The Hopf superalgebra of AdS/CFT

Alessandro Torrielli

Institute for Theoretical Physics and Spinoza Institute,
Leuvenlaan 4, 3584 CE Utrecht, The Netherlands

Abstract

We describe the unconventional infinite-dimensional Hopf superalgebra related to the integrable S-matrix of the AdS/CFT correspondence, and discuss its typical and atypical representations.

Keywords:
Infinite-dimensional Hopf superalgebras, integrable systems

1. Introduction

In recent years there has been a remarkable progress in Theoretical Physics towards a possible proof of the so-called AdS/CFT conjecture [1][1]. This progress has been generated by the discovery of the following fact. Part of the proof of the conjecture can be translated into the calculation of the spectrum of the Hamiltonian of a certain effective two-dimensional integrable model [3]. This model possesses a symmetry which is based on a particular superalgebra, which will be the topic of our concern. There are by now extensive reviews on the subject [4, 5, 6, 7, 8], but the physical background is needed here only as a motivation. In fact, soon after the appearance of integrability, it has been possible to rephrase a large fraction of the problem in the language of Hopf (super-)algebras. From that moment on, one step towards the solution

Email address: `A.Torrielli@uu.nl` (Alessandro Torrielli)

[1]This has also been linked to a possible advance towards the solution of the Yang-Mills Millennium Prize Problem [2].

has become expressible as a purely mathematical problem, namely, the study of the corresponding Hopf superalgebra, its quasi-triangular structure, and its representation theory.

Such Hopf superalgebra turns out to be quite unconventional, and, as of today, its properties are only partially understood. This motivates the attempt we will make here to a description of its essential mathematical features, with almost complete silence on their physical origin (which the interested reader will be able to find in the upcoming review [9]). The aim is to stimulate the interest of pure mathematicians in the subject, and in attempting a formal approach.

Given the fact that the Hopf superalgebra we will be discussing is infinite-dimensional, and has a structure similar to Yangians [10, 11, 12, 13, 14] (with a "level zero" consituted by an ordinary Lie superalgebra, and a "level one" set of generators which serve as a seed to generate an infinite-dimensional algebra), we have divided our presentation everywhere in "level zero" and "level one", wherever it is not a source of confusion[2]. In the Conclusions, we list the many open problems which we hope will stimulate the mathematical curiosity of the reader[3].

2. The Hopf superalgebra: Level zero

The Hopf superalgebra we will discuss is based on the Lie superalgebra $A(1,1) = \mathfrak{psl}(2|2)$ with three-fold central extension. We will denote this centrally-extended Lie superalgebra with $\mathfrak{psl}(2|2)_c$. The possibility of such a large central extension is a unique occurrence among the basic classical simple Lie superalgebras [19]. The even part of $\mathfrak{psl}(2|2)_c$ consists of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and of the space generated by the three central elements, which we will de-

[2]For Yangians based on Lie superalgebras, see *e.g.* [15, 16, 17, 18].

[3]References are also reduced to the bone for readability reasons. The author apologizes for any omission and he will be happy to put remedy when informed.

note[4] as $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$. The odd part forms a basis for the representation $(2,\bar{2})\oplus(\bar{2},2)$ of the even part, where the entries in brackets correspond to the $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ decomposition [20]. Odd generators transforming as $(2,\bar{2})$ will be denoted by $\mathbb{Q}_\alpha^{\ a}$, while those transforming as $(\bar{2},2)$ will be denoted by $\mathbb{G}_a^{\ \alpha}$, for $a=1,2$ and $\alpha=3,4$. Latin indices refer to the first $\mathfrak{sl}(2)$, generated by $\mathbb{L}_a^{\ b}$ with the constraint $\sum_{a=1}^2 \mathbb{L}_a^{\ a}=0$, while greek indices refer to the second $\mathfrak{sl}(2)$, generated by $\mathbb{R}_\alpha^{\ \beta}$ with the constraint $\sum_{\alpha=3}^4 \mathbb{L}_\alpha^{\ \alpha}=0$.

