [7, Theorem A.1] or [3, Theorem 2.4(ii)] to deduce (3.6).

Statement (3.7) follows by first noting that $|\nabla u|^q \in L^p(\Omega)$ for all $p \in (1,\infty)$ and each $\Omega' \Subset \Omega$ whenever $u \in C^1(\Omega)$ and then by applying [6, Lemma 2.4], which states that

$$||(\det \nabla u)^{-s}||_{L^p(\Omega')} \leq C(1 + I(u) + |||\nabla u|^q||_{L^p(\Omega)}).$$

Here, q is the exponent which controls the growth of F in the definition of the stored-energy function W.

□

We remark that $u_\delta - u$ has compact support in Ω, and hence

$$K(u_\delta) = K(u)$$

for all small enough δ. In particular, $K(u_\delta) < \infty$ for all sufficiently small δ whenever $K(u) < \infty$.

Lemma 3.3. *Let u be a C^1 weak local minimizer of I. Let $t < 1$ be such that*

$$\liminf_{s\to\infty} \int_\Omega \sigma(\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)})\, dx < \infty. \tag{3.8}$$

Then

$$\int_{\Omega_t} Df(\nabla u) \cdot \nabla u \, dx \leq \int_{\partial\Omega_t} Df(\nabla u) \cdot u \otimes \frac{\alpha N}{d} \, d\mathcal{H}^{n-1}.$$

Remark 3.4. Condition (3.8) necessarily holds for t in a set of positive measure whenever $K(u) < \infty$. Without loss of generality, therefore, we may assume that (3.14) below and (3.8) hold simultaneously.

Proof. Let

$$\eta_t^{(s)}(x) = \begin{cases} 1 & \text{if } 0 \leq \frac{|x|}{d} \leq t - \frac{1}{s}, \\ s\left(t - \frac{|x|}{d}\right) & \text{if } t - \frac{1}{s} \leq \frac{|x|}{d} \leq t, \\ 0 & \text{if } \frac{|x|}{d} \geq t \end{cases}$$

and note that

$$\nabla \eta_t^{(s)} = -s e_t^{(s)}. \tag{3.9}$$

Let $u^\epsilon(x) = (1 + \epsilon\eta_t^{(s)})u(x)$. Then

$$\det \nabla u^\epsilon = (1 + \epsilon\eta_t^{(s)})^n \det \nabla u - \epsilon s(1 + \epsilon\eta_t^{(s)})^{n-1} \operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)}. \tag{3.10}$$

In view of (3.8), we may assume that

$$\int_\Omega \sigma(\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)})\, dx < \infty$$

for infinitely many s; therefore, for each such s,

$$\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)} > 0$$

for almost every x. In particular, provided $\epsilon < 0$,

$$-\epsilon s(1 + \epsilon\eta_t^{(s)})^{n-1} \operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)} > 0 \text{ a.e.},$$

from which it follows that

$$\det \nabla u^\epsilon > \frac{1}{2} \det \nabla u \text{ a.e.}$$

Since u is a weak local minimizer of I it follows that

$$\limsup_{\epsilon\to 0-} \frac{I(u^\epsilon)-I(u)}{\epsilon} \leq 0. \tag{3.11}$$

The rest of the proof consists in calculating this difference quotient. Now $f(\nabla u)$ is the sum of $F(\nabla u)$ and $h(\det \nabla u)$. The calculation of the quotient

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega (F(\nabla u^\epsilon) - F(\nabla u))\,dx = \int_\Omega DF(\nabla u)\cdot(\eta_t^{(s)}\nabla u + u\otimes\nabla\eta_t^{(s)})\,dx \tag{3.12}$$

is straightforward. We focus on calculating

$$\lim_{\epsilon\to 0-}\int_B \frac{h(\det\nabla u^\epsilon)-h(\det\nabla u)}{\epsilon}\,dx$$

by writing

$$\int_\Omega \frac{h(\det\nabla u^\epsilon)-h(\det\nabla u)}{\epsilon}\,dx = I + II,$$

where

$$\begin{aligned} I &= \int_\Omega \frac{1}{\epsilon}\int_0^\epsilon h'(\det\nabla u^\lambda)(n\eta\det\nabla u + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta)\,d\lambda\,dx,\\ II &= \int_\Omega \frac{1}{\epsilon}\int_0^\epsilon n\lambda\eta h'(\det\nabla u^\lambda)((1+\lambda\eta)^{n-1}-1)\big[n\eta\det\nabla u \\ &\qquad\qquad + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta\big]\,d\lambda\,dx. \end{aligned}$$

