High Precision Relative Motion Modelling

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Summary

There is an increasing interest in missions employing groups of satellites flying in formations and constellations. Such missions require highly autonomous and accurate orbit determination and control systems.

The Disaster Monitoring Constellation (DMC) is an international consortium, consisting currently of four remote-sensing satellites equally spaced around a sun-synchronous circular low Earth orbit. We have developed a simple yet flexible and accurate control algorithm for the 3D orbit acquisition of the constellation, inserting the satellite in their designed orbits with respect to each other. While we did not carry out an optimisation in the strictest sense, real world limitations and concerns override the need for a such a scheme. We report our experience in detail, demonstrating the successful in-orbit results of the orbit acquisition, where the flexibility of the algorithm proved invaluable to deal with unforeseen operational issues.

Another important aspect of relative dynamics is the relative orbit propagation. Relative navigation systems require high precision relative dynamics models to alleviate the need for relying on highly accurate relative navigation sensors. The existing literature relies on analytic relative orbit propagation schemes that become extremely complicated even for a simple geopotential model employing Earth oblateness effects, not least because of the choice of rotating local coordinate frame they work in.

We first present novel and simple analytic solutions which conserve relevant quantities related to the Keplerian motions and we discuss in detail the choice of initial conditions to improve the order of the approximations involved. Comparing to exact Keplerian models, our results show accuracies much greater than anticipated. Unlike previous work in this area, we describe the relative motion in an inertial frame, enabling the effects of perturbations on the relative motion to be incorporated in a straight-forward manner.

Finally we extended this methodology to set up a symplectic relative orbit propagator which can handle an arbitrary number of zonal and tesseral geopotential terms and can be extended to accommodate the effects of atmospheric drag. We exploited the separability of the solution, for much higher computational efficiency. The method is designed to conserve the constants of the motion, resulting in better long term accuracy.

The results show that sub-metre accuracy is possible over five days of propagation with a $36 \times 36$ geopotential model, even for large eccentricities. Furthermore, the relative propagation scheme is significantly faster than differencing two absolute orbit propagations.

Key words: Constellation Control, Formation Flying, Orbit Propagation, Orbit Estimation, Relative Motion.
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I am forever indebted to my friends, my parents and my brother in Turkey, who have always been extremely supportive throughout all these years. And without the people that have come and gone through my life here in Britain, it would not be the memorable and instructive experience that it was.


*Samuel Beckett*
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<tr>
<td>CDGPS</td>
<td>Carrier-phase Differential Global Positioning System</td>
</tr>
<tr>
<td>COTS</td>
<td>Commercial-off-the-shelf</td>
</tr>
<tr>
<td>CW</td>
<td>Clohessy-Wiltshire</td>
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<tr>
<td>DMC</td>
<td>Disaster Monitoring Constellation</td>
</tr>
<tr>
<td>ECEF</td>
<td>Earth-Centred, Earth-Fixed Co-ordinate System</td>
</tr>
<tr>
<td>ECI</td>
<td>Earth-Centred, Equatorial, Inertial Co-ordinate system</td>
</tr>
<tr>
<td>GEO</td>
<td>Geosynchronous Orbit</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
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<tr>
<td>LEO</td>
<td>Low Earth Orbit</td>
</tr>
<tr>
<td>LTAN</td>
<td>Local Time (or Longitude) of the Ascending Node</td>
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<tr>
<td>WGS84</td>
<td>World Geodetic System of 1984</td>
</tr>
<tr>
<td>MEMS</td>
<td>Micro-Electro-Mechanical Systems</td>
</tr>
<tr>
<td>MEO</td>
<td>Medium Earth Orbit</td>
</tr>
<tr>
<td>MJD</td>
<td>Modified Julian Date</td>
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<tr>
<td>RK</td>
<td>Runge-Kutta</td>
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<tr>
<td>RKF</td>
<td>Runge-Kutta-Fehlberg</td>
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<tr>
<td>SAR</td>
<td>Synthetic Aperture Radar</td>
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<tr>
<td>SSTL</td>
<td>Surrey Satellite Technologies Limited</td>
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<tr>
<td>TDRSS</td>
<td>Tracking and Data Relay Satellite System</td>
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Notation

\( a \)  semimajor axis
\( e \)  eccentricity
\( p \)  the parameter
\( \mu \)  gravitational constant
\( n \)  mean motion
\( L \)  angular momentum vector
\( L \)  longitude of the ascending node
\( \theta \)  true anomaly
\( r \)  radius
\( \Omega \)  right ascension of the ascending node
\( I \)  inclination
\( \alpha \)  epicycle phase
\( \lambda \)  argument of latitude
\( M \)  mean anomaly
\( J_i \)  Zonal harmonic coefficient in description of planetary gravitational field
\( J_{nm} \)  Tesseral harmonic coefficient in description of planetary gravitational field
\( \psi_{nm} \)  Tesseral harmonic phase in description of planetary gravitational field
\( P_i \)  Legendre polynomial
\( P_n^m \)  Associated Legendre polynomial
\( C_D \)  drag coefficient
\( t_w \)  wait time between firings
\( t_{tot} \)  total time allocated for phasing
\( \Delta V \)  firing magnitude
\( h \)  integration stepsize
\( H \)  Hamiltonian of a satellite
\( U \)  full potential function
\( \delta X \)  difference of two \( X \) quantities
\( H_R \)  relative Hamiltonian
\( L_R \)  relative angular momentum
\( \delta r \)  relative position vector
\( \delta v \)  relative velocity vector
Chapter 1

Introduction

1.1 Scope of Research

Investigating the dynamics of spacecraft with respect to each other has been a very active area of research within the last decade as a result of the increasing number of proposed constellation and formation flying missions. Numerous simplified analytical models have been proposed to model the relative dynamics of such missions. These analytical methods are extremely useful in the design, analysis and control of relative motion, however, they are poor representations of the motion when high precision position solutions are required due to the complexity of the effects which must be modelled.

The scope of this research is to present the design and analysis of a simple, practical algorithm for constellation establishment and demonstrate its application on a real world problem. A numerical model is also presented that retains the geometric properties of the relative motion. It employs a full geopotential model with zonal and tesseral harmonics, enabling high accuracy long-term predictions of position and velocity.

1.2 Motivation

While constellations of satellites have been in widespread use for many years, there has been an enormous increase in interest in formation flying over the past decade. Perhaps
the most important reason behind this is the trend towards smaller and more capable satellites, with increasing availability of computing power, miniaturised electro-mechanical components (such as Micro-Electro-Mechanical Systems (MEMS)) and commercial-off-the-shelf (COTS) components. These technologies enable the use of varied multiple satellite architectures, comprising of small platforms which may ultimately replace single, large, monolithic satellites. The main advantages of these multiple satellite missions are:

- Existing mission capabilities are dramatically increased. Perhaps the best example is the synthetic aperture radar (SAR) applications where the resolution is proportional to the aperture size. Rather than relying on a large monolithic antenna on a single platform, it is possible to keep the satellites on a tightly controlled formation with a regulated baseline becoming the effective antenna size.

- New mission concepts can be introduced, such as remote spacecraft inspection using small, autonomous satellites; fully autonomous rendezvous and docking employing stereo imaging; stellar interferometry with reasonable resolution will also become viable.

- Increased redundancy, reliability and survivability is introduced to the system, as well as a graceful degradation of performance in the case of a failure. This compares favourably to the vulnerability of a single, large satellite to single point failures or other catastrophic mission failures, such as collision with space debris and single event upsets.

- Inexpensive and better performance alternatives can be proposed. The launch and production costs for a single, larger spacecraft will be arguably much higher than an equivalent distributed system. Common parts used and the similar architecture in a distributed system can drive the design and production costs even lower.

With the increasing onboard computational capabilities, more autonomous satellites are becoming more and more feasible. This will enable much less reliance on the costly ground-station based operations. Multiple satellite missions, particularly those involving formation flying, are characterised by frequent application of control to maintain a certain configuration and prevent possible collisions. Hence it is practically impossible to continuously
1.2. Motivation

monitor the satellites and issue control commands from the groundstation. Evidently, a certain degree of autonomous operation capabilities are extremely important if these missions are to be realised. The satellites should be able to estimate precisely and continuously where they are with respect to the other elements of the formation and in the presence of any errors, compute an appropriate control command to correct them or execute emergency manoeuvres if collision risks are present. In addition to this on-the-fly relative orbit estimation problem which requires high accuracies in relatively short durations (perhaps up to several days), long-term mission planning and design calls for high-accuracy models of the motion that should be usable for durations up to months and years. The propagation methods that take into account certain physical properties of the motion (such as conservation of energy) are more desirable than those that do not even if they provide better accuracy in the short run.

Seen from this perspective, perhaps it is not surprising that developing relative navigation algorithms is crucial for the success of these proposed mission concepts. The motivation for developing such a model for the relative motion of satellites is threefold. The first aim is to “simulate” the motion as realistically as possible, so that it is possible to predict the states and relative behaviour of satellites over time. The second is “analysis” of this motion, so that different properties of this motion can be identified and better exploited. The third is to investigate the “control” of the satellites, where the equations of motion can be manipulated to alter or keep the relative motion within a certain configuration.

In practical terms, better ability to simulate the motion is a prerequisite for better orbit propagators and estimators, enabling better mission planning and better utilisation of resources. Analysis of the motion enables missions to be designed, which will carry out certain tasks relying on a very specific configuration of the satellites with respect to each other. Furthermore, it becomes possible to set up initial conditions that harness natural dynamics to retain this configuration. Finally, control enables the satellites to retain the desired configuration in the most time or fuel efficient manner possible but without losing sight of real world limitations.

As a practical example on the last point, the Disaster Monitoring Constellation (DMC) is a five-satellite, Low Earth Orbit (LEO) imaging constellation, built by Surrey Satellite
Technologies Ltd and operated by five different countries [21, 22, 104]. Following the launch of the first satellite, three further satellites were launched, leading to a requirement to distribute the satellites to their relative stations, located on a single orbital plane with 90 degree phase separations. Even though a number of publications exist on complicated time or fuel optimised schemes or limited in-plane solutions to achieve this, practical, flexible and robust methods take precedence over strict optimisation schemes for real missions. There are also other technical issues (such as launch injection errors, thruster inaccuracies and balanced propellant consumption) as well as management of an international consortium and coordination of operations in four countries based in two continents. Seen from this perspective, there is a gap between the proposed purely theoretical control algorithms and the practical requirements of real world operations.

While it may be argued that relative motion can be modelled simply as the motion of two individual satellites, there are a number of compelling reasons to define it as motion of one satellite with respect to the other.

One practical reason is that the performance criteria as well as mission requirements are defined in relative terms. For example, in an imaging constellation, the angular separation required between two satellites is usually specified; this is more important if the satellites are to be used for stereo-imaging with one satellite closely following the other. In NASA’s first formation flying experiment, EO-1 was to follow LandSat keeping a distance of 450km [31, 38]. Similarly, the GRACE mission relies on how the unforced relative motion of the two satellites change in order to measure the Earth’s gravity field [105]. For SAR missions, the formation has to maintain a certain shape with very high precision [66, 71, 92]. Therefore, for multiple satellite missions, working with relative dynamics not only makes mission design and analysis easier and more intuitive, in some cases it is critical for mission success.

Another reason is computational efficiency. In formation flying scenarios with baselines from metres to kilometres, the two satellites are on very similar orbits and they experience very similar forces. Rather than calculating these similar forces twice, it is possible to calculate them once and extend the solution to the two satellites. Such a scheme could bring about significant advantages in terms of computation time with minimal losses in
1.2. Motivation

accuracy. This argument holds to a great extent for constellations as well; even if the satellites are not in close proximity, they will be on very similar orbits.

Another important point is the computational accuracy in relative orbit estimation and propagation. The relative position is the difference of two large quantities (thousands of kilometres or more) and is a small quantity (from kilometres to metres). This poses a badly conditioned numerical problem, particularly for scenarios with small separations, where there is a risk that the "roundoff" and truncation errors in the computation could compromise the accuracy of the smaller relative quantities. This risk is even greater in the case of highly accurate relative positioning navigation schemes. The Carrier-Phase Differential Global Positioning System (CDGPS) is reported to potentially provide ranging accuracy on the order of millimetres and is a very promising technology for formation flying [53, 57]. However, mixing such small quantities with the magnitudes of quantities involved in absolute motion has the potential to deteriorate the accuracy of the relative motion. It has to be emphasised that this problem will be much more pronounced for onboard applications where the numerical accuracy in floating point arithmetics is limited to single rather than double precision i.e., 32 bits rather than 64 (or 4 bytes rather than 8).

Just like a single satellite receiving GPS measurements and combining it with a dynamical model to compute a smoothed orbit estimate, these high precision relative measurements are used to estimate the relative orbits. The dynamical model in these estimators should therefore be defined via a relative dynamics model. It is obvious that, the better the dynamics model, the better the estimation accuracy will be. On the other hand, it is possible to view the problem from a different angle; for a given level of estimation accuracy, a better dynamical model will enable us to rely less on measurement data. As mentioned above, the current trend in the satellite design involves miniaturisation of the platforms, therefore power remains at a premium. For the absolute orbit estimation requirements of the Disaster Monitoring Constellation (DMC), the onboard GPS receivers are turned on for several minutes each orbit. By contrast, for the relative navigation requirements of the TechSat 21 formation flying mission, Inalhan et al [53] mention the need for a constant stream of high precision CDGPS data at 1Hz. Considering that the DMC platform has an orbit average of 30W [23], a saving of several watts can be had by simply turning the GPS off for long periods and using better dynamic models to compensate. It must be
emphasised that, in the close proximity formation flying missions, unlike single satellite missions or constellations, the satellites should have continuous access to accurate relative orbit information to execute any planned or emergency manoeuvres. Using a steady stream of high-precision relative positioning measurements is an extremely expensive means of achieving this target.

This is also a strong argument in favour of high precision relative orbit propagators. As will be shown in the literature review chapter, virtually all the relative motion literature relies on analytical definitions of relative dynamics. However, the analytical models become extremely complicated even with a very simple geopotential model. Furthermore, these formulations do not take into account the constants of the motion, which results in secular errors. To remedy these problems, numerical propagation schemes that conserve the constants of the motion are to be investigated. Such a numerical scheme is also crucial for long-term mission planning where the shape and configuration of the formation need to be predicted as accurately as possible for as long as possible. This enables mission planners to estimate how much propellant will be consumed throughout the mission life to maintain the predetermined configuration and develop control algorithms.

When discussing the accuracy in computing relative motion, it should be borne in mind that even small errors are crucial in examining close proximity flight. A residual error in relative velocity as small as millimetres per second (which is two orders of magnitude smaller than the absolute velocity of a LEO satellite) results in a positional error of tens of metres within an orbit and hundreds of metres within a day. This may easily be comparable to formation baseline and constitutes a collision risk.

1.3 Aims and Objectives

1.3.1 Aims

In view of the summary of the literature survey presented in Chapter 2, the aims of this study have been determined as:

1. Analysis and execution of a practical method for constellation acquisition.
1.4. Structure of Thesis

2. To develop a high-precision, high-efficiency relative orbit propagator that conserves geometric properties of the relative motion.

1.3.2 Objectives

The above aims translate to the following objectives for this research:

1. Extension and in depth analysis of Kormos' [63] preliminary analysis.

2. Successful completion of the constellation acquisition experiment for Surrey's Disaster Monitoring Constellation.

3. Development and comparison of relative orbit propagators with a simple geopotential model.

4. Extension of the dynamics model to include a full geopotential model.

1.4 Structure of Thesis

This chapter is intended to provide a general overview of the issues addressed in this thesis. Chapter 2 presents a survey of the state-of-the-art in certain aspects of the satellite constellations and formation flying, in particular constellation orbit initialisation as well as modelling absolute and relative motion in orbit. In light of the existing literature, we have identified certain areas to focus our research as well as our methodology. These decisions with the rationales behind them are also outlined in Chapter 2. Chapter 3 provides background information on the fundamentals of modelling the motion of a satellite. It introduces the forces acting on a satellite in orbit and how these forces can be modelled to set up a high-precision orbit propagator. These concepts are then extended to the modelling of the relative motion.

Chapter 4 presents the method employed in the orbit initialisation of the DMC. We first present a simplified version of the Epicycle model of the satellite motion and use this to describe the relative motion between the satellites of the constellation. This description of the relative motion is then harnessed to put together a control algorithm as presented
Chapter 1. Introduction

by Kornos [63], but with a better modelling of the motion and a more in-depth analysis. Finally, we report in detail the real world experience and challenges faced in successfully applying this algorithm to the orbit initialisation of the DMC.

Chapter 5 addresses the issue of high-precision numerical modelling of the relative orbit. We describe the relative motion via a 'relative Hamiltonian', defined with an extended phase space. A symplectic numerical scheme to propagate these equations of motion is then developed, which includes an arbitrary number of zonal and tesseral harmonics. An analytical method to model the Keplerian relative motion, based on the 'universal formulation' is also presented. We discuss Hamiltonian splitting methods to improve the efficiency of the propagator. Extensive tests are conducted to demonstrate the high accuracy and stability of the propagator.

Chapter 6 aims to develop a novel analytical relative motion model for two-body motion. This method not only highlights the importance of the constants of the motion (energy and angular momentum) in relative dynamics, it also uses them to derive a novel set of equations of relative motion using these constants. Integrating these equations, we obtain very simple expressions that describe the relative motion. They show the effects of relative energy explicitly as well as clearly separating periodic and secular parts of the motion. Such a formulation enables us to identify and largely correct a potential source of relative positioning error, which also exists in the other models proposed by researchers. We then integrate this model into the symplectic scheme and show that a surprising level of accuracy is possible for several day or longer term scenarios.

Finally, Chapter 7 outlines the conclusions that can be drawn from the work presented in this thesis and discusses the goals achieved with respect to the original research aims. It goes on to summarise contributions made to the state of the art and provides a number of possible areas of future work to further the achievements of this thesis.

1.5 Novelty of Research

1. In-depth analysis of an analytical model for constellation acquisition and maintenance has been carried out. Demonstration of the practicality, flexibility and accuracy of
such a solution in a real world constellation acquisition application has never been performed before.

2. A novel, linearised analytical solution to two-body relative motion is provided. The novelty is that it conserves the geometric constants of relative motion.

3. A computationally efficient numerical relative orbit propagator is developed that can potentially accommodate an arbitrary number of geopotential harmonics and other perturbations as well as conserve the geometric constants of relative motion. The propagator is scalable to meet possible computational limitations of the application, simple in definition and provides high precision long-term relative position solutions. No numerical relative orbit propagation scheme exists in the literature as far as the author is aware, with the singular exception of an unpublished method by Mikkola, based on his paper [79]. Utilising the concept of symplecticness within a numerical relative orbit propagation scheme, given the advantages of this concept, is also a significant contribution.
Chapter 2

Literature Review

This chapter presents an overview of the literature in the areas of interest for this research. Different classes of existing constellations and proposed formation flying missions are introduced. This is followed by a summary of the key contributions to the relative dynamics modelling, with regards to the complexity of the geopotential model they employ. Potential uses as well as shortcomings of the analytical models have also been highlighted. As an application of the control aspect of multiple satellite dynamics, theoretical and practical approaches constellation acquisition problem are then presented. In light of the absence of relative numerical propagators, we investigate absolute propagation schemes which could have extended to model relative motion dynamics. Finally a brief discussion of the existing methods is presented and the approach in this is justified.

2.1 Formation Flying and Constellation Missions

While being a focus of research within the last decade, formation flying mission launches only began in recent years, but they are set to increase. Xiang and Jørgensen [117] very recently published a survey of planned and existing formation flying missions, listing as many as 15 missions from the year 2000 to year 2015 and beyond.

Among the few missions launched, EO-1 [31, 38] is NASA’s first formation flying experiment, where LEO Earth observation satellite EO-1 flew 1 minute (450km) behind LandSat
in a leader-follower formation, maintaining the alongtrack separation to within 2 seconds. The GRACE mission [105] comprises twin satellites launched in March 2002 to a high inclination LEO, to map Earth's gravity field. The satellites stay within about 220km of each other in a simple leader-follower formation, while measuring the precise distance between them via microwave. The absolute and relative navigation data from the satellites is still handled on the ground. For both of these missions, the operations were handled from the groundstation, with limited autonomous operation by the satellites.

ESSAIM [18] satellites were developed jointly by EADS Astrium and the French Space Agency CNES and were launched in December 2004. It is a technology demonstration mission, comprising a system of four micro-satellites for analysis of the electro-magnetic environment of the Earth's surface. However, as they are military satellites, there is very little information published regarding the formation architecture or the technologies involved.

The above missions can be regarded as initial steps in formation flying, the baselines involved are on the order of several hundred kilometres, with the acceptable errors also on the order of kilometres. On the other hand, many other proposed formation flying missions require much shorter baselines and much higher accuracy in relative navigation. One of the most promising future uses of formation flying is Synthetic Aperture Radar (SAR) interferometry where the satellites in a kilometre-level baseline formation keep their relative positions ideally with an accuracy of one tenth of radar wavelength, corresponding to a few centimetres or less [15] (c.f [9]).

A recent example of such missions was the Air Force Research Laboratory (AFRL) proposed TechSat 21 [71], which was a three satellite SAR and relative navigation technology demonstration mission. Featuring centimetre level relative positioning accuracy, it would have been a testbed for several important relative navigation technologies, but the mission was cancelled in early 2003, as the technical issues on the project were "far more challenging than originally thought" [65]. This illustrates the challenge that close proximity precision formation flying presents for researchers.

Nevertheless, there are other planned missions involving the concept of interferometry. CNES' Interferometric Cartwheel [92] (due for launch in 2006) is a SAR mission with
the formation acting as an interferometer. NASA and ESA are working jointly on The Laser Interferometer Space Antenna (LISA) technology demonstrator LISA Pathfinder [66] (also known as SMART-2) due for launch in 2007, which is another interferometry mission but at one of the Sun-Earth Lagrange points. All of these interferometry missions require extremely fine precision relative navigation. LISA Pathfinder is the technology demonstration for the LISA mission, in which it is hoped that the relative motion of two spacecraft located 5 million kilometres apart will be measured with an accuracy of 10 picometres.

In summary, while there are existing formation flying missions with very large separations (or baselines), high precision, close proximity formation flying has yet to be proven in space. However, the number and the nature of the planned missions demonstrate the requirement for practical and accurate relative navigation solutions.

In contrast to formation flying, constellations are a relatively mature technology. Constellations for communication, satellite navigation and remote sensing have been in orbit for some decades.

Most constellations are on high inclination circular orbits, with the satellites distributed along the orbital plane with some angular separation. There are sometimes a number of such orbital planes to achieve the desired coverage frequency, depending on the mission.

For example, Iridium communication constellation [55] comprises 66 satellites (plus 6 on-orbit backup satellites) on 6 orbital planes, at 780km altitude and 86.4 degree inclination circular orbits. The orbit and the number of orbital planes enable the constellation to have near-global coverage of the Earth at all times. Navigation constellations such as Global Positioning System [58], Glonass [97] and the proposed European Galileo constellation [3] are also examples of this multiple orbital plane configuration, at Medium Earth Orbit (MEO) altitudes.

Optical remote sensing constellations are almost always on sun-synchronous low Earth orbits due to imaging requirements. Surrey Satellite Technologies' Disaster Monitoring Constellation [21, 98, 104] is a recently launched constellation of five satellites at 685km altitude, distributed along an orbital plane. A similar commercial remote sensing constellation currently in production is Rapideye [109, 110], comprising five satellites, equally spaced
Chapter 2. Literature Review

on a single orbital plane. The constellation will be on a 620km altitude sun-synchronous orbit.

Another remote sensing constellation, the NASA 'AM/PM constellation' comprises two sets of satellites - the morning and afternoon 'trains' - with four satellites in the former and five in the latter [17]. The constellation is designed to be on a circular, sun-synchronous and 16-day repeating orbit at 705km altitude.

The Italian Skymed-Cosmo [13] is a 'dual use' constellation with both optical and SAR payloads. It comprises four satellites evenly spaced in a circular 620km altitude sun-synchronous orbit. The constellation is currently under construction.

While for most applications circular orbits provide the best option, there are other useful orbit types for constellations. Perhaps the most well known of them is the Molniya orbit, where the satellites are placed on a 63.4 degree, 0.7 eccentricity and 12-hour period orbit [112]. The satellites (usually with telecommunications payloads) are active around the apogee where they slow down and so spend larger time on the target. They are preferable over geostationary orbits for high latitudes as they offer better visibility.

ESA SWARM mission [25] is a constellation of three satellites to investigate the geomagnetic field of the Earth and its temporal evolution. The three satellites are in three different polar orbits between 400 and 550 km altitude. The planned launch date is 2009. While SWARM constellation is on polar circular orbits, Stern [99, 100] points out that such orbits are useful to investigate the low altitude polar regions only. To investigate the higher and active part of the magnetosphere as well as to facilitate data download, he argues that high eccentricity constellations are much more beneficial, with the apogees reaching 20 to 25 times the Earth radius.

In summary, most constellations are on single or multiple plane circular orbit configurations. Furthermore each satellite is at a prescribed slot with respect to another. Therefore, 'relative navigation' becomes a crucial concern for the constellations; the satellites should maintain a certain station with respect to the others for the success of the mission.
2.2 Modelling Relative Motion

In recognition of the importance of relative navigation, there has been a heightened interest in this area over the last decade. The accumulated literature can be divided into three groups, with respect to the complexity of the dynamic model employed: linearised Keplerian force model only, linearised $J_2$ (primary Earth oblateness term) effects included and other higher order effects included.

2.2.1 Keplerian Force Models

Perhaps the most well-known work in the field of relative motion is the 'Hill’s equations' [49], originally developed in 1878 to describe the motion of the Moon. In 1960, Clohessy and Wiltshire [20] applied these equations for satellite rendezvous problems, which found widespread use in real world operations. The approach is simply to take the difference of the accelerations of two satellites in a spherically symmetric force field and linearise this relative acceleration around a circular reference orbit (see Section 3.5.1). This linearised, relative acceleration is then projected onto a local rotating Cartesian frame centred on the reference satellite (usually called the Hill’s frame). This can be visualised as the acceleration of the chaser satellite as seen from the reference satellite. Since the reference orbit parameters are constant for a circular orbit, the relative acceleration becomes a constant coefficient ordinary differential equation, which can be solved easily.

The resulting equations of motion are simple to analyse and are well understood. However, they do have their limitations; the circular reference orbit assumption and the linearised dynamics are not entirely realistic. They also do not take into account the effect of the oblateness of the Earth or other perturbations such as drag. While for short term, close-proximity applications (like space shuttle rendezvous) these errors may not pose a significant problem, when mission duration is on the order of days, the accumulated errors quickly render these equations useless.

Interestingly, the first attempt at linearised relative motion modelling for eccentric orbits was made by Lawden [67] (c.f. [11]) 6 years before Clohessy and Wiltshire ‘rediscovered’ Hill’s equations for use with near-zero eccentricity formation and rendezvous missions.
His approach is to directly solve the linearised equations of motion for the general case of eccentric orbits, rather than a circular one like Clohessy-Wiltshire equations. In the 1960's, Tschauer and Hempel [108] independently formulated similar solutions to the rendezvous problem for eccentric orbits. These approaches still use Hill's frame but they employ true anomaly rather than time as the independent variable. As the orbit is not a circle but an ellipse, this frame no longer rotates with a constant angular rate. In the late 1980s, Carter and Humi [16], and recently Inalhan et al [54], published papers on how to utilise these equations within modern control architectures. However, they did not extend the basic equations as described by Lawden.

Only fairly recently, when formation flying missions became more popular was there renewed interest in modelling relative eccentricity effects and solving the elliptic rendezvous problem. Broucke [11] solved the linearised relative motion equations using time as the independent variable. Rather than starting with the relative accelerations linearised around an eccentric orbit as Lawden (and Carter and Humi [16]) did, he formulates the problem differently and derives a time-explicit state transition matrix via partial derivatives of the reference orbit coordinates with respect to its orbital elements. The end result is defined in the usual rotating local frame and reduces to Clohessy-Wiltshire equations when the eccentricity is set to zero.

Note that neither Carter and Humi's [16], nor Broucke’s [11] papers report any results to assess their accuracy. In addition, all of these methods are rather complicated and they do not reveal much about the nature of the relative motion.

Melton [73] attempted to provide time-explicit solutions for the relative motion. His method is similar to the Hill's equations approach but, rather than assuming a perfectly circular reference orbit, he includes $e$ and $e^2$ terms in the relative acceleration. While his solution could potentially be instrumental in demonstrating the effects of eccentricity, the state transition matrix is very complicated and impractical to implement. It is also difficult to tell the accuracy of the algorithm from his results.

In 2003, Karlgaard and Lutze [59] published a paper that included not only the first order accelerations like its predecessors, but second order terms as well. They report the existence of a secular cross-track term that precesses the orbit, even though it is actually an artefact
of the model they use. They then employ a "method of multiple scales, which assumes
that the solution will be a function of several timescales ([84, 60] c.f. [59]), each of which
is independent of the others" to get rid of this secular term. Perhaps not surprisingly for a
second-order method, their results are extremely long and unwieldy. There are also some
residual secular terms they cannot eliminate from the solution. This means that angular
momentum and/or energy vary with time in their solution, which violates the dynamics
of the two-body problem. While they claim two orders of magnitude better accuracy than
the linear model, it is difficult to assess this, as they provide results for a single period
only.

Also in 2003 Melton [74] published a comparison of four relative motion models for elliptic
orbits. These methods are Lawden's, Broucke's, his own as well as an earlier version of
Karlgaard and Lutze's work. His experiments confirmed that, as expected, for eccentric
orbits the Clohessy-Wiltshire yielded poor results and the errors in relative positioning
increased as the eccentricity became larger. These errors manifest themselves as alongetrack
drift. He also seems to suggest that Lawden and Broucke's methods are good for high
eccentricity cases whereas below 0.2 eccentricity, his method yields slightly better results.

At the same time as Karlgaard and Lutze, Vaddi et al [111] published their work which takes
into account nonlinearity effects and eccentricity, using them to set up bounded formations.
Rather than just taking into account a linearised model of the Keplerian geopotential, they
use quadratic terms as well. For eccentricity, they utilise Melton's eccentricity expansion
method and extend it to include $e^3$ terms. However, their method cannot fully eliminate
the significant residual alongetrack drift. Furthermore, the inherently cumbersome nature
of Melton's solution is only increased by the inclusion of non-linear terms.

In a very recent publication, Schaub [94] approached the problem from a different angle
and defined the relative geometry through orbital element differences, even though he then
reconverted them into the rotating Hill's frame. He presented results with eccentricities
only up to 0.13. The results are not very clear but it appears that, compared to the fully
nonlinear Keplerian solution, the errors are about 20 to 40m for a 10-20km baseline LEO
formation after only 8 orbits or half a day. His expressions are simple enough, but they
are not time-explicit for the most general case. Perhaps more importantly, the question
remains as to how this method can be extended, as it is not clear how these relative orbital elements can be measured and how they can be described in the presence of higher order geopotentials.

Another very recent and interesting paper by Gurfil and Kasdin [39] formulates the co-orbital relative motion problem via a Hamiltonian approach, originally for use with Hill’s restricted three-body problem with applications to planets and satellites. They identify that such a formulation facilitates the modelling of higher order terms and orbital perturbations. Rather than a linearisation approach, they take into account three terms in the expansion of the geopotential. They then derive the relative motion equations via this ‘relative Hamiltonian’ defined in Hill’s frame. However, their canonical coordinates and the subsequent coordinate transformation make the solutions rather unwieldy and difficult to interpret. Furthermore, their results are limited to co-orbital circular cases only.

The problem common to Karlgaard-Lutze, Vaddi et al and Gurfil-Kasdin is that, even though the Keplerian relative motion is modelled to high accuracy via the inclusion of second-order terms, these effects are of the same order of magnitude as the first order differential $J_2$ effects. Therefore, their inclusion will lead to inconsistency and errors unless first order $J_2$ effects are calculated as well.

Another problem common to many papers in the field, including the $J_2$ and higher order models, is that most present the results for extremely short durations, ranging from a single orbit to a single day. This makes assessing and comparing the accuracies extremely difficult.

### 2.2.2 Linearised $J_2$ Models

In recent years, the impetus of the relative dynamics field has been towards incorporating better geopotential models, chiefly the effect of $J_2$. As Kormos and Palmer [63] note, small satellite propulsion systems are capable of applying control thrusts on the order of $1 \text{cm/s}$, which is comparable to the differential $J_2$ effects [35], highlighting the importance of second order terms. The modelling accuracy should therefore be able to accommodate this level of accuracy to enable satellite control. The formation-keeping controller should also take
2.2. Modelling Relative Motion

into account the effect of $J_2$, so as to work in harmony with and not against the natural dynamics, causing unnecessary propellant consumption.

Interestingly, the first publication to compare Keplerian modelling against more realistic gravitational models was as late as year 2000, by Alfriend et al [2]. They quantified these modelling errors for a sample LEO 1km radius circular formation at a very small eccentricity (0.005) against a fully nonlinear model incorporating $J_2$ and $J_4$. They demonstrate that, in less than a day, Hill's equations' results are off by hundreds of metres and his geometric method incorporating small eccentricities is in error by tens of metres. Obviously, such a rapid growth of errors limits the usefulness of even the best two-body model to a very short duration.

Schaub and Alfriend [95] recognised that the secular effects of $J_2$ are much more important than the periodic ones to describe the motion of the formation in the long run. Therefore they developed a method to take into account these secular changes in mean orbital elements and used it to define initial conditions such that the formation would not drift apart due to $J_2$ effects. While this gives a good starting point, practical issues like determining the relative orbital elements still pose a problem. Following on from their work, Koon et al [116] developed a method for locating a family of orbit configurations which enabled the formation to stay together without applying any control, in effect 'synchronising' the secular $J_2$ effects for the two satellites. While the results are very promising, they are only valid for very specific applications and it is inevitable that higher order zonal terms will force the formation apart.

Recently, another major contribution came from Schweighart and Sedwick [96] to accommodate the effects of $J_2$. By taking the time average of the effect of the $J_2$ potential, they derived a new set of constant-coefficient and linearised differential equations of motion, which is rather similar to CW equations in form. While the results they present claim high accuracy compared to nonlinear $J_2$ results, their method is rather complicated by the numerous corrections they apply to the reference orbit. As the linearisation takes place around the reference orbit, it is crucial to correct for the errors they incur with their definition of the reference orbit. Finally, like the CW equations they extend, they are limited to near-circular cases. Roberts and Roberts [91], presented the extended version
of Schweighart and Sedwick's work, keeping $J_2$ terms time varying rather than averaged. While accuracy is improved, they report that their model "captures relative motion the most accurately, but only for given specific initial conditions."

Also recently, Gim and Alfriend [26, 36] described a method to calculate a state transition matrix that includes the first order $J_2$ effects. They use equinoctial orbital elements to avoid singularities in circular orbit cases and describe the motion in curvilinear local coordinates, rather than the usual Cartesian. The results they present (which they compare against a $J_5$ inclusive nonlinear geopotential model) are an improvement over the previous work.

Mikkola et al [79] had proposed a time regularised absolute orbit propagator based on the logarithm of the Hamiltonian. This 'logarithmic Hamiltonian' has all the properties associated with a Hamiltonian. In an unpublished work, he then extended this formulation into relative motion via the exact difference of the equations of motion for two satellites. His method is a first attempt to develop a numerical relative orbit propagator, where the Keplerian relative motion is propagated numerically and the $J_2$ perturbations are evaluated by converting the relative coordinates into the absolute ones. As the time regularisation is a function of the energy, the two satellites at slightly different energies should be propagated at slightly different timesteps. However, this is not practically possible, therefore there is a small time mismatch between the two satellites, which causes small but growing errors in positioning.

### 2.2.3 Higher Order Models

While there had been numerous attempts to incorporate the effects of $J_2$ into the analytical modelling of relative motion, which resulted in significantly increased accuracy, the higher order zonal and tesseral harmonics as well as differential drag, lunisolar attraction and solar radiation pressure cause periodic and secular effects on the relative motion. As the baselines increase or the drag profiles between the two spacecraft differ considerably, these higher order effects get more pronounced. In 2001 Sabol et al [93] showed how these effects distort sample formations initialised using Hill's equations. They also calculated the propellant costs over the course of a year to maintain the formations, which can be as high as tens of metres per second, depending on the formation type.
2.2. Modelling Relative Motion

However, considering that incorporating $J_2$ in a simple and efficient manner has proved extremely difficult, it is perhaps not surprising that there are not many publications that actually include higher order geopotentials in their modelling.

Recently, Kormos [64, 107] utilised the analytical orbit model developed by Hashida and Palmer [44, 47, 48] to incorporate the $J_2^2$ as well as $e^2$ effects in the relative motion. As previously mentioned, these effects are of the same order of magnitude for small eccentricity cases and a significant part of literature seems to overlook this, accounting for only the eccentricity. Kormos defines a guidance circle around which the formation moves and which retains the secular effects of the $J_2$ terms (or any number of zonal harmonics for same semimajor axis and same inclination cases). He reports high accuracy compared to a $36 \times 36$ geopotential model numerical propagator. It is also interesting to note that he reports a resonance case on certain orbit configurations due to $15^{th}$ order tesseral terms, which cause an alongtrack oscillation of 20-30m. This is an interesting example illustrating the importance of modelling higher order geopotentials for the relative motion dynamics.

Wiesel [114] made a major contribution where he defines a near-circular periodic reference orbit around which the relative motion is defined. This periodicity constraint enables him to apply Floquet's solution, which reduces the problem to a much simpler linear system with constant coefficients. His resulting relative motion equations are not entirely clear but he reports incorporating 14 zonal harmonic terms into the relative motion dynamics. While this approach is shown to yield very good accuracy in the short run, the numerical search for a periodic orbit, complexity of the method and the limitations on the eccentricity are restrictive to some extent.

In their recent publication, Bordner and Wiesel [10] extended the work of Wiesel [114] to a relative orbit estimator setting. They illustrate that, compared to orbit estimation algorithms utilising CW equations, their significantly better relative motion dynamics enable them to reduce the frequency of measurements dramatically. At 200 measurements/day, their results are comparable to the 86400 measurements/day utilising an Extended Kalman Filter [53] and 1440 measurements/day via an Unscented Kalman Filter [81].

Also very recently, Halsall [42] proposed another analytical method based on the epicycle equations developed at Surrey Space Centre by Palmer et al [47, 48]. His method includes
Chapter 2. Literature Review

$J_2$, $J_3$ and secular $J_2^2$ terms, which is an improvement over the $J_2$ only methods presented in the previous section. While the method is limited to small eccentricity cases, it can be used as a design tool to set up formations of required geometries. O'Donnell [85] extended this approach to describe the relative motion for resonant orbits, providing insights into harnessing the resonance effects for formation flying.

While some researchers focussed on the higher order geopotentials, others have incorporated drag into their analytical models. Humi and Carter [51, 52] added first a linear and later a quadratic drag term into the CW equations. While the resulting expressions are simple enough, they make the assumption of equal drag profiles for all the satellites in the formation. Recognising the limitations of the CW equations, they emphasise that the main use of their equations are for rendezvous scenarios. However, given such short mission profiles, these equations could only be useful for scenarios where there is a significant differential drag between the two satellites. The effect of $J_2$ and higher order geopotentials would dominate the results for long-term scenarios and the question still remains as to how such effects can be accommodated.

Recognising this problem, Mishne [80] recently extended the methodology laid out in Schaub and Alfriend's [95] work to include the effect of differential drag and added an optimality condition to calculate the firings required for formationkeeping as well as staying on the $J_2$ invariant orbit. The use of osculating orbital elements enables him to describe the relative motion easily and derive a control algorithm. However, as with Schaub's method, how these relative orbital elements will be determined in a relative navigation setting is an open issue.

2.3 Constellation Orbit Initialisation

The problem of how the satellites in a constellation should acquire their designated orbital slots with respect to each other is in fact a subset of the trajectory optimisation field. However, the existence of more than one satellite make the problem significantly more complicated than just an orbital transfer for a single satellite; collision risks, balanced propellant consumption through the constellation, groundstation availability as well as the relative positioning of the elements of the constellation should be taken into account.
The existing literature can be divided into two groups, those that deal with the in-plane orbit acquisition and those that provide solutions for full 3D orbit acquisition. The former puts the satellite on the right phase with no in-plane drift with respect to the other satellites in the constellation. The latter, in addition to the in-plane phasing, matches the inclination and RAAN of the orbit to the target orbit parameters as well.

Among the first group, Hall et al. [40, 41] described a minimum time solution using Keplerian dynamics. Their method assumes constant thrust magnitude, but varying the thrust angle the satellite accelerates and decelerates to reach the designated orbital phase. Such an approach is more suited to electric propulsion systems. Minimum-time solutions might be desirable for some missions, but with the phasing taking place within two orbital periods, the propellant cost will be high. This solution also potentially poses some challenges for the attitude control system.

By contrast, Nagarajan et al. [83] took a different route and examined three different strategies for in-plane orbit acquisition for longer duration (on the order of days to weeks) scenarios. The first is a continuous thrust case, where the control force is calculated via feedback law. The second is an equal impulse strategy to augment the semimajor axis to achieve a certain phase drift rate and then correct this small semimajor axis difference with a second firing. They solve for the number of firings and the required $\Delta V$, given the phasing duration. The third is a Fuzzy Logic based algorithm for a constrained path scenario, to steer clear of other satellites or space debris. The phasing strategies are particularly simple to implement and the equal impulse strategy is very similar to what is investigated in this research. Similarly, Aorpimai et al. [4, 5] offered a solution to the phase acquisition problem using Hashida and Palmer’s [44, 47] epicycle description of the motion. While their method of augmenting the semimajor axis to induce relative phase drift is very similar to that of Nagarajan’s equal impulse technique [83], their solution takes into account Earth oblateness effects as well as differential drag.

Other researchers tackled the constellation establishment problem in three dimensions, providing full station acquisition solutions. Tebbani et al. [106] offered a solution that also takes into account the RAAN and argument of perigee secular drifts due to $J_2$. They describe three sets of manoeuvres to match the target orbital elements. However, they
then proceed to utilise several layers of optimisation iteratively to minimise the propulsion cost and find the best times to fire, which complicates the solution to a large extent.

Geffroy [34] offered a solution to the constellation establishment problem using continuous thrust, supplied by electric propulsion. Like Hall [40, 41] and Tebbani [106], she derives a control algorithm to continuously adjust the thrust orientation, under the constraints such as sunlight and maximum allowable pitch and yaw rotation. She also discusses possible time and propellant tradeoffs.

Another interesting 3D approach formulates the orbit acquisition as the well-known Lambert's problem. Lambert's problem stipulates that, given initial and target position vectors and the transfer time, it is possible solve for a transfer orbit analytically using a two-body gravitational potential assumption [112]. Prado [89] provided solutions for the case with two impulses and minimum propellant consumption, for a fixed time constraint. While for such a short duration Keplerian dynamics without drag or other perturbations is sufficiently accurate, the associated propellant costs will be prohibitive for constellations comprising smaller satellites.

Kormos [63] presented a detailed software structure as well as basic equations for the 3D orbit acquisition for SSTL's Disaster Monitoring Constellation (DMC). His in-plane phasing strategy, like many others, relies on altering the semimajor axis difference between the satellites to drift them to their respective stations. For the out-of-plane motion, however, he presented a simple method that harnesses the natural dynamics. Via an inclination difference between two satellites, he shows that it is possible to control the rate of the orbital plane drift due to Earth oblateness. While his method is very promising, he did not include the effect of drag or provide a detailed analysis of these equations.

