MECHANICAL VIBRATION AND DYNAMIC STABILITY OF
COMPLEX STRUCTURES BY FINITE ELEMENT METHOD

A Thesis Submitted for the Degree of Doctor of Philosophy
in Mechanical Engineering

by

BAHA AL DIN HUSSAIN ABBAS
B.Sc. Honours (London),
M.Sc. (Surrey)

Department of Mechanical Engineering
University of Surrey
Guildford GU2 5XH
England

April 1977
SUMMARY

During the course of the past twenty years, there has been a significant surge of interest in the problem of stability of structures subjected to time-dependent loads. The purpose of this Thesis is to investigate three new problem areas of dynamic stability of complex structures. Timoshenko beams, three dimensional space frames and eccentrically stiffened plates are considered for the first time.

In this Thesis, a new finite element model for the dynamic stability analysis of Timoshenko beams is developed. The effect of shear deformation and rotary inertia on the natural frequencies, the effect of shear deformation on the static buckling loads and, for the first time, the effect of shear deformation and rotary inertia on the regions of dynamic instability are investigated. The effects of elastic foundation on the natural frequencies, the static buckling loads and the regions of dynamic instability are also investigated.

A finite element model for the dynamic stability analysis of three dimensional frame structures is developed and successfully applied to the analysis of three dimensional frame structures similar to offshore oil rigs subjected to periodic sea wave forces. This approach is tested on plane frames and correlation of results obtained with the available exact results indicates excellent agreement. To verify the accuracy of the computed natural frequencies of vibration of oil rigs, a small scale model is built and free vibration tests are carried out. Agreement between the computed and measured frequencies of vibration of the model is within five per cent averaged for the first ten modes.
A fully conforming finite element model is developed for the dynamic stability analysis of eccentrically stiffened plates. The accuracy of the proposed finite element method is demonstrated before presenting results of its application to cases of complex stiffened plates. To this end, a series of results are presented for the dynamic stability analysis of unstiffened plates with various boundary and loading conditions. The problem of eccentrically stiffened plates is then considered and the effects of the stiffeners' depth on the natural frequencies of vibration, the static buckling loads and the regions of dynamic instability of plates are investigated.
DEDICATION

To my mother
ACKNOWLEDGEMENTS

The author wishes to express his gratitude and sincere appreciation to Dr. J. Thomas for his guidance, valuable suggestions and critical comments throughout the course of this investigation.

The author is deeply indebted to the Iraqi Ministry of Higher Education and Scientific Research, Baghdad, for the scholarship award which enabled him to undertake this research.

The author wishes to express his thanks to the staff of the Computing Unit for their co-operation and to the workshop and technical staff for their assistance in the preparation of the experimental models.

Thanks are also due to the Departmental Secretaries for their help and in particular Mrs. Babs Armstrong for typing the manuscript.
CONTENTS

Title page 1
Summary 2
Dedication 4
Acknowledgements 5
Contents 6
List of Tables 11
List of Figures 15
List of Plates 24
Nomenclature 25

CHAPTER 1 INTRODUCTION 33

CHAPTER 2 THE FINITE ELEMENT METHOD 42
  2.1 Introduction 42
  2.2 Finite Element Method in Structural Analysis 43
    2.2.1 Strain-displacement relations 43
    2.2.2 Stress-strain relations 45
    2.2.3 Strain energy of a finite element 46
    2.2.4 Kinetic energy of a finite element 47
    2.2.5 Strain and kinetic energies of a complete system 48
    2.2.6 Equations of motion 48
### CHAPTER 3  THEORY OF DYNAMIC STABILITY

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Introduction</td>
<td>51</td>
</tr>
<tr>
<td>3.2 Regions of Dynamic Instability</td>
<td>54</td>
</tr>
<tr>
<td>3.3 Types of Parametric Excitation</td>
<td>55</td>
</tr>
<tr>
<td>3.3.1 Periodic change in rigidity</td>
<td>55</td>
</tr>
<tr>
<td>3.3.2 Periodic change in inertia</td>
<td>56</td>
</tr>
<tr>
<td>3.3.3 Periodic change in loading</td>
<td>56</td>
</tr>
<tr>
<td>3.4 Dynamic Stability of Multi-degree of Freedom System</td>
<td>57</td>
</tr>
</tbody>
</table>

### CHAPTER 4  DYNAMIC STABILITY OF TIMOSHENKO BEAMS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Introduction</td>
<td>73</td>
</tr>
<tr>
<td>4.2 Formulation of Elastic Stiffness and Geometric Stiffness Matrices</td>
<td>83</td>
</tr>
<tr>
<td>4.3 Formulation of the Inertia Matrix</td>
<td>85</td>
</tr>
<tr>
<td>4.4 Matrix Equations for Timoshenko Beams</td>
<td>86</td>
</tr>
<tr>
<td>4.4.1 Free vibration</td>
<td>86</td>
</tr>
<tr>
<td>4.4.2 Static stability</td>
<td>86</td>
</tr>
<tr>
<td>4.4.3 Dynamic stability</td>
<td>86</td>
</tr>
<tr>
<td>4.5 Applications</td>
<td>86</td>
</tr>
<tr>
<td>4.5.1 Free vibration</td>
<td>87</td>
</tr>
<tr>
<td>4.5.1.1 Analytical solutions</td>
<td>87</td>
</tr>
<tr>
<td>4.5.1.1.1 Bernoulli-Euler beam</td>
<td>87</td>
</tr>
<tr>
<td>4.5.1.2 Simple shear beam</td>
<td>88</td>
</tr>
<tr>
<td>4.5.1.3 Pure shear beam</td>
<td>89</td>
</tr>
<tr>
<td>4.5.1.2 Timoshenko beam</td>
<td>90</td>
</tr>
<tr>
<td>4.5.2 Static stability</td>
<td>91</td>
</tr>
<tr>
<td>4.5.3 Dynamic stability</td>
<td>92</td>
</tr>
</tbody>
</table>
CHAPTER 4 (contd:-)

4.6 General Discussions
  4.6.1 Free vibration 93
  4.6.2 Static stability 101
  4.6.3 Dynamic stability 103

CHAPTER 5 DYNAMIC STABILITY OF TIMOSHENKO BEAMS
RESTING ON ELASTIC FOUNDATION 144

5.1 Introduction 144

5.2 Formulation of Elastic Stiffness
and Geometric Stiffness Matrices 146

5.3 Applications 148
  5.3.1 Free vibration 148
    5.3.1.1 Analytical solution 148
  5.3.2 Static stability 150
  5.3.3 Dynamic stability 150

5.4 General Discussions 151
  5.4.1 Free vibration 151
  5.4.2 Static stability 153
  5.4.3 Dynamic stability 156

CHAPTER 6 DYNAMIC STABILITY OF SPACE FRAMES 201

6.1 Introduction 201

6.2 Formulation of Element Matrices 202

6.3 Matrix Equations for Space Frames 207
  6.3.1 Free vibration 207
  6.3.2 Static stability 207
  6.3.3 Dynamic stability 207
# CHAPTER 6 (contd:-)

**6.4 Applications**  
- **6.4.1 Application to plane frames**  
- **6.4.2 Application to space frames**  
- **6.4.3 Application to an offshore oil rig**  
  - **6.4.3.1 Introduction**  
  - **6.4.3.2 Loading conditions**  
  - **6.4.3.3 Oil rig model**  
  - **6.4.3.4 Experimental procedure**  
  - **6.4.3.5 Results**

**6.5 General Discussions**
- **6.5.1 Plane frames**
- **6.5.2 Space frames**
- **6.5.3 An offshore oil rig**

# CHAPTER 7 DYNAMIC STABILITY OF PLATES

**7.1 Introduction**

**7.2 Formulation of Elastic Stiffness and Geometric Stiffness Matrices**

**7.3 Formulation of Inertia Matrix**

**7.4 Matrix Equations for Plates**
  - **7.4.1 Free vibration**
  - **7.4.2 Static stability**
  - **7.4.3 Dynamic stability**

**7.5 Applications**

**7.6 General Discussions**
<table>
<thead>
<tr>
<th>Chapter 8</th>
<th>Dynamic Stability of Stiffened Plates</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>301</td>
</tr>
<tr>
<td>8.2</td>
<td>Theoretical Analysis</td>
<td>305</td>
</tr>
<tr>
<td>8.3</td>
<td>Matrix Equations for Stiffened Plates</td>
<td>306</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Free vibration</td>
<td>306</td>
</tr>
<tr>
<td>8.3.2</td>
<td>Static stability</td>
<td>306</td>
</tr>
<tr>
<td>8.3.3</td>
<td>Dynamic stability</td>
<td>307</td>
</tr>
<tr>
<td>8.4</td>
<td>Applications</td>
<td>307</td>
</tr>
<tr>
<td>8.4.1</td>
<td>Analytical solution of free vibration of a stiffened plate</td>
<td>308</td>
</tr>
<tr>
<td>8.5</td>
<td>General Discussions</td>
<td>311</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 9</th>
<th>Conclusions</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.1</td>
<td>Conclusions</td>
<td>345</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scope of Future Work</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2</td>
<td>350</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>References</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.3</td>
<td>351</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table | Page
--- | ---
4.1 Convergence of frequency parameters of a hinged-hinged Timoshenko beam. | 106
4.2 Convergence of frequency parameters of a fixed-free Timoshenko beam. | 106
4.3 Convergence of frequency parameters of a free-free Timoshenko beam. | 107
4.4 Convergence of frequency parameters of a fixed-fixed Timoshenko beam. | 107
4.5 Comparison of frequency parameters of a hinged-hinged Timoshenko beam. | 108
4.6 Comparison of frequency parameters of a fixed-free Timoshenko beam. | 108
4.7 Comparison of frequency parameters of a fixed-free Timoshenko beam. | 109
4.8 Comparison of frequency parameters of a fixed-free Timoshenko beam. | 109
4.9 Natural frequency parameters of a free-free Timoshenko beam. | 110
4.10 Convergence of buckling load parameters of a hinged-hinged Timoshenko beam. | 111
4.11 Convergence of buckling load parameters of a fixed-free Timoshenko beam. | 111
4.12 Convergence of buckling load parameters of a free-free Timoshenko beam. | 112
4.13 Convergence of buckling load parameters of a fixed-fixed Timoshenko beam. | 112
5.1 Natural frequency and static buckling load parameters of a fixed-fixed Bernoulli-Euler beam resting on elastic foundation. | 159
Table 6.1 Convergence of frequency parameters of a plane frame. 220
Table 6.2 Convergence of static buckling load parameters of a plane frame. 220
Table 6.3 Convergence of frequency parameters of a plane frame. 221
Table 6.4 Convergence of static buckling load parameters of a plane frame. 221
Table 6.5 Convergence of frequency parameters of a plane frame. 222
Table 6.6 Convergence of static buckling load parameters of a plane frame. 222
Table 6.7 Natural frequency parameters and static buckling load parameters of a plane frame. 223
Table 6.8 Convergence of natural frequencies of a space frame. 224
Table 6.9 Convergence of static buckling loads of a space frame. 224
Table 6.10 Frequencies of free vibration of an oil rig model. 225
Table 6.11 Static buckling loads of an oil rig model. 225

Table 7.1 Values of buckling parameter of a square plate simply supported on all edges and compressed uniaxially. 277
Table 7.2 Values of buckling parameter of a square plate clamped at all edges and compressed biaxially. 278
Table 7.3 Values of buckling parameter of a rectangular plate simply supported on all edges and subjected to pure shear. 279
Table 7.4 Frequency parameters of a rectangular plate simply supported on all edges. 280
<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>Buckling load parameters of a rectangular plate simply supported on all edges and subjected to normal load.</td>
</tr>
<tr>
<td>7.6</td>
<td>Buckling load parameters of a rectangular plate simply supported on all edges and subjected to pure shear.</td>
</tr>
<tr>
<td>7.7</td>
<td>Buckling load parameters of a rectangular plate simply supported on all edges and subjected to normal loads and shear load.</td>
</tr>
<tr>
<td>7.8</td>
<td>Frequency parameters of a rectangular plate clamped at all edges.</td>
</tr>
<tr>
<td>7.9</td>
<td>Buckling load parameters of a rectangular plate clamped at all edges and subjected to normal loads.</td>
</tr>
<tr>
<td>7.10</td>
<td>Frequency parameters of a cantilevered rectangular plate.</td>
</tr>
<tr>
<td>7.11</td>
<td>Buckling load parameters of a cantilevered rectangular plate subjected to normal load.</td>
</tr>
<tr>
<td>8.1</td>
<td>Comparison of mode $f_{11}$ frequency of a simply supported stiffened plate.</td>
</tr>
<tr>
<td>8.2</td>
<td>Comparison of mode $f_{21}$ frequency.</td>
</tr>
<tr>
<td>8.3</td>
<td>Comparison of mode $f_{12}$ frequency.</td>
</tr>
<tr>
<td>8.4</td>
<td>Comparison of mode $f_{22}$ frequency.</td>
</tr>
<tr>
<td>8.5</td>
<td>Values of buckling parameter of a stiffened square plate simply supported on all edges.</td>
</tr>
<tr>
<td>8.6</td>
<td>Comparison of present buckling results with exact results.</td>
</tr>
<tr>
<td>8.7</td>
<td>Effects of stiffener depth on the static buckling load parameters of a simply supported plate.</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>8.8</td>
<td>Effects of stiffeners depth on the natural frequencies of a rectangular plate.</td>
</tr>
<tr>
<td>8.9</td>
<td>Comparison of buckling parameter of a stiffened plate.</td>
</tr>
<tr>
<td>8.10</td>
<td>Effects of stiffeners depth on the static buckling load parameters of a rectangular plate.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>3.1</td>
<td>Mechanical system.</td>
</tr>
<tr>
<td>3.2</td>
<td>Two solutions of Mathieu's equation.</td>
</tr>
<tr>
<td>3.3</td>
<td>Part of Haines-Strett diagram.</td>
</tr>
<tr>
<td>3.4</td>
<td>System with periodic change in torsional rigidity.</td>
</tr>
<tr>
<td>3.5</td>
<td>System with periodic change in inertia.</td>
</tr>
<tr>
<td>3.6</td>
<td>System with periodic change in loading.</td>
</tr>
<tr>
<td>3.7</td>
<td>Unit circle in the complex plane.</td>
</tr>
<tr>
<td>4.1</td>
<td>Timoshenko beam element.</td>
</tr>
<tr>
<td>4.2</td>
<td>Natural frequencies of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.3</td>
<td>The two spectra of natural frequencies of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.4</td>
<td>Free vibration of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.5</td>
<td>Third mode shape of first spectrum of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.6</td>
<td>Third mode shape of second spectrum of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.7</td>
<td>Natural frequencies of a fixed-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.8</td>
<td>Fourth mode shape of a fixed-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.9</td>
<td>Natural frequencies of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.10</td>
<td>Natural frequencies of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.11</td>
<td>Natural frequencies of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>4.12</td>
<td>Fourth mode shape of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.13</td>
<td>Natural frequencies of a fixed-fixed Timoshenko beam.</td>
</tr>
<tr>
<td>4.14</td>
<td>Fourth mode shape of a fixed-fixed Timoshenko beam.</td>
</tr>
<tr>
<td>4.15</td>
<td>Buckling load parameter ratio versus shear deformation parameter for a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>4.16</td>
<td>Buckling load parameter ratio versus shear deformation parameter for a fixed-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.17</td>
<td>Buckling load parameter ratio versus shear deformation parameter for a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>4.18</td>
<td>Buckling load parameter ratio versus shear deformation parameter for a fixed-fixed Timoshenko beam.</td>
</tr>
<tr>
<td>4.19</td>
<td>Regions of dynamic instability for a hinged-hinged Timoshenko beam – R = 0.0000.</td>
</tr>
<tr>
<td>4.20</td>
<td>Regions of dynamic instability for a hinged-hinged Timoshenko beam – R = 0.0064.</td>
</tr>
<tr>
<td>4.21</td>
<td>Regions of dynamic instability for a hinged-hinged Timoshenko beam – R = 0.0256.</td>
</tr>
<tr>
<td>4.22</td>
<td>Regions of dynamic instability for a fixed-free Timoshenko beam – R = 0.0000.</td>
</tr>
<tr>
<td>4.23</td>
<td>Regions of dynamic instability for a fixed-free Timoshenko beam – R = 0.0064.</td>
</tr>
<tr>
<td>4.24</td>
<td>Regions of dynamic instability for a fixed-free Timoshenko beam – R = 0.0256.</td>
</tr>
<tr>
<td>4.25</td>
<td>Regions of dynamic instability for a free-free Timoshenko beam – R = 0.0000.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>4.26</td>
<td>Regions of dynamic instability for a free-free Timoshenko beam - R = 0.0064.</td>
</tr>
<tr>
<td>4.27</td>
<td>Regions of dynamic instability for a free-free Timoshenko beam - R = 0.0256.</td>
</tr>
<tr>
<td>4.28</td>
<td>Regions of dynamic instability for a fixed-fixed Timoshenko beam - R = 0.0000.</td>
</tr>
<tr>
<td>4.29</td>
<td>Regions of dynamic instability for a fixed-fixed Timoshenko beam - R = 0.0016.</td>
</tr>
<tr>
<td>4.30</td>
<td>Regions of dynamic instability for a fixed-fixed Timoshenko beam - R = 0.0064.</td>
</tr>
<tr>
<td>5.1</td>
<td>Effect of elastic foundation on the natural frequencies of a hinged-hinged Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.2</td>
<td>Effect of elastic foundation on the natural frequencies of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>5.3</td>
<td>Effect of elastic foundation on the natural frequencies of a fixed-free Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.4</td>
<td>Effect of elastic foundation on the natural frequencies of a fixed-free Timoshenko beam.</td>
</tr>
<tr>
<td>5.5</td>
<td>Effect of elastic foundation on the natural frequencies of a free-free Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.6</td>
<td>Effect of elastic foundation on the natural frequencies of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>5.7</td>
<td>Effect of elastic foundation on the natural frequencies of a fixed-fixed Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>5.8</td>
<td>Effect of elastic foundation on the natural frequencies of a fixed-fixed Timoshenko beam.</td>
</tr>
<tr>
<td>5.9</td>
<td>Effect of elastic foundation on the static buckling loads of a hinged-hinged Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.10</td>
<td>Effect of elastic foundation on the static buckling loads of a hinged-hinged Timoshenko beam.</td>
</tr>
<tr>
<td>5.11</td>
<td>Effect of elastic foundation on the static buckling loads of a fixed-free Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.12</td>
<td>Effect of elastic foundation on the static buckling loads of a fixed-free Timoshenko beam.</td>
</tr>
<tr>
<td>5.13</td>
<td>Effect of elastic foundation on the static buckling loads of a free-free Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.14</td>
<td>Effect of elastic foundation on the static buckling loads of a free-free Timoshenko beam.</td>
</tr>
<tr>
<td>5.15</td>
<td>Effect of elastic foundation on the static buckling loads of a fixed-fixed Bernoulli-Euler beam.</td>
</tr>
<tr>
<td>5.16</td>
<td>Effect of elastic foundation on the static buckling loads of a fixed-fixed Timoshenko beam.</td>
</tr>
<tr>
<td>5.17</td>
<td>Regions of dynamic instability of a hinged-hinged Bernoulli-Euler beam on elastic foundation.</td>
</tr>
<tr>
<td>5.18</td>
<td>Regions of dynamic instability of a hinged-hinged Timoshenko beam on elastic foundation.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>5.19</td>
<td>Regions of dynamic instability of a fixed-free Bernoulli-Euler beam on elastic foundation.</td>
</tr>
<tr>
<td>5.20</td>
<td>Regions of dynamic instability of a fixed-free Timoshenko beam on elastic foundation.</td>
</tr>
<tr>
<td>5.21</td>
<td>Regions of dynamic instability of a free-free Bernoulli-Euler beam on elastic foundation.</td>
</tr>
<tr>
<td>5.22</td>
<td>Regions of dynamic instability of a free-free Timoshenko beam on elastic foundation.</td>
</tr>
<tr>
<td>5.23</td>
<td>Regions of dynamic instability of a fixed-fixed Bernoulli-Euler beam on elastic foundation.</td>
</tr>
<tr>
<td>5.24</td>
<td>Regions of dynamic instability of a fixed-fixed Timoshenko beam on elastic foundation.</td>
</tr>
<tr>
<td>6.1</td>
<td>A beam element.</td>
</tr>
<tr>
<td>6.2</td>
<td>A plane frame.</td>
</tr>
<tr>
<td>6.3</td>
<td>First four mode shapes of free vibration of a plane frame.</td>
</tr>
<tr>
<td>6.4</td>
<td>First four mode shapes of static buckling of a plane frame.</td>
</tr>
<tr>
<td>6.5</td>
<td>Regions of dynamic instability of a plane frame.</td>
</tr>
<tr>
<td>6.6</td>
<td>Plane frames.</td>
</tr>
<tr>
<td>6.7</td>
<td>Regions of dynamic instability of a plane frame.</td>
</tr>
<tr>
<td>6.8</td>
<td>Regions of dynamic instability of a plane frame.</td>
</tr>
<tr>
<td>6.9</td>
<td>First three mode shapes of free vibration of a plane frame.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>6.10</td>
<td>First mode shape of static buckling.</td>
</tr>
<tr>
<td>6.11</td>
<td>A three dimensional frame structure.</td>
</tr>
<tr>
<td>6.12</td>
<td>First four mode shapes of free vibration of a space frame.</td>
</tr>
<tr>
<td>6.13</td>
<td>First four mode shapes of static buckling of a space frame.</td>
</tr>
<tr>
<td>6.14</td>
<td>Regions of dynamic instability of a space frame.</td>
</tr>
<tr>
<td>6.15</td>
<td>An oil rig subjected to vertical static and periodic loads.</td>
</tr>
<tr>
<td>6.16</td>
<td>An oil rig subjected to horizontal static and periodic loads.</td>
</tr>
<tr>
<td>6.17</td>
<td>Prying and squeezing forces.</td>
</tr>
<tr>
<td>6.18</td>
<td>An oil rig model.</td>
</tr>
<tr>
<td>6.19</td>
<td>Schematic representation of experimental set-up.</td>
</tr>
<tr>
<td>6.20</td>
<td>Fundamental mode shape of free vibration of oil rig model.</td>
</tr>
<tr>
<td>6.21</td>
<td>Fundamental mode shape of static buckling of oil rig model subjected to vertical loads.</td>
</tr>
<tr>
<td>6.22</td>
<td>Fundamental mode shape of static buckling of oil rig model subjected to horizontal loads.</td>
</tr>
<tr>
<td>6.23</td>
<td>Regions of dynamic instability of oil rig model subjected to periodic vertical loads.</td>
</tr>
<tr>
<td>6.24</td>
<td>Regions of dynamic instability of oil rig model subjected to periodic horizontal loads.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>7.1</td>
<td>A plate element.</td>
</tr>
<tr>
<td>7.2</td>
<td>Plates subjected to periodic loads.</td>
</tr>
<tr>
<td>7.3</td>
<td>First five mode shapes of free vibration of a plate simply supported on all edges.</td>
</tr>
<tr>
<td>7.4</td>
<td>First four mode shapes of static buckling of a plate simply supported on all edges and subjected to normal axial load.</td>
</tr>
<tr>
<td>7.5</td>
<td>Regions of dynamic instability of a simply supported plate subjected to periodic axial load.</td>
</tr>
<tr>
<td>7.6</td>
<td>First four mode shapes of static buckling of a plate simply supported on all edges and subjected to pure shear load.</td>
</tr>
<tr>
<td>7.7</td>
<td>Regions of dynamic instability of a simply supported plate subjected to periodic pure shear load.</td>
</tr>
<tr>
<td>7.8</td>
<td>First mode shape of buckling of a plate subjected to pure shear load.</td>
</tr>
<tr>
<td>7.9</td>
<td>Regions of dynamic instability of a simply supported plate subjected to periodic axial loads.</td>
</tr>
<tr>
<td>7.10</td>
<td>First five mode shapes of free vibration of a plate clamped at all edges.</td>
</tr>
<tr>
<td>7.11</td>
<td>First four mode shapes of static buckling of a plate clamped at all edges and subjected to axial loads.</td>
</tr>
<tr>
<td>7.12</td>
<td>Regions of dynamic instability of a clamped plate subjected to periodic axial loads.</td>
</tr>
<tr>
<td>7.13</td>
<td>First four mode shapes of free vibration of a cantilevered plate.</td>
</tr>
<tr>
<td>7.14</td>
<td>First four mode shapes of static buckling of a cantilevered plate subjected to axial load.</td>
</tr>
</tbody>
</table>
Figure 7.15 Regions of dynamic instability of a cantilevered plate subjected to periodic axial load.

Page 299

8.1 A stiffened plate.

Page 321

8.2 Comparison of mode $f_{11}$ frequency of a simply supported rectangular plate stiffened with one stiffener.

Page 322

8.3 Comparison of mode $f_{12}$ frequency.

Page 323

8.4 Comparison of mode $f_{21}$ frequency.

Page 324

8.5 Comparison of mode $f_{12}$ frequency.

Page 325

8.6 Comparison of mode $f_{22}$ frequency.

Page 326

8.7 Effect of stiffener depth on the natural frequencies of a stiffened plate.

Page 327

8.8 Mode shapes of free vibration of a stiffened plate.

Page 328

8.9 Effect of stiffener depth on the static buckling loads of a stiffened plate.

Page 330

8.10 Mode shapes of static buckling of a stiffened plate.

Page 331

8.11 Regions of dynamic instability of a plate with one stiffener subjected to periodic axial load.

Page 332

8.12 Regions of dynamic instability - $d = 0.5$ in.

Page 333

8.13 Regions of dynamic instability - $d = 1.0$ in.

Page 334

8.14 Regions of dynamic instability - $d = 1.5$ in.

Page 335

8.15 Regions of dynamic instability - $d = 2.0$ in.

Page 336

8.16 Regions of dynamic instability - $d = 2.0$ in.

Page 337

8.17 Comparison of frequencies of a simply supported plate with three stiffeners.

Page 338
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.18</td>
<td>Effect of stiffeners depth on the natural frequencies of a stiffened plate.</td>
</tr>
<tr>
<td>8.19</td>
<td>Effect of stiffeners depth on the static buckling loads of a stiffened plate.</td>
</tr>
<tr>
<td>8.20</td>
<td>Regions of dynamic instability of a plate with three stiffeners subjected to periodic load - d = 0.5 in.</td>
</tr>
<tr>
<td>8.21</td>
<td>Regions of dynamic instability - d = 1.0 in.</td>
</tr>
<tr>
<td>8.22</td>
<td>Regions of dynamic instability - d = 1.0 in.</td>
</tr>
<tr>
<td>A.1</td>
<td>Variation of frequencies of the oil rig model for various member failures.</td>
</tr>
</tbody>
</table>
# LIST OF PLATES

<table>
<thead>
<tr>
<th>Plate</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Three dimensional frame structure.</td>
<td>250</td>
</tr>
<tr>
<td>6.2</td>
<td>An oil rig model.</td>
<td>251</td>
</tr>
<tr>
<td>6.3</td>
<td>Instrumentation.</td>
<td>252</td>
</tr>
<tr>
<td>6.4</td>
<td>Experimental set-up.</td>
<td>253</td>
</tr>
</tbody>
</table>
NOMENCLATURE

Notations used have been defined as they first appear in the text. A dot over a variable denotes a derivative with respect to time. The superscript $T$ over a matrix denotes a transpose of that matrix.

$A$ area of cross-section
$A$ amplitude of oscillation (Chapter 3)
$A_s$ area of cross-section of stiffener
$[A]$ matrix of linear operators
$a$ length

$a_1, a_2$ parameters corresponding to points 1 and 2 respectively
$a_i$ real part of eigenvalue
$a_r$ coefficients
$\{a\}_k$ assumed displacement vector
$\{a\}_k^*$ time-independent vectors

$B$ $P^2\ell^2/EI$ - buckling load parameter
$B_x$ $N_x^{ab}/D$ - static buckling load parameter of plate
$B_y$ $N_y^{ab}/D$ - static buckling load parameter of plate
$B_{xy}$ $N_{xy}^{ab}/D$ - static buckling load parameter of plate
$b$ width
$b_0$ constant
$b_1, b_2$ parameters corresponding to points 1 and 2 respectively
$b_i$ imaginary part of eigenvalue $\rho_i$
$b_{ij}$ buckling mode of plate where $i$ and $j$ are the numbers of half-waves along the $x$ and $y$ directions respectively.

$\{b\}_k$ time-independent vectors

$C_r$ constants

c width of eccentric stiffener

$D$ $Eh^3/12(1-v^2)$ - flexural rigidity of plate

d depth of eccentric stiffener

$E$ modulus of elasticity

$[E]$ elasticity matrix

e distance between stiffeners

$F$ $k_f\ell^4/EI$ - elastic foundation parameter

$F_i$ interpolation functions

$f_{ij}$ free vibration mode of plate where $i$ and $j$ are the numbers of half-waves along the $x$ and $y$ direction respectively

$\{f\}$ given element in functional space

$G$ $E/2(1+v)$ - modulus of rigidity

g acceleration due to gravity

$[\mathbf{H}(t)]$ eigenvectors matrix

$h$ thickness of plate

$\{\mathbf{H}(t)\}_i$ $i$th eigenvector

$I$ second moment of area of cross-section
\( I_p \)  
**polar moment of area**

\( I_s \)  
**effective second moment of area of cross-section of eccentric stiffener**

\( I_x, I_y \)  
**second moments of area of cross-section about the principal axes \( xx \) and \( yy \) respectively.**

\([I]\)  
**identity matrix**

\( J \)  
**St. Venant's torsion constant**

\( j \)  
\( \sqrt{\mathbf{I}} \)

\( K \)  
**buckling parameter of plate**

\( K_1 \)  
\((\rho A p^2 / D)^{\frac{1}{2}} \)  
**stiffened plate parameter**

\( K_2 \)  
\((K_1 a / \pi)^2 \)

\( K_3 \)  
\((ma/b)^2 \)

\([K]\)  
**stiffness matrix**

\([K_e]\)  
**elastic stiffness matrix**

\([K_g]\)  
**geometric stiffness matrix**

\([K_{g_s}]\)  
**geometric stiffness matrix reflecting the influence of \( P_0 \)**

\([K_{g_t}]\)  
**geometric stiffness matrix reflecting the influence of \( P_t \)**

\([K_{g_x}]\)  
**geometric stiffness matrix reflecting the influence of \( N_x \)**

\([K_{g_y}]\)  
**geometric stiffness matrix reflecting the influence of \( N_y \)**

\([K_{g_{xy}}]\)  
**geometric stiffness matrix reflecting the influence of \( N_{xy} \)**

\([K_i]\)  
**stiffness matrix of an element**

\( k \)  
**shear coefficient**

\( k \)  
**coefficient of rigidity (Chapter 3)**

\( k_f \)  
**foundation stiffness/unit length**
L \quad \text{length of beam}

l \quad \text{length of element}

[M] \quad \text{inertia or mass matrix}

[M_1] \quad \text{inertia matrix of element}

m \quad \text{mass}

N^* \quad \text{static buckling load}

N_x \quad \text{in-plane force in } x \text{ direction}

N_y \quad \text{in-plane force in } y \text{ direction}

N_{xy} \quad \text{in-plane shear force}

N_x^* \quad \text{in-plane static buckling load}

N_y^* \quad \text{in-plane static buckling load}

N_{xy}^* \quad \text{in-plane shear static buckling load}

\alpha_x N_x^* + \beta_x N_x \cos \omega t

\alpha_y N_y^* + \beta_y N_y \cos \omega t

\alpha_{xy} N_{xy}^* + \beta_{xy} N_{xy} \cos \omega t

[N] \quad \text{shape function matrix}

n \quad \text{constant}

P \quad \text{axial load}

P(t) \quad \text{periodic load}

P_0 \quad \text{static component of load}

P_t \quad \text{time-dependent component of load}

P^* \quad \text{static buckling load}

P_e^* \quad \text{fundamental static buckling load of Bernoulli-Euler beam}

P_t^* \quad \text{fundamental static buckling load of Timoshenko beam}

p \quad \text{circular frequency of vibration}

p_e \quad \text{circular frequency of vibration of beam on elastic foundation (Chapter 5)}
$P_e$ fundamental natural frequency of vibration of Bernoulli-Euler beam

$P_t$ fundamental natural frequency of vibration of Timoshenko beam

$P_i$ fundamental natural frequency of space frames and plates.

$\{Q\}$ generalised force vector

$\{q\}$ generalised co-ordinate vector

$R$ \( I/A\ell^2 \) - rotary inertia parameter

$[R]$ transformation or direction cosine matrix

$S$ \( (kG/E)(A\ell^2/I) \) - shear deformation parameter

$T$ kinetic energy

$T$ period (Chapter 3)

$t$ time

$t_1, t_2$ time limits

$U$ strain energy

$U_i$ strain energy of plate element

$U_b$ strain energy of plate due to bending

$U_s$ strain energy of plate due to in-plane forces

$U_x, U_y, U_z$ displacements in $x$, $y$ and $z$ directions

$u$ displacement in $x$ direction

$\{u\}$ nodal displacement vector

$V$ potential energy

$V$ volume (Chapter 2)

$v$ displacement in $y$ direction

$w$ displacement in $z$ direction

$w$ deflection surface of plate (Chapter 7)
$x$  co-ordinate axis

$\{x\}$ element in functional space

$y$  co-ordinate axis

$\{Z(t)\}$ periodic vector

$z$  co-ordinate axis

$\alpha$ fraction representing static component of load

$\alpha_i$ constants

$\beta$ fraction representing time dependent component of load

$\gamma$ $(\psi' - \phi)$ - shear slope

$\gamma$ $k_f l^a/\pi^b EI$ - elastic foundation constant (Chapter 5)

$\delta$ arbitrary constant

$\varepsilon_{xx}$ normal strains

$\varepsilon_{yy}$

$\varepsilon_{zz}$

$\varepsilon_{xy}$ shear strains

$\varepsilon_{yz}$

$\varepsilon_{zx}$

$\{\varepsilon\}$ strain vector

$\{\zeta\}$ nodal co-ordinate vector

$\eta$ $x/l$ - non-dimensional co-ordinate

$\theta_x$ rotation about x axis

$\theta_y$ rotation about y axis

$\theta_z$ rotation about z axis

$\theta$ $(k_2 + k_3)^{\frac{1}{2}}$ - stiffened plate parameter (Chapter 8)
\( \lambda \) \( \rho AL^4 p^2 / EI \) - frequency parameter of beam

\( \lambda_d \) \( \rho AL^4 \omega^2 / EI \) - disturbing frequency parameter

\( \lambda_e \) \( \rho AL^4 p_e^2 / EI \) - frequency parameter of beam on elastic foundation

\( \nu \) Poisson's ratio

\( \rho \) mass density

\( \rho_i \) complex roots or eigenvalues (Chapter 3)

\( \sigma_{xx} \) normal stresses

\( \sigma_{yy} \) normal stresses

\( \sigma_{zz} \) normal stresses

\( \sigma_{xy} \) shear stresses

\( \sigma_{yz} \) shear stresses

\( \sigma_{zx} \) shear stresses

\( \{ \sigma \} \) stress vector

\( \tau \) dimensionless time

\( \phi \) bending slope

\( \phi (K_2 - K_3)^{1/2} \) - stiffened plate parameter (Chapter 8)

\( \dot{\phi}(t) \) even function of time

\( \psi \) \( y/l \) - non-dimensional co-ordinate

\( \psi' \) \( (\phi + \gamma) \) - total slope

\( \omega \) disturbing frequency

\( \Omega \) specified region
CHAPTER 1
CHAPTER 1

INTRODUCTION

In recent years the study of dynamic stability has gained importance in virtue of the complex nature of the forces acting on the mechanical structures. A number of catastrophic incidents can be traced to parametric resonance. Parametric resonance may serve as a triggering mechanism to other phenomenon such as flutter, or it may bring about a fatigue failure. When a rod is subjected to the action of longitudinal compressive force varying periodically with time, then for a definite frequency, the transverse vibration of the rod will have rapidly increasing amplitude. Thus, the study of the formation of these types of vibrations and the formulation of the method for the prevention of their occurrence are necessary in various areas of machine design.

Mechanism linkages are frequently subjected to axial periodic forces. These forces, if allowed to persist, may cause parametric vibrations which because of their large amplitude can threaten or impair its normal operation.

Bridge design problems have motivated a number of analytical studies of the dynamic influence of travelling loads on beams. The problems encountered in the designs of modern rocket test tracks and submarines are of the same nature. Similar analytical study is required to determine the critical speed of a fluid passing through a flexible pipe.

