# Projective Systems and Higher Weights

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*Abstract* — **We use projective multisets (projective systems) to find upper bounds on the weight hierarchies for a special class of codes, namely the extremal non-chain codes. Several code constructions exist meeting the bounds with equality.**

## I. Introduction

Let $C$ be a linear $q$-ary code of dimension $k$ and length $n$. The weight $w(S)$ of a subcode $S \subseteq C$, is the number of positions where at least one word in $S$ differs from zero. The $r$th generalised Hamming weight $d_r$ of $C$ is the least weight of an $r$-dimensional subcode of $C$. The sequence $(d_1, d_2, \ldots, d_k)$ is called the weight hierarchy of $C$ [6].

## II. Extremal Non-Chain Codes

The chain condition was introduced in [7], and states that there is a chain $D_0 \subset \ldots \subset D_k$ of subcodes, where $D_i$ has dimension $i$ and weight $d_i$.

The opposite extreme are the extremal non-chain codes, defined as follows. For each pair $(i,j)$ where $1 \leq i < j < k$, there are no subcodes $D_i \subseteq D_j$ of dimensions $i$ and $j$ respectively such that $w(D_i) = d_i$ and $w(D_j) = d_j$. The extremal non-chain codes were introduced by Chen and Kløve [1], and this study continues their work.

## III. Projective Multisets

Let $G$ be a $k \times n$ generator matrix of $C$. By permuting columns of $G$ or by multiplying certain columns by non-zero scalars, we get an equivalent code. Equivalent codes have the same weight hierarchy.

Let $\mathsf{PG}(k-1,q)$ be the projective $(k-1)$-space over the finite field with $q$ elements. The code $C$ is determined up to equivalence by giving the map $\gamma : \mathsf{PG}(k-1,q) \to \{0,1,\ldots\}$, saying how many times each projective point occurs as a column in $G$. Such a map is called a projective multiset [2], a projective system [5], or a value assignment [1, 4]. The definition of $\gamma$ is extended by $\gamma(S) = \sum_{x \in S} \gamma(x)$ for all $S \subseteq \mathsf{PG}(k-1,q)$. The number $\gamma(S)$ is called the value of $S$.

We know [3, 5] that a subcode $D_r$ of dimension $r$ and weight $w$, corresponds to a subspace $S_r \subseteq \mathsf{PG}(k-1,q)$ of dimension $k-r-1$ and value $\gamma(S_r) = n - w$. Hence a subcode $D_r$ of minimum value corresponds to a projective subspace $S_r$ of maximum value. Also if $D_r \subseteq D_{r'}$, then $S_r \supseteq S_{r'}$.

The difference sequence $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ is defined by $\delta_i = d_{k-1} - d_{k-1-i}$. The difference sequence is easily computed from the weight hierarchy and vice versa. If $S$ is an $i$-space of maximum value, then $\gamma(S) = \delta_0 + \delta_1 + \ldots + \delta_i$. A difference sequence corresponding to an extremal non-chain code is called an ENDS (Extremal Nonchain Difference Sequence).

## IV. Results

**Theorem 1 (General Bound)** *If* $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ *is an ENDS,* $1 \leq m \leq k-2$, *then*

$$\delta_m \leq q^m \delta_0 - \frac{q^{m+1}-1}{q-1}.$$

*If equality holds for* $m = \bar{m}$, *then equality holds for all* $m < \bar{m}$.

**Theorem 2 (Binary Codes)** *If* $(\delta_0, \delta_1, \ldots, \delta_{k-1})$, $k \geq 4$ *is a binary ENDS, then*

$$\delta_{k-2} \leq 2^{k-3}\delta_1 - 2 - 2^{k-3}.$$

**Theorem 3 (Total Value)** *If* $(\delta_0, \delta_1, \ldots, \delta_{k-1})$, $k \geq 3$ *is an ENDS, then*

$$\gamma(\mathsf{PG}(k-1,q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1)\frac{q^{k-m}-1}{q-1},$$

*for all* $m$ *such that* $1 \leq m \leq k-2$.

Explicit constructions meeting the bounds with equality exist in dimension 5 and less, provided $\delta_0$ is sufficiently large; $\delta_0 \geq 5$ is sufficient in all cases.

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