While there are quite a few publications dealing with the theoretical aspects of the constellation orbit design, acquisition and control, there are not many dealing with the engineering and operational aspects of the phasing problem. These include collision avoidance, fault tolerance, sunlight and power requirements, groundstation visibility requirements, attitude determination and control system constraints and other operational priorities such as imaging. Apart from Kormos [63], one exception is the AM/PM constellation, which is well documented in the literature.
2.4. Numerical Orbit Propagation

Filici and Suarez [29, 30] reported in detail their experience of inserting SAC-C satellite into the morning 'train' of the NASA 'AM/PM constellation'. Demarest et al. [24] reported the mission design for the insertion of Aqua satellite into the constellation. A more detailed look into their experience will be presented in Section 4.6.1. But it must be emphasised that they highlight the numerous real world challenges of carrying out theoretically simple in-plane phase insertion of satellites, into an international constellation with a decentralised architecture. Such challenges usually go unreported in papers limiting themselves to the theoretical aspects.

2.4 Numerical Orbit Propagation

The motion of a satellite under the effect of the spherical geopotential can be described by Keplerian dynamics fully analytically. However, there are numerous other smaller forces that affect the motion of the satellite, such as atmospheric drag, the non-spherical mass distribution of the primary body, tides, lunisolar attraction and solar pressure. While these perturbations are usually at least an order of magnitude smaller than two-body forces, for accurate long term prediction of the motion of the satellite, they should be taken into account. In many cases, while it is possible to calculate these forces, it is practically impossible to integrate these accelerations analytically. Therefore, as Vallado [112] points out, the most accurate way to analyse perturbations is numerically. These methods are called special perturbation techniques.

The relative motion propagation literature uses analytical approaches overwhelmingly, which results in rather complicated solutions, even without the addition of higher order geopotentials or other perturbations. An interesting example, on the other hand, to show how a numerical scheme might work is by Encke [6, 112]. His scheme has originally been conceived for the absolute propagation of a single satellite, where the perturbations are small with respect to the two-body motion. He defines an osculating Keplerian orbit and a perturbed orbit. Rather than integrating all the forces on the satellite, he integrates just the difference between the perturbed acceleration and Keplerian acceleration. This enables to work with smaller magnitudes and retain the precision. In this sense, his approach lays the foundations for numerical relative orbit propagation as well.
However, it was not until Cowell [112] that numerical techniques have started seeing widespread use in astrodynamics. He simply defines all the accelerations from the perturbing forces as an additional acceleration term to the two-body acceleration. This came to be known as 'Cowell's formulation'. ‘Cowell's method’, on the other hand, integrates this equation of motion via a finite differences method [112].

While Cowell's method has seen widespread use, other techniques have been described by many researchers. Perhaps one of the most well-known methods is the Runge-Kutta (RK) family of numerical integrators [112]. To calculate the next integration step, RK methods calculate the forces at several intermediate steps and take a weighted average. As they use the single-state value and several intermediary values around it, it is said to be a family of 'single-step' methods. They are extremely simple to set up and do not require a sequence of back values.

The most computationally expensive step of any numerical orbit propagator is the calculation of forces. Therefore, minimisation of force calculations (via lower order methods or increasing the integration stepsize) with minimal losses in accuracy is the primary goal in any numerical propagation problem.

In this context, for elliptic orbits, a constant stepsize will constitute a poor utilisation of computational resources. The satellite velocity changes considerably around the orbit; a small stepsize is required to capture the motion around the periapsis, whereas large stepsizes will be sufficient around the apoapsis. Therefore, Adaptive Runge-Kutta methods (also known as Runge-Kutta-Fehlberg (RKF) methods [27, 28]) have been utilised for better computation efficiency, where the stepsize is varied with respect to the satellite velocity.

Another family of methods is called 'multi-step' techniques, the most well-known of which is Adams-Bashforth-Moulton method [112]. They are called predictor-corrector methods, as they predict a forward step using previous steps and then recalculate this forward step as a refined estimate, using the predicted state. While they offer better efficiency and accuracy over single-step methods, they require a series of back values for use in the algorithm. These back values are generally calculated via RK methods. It is therefore significantly more difficult to use variable stepsize in multi-step techniques.

An alternative approach has been presented by Bulirsch and Stoer [12, 102] that uses fewer
introduction steps than predictor-corrector methods. They assume that the differential
equation (in our case for satellite acceleration) varies smoothly. The basic principle is
that the end state of the satellite is a function of the integration stepsize that is smooth
and, in theory, that can be described analytically. Computing predictions (of rational or
polynomial functions) varying with this stepsize and extrapolating it to infinitely small
stepizes it is, in theory, possible to reach exact results. In reality, it is limited by the
round-off errors in the computation.

Since the publication of the Wisdom-Holman mapping method [115] (c.f. [76]) in 1991,
symplectic methods have gained increasing popularity in Solar System studies. Symplectic
integration methods take into account geometric properties of the problem, so that the
energy and, for an axisymmetric geopotential, angular momentum of the satellite is con­
served [90]. This has the immediate consequence that, unlike other numerical integration
methods, the shape of the orbit does not get distorted in time and the accuracy can be
maintained for longer duration propagations. The associated error can be described with
another Hamiltonian that is also conserved, therefore the error is bounded [77]. In other
words, a symplectic integrator solves a slightly perturbed Hamiltonian problem “exactly”
[69]. If the qualitative solution behaviour of the given problem is “stable” under small
perturbations of the Hamiltonian, then, roughly speaking, a symplectic method will repro­
duce this qualitative solution behaviour (for example periodic solution in Kepler problem).
Leimkuhler and Reich [69] also emphasise that while there is in fact a drift in energy, “it
remains exponentially small over exponentially long time intervals”.

While these symplectic methods have been used extensively in N-body simulations [43, 90],
their use in the satellite orbit propagation field has not been very widespread. The satellite
orbit propagation problem differs from the planetary studies in that the oblateness potential
is usually the most important perturbation and it is strong near the Earth [76]. There is
also atmospheric drag to be considered, which is a non-conservative force.

Mikkola et al [76] proposed a time-regularisation scheme, which they later improved via
the use of ‘Logarithmic Hamiltonian’ [79]. Rather than the sum of kinetic and potential
energy, the Logarithmic Hamiltonian is found by taking the the natural logarithm of the
Hamiltonian; this form retains all the properties of the Hamiltonian. Time is no longer the
independent variable but it is a coordinate like the position and velocity. Its advantage is that, for the two-body problem, the method is exact in defining the satellite orbit and the error is actually in time. They also show how higher order geopotentials and drag can be incorporated into the method.

Recently, Mikkola et al [78] also proposed a composite symplectic scheme, where two-body motion is solved via exact analytical equations of motion and the effect of higher order geopotentials can be added as momentum jumps. They demonstrate that this scheme is not only very accurate, but can also be used within an orbit estimator to run onboard a satellite.

While there are numerous publications detailing many variations on the numerical propagators, there are some that compare and contrast different methods. Palmer et al [88] compared four different first and second order predictor-corrector and Bulirsch-Stoer methods. Two of these methods were regularised, where the time is no longer the independent variable but another state like position and with its corresponding momenta. This effectively converts the propagator into an adaptive timestep scheme similar in principle to RKF methods. Their results confirm that, for eccentric orbits, time regularisation leads to a doubling of computational speed, with respect to fixed timestep schemes and that a predictor-corrector method is faster than a Bulirsch-Stoer scheme. They also report that second order schemes are twice as fast as first order schemes for a given level of accuracy. Palacios [87] published a comparison between his symplectic method based on a fixed-stepsize Runge-Kutta scheme, a standard RK and an Adams-Bashforth predictor-corrector. He conducted tests on LEO and high inclination Molniya cases. His results show that the symplectic scheme provides equal or better efficiency, whereas the Adams-Bashforth method requires smaller stepsizes than the other two. Long term accuracy of the symplectic method is also better. Calvo et al [14] compared three variable stepsize geometric integrators for the 2D Kepler problem. They concluded that the symplectic Variable Stepsize method yields higher accuracy than Implicit and Explicit Adaptive Verlet methods.
2.5 Conclusions

In Section 2.3 we have presented an overview of the existing methods for the constellation establishment problem. While there are a number of methods that present minimum-time or minimum-propellant solutions, the mathematics of the optimisation make it a practical necessity to limit the calculations to two-body dynamics. Even though it is possible to execute the constellation acquisition at a very short time with a very large propellant cost, it is much more appealing to find practical, longer duration, fuel-inexpensive solutions for small satellite applications with limited propellant budgets. For such long durations (on the order of weeks to months) the effects of drag and the major Earth oblateness term $J_2$ will become significant and invalidate any two-body optimisation scheme. Unfortunately, the methods that take into account these perturbations tend to limit themselves to in-plane solutions (such as [4, 40, 41, 83]); any inclination difference between the satellites of the constellation will induce differential drift of the orbital planes due to Earth oblateness. Therefore it is crucial to develop solutions that enable full 3D orbit acquisition.

On the other hand, the engineering issues regarding the orbit acquisition problem have rarely been mentioned in the literature. Therefore, there seems to be a gap between complicated optimisation schemes and practical applications. While these theoretical schemes remain valid contributions, real-life satellite operations put more emphasis on practicality, flexibility and robustness rather than strict propellant optimisation, as shown by [24, 29, 30]. This work aims to bridge this gap with the detailed treatment of a practical and fuel-inexpensive method and demonstrate its practical application to a real constellation. To this end, Kormos’ [63] work in epicycle equations has been chosen for further investigation for its simplicity and decoupled in-plane and out-of-plane dynamics.

The literature on modelling the relative motion is very rich, as can be seen in Section 2.2. A simple, linearised two-body model of relative motion on a near-circular orbit has been presented by Hill as early as late 19th century [20, 49]. As applied to artificial satellites, the shortcomings of this model, namely the eccentricity, the negation of non-linear terms and higher order geopotential terms (as well as drag) has been widely recognised, but a definitive solution that encompasses all of these effects has remained elusive.

Many publications offer dynamic models for formations on eccentric orbits. But, unable to
incorporate the major Earth oblateness term $J_2$, they are limited to short term rendezvous applications. Others have argued that since most missions are on near-circular orbits, it is more sensible to concentrate on these cases. They provided linearised solutions with the effects of $J_2$ included. Such an approach has the added benefit of the ability to design so called ‘$J_2$ invariant orbits’ i.e., orbits where the formation remains together even in the presence of primary Earth oblateness effects [95, 116]. There are also rare examples where authors have attempted to model higher order geopotentials and the effects of drag, limited to near-circular formations only [114]. Some attempts have also been made to include second order two-body effects but these effects are the same order as differential $J_2$ effects and cannot be used without them.

What unites all of these approaches is that all of them describe an analytical solution to the relative motion. While analytical solutions are extremely useful for developing control algorithms, designing formations and understanding the nature of the motion, they quickly become very complicated as new perturbation terms are added. It is not uncommon to see many pages of appendices in these papers, detailing the elements of the state transition matrices.

As Kormos [63] rightly points out, most of these papers compare their results against a numerical propagation of the same geopotential or perturbation model. While such an approach is useful in assessing how accurately the analytical model simulates the same effects, it does not give any indication as to how it would perform in a real world application. It may be possible to show that certain initial conditions yield $J_2$ invariant formations that stay together for years in a purely $J_2$ force field, but in reality, other zonal terms of the geopotential will force the satellites apart while tesseral terms will induce further oscillations. Therefore, current analytical models are inadequate to predict the relative positioning of the elements of the formation when long term accuracy is required.

Interestingly, all of the papers dealing with modelling the relative motion work in the local-vertical-local-horizontal frame. Although this accelerating frame is useful for analysing formations and rendezvous scenarios, it is difficult to work with once non-Keplerian forces, particularly those due to the geopotential are introduced. An inertial frame may be be easier to analyse if higher order terms are to be investigated.
Finally, many of these papers (most prominently Karlgaard and Lutze [59]) report residual secular motions that are artifacts of the analytical model they employ, which do not appear in reality. In many cases, the residual errors and secular effects are due to the fact that the relative energy is not exactly conserved; any such error would manifest itself as alongtrack drift. Some papers try to rectify this by adding 'period matching' correction terms after the fact, further complicating the method, but none of them address this issue let alone make it a design goal. The only methods that implicitly get around this issue are the ones that work with relative orbital elements; however these methods lose their elegance and simplicity once a better geopotential model than spherically symmetric force field is employed.

In this context, among the numerical orbit propagators, symplectic propagators are of special interest. If the geometric properties of the problem can be accommodated in the solution, it should be possible to correctly model the relative motion in the long term with high accuracy.

In light of the above discussion, we can re-iterate the three main contributions of this thesis as:

- Development and analysis of a practical, flexible and robust orbit acquisition algorithm with low propellant cost, for use with the Disaster Monitoring Constellation: a novel approach and its real-world application to an established problem.

- Development of a novel, linearised analytical solution to the two-body relative motion, that conserves the geometric constants of relative motion: a novel approach to an established problem.

- Development of a high-precision numerical relative orbit propagator that can accommodate detailed geopotential models, with long-term accuracy and stability: an entirely novel problem.
Chapter 3

Modelling the Satellite Motion

3.1 Introduction

To be able to model the motion of a single satellite in space accurately, an understanding of the forces acting on the satellite is crucial. Recognising the difference in magnitudes of these forces for different orbits will enable us to decide which force needs to be modelled at which accuracy or whether it needs to be modelled at all. In this way, a compromise can be reached between modelling detail and required accuracy for the mission.

In this chapter we highlight various perturbations to the two-body dynamics, for the motion of a single satellite as well as relative motion. We then introduce the mathematical models of these perturbations as well as other numerical integration issues in the context of the design of a symplectic orbit propagator. This approach will also form the blueprint for the development of the relative orbit propagator in later chapters.

3.2 Keplerian Orbital Elements

Keplerian elements describe the orbit of a satellite fully and are widely used in the astrodynamics field. Figure 3.1 illustrates the orbital elements that describe the shape and size of the orbit ellipse as well as where the satellite is on this ellipse.
Chapter 3. Modelling the Satellite Motion

The semimajor axis \( a \) is a measure of how big the orbit ellipse is. Eccentricity \( e \) describes how 'elliptic' or 'circular' the orbit is. \( e = 0 \) corresponds to a circular orbit, whereas \( 0 < e < 1 \) is an elliptic orbit. \( e = 1 \) corresponds to a parabolic orbit and \( 1 < e \) is a hyperbolic orbit, both of which are not closed orbits. These are of more interest when considering gravitational assist manoeuvres and planetary pass-by missions. The parameter \( p \) is also a measure of the size of the conic section and is given by \( p = a(1-e^2) \). The mean motion \( n \) is given as \( n = \sqrt{\mu/a^3} \), where \( \mu \) is the gravitational constant. Mean motion is the orbital frequency and the orbital period is given by \( 2\pi/n \). Angular momentum vector \( \mathbf{L} \) points in the orbit normal and has a magnitude of \( \sqrt{\mu p} \).

In addition to the two parameters \( a \) and \( e \) describing the size and shape of the orbital ellipse, we need a third parameter to define where the satellite is on this ellipse. One way to do it is to define the angle it travelled from the perigee, which is called true anomaly \( (\theta) \).

Three additional parameters are required to describe the orientation of the orbit ellipse in space. Figure 3.2 shows the orbit in 3D space. Suppose that the \( \mathbf{IJK} \) describes an inertial frame of reference, where \( \mathbf{I} \) points to fixed location in space (for example first point of Aries). \( \mathbf{IJ} \) plane is coplanar with the equator and the \( \mathbf{K} \) vector points towards the North pole. Inclination \( I \) is the angle between the orbit normal or angular momentum vector...
3.2. Keplerian Orbital Elements

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Orbit normal

Figure 3.2: 3D orbit geometry

and the K vector. It is a measure of how tilted the orbital plane is. Line of nodes is the vector from the origin to the point on orbit where it intersects the equatorial plane. The angle from the I vector to this line of nodes is the Right Ascension of the Ascending Node (RAAN) \( \Omega \). \( \Omega \) and \( I \) therefore define the orientation of the orbital plane in 3D space. The orientation of the orbit within the orbital plane is described by the argument of perigee \( \omega \), which is the angle from the line of nodes to the perigee. These three parameters complete the six Keplerian elements that define the orbit.

There are a few more useful parameters. Mean anomaly \( M \) describes the angle the satellite travelled from the perigee, assuming it travels at the mean rotation rate i.e., \( M = n(t - t_p) \), where \( t_p \) is the time at perigee passage. Mean anomaly is more useful as a measure of time rather than an angular measure of the location of the satellite on orbit. Another useful measure of the location of the satellite on the orbit is the eccentric anomaly \( E \), which can be found from the solution of Kepler's Equation, \( M = E - e \sin E \).

Finally, Local Time (or Longitude) of the Ascending Node (LTAN) is the local time the satellite crosses the node on its ascending pass, when travelling from the Southern Hemi-
sphere to the Northern Hemisphere. This is a useful measure particularly for imaging missions, where the lighting conditions on the ground is a significant concern. As will be discussed shortly, the oblateness of the Earth causes certain effects on the orbital plane. One of these effects is the precession of the orbital plane. With a careful choice of inclination and semimajor axis, the rate of this precession can be equated to the apparent rotation rate of the Sun. Such orbits are called sun-synchronous orbits and their LTANs ideally stay constant. This enables the satellite cross the Equator at a certain local time, yielding consistent lighting conditions and easy to interpret shadow data.

3.3 Orbit Perturbations

The dynamics of two bodies in space can be described via equations of motion laid out by Keplerian dynamics. In the case of an artificial satellite orbiting a central body, while this description captures the essence of the motion, there are numerous additional forces that act as perturbations to the simple two-body potential.

Typical magnitudes of the forces acting on a satellite around the Earth are given in Table 3.1 [8], where the \( J_n \) terms are the effect of zonal harmonics of the geopotential of the Earth, which will be explained in more detail in the next section.

<table>
<thead>
<tr>
<th>Source</th>
<th>150km</th>
<th>750km</th>
<th>1500km</th>
<th>geosynch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central gravity</td>
<td>9.35</td>
<td>7.85</td>
<td>6.42</td>
<td>0.22</td>
</tr>
<tr>
<td>Earth oblateness ( J_2 )</td>
<td>( 30 \times 10^{-3} )</td>
<td>( 20 \times 10^{-3} )</td>
<td>( 14 \times 10^{-3} )</td>
<td>( 160 \times 10^{-7} )</td>
</tr>
<tr>
<td>( J_3 )</td>
<td>( 0.09 \times 10^{-3} )</td>
<td>( 0.06 \times 10^{-3} )</td>
<td>( 0.04 \times 10^{-3} )</td>
<td>( 0.08 \times 10^{-7} )</td>
</tr>
<tr>
<td>( J_4 )</td>
<td>( 0.07 \times 10^{-3} )</td>
<td>( 0.04 \times 10^{-3} )</td>
<td>( 0.02 \times 10^{-3} )</td>
<td>( 0.01 \times 10^{-7} )</td>
</tr>
<tr>
<td>Equatorial ellipticity ( J_{2,2} )</td>
<td>( 0.09 \times 10^{-3} )</td>
<td>( 0.07 \times 10^{-3} )</td>
<td>( 0.04 \times 10^{-3} )</td>
<td>( 0.05 \times 10^{-7} )</td>
</tr>
<tr>
<td>Atmospheric drag</td>
<td>( 3 \times 10^{-3} )</td>
<td>( 10^{-7} )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Luni-solar attraction</td>
<td>( 10^{-6} )</td>
<td>( 10^{-6} )</td>
<td>( 10^{-6} )</td>
<td>( 10^{-6} )</td>
</tr>
<tr>
<td>Solar radiation pressure</td>
<td>( 10^{-7} )</td>
<td>( 10^{-7} )</td>
<td>( 10^{-7} )</td>
<td>( 10^{-7} )</td>
</tr>
</tbody>
</table>
3.3. Orbit Perturbations

When designing a propagator, to strike a balance between accuracy and computational time, it is crucial to decide what level of modelling accuracy is required for a specific application. In this context, Table 3.1 gives a very good starting point; for a LEO satellite, the effects of luni-solar attraction and solar radiation pressure are periodic and significantly smaller than $J_2$. On the other hand, the effects of drag, while small, will accumulate over time and will cause secular effects on the satellite orbit. Obviously, for a geostationary satellite solar radiation pressure and luni-solar attraction attain significant importance.

Detailed treatments of non-spherical geopotential, drag and third body effects can be found in King-Hele [61], Vallado [112] and Blitzer [8].

3.3.1 Non-spherical Gravitational Potential

As mentioned in the preceding section, the effect of the Earth oblateness has a significant effect on the motion of the satellite. This can be generalised to investigate how the non-uniform mass distribution of the central body can be modelled and how it affects the motion of the satellite.

The gravitational potential can be described by superposing spherical harmonics with gravitational coefficients; this can be visualised as describing the potential via a series expansion. This set of coefficients, computed via on-orbit measurements, therefore fully describe the shape of the central body. This definition can also be interpreted as adding or subtracting 'bands' of mass onto a perfectly spherical central body (See Figure 3.3). The 'latitudinal bands' are called zonal harmonics and therefore are symmetrical about the polar axis. Tesseral harmonics can be visualised as 'tiles' on the sphere. The 'longitudinal bands' are called sectorial harmonics. They can be classified as a subset of tesseral harmonics and we will adopt this convention.

These harmonics can be further divided into subgroups with respect to the period of the effects they cause. Zonal harmonics cause secular, short-periodic (orbital period) and long-periodic (longer than an orbital period) effects. The secular effects due to zonal harmonics are what cause the well-known precession and rotation of the orbital plane. As the first zonal harmonic $J_2$ is about 1000 times larger than any other harmonics, modelling its effect is a significant step in modelling both absolute and relative motion.
Chapter 3. Modelling the Satellite Motion

Figure 3.3: Zonal, tesseral and sectorial harmonics

Tesseral harmonics cause $m$-daily variations, which happen $m$ times a day, where $m$ is the order of the tesseral harmonic, and resonance where the period is weeks to years.

The mathematical expression of the full geopotential is given as [112]:

$$
U = -\frac{\mu}{r}
\left[ 1 - \sum_{l=2}^{\infty} J_l \left( \frac{R_\oplus}{r} \right)^l P_l(\cos \theta) + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R_\oplus}{r} \right)^n P^m_n(\cos \theta) J_{nm} \cos m(\varphi - \psi_{nm}) \right]
$$

where $\mu$ is the gravitational parameter, $R_\oplus$ the maximum equatorial radius of the central body and $(r, \theta, \varphi)$ are spherical polar co-ordinates fixed and rotating with the central body.

For the case of the Earth, they are measured from the rotation axis and the first point of Aries. $P_l$ is a Legendre polynomial of degree $l$, $P^m_n$ is an associated Legendre polynomial of degree $n$ and order $m$. $J_l$ is the zonal harmonic coefficient of order $l$, $J_{nm}$ and $\psi_{nm}$ are the tesseral harmonic coefficient and phase of degree $n$ and order $m$. These $J$ and $\psi$ coefficients essentially define the mass distribution of the central body.

### 3.3.2 Atmospheric Drag

Atmospheric drag is one of the largest sources of perturbation of LEO satellites. While the atmosphere is very thin at higher altitudes, its effect is still enough to take energy away from the satellite and cause it to slowly lose altitude. As the satellite loses altitude, the atmosphere becomes more dense and this rate of fall increases.

The force acting on the satellite due to drag can be simply written as:

$$
F = \frac{1}{2} \rho C_D A_L V^2
$$

(3.2)
3.3. Orbit Perturbations

where $C_D$ is the drag coefficient, $\rho$ is the local density of the atmosphere, $A_L$ is the cross-sectional area of the satellite and $V$ is the velocity of the satellite relative to the atmosphere.

Primarily, the effect of this force is to decrease the semimajor axis and eccentricity of the orbit. At periapsis, the drag is greater and the satellite loses energy, causing the apoapsis distance to decrease secularly. The semimajor axis decay is essentially linear in time and therefore can be modelled fairly easily. Figure 3.4 shows filtered GPS data from the sun-synchronous low Earth orbit (LEO) of the satellite AlSat, illustrating the real semimajor axis decay over time. There are other, much smaller effects on the other orbital elements as well, particularly cross-track effects due to the rotation of the atmosphere.

While the drag force acting on the satellite can be described with a very simple expression, in reality, determining and projecting ahead the terms inside this expression is extremely difficult. The cross-sectional area $A_L$ is a function of the satellite attitude and its long term forward prediction is very difficult. The drag coefficient $C_D$ is also another source of complication as it is not possible to test it on the ground. However the most difficult to predict term remains the local density of the atmosphere. Variations in Solar fluxes and geomagnetic activity heat and 'swell' the atmosphere. Similarly, atmospheric temperature changes considerably during day and night time (called 'diurnal variations'), changing the
density profile. The 27-day solar rotation cycle, 11-year Sun spot cycle and other cyclical variations also cause strong variations on Solar radiation. Rotating atmosphere, wind patterns and even ocean tides cause variations of different magnitudes in the temperature and density profiles. To illustrate how a Solar flare dramatically changes the drag, see Figure 3.5 showing sun-synchronous LEO satellite AlSat semimajor axis profile from filtered GPS data.

The static atmosphere models assume a constant atmospheric profile that does not change with time. By contrast, the time-varying models take into account a number of the above effects. Complicated geomagnetic models also exist with tables detailing the global magnetic field profile. Different models for different accuracy requirements and different altitudes exist, but as can be expected, the most detailed models (such as Jacchia-Roberts) require immense amounts of computational power, which would be impractical and unfeasible for onboard or real-time operation. Even then, the inherent difficulties of predicting the atmosphere density over time still remains and makes long-term, accurate position prediction extremely difficult.

Due to the above complications, it is also difficult to assess and compare the accuracy of any given model, as “different models are required for different applications and altitudes” [112]. The problem is further complicated by the absence of ‘true orbit’ to compare against.
3.3. Orbit Perturbations

"In general, we know that atmospheric models tend to introduce about 10-15% error [32, 86, 112]. We also know the ballistic coefficient varies about 10-12% [50], given that our
differential corrections have gravity fields of about 40x40 for LEO satellites". [113]

Nevertheless, there are some publications that compare different atmospheric models such
as Akins et al. [1] where orbital data from a large number of missions has been tested
against the detailed Jacchia and MSIS atmospheric models though their results are incon­
cclusive. They report orbit-to-orbit accuracies ranging between 2km to 17km and the error
is mostly in the alongtrack direction.

3.3.3 Third Body Interactions

The gravitational force of the central body causes an acceleration on the satellite but there
are many other celestial bodies that attract the satellite and disturb its orbit. In its most
general case, there are no analytical solutions to the so-called ‘three-body problem’ unless
certain assumptions and simplifications are made. In reality, considering the attraction of
the Moon and the Sun on a satellite orbiting the Earth (which constitutes a ‘four-body
problem’) numerical integration methods have to be used. Note that these forces are
conservative.

For the case of a LEO satellite there are much more significant forces affecting the motion
of the satellite (such as drag and Earth oblateness) therefore the lunisolar attraction is
sometimes ignored. For higher altitude orbits, drag and the perturbations due to non­
spherical geopotential terms become very small; however the lunisolar attraction remains
largely constant (see Table 3.1) and therefore becomes increasingly significant.

The effect of these perturbations are extremely complicated due to the dynamic nature
of the problem. There are secular changes in the node and the perigee, as well as other
long-period changes in all the orbital elements except for the semimajor axis.

3.3.4 Solar Radiation Pressure

Solar radiation pressure is dependent upon the attitude of the satellite, the reflectivity of
the materials and the magnitude of the Solar flux. Evidently, this disturbance vanishes
whenever the satellite enters into eclipse. This constant push is a non-conservative force and it mainly affects the eccentricity of the satellite.

The force exerted on the satellite can be described as:

\[ F_{SR} = p_{SR} C_R A_0 \mathbf{r}_\odot \]  \hspace{1cm} (3.3)

where \( p_{SR} \) is the radiation pressure, \( C_R \) the surface reflectivity, \( A_0 \) the exposed surface area and \( \mathbf{r}_\odot \) is the unit sun-satellite vector. As noted in the preceding section dealing with drag, calculation of the solar flux over time or its forward prediction is extremely difficult, as well as accounting for variations in satellite pointing.

The effect is to cause oscillations in all the orbital elements with periods reaching years [112]. Blitzer [8] states that for satellites over 800km altitude, its effect can be more significant than drag for many satellite platforms. While the radiation pressure is small, like drag, larger area and lighter mass satellites are affected more.

3.4 Numerical Orbit Propagation

3.4.1 Orbit Propagation Basics

Orbit propagation, by its definition, is to integrate the governing equations of motion of a satellite. Integrating the acceleration equations, one can calculate the velocity and position of the satellite at a later time.

For single satellite missions, one of the primary determinants of the mission success is the precision with which the position and the orbit of the satellite can be estimated, predicted ahead and, in some cases, controlled. Imaging missions are a case in point, where long-term scheduling (on the order of a few weeks) require a high precision knowledge of the current and the future trajectory of the satellite. Therefore, the modelling accuracy should reflect these accuracy requirements.

While the forces acting on a satellite are well known, the force models, particularly for the perturbations introduced in the preceding sections are rather complicated and therefore cannot be integrated analytically. One way to get around this problem is to make certain
3.4. Numerical Orbit Propagation

assumptions and simplifications, with penalties to the accuracy of the method. The second way is to carry out this integration numerically, which enables us to model the complicated nature of the perturbations more accurately.

In an analytical propagator, to be able to find the positions and velocities at a time $t^*$, the initial conditions (such as position and velocity at $t = t_0$) are inserted into the position and velocity equations; this yields the position and velocity at the required time $t^*$ directly. In a numerical propagator, on the other hand, one cannot calculate the end positions and velocities in a single step.

To illustrate how a simple numerical propagator works, consider a satellite with an initial position and velocity of $(r_0, v_0)$ at time $t_0$. The initial acceleration $a_0$ is a function of the location of the satellite.

The differential equations that describe the motion can be written as:

$$\dot{r} = \frac{dr}{dt}, \quad \ddot{r} = \frac{d\dot{r}}{dt} \quad (3.4)$$

As Leimkuhler and Reich [69] put it, “given a differential equation and an initial value, a discrete version of a trajectory of the system could be obtained by taking snapshots of the solution at equally spaced points in time.” The discrete form of these differential equations can be written as:

$$\frac{\dot{r}_{k+1} - \dot{r}_k}{t_{k+1} - t_k} = \ddot{r}_k \quad (3.5)$$

$$\frac{r_{k+1} - r_k}{t_{k+1} - t_k} = \dot{r}_k \quad (3.6)$$

Given the initial conditions at $t = t_{k+1}$, these discretised equations can be used to estimate the state at a short timestep of $h$ later, where $t_{k+1} - t_k = h$.

$$\dot{r}_{k+1} = \dot{r}_k + h\ddot{r}_k \quad (3.7)$$

$$r_{k+1} = r_k + h\dot{r}_k \quad (3.8)$$

Therefore, using the initial conditions at $t = 0$, one can estimate the positions and velocities at $t = h$ via the equations above. The acceleration is purely a function of the position vector, therefore using $r_{k+1}$, the acceleration vector $\ddot{r}_{k+1}$ can be calculated. Hence, using the state at $t = h$, it is possible to calculate the next state at $t = 2h$. 
Carrying out the above procedure for all the small timesteps progressively until \( t = t^* \), we find the final positions and velocities. This is the most basic numerical integration scheme, called Euler's Method. As the discretised equations take into account only the first order terms (i.e., \( O(h^2) \)), it is called a 1st order propagator. As can be expected, finer stepsizes will increase the accuracy of the propagator and in the limit, as \( h \to 0 \), the numerical model will be free of errors. In practice, very small stepsizes will cause rapid growth of the roundoff and truncation errors.

In reality, such a scheme would not yield the high accuracy, efficiency or stability required by real world applications. To increase the accuracy of the numerical integration, the simplest way is to make the timesteps smaller, even though this is a rather inefficient way. The second method is to incorporate a 'higher order' integration scheme, which uses a more complicated algorithm to compute the position and velocity advance over a single timestep.

Before we describe some of the higher order schemes employed in orbit propagation, a few points have to be emphasised. Computational efficiency and speed as well accuracy are the ultimate aims in any propagator. The most computationally expensive part of the propagation is the force calculation, hence the number of such force evaluations should be minimised for computational efficiency. Both decreasing the stepsizes and employing higher order schemes will increase the total number of force calculations per step. The design of a propagator for a specific application therefore involves striking a balance between the competing needs of speed and accuracy within the limits of the computational power available.

**Runge-Kutta Methods**

Runge-Kutta (RK) methods are probably the most well known and most widely used among the numerical integration schemes [112]. Consider the example of a known acceleration function, which is to be integrated numerically to obtain the velocity. Euler method assumes that the average acceleration throughout a timestep is determined purely by the initial conditions at the beginning of the stepsise. RK schemes, however, evaluate the acceleration at a number of points within a timestep and take a weighted average of
3.4. Numerical Orbit Propagation

these values, thereby obtaining a more accurate snapshot of the acceleration variation. As they use a single calculation to propagate the state by one timestep and do not use any corrections for the past steps, these methods are called ‘single-step’ methods.

The change in the position vector at each timestep $h$ can therefore be written as:

$$r_{n+1} = r_n + h \sum_{i=1}^{s} b_i k_i$$  \hspace{1cm} (3.9)

where $k_i$ is given by,

$$k_1 = f(t_n, r_n)$$
$$k_2 = f(t_n + c_2 h, r_n + c_{21} h k_1)$$
$$k_3 = f(t_n + c_3 h, r_n + a_{31} h k_1 + a_{32} h k_2)$$
$$\vdots$$
$$k_s = f(t_n + c_s h, r_n + a_{s1} h k_1 + a_{s2} h k_2 + \ldots + a_{s,s-1} h k_{s-1})$$  \hspace{1cm} (3.10)

To specify a particular method, one needs to provide the integer $s$ (the number of stages), and the coefficients $a_{ij}$ (for $1 \leq j < i \leq s$), $b_i$ (for $i = 1, 2, \ldots, s$) and $c_i$ (for $i = 2, 3, \ldots, s$). This method is called a $s$-stage (or $s^{th}$ order) Runge-Kutta method, as it can be shown that the method has a local error of size proportional to $h^{s+1}$ (i.e., $O(h^{s+1})$). Evidently, the higher the order of the integrator, the higher will be the accuracy, but at a cost of increased computational load per timestep.

For problems with rapidly changing dynamics, the Adaptive Runge-Kutta schemes (also known as Runge-Kutta-Fehlberg (RKF)) have been proposed [27, 28]. These methods vary the stepsize to better capture abrupt changes in the function to be integrated. Consider a satellite on a highly elliptic orbit. The satellite lingers on for a long time around apoapsis with small changes in position and velocity. However, around periapsis, the velocity and positions change rapidly. Therefore, a constant stepsize scheme would either have small stepsizes to capture periapsis motion or large stepsizes to have better efficiency around apoapsis. An RKF scheme would decrease the timesteps around periapsis and increase them around apoapsis, providing both accuracy and efficiency, though at a cost of slightly more complicated implementation.
Adams-Bashforth Method

Unlike the ‘single-step’ ones, ‘multi-step’ schemes utilise previously determined back values of the states, in addition to the initial state. Rather than evaluating the forces at several intermediary steps within a timestep (as in RK schemes), they use a single force calculation to ‘predict’ the next state, and another force calculation to ‘correct’ this predicted state; hence, these methods are also called ‘predictor-corrector’ schemes. As they require back values to propagate the step forward, we need the values of some states before the initial state to be able to initialise the propagation. These methods can be thought as fitting a polynomial through the back points of the trajectory to estimate the next point.

The $s^{th}$ order Adams-Bashforth method is an explicit $s+1$ step method and is of the form,

$$r_{n+1} = r_n + h \sum_{j=0}^{s} b_j f(t_{n-j}, r_{n-j}) \quad (3.11)$$

where the coefficient $b_j$ is given as,

$$b_j = \frac{(-1)^j}{j!(s-j)!} \int_0^1 \prod_{i=0,i\neq j}^s (u+i) \, du, \quad j = 0, \ldots, s \quad (3.12)$$

As can be seen, both $r_n$ and $r_{n-1}$ are necessary to determine $r_{n+1}$, making it a multi-step method.

Therefore, the first and second order methods can be written as:

$$s = 1 \quad r_{n+1} = r_n + h \left( \frac{3}{2} f(t_n, r_n) - \frac{1}{2} f(t_{n-1}, r_{n-1}) \right) \quad (3.13)$$

$$s = 2 \quad r_{n+1} = r_n + h \left( \frac{23}{12} f(t_n, r_n) - \frac{16}{12} f(t_{n-1}, r_{n-1}) + \frac{5}{12} f(t_{n-2}, r_{n-2}) \right) \quad (3.14)$$

Note that, the Gauss-Jackson (or Störmer-Cowell) method is simply a variant of the multi-step methods, but designed especially for second-order systems. While the formulation is very similar to Adams-Bashforth, this method uses the accelerations directly to compute the position updates, without resorting to the velocities. This yields better accuracy than the Adams-Bashforth method.

Bulirsch-Stoer Method

We have already introduced the basic principle behind Bulirsch-Stoer in the literature survey chapter. This method relies on using a modified midpoint rule along with an
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extrapolation algorithm to reach a solution.

Consider a large integration stepsize of $H$, which we can evaluate via $m$ substeps of length $h$ i.e., $h = H/m$. The modified midpoint method is therefore given as:

\begin{align}
    r_0 &= r(t_0) \\
    r_1 &= r_0 + h\dot{r}(t_0) \\
    r_{n+1} &= r_{n-1} + 2h\dot{r}(t_n), \quad n = 1, 2, \ldots, m - 1 \\
    r(t + H) &= \frac{1}{2}[r_m + r_{m-1} + h\dot{r}(t + H)]
\end{align}

A similar expression can be written for the propagation of the velocity via the acceleration terms. Except for the end points, this is essentially the standard midpoint method, where the rate of change of the function is sampled not from the beginning of the timestep like the Euler method but in the middle of the timestep. It can be shown that, such a formulation yields two orders of accuracy gain for the calculation of each extra intermediary step.

At this point we introduce the Richardson extrapolation. One of the key ideas is the assumption that the end state is an unknown and complicated function of the stepsize, such that, as the number of intermediate timesteps goes to infinity, we will converge to the true end state. However, we can try several different intermediate stepsizes, which are not necessarily very small, so that we can get an idea of the behaviour of this complicated function and eventually fit a function to estimate the end state i.e., an extrapolation to infinite number of intermediary timesteps. This fitted function can be a polynomial or a rational function. In Bulirsch-Stoer method, the integrations within the Richardson extrapolation are carried out via the modified midpoint method.

As Vallado [112] stresses, while this method uses even fewer force calculations than Adams-Bashforth, there are some possible issues with stability. Secondly, it works best with smooth and well-behaved functions and not with functions with singularities, discontinuities and abrupt changes.

Notes on the Conventional Numerical Propagation Techniques

In the previous sections, we have introduced three widely used conventional numerical integration techniques for orbit propagation. While RK methods enjoy widespread use in
the field, as Leimkuhler and Reich [69] note, in long-term simulations or at large stepsizes, nonphysical effects generally become apparent, such as energy drift or artificial dissipation. Hence, when the motion of a satellite in a spherically symmetric force field is computed using a RK scheme, the constants of the motion i.e., energy and angular momentum are not kept constant. Therefore, the semimajor axis and eccentricity of the modelled orbit change secularly, which obviously is a nonphysical artifact of the numerical scheme. As shown in [69], as the order of the RK scheme increases, this effect is dampened but not eliminated. This discussion is also valid for the Adams-Bashforth and Bulirsch-Stoer methods.

Vallado [112] notes that RK schemes as well as Adams-Bashforth and Gauss-Jackson schemes have been employed in the field with excellent stability, however, there are certain stability issues with the Bulirsch-Stoer method.

Another remark that concerns the three methods is about the effect of different sources of accelerations. We have already mentioned that the calculation of the acceleration is the most expensive part of the computation in terms of processor time, hence for a similar accuracy, a scheme with less number of acceleration calculations will yield the most efficient propagator. This becomes particularly important as more and more perturbations are calculated at each step. While different perturbation sources cause accelerations of different orders of magnitude, these schemes have no way of 'prioritising' different perturbations. Therefore, the effect of the Keplerian potential is calculated as frequently as that of a high order geopotential, which may be some orders of magnitude smaller and therefore can be calculated more sparsely without a significant penalty in accuracy. It is evident that this is not a very efficient way of handling forces of different magnitudes. Hence, it is desirable to uncouple the effects of the forces and evaluate them at different frequencies.

### 3.4.2 Symplectic Integration

In the preceding sections, we have introduced a number of common numerical integration schemes used in orbit propagation. While these methods can be tuned to yield high accuracies, as explained in the RK scheme, they are insensitive to the physical properties of the problem such as constant energy and angular momentum. Therefore, they will invariably introduce growing errors in these quantities, changing crucial properties of the motion,
such as rotation rate, semimajor axis and eccentricity. It is then logical to investigate a numerical integration scheme that observes the conservation of these constants of the motion, which would possibly yield better accuracy in the long term. To this end, we will have to briefly mention Hamiltonian systems.

A Hamiltonian system can be defined as a system of variables which can be written in the form of Hamilton's Equations, which are given as:

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \]  

(3.19)

where \( H \) is the Hamiltonian, \( q \) is the position and \( p \) is the momentum of the system. Hamiltonian systems are therefore a subset of 'conservative systems of ordinary differential equations'.

As a general rule, mechanical systems resulting from physical principles are Hamiltonian systems until they are subjected to truncations and simplifications, usually for modelling purposes [69]. A simple harmonic oscillator, a satellite in a geopotential field or planets in the Solar System are examples of such Hamiltonian systems. The Hamiltonian has a physical significance and corresponds to the energy of the system.

'Phase space' for such systems is simply all the possible values of the state i.e, position and momenta. Each point in the phase space corresponds to a unique state. For example, for a simple pendulum, the phase space has two dimensions, the angle and the angular velocity. The trajectory defined by the equations of motion is therefore a curve in the phase space, parametrised by the time variable. The initial conditions on the phase space correspond to a certain energy level or Hamiltonian and this uniquely determines the trajectory; only another trajectory of the same energy can intersect this trajectory. Furthermore, the initial state determines the solution at all later points on the trajectory; the solution of the differential equations of the motion effectively determine a 'flow map' that maps the initial state to the final state (similar in concept to a coordinate transformation or a state transition matrix of a linear system).

For such conservative mechanical systems the flow map inherits certain geometric properties of the motion itself [69]. Consider a set of initial conditions in the phase space that form an enclosed area. If we propagate all of these points to a future time through their
trajectories, the enclosed area should be conserved; this property is inherent to the conservative motion. A numerical integrator can be visualised as a mapping, which approximates the flow map of a given system of differential equations. Ideally, a perfect numerical integration scheme would correspond to discrete points exactly on the trajectory on the phase space. Most numerical schemes, however, cannot conserve the energy, therefore slowly diverge from the real trajectory; the aforementioned area conservation property of the motion will no longer be satisfied. A symplectic integrator, on the other hand, maps a discrete trajectory which stays on or near the real trajectory of the motion in the phase space. Consequently, the enclosed area of the family of states is conserved throughout the motion.