Structures, during an earthquake, are subjected to axial dynamic motions. These motions can excite some structures,
having certain natural frequencies, to become dynamically unstable due to large deflections. The understanding of the nature of these parametric vibrations will be the first step in the process of preventing their occurrence.

In the design of offshore structures, it is suggested here, that these structures must be designed not only for the freak 50-year or 100-year wave, but also for the continuous battering by sea waves. Sea wave forces can be assumed to be periodic in nature and their influence on the dynamic stability of offshore drilling units can be investigated. Devices such as drill strings and others in ocean engineering where the suspension point moves up and down periodically may become dynamically unstable and the amplitude growth becomes a serious problem.

Every large liquid rocket developed so far has experienced some form of vibrational instability. The vibration, which is caused by a regenerative feedback interaction between the vehicle's propulsion system and structure, can be critical because it would seriously degrade the astronaut's ability to perform his functions and could lead to a physical injury. Also, to assure the precise location and direction of a space rocket or a missile at any moment during the flight, the vibration and stability behaviour are obviously of primary importance.

The finite element method is, nowadays, the most general and one of the most powerful tool for the analysis of structures. Procedures for the formulation of finite element force-displacement equations can be classed as either direct or
variational methods. Today, it is popular to characterise
the finite element method as a generalization of the Ritz
method in the calculus of variations. The original development
of the direct stiffness method by Turner, Clough, Martin and
Topp [1] in fact makes no reference to variational consider-
ations. The theoretical foundations of the finite element
method are presented in Chapter 2.

Bolotin's work [2] is the most comprehensive one on the
theory of dynamic stability. The regions of dynamic instability
are determined for several types of structural systems.
Bolotin [2] also indicates the method of solution of systems
with several degrees of freedom. The basic theory of dynamic
stability is presented in Chapter 3.

The problem of dynamic stability of a simply supported
bar was first investigated by Baliaev [3]. Additional terms
were introduced by Mettler [4] to take into account the inertia
forces. This problem was also investigated by Bolotin [2].
As in the case of the applied theory of vibration, Bolotin [2]
did not include the inertia forces associated with the rotation
of the cross sections of the rod with respect to its own
principal axes. The finite element method was used by Brown,
Hutt and Salama [5] to study the dynamic stability of bars with
various end conditions. In Reference [5] the Euler beam theory
was employed and the effects of shear deformation and rotary
inertia were neglected.

In Chapter 4, a finite element model is developed for the
dynamic stability analysis of Timoshenko beams subjected to
periodic axial forces. The effect of shear deformation and
rotary inertia on the natural frequencies of vibration is investigated and the reported existence of a second spectrum of frequencies is examined. The effect of shear deformation on the static buckling loads and for the first time the effect of shear deformation and rotary inertia on the regions of dynamic instability are investigated.

The elastic stiffness, geometric stiffness and inertia matrices are developed for a Timoshenko beam and the matrix equation for the dynamic stability is solved for hinged-hinged, fixed-free, free-free and fixed-fixed beams. A literature survey of the problem of dynamic stability of beams is also presented in Chapter 4.

The effects of elastic foundation of the Winkler type on the natural frequencies, static buckling loads and the regions of dynamic instability of Timoshenko beams with various boundary conditions are investigated in Chapter 5.

The problem of dynamic stability of plane frames was investigated by Bolotin [2]. Stability coefficients and corresponding mass coefficients were determined for a number of predefined system co-ordinates. The construction of the basic equations for the chosen co-ordinate functions was preceded by many calculations. Furthermore the choice of co-ordinates may have a serious effect on the solution of the dynamic stability problems. For these reasons the range of application of the method reported by Bolotin [2] is almost limited to simple problems. A similar approach with the same limitation was used by Roberts [6] to study the dynamic stability of plane frames.
In Chapter 6, a finite element model for the dynamic stability analysis of three dimensional frame structures is developed. Complete generality is maintained in the description of the structure co-ordinates. Regions of dynamic instability as well as natural frequencies and static buckling loads are determined. The computer program developed is tested on plane frames and correlation of results obtained with the available exact results indicates excellent agreement. A literature survey of the development and application of the finite element method to the dynamic stability of frames is presented.

The finite element approach is successfully applied to the analysis of offshore oil rigs subjected to periodic sea wave forces. To verify the accuracy of the computed natural frequencies of vibration a small scale model is built and free vibration tests are carried out. Agreement between the computed and measured frequencies of the oil rig model is found to be very good.

The dynamic stability of plates under compressive inplane forces was first investigated by Einaudi [7]. Approximate solution was obtained by Bolotin [2] using Galerkin's method. The form of vibration was approximated with the help of some suitable functions which satisfied the boundary conditions. The finite element method was used by Hutt and Salama [8] to study this problem. The element used in Reference [8] is the Melosh [9] rectangular plate element. The elastic stiffness matrix derived by Melosh [9] contained several errors and it was corrected by Tocher and Kapur [10]. The same elastic stiffness matrix was developed independently by Zienkiewicz.
and Cheung [11]. The mass matrix was developed by Dawe [12] and the geometric stiffness matrix by Kapur and Hartz [13]. The Melosh [9] element is based on an assumed deflection function which does not ensure compatibility of the normal slopes along coincident boundaries of the elements. Owing to this violation of compatibility a lower bound solution was obtained.

The application of smooth surface interpolation in finite element analysis was suggested by Birkhoff and Garabedian [14]. The values of the displacement are interpolated by means of the fourth-order Hermitian polynomial along the edges, whereas the slopes normal to each side are interpolated as varying linearly along that side. These conditions permit a fully compatible displacement field when elements are assembled together. Deak and Pian [15] derived the mass and elastic stiffness matrices using this displacement function giving a 12 degree-of-freedom rectangular bending element and used them to study the free vibration problem of rectangular plates.

In Chapter 7, the geometric stiffness matrices of this conforming element are developed for the first time and employed to investigate the dynamic stability problems of rectangular plates with various boundary and loading conditions. The rapid convergence of buckling solutions based on the derived geometric stiffness matrices is demonstrated. Comparison of results shows that the present conforming element gives in all cases better results than the well known 12 degree-of-freedom non-conforming rectangular element. A literature survey of the development and application of the
finite element for the static stability and dynamic stability of plates is presented.

Stiffened plates are a structural element of practical importance in many applications like ship superstructures, bridge decks and aircraft structures. The knowledge of the dynamic characteristics of stiffened plates is important in the prediction of fatigue failures. In engineering practice the plates are eccentrically stiffened by attaching the ribs on one side of the plates only so that the stiffness of the combined system is a maximum.

The free vibration of eccentrically stiffened plates was investigated by Long [16]. The natural frequencies were obtained by solving a set of homogeneous equations. The finite element method was used by Davis [17] where the plate was idealized by a non-conforming plate bending element.

The static stability problem of eccentrically stiffened plates was first solved by Timoshenko [18]. Seid and Stein [19] used the Rayleigh-Ritz method and Kapur and Hartz [13] used the finite element method.

The dynamic stability analysis of eccentrically stiffened plates subjected to periodic axial loads is presented in Chapter 8 for the first time. The effects of the stiffeners depth on the regions of dynamic instability are investigated. Results obtained for the natural frequencies and the static buckling loads are compared with the results obtained by other investigators. A literature survey of the development of the free vibration, static stability and dynamic stability analyses
of stiffened plates is presented in Chapter 8.

Final conclusions and possible further developments are presented in Chapter 9.
CHAPTER 2
2.1 Introduction

The concept of a "finite element" initially proposed by Turner, Clough, Martin and Topp [1] in 1956 for the analysis of aircraft structures has evolved into a powerful numerical tool for the solution of a wide range of problems in continuum mechanics. In early 1960's attention was focused on solid mechanics and the method was related to variational principles. Thus the finite element displacement method was interpreted as a piece-wise Rayleigh-Ritz approximation for the potential energy functional.

An important class of problems in solid mechanics can be described by a differential equation of the form

$$[A]\{x\} = \{f\} \quad (2.01)$$

in some region $\Omega$, where $\{x\}$ is an element sought in some functional space, $[A]$ is a matrix of linear operators and $\{f\}$ is a given element in the same functional space.

Solving equation (2.01) means determining the element $\{x\}$ which the operator $[A]$ transforms into $\{f\}$.

There exist several approximate methods of solution of equation (2.01) based on the variational principles like Rayleigh-Ritz and finite difference methods. Full details of the variational methods have been presented by Mikhlin [20].

The finite element method is similar to the Rayleigh-Ritz method and the only difference is in the manner in which the
displacements are prescribed. The theoretical foundations of the finite element method has been presented by Arantes Oliveira [21].

2.2 Finite Element Method in Structural Analysis

The finite element method is a general technique of numerical analysis which provides an approximate solution for equation (2.01). In this method, the domain $\Omega$ is considered to be composed of a finite number of subdomains. The discrete subdomain or element is defined by specific boundaries with a number of nodal points. A convenient local co-ordinate system is chosen and each node is allowed a number of translations or rotations, known as nodal displacement. To derive the element properties, displacement functions are assumed to define the displacement at any point within the element. It is written in a matrix form as

$$\{a\} = [N]\{u\} \quad (2.02)$$

where $\{a\}$ is the assumed displacement vector

$\{u\}$ is the nodal displacement vector

$[N]$ is the shape function matrix.

2.2.1 Strain-displacement relations

The deformed shape of an elastic structure under a given system of loads can be described completely by the three displacements

$$u = u(x, y, z)$$
$$v = v(x, y, z) \quad (2.03)$$
$$w = w(x, y, z)$$

The vectors representing these three displacements at a point
in the structure are mutually orthogonal and their positive
directions correspond to the positive directions of the co-
ordinate axes x, y and z. The strains in the deformed
structure can be expressed as partial derivatives of the
displacements u, v and w. For small deformation the strain-
displacement relations are linear and the strain components
are given by

$$
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} \\
\varepsilon_{xy} &= \varepsilon_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\varepsilon_{yz} &= \varepsilon_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\
\varepsilon_{zx} &= \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{align*}
$$

(2.04)

where $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\varepsilon_{zz}$ represent normal strains and $\varepsilon_{xy}$, $\varepsilon_{yz}$ and $\varepsilon_{zx}$ represent shearing strains.

Equations (2.04) can be presented in matrix form as

$$
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{zx}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
$$

(2.05)

which can be written as

$$
\{\varepsilon\} = [G] \{u\}
$$

(2.06)

where $\{\varepsilon\}$ is the strain vector

$\{u\}$ is the displacement vector
2.2.2 Stress-strain relations

The elastic strains are related to the stresses by means of Hooke's law for linear isothermal elasticity.

\[ \varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \right] \]
\[ \varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx}) \right] \]
\[ \varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) \right] \]
\[ \varepsilon_{xy} = \frac{1}{G} \cdot \sigma_{xy} \]
\[ \varepsilon_{yz} = \frac{1}{G} \cdot \sigma_{yz} \]
\[ \varepsilon_{zx} = \frac{1}{G} \cdot \sigma_{zx} \]

where \( E \) is the modulus of elasticity, \( \nu \) is Poisson's ratio and \( G = \frac{E}{2(1+\nu)} \) is the modulus of rigidity.

Equations (2.07) can be solved for the stresses and the following stress-strain relationships are then obtained.

\[ \sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz}) \right] \]
\[ \sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{zz} + \varepsilon_{xx}) \right] \]
\[ \sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy}) \right] \]
\[ \sigma_{xy} = \frac{E}{2(1+\nu)} \varepsilon_{xy} \]
\[ \sigma_{yz} = \frac{E}{2(1+\nu)} \varepsilon_{yz} \]
\[ \sigma_{zx} = \frac{E}{2(1+\nu)} \varepsilon_{zx} \]

Equations (2.08) can be written in matrix form as
which can be written as

\[
\{\sigma\} = [E] \{\epsilon\} \quad (2.10)
\]

where

\{\sigma\} is the stress vector

\{\epsilon\} is the strain vector

\[E\] is the elasticity matrix

### 2.2.3 Strain energy of a finite element

The strain energy \( U_i \) of the element can be presented in terms of the strains and stresses as

\[
U_i = \frac{1}{2} \int_V \{\epsilon\}^T \{\sigma\} \, dV \quad (2.11)
\]

The integration being carried out over the volume of the element \( V \) and the superscript \( [\ ]^T \) denotes the transpose of the matrix.

Substituting the stress-strain relationship (2.10) into equation (2.11) the energy expression becomes

\[
U_i = \frac{1}{2} \int_V \{\epsilon\}^T [E] \{\epsilon\} \, dV \quad (2.13)
\]
Substituting the strain-displacement relationship (2.06) into (2.13) the expression becomes

\[ U_i = \frac{1}{2} \int_V \{u\}^T [G] [E] [G] \{u\} \, dV \quad (2.14) \]

\[ U_i = \frac{1}{2} \{u\}^T [k_i] \{u\} \quad (2.15) \]

where the matrix \([k_i]\) is known as the stiffness matrix of the element and is given by

\[ [k_i] = \int_V [G]^T [E] [G] \, dV \quad (2.16) \]

2.2.4 Kinetic energy of a finite element

The kinetic energy \(T_i\) of the element can be written as

\[ T_i = \frac{1}{2} \int_V \rho \{\ddot{u}\}^T \{\ddot{u}\} \, dV \quad (2.17) \]

where dots denote partial differentiation with respect to time and \(\rho\) is the mass density of the material.

Equation (2.17) can be expressed in terms of the nodal velocity \(\{\ddot{u}\}\) obtained from equation (2.02) thus

\[ T_i = \frac{1}{2} \int_V \{\ddot{u}\}^T [N]^T \rho [N] \{\ddot{u}\} \, dV \quad (2.18) \]

\[ T_i = \frac{1}{2} \{\ddot{u}\}^T [M_i] \{\ddot{u}\} \quad (2.19) \]

where the matrix \([M_i]\) is known as the mass or inertia matrix of the element and is given by

\[ [M_i] = \int_V [N]^T \rho [N] \, dV \quad (2.20) \]
2.2.5 Strain and kinetic energies of a complete system

As some of the elements may be differently orientated with each other, a common co-ordinate system may be arbitrarily chosen as a datum for the whole system. The element stiffness and mass matrices can be transformed from local co-ordinates \{u\} into common co-ordinates \{q\} by using

\[ \{u\} = [R] \{q\} \quad (2.21) \]

where \([R]\) is known as the transformation matrix or the direction cosine matrix.

The total strain and kinetic energies of the system are equal to the sum of the energies of all the elements and can be expressed in general quadratic forms as

\[ U = \frac{1}{2} \{q\}^T [K] \{q\} \quad (2.22) \]
\[ T = \frac{1}{2} \{\dot{q}\}^T [M] \{\dot{q}\} \quad (2.23) \]

where \([K]\) and \([M]\) are the global stiffness and mass matrices of the whole system.

2.2.6 Equations of motion

The equations of motion of the system can be derived from Hamilton's variational principle, which states that the variation of the system's energies vanishes between prescribed time limit \(t_1\) to \(t_2\)

\[ \int_{t_1}^{t_2} (T-U) \, dt = 0 \quad (2.24) \]

equation (2.24) leads to Lagrange's equation.
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q} \right) - \frac{d}{dt} \left( \frac{\partial U}{\partial q} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = \{Q\} \tag{2.25}
\]

where \(\{Q\}\) is the generalized nodal force corresponding to \(\{q\}\).

Applying Lagrange's principle to the energies yields the equation of dynamic equilibrium of the system

\[
[M] \{\ddot{q}\} + [K] \{q\} = \{Q\} \tag{2.26}
\]

Upon assuming simple harmonic motion the free vibration equation becomes

\[
\left[ [K] - \lambda [M] \right] \{q\} = 0 \tag{2.27}
\]

where \(\lambda\) is the frequency parameter.

The eigenvalue problem \((K - \lambda M) = 0\), where \(K\) is real symmetric and \(M\) is real symmetric positive definite, is solved by Householder's method and QL algorithm.
CHAPTER 3
3.1 Introduction

As a simple introduction, the elastic system shown in Figure 3.1 is considered. The concentrated mass \( m \) is fastened to the end of the weightless rod \( r \) where the free displacement of the rod is further limited by the bushing \( b \) at a distance \( l \) from the lower end of the rod. If the displacement of the mass at any instant of time \( t \) is \( y \), the restoring force of the elasticity of the rod is given by \(-ky\) and the equation of motion of the mass is of the form \(-ky = my\) \( (3.01) \)

where \( k \) is the coefficient of rigidity of the system.

If the bushing is long enough to give practically complete clamping of the lower part of the rod, then the coefficient \( k \) may be found from the formula

\[ k = \frac{3EI}{l^3} \]

where \( E \) is the modulus of elasticity and \( I \) is the moment of inertia. Thus, the equation of motion can be written as

\[ \ddot{y} + \frac{3EI}{ml^3} y = 0 \] \( (3.02) \)

If the distance \( l \) is constant, Equation (3.02) describes the free vibration of the mass about its mean position and the fraction \( 3EI/ml^3 \) is the square of the characteristic frequency of the vibration.
Supposing that the bushing slides along the rod according to the fixed law

\[ z = A \cos \omega t \]

i.e., it executes harmonic oscillations with amplitude \( A \) and angular frequency \( \omega \), then the coefficient of rigidity is a function of the time

\[
k = \frac{3EI}{(l+z)^3} = \frac{3EI}{(l+A \cos \omega t)^3} \quad (3.03)
\]

The differential equation (3.02) becomes an equation with variable coefficients

\[
\ddot{y} + \frac{3EI}{m(l+A \cos \omega t)^3} y = 0 \quad (3.04)
\]

The vibrations of the mass \( m \) can now no longer be called free, since they occur under the time-dependent external force consisting of the periodic change in the rigidity of the system. On the other hand, these vibrations can not be called forced, since the external force is not a driving force. The vibrations of such systems, which occur for a fixed type of change in the parameters of the system (in this case the rigidity) are said to be parametrically excited.

There are some ranges of the exciting frequencies in which the amplitudes of the parametric oscillation increase monotonically. This phenomenon of parametric resonance or dynamic instability is more dangerous than ordinary resonance which only occurs for clearly defined values of the external driving frequency.
There are many mechanical systems that are subjected to parametric excitation and in the majority of cases of practical importance, the differential equation of the parametric vibration may be reduced to the standard form

\[
\frac{d^2y}{d\tau^2} + (a - 2b \cos 2\tau) y = 0 \quad (3.05)
\]

where \(a\) and \(b\) are constants.

In the mechanical system shown in Figure 3.1, if we assume that the amplitude \(A\) of the vibrations of the bushing is very small in comparison with the length \(l\), then instead of equation (3.03) we obtain approximately

\[
k = \frac{3EI}{(l + A \cos \omega t)^3} \approx \frac{3EI}{l^3 + 3Al^2 \cos \omega t} = \frac{3EI}{l^3} \left(1 - \frac{3A}{l} \cos \omega t\right) \quad (3.06)
\]

and the differential equation (3.04) takes the form

\[
\ddot{y} + \frac{3EI}{ml^3} \left(1 - \frac{3A}{l} \cos \omega t\right) y = 0 \quad (3.07)
\]

Expressing in terms of the dimensionless time \(\tau\)

where \(2\tau = \omega t\)

the equation (3.07) becomes

\[
\frac{d^2y}{d\tau^2} + \left(\frac{12EI}{momega^2l^3} - \frac{36EIA}{momega^2l^4}\right) \cos 2\tau y = 0 \quad (3.08)
\]

Now the differential equation (3.08) takes on the standard form (3.05) with

\[
a = \frac{12EI}{momega^2l^3} \quad \text{and} \quad b = \frac{18EIA}{momega^2l^4} \quad (3.09)
\]

Transformations of this type are typical of cases of small pulsations of the variable parameter of the system.
Equation (3.05) is called Mathieu's equation and its solutions are oscillatory and depend in a decisive way on the actual values of the parameters \(a\) and \(b\). In some cases, a given combination of \(a\) and \(b\) corresponds to combinations of limited amplitude, while in other cases, the vibrations are of increasing amplitude. Essentially, the subsequent behaviour of the vibrations is of little importance, since the important thing is the tendency followed by the vibrational process: if the amplitudes remain bounded, the system is dynamically stable; otherwise, parametric resonance occurs, and the system is dynamically unstable.

### 3.2 Regions of Dynamic Instability

The results of solving Mathieu's equation (3.05) for two different combinations of \(a\) and \(b\) are shown in Figure 3.2. Although the parameter \(b\) of the system is the same in both cases \((b=0.1)\), the vibrations are greatly different because of the difference between the values of the parameter \(a\) \((a=1; a=1.2)\). In the first case, they increase, i.e., the system is dynamically unstable, while in the second case they remain bounded, i.e., the system is dynamically stable.

The greatest importance, for practical purpose, is attached to the boundaries between the regions of stable and unstable solutions. This problem has been well studied, and the final results have been presented in the form of a diagram plotted in the plane of the parameters \(a\) and \(b\). It is called the Haines-Strett diagram. Figure 3.3 shows part of a Haines-Strett diagram for small values of the parameter \(b\). Any given system
having the parameters a and b corresponds to the point with
the co-ordinates a and b on the Haines-Strett diagram. If
the representative point is in the shaded parts of the diagram,
the system is dynamically stable, while unstable systems
 correspond to representative points in the unshaded parts. The
unshaded regions are called the regions of dynamic instability.

As an example, the diagram in Figure 3.3 shows the points
1 and 2 corresponding to the parameter a₁ = 1.0 and b₁ = 0.1,
and a₂ = 1.2 and b₂ = 0.1. The point 1 is in the region of
dynamic instability and the vibration occurs with increasing
amplitude as shown in Figure 3.2. The point 2 is in the
stable region and it corresponds to motion with a limited
amplitude.

3.3 Types of Parametric Excitation

Three reasons for parametric excitation may be observed as
a) periodic change in rigidity    b) periodic change in inertia
of the system and  c) periodic change in the loading of the
system.

3.3.1 Periodic change in rigidity

The elastic part of the system shown in Figure 3.4
consists of the slotted shaft s; on the lower end of the shaft
is the disc d. Connected to the shaft is the massive slotted
bushing b which slides along the axis of the shaft and
executes harmonic oscillations in the vertical direction. It
is possible in this system to have parametric excitation of
both flexural and torsional vibrations. Let the free length
of the shaft at the instant of time t be
The coefficient of torsional rigidity of the rod is given by

\[ k = \frac{GJ}{\ell} = \frac{GJ}{\ell_0 + A \cos \omega t} \]  

(3.11)

where \( G \) is the modulus of rigidity and \( J \) is St. Venant torsion constant.

If the amplitude \( A \) of the vibrations is considerably less than the mean value of the length \( \ell_0 \), equation (3.11) may be written in the form

\[ k = \frac{GJ}{\ell_0} \left( 1 - \frac{A}{\ell_0} \cos \omega t \right) \]  

(3.12)

which is of exactly the same structure as equation (3.06). Accordingly, the torsional vibrations of the system are likewise described by the Mathieu equation (3.05) with

\[ a = \frac{4GJ}{\ell_0 \omega^2}, \quad b = \frac{2GJA}{\ell_0^2 \omega^2} \]  

(3.13)

### 3.3.2 Periodic change in inertia

The simplest system of this type is shown in Figure 3.5. The point mass \( m \) moves alternately up and down along a weightless rod, freely rotating in the plane of the paper about the hinge \( h \). The length of a pendulum of this sort is a periodic function of the time.

### 3.3.3 Periodic change in loading

Consider the system shown in Figure 3.6. The mass \( m \) is fastened to the upper end of the vertical, perfectly rigid rod \( r \), and at the bottom of the rod there is the support \( s \),
which offers elastic resistance to rotation. The vertical force \( P \) acts at the upper end of the rod.

The equation of free vibration of the system is given by

\[
P\ddot{\phi} - k\phi = m\omega^2 \phi
\]  

(3.14)

where \( k\phi \) is the restoring moment (the moment of the elastic hinge) and the gravitational force due to the mass \( m \) is included in the force \( P \).

If the force \( P \) varies according to the harmonic law

\[
P = P_0 + P_t \cos \omega t
\]  

(3.15)

then equation (3.14) becomes

\[
\ddot{\phi} + \frac{1}{m\omega^2} (k - P_0 \ell - P_t \ell \cos \omega t) \phi = 0
\]  

(3.16)

This equation reduces to the standard form of the Mathieu equation (3.05) with

\[
2\tau = \omega t , \quad a = \frac{4}{m\omega^2 \ell} \left( \frac{k}{\ell} - P_0 \right) , \quad b = \frac{2P_t}{m\omega^2 \ell}
\]  

(3.17)

3.4 Dynamic Stability of Multi-degree of Freedom Systems

The matrix equation for free vibration of axially loaded system can be written as [2] :

\[
[M] \{\ddot{q}\} + [K_e] \{q\} - [K_g] \{q\} = 0
\]  

(3.18)

where

\( \{q\} \) is the generalized co-ordinates

\( [M] \) is the inertia matrix

\( [K_e] \) is the elastic stiffness matrix

\( [K_g] \) is the geometric stiffness matrix which is a function of the axial load \( P \).
For a system subjected to a periodic force

\[ P = P_0 + P_t f(t) \]  

(3.19)

the static and time dependent components of the load can be represented as a fraction of the fundamental static buckling load \( P^* \), hence by writing \( P = \alpha P^* + \beta P^* f(t) \) equation (3.18) becomes

\[
[M] \{\ddot{q}\} + \left[\begin{array}{c}
K_e - \alpha P^* [K_g] - \beta P^* f(t) [K_{g_t}] 
\end{array}\right] \{q\} = 0 \tag{3.20}
\]

where the matrices \( [K_{g_s}] \) and \( [K_{g_t}] \) reflect the influence of \( P_0 \) and \( P_t \) respectively. \( f(t) \) is periodic with period \( T \). Therefore

\[ f(t + T) = f(t) \]  

(3.21)

Equation (3.20) is a system of \( n \) second order differential equations which may be written as

\[
\{\ddot{q}(t)\} + [Z] \{q(t)\} = 0 \tag{3.22}
\]

where

\[
[Z] = [M]^{-1} \left[\begin{array}{c}
K_e - \alpha P^* [K_g] - \beta P^* f(t) [K_{g_t}] 
\end{array}\right] \tag{3.23}
\]

It is convenient to replace the \( n \) second order equations with 2\( n \) first order equations by introducing

\[
\{h\} = \begin{bmatrix} \{q\} \\ \{\dot{q}\} \end{bmatrix} \tag{3.24}
\]

then, equation (3.22) becomes

\[
\{\ddot{h}(t)\} + [\phi(t)] \{h(t)\} = 0 \tag{3.25}
\]

where

\[
[\phi] = \begin{bmatrix} 0 & -[I] \\ [Z] & 0 \end{bmatrix}
\]
Equation (3.25) needs not be solved completely in order to determine the stability of the system. It is merely necessary to determine whether the solution is bounded or unbounded.

It is assumed that the 2n linearly independent solutions of equation (3.25) are known over the interval \( t = 0 \) to \( t = T \). Then they may be represented in matrix form as

\[
[H(t)] = \begin{bmatrix}
h_{11} & \cdots & h_{1,2n} \\
\vdots & \ddots & \vdots \\
h_{2n,1} & \cdots & h_{2n,2n}
\end{bmatrix}
\]

Since \( f(t) \), and therefore \([\phi(t)]\) is periodic with period \( T \), then the substitution

\[ t = t + T \]

will not alter the form of the equations, and the matrix solutions, at time \( t + T \), \([H(t+T)]\) may be obtained from \([H(t)]\) by a linear transformation

\[
[H(t+T)] = [R] [H(t)]
\]

where \([R]\) is the transformation matrix and is composed only of constant coefficients.

It is desirable to find a set of solutions for which the matrix \([R]\) can be diagonalized. Hence the \( i \)th solution vector after period \( T \), \( \{\tilde{h}(t+T)\}_i \), may be determined from \( \{\tilde{h}(t)\}_i \) using the simple expression

\[
\{\tilde{h}(t+T)\}_i = \rho_i \{\tilde{h}(t)\}_i \quad (3.26)
\]

The behaviour of the solution is determined by \( \rho_i \).
If $\rho_i > 1$, then the amplitude of vibration will increase with time. If $\rho_i < 1$, then the amplitude will decrease. For $\rho_i = 1$, the amplitude will remain unchanged, and this represents the stability boundary.

In order to diagonalize the matrix $[R]$, the characteristic equation

$$\left| [R] - \rho [I] \right| = 0 \quad (3.27)$$

must be solved for its $2n$ roots, where $[I]$ is the Identity matrix. The roots of the equation, $\rho_i$, are eigenvalues, each having a corresponding eigenvector. The $i^{th}$ eigenvector, $\{\bar{h}(t)\}_i$, is the solution which will satisfy equation (3.26). The $2n$ resulting eigenvectors are chosen as the $2n$ solutions to equation (3.25). They can be placed in a matrix, $[\bar{H}(t)]$, which will then satisfy the expression

$$[\bar{H}(t)] = [\bar{R}][\bar{H}(t+T)] \quad (3.28)$$

where $[\bar{R}]$ is the diagonalized $[R]$ matrix composed of the $2n$ eigenvalues of equation (3.27)

The periodic vector, $\{Z(t)\}_i$, with period $T$ is introduced so that

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i \, e^{(t/T)\log\rho_i} \quad (3.29)$$

For an even function of time, like, $[\phi(t)]$, it is true that

$$[\phi(t)] = [\phi(-t)] \quad (3.30)$$

hence equation (3.29) can be written as
\[
\{\mathbf{h}(-t)\}_i = \{Z(-t)\}_i e^{-(t/T) \log \rho_i}
\]
then
\[
\{\mathbf{h}(-t)\}_i = \{Z(-t)\}_i e^{(t/T) \log (1/\rho_i)} \quad (3.31)
\]

It is clear from (3.31) that \(1/\rho_i\) is also an eigenvalue. This property is not restricted to even functions, but is also preserved in the case of arbitrary periodic functions as shown by Bolotin [2].

In general, the eigenvalues \(\rho_i\) are complex numbers of the form
\[
\rho_i = a_i + j b_i \quad (3.32)
\]
and the natural logarithm of a complex number is given by
\[
\log \rho = \log |\rho| + j \text{(argument } \rho) \quad (3.33)
\]
or in this case
\[
\log \rho_i = \log \sqrt{a_i^2 + b_i^2} + j \tan^{-1}(b_i/a_i) \quad (3.34)
\]
where \(j = \sqrt{-1}\)

From equation (3.29), it is clear that if the real part of \(\log \rho_i\) is positive for any of the solutions, then that solution will be unbounded with time. A negative real part means that the corresponding solution will damp out with time. It therefore follows that the boundary case for a given solution is that for which the characteristic exponent has a zero real part. This is identical to saying that absolute value of \(\rho_i\) is unity.

For the system to remain stable, every one of the solutions
must remain bounded. If even one of the solutions has a characteristic exponent which is positive, then the corresponding solution is unbounded and therefore the system is unstable.

It has been shown that if \( \rho_i \) is a solution, then \( 1/\rho_i \) is also a solution. These two solutions can be written as

\[
\rho_i = a_i + j b_i
\]

\[
\rho_{i+n} = (a_i - j b_i)/(a_i^2 + b_i^2)
\]

Another restriction on the solutions of the characteristic equation is that the complex eigenvalues must occur in complex conjugate pairs. Hence it follows that \( \rho_{i+1} \) and \( \rho_{i+n+1} \) are also solutions where

\[
\rho_{i+1} = a_i - j b_i
\]

\[
\rho_{i+n+1} = (a_i + j b_i)/(a_i^2 + b_i^2)
\]

These solutions are presented in Figure 3.7 which shows a unit circle in the complex plane. The area inside the unit circle represents stable or bounded solutions, while the area outside the unit circle represents unstable or unbounded solutions. For each stable solution which lies inside the circle, there corresponds an unstable solution outside the circle due to the reciprocity constraint. Therefore the only possible stable solutions must lie on the unit circle.

Points on this unit circle may be represented in polar co-ordinates by \( r = 1 \) and \( \theta = \tan^{-1} b/a \) where \(-\pi \leq \theta \leq \pi\). For each root on the upper semicircle, there is a corresponding root on the lower semicircle due to the fact that the roots
occur in complex conjugate pairs. The logarithm of $\rho_i$, when $\rho_i$ lies on the unit circle will be

$$\log \rho_i = j\theta$$

and equation (3.29) becomes

$$\{\bar{H}(t)\} = \{Z(t)\} e^{j\theta t/T}$$

(3.35)

Since the eigenvalues occur in complex conjugate pairs, the limiting values of $\theta$ are zero and $\pi$.

when $\theta = 0$, equation (3.35) becomes

$$\{\bar{H}(t)\} = \{Z(t)\}$$

(3.36)

and, therefore, the solution $\{\bar{H}(t)\}$ is periodic with period $T$

when $\theta = \pi$, equation (3.35) becomes

$$\{\bar{H}(t)\} = \{Z(t)\} e^{j\pi t/T}$$

(3.37)

or

$$\{\bar{H}(t)\} = \{Z(t)\} e^{j2\pi t/2T}$$

(3.38)

it is clear from equation (3.38) that the solution $\{\bar{H}(t)\}$ is also periodic with period $2T$.

It can be concluded that equation (3.20) has periodic solutions of period $T$ and $2T$. Also the boundaries between stable and unstable regions are formed by periodic solutions of period $T$ and $2T$.

For a system subjected to the periodic force

$$P = P_0 + P_t \cos \omega t$$

(3.39)

where $\omega$ is the disturbing frequency, equation (3.20) becomes
\[ [M]\{\ddot{q}\} + \left[ [K_e] - \alpha P^*[K_{g_S}] - \beta P^* \cos \omega t [K_{g_t}] \right] \{q\} = 0 \quad (3.40) \]

Now we seek periodic solutions of period \( T \) and \( 2T \) of equation (3.40) where \( T = 2\pi/\omega \).

When a solution of period \( 2T \) exists, it may be represented by the Fourier series

\[ \{q\} = \sum_{k=1,3,5}^{\infty} \{a\}_k \sin k\omega t/2 + \{b\}_k \cos k\omega t/2 \quad (3.41) \]

where \( \{a\}_k \) and \( \{b\}_k \) are time-independent vectors.