We have suppressed the dependence of η on s and t, and $\det\nabla u^\lambda$ is exactly (3.10) with λ in place of ϵ. In each case the integrand is dominated by

$$C(|h'(\det\nabla u)||\det\nabla u| + |h'(\det\nabla u)||\nabla u||\nabla\eta|), \tag{3.13}$$

where C is a constant independent of ϵ and λ. The first term $|h'(\det\nabla)||\det\nabla u|$ in (3.13) is $L^1(\Omega)$ by the inequality $y|h'(y)| \leq C(1+y+h(y))$, which holds for all positive y and which follows from the growth hypotheses on h expressed in (2.2). The second is in $L^1(\Omega)$ by applying (3.7) with $p = s+1$. Note that this reasoning also shows that $Df(\nabla u)\in L^1(\Omega)$. By dominated convergence, $\lim_{\epsilon\to 0} II = 0$ and

$$\lim_{\epsilon\to 0} I = \int_\Omega h'(\det\nabla u)(n\eta\det\nabla u + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta)\,dx.$$

The latter may be rewritten as

$$\int_\Omega Dh(\det\nabla u)\cdot(\eta\nabla u + u\otimes\nabla\eta)\,dx.$$

Thus, in view of (3.12),

$$\lim_{\epsilon\to 0-}\frac{I(u^\epsilon)-I(u)}{\epsilon} = \int_\Omega Df(\nabla u)\cdot(\eta_t^{(s)}\nabla u + u\otimes\nabla\eta_t^{(s)})\,dx.$$

Finally, and bearing in mind (3.9) and (3.11), let $s\to\infty$ to obtain

$$\int_{\Omega_t} Df(\nabla u)\cdot\nabla u\,dx \leq \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1}.$$

Here we used the observation made above that $Df(\nabla u) \in L^1(\Omega)$ together with the fact that

$$\lim_{s\to\infty} s\int_{\Omega_t\setminus\Omega_{t-\frac{1}{s}}} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,dx = \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1} \tag{3.14}$$

for a.e. t; see [10] for details of the latter. This concludes the proof of Lemma 3.3. □

4. Uniqueness subject to linear boundary conditions

4.1. **Comparing $I(u^{\mathbf{hom}})$ and $I(u)$.** Assume for now that u is a weak local minimizer of I in $\mathcal{A}_{u_0}$ and is such that $K(u) < \infty$. Recall that for each $t \in (0,1]$,

$$u_t^{\text{hom}}(x) = \frac{|x|}{td}u(t\theta d)$$

and that

$$\nabla u_t^{\text{hom}}(x) = \nabla u(t\theta d) + \left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N.$$

The fact that the right-hand side is a function of the angular variable θ only suggests that a suitable version of the coarea formula can be used to evaluate $\int_{\Omega_t} f(\nabla u_t^{\text{hom}})\,dx$. One can apply [22, Equation 2.1], or else use a variant of [10, Proposition 3.4.4], to obtain

$$n\int_{\Omega_t} f(\nabla u_t^{\text{hom}})\,dx = t\int_{\partial\Omega_t} f\left(\nabla u(t\theta d) + \left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N\right)d\mathcal{H}^{n-1}. \tag{4.1}$$

Now $f(A)$ is the sum of the everywhere finite quasiconvex function $F(A)$ and the function $h(\det A)$. The former is rank-one convex on $\mathbb{R}^{n\times n}$ by standard results (see, for example, [9, Theorem 5.3 (i)]). The latter is rank-one convex on the half-lines

$$\left\{C_\lambda := \nabla u(t\theta d) + \lambda t\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N : \quad \lambda \geq 0\right\}.$$