To understand the concept of conservation of geometric properties' and symplectic integration, consider a simple harmonic oscillator with a 'cloud' of initial data points shown on the phase space in Figure 3.6 [56]. This initial set of data points (at '3 o'clock' position) are propagated in time and the figure shows how this set of points evolve at discrete points in time. Since we know the exact analytical solution, we know that the 'smiley face' cloud of states in the phase space should rotate as well as translate at each step in the anti-clockwise direction, without changing its shape. Furthermore, each circle in the 'smiley face' is an enclosed area and these areas should be conserved throughout the propagation. In this example we carry out the propagation numerically via simple Forward Euler Method and Implicit Midpoint Method, which is a symplectic integration scheme. In the Euler Method (non-symplectic), the image on the phase space grows bigger, illustrating a secular change in the energy and an unbounded deviation from the real solution. By contrast, with the Implicit Midpoint Method (symplectic), the geometric properties of the solution are conserved, with each cloud of states keeping its area constant.

In summary, a Hamiltonian system, propagated via a symplectic scheme will conserve the constants of the motion. The reason this approach works so well is because at each step of the procedure the error has a Hamiltonian form. This causes the energy to oscillate but never to diverge, so even for reasonably large timesteps the energy is conserved. Unsurprisingly, as the timestep continues to increase, the system starts to become chaotic and the stability of the method collapses [115].
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(a) Forward Euler Method  
(b) Implicit Midpoint Method

Figure 3.6: Phase Space for a symplectic and a non-symplectic method (position on horizontal and momentum on vertical axes) [56]

Construction of Symplectic Methods and Hamiltonian Splitting

We have already mentioned some properties and advantages of symplectic methods over conventional numerical integration schemes. Nevertheless, construction of symplectic integrators are not much different than other numerical schemes.

At this point, it is useful to introduce the concept of ‘Hamiltonian splitting’ which we will exploit to great effect later on when we are putting together the numerical integration scheme to achieve very significant savings in the computational power. Hamiltonian splitting, as the name implies, divides the Hamiltonian into the sum of a number of Hamiltonians, each of them having a flow map that is symplectic. In fact, any composition of symplectic maps yield a symplectic map and the resulting numerical integration is therefore symplectic [69].

Suppose that the Hamiltonian is the sum of \( n \) Hamiltonians, such that \( H = \sum_{i=1}^{n} H_i \). A symplectic integrator can therefore be derived as an appropriate combination of the flow maps; as each flow map is symplectic, the resulting method will also be symplectic.

For example, the Hamiltonian of a satellite can be separated into potential and kinetic energy terms. For each of these energy terms, one can write Hamilton’s Equations to yield two differential equations. These four differential equations can be discretised to yield
position and velocity updates:

\[
v_{k+1} = v_k - h \left( \frac{\partial U}{\partial r} \right)_k \tag{3.20}
\]

\[
r_{k+1} = r_k + h \left( \frac{\partial (1/2 v^2)}{\partial v} \right)_{k+1} \tag{3.21}
\]

where \( h \) is the timestep. This can be thought as keeping the satellite stationary and applying a velocity jump, followed by a position jump using this updated velocity. The resulting scheme is a variation of the first order Euler method.

Similarly, it is possible to split the Hamiltonian into three parts, with \( H_1 = 1/2 \ U(r) \), \( H_2 = 1/2 \ v^2 \) and \( H_3 = 1/2 \ U(r) \). The equations of motion coming from these three Hamiltonians can be combined to yield:

\[
v_{k+1/2} = v_k - 1/2 \ h \left( \frac{\partial U}{\partial r} \right)_k \tag{3.22}
\]

\[
r_{k+1} = r_k + h \left( \frac{\partial (1/2 v^2)}{\partial v} \right)_{k+1/2} \tag{3.23}
\]

\[
v_{k+1} = v_k - 1/2 \ h \left( \frac{\partial U}{\partial r} \right)_{k+1} \tag{3.24}
\]

This is the second order leapfrog method [62, 115]. Note that, the leapfrog method is only a combination of half-step Euler methods.

Finally, we can introduce the Lie operator notation. The Lie operator \( \dot{H} \) gives the time derivative of an arbitrary function \( f(p, q, t) \) (with \( q \) and \( p \) being position and momentum), which is moving under a Hamiltonian \( H \) [37, 75].

\[
f = f \dot{H} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \tag{3.25}
\]

It can be shown that the \( n^{th} \) derivative can be written as:

\[
\frac{d^n f}{dt^n} = \dot{H}^n f , \quad \dot{H}^n = \dot{H}(\dot{H}^{n-1} f) \tag{3.26}
\]

The Taylor expansion for \( f \) can be written as:

\[
f(t_0 + h) = \exp(h\dot{H})f(t_0) = f + h \dot{H} f + \frac{h^2}{2} \dot{H}^2 f + \ldots \tag{3.27}
\]

This can be used to describe how this function moves forward in time, under the motion defined by this Hamiltonian, with the notation \( \exp(h\dot{H})f(p, q, 0) = f(p, q, h) \) i.e, propagating the state by a timestep \( h \).
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Hence, as the leapfrog is made up of three split Hamiltonians, it can be written in a symbolic form using Lie operators as:

\[ \exp\left(\frac{1}{2}hV\right) \exp(hX) \exp\left(\frac{1}{2}hV\right) \]  

(3.28)

where \( \exp(hX) \) represents a position update step and \( \exp\left(\frac{1}{2}hV\right) \) represents a velocity update step.

Note on Dissipative Systems

Up until now, we have seen the basics of how a symplectic propagator can be put together. However, a satellite at LEO will experience some drag force which will cause the satellite to slowly lose energy and altitude. Obviously, a numerical model that incorporates this effect will not conserve the Hamiltonian and cannot, strictly speaking, be symplectic.

The Hamiltonian splitting technique is well suited to separate out Keplerian part of the motion from other perturbations (such as higher order geopotential terms). The Hamiltonian for the former is a function of position and velocity, whereas the latter is purely a function of position and is therefore trivially integrable. The dissipative forces that depend on velocity (such as drag) is therefore not well suited for this sort of treatment. However, as Christou et al. [19] remarked, “we can go around this difficulty by incorporating the dissipation into the ‘main’ [Keplerian] Hamiltonian”.

In summary, while the dissipation makes the scheme non-symplectic, it is not rendered useless by any means. It is still possible to add in the effect of drag. And, unlike explicit RK schemes, artificial dissipation is not observed when drag is taken out of the model.

3.4.3 A Symplectic Orbit Propagator: SPSAT

In this section we will describe a symplectic scheme to propagate the orbit of a satellite, developed by Palmer et al [19, 45, 78]. Its modelling of the full geopotential model will be described as well as the addition of dissipative forces. With the latter, even though the scheme will no longer be symplectic in the strictest sense, the method will still be valid, as the non-conservative forces are small.
Description of Motion in Inertial Space

Construction of a symplectic method begins with the Hamiltonian \( H \) of the motion, which corresponds to the total energy. Consider the motion of a satellite in inertial space, orbiting around a planet. The motion of the satellite can be described using the Hamiltonian:

\[
H(r, v) = \frac{1}{2}(v \cdot v) + U(r)
\]  

(3.29)

where \( v \) is the magnitude of the velocity of the satellite, \( r \) is the position vector and \( U(r) \) is the potential function given in Equation (3.1).

Equation (3.29) can also be written in terms of the Keplerian and the perturbation parts:

\[
H(r, v) = K(r, v) + R(r) = \frac{1}{2}v^2 - \frac{k}{r} + R(r)
\]  

(3.30)

where \( K(r, v) \) is the Hamiltonian describing Keplerian motion and \( R(r) \) is the perturbing function due to the remaining terms in the spherical harmonic expansion of the gravitational field. This perturbing function \( R \) is an order of magnitude smaller than the Keplerian potential \( K \). Both of the forms given in (3.29) and (3.30) will be useful later on.

The equations of motion can be defined via Hamilton's Equations:

\[
\dot{r} = \frac{\partial H}{\partial v} = v
\]

\[
\dot{v} = -\frac{\partial H}{\partial r} = -\frac{\partial U(r)}{\partial r}
\]  

(3.31)  

(3.32)

At this stage, we will employ a 'Hamiltonian splitting' technique, which we will exploit to great effect later on when we are putting together the numerical integration scheme to achieve very significant savings in the computational power. Hamiltonian splitting, as explained in the previous section, divides the Hamiltonian into the sum of a number of Hamiltonians, each of them having a flow map that is symplectic. Hence, it is possible to divide the Hamiltonian as well as the resulting equations of motion into the Keplerian and perturbations parts as shown in (3.30) i.e., \( H = K + R \), where \( R \) is about 1000 times smaller than \( K \).

To demonstrate how a simple symplectic scheme works with this Hamiltonian splitting, the leapfrog method can be employed. Figure 3.7 illustrates how this scheme works. The
Keplerian part of the propagation is carried out for half the timestep, ignoring the perturbation term completely. This is followed by a propagation ignoring $K$ completely over a full timestep. $R$ is independent of velocity and it causes a jump in velocity with no change in position:

$$\Delta v = -h \frac{\partial R}{\partial r}$$

(3.33)

where $h$ is the integration timestep. Section A.1 in Appendix details the derivation of $\partial R/\partial r$.

This leapfrog scheme can also be written in the Lie notation as:

$$\exp\left(\frac{1}{2}hK\right)\exp(hR)\exp\left(\frac{1}{2}hK\right)$$

(3.34)

where $K$ denotes a Keplerian propagation and $R$ denotes a velocity jump due to the perturbation term.

While the leapfrog scheme is too simple to be of any real use for the satellite propagation problem, it highlights a few important points about the symplectic integration scheme. It is evident that, the splitting has enabled us to effectively decouple Keplerian motion and the perturbations, largely simplifying the method. Note that, we calculate the more important Keplerian force twice as frequently than the less important (but computationally more expensive) perturbations, which increases the computational efficiency significantly.

Finally, before we proceed to setting up the actual numerical integration scheme, we can show that the equations of motion presented in (3.31) actually conserve the Hamiltonian.

The total time derivative of this Hamiltonian is given as:

$$\dot{H} = \frac{\partial H}{\partial v} \frac{dv}{dt} + \frac{\partial H}{\partial r} \frac{dr}{dt} = \frac{\partial H}{\partial v} \dot{v} + \frac{\partial H}{\partial r} \dot{r}$$

(3.35)
Substituting the suitable expressions from (3.31),

\[ \dot{H} = \frac{\partial H}{\partial v} \left( -\frac{\partial H}{\partial r} \right) + \frac{\partial H}{\partial r} \frac{\partial H}{\partial v} = 0 \]  

(3.36)

This proves that the equations of motion indeed conserve the Hamiltonian.

Let \( F(q, p) \) and \( G(q, p) \) be two functions. Therefore, the Poisson bracket is defined as:

\[ \{F, G\} = \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} \]  

(3.37)

It can be easily shown that \( \{F, F\} \) is equal to zero, which is a well known property of the Poisson brackets. Consequently, the \( \dot{H} \) expression above can also be written in terms of the Poisson brackets, such that \( \dot{H} + \{H, H\} = 0 \).

Analytical Modelling of the Keplerian Dynamics and Drag

We have already demonstrated in the previous section how the computation of the Keplerian motion is decoupled from the perturbations. While it is possible to model the Keplerian motion numerically, the existence of the exact solution for the two-body motion can be exploited. It should be emphasised that the Keplerian part of the potential is by far the largest and any error in its evaluation will therefore be more amplified than the perturbation terms. As the exact solution obviously conserves the energy and the angular momentum, the symplecticness of the total solution is not disturbed.

For this we use the Gauss' \( f-g \) functions (see Vallado [112] and Battin [7] for a particularly detailed treatment). This method is particularly appealing as it is free of singularities and does not suffer from small eccentricity effects. In fact, used in the generalised form, it is valid for even parabolic and hyperbolic orbits.

To evaluate these functions we employ the Stumpff \( c \) functions [103] and introduce a set of \( C \) functions for simplicity [101]. The details of this calculation can be found in many textbooks, but we use the notation consistent with Mikkola et al. [78]. In one sense, the method finds the state transition matrix to relate the initial position and velocity \((r_0, v_0)\) to the position and velocity \((r, v)\) at a later time.
3.4. Numerical Orbit Propagation

We propagate forwards in time the nominal position and velocity through the relations:

\[
\begin{align*}
    f &= 1 - \mu G_2 / r_0 \\
    g &= t - \mu G_3 \\
    \dot{f} &= -\mu G_1 / (r_0 r) \\
    \dot{g} &= 1 - \mu G_2 / r \\
    r &= f r_0 + g v_0 \\
    v &= \dot{f} r_0 + \dot{g} v_0
\end{align*}
\]  

(3.38)

While the \( G \) functions provide generality, the equations are not entirely intuitive in the above form. The coefficients \( f, g \) and \( \dot{f}, \dot{g} \) can be written in terms of more familiar orbital elements (valid for elliptic orbits only).

\[
\begin{align*}
    f &= 1 - \left( \frac{r}{p} \right) (1 - \cos(\Delta \theta)) \\
    g &= \frac{r r_0 \sin(\Delta \theta)}{\sqrt{\mu p}} \\
    \dot{f} &= \frac{\mu}{p} \tan \left( \frac{\Delta \theta}{2} \right) \left( \frac{1 - \cos(\Delta \theta)}{p} - \frac{1}{r} - \frac{1}{r_0} \right) \\
    \dot{g} &= 1 - \left( \frac{r_0}{p} \right) (1 - \cos(\Delta \theta))
\end{align*}
\]  

(3.39)

where \( p \) is the semi-parameter and \( \Delta \theta \) is the true anomaly difference between the initial and the final time. Note that, while this form of the equations makes them easier to comprehend, for programming it is vulnerable to implementation errors such as small angle effects.

Drag modelling in SPSAT is fairly simple and is explained in the paper by Christou et al. [19], who in turn use the method proposed by Malhotra [70]. The basic idea is to modify the \( f-g \) functions such that an in-plane drag force acts on the satellite which can be integrated analytically. Parameters such as atmospheric density as well as satellite drag coefficient and attitude profile are taken as constants, making it more suited to long-term analysis of near-circular orbits. Obviously, addition of any dissipative force such as drag means that the Hamiltonian is no longer a constant. While this does not cause any issues in the analytical part of the propagation, the overall scheme is, strictly speaking, no longer symplectic. However, as the drag force is small, the method remains useful and does not simply topple over.
Setting up the Numerical Integrator

In the previous sections we have shown how a simple $2^{nd}$ order symplectic scheme would work with the Keplerian state update and the first order momentum update due to perturbations. However, in reality, a second order scheme would require very small timesteps for high-precision orbit propagation. Clearly, we need to construct higher order schemes for better efficiency. Using the same technique as leapfrog but with many more intermediary steps, one can construct higher order methods [72]. Figure 3.8 illustrates such a scheme, where $x_i, w_j$ are the coefficients for the lengths of intermediary stepsizes.

For example, the fourth order Simpson’s rule can be written in Lie operator notation as:

$$\exp\left(\frac{h}{3}R\right) \exp(hK) \exp\left(\frac{2h}{3}R\right) \exp(hK) \exp\left(\frac{h}{3}R\right)$$

which propagates for two timesteps $(2h)$. $K$ corresponds to a Keplerian propagation, for which an exact analytical solution was detailed in the last section. $R$ corresponds to a velocity jump due to the perturbations to the Keplerian motion, as given in (3.33).

Methods constructed in this way have similar orders to the error of the associated numerical integration formula, as long as these perturbations are first order. Clearly, this is the case in the perturbations to the two-body potential. [78]

While this higher order scheme will result in a high precision integration, it is possible to take the Hamiltonian splitting approach one step further for increased efficiency. $J_2$ effects are several hundred times larger than the next harmonics, namely $J_3$ zonal and $J_{2,2}$ tesseral terms. Therefore, just in the same way Keplerian term is separated from the rest of the
perturbations, it is possible to further decouple $J_2$ from the rest of the perturbations. In this way, we can construct a scheme where more important but easy to evaluate terms are calculated very frequently and less important but computationally expensive terms are calculated more sparsely.

Such an approach is shown in Figure 3.9, where we have a composite of two leapfrog schemes. We split the Hamiltonian such that \( H = K + R_1 + R_2 \), where \( K \) is the Keplerian potential, \( R_1 \) is the \( J_2 \) potential and \( R_2 \) is the potential for the remaining perturbations. With this composite leapfrog scheme, we calculate the Keplerian forces four times, \( J_2 \) twice and the forces due to remaining geopotentials only once.

For higher accuracy, in the real propagator, the above \( H = K + R_1 + R_2 \) splitting is used for the higher order Simpson’s rule method as given in (3.40), with the split potential function:

\[
\exp\left(\frac{h}{3}R_2\right)\exp(h(K + R_1))\exp\left(\frac{4h}{3}R_2\right)\exp(h(K + R_1))\exp\left(\frac{h}{3}R_2\right)
\]  

(3.41)

where each \( \exp(h(K + R_1)) \) operator corresponds to the sixth order scheme described earlier on.

Along with the conservation of the constants of the motion, such a hierarchical approach to the perturbations is the other significant advantage of our symplectic scheme. Other
numerical schemes used in orbit propagation are forced to use a single timestep to calculate all the forces at the same time; compared to the composite symplectic schemes, this constitutes a very significant computational inefficiency.

This scheme is similar to multiple timestep methods, where the Hamiltonian can be separated into two parts with fast-changing and slow-changing dynamics, which would ideally be solved with two different stepsizes [69]. For example, in a leapfrog scheme, firstly the 'slow Hamiltonian' is propagated for half a timestep, followed by $N$ steps of 'fast Hamiltonian' and then another half a timestep of the slow one to complete a full timestep. Note that, for certain values of the timestep, 'numerically induced resonances' may occur, where sharp increases in energy are observed. These are usually seen when $h \omega$ is an integer number, where $h$ is the integration stepsize and $\omega$ is the highest frequency in the system. This is less of a problem in nonlinear systems as the frequency changes with time.

To get around this problem Leimkuhler and Reich [68, 69] proposed the 'reversible averaging' method, which is difficult to generalise for systems with more than one fast degree of freedom. García-Archilla et al [33], on the other hand, proposed the MOLLY scheme, where the 'slow Hamiltonian' is replaced by a time averaged Hamiltonian in a multiple timestep scheme. Obviously, this would require the dynamics associated with the 'slow Hamiltonian' to be nearly linear. It has to be emphasised that, the orbit propagation problem is not limited to simple fast and slow dynamics and can be better described as a multiple frequency system. Therefore the above two schemes are not very useful in our case, nevertheless they underline the need to be careful in the choice of timesteps so as to avoid resonance effects.

A final remark is regarding the number of geopotentials used. SPSAT uses 36 zonal and tesseral harmonics in the orbit propagation. This is in line with Vallado's [112] remark that while more than two hundred terms are known in the geopotential of the Earth with good accuracy, only about 40 is used in high-precision orbit propagation applications.
3.5 Modelling the Relative Motion

3.5.1 Hill's Equations

As we have briefly explained in the literature survey, Hill's Equations are perhaps the most well-known solutions for the relative motion. In this section we will present a summary of how these equations are derived. While the assumptions involved limit the usefulness of these equations, the solution procedure is representative of many other methods in the literature.

We start with the equation of motion for a satellite in a spherically symmetric force field:

\[ \ddot{r} = -\mu \frac{r}{r^3} \quad (3.42) \]

The relative acceleration is given as \( \delta \ddot{r} = \ddot{r}_1 - \ddot{r}_0 \), the difference of the accelerations between target and interceptor satellites. Writing this explicitly:

\[ \delta \ddot{r} = \left( -\mu \frac{\dot{r}_1}{r_1^3} \right) - \left( -\mu \frac{\dot{r}_0}{r_0^3} \right) \quad (3.43) \]

We can then make the substitution \( r_1 = \delta r + r_0 \):

\[ \delta \ddot{r} = \left( -\mu \frac{\delta \dot{r} + r_0}{|\delta r + r_0|^3} \right) - \left( -\mu \frac{\dot{r}_0}{r_0^3} \right) \quad (3.44) \]

We will now make our first approximation and linearise this relative acceleration, assuming that \( \delta r \ll r_0 \) i.e., the satellites are in close proximity. Therefore, we can rewrite the relative acceleration, to the first order as:

\[ \delta \ddot{r} = \left( -\mu \frac{\delta \dot{r} + r_0}{r_0^3 \left( 1 + \frac{2 \delta \dot{r} r_0}{r_0^3} \right)^{3/2}} \right) - \left( -\mu \frac{\dot{r}_0}{r_0^3} \right) \]

\[ = \left( -\mu \frac{\delta \dot{r} + r_0}{r_0^3} \right) \left( 1 \right) - \left( -\mu \frac{\dot{r}_0}{r_0^3} \right) \quad (3.45) \]

Finally, we can rearrange the terms to get:

\[ \delta \ddot{r} = -\mu \frac{r_0^3}{r_0^3} \left[ \delta r - \frac{3 \dot{r}_0 (\delta r r_0)}{r_0^2} \right] \quad (3.46) \]

This is the first order inertial relative acceleration for two satellites in close proximity.

Note that we have not yet made any assumptions as to the shape of the orbit. Integrating
this relative acceleration, it is possible to find the evolution of the relative position and velocity.

However, Hill's method, as well as most of the other methods following it, finds it more useful to convert this relative acceleration vector from the inertial frame to a rotating frame that rests on target satellite. This frame has its $x$ axis collinear with the position vector $r_0$. The $z$ axis is parallel to the angular momentum vector and $y$ axis simply completes the orthogonal system. The relative motion in this rotating coordinate system can be interpreted as the motion of interceptor satellite as seen from the target satellite.

To continue the derivation of Hill's Equations, we further assume that the target satellite is rotating on a circular orbit. This ensures that this local frame is rotating at a constant angular rate, simplifying the equations significantly. As the true anomaly is then equivalent to mean anomaly, the relative equations of motion can be easily written in a time explicit form. Finally, this assumption causes $y$ vector to be in the direction of the target satellite velocity vector.

The resulting set of linearised relative accelerations after this coordinate conversion can be written as:

\[
\begin{align*}
\ddot{x} - 2n\dot{y} - 3n^2 x &= 0 \quad (3.47) \\
\ddot{y} - 2n\dot{x} &= 0 \quad (3.48) \\
\ddot{z} + n^2 z &= 0 \quad (3.49)
\end{align*}
\]

where $n$ is the mean motion of target satellite. Note that the out-of-plane component of the motion is decoupled from the in-plane motion.

The solution of these equations can be found in many orbital mechanics textbooks and can be written as:

\[
\begin{align*}
x(t) &= \dot{x}_0/n \sin(nt) - (3x_0 + 2\dot{y}_0/n) \cos(nt) + (4x_0 + 2\dot{y}_0/n) \\
y(t) &= (6x_0 + 4\dot{y}_0/n) \sin(nt) + 2\dot{x}_0/n \cos(nt) - (6nx_0 + 3\dot{y}_0/n)t + (y_0 - 2\dot{x}_0/n) \\
z(t) &= z_0 \cos(nt) + \dot{z}_0/n \sin(nt)
\end{align*}
\]
3.5. Modelling the Relative Motion

\[ x(t) = x_0 \cos(nt) + (3nx_0 + 2y_0) \sin(nt) \]  
(3.54)

\[ y(t) = (6nx_0 + 4y_0) \cos(nt) - 2x_0 \sin(nt) - (6nx_0 + 3y_0) \]  
(3.55)

\[ z(t) = -z_0 \sin(nt) + z_0 \cos(nt) \]  
(3.56)

While most of the terms in these equations are periodic, there are some secular terms in the alongtrack (y) components of relative position and velocity. These terms correspond to the satellites drifting apart due to a difference in energies, hence a difference in rotation rates.

It is possible to harness these secularly increasing terms to set the initial conditions of the formation such that the two satellites do not drift apart. If the satellites drift apart too much, the first order approximation begins to fail. Obviously, in reality, the forces acting on the satellites are not limited to a spherically symmetric force field and the perturbations due to oblateness effects and drag will eventually cause the satellites to drift apart unless some control is applied.

3.5.2 Perturbations and Relative Motion

In the preceding sections we have explained in detail the perturbations affecting the orbit of a single satellite. While Hill’s Equations describe the relative motion of two satellites in a Keplerian potential, in reality, these perturbations affect the relative motion to a significant extent. As Alfriend et al [2] demonstrated, when compared to numerical simulations including the effect of \( J_2 \) to \( J_4 \) perturbations, the Hill’s Equations solution for a small eccentricity formation scenario has errors on the order of hundreds of metres within a day. Within the last decade many researchers recognised this shortcoming of the Hill’s Equations and proposed various methods to incorporate the effect of \( J_2 \) as well as higher order geopotentials and atmospheric drag to the relative motion (see Sections 2.2.2 and 2.2.3).

For most formation flying missions the success of the mission is dependent upon how accurately a certain relative motion (or distance) is maintained between the two satellites, therefore better modelling of the relative motion is required. In formation flying settings, the separations usually vary from several kilometres to tens of metres. As the satellites are in such close proximity, the forces affecting the satellites have very similar magnitudes.
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As seen in the derivation of Hill's Equations, relative positioning vector is the difference of the position vectors of the satellites. The magnitude of this relative positioning vector is at least an order of magnitude smaller than the position vectors (measured from the centre of the Earth) of the individual satellites. Therefore, the difference of the forces affecting the motion of the satellites will be at least an order of magnitude smaller than the forces affecting the motion of a single satellite.

In light of this discussion, we can write the accelerations of one satellite as a linear expansion around the other satellite in formation; the difference of the accelerations will therefore yield the linearised relative acceleration, as seen in (3.46). Such a linearisation simplifies the treatment of the relative motion to a great extent and nearly all the publications employ similar first order techniques.

Furthermore, the perturbations presented in Table 3.1 can simply be scaled down by an order of magnitude to get an idea on the accelerations that drive the relative motion. If the satellites are similar in shape and mass (which is the case in many formation flying scenarios) and are following similar attitude profiles, the effect of differential drag and solar radiation pressure will be even smaller than their scaled down versions. Similarly, the distance between the two satellites, as seen from the Moon or the Sun will be negligible, therefore it is reasonable to assume that there are no differential third-body effects. Evidently, if the two satellites have a significantly different surface area and mass (as in the case of a small inspection satellite orbiting a bigger mother ship) differential drag and solar radiation effects will be considerably higher and it may be necessary to take these into account.

The effect of geopotentials on the satellite is a significantly more complicated matter. The number and the type of the geopotentials to take into account in the modelling will depend on the accuracy requirements of the application. Even zonal harmonics are crucial as they induce secular effects on the orbit and their absence from the model will immediately result in differential drift between the satellites. That is the primary reason why many researchers have focussed on including $J_2$ dynamics. Wiesel [114] has developed a method that takes into account all the zonal harmonics. However, the tesseral harmonics have been ignored and will still have a significant effect in the separation.
3.5. Modelling the Relative Motion

There are other important effects of the higher order geopotentials that affect the relative motion. Kormos [63] and O'Donnell [85] report the existence of a resonance due to the 15th order terms in the geopotential, manifested as a long periodic along track oscillation. Furthermore, O'Donnell [85] emphasises that, because of the resonance, the 15th order tesseral terms cause a large amplitude oscillation in the relative motion, as large as those cause by the $J_2$ for kilometre size formations. The period of the oscillations caused by the resonance can be as long as several weeks, therefore for short durations they appear as secularly growing. It can be easily seen that a relative navigation and control algorithm with a very simple geopotential model would interpret these as perturbations that need to be corrected, causing unnecessary propellant consumption.

As mentioned in the literature survey section, for relative motion modelling the usual approach is to develop analytical models that take into account the effects of $J_2$ and perhaps a few more geopotentials in some rare cases. Using a numerical propagator we can quantify how the error increases as the geopotential model becomes simpler and for how long such a $J_2$ level propagation would stay accurate as compared to the real motion.

For this experiment, we use the SPSAT software introduced in the previous sections. First, we will investigate the effect of zonal and tesseral harmonics on the motion of the satellite. We define a near-circular orbit with a semimajor axis of 7653.8km. The truth model is a 1000 step/orbit propagation with the $36 \times 36$ geopotential model (i.e., 36 terms in both zonal and tesseral harmonics). The propagations are run for a duration of five days with 100 steps/orbit and they include a various number of zonal and tesseral harmonics. Figure 3.10 illustrates the positioning error evolution in time. As expected, the $2 \times 2$ model has large errors (about 17.5km), whereas the $4 \times 4$ model has kilometre level errors. Unsurprisingly, better modelling reduces the errors significantly. Finally, the errors in the $36 \times 36$ case is caused by the difference in stepsizes.

On the other hand, Figure 3.11 illustrates the errors with respect to this $36 \times 36$ geopotential truth model, for the cases with an axisymmetric geopotential model i.e, with the tesseral terms disregarded. As with the previous case, the more terms in the model, the better will be the modelling. However there is a secularly growing large offset in all cases.

These errors can be explained through these ignored tesseral terms of the energy. The total
Figure 3.10: Absolute positioning error (log scale) with various zonal and tesseral harmonics

Figure 3.11: Absolute positioning error (log scale) with various zonal harmonics
3.5. Modelling the Relative Motion

energy can be written as a summation of smaller Hamiltonians due to various geopotential terms i.e., \( H = H_K + H_2 + H_3 + \ldots + H_{tess} \), where \( H_K \) is the Keplerian potential, \( H_2 \) is the \( J_2 \) potential, \( H_3 \) due to \( J_3 \) and \( H_{tess} \) due to all tesseral terms. Given the initial coordinates of a satellite, different models will obviously yield different \( H \) values. The energy of a satellite determines the mean motion. Evidently, a simple model that includes 4 terms in the geopotential will have a slightly different rotation rate with respect to a 36 term model. Similarly, a model with 36 zonal harmonics will be missing all the tesseral terms in the Hamiltonian when compared to a 36 \( \times \) 36 geopotential model. This difference in energy, leading to a difference in the rotation rates, causes the satellite with an inferior geopotential model to drift away from the true position of the satellite.

We can repeat the above experiment for relative motion to investigate the effect of modelling the potential to the relative positioning accuracy. We define the formation by adding another satellite with a very similar orbit to the previous scenario. We also introduce a semimajor axis difference of 80m, such that the initial separation of about 4km increases to about 45 km after a five day propagation.

Figure 3.12 illustrates the relative positioning error for various zonal and tesseral harmonics included in the model. The graph resemble that of the absolute motion very closely, however, as expected, the errors are about an order of magnitude smaller. The relative positioning error of the 2 \( \times \) 2 model exceeds metre level before the end of the first day, reaching 90m at the end of the five day propagation, corresponding to about 0.2\% of the final separation. While small, this error is still very large in comparison to the stringent accuracy requirements of many formation missions presented in the previous chapter. Metre level errors at the end of the fifth day require at least a 20 \( \times \) 20 model.

Similarly, Figure 3.13 is very similar to its absolute error counterpart given in Figure 3.11. The error due to the absence of axisymmetric terms in the absolute motion was about 25km, whereas for the relative motion this offset is about 0.14km. The above discussion regarding the absolute positioning error and error in the energy is fully valid for the relative motion case as well. The error in the energy difference causes an error in the differential mean motion i.e., the rate at which the satellites are drifting away from each other.

Noting that the literature usually provides analytical solutions including \( J_2 \) zonal harmonic
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Figure 3.12: Relative positioning error (log scale) with various zonal and tesseral harmonics.

Figure 3.13: Relative positioning error (log scale) with various zonal harmonics.
only, the most important conclusion of these experiments is that such analytical methods will be woefully inadequate for high-accuracy long term propagation needs of the state-of-the-art formation missions proposed. The relative orbit estimation accuracy requirements are usually significantly more stringent than the absolute orbit knowledge. The GPS sensors can provide position estimates for the satellite with an accuracy of tens of metres [57]. By contrast, the SAR missions require relative positioning accuracies on the order of centimetres. It may be argued that even a limited accuracy model of the relative motion will be useful within a filter, coupled with high accuracy relative positioning measurements. In fact, many researchers use the simple Hill's Equations within an Extended Kalman Filter [53, 54] or an Unscented Kalman Filter [81] structure for their relative orbit estimators. Similarly, Montenbruck et al [82] proposed a relative navigation algorithm that does not include a dynamic model. However, the low fidelity model of the motion means that a constant stream of these high accuracy measurements are required, using up precious resources of the satellite. We therefore reiterate our claim that significantly better models of the motion will be beneficial in providing high accuracy relative navigation solutions, while enabling us to switch off the relative navigation sensors for extended durations. Secondly, such models will provide long-term accuracy to plan and design missions which can span months and years, as well as to aid in control architecture testing and design.

3.6 Conclusions

This chapter outlined different aspects of the problem of modelling the motion of a satellite orbiting a central body. Firstly, we have discussed the effect of dominant perturbations to the simple two-body motion. This was followed by the introduction of the concept of symplectic integration and the development of a fast, high-precision symplectic orbit propagator which will act as a road map for the relative orbit propagation we will develop in the later chapters.

We have also briefly investigated the effect of the perturbations on relative motion, with a view to form our assumptions in developing the relative orbit propagator. These 'differential perturbations' are an order of magnitude smaller than their counterparts on the absolute motion. The result is that the zonal as well as tesseral harmonics are to be
modelled for high precision close proximity flight. Drag modelling may or may not be crucial, depending on the differential drag profiles between the satellites. For close proximity scenarios with very similar satellite platforms, drag can be neglected. If required, the modularity of the numerical method allows us to plug in a drag model as complicated as is necessary. The effect of solar radiation and lunisolar attractions on the relative motion are also negligible, particularly for the close proximity flight; as seen from the Sun or the Moon, there is virtually no distance between the two satellites. We have also demonstrated that, while the existing literature mostly limits itself to $J_2$ level propagations, high-precision relative orbit propagation is possible with the inclusion of significantly more geopotentials. In addition, tesseral harmonics are important as well.

The symplectic scheme by Palmer et al [19, 45, 78] brings a completely novel approach to orbit propagation. Symplecticness ensures that even the long term propagations will not introduce errors due to poor modelling of the geometric properties of the motion and ensure that energy and angular momentum conservation are observed. Splitting the Hamiltonian and constructing composite schemes enable an unprecedented modularity, flexibility and scalability in tailoring schemes that are suited to the needs of the application. In the development of our relative orbit propagator we will aim to preserve these desirable characteristics.
Chapter 4

Design and Application of a Constellation Initialisation Algorithm

4.1 Introduction

From an orbital control point of view, arguably the most important question for a constellation is how the elements of the constellation will come together (orbit initialisation) and how this configuration can be maintained throughout the mission (station-keeping). This should be achieved in the least time and the most fuel efficient manner possible, without sacrificing practicality. It is worthwhile at this point to elaborate on these three parameters further.

The duration of the manoeuvre is extremely important; for a quick orbit initialisation a simple model would be adequate, as the effects of orbital perturbations take time to accumulate. On the other hand, propellant cost will increase dramatically for shorter manoeuvre durations. For a longer term manoeuvre, the secular effects of $J_2$ and drag will need to be considered and incorporated into the controller algorithm.

Similarly, propellant cost will be an important design driver. Not only will lighter spacecraft cost less to launch, they will also achieve a larger $\Delta V$ per unit of propellant mass.
Furthermore, the constellation lifetime depends largely on the propellant availability for station-keeping manoeuvres.

Finally but no less importantly, practicality of the algorithm for use in the real world should be considered to put the propellant and time requirements into perspective. Satellite platforms usually have maximum and minimum $\Delta V$ limitations within a single firing, attitude determination and control cannot be assumed to be perfect and there may be other mission priorities or unforeseen problems resulting in manoeuvres executed at less than ideal times. The operations team cannot be expected to monitor the satellites constantly, hence a certain level of autonomous operation should also be built-in. Consequently, the orbit initialisation needs to be built around these requirements and limitations; rather than a rigid optimal firing scheme, a sub-optimal yet more flexible scheme may be more desirable.

Disaster Monitoring Constellation (DMC) is a unique international collaboration of five countries: United Kingdom, Algeria, Turkey, Nigeria and China. Each country contributes a remote-sensing minisatellite to the constellation although all the platforms have been built by the Surrey Satellite Technologies Limited (SSTL) in the UK. The satellites are on a sun-synchronous LEO orbit with a capability to take images of anywhere on the Earth with slightly longer than 24 hour intervals.

This chapter details the derivation of a controller algorithm for 3D constellation orbit initialisation, bearing in mind the above design issues. The emphasis of the first half of this chapter will be on the mathematics of the controller algorithm. The second half will introduce the overall software for the Disaster Monitoring Constellation (DMC) as well as providing the details of the real world results. We will very briefly look into the AM/PM constellation as they deal with a somewhat similar problem. Then we will introduce the DMC operational history and structure as well as the unique practical challenges it poses. Finally we will present the real world results from the DMC orbit initialisation experience.

4.1.1 The Disaster Monitoring Constellation

The Disaster Monitoring Constellation (DMC) is a typical Low Earth Orbit remote sensing constellation comprising originally four small satellites equally spaced in phase around a
4.1. Introduction

Figure 4.1: Algerian DMC satellite AlSat

circular sun-synchronous low Earth orbit. Its mission is to monitor natural and man-made disasters and it can provide images of anywhere on the Earth on a near-daily basis [21, 22, 104]. The satellites were built by the Surrey Satellite Technologies Limited (SSTL). The DMC is an international collaboration and therefore the satellites belong to their respective countries, though they share data. The constellation has a planned lifetime of 5 years.

The first satellite of the constellation is owned and operated by Algeria and is called AlSat (see Figure 4.1). It features a 32m resolution three band pushbroom multispectral imager with a 660km swath width. Its mass is approximately 85kg. It was launched in November 2002.

Three other satellites were launched in September 2003, owned and operated by Nigeria, Britain and Turkey, called NigeriaSat, UKDMC and BilSat, respectively. NigeriaSat and UKDMC platforms are essentially identical to AlSat. The Turkish satellite BilSat is a larger and more complicated platform (see Figure 4.2). It carries a 14m resolution panchromatic and a 26m resolution four-band multispectral imager. Launch mass is approximately 125kg.

An additional satellite has since been added to the constellation design to share the phase slot of BilSat. This satellite is owned and operated by China. Named DMC+4, it is a
significantly more complicated platform than the standard DMC platforms; it carries a 4m resolution panchromatic camera as well as the standard pushbroom 32m resolution multispectral imagers of the standard DMC platforms. It was launched successfully in October 2005, although its results will not be presented here.

The satellites are equipped with Global Positioning System (GPS) receivers to collect navigation data. While during normal operations this data is downloaded to the groundstation for processing, there is also provision for onboard orbit determination.

All the platforms employ 3-axis attitude stabilisation with gravity gradient assistance. They also carry an identical butane based low-thrust propulsion system with a total propulsion mass of about 2.3kg. It provides the satellites with a total $\Delta V$ capability of approximately 24m/s, although due to the mass difference this figure is about 16m/s for the case of BilSat.

The constellation is on a sun-synchronous orbit at 686km altitude, with an LTAN (Longitude of Ascending Node) of 10AM. The constellation design dictates that four satellites should be on a single orbital plane, distributed with a 90 degree phase separation. This phase separation between the satellites should be achieved and maintained with an accuracy of $\pm 3.6$ degrees as a design requirement. This design enables the constellation to maintain a near-daily global coverage of the Earth for imaging purposes. Figure 4.3 illus-
4.1. Introduction

Figure 4.3: DMC daily groundtrack and swath width

trates the groundtrack and imager footprint (i.e., swath width) of the constellation and how it covers the globe in 24 hours. Note that, all imagers overlap around polar region but the constellation design ensures that there is no overlap on the equator region.

The operating conditions for the DMC satellites were set at ±30 minutes of the 10AM requirement. This has to be maintained over the 5 year lifetime of the DMC constellation. This choice of LTAN stems from the imaging requirements of the mission. The atmospheric conditions are more favourable in the morning and the long shadows in the earlier hours facilitate the distinguishing of the features on the ground. On the other hand, the imagers require a certain amount of incident light from the ground to register correctly exposed and sharp images; light levels are obviously much less nearer the local sunrise than noon. A final constraint is the average power onboard a satellite; the side panels of a satellite on an early morning orbit receive more sunlight than an orbit on a noon orbit, in which the sunlight will illuminate the top panel rather than the sides. To accommodate these competing requirements, 10AM LTAN with a ±30 minute window has been chosen as the best compromise.

As mentioned before, the near-daily global coverage of the DMC relies on the satellites being on a single plane with equal spacing in phase. Inspecting Figure 4.3, it can be easily seen that any errors in phase or LTAN in this configuration will lead to overlaps in sensor footprint. This results in some locations on the globe, particularly on the equator region,
to not be imaged. The imagers on the DMC have a swath width of 660km and each day the satellites make 14.6 rotations, thereby covering 9636km of the equator. The four satellite constellation therefore covers 38544km of the 40075km equator in 24 hours. Hence, at each orbit, there is a total gap of about 105km left between the swath widths that remains uncovered. Between each satellite, this corresponds to about 29km between each swath width or 62 seconds in LTAN before the image swaths overlap. This quick calculation illustrates the importance of keeping the individual orbital planes as close as possible to the mean constellation orbital plane.

There are other aspects of the DMC that affect constellation control throughout the mission. The satellites are owned by four different countries and operated by groundstations in each country. Hence, the advantages of centralised operations are somewhat sacrificed; although the critical commands such as satellite manoeuvre data for orbit corrections are issued from the UK based SSTL groundstation, actual upload of the files are carried out by the respective groundstations, each to their respective satellite.

In addition to the certain practical limitations imposed by this decentralised structure, there are other restrictions as to what manoeuvres the satellites can execute. For example, the firings are allowed in the alongtrack or crosstrack direction only, due to attitude determination and control limitations. The thrusters can be fired for a certain duration. A certain time has to be left between firings for the attitude to stabilise and GPS data to be collected to evaluate the effects of the firing. Perhaps more importantly, the constellation has a prescribed long term LTAN behaviour and the controller algorithm should be able to generate the crosstrack firings to comply with this criterion.

### 4.2 Orbit Acquisition Problem

There are many problems to be addressed in the design and operation of the DMC, a typical Low Earth Orbit Earth Observation constellation. The tight, near-24 hour global coverage and the fixed pass time requires precise orbit acquisition and maintenance.

To illustrate the concept of orbit acquisition better, we can consider an example. Figure 4.4 shows the configuration of the DMC about a week after the launch of the three satellites.
4.2. Orbit Acquisition Problem

The three satellites are seen to trail about 160 degrees behind AlSat and, although not visible on this graph, they have small inclination and LTAN difference with respect to AlSat. The figure on the right shows the design configuration following a successful orbit acquisition, with the four satellites at 90 degree phase separation and on the same orbital plane.

As we described in Section 3.3, the non-spherical Earth effects, chiefly $J_2$ zonal harmonics, and the secular effects of the solar attraction at the sun-synchronous orbits, cause a significant drift of the orbital plane, hence LTAN. The zonal harmonics also alter the mean motion, while drag causes the semimajor axis to decay slowly, inducing in-plane phase drift and secular out-of-plane motion of the orbital plane. Ideally, the individual satellites of the constellation stay on the same orbital plane, with some phase separation. In this ideal case the aforementioned natural forces will have minimal effect on the constellation, as the constellation will experience perturbation effects equally and will drift as a whole. However, if the semimajor axes or inclinations of the satellites differ even by a small amount, this will cause the satellites to drift at different rates. The semimajor axis difference between one satellite and the rest of the constellation will cause it to have an in-plane relative drift out of its designated phase. The inclination difference will cause the orbital plane of the satellite to separate from the constellation orbital plane.
In addition to any variation in orbital parameters, if the satellites of the constellation differ in their mass and physical dimensions, as is the case in DMC, the effects of the drag force will also differ for each satellite, resulting in different semimajor axis decay rates. This is particularly of importance for a constellation with a planned lifetime of 5 years, as any small error at the beginning of the lifetime will have significant effects at later stages unless corrected.

