

The frame bundle picture of Gaussian wave packet dynamics in semiclassical mechanics

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Abstract

Recently Ohsawa [Ohs15] has studied the Marsden-Weinstein-Meyer quotient of the manifold $T^*\mathbb{R}^n \times T^*\mathbb{R}^{2n^2}$ under a $O(2n)$ -symmetry, and has used this quotient to describe the relationship between two different parametrisations of Gaussian wave packet dynamics commonly used in semiclassical mechanics. In this paper we suggest a new interpretation of (a subset of) the unreduced space as being the frame bundle $\mathcal{F}(T^*\mathbb{R}^n)$ of $T^*\mathbb{R}^n$. We outline some advantages of this interpretation, and explain how it can be extended to more general symplectic manifolds using the notion of the diagonal lift of a symplectic form due to Cordero and de León [CdL83].

1 Introduction

1.1 Motivation

The Gaussian wave packet ansatz is frequently used in the study of the time-dependent Schrödinger equation, and its semiclassical limit. In particular, the wavefunction

$$\psi(x) = \exp \left\{ \frac{i}{\hbar} \left[\frac{1}{2} (x - q)^\top (\mathcal{A} + i\mathcal{B})(x - q) + p^\top (x - q) + (\phi + i\delta) \right] \right\},$$

parametrised by $(q, p) \in T^*\mathbb{R}^n$, $\phi, \delta \in \mathbb{R}$, and $\mathcal{A} + i\mathcal{B} \in \Sigma_n = \{W \in M_n(\mathbb{C}) \mid W^\top = W, \text{Im } W > 0\}$, is well known to be an *exact* solution of the Schrödinger equation for quadratic Hamiltonians, provided the parameters satisfy certain ODEs [Hel76].

Faou and Lubich [FL06, Lub08] have shown that these parameter ODEs constitute a Hamiltonian system, and recently Ohsawa and Leok [OL13] have clarified the symplectic structure underlying these Hamiltonian dynamics. Their main observation is that Σ_n is a symplectic manifold, the well-studied *Siegel upper half plane* [Sie73].

In [Hag80], Hagedorn introduced another parametrisation of Gaussian wavefunctions, by replacing $\mathcal{A} + i\mathcal{B} \in \Sigma_n$ by PQ^{-1} , where $Q, P \in M_n(\mathbb{C})$ satisfying certain algebraic relations. In this parametrisation, the ODEs governing the parameter evolution are somewhat simpler. In [Ohs15], Ohsawa explained how to interpret the relation between Q, P and $\mathcal{A} + i\mathcal{B}$ as an instance of symplectic reduction on the symplectic manifold $T^*\mathbb{R}^n \times T^*\mathbb{R}^{n^2}$ under a certain right

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$O(2n)$ -action. The matrices Q, P are seen to be coordinates on a level set of the momentum map corresponding to the $O(2n)$ -action, and symplectic reduction at this momentum level gives the Siegel upper half plane Σ_n . The dynamics in the unreduced and reduced spaces are then related by the quotient map in the usual way, explaining how the Hagedorn ODEs can also be interpreted as a Hamiltonian system.

1.2 Main Results and Outline

The purpose of the present paper is to reinterpret Ohsawa's results, and to suggest some possible extensions. Specifically, we interpret an open subset of Ohsawa's unreduced space $T^*\mathbb{R}^n \times T^*\mathbb{R}^{2n^2}$ as the *frame bundle* $\mathcal{F}(T^*\mathbb{R}^n) \simeq T^*\mathbb{R}^n \times \mathrm{GL}(2n, \mathbb{R})$ of $T^*\mathbb{R}^n$, and give an intrinsic description of the symplectic structure on $\mathcal{F}(T^*\mathbb{R}^n)$. There are several advantages to this picture:

- there exists a dual pair structure on $\mathrm{GL}(2n, \mathbb{R})$, generated by left multiplication by $\mathrm{Sp}(2n, \mathbb{R})$ and right multiplication by $O(2n)$. The existence of this dual pair allows one to realise the symplectic reduced space of $\mathcal{F}(T^*\mathbb{R}^n)$ at *all* values of $O(2n)$ -momentum in terms of adjoint orbits of $\mathfrak{sp}(2n, \mathbb{R})$.¹
- the physical picture of moving frames on $T^*\mathbb{R}^n$ is intuitive, and gives insight into the definition of the Hamiltonian on the unreduced space. It also has overlap with other work on semiclassical mechanics, in particular the *nearby orbit approximation* of Littlejohn [Lit86].
- the frame bundle picture can be extended to more general symplectic manifolds than $T^*\mathbb{R}^n$, using the results of Cordero et al. [CdL83] on the lifting of symplectic structures to frame bundles.