This can be verified directly by noting that $\det C_\lambda = \lambda \operatorname{cof}\nabla u(t\theta d)\cdot u(t\theta d)\otimes\frac{\alpha N}{d}$, which by (3.2) implies that $\det C_\lambda > 0$ for $\mathcal{H}^{n-1}$-a.e. θ and all $\lambda > 0$. Since h is convex on $(0,\infty)$, it follows in particular that

$$h(\det C_1) \geq h(\det C_0) + Dh(\det C_0)\cdot t\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N.$$

The rank-one convexity of F implies that exactly the same inequality holds with $F(A)$ in place of $h(\det A)$. Hence, from (4.1),

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u_t^{\text{hom}})\,dx \geq\ & t\int_{\partial\Omega_t} f(\nabla u(t\theta d)) \\ & + Df(\nabla u(t\theta d))\cdot\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.2}$$

Following Taheri's [22] argument, we set $\phi(x) = \eta_t^{(s)}(x)x$ in the energy-momentum equations

$$\int_\Omega \left(f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)\right)\cdot\nabla\phi\,dx = 0.$$

This gives

$$\begin{aligned} 0 &= \int_\Omega n f(\nabla u)\eta_t^{(s)}\,dx - s\int_\Omega f(\nabla u)x\cdot e_t^{(s)}\,dx \\ &\quad + s\int_\Omega \nabla u^T Df(\nabla u)\cdot x\otimes e_t^{(s)}\,dx - \int_\Omega \nabla u\cdot Df(\nabla u)\eta_t^{(s)}\,dx. \end{aligned}$$

Sending $s\to\infty$, applying the result from [4] that $\nabla u^T Df(\nabla u)\in L^1(\Omega)$, and rearranging give

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u)\,dx &= \int_{\Omega_t}\nabla u\cdot Df(\nabla u)\,dx \\ &\quad + t\int_{\partial\Omega_t} f(\nabla u) - \nabla u^T Df(\nabla u)\cdot\theta\otimes\alpha N\,d\mathcal{H}^{n-1} \end{aligned}$$

for a.e. t. Since $K(u)<\infty$, we may assume without loss of generality that condition (3.8) holds for a sequence of t to which the above reasoning also applies. Without relabelling these t, we apply Lemma 3.3 to deduce

$$\int_{\Omega_t} Df(\nabla u)\cdot\nabla u\,dx \le \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1}. \tag{4.3}$$

Therefore

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u)\,dx &\le \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1} \\ &\quad + t\int_{\partial\Omega_t} f(\nabla u) - \nabla u^T Df(\nabla u)\cdot\theta\otimes\alpha N\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.4}$$

When compared with the right-hand side of (4.2), inequality (4.4) implies that

$$\int_{\Omega_t} f(\nabla u)\,dx \le \int_{\Omega_t} f(\nabla u_t^{\mathrm{hom}})\,dx.$$

The above reasoning proves:

Proposition 4.1. *Let $u\in\mathcal{A}_{u_0}$ be a C^1 weak local minimizer of I such that $K(u)<\infty$. Then*

$$\int_{\Omega_{t_n}} f(\nabla u)\,dx \le \int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\,dx \tag{4.5}$$

for a sequence $t_n\to 1$.

Remark 4.1. The calculation shown above is clearly inspired by that given in [22]. But there are two key differences, the main one being that the full Euler-Lagrange equation is not assumed to hold for the weak local minimizer u. Instead, we rely on Lemma 3.3 for the inequality (4.3). Also, the p-growth assumption made in [22] easily supplies the inclusion $Df(\nabla u)\in L^1(\Omega)$ for all $u\in W^{1,p}$. Our route is more circuitous: it relies on estimates in [6] derived from the energy-momentum equations and which only apply to solutions of these equations.