To illustrate the relative in-plane effect of $J_2$ zonal harmonics and drag (as well as the shortcomings of a purely Keplerian model), a numerical example will be helpful. Consider a 98 degree inclination LEO satellite at 600km altitude with a modest drag drop rate of 2m/day. One can predict the phase drift due to an initial alongtrack firing of 1m/s. The phase drift prediction after 90 days under two-body model is -191.54 degrees. With $J_2$, it is -191.34 degrees and with $J_2$+drag -182.01. It is important to bear in mind that the effect of $J_2$ on the phase increases linearly with time and the effect of a linear drop in altitude due to drag changes the phase quadratically in time. These effects therefore will accumulate over time and cannot be ignored, except for very short timescales.

Not only the in-plane parameters phase and semimajor axis, but also the orientation of the orbital plane (inclination and Right Ascension of Ascending Node (RAAN)) should also be accurately fixed at the orbit acquisition and maintained throughout the mission. This changes the problem from a relatively well known and simple 2D case to a more complex full 3D case. The problem is further complicated by the fact that the drift of the orbital plane is coupled to the semimajor axis via the $J_2$ secular drift term.

In any mission, lowering the propellant consumption is an important design driver and all the more so for small satellites where mass and volume constraints are very restrictive. The propellant requirements of the orbit acquisition varies to a great extent with the duration of the manoeuvre. The choice of duration should be based on the other mission parameters but a budget of a few m/s per satellite means the manoeuvre duration will be on the order of some months, rather than days or weeks.

In addition to these considerations, in developing a control architecture for the DMC, there are real world limitations as described in the previous section that need to be accommodated. This control architecture should require minimal input from groundstation
4.2. Orbit Acquisition Problem

staff to reduce their workload. Furthermore it should be robust enough to be able to
easily deal with unforeseen problems. These practical issues take higher precedence than
putting together a very complicated optimal scheme with limited modelling accuracy (such
as two-body only) and little applicability.

To summarise, the orbit acquisition problem should be tackled in three dimensional space
and the solution should be flexible enough to accommodate different scenarios so as to
maximise real world applicability. Secular Earth oblateness effects up to the $J_2$ level as
well as the drag effects should be included, as their effects are cumulative. The choice for
the $J_2$ level is based on the fact that their effects are 1000 times bigger than the nearest
oblateness terms. The eccentricity is assumed very small (same order as $J_2$), which is a
valid assumption for many, if not most of, LEO constellations. Its effects, along with the
short and long periodic Earth oblateness terms, are also ignored, as they do not cause an
error accumulation and therefore are deemed much less important. Note that, the study
will be limited to satellite platforms with low thrust, impulsive (a few tens of metres per
second level) thrusters, hence the continuous or very large thrusters will be out of the scope
of this study.

4.2.1 Epicycle Description of an Orbit

Before delving into the orbit correction issues, it is imperative to introduce the set of
equations used in this work. This short introduction draws mainly from [46] and [47].
For further information on epicycle theory, the reader is referred to also [48]. This new
coordinate set simplifies the derivation of the controller equations significantly.

The epicycle equations describe the position of the satellite using four, rather than the
usual three, parameters. The first two are the radial and azimuthal angle polar coordinates,
denoted $r$ and $\lambda$, respectively. To describe the out-of-plane coordinates the orbital elements
inclination $I$ and ascending node $\Omega$ are used. $\lambda$ is chosen to be the argument of latitude -
angle measured from the ascending node to the satellite.

The description of the near circular motion of the satellite under an axisymmetric potential
can be summarised by the epicycle expressions in Equation (4.1). However, the derivation of
the orbit acquisition equations require that the periodic perturbations such as eccentricity
effects as well as short and long periodics due to Earth oblateness are disregarded (these effects are shown collectively as $\Delta$ terms in the equations). Therefore, the simplified equations are given by:

$$
\begin{align*}
\begin{cases}
r = a(1 + \varrho) + \Delta_r \\
I = I_0 + \Delta_I \\
\Omega = \Omega_0 + \vartheta \alpha + \Delta_\Omega \\
\lambda = (1 + \kappa) \alpha + \Delta_\lambda
\end{cases}
\end{align*}
$$

where $a$ is the mean semimajor axis defined through the conserved orbital energy $\varepsilon$ by $a = -\mu/(2\varepsilon)$, and $\mu$ is the gravitational parameter. The epicycle phase, $\alpha$, is defined as $\alpha = n(t - t_E)$ where $n$, is the epicycle frequency, obtained through $a^3n^2 = \mu$. $t_E$ is time of equator crossing at ascending node. Hence, this is simply mean anomaly defined from the equator crossing. $I_0$ and $\Omega_0$ inclination and right ascension of the ascending node at $t = t_E$.

The secular variations are described by the quantities $\varrho$, $\vartheta$ and $\kappa$ which are caused by even zonal harmonics. The first of these describes a secular shift in the mean orbital radius. The secular change in the ascending node is described by $\vartheta$ which gives a linear variation of $\Omega$ with time. The secular drift in the argument of perigee is described by $\kappa$ which causes a drift in the argument of latitude. The secular $J_2$ coefficients are given as:

$$
\begin{align*}
\varrho &= -\frac{1}{4} J_2 \left( \frac{R}{a} \right)^2 (2 - 3 \sin I_0^2) \\
\vartheta &= -\frac{3}{2} J_2 \left( \frac{R}{a} \right)^2 \cos I_0 \\
\kappa &= \frac{3}{4} J_2 \left( \frac{R}{a} \right)^2 (4 - 5 \sin I_0^2)
\end{align*}
$$

For LEO satellites, the orbit perturbation caused by the atmospheric drag may not be negligible when longer time scale prediction is required. Major contribution of the drag occurs to the semi-major axis and epicycle phase which is approximated such that,

$$
\begin{align*}
a^* &= a \left( 1 - 2 \frac{B^*}{R} \alpha \right) \\
\alpha^* &= \alpha + \frac{3}{2} \frac{B^*}{R} \alpha^2
\end{align*}
$$

where $R = 6378.137$ km the radius of the Earth and $B^*$, the normalised drag coefficient, given by:

$$
B^* = \frac{1}{2} C_D \frac{A}{m_s} \rho_0 R
$$
4.2. Orbit Acquisition Problem

where $C_D$ is the drag coefficient of the satellite, $A_\perp$ is the time-averaged cross-sectional area of the satellite normal to the velocity vector, $m_s$ is the mass of the satellite and $\rho_0$ is the atmospheric density at the nominal altitude of the orbit. Equation (4.3) is thus solved by assuming $\rho_0$ and $A_\perp$ to be constants, implying coefficient $B^*$ is also constant.

4.2.2 Adjustment of LTAN and Orbital Phase

The underlying principle of the in-plane phase acquisition is to attain the required relative phase drift rate with respect to a reference satellite, so that the ‘firing’ satellite will travel to the desired phase slot within the required amount of time. This relative drift rate can be reached by altering the semimajor axis difference between the ‘firing’ and the reference satellites, via alongtrack firings. Once the relative phase difference is equal to the desired value, another set of alongtrack firings are to be executed to negate the semimajor axis difference, hence the relative phase drift. The motion therefore can be viewed as made up of three building blocks: ‘firing’, ‘coasting’ and ‘firing.’ Once the dynamics of these building blocks are calculated, a more realistic motion profile of several ‘firing’s with ‘coasting’s of varying duration in between can be generated, by simply putting together these building blocks.

The basic principle of the inclination correction and LTAN acquisition (out-of-plane part of the orbit acquisition) is quite similar to that of the semimajor axis correction and phase acquisition. The inclination difference between the reference and the ‘firing’ satellites can be manipulated, which in turn result in a differential $J_2$ induced drift of the orbital plane, much in the same way as manipulating semimajor axis difference to attain a desired relative phase drift rate. Setting the inclination difference to the required drift rate, the desired LTAN can be reached within a given time. Note that, this differential drift effect is small and takes time to change the LTAN substantially. For a sun-synchronous LEO, a 0.1 degree inclination difference causes a differential LTAN drift rate of approximately 0.01 deg/day. While it is possible to reach the same effect by manipulating the semimajor axis difference, executing crosstrack manoeuvres only ensures that LTAN acquisition is uncoupled from relative phasing, which is very sensitive to semimajor axis differences.

The other possible method of LTAN change is to execute crosstrack firings at the poles,
changing LTAN 'directly,' rather the aforementioned 'indirect method' of drifting towards the right LTAN. A comparison of the two methods are given in Section 4.4.4.

It may not always be easy to visualise the complete manoeuvre sequence, which is complicated by the coupled semimajor axis and inclination effects on the LTAN. The LTAN and phase relative drift rates are crucial for the problem and a graphical approach simplifies the understanding of the variation of these two parameters and their interaction.

Figure 4.5 illustrates the effect of a unit $\Delta V$ on the in-plane and out-of-plane relative drift rates. All the possible firing directions, from full alongtrack to full crosstrack, can be visualised as a circle on the alongtrack-crosstrack plane. However, the projection of this circle on the relative drift rates is an ellipse, as seen in the figure. Its orientation is a function of semimajor axis and inclination.

Figure 4.5 illustrates the effect of a unit $\Delta V$ on the in-plane and out-of-plane relative drift rates. All the possible firing directions, from full alongtrack to full crosstrack, can be visualised as a circle on the alongtrack-crosstrack plane. However, the projection of this circle on the relative drift rates is an ellipse, as seen in the figure. Its orientation is a function of semimajor axis and inclination.

Suppose that, initially the relative drift rates are zero, which corresponds to the origin in Figure 4.5. To attain a particular drift rate, a firing should be executed. A crosstrack firing will change only the LTAN drift rate, hence it will move the current location on the graph along the vertical axis only. On the other hand, an alongtrack firing changes both the phase and LTAN drift rates as the natural out-of-plane drift due to $J_2$ is a function of semimajor axis as well. Therefore, an alongtrack firing will signify a move on a skewed axis on the graph. The slope of this skewed axis is a function of semimajor axis and inclination but is essentially constant as long as the firings involved are small. In theory, it is possible to execute a firing in any direction, but this is merely an arbitrary combination of crosstrack
4.2. Orbit Acquisition Problem

Figure 4.6: Relative phase and LTAN drift rate profile

and alongtrack directions, therefore we will focus on the alongtrack and crosstrack firings only.

Now consider a combined manoeuvre case with two sets of alongtrack firings and a single set of crosstrack firings to lock the satellite on the correct orbital plane and correct phase slot. The relative LTAN and phase drift rates are illustrated in Figure 4.6. The initial conditions (point O) dictate that the satellite has an initial relative drift in both axes, as there will invariably be an initial semimajor axis and inclination difference with respect to the reference satellite. The aim is to move to the correct drift rates on this graph via firings, wait for the required waiting duration and fire again to end up at the origin, locking the satellite on the correct orbital plane and at the correct phase slot. Figure 4.7 shows the change in relative phase and relative LTAN. This manoeuvre is plotted on the figure, where the satellite ends up at a certain relative phase and zero LTAN difference.

The satellite should therefore follow the trajectory OABE (Figure 4.6), with waits at A and/or B, to reach the required LTAN and phase slots. Hence, an alongtrack firing is executed to change the semimajor axis from $a_0$ (point O) to $a_c$ (point A). Since there is a single crosstrack firing opportunity, it has to move the satellite from point A to point B.
The timing of this AB manoeuvre is critical and is dictated by the initial and the required LTANs. Consider the case where the initial time is \( t = 0 \), the first alongtrack firings are executed at time \( t = T_1 \) and the second alongtrack firings at \( t = T_2 \) and the time spent during the firings is disregarded. If the crosstrack manoeuvre AB is executed immediately after the first alongtrack firing, the total relative LTAN change will be \( \delta \Omega_1 T_1 \). Similarly, if the crosstrack firings are executed immediately before the second alongtrack firings, at the end of the coasting period, it will be \( \delta \Omega_2 T_2 \). These two extremes illustrate the boundaries of possible relative LTAN change. If the required LTAN change lies between the two boundaries, it is possible to attain the required LTAN via a single crosstrack firing which also corrects the inclination. This would be apparent on Figure 4.7 rather than Figure 4.6.

Note that, in reality, each jump on Figure 4.6 could be broken down into a series of smaller firings in either alongtrack or crosstrack direction.

### 4.3 Controller Algorithm

#### 4.3.1 Assumptions

The basic assumptions (or the constraints) adopted within the derivation are as follows:

- The terms at order \( O(J^2) \) are ignored in the derivation, however, some small secular terms (such as the effects of drag) are still kept as the phasing duration can run into months and these effects accumulate with time. This ordering scheme is the rationale behind most of the assumptions below.
4.3. Controller Algorithm

- $J_2$ secular drift terms are kept in the phase equations, while eccentricity, short and long periodics are assumed negligible. While they have significant amplitudes, their effects do not accumulate over time.

- Firings are on the alongtrack direction only for the phase acquisition and on the crosstrack direction for the LTAN acquisition, impulsive and are of small magnitude (a few tens of metres per second maximum). This is consistent with our above approximation of keeping the $O(J_2)$ terms only.

- The initial orbits of the firing and the reference satellites are assumed in close proximity (a few tens of kilometres of semimajor axis difference). Therefore, $J_2$ coefficients $\rho$, $\kappa$ and $\vartheta$ are assumed constant and equal for both satellites.

- Each alongtrack and crosstrack $\Delta V$ changes the semimajor axis and inclination by a constant amount which depends only on the reference orbit.

- As the crosstrack firings are small, they are assumed to have negligible effect on the alongtrack velocity and semimajor axis.

4.3.2 Phase and Semimajor Axis Acquisition

As can be seen from the firing profile given in Figure 4.8, the strategy is to change the semimajor axis by $N$ firings, with $t_w$ long coasting times in between; wait for the second set of firings and execute $M$ firings to correct the semimajor axis. Note that, the area under this curve is proportional to the total phase change, therefore this figure is a convenient means to visualise the phasing manoeuvre and the interplay between the semimajor axis difference and phase drift. Similarly, the magnitude of the semimajor axis change is proportional to the $\Delta V$ cost of the manoeuvre.

Therefore, the whole phase acquisition process has two basic building blocks: coasting and firing.

In the coasting phase (disregarding the eccentricity effects), the mean relative phase drift rate is:

$$\delta \dot{\lambda} = \delta n (1 + \kappa) \quad (4.5)$$
where the $\delta$ terms refer to the difference between the firing and the reference satellites, e.g. $\delta n$ refers to the difference between the mean motions of the firing and the reference satellite ($n_{\text{fire}} - n_{\text{ref}}$).

On the other hand, the alongtrack firings change the semimajor axis, hence the mean motion:

$$
\Delta n_1 = -\frac{3n_0 \Delta a_1}{2a_0} \\
\Delta n_2 = \frac{3n_0 \Delta a_2}{2a_0}
$$

(4.6)

where $\Delta$ terms refer to the difference between the before and after firing values, e.g. $\Delta n$ refers to the mean motion difference between before and after the firing. $a_0$ and $n_0$ are the initial semimajor axis and mean motion of the reference satellite, respectively. $\Delta a_1$ is the semimajor axis change with each firing of the first set and $\Delta a_2$ is the semimajor axis change with each firing in the second set. Note that, as the sign convention the first set of firings are assumed to be in positive alongtrack and the second set is assumed to be in the negative alongtrack directions, respectively, although the correct firing directions will be determined by the ultimate solution of the velocities.

Using these two building blocks, we will now derive the variation of the mean motion difference, semimajor axis difference and phase difference throughout the manoeuver.

The relative mean motion at the end of the first set of firings,

$$
\delta n_N = \delta n_0 + N \Delta n_1
$$

(4.7)
and at the end of the second set of firings,

$$\delta n_{N+M} = \delta n_0 + N\Delta n_1 + M\Delta n_2$$  \hspace{1cm} (4.8)

The semimajor axis difference \(\delta a^* = a^*_\text{ture} - a^*_\text{ref}\) after \(N\) and \(M\) firings in two sets of alongtrack firings:

$$\delta a^* = \delta a_0 + N\Delta a_1 - M\Delta a_2 + \delta a_{\text{drag}}$$  \hspace{1cm} (4.9)

where \(\delta a_0\) is the initial semimajor axis difference, \(\delta a^*\) the final semimajor axis difference and \(\delta a_{\text{drag}}\) the semimajor axis change due to differential drag. The \((.)^*\) notation means the parameter takes drag into account, consistent with (4.3). Note, once again, \(\Delta a_1\) and \(\Delta a_2\) terms are the semimajor axis changes with each firing of the first and the second sets.

The phase difference between the firing and the reference satellites comprises the following parts (see Figure 4.8):

- initial phase difference and the phase difference accumulated due to drag during the whole \(t_{tot}\)
  $$\delta \lambda_0 + \delta \lambda_{\text{drag}}$$

- phase difference accumulated during the first set of firings \((N\) firings, with \(t_w\) time in between)
  $$\delta n_1 t_w + \delta n_2 t_w + \delta n_3 t_w + \ldots + \delta n_{N-1} t_w)(1 + \kappa)$$

- phase difference accumulated during coasting time between two sets of firings
  $$\delta n_N(t_{tot} - (N - 1)t_w - (M - 1)t_w)(1 + \kappa)$$

- phase difference accumulated during the second set of firings \((M\) firings)
  $$(\delta n_{(N+1)} t_w + \delta n_{(N+2)} t_w + \delta n_{(N+3)} t_w + \ldots + \delta n_{(N+M-1)} t_w)(1 + \kappa)$$

Putting these terms together, we can write the relative phase evolution as:

$$\frac{\delta \lambda^*}{1 + \kappa} = \frac{\delta \lambda_0}{1 + \kappa} + \frac{\delta \lambda_{\text{drag}}}{1 + \kappa} + \left[ \delta n_1 t_w + \delta n_2 t_w + \ldots + \delta n_{N-1} t_w \right]$$

$$+ \left[ \delta n_N(t_{tot} - (N - 1)t_w - (M - 1)t_w) \right]$$

$$+ \left[ \delta n_{N+1} t_w + \delta n_{N+2} t_w + \ldots + \delta n_{N+M-1} t_w \right]$$  \hspace{1cm} (4.10)
Summing these series, (4.10) can be written in a more compact form:

\[
\frac{\delta \lambda^*}{1 + \kappa} = \frac{\delta \lambda_0}{1 + \kappa} + \frac{\delta \lambda_{drag}}{1 + \kappa} - \left[ \frac{N(N-1)}{2} \Delta n_1 + \frac{M(M-1)}{2} \Delta n_2 \right] t_w + [\delta n_0 + N \Delta n_3] t_{tot}
\] (4.11)

A simplified drag model will be employed for the formulation [46]. The effect of drag is chiefly on phase and semimajor axis, given by Equation (4.3) but using \( \lambda \) instead of \( \alpha \). As the semimajor axes of the reference and firing satellites are assumed very close, as well as the firings very small, the variation in drag due to semimajor axis changes is assumed negligible. Furthermore, because of the closely separated orbits assumption, \( n_{0fire} \), can be assumed to be approximately equal to \( n_{0ref} \) (the difference in mean motions is no more than \( \%0.1 \)). Consequently, so far as the drag term in the firing satellite is concerned:

\[
\lambda_{fire} \cong n_{0ref} t_{tot} (1 + \kappa)
\] (4.12)

All the above assumptions can be easily validated; since the differential drag effects are already small \( (O(J_2)) \) or smaller, the ‘closely separated orbits’ and ‘small firings’ assumptions can only cause a second order error \( (O(J_2^2)) \).

Inserting the above definition of \( \lambda_{fire} \) into the expressions for \( \lambda^* \) and \( a^* \) (Equation (4.3)) for the reference and firing satellites and taking the difference, the greatly simplified differential drag effect equations can be obtained.

\[
\delta a_{drag} = -2 \left( \frac{a_{ref}}{R} \right) [n_{0ref} t_{tot}] \delta B^* 
\] (4.13)

\[
\delta \lambda_{drag} = \frac{3}{2} \left( \frac{a_{ref}}{R} \right) [n_{0ref} t_{tot}]^2 \delta B^* 
\] (4.14)

where \( \delta B^* = B^*_{fire} - B^*_{ref} \), the normalised drag coefficients, given in (4.4). All the terms in the above equations are known or specified, the resultant drag terms are not a function of the firings.

Equations (4.9) and (4.11) are the two necessary equations to solve for the unknowns; namely, semimajor axis changes due to firings, \( \Delta a_1 \) and \( \Delta a_2 \).

The semimajor axis and the phase differences at the left hand side of these two equations are defined as the ‘required’ values for the controller equations. The ‘required’ semimajor
axis difference can be assumed zero, as the reference semimajor axis is usually defined as the target value. The required phase difference \( \delta \lambda_{\text{req}} \), however, is one of the inputs and is to be specified.

Combining the two equations in the matrix form and solving for the two unknowns, the controller equation finally becomes:

\[
\begin{bmatrix}
\Delta a_1 \\
\Delta a_2 
\end{bmatrix} = C_1 \begin{bmatrix} 1 \\ C_2 \\ C_3 \end{bmatrix}
\]

The known coefficients \( C_1, C_2 \) and \( C_3 \) are given by,

\[
C_1 = \frac{1}{\frac{1}{2} MN \left[ (-M - N + 2) t_w + 2 t_{\text{tot}} \right]}
\]

\[
C_2 = (\delta a_0 + \delta a_{\text{drag}}) \left( \frac{1}{2} M (M - 1) \right) t_w
\]

\[
+ M \left( \frac{-2 a_0}{3 n_0} \right) \left( \frac{\delta \lambda_{\text{req}} - \delta \lambda_0 - \delta \lambda_{\text{drag}}}{1 + \kappa} - \delta n_0 t_{\text{tot}} \right)
\]

\[
C_3 = - (\delta a_0 + \delta a_{\text{drag}}) \left( \frac{1}{2} N (N - 1) t_w - N t_{\text{tot}} \right)
\]

\[
+ N \left( \frac{-2 a_0}{3 n_0} \right) \left( \frac{\delta \lambda_{\text{req}} - \delta \lambda_0 - \delta \lambda_{\text{drag}}}{1 + \kappa} - \delta n_0 t_{\text{tot}} \right)
\]

In summary, the main controller equation is given in (4.15) (with the coefficients in (4.16)), while the required drag terms are given in (4.13).

The transition between the semimajor axis change to required firing magnitude \( \Delta V_{\theta_1} \) is given by the equation (c.f. [4]):

\[
\Delta V_{\theta_1} = \frac{n_0 \Delta a}{2(1 + \rho + \kappa)}
\]

Note that, due to the sign convention employed, \( \Delta V_{\theta_2} \) is to be multiplied by (-1).

4.3.3 LTAN and Inclination Acquisition

The similarity of LTAN/inclination and phase/semimajor solutions is evident upon comparison of Figure 4.8 and Figure 4.9. Instead of changing the semimajor axis via alongtrack
firings at perigee/apogee, the inclination is to be changed via crosstrack firings at the ascending node. The linear relative phase change is thus substituted by the linear relative LTAN change. The effect of uncontrolled semimajor axis differences and drag are also taken into account. As per the case in the in-plane phasing, there is a $t_w$ long waiting time in between the firings and a long coasting time (usually weeks to months) in between the two sets of crosstrack firings. The second set of firings equate the inclination to the reference value, thereby 'killing' the relative drift.

The LTAN acquisition process has two basic building blocks: coasting and firing. Similar to the in-plane phasing in the preceding section, we will now derive the evolution of relative inclination and relative LTAN throughout the manoeuver, using these two building blocks.

The effect of each crosstrack firing to modify the inclination can be summarised with the following equation [4]:

$$\Delta I = \frac{\cos \lambda}{\alpha_0 \beta_0} \Delta V_c$$  \hspace{1cm} (4.18)

where $\Delta$ terms refer to the difference between the before and after firing values. Assuming all the crosstrack firings take place at the ascending node, the $\cos \lambda$ term vanishes. Firing at the node enables to impart maximum change to the inclination.

The inclination, after $P$ and $Q$ firings in two sets of crosstrack firings:

$$I_{fire} - I_{ref} = \delta I = \delta I_0 + P\Delta I_1 - Q\Delta I_2$$  \hspace{1cm} (4.19)

where $\delta I_0$ is the initial inclination difference and $\delta I$ is the inclination difference at the end of the firings. $\Delta I_1$ and $\Delta I_2$ are the inclination changes by each crosstrack firing in the first
and the second sets, respectively.

In the coasting phase, the mean LTAN (or equivalently Right Ascension of Ascending Node (RAAN)) drift rate is:

$$\delta \frac{\delta \omega}{\delta I} (\delta I) + \frac{\partial \omega}{\partial a} (\delta a)$$

$$= \left( n_0 \frac{\partial \theta}{\partial I} \right) \delta I + \left( n_0 \frac{\partial \theta}{\partial a} + \nu \frac{\partial n_0}{\partial a} \right) \delta a$$

$$= \left( -n_0 \theta \tan I_0 \right) \delta I + \left( -\frac{7 n_0 \theta}{2 a_0} \right) \delta a$$

where the $\delta$ terms refer to the difference between the manoeuvring satellite and the reference orbit elements.

From (4.19), the inclination difference at any given time,

$$\delta I_i = \begin{cases} 
\delta I_0 + i \Delta I_1 ; & i \leq P \\
\delta I_0 + P \Delta I_1 - (i-P) \Delta I_2 ; & P < i \leq P + Q 
\end{cases}$$

Integrating (4.20) and using the above shorthand notation, the LTAN difference between firing and reference satellites at any given time can be written as:

- initial LTAN difference and the LTAN difference accumulated due to semimajor axis difference
  $$\delta L_0 + \delta L_\alpha$$

- LTAN difference accumulated during the first set of firings ($P$ firings)
  $$\delta I_1 t_w + \delta I_2 t_w + \delta I_3 t_w + \ldots + \delta I_{P-1} t_w$$

- LTAN difference accumulated during coasting time between two sets of firings
  $$\delta I_P (t_{tot} - (P - 1)t_w - (Q - 1)t_w)$$

- LTAN difference accumulated during the second set of firings ($Q$ firings)
  $$\delta I_{P+1} t_w + \delta I_{P+2} t_w + \delta I_{P+3} t_w + \ldots + \delta I_{(P+Q-1)} t_w$$
More explicitly, the LTAN expression can be written as:

\[ \delta L = \delta L_0 + \delta L_{\delta a} + \left[ \delta I_1 D t_w + \delta I_2 D t_w + \ldots + \delta I_{P-1} D t_w \right] + \left[ \delta I_P D (t_{tot} - (P - 1) t_w - (Q - 1) t_w) \right] + \left[ \delta I_{P+1} D t_w + \delta I_{P+2} D t_w + \ldots + \delta I_{P+Q-1} D t_w \right] \]  

(4.22)

where \( D = -\dot{\theta}_n \tan I_0 \). Note that the first set of firings are assumed to be in positive crosstrack and the second set is in the negative crosstrack directions, respectively, although the correct firing directions will be determined by the ultimate solution of the velocities.

Summing the series, (4.22) can be written in a more compact form:

\[ \delta L = \delta L_0 + \delta L_{\delta a} + \left[ \frac{-P(P-1)}{2} \Delta I_1 + \frac{-Q(Q-1)}{2} \Delta I_2 \right] D t_w \]  

(4.23)

Now that the basic relative LTAN evolution is derived, it is convenient to add in the effect of differential semimajor axes at this point. Extracting the semimajor difference induced RAAN/LTAN drift from (4.20) and integrating yields the total LTAN offset caused by the semimajor axis difference \( \delta a \) during the time \( t_{tot} \):

\[ \delta L_{\delta a} = \frac{\partial \Delta \dot{\Omega}_2}{\partial \delta a} (\delta a) t_{tot} \]

(4.24)

The drag affects the LTAN drift rate through the change it causes on semimajor axis. It is possible to rewrite (4.24) as,

\[ \delta L_{\delta a} = \left( \frac{-7 n_0 \dot{\vartheta}}{2 a_0} \right) (\delta a_0 + \frac{\delta a_{\text{drag}}}{2}) t_{tot} \]

(4.25)

where \( \delta a_{\text{drag}} \) is given by (4.13). Note that, the drag effects are halved, as the semimajor axis decay is assumed to be linear with respect to time. It is important to emphasise at this point that, while the effect of drag on the crosstrack motion might seem very small, it does accumulate with time. Also we make no assumptions as to the allowable drag difference between the satellites. We are therefore including it for the sake of generality.
4.3. Controller Algorithm

The above equation accounts for the effect of semimajor axis difference between the two satellites. Therefore, the LTAN values will be correct as of the end of $t_{tot}$. Obviously, if they are not corrected (hence the relative drift not neutralised), the relative LTAN drift will continue after this time as well. In Section 4.4.1, the effect of the semimajor axis difference, which may be substantial particularly for high inclination cases, is investigated in greater detail.

Equations (4.19) and (4.23) are the two equations necessary to solve for the inclination changes due to each firing, $\Delta I_1$ and $\Delta I_2$. The required inclination difference, which is zero, and the required LTAN difference $\delta L_{req}$ are set to be the values at the left hand side of the two evolution equations. Writing the two equations in matrix form and solving for the two unknowns $\Delta I_1$ and $\Delta I_2$, the controller equation becomes:

$$\begin{bmatrix} \Delta I_1 \\ \Delta I_2 \end{bmatrix} = C_1 \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$$  \hspace{1cm} (4.26)

The coefficients $C_1$, $C_2$ and $C_3$ are given by:

$$C_1 = \frac{2}{PQ\left(\left(-P - Q + 2\right)t_w + 2t_{tot}\right)}$$

$$C_2 = \delta I_0 \frac{Q(Q-1)}{2} t_w$$

$$+ Q \left( \frac{\delta L_{req} - \delta L_0 - \delta L_{6a}}{D} - \delta I_0 t_{tot} \right)$$

$$C_3 = -\delta I_0 \frac{P(P-1)}{2} t_w$$

$$+ P \left( \frac{\delta L_{req} - \delta L_0 - \delta L_{6a}}{D} \right)$$

where, once again, $D = -\delta n_0 \tan I_0$ shorthand is used.

The transition between the inclination change to required crosstrack firing magnitude ($\Delta V_{1,1}$) is given by the equation:

$$\Delta V_{1,1} = a_n n_0 \Delta I$$ \hspace{1cm} (4.28)

The firings are assumed to take place at the ascending node crossing. Because of the sign convention employed, $\Delta V_{1,2}$ should be multiplied by (-1).
4.3.4 Phase and LTAN Controllers - ‘Single Manoeuvre’ Case

So far, the two sets of controller equations derived have taken into account a scenario comprising two sets of firings, with a coasting time in between (hence a ‘double manoeuvre’ case). In reality, however, a ‘single manoeuvre’ case i.e., a single set of firings, may mean better fuel utilisation, provided that the initial conditions are favourable. More importantly, in a real application, when the first set of firings is complete, the firetable generated should be updated regularly, to take into account the effect of firing anomalies and the unaccounted perturbations. This is analogous to this so-called ‘single manoeuvre’ case. Thus, the controller equations for this case are required.

Unfortunately, it is not possible to simply insert ‘the number of firings ($N$ or $P$) in the first set is equal to zero’ into the equations, as this would cause a singularity, evident upon cursory inspection of the equations (4.15) and (4.26). This means the equations have to be derived from the beginning.

This leads to another question: what should the unknowns be? They used to be the semimajor axis or the inclination ‘jumps’ due to firings for the ‘double manoeuvre’ case. However, as the first set of firings is now nonexistent, there should be another parameter to be solved for. It is apparent that, the problem is now constrained by time; to be able to bring the satellite to the required LTAN/phase, one should wait for the correct amount of time and then bring the relative drift rate to zero via a set of firings.

It is also evident from Figure 4.10 that the time parameter to be solved for is either $t_{fw}$ or $t_c$, the final waiting time or the coasting time, respectively. Solving for either one is equivalent, as their sum (along with the time required for the firings) makes up the total $t_{tot}$, which is known. Obviously, this discussion is valid for the case of crosstrack firings (LTAN/inclination control case) as well.

Single Manoeuvre Case – Phase Control

The solution procedure is virtually the same as the double manoeuvre case; the equations for semimajor axis and phase evolution are to be derived and the two unknowns, namely
4.3. Controller Algorithm

\[ \delta a^* = \delta a_0 - M \Delta a_2 + \delta a_d \] 

(4.29)

\( \delta a_d \) is the drag term and it is to be evaluated for the whole duration i.e., total time, \( t_{tot} \), so that the relative phasing and semimajor axis will be correct as of the end of the total time. \( \delta a^* \) is the required semimajor axis difference, hence is zero. Solving for the semimajor axis jump due to a single firing,

\[ \Delta a_2 = \frac{\delta a_0 + \delta a_d}{M} \] 

(4.30)

Note that, the velocity change with each along-track firing is given by:

\[ \Delta V_{\phi 2} = \frac{-n_0 \Delta a_2}{2(1 + \rho + \kappa)} \] 

(4.31)

The evolution of relative phase is given by the equation:

\[ \frac{\delta \lambda^*}{1 + \kappa} = \frac{\delta \lambda_0}{1 + \kappa} + \frac{\delta \lambda_d}{1 + \kappa} + [\delta n_0 (t_c + (M-1)t_w) + \Delta n_2 t_w] + 2\Delta n_2 t_w + \cdots + (M-1)\Delta n_2 t_w \] 

(4.32)
where the $\Delta n_2$ notation is the one defined in (4.6). $\delta \lambda_d$ is the drag term and it is to be evaluated for the whole duration i.e., $t_{tot}$. Summing the series and solving for the coasting time,

$$t_c = \frac{1}{\delta n_0} \left[ \frac{\delta \lambda_{eq} - \delta \lambda_0}{1 + \kappa} - (M - 1)\delta n_0 t_w \right. \right.$$

$$\left. - \left( \frac{M - 1}{2} \right) \left( \frac{3n_0 f \Delta a_2}{2a_0 f} \right) t_w \right]^{(4.33)}$$

Using equations (4.30), (4.31) and (4.33), the coasting time and the firing magnitudes can be calculated easily.

The final waiting time is given by the equation:

$$t_{fw} = t_{tot} - (M - 1)t_w - t_c \tag{4.34}$$

If $t_{fw}$ is found to be negative, this means the total waiting time is not enough to complete this manoeuvre; the initial conditions are not appropriate.

Single Manoeuvre Case — LTAN Control

The solution procedure is again virtually the same as the double manoeuvre case; the equations for inclination and LTAN evolution are to be derived and solved for the two unknowns, namely $t_c$ and $\Delta I_2$. Once again, the same assumptions as the double manoeuvre case hold for this case as well.

The change in inclination is given by the equation:

$$\delta I = \delta I_0 - Q\Delta I_2 \tag{4.35}$$

where $\delta I$, is the required final relative inclination and is equal to zero. The inclination jump due to a single firing is thus,

$$\Delta I_2 = \frac{\delta I_0}{Q} \tag{4.36}$$

The velocity change with each crosstrack firing is given by:

$$\Delta V_{x2} = -a_0 n_0 \Delta I_2 \tag{4.37}$$
4.3. Controller Algorithm

The evolution of relative LTAN difference is given by the equation:

\[
\delta L_{req} = \delta L_0 + \delta L_{\delta a} \\
+ \left[ \delta I_0 D (t_c + (Q - 1)t_w) \\
- \Delta I_2 D t_w - \cdots - (Q - 1) \Delta I_2 D t_w \right]
\] (4.38)

The semimajor difference effect on LTAN, \(\delta L_{\delta a}\) is to be calculated using (4.25) and for the total duration, \(t_{tot}\).

Summing the series and solving for the coasting time,

\[
t_c = \frac{\delta L_{req} - \delta L_0 - \delta L_{\delta a}}{\delta I_0 D} - \frac{1}{2} (Q - 1) t_w
\] (4.39)

The final waiting time is given by the equation:

\[
t_{fw} = t_{tot} - (Q - 1) t_w - t_c
\] (4.40)

Once again, if \(t_{fw}\) is found to be negative, this means the total waiting time is not enough to complete this manoeuvre; the initial conditions are not appropriate.

4.3.5 Combined LTAN and Phase Controller

With the independent LTAN and phase controllers at hand, it is possible to design a combined controller, which, once initiated, is capable of outputting the required firing phase/time and the firing magnitude (both alongtrack and crosstrack) to attain a specific orbit.

There are several remarks to be made with regards to implementation.

- The order of the firings does not change the total fuel required appreciably, assuming the time consumed for the firings are small compared to the total manoeuvre time.

- To lighten the burden on the attitude subsystem, it is desirable to execute crosstrack and alongtrack firings separately (i.e., not firing on a combined axis – fire alongtrack or crosstrack only at a time). This will also uncouple the out-of-plane corrective manoeuvres from the in-plane ones.
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- In addition to the above, it may also be desirable to keep the firings in groups, such that, for example, the first set of alongtrack firings will not be executed before the first set of crosstrack firings are complete, or vice versa.

Initially, the first set of the alongtrack firings are executed to correct the phase difference, followed immediately by the first set of crosstrack firings to correct the LTAN. When the coasting time (the time between the first and the second sets of firings) is over, the second set of crosstrack firings is executed, immediately followed by the second set of alongtrack firings.

Care is required with the notation at this point; so far, the expression $t_{tot}$ has been used for both LTAN and phase control equations. However, these two times are not equal for the 'combined controller' case. Therefore, the notation $t_{phase}$ and $t_{titan}$ will be adopted.

Clearly, for the firing scheme described above, the time between two sets of alongtrack firings is the time available for the total crosstrack manoeuvre and is equal to $t_{titan}$.

Another point worth mention is the addition of some terms into the 'differential semimajor axis' part of the LTAN equations, given in (4.25). These terms are inserted to correct for the semimajor axis change due to the phase control alongtrack firings. Note that the waiting times between alongtrack firings are equal to $t_w$ (although this is not necessarily equal to that of crosstrack firings), and the number of firings in the phase control are $N$ and $M$, respectively;

$$
\delta L_{6a} = \left( -\frac{7 n_0 q}{2 a_0} \right) \left[ \left( \delta a_0 + \frac{\delta a_{drag}}{2} \right) t_{phase} + \frac{1}{2} (N(N - 1) \Delta a_1 - M(M - 1) \Delta a_2) t_w \right] + N \Delta a_1 \left( t_{phase} - (N - 1) t_w \right) \quad (4.41)
$$

4.4 Orbit Acquisition Strategy and Tradeoffs

4.4.1 Alongtrack or Crosstrack Firings - More Efficient LTAN Control

It has previously been noted that the LTAN control strategy relies primarily on changing the inclination via crosstrack firings, while correcting for (or taking advantage of) the effect
of the initial semimajor axis difference. This scheme has the advantage of uncoupling in-plane orbit acquisition from the out-of-plane part; LTAN control is carried out with the phase virtually unaffected.

However, the question then arises whether it is more advantageous to carry out the LTAN control via alongtrack firings, disregarding the potential effects on the phase controller for the time being. Recall the LTAN/RAAN relative drift rate equation, \((4.20)\). Rewriting this equation in terms of \(\Delta V\) magnitudes applied, it is possible to find a threshold inclination angle for a nominal LEO scenario. This threshold inclination angle will indicate the transition where it is more advantageous to fire alongtrack or vice versa, for a unit \(\Delta V\) applied. Assuming the initial inclinations and the semimajor axes of the reference and firing satellite are equal:

\[
\dot{\Omega}_0 = (-n_0 \tan I_0) \Delta I + \left(-\frac{7 n_0 \theta}{2 a_0}\right) \Delta a
\]

\[
= \dot{\Omega}_{\text{eff}} + \dot{\Omega}_{\text{sa}} \tag{4.42}
\]

\[
\dot{\Omega}_{\text{eff}} = -n_0 \theta \tan I_0 \Delta V
\]

\[
\dot{\Omega}_{\text{sa}} = \left(-\frac{7 n_0 \theta}{2 a_0}\right) \frac{2}{n_0} \Delta V
\]

Also notice the crosstrack firings are assumed to be at the node and the \(J_2\) terms appearing in the alongtrack \(\Delta V\) are ignored.

The region where \(\dot{\Omega}_{\text{eff}} > \dot{\Omega}_{\text{sa}}\), from the above expressions, can be found to be \(\tan I_0 > 7\) for a given magnitude of \(\Delta V\). In other words, per unit velocity change, more LTAN change is possible via crosstrack firings than the alongtrack firings in this region. At the boundaries, inclinations 81.87° and 98.13°, firing alongtrack or crosstrack are equivalent, though the \(\Delta V\) signs are different in the 98.13° case. Clearly, for the lower inclinations an alongtrack firing is a better option. The results are also irrespective of the semimajor axis and are valid within the bounds of \((4.20)\) i.e., LEO settings where the LTAN drift is dominated by \(J_2\) effects.

As far as LEO sun-synchronous orbits like that of DMC are concerned, firing alongtrack and crosstrack are almost equivalent in terms of LTAN correction. That is why including the
effects of semimajor axis differences is crucial in LTAN control. For the other orbit settings, the mission and propellant requirements and constraints would be the final determinant; however, it should be emphasised that the proposed scheme has the significant advantage of essentially decoupling phasing from the LTAN/inclination acquisition.

4.4.2 Feasibility of Firing in a Combined Axis

We have already explained that a nominal full 3D orbit correction manoeuvre would require two sets of alongtrack and two sets of crosstrack firings. The first of these sets are executed within a couple of days for a typical case, assuming there are four or five orbits between the firings. This is followed by a long coasting time, on the order of some months. Subsequently a second set of crosstrack and alongtrack firings conclude the manoeuvre. This corresponds to the path $OAA'BE$ in Figure 4.6 where the satellite coasts for a long time at $A'$. Perhaps one of the most obvious ways to improve performance is, at least in theory, to combine each set of alongtrack and crosstrack manoeuvres in a single set. Previously, the movement on this plot was said to be limited to the skewed axis (alongtrack firings) or the vertical axis (crosstrack firings). However, as long as the ideal inclination and semimajor axis values, hence the drift rates, are reached before the beginning of the coasting time, any combination of firings is possible. In the same vein, it is clear from Figure 4.6 that taking the shortest route ($OA'$ rather than $OAA'$) will result in the lowest possible propulsion consumption.

Consider the case where two sets of alongtrack and crosstrack firings are executed to correct the orbit fully. The propellant cost of the first set of firings will then be,

$$\sqrt{(N\Delta V_{\theta})^2 + (P\Delta V_{\perp})^2}$$

(4.43) rather than $N\Delta V_{\theta} + P\Delta V_{\perp}$. For a nominal four satellite scenario using DMC initial conditions, this brings the total $\Delta V$ (or propellant) cost for the whole constellation from $5.25\text{m/s}$ down to $4.12\text{m/s}$, a saving of more than 20%.