The commutation relations are as follows [21]:

$$
\begin{aligned}
&[\mathbb{L}_a^{\ b},\mathbb{J}_c]=\delta_c^b\mathbb{J}_a-\tfrac{1}{2}\delta_a^b\mathbb{J}_c, && [\mathbb{R}_\alpha^{\ \beta},\mathbb{J}_\gamma]=\delta_\gamma^\beta\mathbb{J}_\alpha-\tfrac{1}{2}\delta_\alpha^\beta\mathbb{J}_\gamma,\\
&[\mathbb{L}_a^{\ b},\mathbb{J}^c]=-\delta_a^c\mathbb{J}^b+\tfrac{1}{2}\delta_a^b\mathbb{J}^c, && [\mathbb{R}_\alpha^{\ \beta},\mathbb{J}^\gamma]=-\delta_\alpha^\gamma\mathbb{J}^\beta+\tfrac{1}{2}\delta_\alpha^\beta\mathbb{J}^\gamma,\\
&\{\mathbb{Q}_\alpha^{\ a},\mathbb{Q}_\beta^{\ b}\}=\varepsilon_{\alpha\beta}\varepsilon^{ab}\mathbb{C}, && \{\mathbb{G}_a^{\ \alpha},\mathbb{G}_b^{\ \beta}\}=\varepsilon^{\alpha\beta}\varepsilon_{ab}\mathbb{C}^\dagger,\\
&\{\mathbb{Q}_\alpha^a,\mathbb{G}_b^\beta\}=\delta_b^a\mathbb{R}_\alpha^{\ \beta}+\delta_\alpha^\beta\mathbb{L}_b^{\ a}+\tfrac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbb{H}.
\end{aligned}
$$

where $\mathbb{J}$ denotes any odd generator with the appropriate index displayed. The $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ commutation relations are straightforward in this notation and we do not report them explicitly. The elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$ commute with all the generators.

The algebra $\mathfrak{psl}(2|2)_c$ can be obtained as a certain contraction of the simple Lie superalgebra $D(2,1;\alpha)$ (see for instance [21, 22, 23, 24]), and for this reason it is sometimes indicated with the symbol $D(2,1;-1)$. The Killing form vanishes identically. The algebra admits a large outer automorphism group $\mathfrak{sl}(2)$ [25], which is inherited from the simple version $A(1,1)$ [26]. The action of this outer automorphism on the central elements is an $\mathfrak{sl}(2)$ rotation on the three-vector $(\mathbb{H},\mathbb{C},\mathbb{C}^\dagger)$ preserving the "norm" $\mathbb{H}^2-\mathbb{C}\mathbb{C}^\dagger$ equal to a constant.

One can put a non-trivial Hopf algebra structure on $\mathfrak{psl}(2|2)_c$, defined by the following coproduct [27, 28]. For any $\mathfrak{J}^A\in\mathfrak{psl}(2|2)_c$, $A=1,\dots,17$

[4] The 'dagger' is an abuse of notation to remind that, in unitary representations, the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ are hermitean conjugate to each other.

labeling the independent generators of $\mathfrak{psl}(2|2)_c$,

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes e^{i[[A]]p} + \mathbb{1} \otimes \mathfrak{J}^A,$$
$$\Delta(e^{ip}) = e^{ip} \otimes e^{ip}, \tag{2.1}$$

where p is a central element. The number $[[A]]$ equals 0 for generators in $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and for $\mathbb{H}$, $\frac{1}{2}$ for $\mathbb{Q}^a_\alpha$, $-\frac{1}{2}$ for $\mathbb{G}^\alpha_a$, 1 for $\mathbb{C}$ and -1 for $\mathbb{C}^\dagger$. The above coproduct can be easily shown to be a Lie algebra homomorphism. The corresponding counit and antipode are straightforwardly derived from the Hopf algebra axioms. A complete description is presented in [28].