4.2. **Uniqueness of C^1 weak local minimizers.** We now apply the foregoing analysis to the case $u_0(y) = \xi y$, where ξ is a constant $n \times n$ matrix. It is straightforward to check that any $u \in C^1(\bar{\Omega}) \cap \mathcal{A}_{u_0}$ is such that

$$K(u) < \infty \text{ if } \det \xi > 0.$$

Since u is C^1, and in view of the boundary condition, it is the case that

$$\int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\, dx \to \int_{\Omega} f(\xi)\, dx$$

as $n \to \infty$ for any sequence $t_n \to 1^-$. If, in addition, u is a weak local minimizer, then Proposition 4.1 applies, giving

$$\int_{\Omega_{t_n}} f(\nabla u)\, dx \leq \int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\, dx$$

for each n, and hence on letting $n \to \infty$,

$$\int_{\Omega} f(\nabla u)\, dx \leq \int_{\Omega} f(\xi)\, dx. \tag{4.6}$$

We now assume that f is strictly quasiconvex at ξ, implying in particular that

$$\int_{\Omega} f(\xi)\, dx \leq \int_{\Omega} f(\nabla u)\, dx \tag{4.7}$$

with equality if and only if $u(x) = \xi x$ on Ω. Putting (4.6) and (4.7) together yields:

Proposition 4.2. *Let $u \in C^1(\bar{\Omega})$ be a weak local minimizer of I in $\mathcal{A}_{u_0}$, where $u_0(y) = \xi y$, $\det \xi > 0$, and f defined in* (1.4) *is strictly quasiconvex at ξ. Then $u(x) = \xi x$ for all x in Ω.*

4.3. **Concluding remarks.** We briefly address the question of whether (3.4) is the only or right choice for the auxiliary functional K. Clearly, the K defined by (3.4) suffices in the situation that u is $C^1(\bar{\Omega})$. Thus the following remarks apply primarily to weak local minimizers that are not *a priori* assumed to be C^1.

(i) Ideally, any replacement for K (again denoted K) would be sequentially lower semicontinuous with respect to weak convergence in $W^{1,n}$, say. One could then (locally) minimize $I + K$, and the conclusion $K(u) < \infty$ would be automatic rather than imposed.

(ii) Potentially, one could allow the set

$$E := \{x \in \Omega : \ \operatorname{cof} \nabla u \cdot u \otimes \alpha N \leq 0\}$$

to approach $\partial\Omega$ in a less restrictive manner than is prescribed by the condition $K(u) < \infty$, where K is as per (3.4). Indeed, if $K(u)$ is finite, then for t in a set of positive measure,

$$\operatorname{cof} \nabla u \cdot u \otimes \alpha N > 0 \text{ a.e. } x \in \Omega_t \setminus \Omega_{t-\frac{1}{s}}$$

for at least one $s = s(t)$. Moreover, one can take t for which this holds arbitrarily close to 1. So E is trapped in a specific sequence of sets which approach $\partial\Omega$. But it is possible to imagine a set E for which $K(u) = +\infty$ but which might nevertheless admit an analysis similar to that given in Sections 3 and 4 above. This will be investigated in a future paper.

(iii) K should not depend on values of u in the interior of the domain. Energy functionals for elastic materials typically depend only on the gradient of the deformation in the interior. The K proposed in (3.4) does this to an extent; any modifications with (i) and (ii) above in mind should preserve this property. It would not do, for example, to require that for fixed $l < 1$,

$$\hat{K}(v) := \int_{\Omega\setminus\Omega_l} \sigma(\mathrm{cof}\,\nabla u \cdot u \otimes \alpha N)\,dx$$

be finite. Although $\hat{K}$ would be sequentially weakly lower semicontinuous (by [5, Proposition A.3], for example), its value would still depend on $u|_{\Omega\setminus\Omega_l}$.

(iv) Dropping the assumption that u is C^1 is problematic for the reasons pointed out in [22]. See [14, Section 7] for examples of nowhere C^1 weak local minimizers based on the construction of [17]. The assumption $K(v) < \infty$ would appear to limit possible oscillations of ∇u in the direction tangent to $\partial\Omega$, say, but there is still room for bad behaviour in the directions normal to $\partial\Omega$. Any modification of (3.4) should take these difficulties into account.

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