However, in reality, this poses significant potential problems with the Attitude Determination and Control System. To attain and maintain an orientation on this combined axis accurately is much more difficult than 'easier' orientations like pure alongtrack or
4.4. Orbit Acquisition Strategy and Tradeoffs

Figure 4.11: Variation of total $\Delta V$ with the number of firings

crosstrack. For the case of DMC, in the end, the decision was made against firing on this combined axis, as the risks were deemed not justified by the relatively small potential gains. Assuming one of the satellites would be the 'reference' satellite, the remaining three satellites would have to consume this extra 20\% propellant, which is approximately 0.4m/s per satellite. Since the majority of the DMC satellites has 24m/s of propellant, this extra was considered to be an acceptable price.

On the other hand, the equations can be very easily modified for combined axis firings if the specific scenario allows.

4.4.3 Tradeoff Analysis and Choice of Manoeuvre Parameters

Up until now, no guidelines have been set as to how the variables such as the number of firings, total manoeuvre time or wait time between firings should be chosen.

To shed some light on the effect of the first variable, number of firings, consider a 80 degree phase acquisition case with 5 alongtrack firings in the first set while the number of firings in the second set is varied (7065km semimajor axis, sun-synchronous orbit).

Figure 4.11 illustrates the outcome of this configuration. The total $\Delta V$ increases linearly with the increasing number of firings, as the time lost during firings has to be compensated with a higher phase drift rate. This suggests the minimum number of firings is desirable. However, from a practical point of view, the propellant consumption is usually not linearly
varying with the resulting velocity change and depends heavily on thruster characteristics. In many cases the propulsion system has an optimal firing duration (i.e., an optimal $\Delta V$) and exceeding or falling below this value significantly may prove to be an inefficient firing strategy or simply impossible with the given propulsion system. Therefore, the optimal number of firings is to be decided with the limitations and the characteristics of the propulsion system in mind. The total firing magnitude change is less than 2% between a single firing case and 10 firings, therefore the loss is ultimately not extremely important.

The second parameter to analyse is the waiting time between firings. As can be seen in Figure 4.12, as the waiting time between the orbits increases, the $\Delta V$ cost increases linearly. The reason is the same as increasing the number of firings; since the total time to phase is fixed, the longer the time spent during the firings, the higher the drift required. However, it has to be emphasised that, this increase is very small; the increase in $\Delta V$ is less than 2% for every 5 orbits waiting time for this scenario. From a practical point of view, there is a minimum time that can be left between the firings, as the attitude control stabilises the satellite for the next firing and possibly the GPS takes some measurements to assess the success of the previous firing. Nominally, four orbits have been allocated as a starting value, although the algorithm allows an arbitrary amount of time to be chosen.

Finally, Figure 4.13 illustrates the change in total $\Delta V$ required with respect to the total allocated time, one of the most crucial parameters of the manoeuvre. As confirmed by Equation (4.16), the total manoeuvre time is essentially inversely proportional to the total
4.4. Orbit Acquisition Strategy and Tradeoffs

\[ \Delta L = \frac{\sin \lambda}{a \sin J} \Delta V_\perp \] (4.44)

This would induce no drift of the orbital plane and change the LTAN directly.

On the other hand, the proposed method depends heavily on the duration between two
firings. Writing the LTAN change due to a change in inclination via (4.20) explicitly,

\[ \Delta L = n \frac{3}{2} J_2 \left( \frac{R}{a} \right)^2 \sin I \frac{\cos \lambda}{an} \Delta V_{1.3} \]

(4.45)

It is possible to solve for the manoeuvre duration required to justify the utilisation of the second method for equal amounts of propellant expended. Equating (4.44) and (4.45) and solving for time,

\[ t = \frac{4a^2}{3J_2 R^2 \sin I} \]

(4.46)

If the manoeuvre duration is longer than this time, it is more advantageous to use the drift method rather than the direct change method.

To give a more concrete example, for the case of a LEO at 7000km altitude and 98.2 degree inclination, the gains from a 1m/s firing with the direct method is matched by the proposed method within 16 days. For a 3 month manoeuvre period, five times more LTAN change than the direct method is possible for the same propellant cost.

4.5 Results of the Controller Simulations

4.5.1 Additional Notes on Implementation

There are additional potential gains to be had while executing the orbit acquisition manoeuvre. The firing times are manipulated by the orbit acquisition software, so that the alongtrack firings occur at the perigee or apogee to reduce the eccentricity. However, in reality, reducing the eccentricity might be more involved than that, as its long term behaviour is periodic and more complicated. More importantly, the already small eccentricity make it difficult to assess the location of the perigee clearly. Although this was not investigated in detail, at least trying to fire at the apogee or perigee was deemed more beneficial than firing at a random phase.

To achieve this, a very simple propagator, using the phase evolution equations presented in (4.1), tracks the perigee and the nodes as well as the current phase of the satellite. This enables to fire within a degree of the apogee/perigee or the ascending node as necessary. It is estimated that a 1m/s firing decreases the eccentricity by about $2.5 \times 10^{-4}$. 
4.5. Results of the Controller Simulations

Rather than the circularisation of the orbit, it may also be desirable to shift the eccentricity vectors such that reference and firing satellite perigees coincide and the relative motion in the radial and alongtrack directions due to eccentricity are minimised. However, this issue has not been addressed in this study.

Another important point to note is the direction of the phasing. For example, if the satellite is to travel from the 0 degree slot to 270 degree slot, both phasing forwards (travelling full 270 degrees) and backwards (travelling -90 degrees) options are evaluated and the inexpensive route is chosen. This enables to use initial semimajor axis difference (or the injection error) to maximum benefit.

The simulation results presented below demonstrate a fully open-loop case i.e., the firings on both the first and the second sets are executed with the initial knowledge of orbits. In reality, depending on operational practices, after each firing or at the end of the set of firings the controller software has to be run again with the new orbit information. Therefore, in a real scenario, one can expect even better accuracy than that is presented.

4.5.2 Controller Simulation Software

The controller/simulation software simulates the elements of the constellation, provides the initial conditions for the controller, applies the controller commands (scheduled firings) to the satellites and propagates them in time in a realistic setting. This simulation software forms the core of the DMC Orbit Acquisition Software; a more detailed treatment will be given in the next sections in the context of DMC.

The software comprises a number of different modules, the most important of which are the main controller that takes in the initial conditions and outputs the firing schedule, the symplectic propagator and the epicycle equations based filter to extract the orbit. The details of the last two will not be repeated here and the reader is referred to [78] and [45]. The drag modelling can be found in the paper by Christou et al [19].

The simulation commences with the satellite initial conditions propagated forward to a synchronised time. This, in turn, is input into the orbit acquisition module, which outputs a firing schedule for each satellite. The propagator then executes these firing schedules,
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<td>0.018</td>
<td>7.8E-05</td>
<td>71.098</td>
<td>0</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-1.311</td>
<td>0.041</td>
<td>-0.039</td>
<td>-1.63E-04</td>
<td>74.211</td>
<td>0</td>
</tr>
<tr>
<td>BilSat</td>
<td>-1.048</td>
<td>-0.028</td>
<td>0.007</td>
<td>9.00E-05</td>
<td>62.965</td>
<td>7.04E-10</td>
</tr>
</tbody>
</table>

Table 4.1: Simulated constellation initial conditions (with respect to AlSat)

while the propagated orbits are filtered at the time of the firings. The propagations take into account a 36 x 36 geopotential model as well as the effect of drag. The effects of lunisolar gravitation are not included in the simulation but are included when the software is implemented in the real world DMC application.

4.5.3 Simulation Results

The initial conditions has been created to simulate a possible injection scenario for the Disaster Monitoring Constellation (DMC), to assess the potential accuracy of the controller algorithm. The satellites are separated from the launcher in different directions and slightly different initial velocities. Nominally, about a month is required for the satellite commissioning, during which time the satellites are free to drift apart. The specifics of the initial conditions for the individual satellites are given in Table 4.1.

The reference satellite has been taken as AlSat, while the newly launched NigeriaSat, BilSat and UKDMC executed a number of firings for injection error correction and orbit acquisition. The firing magnitudes are limited to 0.2 m/s each, to simulate possible limitations stemming from the propulsion system. The maximum firing magnitude, therefore, determined the number of firings for each satellite.

The simulation has been run for the ‘combined control’ case, demonstrating combined phasing and LTAN control with two sets of firings in alongtrack direction but only one set of firing in the crosstrack direction. In the ideal conditions, the three firing satellites should bring their semimajor axes and inclinations equal to those of AlSat. Additionally, they should end up in their allocated phase slots with respect to AlSat. As for the LTANs,
4.5. Results of the Controller Simulations

<table>
<thead>
<tr>
<th>Name</th>
<th>SM Axis(km)</th>
<th>LTAN(deg)</th>
<th>Incl(deg)</th>
<th>phase(deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UKDMC</td>
<td>1.1689E-03</td>
<td>4.93073E-03</td>
<td>-5.7500E-06</td>
<td>0.382</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-4.8168E-03</td>
<td>-6.53673E-03</td>
<td>1.6124E-04</td>
<td>0.147</td>
</tr>
<tr>
<td>BilSat</td>
<td>-2.2860E-03</td>
<td>2.44833E-02</td>
<td>9.1390E-05</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 4.2: Residual orbit acquisition error – Combined case

no firm criteria had been set. The aim is to test the performance of the LTAN controller as it times the crosstrack firings to minimise the LTAN difference with AlSat.

In the second case, ‘pure LTAN/inclination control’, only inclination and LTAN were to be corrected to test the full LTAN acquisition accuracy. Although the LTAN values will ideally be correct at the end of the manoeuvre, they will continue to drift at different rates as the semimajor axis differences are accounted for, but not corrected.

The basic constraints imposed was the total time allocated for the manoeuvre and the number of firings (determined by the maximum possible firing magnitude). The time allocated was 84 days. A duration of 4 orbits has been left between the firings.

Combined LTAN Control and Phasing

Table 4.2 summarises the errors with respect to the desired values in the orbit acquisition. The semimajor axis error was less than 5 metres in all cases (which is the accuracy limit of the filter), whereas all the satellites were inserted in their allocated slots with less than 0.5 degree error. The inclination error was limited to $1.6 \times 10^{-4}$ degrees. Semimajor axis, inclination and LTAN evolutions are presented in Figures 4.14, 4.15 and 4.16.

The LTAN results are particularly interesting, since the controller has not had the full freedom to correct them. Nonetheless, the LTAN error, even in the case of BilSat, was less than 0.025 degrees. This shows that initial LTAN difference of 0.028 degrees has not worsened. Inspection of Figure 4.16 illustrates how the LTAN controller manipulates the time of the crosstrack firings to achieve the required LTAN. In fact, the controller reported that desired LTAN was achievable for the case of NigeriaSat and UKDMC, whereas, in BilSat, the crosstrack firings had been executed immediately so as to change the initial
Chapter 4. Design and Application of a Constellation Initialisation Algorithm

Figure 4.14: Semimajor axis evolution for the constellation

Figure 4.15: Inclination evolution for the constellation

Figure 4.16: LTAN evolution for the constellation
4.5. Results of the Controller Simulations

Figure 4.17: Inclination evolution for the constellation for the LTAN-only control
drift direction to a more favourable one. The crosstrack firings UKDMC and NigeriaSat
marked at about 52970 and 52950 days, respectively. These firings are more clearly visible
in Figure 4.15, illustrating the inclination evolution for the constellation.

In Figure 4.14, the semimajor axis evolution of the constellation is presented. The satellites
are clearly seen to execute two sets of alongtrack firings, with a drift duration in between.
The slow decrease in the semimajor axes are due to drag. Note that, BilSat experiences
significantly less drag but the controller takes this into account and all satellites settle on
the reference semimajor axis at the end of the phasing.

The accuracy reached well exceeds the mission requirements for DMC, which state that
for optimal, daily global coverage the satellite constellation should stay within 3.6 degrees
of relative phase windows and 0.2 degrees of relative LTAN windows.

Pure LTAN Control

The second test case is the ‘pure LTAN control’ setting, where only crosstrack firings are
executed to correct the LTAN and inclination fully. The requirement from the simulation
was to match the LTAN and inclination of the satellites to that of AlSat at the end of the
manoeuvre.

Figure 4.17 and Figure 4.18 demonstrate how the satellites manoeuvre to match their
inclinations and LTANs to that of AlSat. The inclination profile is now similar to the
Figure 4.18: LTAN evolution for the constellation for the LTAN-only control

<table>
<thead>
<tr>
<th>Name</th>
<th>Incl error(deg)</th>
<th>LTAN error(deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UKDMC</td>
<td>1.0115E-04</td>
<td>9.4670E-04</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>9.8740E-05</td>
<td>-1.1106E-03</td>
</tr>
<tr>
<td>BilSat</td>
<td>9.8740E-05</td>
<td>5.9740E-04</td>
</tr>
</tbody>
</table>

Table 4.3: Residual orbit acquisition error – LTAN-only case
4.6. Orbit Initialisation of the DMC

The semimajor axis profile of the previous case. The initial set of firings are executed to attain the desired drift rate. Following the long drift period, where the LTANs converge (Figure 4.18), the second set of firings are executed to stop the out-of-plane drift.

Table 4.3 summarises the results. The maximum error for the inclination is about $10^{-4}$ degrees, whereas the maximum error for the LTAN is approximately $10^{-3}$ degrees.

4.6 Orbit Initialisation of the DMC

Up until now we have presented the controller architecture and the open-loop simulation results. However, the real test for the controller is how well it is going to cope with the real world application, namely the orbit acquisition of the Disaster Monitoring Constellation. The following and the next sections detail the operational aspects of the DMC and our experience in phasing the satellites into their respective orbital slots.

The DMC software has been authored by Tamas Kormos, Egemen Imre and Dr Phil Palmer. Once the software was installed in the SSTL groundstation, it was configured to run every day in an automated fashion with the most up-to-date GPS information downloaded from the satellites. Throughout the phasing manoeuvre, the software team worked closely with the SSTL groundstation staff, who gradually took over the day-to-day supervision of the constellation through the controller software. Any anomalies encountered has been dealt with in a joint effort.

4.6.1 AM/PM Constellation Phasing Experience

The NASA mission of 'AM/PM constellation' comprises two sets of satellites - the morning and afternoon 'trains' - with four satellites on the former and five in the latter [17]. The constellation is designed to be on a circular, sun-synchronous and 16-day repeating orbit at 705km altitude.

From early 2005, the morning train was fully in orbit as well as Aqua from the afternoon train. Filici and Suarez [29, 30] reported in detail their experience of inserting SAC-C satellite into the morning constellation.
The planned SAC-C satellite launch was at 7065 km, 13km below the nominal altitude of the constellation. This is to guarantee a large initial phase drift with respect to the remaining elements of the constellation. However, this comes with a significant propellant cost attached. Correcting the initial semimajor axis difference of 12.5km has a propellant cost of about 6.65m/s. Filici and Suarez [29, 30] report that their overall $\Delta V$ consumption, including calibration and engineering firings, has been 7.29m/s. To put this figure into perspective, it has to be emphasised that the pre-launch propellant budget allocated to the whole Disaster Monitoring Constellation for initial orbit injection error correction was about 10m/s. After a particularly successful launch for the DMC, the total $\Delta V$ cost has been recalculated as about 3m/s for the overall constellation.

Filici and Suarez [29, 30] also report that their whole phasing sequence took nearly 50 days, with phasing proper (following the engineering firings) taking about 20 days. During this 50 days, SAC-C satellite travels about 470 degrees in phase and in the last 20 days, about 100 degrees. Arguably, the overall propellant cost would have been much lower if the satellite was launched directly to its nominal orbit. Assuming no semimajor axis errors during the launch, such a strategy would have completed a 100 degree phasing in 20 days with a total cost of 4.8m/s.

In addition, they have other operational requirements such as groundstation visibility and illumination while thrusters are firing as well as a maximum of 5 minutes firing duration. As there were existing satellites in the constellation and a number of close passes anticipated, collision avoidance was also a crucial concern.

Demarest et al. [24] reported the mission design for the insertion of Aqua into the constellation. Their strategies, as could be expected, are very similar to those of SAC-C. Rather than simply optimising or minimising the propellant cost, meeting a set of practical and operational requirements (such as GEO relay satellite TDRSS visibility during firings) has taken precedence in their solutions. Note that, while their problem is similar to that analysed in this chapter, these papers limit themselves strictly to in-plane manoeuvres.

In summary, AM/PM constellation highlights the numerous real world challenges of carrying out theoretically simple in-plane phase insertion of satellites, into an international constellation with a decentralised architecture.
4.6. Operational Aspects of the DMC Phasing

The first satellite of the DMC was AlSat, launched in November 2002. The second batch included BilSat, UKDMC and NigeriaSat, which was launched in September 2003. The DMC is an international collaboration and therefore the satellites are owned and operated by their respective countries even though they share the data. It follows that, all major decisions have to be accepted by all the participants. This results in a distributed rather than centralised control structure. For the thruster firings, the standard procedure is to calculate the required manoeuvres at the Surrey Satellite Technologies Limited (SSTL) groundstation and send them off to the respective agencies at each country (UKDMC was controlled by SSTL). The files are then inspected and scheduled for upload. The firing results, in the form of GPS data and other telemetry information are downloaded from each satellite by SSTL groundstation as well as by the owner of the satellite.

This distributed structure is one of the significant challenges of the DMC – the operations became less flexible but the upside is the redundancy in the groundstations in the case of a failure.

As mentioned in Section 4.1.1, the nominal orbit for DMC had already been designated as 680km altitude, sun-synchronous orbit with the ascending node at 10:00 AM local time (LTAN). The operating conditions for the DMC satellites were set at ±30 minutes of the 10:00AM requirement. This has to be maintained over the 5 year lifetime of the DMC constellation. The 90 degree phase separation between the satellites should be achieved and maintained with an accuracy of ±3.6 degrees as a design requirement. Table 4.4 summarises the 2σ launch errors for the DMC satellites.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>semi-major axis</td>
<td>+/- 10 km</td>
</tr>
<tr>
<td>LTAN</td>
<td>+/- 20 sec (0.07 deg)</td>
</tr>
<tr>
<td>inclination</td>
<td>+/- 0.03 deg</td>
</tr>
<tr>
<td>eccentricity</td>
<td>up to 0.002</td>
</tr>
</tbody>
</table>

Table 4.4: Estimated launch errors for DMC satellites
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After a successful launch of the three satellites in September 2003 by a Cosmos rocket, the initial commissioning period began, which lasted for more than a month. Satellite attitudes were stabilised and the subsystems were checked before handing the satellites over. Shortly after the commissioning, preparations for the phasing sequence began.

The orbit injection of AlSat in November 2002 resulted in a significant semimajor axis overshoot. To make sure the orbit stays sun-synchronous, a series of firings had to be executed between January and April 2003. This costed about 17m/s $\Delta V$ to AlSat and had profound effects to the orbit acquisition planning; whatever the overall propellant cost of the orbit acquisition, AlSat had to be spared most of the burden.

Balanced propellant consumption was therefore the next issue that needed to be addressed. The satellites rely on their propulsion systems for stationkeeping and decommissioning, therefore the propellant consumption is a key parameter in determining the operational lifetime of the constellation. Consequently, one of the challenges was to determine a phasing strategy and to decide which satellite would take which slot while keeping a balance between the propellant consumptions of the satellites. Once again, the international nature of the constellation and the decision-making procedure which requires the agreement of each member puts more importance on this requirement.

For each of the newly launched satellites, a propellant budget of about 2m/s had been allocated for initial orbit acquisition and injection error corrections. A further 2m/s has been allocated for stationkeeping operations. Therefore, while allowing some flexibility, these constituted upper boundaries for the propellant consumption.

Since the satellite orbits at the end of phasing should be extremely close, only small stationkeeping firings over the mission lifetime of 5 to 7 years are to be expected. However, the DMC consortium had to make a decision on the drag compensation strategy of the constellation. The first option was to maintain the altitude in an absolute sense, correcting for the semimajor axis decay regularly and fully. The second option was to let all the elements of the constellation to drop at a predetermined rate to minimise propellant costs. Another complication is that, while three of the DMC satellites have practically identical platforms with an estimated drag-drop rate of 2.5m/day, BilSat drops at a rate of about 1.5m/day. This differential drag has to be corrected for, otherwise BilSat will not only drift from its
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Phase slot, its orbital plane will also drift away from the rest of the constellation as well. Given the low propellant levels of AlSat (which has the same drag coefficient as NigeriaSat and UKDMC), it was decided that BilSat should correct for its differential drag. Therefore for the second strategy, 'average decay rate' for the constellation would be set as that of AlSat. Table 4.5 summarises the propellant costs of the two drag compensation strategies for BilSat and a standard DMC platform.

<table>
<thead>
<tr>
<th></th>
<th>BilSat</th>
<th>standard DMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drop rate (m/day)</td>
<td>1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>Total drop after one year (m)</td>
<td>547</td>
<td>912</td>
</tr>
<tr>
<td>Full drag correction strategy cost (m/s)</td>
<td>0.29</td>
<td>0.48</td>
</tr>
<tr>
<td>Average drag correction strategy cost (m/s)</td>
<td>0.19</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 4.5 reveals that the full drag compensation strategy would cost about 1.73m/s for the whole constellation whereas enforcing an average decay rate costs only 0.19m/s. As correcting for the full drag had no obvious benefits but a hefty propellant cost, the consortium opted for the 'average decay rate' strategy.

The consortium also decided against executing crosstrack firings as the initial inclination and LTAN differences were deemed small. AlSat, however, did need to make some small corrections to bring it into line with the other satellites.

Another constraint imposed was the firing direction and duration. The attitude subsystem determines and keeps the attitude in the alongtrack and crosstrack directions with much better accuracy than an angle that is a combination of these directions (see Section 4.4.2). Even though during the firings the attitude system works in the 'fine attitude-keeping' mode (using a significant amount of power), the firings induce attitude oscillations. Therefore to keep attitude related firing errors to a minimum, the alongtrack firing duration was limited to 180 seconds for the satellites with gravity gradient booms deployed. In the case of BilSat, which had not deployed its boom and AlSat which was to execute crosstrack firings, the firings were limited to 90 seconds.

When the constellation initialisation is discussed in the literature, complicated strategies
are often proposed but the engineering issues as discussed above are rarely addressed. As seen in the cases of DMC and AM/PM constellation, practical considerations frequently override optimal manoeuvring schemes.

### 4.6.3 Evaluation of the Launch Results and Reference Orbit Parameters

In accordance with the mission requirements, the DMC satellites were to be placed on a single sun-synchronous orbital plane at 686km altitude with 90 degree phase separation between the satellites. The second launch carrying three satellites was very successful with sub-kilometre level semimajor axis errors and an estimated $-5 \times 10^{-3}$ degree inclination error with respect to AlSat. The three satellites were injected into their orbits at roughly 160 degrees behind AlSat. Table 4.6 summarises the initial conditions of the constellation prior to the phasing manoeuvres. Note that, the reference values are absolute, whereas the others are relative with respect to the reference orbit parameters.

Table 4.6: DMC initial conditions on 11 November 2003, 12am. (Reference parameters are absolute, the remaining are relative to the reference)

<table>
<thead>
<tr>
<th>SM Axis</th>
<th>LTAN (deg)</th>
<th>Incl (deg)</th>
<th>phase (λ) (deg)</th>
<th>drag drop (m/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>7063.517</td>
<td>151.386</td>
<td>98.191</td>
<td>-64.589</td>
</tr>
<tr>
<td>AlSat</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.005</td>
<td>0.0</td>
</tr>
<tr>
<td>UKDMC</td>
<td>-0.203</td>
<td>0.021</td>
<td>0.0</td>
<td>2.9E-4</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-0.392</td>
<td>0.022</td>
<td>0.0</td>
<td>2.7E10-4</td>
</tr>
<tr>
<td>BilSat</td>
<td>-0.232</td>
<td>0.022</td>
<td>0.0</td>
<td>2.7E-4</td>
</tr>
</tbody>
</table>

The three newly launched satellites were released from the launcher by loaded springs as it rotated along its symmetry axis. Therefore, they ended up at slightly different semimajor axes. Nevertheless, their altitudes were still below AlSat by about 200 to 400m and they had already drifted towards AlSat by several degrees in the time between separation and the onset of phasing manoeuvres. The three satellites are also seen clustered in a slightly different orbital plane than that of AlSat, with small differences in LTAN and inclination.
Finally, BilSat, as expected, has a significantly smaller drag rate in comparison to the others and this needs to be accounted for both during the phasing and the stationkeeping phase afterwards.

It should be underlined that the batch filter [78] which processes the GPS data to compute the epicycle elements can determine the semimajor axis with an estimated accuracy of 10m while it can calculate the inclination to within $1 \times 10^{-3}$ degrees. To correct for any small errors after the phasing and to account for the effects of differential drag, maintenance firings are scheduled every two months, which uses the same controller structure to compute the firings.

It should also be noted that due to power requirements and other operational priorities, the GPS is turned on about 10% of the time during each orbit, which shifts slowly to minimise the errors.

Phase slots were allocated with AlSat at 0 degrees, UKDMC at 90, BilSat at 180 and NigeriaSat at 270. This meant that AlSat maintained its present phase location while other satellites were phased with respect to it. This decision was made in view of the AlSat's low propellant as explained above. The semimajor axes were to be equated to that of AlSat's, which also became the reference semimajor axis. This reference semimajor axis was assumed to decay at the same rate as AlSat.

The reference inclination was to be chosen with the long term LTAN behaviour in mind. In sun-synchronous orbits, the attraction of the Sun, although small, causes a secular change in inclination directly proportional to time, which in turn causes a quadratic change in LTAN [44]. Therefore, even with a careful selection of semimajor axis and inclination that minimises its variation, LTAN can change as much as an hour within five years. Simulations demonstrated that the inclinations of BilSat, UKDMC and NigeriaSat, which differed only marginally, offered better long term LTAN performance than that of AlSat's (see Section 4.6.4). Therefore their inclination was chosen as the reference inclination for all four satellites. The reference LTAN was chosen as that of AlSat's, although exactly matching the orbital planes was not a strict mission requirement. Furthermore, the final LTAN differences were expected to be less than 20 seconds. To limit the dispersion of LTAN, AlSat was to execute its inclination change manoeuvre whenever operationally
feasible and the remaining three satellites were to execute them at the end of their phasing manoeuvres, if necessary.

The total phasing time determination is, ultimately, a compromise between time and propellant consumption. The time allocated for phasing is inversely proportional to the $\Delta V$ required for the whole manoeuvre. To keep the propellant consumption low, a phasing duration of 85 days has been selected. This number is based on the fact that, approximately 1m/s of firings is enough to phase the satellite by 90 degrees in 85 days, assuming no initial semimajor axis difference. The phasing duration allowed a margin for flexibility, as the mission was not totally dependent on exact phasing and the total duration would be determined after the first set of firings.

Rather than programming a fixed number of orbits between firings, each firing has been evaluated individually and its results used to update the parameters for the next firing. For the first set of firings, on average a single firing per day was executed. The reasons were threefold; firstly, the actual propulsion performance cannot be clearly known prior to launch and we needed some data to accurately relate $\Delta V$ values to the firing duration. Secondly, the results from the initial test firings are not fully trustworthy, the propulsion system cannot achieve its nominal performance before using up some propellant. And finally, the firing schedule should not interfere with the imaging schedule.

### 4.6.4 Long Term LTAN Evolution for DMC

The effect of the lunisolar perturbations is generally small enough to be ignored in constellation orbit acquisition and maintenance problem. These are periodic in their nature and as all the satellites are on the same plane, they are subjected to equal forces, therefore there is no dispersion of the orbital planes.

However, sun-synchronous orbits constitute a special case. Because their orbital precession rates are interlocked with the apparent motion of the Sun, they undergo a constant pull by the Sun. This results in a linear change in the inclination, adding a quadratic term into the LTAN variation as well as forcing the satellite (in our case, the whole constellation) out of sun-synchronicity. While this effect is small in the short run, it can lead to an LTAN shift of an hour or more after five years. In the case of DMC, this would mean an unacceptable
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level of change in imaging conditions. The LTAN window throughout its mission duration is defined as ±30 minutes.

This linear change in inclination is given as [44]:

$$\Delta I = -\frac{3}{16}(n_\odot/n)^2 \sin I (1 + \cos I_\odot)^2 \left(\sqrt{3}nt\right)$$  \hspace{1cm} (4.47)

where \((n_\odot/n)\) is the ratio of the mean motion of the satellite to that of the apparent motion of the Sun. \(I_\odot\) is the apparent inclination of the Sun.

The solution to this problem is to initiate the constellation at a slightly off-sun-synchronous state and let the orbital plane slowly tilt and drift. The constellation then changes its LTAN slowly, first going towards the upper end of its LTAN window towards 10:30AM, then slowly decreasing as the inclination changes the drift rate. The aim is to ensure that at the end of its lifetime, the constellation stays within the lower end of its LTAN window at 9:30AM. Figure 4.19 illustrates this behaviour.

Therefore, when choosing the reference orbit semimajor axis and inclination parameters, this long term LTAN behaviour is to be taken into account. While it is possible to change the inclination (hence the drift rate) via crosstrack firings at any time throughout the mission, clearly the earlier the firings, the smaller will be the propellant cost.
4.6.5 Software Structure

Figure 4.20 depicts the major components of the controller software. This software is designed to run at the SSTL groundstation in an automated fashion, although it allows easy human intervention through an interface to modify parameters of phasing as well as the reference orbit. Each time a satellite passes over the groundstation, the GPS data is downloaded automatically and the controller software runs to refresh the ΔV estimates for the future manoeuvres.

Figure 4.20: Orbit Acquisition Software Structure

1. Orbit Determination: This module processes the GPS input file and performs sanity checks. The data is processed through a batch filter to evaluate the orbital parameters (before and after the firings, if any)

2. Controller: This involves the user interface (through which the controller parameters are manipulated) as well as the controller software itself. The number of satellites in the constellation, final date for the manoeuvres, the type of the firings (in-plane, out-of-plane or a combination of both), required phase and LTANs for each satellite and waiting time between firings are among the most crucial parameters that can be manipulated via the interface.
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The controller is the 'nervous system' of the algorithm. It handles the transfer of the correct variables as dictated by the requirements into other modules of the algorithm, determines the reference orbit and calls the other modules such as the filter and the propagator as necessary. It also outputs the results of the simulations into a series of graphs for easy interpretation and analysis by the groundstation staff.

The user interface is a configuration file where certain parameters pertinent to the phasing manoeuvres are set. A sample file is given below. As can be seen the interface is divided into sets of parameters defining different aspects of the manoeuvre such as the total mission duration, maximum limit $\Delta V$ as well as the phase slots the satellites should stay on.

```
#---------------------------------------------
StartEpoch 01-05-2004

# Acquisition Parameters
#ControlMode: 0 - only alongtrack; 1 - only crosstrack; 2 - combined
PhasingDuration: 230.0  # days
ControlMode: 0
MinDelay: 6.17  # days
AWaitTime: 4  # Interfiring delay (orbits)
LWaitTime: 4  # Cross track delay (orbits)
MaxDV: 0.043  # Max DV allowed (m/sec)
MinDV: 0.000  # Min DV allowed (m/sec)

# Simulation Parameters
GeoPotentialTerms: 4
SunandMoon: 1  # Should always be 1 !
ReportFrequency: 7.0  # days
TargetLTAN: 9:44:00
TargetDate: 1-09-2008
DosBit 1
```
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| Sat1Phase | 0       | #AlSat |
| Sat2Phase | 270     | #NiSat |
| Sat3Phase | 180.4   | #BilSat|
| Sat4Phase | 90      | #UKDMC |
| Sat1Drag  | 1.288e-9|        |
| Sat2Drag  | 1.288e-9|        |
| Sat3Drag  | 0.742e-9|        |
| Sat4Drag  | 1.288e-9|        |

---

3. Phasing: This module is the 'brain' of the orbit acquisition software. Given the reference and the satellite orbital elements as well as the 3D orbit acquisition requirements, it outputs the necessary in-plane or out-of-plane firings as well as the firing times in a 'firetable', which are then executed by the controller via the simulator module.

4. Simulator: This is the high precision propagator and the filter combination which executes the 'firetable' and simulates the results. The output is a series of orbital elements at regular intervals as well as before and after the firings. This module enables the groundstation staff to check the results of the firings.

5. Command: This module collects the generated 'firetables' for each satellite and reads them for upload to individual satellites. It generates a series of firing opportunities for a number orbits to enable the groundstation staff to schedule the firings according to other operational needs and requirements at the time, providing an extra layer of flexibility.

6. Telemetry Processing: The downloaded data from the telemetry files include the
measurements from the propulsion system, aiding the evaluation of the results of the firing and the propellant consumption.

While the software can operate in a closed-loop, fully autonomous fashion, real operational requirements favour a human-in-the-loop feedback control system. Nonetheless, human intervention to the system is minimal, usually to act as a final verification or to accommodate other mission priorities.

It should be underlined that, the whole controller software has been designed to be flexible enough to accommodate addition of other satellites to the constellation with significantly differing initial conditions (such as semimajor axis differences of tens of kilometres, large drag differences or orbital planes of different orientation). Simply recalculating the reference orbit and entering new phasing requirements would yield the set of firings to achieve the new orbit configuration.

4.7 The DMC Orbit Acquisition Results

4.7.1 First Set of Firings and Initial Estimates

UKDMC commenced firings on 11 November 2003 and BilSat and NigeriaSat on 20 November 2003, with the initial aim of completing the phasing by 12 February 2004. AlSat was to execute its cross-track firings when operationally feasible. On average a single firing per day had been scheduled, as it enabled the groundstation staff to analyse the results and did not interfere with the imaging tasks. The initial propellant consumption estimates for the DMC orbit acquisition as of the beginning of the firings are presented in Table 4.7. Note that, while the small inclination differences require small crosstrack firings, only AlSat was scheduled to execute any crosstrack firings.

The actual thruster performance achieved was monitored both via the filtered GPS data and the propulsion system telemetry to determine if any adjustments to firing magnitudes would be necessary. It was concluded that, for the purposes of this study, GPS data yielded more reliable results.
Table 4.7: DMC estimated $\Delta V$ cost at the beginning of the firings (in m/s)

<table>
<thead>
<tr>
<th></th>
<th>$\Delta V_{01}$</th>
<th>$\Delta V_{02}$</th>
<th>$\Delta V_{1}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>AlSat</td>
<td>0.0</td>
<td>0.0</td>
<td>0.649</td>
<td>0.649</td>
</tr>
<tr>
<td>UKDMC</td>
<td>0.732</td>
<td>-0.624</td>
<td>-0.007</td>
<td>1.426</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-0.144</td>
<td>0.341</td>
<td>0.015</td>
<td>0.500</td>
</tr>
<tr>
<td>BilSat</td>
<td>0.258</td>
<td>-0.207</td>
<td>-0.020</td>
<td>0.485</td>
</tr>
</tbody>
</table>

Although the algorithm outputs very specific $\Delta V$ values for accurate phasing, it is difficult to convert them to precise firing durations consistently and accurately. The chief reason is that, the tests on the ground provide limited knowledge of the thruster performance in space. In fact, throughout the manoeuvre sequence, the thrusters tended to overfire by up to 10%, potentially highlighting inaccuracies with the calibration. In addition, there are many other elements which make long term prediction difficult, one of the most significant being aerodynamic drag. Nevertheless, this level of accuracy is rarely necessary in practice; the algorithm can be run with the updated orbit parameters throughout the phasing, providing a closed-loop operation. This enables propellant consumption to be predicted with good accuracy for the whole orbit acquisition at the beginning and allow this prediction to adapt to the changing circumstances as the phasing progresses.

Another possible source of error is the location of the firings along the orbit. While this does not change the phasing performance, our aim in the alongtrack firings has been to fire at the perigee or apogee to circularise the orbit. However, it is extremely difficult to locate these extreme points of the orbit when the eccentricity is very small ($10^{-3}$ level in our case). More importantly, the propulsion system required the firings to take place while the satellite receives sunlight. This occasionally meant firing at slightly off-perigee/apogee.

Table 4.8 summarises the apogee firing performance for UKDMC in its first set of firings. This 'no firing in eclipse' requirement caused only small penalties, as the firing phase has stayed to within 10 degrees of the apogee. Note that, as a result of these firings, the eccentricity decreases from $1.32 \times 10^{-3}$ to about $1.10 \times 10^{-3}$.

Following the first set of firings, during the phasing, the evolution of the orbits have been monitored closely. Using the fresh GPS data, the orbit control software has been run every
The DMC Orbit Acquisition Results

Table 4.8: UKDMC Estimated Apogee and Real Firing Phase

<table>
<thead>
<tr>
<th>Date</th>
<th>Duration (sec)</th>
<th>Firing phase (λ)</th>
<th>Calculated apogee phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 Nov '03</td>
<td>180</td>
<td>266.89</td>
<td>262.99</td>
</tr>
<tr>
<td>12 Nov '03</td>
<td>180</td>
<td>265.47</td>
<td>259.95</td>
</tr>
<tr>
<td>13 Nov '03</td>
<td>180</td>
<td>265.30</td>
<td>255.55</td>
</tr>
<tr>
<td>15 Nov '03</td>
<td>146</td>
<td>250.85</td>
<td>251.11</td>
</tr>
<tr>
<td>15 Nov '03</td>
<td>146</td>
<td>248.79</td>
<td>248.40</td>
</tr>
<tr>
<td>16 Nov '03</td>
<td>146</td>
<td>246.93</td>
<td>246.93</td>
</tr>
<tr>
<td>18 Nov '03</td>
<td>111</td>
<td>244.09</td>
<td>241.70</td>
</tr>
</tbody>
</table>

day, providing up-to-date information on the projected magnitude of the firings.

AlSat's cross-track firings commenced on 13 January, increasing its inclination from 98.183 to 98.189 degrees, to within about $0.5 \times 10^{-3}$ degrees of the inclination of the other three satellites. As the inclination differences between the satellites were very small (smaller than the estimated filter accuracy) at this point, no further corrections were necessary until a more detailed study was possible.

4.7.2 Second Set of Firings and Final Results

The second set of firings requires greater precision than the first set, as the timing and the magnitude of the firings become crucial for correct phase and semimajor axis matching. Unfortunately, at this precise time, there were unforeseen operational problems which provided more challenges for the algorithm.

In late January and early February, the SSTL groundstation antenna developed a problem with the driving motor and finally malfunctioned, causing some operational inconvenience, just before the second set of firings. The GPS and telemetry data download and command uploads for the UKDMC have been executed via the CNES groundstation in Algeria until 10 February. For this reason, it was deemed useful to decrease the semimajor axes of UKDMC and BilSat to dampen the phase drift significantly or kill it altogether to prevent a serious overshoot of the target phases. This would relax the final date of phasing operation
by several days. The plan was to execute a final set of corrective firings after the antenna was fixed.

The end result was that, BilSat ended up at its phase slot successfully following its along-track firings between 3 and 5 February.

UKDMC, on the other hand, was scheduled to execute a set of firings on 1 February. However, problems with the attitude control system resulted in the satellite spinning along its yaw axis at a very rapid rate. Therefore, although the firings were executed, their overall effect was very small. Since the firings were scheduled to take place in an open-loop fashion and the downloaded data had to be transferred from the groundstation at Algeria, the UK groundstation became aware of this only after all the firings were executed. In effect, only one of the five 8cm/s firings took place.

At this point, the priority was to negate the large rotational rate of the satellite and it was decided to leave the satellite to drift at a rate of 1.1 degrees per day; the corrective firings were to take place after the UK groundstation antenna at SSTL facilities was fixed and the attitude stabilised. Taking into account the successful launch and comparatively small total propellant cost of the overall phasing operation, this was reckoned to be a small extra. In the end, UKDMC overshot its phase slot by no more than 5 degrees. Immediately after the SSTL satellite dish was fixed, additional firings have been scheduled for UKDMC to correct this phase overshoot.

Considering that the phase window has been defined as +/-3.6 degrees at the mission planning stage, the errors can be said to be fairly small. At this point, the flexibility of the algorithm proved extremely valuable and a new strategy was quickly devised for NigeriaSat and UKDMC to refine the phasing, simply using the new orbital parameters as the initial conditions.

Table 4.9 summarises the results of the firings as of 18 February (day 53053 in MJD) in terms of the error with respect to AlSat, which became the reference orbit after the cross-track firings. The expected phases for UKDMC, NigeriaSat and BilSat are 90, 270 and 180 degrees, respectively.

In Table 4.9, AlSat is the same as the reference satellite. NigeriaSat is adrift from its designated semimajor axis by 191m, causing a drift of approximately 0.2 degrees per day.
4.7. The DMC Orbit Acquisition Results

Table 4.9: DMC orbital elements as of 18 February 2004. (AlSat parameters absolute, the remaining are relative to AlSat)

<table>
<thead>
<tr>
<th></th>
<th>SM Axis (km)</th>
<th>LTAN (deg)</th>
<th>Incl (deg)</th>
<th>e</th>
<th>phase (λ) (deg)</th>
<th>drag drop (m/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AlSat</td>
<td>7063.234</td>
<td>152.018</td>
<td>98.186</td>
<td>0.00122</td>
<td>-145.813</td>
<td>2.5</td>
</tr>
<tr>
<td>UKDMC</td>
<td>-0.155</td>
<td>0.010</td>
<td>0.0</td>
<td>-3.4 × 10^{-5}</td>
<td>86.324</td>
<td>0.0</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-0.191</td>
<td>0.091</td>
<td>0.001</td>
<td>2.4 × 10^{-5}</td>
<td>271.021</td>
<td>0.0</td>
</tr>
<tr>
<td>BilSat</td>
<td>-0.013</td>
<td>0.050</td>
<td>0.001</td>
<td>7.2 × 10^{-5}</td>
<td>180.025</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

in phase. This is because the very last firing has been delayed due to other operational issues as described below. UKDMC is also seen to be below the reference semimajor axis by 155m, as the manoeuvres to refine the phase acquisition had been initiated at that stage and the satellite was drifting in the correct direction. BilSat, on the other hand, is seen to be in its correct station with phase errors below 0.05 degrees and semimajor axis errors below 15m. Since there had not been any significant operational issues with BilSat, its results are more representative of the potential accuracy expected from the algorithm.

Like the above tables, the following figures are compiled from the filtered onboard GPS sensor output. Figure 4.21 illustrates the semimajor axis evolution of the constellation. The satellites can be seen commencing their firings around the MJD day number 52960-52965 marks. AlSat's crosstrack firings appear as a series of spikes around MJD day 53020.

The linear semimajor axis decay due to drag is visible for all satellites; BilSat experiences less drag, despite being physically slightly larger than the other satellites. This is because it is about 60% heavier than others and it can 'penetrate' the atmosphere better. From the graph, it is also obvious that, while there are daily variations in drag, in the long term, it manifests itself as a steady linear decay in semimajor axis. This demonstrates the validity of the simple drag modelling scheme for long-term planning and control purposes.

Around MJD day 53040, BilSat successfully executes its firings to negate the semimajor axis difference while the groundstation hardware problems prevent NigeriaSat and UKDMC to complete their second set of firings. Near MJD day 53035 mark (1 February 2004), the failed firings of UKDMC are also visible as a set of spikes. To compensate for the phase drift...
Chapter 4. Design and Application of a Constellation Initialisation Algorithm

Figure 4.21: Semimajor axis evolution of the DMC
due to the delayed firings, the algorithm was re-run using new GPS data. Consequently, the small corrective phasing sequence between MJD days 53050 and 53073 can be seen for UKDMC. On 9 March (MJD day 53073 on the graph), UKDMC executed the final set of corrective firings to lock itself in its final and correct phase.