We now outline the structure of the paper. In Section 2, we describe a symplectic structure on the set $M_{2n}(\mathbb{R})$ of $2n \times 2n$ dimensional real matrices, demonstrate that the obvious left $\mathrm{Sp}(2n, \mathbb{R})$ - and right $O(2n)$ -actions are Hamiltonian, and calculate the corresponding momentum maps, and their Lie algebra-valued counterparts. In Section 3, we review some properties of dual pairs that will be needed in the sequel. In Section 4, we restrict our symplectic manifold to $\mathrm{GL}(2n, \mathbb{R}) \subset M_{2n}(\mathbb{R})$, and demonstrate that the two group actions give a dual pair structure on this restricted space. In Section 5, we discuss reduction of $\mathrm{GL}(2n, \mathbb{R})$ under the $O(2n)$ -action at a particular value of momentum, use the results of Section 3 to describe the reduced space as an adjoint orbit in $\mathfrak{sp}(2n, \mathbb{R})$, and give a geometric description of this adjoint orbit as the set of $\omega_{\mathbb{R}^{2n}}$ -compatible complex structures, where $\omega_{\mathbb{R}^{2n}}$ is the canonical symplectic structure on \mathbb{R}^{2n} . In Section 6, we use the standard isomorphism of $\omega_{\mathbb{R}^{2n}}$ -compatible complex structures with the positive Lagrangian Grassmannian to introduce coordinates on the adjoint orbit, and demonstrate that in these coordinates the symplectic projection and reduced symplectic form agree with those obtained by Ohsawa. In Section 7, starting with the cotangent symplectic

¹Although we consider one particular adjoint orbit in this paper, it should be possible to associate semiclassical wavefunctions with arbitrary integral adjoint orbits. We plan to explore this possibility in future work.

structure $\omega = \sum_{\alpha=1}^n dq^\alpha \wedge dp_\alpha$ and an arbitrary Hamiltonian H on $T^*\mathbb{R}^n$, we describe the \hbar -diagonal symplectic structure Ω^\hbar and \hbar -lifted Hamiltonian H^\hbar on $\mathcal{F}(T^*\mathbb{R}^n)$, which agree with those defined by Ohsawa on their common domain. We give a geometric interpretation of H^\hbar , demonstrate that for quadratic H one obtains the dynamics of Hagedorn, and emphasise that the $O(2n)$ -invariance of H^\hbar leads to conservation of *symplectic* frame. We also describe the dynamics on the reduced space, and show it reproduces those of Heller. Finally in Section 8, we explain how the results of Cordero et al. [CdL83] can be used to extend the construction of a symplectic structure on the frame bundle to arbitrary symplectic manifolds.

2 The symplectic geometry of $M_{2n}(\mathbb{R})$

In this section, we outline the symplectic structure on $M_{2n}(\mathbb{R})$, describe two natural symplectic group actions on $M_{2n}(\mathbb{R})$, and calculate the momentum maps associated with these actions.

2.1 The symplectic form on $M_{2n}(\mathbb{R})$

Take \mathbb{R}^{2n} with its canonical symplectic structure

$$\omega_{\mathbb{R}^{2n}}(v, w) := v^\top \mathbb{J} w,$$

where $\mathbb{J} := \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$. The $2n$ -fold direct sum of $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ is naturally isomorphic to the space $M_{2n}(\mathbb{R})$ of real $2n \times 2n$ matrices with the symplectic form

$$\Omega(E, F) := \sum_{a=1}^{2n} \omega_{\mathbb{R}^{2n}}(E_a, F_a) = \text{Tr}(E^\top \mathbb{J} F),$$

where E_a denotes the a th column of $E \in M_{2n}(\mathbb{R})$, considered as a vector in \mathbb{R}^{2n} . We can consider $(\Omega, M_{2n}(\mathbb{R}))$ as a symplectic *manifold*, using the canonical isomorphism $T_E M_{2n}(\mathbb{R}) \simeq M_{2n}(\mathbb{R})$, and we denote the induced symplectic form by Ω also. Explicitly, if $V_E = \left. \frac{d}{dt} \right|_{t=0} (E + tV)$ for $V \in M_{2n}(\mathbb{R})$ etc., then

$$\Omega_E(V_E, W_E) := \Omega(V, W) = \text{Tr}(V^\top \mathbb{J} W).$$

2.2 Commuting symplectic actions

Let $L_S : M_{2n}(\mathbb{R}) \rightarrow M_{2n}(\mathbb{R})$ denote left multiplication by $S \in \text{Sp}(2n, \mathbb{R})$, and $R_O : M_{2n}(\mathbb{R}) \rightarrow M_{2n}(\mathbb{R})$ denote right multiplication by $O \in O(2n)$. Using the expressions for the differentials of these actions

$$T_E L_S(V_E) = (SV)_{SE} \quad \text{and} \quad T_E R_O(V_E) = (VO)_{EO}$$

it is straightforward to check that L and R define left and right symplectic actions respectively, i.e., for any $E \in M_{2n}(\mathbb{R})$,

$$(L_S^* \Omega)_E = \Omega_E \quad \text{and} \quad (R_O^* \Omega)_E = \Omega_E.$$

The momentum maps corresponding to these two symplectic actions have a standard form.