Differentiating equation (3.41) twice with respect to time yields

\[ \{\ddot{q}\} = \sum_{k=1,3,5}^{\infty} -\left(\frac{k\omega}{2}\right)^2 \left[ \{a\}_k \sin k\omega t/2 + \{b\}_k \cos k\omega t/2 \right] \quad (3.42) \]

Substituting into equation (3.40) and using the trigonometric relations

\[
\begin{align*}
\sin A + \sin B &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\
\sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \\
\cos A + \cos B &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\
\cos A - \cos B &= 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}
\end{align*}
\]

and comparing the coefficients of \( \sin k\omega t/2 \) and \( \cos k\omega t/2 \) lead to the following matrix equations relating the vectors \( \{a\}_k \) and \( \{b\}_k \):
\[
\begin{bmatrix}
[K_e] - \alpha P*[K_{gs}] + \frac{\beta P*[K_{gt}]}{4} - \frac{\omega^2}{4} [M] & -\frac{\beta P*[K_{gt}]}{4} & 0 \\
-\frac{\beta P*[K_{gt}]}{4} & [K_e] - \alpha P*[K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{\beta P*[K_{gt}]}{4} \\
0 & -\frac{\beta P*[K_{gt}]}{4} & [K_e] - \alpha P*[K_{gs}] - \frac{25\omega^2}{4} [M]
\end{bmatrix}
\begin{bmatrix}
\{a\}_1 \\
\{a\}_2 \\
\{a\}_3
\end{bmatrix} = 0 \quad (3.43)
\]

and

\[
\begin{bmatrix}
[K_e] - \alpha P*[K_{gs}] - \frac{\beta P*[K_{gt}]}{4} - \frac{\omega^2}{4} [M] & -\frac{\beta P*[K_{gt}]}{4} & 0 \\
-\frac{\beta P*[K_{gt}]}{4} & [K_e] - \alpha P*[K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{\beta P*[K_{gt}]}{4} \\
0 & -\frac{\beta P*[K_{gt}]}{4} & [K_e] - \alpha P*[K_{gs}] - \frac{25\omega^2}{4} [M]
\end{bmatrix}
\begin{bmatrix}
\{b\}_1 \\
\{b\}_2 \\
\{b\}_3
\end{bmatrix} = 0 \quad (3.44)
\]
The order of matrices in equations (3.43) and (3.44) are infinite. If solutions of period $2T$ exist, then the determinants of these matrices must vanish. Combining these two determinants, the condition may be written as

\[
\begin{vmatrix}
K_e - aP[K_{gs}] + \frac{1}{2} \beta P[K_{gt}] - \frac{\omega^2}{4} [M] & -\frac{1}{2} \beta P[K_{gs}] & 0 \\
-\frac{1}{2} \beta P[K_{gs}] & K_e - aP[K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{1}{2} \beta P[K_{gt}] \\
0 & -\frac{1}{2} \beta P[K_{gs}] & K_e - aP[K_{gs}] - \frac{25\omega^2}{4} [M]
\end{vmatrix} = 0 \quad (3.45)
\]

If a solution to equation (3.40) exists with a period $T = 2\pi/\omega$ then it may be expressed as a Fourier series

\[
\{q\} = \frac{1}{2} b_0 + \sum_{k=2,4,6} a_k \sin \frac{k\omega t}{2} + b_k \cos \frac{k\omega t}{2} \quad (3.46)
\]

Substituting into equation (3.40), the following conditions for the existence of solution with period $T$ are obtained:
\[
\begin{bmatrix}
K_e - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2} \beta P^*[K_{gt}] \\
-\frac{1}{2} \beta P^*[K_{gt}] & K_e - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2} \beta P^*[K_{gt}] \\
0 & -\frac{1}{2} \beta P^*[K_{gt}] & K_e - \alpha P^*[K_{gs}] - 9\omega^2[M]
\end{bmatrix} = 0
\]

(3.47)

and

\[
\begin{bmatrix}
K_e - \alpha P^*[K_{gs}] & -\frac{1}{2} \beta P^*[K_{gt}] \\
-\frac{1}{2} \beta P^*[K_{gt}] & K_e - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2} \beta P^*[K_{gt}] \\
0 & -\frac{1}{2} \beta P^*[K_{gt}] & K_e - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2} \beta P^*[K_{gt}] \\
0 & 0 & -\frac{1}{2} \beta P^*[K_{gt}] & K_e - \alpha P^*[K_{gs}] - 9\omega^2[M]
\end{bmatrix} = 0
\]

(3.48)
It has been shown by Bolotin [2] that solutions with period $2T$ are the ones of greatest practical importance and that as a first approximation the boundaries of the principal regions of dynamic instability can be determined from the equation

$$\left[Ke - \alpha P^* [K_g] + \beta P^* [K_{gt}] - \frac{\omega^2}{4} [M]\right] \{q\} = 0 \quad (3.49)$$

The two matrices $[K_{gs}]$ and $[K_{gt}]$ will be identical if the static and time dependent components of the loads are applied in the same manner. If $[K_{gs}] = [K_{gt}] = [K_g]$, then equation (3.49) becomes

$$\left[Ke - (\alpha \pm \beta) P^* [K_g] - \frac{\omega^2}{4} [M]\right] \{q\} = 0 \quad (3.50)$$

Equation (3.50) represents solutions to three related problems

(i) Free vibration with $P^* = 0$ and $p = \omega/2$ the natural frequency

$$\left[Ke - p^2 [M]\right] \{q\} = 0 \quad (3.51)$$

(ii) Static stability with $\alpha = 1$, $\beta = 0$ and $\omega = 0$

$$\left[Ke - P^* [K_g]\right] \{q\} = 0 \quad (3.52)$$

(iii) Dynamic stability when all terms are present

$$\left[Ke - (\alpha \pm \beta) P^* [K_g] - \frac{\omega^2}{4} [M]\right] \{q\} = 0 \quad (3.53)$$
FIG. 3.1 MECHANICAL SYSTEM

![Mechanical System Diagram](image)

FIG. 3.2 TWO SOLUTIONS OF MATHIEU'S EQUATION

![Graph of Two Solutions](image)

FIG. 3.3 PART OF HAINES-STRETT DIAGRAM

![Haines-Strett Diagram](image)

THE POINTS 1 AND 2 CORRESPOND TO THE SOLUTIONS 1 AND 2 IN FIGURE 3.2
FIG. 3.4  SYSTEM WITH PERIODIC CHANGE IN TORSIONAL RIGIDITY

FIG. 3.5  SYSTEM WITH PERIODIC CHANGE IN INERTIA

FIG. 3.6  SYSTEM WITH PERIODIC CHANGE IN LOADING
FIG. 3.7 UNIT CIRCLE IN THE COMPLEX PLANE
CHAPTER 4
CHAPTER 4

DYNAMIC STABILITY OF TIMOSHENKO BEAMS

4.1 Introduction

The classical Bernoulli-Euler theory for bending vibrations is known to give higher frequency values than those obtained by experiment for higher modes and even in the lower modes for thick beams. Rayleigh [22] improved the classical theory by allowing for the effects of rotary inertia of the cross-sections of the beam. Timoshenko [23] extended the theory to include the effects of shear deformation. The resulting equations are known as the Timoshenko beam equations. Prescott [24] and Volterra [25] suggested, by independent reasoning, various Timoshenko-type beam models. The Timoshenko theory is considered to be satisfactory due to the close agreement it gives with the exact elasticity solutions obtained by Pochhammer [26] for circular beams. Solutions of Timoshenko equations for a cantilever beam of rectangular cross section have been given by Sutherland and Goodman [27] and also by Huang [28, 29].

Various methods of solution have been applied to this problem. Anderson [30] and Dolph [31] gave a general solution and complete analysis of a simply supported uniform beam. Huang [29] gave frequency equations and normal modes of vibration for various cases of a uniform beam using homogeneous boundary conditions. Ritz and Galerkin methods were used by Huang [28] to obtain frequencies of simply supported beams. The finite difference method was used by Thomas [32] to obtain frequencies of a fixed-free beam.
A number of finite element models have been presented for the analysis of a Timoshenko beam by various investigators. Many of these experienced difficulty in incorporating all the boundary conditions. Although some authors claimed that their finite element model was designed to incorporate all the boundary conditions, none of them so far have been able to apply all the boundary conditions. The true boundary conditions associated with various end conditions are as follows:

(a) free end - zero bending moment and zero shear force;
(b) hinged end - zero total deflection and zero bending moment;
(c) fixed end - zero total deflection and zero bending slope.

The conditions of deflection and slope are caused by restraints or external forces applied and can be referred to as forced or geometric boundary conditions while the others can be referred to as the natural boundary conditions.

McCalley [33] derived consistent mass and stiffness matrices by selecting total deflection and bending slope as nodal co-ordinates. Archer [34] used these matrices to obtain frequencies of a cantilevered Timoshenko beam. Boundary conditions at the free end were not and could not be applied in this model.

Kapur [35] improved on this model by taking bending deflection, shear deflection, bending slope and shear slope as the nodal co-ordinates, and derived the stiffness and mass matrices. Frequency parameters also were obtained for cantilevered and simply supported beams. The true boundary conditions were applied only at the fixed end. At the free end the shear force is assumed to be zero but the condition of zero bending moment could not be imposed. For the hinged end, the bending
deflection and shear deflection were assumed to be zero but again the zero bending moment condition could not be applied. Although it was an improvement over Archer's [34] model, it still lacked the facility of applying the true boundary conditions in all cases.

Carnegie, Thomas and Dokumaci [36] presented an internal node element considering the total deflection and bending slope as the co-ordinates at the two terminal nodes and two internal nodes, thus giving eight degrees of freedom element. This, however, lacked the facility to impose the natural boundary conditions at the free end.

Egle [37] presented an approximate Timoshenko beam theory designed to eliminate the coupling between the shear deformation and rotary inertia. He postulated a constraint, consistent with Bernoulli-Euler theory, that the shear force to be given by the first-derivative of bending moment. This constraint implies that this theory is valid only when shear deformation is negligible in comparison to the bending deformation.

Nickel and Secor [38] derived stiffness and mass matrices for what they called TIM7 element, using total deflection, total slope and bending slope as the nodal co-ordinates and the bending slope at mid point, giving rise to matrices of order seven. The boundary conditions used for a cantilever beam were the same as Kapur's [35], and thus again the zero bending moment condition at the free end was missing. Nickel and Secor [38] further reduced the order of the matrix from seven to four by using the constraint postulated by Egle [37]. This element was referred to as TIM4. The natural boundary
conditions at the free end could not be applied to this element. Davis, Henshell and Warburton [39] used an element model similar to McCalley's [33] using the constraint postulated by Egle [37]. The stiffness matrix was obtained from the static equilibrium condition based on a cubic polynomial for total deflection. This has the same limitations as McCalley model [33] in that the natural boundary conditions at the free end or hinged end could not be applied.

Thomas, Wilson and Wilson [40] derived an element with three degrees of freedom at each mode. These are the total deflection, bending slope and shear slope. At the free end the shear force is assumed to be zero but the condition of zero bending moment could not be imposed.

In this chapter an element model with total deflection $\psi$, total slope $\psi'$, bending slope $\phi$, and the first derivative of bending slope $\phi'$, as nodal co-ordinates is presented. This model [41,42] is capable of incorporating all the geometric and natural boundary conditions associated with various end conditions.

Most of the previous investigators were concerned with a few of the lower modes of vibration and studied the variation of the frequencies from those predicted by the Bernoulli-Euler theory. Goens [43] studied the vibration of free-free beams by obtaining a solution of the differential equation of motion in terms of hyperbolic and trigonometric functions and noted that there will be a critical frequency above which the hyperbolic functions of the solution become trigonometric functions and remarked that a change in the vibration mode may be expected.
The presence of a new spectrum of natural frequencies in a Timoshenko beam was first claimed by Traill-Nash and Collar [44]. The differential equation governing the dynamic motion of the Timoshenko beam was derived and the characteristic equations for various boundary conditions were obtained and it was noticed that the expression giving the characteristic equations changed depending on the values of shear deformation and rotary inertia parameters. This change was interpreted as an introduction of a new spectrum of frequencies. In the case of a hinged-hinged beam there are two families of curves and they exist as two separate families exhibiting their own independent mode shapes. For the case of free-free beam Traill-Nash and Collar [44] made an error in classifying the numerical results and this resulted in claiming the existence of a second spectrum. Anderson [30], Dolph [31], Kapur [35] and Dong and Wolf [45] showed the presence of the second family of frequencies for the hinged-hinged Timoshenko beams. Tobe and Sato [46] expected the existence of the second spectrum of frequencies while analysing the vibration of short cantilever beams but experienced difficulty in classifying the frequencies into two spectra.

Mindlin [47,48] in his study of piezoelectric crystal plates which resemble free-free Timoshenko beams made a passing reference to the frequencies of higher spectra but it appears that he considered the frequencies above that of pure shear mode frequencies as higher spectrum frequencies. Barr [49] noticed two frequencies with the same number of nodal points in his experimental study of free-free vibration of a thick beam. He classified the higher values of these frequencies to belong to a second spectrum and noticed that these frequencies occurred
only above the frequencies of pure shear vibration.

In this chapter the frequencies of vibration of Timoshenko beam with various end conditions are investigated. The behaviour of the frequency curves are explained by considering the beam to be executing a coupled mode of vibration coupling between the independent modes [50]. The results presented by various investigators are studied in detail and explanations are given for the mistaken classification of the frequency values obtained and for the mistaken claims of a second spectrum. The examination of lateral deflection, bending slope and shear slope presented in this chapter helps to form a clear picture of the behaviour of Timoshenko beams.

The theory of elastic stability, which originated in the works of Euler, is now a very well-developed branch of applied mechanics employing many effective techniques and possessing a large number of problems already solved, as well as a very large body of literature. One of the major factors which contributed to the rapid accumulation of material in the field of elastic stability was undoubtedly the extremely successful concept of stability and critical force. In the theory of elastic stability it is assumed that for sufficiently small loads the equilibrium of an elastic system is stable, and that it remains so up to the first point of bifurcation of equilibrium forms, thereafter, the initial form of equilibrium becomes unstable. The critical force is then defined as the smallest value of the force at which, in addition to the initial form of equilibrium, there can exist others which are very close to the initial form. This concept is to be found as far back as
the work of Euler, who defined the critical force as the force required to cause the smallest inclination of a column.

The critical force of a slender, ideal column built in at the base, free at the upper end and subjected to a constant axial load was first obtained by Euler [51] in 1744. Critical loads of beam with other boundary conditions were obtained by Timoshenko [18].

The development and application of the direct stiffness method to solve the elastic stability problem started in 1960 by Turner, Dill, Martin and Melosh [52]. The geometric stiffness matrix for a beam-column was derived by several authors using several approaches, these were Greene [53] Gallagher and Padlog [54], Bolotin [2] and Hartz [55]. Martin [56] presented a basic and unified approach to derive the geometric stiffness matrices by considering the total strain energy. Geometric stiffness matrices for the torsional and lateral stability analysis of prismatic elements subjected to constant axial loads were formulated by Barsoum and Gallagher [57]. The stability behaviour of a flexible missile idealized as a free-free Bernoulli-Euler beam was studied by Wu [58] using finite elements.


The influence of axial compressibility on the stability of elastic columns subjected to non-conservative forces has been investigated by several authors. Nemat-Nasser [60], Farshad [61], Hauger [62], Anderson [63], Rao and Rao [64,65]
and Sundararajan [66] are some of these authors.

The effect of shear deformation on the static buckling loads was investigated by Timoshenko [18] by solving the differential equation of the deflection curve in which the effect of shearing force was included. A matrix formulation of this problem was obtained by Rodden, Jones and Bhuta [67] by calculating the structural influence coefficients for a simply supported beam. The method was limited to simple cases and the results obtained differed by about ten per cent from the analytical solution [18].

In this chapter, the finite element method is employed to investigate the effect of shear deformation on the static buckling loads of hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams subjected to axial loads. The excellent agreement of the results obtained with the analytical solutions is demonstrated.

The first observation of parametric resonance is attributed to Faraday [68] in 1833. Faraday [68] observed that the wine in a wineglass oscillated with half the frequency of the exciting force movement of moist fingers around the glass edge. Melde [69] in 1859 provided a more conspicuous demonstration of parametric resonance by tying a string to a tuning fork. The string vibrated in the vertical direction while the tuning fork vibrated in the horizontal direction. Lord Rayleigh [70] explained this phenomenon mathematically in 1883.

The problem of dynamic stability of a simply supported bar was first investigated by Baliaev [3]. The resulting governing
equation was a linear differential equation with periodic coefficients of the Mathieu-Hill type. Baliaev [3] calculated the first parametric resonance frequency of the column to be half of the frequency of the disturbing force. The physical meaning of this result is that the column executes one cycle of vibration while the disturbing force executes two cycles up and down.

Additional terms were introduced by Mettler [4] to take into account the inertia forces in Baliaev formulation. This problem was investigated by Bolotin [2] also. As in the case of the applied theory of vibration, Bolotin [2] did not include the inertia forces associated with the rotation of the cross sections of the rod with respect to its own principal axes.

Theoretical and experimental work on the dynamic stability of columns was carried out by Utida and Sezawa [71]. The Mathieu equation was investigated and the regions of dynamic instability of columns were determined by Lubkin and Stoker [72].

A detailed review of the literature on the theory of dynamic stability up to 1951 was presented by Beilin and Dzhanelidze [73]. Another survey up to 1964 was presented in an article by Evan-Iwanowski [74]. Proceedings of international conferences on dynamic stability of structures have been edited by Herrmann [75] and by Leipholz [76].

In recent years, various methods of solution have been applied to the dynamic stability problem of Euler beam subjected to axial periodic loads. Burney and Jaeger [77] determined the regions of dynamic instability of hinged-hinged and fixed-
free columns using a numerical method where the column is idealized as consisting of massless springs and lumped masses. Stevens [78] solved the hinged-hinged column problem using Fourier transformations, while Krajcinovic and Herrmann [79] used an integral equation technique. Iwatsubo, Sugiyama and Ishihara [80] investigated the Euler column problem using digital simulation where the smooth cosine form of the disturbing force was approximated by successive piecewise constant forces. Iwatsubo, Saigo and Sugiyama [81] determined the parametric resonance of elastic columns with clamped-clamped and clamped-hinged ends using the finite difference method. Hsu [82] investigated the response of a parametrically excited, hanging string by making simplifying assumptions like neglecting the bending stiffness of the column.

Other methods of solution have been used in the dynamic stability studies of pipes conveying fluid. Analytical solution of the differential equation of a hinged-hinged pipe was given by Housner [83]. Bolotin's method and a numerical Floguet analysis were used by Paidoussis and Sundararajan [84] to investigate the stability of pipes conveying fluid with pulsating flow. Galerkin method was used by Plaut and Huseyin [85] to study the effect of axial loading on the stability of pipes with hinged or clamped ends.

The finite element method was first used by Brown, Hutt and Salama [5] to study the dynamic stability of bars with various boundary conditions. In [5] and in all the previous investigations, the Bernoulli-Euler beam theory was employed and the effects of shear deformation and rotary inertia were neglected.
In this chapter, a finite element model is developed for the dynamic stability analysis of Timoshenko beams subjected to periodic axial loads. The effects of shear deformation and rotary inertia on the regions of dynamic instability are investigated for the first time [86].

The elastic stiffness, geometric stiffness and inertia matrices are developed for a Timoshenko beam and the matrix equation for the dynamic stability is solved for hinged-hinged, fixed-free, free-free and fixed-fixed beams.

4.2 Formulation of Elastic Stiffness and Geometric Stiffness Matrices

The strain energy $U$ of an elemental length $\ell$ of a Timoshenko beam shown in Figure 4.1, subjected to an axial force $P$ is given by $[13, 22, 73]$

$$U = \frac{1}{2} EI \left[ \int_0^\ell \left( \frac{d\phi}{dx} \right)^2 dx + \int_0^\ell kAG \left( \frac{dy}{dx} - \phi \right)^2 dx - \int_0^\ell P \left( \frac{dy}{dx} \right)^2 dx \right] \quad (4.01)$$

where $y$ is the deflection and $\phi$ is the bending slope

On non-dimensionalising by substituting

$$\eta = \frac{x}{\ell} \quad \text{and} \quad \psi = \frac{y}{\ell}$$

the expression becomes

$$U = \frac{1}{2} \frac{EI}{\ell} \left[ \int_0^1 \left( \frac{d\phi}{d\eta} \right)^2 d\eta + \int_0^1 kAG \left( \frac{d\psi}{d\eta} - \phi \right)^2 d\eta - \int_0^1 P \left( \frac{d\psi}{d\eta} \right)^2 d\eta \right] \quad (4.02)$$

Assuming cubic polynomial expansions for $\psi$ and $\phi$ to be of the form
\[ \psi = \sum_{r=0}^{3} a_r \eta^r \quad \text{and} \quad \phi = \sum_{r=0}^{3} b_r \eta^r \tag{4.03} \]

and substituting into equation (2) and replacing the coefficients \( a_r \) and \( b_r \) \((r=0,1,2,3)\) the strain energy expression becomes

\[ U = \frac{1}{2} \frac{EI}{\lambda} \{\zeta\}^T [K] \{\zeta\}. \tag{4.04} \]

where

\[ [K] = [K_e] - \frac{P^2}{EI} [K_g] \]

\([K_e]\) is the elastic stiffness matrix

\([K_g]\) is the geometric stiffness matrix

\[
[K_e] = \frac{1}{420} \begin{bmatrix}
156S+504 & -42S & 22S+42 & -210S & 54S-504 & 42S & -13S+42 & \\
56S & 0 & -42S & 42S & -14S & -7S & \\
4S+56 & -42S & 13S-42 & 7S & -3S-14 & \\
504S & -210S & -42S & 42S & \\
symmetric & 156S+504 & -42S & -22S-42 & \\
56S & 0 & \\
4S+56
\end{bmatrix}
\]

\[ S = \frac{kG}{E} \cdot \frac{A \delta^2}{I} \] is the shear deformation parameter of the element

\[
[K_g] = \frac{1}{420} \begin{bmatrix}
504 & 0 & 42 & 0 & -504 & 0 & 42 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56 & 0 & -42 & 0 & -14 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
504 & 0 & -42 & 0 & \\
symmetric & 0 & 0 & 0 & \\
56 & 0 & \\
0
\end{bmatrix}
\]
4.3 Formulation of the Inertia Matrix

The kinetic energy $T$ of an elemental length $l$ of a Timoshenko beam is given by

$$T = \frac{1}{2} \rho I \int_0^l \dot{\phi}^2 \, dx + \frac{1}{2} \rho A \int_0^l \dot{y}^2 \, dx$$  \hspace{1cm} (4.05)

where $\rho$ is the mass density of the material of the beam and $I$ is the second moment of area of cross-section.

On non-dimensionalising the kinetic energy expression becomes

$$T = \frac{1}{2} \rho IL \left[ \dot{\phi}^2 \right]_0^1 + \frac{1}{2} \rho Al^3 \left[ \dot{y}^2 \right]_0^1$$  \hspace{1cm} (4.06)

substituting for $\phi$ and $\psi$ from equation (4.03) and replacing the coefficients $a_r$ and $b_r$ ($r = 0, 1, 2, 3$) by the nodal co-ordinates the expression becomes

$$T = \frac{1}{2} \rho Al^3 \{ \zeta \}^T \begin{bmatrix} M \end{bmatrix} \{ \zeta \}$$  \hspace{1cm} (4.07)

where

$$\begin{bmatrix} M \end{bmatrix} = \frac{1}{420} \begin{bmatrix} 156 & 0 & 22 & 0 & 54 & 0 & -13 & 0 \\ 156R & 0 & 22R & 0 & 54R & 0 & -13R \\ 4 & 0 & 13 & 0 & -3 & 0 \\ 4R & 0 & 13R & 0 & -3R \\ 156 & 0 & -22 & 0 \\ \text{Symmetric} & 156R & 0 & -22R \\ 4 & 0 \\ 4R \end{bmatrix}$$
\[ R = I/Ax^2 \] is the rotary inertia parameter of the element

\[
\{\xi\}_T = \begin{bmatrix}
\dot{\psi}_i & \dot{\phi}_i & \dot{\psi}_i & \dot{\phi}_i & \dot{\psi}_{i+1} & \dot{\phi}_{i+1} & \dot{\psi}_{i+1} & \dot{\phi}_{i+1}
\end{bmatrix}
\]

4.4 Matrix Equations for Timoshenko Beams

4.4.1 Free vibration

\[
[K_e] - \lambda [M] \{\xi\} = 0 \quad (4.08)
\]

where \( \lambda = \rho AL^4 p^2 / EI \) is the frequency parameter of the beam

4.4.2 Static stability

\[
[K_e] - B[K_g] \{\xi\} = 0 \quad (4.09)
\]

where \( B = P*L^2 / EI \) is the buckling load parameter

4.4.3 Dynamic stability

\[
[K_e] - (\alpha \pm i\beta) B[K_g] - \frac{\lambda d}{4} [M] \{\xi\} = 0 \quad (4.10)
\]

where \( \lambda d = \rho AL^4 \omega^2 / EI \) is the disturbing frequency parameter, \( \alpha \) and \( \beta \) are fractions to represent the static and time dependent components of the load respectively, and \( \omega \) is the disturbing frequency.

4.5 Applications

The elastic stiffness, geometric stiffness and inertia matrices developed are used to solve the matrix equations for hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams. The boundary conditions are

(i) hinged end - total deflection \( \psi = 0 \) and bending moment \( \phi' = 0 \)

(ii) fixed end - total deflection \( \psi = 0 \) and bending slope \( \phi = 0 \)
(iii) free end - bending moment $\phi' = 0$ and shear force $\psi' - \phi = 0$.

4.5.1 Free vibration

The elastic stiffness and inertia matrices are used to solve the matrix equation (4.08) of free vibration of Timoshenko beams with $k = 0.85$, $v = 0.3$ and $R = I/AL^2 = (0.08)^2$.

The convergence of the frequency parameters with increasing number of elements are shown in Tables 4.1, 4.2, 4.3 and 4.4 for the hinged-hinged, fixed-free, free-free and fixed-fixed beams respectively.

Comparisons of the percentage errors in the square root of the frequency parameters of a hinged-hinged and fixed-free Timoshenko beams obtained by the present element with those obtained by McCalley [33], Kapur [35], Thomas, Wilson and Wilson [40] (TWW) and Carnegie, Thomas and Dokumaci [36] (CTD) are given in Tables 4.5, 4.6, 4.7 and 4.8.

4.5.1.1 Analytical solutions

The vibration of a Timoshenko beam can be considered to be a coupled vibration, coupling various independent modes of vibration of constituent beams. These constituent beams are the Bernoulli-Euler beam, the simple shear beam and the pure shear beam.

4.5.1.1.1 Bernoulli-Euler beam

In the Bernoulli-Euler beam dynamic equilibrium is maintained between the bending energy and the energy due to lateral displacement. The equation for lateral vibration is

$$\frac{d^4 \psi}{d\eta^4} - \lambda \psi = 0 \quad (4.11)$$
where

\[ \lambda = \rho AL^4 p^2 / EI \]

The general solution of equation (4.11) is

\[ \psi = C_1 \cosh \lambda \eta + C_2 \sinh \lambda \eta + C_3 \cos \lambda \eta + C_4 \sin \lambda \eta \]

(4.12)

The values of \( \sqrt{\lambda} \) for beams with various end conditions are:

<table>
<thead>
<tr>
<th>End conditions</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
<th>Mode 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>hinged-hinged</td>
<td>9.870</td>
<td>39.478</td>
<td>88.826</td>
<td>157.914</td>
<td>246.740</td>
</tr>
<tr>
<td>fixed-free</td>
<td>3.516</td>
<td>22.034</td>
<td>61.697</td>
<td>120.902</td>
<td>199.860</td>
</tr>
<tr>
<td>free-free</td>
<td>22.373</td>
<td>61.673</td>
<td>120.903</td>
<td>199.859</td>
<td>298.556</td>
</tr>
<tr>
<td>fixed-fixed</td>
<td>22.373</td>
<td>61.673</td>
<td>120.903</td>
<td>199.859</td>
<td>298.556</td>
</tr>
</tbody>
</table>

4.5.1.1.2 Simple shear beam

In the simple shear beam dynamic equilibrium is maintained between energies due to shear deformation and lateral displacement and it is assumed that no rotation of cross-sections takes place. The equation of motion is

\[ kAG \frac{d^2 \psi}{d\eta^2} = \rho AL^2 \frac{d^2 \psi}{dt^2} \]

(4.13)

Equation (4.13) can be reduced to:

\[ \frac{d^2 \psi}{d\eta^2} + m^2 \psi = 0 \]

(4.14)

where \( m = \sqrt{\lambda R \frac{E}{kG}} \) and \( R = \frac{I}{AL^2} \)
The general solution of equation (4.14) is

$$\psi = C_1 \sin m\eta + C_2 \cos m\eta \quad (4.15)$$

The values of $\sqrt{\lambda R}$ for beams with various end conditions are:

<table>
<thead>
<tr>
<th>$\nu = 0.5 \quad \kappa = 0.85$</th>
<th>$\sqrt{\lambda R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>End conditions</strong></td>
<td><strong>Mode</strong></td>
</tr>
<tr>
<td></td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>hinged-hinged</td>
<td>1.796 3.592 5.388 7.184 8.980</td>
</tr>
<tr>
<td>free-free</td>
<td></td>
</tr>
<tr>
<td>fixed-fixed</td>
<td>0.898 2.694 4.491 6.287 8.083</td>
</tr>
<tr>
<td>fixed-free</td>
<td></td>
</tr>
</tbody>
</table>

4.5.1.1.3 Pure shear beam

In the pure shear beam, dynamic equilibrium is maintained between the strain energy due to rotation and the energy due to rotary inertia and it is assumed that no bending deflection takes place. The equation of motion is

$$\frac{EI}{L^2} \frac{d^2\phi}{d\eta^2} - kAG \phi = \rho I \frac{d^2\phi}{dt^2} \quad (4.16)$$

Equation (4.16) can be reduced to

$$\frac{d^2\phi}{d\eta^2} + (\lambda R - \frac{kG}{E} \cdot \frac{1}{R}) \phi = 0 \quad (4.17)$$

One obvious solution of equation (4.17) is

$$\lambda R - \frac{kG}{E} \cdot \frac{1}{R} = 0$$

then

$$\sqrt{\lambda R} = \sqrt{\frac{kG}{E} \cdot \frac{1}{R}} \quad (4.18)$$
The general solution of equation (4.17) is

$$\phi = C_1 \sin \mu \eta + C_2 \cos \mu \eta$$  \hspace{1cm} (4.19)

where

$$\mu^2 = \lambda R - \frac{kG}{E} \cdot \frac{1}{R}$$  \hspace{1cm} (4.20)

Equation (4.20) can be written as

$$\sqrt{\lambda R} = \sqrt{kG} \cdot \frac{1}{R} + \mu^2$$  \hspace{1cm} (4.21)

The values of $\mu$ for beams with various end conditions are:

<table>
<thead>
<tr>
<th>End condition</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hinged-hinged</td>
<td>$n\pi$</td>
</tr>
<tr>
<td>free-free</td>
<td>$n\pi$</td>
</tr>
<tr>
<td>fixed-fixed</td>
<td>$n\pi$</td>
</tr>
<tr>
<td>fixed-free</td>
<td>$(2n+1)\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

where $n = 0, 1, 2, 3, \ldots$

It can be seen that, for hinged-hinged, free-free and fixed-fixed beams, for $n = 0$, equation (4.21) is reduced to equation (4.18) which is the solution for the fundamental pure shear mode.

4.5.1.2 Timoshenko beam

The free vibration analysis of Timoshenko beam had been carried out by several investigators using many different methods. Although algebraic expressions for characteristic equations were derived the solution to obtain frequency values resorted to numerical or graphical methods with the inherent approximations.
Results presented for the free vibration analysis shown in Figures 4.2 to 4.14 are obtained by using twenty finite elements. This number of elements is necessary to get high accuracy for the frequencies of higher modes. The values used for $k$ and $\nu$ are 0.85 and 0.3 respectively.

4.5.2 Static stability

The elastic stiffness and geometric stiffness matrices are used to solve the matrix equation (4.09) of static stability of Timoshenko beams with $k = 0.85$, $\nu = 0.3$ and

$$S = \frac{kG}{E} \cdot \frac{AL^2}{L} = 51.0817.$$  

The convergence of the static buckling load parameters with increasing number of elements are shown in Tables 4.10, 4.11, 4.12 and 4.13 for the hinged-hinged, fixed-free, free-free and fixed-fixed beams respectively.

The effects of shear deformation on the critical buckling loads of Timoshenko beams with various boundary conditions are investigated. The first five buckling load parameters are obtained for various values of the shear deformation parameter ($S$) using six element idealization. These results are shown in Figures 4.15, 4.16, 4.17 and 4.18 for hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams respectively. The buckling load parameter ratio is the ratio of the buckling load parameter according to Timoshenko beam theory to the buckling load parameter according to Euler beam theory. Results obtained by Timoshenko and Gere [18] are also shown in these figures.

In solving the static stability problem the matrix equation (4.09) is written as
\[
\left[ [K_g] - \frac{1}{B} [K_e] \right] \{\zeta\} = 0 \quad (4.22)
\]
since \([K_g]\) is not positive definite. Alternatively the degrees of freedom associated with \(\phi\) and \(\phi'\) can be eliminated by the use of the Eigenvalue Economizer Technique [87] so that the global geometric stiffness matrix becomes positive definite [86].

4.5.3 Dynamic stability

The effects of rotary inertia and shear deformation on the regions of dynamic instability are investigated by solving equation (4.10) using six element idealization.

The regions of dynamic instability for a hinged-hinged Timoshenko beam are shown in Figures 4.19, 4.20 and 4.21 for \(R = 0.0000, 0.0064\) and \(0.0256\) respectively. Regions associated with the pure shear and second spectrum modes of vibration of thick hinged-hinged beams are also shown. These regions are degenerated into straight lines in Figures 4.20 and 4.21.

The regions for a fixed-free Timoshenko beam are shown in Figures 4.22, 4.23 and 4.24 for \(R = 0.0000, 0.0064\) and \(0.0256\) respectively.

The regions for a free-free Timoshenko beam are shown in Figures 4.25, 4.26 and 4.27 for \(R = 0.0000, 0.0064\) and \(0.0256\) respectively.

The regions for a fixed-fixed Timoshenko beam are shown in Figures 4.28, 4.29 and 4.30 for \(R = 0.0000, 0.0016\) and \(0.0064\) respectively.
4.6 General Discussions

4.6.1 Free vibration

Various finite element models were presented by several investigators but none of them represented the Timoshenko beam accurately. The difficulty experienced by many was the inability of the element model to cater for all the boundary conditions of the Timoshenko beam. In most models the geometric boundary conditions could be applied correctly but the natural conditions could not be either partly or fully applied. The element accepted as the best up to now was the one presented by Kapur [35]. In his model, although most of the boundary conditions could be applied, one condition, namely that the bending moment be zero at the free and hinged ends could not be applied.

The element developed in this chapter is the only one presented so far in which both geometric and natural boundary conditions can be correctly applied. The rapid convergence of the results obtained from this element is demonstrated in Tables 4.1, 4.2, 4.3 and 4.4 for hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams respectively with $v = 0.3$, $k = 0.85$ and $\sqrt{R} = 0.08$.

Comparison of the frequency parameters obtained by the present element with those obtained by McCalley [33], Kapur [35], Thomas, Wilson and Wilson [40,88] and Carnegie, Thomas and Dokumaci [36] are shown in Tables 4.5, 4.6, 4.7 and 4.8. The present finite element model is seen to give a more accurate result than any of the models previously presented by other investigators. The total number of degrees of freedom of the present model is no greater than those presented by some other
authors. A smaller number of elements is required to obtain good acceptable results for the first few modes of vibration. Also this element is proved to be very efficient when investigating the higher modes of vibration.

The vibration of a Timoshenko beam can be considered to be a coupled vibration coupling various independent modes of vibration of constituent beams. A Timoshenko beam is considered to be a combination of a Bernoulli-Euler beam, a simple shear beam and a pure shear beam.

The effect of the combination of these three types of beams on the frequencies of vibration of the hinged-hinged Timoshenko beam is shown in Figure 4.2 for various values of the square root of the rotary inertia parameter ($\sqrt{I}$). The lines radiating from the origin are the frequencies of vibration of Bernoulli-Euler beam, while the horizontal lines are the frequencies of vibration of simple shear beam. The dotted hyperbolic lines show the frequencies of vibration of the pure shear beam.

It can be seen that the frequencies of vibration of the Timoshenko beam, shown as continuous lines in Figure 4.2 and obtained by using twenty finite elements, form two distinct spectra. The lower spectrum of frequencies is the result of coupling between the Bernoulli-Euler and simple shear beams and starts off as a Bernoulli-Euler beam frequencies for small values of $\sqrt{I}$ but approach asymptotically the simple shear beam frequencies as $\sqrt{I}$ increases. The second spectrum of frequencies is the result of coupling between the pure shear and simple shear beams and starts off as pure shear beam frequencies for small $\sqrt{I}$ but approach asymptotically the simple shear beam
frequencies as $\sqrt{R}$ increases.

This can be shown clearly by presenting the plots as in Figure 4.3. Here a coordinate transformation is used such that frequencies of all modes of vibration coincide to give one line for each type of beam. In the pure shear mode the lateral deflection is zero and the rotation $\phi$ of the cross-section is equal to the shear deformation $\gamma$ but is in antiphase as shown in Figure 4.4.

The mode shapes of the third mode of vibration of the Timoshenko beam for various values of $\sqrt{R}$ are shown in Figure 4.5 and Figure 4.6 for the first and second spectra respectively. From Figure 4.5 it can be seen that for the first spectrum, as $\sqrt{R}$ increases the shear slope increases in comparison to the bending slope and the effect due to shear deformation becomes appreciable. Figure 4.6 shows the effect of coupling of mode shapes in the second spectrum which is illustrated by the relative increase in the shear deflection as $\sqrt{R}$ increases.

The coupling observed in the case of a fixed-free Timoshenko beam is much more complex. The frequencies of vibration shown in Figure 4.7 can be considered to be the result of coupling between Bernoulli-Euler beam and simple shear beam for low values of $\sqrt{R}$ and between the modes when rotary inertia is ignored and the pure and simple shear modes for higher values of $\sqrt{R}$. The modes when rotary inertia is ignored represent the resulting frequencies of the coupling between the Bernoulli-Euler beam and simple shear beam and are shown as dotted lines in Figure 4.7. It can be seen that the variation of the frequency parameter with rotary inertia parameter is continuous.
and there is no discontinuity or separation and hence there is no second spectrum as claimed by other investigators.