The belated second set of firings has been executed for NigeriaSat on 10 and 11 February (near day 53045 mark in Figure 4.21). While these firings lowered the drift to a large extent, they did not fully negate it, as the final firing had been delayed due to some anomalies in the attitude telemetry. Following the investigation of these anomalies, the new firings have been scheduled about two weeks later.

Figure 4.22 and 4.23 show the inclination and LTAN evolutions of the DMC, respectively. AlSat’s successful crosstrack firing near day 53020 (16 January 2004) and the overall linear decrease in the inclinations due to lunisolar attraction is clearly visible on the former graph. The occasional spikes on the data (such as the one at day 53005) show that care must still be taken when using GPS data in isolation.

Recall that the consortium had decided that the out-of-plane errors for the constellation were not large and that no corrective action was to be taken, except for AlSat matching its inclination with the remainder of the constellation.

Figure 4.23 shows these ‘uncorrected’ LTAN values and the slowly diverging orbital planes.
4.7. The DMC Orbit Acquisition Results

Figure 4.22: Inclination evolution of the DMC

Figure 4.23: LTAN evolution of the DMC
during phasing. However, as can be seen, once the inclinations and the semimajor axes are matched, this divergence practically stops and the LTAN differences are frozen. In any case, the maximum LTAN difference is limited to about 0.1 degrees or about 25 seconds.

4.7.3 Total ΔV Cost

Table 4.10 summarises the total propellant cost for the constellation. The numbers in brackets are the extra ΔV consumption due to corrective firings and, in the case of UKDMC, failed firings.

<table>
<thead>
<tr>
<th></th>
<th>ΔVₖ₁</th>
<th>ΔVₖ₂</th>
<th>ΔVₗ</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>AlSat</td>
<td>0.0</td>
<td>0.0</td>
<td>0.655</td>
<td>0.655</td>
</tr>
<tr>
<td>UKDMC</td>
<td>0.695</td>
<td>-0.636 (+0.178)</td>
<td>0.0</td>
<td>1.509</td>
</tr>
<tr>
<td>NigeriaSat</td>
<td>-0.170</td>
<td>0.255 (+0.171)</td>
<td>0.0</td>
<td>0.596</td>
</tr>
<tr>
<td>BilSat</td>
<td>0.268</td>
<td>-0.211</td>
<td>0.0</td>
<td>0.479</td>
</tr>
</tbody>
</table>

BilSat's results can be used to investigate the long-term, predictive accuracy of the algorithm. In Table 4.7, the initial alongtrack ΔV estimate is 25.8cm/s in the first set and 20.7cm/s in the second. In reality, the first set of firings were 26.8cm/s in total, the difference being mainly due to thruster overfiring. The second firing was 21.1cm/s. The total percentage difference between the estimated and the real values is 3.0%.

A quick comparison with Table 4.7 reveals that, despite the problems in the phasing, the projected propellant costs for UKDMC are within about 5% that of initial calculations. For NigeriaSat the error is nearer 20%. While this is a significant error, the overall propellant costs are still small; even with the operational problems, the total propellant cost for the whole constellation has been 3.24m/s. This compares well with the initial estimates of 3.06m/s.

An important factor in increasing the precision is to have as accurate as possible thruster calibration data to be able to relate the ΔV requirements of the firing accurately into the thruster 'on' time. Unfortunately, the propulsion system characteristics are very difficult
to measure until the spacecraft is actually in orbit and compiling such data will invariably lead to a significant number of test firings, causing unnecessary propellant consumption. An obvious practical solution has been to process the GPS data after the penultimate firing of the set and alter the final firing accordingly.

### 4.8 Conclusions

A low-propellant-cost and practical solution to the 3D constellation orbit acquisition problem has been presented, primarily for a single plane, high inclination, LEO constellation. The highlight of the method is the harnessing of the natural orbital plane drift due to $J_2$ zonal harmonics to carry out the plane changes. The solution method is valid for full 3D orbit acquisition, very simple to implement and causes minimal computational burden.

This method is developed with the orbit acquisition of the Disaster Monitoring Constellation (DMC) in mind. Therefore, some of the unique operational and satellite platform challenges posed by the DMC had to be incorporated into the result, although similar limitations exist for all constellations and the controller can easily be adapted for other practical restrictions or requirements. The controller is flexible enough to be used in other possible scenarios involving higher $\Delta V$ values, larger differential drag effects and different combinations of in-plane and out-of-plane manoeuvres.

The simulation results show that the proposed method can be successfully applied to the orbit acquisition problem of the DMC. This method can be used in a semi-autonomous fashion on board the satellites, where the feedback part of the controller is supplied to the satellites via precise calculation of orbits from the groundstation. Furthermore, it seems feasible to investigate the possible uses of this algorithm in constellation maintenance and formation initiation tasks, where considerable accuracy is needed within a much shorter time frame.

In the second half of the chapter, we have explained in detail how this controller is put into practice for the Disaster Monitoring Constellation, for which the phasing was crucial to achieve near-24-hour global coverage target.

The design of the DMC presented some unique technical (propulsion, attitude determi-
nation and control etc) as well as operational (distributed and international nature) challenges. Furthermore, the presence of AlSat already in-orbit provided another complication; this case is significantly more difficult and restrictive than simply launching four satellites in orbit and deploying them to their respective stations. We do recognise that, while our solutions were not optimal in the strict sense, these real-world challenges make it nearly impossible to implement a straightforward, off-the-shelf propellant optimised solution for this problem.

On the other hand, the controller had been designed to cope with more challenging phasing scenarios, complete with full LTAN correction and much higher semimajor axis or drag differences. Therefore it retains its wider applicability for other possible constellation acquisition problems.

The second set of real world challenges presented themselves during the course of the mission, while unforeseen operational circumstances delayed the firings by some weeks in some cases. The flexible design of the controller proved invaluable during this phase; the controller was simply instructed to re-run with the new initial conditions, providing a new strategy for corrective firings.

In the end, the DMC phasing has been completed successfully with all the satellites in their designated orbits well within the requirements. The operational issues also provided us with a significant hands-on experience. When the constellation initialisation is discussed in the literature, complicated strategies are often proposed but the engineering issues as discussed above are rarely addressed. As seen in the cases of DMC and AM/PM constellation, practical considerations override optimal manoeuvring schemes.

Although the results are not reported here, the DMC software was deemed accurate enough for use with the orbit maintenance operations as well.
Chapter 5

Design of a Symplectic Relative Orbit Propagator

5.1 Introduction

In the first chapters we have already argued the need for a high-precision numerical relative propagator. In Section 3.4, we have also introduced the basics of numerical integration and the advantages of symplectic algorithms for orbit propagation which conserve the constants of the motion such as angular momentum and energy. Finally, in Section 3.4.3 we have explained in detail how a symplectic orbit propagator for a single satellite is constructed.

In this chapter, we will use these building blocks to put together a symplectic relative propagation scheme that closely resembles the single satellite symplectic orbit propagator. However, rather than propagating two 'absolute' orbits, our algorithm will need to propagate one 'absolute' and one 'relative' orbit. As will be shown in this chapter, this is more complicated than just propagating the orbit of a single satellite.

Following the example of a single satellite orbit propagation, we will first derive the Hamiltonian description of the relative motion. Using the resulting Hamilton's Equations, we will construct the symplectic integration scheme and present the results.
5.2 Hamiltonian Description of Relative Motion

5.2.1 Relative Motion in a Keplerian Field

In this section we shall consider the relative motion between two satellites moving in a Keplerian potential. Our approach is based on the Hamiltonian description of the motion and our discussions focus upon the conserved quantities of the motion. We start by considering a satellite at position $\mathbf{r}$ moving with velocity $\mathbf{v}$ in a Keplerian potential. The Hamiltonian for this satellite is given by:

$$ H = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v}) - \frac{\mu}{r} \tag{5.1} $$

where $\mu$ is the gravitational parameter defining the potential. The position and velocity of this satellite defines co-ordinates in a 6 dimensional phase space, and Hamilton's equations define the motion of the satellite through this phase space at all later times. Now suppose that instead of a single satellite there are two satellites in close proximity to each other in this phase space. We can define the position and velocity of these two satellites as $(\mathbf{r} \pm \frac{1}{2}\delta \mathbf{r}, \mathbf{v} \pm \frac{1}{2}\delta \mathbf{v})$. This description locates the midpoint in phase space as defined by $(\mathbf{r}, \mathbf{v})$ and the deviation from this midpoint for each of the two satellites. As will be shown shortly, the motion of the midpoint is equivalent to the motion of a satellite to the first order, therefore it can also be called the imaginary reference satellite.

Consider the Hamiltonian that describes the motion of the satellite for which the small increments in phase space co-ordinates are added to the midpoint co-ordinates:

$$ H_1 = \frac{1}{2} \left( \mathbf{v} + \frac{1}{2}\delta \mathbf{v} \right) \cdot \left( \mathbf{v} + \frac{1}{2}\delta \mathbf{v} \right) - \frac{\mu}{|\mathbf{r} + \frac{1}{2}\delta \mathbf{r}|} \tag{5.2} $$

Similarly, the Hamiltonian for the second satellite can be written as:

$$ H_2 = \frac{1}{2} \left( \mathbf{v} - \frac{1}{2}\delta \mathbf{v} \right) \cdot \left( \mathbf{v} - \frac{1}{2}\delta \mathbf{v} \right) - \frac{\mu}{|\mathbf{r} - \frac{1}{2}\delta \mathbf{r}|} \tag{5.3} $$

We can rewrite these Hamiltonians as expansions up to second order terms as:

$$ H_1 = \frac{1}{2} \left[ (\mathbf{v} \cdot \mathbf{v}) + (\mathbf{v} \cdot \delta \mathbf{v}) + \frac{1}{4}(\delta \mathbf{v} \cdot \delta \mathbf{v}) \right] - \frac{\mu}{|\mathbf{r}|} \left[ 1 - \frac{1}{2} \frac{\mathbf{r} \cdot \delta \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} - \frac{1}{8} \frac{\mathbf{r} \cdot \delta \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} + \frac{3}{8} \frac{(\delta \mathbf{r})^2}{\mathbf{r} \cdot \mathbf{r}} \right] \tag{5.4} $$

$$ H_2 = \frac{1}{2} \left[ (\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \delta \mathbf{v}) + \frac{1}{4}(\delta \mathbf{v} \cdot \delta \mathbf{v}) \right] - \frac{\mu}{|\mathbf{r}|} \left[ 1 + \frac{1}{2} \frac{\mathbf{r} \cdot \delta \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} - \frac{1}{8} \frac{\mathbf{r} \cdot \delta \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} - \frac{3}{8} \frac{(\delta \mathbf{r})^2}{\mathbf{r} \cdot \mathbf{r}} \right] \tag{5.5} $$
5.2. Hamiltonian Description of Relative Motion

According to the theory of Hamiltonian systems, both of these quantities \( H_1 \) and \( H_2 \) are conserved by the motion. We would therefore like to find a description of the relative motion that also conserves these quantities and exploits the fact that the separations in phase space are small. If we add these two Hamiltonians together and ignore the second order terms, we obtain:

\[
H_1 + H_2 = 2 \left( \frac{1}{2} v \cdot v - \frac{\mu}{|r|} \right) = 2H
\]  

(5.6)

where \( H \) is the Hamiltonian associated with the motion of the midpoint through phase space and is equal to the average of the Hamiltonians of the two satellites to the first order. Note that, we can ignore the second order terms as we are more interested in the relative motion of the two satellites rather than the absolute motion of the reference satellite. We will investigate the effect of the second order terms in greater detail in Section 6.2.3.

The relative energy is defined as the difference between these two energies \( H_1 \) and \( H_2 \):

\[
H_R = H_1 - H_2 = v \cdot \delta v + \frac{\mu}{|r|^3} (r \cdot \delta r)
\]  

(5.7)

The important point to note in this expression is that by our choice of describing the motion in terms of the phase space midpoint, the second order terms in \( H_R \) cancel. Hence the relative energy is accurate to third order.

We can think of the relative motion of the two satellites as a motion in a 12 dimensional phase space defined by the position and velocity of the midpoint and the separation positions and velocities. In this context we may generalise the set of Hamilton's equations to obtain the following set in 12 dimensions:

\[
\dot{r} = \frac{\partial H_R}{\partial v} \quad \dot{v} = -\frac{\partial H_R}{\partial r}
\]  

(5.8)

\[
\delta \dot{r} = \frac{\partial H_R}{\partial v} \quad \delta \dot{v} = -\frac{\partial H_R}{\partial r}
\]  

(5.9)

These equations are an extension of a Hamiltonian system in 6 dimensions, but there is a cross coupling between the relative motion and absolute motion of the midpoint.

It is important to note that, like Hamilton's equations in 6 dimensions, these equations yield the equations of motion. In other words, the relative Hamiltonian contains both 'absolute' and 'relative' motion information. The equations given in (5.8) correspond to the equations of motion for a satellite in a Keplerian orbit:

\[
\dot{r} = v \quad \dot{v} = -\frac{\mu}{r^3} r
\]  

(5.10)
The solution of this equation can be described in terms of two constant vectors: \( L \), the angular momentum vector and \( e \), the eccentricity vector. The solution satisfies the equation of an ellipse:

\[
\mathbf{r} + e \mathbf{r} = \mathbf{p} \tag{5.11}
\]

where \( p \), the parameter which is dependent upon the magnitude of \( L \).

Similarly, (5.9) can be written as:

\[
\delta \mathbf{r} = \delta \mathbf{v} \quad \delta \mathbf{\dot{r}} = \frac{\mu}{r^3} \delta \mathbf{r} + \frac{3\mu}{r^5} (\mathbf{r} \cdot \delta \mathbf{r}) \mathbf{r} \tag{5.12}
\]

which is simply equations of the relative motion. Note that this form is exactly the same as Equation (3.46) we derived in Hill’s Equations.

If we now consider the relative motion associated with these equations, then we can easily show that \( H_R \) is a conserved quantity:

\[
\frac{dH_R}{dt} = \frac{\partial H_R}{\partial \mathbf{r}} \cdot \mathbf{\dot{r}} + \frac{\partial H_R}{\partial \mathbf{v}} \cdot \mathbf{\dot{v}} + \frac{\partial H_R}{\partial \delta \mathbf{r}} \cdot \delta \mathbf{r} + \frac{\partial H_R}{\partial \delta \mathbf{v}} \cdot \delta \mathbf{v} = 0 \tag{5.13}
\]

A second quantity that is conserved in a Keplerian orbit is the orbital angular momentum \( L \). Consider then the angular momentum associated with the first satellite:

\[
L_1 = (\mathbf{r} + \frac{1}{2} \delta \mathbf{r}) \wedge (\mathbf{v} + \frac{1}{2} \delta \mathbf{v}) \tag{5.14}
\]

Expanding this to first order gives:

\[
L_1 = \mathbf{L} + \frac{1}{2} (\mathbf{r} \wedge \delta \mathbf{v}) - \frac{1}{2} (\mathbf{v} \wedge \delta \mathbf{r}) \tag{5.15}
\]

Again the angular momentum of the second satellite is found by reversing the sign of the terms in the phase space separation. As with the energy, we can consider the sum of these terms to the first order as:

\[
L_1 + L_2 = 2\mathbf{L} \tag{5.16}
\]

which is just the angular momentum of the Keplerian motion associated with the midpoint location. The relative angular momentum is then:

\[
L_R = L_1 - L_2 = \mathbf{r} \wedge \delta \mathbf{v} - \mathbf{v} \wedge \delta \mathbf{r} \tag{5.17}
\]
5.2. Hamiltonian Description of Relative Motion

We can show that this quantity is also conserved by our equations of motion. Taking the time derivative and using Equation (5.7) we have:

\[
\frac{d}{dt}(r \wedge \delta v) = v \wedge \delta v - \frac{\mu}{r^3}(r \wedge \delta r)
\]  

(5.18)

Similarly,

\[
\frac{d}{dt}(v \wedge \delta r) = -\frac{\mu}{r^3}(r \wedge \delta r) + v \wedge \delta v
\]  

(5.19)

Hence substituting into Equation (5.17) shows that \( L_{R} \) is also a conserved quantity.

5.2.2 Relative Motion in a Nonspherical Potential Field

In the previous section we described how to define the relative Hamiltonian and derived the equations of absolute and relative motion. While this was limited to Keplerian motion, it is possible to generalize these solutions to a nonspherical potential field in a similar fashion.

The Hamiltonian for a satellite in a nonspherical potential field can be written as:

\[
H(r, v) = \frac{1}{2}(v \cdot v) + U(r)
\]

(5.20)

where \( U(r) \) is the function defining the gravitational potential in terms of spherical harmonics, as given in Equation (3.1).

As with the previous section, consider the case of two satellites with their positions and velocities given as \((r \pm \frac{1}{2}\delta r, v \pm \frac{1}{2}\delta v)\). The Hamiltonians for these two satellites can be written as:

\[
H_1 = \frac{1}{2}(v + \delta v \cdot v) + \frac{1}{2}\delta v + U(r + \frac{1}{2}\delta r)
\]

(5.21)

\[
H_2 = \frac{1}{2}(v - \delta v \cdot v) - \frac{1}{2}\delta v + U(r - \frac{1}{2}\delta r)
\]

(5.22)

Expanding these Hamiltonians to second order:

\[
H_1 = \frac{1}{2} \left[ v \cdot v + (v \cdot \delta v) + \frac{1}{4}(\delta v \cdot \delta v) \right]
\]

\[
+ \left[ U(r) + \frac{1}{2} \left[ \frac{\partial U(r)}{\partial r} \delta r + \frac{1}{8} \frac{\partial^2 U(r)}{\partial r^2} (\delta r)(\delta r)^T \right] \right]
\]

(5.23)

\[
H_2 = \frac{1}{2} \left[ v \cdot v - (v \cdot \delta v) + \frac{1}{4}(\delta v \cdot \delta v) \right]
\]

\[
+ \left[ U(r) - \frac{1}{2} \left[ \frac{\partial U(r)}{\partial r} \delta r + \frac{1}{8} \frac{\partial^2 U(r)}{\partial r^2} (\delta r)(\delta r)^T \right] \right]
\]

(5.24)
Carrying out the summation $H_1 + H_2$ to the first order, the Hamiltonian of the midpoint or the reference satellite is seen to be the average of the two satellites:

$$H_1 + H_2 = 2 \left( \frac{1}{2} (v \cdot v) + U(r) \right) = 2H$$

(5.25)

Similarly, if we subtract the two Hamiltonians, we obtain the relative Hamiltonian $H_R$:

$$H_R = (v \cdot \delta v) + \frac{\partial U(r)}{\partial r} \delta r$$

(5.26)

Like the Keplerian case, this is completely accurate in the differences in the kinetic energy term and accurate to third order in the potential energy term. We can write down Hamilton’s equations for this twelve dimensional phase flow:

$$\dot{\mathbf{r}} = \frac{\partial H_R}{\partial \mathbf{v}} \quad \dot{\mathbf{v}} = -\frac{\partial H_R}{\partial \mathbf{r}}$$

(5.27)

$$\dot{\delta \mathbf{r}} = \frac{\partial H_R}{\partial \delta \mathbf{r}} \quad \dot{\delta \mathbf{v}} = -\frac{\partial H_R}{\partial \delta \mathbf{r}}$$

(5.28)

From these equations, we can write the absolute and relative equations of motion as:

$$\dot{\mathbf{r}} = \mathbf{v} \quad \dot{\mathbf{v}} = -\frac{\partial U(r)}{\partial r}$$

(5.29)

$$\dot{\delta \mathbf{r}} = \delta \mathbf{v} \quad \dot{\delta \mathbf{v}} = -\frac{\partial^2 U(r)}{\partial r^2} \delta \mathbf{r}$$

(5.30)

Note that, the potential can be written as $U = -\mu/r + R(r)$, which shows the Keplerian part of the potential and the perturbations. Sections A.1 and A.2 in Appendix detail the derivation of $\partial R/\partial r$ and $\partial^2 R/\partial r^2$.

For this generalised case, conservation of energy can be shown in the same way as the Keplerian case. Taking the time derivative of $H_R$:

$$\frac{dH_R}{dt} = \frac{\partial H_R}{\partial \mathbf{r}} \dot{\mathbf{r}} + \frac{\partial H_R}{\partial \mathbf{v}} \dot{\mathbf{v}} + \frac{\partial H_R}{\partial \delta \mathbf{r}} \dot{\delta \mathbf{r}} + \frac{\partial H_R}{\partial \delta \mathbf{v}} \dot{\delta \mathbf{v}} = 0$$

(5.31)

Substituting the relevant equations from (5.27) and (5.28) to replace the time derivatives, $\dot{H}_R$ is shown to be equal to zero i.e., $H_R$ is a constant of the motion.

Before we proceed further a small remark has to be made about why we have introduced an imaginary reference satellite at the midpoint, rather than simply using one of the satellites as the origin and defining the relative motion with respect to this satellite, such that the two satellites are defined at the coordinates $(\mathbf{r}, \mathbf{v})$ and $(\mathbf{r} + \delta \mathbf{r}, \mathbf{v} + \delta \mathbf{v})$. In that case, it
can be easily shown that the second order terms of the Taylor expansion would have to be ignored in the definition of the relative Hamiltonian (i.e., $O(\delta^2)$). On the other hand, the midpoint definition actually enables us to exploit the symmetry of the problem and we were able to include the second order terms in the Taylor expansion, which conveniently cancel out.

5.3 Symplectic Relative Integration Scheme

In the preceding section, we derived the equations of motion for the imaginary reference satellite as well as the relative motion between the two satellites. In this section we will demonstrate how these equations of motion can be used within a sophisticated symplectic integration scheme.

In Section 3.4.3, we showed how a higher order method can be constructed that uses different order numerical integration schemes for perturbations of different magnitudes. The Hamiltonian Splitting technique enabled us to separate the effect of different potential sources. The Keplerian potential causes by far the largest acceleration on the satellite, therefore the Keplerian part of the motion needed to be calculated very accurately, which would require a high order numerical model or very fine timesteps, costing precious processor time. Therefore, we opted to evaluate this term analytically.

The remaining acceleration terms which are due to perturbations are at least an order of magnitude smaller. We therefore further split these perturbations into $J_2$ and remaining terms. Higher order perturbations are computationally expensive to calculate yet have significantly smaller effects on the satellite orbit than Keplerian and $J_2$ effects. Therefore, such a splitting enabled us to compute these complicated terms much less frequently. This discussion is fully valid for the case of relative motion, hence, for the relative orbit propagator, we will employ the same methodology.

5.3.1 Keplerian Motion

While it may be possible to solve the Keplerian motion numerically, the existence of an exact analytical solution can be exploited. If a highly accurate analytical solution to
the relative Keplerian motion is found, the perturbations can be evaluated with larger
timesteps, saving processor time.

To this end, we will use the Gauss' \( f-g \) functions (see Vallado [112] and Battin [7] for
a particularly detailed treatment). This method is particularly appealing as it is free of
singularities and does not suffer from small eccentricity effects. To evaluate the relative
Keplerian motion we will simply use the variations. These variations to the \( f-g \) functions
were first proposed by Mikkola et al. [78] though they have used it within the construction
of an orbit determination filter for a single satellite. Hence, their use in the relative motion
field is novel. We will follow their derivation and notation closely in this section deriving the
Keplerian motion, however it is possible to obtain the resulting relative motion equations
via applying the same methodology to the forms presented by Vallado [112] and Battin [7].

Stumpff \( c \) functions [103] (see also [101]) enable us to write trigonometric functions as series
expansions and are widely used in celestial mechanics. The \( \cos \) and \( \sin \) functions can be
written as:

\[
\cos(x) = c_0(x^2) = 1 - x^2c_2(x^2) = 1 - x^2/2! + x^4c_4(x^2) = \ldots \tag{5.32}
\]

\[
\sin(x) = xc_1(x^2) = x - x^3c_3(x^2) = x - x^3/3! + x^5c_5(x^2) = \ldots \tag{5.33}
\]

The \( c \)-functions can be defined more generally by the relation:

\[
c_n(z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{(n+2j)!} \tag{5.34}
\]

which also satisfy

\[
c_n(z) = \frac{1}{n!} - zc_{n+2} \tag{5.35}
\]

Mikkola et al.[78] report that, for small values of the argument these can be evaluated
using the above recursion starting with \( c_n \approx 1/n! \), provided \( n \) is sufficiently large. For
larger argument values the 4-folding formulae are useful:

\[
c_5(z) = (c_5(z/4) + c_4(z/4) + c_3(z/4)c_2(z/4))/16
\]

\[
c_4(z) = c_3(z/4)(1 + c_1(z/4))/8
\]

\[
c_m(z) = 1/m! - zc_{m+2}(z), \quad m = 3, 2, 1, 0. \tag{5.36}
\]

For convenience we also introduce the \( G \)-functions [101]

\[
G_n(\beta, X) = X^n c_n(\beta X^2) \tag{5.37}
\]
which satisfy the relations:

\[ \begin{align*}
G_n &= X^n / n! - \beta G_{n+2} \\
G'_{n+1} &= G_n \\
G''_{n+2} &= -\beta G_{n+2} + X^n / n! \\
G_{n,\beta} &= \frac{1}{2}(nG_{n+2} - XG_{n+1})
\end{align*} \] (5.38)

where \( G_{n,\beta} = \partial G_n / \partial \beta \) and the primes denote differentiation with respect to \( X \).

Mikkola et al. [78] list the equations to evaluate the \( f-g \) formulation of the Keplerian motion for a single satellite as follows:

\[
\begin{align*}
\beta &= 2\mu / r_0 - v_0^2 = (\mu/a) \\
\eta_0 &= r_0 \cdot v_0 \\
\zeta_0 &= \mu - \beta r_0 \\
t(X) &= r_0 X + \eta_0 G_2 + \zeta_0 G_3 = h \quad \text{[Kepler's eq.]} \\
r &= r_0 + \eta_0 G_1 + \zeta_0 G_2 \\
f &= 1 - \mu G_2 / r_0 \\
g &= t - \mu G_3 \\
f' &= -\mu G_1 / (r_0 r) \\
g' &= 1 - \mu G_2 / r \\
r &= f r_0 + g v_0 \\
v &= f' r_0 + g' v_0
\end{align*}
\] (5.39)

The solution procedure starts with the initial positions and velocities \((r_0, v_0)\), from which we calculate \( \beta, \eta_0 \) and \( \zeta_0 \). The independent variable \( X \) needs to be solved for, where:

\[ X = \int \frac{dt}{r} \] (5.40)

This corresponds to solving Kepler's Equation in universal form.

Once \( X \) is calculated, the \( c \) and \( G \)-functions can be evaluated easily. This is followed by the \( f, g \) and \( f', g' \) parameters for the Kepler propagation that yield the new positions and velocities.

The evaluation of the relative motion is very similar to the absolute motion, by simply
calculating variations to each equation in (5.39). The equations are given below [78]:

\[
\begin{align*}
\delta r_0 &= r_0 \cdot \delta r_0 / r_0 \\
\delta \beta &= -2 \mu \delta r_0 / r_0^2 - 2 v_0 \cdot \delta v_0 \\
\delta \eta_0 &= v_0 \cdot \delta r_0 + r_0 \cdot \delta v_0 \\
\delta \zeta_0 &= -\beta \delta r_0 - r_0 \delta \beta \\
G_{3,\beta} &= (3 G_5 - X G_4) / 2 \\
G_{2,\beta} &= (2 G_4 - X G_3) / 2 \\
G_{1,\beta} &= (G_3 - X G_2) / 2 \\
t_{\beta} &= \eta_0 G_{2,\beta} + \zeta_0 G_{3,\beta} \\
\delta X &= -(X \delta r_0 + G_2 \delta \eta_0 + G_3 \delta \zeta_0 + t_{\beta} \delta \beta) / r \\
\delta G_1 &= G_0 \delta X + G_{1,\beta} \delta \beta \\
\delta G_2 &= G_1 \delta X + G_{2,\beta} \delta \beta \\
\delta G_3 &= G_2 \delta X + G_{3,\beta} \delta \beta \\
\delta r &= \delta r_0 + G_1 \delta \eta_0 + G_2 \delta \zeta_0 + \eta_0 \delta G_1 + \zeta_0 \delta G_2 \\
\delta f &= \mu G_2 \delta r_0 / r_0^2 - \mu \delta G_2 / r_0 \\
\delta g &= -\mu \delta G_3 \\
\delta \dot{f} &= -\mu \delta G_1 / (r_0 r) + \mu G_1 (\delta r_0 / r_0 + \delta r / r) / (r r_0) \\
\delta \dot{g} &= -\mu \delta G_2 / r + \mu G_2 \delta r / r^2 \\
\delta \ddot{r} &= f \delta v_0 + g \delta v_0 + r_0 \delta \dot{f} + v_0 \delta g \\
\delta \ddot{v} &= \dot{f} \delta r_0 + \dot{g} \delta v_0 + r_0 \delta \dot{f} + v_0 \delta \dot{g}
\end{align*}
\]

where $G_n = G_n(\beta, X)$ and the subscript $\beta$ denotes partial derivative with respect to that quantity.

The solution procedure starts with the initial relative positions and velocities $(\delta r_0, \delta v_0)$, from which we calculate $\delta \beta$, $\delta \eta_0$ and $\delta \zeta_0$. As independent variable $X$ has already been solved for, we can calculate the $\beta$ derivatives of the $G$ functions and the variations $\delta G$ directly. Finally the final relative positions and velocities $(\delta r$ and $\delta v)$ are calculated via the $(\delta f, \delta g)$ and $(\delta \dot{f}, \delta \dot{g})$ terms.

While the original $f$-$g$ equations yield exact positions and velocities, the $\delta f$-$\delta g$ for the relative motion are just first order variations and therefore are not exact. One problem with these equations is that, while they are simple to evaluate, they do not reveal much
5.3. Symplectic Relative Integration Scheme

Table 5.1: Test case formation initial conditions in Keplerian elements

<table>
<thead>
<tr>
<th></th>
<th>a(km)</th>
<th>e</th>
<th>I(deg)</th>
<th>Ω(deg)</th>
<th>ω(deg)</th>
<th>θ(deg)</th>
<th>H</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat1</td>
<td>15945.80</td>
<td>0.3500</td>
<td>60.00</td>
<td>40.03</td>
<td>20.00</td>
<td>70.00</td>
<td>-0.199994</td>
<td>1.481152</td>
</tr>
<tr>
<td>Sat2</td>
<td>15945.65</td>
<td>0.3501</td>
<td>60.03</td>
<td>40.03</td>
<td>19.95</td>
<td>70.05</td>
<td>-0.199996</td>
<td>1.481086</td>
</tr>
<tr>
<td>Diff</td>
<td>-0.15</td>
<td>0.0001</td>
<td>0.030</td>
<td>0.030</td>
<td>-0.050</td>
<td>0.050</td>
<td>-1.88 × 10^{-6}</td>
<td>-6.60 × 10^{-5}</td>
</tr>
</tbody>
</table>

about the nature of the motion. Furthermore, it is not immediately obvious whether $H_R$ or $L_R$ is conserved.

To test the performance of this method, we set up a sample formation flying case, with one satellite at about 15945.8km semimajor axis and the other 150m lower (see Table 5.1). The energy difference causes the initial separation of 7.5km to grow to 46km by the end of a 5 day propagation. Note that, this is a large separation for the usual formation missions and is challenging for the algorithm, which assumes kilometre level separations. The model to represent the truth is the full nonlinear exact two-body equations of motion for each satellite; the difference of the two absolute orbits yields the exact relative orbit. The energy and angular momentum terms are non-dimensionalised, where unit distance is the average radius of the Earth, 6378.137km and the gravitational parameter $\mu$ is taken as unity.

The simplest way to demonstrate the conservation of energy and angular momentum is to recalculate the $H_R$ and $L_R$ values throughout the propagation. Figure 5.1 illustrates how much the recalculated $H_R$ and $L_R$ values deviate from the initial values. As can be seen, this variation is limited to roundoff errors, therefore this method conserves the relative energy and angular momentum. This result is not very surprising, as $\delta f - \delta g$ equations are simply a first order approximation to the Keplerian relative motion and therefore the resulting equations conserve a first order approximation to the energy difference of the two satellites.

The second test is to evaluate the relative positioning accuracy for a range of eccentricities. Figure 5.2 shows the peak relative positioning errors after a 5 day (21.5 orbit) propagation. The initial separations range from 1.6km from highest eccentricity to 9.3km for the lowest eccentricity and the peak separations range from 70km for $e = 0.9$ to 35.4km for near-circular cases. The graph shows that metre level accuracy is possible, but with increasing
eccentricity the accuracy seems to suffer. The reason for this has been identified as the distance between the geometric midpoint of the two satellites and the reference satellite. Our equations assume that they follow the same trajectory to the first order; if this assumption is stretched, as is the case in here, errors in relative positioning start to appear. Nevertheless, the errors as a percentage of the separation stay around 0.03% for most cases and reach 0.35% for the $e = 0.9$ case only.

The final test is for the stability of the algorithm and how quickly the errors develop once the close proximity as well as coincident reference satellite and geometric midpoint assumptions break down. Figure 5.3 illustrates a 50 day (215 orbit) simulation with the same initial conditions as before with a semimajor axis difference of 2km. Therefore, peak separation reaches 5770km whereas peak error reaches 1%. This shows that the method has excellent stability and a graceful degradation of performance even when the linear assumptions are no longer valid.

From these tests, we conclude that the analytic propagation scheme conserves relative angular momentum and relative energy. Furthermore, its performance is very promising, even for relatively large separations.
5.3. Symplectic Relative Integration Scheme

Figure 5.2: Peak relative positioning error (log scale) variation with eccentricity after 5 days (21.5 orbits)

Figure 5.3: Relative positioning error evolution as percentage of the separation in 50 days (215 orbits)
5.3.2 Motion Due to Perturbations

The relative motion due to perturbations are evaluated in a very similar method to the absolute orbit propagator we have described in Section 3.4.3.

The leapfrog scheme is a second order method and can be written succinctly via Lie notation:

\[
\exp\left(\frac{1}{2}hK\right)\exp(hR)\exp\left(\frac{1}{2}hK\right)
\]

(5.42)

where \( K \) is a Keplerian propagation step, as explained in the previous section. \( R \) is the propagation due to other terms in the geopotential. The effect of these remaining terms is to modify the absolute and relative accelerations, which are purely functions of \( r \) and \( \delta r \). Therefore, they represent jumps in the velocity. Using the acceleration expressions given in Equations (5.29) and (5.30), we can write these velocity jumps as:

\[
\begin{align*}
\mathbf{v}_{k+1} &= \mathbf{v}_k - h \left( \frac{\partial R}{\partial r} \right)_k \\
\delta \mathbf{v}_{k+1} &= \delta \mathbf{v}_k - h \left( \frac{\partial^2 R}{\partial r^2} \right)_k \delta r_k
\end{align*}
\]

(5.43)

(5.44)

where \( R \) is given as:

\[
R = \mu \left[ \sum_{l=2}^{\infty} J_l \left( \frac{R_\odot}{r} \right)^l P_l(\cos \theta) + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R_\odot}{r} \right)^n P_n^m(\cos \theta) J_{nm} \cos m(\varphi - \psi_{nm}) \right]
\]

(5.45)

This is simply the perturbations part of the full geopotential given in Equation (3.1). The terms containing \( J_l \) are the zonal harmonics and the ones with \( J_{nm} \) are the tesseral harmonics.

It is possible to construct higher order schemes in the following form [77]:

\[
\exp(x_m hK) \exp(w_m hR) \ldots \exp(x_0 hK) \exp(w_0 hR) \exp(x_0 hK) \ldots \exp(w_m hR) \exp(x_m hK)
\]

(5.46)

where \( x_m = w_m/2, \ x_{m-1} = (w_m + w_{m-1})/2, \ldots, \ x_0 = (w_1 + w_0)/2. \)

As long as the perturbations are first order, these methods will have similar orders to the error of the associated numerical integration formula [77]. This condition is satisfied for the case of satellites orbiting the Earth, where the largest perturbation term in the potential is \( J_2 \), which is about \( 10^3 \) times smaller than the Keplerian potential.
For the $6^{th}$ order scheme, Yoshida [118] reports that there are three solutions for $w_m$ but the one with smallest error is:

$$
\begin{align*}
  w_1 &= -1.17767998417887100695 \\
  w_2 &= 0.23557321335935813368 \\
  w_3 &= 0.78451361047755726382 \\
  w_0 &= 1 - 2(w_1 + w_2 + w_3)
\end{align*}
$$

Yoshida [118] obtained these solutions via numerically solving a set of three algebraic equations simultaneously. We have fully reproduced the coefficients from Leimkuhler and Reich [69] who presented higher precision results for the same coefficients in comparison to Yoshida [118].

Obviously, while yielding better accuracy, higher order schemes require a greater number of force calculations per integration step. In fact, considering that the force calculations due to perturbations are computationally expensive, it is instructive to note that the $2^{nd}$ order scheme requires a single force calculation whereas this increases to seven force calculations for the $6^{th}$ order scheme. Therefore the design of the propagation scheme should strike a balance between computational load and accuracy requirements.

As mentioned earlier in this section, the effects of $J_2$ are at least an order of magnitude larger than the remaining terms in the geopotential. Hence, it is not computationally efficient to calculate the effect of both $J_2$ and less significant terms at the same frequency within the propagation. Composite schemes enable us to achieve precisely this, by combining high order integration for more significant terms with a low order integration for higher order terms. Therefore, separating the Hamiltonian further, one obtains Keplerian part ($K$), $J_2$ part ($R_2$) and the remaining geopotential terms ($R_j$). Rewriting the leapfrog, the composite integrator is constructed,

$$
\exp\left(\frac{1}{2}h(K + R_2)\right)\exp(hR_4)\exp\left(\frac{1}{2}h(K + R_2)\right)
$$

so that the higher order terms (denoted $\exp(hR_4)$) are propagated via longer timesteps within a $2^{nd}$ order scheme, while the more significant Keplerian and $J_2$ effects (denoted $\exp(\frac{1}{2}h(K + R_2))$) are calculated via a higher order scheme such as the $6^{th}$ order method above, saving precious processor time. Evidently, such a modular design has significant
practical advantages, particularly the ease with which it could be tailored to adapt to
different accuracy requirements of on-board and groundstation based applications.

Finally, a remark needs to be made regarding the integration order and error of this com­
posite scheme. When we split a Hamiltonian \((H)\) into two parts of similar magnitude
\((H_1, H_2)\) and use a leapfrog scheme, the error is said to be \(O(h^2)\). However, if one Hamil­
tonian is significantly larger than the other, such that \(H_1 = \epsilon H_2\), the error is actually of
order \(O(\epsilon h^2)\) \cite{75}. Therefore, for the case of 6th order method for \(J_2\) propagation, the error
is actually of order \(O(J_2 h^6)\), where \(J_2\) coefficient is about \(10^{-3}\). Similarly, for the propa­
gation of the higher order terms, even a low accuracy second order method still yields high
accuracy in positioning, as the next terms in the zonal harmonics are \(J_3, J_4\) and tesseral
harmonics is \(J_2, J_2\), all of which are an order of magnitude smaller than the already small \(J_2\)
term.

### 5.3.3 Order of Integration

In the preceding sections we have explained how we can construct a composite scheme
that treats Keplerian, \(J_2\) and remaining perturbations separately and uses different orders
of integration to handle each of them. However, we have not yet tested and quantified
the differences between various orders of integration. While the acceptable errors, hence
the integration method, will depend on the specific application, metre to centimetre level
errors are a good benchmark for formation scenarios, where the separations are usually an
order of magnitude larger, on kilometre level.

For this test, we set up a sample formation flying case, with one satellite at about 7653km
semimajor axis and the other 80m lower (see Table 5.2). The energy difference causes the
initial separation of 1.75km to increase to 63 km by the end of a 5 day propagation. The
truth model in our test cases is a composite symplectic integrator for a single satellite,
which evaluates the Keplerian part using exact analytical equations, \(J_2\) with a 6th order
scheme and higher order terms with Simpson’s Rule. We run the propagator once for each
satellite at 1000 steps/orbit and simply take the difference of the two orbits to obtain the
true relative orbit.

The first test case is to compare the performance of a 2nd order integration with a 6th
5.3. Symplectic Relative Integration Scheme

<table>
<thead>
<tr>
<th>$a$ (km)</th>
<th>$e$</th>
<th>$I$ (deg)</th>
<th>$\Omega$ (deg)</th>
<th>$\omega$ (deg)</th>
<th>$\theta$ (deg)</th>
<th>$H$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat1 7653.780</td>
<td>0.0050</td>
<td>60.00</td>
<td>40.0</td>
<td>20.0</td>
<td>240.0</td>
<td>-0.416665</td>
<td>1.095432</td>
</tr>
<tr>
<td>Sat2 7653.700</td>
<td>0.0055</td>
<td>60.01</td>
<td>40.0</td>
<td>19.0</td>
<td>241.0</td>
<td>-0.416667</td>
<td>1.095426</td>
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<tr>
<td>Diff</td>
<td>-0.080</td>
<td>0.0005</td>
<td>0.010</td>
<td>0.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>(-2.17 \times 10^{-5})</td>
</tr>
</tbody>
</table>

Table 5.2: Integration test case initial conditions in Keplerian elements

order scheme, for the case of a Keplerian and $J_2$ integration. Obviously, the Keplerian propagation is analytical, therefore the error is independent of the stepsize, whereas the $J_2$ effects are propagated numerically, so with decreasing stepsize the error should diminish. Figure 5.4 demonstrates the error in relative positioning of these two methods after a 5 day propagation, for various step sizes. The 2$^\text{nd}$ order method accuracy decreases for larger step sizes; even though it seems to yield better accuracy than the 6$^\text{th}$ order method for lower step sizes, it is actually the interplay between second order errors due to $J_2$ and analytical Keplerian parts of the propagation. The accuracy of the 6$^\text{th}$ order method seems to be independent of the step size, suggesting that the primary source of error is not the $J_2$ part of the propagation but the analytical Keplerian propagation. This residual error at the end of 5 days is about 2m, which corresponds to $3.5 \times 10^{-3}\%$ of the final formation separation.

We conclude that the 6$^\text{th}$ order scheme to handle $J_2$ and Keplerian parts of the propagation yields remarkably good accuracy more reliably; the coincidence that enabled the 2$^\text{nd}$ order scheme to have slightly smaller errors than the 6$^\text{th}$ order will not exist for most cases. On an AMD2400+ computer with 512MB RAM and MS Windows operating system, the run time for 2$^\text{nd}$ order methods is less than a second. For the 6$^\text{th}$ order scheme, run times are about a second for propagations with timesteps less than 100 steps/orbit and about 2 seconds for propagations with step sizes between 100 steps/orbit to 300 steps/orbit. From this discussion, it can be concluded that using a 6$^\text{th}$ order scheme with large step sizes gives best accuracy without causing significant computational burden.

The second test case is to evaluate the performance of a 2$^\text{nd}$ order method for higher order geopotentials. In this case, we will employ a 6$^\text{th}$ order scheme to compute the motion due to Keplerian and $J_2$. We include 36 $\times$ 36 geopotential model (i.e., 36 terms in zonal and tesseral geopotential terms in the spherical harmonic) for this 5-day test run. Once again,
Figure 5.4: Relative positioning error (log scale) (Kepler + $J_2$)

the truth model is the difference between two absolute propagations with 1000 steps/orbit and 36 x 36 geopotential model.