From the point of view of physics, one would like to render this Hopf superalgebra quasi-cocommutative. This would imply the existence of an invertible element $R \in U(\mathfrak{psl}(2|2)_c) \otimes U(\mathfrak{psl}(2|2)_c)$ such that

$$\Delta^{op} R = R\Delta. \tag{2.2}$$

Since $\Delta(\mathbb{C})$ is central in $U(\mathfrak{psl}(2|2)_c) \otimes U(\mathfrak{psl}(2|2)_c)$, this would require as a necessary condition

$$\Delta^{op}(\mathbb{C})R = R\Delta(\mathbb{C}) = \Delta(\mathbb{C})R \qquad \Longrightarrow \qquad \Delta^{op}(\mathbb{C}) = \Delta(\mathbb{C}) \tag{2.3}$$

(analogously for $\mathbb{C}^\dagger$). This is guaranteed by the physicality conditions[5] [21]

$$e^{ip} = \mathbb{C} + \mathbb{1} \qquad \textit{and} \qquad e^{-ip} = \mathbb{C}^\dagger + \mathbb{1}. \tag{2.4}$$

With these conditions, one has

$$\Delta(\mathbb{C}) = \mathbb{C} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{C} + \mathbb{C} \otimes \mathbb{C} = \Delta^{op}(\mathbb{C}) \tag{2.5}$$

and similarly for $\Delta(\mathbb{C}^\dagger)$. The conditions (2.4) imply the quadratic relation $\mathbb{C}\mathbb{C}^\dagger + \mathbb{C} + \mathbb{C}^\dagger = 0$ in the universal enveloping algebra $U(\mathfrak{psl}(2|2)_c)$. From now on, we will always assume that we are dealing with $U(\mathfrak{psl}(2|2)_c)$ modulo the equivalence relations imposed by (2.4).

[5]In physical terms, these conditions guarantee "momentum conservation" of the two-particle scattering governed by the scattering matrix R.

3. The Hopf superalgebra: Level one

One can add another set of generators to the ones described in the previous section, and generate an infinite-dimensional Hopf algebra (which we will call Y) in a way similar to what happens for Yangians [29]. The presentation of this infinite-dimensional Hopf algebra we adopt here is the following, in the spirit of Drinfeld's second realization of the Yangian [30]. It is given in terms of Cartan generators $\kappa_{i,m}$ and fermionic simple roots $\xi^{\pm}_{i,m}$, $i = 1,2,3$, $m = 0,1,2,\ldots$, subject to the following relations [31]:

$$
\begin{aligned}
&[\kappa_{i,m},\kappa_{j,n}] = 0, \quad [\kappa_{i,0},\xi^{+}_{j,m}] = a_{ij}\,\xi^{+}_{j,m},\\
&[\kappa_{i,0},\xi^{-}_{j,m}] = -a_{ij}\,\xi^{-}_{j,m}, \quad \{\xi^{+}_{i,m},\xi^{-}_{j,n}\} = \delta_{i,j}\,\kappa_{j,n+m},\\
&[\kappa_{i,m+1},\xi^{+}_{j,n}] - [\kappa_{i,m},\xi^{+}_{j,n+1}] = \frac{1}{2}a_{ij}\{\kappa_{i,m},\xi^{+}_{j,n}\},\\
&[\kappa_{i,m+1},\xi^{-}_{j,n}] - [\kappa_{i,m},\xi^{-}_{j,n+1}] = -\frac{1}{2}a_{ij}\{\kappa_{i,m},\xi^{-}_{j,n}\},\\
&\{\xi^{+}_{i,m+1},\xi^{+}_{j,n}\} - \{\xi^{+}_{i,m},\xi^{+}_{j,n+1}\} = \frac{1}{2}a_{ij}[\xi^{+}_{i,m},\xi^{+}_{j,n}],\\
&\{\xi^{-}_{i,m+1},\xi^{-}_{j,n}\} - \{\xi^{-}_{i,m},\xi^{-}_{j,n+1}\} = -\frac{1}{2}a_{ij}[\xi^{-}_{i,m},\xi^{-}_{j,n}],
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&i \neq j, \;\; n_{ij} = 1+|a_{ij}|, \;\; Sym_{\{k\}}[\xi^{+}_{i,k_1},[\xi^{+}_{i,k_2},\ldots\{\xi^{+}_{i,k_{n_{ij}}},\xi^{+}_{j,l}\}\ldots\}\} = 0,\\
&i \neq j, \;\; n_{ij} = 1+|a_{ij}|, \;\; Sym_{\{k\}}[\xi^{-}_{i,k_1},[\xi^{-}_{i,k_2},\ldots\{\xi^{-}_{i,k_{n_{ij}}},\xi^{-}_{j,l}\}\ldots\}\} = 0,\\
&\text{except for} \quad \{\xi^{+}_{2,n},\xi^{+}_{3,m}\} = \mathbb{C}_{n+m}, \qquad \{\xi^{-}_{2,n},\xi^{-}_{3,m}\} = \mathbb{C}^{\dagger}_{n+m},
\end{aligned}
\tag{3.2}
$$