The variation in the relative magnitude of the various components of mode shape with $\sqrt{\bar{R}}$ illustrates the phenomenon of coupling and the consequent effect on the frequency of vibration. Figure 4.8 shows the mode shape for the fourth mode of vibration of a Timoshenko beam for various values of $\sqrt{\bar{R}}$. The mode shape starts off as a combination of fourth mode of vibration of Bernoulli-Euler beam and fourth mode of vibration of simple shear beam. As $\sqrt{\bar{R}}$ increases the frequency plot tends to approach and follow the pure shear mode of vibration and this results in an increase in the shear slope component of the mode shape. Further increase in the value of $\sqrt{\bar{R}}$ makes the frequency plot in Figure 4.7 to follow the third mode of simple shear beam and as a result the number of nodes along the beam is reduced as can be seen from Figure 4.8.

The variation of the frequency parameter of the free-free Timoshenko beam with rotary inertia parameter is shown in Figure 4.9. As in the previous cases, for small values of $\sqrt{\bar{R}}$ the coupling is taking place between the Bernoulli-Euler beam and the simple shear beam modes, and the resulting frequencies are the frequencies of the Timoshenko beam when the effects of rotary inertia are neglected. When this frequency approaches the pure shear beam frequency a further coupling takes place and the frequency curves tend to follow the pure shear beam frequencies. The mode shape of vibration of the beam will therefore have changed due to the coupling and may result in a change in the number of nodes along the beam.
The change in the number of nodes can be clearly explained with reference to Figure 4.10. The symmetric modes with an even number of nodes along the beam and the asymmetric modes with an odd number of nodes should be considered separately as a symmetric mode can not change into an asymmetric mode due to the symmetry in the end conditions of the free-free beam. Consider the frequency curve of an asymmetric mode, say, with fifteen nodes along the beam in Figure 4.10. As the value of $\sqrt{\lambda}$ increases the frequency curve starts off as the fifteen-node line and follows it closely until it reaches the pure shear beam frequency. Due to coupling of the mode shapes the fifteen-node curve follows the pure shear beam frequency until it reaches the thirteen-node line. Here another coupling of the modes takes place and the frequency curve follows the thirteen-node line. So the mode of vibration that started off with fifteen nodes has degenerated into a mode of vibration with thirteen nodes by dropping two nodes, one from each end, due to the effect of coupling of various modes, as the value of $\sqrt{\lambda}$ increases. It must be noticed that the frequency curve of the fifteen-node mode has crossed the frequency curves of the fourteen-node mode without altering it in any way. A similar argument can be provided for all the other frequency curves where the number of nodes along the beam is reduced by two due to coupling.

For symmetric modes the coupling is taking place between the combined Bernoulli-Euler and simple shear beams and the second mode of the pure shear beam as the symmetry of end conditions of the free-free beam restricts the slope at the midpoint to be zero, which is satisfied by the second pure shear beam mode. For the symmetric modes also the number of nodes
along the beam decreases by two as the value of \( \sqrt{R} \) increases.

This change in the nodes number was not appreciated by any of the previous investigators and has led to several erroneous conclusions. Traill-Nash and Collar [44] analysed a free-free beam with one value of \( R \) and presented some symmetric mode frequency values for a Timoshenko beam with and without the rotary inertia effects considered. These are shown in Table 4.9 along with the results obtained from the present analysis. Traill-Nash and Collar [44] did not understand the behaviour of certain modes and hence invented a second spectrum to classify these frequencies. They based their argument on

(i) "they (frequencies) interrupt the steady (almost arithmetical) progression of the natural frequencies";
(ii) "the nodal functions are unusual";
(iii) "the number of nodal points also interrupts the usual progressive increase".

In fact all these aspects of the behaviour arose out of the coupling of the various modes. The frequencies that puzzled Traill-Nash and Collar [44] are indicated in Figure 4.10 by circles and one of those claimed to belong to the second spectrum is shown as a cross within a circle. The appearance of two modes each with fourteen nodal points is due to the fact that at higher modes the coupling takes place for smaller values of \( \sqrt{R} \) and the sixteen-node mode has already reduced to a fourteen-node mode while the original fourteen-node mode is just starting to change. In Table 4.9 the comparisons of the results given by Traill-Nash and Collar [44] for the beam with \( R = 0.0007625 \) and \( R = 0 \) (that is when rotary inertia effect is neglected) imply that there is very
little reduction in the frequencies of vibration due to rotary inertia effects. This is because the comparison is made between frequencies of very different modes. For example for the twenty first mode with twenty nodal points the frequency parameter of the Timoshenko beam is 282.0. The equivalent mode of vibration when rotary inertia is ignored has twenty two nodal points and has a frequency parameter of 310.4. Hence the comparison should have been between the values of 282.0 and 310.4 showing a considerable drop in the frequency parameter due to rotary inertia. Instead a comparison is made between the frequency parameter of the beam with the values to which the curve has asymptotically approached, thus comparing values of 282.0 and 282.0 and concluding that the rotary inertia has no effect on the frequency of vibration. To correct the table the values of Traill-Nash and Collar [44] under the column for $R = 0$ should be shifted up. The values for mode number seventeen up to twenty seven should be shifted up by one row and the values for mode number thirty one onwards by two rows to fill up all the gaps left in the table. This will then compare very favourably with the results in the last two columns obtained from the present analysis using a twenty element idealisation of the beam.

Barr [49] has also experienced difficulty when he tried to classify the results he obtained from his experimental work on a free-free beam. He noticed pairs of mode shapes with the same number of nodal points. The higher set of these frequencies were then classified as belonging to a different (second) spectrum. The value of $R$ for the beam he considered is 0.001154375 and the results obtained are shown by circles
and squares in Figure 4.11. Due to the coupling and the reduction of the number of nodal points there were several pairs of modes with the same number of nodal points. He has rightly noted that this occurs only for frequencies above the pure shear beam frequency.

Figure 4.12 shows the gradual change in the mode shape of the fourth mode of vibration of a Timoshenko beam as the value of $\sqrt{\eta}$ increases. The mode with five nodal points has changed into a mode with three nodal points. There is also a considerable change in the bending and shear slopes. Similar changes in the mode shapes were observed for all the modes.

The frequency parameters for a fixed-fixed Timoshenko beam also exhibit the coupling characteristics. Again due to the symmetry in the end conditions the symmetric and asymmetric modes of vibration can be separated and considered separately. The reduction in the number of nodal points is again by two. The coupling takes place at various points where the combined frequency curves of the Bernoulli-Euler and simple shear beams cross the pure shear beam frequencies. In some of the curves it can be seen that coupling takes place several times giving a complex shape of the curve, as seen from Figure 4.13.

The variation of the mode shape for the fourth mode of vibration is shown in Figure 4.14. It can be seen that the mode with five nodal points gradually changes to a mode with three nodal points with the associated change in the bending and shear slopes.
Considering the geometric and natural boundary conditions of Timoshenko beams with various end conditions:

<table>
<thead>
<tr>
<th>End condition</th>
<th>Total deflection</th>
<th>Bending slope</th>
<th>Total slope</th>
<th>Bending moment</th>
<th>Shear slope</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \psi = 0 )</td>
<td>( \phi \neq 0 )</td>
<td>( \psi' \neq 0 )</td>
<td>( \phi' = 0 )</td>
<td>( \gamma \neq 0 )</td>
</tr>
<tr>
<td>hinged end</td>
<td>( \psi = 0 )</td>
<td>( \phi = 0 )</td>
<td>( \psi' \neq 0 )</td>
<td>( \phi' = 0 )</td>
<td>( \gamma \neq 0 )</td>
</tr>
<tr>
<td>Fixed end</td>
<td>( \psi = 0 )</td>
<td>( \phi = 0 )</td>
<td>( \psi' \neq 0 )</td>
<td>( \phi' = 0 )</td>
<td>( \gamma = 0 )</td>
</tr>
<tr>
<td>Free end</td>
<td>( \psi \neq 0 )</td>
<td>( \phi \neq 0 )</td>
<td>( \psi' \neq 0 )</td>
<td>( \phi' = 0 )</td>
<td>( \gamma = 0 )</td>
</tr>
</tbody>
</table>

It can be seen from the table that the necessary conditions for the separation into two distinct spectra of frequencies are that the bending slope \( \phi \) and the shear slope \( \gamma \) are not restricted at either ends. These conditions are satisfied by the hinged-hinged beam and hence two distinct spectra of frequencies are obtained.

4.6.2 Static stability

The rapid convergence of the static buckling results obtained by using the present finite element model is demonstrated in Tables 4.10, 4.11, 4.12 and 4.13 for hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams respectively with \( v = 0.3 \), \( k = 0.85 \) and \( \sqrt{\kappa} = 0.08 \). The excellent agreement of these results with the analytical results of Timoshenko and Gere [18] clearly demonstrates the good accuracy of the developed finite element model in representing the Timoshenko beam.

In solving the static stability problem the matrix equation (4.09) is written as
\[
\begin{bmatrix}
[K_g] - \frac{1}{B} [K_e]
\end{bmatrix} \{\zeta\} = 0
\] (4.22)

since the global geometric stiffness matrix \([K_g]\) is not positive definite. However for the free-free support conditions the global elastic stiffness matrix \([K_e]\) becomes singular and therefore does not possess an inverse. To preserve the applicability of the general method used in this chapter, equation (4.22) is modified to

\[
\begin{bmatrix}
[K_g] - \frac{1}{B+\delta} \left( [K_e] + \delta [K_g] \right)
\end{bmatrix} \{\zeta\} = 0
\] (4.23)

where \(\delta\) is an arbitrary constant of the same order as \(B\).

The new matrix \(\left( [K_e] + \delta [K_g] \right)\) can be inverted and values of \(\frac{1}{B+\delta}\) can be found.

The variations of the first five buckling load parameter ratios with the shear deformation parameter are shown in Figures 4.15, 4.16, 4.17 and 4.18 for hinged-hinged, fixed-free, free-free and fixed-fixed Timoshenko beams respectively. These results are obtained using six element idealisations. Analytical solutions obtained by Timoshenko and Gere \([18]\) are also shown in these figures. The agreement of the finite element results with the analytical solutions is excellent for all modes.

It is seen that due to the action of shear forces in the beams, the static buckling loads are diminished. The fixed-fixed beam is more sensitive to the shear deformation variation than the hinged-hinged beam or the free-free beam, while the fixed-free beam is least sensitive to this variation.

The percentage decreases in the static buckling loads of a fixed-free beam due to the variation of shear deformation
parameter from the Euler beam value \((S = \infty)\) to the value of \(S = 817.3\) (which corresponds to a beam with \(\sqrt{R} = 0.02\)) are 0.3, 2.5, 6.6, 12.1 and 18.2 for the first five modes respectively. For the same variation of the shear deformation parameter, the percentage decreases in the first five static buckling loads of other beams are as follows:

- for a hinged-hinged beam: 1.2, 4.6, 9.6, 15.8 and 22.5,
- for a free-free beam: 1.1, 4.2, 8.8, 14.3 and 20.4,
- and for a fixed-fixed beam: 4.6, 9.7, 15.8, 22.6 and 29.7.

It must be noted that the present analysis ignores the possibility of yielding of the material of the beam prior to the occurrence of instability. To include such a possibility, the shear deformation parameter \((S)\) must be bounded by a minimum value which corresponds to the yield stress of the beam material.

4.6.3 Dynamic stability

The effects of rotary inertia and shear deformation on the regions of dynamic instability of a hinged-hinged Timoshenko beam are shown in Figures 4.19, 4.20 and 4.21.

Figure 4.19 shows the regions of dynamic instability for a hinged-hinged beam with \(R = 0\) and \(S = \infty\). This case represents a beam without the effects of shear deformation and rotary inertia, which is the classic beam.

Figure 4.20 shows the regions of dynamic instability for a hinged-hinged Timoshenko beam with \(\sqrt{R} = 0.08\) and \(S = 51.0817\). The regions associated with the fundamental pure shear mode and second spectrum have degenerated into
vertical lines of zero width.

Figure 4.21 shows the regions of dynamic instability associated with the first spectrum together with the regions associated with the pure shear mode and the second spectrum modes for a hinged-hinged Timoshenko beam with $\sqrt{R} = 0.16$ and $S = 12.8$.

From these figures, it can be seen that as the value of $R$ increases, the regions of dynamic instability associated with the first spectrum are shifted towards the vertical axis which represents the static buckling case ($\omega = 0$), and the width of these regions is increased, thus making the beam more sensitive to periodic forces.

The value of $\beta$ for static buckling is reduced to about 0.675 for a Timoshenko beam of $\sqrt{R} = 0.08$ and to about 0.127 for a Timoshenko beam of $\sqrt{R} = 0.16$.

The regions of dynamic instability associated with the pure shear mode and the second spectrum modes have all degenerated into vertical straight lines of zero width and hence have no influence on the stability characteristics of the hinged-hinged Timoshenko beam. It is seen that there is no second spectrum of static buckling loads since in this case $\omega = 0$ and the influence of rotary inertia is eliminated.

The effects of rotary inertia and shear deformation on the regions of dynamic instability of a fixed-free Timoshenko beam are shown in Figures 4.22, 4.23 and 4.24. The shift of the regions towards the vertical axis and the increase in their width are clear though not as rapid as in the case of the
hinged-hinged beam.

The dynamic stability characteristics of a free-free Timoshenko beam are shown in Figures 4.25, 4.26 and 4.27. As the value of $R$ increases the regions of dynamic instability are shifted and their widths are increased thus making the beam more sensitive to periodic forces.

The effect of coupling on these regions is very clear and can be discussed with respect to Figure 4.9 which shows the vibrational modes of the free-free Timoshenko beam. The corresponding vibration modes in Figures 4.25, 4.26 and 4.27 are defined by the intersections of the right hand boundaries (or branches) of the dynamic instability regions with the horizontal line $\beta = 1.0$. As an example, Figure 4.26 ($\sqrt{R} = 0.08$) is considered where it can be seen that the fifth and sixth modes of vibration are coinciding with each other, so are the seventh and eighth modes to a certain degree. This can be verified in Figure 4.9 where these modes cross each other for $\sqrt{R} = 0.08$.

The rotary inertia parameter and shear deformation parameter have a significant effect on the regions of dynamic instability of a fixed-fixed Timoshenko beam. The regions are shown in Figures 4.28, 4.29 and 4.30 for the values of $\sqrt{R} = 0, 0.04$ and 0.08 respectively. As the value of $R$ increases the regions are shifted and their widths are increased very rapidly. As in the case of a free-free Timoshenko beam, the effect of coupling on the characteristics of the regions boundaries is quite clear.

It is seen that, for all cases, as the rotary inertia parameter increases, the regions of dynamic instability are
shifted towards the vertical axis and the widths of these regions are increased thus making the beams more sensitive to periodic forces.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Square root of frequency parameter ( \lambda^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>9.1691</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>8.8459</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>8.8405</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>8.8398</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>8.8397</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>8.8397</td>
</tr>
<tr>
<td>Exact solution</td>
<td></td>
<td>8.8397</td>
</tr>
</tbody>
</table>

**TABLE 4.1** CONVERGENCE OF FREQUENCY PARAMETERS OF A HINGED-HINGED TIMOSHENKO BEAM WITH \( \nu = 0.3, \ k = 0.85 \) AND \( \sqrt{R} = 0.08 \).

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Square root of frequency parameter ( \lambda^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>3.3340</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>3.3244</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>3.3241</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>3.3241</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>3.3241</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>3.3241</td>
</tr>
<tr>
<td>Exact solution</td>
<td></td>
<td>3.3241</td>
</tr>
</tbody>
</table>

**TABLE 4.2** CONVERGENCE OF FREQUENCY PARAMETERS OF A FIXED-FREE BEAM WITH \( \nu = 0.3, \ k = 0.85 \) AND \( \sqrt{R} = 0.08 \).
**TABLE 4.3** CONVERGENCE OF FREQUENCY PARAMETERS OF A FREE-FREE TIMOSHENKO BEAM WITH $\nu = 0.3$, $k = 0.85$ and $\sqrt{\alpha} = 0.08$.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Square root of frequency parameter $\lambda^\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>25.8129</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>18.2620</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>18.2221</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>18.2163</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>18.2150</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>18.2147</td>
</tr>
</tbody>
</table>

**TABLE 4.4** CONVERGENCE OF FREQUENCY PARAMETERS OF A FIXED-FIXED TIMOSHENKO BEAM WITH $\nu = 0.3$, $k = 0.85$ and $\sqrt{\alpha} = 0.08$.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Square root of frequency parameter $\lambda^\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>17.6156</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>15.7709</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>15.7450</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>15.7415</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>15.7408</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>15.7407</td>
</tr>
<tr>
<td>Mode number</td>
<td>Number of elements</td>
<td>Percentage error in $\lambda^\frac{1}{2}$</td>
</tr>
<tr>
<td>-------------</td>
<td>--------------------</td>
<td>----------------------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Kapur [35]</td>
</tr>
<tr>
<td>Mode 1</td>
<td>1</td>
<td>8.80</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 8.645$</td>
<td>2</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.01</td>
</tr>
<tr>
<td>Mode 2</td>
<td>1</td>
<td>14.40</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 26.960$</td>
<td>2</td>
<td>6.00</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.23</td>
</tr>
<tr>
<td>Mode 3</td>
<td>2</td>
<td>9.50</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 47.680$</td>
<td>4</td>
<td>0.74</td>
</tr>
<tr>
<td>Mode 4</td>
<td>4</td>
<td>3.12</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 68.726$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4.5 COMPARISON OF FREQUENCY PARAMETERS OF A HINGED-HINGED TIMOSHENKO BEAM WITH $G/E = 3/8$, $k = 2/3$ AND $\sqrt{R} = 0.08$.  

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Number of degrees of freedom</th>
<th>Percentage error in $\lambda^\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>McCalley [33]</td>
<td>TWW [40]</td>
</tr>
<tr>
<td>Mode 1</td>
<td>12</td>
<td>0.003 (6)</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 3.500$</td>
<td>18</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.000(12)</td>
</tr>
<tr>
<td>Mode 2</td>
<td>12</td>
<td>0.101 (6)</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 21.353$</td>
<td>18</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.020(12)</td>
</tr>
<tr>
<td>Mode 3</td>
<td>12</td>
<td>0.701 (6)</td>
</tr>
<tr>
<td>$\lambda^\frac{1}{2} = 57.474$</td>
<td>18</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.148(12)</td>
</tr>
</tbody>
</table>

T: FREQUENCY PARAMETERS OF A FIXED-FREE TIMOSHENKO BEAM WITH $2/3$ AND $\sqrt{R} = 0.02$ ( ) Number of elements)
<table>
<thead>
<tr>
<th>Mode number</th>
<th>Number of degrees of freedom</th>
<th>Percentage error in $\lambda^{\frac{1}{4}}$</th>
<th>McCalley [33]</th>
<th>TWW [40]</th>
<th>CTD [36]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 3.419$</td>
<td>12</td>
<td>0.008 (6)</td>
<td>-</td>
<td>0.000(2)</td>
<td>0.000(3)</td>
<td>0.000(3)</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.000(6)</td>
<td>0.000(3)</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.000(12)</td>
<td>-</td>
<td>0.000(4)</td>
<td>0.000(6)</td>
<td></td>
</tr>
<tr>
<td>Mode 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 18.613$</td>
<td>12</td>
<td>0.389 (6)</td>
<td>-</td>
<td>0.208(2)</td>
<td>0.056(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.031(6)</td>
<td>0.029(3)</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.099(12)</td>
<td>-</td>
<td>0.006(4)</td>
<td>0.004(6)</td>
<td></td>
</tr>
<tr>
<td>Mode 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 44.624$</td>
<td>12</td>
<td>2.137 (6)</td>
<td>-</td>
<td>3.824(2)</td>
<td>0.320(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.231(6)</td>
<td>0.233(3)</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.542(12)</td>
<td>-</td>
<td>0.057(4)</td>
<td>0.011(6)</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.7** COMPARISON OF FREQUENCY PARAMETERS OF A FIXED-FREE TIMOSHENKO BEAM WITH $\nu = 0.3$, $k = 0.65$ AND $\sqrt{\nu} = 0.05$

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Number of degrees of freedom</th>
<th>Percentage error in $\lambda^{\frac{1}{4}}$</th>
<th>McCalley [33]</th>
<th>TWW [40]</th>
<th>CTD [36]</th>
<th>Kapur [35]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 3.284$</td>
<td>8</td>
<td>0.06 (4)</td>
<td>-</td>
<td>-</td>
<td>0.03(2)</td>
<td>0.00(2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>0.00(2)</td>
<td>-</td>
<td>0.00(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.00(6)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>-</td>
<td>0.00(4)</td>
<td>-</td>
<td>-</td>
<td>0.00(6)</td>
<td></td>
</tr>
<tr>
<td>Mode 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 15.488$</td>
<td>8</td>
<td>1.44 (4)</td>
<td>-</td>
<td>-</td>
<td>0.57(2)</td>
<td>0.19(2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>0.12(2)</td>
<td>-</td>
<td>0.03(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.03(6)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>-</td>
<td>0.00(4)</td>
<td>-</td>
<td>-</td>
<td>0.00(6)</td>
<td></td>
</tr>
<tr>
<td>Mode 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda^{\frac{1}{4}} = 34.301$</td>
<td>8</td>
<td>6.45 (4)</td>
<td>-</td>
<td>-</td>
<td>7.88(2)</td>
<td>2.80(2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>2.19(2)</td>
<td>-</td>
<td>0.20(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-</td>
<td>0.22(6)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>-</td>
<td>0.03(4)</td>
<td>-</td>
<td>-</td>
<td>0.01(6)</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.8** COMPARISON OF FREQUENCY PARAMETERS OF A FIXED-FREE TIMOSHENKO BEAM WITH $G/E = 3/8$, $k = 2/3$ AND $\sqrt{\nu} = 0.08$
\[ \lambda^{\frac{1}{2}} = \left( \frac{pAL^4p^2}{EI} \right)^{\frac{1}{2}} \]

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Number of nodes along the beam</th>
<th>Traill-Nash and Collar [44]</th>
<th>Present 20 finite elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R = 0.0007625, S = 20.3097</td>
<td>R = 0.0007625, S = 20.3097</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>16.8</td>
<td>16.8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>50.0</td>
<td>50.0</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>80.4</td>
<td>80.4</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>109.6</td>
<td>109.6</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>138.4</td>
<td>138.4</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>165.6</td>
<td>167.6</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>190.8</td>
<td>196.4</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>204.4</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>16</td>
<td>226.8</td>
<td>225.2</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>254.0</td>
<td>253.6</td>
</tr>
<tr>
<td>21</td>
<td>20</td>
<td>282.0</td>
<td>282.0</td>
</tr>
<tr>
<td>23</td>
<td>22</td>
<td>310.4</td>
<td>310.4</td>
</tr>
<tr>
<td>25</td>
<td>24</td>
<td>338.8</td>
<td>338.8</td>
</tr>
<tr>
<td>27</td>
<td>26</td>
<td>366.0</td>
<td>367.2</td>
</tr>
<tr>
<td>29</td>
<td>26</td>
<td>379.2</td>
<td>-</td>
</tr>
<tr>
<td>31</td>
<td>28</td>
<td>396.8</td>
<td>395.6</td>
</tr>
</tbody>
</table>

TABLE 4.9 NATURAL FREQUENCY PARAMETERS OF A FREE-FREE TIMOSHENKO BEAM. (SYMMETRIC MODES)

Values presented in Reference [44] are multiplied by a factor of four to get the values of Traill-Nash and Collar in column 3 and 4 since the length of the beam in Reference [44] was 2L.
### TABLE 4.10 CONVERGENCE OF BUCKLING LOAD PARAMETERS OF A HINGED-HINGED TIMOSHENKO BEAM WITH $S = 51.0817$

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Buckling load parameter $B = \frac{P\times L^2}{EI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>8.784</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>8.281</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>8.273</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>8.272</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>8.272</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>8.271</td>
</tr>
<tr>
<td>Analytical solution by Timoshenko and Gere [18]</td>
<td>8.271</td>
<td>22.268</td>
</tr>
</tbody>
</table>

### TABLE 4.11 CONVERGENCE OF BUCKLING LOAD PARAMETERS OF A FIXED-FREE TIMOSHENKO BEAM WITH $S = 51.0817$

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Buckling load parameter $B = \frac{P\times L^2}{EI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2.397</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>2.367</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>2.362</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>2.360</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>2.359</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>2.358</td>
</tr>
<tr>
<td>Analytical solution by Timoshenko and Gere [18]</td>
<td>2.354</td>
<td>15.478</td>
</tr>
</tbody>
</table>
### TABLE 4.12 CONVERGENCE OF BUCKLING LOAD PARAMETERS OF A FREE-FREE TIMOSHENKO BEAM WITH $S = 51.0817$

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Buckling load parameter $B = \frac{P \cdot L^2}{EI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>17.319</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>8.640</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>8.466</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>8.409</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>8.380</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>8.361</td>
</tr>
<tr>
<td>Analytical solution by Timoshenko and Gere [18]</td>
<td></td>
<td>8.271</td>
</tr>
</tbody>
</table>

### TABLE 4.13 CONVERGENCE OF BUCKLING LOAD PARAMETERS OF A FIXED-FIXED TIMOSHENKO BEAM WITH $S = 51.0817$

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Number of degrees of freedom</th>
<th>Buckling load parameter $B = \frac{P \cdot L^2}{EI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>31.459</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>22.433</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>22.300</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>22.276</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>22.271</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>22.269</td>
</tr>
<tr>
<td>Analytical solution by Timoshenko and Gere [18]</td>
<td></td>
<td>22.268</td>
</tr>
</tbody>
</table>
FIG. 4.1 TIMOSHENKO BEAM ELEMENT
4.2 Natural Frequencies of a Hinged-Hinged Timoshenko Beam
FIG. 4.3 THE TWO SPECTRA OF NATURAL FREQUENCIES OF A HINGED - HINGED TIMOSHENKO BEAM
FIG. 4.4 FREE VIBRATION OF A HINGED-HINGED TIMOSHENKO BEAM
(a) SPECTRUM I  (b) PURE SHEAR  (c) SPECTRUM II
FIG. 4.5 THIRD MODE SHAPE OF FIRST SPECTRUM OF A HINGED-HINGED TIMOSHENKO BEAM FOR VARIOUS VALUES OF ROTARY INERTIA PARAMETER (R)
FIG. 4.6
THIRD MODE SHAPE OF SECOND SPECTRUM OF A HINGED-HINGED TIMOSHENKO BEAM FOR VARIOUS VALUES OF ROTARY INERTIA PARAMETER (R)
4.7 NATURAL FREQUENCIES OF A FIXED-FREE TIMOSHENKO BEAM
FIG. 4.8 FOURTH MODE SHAPE OF A FIXED-FREE TIMOSHENKO BEAM FOR VARIOUS VALUES OF ROTARY INERTIA PARAMETER (R)
4.9 NATURAL FREQUENCIES OF A FREE-FREE TIMOSHENKO BEAM

- Rotary inertia neglected
- Pure shear + simple shear
- Timoshenko beam $\nu = 0.3$, $k = 0.85$
No. of nodes along the beam = 19

\[ \sqrt{R} \]

<table>
<thead>
<tr>
<th>( R )</th>
<th>( 0.010 )</th>
<th>( 0.020 )</th>
<th>( 0.030 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Traill-Nash and Collar [44]
  \( R = 0.0007625 \) \( \frac{E}{K_G} = 64.5737705 \)

- Frequency claimed to be of the second spectrum in reference [44]

- Rotary inertia neglected

- Timoshenko beam

3.4.10 NATURAL FREQUENCIES OF A FREE-FREE TIMOSHENKO BEAM
NATURAL FREQUENCIES OF A FREE-FREE TIMOSHENKO BEAM
FIG. 4.12  FOURTH MODE SHAPE OF A FREE-FREE TIMOSHENKO BEAM FOR VARIOUS VALUES OF ROTARY INERTIA PARAMETER (R)
4.13 NATURAL FREQUENCIES OF A FIXED-FIXED TIMOSHENKO BEAM
FIG. 4.14
FOURTH MODE SHAPE OF A FIXED-FIXED TIMOSHENKO BEAM FOR VARIOUS VALUES OF ROTARY INERTIA PARAMETER (R)
FIG. 4.15  BUCKLING LOAD PARAMETER RATIO VERSUS SHEAR DEFORMATION PARAMETER FOR A HINGED-TIMOSHENKO BEAM
FIG. 4.16  BUCKLING LOAD PARAMETER RATIO VERSUS SHEAR DEFORMATION PARAMETER FOR A FIXED - FREE TIMOSHENKO BEAM
FIG. 4.17  BUCKLING LOAD PARAMETER RATIO VERSUS SHEAR DEFORMATION PARAMETER FOR A FREE-FREE TIMOSHENKO BEAM
FIG. 4.18  BUCKLING LOAD PARAMETER RATIO VERSUS SHEAR DEFORMATION PARAMETER FOR A FIXED - FIXED TIMOSHENKO BEAM
\[ P(t) = \alpha P^*_e + \beta P^*_e \cos \omega t \]

\[ P^*_e = 9.869 \frac{EI}{L^2} \] Fundamental static buckling load of Bernoulli-Euler beam

\[ P_e = 9.851\sqrt{EI/\rho AL^4} \] Fundamental natural frequency of Bernoulli-Euler beam

\[ \omega \] Disturbing frequency

\[ \alpha = 0.5 \]

FIG. 4.19 REGIONS OF DYNAMIC INSTABILITY FOR A HINGED - HINGED TIMOSHENKO BEAM. (R = 0.0000)
FIG. 4.20  REGIONS OF DYNAMIC INSTABILITY FOR A HINGED - HINGED TIMOSHENKO BEAM. (R=0.0064)
FIG. 4.21
REGION OF DYNAMIC INSTABILITY FOR A HINGED - HINGED TIMOSHENKO BEAM. (R=0.0256)
\[ \kappa(t) = \alpha P_e + \beta P_e \cos \omega t \]

- Fundamental static buckling load of Bernoulli-Euler beam
  \[ P_e^* = 2.467 \frac{EI}{L^2} \]
- Fundamental natural frequency of Bernoulli-Euler beam
  \[ \omega = \text{Disturbing frequency} \]
  \[ \alpha = 0.5 \]

**FIG. 4.22** REGIONS OF DYNAMIC INSTABILITY FOR A FIXED - FREE TIMOSHENKO BEAM. (R = 0.0000)
FIG. 4.23 REGIONS OF DYNAMIC INSTABILITY FOR A FIXED - FREE TIMOSHENKO BEAM. (R = 0.0064)
FIG. 4.24 REGIONS OF DYNAMIC INSTABILITY FOR A FIXED - FREE TIMOSHENKO BEAM. (R = 0.0256)
\[ P(t) = \alpha P_e^* + \beta P_e^* \cos \omega t \]

\[ p_e^* = 9.870 \frac{EI}{L^2} \] Fundamental static buckling load of Bernoulli-Euler beam

\[ p_e = 22.380 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of Bernoulli-Euler beam

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

**FIG. 4.25** REGIONS OF DYNAMIC INSTABILITY FOR A FREE-FREE TIMOSHENKO BEAM. (\( R = 0.0000 \))
FIG. 4.26 REGIONS OF DYNAMIC INSTABILITY FOR A FREE-FREE TIMOSHENKO BEAM. \((R = 0.0064)\)
FIG. 4.27

REGIONS OF DYNAMIC INSTABILITY FOR A FREE-FREE TIMOSHENKO BEAM, \( R = 0.0256 \)
\[ P(t) = \alpha P_e^* + \beta P_e^* \cos \omega t \]

\[ P_e^* = 39.537 \frac{EI}{L^2} \] Fundamental static buckling load of Bernoulli-Euler beam

\[ P_e = 22.372 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of Bernoulli-Euler beam

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

**FIG. 4.28** REGIONS OF DYNAMIC INSTABILITY FOR A FIXED - FIXED TIMOSHENKO BEAM. (R = 0.0000)
FIG. 4.29

REGIONS OF DYNAMIC INSTABILITY FOR A FIXED TIMOSHENKO BEAM. (R = 0.0016)
FIG. 4.30  REGIONS OF DYNAMIC INSTABILITY FOR A FIXED - FIXED TIMOSHENKO BEAM. (R = 0.0064)
CHAPTER 5
CHAPTER 5

DYNAMIC STABILITY OF TIMOSHENKO BEAMS

RESTING ON ELASTIC FOUNDATION

5.1 Introduction

The dynamic response of beams resting on an elastic foundation of the Winkler type and subjected to various loading conditions is of great practical importance, thus many investigations have been carried out in the field of such problems.

The stability of a fixed-free Bernoulli-Euler beam resting on an elastic foundation and subjected to a non-conservative force has been investigated by Smith and Herrmann [89], Sundararajan [90] and later by Hauger and Vetter [91]. The influence of rotary inertia was recently added to this analysis by Anderson [92].

The response of a Bernoulli-Euler beam on elastic foundation and subjected to moving transverse load has been studied by Kenney [93], Steele [94] and recently by Torby [95]. Considering high load velocities, Achenbach and Sun [96] found it necessary to improve the model from Bernoulli-Euler beam to Timoshenko beam. Achenbach and Sun's work [96] motivated a study by Chonan [97] to obtain a solution for the flexural vibration of an elastically supported Timoshenko beam which is subjected to moving transverse load and a constant axial force.

The stability of the lateral response of a simply-supported Bernoulli-Euler beam resting on an elastic foundation and carrying a continuous series of equally spaced mass particles has been analysed by Nelson and Conover [98] and
The optimization of the fundamental frequency of a Timoshenko beam resting on an elastic foundation has been considered by Kamat [100].

The effect of elastic foundation of the Winkler type on the dynamic stability of a hinged-hinged bar subjected to periodic axial loads was first investigated by Goldenblat [101] in 1944. Bolotin [2] investigated the effect of elastic foundation on the natural frequencies and the static buckling loads of a hinged-hinged Bernoulli-Euler beam by solving the differential equations of free vibration and static stability. Bolotin [2] determined the boundaries of the principal regions of dynamic instability also. The effect of elastic foundation on the buckling of a hinged-hinged Bernoulli-Euler beam was analysed by Timoshenko and Gere [18] using the energy method. The expression obtained for the static buckling loads, was identical to that reported by Bolotin [2].

The static stability of a Bernoulli-Euler beam on elastic foundation was also investigated by Sundararajan [90] and recently by Avent, Bounin and Rhee [102] using a Fourier series solution.

Brown, Hutt and Salama [5] used the finite element method to study the effect of elastic foundation on the dynamic stability of Bernoulli-Euler beam with various boundary conditions. The results reported for the case of a fixed-fixed beam are found to be inaccurate and the vibration and buckling mode shapes are seen to be incorrect. The values of the natural frequencies and the static buckling loads obtained by
Brown, Hutt and Salama [5] are proved to be far from the results obtained in this chapter using an analytical method and a finite element method.

The dynamic stability of a hinged-hinged Bernoulli-Euler beam resting on an elastic foundation was investigated by Ahuja and Duffield [103] using a slightly modified Galerkin method.

However, no consideration has been given to Timoshenko beams resting on an elastic foundation and subjected to periodic axial loads [104]. The purpose of the work in this chapter is to investigate the effect of elastic foundation on the natural frequencies, static buckling loads and the regions of dynamic instability of Timoshenko beams with hinged-hinged, fixed-free, free-free and fixed-fixed ends. The results obtained for the Bernoulli-Euler beam, which is a special case of the present analysis, show excellent agreement with the available results.