Figure 5.5 shows the evolution of the relative positioning error for different stepsizes; the figure caption shows the integration timestep in steps/orbit. We can see that for low stepsizes such as 20 steps/orbit, the integration starts to lose stability. However, at 75 steps/orbit the integration is stable and provides relative positioning accuracies around 2.2m at the end of the 5 day propagation. It can be seen that having more than 100 steps/orbit brings little benefit as the error stabilises around 2.2m metre mark. This corresponds to $3.4 \times 10^{-3}\%$ of the final formation separation.

On an AMD2400+ computer with 512MB RAM and MS Windows operating system, the maximum run time for this method is about 7 seconds for the 300step/orbit run and the minimum is about one second for the 20steps/orbit. It appears that an increase of about 40steps/orbit increases the run time by a second. Combining this with the accuracies obtained, it can be concluded that for high order propagations, 100 to 150 steps/orbit is a reasonable timestep choice to obtain accuracies at metre level or better.
5.4. Results

5.4.1 Conservation of Energy

The conservation of energy for the relative motion can be demonstrated via calculating $H_R$ via Equation (5.26) at each output step. Note that, the energy is constant in time for both the Keplerian potential and zonal harmonics in the geopotential, as the potential field is axisymmetric. On the other hand, the Hamiltonian oscillates as the potential is no longer axisymmetric and becomes a function of time. Therefore, within the propagator, we would expect the energy oscillation to become smaller as the integration timesteps get smaller. This is in contrast to the tesseral harmonics which by their definition correspond to the variations in the geopotential along a latitude band; the potential is therefore a function of the rotation of the Earth. This variation manifests itself as an oscillation in the potential (and the Hamiltonian) which is independent of the integration stepsize.

To compare and contrast these two cases, consider a satellite at 9567.2km semimajor axis and $e = 0.3$ eccentricity. Figure 5.6 illustrates the variation of the energy from its initial value for a 5 day simulation with 4 terms in zonal harmonics (shown as $n = 4A$ in the figure caption) and another with 4 terms in both zonal and tesseral harmonics (shown as $n = 4 \times 4$...
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Figure 5.6: Variation of the energy with respect to the initial values, with \( J_4 \) level zonal and tesseral harmonics, single satellite case

in the figure caption). For both cases we use 100 steps/orbit as integration timestep. The energy oscillation amplitude is less than \( 10^{-8} \) for the axisymmetric case (seen on the left hand side axis in the figure), with a period matching the period of the satellite orbit. However, for the case with tesseral harmonics, the energy oscillation amplitude is \( 10^{-6} \) (seen on the right hand side axis in the figure), which peaks twice every day. These are the 2\(^{nd}\) daily variations due to 2\(^{nd}\) order tesseral harmonics.

A similar case can be demonstrated for the relative energy. We carry out the above propagation, this time adding another satellite in a similar orbit but at 23m semimajor axis difference. Figure 5.7 shows the \( H_R \) value calculated at each step for both cases, but zoomed in to the first 5 days. The \( 10^{-11} \) level relative energy oscillation of the axisymmetric case is shown on the left hand side axis. The \( 4 \times 10^{-9} \) level relative energy oscillation of the non-axisymmetric case is shown on the right hand side axis. The axisymmetric case has an oscillation magnitude equal to the period of the reference satellite, as is the case in single satellite case above. However, the 2\(^{nd}\) and 4\(^{th}\) daily variations due to tesseral harmonics are seen to dominate the non-axisymmetric case curve. It should also be noted that Figures 5.6 and 5.7 demonstrate the conservation of energy for the absolute and relative motion, respectively.
Having established the behaviour of the propagator for the axisymmetric and non-symmetric geopotential cases and having demonstrated the conservation of energy, one last interesting experiment is to demonstrate the rate at which the relative energy oscillation amplitude decreases with various stepsizes. To this end, we can use the above test case but plot the oscillation amplitude in the absolute energy $H$ and relative energy $H_R$ for various stepsizes. Note that the oscillation amplitude is a measure of the positional accuracy, therefore this graph should give us another indication of the diminishing returns of the increased positioning accuracy with decreasing stepsizes.

Figure 5.8 illustrates the oscillation amplitudes for absolute and relative propagation (shown as $H$ and $H_R$ in the figure caption, respectively) with various stepsizes. The first thing to observe is the similarity in the behaviour of the two curves; as the stepsizes shrink, the oscillation amplitudes become smaller. This is precisely the behaviour we expect from the numerical integration scheme. In theory, as the timesteps go to infinity, the energy oscillation amplitude goes to zero; in reality, the results from extremely small timesteps will be dominated by roundoff errors. On the other hand, the gains from halving the stepsize diminishes for smaller stepsizes, while the computational cost grows rapidly; therefore there is little advantage to use very small timesteps.

Another important point is that there is an order of magnitude difference between the $H$
and $H_R$ oscillation amplitudes. This shows that, while there is a fairly large oscillation amplitude in the absolute energy (hence low absolute positioning accuracy), we can still get low oscillation amplitudes in the relative energy (hence high relative positioning accuracy).

We have already mentioned that the $z$ component of the angular momentum is conserved for a satellite in an axisymmetric geopotential field. The same holds true for the relative motion, where the $z$ component of the 'relative' angular momentum is conserved. To test this, we repeat the above formation example for 15 days, in a $J_4$ axisymmetric geopotential field. Figure 5.9 shows the error in the $z$ component of the relative angular momentum, compared to its initial value. We can see that the error is at $10^{-17}$ level, which is down to machine accuracy. Therefore, conservation of the $z$ component of the angular momentum is shown for the relative motion.

5.4.2 Number of Geopotentials

One important parameter to decide upon for the propagator is the number of geopotentials in the spherical harmonic to use in the calculations. Obviously, the higher the number of geopotentials, the better will be the accuracy but also the higher will be the computational burden. In reality, the specific application, its accuracy requirements and computational
5.4. Results

resources will be the determinants.

To test the effect of including various number of geopotentials, we used the initial conditions based on Table 5.2. however, we halved the semimajor axis difference and doubled the integration time to better observe the differences. We used the true anomalies at 30 and 31 degrees. In this case, for a 10-day propagation, the separation increases from a few kilometres to about 63km. The truth model is a 1000 steps/orbit composite symplectic scheme to calculate the absolute orbit with a $36 \times 36$ geopotential field model. We employ a conservative stepsize choice of 120 steps/orbit; as shown in Section 5.3.3, even with 36 terms in zonal and tesseral harmonics this yields metre level accuracy for this example.

Figure 5.10 demonstrates the evolution of the error in relative positioning with various zonal and tesseral harmonics included in the modelling. As expected, as the number of terms in the spherical harmonic increase, the accuracy increases dramatically. At the end of the 10 day propagation, metre level accuracy is possible with 10 terms in the geopotential model. Note that, even the 36 geopotential propagation takes two seconds on an AMD2400+ computer with 512MB RAM for this 120 steps/orbit case.

Figure 5.11 demonstrates the evolution of the error in relative positioning with various zonal harmonics included in the modelling but no tesseral harmonics. The truth model, however, includes 36 zonal and tesseral terms in the geopotential model. The results show
Figure 5.10: Relative positioning error (log scale) with various zonal and tesseral harmonics

that, while different number of geopotentials induce different levels of errors, there is a very large bias of 250m in all cases, simply because the tesseral terms are not taken into account.

The modelling errors presented above can be explained via the energy differences, in a similar fashion to Section 3.5.2. The total energy difference can be written as a summation of smaller Hamiltonians due to other geopotential terms i.e.,

\[ \delta H = \delta H_K + \delta H_2 + \delta H_3 + \ldots + \delta H_{tess}, \]

where \( \delta H_K \) is the difference in Keplerian potential, \( \delta H_2 \) is the difference in \( J_2 \) potential, \( \delta H_3 \) due to \( J_3 \) and \( \delta H_{tess} \) due to all tesseral terms. Given the coordinates of two satellites, different models will obviously yield different \( \delta H \) values. The energy of a satellite determines the mean motion and the energy difference thus determines the relative mean motion or relative drift rate. Evidently, a simple model that includes 4 terms in the geopotential will have a significantly different drift rate with respect to a 36 term model due to these truncated geopotential terms, hence a large relative positioning error results.

We note that, while other researchers usually limited themselves to analytical models of the motion with \( J_2 \) level perturbations, such models will yield limited accuracy when compared to a much more realistic model of the motion. For this example, a \( 2 \times 2 \) geopotential field model compared to a \( 36 \times 36 \) model, the error in relative positioning exceeds 10m in three days and it reaches 100m in about 8 days. Therefore, modelling of high order geopotentials...
5.4. Results

Figure 5.11: Relative positioning error (log scale) with various zonal harmonics only is crucial if long-term, high accuracy relative positioning solutions are required. However, it should also be noted that, in many cases, the propagation algorithms are not used independently, but within a filtering scheme such as a Kalman Filter. Within the filter, the propagators constitute the dynamic model of the motion and sensors provide (usually high-precision) range or relative position measurements. The filter uses the propagator in the absence of sensor measurements to propagate the state and modify this propagated state via the measurements. Therefore, while the dynamic model may not be very accurate, the resulting estimate of the trajectory is more accurate. It may be therefore more instructive to talk about the fitting accuracy of the propagator as a measure of the precision, demonstrating how much a trajectory propagated via a simple model deviates from this filtered trajectory. Nevertheless, it can be said that, a better propagator would yield a better fit to the estimated trajectory. In other words, a given level of orbit fitting accuracy could be maintained via sparser sensor measurements.

5.4.3 Eccentricity

An important aspect when assessing the performance of the orbit propagation scheme is its behaviour in various eccentricities. For near circular cases, the magnitude of the acceleration acting on a satellite is fairly uniform, therefore constant stepsize algorithms
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Figure 5.12: Peak relative positioning error (log scale) with respect to eccentricity (5 days)

capture the motion successfully. However, for highly eccentric orbits, the acceleration varies greatly throughout an orbit. Around the periapsis, the satellite moves very fast, requiring very fine stepsizes. On the other hand, around the apoapsis, the satellite slows down and spends a longer time with only small changes in the acceleration; large stepsizes will therefore be adequate for accurate modelling of the motion. Adaptive stepsize schemes such as RKF are usually utilised for such scenarios for greater efficiency.

However, in the absence of such a scheme, we will demonstrate the accuracy of our method for different stepsizes. We will use the initial conditions as presented in Table 5.1 and run 5 day (21.5 orbit) simulations with stepsizes of 100 steps/orbit and 300 steps/orbit. The geopotential model will be $36 \times 36$ and the truth model will use 1000 steps/orbit. Figure 5.12 illustrates the peak relative positioning error for 100 steps/orbit and 300 steps/orbit cases. For relatively lower eccentricities there is little difference between the two schemes and the error increases slowly but steadily with increasing eccentricity. However, at around $e = 0.5$ the 100 steps/orbit case begins to become unstable and the relative positioning error increases very rapidly, while the 300 steps/orbit case remains remarkably accurate, with metre level errors.

The conclusion from this test is that, while eccentric cases can be handled accurately with moderate stepsizes, for high eccentricities the stepsize should be chosen carefully to ensure
high precision.

5.4.4 Stability of the Propagator

Up to now, we have successfully demonstrated the performance and accuracy of the propagator for scenarios with several day duration and challenging but reasonable separations, at less than 100km. We can test the stability of the algorithm by running the simulation for longer timescales and with larger semimajor axis differences. When the separation becomes very large, close proximity assumption breaks down and we would like the investigate how quickly the propagation results become unusable.

The test case uses the initial conditions given in Table 5.2, but instead of a 80m semimajor axis difference we introduce a 1200m semimajor axis difference for the satellites to drift away rapidly. The initial 1.3km separation steadily increases to 4360km at the end of a 30 day propagation run.

The first test is to evaluate the stability of the analytical Keplerian algorithm, therefore we do not include the effect of other geopotentials. Figure 5.13 shows the relative positioning error in kilometres as well as a percentage of the actual separation. It can be seen that, the relative positioning error reaches about 60km and it grows at a rate of \((t^3)\). However, given that the total separation reaches 4357km, the relative positioning error is merely 1.4% of the total separation. Therefore, we conclude that the accuracy of the analytical algorithm degrades gracefully even when the separations become extremely large.

The second test is to evaluate the stability of the symplectic method employing a full geopotential of 36 x 36 terms in the spherical harmonics. We run the symplectic propagator with 120 steps/orbit, which takes 10 seconds to execute. Figure 5.14 shows the relative positioning error in kilometres as well as a percentage of the actual separation. As can be seen, the results are almost the same as the analytical example, suggesting that the errors are dominated by the errors in the analytical part of the propagation. However, even with the maximum separation at thousands of kilometres and a complicated geopotential model, the total percentage error is still about 1.4%. Hence we conclude that the addition of the numerical integration of the high order geopotentials does not degrade the stability of the algorithm.
Figure 5.13: Relative positioning error evolution (Keplerian case)

Figure 5.14: Relative positioning error evolution (full geopotential case)
5.5 Conclusions

In this chapter we have detailed the derivation of a symplectic relative orbit propagator. The propagator uses a sophisticated composite symplectic integration scheme that uses different integration schemes for perturbations of different magnitude. Therefore integration stepsize, number of geopotentials as well as the order of integration for the motion due to Keplerian, $J_2$ and higher order geopotentials can be specified. Due to the flexibility of the numerical scheme, it is possible to add other perturbations such as differential drag. The scheme can accommodate an arbitrary number of geopotentials.

Such a modular design has an added advantage of scalability and adaptability for any application; for very high-precision applications to be run on the ground, high order numerical schemes can be used with many geopotential terms, whereas onboard a satellite with limited computational resources the numerical scheme can be tailored to the accuracy requirements and processor load.

We have also demonstrated the adaptation of an analytical model for the relative Keplerian motion, based on the $f$-$g$ equations and their variations.

We have presented detailed results illustrating the conservation of energy, computational performance and accuracy of the analytical Keplerian scheme as well as the $36 \times 36$ geopotential model numerical scheme, for a wide range of eccentricities and semimajor axes. In most cases we used formation separations that would be considered large and challenging for relative propagation algorithms. We conclude that this novel numerical scheme has very good accuracy, employing a vastly superior modelling than any other relative motion models in the literature.
Chapter 6

Analytical Relative Orbit Propagation - \( H_R L_R \) Method

6.1 Introduction

In the previous chapter, we have described the design of a symplectic numerical relative orbit propagator in detail. The relative orbit propagator uses a composite symplectic scheme as the numerical integration algorithm.

We have also shown how the variations of \( f-g \) equations [78] can be used as an analytical relative orbit propagation scheme for the Keplerian part of the motion. This method works in the inertial frame rather than a rotating accelerating local frame, which facilitates adding the perturbation terms, as they are defined in the inertial frame. Despite these advantages, this method provides little insight into the nature of the relative motion. In the last chapter we have derived the relative Hamiltonian \( H_R \) and the relative angular momentum \( L_R \), which are crucial constants of the motion but they do not appear explicitly in the variations of \( f-g \) equations.

In this chapter, we will derive the equations for the Keplerian relative motion using the Hamiltonian description of the relative motion. We will show that the resulting equations of motion in the inertial frame is remarkably simple and reveals important properties of the motion. Using these equations of motion, we will show that second order modifications to
the reference satellite initial conditions will enable us to make very significant improvements to the relative positioning accuracy. Finally, we will carry out some tests to examine the performance, when used within the symplectic numeric integration scheme described in the previous chapter.

6.2 Relative Motion in a Keplerian Potential

6.2.1 Description of the Relative Motion

In Section 5.2.1, we have analysed the two-body motion of two satellites at coordinates \((r \pm \frac{1}{2}\delta r, v \pm \frac{1}{2}\delta v)\). The trajectory of this geometric midpoint follows that of a real satellite, as a first order approximation, as shown in Equation (5.6). Therefore, its motion can be described analytically. We wrote the relative Hamiltonian \(H_r\) in Equation (5.7), which is an approximation to the energy difference between the two satellites. Similarly, we defined a relative angular momentum \(L_r\), which is an approximation to the angular momentum difference between the two satellites. As expected, \(H_r\) and \(L_r\) are constants of the motion. Finally, in Equations (5.8) and (5.9) we have described the equations of absolute and relative motions through Hamilton’s Equations for this 12 dimensional phase space.

In this section, we will solve these relative equations of motion. We have, however, demonstrated that these solutions conserve a relative energy and a relative angular momentum. These conservation laws are to be expected following conservation of energy and angular momentum of the individual satellite orbits.

We shall consider the solution of the relative equations, and first we need to decide a coordinate frame to use. Since the midpoint moves on a Keplerian orbit, then it makes sense to use one co-ordinate axis (z-axis) along the angular momentum vector of this orbit, \(L\). Another direction (x-axis) points along the eccentricity vector of this orbit, \(e\), and then the third to make a right-handed set. This coordinate set is also called the ‘perifocal frame’ [112]. The equations of relative motion are:

\[
\delta \dot{r} = \delta v \\
\delta \dot{v} = -\frac{\mu}{r^3} \delta r + \frac{3\mu}{r^5} (r \cdot \delta r) r
\]  
(6.1)
6.2. Relative Motion in a Keplerian Potential

If we now take the dot product of these equations with the unit vector along the z axis, then since \( r \) lies on the plane \( z = 0 \) we have:

\[
\delta \ddot{z} + \frac{\mu}{r^3} \delta z = 0 \tag{6.2}
\]

where \( \delta z = \delta r \cdot \mathbf{k} \). If the angle between \( r \) and \( e \) is \( \theta \) (true anomaly) then we can replace time as the independent variable by \( \theta \). This equation becomes:

\[
(1 + e \cos \theta) \frac{d^2 \delta z}{d\theta^2} - 2e \sin \theta \frac{d\delta z}{d\theta} + \delta z = 0 \tag{6.3}
\]

By changing variable to \( R(\theta) \) such that: \( R = \delta z/r \) (not to be confused with the perturbation potential, \( R(r) \)) this equation simplifies to:

\[
\frac{d^2 R}{d\theta^2} + R = 0 \tag{6.4}
\]

which provides the solution:

\[
\frac{\delta z}{r} = C \cos \theta + D \sin \theta \tag{6.5}
\]

Hence the motion of the satellites around the orbit plane of the midpoint is a simple oscillation, which is the same solution as obtained from Hill's equations for the limit of zero eccentricity (see Equation (3.51)).

Now consider the other components of \( \delta r \). We note that in the co-ordinates we are using then both \( r \) and \( v \) have no \( z \) component. We can then expand out Equation (5.7) as:

\[
H_R = \dot{x} \ddot{x} + \dot{y} \ddot{y} + \frac{\mu}{r^3} (x\delta x + y\delta y) \tag{6.6}
\]

Similarly the expression for the relative angular momentum can be written:

\[
L_R = (x\dot{y} - y\dot{x}) + (\delta x \dot{y} - y\delta \dot{x}) \tag{6.7}
\]

Since both these quantities are conserved then we have the first integrals of the relative motion equations and only need to solve these coupled first order equations for \( \delta x \) and \( \delta y \). Once again we express the components of the midpoint position and velocity in terms of its true anomaly \( \theta \) and use this as the independent variable. These equations for \( H_R \) and \( L_R \) then reduce to the form:

\[
- \sin \theta \frac{d\delta x}{d\theta} + (e + \cos \theta) \frac{d\delta y}{d\theta} + (\delta x \cos \theta + \delta y \sin \theta) = \frac{\mu^3 H_R}{L^2 (1 + e \cos \theta)^2} \tag{6.8}
\]
\[ -(1 + e \cos \theta) \sin \theta \frac{d\delta x}{d\theta} + (1 + e \cos \theta) \cos \theta \frac{d\delta y}{d\theta} + (e + \cos \theta) \delta x + \sin \theta \delta y = \frac{p_{LR}}{L} \]  \hspace{1cm} (6.9)

We need to disentangle these equations to remove \( \delta x \) and its derivative. It is convenient to introduce \( w = \cos \theta \) as independent variable and introduce new unknowns: \( P = \delta x/r \) and \( Q = \delta y/r \). These equations can then be rewritten in the form:

\[ \frac{L_R}{L} = \left[ (1 - w^2) \frac{dP}{dw} + wP \right] - \sqrt{1 - w^2} \left[ \frac{dQ}{dw} - Q \right] \]  \hspace{1cm} (6.10)

and

\[ \frac{p^2 H_R}{L^2} = (1 + ew) \left[ (1 - w^2) \frac{dP}{dw} + wP \right] - e(1 - w^2)P \]
\[ - e \sqrt{1 - w^2} \left[ \frac{dQ}{dw} - wQ \right] - \sqrt{1 - w^2} (1 + ew + e^2) \left[ \frac{dQ}{dw} - Q \right] \]  \hspace{1cm} (6.11)

We can now substitute the first bracket in Equation (6.10) into Equation (6.11) and solve for \( P \):

\[ (1 - w^2)P = \frac{1}{eL} \left[ (1 + ew)L_R - \frac{p^2 H_R}{L} \right] - \sqrt{1 - w^2} \left[ (1 + ew) \frac{dQ}{dw} - (e + w)Q \right] \]  \hspace{1cm} (6.12)

We can also rewrite Equation (6.10) in the form:

\[ (1 - w^2)^{3/2} \frac{d}{dw} \left[ \frac{P}{\sqrt{1 - w^2}} \right] = \frac{L_R}{L} + \sqrt{1 - w^2} \left[ \frac{dQ}{dw} - Q \right] \]  \hspace{1cm} (6.13)

To complete the elimination we divide Equation (6.12) by \( (1 - w^2)^{3/2} \) and differentiate. Substituting the result into Equation (6.13) then leaves us with a differential equation for \( Q(w) \):

\[ (1 - w^2) \frac{d^2Q}{dw^2} + 2w \frac{dQ}{dw} - 2Q = \frac{3w}{\sqrt{1 - w^2}} \left[ \frac{L_R}{eL} - \frac{p^2 H_R}{eL^2} \frac{1}{1 + ew} \right] \]  \hspace{1cm} (6.14)

The solution of this differential equation is straight-forward, if lengthy, but the general solution is:

\[ Q(w) = A(1 + w^2) + Bw - \frac{L_R}{eL} w \sqrt{1 - w^2} \]
\[ + \frac{p^2 H_R}{eL^2} \frac{1}{1 - e^2} \left[ \frac{e(2 + e^2) + w(1 + 2e^2)}{1 - e^2} \right] \sqrt{1 - w^2} - \frac{3eE}{(1 - e^2)^{3/2}} (e + w)(1 + ew) \]  \hspace{1cm} (6.15)

where \( E \) is the eccentric anomaly of the midpoint orbit. Some comment needs to be made on how the eccentric anomaly enters this equation. It appears through the evaluation of
6.2. Relative Motion in a Keplerian Potential

the following integral:

\[
\int \frac{dw}{\sqrt{1 - w^2(1 + e\omega)}} = \frac{\pi - E}{\sqrt{1 - e^2}}
\]  

(6.16)

Having found the solution for \(Q\) we can substitute this result into Equation (6.12) to find \(P\). The result needs to be manipulated to remove apparent singularities which cancel when terms are grouped appropriately, leaving the result:

\[
P = \sqrt{1 - w^2}[(e - w)A - B] + \frac{L_R}{eL}[2 + w(e - w)]
\]

\[
+ \frac{p^2H_R}{eL^2} \frac{1}{1 - e^2} \left[ (1 + e\omega + w^2) - \frac{3(1 - e^2w^2)}{1 - e^2} + \frac{3eE}{(1 - e^2)^{3/2}} \sqrt{1 - w^2(1 + e\omega)} \right]
\]

(6.17)

The emergence of the eccentric anomaly in these solutions requires us to solve Kepler’s problem to find the relative positions at a given time. We note, however, that the components of velocity of the midpoint have the form:

\[
v_x = -\frac{L}{p} \sin \theta
\]

(6.18)

and

\[
v_y = \frac{L}{p}(e + \cos \theta)
\]

(6.19)

This shows that the factors multiplying the eccentric anomaly in (6.15) and (6.17) are the components of the midpoint velocity. If we further use the Kepler relation:

\[
\sin \theta = \frac{p}{r(1 - e^2)^{1/2}}
\]

(6.20)

we can replace the eccentric anomaly in the above with the mean anomaly, \(M\). Our results then simplify to:

\[
\delta x = Ar \sin \theta (e - \cos \theta) - Br \sin \theta + \frac{L_R}{eL} [2 + \cos \theta (e - \cos \theta)] + \frac{a^2H_R}{eL^2} (1 - e^2) \mathcal{P}
\]

\[
\delta y = Ar(1 + \cos^2 \theta) + Br \cos \theta - \frac{L_R}{eL} \sin \theta \cos \theta + \frac{a^2H_R}{eL^2} (1 - e^2) \mathcal{Q}
\]

(6.21)

where,

\[
\mathcal{P} = r \cos \theta (e + \cos \theta) - 2r - \frac{3a^2}{L} e\sqrt{1 - e^2} v_x M
\]

(6.22)

and

\[
\mathcal{Q} = r \sin \theta (2e + \cos \theta) - \frac{3a^2}{L} e\sqrt{1 - e^2} v_y M
\]

(6.23)

We can rearrange these expressions in a more convenient form:

\[
\mathcal{P} = 2ex - r [2 + \cos \theta (e - \cos \theta)] - \frac{3a^2}{L} e\sqrt{1 - e^2} v_y M
\]

(6.24)
We can then simplify our expressions for the relative position of the satellites to:

\[ \delta r = \delta r_p - \frac{H_R}{H} (r - \frac{3}{2} vt) \]  \hspace{1cm} (6.26)

where \( \delta r_p = \delta x_p + \delta y_p \) is the periodic variation of the separation given by:

\[ \delta x_p = (eA - B)y - Ax \sin \theta + \frac{1}{eL} \left( L_R + \frac{H_R}{2H} L \right) (p + y \sin \theta) \]

\[ \delta y_p = 2A + Bx - Ay \sin \theta - \frac{1}{eL} \left( L_R + \frac{H_R}{2H} L \right) x \sin \theta \]  \hspace{1cm} (6.27)

This provides a very simple expression for the relative position of the satellites.

To complete our analysis we now consider the relative velocity between the satellites. Differentiating Equation (6.26) above gives:

\[ \delta v = \delta v_p + \frac{H_R}{2H} \left( v - \frac{3 \mu}{r^3} vt \right) \]  \hspace{1cm} (6.28)

and the periodic part of the velocity reduces to:

\[ \delta x_p = (eA - B) v_y - A \left( v_x \sin \theta + \frac{L}{r^2} x \cos \theta \right) + \frac{1}{eL} \left( L_R + \frac{H_R}{2H} L \right) \left( v_y \sin \theta + \frac{L}{r^3} y \cos \theta \right) \]

\[ \delta y_p = (B - 2eA) v_x - \frac{1}{eL} \left( L_R + \frac{H_R}{2H} L \right) \left( v_x \sin \theta + \frac{L}{r^2} x \cos \theta \right) - A \left( v_y \sin \theta + \frac{L}{r^3} y \cos \theta \right) \]  \hspace{1cm} (6.29)

This completes the solution for the relative motion.

Note that, the time term \( t \) in both relative position and velocity equations come from the mean anomaly i.e., \( t = M/n \). Therefore it corresponds to the time since perigee. If the propagation lasts for more than one orbit, we add \( 2K \pi \) to \( M \) when evaluating the time, where \( K \) is the number of orbits.

### 6.2.2 Circular Case

We note that these solutions have still a singularity as \( e \to 0 \), which corresponds to Hill's problem. We briefly show how these equations reduce to the usual Hill's equations in this limit. The reason the singularity arises in the solutions is because if the midpoint moves around a circular orbit then \( H_R \) and \( L_R \) are related by:

\[ H_R = nL_R \]  \hspace{1cm} (6.30)
where $n$ is the angular rate of the motion of the midpoint. This means that in this case we no longer have two independent differential equations. The above condition follows from the fact that for circular orbits: $HL^2 = -\mu^2/2$. Differentiating this result gives the relationship: $\delta H = n\delta L$.

The energy vs angular momentum plane has a forbidden region where no orbits are possible as this would imply $e < 0$ (see Figure 6.1). The bounding curve of this region is the set of circular orbits and since the energy and angular momenta of our two satellites must be symmetrically placed about that of the midpoint, then they must lie along the tangent to this curve. This is described by the above relation. If the two satellites are both on circular orbits, as shown in the figure as $sat_1$ and $sat_2$, the midpoint orbit actually lies in the forbidden region. For the reference satellite to be on a circular orbit, the line connecting the two satellites on this graph should be tangent to this curve, as seen in the case of $sat_3$ and $sat_4$ on this curve.

In short, when the midpoint moves along a circular orbit we cannot use the angular momentum as independent information. From Equation (6.5) we see that the $z$ motion satisfies the solution of Hill's problem when $r$ is constant. The in-plane motion can be found from the equations in (5.8), which in component form reduce to:

$$\delta z = -n^2 \delta x - 3n^2 \cos \theta (\delta x \cos \theta + \delta y \sin \theta) \quad (6.31)$$
Chapter 6. Analytical Relative Orbit Propagation - $H_R L_R$ Method

\[
\delta y = -n^2 y - 3n^2 \sin \theta (\delta x \cos \theta + \delta y \sin \theta) \tag{6.32}
\]

If we use rotated co-ordinates:

\[
u = \delta x \cos \theta + \delta y \sin \theta \quad v = -\delta x \sin \theta + \delta y \cos \theta \tag{6.33}
\]

then the accelerations in (6.31) can be written as:

\[
\ddot{u} - 2n \dot{v} + 3n^2 u = 0 \tag{6.34}
\]

\[
\ddot{v} + 2n \dot{u} = 0 \tag{6.35}
\]

These accelerations are the same as those defined in Hill's Equations (see Equation (3.47)). Therefore, the analytic solutions developed here match directly onto the more familiar Hill solutions.

6.2.3 Initial Conditions

In the previous sections we derived a solution for the relative motion between two satellites following arbitrary elliptic orbits. We now consider the twelve initial conditions: the position and velocity of the reference and the relative position and velocity. For the relative motion, we replace these initial conditions by the constant quantities: $A, B, C, D, H_R$ and $L_R$. We note, however, that when $\theta = 0, \pi$ then $y$ vanishes and from Equation (6.27) $\delta x_p$ becomes independent of both $A$ and $B$. This is along the line of nodes of the reference orbit. Along this line $\nu = 0$ also, and hence from (6.29) $\delta y_p$ also becomes independent of $A$ and $B$. It is therefore more useful to determine $A$ and $B$ from $y_p$ and $\delta z_p$, which is a straightforward calculation with two equations and two unknowns. The constants $C$ and $D$ describing the cross track motion are easily determined from:

\[
C = \delta z_0 p (e + \cos \theta_0) - \delta z_0 p^3 \sin \theta_0 \left( \frac{1}{1 + e \cos \theta_0} \right) \tag{6.36}
\]

\[
D = \delta z_0 p \sin \theta_0 + \delta z_0 p^3 \cos \theta_0 \left( \frac{1}{1 + e \cos \theta_0} \right) \tag{6.37}
\]

The choice of the reference satellite orbit allows for some flexibility. Our analysis suggests we should average the two satellite positions and velocities. This will induce a second order error in the energy $H$ (see Equations (5.4), (5.5) and (5.6)) and a third order error the
relative energy $H_R$ (see Equation (5.7)). Equation (6.26), however, has a secularly growing term which is dependent upon the difference in orbital energy. In order to maintain the accuracy of our linear approximations, we must minimise any errors in this secular term. This suggests we should ensure that $H_R$ is as close as possible to $H_1 - H_2$. The relative coordinates are fixed by the initial conditions, therefore we should examine how we can modify the reference orbit to satisfy this condition. The fact that $1/H \propto a$ appears in this expression suggests we may find it better to set the semi-major axis of the reference orbit as the average of the values of $a$ for the two satellite orbits.

What difference does this choice of setting the midpoint orbit have on our analysis? If we compute the average of the energies then the semi-major axis of the midpoint orbit is given by $<a>$, where:

$$<a> = \frac{1}{2} \left( \frac{1}{a_1} + \frac{1}{a_2} \right)$$

If $\bar{a} = (a_1 + a_2)/2$ then one can easily show that:

$$<a> = \bar{a} - \frac{1}{4}(a_1 - a_2)^2$$

Hence the difference in the choice is second order, and to the order of accuracy of our analysis, both should be valid. This inclusion of second order in the initial conditions changes how long our analytic solution is valid for, and how large our formation can be.

We have considered a number of alternative ways to choose the reference orbit, and compared the accuracy of our analytic approximation to the difference of the two Keplerian orbits in the next sections.

**Averaging satellite positions and velocities**

Averaging the position and velocity vectors of the two satellites is probably the most straightforward method to compute the initial conditions of the reference satellite. However, there is a second order energy difference between the resulting reference orbit and the average of the two energies, which can be found from recalculating $H_1 + H_2$ from Equations (5.4) and (5.5):

$$H = \frac{H_1 + H_2}{2} - \frac{1}{8} \left[ (\delta v \cdot \delta v) + \frac{\mu}{|\mathbf{r}|} \frac{\delta r \cdot \delta r}{|\mathbf{r} \cdot \mathbf{r}|} \right]$$  

(6.40)
As can be seen from the secular part of Equation (6.26), a second order error in both $H$ and $H_R$ cause secular relative positioning errors of similar magnitude.

This shows that, while the reference satellite starts at the geometric midpoint of the two satellites, it drifts away due to this second order energy difference. For a satellite at 7650km semimajor axis, the non-dimensionalised energy is -0.416 and an energy error as small as $0.5 \times 10^{-6}$ results in a drift in excess of 1km/day. Inspecting Equation (6.26), this translates into a linearly growing error in $v$ and therefore a quadratically growing error in the overall secular term.

**Averaging the orbital elements of each satellite**

The averaged orbital elements define the orbit and the phase of the reference satellite fully. The Hamiltonian of the reference satellite is given as:

$$H = \frac{H_1 + H_2}{2} - \frac{(H_1 - H_2)^2}{2(H_1 + H_2)}$$

The ratio $(H_1 - H_2)/(H_1 + H_2)$ is a measure of how fast the formation separates. Consider a case where this term is $10^{-5}$, corresponding to about 8-10km/day separation rate, which is quite large for a formation scenario. It follows that, for this example this second order term $(H_1 - H_2)^2/(2(H_1 + H_2))$ is around $10^{-11}$, which corresponds to a very small drift of the reference satellite from the geometric midpoint, on the order of several millimetres/day. We can conclude that this method ensures that the reference satellite stays around the geometric midpoint much longer than averaging the coordinates.

**Averaging orbital energies and angular momentum vectors**

The energy, coupled with the angular momentum vector, defines the shape and the orientation of the reference orbit. To locate the satellite on this orbit, we simply take the average of the true anomalies of the two satellites.

We have already shown the semimajor axis of the reference satellite for this method in Equation (6.39), as well as its second order difference from the average semimajor axis. As the semimajor axis difference should be small for the keep the formation closer, it is
obvious that the second order effects will be much smaller. In practice, the difference between averaging the energies and semimajor axes is not very significant; the semimajor axis difference between the satellites is about two orders of magnitude smaller than the semimajor axes themselves and therefore the second order terms usually correspond to about $10^{-8}$ kilometres difference in semimajor axis.

**Averaging Eccentricity and Angular Momentum Vectors**

Another way to define the reference orbit is to average the eccentricity and angular momentum vectors of the two satellites. These two vectors together define the shape and orientation of the orbit, but not where the satellite is on this orbit i.e., true anomaly. A simple way to determine this is to simply average the true anomalies of the two satellites.

To calculate the Hamiltonian of the reference satellite, assume that there is an angular difference of $\delta \alpha$ between the two angular momenta vectors $L_1, L_2$ and an angular difference of $\delta \beta$ between the two eccentricity vectors $e_1, e_2$. As the two orbits are by their definition similar, these angular differences are small. Therefore we can write the expressions for $L^2$ and $e^2$, with second order terms:

$$L^2 = L \cdot L = \frac{(L_1^2 + L_2^2)}{4} + \frac{L_1 L_2}{2} \left( 1 - \frac{\delta \alpha^2}{2} \right)$$

$$e^2 = e \cdot e = \frac{(e_1^2 + e_2^2)}{4} + \frac{e_1 e_2}{2} \left( 1 - \frac{\delta \beta^2}{2} \right)$$

Using these two terms and after some rearranging, the Hamiltonian can be written as:

$$H = \frac{-\mu^2}{2L^2(1 - e^2)}$$

$$H = \frac{-\mu^2}{2 \left( \frac{L_1 + L_2}{2} \right)^2 \left( 1 - \left( \frac{e_1^2 + e_2^2}{2} \right) \right)^2} \left[ 1 + \frac{L_1 L_2 \delta \alpha^2}{(L_1 + L_2)^2} - \frac{e_1 e_2 \delta \beta^2 / 4}{1 - \left( \frac{e_1 + e_2}{2} \right)^2} \right]$$

Once again, while this Hamiltonian is equivalent to those defined by the other three methods to the first order, there is a second order difference. It is important to note that, like 'position and velocity averaging' method, the error in the Hamiltonian is a function of not only the energy difference between the two satellites but also the difference in orbital elements, through $\delta \alpha$ and $\delta \beta$ terms.
Comparing the initialisation methods

While the above discussion provides some insight into different methods, it is not immediately obvious which one provides the highest accuracy in the long run. Ultimately, our problem is how to handle the second order errors and which terms should have them, so as to have a high long-term relative positioning accuracy.

A simple formation scenario was used to compare these initialisation methods, based on the initial conditions given in Table 5.1. We considered the accuracy of the predicted relative position after 5 days, corresponding to 21.5 orbital periods. The difference in eccentricity of the two satellites was fixed at 0.001, although their eccentricities were allowed to vary. The formation starts with an initial separation of 8 to 9 km, depending upon the eccentricity. The 150 m semi-major axis difference causes the satellites to drift apart, reaching about 40 km separation by the end.

In Figure (6.2) we see the relative position error as a function of the eccentricity of the orbits. We see that all the methods perform well. Averaging the position and velocity vectors (shown as $\text{pos.vel}$ in the figure caption) yield accuracies around a few metres. Continuation of the propagation beyond 5 days, however, shows that this error grows quadratically with time. The same quadratic growth in error is also seen in averaging the eccentricity and angular momentum vectors (shown as $e_h$ in the caption). The other two methods, namely averaging the energies and angular momentum vectors (shown as $e_n h$ in the caption) and averaging the orbital elements (shown as $\text{elems}$ in the caption), clearly outperform these, reducing the error to between 1% - 10% of the first two methods. The best method is averaging the orbital element sets where an accuracy around 10 cm is achieved after 5 days.

6.2.4 Correcting the Relative Hamiltonian and Angular Momentum

The second order changes to the reference will change the quantity in Equation (5.7) that is conserved by the equations of motion. This value will differ from $H_1 - H_2$, which is the correct conserved quantity. The error will become second order due to the second order changes made to $r$ and $v$. This can be seen in Figure 6.3 along with the resulting
6.2. Relative Motion in a Keplerian Potential

Figure 6.2: Peak relative positioning error magnitude (in log scale) variation with eccentricity (21.5 orbits)

Table 6.1: Formation initial conditions in Keplerian elements.

<table>
<thead>
<tr>
<th>$a$(km)</th>
<th>$e$</th>
<th>$I$(deg)</th>
<th>$\Omega$(deg)</th>
<th>$\omega$(deg)</th>
<th>$\theta$(deg)</th>
<th>$H$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat1</td>
<td>7653.780</td>
<td>$\leq 0.0050$</td>
<td>60.00</td>
<td>40.0</td>
<td>20.0</td>
<td>61.0</td>
<td>-0.416665</td>
</tr>
<tr>
<td>Sat2</td>
<td>7653.700</td>
<td>$\leq 0.0055$</td>
<td>60.01</td>
<td>40.0</td>
<td>19.0</td>
<td>62.0</td>
<td>-0.416670</td>
</tr>
<tr>
<td>Diff</td>
<td>-0.080</td>
<td>0.0005</td>
<td>0.0</td>
<td>0.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>$-4.35 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

error in relative position after 5 days. The initial conditions for this experiment are given in Table 6.1 where the two satellites started at true anomalies of 301 degrees and 302 degrees respectively. We see from this figure that the relative position errors are directly proportional to the error in $H_R$. What is not seen in this figure is that this relative position error grows linearly with time. This means we can predict from the outset what the positional errors will be at any future time just based upon how accurately $H_R$ matches the difference in orbital energies of the two satellites.

As shown in this example, 15cm relative positioning accuracy is possible for a 5 day propagation. To put the errors in perspective, it should be emphasised that the initial separations vary from about 10km for higher eccentricities to about 1km for lower eccentricities and peak separations are about 65km for high eccentricities and 50km for low eccentricities.

To ensure a better match for $H_R$ we need to reconsider Equation (5.7). Since the relative
position and velocity of the two satellites are fixed, we need to adjust the position and velocity of the reference point. The orbital parameters for the reference, however, were fixed in the previous section. Nevertheless, we have freedom to adjust the true anomaly of the reference point along that orbit and we can adjust the argument of perigee of the orbit, $\omega$.

Consider first $H_R$ and $L_R$ expressed in terms of the true anomaly along the reference orbit:

\[
H_R = \frac{L}{p} \left[ -\sin \delta \dot{x} + (e + \cos \theta)\delta \dot{y} \right] + \frac{L^2(1 + e \cos \theta)^2}{p^3} \left[ \cos \theta \delta \dot{x} + \sin \theta \delta \dot{y} \right] \tag{6.45}
\]

\[
L_R = \frac{-p}{(1 + e \cos \theta)} \left[ \sin \theta \delta \dot{x} - \cos \theta \delta \dot{y} \right] + \frac{L}{p} \left[ (e + \cos \theta) \delta \dot{x} + \sin \theta \delta \dot{y} \right] \tag{6.46}
\]

By making small adjustments to the starting value of $\theta$ we can bring these quantities closer to the correct values. If we introduce a set of inertial relative co-ordinates $(\delta x_I, \delta y_I)$ related to the perifocal co-ordinates we have used through:

\[
\delta x = \delta x_I \cos \omega + \delta y_I \sin \omega \tag{6.47}
\]

\[
\delta y = -\delta x_I \sin \omega + \delta y_I \cos \omega \tag{6.48}
\]

then we can also adjust $\omega$.