where the symmetric Cartan matrix a_{ij} of $\mathfrak{psl}(2|2)_c$ has all zeroes except for $a_{12} = a_{21} = 1$ and $a_{13} = a_{31} = -1$ (and is, therefore, degenerate). We call the index n of the generators in this realization the *level*. The level $n = 0$ is a subalgebra which coincides with the original $\mathfrak{psl}(2|2)_c$ Lie superalgebra. The choice corresponds to one of the Chevalley-Serre presentations of $\mathfrak{psl}(2|2)_c$ [32], based on Cartan generators $\kappa_{i,0}$, and positive (negative) simple odd roots

$\xi^+_{i,0}$ ($\xi^-_{i,0}$, respectively), as follows:

$$\begin{aligned}
\xi^+_{1,0} &= \mathbb{G}^4_2, & \xi^-_{1,0} &= \mathbb{Q}^2_4, & \kappa_{1,0} &= -\mathbb{L}^1_1 - \mathbb{R}^3_3 + \frac{1}{2}\mathbb{H}, \\
\xi^+_{2,0} &= i\mathbb{Q}^1_4, & \xi^-_{2,0} &= i\mathbb{G}^4_1, & \kappa_{2,0} &= -\mathbb{L}^1_1 + \mathbb{R}^3_3 - \frac{1}{2}\mathbb{H}, \\
\xi^+_{3,0} &= i\mathbb{Q}^2_3, & \xi^-_{3,0} &= i\mathbb{G}^3_2, & \kappa_{3,0} &= \mathbb{L}^1_1 - \mathbb{R}^3_3 - \frac{1}{2}\mathbb{H}.
\end{aligned} \tag{3.3}$$

The generators $\mathbb{C}_n$ and $\mathbb{C}^\dagger_n$ are central in Y for all n.

The coproduct map compatible with the level zero coproduct (2.1) is quite cumbersome and can be found in the literature. As for ordinary Yangians, it is enough to specify the coproduct on the generators at levels 0 and 1. Recursive use of the defining relations and of the algebra-homomorphism property of the coproduct allows one to obtain the coproducts for all the other levels.

The coproduct for the central elements $\mathbb{C}_1$ and $\mathbb{C}^\dagger_1$ turns out to be quite non-trivial, although central in $U(Y)\otimes U(Y)$, $U(Y)$ being the universal enveloping algebra of Y. Following the same argument described in the previous section, in order to have quasi-cocommutativity one needs to have $\Delta(\mathbb{C}_1) = \Delta^{op}(\mathbb{C}_1)$, and the same for $\Delta(\mathbb{C}^\dagger_1)$. This implies extra constraints (which we will call here *hatted constraints*) to be added to the physical constraints (2.4). We will from now on always assume we are dealing with $U(Y)$ modulo the equivalence relations imposed by (2.4) and by the hatted constraints.

4. Representations: Level zero

We first observe that we can always use an $\mathfrak{sl}(2)$ outer automorphism to put the three-vector of central charges $(\mathbb{H},\mathbb{C},\mathbb{C}^\dagger)$ into the form $(\mathbb{H}',0,0)$, corresponding to the Lie superalgebra $\mathfrak{sl}(2|2)$. In turn, $\mathfrak{sl}(2|2)$ is strictly related to the Lie superalgebra $\mathfrak{gl}(2|2)$, which will therefore be our starting point to study representations.