5.2 Formulation of Elastic Stiffness and Geometric Stiffness Matrices

The strain energy \( U \) of an elemental length \( \ell \) of a Timoshenko beam resting on an elastic foundation and subjected to an axial load \( P \) is given by

\[
U = \frac{1}{2} EI \int_0^\ell \left( \frac{d\phi}{dx} \right)^2 dx + \frac{1}{2} kAG \int_0^\ell \left( \frac{dy}{dx} - \phi \right)^2 dx + \frac{1}{2} k_f \int_0^\ell y^2 dx
\]

\[
- \frac{1}{2} P \int_0^\ell \left( \frac{dy}{dx} \right)^2 dx \quad (5.01)
\]

where \( y \) is the deflection, \( \phi \) is the bending slope and \( k_f \) is the foundation stiffness/unit length.
On nondimensionalising by substituting

\[ \eta = \frac{x}{\ell} \quad \text{and} \quad \psi = \frac{y}{\ell} \]

the expression becomes

\[
U = \frac{\text{EI}}{\ell^2} \left[ \int_0^1 \left( \frac{d\phi}{d\eta} \right)^2 d\eta + \frac{1}{2} k_1 \ell^3 \right] \left[ \int_0^1 \left( \frac{d\psi}{d\eta} - \phi \right)^2 d\eta + \frac{1}{2} k F \ell^3 \right] \left[ \int_0^1 \psi^2 d\eta \right]
\]

- \frac{\text{P}}{\ell} \left[ \int_0^1 \left( \frac{d\psi}{d\eta} \right)^2 d\eta \right] \quad (5.02)

Substituting for \( \phi \) and \( \psi \) from equation (4.03) and replacing the coefficients \( a_r \) and \( b_r \) (\( r = 0,1,2,3 \)) by the nodal co-ordinates the expression becomes

\[
U = \frac{\text{EI}}{\ell^2} [\zeta]^T [K] [\zeta] \quad (5.03)
\]

where

\[
[K] = [K_e] - \frac{P\ell^2}{EI} [K_g]
\]

\[
[K_e] = \frac{1}{420}
\]

\[
\begin{bmatrix}
504S+156F & 210S & 42S+22F & 42S & -504S+54F & 210S & 42S-13F & -42S \\
156S+504 & -42S & 22S+42 & -210S & 54S-504 & 42S & -13S+42 \\
56S+4F & 0 & -42S+13F & 42S & -14S-3F & -7S \\
4S+56 & -42S & 13S-42 & 7S & -3S-14 \\
504S+156F & -210S & -42S-22F & 42S & \\
\text{Symmetric} & 156S+504 & -42S & -22S-42 \\
56S+4F & 0 & & & & 4S+56
\end{bmatrix}
\]

\[ S = \frac{kG}{E} \cdot \frac{A\ell^2}{I} \quad \text{is the shear deformation parameter} \]

\[ F = \frac{kF\ell^4}{EI} \quad \text{is the elastic foundation parameter} \]
The geometric stiffness matrix $[K_g]$ and the inertia matrix $[M]$ are derived in Chapter 4.

5.3 Applications

The effects of elastic foundation on the natural frequencies, static buckling loads and regions of dynamic instability of Bernoulli-Euler and Timoshenko beams with various end conditions are investigated using six element idealization.

5.3.1 Free vibration

The effects of elastic foundation constant ($\gamma = \frac{k_f \ell^4}{\pi^4 \text{EI}}$) on the natural frequencies are shown in Figures 5.1 to 5.8.

Figures 5.1 and 5.2 show the effects of elastic foundation on the natural frequencies of a hinged-hinged Bernoulli-Euler beam and a hinged-hinged Timoshenko beam with $\sqrt{\text{EI}} = 0.08$ respectively.

Figures 5.3 and 5.4 show the effects of elastic foundation on the natural frequencies of a fixed-free Bernoulli-Euler beam and a fixed-free Timoshenko beam respectively.

Figures 5.5 and 5.6 show the cases of the free-free beams, while Figures 5.7 and 5.8 show the cases of the fixed-fixed beams.

5.3.1.1 Analytical solution

The governing differential equation of free vibration of Bernoulli-Euler beam without elastic foundation is

$$\frac{d^n \psi}{d\eta^n} - \lambda \psi = 0$$  \hspace{1cm} (5.04)

where $\lambda = \rho A L^4 p^2 / \text{EI}$ is the frequency parameter.
The governing differential equation of a beam with elastic foundation can be written as

\[
\frac{d^4\psi}{d\eta^4} - \lambda_e \psi + \frac{k_f L^4}{EI} \psi = 0 \tag{5.05}
\]

where \( k_f \) is the foundation stiffness/unit length and \( \lambda_e = \frac{\rho A L^4 p_e^2}{EI} \) is the frequency parameter of a beam with elastic foundation.

Equation (5.05) can be reduced to

\[
\frac{d^4\psi}{d\eta^4} - (\lambda_e - \frac{\pi^4}{\lambda} \gamma) \psi = 0 \tag{5.06}
\]

where \( \gamma = \frac{k_f L^4}{\pi^4 EI} \) is the elastic foundation constant.

It is seen from Equations (5.04) and (5.06) that the only parameter affected by the presence of the elastic foundation is the frequency and hence it can be written that

\[
\lambda = \lambda_e - \frac{\pi^4}{\lambda} \gamma \tag{5.07}
\]

which reduces to

\[
p_e = p \sqrt{1 + \frac{\pi^4 \gamma}{\lambda}} \tag{5.08}
\]

Equation (5.08) is true for Bernoulli-Euler beams with any end conditions. It has been found that Equation (5.08) is also true for Timoshenko beams.

For a hinged-hinged Bernoulli-Euler beam \( \lambda = n^4 \pi^4 \) and Equation (5.08) is reduced to the equation derived by Bolotin [2] for the hinged-hinged beam, namely

\[
p_e = p \sqrt{1 + \frac{\gamma}{n^4}}
\]
Results obtained from Equation (5.08) are shown in Figures 5.1 to 5.8. Also shown, are the available results of Brown, Hutt and Salama [5].

5.3.2 Static stability

The effects of elastic foundation on the critical buckling loads of various beams are shown in Figures 5.9 to 5.16.

Figures 5.9 and 5.10 show the effects of elastic foundation on the static buckling loads of a hinged-hinged Euler beam and a hinged-hinged Timoshenko beam with $\sqrt{R} = 0.08$ respectively.

These effects are shown in Figures 5.11 and 5.12 for fixed-free Euler and Timoshenko beams, Figures 5.13 and 5.14 for free-free Euler and Timoshenko beams and Figures 5.15 and 5.16 for fixed-fixed Euler and Timoshenko beams.

Results obtained by Timoshenko and Gere [18] and by Brown, Hutt and Salama [5] for Euler beams are also shown in the figures.

5.3.3 Dynamic stability

The effects of elastic foundation on the regions of dynamic stability of hinged-hinged, fixed-free, free-free and fixed-fixed Bernoulli-Euler and Timoshenko beams subjected to periodic axial loads are investigated.

Figures 5.17.a to c show the regions of dynamic instability for a hinged-hinged Bernoulli-Euler beam resting on elastic foundation with constant equal to 0, 2 and 4 respectively.

Figures 5.18.a to c show the regions for a hinged-hinged Timoshenko beam with $\sqrt{R} = 0.08$ resting on elastic foundation.
with constant equals to 0, 2 and 4 respectively.

Figures 5.19 and 5.20 show the regions for a fixed-free Bernoulli-Euler beam and a fixed-free Timoshenko beam respectively.

Figures 5.21 and 5.22 show the regions for a free-free Bernoulli-Euler beam and a free-free Timoshenko beam respectively.

Figures 5.23 and 5.24 show the regions for a fixed-fixed Bernoulli-Euler beam and a fixed-fixed Timoshenko beam respectively.

5.4 General Discussions

5.4.1 Free vibration

Figure 5.1 shows the variation of the frequencies of the first three modes of vibration of a hinged-hinged Bernoulli-Euler beam on elastic support of various elastic foundation constant $\gamma$. As the stiffness of the elastic foundation increases the frequencies of all modes of vibration are increased. The effect is more marked in the first mode than in higher modes of vibration. As the mode order increases the effect of elastic foundation on the frequencies of vibration becomes negligible. It is seen that the mode shapes are independent of whether or not elastic foundation support is provided and the only parameter affected by the presence of an elastic foundation is the frequency of vibration. Results obtained by Bolotin [2] using analytical solution and by Brown, Hutt and Salama [5] are also shown in Figure 5.1 and very close agreement is observed.

Figure 5.2 shows the variation of the frequencies of the
first five modes of vibration of a hinged-hinged Timoshenko beam with $R = 0.08$ with the elastic foundation constant. Also shown are the pure shear mode and the first mode of the second spectrum, where it is clear that these two modes are not affected by the presence of the elastic foundation. The frequency of vibration of any one mode of Timoshenko beam for any one value of elastic foundation constant is lower than that for the Bernoulli-Euler beam. This is expected as there is a reduction in the frequencies of vibration due to shear deformation and rotary inertia effects. The effect of elastic foundation is to increase the frequencies of vibration of the Timoshenko beam and this effect becomes less marked as the mode order increases. The percentage increases in the frequencies are 140.0, 20.0, 6.5, 3.1 and 1.8 for the first five modes respectively as the elastic foundation constant increases from zero to four.

Figures 5.3 and 5.4 show the variation of the frequencies of vibration with elastic foundation constant for a fixed-free Bernoulli-Euler beam and a fixed-free Timoshenko beam respectively. The results are in close agreement with those obtained by Brown, Hutt and Salama [5] and by the analytical solution. The increase in the frequencies of vibration with the increase in elastic foundation constant for fixed-free beams is more than for hinged-hinged beams. The percentage increases in the frequencies of the Timoshenko beam are 494.0, 51.7, 11.7, 4.5 and 2.3 for the first five modes respectively as the elastic foundation constant increases from zero to four.

Figures 5.5 and 5.6 show the variation of the frequencies of vibration with the elastic foundation constant for free-free
Bernoulli-Euler and Timoshenko beams respectively. For the special case without any elastic support, two zero eigenvalues are obtained representing two rigid body modes of vibration. With the presence of elastic foundation the zero frequency modes disappear and frequency values are obtained for the vibration of the beams on the elastic foundation. These two coincident frequencies are the same for the slender (Bernoulli-Euler) beam and the thick (Timoshenko) beam except the fact that one frequency is slightly reduced in the case of the thick beam. It can be concluded that the increase in the rotary inertia parameter $R$ has little effect on the frequencies of the rigid body modes of free-free beams on elastic foundation.

Figures 5.7 and 5.8 show the variation of the frequencies of vibration with the elastic foundation constant for fixed-fixed Bernoulli-Euler and Timoshenko beams respectively. It is seen very clearly that the results reported by Brown, Hutt and Salama [5] for the fixed-fixed Bernoulli-Euler beam are in error. The corresponding mode shapes given in Reference [5] are also incorrect. The correct numerical results obtained by the author using the finite element method and the analytical solution are given in Table 5.1.

5.4.2 Static stability

The variation of the elastic foundation constant has a very marked effect on the static buckling loads of both Euler beams and Timoshenko beams with any end condition. For a hinged-hinged Euler beam the buckling load increases steeply for the first mode. As the elastic foundation constant equals 4 the first mode shape and the second mode shape are interchanged as seen from Figure 5.9. This is more marked in the
case of a hinged-hinged Timoshenko beam as can be seen from Figure 5.10 where the first mode characteristics are interchanged for lower elastic foundation constant. As the value of rotary inertia parameter increases the buckling load in the higher modes decreases but the first mode is not very much affected. Hence the crossing point of the first mode with the other modes occurs for much lower elastic foundation constant. The results obtained for a hinged-hinged Euler beam agree very closely with those given by Timoshenko and Gere [18] and by Brown, Hutt and Salama [5] as shown in Figure 5.9.

For a fixed-free Euler beam, the increase in the stiffness of the elastic foundation has a significant effect on the mode shapes of static buckling and hence the variation of the static buckling loads with the elastic foundation constant is not linear as can be seen from Figure 5.11. The results obtained are in good agreement with the results obtained for particular values of elastic foundation constant by Timoshenko and Gere [18] and Brown, Hutt and Salama [5].

Figure 5.12 shows the buckling loads for a Timoshenko beam \( \sqrt{R} = 0.08 \) resting on elastic foundation of various constants. The effect of elastic foundation on the buckling loads is more marked for small values of the elastic foundation constant and for lower modes. Very little change in the buckling load is observed for higher modes and for large elastic foundation constants.

Comparison between Figure 5.11 and Figure 5.12 shows that the shear deformation have a large effect on the static buckling loads in higher modes while the first mode is not very
much affected. Furthermore, it is seen that for the fixed-free beams the relationship between the static buckling loads and the elastic foundation constant is not linear as in the case of the hinged-hinged beams. This indicates that there is a significant change in the buckling mode shapes due to the presence of the elastic support.

Figures 5.13 and 5.14 show the variation of the static buckling loads with the elastic foundation constant for a free-free Euler beam and a free-free Timoshenko beam respectively. For the special case of no elastic support, one zero eigenvalue is obtained representing one rigid body mode. With the presence of the elastic foundation the zero mode disappears and a buckling load value is obtained. This value is associated with the rigid body mode and becomes higher than the first buckling load value as the elastic foundation becomes stiffer. It is seen that the value of the elastic foundation constant corresponding to the crossing of the rigid body mode and the first buckling mode is the same for Euler and Timoshenko beams.

It is seen that, for the symmetric modes of the free-free beams, no change takes place in the buckling mode shapes. While for the asymmetric modes, an apparent change in the mode shapes is observed and hence the variation of the static buckling loads with the elastic foundation constant is not linear.

Figures 5.15 and 5.16 show the variation of the static buckling loads with the elastic foundation constant for fixed-fixed Euler and Timoshenko beams respectively. It is seen that the results reported by Brown, Hutt and Salama [5] for the fixed-fixed Euler beam are in error. Also the corresponding
mode shapes given in Reference [5] are incorrect since the first mode shape is expected to be symmetrical due to the symmetry of the end conditions. The correct results are shown in Table 5.1.

For the fixed-fixed Euler beam, the buckling mode shapes are effected very slightly with the increase in the stiffness of the elastic foundation. The effect is more marked for the symmetric modes of the Timoshenko beam shown in Figure 5.16, where it is seen that the variation of the static buckling loads of these modes with the elastic foundation constant is not linear.

Comparison between Figure 5.15 and Figure 5.16 shows that the static buckling loads of a fixed-fixed beam is very sensitive to the variation in the shear forces and a significant drop in the critical loads is observed.

5.4.3 Dynamic stability

The effect of elastic foundation on the regions of dynamic instability of a hinged-hinged Bernoulli-Euler beam is shown in Figures 5.17.a to c. As the elastic foundation constant increases the regions are shifted away from the vertical axis which represents the static buckling case ($\omega = 0$) and the width of these regions is decreased thus making the beam less sensitive to periodic forces.

Figures 5.18.a to c show the regions of dynamic instability of a hinged-hinged Timoshenko beam resting on elastic foundation with constant equals to 0, 2 and 4 respectively. The regions associated with the pure shear mode and the first mode of the
second spectrum have degenerated into vertical lines with zero width and the variation in the elastic foundation has no effect at all on these regions. The first five regions associated with the first spectrum are shifted away from the static buckling axis \((\omega = 0)\) and their widths are decreased thus improving the stability characteristics of the beam.

The variations of the regions of dynamic instability of a fixed-free Bernoulli-Euler beam and a fixed-free Timoshenko beam with the elastic foundation constant are shown in Figures 5.19 and Figures 5.20 respectively. The shifting of the regions and the decrease in their width are more marked than in the hinged-hinged case.

Figures 5.21.a to c show the regions of a free-free Bernoulli-Euler beam resting on elastic foundation with constant equals to 0, 2 and 4 respectively. It is seen that one of the regions associated with the rigid body modes has degenerated into a line of zero width and hence it has no effect on the stability characteristics of the beam. However an interesting result is obtained regarding the other rigid body mode corresponding to rotation. Here a region of instability exists with a relatively large width for values of disturbing frequencies lower than that associated with the fundamental mode. The degeneration of one of the two regions associated with the rigid body modes into a vertical line explains the reasons for having two rigid body modes of vibration and only one rigid body mode of static buckling.

For the special case of no elastic support and no axial thrust, two zero eigenvalues are obtained representing two
rigid body modes of vibration of the free-free beam. It is seen that when the beam is subjected to constant or periodic thrusts, the zero eigenvalue associated with the rotational rigid body mode disappears and a positive value is obtained for the frequency. Thus in spite of a numerical error in Wu's work [58], which has been pointed out very recently by Sundararamaiah and Johns [105], the claim made by Wu that for a free-free beam subjected to a constant thrust there exists only one zero eigenvalue, is confirmed by the present analysis.

The regions of dynamic instability associated with the rigid body modes of a free-free Timoshenko beam are shown in Figures 5.22.a to c. It is noticed that as the elastic foundation becomes more stiff these regions together with the normal regions of dynamic instability are all shifted and their widths are all decreased.

The variation of the instability regions of fixed-fixed Bernoulli-Euler and Timoshenko beams with the elastic foundation constant are shown in Figures 5.23 and Figures 5.24 respectively. Figures 5.24 a to c show the effect of coupling on the characteristics of the regions of dynamic instability of Timoshenko beam.

It is seen that, for all beams, as the elastic foundation constant increases the regions of dynamic instability are shifted away from the axis representing the static buckling case ($\omega = 0$) and the width of these regions is decreased thus making the beams less sensitive to periodic forces.
### Table 5.1 Natural Frequency and Static Buckling Load Parameters of a Fixed-Fixed Bernoulli-Euler Beam Resting on Elastic Foundation with $\gamma = 4.5$

<table>
<thead>
<tr>
<th>Natural frequencies</th>
<th>Static buckling loads</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^\frac{1}{3} = (pAL^4\eta^2/EI)^\frac{1}{3}$</td>
<td>$B = p* L^2/EI$</td>
</tr>
<tr>
<td><strong>First mode</strong></td>
<td><strong>Second mode</strong></td>
</tr>
<tr>
<td>Present using six elements</td>
<td>30.643</td>
</tr>
<tr>
<td>Brown, Hutt and Salama[5]</td>
<td>35.92</td>
</tr>
<tr>
<td>Analytical solution</td>
<td>30.641</td>
</tr>
</tbody>
</table>
FIG. 5.1 EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A HINGED - HINGED BERNOULLI-EULER BEAM
FIG. 5.2 EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A HINGED - HINGED TIMOSHENKO BEAM. (R = 0.0064)
Elastic foundation constant ($\gamma$) vs. $\sqrt{EL/\rho A L^4}$

FIG. 5.3 EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A FIXED-FREE BERNOULLI-EULER BEAM
FIG. 5.4  EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A FIXED-FREE TIMOSHENKO BEAM. (R=0.0064)
FIG. 5.5 EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCY OF A FREE - FREE BERNOULLI-EULER BEAM
FIG. 5.6  EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A FREE-FREE TIMOSHENKO BEAM. (R = 0.0064)
FIG. 5.7  EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A FIXED - FIXED BERNOULLI-EULER BEAM
FIG. 5.8  EFFECT OF ELASTIC FOUNDATION ON THE NATURAL FREQUENCIES OF A FIXED - FIXED TIMOSHENKO BEAM. (R = 0.0064)
FIG. 5.9  EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A HINGED - HINGED BERNOULLI-EULER BEAM
FIG. 5.10 EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A HINGED - HINGED TIMOSHENKO BEAM. \( (R = 0.0064) \)
FIG. 5.11  EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FIXED - FREE BERNOUlli-EULER BEAM
FIG. 5.12 EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FIXED-FREE TIMOSHENKO BEAM. \((R = 0.0064)\)
FIG. 5.13  EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FREE-FREE BERNOULLI-EULER BEAM
FIG. 5.14  EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FREE-FREE TIMOSHENKO BEAM
FIG. 5.15 EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FIXED - FIXED BERNOULLI-EULER BEAM
FIG. 5.16 EFFECT OF ELASTIC FOUNDATION ON THE STATIC BUCKLING LOADS OF A FIXED - FIXED TIMOSHENKO BEAM. (R = 0.0064)
\[ P(t) = \alpha P_e^* + \beta P_e^* \cos \omega t \]

\[ P_e^* = 9.874 \frac{EI}{L^2} \] Fundamental static buckling load of a Bernoulli-Euler beam with zero elastic foundation constant

\[ P_e = 9.844 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of a Bernoulli-Euler beam with zero elastic foundation constant

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

FIG.5.17.a REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. \((R = 0.0000, \gamma = 0.0)\)
FIG. 5.17.b  REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED BERNOULLI-
EULER BEAM ON ELASTIC FOUNDATION. (R = 0.0000, γ = 2.0)
FIG. 5.17.c  REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED BERNOUlli-
EULER BEAM ON ELASTIC FOUNDATION, (R = 0.0000, \( \gamma = 4.0 \))
\( P(t) = \alpha P^*_t + \beta P^*_t \cos \omega t \)

\( P^*_t = 8.272 \frac{EI}{L^2} \) Fundamental static buckling load of a Timoshenko beam with zero elastic foundation constant

\( p_t = 8.840 \frac{\sqrt{EI/\rho A L^4}}{\gamma} \) Fundamental natural frequency of a Timoshenko beam with zero elastic foundation constant

\( \omega = \text{Disturbing frequency} \)

\( \alpha = 0.5 \)

**FIG. 5.18.a** REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (\( R = 0.0064 \), \( \gamma = 0.0 \))
FIG. 5.18.b  REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (R = 0.0064, γ = 2.0)
FIG. 5.18.c REGIONS OF DYNAMIC INSTABILITY OF A HINGED-HINGED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. \( R = 0.0064, \gamma = 4.0 \)
\[ P(t) = \alpha p_e^* + \beta p_e^* \cos \omega t \]

\[ p_e^* = 2.467 \frac{EI}{L^2} \] Fundamental static buckling load of Bernoulli-Euler beam with zero elastic foundation constant

\[ p_e = 3.504 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of Bernoulli-Euler beam with zero elastic foundation constant

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

FIG. 5.19.a REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. \((R = 0.0000, \gamma = 0.0)\)
FIG. 5.19.b  REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (R = 0.0000, $\gamma = 2.0$)
FIG. 5.19.c  REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (R=0.0000, $\gamma = 4.0$)
\[ P(t) = \alpha P^*_t + \beta P^*_t \cos \omega t \]

\[ P^*_t = 2.358 \frac{EI}{L^2} \] Fundamental static buckling load of a Timoshenko beam with zero elastic foundation constant

\[ p_t = 3.324 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of Timoshenko beam with zero elastic foundation constant

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

---

**FIG. 5.20.a** REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. \((R = 0.0064, \gamma = 0.0)\)
FIG. 5.20.b  REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (R = 0.0064, \( \gamma = 2.0 \))
FIG. 5.20.c REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. \( R = 0.0064, \gamma = 4.0 \)
\[ P(t) = \alpha P_e^* + \beta P_e^* \cos \omega t \]

\[ P_e^* = 9.870 \frac{EI}{L^2} \] Fundamental static buckling load of Bernoulli-Euler beam with zero elastic foundation constant

\[ P_e = 22.380 \sqrt{\frac{EI}{\rho AL^4}} \] Fundamental natural frequency of Bernoulli-Euler beam with zero elastic foundation constant

\( \omega \) = Disturbing frequency

\( \alpha = 0.5 \)

**FIG. 5.21.a** REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (\( R = 0.0000 \), \( \gamma = 0.0 \))
FIG. 5.21.b REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. \( R = 0.0000, \gamma = 2.0 \)
FIG. 5.21.c REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (R = 0.0000, $\gamma = 4.0$)
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]

\( P^*_t = 8.361 \frac{EI}{L^2} \) Fundamental static buckling load of Timoshenko beam with zero elastic foundation constant

\( p^*_t = 18.214 \sqrt{\frac{EI}{\rho AL^4}} \) Fundamental natural frequency of Timoshenko beam with zero elastic foundation constant

\( \omega = \) Disturbing frequency

\( \gamma = 0.5 \)

\( \beta \)

\( \omega/p_t \)

**FIG. 5.22.a** REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. \((R = 0.0064, \gamma = 0.0)\)
FIG. 5.22.b REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (R = 0.0064, χ = 2.0)
FIG. 5.22.c  REGIONS OF DYNAMIC INSTABILITY OF A FREE-FREE TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (R = 0.0064, \( \gamma = 4.0 \))
\[ P(t) = \alpha P_e^* + \beta P_e^* \cos \omega t \]

\[ P_e^* = 39.537 \frac{EI}{L^2} \] Fundamental static buckling load of a Bernoulli-Euler beam with zero elastic foundation constant

\[ p_e = 22.372 \sqrt{\frac{E}{\rho A}} L^4 \] Fundamental natural frequency of a Bernoulli-Euler beam with zero elastic foundation constant

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

**FIG. 5.23.a** REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. \((R = 0.0000, \gamma = 0.0)\)
FIG. 5.23,b  REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (R = 0.0000, γ = 2.0)
FIG. 5.23: REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED BERNOULLI-EULER BEAM ON ELASTIC FOUNDATION. (R=0.0000, γ = 4.0)
\[ P(t) = \alpha P_t^* + \beta P_t^* \cos \omega t \]
\[ P_t^* = 22.269 \frac{EI}{L^2} \] Fundamental static buckling load of a Timoshenko beam with zero elastic foundation constant
\[ P_t = 15.740 \sqrt{EI/\rho AL^4} \] Fundamental natural frequency of a Timoshenko beam with zero elastic foundation constant
\[ \omega = \text{Disturbing frequency} \]
\[ \alpha = 0.5 \]

**FIG. 5.24.a** REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. *(R = 0.0064, \gamma = 0.0)*
FIG. 5.24.b  REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. (R = 0.0064, γ = 2.0)
FIG. 5.24.c REGIONS OF DYNAMIC INSTABILITY OF A FIXED-FIXED TIMOSHENKO BEAM ON ELASTIC FOUNDATION. \((R = 0.0064, \gamma = 4.0)\)
CHAPTER 6
CHAPTER 6

DYNAMIC STABILITY OF SPACE FRAMES

6.1 Introduction

The application of the finite element method to study the free vibration of frame structures has been dealt with in several papers and text books [106, 107, 108].

Matrix formulation for the static stability of plane frames under axial loads has been presented by Hartz [55], Halldorsson and Wang [109] and by Yang and Shinozuka [110].

The problem of dynamic stability of plane frames has been investigated by Bolotin [2]. Stability coefficients and corresponding mass coefficients are determined for a number of predefined system co-ordinates. The co-ordinate functions are taken as the static deflection forms of the frames under the influence of certain forces. These forces are assumed in such a way that the deflection forms sufficiently resemble the forms of free or forced vibration of the frame. The construction of the basic equations for the chosen co-ordinates is preceded by many calculations. Furthermore, the choice of co-ordinates have a serious effect on the solution of the dynamic stability problems. For these reasons the range of application of the method reported by Bolotin [2] is almost limited to simple problems. A similar approach, with the same limitation was used by Roberts [6] to study the dynamic stability of plane frames.

Gorzynski and Thornton [111] calculated a set of coefficients from which a stiffness matrix could be assembled for plane frames. The matrix was called the dynamic stability
stiffness matrix. This terminology is inaccurate since the plane frames considered by Gorzynski and Thornton [111] were subjected to static loads only. Although the natural frequencies and the static buckling loads were extracted from the matrix, the problem of determining the regions of dynamic instability had not been considered.

This chapter presents for the first time a finite element method for the dynamic stability analysis of three dimensional frame structures subjected to periodic axial loads [112]. Complete generality is maintained in the description of the structure co-ordinates. Regions of dynamic instability, as well as natural frequencies and static buckling loads, are determined. The computer program developed is tested on plane frames and correlation of the results obtained with the available exact results indicates excellent agreement.

The present approach is successfully applied to the analysis of an offshore oil rig structure subjected to periodic sea wave forces [117]. To verify the accuracy of the computed natural frequencies, a small scale model is built and free vibration tests are carried out.

6.2 Formulation of Element Matrices

For the beam element shown in Figure 6.1, the strain energy \( U \) is given by

\[
U = \frac{1}{2} \left[ \int_0^L AE \left( \frac{dU}{dz} \right)^2 dz + \int_0^L EI_y \left( \frac{d^2U}{dz^2} \right)^2 dz + \int_0^L EI_x \left( \frac{d^2U}{dz^2} \right)^2 dz + \int_0^L GJ \left( \frac{d\theta}{dz} \right)^2 dz \right]
\]

(6.01)
where \( E \) is the modulus of elasticity,
\( A \) is the cross-sectional area,
\( G \) is the modulus of rigidity,
\( J \) is St. Venant torsion constant
\( I_y \) and \( I_x \) are moments of area of cross-section about its principal axes \( yy \) and \( xx \) respectively.

The potential energy \( V \) of the element is given by
\[
V = \frac{1}{2} \int_0^L \frac{dU_y}{dz}^2 \, dz + \frac{1}{2} \int_0^L \frac{dU_x}{dz}^2 \, dz + \frac{1}{2} \int_0^L \frac{IP_z}{A} \left( \frac{d\theta_z}{dz} \right)^2 \, dz \quad (6.02)
\]
where \( P \) is the axial load and \( I_p \) is polar moment of area of cross-section.

The kinetic energy \( T \) of the element is given by
\[
T = \frac{1}{2} \int_0^L \rho A \left( \dot{U}_x^2 + \dot{U}_y^2 + \dot{U}_z^2 \right) \, dz + \int_0^L \rho I_p \dot{\theta}_z^2 \, dz \quad (6.03)
\]
where \( \rho \) is the mass density.

Assuming cubic polynomial expansions for \( U_y \) and \( U_x \) and linear polynomial expansions for \( U_z \) and \( \theta_z \) to be of the forms
\[
U_y = \sum_{r=0}^{3} a_r \, z^r \quad U_x = \sum_{r=0}^{3} b_r \, z^r \quad (6.04)
\]
\[
U_z = \sum_{r=0}^{1} c_r \, z^r \quad \text{and} \quad \theta_z = \sum_{r=0}^{1} d_r \, z^r
\]
and substituting into equations (6.01), (6.02) and (6.03) and replacing the coefficients \( a_r, b_r, c_r, \text{ and } d_r \), the element elastic stiffness, geometric stiffness and inertia matrices.
are found to be:

\[
\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
\frac{12EI_x}{L^3} & 0 & 0 & 0 & \frac{6EI_x}{L^2} & 0 & -\frac{12EI_x}{L^3} & 0 & 0 & 0 & \frac{6EI_x}{L^2} & \\
\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & \\
\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & \\
\frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & 2EI_y & 0 & \\
\frac{4EI_x}{L} & 0 & -\frac{6EI_x}{L^2} & 0 & 0 & 0 & 2EI_x & 0 \\
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \\
\end{bmatrix}
\]
\[
[K_g] =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{6}{5L} & 0 & 0 & 0 & \frac{1}{10} \\
\frac{6}{5L} & 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{6}{5L} & 0 & -\frac{1}{10} & 0 \\
\frac{I_p}{AL} & 0 & 0 & 0 & 0 & 0 & -\frac{I_p}{AL} & 0 & 0 & 0 \\
\frac{2L}{15} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{L}{30} & 0 \\
\frac{2L}{15} & 0 & -\frac{1}{10} & 0 & 0 & 0 & -\frac{L}{30} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\text{Symmetric}
\begin{bmatrix}
\frac{6}{5L} & 0 & 0 & 0 & -\frac{1}{10} \\
\frac{6}{5L} & 0 & \frac{1}{10} & 0 \\
\frac{I_p}{AL} & 0 & 0 \\
\frac{2L}{15} & 0 \\
\frac{2L}{15} \\
\end{bmatrix}
The nodal displacement vector is

$$\{u_i, u_j, x_i, z_i, \theta_j, \theta_i, x_i, z_i+1, \theta_j+1, \theta_i+1, x_i+1, z_i+1, \theta_j+1, \theta_i+1, x_i+1\}^T$$
6.3 Matrix Equations for Space Frames

6.3.1 Free vibration

From equation (3.51) the equation of motion for free vibration becomes

\[[K_e] - p^2[M] \{\xi\} = 0\] (6.05)

where \( p \) is the natural frequency.

6.3.2 Static stability

From equation (3.52) the equation of motion for static stability becomes

\[[K_e] - P*[K_g] \{\xi\} = 0\] (6.06)

where \( P* \) is the static buckling load.

6.3.3 Dynamic stability

From equation (3.53) the equation of motion for dynamic stability becomes

\[[K_e] - (\alpha \pm i\beta) P*[K_g] - \frac{\omega^2}{4}[M] \{\xi\} = 0\] (6.07)

where \( \alpha \) and \( \beta \) are fractions representing the static and time dependent components of the load respectively and \( \omega \) is the disturbing frequency.

6.4 Applications

Since there are no exact theoretical results available for three dimensional space frames, the analysis and the computer program developed are first applied to plane frames for which exact values of natural frequencies and static buckling loads are available. The comparison of results will enable the
accuracy of the present analysis to be ascertained.

6.4.1 Application to plane frames

The developed mass and elastic stiffness matrices are employed to solve the matrix equation (6.05) of free vibration for the plane frame shown in Figure 6.2. The convergence of the frequency parameter with the number of beam elements used to idealize the plane frame is shown in Table 6.1. Also shown are the results obtained by Bolotin [2] and the exact results reported by Hohenemser and Prager [113].

The developed elastic stiffness and geometric stiffness matrices are employed to solve the matrix equation (6.06) of static stability for the plane frame shown in Figure 6.2. The convergence of the static buckling load parameter is shown in Table 6.2. The fundamental static buckling load reported by Bolotin [2] was in error and the value obtained after correcting Bolotin's [2] solution is shown in Table 6.2.

The mode shapes of free vibration and static buckling are shown in Figures 6.3 and 6.4 respectively.

The developed matrices are employed to solve the dynamic stability problem of the plane frame of Figure 6.2. The regions of dynamic instability are determined. The first four regions are shown in Figure 6.5.

The present analysis is also applied to various types of plane frames. Some of these frames are shown in Figure 6.6.

The convergences of the natural frequency parameters and the static buckling load parameters of the plane frame of
Figure 6.6.a are shown in Tables 6.3 and 6.4 respectively. Results obtained by other investigators are also shown in these tables.

The convergences of the natural frequency parameters and the static buckling load parameters of the plane frame of Figure 6.6.b are shown in Tables 6.5 and 6.6 respectively. The regions of dynamic instability are shown in Figure 6.7.

Finally the plane frame of Figure 6.6.c is considered. The natural frequency parameters and the static buckling load parameters are shown in Table 6.7 for two element idealization of each member of the frame. The regions of dynamic instability are shown in Figure 6.8.

6.4.2 Application to space frames

The present analysis is applied to the three dimensional frame structure shown in Figure 6.11 and Plate 6.1. The convergence of the natural frequencies is shown in Table 6.8. The experimental results obtained for the frequencies are also shown in this table. The static buckling loads are shown in Table 6.9.

The first four mode shapes of free vibration and static buckling are shown in Figures 6.12 and 6.13 respectively.

The regions of dynamic instability are shown in Figure 6.14.

6.4.3 Application to an offshore oil rig

This section presents the application of the finite element solution to the dynamic stability problem of frame structures similar to the offshore oil rig structures subjected to periodic vertical and horizontal forces. To verify the
accuracy of the computed natural frequencies of vibration, a small scale model has been built and free vibration tests are carried out.

6.4.3.1 Introduction

A large number of offshore drilling units have been constructed since 1950. Some of these units have been lost or suffered damages. The high failure rate was not acceptable and continuous efforts have been made to improve the integrity of the designs. Although these failures were by no means exclusively of a structural nature, structural inadequacies have figured greatly in a number of failures and near failures.

These structures are designed for a variety of loading conditions. The loads had to be known in great detail for any extensive structural analysis to be carried out. To ensure a proper balance between the accuracy of load evaluation and structural analysis is fundamental in any structural design, and when the finite element method is used it is the load evaluation which sets the standard. The waves forces present the greatest problem in this connection.

In the Gulf of Mexico, the platforms were designed to withstand the biggest waves that could be expected in a 50-year period. The same reasoning was applied to the North Sea, and some of the gas platforms in southern sector were designed to withstand waves of maximum height of 49 feet. Subsequent experience has revised this estimate to 55 feet and it is up to 100 feet in the northern sector of the North Sea.

It is suggested here that the offshore structures must be designed not only for the freek 50 year or 100 year wave, but
also for the continuous battering by sea waves.

The sea waves are assumed to be periodic in nature and their influence on the dynamic stability is investigated.

6.4.3.2 Loading conditions

The oil rig is basically a large three-dimensional frame having few structurally redundant members. The unit is subjected to vertical and horizontal dead and live loads. Two cases of loading are considered.

Case 1 Unit subjected to vertical static and periodic loads as shown in Figure 6.15. The vertical static loads are from self weight of the unit, the drilling equipment, accommodation, mud pumps, etc. These loads are carried by the vertical trusses and legs which are supported on the seabed. The live loads are assumed to be periodic in nature with varying magnitude and frequency.

Case 2 Unit subjected to horizontal static and periodic loads as shown in Figure 6.16. If the rig is located astride a crest, one leg will be subjected to inertia forces in one direction and the other leg will be subjected to inertia forces in the opposite direction. The net result is a prying force tending to force the legs apart as shown in Figure 6.17.a. Alternatively, if the rig is sitting in the wave trough, there will again be opposite forces and the net result is a squeezing force tending to force the legs together as shown in Figure 6.17.b.