The $H_R$ and $L_R$ terms can thus be adjusted via first order corrections:

\[
H_1 - H_2 = \delta H = H_R + \frac{\partial H_R}{\partial \theta} \Delta \theta + \frac{\partial H_R}{\partial \omega} \Delta \omega \tag{6.49}
\]
where $\Delta \theta$ and $\Delta \omega$ are small true anomaly and argument of perigee corrections. As all the other terms are known, we can solve for the $\Delta \theta$ and $\Delta \omega$ and calculate the corrections required via these two equations. The partial derivatives are given as:

\[
\frac{\partial H_R}{\partial \theta} = \frac{-L}{p} [\delta \dot{x}_I \cos(\theta + \omega) + \delta \dot{y}_I \sin(\theta + \omega)] + \frac{\mu}{p^2} [\delta \dot{x}_I (\sin(\theta + \omega) + e \sin \theta \cos(\theta + \omega) + e \sin(2\theta + \omega)) + \delta \dot{y}_I (\cos(\theta + \omega) - e \sin \theta \sin(\theta + \omega) + e \cos(2\theta + \omega))]
\]

\[
\frac{\partial L_R}{\partial \theta} = \frac{-r^2 e \sin \theta}{p} [\delta \dot{x}_I \sin(\theta + \omega) - \delta \dot{y}_I \cos(\theta + \omega)] - r [\delta \dot{x}_I \cos(\theta + \omega) + \delta \dot{y}_I \sin(\theta + \omega)] + \frac{L}{p} [-\delta \dot{x}_I \sin(\theta + \omega) + \delta \dot{y}_I \cos(\theta + \omega)]
\]

\[
\frac{\partial H_R}{\partial \omega} = \frac{-L}{p} [\delta \dot{x}_I (e \cos \omega + \cos(\theta + \omega)) + \delta \dot{y}_I (e \sin \omega + \sin(\theta + \omega))] + \frac{\mu}{p r^2} [-\delta \dot{x}_I \sin(\theta + \omega) + \delta \dot{y}_I \cos(\theta + \omega)]
\]

\[
\frac{\partial L_R}{\partial \omega} = -r [\delta \dot{x}_I \cos(\theta + \omega) + \delta \dot{y}_I \sin(\theta + \omega)] + \frac{L}{p} [-\delta \dot{x}_I (e \sin \omega + \sin(\theta + \omega)) + \delta \dot{y}_I (e \cos \omega + \cos(\theta + \omega))]
\]

The effect of these adjustments is shown in Figure 6.4. Obviously for smaller eccentricities, the effect of $\omega$ becomes very small and we are unable to match both $H_R$ and $L_R$ at the same time. In this case we may need to restrict ourselves to correcting $H_R$ only. If we fix the value of $H_R$ to the correct value then we need to solve a non-linear equation for $\theta$ and $\omega$, but in practice a linearised approximation will suffice as the adjustments in these angles will be small.

We tested the effectiveness of this correction term using the initial conditions given in Table 6.1. Figure 6.5 illustrates $H_R$ and relative positional errors for both corrected (shown as corr) and uncorrected cases. As expected, $H_R$ errors grow parallel to the relative positioning errors. $H_R$ error is significantly lower in the corrected case and this results in relative positioning errors to be cut by about a factor of three, increasing to a factor of 20 to 40 for larger eccentricities. Therefore, errors on the order of 30 to 60m are cut to about 0.7m to 2.5m, for a five day propagation. This illustrates the effectiveness of correcting
Figure 6.4: True anomaly and argument of perigee correction for large and small eccentricity cases

the phase of the reference orbit.

Figure 6.5: $H_R$ error variation and peak relative positioning error magnitude with/without corrections (in log scale) with eccentricity

Similarly, Figure 6.6 demonstrates the variation of the $L_R$ error, where the relative angular momentum varies between $-5.7 \times 10^{-6}$ for the smallest eccentricity to $-1.7 \times 10^{-5}$ for the largest. As can be seen, the corrected $L_R$ yields vastly superior performance, practically matching the relative angular momentum down to machine accuracy.

Figure 6.7 illustrates the angular correction required to modify the $H_R$ and $L_R$ values for
6.2. Relative Motion in a Keplerian Potential

Figure 6.6: $L_R$ error variation with/without corrections (in log scale) with eccentricity

In this case, where true anomaly correction is shown as $\theta_{\text{corr}}$ and argument of perigee correction is shown as $\omega_{\text{corr}}$. For larger eccentricities, small angular corrections are adequate as it is easier to manipulate the along-track and radial coordinates of the reference satellite with a small change. However, as the orbit becomes more circular, it takes progressively larger angular changes to alter $H_R$ and $L_R$ sufficiently. Such a large angular change in true anomaly was countered by a similar and opposite direction change in argument of perigee, in agreement with the discussion regarding Figure 6.4.

Figure 6.7: True anomaly and argument of perigee correction with eccentricity (21.5 orbits)

We also considered the case when the difference in semi-major axis was larger, as per Table 5.1. Here we used true anomalies of the satellites as 73 and 73.05 degrees, respectively.
Initial separations vary between 1.7km at large eccentricities to about 9.6km at small eccentricities. Due to the 150m semi-major axis difference, separations increase rapidly in time, with peak separations between 35 to 70km. Figure 6.8 presents a comparison of the \(H_R\) error with and without the correction terms applied (shown in the figure caption as \(H_R\) err corr and \(H_R\) err, respectively). When compared to the energy difference of \(-1.8 \times 10^{-6}\), both corrected and uncorrected \(H_R\) values are very accurate. However, the correction term is seen to reduce the \(H_R\) error even further across the eccentricity spectrum. The figure also shows the relative positioning error when the correction terms are applied. As expected, the relative positioning error profile follows the \(H_R\) error and it exceeds 1m mark only for the \(e = 0.9\) case.

Figure 6.8: \(H_R\) error variation with/without corrections and peak relative positioning error magnitude (in log scale) with eccentricity

Figure 6.9 shows the variation of the error in the \(L_R\) term. The relative angular momentum starts at about \(-9 \times 10^{-6}\) for the 10\(^{-2}\) level eccentricities, increasing to \(-3 \times 10^{-4}\) for the \(e = 0.9\) case. While the error in the uncorrected \(L_R\) (shown as \(L_R\) err in the figure caption) is already small compared to the real relative angular momentum, the corrected \(L_R\) is seen to practically match it.
6.2. Relative Motion in a Keplerian Potential

6.2.5 Long-Term Stability

For most formation flying scenarios the satellites will remain within a few kilometres of each other over long times. To evaluate the approximations we have used for such close proximity scenarios we used the initial conditions in Table 6.1, but without any difference in semi-major axis, to ensure that the satellites do not drift away. Again we considered a range of eccentricities and evaluated the relative position after 50 days (650 orbits). The satellites stay within less than 3.5km of each other. Figure 6.10 demonstrates the relative positioning error variation with time for various eccentricities. While the errors grow with increasing eccentricity, it takes 50 days for the largest error to exceed 1m. For eccentricities around 0.001 the error is around 10cm. This demonstrate the high accuracy of our approximation once correction of initial conditions is made.

Encouraged by the achievements of this approximation over several days, we considered testing it over very long timescales when the satellites had drifted apart substantially. Once again, we set up initial conditions as in Table 5.1 with an eccentricity of 0.45 and an eccentricity difference of 0.001. We increased the semi-major axis difference to 1150m, such that the separation between the satellites reach thousands of kilometres in 50 days (217 orbits). Figure 6.11 shows the separation between the satellites, which exceeds 3500km, as well as the relative positioning errors as a percentage of the separation. Even though the relative position error no longer grows linearly, it only reaches 0.5% of the total separation.
corresponding to about 20km. The initial $H_R$ error is only $-2.1 \times 10^{-13}$ however, the distance between geometric midpoint of the satellite positions and the reference point reaches 140km. Since we assume that the reference satellite stays near the geometric midpoint to the first order, the force acting on the satellites is evaluated at this location, then the increasing separation between these two points is likely to be the main source of error.
6.3 Results

In the preceding sections, we have explained in detail the derivation of an analytical scheme to evaluate and analyse the relative motion of two satellites in a Keplerian potential. These equations of relative motion can be used within a composite symplectic relative propagation scheme, just like the variations of $f-g$ equations we presented in the previous chapter.

In this section we will compare these relative propagation schemes and characterise their behaviour. To initialise the reference satellite, we will use the $H_R$ and $L_R$ correction method we introduced in the last section in all of the test cases. We will also compare the accuracy and speed of the relative propagators with a symplectic absolute orbit propagation scheme. As a shorthand, the variations of $f-g$ equations will be called $\delta f-\delta g$ equations and the analytical scheme presented in this chapter $H_R L_R$ equations.

6.3.1 Conservation of Energy

As explained in detail in Section 5.4.1 we can show that the relative energy is an oscillation around a stable mean and this oscillation amplitude goes to zero as the timesteps get smaller. The condition for this is that the geopotential model is axisymmetric i.e., we take into account zonal harmonics only. Similarly, for such a geopotential model, the $z$ component of the relative angular momentum is conserved as well.

Firstly, we will show that the relative energy and the $z$ component of the relative angular momentum are zero mean oscillations. We will use the example in Section 5.4.1, where we have one satellite at 9567.2km semimajor axis and $e = 0.3$ eccentricity and another in a similar orbit at 23m semimajor axis difference. We use an axisymmetric geopotential containing terms up to $J_4$ only and run the propagation at 100 steps/orbit, for 5 days (46 orbits). Figure 6.12 shows the variation of $H_R$, which is seen to be oscillating around a stable mean, with an amplitude of $4.8 \times 10^{-11}$. This shows that $H_R$ is indeed conserved.

Secondly, we will show that the $z$ component of the relative angular momentum is conserved. Figure 6.13 shows its variation for the same example as above, compared to its initial value. This is just a random walk with an error at $10^{-16}$ level, which is down to
machine accuracy. Therefore, conservation of the $z$ component of the angular momentum is shown for the relative motion.

We can use this test setup to illustrate how fast the relative energy oscillation amplitude decreases as integration timesteps are made smaller. We also would like to investigate whether this decrease rate is any different as compared to running two absolute propagations and taking the difference of the calculated energies. It should be emphasised that this oscillation amplitude is an indicator of the positional accuracy as well.

Figure 6.14 shows the variation of oscillation amplitudes in $H$, $H_1 - H_2$ and $H_R$ for various stepsizes. As can be expected, the oscillation in $H$ is about an order of magnitude larger than that of the relative motion. For all cases, with increasing number of steps per orbit, we can see the diminishing returns in oscillation amplitude decrease rate. This suggests that, in practice, very small stepsizes will be of limited use.

Comparing the difference in energies $H_1 - H_2$ and the relative Hamiltonian $H_R$, we see that their oscillation amplitudes decrease at a very similar rate, which suggests that very similar accuracies can be obtained by running the relative orbit propagator or differencing the results of two absolute propagations. We will further investigate possible computational speed benefits of this in the next sections.
6.3. Results

Figure 6.13: Conservation of relative angular momentum - z component

Figure 6.14: Energy oscillation amplitude (log scale) in $H$, $H_1 - H_2$ and $H_R$ for various integration stepsizes
6.3.2 Number of Geopotentials

In Section 5.4.2 we carried out some experiments to demonstrate the performance of the \( \delta f-\delta g \) equations based symplectic relative orbit propagation for various number of terms in the geopotential. However, having developed a means to correct the \( H_R \) and \( L_R \), we can apply it to this \( \delta f-\delta g \) method and compare it to the \( H_R L_R \) method presented in this chapter.

We will essentially repeat the experiment in Section 5.4.2, using the initial conditions given in Table 5.2 for a 5-day propagation at 120 steps/orbit. The truth model is a 1000 steps/orbit composite symplectic scheme with a 36 \( \times \) 36 geopotential field model; as before, we calculate the absolute orbit for each satellite with this scheme and take the difference to obtain the relative orbit.

Figure 6.15 summarises the relative positioning errors after a five day propagation with axisymmetric and non-axisymmetric geopotential models containing various number of geopotentials. Note that, 'zero geopotentials' in the figure correspond to the Keplerian case. The error behaviours for both methods are the same as the ones explained in Section 5.4.2; the model without the tesseral terms suffer significantly due to the missing terms in the relative energy.

However, the most striking feature of this graph is that \( \delta f-\delta g \) and \( H_R L_R \) methods yield virtually the same errors. Figure 6.16 shows the evolution of the difference in errors between the two methods for various number of geopotentials. Even in the Keplerian case where the difference grows proportional to \( t^3 \), it is less than \( 10^{-5} \) metres. The addition of the higher order geopotentials seems to affect the difference by the same order of magnitude, thereby effectively cancelling it for this example.

From these tests we conclude that, for a given set of formation initial conditions, it is possible to obtain metre level accuracy after 5 days with about 20 geopotentials included in the model. More importantly, we showed that the two analytical methods for Keplerian relative orbit propagation are practically equivalent.
6.3. Results

Figure 6.15: Relative positioning error (log scale) with various geopotentials - 5 days (65 orbits)

Figure 6.16: Relative positioning error difference (log scale) between $\delta f-\delta g$ and $H_R L_R$ methods with various geopotentials - 5 days (65 orbits)
6.3.3 Eccentricity

The previous set of tests proved that very high accuracy is indeed possible with the proposed algorithm for a near-circular case. In this section we will demonstrate the accuracy for a range of eccentricities. We also would like to test whether the two relative propagation methods will continue to yield practically the same results. We will follow the same methodology as Section 5.4.3.

The first set of tests will use the initial conditions given in Table 5.1 to run 5 day (21.5 orbit) simulations. We will start with a Keplerian model to compare the two analytical relative orbit models. The initial separations range from 5.8km for $e = 0.55$ to 9km for near-circular cases. Peak separations range from 54km for $e = 0.55$ to 34km for near-circular cases.

Figure 6.17 illustrates the relative positioning error for this case through a range of eccentricities. As can be seen, the two methods yield virtually the same results, proving further that they are practically equivalent. The difference between the two remains around $10^{-7}$ metres, regardless of the eccentricity.

The second set of tests is to repeat the above but with a $36 \times 36$ geopotential model and
6.3. Results

Figure 6.18: Relative positioning error (log scale) with eccentricity for $36 \times 36$ potential - 5 days (21.5 orbits)

Two stepsizes: 100 steps/orbit and 300 steps/orbit. The truth model is 1000 steps/orbit. Figure 6.18 shows the results for this test case, where the $H_R L_R$ or $df - dg$ denotes the method and 100 or 300 denotes the number of steps/orbit in the figure caption. As expected, the two methods yield practically identical results with differences around $10^{-7}$ metres or less. While the relative positioning accuracy is very high for both 100 and 300 steps/orbit cases, the former start to become unstable at around $e = 0.5$. For the latter, while the errors increase with eccentricity, they stay well below metre level.

Furthermore, we can compare these methods where we use the orbital elements and a correction to $H_R$ and $L_R$ to initialise the reference satellite, to the one presented in the previous chapter, where we average the coordinates of the two satellites. The results of the latter method was presented in Figure 5.12. Comparing the two figures, we see that the orbit element averaging method with corrections vastly outperforms the coordinate averaging, reducing the error to about 1% for this case.

6.3.4 Speed Tests

As we mentioned in the earlier chapters, a strong argument in favour of using high-precision relative orbit propagators instead of differencing two absolute orbit propagations is that
significant gains in computational time can be had for little or no loss in relative positioning accuracy. While our priority in developing the relative orbit propagator code has not been speed, we would like to compare two absolute orbit propagations to a relative orbit propagation.

Also in the previous sections we demonstrated that $H_{RLR}$ and $\delta f - \delta g$ methods are practically equivalent. However, in terms of implementation, the latter is more efficient as there is no need for rotation to a perifocal frame or finding the orbital elements. Therefore, we will use this method for speed tests.

We will use the example given in Section 5.4.1, where we have one satellite at 9567.2km semimajor axis and $e = 0.3$ eccentricity and another in a similar orbit at 23m semimajor axis difference in the Keplerian orbital elements. The relative orbit propagation as well as the SPSAT propagations for two absolute orbits are run for 30 days (276 orbits) at 200 steps/orbit. The truth model is the difference of two absolute propagations at 1000 steps/orbit.

At this stepsize, a propagation with $36 \times 36$ geopotential model yields a relative positioning error of 1.1m with the difference of two SPSAT propagated orbits and 1.5m with the relative orbit propagator. This shows that, with respect to the true orbit, a similar level of accuracy can be expected from both of these propagators for the same stepsize.

Figure 6.19 shows the variation of computational speed for this case, with various geopotentials. As can be seen, at a high number of geopotentials, the relative orbit propagator provides substantial gains in speed. On the other hand, when less geopotentials are taken into account, these gains are negated. The basic reason is that there are some overheads associated with the relative orbit propagator in comparison to two absolute orbit propagations. For example, in addition to the absolute and relative orbits in inertial coordinates, we calculate the relative motion in local coordinates as well. Such operations need to be carried out regardless of the number of geopotentials. Secondly, there probably is room for improvement in computational speed in our code.

Figure 6.19 is also useful in illustrating the increasing computational cost with the increasing number of geopotentials. Nevertheless, the relative orbit propagator is very fast even for this long term propagation case and the 36 term geopotential model. This makes it an
6.4 Conclusions

In this chapter we have presented an analytical solution for the relative motion of two satellites moving along similar Keplerian elliptic orbits. These orbits can have arbitrary eccentricity and no assumption about the satellites being at similar phase appears to be required. Our analysis differs significantly from other approaches published in that we employ an inertial frame of reference rather than a rotating frame (which rotates at varying rates for elliptic orbits) and yet still generalises directly from the results of Hill's equations. We also introduce a formal methodology based upon a 'relative Hamiltonian' and a 'relative angular momentum', which are exactly conserved by the equations of motion we derive. The solutions to these dynamical equations are not expressed fully explicitly in time, but are very simple expressions and separate out the secular drift between the satellites from periodic oscillations in relative position around the reference orbit.

We have analysed the choice of reference orbit about which our linearisation is made, and shown that by appropriately choosing second order corrections in the initial conditions the

Figure 6.19: SPSAT and relative orbit propagator speed tests for 200 steps/orbit propagation - 30 days (276 orbits)

ideal candidate for onboard applications.
resulting analytic equations remain accurate to very high precision over periods of time lasting several months. This level of accuracy is another key feature of our analysis that will make the analytic solutions valuable for modelling formation flying scenarios over mission lifetimes. We have argued that the reference orbit should be chosen by averaging the orbital elements of the two satellites and then adjusting the phase and orientation of the reference ellipse to ensure accurate matching of the relative energy and angular momentum.

We have presented a detailed set of tests of the accuracy and robustness of our model for various scenarios, investigating the behaviour for a wide range of eccentricities, proximities and semi-major axes. Most of these scenarios involved formation separations varying from several kilometres to about 30-60km, which would be deemed challenging for any linearised relative motion algorithm. We demonstrate accuracies between metre and centimetre level after several days, outperforming second order models such as [59]. The method we have developed is seen to yield practically identical results to the $\delta f - \delta g$ method detailed in the previous chapter.

Finally, we have also showed that similar accuracies can be reached when the analytical model is used within a numerical relative orbit propagation scheme as presented in the preceding chapter.
Chapter 7

Conclusions

In this chapter we will present our conclusions from this thesis and possible areas to extend this research. We will first highlight the key conclusions drawn throughout the thesis. This is followed by a summary of our contributions to the state-of-the-art, in the context of our initial research goals. Finally, we will present possible areas of further work that could provide valuable extensions to this research.

7.1 Summary of Conclusions

7.1.1 Constellation Initialisation Algorithm

In Chapter 4, we described a low-propellant-cost, practical and robust 3D constellation initialisation algorithm based on the epicycle equations. The algorithm extended the basic equations derived by Kormos [63], describing the relative motion between satellites in the constellation. We then used this algorithm to carry out the real world orbit initialisation of the Disaster Monitoring Constellation (DMC).

The benefits of this algorithm can be summarised as follows:

- As its design is driven by real world constraints, it provides a practical, robust and low-propellant solution to the 3D orbit initialisation problem.

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• It harnesses the natural orbital plane drift due to $J_2$ zonal harmonics to carry out the plane changes, which provides significant propellant savings for orbit initialisation manoeuvres spanning several months.

• With the inclusion of the effects of drag and $J_2$, it utilises a better orbit modelling than the existing literature.

• It improves over Kormos’ [63] work by adding drag as well as carrying out a detailed analysis on how to choose certain parameters of the manoeuvre.

• It can be easily adapted to constellation orbit maintenance as well as coarse formation establishment and rendezvous missions.

• While it has been developed with the unique design and operational challenges and limitations of DMC in mind, the resulting controller is flexible enough to be used in other possible scenarios involving higher $\Delta V$ values, larger differential drag effects and different combinations of in-plane and out-of-plane manoeuvres.

While we have not devised a formal optimisation algorithm, we carried out detailed analyses into how the parameters such as number of firings and total duration of the manoeuvre should be chosen. We showed that the most important parameter that affects the propellant consumption is the total duration of the manoeuvre, which should be on the order of a few months rather than days if the satellites has to change their phase and LTANs considerably and still remain within a low propellant budget of a few metres per second.

Despite the simple model, the simulation results showed that the proposed method can be successfully applied to the orbit acquisition problem of the DMC.

In the second half of the chapter, we explained in detail how this controller is put into practice for the Disaster Monitoring Constellation, for which the phasing was crucial to achieve near-24-hour global coverage target.

The flexibility and robustness of the algorithm proved invaluable throughout the orbit initialisation, where other real-world challenges presented themselves during the course of the mission. Even when unforeseen operational circumstances delayed the firings by some
weeks in some cases, the controller was simply instructed to re-run with the new initial conditions, providing a new strategy for corrective firings.

In the end, the DMC phasing has been completed successfully with all the satellites in their designated orbits well within the requirements. The operational issues also provided us with a significant hands-on experience. When the constellation initialisation is discussed in the literature, complicated strategies are often proposed but the engineering issues as discussed above are rarely addressed. As seen in the cases of DMC and AM/PM constellation, practical considerations override optimal manoeuvring schemes.

7.1.2 Analytical Models of Keplerian Relative Motion

In Chapter 5 we employed variations of the well-known ‘universal’ \( f-g \) equations, which we called \( \delta f - \delta g \) equations. The \( f-g \) equations provide exact position and velocity solutions for a satellite moving under a Keplerian geopotential. Likewise, these variations, first proposed by Mikkola et al. [78] for use within an orbit determination filter for a single satellite, provide first order approximate relative position and velocity solutions for two satellites in close proximity, regardless of the eccentricity.

These relative motion equations demonstrate excellent accuracy in relative positioning, even when the close proximity assumption is stretched. Relative positioning accuracies less than 20m and usually around several metres for a 5 day formation scenario with separations starting from a few kilometres to 30-60 kilometres are commonplace for various eccentricities and semimajor axes. We have also verified that these equations conserve the relative energy and angular momentum, as expected. They also show excellent stability when the separations reach thousands of kilometres and the method degrades in accuracy gracefully.

While providing good performance and accuracy, the \( \delta f - \delta g \) method is rather opaque when it comes to analysis of the relative motion. For this reason, in Chapter 6 we developed a novel formulation for the analytical Keplerian relative motion which is not limited in eccentricity. Rather than solving the relative accelerations in a local accelerating frame like Clohessy and Wiltshire [20, 49] or Lawden [67], we derived a ‘relative Hamiltonian’ and a ‘relative angular momentum’, which are simply first order approximations to the difference
of the energies and angular momenta of the two satellites. From these conserved quantities, we derived the equations of relative motion. The advantage of this formulation over existing literature is that it is extremely simple in form and it separates periodic and secular relative motions, clearly illustrating the effect of relative energy and angular momentum. That it is in an inertial frame also facilitates potential addition of perturbations which are usually defined in an inertial rather than local frame.

This simple formulation also alerted us to the importance of matching the relative Hamiltonian to the relative energy more accurately, for greatly increased accuracy. We achieved this via a small modification to the reference satellite orbit orientation. The existing literature fails to take into account the importance of this relative energy error, resulting in large alongtrack drifts that cannot be accounted for. In fact, with this correction, our method seems to outperform second order models such as [59].

From the tests we conducted, we showed that this algorithm (named $H_{RLR}$ method) yields very good accuracy; metre to centimetre level relative positioning accuracy for a 5 day formation scenario with separations starting from a few kilometres to 30-60 kilometres is the norm for various eccentricities and semimajor axes. Similarly, it has excellent stability characteristics.

An interesting point to note is that, even though their derivations are very different, the $H_{RLR}$ and $\delta f - \delta g$ methods yield practically identical results. This is not very surprising as both of them are first order approximations to the Keplerian relative motion.

### 7.1.3 Development of a Symplectic Relative Orbit Propagator

In Chapter 5, we also demonstrated the development of a sophisticated symplectic relative orbit propagator. It is an extension of the absolute orbit propagator SPSAT presented in Section 3.4.3. To model the relative motion, we define the relative Hamiltonian, which is a first order approximation to the difference in energies of the two satellites. From this relative Hamiltonian defined in the extended phase space including absolute and relative coordinates, we can derive Hamilton's equations. This provides us with the equations of motion for both the relative motion between the two satellites and the absolute motion of the imaginary reference satellite that stays at the geometric midpoint of the two satellites.
The motion of the geometric midpoint was shown to be first order equivalent to the motion of a satellite. Therefore the relative Hamiltonian contains both the absolute and relative motion information.

The equations of motion can accommodate an arbitrary number of geopotentials and can potentially include other perturbations such as drag. This is a major benefit over the existing literature, as most publications limit themselves to complicated analytical equations containing terms up to $J_2$ or $J_3$, usually with restrictions on eccentricity.

To set up the symplectic scheme, we used the Hamiltonian splitting technique extensively. The effect of the Keplerian potential is an order of magnitude larger than that of $J_2$ and the effect of the next geopotential in the spherical harmonic is an order of magnitude smaller than $J_2$. Therefore we can evaluate their effects at different frequencies without compromising the accuracy or conservation of the constants of the motion. This makes the method extremely efficient in comparison to other numerical orbit propagation schemes in the literature, where all forces are calculated at the same frequency at each step. Furthermore, the symplecticness ensures that the constants of the motion are conserved, preventing a secular distortion of the orbit in the long run, which would be apparent if we were to use for example a Runge-Kutta scheme instead.

For the Keplerian part of the propagation, we successfully utilised both of the analytical relative orbit propagation schemes we developed. This enabled us to use larger stepsizes to evaluate other perturbations as their effects are smaller.

Finally, we demonstrated the accuracy of this composite symplectic scheme for various stepsizes, which is around metre to centimetre level for the examples we mentioned in the previous section. However, our geopotential model takes into account 36 terms in zonal and tesseral harmonics. We also showed that, particularly for propagations with a high number of geopotentials, savings of computational time up to 40% are possible without significant penalties in accuracy when the relative propagator is used instead of the difference of two absolute propagations.
7.2 Key Contributions

The objectives outlined in Chapter 1 have been met as below:

- Development and in-depth analysis of an analytical model for constellation acquisition and maintenance has been carried out, taking Kormos' [63] preliminary treatment further. The resulting method is flexible, robust and practical, well suited for a low-propellant constellation initialisation scenario.

- This orbit initialisation method proved itself with the successful orbit acquisition of the Disaster Monitoring Constellation. This experiment illustrated the real-world challenges of working within an existing satellite and operations architecture as well as an international collaboration.

- We adapted an existing analytical method ($\delta f - \delta g$ algorithm) to the two-body relative orbit modelling problem. We also developed a novel, linearised analytical solution to the same problem ($H_R L_R$ method), which yielded a very simple final expression describing the relative motion. This method differs from the existing literature as it is based upon the constants of the motion that should be conserved. Secondly, it expresses the relative motion in an inertial rather than the customary rotating accelerating frame.

- Using this simple form of the equations, for the first time in relative motion field, we were able to clearly identify the direct relationship between relative energy and relative positioning accuracy. This enabled us to claim better accuracy than second order methods (such as [59]) without any limitations on eccentricity.

- A computationally efficient numerical relative orbit propagator is developed that can potentially accommodate an arbitrary number of geopotential harmonics and other perturbations. No numerical relative orbit propagation scheme exists in the literature as far as the author is aware, with the singular exception of an unpublished method by Mikkola, based on his paper [79]. The symplectic scheme ensures that constants of the motion are conserved. The sophisticated integration scheme is extremely flexible and scalable to be tailored to the accuracy needs and computational limitations of
7.3. Further Work

We have identified certain areas of further research that are potentially valuable:

- The constellation initialisation algorithm described in Chapter 4 assumes that the satellite executes small magnitude discrete firings to achieve its orbit slot. A logical extension of this would be to derive similar equations for the case of a continuous thrust model.

- The constellation initialisation algorithm also has the potential to work onboard a satellite constellation in an autonomous and closed-loop fashion. Even in the absence of inter-satellite links or no line-of-sight, the groundstation can simply act as a relay to exchange GPS data between the satellites, which does not require human intervention. This would significantly reduce the workload for groundstation staff, which would be simply overseeing the manoeuvres.

- As our relative orbit propagator described in Chapter 5 has a vastly superior dynamic model than simple Hill’s Equations, it is reasonable to expect that it will provide a better accuracy when combined with sensor measurements within an estimator. It would be particularly useful to assess its performance in a state-of-the-art estimator such as an Unscented Kalman Filter that is well suited for nonlinear problems. Ideally, sensor measurements can be simulated via hardware-in-the-loop GPS simulators.

- Once such a complete relative navigation setup is successfully demonstrated on the ground, a simple formation can act as a testbed in space. Even if the formation does not have any sophisticated control capability for long-term formation-keeping, it can be initialised such that the formation stays together for a long duration.

- Some researchers suggested formations that are initialised such that they harness the effect of $J_2$ to stay together i.e., the two satellites have the same energy up to $J_2$
level. However, for the satellites to stay together for a substantially long time, we know that their relative energy should be equal to zero, taking into account as many geopotential terms as possible. Therefore, the relative orbit propagator can be used within an intelligent search algorithm to yield initial conditions for such a formation.

- While the relative orbit propagator yields very high accuracy even for highly eccentric cases, it is not very efficient as it uses a fixed timestep scheme. A time-regularised scheme would significantly increase the speed and efficiency of the algorithm for such scenarios.

- In Chapter 6, we reported that, upon inspection of the results, $\delta f - \delta g$ and $H_R L_R$ methods are identical. However, a more in-depth analysis should reveal how these two are related and which terms correspond to which part of the motion. That should enable a better understanding of the $\delta f - \delta g$ method.

- Also in Chapter 6, when doing the computational speed comparisons, we mentioned that the relative orbit propagation scheme, while fast, is not optimised for speed. Therefore, there is room for improvement to obtain even higher gains when compared to the differencing of two absolute orbits.

- In Chapter 6, it is possible to add in the effects of $J_2$ to the derivation of the analytical relative motion model. While such a model would still suffer from the shortcomings of other $J_2$ inclusive analytical methods, it would certainly further our understanding of the motion under oblateness effects. Such solutions are extremely useful in designing control algorithms.
Bibliography


Bibliography


Bibliography
Appendix A

Partial Derivatives of the Potential Function

A.1 First Derivative of the Potential Function

In Equation (3.31) we had utilised the partial derivative of the full potential function $U(r)$ with respect to $r$. Afterwards, in Section 3.4.3, we divided this potential function into the Keplerian part and perturbations. As the Keplerian part of the motion is handled via $f-g$ equations, we will need to evaluate the derivative of the perturbation potential with respect to $r$. This potential can be written as:

$$R(r, \varphi, \theta) = \frac{\mu}{r} \sum_{n=2}^{N} \sum_{m=0}^{n} \left( \frac{R_\infty}{r} \right)^n P_n^m(\cos \theta) [C_{nm} \cos m \varphi + S_{nm} \sin m \varphi]$$  \hspace{1cm} (A.1)

where $\mu$ is the gravitational parameter, $R_\infty$ the maximum equatorial radius of the central body and $(r, \theta, \varphi)$ are spherical polar co-ordinates fixed and rotating with the central body. For the case of the Earth, they are measured from the rotation axis and the first point of Aries. $P_l$ is a Legendre polynomial of degree $l$, $P_n^m$ is an associated Legendre polynomial of degree $n$ and order $m$. $N$ is the number of geopotentials taken into account.

Note that, the notation is slightly different than Equation (3.1), where we have used $J_l$ for the zonal harmonic coefficient of order $l$ and $(J_{nm}, \psi_{nm})$ for the tesseral harmonic coefficient and phase of degree $n$ and order $m$. $C_{nm}, S_{nm}$ is an alternative notation for
these parameters. The zonal harmonics are written as \(-C_{n,0} = J_n\). (\(J_{nm}, \psi_{nm}\)) are simply the constants of the tesseral harmonics \((C_{nm}, S_{nm})\) written in the phase and amplitude notation.

The first derivative of \(R\) in rotating spherical coordinates is,

\[
\frac{\partial R}{\partial r} = -(n + 1) \frac{\mu}{r} \sum_{n=2}^{N} \sum_{m=0}^{n} \left( \frac{R_{\Phi}}{r} \right)^n p_n^m [C_{nm} \cos m\varphi + S_{nm} \sin m\varphi]
\]

(A.2)

\[
\frac{\partial R}{\partial \theta} = \mu \sum_{n=2}^{N} \sum_{m=0}^{n} \left( \frac{R_{\Theta}}{r} \right)^n \left[ m \frac{\cos \theta}{\sin \theta} p_n^m - p_n^{m+1} \right] [C_{nm} \cos m\varphi + S_{nm} \sin m\varphi]
\]

(A.3)

\[
\frac{\partial R}{\partial \varphi} = -m \mu \sum_{n=2}^{N} \sum_{m=0}^{n} \left( \frac{R_{\Phi}}{r} \right)^n p_n^m [C_{nm} \sin m\varphi - S_{nm} \cos m\varphi]
\]

(A.4)

The forces (i.e., accelerations) in the three rotating spherical coordinates are:

\[
F_r = -\frac{\partial R}{\partial r}
\]

\[
F_\theta = -\frac{1}{r} \frac{\partial R}{\partial \theta}
\]

\[
F_\varphi = -\frac{1}{r \sin \theta} \frac{\partial R}{\partial \varphi}
\]

(A.5)

where \(F = F_re_r + F_\theta e_\theta + F_\varphi e_\varphi\).

These equations express the accelerations in a rotating spherical coordinate frame, which is not very useful. We would like to convert them into the standard Earth Centred Inertial (ECI) cartesian frame.

The conversion from rotating spherical to rotating cartesian is carried out as follows:

\[
\begin{pmatrix}
  F_x \\
  F_y \\
  F_z 
\end{pmatrix}
=
\begin{pmatrix}
  \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi \\
  \sin \theta \sin \varphi & \cos \theta \sin \varphi & \cos \varphi \\
  \cos \theta & -\sin \theta & 0 
\end{pmatrix}
\begin{pmatrix}
  F_r \\
  F_\theta \\
  F_\varphi 
\end{pmatrix}
\]

(A.6)

where the \(F_x\) notation represents the force component along the coordinate \(x\). The subscript notation \((\cdot)_C\) denotes the rotating frame.

The next step is to convert the rotating frame into the inertial one (denoted by the notation \((\cdot)_I\)). Consider the effect of a rotation of \(\beta\) around the z axis:

\[
\begin{pmatrix}
  F_x \\
  F_y \\
  F_z 
\end{pmatrix}
_I
=
\begin{pmatrix}
  -\sin \beta & \cos \beta & 0 \\
  \cos \beta & \sin \beta & 0 \\
  0 & 0 & 1 
\end{pmatrix}
\begin{pmatrix}
  F_x \\
  F_y \\
  F_z 
\end{pmatrix}
_C
\]

(A.7)
A.2 Second Derivative of the Potential Function

Multiplying the two conversion matrices to convert from rotating spherical to inertial cartesian, the coordinate transformation for the forces can be written as:

\[
\begin{pmatrix}
F_x \\
F_y \\
F_z
\end{pmatrix}_I = \begin{pmatrix}
\frac{x_I}{r} & \frac{\cot \theta}{r} & \frac{-y_I}{r \sin \theta} \\
\frac{y_I}{r} & \frac{\cot \theta}{r} & \frac{x_I}{r \sin \theta} \\
\frac{z_I}{r} & \frac{\tan \theta}{r} & 0
\end{pmatrix}
\begin{pmatrix}
F_r \\
F_\theta \\
F_\phi
\end{pmatrix}_C
\]  
(A.8)

where the inertial cartesian coordinates are given by,

\[
x_I = r \sin \theta \cos(\varphi + \beta)
\]
\[
y_I = r \sin \theta \sin(\varphi + \beta)
\]
\[
z_I = r \cos \theta
\]  
(A.9)

It is also possible to write the force along to the inertial cartesian coordinates, in vector notation:

\[
\mathbf{F} = \left[ (F_r + F_\theta \cot \theta) \frac{x_I}{r} - \frac{F_\theta}{r \sin \theta} \right] \mathbf{i} + \left[ (F_r + F_\theta \cot \theta) \frac{y_I}{r} - \frac{F_\phi}{r \sin \theta} \right] \mathbf{j} + \left[ (F_r - F_\theta \tan \theta) \frac{z_I}{r} \right] \mathbf{k}
\]  
(A.10)

A.2 Second Derivative of the Potential Function

As shown in Section 5.2.2, to calculate the relative accelerations, we need the second derivatives of the potential. Similar to the absolute propagation case presented above, we separate the Keplerian part from the potential and calculate the second derivative of the perturbation to the Keplerian potential \( R \) only. Note that this derivation is based on the formulation given in [78].

The first derivative of the geopotential in the rotating coordinates can be written as,

\[
[R(x, \gamma, z),_I]_C = A [R(r, \theta, \varphi),_1]_C
\]  
(A.11)

where \( A \) is the matrix for spherical to cartesian coordinate conversion and \( R, \varphi \) denotes the first derivative. The \( A \) matrix is given by,

\[
A = \begin{pmatrix}
\sin \theta \cos \varphi & \frac{\cos \theta \cos \varphi}{r} & -\frac{\sin \varphi}{r \sin \theta} \\
\sin \theta \sin \varphi & \frac{\cos \theta \sin \varphi}{r} & \frac{\cos \varphi}{r \sin \theta} \\
\cos \theta & -\frac{\sin \theta}{r} & 0
\end{pmatrix}
\]  
(A.12)
Appendix A. Partial Derivatives of the Potential Function

Taking the derivative of \( R_{ij} \) with respect to the cartesian coordinates,

\[
[R(x, y, z)_C] = A[R(r, \theta, \varphi)_C] A^T + AA_A [R(r, \theta, \varphi)_C]_C
\]  \hspace{1cm} (A.13)

where \( A_{ij} \) is the derivative of \( A \) in spherical coordinates.

The derivative of the \( A \) matrix with respect to the spherical coordinates is written with the shorthand notation \( A_{ij} = (B_1|B_2|B_3) \), where \( B_i \) are \( 3 \times 3 \) matrices each.

\[
B_1 = \begin{bmatrix}
0 & -\frac{\cos \theta \cos \varphi}{r^2} & \frac{\sin \varphi}{r^2 \sin \theta} \\
\cos \theta \cos \varphi & -\frac{\sin \theta \cos \varphi}{r} & \frac{\cot \theta \sin \varphi}{r \sin \theta} \\
-\sin \theta \sin \varphi & -\frac{\cos \theta \sin \varphi}{r} & -\frac{\cos \varphi}{r \sin \theta}
\end{bmatrix}
\]  \hspace{1cm} (A.14)

\[
B_2 = \begin{bmatrix}
0 & -\frac{\cos \theta \sin \varphi}{r^2} & -\frac{\cos \varphi}{r^2 \sin \theta} \\
\cos \theta \sin \varphi & -\frac{\sin \theta \sin \varphi}{r} & \frac{\cot \theta \cos \varphi}{r \sin \theta} \\
\sin \theta \cos \varphi & \frac{\cos \theta \cos \varphi}{r} & -\frac{\sin \varphi}{r \sin \theta}
\end{bmatrix}
\]

\[
B_3 = \begin{bmatrix}
0 & \frac{\sin \theta}{r^2} & 0 \\
-\sin \theta & -\frac{\cos \theta}{r} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The last unevaluated term in (A.13) is the second derivative of the geopotential, \( R(r, \theta, \varphi)_C \), in rotating spherical coordinates. Though this is not particularly difficult to evaluate, it is much more practical to express the second derivative in terms of the already known first derivatives. The second derivatives in \( r \) and \( \varphi \) are given as:

\[
R_{rr} = \frac{(n+1)(n+2)}{r^2} R
\]  \hspace{1cm} (A.15)

\[
R_{\varphi\varphi} = -m^2 R
\]  \hspace{1cm} (A.16)

\( R_{\theta\theta} \) can be computed via Legendre’s equation (for \( P_n^m = P_n^m(z) \)),

\[
\frac{d}{dz} \left[ (1-z^2) \frac{dP_n^m}{dz} \right] + \left[ n(n+1) - \frac{m^2}{1-z^2} \right] P_n^m = 0
\]  \hspace{1cm} (A.17)
A.2. Second Derivative of the Potential Function

Substituting \( z = \cos \theta \) and \( d/dz = (\sin \theta)^{-1}d/d\theta \),
\[
- \frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \frac{\sin^2 \theta dP_n^m}{d\theta} \right] + \left[ n(n + 1) - \frac{m^2}{\sin^2 \theta} \right] P_n^m = 0 \tag{A.18}
\]

Further substituting the first derivative \( dP_n^m / d\theta = m \cot \theta P_n^m - P_n^{m+1} \) and rewriting the resulting expression in terms of the geopotential and its first derivative, the second derivative with respect to \( \theta \) is found as,
\[
R_{,\theta \theta} = - \cot \theta R_{,\theta} - \left[ n(n + 1) - \frac{m^2}{\sin^2 \theta} \right] R \tag{A.19}
\]

The off-diagonal terms are,
\[
R_{,r \theta} = - \frac{n + 1}{r} R_{,\theta} \tag{A.20}
\]
\[
R_{,\phi \theta} = - \frac{n + 1}{r} R_{,\phi} \tag{A.21}
\]

For \( R_{,\phi \phi} \) there are no simplifications possible, though it is more convenient to express this in terms of the forces,
\[
R_{,\phi \phi} = \frac{\partial (R_{,\phi})}{\partial \phi} = - \frac{r}{\sin \theta} \frac{\partial (F_{\phi} \sin \theta)}{\partial \phi} \tag{A.22}
\]

In summary, in simple matrix form, the second derivatives are given as,
\[
[R_2]_C = \begin{pmatrix}
\frac{(n + 1)(n + 2)}{r^2} R & \frac{(n + 1)}{r} R_{,\theta} & \frac{(n + 1)}{r} R_{,\phi} \\
- \frac{(n + 1)}{r} R_{,\theta} & - \cot \theta R_{,\theta} - \left[ n(n + 1) - \frac{m^2}{\sin^2 \theta} \right] R & \frac{r}{\sin \theta} (F_{\phi} \sin \theta)_{,\phi} \\
- \frac{(n + 1)}{r} R_{,\phi} & \frac{r}{\sin \theta} (F_{\phi} \sin \theta)_{,\phi} & -m^2 R
\end{pmatrix}^C \tag{A.23}
\]

At this stage, all the components of (A.13) are evaluated, which converts the second derivative of the potential in rotating spherical coordinates to rotating cartesian coordinates. The final touch is to convert the resulting matrix in the rotating frame into inertial frame. This is carried out via the conversion matrix given in (A.7), which can be called matrix \( Z \).
\[
[R_r(x, y, z),_2]_I = (ZA) \left( [R(r, \theta, \phi),_2]_C + A_{,1} [R(r, \theta, \phi),_1]_C A^{-T} \right) (ZA)^T \tag{A.24}
\]
For convenience, the expression \( \left( A_1 R(r, \theta, \varphi), 2 \right) \) can be evaluated analytically to give the matrix:

\[
\begin{pmatrix}
\frac{(n + 2)}{r} F_r & (n + 2) F_\theta & (n + 2) F_{\varphi} s \theta \\
(n + 2) F_\theta & \frac{r}{s^2 \theta} \left( c \theta s \theta F_{\varphi} + \frac{m^2}{(n + 1)} F_r \right) - (n + 1) r F_r & r c \theta F_{\varphi} - r F_{\varphi \varphi} \\
(n + 2) F_{\varphi} s \theta & r c \theta F_{\varphi} - r F_{\varphi \varphi} & -\left( s^2 \theta + \frac{m^2}{(n + 1)} r F_r \right) - r c \theta s \theta F_{\varphi}
\end{pmatrix}
\]

(A.25)

where \( c \theta \) and \( s \theta \) are shorthands for \( \cos \theta \) and \( \sin \theta \), respectively.