The paper [33] (see also [34] and [35]) explicitly constructs all finite-dimensional irreducible representations of $\mathfrak{gl}(2|2)$ in an oscillator basis. Generators of $\mathfrak{gl}(2|2)$ are denoted by E_{ij}, with commutation relations

$$[E_{ij},E_{kl}] = \delta_{jk}E_{il} - (-)^{(d[i]+d[j])(d[k]+d[l])}\delta_{il}E_{kj}. \tag{4.1}$$

Indices i,j,k,l run from 1 to 4, and the fermionic grading is assigned as $d[1] = d[2] = 0$, $d[3] = d[4] = 1$. The quadratic Casimir of this algebra is $C_2 = \sum_{i,j=1}^{4}(-)^{d[j]}E_{ij}E_{ji}$. Finite dimensional irreps are labelled by two half-integers $j_1, j_2 = 0, \frac{1}{2}, ...$, and two complex numbers q and y. These numbers correspond to the values taken by appropriate generators on the highest weight state $|\omega\rangle$ of the representation, defined by the following conditions:

$$H_1|\omega\rangle = (E_{11}-E_{22})|\omega\rangle = 2j_1|\omega\rangle, H_2|\omega\rangle = (E_{33}-E_{44})|\omega\rangle = 2j_2|\omega\rangle,$$
$$I|\omega\rangle = \sum_{i=1}^{4} E_{ii}|\omega\rangle = 2q|\omega\rangle, \qquad N|\omega\rangle = \sum_{i=1}^{4}(-)^{[i]}E_{ii}|\omega\rangle = 2y|\omega\rangle,$$

and

$$E_{ij}|\omega\rangle = 0, \qquad \forall\, i<j. \tag{4.2}$$

The generator N never appears on the right hand side of the commutation relations, therefore it is defined up to the addition of a central element βI, with β a constant[6]. This also means that we can consistently mod out the generator N, and obtain $\mathfrak{sl}(2|2)$, the algebra of supertraceless matrices with two odd and two even entries, as a subalgebra of the original $\mathfrak{gl}(2|2)$ algebra[7]. In order to construct representations of the centrally-extended $\mathfrak{sl}(2|2)$ Lie superalgebra we then first mod out N, and subsequently perform an $\mathfrak{sl}(2)$ rotation by means of the outer automorphism.

Irreps of $\mathfrak{gl}(2|2)$ are divided into typical (also referred to as "long" in physical terms), which have generic values of the labels j_1, j_2, q and dimension $16(2j_1+1)(2j_2+1)$, and atypical (also referred to as "short" in physical

[6]We decided to drop the term βI since it will not affect our discussion.

[7]Further modding out of the center I produces the simple Lie superalgebra $\mathfrak{psl}(2|2)$. Its representations can be understood as that of $\mathfrak{sl}(2|2)$ for which $q=0$ [36].

terms), for which special relations are satisfied by the labels. Short representations occur here for $\pm q = j_1 - j_2$ and $\pm q = j_1 + j_2 + 1$. When these relations are satisfied, the dimension of the representation is smaller than $16(2j_1+1)(2j_2+1)$.

Certain representations will be of special importance. In physical terms, they will correspond to representations of $\mathfrak{psl}(2|2)_c$ describing "fundamental particles" and their "bound states". The "fundamental" representation [21] corresponds to $j_1 = \frac{1}{2}, j_2 = 0$ and $q = \frac{1}{2}$, and it is 4-dimensional. The "symmetric bound state" representations [37, 38, 39, 40, 25, 41] are given by $j_2 = 0, q = j_1$, with $j_1 = \frac{1}{2}, 1, \ldots$. The "antisymmetric bound state" representations are given by $j_1 = 0, q = 1 + j_2$, with $j_2 = 0, \frac{1}{2}, \ldots$. Symmetric and antisymmetric bound state representations have dimension $4M$, with $M = 2j_1$ for symmetric, $M = 2(j_2+1)$ for antisymmetric. Symmetric and antisymmetric bound state representations are associated with two different shortening conditions, namely $q = j_1 - j_2$ and $q = 1 + j_1 + j_2$ respectively.