It is clear that the horizontal frame of the rig takes the bulk of the wave forces. These forces are essentially of an
alternating nature and the members of the horizontal frame will therefore be continuously subjected to alternating forces whenever the unit is at sea.

6.4.3.3 Oil rig model

The present analysis and the computer program developed are employed to study the free vibration, static stability and dynamic stability of the oil rig model shown in Figure 6.18. The size and complexity of the model are reduced to cut down cost and time necessary to build such a model. The experimental model is tested and the results are compared with those obtained from the computer program to ascertain the accuracy obtained from the theoretical analysis.

6.4.3.4 Experimental procedure

The model is made of mild steel. Welding is carried out carefully to minimise distortion and residual stresses. Dimensions of the model are shown in Figure 6.18. The schematic representation of the general arrangement of the instrumentation used is shown in Figure 6.19. The model, the instruments and the general arrangement are shown in Plates 6.2, 6.3 and 6.4 respectively.

The oil rig model is excited by means of a piezo-electric crystal fixed close to the root cross-section of one of the main four legs. An input voltage is supplied to the piezo-electric crystal by a variable frequency oscillator through an amplifier. The input frequency is varied until resonance occurs. Resonance is detected by another piezo-electric crystal, similarly mounted, which functions as a pick up. The voltage produced in this piezo-electric crystal is displayed on an oscilloscope and a
frequency counter. The resonance is taken to occur when the displayed signal has a maximum amplitude.

6.4.3.5 Results

The first ten frequencies of free vibration of the oil rig model obtained theoretically and experimentally are shown in Table 6.10. Fundamental mode shape of vibration is shown in Figure 6.20.

The first ten static buckling loads of the oil rig model subjected to vertical loads in one case and to horizontal loads at the middle frame in another case are shown in Table 6.11. Fundamental mode shape of static buckling for case 1 is shown in Figure 6.21 and for case 2 in Figure 6.22.

The regions of dynamic instability of the oil rig model subjected to periodic vertical loads are shown in Figure 6.23, while Figure 6.24 shows the regions of dynamic instability when the model is subjected to periodic horizontal loads.

6.5 General Discussions

The co-ordinate axes and the displacement co-ordinates of the element model employed in this analysis are shown in Figure 6.1. Six degrees of freedom are assigned to each node of the element, thus allowing complete freedom of displacements. The present analysis is first applied to plane frames for which exact results are available.

6.5.1 Plane frames

Table 6.1 shows the comparison of the natural frequency parameters of the plane frame, shown in Figure 6.2, with the
results given by Bolotin [2] and the exact results reported by Hohenemser and Prager [113]. It is seen that the present results are in very close agreement with the exact results. The values obtained for the static buckling loads of this plane frame are shown in Table 6.2. Bolotin [2] reported the value 6.92 EI/L^2 for the fundamental buckling load. This value is found to be incorrect since it is not a solution of the equation given by Bolotin (equation 20.32 in ref. 2). Solving Bolotin equation correctly a value of 8.12 EI/L^2 is obtained which compares with the value of 8.04 EI/L^2 obtained in the present analysis. The relative displacements of the frame for the first four modes of vibration are shown in Figure 6.3, while Figure 6.4 shows the relative displacements for the first four modes of static buckling. The similarity between the free vibration and static buckling mode shapes is very clear from these figures. The first four regions of dynamic instability of the frame for \( \alpha = 0.5 \) are shown in Figure 6.5. The intersection of the right hand side boundary of the first region with the \( \beta = 1 \) line defines the fundamental mode of free vibration. In this case \( \omega/p_1 \) is 2 and \( \alpha - \frac{1}{2} \beta = 0 \). Inserting these values into equation (6.07) results
\[
\left| \left[ K_e \right] - p^2 \left[ M \right] \right| = 0 \quad \text{which is the equation of free vibration.}
\]
The fundamental static buckling mode is defined by the intersection of the left hand side boundary of the first region with the \( \beta \) axis. This gives the values of \( \omega/p_1 = 0 \) and \( \beta = 1 \). Substituting these values into equation (6.07) results
\[
\left| \left[ K_e \right] - P^* [K_e] \right| = 0 \quad \text{which is the equation of static stability.}
\]
Figure 6.5 also confirms the similarity between the free vibration and static buckling mode shapes.
The rapid convergence of the vibration and buckling solutions based on the derived matrices is demonstrated for various types of plane frames. Tables 6.3 and 6.4 show the rapid convergence of the frequency parameters and the static buckling load parameters of the plane frame shown in Figure 6.6.a respectively. Comparison of results shows that the present analysis gives extremely good results even for two element idealization of each member of the frame. The number shown in these tables is the number of elements used to idealize the complete frame. Tables 6.5 and 6.6 show the convergence of the frequency and buckling parameters respectively of the frame shown in Figure 6.6.b. The comparison of the buckling results shows excellent agreement with the exact results reported by Hohenemser and Prager [113]. The first five regions of dynamic instability of the frame subjected to periodic force with $\alpha = 0.5$ are shown in Figure 6.7. It is seen that the third region is degenerated into a vertical line concluding that for this case of loading, the frame does not buckle in a mode shape similar to the third mode shape of free vibration.

Roberts [6] considered the plane frame shown in Figure 6.6.c. The components of the axial loads in the columns taken by Roberts [6] are shown in Figure 6.6.d. The values obtained by the present analysis for the natural frequency and buckling load parameters are shown in Table 6.7. For the fundamental buckling load a value of $52.5 \, \text{EI}/L^2$ is obtained, compared to the value of $55.1 \, \text{EI}/L^2$ reported by Roberts [6]. The regions of dynamic instability are shown in Figure 6.8. It is seen that the third region is the most important one and that the third mode shape of free vibration is similar to the first mode shape of static
buckling. This is confirmed by plotting the mode shapes of free vibration and static buckling. These are shown in Figures 6.9 and 6.10 respectively, where it is very clear that the third mode shape of free vibration is similar to the fundamental mode shape of static buckling.

It can be concluded that the present analysis gives extremely good results for the natural frequencies and static buckling loads of plane frames. Since the present analysis is developed for three-dimensional frame structures but applied to special cases of plane frames, the accuracy of the results obtained by the present analysis is ascertained.

6.5.2 Space frames

The natural frequencies of free vibration of the three dimensional frame structure shown in Figure 6.11 are given in Table 6.8 for one, two and three element idealization of each member. It is seen that the convergence is very rapid and the results obtained from two element idealization (96 degrees of freedom) are very close to the experimental values of the frequencies.

The components of the static forces acting on each member are shown in Figure 6.11.c, and the values obtained for the static buckling loads are given in Table 6.9. Figures 6.12 and 6.13 show the first four mode shapes of free vibration and static buckling respectively. It can be seen that:

(i) The first mode shape of static buckling is similar to the third mode shape of free vibration.

(ii) The second mode shape of static buckling is similar to the first mode shape of free vibration.
(iii) The third mode shape of static buckling is similar to the second mode shape of free vibration, and
(iv) The fourth mode shape of static buckling is similar to the fourth mode shape of free vibration.

The dynamic stability characteristics are shown in Figure 6.14 for $\alpha = 0$. The intersection of the boundary of the first region with the $\beta = 0$ axis defines the fundamental mode of free vibration, while the fundamental static buckling mode is defined by the intersection of the left hand side boundary of the first region with the $\beta$ axis. Figure 6.14 can also be used to verify the relationships between the mode shapes of free vibration and the mode shapes of static buckling.

6.5.3 An offshore oil rig

Table 6.10 shows the first ten frequencies of free vibration of the oil rig model. The agreement between theoretical and experimental values is averaged within 4.5 per cent. Since the present analysis gives extremely good results for natural frequencies of free vibration, it is fair to assume that the accuracy of the buckling results will be of the same order.

Figure 6.20 shows the fundamental mode shape of free vibration of the oil rig model. The three views show the front elevation, the end elevation and the plan of the model. To make the plan view clear, only the members of the lower horizontal frame are shown. It can be seen that the major deflection takes place in the horizontal frames, giving an ovalised mode shape.

Table 6.11 shows the first ten static buckling loads of the oil rig model subjected to vertical loads for the first case.
and to horizontal loads at the mid-frame for the second case. The middle horizontal frame represents in this analysis the horizontal struts just beneath the water line in the splash zone which take the bulk of the sea wave forces.

Figure 6.21 shows the fundamental static buckling mode shape of the oil rig model subjected to vertical loads. Major deformations take place in the x-direction, while deformations in the y and z directions are relatively small. The mode shape for the three horizontal frames are similar and only the lower frame is shown in Figure 6.21.

Figure 6.22 shows the fundamental static buckling mode shape of the model subjected to horizontal loads at the middle horizontal frame. As expected, the major deformation takes place at this frame while other deformations are relatively small.

Figure 6.23 shows the regions of dynamic instability of the oil rig model subjected to periodic vertical loads. The intersection of the first region with the \( \omega/p_1 \) axis defines the fundamental mode of free vibration. In this case the value of \( \omega/p_1 \) is 2 and that of \( \beta \) is zero. Inserting these values into equation (6.07) results in the equation of free vibration. The fundamental static buckling mode is defined by the intersection of the first region with the \( \beta \) axis. This gives the values of \( \omega/p_1 \) to be zero and that of \( \beta \) to be 2. Substituting these values into equation (6.07) gives the equation of static stability. From Figure 6.23 it can be seen that as the disturbing frequency becomes equal to twice the fundamental
natural frequency of the structure, the greatest part of the regions becomes unstable even under the influence of periodic forces of small magnitude.

Figure 6.24 shows the regions of dynamic instability when the model is subjected to periodic horizontal loads acting at the middle horizontal frame. The loading conditions in this case excite very few modes of instability and it is seen from this figure that the fundamental mode shapes of free vibration and static buckling are similar. This is verified by the obvious similarity between these mode shapes as shown in Figures 6.20 and 6.22.
<table>
<thead>
<tr>
<th>Total number of elements to idealize the whole plane frame</th>
<th>Frequency parameter $\lambda = \rho AL^4 p^2 / EI$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
<td>mode 2</td>
</tr>
<tr>
<td>3</td>
<td>6.956</td>
<td>287.053</td>
</tr>
<tr>
<td>6</td>
<td>6.946</td>
<td>205.931</td>
</tr>
<tr>
<td>9</td>
<td>6.945</td>
<td>203.861</td>
</tr>
<tr>
<td>12</td>
<td>6.945</td>
<td>203.477</td>
</tr>
<tr>
<td>15</td>
<td>6.945</td>
<td>203.392</td>
</tr>
<tr>
<td>Bolotin [2]</td>
<td>6.900</td>
<td>222.0</td>
</tr>
<tr>
<td>Exact Hohenemser and Prager [113]</td>
<td>6.950</td>
<td>204.2</td>
</tr>
</tbody>
</table>

**TABLE 6.1. CONVERGENCE OF FREQUENCY PARAMETERS OF THE PLANE FRAME SHOWN IN FIG.6.2.**

<table>
<thead>
<tr>
<th>Total number of elements to idealize the whole plane frame</th>
<th>Static buckling load parameter $B = P*\lambda^2 / EI$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
<td>mode 2</td>
</tr>
<tr>
<td>3</td>
<td>8.116</td>
<td>52.500</td>
</tr>
<tr>
<td>6</td>
<td>8.075</td>
<td>27.546</td>
</tr>
<tr>
<td>9</td>
<td>8.050</td>
<td>27.179</td>
</tr>
<tr>
<td>12</td>
<td>8.045</td>
<td>27.036</td>
</tr>
<tr>
<td>15</td>
<td>8.044</td>
<td>26.992</td>
</tr>
<tr>
<td>Bolotin [2]</td>
<td>8.12*</td>
<td>-</td>
</tr>
</tbody>
</table>


**TABLE 6.2. CONVERGENCE OF STATIC BUCKLING LOAD PARAMETERS OF THE PLANE FRAME SHOWN IN FIG.6.2.**
### TABLE 6.3. CONVERGENCE OF FREQUENCY PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.a.

<table>
<thead>
<tr>
<th>Total number of elements</th>
<th>Frequency parameter $\lambda = \rho AL^2p^2/EI$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>3</td>
<td>10.306</td>
</tr>
<tr>
<td>6</td>
<td>10.272</td>
</tr>
<tr>
<td>9</td>
<td>10.270</td>
</tr>
<tr>
<td>12</td>
<td>10.269</td>
</tr>
<tr>
<td>15</td>
<td>10.269</td>
</tr>
<tr>
<td>Bishop and Johnson [114]</td>
<td>9.94</td>
</tr>
<tr>
<td>Gorzynski and Thornton [111]</td>
<td>10.28</td>
</tr>
</tbody>
</table>

### TABLE 6.4. CONVERGENCE OF STATIC BUCKLING LOAD PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.a.

<table>
<thead>
<tr>
<th>Total number of elements</th>
<th>Static buckling load parameter $B = P^*z^2/EI$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>3</td>
<td>7.444</td>
</tr>
<tr>
<td>6</td>
<td>7.401</td>
</tr>
<tr>
<td>9</td>
<td>7.384</td>
</tr>
<tr>
<td>12</td>
<td>7.380</td>
</tr>
<tr>
<td>15</td>
<td>7.379</td>
</tr>
<tr>
<td>Yang and Shinozuka [110]</td>
<td>7.40</td>
</tr>
<tr>
<td>Gregory [115]</td>
<td>7.40</td>
</tr>
<tr>
<td>Gorzynski and Thornton [111]</td>
<td>7.373</td>
</tr>
<tr>
<td>Exact Timoshenko [18]</td>
<td>7.38</td>
</tr>
</tbody>
</table>

TABLE 6.3. CONVERGENCE OF FREQUENCY PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.a.

TABLE 6.4. CONVERGENCE OF STATIC BUCKLING LOAD PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.a.
### TABLE 6.5. CONVERGENCE OF FREQUENCY PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.b.

<table>
<thead>
<tr>
<th>Total number of elements</th>
<th>Frequency parameter $\lambda = \rho AL^4p^2/EI$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>4</td>
<td>6.645</td>
</tr>
<tr>
<td>8</td>
<td>6.637</td>
</tr>
<tr>
<td>12</td>
<td>6.636</td>
</tr>
<tr>
<td>16</td>
<td>6.636</td>
</tr>
</tbody>
</table>

### TABLE 6.6. CONVERGENCE OF STATIC BUCKLING LOAD PARAMETERS OF THE PLANE FRAME SHOWN IN FIG. 6.6.b.

<table>
<thead>
<tr>
<th>Total number of elements</th>
<th>Static buckling load parameter $B = P*\lambda^2/EI$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode 1</td>
</tr>
<tr>
<td>4</td>
<td>15.383</td>
</tr>
<tr>
<td>8</td>
<td>15.231</td>
</tr>
<tr>
<td>12</td>
<td>15.124</td>
</tr>
<tr>
<td>16</td>
<td>15.106</td>
</tr>
<tr>
<td>Bolotin [2]</td>
<td>15.40</td>
</tr>
<tr>
<td>Exact by Smirnov [116]</td>
<td>15.10</td>
</tr>
<tr>
<td>Frequency parameter $\lambda = \rho AL^4 p^2/EI$</td>
<td>mode 1</td>
</tr>
<tr>
<td>-------------------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>4.851</td>
<td>34.498</td>
</tr>
<tr>
<td>Static buckling load parameter $B = P*\xi^2/EI$</td>
<td>mode 1</td>
</tr>
<tr>
<td>52.534</td>
<td>139.98</td>
</tr>
<tr>
<td>Roberts [6]</td>
<td>55.13</td>
</tr>
</tbody>
</table>

TABLE 6.7. NATURAL FREQUENCY PARAMETERS AND STATIC BUCKLING LOAD PARAMETERS OF THE PLANE FRAME SHOWN IN FIG.6.6.c. USING TWO ELEMENT IDEALIZATION FOR EACH MEMBER.
<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural frequencies Hz</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of elements along each member</td>
<td>1 (24 d.o.f.)</td>
</tr>
<tr>
<td>1</td>
<td>638.1</td>
<td>416.9</td>
</tr>
<tr>
<td>2</td>
<td>658.0</td>
<td>424.1</td>
</tr>
<tr>
<td>3</td>
<td>673.9</td>
<td>427.5</td>
</tr>
<tr>
<td>4</td>
<td>701.0</td>
<td>435.5</td>
</tr>
<tr>
<td>5</td>
<td>979.4</td>
<td>474.7</td>
</tr>
<tr>
<td>6</td>
<td>1038.3</td>
<td>481.9</td>
</tr>
<tr>
<td>7</td>
<td>1070.0</td>
<td>507.7</td>
</tr>
<tr>
<td>8</td>
<td>1092.6</td>
<td>507.7</td>
</tr>
<tr>
<td>9</td>
<td>1174.0</td>
<td>616.9</td>
</tr>
<tr>
<td>10</td>
<td>1212.21</td>
<td>625.3</td>
</tr>
</tbody>
</table>

**TABLE 6.8. CONVERGENCE OF NATURAL FREQUENCIES OF THE SPACE FRAME SHOWN IN FIG.6.11.**

| Mode | Static buckling loads lbf |  |
|------|----------------------------|  |
|      | Number of elements along each member | 1 (24 d.o.f.) | 2 (96 d.o.f.) | 3 (168 d.o.f.) |
| 1    | 8575.9 | 3930.6 | 3899.2 |  |
| 2    | 8880.2 | 3977.9 | 3948.9 |  |
| 3    | 10698.8 | 4170.5 | 4152.2 |  |
| 4    | 10826.6 | 4185.6 | 4168.3 |  |
| 5    | - | 4540.8 | 4548.0 |  |
| 6    | - | 4557.9 | 4566.1 |  |
| 7    | - | 4597.6 | 4608.5 |  |
| 8    | - | 4626.1 | 4639.1 |  |
| 9    | - | 11429.0 | 8702.3 |  |
| 10   | - | 11533.8 | 8748.6 |  |

**TABLE 6.9. CONVERGENCE OF STATIC BUCKLING LOADS OF THE SPACE FRAME SHOWN IN FIG.6.11.**
<table>
<thead>
<tr>
<th>Mode</th>
<th>Analytical frequency (Hz)</th>
<th>Experimental frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>78.4</td>
<td>81.0</td>
</tr>
<tr>
<td>2</td>
<td>113.4</td>
<td>101.0</td>
</tr>
<tr>
<td>3</td>
<td>115.3</td>
<td>108.0</td>
</tr>
<tr>
<td>4</td>
<td>118.1</td>
<td>112.0</td>
</tr>
<tr>
<td>5</td>
<td>118.8</td>
<td>117.0</td>
</tr>
<tr>
<td>6</td>
<td>130.2</td>
<td>122.0</td>
</tr>
<tr>
<td>7</td>
<td>132.5</td>
<td>124.0</td>
</tr>
<tr>
<td>8</td>
<td>132.8</td>
<td>133.0</td>
</tr>
<tr>
<td>9</td>
<td>138.5</td>
<td>141.0</td>
</tr>
<tr>
<td>10</td>
<td>146.5</td>
<td>147.0</td>
</tr>
</tbody>
</table>

**TABLE 6.10. FREQUENCIES OF FREE VIBRATION OF THE OIL RIG MODEL**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Buckling loads case (1) lbf</th>
<th>Buckling loads case (2) lbf</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>103.2</td>
<td>26.0</td>
</tr>
<tr>
<td>2</td>
<td>105.7</td>
<td>89.6</td>
</tr>
<tr>
<td>3</td>
<td>106.3</td>
<td>89.7</td>
</tr>
<tr>
<td>4</td>
<td>106.8</td>
<td>100.9</td>
</tr>
<tr>
<td>5</td>
<td>108.7</td>
<td>106.2</td>
</tr>
<tr>
<td>6</td>
<td>110.8</td>
<td>109.6</td>
</tr>
<tr>
<td>7</td>
<td>112.1</td>
<td>113.3</td>
</tr>
<tr>
<td>8</td>
<td>116.4</td>
<td>142.0</td>
</tr>
<tr>
<td>9</td>
<td>131.0</td>
<td>149.5</td>
</tr>
<tr>
<td>10</td>
<td>205.5</td>
<td>151.8</td>
</tr>
</tbody>
</table>

**TABLE 6.11. STATIC BUCKLING LOADS OF THE OIL RIG MODEL**
FIG. 6.1  A BEAM ELEMENT
FIG. 6.2  A PLANE FRAME
FIG. 6.3 FIRST FOUR MODE SHAPES OF FREE VIBRATION OF A PLANE FRAME
FIG. 6.4 FIRST FOUR MODE SHAPES OF STATIC BUCKLING OF A PLANE FRAME
\[ P(t) = \sigma P^* + \beta P^* \cos \omega t \]

\[ P^* = 8.044 \frac{EI}{L^2} \quad \text{Fundamental static buckling load} \]

\[ p_1 = 2.635 \sqrt{\frac{EI}{\rho AL}} \quad \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

**FIG. 6.5** REGIONS OF DYNAMIC INSTABILITY OF A PLANE FRAME
FIG. 6.6 PLANE FRAMES
FIG. 6.6.d  AXIAL LOADS IN COLUMNS OF THE PLANE FRAME SHOWN IN FIG. 6.6.c
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]

\[ P^* = 15.106 \frac{EI}{L^2} \quad \text{Fundamental static buckling load} \]

\[ p_1 = 2.576 \sqrt{\frac{EI}{\rho AL^4}} \quad \text{Fundamental natural frequency} \]

\( \omega \) = Disturbing frequency

\( \alpha = 0.5 \)

---

**FIG. 6.7** REGIONS OF DYNAMIC INSTABILITY OF THE PLANE FRAME SHOWN IN FIG. 6.6.b
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]

\[ P^* = 52.534 \frac{EI}{L^2} \quad \text{Fundamental static buckling load} \]

\[ p_1 = 2.202 \sqrt{\frac{EI}{\rho AL^4}} \quad \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.544 \]

FIG. 6.8 REGIONS OF DYNAMIC INSTABILITY OF THE PLANE FRAME SHOWN IN FIG. 6.6.c
FIG. 6.9 FIRST THREE MODE SHAPES OF FREE VIBRATION OF THE PLANE FRAME SHOWN IN FIG. 6.6.c

Third mode

FIG. 6.10 FIRST MODE SHAPE OF STATIC BUCKLING
FIG. 6.11 A THREE DIMENSIONAL FRAME CTURE

Height = 8"
Cross-section of all members is a square of 1/4" side
\[ P_1 = P_2 = P_3 = P_4 = 1.06P \]
\[ P_5 = P_6 = P_7 = P_8 = 0.25P \]
\[ P_9 = P_{10} = P_{11} = P_{12} = 0 \]

**FIG. 6.11.c** COMPONENTS OF STATIC FORCES
FIG. 6.12 FIRST FOUR MODE SHAPES OF FREE VIBRATION
FIG. 6.13  FIRST FOUR MODE SHAPES OF STATIC BUCKLING
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]

\( P^* = 3899.2 \text{ lbf} \) \hspace{1em} \text{Fundamental static buckling load}

\( p_1 = 413.1 \text{ Hz} \) \hspace{1em} \text{Fundamental natural frequency}

\( \omega = \text{Disturbing frequency} \)

\( \alpha = 0.0 \)

**FIG. 6.14** \hspace{1em} \text{Regions of dynamic instability of the three dimensional structure shown in FIG. 6.11}
UNIT SUBJECTED TO HORIZONTAL STATIC AND PERIODIC LOADS

FIG. 6.16

UNIT SUBJECTED TO VERTICAL STATIC AND PERIODIC LOADS

FIG. 6.15
FIG. 6.17.a  PRYING FORCE  

FIG. 6.17.b  SQUEEZING FORCE
The four main legs have a square cross section 1/8" x 1/8"

All other members have a circular cross section 1/16" diameter

FIG. 6.18  OIL RIG MODEL
FIG. 6.19 SCHEMATIC REPRESENTATION OF EXPERIMENTAL SET-UP
FIG. 6.20  FUNDAMENTAL MODE SHAPE OF FREE VIBRATION OF THE OIL RIG MODEL
FIG. 6.21  FUNDAMENTAL BUCKLING MODE SHAPE OF THE OIL RIG MODEL SUBJECTED TO VERTICAL LOADS

Components of static forces

\[ P_1 = 1.0151 \, P \]
\[ P_2 = 0.13636 \, P \]
\[ P_3 = 0.10908 \, P \]
\[ P_4 = 0.0 \]
FIG. 6.22  FUNDAMENTAL BUCKLING MODE SHAPE OF THE OIL RIG MODEL SUBJECTED TO HORIZONTAL LOADS
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]
\[ P^* = 103.2 \text{ lbf} \quad \text{Fundamental static buckling load} \]
\[ P_1 = 78.4 \text{ Hz} \quad \text{Fundamental natural frequency} \]
\[ \omega = \text{Disturbing frequency} \]
\[ \alpha = 0 \]

FIG. 6.23 REGIONS OF DYNAMIC INSTABILITY OF THE OIL RIG MODEL SUBJECTED TO PERIODIC VERTICAL LOADS
\[ P(t) = \alpha P^* + \beta P^* \cos \omega t \]

\[ P^* = 26.0 \text{ lbf } \text{Fundamental static buckling load} \]

\[ p_1 = 78.4 \text{ Hz } \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0 \]

FIG. 6.24 REGIONS OF DYNAMIC INSTABILITY OF THE OIL RIG MODEL SUBJECTED TO PERIODIC HORIZONTAL LOADS
PLATE 6.1 A THREE DIMENSIONAL FRAME STRUCTURE
PLATE 6.2 AN OIL RIG MODEL
CHAPTER 7
CHAPTER 7

DYNAMIC STABILITY OF PLATES

7.1 Introduction

The problem of free vibration of plates has long remained one of both practical and academic importance. Many cases have been solved either exactly or approximately. A vast number of papers have been published on this subject [118-125]. The finite element method was used by Dawe [12], Deak and Pian [15], Anderson, Irons and Zienkiewicz [126], and Dickinson and Henshell [127].

The first published work on static stability of plates using the finite element method was by Kapur and Hartz [13]. The element used by Kapur and Hartz was the Melosh rectangular plate bending element [9]. This element was based on an assumed deflection function which did not ensure compatibility of normal slopes along coincident boundaries of the elements. Owing to this violation of compatibility a lower bound for the buckling load was obtained. However a reasonable agreement with analytical results was reported for a variety of rectangular plate problems.

Bogner, Fox and Schmit [128] developed a 12 degree-of-freedom rectangular plate bending element using Hermitian polynomial interpolation formula. Certain deficiencies which affected the monotonicity of convergence of this element were reported by the same authors [128]. In order to correct these deficiencies, an added degree of freedom was included at each of the four nodes thus increasing the number of degrees-of-freedom from 12 to 16. This 16 degree-of-freedom element was
used by Carson and Newton [129] to solve plate buckling problems.

A refined quadrilateral bending element composed of four triangular bending elements has been developed by Clough and Felippa [130]. Each triangular element has 12 degree-of-freedom and prior to the combination of these elements to form the quadrilateral constraints are placed on sides which are to be exterior boundaries so as to remove a degree-of-freedom at the mid-points of each side. The resulting element has 19 degrees-of-freedom and in order to reduce them to 12, the 7 internal degrees-of-freedom are eliminated by a static condensation process. No explicit algebraic forms of matrices were presented for this element.

A triangular bending element was used by Anderson, Irons and Zienkiewicz [126] to solve plate buckling problems. This element was based on a displacement function [131] which did not ensure the continuity of slopes.

A solution for the static stability problem of plates, based on a mixed variational principle was used by Allman [132]. The element elastic stiffness matrix used was identical to that derived in Reference [133], where a linear bending moment field, defined inside a triangular element, was used in conjunction with a cubic displacement field on the element boundary. The geometric stiffness matrix was calculated using the non-compatible cubic displacement field given by Bazeley, Cheung, Irons and Zienkiewicz [131]. This approach has been previously considered by Cook [134] using a mixed formulation [135] of the constant bending moment equilibrium element [132, 136]
but the accuracy of the numerical results was found to be poor.

The application of smooth surface interpolation in finite element analysis was suggested by Birkhoff and Garabedian [14]. The values of the displacement are interpolated by means of the fourth-order Hermitian polynomial along the edges, where as the slopes normal to each side are interpolated as varying linearly along that side. These conditions permit a fully compatible displacement field when elements are assembled together. Deak and Pian [15] derived the mass and stiffness matrices using this displacement function giving a 12 degree-of-freedom rectangular bending element and used them to study the free vibration problem of rectangular plates.

In this chapter, the geometric stiffness matrices of this conforming element are developed for the first time and used to solve the static stability problem of rectangular plates under combined in-plane forces [137]. Results are compared with those obtained by using the well-known 12 degree-of-freedom non-conforming rectangular element [13] and the non-conforming triangular element [126].

The lateral vibration of plates subjected to static in-plane forces has been investigated by several authors. Weinstein and Chien [138], Luria [139] and Kaul and Tewari [140] considered the vibrations of plates under normal uniaxial or biaxial in-plane loads. Dickinson [141], and Simons and Leissa [142] studied the vibrations of plates under combinations of uniform shear and normal stresses. Lateral vibration of plates with various boundary and loading conditions were investigated by Bassily and Dickinson [143, 144, 145] using the Ritz method,
the perturbation method and a multiparameter perturbation method.

The finite element method was used by Henshell, Walters and Warburton [146] to investigate the vibration of a simply supported square plate subjected to a uniaxial compressive stresses. Non-conforming plate elements with four, eight and twelve nodes were derived and used in the analysis. Henshell, Walters and Warburton [146] concluded that the eight nodes element is the best of the three elements.

Mei and Yang [147] studied the lateral vibration of plates with various loading conditions using the 16 degree-of-freedom plate element derived by Bogner, Fox and Schmit [128]. As has been shown by Lurie [139], the load corresponding to zero frequency was extrapolated by Mei and Yang [147] to yield the critical buckling load.

When a plate is subjected to periodic in-plane loads, lateral vibration is induced in the plate and resonance will occur when a certain relationship exists between the natural frequency of transverse vibration of the plate and the frequency of the disturbing force. The range of values of the parameters causing unstable motion is referred to as the regions of dynamic instability. The dynamic stability of plates subjected to periodic compressive in-plane forces was first investigated by Einaudi [7]. Approximate solution was obtained by Bolotin [2] using Galerkin method. The form of vibration was approximated with the help of some suitable function which satisfied the boundary conditions. Krajcinovic and Herrmann [79] used a numerical integration technique to obtain the first region of dynamic instability of a simply supported rectangular plate.
stressed in its plane by an axial periodic load.

The finite element method was used by Hutt and Salama [8, 148] to study the dynamic stability problem of rectangular plates. The element used by Hutt and Salama [8] was the Melosh [9] non-conforming rectangular plate bending element. The elastic stiffness matrix derived by Melosh [9] contained several errors and it was corrected by Tocher and Kapur [10]. An identical elastic stiffness matrix was developed independently by Zienkiewicz and Cheung [11]. The mass matrix for this element was derived by Dawe [12] and the geometric stiffness matrix was derived by Kapur and Hartz [13].

Roberts [6] used the conjugate gradient technique to solve the case of a simply supported square plate. The technique involves an iteration procedure and the number of steps required for convergence may be as high as 15 iterations. The accuracy of the higher frequencies is dependent on the accuracy of the lower modes, and any error in a lower mode will be reflected in the higher modes.

Parametric stability of plates in nonlinear formulation was considered by Bolotin [2] and Somerset and Evan-Iwanowski [149]. Flutter analysis of plates in a potential flow was investigated by Rossettos and Tong [150] and Yang [151].

In this chapter, a series of results are presented for the dynamic stability analysis of rectangular plates with various boundary and loading conditions. The degeneration of some regions of dynamic instability into lines is explained and the relationship between the vibration modes and the buckling modes is examined. Comparison of results shows that the present
fully conforming element gives in all cases better results than
the well known 12 degree-of-freedom non-conforming element used
by Hutt and Salama [8].

7.2 Formulation of Elastic Stiffness and Geometric Stiffness
Matrices

The strain energy of a plate element subjected to in-plane
forces [18] can be written as:

\[ U = U_b + U_s \]

where \( U_b \) is the strain energy due to plate bending and is
given by:

\[ U_b = \frac{D}{2} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \, \text{dx dy} \quad (7.01) \]

and \( U_s \) is the strain energy due to the in-plane forces and
is given by:

\[ U_s = \frac{1}{4} \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) \right] \, \text{dx dy} \quad (7.02) \]

where \( D = \frac{Eh^3}{12(1-\nu^2)} \) is the flexural rigidity of the plate,
\( N_x \) and \( N_y \) are in-plane forces in \( x \) and \( y \) directions
respectively and \( N_{xy} \) is the in-plane shear force.

The displacement functions used are based on smooth surface
interpolation [14]. The lateral displacement of the rectangular
plate shown in Figure 7.1 of a dimension 2a x 2b is expressed
in terms of the following interpolation functions:

\[ w(x, y) = \sum_{i=1}^{12} a_i F_i (x, y) \quad (7.03) \]
where the independent \( \alpha \)'s can be expressed in terms of the 12 corner displacements, i.e. one deflection and two slopes at each of the four corners.

The functions \( F_i \) are given by:

\[
\begin{align*}
F_1 &= 1 & F_2 &= x^2 & F_3 &= y^2 \\
F_4 &= x & F_5 &= x^3 & F_7 &= y \\
F_8 &= y^3 & F_{10} &= xy \\
F_{11} &= 3x^3y + 3y^3x - x^3y^3 - 5xy
\end{align*}
\]

\[ \text{in regions 1, 2, 3 and 4} \]

<table>
<thead>
<tr>
<th>Region 1</th>
<th>Region 2</th>
<th>Region 3</th>
<th>Region 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_6 )</td>
<td>( x^2 - 2x + y^2 )</td>
<td>( 2xy - 2x )</td>
<td>(-x^2 - 2x - y^2 )</td>
</tr>
<tr>
<td>( F_9 )</td>
<td>( 2xy - 2y )</td>
<td>( y^2 - 2y + x^2 )</td>
<td>(-2xy - 2y )</td>
</tr>
<tr>
<td>( F_{12} )</td>
<td>( \frac{1}{4}(x^3y^5 - yx^5 - 3xy^3 + 3yx^5) )</td>
<td>( \frac{1}{4}(xy^5 - x^3y^3 - 3xy^3 + 3yx^3) )</td>
<td>( \frac{1}{4}(xy^5 - x^3y^3 - 3xy^3 + 3yx^3) )</td>
</tr>
</tbody>
</table>

The Birkhoff-Garabedian function gives an expression for the displacement over the whole element and therefore it can be used for deriving the elastic and geometric stiffness matrices.