We will also focus our attention on a particular long representation, the 16-dimensional long representation characterized by $j_1 = j_2 = 0$, and arbitrary q. It is instructive to see how it branches under the $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ algebra. We denote as $[l_1, l_2]$ the subset of states which furnish a representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ with label l_1 w.r.t the first $\mathfrak{sl}(2)$, and l_2 w.r.t the second $\mathfrak{sl}(2)$, respectively. The branching rule is

$$(2,2) \;\rightarrow\; 2 \times [0,0] \oplus 2 \times [\frac{1}{2}, \frac{1}{2}] \oplus [1,0] \oplus [0,1]. \tag{4.3}$$

One can verify that the total dimension adds up to 16, since $[l_1, l_2]$ has dimension $(2l_1+1) \times (2l_2+1)$. For $q = 1$ the 16-dimensional representation becomes reducible but indecomposable. The corresponding subrepresentation is the 8-dimensional antisymmetric bound state representation, while the factor representation is the 8-dimensional symmetric bound state representation.

Once one has constructed the oscillator representation by using the formulas of [33], and derived from it the matrix realization of the algebra generators, a subsequent $\mathfrak{sl}(2)$ rotation provides an explicit matrix representation of $\mathfrak{sl}(2|2)_c$. The way the outer automorphism is implemented is by mapping the $\mathfrak{gl}(2|2)$ non-diagonal generators into new generators as follows:

$$\mathbb{L}_a^b = E_{ab} \;\forall\, a \neq b, \qquad \mathbb{R}_\alpha^\beta = E_{\alpha\beta} \;\forall\, \alpha \neq \beta,$$
$$\mathbb{Q}_\alpha^a = a\, E_{\alpha a} + b\, \varepsilon_{\alpha\beta}\varepsilon^{ab} E_{b\beta} \qquad \mathbb{G}_a^\alpha = c\, \varepsilon_{ab}\varepsilon^{\alpha\beta} E_{\beta b} + d\, E_{a\alpha}, \quad (4.4)$$

subject to the constraint

$$ad - bc = 1. \quad (4.5)$$

Diagonal generators are automatically obtained by commuting positive and negative roots.

We still need to impose the constraints (2.4). This results in further conditions on a,b,c,d which altogether define a certain algebraic curve. We will not report this parametrization here but it can be found in the literature.

5. Representations: Level one

All short representations can be extended to matrix evaluation representations of $U(Y)$ for which (2.2) holds in $U(Y) \otimes U(Y)$ [29, 31]. This means, in particular, that the constraints (2.4) and the hatted constraints are satisfied in these representations, and one can also prove that indeed

$$\Delta^{op}(\mathbb{C}_n) = \Delta(\mathbb{C}_n), \qquad \Delta^{op}(\mathbb{C}_n^\dagger) = \Delta(\mathbb{C}_n^\dagger) \quad (5.1)$$

$\forall n$ in these representations. *Evaluation* here means that the level n generators are obtained by multiplying the corresponding level zero matrix representation by certain polynomials of degree n in a spectral parameter u [31]. The hatted constraints boil down in these representation to a constraint that fixes the spectral parameter u to be a definite function of the eigenvalue of the level zero central charges [29].

Furthermore, after computing the element R for all short representations and verify it satisfies the Yang-Baxter (or star-triangle) equation [21, 41, 42], one notices that it automatically solves the additional equation [43, 44, 45]

$$\Delta^{op}(\widehat{\mathbb{B}})R = R\Delta(\widehat{\mathbb{B}}), \tag{5.2}$$

with

$$\Delta(\widehat{\mathbb{B}}) = \widehat{\mathbb{B}} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathbb{B}} + \sum_{a=1,2;\alpha=3,4} (\mathbb{S}^{\alpha}_{a} \otimes \mathbb{Q}^{a}_{\alpha} + \mathbb{Q}^{a}_{\alpha} \otimes \mathbb{S}^{\alpha}_{a}),$$
$$\widehat{\mathbb{B}} = B_0 \, diag(1, 1, ..., 1, -1, -1..., -1). \tag{5.3}$$

The "1"s run over the even subspace of the representation module, the "-1"s over the odd subspace, and B_0 is a definite function of the eigenvalues of the level zero central charges in each specific short representation. One can notice that $\widehat{\mathbb{B}}$ is not supertraceless, unlike all the generators of Y.

For certain long representations, the situation is different. In [46], a specific long representation[8] was studied which extends to a representation of $U(Y)$ (therefore, the constraints (2.4) and the hatted constraints are satisfied in this representation), but for which

$$\Delta^{op}(\mathbb{C}_2) \neq \Delta(\mathbb{C}_2). \tag{5.4}$$

despite $\Delta(\mathbb{C}_2)$ and being proportional to the indentity matrix. Hence, one concludes that $U(Y)$ does not admit a universal R-matrix.

6. Conclusions

We have tried to present the mathematical features of the Hopf superalgebras emerging in the context of certain integrable models of Theoretical Physics. These models are related to the so-called AdS/CFT correspondence.

[8]This long representation is of dimension 16, and it corresponds to $j_1 = j_2 = 0$ (with q generic) in the classification of Section 4.

The aim is to set the ground for a formal approach to be attempted, which may bring to the full development of this young and rich subject. In this same spirit, we list here a few of what we see as main open problems to be addressed in the immediate future.

- The first problem (mostly dear to physicists which naturally look for a notion of scattering) is to find an extension of the algebra $U(Y)$ for which a universal R-matrix may exist. There are proposals in the literature [47, 25, 29, 48] where additional relations are formulated, which rule out for instance the representation responsible for (5.4). The complete analysis of these algebraic extensions is a very interesting open problem.

- In all short representations there exists a notion of classical limit in a small parameter $\hbar$ [49, 50, 45, 42]. This limit involves a scaling of the eigenvalues of the central elements, since they depend on $\hbar$ as well. The element R can be expanded in a Taylor series, and the first order r is a solution of the classical Yang-Baxter equation [51, 52, 53, 54, 55, 56, 57, 58]. The element r displays a single pole at the origin in some appropriate classical spectral variables, with residue the quadratic Casimir element of the superalgebra $\mathfrak{gl}(2|2) \otimes \mathfrak{gl}(2|2)$. There exists an infinite-dimensional Lie bialgebra [44], formulated purely in abstract terms, which, when projected in these short representations, admits r as coboundary structure. Its nature is quite unconventional, and its quantization is a fascinating open problem, since it may naturally bring to a quasi-cocommutative algebraic extension of $U(Y)$ (see the previous point). This Lie bialgebra accomodates also a class of generators of the type $\widehat{\mathbb{B}}$, which appear naturally in the classical limit [59]. The quantization of this Lie bialgebra may also therefore resolve the open problem of embedding the additional symmetry $\widehat{\mathbb{B}}$ one observes in short representations into the full algebra. In fact, a naive computation of the commutant of this additional symmetry with $U(Y)$ produces an outcome which is hard to interpret in terms of some natural extension of $U(Y)$ (for example, to $U(\mathfrak{gl}(2|2))$.

- In all short representations, the element R appears in the factorized form

$$R = [F^{op}]^{-1} F, \tag{6.1}$$

for some element F only known in matrix representation. By applying a Drinfeld's twist [60] based on F, the element R can be transformed into the identity matrix, and the coproduct can be transformed into a co-commutative one. However, this co-commutative coproduct is highly non-trivial, and its expression is only known in matrix representation. It would be interesting to develop a theory of these twists.

- Interesting related algebraic developments can be found in [61, 62, 63, 64], which represent fascinating new directions of application of the techniques one hopes to develop for the matter we have treated here.

7. Acknowledgements

The author would like to thank the organizers of the Workshop "The Interface of Integrability and Quantization" (Leiden, April 2010) for an inspiring event, and for giving him the opportunity of presenting the subject of these proceedings to an audience of mathematicians. The author would also like to thank his collaborators on this topic, whose names can be deduced from the references, for the most enjoyable time spent throughout these recent years.

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