The energy functionals are obtained by substituting the displacement functions into equation (7.01) and (7.02) and by carrying out the integration over the area. The energy functionals thus obtained can be written in matrix notation as:

\[
\begin{align*}
U_b &= \frac{1}{2} \{\zeta\}^T [K_e] \{\zeta\} \\
U_s &= \frac{1}{2} \{\zeta\}^T [K_g] \{\zeta\}
\end{align*}
\]

where \( \{\zeta\} \) is the nodal displacement vector

\([K_e]\) is the elastic stiffness matrix

\([K_g]\) is the geometric stiffness matrix which is separated
into three parts reflecting individually the influence of $N_x$, $N_y$ and $N_{xy}$. These matrices are derived for the first time in the present analysis.

$$[K_e] = \frac{D}{4ab}$$

Symmetric

$$K_1 = 2.88 + 156 (\zeta + \eta)/35$$
$$K_2 = -2b (-2.1\zeta - 22\eta/35 - (0.22 + \nu))$$
$$K_3 = 2a (-2.1\eta + 22\zeta/35 - (0.22 + \nu))$$
$$K_4 = -2.88 + (54\zeta - 156\eta)/35$$
$$K_5 = 2b (-0.9\zeta + 22\eta/35 + (0.22 + \nu))$$
$$K_6 = 2a (2.1\eta - 13\zeta/35 + 0.22)$$
$$K_7 = -2.88 + (54\eta - 156\zeta)/35$$
$$K_8 = 2b (2.1\zeta - 13\eta/35 + 0.22)$$
$$K_9 = 2a (-0.9\eta + 22\zeta/35 + (0.22 + \nu))$$
$$K_{10} = 2.88 - 54 (\zeta + \eta)/35$$
$$K_{11} = 2b (0.9\zeta + 13\eta/35 - 0.22)$$
\[ K_{12} = 2a \left( 0.9\eta + 13\zeta/35 - 0.22 \right) \]
\[ K_{13} = 4b^2 \left( \frac{83\zeta}{60} + \frac{31\eta}{210} + 0.36 \right) \]
\[ K_{14} = 4ab \left( \nu + 7 \left( \zeta + \eta \right)/24 \right) \]
\[ K_{15} = 4b^2 \left( \frac{37\zeta}{60} - \frac{31\eta}{210} - 0.36 \right) \]
\[ K_{16} = 4ab \left( \frac{5\zeta}{24} - 7\eta/24 \right) \]
\[ K_{17} = 4b^2 \left( -\frac{38\zeta}{60} + \frac{43\eta}{420} + 0.14 \right) \]
\[ K_{18} = 4ab \left( \frac{5\eta}{24} - \frac{7\zeta}{24} \right) \]
\[ K_{19} = 4b^2 \left( -\frac{22\zeta}{60} - \frac{43\eta}{420} - 0.14 \right) \]
\[ K_{20} = -20ab \left( \zeta + \eta \right)/24 \]
\[ K_{21} = 4a^2 \left( \frac{83\eta}{60} + \frac{31\zeta}{210} + 0.36 \right) \]
\[ K_{22} = 4a^2 \left( -\frac{38\eta}{60} + \frac{43\zeta}{420} + 0.14 \right) \]
\[ K_{23} = 4a^2 \left( \frac{37\eta}{60} - \frac{31\zeta}{210} - 0.36 \right) \]
\[ K_{24} = 4a^2 \left( -\frac{22\eta}{60} - \frac{43\zeta}{420} - 0.14 \right) \]

\[ \zeta = (a/b)^2 \quad \eta = (b/a)^2 \quad \nu = \text{Poisson's ratio} \]
\[ \frac{22464}{2360b^2} -931ab -6231b -2360b^2 -931ab \]

\[ 3633a 931ab -2758a^2 -1407a \]

\[ 749ab 3598a^2 1407a -749ab -602a^2 \]

\[ 7942a^2 -1407a 749ab -602a^2 1407a -749ab 3598a^2 \]

\[ 22464 -6231b -3633a -22464 6231b -3633a \]

\[ 2360b^2 931ab 6231b -2360b^2 931ab \]

\[ 9842a^2 3633a -931ab -2758a^2 \]

\[ 22464 -6231b 3633a \]

\[ 2360b^2 -931ab 9842a^2 \]

\[ \frac{N_x b}{a} \frac{1}{50400} \]

---

**Symmetric**
7.3 Formulation of Inertia Matrix

The kinetic energy of the plate element \( T \) is given by

\[
T = \frac{1}{2} \int \rho \left( \frac{\partial w}{\partial t} \right)^2 \, dx \, dy \tag{7.04}
\]

The energy functional is obtained by substituting the displacement functions into equation (7.04) and by carrying out the integration over the area. The energy functional thus obtained can be written in matrix notation as:

\[
T = \frac{1}{2} \{\zeta\}^T [M] \{\zeta\}
\]

where \([M]\) is the inertia matrix and is given by

\[
[M] = \frac{\rho a b p^2}{11025}
\]

\[
\begin{bmatrix}
M1 & -M2 & M3 & M4 & -M5 & -M6 & M4 & M7 & M8 & M9 & M10 & -M11 \\
M1 & -M2 & -M3 & M9 & M10 & M11 & M4 & M7 & -M8 & \\
M1 & M2 & M3 & M4 & M5 & -M6 & \\
\end{bmatrix}
\]

Symmetric

\[
\begin{bmatrix}
M12 & M13 & M5 & M14 & -M15 \\
M19 & M6 & M15 & -M20 & \\
M1 & M2 & -M3 & \\
M12 & -M13 & \\
M19 & \\
\end{bmatrix}
\]

\[
M1 = 6034 \quad M2 = -27294 \, b/16 \quad M3 = -27294 \, a/16 \\
M4 = 2106 \quad M5 = -9666 \, b/16 \quad M6 = -16386 \, a/16 \\
M7 = -16386 \, b/16 \quad M8 = -9666 \, a/16 \quad M9 = 729 \\
M10 = -5454 \, b/16 \quad M11 = -5454 \, a/16 \quad M12 = 20015 \, b^2 \\
M13 = 15337 \, a b \quad M14 = 6865 \, b^2 \quad M15 = 9303 \, a b
\]
7.4 Matrix Equations for Plates

7.4.1 Free vibration

From equation (3.51) the equation of motion for free vibration becomes

\[
[K_e] - \lambda [M] \{\zeta\} = 0 \quad (7.05)
\]

where \( \lambda = a^2 b^2 \rho p^2 / D \) is the frequency parameter of the plate and \( D = E h^3 / 12(1 - \nu^2) \) is the flexural rigidity of the plate.

7.4.2 Static stability

From equation (3.52) the equation of motion for static buckling becomes

\[
[K_e] - B_x [K_{gx}] - B_y [K_{gy}] - B_{xy} [K_{gxy}] \{\zeta\} = 0 \quad (7.06)
\]

where \( B_x = N_x^* ab/D \) is the static buckling load parameter in the x-direction and \( N_x^* \) is the in-plane static buckling load.

\( B_y = N_y^* ab/D \) is the static buckling load parameter in the y-direction and \( N_y^* \) is the in-plane static buckling load.

\( B_{xy} = N_{xy}^* ab/D \) is the static buckling load parameter of pure shear and \( N_{xy}^* \) is the in-plane shear static buckling load.

7.4.3 Dynamic stability

For a rectangular plate subjected to normal and shear in-plane forces of the forms

\[
N_x(t) = \alpha_x N_x^* + \beta_x N_x^* \cos \omega t
\]
the dynamic stability matrix equation (3.53) becomes

\[
\begin{bmatrix}
K_e & -(a_x + \beta_x)N_y^* K_{g_x} & -(a_y + \beta_y)N_x^* K_{g_y} & -(\alpha_{xy} + \beta_{xy})N_{xy}^* K_{g_{xy}} \\
(a_x + \beta_x)N_y^* K_{g_x} & (a_x^2 + \beta_x^2)N_y^* K_{g_x} & (a_y^2 + \beta_y^2)N_y^* K_{g_y} & (\alpha_{xy}^2 + \beta_{xy}^2)N_{xy}^* K_{g_{xy}} \\
(a_y + \beta_y)N_x^* K_{g_y} & (a_y^2 + \beta_y^2)N_x^* K_{g_y} & (a_y^2 + \beta_y^2)N_x^* K_{g_y} & (\alpha_{xy}^2 + \beta_{xy}^2)N_{xy}^* K_{g_{xy}} \\
(\alpha_{xy} + \beta_{xy})N_{xy}^* K_{g_{xy}} & (\alpha_{xy}^2 + \beta_{xy}^2)N_{xy}^* K_{g_{xy}} & (\alpha_{xy}^2 + \beta_{xy}^2)N_{xy}^* K_{g_{xy}} & (\alpha_{xy}^2 + \beta_{xy}^2)N_{xy}^* K_{g_{xy}}
\end{bmatrix}
\]

\[
\frac{\lambda_d}{4[M]} \{\xi\} = 0
\]  

(7.07)

where \(\lambda_d = \frac{\rho a^2 b^2 \omega^2}{D}\) is the disturbing frequency parameter of the plate and \(\alpha'\)s and \(\beta'\)s are fractions representing the static and time dependent components of the loads respectively.

7.5 Applications

Before applying the present analysis to investigate the dynamic stability of plates subjected to periodic axial loads, the convergence of the solutions based on the derived geometric stiffness matrices is tested for plates with various boundary and loading conditions. The results of these tests are shown in Tables 7.1, 7.2 and 7.3 where a comparison is made with the results obtained by using the well-known 12 degrees-of-freedom non-conforming rectangular element [13] and the non-conforming triangular element [126].

The rapid convergence of the buckling solutions is demonstrated and the comparisons of results show that the present fully conforming element gives in all cases better results than the non-conforming elements.

The present analysis is then applied to several problems
of dynamic stability of rectangular plates as shown in Figure 7.2

(i) A plate simply supported on all edges and subjected to periodic normal axial load in one direction.

The frequency parameters of a plate simply supported on all edges are shown in Table 7.4 and the corresponding mode shapes of vibration are shown in Figure 7.3.

The static buckling load parameters of a plate subjected to axial load in the y direction are given in Table 7.5 and the corresponding mode shapes of static buckling are shown in Figure 7.4.

The regions of dynamic instability of the plate subjected to periodic axial load in the y direction are shown in Figure 7.5.

(ii) A plate simply supported on all edges and subjected to periodic pure shear load.

The static buckling load parameters of a plate subjected to pure shear load are given in Table 7.6 and the corresponding mode shapes of static buckling are shown in Figure 7.6.

The regions of dynamic instability of a simply supported plate subjected to periodic pure shear load are shown in Figure 7.7

(iii) A plate simply supported on all edges and subjected to periodic axial normal loads in two directions and shear load.

The static buckling load parameters of a plate subjected
to axial loads in two directions and shear load are given in Table 7.7. The regions of dynamic instability are shown in Figure 7.9.

(iv) A plate clamped at all edges and subjected to periodic axial normal loads in two directions.

The frequency parameters of a plate clamped at all edges are given in Table 7.8 and the corresponding mode shapes of vibration are shown in Figure 7.10.

The static buckling load parameters of a plate subjected to axial loads in the x and y directions are given in Table 7.9 and the corresponding mode shapes of static buckling are shown in Figure 7.11.

The regions of dynamic instability are shown in Figure 7.12.

(v) A cantilevered plate subjected to periodic axial load in one direction.

The frequency parameters of a cantilevered plate are given in Table 7.10 and the corresponding mode shapes of vibration are shown in Figure 7.13.

The static buckling load parameters of a cantilevered plate subjected to axial load in the y direction are given in Table 7.11 and the corresponding mode shapes of static buckling are shown in Figure 7.14.

The regions of dynamic instability of a cantilevered plate subjected to periodic axial load in the y direction are shown in Figure 7.15.
7.6 General Discussions

The convergence of the buckling solutions based on the derived geometric stiffness matrices is tested for three different loading conditions acting on plates with different boundary conditions.

Table 7.1 shows the values of the buckling parameter $K (K = N^* b^2/\pi^2 D)$ for a square plate simply supported on all edges and compressed uniaxially ($N_x = N^*$). It can be seen that the present model gives a lower error and better convergence characteristics than the models used by Kapur and Hartz [13] or Anderson, Irons and Zienkiewicz [126]. Further the results presented in References [13] and [126] are lower than the exact results and approaches the true value from below. This is due to the fact that there is a slope discontinuity at the element boundaries.

Table 7.2 shows the values of the parameter $K$ for a square plate clamped at all edges and compressed biaxially ($N_x = N_y = N^*$). The present model gives a faster convergence characteristics than those given in [13] and [126]. The elements used in [13] and [126] give values of $K$ less than the true values as these elements have slope discontinuity and hence give lower bounds for strain energy.

Table 7.3 shows the values of the parameter $K$ for a rectangular plate of aspect ratio 1.25 simply supported on all edges and subjected to pure shear forces ($N_{xy} = N^*$). The results given by Kapur and Hartz [13] converge slowly approaching the true value from below. The results based on the element used by Anderson, Irons and Zienkiewicz [126] show a diverging
characteristic where the error increases as the plate is subdivided into larger number of elements. The results obtained by the present analysis show a monotonic convergence characteristic.

Comparison of results confirms that the present conforming element gives consistently better results than the non-conforming elements used by Kapur and Hartz [13] or Anderson, Irons and Zienkiewicz [126]. The present conforming element is used to investigate the dynamic stability of plates subjected to periodic axial loads. The regions of dynamic instability, as well as natural frequencies of free vibration and static buckling loads are obtained.

For a rectangular plate of aspect ratio 1.5 simply supported on all edges and subjected to periodic axial loads in one direction, the values of the natural frequency parameter $\lambda^2 (\lambda = a^2 b^2 \rho^2 \rho/D)$ are shown in Table 7.4; the values of the static buckling load parameter $B_y (B_y = N_y^{*} ab/D)$ are shown in Table 7.5 and the first seven regions of dynamic instability are shown in Figure 7.5 for $\alpha = 0.0$. It is seen that the present results are in very close agreement with the exact results of frequencies and buckling loads. The intersections of the regions in Figure 7.5 with the $\omega/\omega_1$ axis represent the free vibration modes which are marked as $f_{mn}$ where $m$ and $n$ are the numbers of half-waves in the $x$ and $y$ directions respectively. The modes associated with the regions of greater widths represent the most dominating modes of static buckling since these regions are the ones most likely to intersect the $\beta$ axis which represents the static buckling case. This can be verified from Figures 7.3 and 7.4 where the first static buckling
mode shape $b_{11}$ is similar to the first free vibration mode shape $f_{11}$, the second static buckling mode shape $b_{12}$ is similar to the third free vibration mode shape $f_{12}$, and the third and fourth static buckling mode shapes $b_{21}$ and $b_{22}$ are similar to the second and fifth free vibration modes $f_{21}$ and $f_{22}$ respectively.

When the simply supported plate is subjected to periodic pure shear load, the regions of dynamic instability degenerate into lines as shown in Figure 7.7. The reason for this is that during a positive half-cycle of the disturbing force, the plate will be deflected in the manner shown in Figure 7.8.a for the first mode, while during the negative half-cycle of the disturbing force, the plate will deflect in a manner similar to the first case but in opposite direction as shown in Figure 7.8.b and as a result of that, the plate does not become dynamically unstable.

From Figure 7.7 it can be seen that there are no buckling modes similar to the third, fifth and seventh modes of vibration, since the regions of dynamic instability associated with these modes move away from the $\beta$ axis. It is seen that the first and second static buckling mode shapes $b_{11}$ and $b_{21}$ are similar to the first and second free vibration mode shapes $f_{11}$ and $f_{21}$ respectively. Also it is expected that the third static buckling mode shape will be similar to the fourth free vibration mode shape $f_{31}$. These results can be verified from Figures 7.3 and 7.6.

For a simply supported plate subjected to periodic normal and shear in-plane loads, the regions of dynamic instability are shown in Figure 7.9. It is seen that the size of the
regions is approximately the same and hence it can be concluded that the present loading conditions cause all the modes to become dynamically unstable.

For a rectangular plate clamped at all edges and subjected to periodic axial loads in two directions, the values of the frequency parameter are given in Table 7.8 while the corresponding mode shapes are shown in Figure 7.10, and the values of the buckling load parameter are given in Table 7.9 while the corresponding mode shapes are shown in Figure 7.11. It can be seen from Tables 7.8 and 7.9 that the results obtained by the present analysis are very close to the results reported by Leissa [125] and Timoshenko [18]. Comparison of the vibration and buckling mode shapes reveals that the first and second mode shapes of static buckling are similar to the first and second mode shapes of free vibration respectively. Also the third mode shape of static buckling is similar to the fourth mode shape of free vibration and the fourth static buckling mode shape is similar to the third free vibration mode shape. Figure 7.12 shows the regions of dynamic instability of the clamped plate where it is clear that the plate will become dynamically unstable for a wide range of the parameters $\beta$ and $\omega$. Figure 7.12 also shows the relationship between the vibration and buckling mode shapes.

Figure 7.15 shows the first four regions of dynamic instability of a cantilevered rectangular plate subjected to periodic axial load in the $y$ direction. It is seen that the first region has degenerated into a vertical straight line parallel to the $\beta$ axis which means that there is no buckling mode similar to the first vibration mode. The second region
is the dominating unstable region and it can be seen that the plate will statically buckle in a mode similar to the second mode of vibration. These observations are found to be true by comparing the mode shapes shown in Figures 7.13 and 7.14 for the free vibration and static buckling of the cantilevered plate respectively.

From the above series of results, it can be concluded that the present analysis gives extremely good results for the dynamic stability of rectangular plates with various boundary and loading conditions. Furthermore it makes the relationships between the free vibration and static buckling mode shapes very clear.
TABLE 7.1 VALUES OF $K$ ($K = \frac{N^*b^2}{\pi^2D}$) FOR A SQUARE PLATE SIMPLY SUPPORTED ON ALL EDGES AND COMPRESSED UNIAXIALY

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K$</td>
<td>$%$ error</td>
<td>$K$</td>
</tr>
<tr>
<td>4 x 4</td>
<td>3.770</td>
<td>-5.75</td>
<td>3.720</td>
</tr>
<tr>
<td>6 x 6</td>
<td>3.887</td>
<td>-2.80</td>
<td>-</td>
</tr>
<tr>
<td>8 x 8</td>
<td>3.933</td>
<td>-1.68</td>
<td>3.900</td>
</tr>
<tr>
<td>10 x 10</td>
<td>3.960</td>
<td>-1.00</td>
<td>-</td>
</tr>
</tbody>
</table>

Exact value of $K = 4.000$ [18]
Known value of $K = 5.315 \ [18]$
\[ \frac{a}{b} = 1.25 \]

\[ \text{Known value of } K = 7.780 \]

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Kapur and Hartz [13]</th>
<th>Anderson, Irons and Zienkiewicz [126]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[ K ]</td>
<td>[ % error ]</td>
<td>[ K ]</td>
</tr>
<tr>
<td>4 x 4</td>
<td>6.945</td>
<td>-10.73</td>
<td>7.692</td>
</tr>
<tr>
<td>6 x 6</td>
<td>7.247</td>
<td>-6.85</td>
<td>7.563</td>
</tr>
<tr>
<td>8 x 8</td>
<td>7.450</td>
<td>-4.24</td>
<td>7.543</td>
</tr>
</tbody>
</table>

**TABLE 7.3** VALUES OF \( K (K = \frac{N^*b^2}{\pi^2D}) \) FOR A RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO PURE SHEAR.
### TABLE 7.4 FREQUENCY PARAMETERS OF A RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Present 8 x 8 mesh size</th>
<th>Exact Timoshenko [118]</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_{11}</td>
<td>21.522</td>
<td>21.384</td>
</tr>
<tr>
<td>f_{21}</td>
<td>41.836</td>
<td>41.123</td>
</tr>
<tr>
<td>f_{12}</td>
<td>66.246</td>
<td>65.797</td>
</tr>
<tr>
<td>f_{31}</td>
<td>75.866</td>
<td>74.022</td>
</tr>
<tr>
<td>f_{22}</td>
<td>87.723</td>
<td>85.536</td>
</tr>
</tbody>
</table>

### TABLE 7.5 BUCKLING LOAD PARAMETERS OF A RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO AXIAL LOAD IN Y-DIRECTION.

<table>
<thead>
<tr>
<th>Mode</th>
<th>B_{y} = N_{y} ab/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present 8 x 8 mesh size</td>
<td>Exact Timoshenko [18]</td>
</tr>
<tr>
<td>b_{11}</td>
<td>31.286</td>
</tr>
<tr>
<td>b_{12}</td>
<td>74.079</td>
</tr>
<tr>
<td>b_{21}</td>
<td>118.406</td>
</tr>
<tr>
<td>b_{22}</td>
<td>129.948</td>
</tr>
</tbody>
</table>

### TABLE 7.6 BUCKLING LOAD PARAMETERS OF A RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO PURE SHEAR.

<table>
<thead>
<tr>
<th>Mode</th>
<th>B_{xy} = N_{xy} ab/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present 8 x 8 mesh size</td>
<td>Exact Timoshenko [18]</td>
</tr>
<tr>
<td>b_{11}</td>
<td>109.006</td>
</tr>
<tr>
<td>b_{21}</td>
<td>124.000</td>
</tr>
<tr>
<td>b_{31}</td>
<td>233.479</td>
</tr>
<tr>
<td>b_{41}</td>
<td>265.948</td>
</tr>
</tbody>
</table>

a/b = 1.5
<table>
<thead>
<tr>
<th>Mode</th>
<th>( B = N^* \frac{ab}{D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present 8 x 8 mesh size</td>
</tr>
<tr>
<td>b_{11}</td>
<td>20.912</td>
</tr>
<tr>
<td>b_{21}</td>
<td>36.880</td>
</tr>
<tr>
<td>b_{12}</td>
<td>61.315</td>
</tr>
<tr>
<td>b_{31}</td>
<td>71.680</td>
</tr>
</tbody>
</table>

**TABLE 7.7**  BUCKLING LOAD PARAMETERS OF A RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO AXIAL LOADS IN X AND Y DIRECTIONS AND SHEAR LOAD.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \lambda^\frac{1}{3} = abp (\rho/D)^\frac{1}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present 8 x 8 mesh size</td>
</tr>
<tr>
<td>f_{11}</td>
<td>41.016</td>
</tr>
<tr>
<td>f_{21}</td>
<td>64.312</td>
</tr>
<tr>
<td>f_{12}</td>
<td>100.283</td>
</tr>
<tr>
<td>f_{31}</td>
<td>103.198</td>
</tr>
<tr>
<td>f_{22}</td>
<td>123.899</td>
</tr>
</tbody>
</table>

**TABLE 7.8**  FREQUENCY PARAMETERS OF A RECTANGULAR PLATE CLAMPED AT ALL EDGES

<table>
<thead>
<tr>
<th>Mode</th>
<th>( B = N^* \frac{ab}{D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present 8 x 8 mesh size</td>
</tr>
<tr>
<td>b_{11}</td>
<td>62.591</td>
</tr>
<tr>
<td>b_{21}</td>
<td>84.495</td>
</tr>
<tr>
<td>b_{31}</td>
<td>128.715</td>
</tr>
<tr>
<td>b_{12}</td>
<td>129.097</td>
</tr>
</tbody>
</table>

**TABLE 7.9**  BUCKLING LOAD PARAMETERS OF A RECTANGULAR PLATE CLAMPED AT ALL EDGES AND SUBJECTED TO AXIAL LOADS IN X AND Y DIRECTIONS
TABLE 7.10 FREQUENCY PARAMETERS OF A CANTILEVERED RECTANGULAR PLATE.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Present 6 x 6 mesh size</th>
<th>Leissa [125]</th>
</tr>
</thead>
<tbody>
<tr>
<td>f₁</td>
<td>2.293</td>
<td>2.318</td>
</tr>
<tr>
<td>f₂</td>
<td>7.821</td>
<td>7.784</td>
</tr>
<tr>
<td>f₃</td>
<td>14.059</td>
<td>14.412</td>
</tr>
<tr>
<td>f₄</td>
<td>26.343</td>
<td>26.328</td>
</tr>
<tr>
<td>f₅</td>
<td>34.595</td>
<td>35.917</td>
</tr>
</tbody>
</table>

TABLE 7.11 BUCKLING LOAD PARAMETERS OF A CANTILEVERED RECTANGULAR PLATE SUBJECTED TO AXIAL LOAD IN Y DIRECTION.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Present 6 x 6 mesh size</th>
</tr>
</thead>
<tbody>
<tr>
<td>b₁</td>
<td>3.490</td>
</tr>
<tr>
<td>b₂</td>
<td>16.814</td>
</tr>
<tr>
<td>b₃</td>
<td>38.645</td>
</tr>
<tr>
<td>b₄</td>
<td>45.047</td>
</tr>
<tr>
<td>b₅</td>
<td>62.046</td>
</tr>
</tbody>
</table>
FIG. 7.1 A PLATE ELEMENT
(i), (ii) and (iii) simply supported on all edges

\[ a/b = 1.5 \]

\[ N(t) = \alpha N^* + \beta N^* \cos \omega t \]

Clamped at all edges
Cantilevered
FIG. 7.3  FIRST FIVE MODE SHAPES OF FREE VIBRATION OF
A PLATE SIMPLY SUPPORTED ON ALL EDGES
$f_{31}$

$p = 75.866 \sqrt{D/\rho}/ab$

$\bar{f}_{22}$

$p = 87.723 \sqrt{D/\rho}/ab$

(Cont.)

FIG. 7.3
$b_{11}$  

$N_y^* = 31.286 \text{ D/ab}$

$N_y^* = 74.079 \text{ D/ab}$

$N_y^* = 118.406 \text{ D/ab}$

$N_y^* = 129.948 \text{ D/ab}$

**FIG. 7.4** FIRST FOUR MODE SHAPES OF STATIC BUCKLING OF A PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO AXIAL LOAD IN ONE DIRECTION
\[ N_y(t) = \alpha N_y^* + \beta N_y^* \cos \omega t \]

\[ N_y^* = 31.286 \frac{D}{ab} \text{ Fundamental static buckling load} \]

\[ p_1 = 21.522 \sqrt{\frac{D}{\rho ab}} \text{ Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 7.5** REGIONS OF DYNAMIC INSTABILITY OF A SIMPLY SUPPORTED PLATE Subjected TO PERIODIC AXIAL LOAD IN ONE DIRECTION
FIG. 7.6 FIRST FOUR MODE SHAPES OF STATIC BUCKLING OF A PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUBJECTED TO PURE SHEAR LOAD

\[ N_{xy}^{*} = 109.006 \text{ D/ab} \]

\[ N_{xy}^{*} = 124.000 \text{ D/ab} \]

\[ N_{xy}^{*} = 233.479 \text{ D/ab} \]

\[ N_{xy}^{*} = 265.948 \text{ D/ab} \]
\[ N_{xy}(t) = \alpha N^* + \beta N^* \cos \omega t \]

\[ N^* = 109.006 \, \text{D}/\text{ab} \quad \text{Fundamental static buckling load} \]

\[ p_1 = 21.522 \sqrt{\text{D}/\rho}/\text{ab} \quad \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 7.7** REGIONS OF DYNAMIC INSTABILITY OF A SIMPLY SUPPORTED PLATE SUBJECTED TO PERIODIC PURE SHEAR LOAD
FIG. 7.8
FIRST MODE SHAPE OF BUCKLING OF A PLATE SUBJECTED TO
PURE SHEAR LOAD
\[ N_x(t) = N_y(t) = N_{xy}(t) = \alpha N^* + \beta N^* \cos \omega t \]

\[ N^* = 20.912 \frac{D}{ab} \quad \text{Fundamental static buckling load} \]

\[ p_1 = 21.522 \sqrt{\frac{D}{\rho}} \frac{1}{ab} \quad \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.5 \]

**FIG. 7.9** REGIONS OF DYNAMIC INSTABILITY OF A SIMPLY SUPPORTED PLATE SUBJECTED TO PERIODIC AXIAL LOADS IN TWO DIRECTIONS AND PURE SHEAR LOAD
FIG. 7.10 FIRST FIVE MODE SHAPES OF FREE VIBRATION OF A PLATE CLAMPED AT ALL EDGES

$f_{11}$  
$p = 41.016 \sqrt{D/\rho/ab}$

$f_{21}$  
$p = 64.312 \sqrt{D/\rho/ab}$

$f_{12}$  
$p = 100.283 \sqrt{D/\rho/ab}$

$f_{31}$  
$p = 103.198 \sqrt{D/\rho/ab}$
\( f_{22} \quad \text{p} = 123.899 \sqrt{D/\rho/ab} \)

(Cont.)

FIG. 7.10
FIG. 7.11 FIRST FOUR MODE SHAPES OF STATIC BUCKLING OF A PLATE CLAMPED AT ALL EDGES AND SUBJECTED TO AXIAL LOADS IN TWO DIRECTIONS
\[ N_y(t) = \alpha N^* + \beta N^* \cos \omega t \]
\[ N_x(t) = \alpha N^* + \beta N^* \cos \omega t \]
\[ N^* = 62.591 \text{ D/ab} \quad \text{Fundamental static buckling load} \]
\[ p_1 = 41.016 \sqrt{D/\rho/ab} \quad \text{Fundamental natural frequency} \]
\[ \omega = \text{Disturbing frequency} \]
\[ \alpha = 0.0 \]

**FIG. 7.12** REGIONS OF DYNAMIC INSTABILITY OF A CLAMPED PLATE SUBJECTED TO PERIODIC AXIAL LOADS IN TWO DIRECTIONS
FIG. 7.13 FIRST FOUR MODE SHAPES OF FREE VIBRATION OF A CANTILEVERED PLATE

- $f_1$, $p = 2.293 \sqrt{D/\rho}/ab$
- $f_2$, $p = 7.821 \sqrt{D/\rho}/ab$
- $f_3$, $p = 14.059 \sqrt{D/\rho}/ab$
- $f_4$, $p = 26.343 \sqrt{D/\rho}/ab$
FIG. 7.14 FIRST FOUR MODE SHAPES OF STATIC BUCKLING OF A CANTILEVERED PLATE SUBJECTED TO AXIAL LOAD IN ONE DIRECTION

\[ N_y^* = 3.490 \text{ D/ab} \]

\[ N_y^* = 16.814 \text{ D/ab} \]

\[ N_y^* = 38.645 \text{ D/ab} \]

\[ N_y^* = 45.047 \text{ D/ab} \]
\[ N_y(t) = \alpha N^* + \beta N^* \cos \omega t \]

\[ N^* = 3.490 \, D/ab \quad \text{Fundamental static buckling load} \]

\[ p_1 = 2.293 \sqrt{D/\rho/ab} \quad \text{Fundamental natural frequency} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 7.15** REGIONS OF DYNAMIC INSTABILITY OF A CANTILEVERED PLATE SUBJECT TO PERIODIC AXIAL LOAD IN ONE DIRECTION
CHAPTER 8
8.1 Introduction

Stiffened plates are a structural element of practical importance in many applications like ship superstructures, bridge decks and aircraft structures. The knowledge of the dynamic characteristics of stiffened panels is essential to the prediction of fatigue failures. In engineering practice the plates are eccentrically stiffened by placing the stiffeners on one side of the plates so that the stiffness of the combined system is highest.

The free vibration of stiffened plates has been investigated by various methods of solution. Lin [152] has determined the natural frequencies and mode shapes of vibration for continuous panels over intermediate stringers which are supported on both sides by rigid structures and the edges of the panels are assumed to be simply supported. The differential equation of motion is solved for a particular case where the stiffeners are identical, equally spaced and in a single direction and the edges of the panels are all simply supported. Wah [153] has used a finite difference approach to calculate the natural frequencies of rectangular plates continuous over identical and equally spaced elastic beams which are simply supported at their ends. Numerical results obtained for a simply supported stiffened plates are compared with those obtained using the orthotropic plate approximation [154, 155].

A more complicated method referred to as the composite beam-plate method has been presented by Long [16] for
eccentrically stiffened plates simply supported along the edges normal to the direction of stiffening. Considerations of equilibrium and of displacement compatibility result in a system of homogeneous transcendental equations involving eight constants of integration for every plate segment and four constants for every stiffener. Natural frequencies and mode shapes are obtained by solving this set of equations. Long [156] has noticed that in-plane displacement is relatively small compared with other displacements and can be neglected without considerable effect on the computed natural frequencies. This simplification has been utilized by Long [156] by formulating the problem in a manner similar to the stiffness method of structural analysis which leads to a considerable saving in computing time. However, the analysis is still limited to a particular class of stiffened plates where the stiffeners must all be identical, equally spaced and in a single direction and the two sides of the plate perpendicular to the stiffeners must be simply supported. A composite beam-plate method similar to Long's method [16] has been used by Smith [157] to analyse ship deck structures.

Lin and Donaldson [158] used a transfer matrix approach to obtain frequencies of free vibration of a row of aircraft panels with supporting stringers. The Ritz method was used by Kirk [159] to determine the natural frequencies of the first symmetric and first antisymmetric modes of a simply supported rectangular plate reinforced by a single stiffener. NG and Kulkarni [160, 161] used an approximate approach based on the orthotropic plate theory [154] for computing the natural frequencies of bridge slabs.
Davis [17] used an 8-node non-conforming plate bending element [146] and an off-set beam element to study the free vibration of eccentrically stiffened plates by finite element method. The frequency values obtained were compared with the frequencies calculated using Long's method [16]. The frequencies of some modes obtained by Davis [17] do not agree with the corresponding values of Long [16] and it has been suggested by Davis [17] that this discrepancy may be due to the non-conformity of the plate element. Very recently Aksu and Ali [162] used the finite difference method to study the free vibration of eccentrically stiffened plates having a single stiffener placed along one of the centre lines.

The elastic stability of plates can be increased by increasing their thickness, but such a design will not be economical in respect to the weight of the material used. A more economical solution is obtained by keeping the thickness of the plate as small as possible and increasing the stability by introducing reinforcing ribs. Eccentrically stiffened plates are frequently employed as structural elements in current aerospace and aircraft vehicles.

The static stability of eccentrically stiffened plates was first investigated by Timoshenko [163]. Analytical solution was presented for a simply supported rectangular plate with longitudinal ribs subjected to compressive forces. Torsional rigidity of the ribs was neglected and the moment of area of the cross section of the stiffener was calculated with respect to an axis at the outer surface of the plate. Seide and Stein [19] used the Rayleigh-Ritz energy method to calculate the fundamental buckling load of uniformly compressed simply
supported rectangular plates with longitudinal stiffeners and the results were presented in graphical forms. The torsional rigidity of the stiffener was assumed to be zero and the flexural rigidity was assumed to be about an axis situated at the middle surface of the plate.

The finite element method was used by Kapur and Hartz [13] to solve the static stability problem of a square plate with one longitudinal rib. The element used was the non-conforming plate bending element which gave values of buckling loads lower than the exact values.

In the case of a large number of equal and equidistant ribs, the stiffened plate may be considered as a plate having two different flexural rigidities in the two perpendicular directions. The differential equation for the deflection surface of such a plate under uniform compression parallel to one axis has been solved by Timoshenko [18] and the smallest value of the critical stress was obtained. Rao [164] used this value of the critical stress in his optimization study of stiffened plates.

It can be seen that most of the present literature is concerned either with the free vibration analysis or with the static stability analysis. No consideration has been given to the dynamic stability problem of eccentrically stiffened plates subjected to periodic axial loads.

In this chapter, the dynamic stability analysis of eccentrically stiffened plates is presented for the first time. The plate is idealized by the fully conforming plate element developed in Chapter 7 and the stiffeners are idealized by the
beam element developed in Chapter 6. The effects of the
stiffeners depth on the regions of dynamic instability are
investigated. Also the effects of the stiffeners depth on the
natural frequencies of vibration and the static buckling loads
are examined. The results obtained for the natural frequencies
are compared with those given by Long [16] and Davis [17], and
the results obtained for the static buckling loads are compared
with those given by Timoshenko [18], Kapur and Hartz [13] and
other investigators. The rapid convergence of the present
solution is demonstrated in all cases.

8.2 Theoretical Analysis

The simply supported stiffened rectangular plate shown in
Figure 8.1 is considered to be performing a sinusoidal
vibrations of small amplitude relative to the plate thickness.
The plate is of uniform thickness \( h \) except in the small
region where the stiffener is attached. Exact theoretical
analysis of the dynamics of such a plate involves cumbersome
mathematical procedures which can not be easily incorporated
into a much needed flow-chart type design procedure. Many types
of simplifying assumptions had been made so as to obtain results
sufficiently accurate for practical design purposes. However,
the degree of accuracy depends on how closely the actual
behaviour of the system follows the assumptions made during the
theoretical analysis. It will be seen later that the assumptions
made in the present analysis have little effect on the accuracy
of the results obtained.

It is assumed that the in-plane displacement of the plate
can be neglected and that the effective moment of area of the
eccentric stiffener is calculated about an axis situated in the
upper surface of the plate, thus

\[ I_s = \frac{cd^3}{3} \]  

(8.01)

where \( c \) and \( d \) are the width and the depth of the stiffener respectively. Another alternative for calculating the effective moment of area of the eccentric stiffeners is that the bending axis is assumed to be situated in the middle surface of the plate. Therefore instead of equation (8.01), \( I_s \) is given by

\[ I_s = \frac{cd^3}{3} \left( 1 + \frac{3h}{2d} + \frac{3h^2}{4d^2} \right) \]  

(8.02)

where \( h \) is the thickness of the plate.

It is clear that (8.01) implies a lower bound while (8.02) implies an upper bound to the stiffener flexural rigidity. Results obtained using the two assumptions (8.01) and (8.02) are compared with each other and with the analytical solution.

8.3 Matrix Equations for Stiffened Plates

8.3.1 Free vibration

From equation (3.51) the equation of motion for free vibration becomes

\[
\begin{bmatrix}
K_e - p^2 [M]
\end{bmatrix}
\{\zeta\} = 0
\]  

(8.03)

where \([K_e]\) is the combined elastic stiffness matrix of the plate and the stiffeners, \([M]\) is the combined inertia matrix of the plate and the stiffeners, and \( p \) is the natural frequency of the stiffened plate structure.

8.3.2 Static stability

From equation (3.52) the equation of motion for static stability becomes
where $N_x^*$, $N_y^*$ and $N_{xy}^*$ are the in-plane normal and shear static buckling loads and $[K_{gx}]$, $[K_{gy}]$ and $[K_{gxy}]$ are the combined geometric stiffness matrices of the stiffened plate.

### 8.3.3 Dynamic stability

For a stiffened plate subjected to normal and shear in-plane loads of the forms

\[
N_x(t) = \alpha_x N_x^* + \beta_x N_x^* \cos \omega t
\]

\[
N_y(t) = \alpha_y N_y^* + \beta_x N_y^* \cos \omega t
\]

\[
N_{xy}(t) = \alpha_{xy} N_{xy}^* + \beta_{xy} N_{xy}^* \cos \omega t
\]

the dynamic stability matrix equation can be written as

\[
\begin{bmatrix}
[K_e] - \alpha_x [K_{gx}] - \alpha_y [K_{gy}] - \alpha_{xy} [K_{gxy}]
\end{bmatrix}
\begin{bmatrix}
\dot{\zeta}
\end{bmatrix} = 0
\]

where $\alpha$'s and $\beta$'s are fractions representing the static and time dependent components of the loads respectively.

### 8.4 Applications

In the present analysis, the plate is idealized by the fully conforming plate element developed in Chapter 7 and the stiffeners are idealized by the beam element developed in Chapter 6. All results reported are obtained by using $8 \times 8$ mesh size to idealize the complete stiffened plates.

The free vibration, the static stability and the dynamic stability of a rectangular plate simply supported on all
edges, stiffened by one and three eccentric stiffeners and subjected to in-plane axial loads are investigated. The stiffened plate is 24 in. x 16 in. x 0.25 in. in size. The material properties of both the plate and the stiffener are $E = 30 \times 10^6$ lb/in$^2$, $\nu = 0.3$ and $\rho = 0.283$ lb/in$^3$.

The results of free vibration of a simply supported rectangular plate stiffened by one stiffener are shown in Tables 8.1 - 8.4 and Figures 8.2 - 8.8. The static buckling results of the stiffened plate subjected to axial load in the $y$ direction are shown in Tables 8.5 - 8.7 and Figures 8.9 - 8.10. The effects of the stiffener depth on the regions of dynamic instability of the stiffened plate subjected to periodic axial load in the $y$ direction are shown in Figures 8.11 - 8.16.

The free vibration results of a rectangular plate stiffened by three stiffeners are shown in Table 8.8 and Figures 8.17 - 8.18. The buckling results are shown in Tables 8.9-10 and Figure 8.19. The effects of the three stiffeners depth on the regions of dynamic instability are shown in Figures 8.20 - 8.22.

8.4.1 Analytical solution of free vibration of stiffened plate

The differential equation for small-amplitude vibrations of the thin rectangular plate shown in Figure 8.1 is given by

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\rho h p^2}{D} w = 0 \quad (8.06)$$

where $\rho$ is the mass density of the plate material

$h$ is the thickness of the plate
D is the flexural rigidity of the plate \( Eh^3/12(1-\nu^2) \)
and \( p \) is the cyclic frequency.

The plate deflection surface is taken in the form

\[
w = W(x) \sin \frac{m\pi y}{b}
\]

(8.07)

and substituting into equation (8.06) gives

\[
\frac{\partial^4 W}{\partial x^4} - 2\left(\frac{m\pi}{b}\right)^2 \frac{\partial^2 W}{\partial x^2} + \left[ \left(\frac{m\pi}{b}\right)^4 - \frac{pAp^2}{D} \right] W = 0
\]

(8.08)

Assuming that \( W = Ae \) and substituting into (8.08) gives

\[
q^4 - 2\left(\frac{m\pi}{b}\right)^2 q^2 + \left[ \left(\frac{m\pi}{b}\right)^4 - \frac{pAp^2}{D} \right] = 0
\]

(8.09)

and therefore

\[
q^2 = \left(\frac{m\pi}{b}\right)^2 \pm \left(\frac{pAp^2}{D}\right)^{\frac{1}{2}}
\]

(8.10)

Equation (8.10) can be written as

\[
q = \pm \frac{\pi}{a} \left[ \left(\frac{ma}{b}\right)^2 \pm \left(\frac{pAp^2}{D}\cdot\frac{a^2}{\pi^4}\right)^{\frac{1}{2}} \right]^{\frac{1}{2}}
\]

(8.11)

Introducing

\[
K_1 = \frac{pAp^2}{D}
\]

\[
K_2 = \left(\frac{Kia}{\pi}\right)^4
\]

\[
K_3 = \left(\frac{ma}{b}\right)^2
\]

(8.12)

then equation (8.11) can be written as

\[
q = \pm \frac{\pi}{a} \left[ K_3 \pm K_2 \right]^{\frac{1}{2}}
\]

(8.13)

Again introducing

\[
\phi^2 = K_2 - K_3 \quad \text{and} \quad \theta^2 = K_2 + K_3
\]

(8.14)
yields

\[ q = \pm \frac{\pi}{a} \theta \]  \hspace{1cm} (8.15)

\[ q = \pm j \frac{\pi}{a} \phi \]

The general solution of equation (8.08) is then written as

\[ W = C_1 \sin \frac{\pi x}{a} + C_2 \cos \frac{\pi x}{a} + C_3 \sinh \frac{\pi \theta x}{a} + C_4 \cosh \frac{\pi \theta x}{a} \]  \hspace{1cm} (8.16)

where \( C \)'s are arbitrary constants.

For the fundamental symmetric mode, the four boundary conditions required to eliminate the constants \( C_1, C_2, C_3 \) and \( C_4 \) are

\[ \text{for } x = 0 \quad w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 \]  \hspace{1cm} (8.17)

\[ \text{for } x = a \quad \frac{\partial w}{\partial x} = 0 \]

and equating the shear force in the plate to the difference between the elastic force and inertia force of the stiffener gives for \( x = a \)

\[ D \left[ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] = \frac{1}{2} \left[ EI \frac{\partial^4 w}{\partial y^4} - \rho A_s p^2 w \right] \]  \hspace{1cm} (8.18)

where \( A_s \) is the stiffener cross-sectional area.

The right-hand side of the equation (8.18) represents the differential equation of free vibration of an independent stiffener or beam. Thus, if the resulting frequency of vibration of the stiffened plate is equal to the natural frequency of an independent stiffener, then the stiffener will exert no reaction on the plate.

Using equations (8.07), (8.16), (8.17) and (8.18) to
eliminate the constants \( C_1, C_2, C_3 \) and \( C_4 \) yields the following frequency equation for the fundamental mode

\[
\frac{\phi \left[ \phi^2 + (a/b)^2 \right] \sqrt{\phi^2 + 2(a/b)^2 \cos \pi \phi}}{\phi \cos \pi \phi \left[ \tanh \left( \sqrt{\phi^2 + 2(a/b)^2} \right) \right] - \sqrt{\phi^2 + 2(a/b)^2 \sin \pi \phi}} + \frac{\pi A_s}{4ah} \left( \phi^2 + (a/b)^2 \right)^2 = 3\pi(1-\nu^2) \left( \frac{a}{b} \right)^4 \frac{I_s}{ah^3} \tag{8.19}
\]

Equation (8.19) is solved numerically on the digital computer for the lowest value of \( \phi \) and the natural frequency \( \omega \) is calculated from

\[
\omega^2 = \frac{D(\pi/a)^4}{\rho h} \left[ \phi^2 + (a/b)^2 \right]^2 \tag{8.20}
\]

The values used are

\[
\begin{align*}
2a &= 24 \text{ in.} \quad 2b = 16 \text{ in.} \quad h = 0.25 \text{ in.} \\
A_s &= c \times d \quad c = 0.5 \text{ in.} \\
I_s &= cd^3/3
\end{align*}
\]

Another set of results is obtained for the value of \( I_s \) given by equation (8.02), namely

\[
I_s = \frac{cd^3}{3} \left( 1 + \frac{3h}{2d} + \frac{3h^2}{4d^2} \right)
\]

In both cases the effect of varying the depth of the stiffener \( d \) on the natural frequency is investigated and the results are shown in Figure 8.2.

8.5 General Discussions

Figure 8.2 shows the value of the fundamental frequency of a simply supported rectangular plate stiffened by one stiffener. Lower and upper bounds of frequencies are obtained depending on the assumptions made in calculating the flexural rigidity of the stiffener. The results obtained by the present
finite elements show very good agreement with the analytical solutions. When the depth of the stiffener is smaller than its width, it is seen that the upper bound frequency is more accurate, but as the stiffener depth increases, the lower bound frequencies become more accurate and yield values close to those reported by Long [16] and Davis [17]. In practice, the depth of the stiffener is made greater than its width so as to have a maximum flexural rigidity and therefore the assumption giving the lower bound of frequencies is more realistic since the centroid of the cross section consisting of the stiffener and the plate will be very near the surface of the plate.

The effects of the stiffener depth on the natural frequencies of free vibration are shown in Tables 8.1 - 8.4 and Figures 8.3 - 8.7. It is seen that the present finite element analysis gives more accurate results than Davis [17] analysis, even though the total number of degrees of freedom employed by Davis [17] to represent one quarter of the stiffened plate is greater than the total number of degrees of freedom employed in the present analysis to represent the complete stiffened plate structure. Figure 8.7 shows that as the depth of the stiffener d increases the mode shapes where the stiffener is deflected become associated with higher frequencies. It is seen that mode $f_{11}$ frequency becomes higher than mode $f_{21}$ frequency as the value of $d$ exceeds 1.2 in. Also mode $f_{12}$ frequency becomes higher than mode $f_{22}$ frequency as the value of $d$ exceeds 0.7 in. The first five mode shapes of vibration are shown in Figure 8.8 for $d = 0.75$ in.

The rapid convergence of the buckling solutions is demonstrated in Table 8.5 which shows the values of buckling
parameter \( K \) for a square plate with one stiffener, simply supported on all edges and compressed uniaxially. It is evident that the present finite element model gives zero error for \( 8 \times 8 \) mesh size and has a stronger convergence characteristics than the model used by Kapur and Hartz [13]. The excellent agreement between the results obtained by the present analysis and the exact results reported by Timoshenko [18] for other cases of stiffened plates is demonstrated in Table 8.6.

The effects of the stiffener depth on the static buckling load parameters of the simply supported rectangular plate are shown in Table 8.7 and Figure 8.9. As the depth of the stiffener \( d \) increases the mode shapes where the stiffener is deflected \( (b_{11} \text{ and } b_{12}) \) become associated with higher buckling parameters. This can be seen clearly from Figure 8.10 which shows the mode shapes of static buckling for \( d = 0.75 \) in.

The effects of the stiffener depth \( d \) on the regions of dynamic instability of the rectangular stiffened plate subjected to periodic axial load in the \( y \) direction are shown in Figures 8.11 - 8.15. As the value of \( d \) increases, the regions of dynamic instability are shifted away from the static buckling axis \( (\omega = 0) \) and the width of these regions is decreased thus making the stiffened plate less sensitive to periodic forces than the unstiffened plate when subjected to the same periodic force. But if the disturbing frequency of the periodic force lies in the vicinity of twice the fundamental natural frequency of the stiffened plate, then the stability characteristics will be as shown in Figure 8.16. The regions of dynamic instability exist over a larger range and result in overlapping. This may
be dangerous since it causes the stiffened plate to be dynamically unstable over a wider range of the disturbing frequency $\omega$. Figure 8.16 shows also the relationships between the buckling and vibration mode shapes for the stiffened plate with $d = 2.0$ in. It is seen that the first buckling mode shape is similar to the first vibration mode shape $f_{21}$, and it is expected that the second buckling mode shape will be similar to the third vibration mode shape $f_{22}$, the third buckling mode shape will be similar to the fourth vibration mode shape $f_{12}$, and the fourth buckling mode shape will be similar to the second vibration mode shape $f_{11}$. These relationships are all found to be true and they are verified by using Figures 8.7 and 8.9.

The accuracy of the results obtained for the natural frequencies of free vibration of the rectangular plate stiffened by three stiffeners is shown in Table 8.8 and Figure 8.17 in relation to the results reported by Long [16] and Davis [17]. It is evident that the present analysis gives more accurate results than Davis's [17]. The total number of degrees of freedom employed in the present analysis is less than one quarter of the total number of degrees of freedom employed by Davis [17]. The effects of the stiffeners depth on the natural frequencies are shown in Figure 8.18, where it is apparent that stiffening increases the natural frequencies of free vibration.

The accuracy of the result obtained for the static buckling parameter of a plate with three stiffeners is shown in Table 8.9 in relation to the result reported by Seide and Stein [19] in
a graphical form. The effects of the three stiffeners on the static buckling load parameters of the simply supported rectangular plate are shown in Table 8.10 and Figure 8.19. It is seen that considerable increase in the buckling loads can be gained by stiffening.

The effects of the stiffeners depth on the regions of dynamic instability are shown in Figures 8.20 - 8.21. As $d$ increases the stiffened plate with three stiffeners becomes less sensitive to periodic forces than the unstiffened plate and also less sensitive than the stiffened plate with one stiffener. Figure 8.22 shows that if the disturbing frequency of the periodic force is in the vicinity of twice the fundamental natural frequency of the stiffened plate then the stiffened plate becomes dynamically unstable over a wider range of the disturbing frequency $\omega$. 
\[ a = 24 \text{ in.} \quad b = 16 \text{ in} \]
\[ h = 0.25 \text{ in.} \quad c = 0.5 \text{ in.} \]
\[ I_s = \frac{cd^3}{3} \]

<table>
<thead>
<tr>
<th>Mode ( f_{11} )</th>
<th>( h )</th>
<th>( c )</th>
<th>( d )</th>
<th>( I_s )</th>
<th>( I_s /d^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td></td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h )</td>
<td>0.25</td>
<td></td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td></td>
<td></td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 8.1 COMPARISON OF MODE \( f_{11} \) FREQUENCY OF A SIMPLY SUPPORTED PLATE

<table>
<thead>
<tr>
<th>Mode ( f_{11} )</th>
<th>Stiffener depth ( d ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Analytical solution</td>
<td>135.7</td>
</tr>
<tr>
<td>Present</td>
<td>136.5</td>
</tr>
<tr>
<td>[ 8 \times 8 \text{ mesh size} ]</td>
<td></td>
</tr>
<tr>
<td>Long [16]</td>
<td></td>
</tr>
<tr>
<td>Davis [17]</td>
<td></td>
</tr>
<tr>
<td>Aksu and Ali. Fig. 9, ref. [162]</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 8.2 COMPARISON OF MODE \( f_{21} \) FREQUENCY

<table>
<thead>
<tr>
<th>Mode ( f_{21} )</th>
<th>( h )</th>
<th>( c )</th>
<th>( d )</th>
<th>( I_s )</th>
<th>( I_s /d^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td></td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h )</td>
<td>0.25</td>
<td></td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td></td>
<td></td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode ( f_{21} )</th>
<th>Stiffener depth ( d ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Present</td>
<td>265.5</td>
</tr>
<tr>
<td>Long [16]</td>
<td></td>
</tr>
<tr>
<td>Davis [17]</td>
<td></td>
</tr>
<tr>
<td>Aksu and Ali. Fig. 9, ref. [162]</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 8.3 COMPARISON OF MODE \( f_{21} \) FREQUENCY
TABLE 8.3 COMPARISON OF MODE $f_{12}$ FREQUENCY

<table>
<thead>
<tr>
<th>Stiffener depth d (in.)</th>
<th>Present</th>
<th>Long [16]</th>
<th>Davis [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>420.4</td>
<td>-466.7</td>
<td>-463.8</td>
</tr>
<tr>
<td>0.25</td>
<td>435.5</td>
<td>542.5</td>
<td>533.9</td>
</tr>
<tr>
<td>0.50</td>
<td>522.5</td>
<td>577.4</td>
<td>568.3</td>
</tr>
<tr>
<td>0.75</td>
<td>582.0</td>
<td>590.2</td>
<td>581.3</td>
</tr>
<tr>
<td>1.00</td>
<td>602.0</td>
<td>595.6</td>
<td>586.9</td>
</tr>
<tr>
<td>1.25</td>
<td>609.3</td>
<td>598.4</td>
<td>591.2</td>
</tr>
<tr>
<td>1.50</td>
<td>612.5</td>
<td>599.9</td>
<td>592.1</td>
</tr>
<tr>
<td>1.75</td>
<td>614.2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.00</td>
<td>615.1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.25</td>
<td>615.6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.50</td>
<td>616.8</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE 8.4 COMPARISON OF MODE $f_{22}$ FREQUENCY

<table>
<thead>
<tr>
<th>Stiffener depth d (in.)</th>
<th>Present</th>
<th>Long [16]</th>
<th>Davis [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>556.7</td>
<td>-549.8</td>
<td>-548.6</td>
</tr>
<tr>
<td>0.25</td>
<td>559.0</td>
<td>558.5</td>
<td>560.7</td>
</tr>
<tr>
<td>0.50</td>
<td>566.5</td>
<td>565.3</td>
<td>571.7</td>
</tr>
<tr>
<td>0.75</td>
<td>573.8</td>
<td>-574.8</td>
<td>583.9</td>
</tr>
<tr>
<td>1.00</td>
<td>579.7</td>
<td>578.1</td>
<td>579.1</td>
</tr>
<tr>
<td>1.25</td>
<td>584.1</td>
<td>580.8</td>
<td>581.1</td>
</tr>
<tr>
<td>1.50</td>
<td>587.4</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.75</td>
<td>590.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.00</td>
<td>592.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.25</td>
<td>593.6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.50</td>
<td>596.2</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE 8.5 VALUES OF PARAMETER $K$ ($K=N^*a^2/\pi^2D$) OF A STIFFENED SQUARE PLATE SIMPLY SUPPORTED ON ALL EDGES.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Kapur and Hartz [13]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>K</td>
<td>%error</td>
</tr>
<tr>
<td>4 x 4</td>
<td>9.46</td>
<td>-2.66</td>
</tr>
<tr>
<td>8 x 8</td>
<td>9.63</td>
<td>-0.93</td>
</tr>
</tbody>
</table>
TABLE 8.6 COMPARISON OF PRESENT BUCKLING RESULTS WITH EXACT RESULTS.

<table>
<thead>
<tr>
<th>Dimensions of stiffened plate</th>
<th>Present 8 x 8 mesh size</th>
<th>Exact by Timoshenko [18]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EI_s/aD = 5$</td>
<td>9.72</td>
<td>9.72</td>
</tr>
<tr>
<td>$A_s/ah = 0.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$EI_s/aD = 5$</td>
<td>11.11</td>
<td>11.1</td>
</tr>
<tr>
<td>$A_s/ah = 0.1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$EI_s/aD = 5$</td>
<td>11.94</td>
<td>12.0</td>
</tr>
<tr>
<td>$A_s/ah = 0.05$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 8.7 EFFECTS OF STIFFENER DEPTH ON THE STATIC BUCKLING LOAD PARAMETERS OF A SIMPLY SUPPORTED PLATE.
a = 24 in.  b = 16 in.  
h = 0.25 in.  c = 0.5 in.  
e = 5.625 in.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural frequency of free vibration (Hz)</th>
<th>Stiffeners depth d (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>f_{11}</td>
<td>Present</td>
<td>140.9</td>
</tr>
<tr>
<td></td>
<td>Long [16]</td>
<td>157.5</td>
</tr>
<tr>
<td></td>
<td>Davis [17]</td>
<td>156.7</td>
</tr>
<tr>
<td>f_{21}</td>
<td>Present</td>
<td>261.5</td>
</tr>
<tr>
<td></td>
<td>Long [16]</td>
<td>273.0</td>
</tr>
<tr>
<td></td>
<td>Davis [17]</td>
<td>267.8</td>
</tr>
<tr>
<td>f_{31}</td>
<td>Present</td>
<td>466.3</td>
</tr>
<tr>
<td></td>
<td>Long [16]</td>
<td>477.6</td>
</tr>
<tr>
<td></td>
<td>Davis [17]</td>
<td>462.3</td>
</tr>
<tr>
<td>f_{41}</td>
<td>Present</td>
<td>789.5</td>
</tr>
<tr>
<td></td>
<td>Long [16]</td>
<td>777.4</td>
</tr>
<tr>
<td></td>
<td>Davis [17]</td>
<td>773.8</td>
</tr>
</tbody>
</table>

**TABLE 8.8 EFFECTS OF STIFFENERS DEPTH ON THE NATURAL FREQUENCIES OF A RECTANGULAR PLATE**
\[
\frac{N^*}{EI_s/aD} = 5 \quad \frac{A_s/ah}{ab} = 0.05
\]

\[K = N^* a^2/\pi^2D\]

<table>
<thead>
<tr>
<th>Seid and Stein [19]</th>
<th>Present 8 x 8 mesh size</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>19.87</td>
</tr>
</tbody>
</table>

**TABLE 8.9 COMPARISON OF PARAMETER K OF A STIFFENED SQUARE PLATE.**

**TABLE 8.10 EFFECTS OF STIFFENERS DEPTH ON THE STATIC BUCKLING LOAD PARAMETERS OF A RECTANGULAR PLATE.**
FIG. 8.1  A STIFFENED PLATE
FIG. 8.2 COMPARISON OF MODE f_{11}, FREQUENCY OF A SIMPLY SUPPORTED RECTANGULAR PLATE STIFFENED WITH ONE STIFFENER

Mode f_{11}

Stiffener depth d (in.)

Frequency

Assumption (1) I_s = cd^2/3
Assumption (2) I_s = cd^2(1+2h^2/4d^2)

Analytical solution using assumption (1)
Analytical solution using assumption (2)
Present finite elements using assumption (1)
Present finite elements using assumption (2)

- Long [16]
- Davis [17]
FIG. 8.3 COMPARISON OF MODE $f_{11}$ FREQUENCY

Mode $f_{11}$

Frequency (Hz)

Stiffener depth $d$ (in.)

- Present
- Long [16]
- Davis [17]
- Aksu and Ali [162]
FIG. 8.4  COMPARISON OF MODE $f_{21}$ FREQUENCY
FIG. 8.5  COMPARISON OF MODE $f_{12}$ FREQUENCY
FIG. 8.6 COMPARISON OF MODE \( f_{22} \) FREQUENCY
FIG. 8.7 EFFECT OF STIFFENER DEPTH ON THE NATURAL FREQUENCIES OF A SIMPLY SUPPORTED PLATE
FIG. 8.8  MODE SHAPES OF FREE VIBRATION OF A STIFFENED PLATE (d = 0.75 in.)
(Cont.)

FIG. 8.8

\[ f_{22} \quad p = 573.8 \text{ Hz} \]

\[ f_{12} \quad p = 582.0 \text{ Hz} \]
FIG. 8.9 EFFECT OF STIFFENER DEPTH ON THE STATIC BUCKLING LOADS OF A SIMPLY SUPPORTED PLATE

a = 24 in.
b = 16 in.
h = 0.25 in.
c = 0.50 in.
FIG. 8.10  MODE SHAPES OF STATIC BUCKLING OF A STIFFENED PLATE (d = 0.75 in.)
\[ N_y(t) = \alpha N^*_y + \beta N^*_y \cos \omega t \]

\[ N^*_y = 4.755 \pi^2 D/a^2 \] Fundamental static buckling load of unstiffened plate

\[ p_1 = 136.5 \text{ Hz} \] Fundamental natural frequency of unstiffened plate

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 8.11** REGIONS OF DYNAMIC INSTABILITY OF A STIFFENED PLATE (d = 0.0 in.)
FIG. 8.12  REGIONS OF DYNAMIC INSTABILITY (d = 0.50 in.)

Stable

$\rho$

$\omega/\rho_1$

$f_{21}$

$f_{12}$

$f_{31}$

$d = 0.50$ in.
Fig. 8.13 Regions of Dynamic Instability (d = 1.0 in.)
FIG. 8.14
REGIONS OF DYNAMIC INSTABILITY (d = 1.5 in.)
FIG. 8.15 REGIONS OF DYNAMIC INSTABILITY (d = 2.0 in.)
\[ N_y(t) = \alpha N_y^* + \beta N_y^* \cos \omega t \]

\( N_y^* = 28.231 \pi^2 D/a^2 \) Fundamental static buckling load of a stiffened plate with \( d=2.0 \) in.

\( p_1 = 332.4 \) Hz Fundamental natural frequency of a stiffened plate with \( d = 2.0 \) in.

\( \omega = \) Disturbing frequency

\( \alpha = 0.0 \)

---

FIG. 8.16 REGIONS OF DYNAMIC INSTABILITY OF A STIFFENED PLATE (d = 2.0 in.)
FIG. 8.17 COMPARISON OF FREQUENCIES OF A SIMPLY-SUPPORTED PLATE WITH THREE STIFFENERS
FIG. 8.18 EFFECT OF STIFFENERS DEPTH ON THE NATURAL FREQUENCIES OF A SIMPLY SUPPORTED PLATE
FIG. 8.19  EFFECT OF STIFFENERS DEPTH ON THE STATIC BUCKLING LOADS OF A SIMPLY SUPPORTED PLATE
\[ N_y(t) = \alpha N_y^* + \beta N_y^* \cos \omega t \]

\[ N_y^* = 4.755 \pi^2 D/a^2 \quad \text{Fundamental static buckling load of unstiffened plate} \]

\[ p_1 = 136.5 \text{ Hz} \quad \text{Fundamental natural frequency of unstiffened plate} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 8.20** REGIONS OF DYNAMIC INSTABILITY OF A SIMPLY SUPPORTED PLATE SUBJECT TO PERIODIC AXIAL LOAD IN ONE DIRECTION (d = 0.5 in.)
FIG. 8.21
REGIONS OF DYNAMIC INSTABILITY (d = 1.0 in.)
\[ N_Y(t) = \alpha N_Y^* + \beta N_Y^* \cos \omega t \]

\[ N_Y^* = 35.529 \pi^2 D/a^2 \quad \text{Fundamental static buckling load of a stiffened plate with } d = 1.0 \text{ in.} \]

\[ p_1 = 373.3 \text{ Hz} \quad \text{Fundamental natural frequency of a stiffened plate with } d = 1.0 \text{ in.} \]

\[ \omega = \text{Disturbing frequency} \]

\[ \alpha = 0.0 \]

**FIG. 8.22** REGIONS OF DYNAMIC INSTABILITY OF STIFFENED PLATE (d = 1.0 in.)
CHAPTER 9

CONCLUSIONS

The finite element model developed in Chapter 4 for the dynamic stability analysis of Timoshenko beams is the only one, presented so far, in which both geometric and natural boundary conditions can be correctly applied. The convergence of the results is very rapid and a small number of elements is required to obtain good results.

The approach of considering the modes of vibration of the Timoshenko beam as one of coupled vibration, the coupling being between the various independent modes of vibration, has helped immensely to understand the dynamic behaviour of the Timoshenko beams. It is now possible to estimate the frequency of vibration of the Timoshenko beam from the knowledge of the independent modes of vibration. This approach of synthesis gives a good qualitative appreciation of the complex mode shape of vibration and hence it is possible to estimate approximately the relative magnitude of the various components of the mode shape.

It is seen that when the boundary conditions, as in the hinged-hinged beam, are such that both the rotation of the cross-section and the shear deformation are not inhibited, the beam can execute a pure shear mode of vibration independent of other modes and as a result of coupling gives two distinct spectra of frequencies of vibration. For the fixed-free, free-free and fixed-fixed end conditions, the frequencies of vibration of Timoshenko beam do not separate into two spectra, but give only one family of continuous curves. The frequencies
of vibration can be considered to be the result of coupling between Bernoulli-Euler beam and simple shear beam for low values of rotary inertia parameter, and for higher values of rotary inertia parameter between the modes when rotary inertia is ignored and the pure and simple shear modes.

The existing concept of the second spectrum of frequencies in the Timoshenko beam must be abandoned and the frequencies must be considered as the result of coupling between different independent modes of vibration.

The convergence of the static buckling results is very rapid. The excellent agreement of these results with the analytical results demonstrates the good accuracy of the developed finite element model in representing the Timoshenko beam.

Due to the action of shear forces in the Timoshenko beams, the static buckling loads are diminished. The fixed-fixed beam is more sensitive to the shear deformation variation than the hinged-hinged or the free-free beams, while the fixed-free beam is least sensitive to this variation.

As the rotary inertia parameter increases, the regions of dynamic instability of Timoshenko beams are shifted towards the static buckling axis and the width of these regions is increased, thus making the beams more sensitive to periodic forces. The regions of dynamic instability associated with the pure shear and the second spectrum modes of the hinged-hinged Timoshenko beam are degenerated into vertical lines and hence have no influence on the stability characteristics of the beam.
The effects of elastic foundation on the natural frequencies of vibration and static buckling loads of Timoshenko beams with various end conditions are investigated in Chapter 5. As the elastic foundation constant increases, the natural frequencies and static buckling loads are increased. The results obtained for Bernoulli-Euler beam, which is a special case of the present analysis, show excellent agreement with those obtained by analytical solution. It is shown that the results reported by previous investigators for the natural frequencies and static buckling loads of a fixed-fixed Bernoulli-Euler beam and the corresponding mode shapes are incorrect. Also the uncertain claim, made recently by another investigator that for a free-free Bernoulli-Euler beam subjected to constant thrust, there exists only one zero eigenvalue representing one rigid body mode of vibration, is confirmed.

As the stiffness of elastic foundation increases, the regions of dynamic instability are shifted away from the static buckling axis and the width of these regions is decreased thus making the beams less sensitive to periodic forces. It is seen that one of the two regions, associated with the rigid body modes of free-free beam, degenerates into a vertical line, while the second region, associated with the rotational rigid body mode, has a considerable width and thus represents a region of dynamic instability of high influence.

The convergence of the vibration and buckling results of the plane frames considered in Chapter 6 is very rapid. The comparison of these results shows excellent agreement with exact results and it is concluded that the developed finite
element model gives extremely good results for the dynamic stability analysis of plane frames.

The problem of dynamic stability which has until now been restricted to flexural behaviour in a single plane, is extended to three dimensional space frames. The analysis includes the combined torsional-flexural instability. This method of analysis provides for the first time a means of obtaining the regions of dynamic instability of space frames subjected to periodic forces. The present approach is successfully applied to the analysis of an offshore oil rig structure subjected to periodic sea wave forces. Agreement between the computed and measured frequencies of the experimental model is found to be very good. It is less than 4.5 per cent averaged for the first ten modes.

Comparison of buckling results of rectangular plates confirms that the fully conforming plate element, developed in Chapter 7, gives consistently better results than the existing non-conforming plate elements.

The modes associated with the wide regions of dynamic instability represent the dominating modes of static buckling. For plates subjected to periodic pure shear loads, the regions of dynamic instability degenerate into lines of zero width and it is concluded that the plate does not become dynamically unstable.

The idealization of the eccentrically stiffened plates in Chapter 8 by the fully conforming plate element and the beam element gives extremely good results.

The results obtained for the natural frequencies of free
vibration of eccentrically stiffened plates are more accurate than those obtained by other investigators. Further, the total number of degrees-of-freedom employed in the present analysis is less than one quarter of the total number of degrees of freedom used in the other analysis.

The accuracy of the results obtained for the static buckling loads of eccentrically stiffened plates is very good. It is seen that the present finite element model gives zero error for 8 x 8 mesh size and has a stronger convergence characteristics than the model used by other investigators.

As the depth of the stiffener increases, the mode shapes where the stiffener is deflected, become associated with higher frequencies of vibration and higher static buckling loads.

The regions of dynamic instability of eccentrically stiffened plates subjected to axial periodic loads are obtained. As the depth of the stiffener increases, the regions are shifted away from the static buckling axis and the width of these regions is decreased thus making the stiffened plate less sensitive to periodic forces than the unstiffened plate when subjected to the same periodic force. But as the disturbing frequency reaches the vicinity of twice the fundamental frequency of the stiffened plate, the regions of dynamic instability exist over a wide range and as a result, an overlapping of the regions occurs. This may be dangerous since it causes the stiffened plate to be dynamically unstable over a wider range of the disturbing frequency.
SCOPE OF FUTURE WORK

Since the values obtained by the present analysis for the frequencies of vibration are extremely close to the experimental values, it is suggested here that a method of detecting failures in space frame structures by measuring the variation in the natural frequencies of free vibration, can be introduced. The proposed method depends on measuring the frequencies and comparing them with the frequencies of the intact structure. Figure A.1 shows the variation of frequencies of the first ten modes of vibration of the oil rig model for various member failures. It is seen that the pattern of variation has a different characteristics for each member failure and by measuring the frequencies a reasonable guess can be made about the location of the structural failure.

![Figure A.1: Variation of Frequencies of the Oil Rig Model for Various Member Failures](image-url)
REFERENCES


3. BALIAEV, N.M. 1924 Engineering Constructions and Structural Mechanics, 149-167. Stability of prismatic rods subjected to variable longitudinal forces.


9. MELOSH, R.J. 1963 AIAA Journal 1, 1631-1637. Basis for
derivation of matrices for the direct
stiffness method.

Comment on "Basis for derivation of matrices
for the direct stiffness method".

11. ZIENKIEWICZ, O.C. and CHEUNG, Y.K. 1964 Proceedings of
the Institution of Civil Engineers 28, 471-488.
The finite element method for analysis of
elastic isotropic and orthotropic slabs.

12. DAWE, D.J. 1965 Journal of Mechanical Engineering Science
7, 28-32. A finite element approach to plate
vibration problems.

13. KAPUR, K.K. and HARTZ, B.J. 1966 Journal of the
Engineering Mechanics Division, ASCE 92, EM2,
177-195. Stability of plates using finite
element method.

and Physics 39, 353-368. Smooth surface
interpolation.

Application of the smooth surface interpolation
to the finite element analysis.

Vibration of eccentrically stiffened plates.

17. DAVIS, R. 1972 Ph.D. Thesis, University of Nottingham,
England. Advances in beam finite elements and
applications to stiffened plates.

Compressive buckling of simply supported plates with longitudinal stiffeners.


23. TIMOSHENKO, S.P. 1921 Philosophical Magazine 41, 744-746.
On the correction for shear of the differential equation for transverse vibrations of prismatic bars.

24. PRESCOTT, J. 1942 Philosophical Magazine 33, 703-754.
Elastic waves and vibrations of thin rods.


27. SUTHERLAND, J. and GOODMAN, L. 1951 Department of Civil Engineering Report, University of Illinois, Urbana. Vibration of prismatic bars including rotary inertia and shear corrections.


30. ANDERSON, R. 1953 Journal of Applied Mechanics, ASME 20, 504-510. Flexural vibrations in uniform beams according to the Timoshenko theory.


60. NEMAT-NASSER, S. 1967 Journal of Applied Mechanics, ASME, 484-485. Instability of a cantilever under a follower force according to Timoshenko beam theory.


68. FARADAY, M. 1831 *Philosophical Transactions of the Royal Society* 121, 299-318. On a peculiar class of acoustical figures; and on certain forms assumed by a group of particles upon vibrating elastic surfaces.
   Uber Erregung Stehender Wellen eines 
   fadenformigen Korpers.

70. LORD RAYLEIGH 1883 Philosophical Magazine 16, 50-53. On 
   the crispations of fluid resting upon a 
   vibrating support.

71. UTIDA, I. and SEZAWA, K. 1940 Tokyo Imperial University, 
   Aeronautical Research Institute Report 15. 
   Dynamical stability of a column under periodic 
   longitudinal forces.

72. LUBKIN, S. and STOKER, J.J. 1943 Quarterly Journal of 
   Applied Mathematics 1, 215-236. Stability 
   of columns and strings under periodically 
   varying forces.

73. BEILIN, E.A. and DZHANELIDZE, G.U. 1952 ASTIA No. AD- 
    264148. Survey of works on dynamic stability 
    of elastic systems.

74. EVAN-IWANOWSKI, R.M. 1965 Applied Mechanics Reviews 18, 
    699-702. On the parametric response of 
    structures.

    Pergamon Press.

76. LEIPHOLZ, H.H.E. 1972 Stability. University of Waterloo, 
    Canada.

77. BARNEY, S.Z.H. and JAEGGER, L.G. 1971 Journal of Sound 
    and Vibration 15, 75-91. A method of 
    determining the regions of instability of a 
    column by a numerical method approach.

78. STEVENS, K.K. 1972 Journal of Applied Mechanics, ASME, 
    1161-1162. The use of Fourier transforms in 
    parametric excitation problems.


95. TORBY, B.J. 1975 Journal of Applied Mechanics, ASME, 738-739. Deflection results from moving loads on a beam that rests upon an elastic foundation reacting in compression only.


97. CHONAN, S. 1975 International Journal of Mechanical Sciences 17, 573-581. The elastically supported Timoshenko beam subjected to an axial force and a moving load.


115. GREGORY, M.S. 1967 Elastic Instability, E and F.N. Spon Ltd.


120. LURIE, H. 1951 Journal of the Aeronautical Sciences 18, 139-140. Vibrations of rectangular plates.