Proposition 2.1. (i) The momentum map $J_{\mathrm{Sp}} : M_{2n}(\mathbb{R}) \rightarrow \mathfrak{sp}(2n, \mathbb{R})^*$ corresponding to the left $\mathrm{Sp}(2n, \mathbb{R})$ -action is given by

$$\langle J_{\mathrm{Sp}}(E), \zeta \rangle = \frac{1}{2} \Omega(\zeta E, E).$$

(ii) The momentum map $J_{\mathrm{O}} : M_{2n}(\mathbb{R}) \rightarrow \mathfrak{o}(2n)^*$ corresponding to the right $\mathrm{O}(2n)$ -action is given by

$$\langle J_{\mathrm{O}}(E), \xi \rangle = \frac{1}{2} \Omega(E\xi, E).$$

Proof. Both results follow from the general expression for the momentum map of a linear symplectic action on a symplectic vector space—see for example [MR99, Section 12.4, Example (a)]. \square

Remark. Both momentum maps are easily seen to be equivariant,

$$J_{\mathrm{Sp}}(SE) = \mathrm{Ad}_{S^{-1}}^*(J_{\mathrm{Sp}}(E)) \quad \text{and} \quad J_{\mathrm{O}}(EO) = \mathrm{Ad}_{O}^*(J_{\mathrm{O}}(E)),$$

(again, a general result for linear symplectic actions).

2.3 Lie algebra-valued momentum maps

For any (real) Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(N, \mathbb{C})$, consider the trace form $\langle\langle \cdot, \cdot \rangle\rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ defined by

$$\langle\langle \xi, \zeta \rangle\rangle := \frac{1}{2} \mathrm{Tr}(\xi\zeta).$$

If \mathfrak{g} is invariant under conjugate transpose, then $\langle\langle \cdot, \cdot \rangle\rangle$ is non-degenerate, as is its real part, since

$$\langle\langle \xi, \xi^\dagger \rangle\rangle = \mathrm{Re} \langle\langle \xi, \xi^\dagger \rangle\rangle > 0 \quad \text{for } \xi \neq 0.$$

In particular, it is non-degenerate and real-valued on $\mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{o}(2n)$. Using the trace form to identify Lie algebras with their duals, we can write down Lie algebra-valued versions of the momentum maps discussed in the previous section.

Proposition 2.2. (i) The $\mathfrak{sp}(2n, \mathbb{R})$ -valued momentum map $j_{\mathrm{Sp}} : M_{2n}(\mathbb{R}) \rightarrow \mathfrak{sp}(2n, \mathbb{R})$ corresponding to J_{Sp} is given by

$$j_{\mathrm{Sp}}(E) = -EE^\top \mathbb{J}.$$

(ii) The $\mathfrak{o}(2n)$ -valued momentum map $j_{\mathrm{O}} : M_{2n}(\mathbb{R}) \rightarrow \mathfrak{o}(2n)$ corresponding to J_{O} is given by

$$j_{\mathrm{O}}(E) = -E^\top \mathbb{J} E.$$

Proof. (i) Denoting the natural pairing of $\mathfrak{sp}(2n, \mathbb{R})^*$ with $\mathfrak{sp}(2n, \mathbb{R})$ by $\langle \cdot, \cdot \rangle$, we have that for $\zeta \in \mathfrak{sp}(2n, \mathbb{R})$,

$$\begin{aligned} \langle J_{\mathrm{Sp}}(E), \zeta \rangle &= \frac{1}{2} \Omega(\zeta E, E) = \frac{1}{2} \mathrm{Tr}(E^\top \zeta^\top \mathbb{J} E) \\ &= \frac{1}{2} \mathrm{Tr}(EE^\top \zeta^\top \mathbb{J}) = -\frac{1}{2} \mathrm{Tr}(EE^\top \mathbb{J} \zeta) \quad \text{since } \zeta \in \mathfrak{sp}(2n, \mathbb{R}) \\ &= \langle\langle -EE^\top \mathbb{J}, \zeta \rangle\rangle. \end{aligned}$$

Also

$$(-EE^\top \mathbb{J})^\top \mathbb{J} + \mathbb{J}(-EE^\top \mathbb{J}) = \mathbb{J}EE^\top \mathbb{J} - \mathbb{J}EE^\top \mathbb{J} = 0.$$

So $-EE^\top \mathbb{J} \in \mathfrak{sp}(2n, \mathbb{R})$, and the result follows.

(ii) Similar. □

Remark. Again, the Lie algebra-valued momentum maps are equivariant

$$j_{\text{Sp}}(SE) = \text{Ad}_S(j_{\text{Sp}}(E)) \quad \text{and} \quad j_{\text{O}}(EO) = \text{Ad}_{O^{-1}}(j_{\text{O}}(E)).$$

3 Mutually transitive actions and dual pairs

In Section 4 we will demonstrate that the left $\text{Sp}(2n, \mathbb{R})$ -action and right $\text{O}(2n)$ -action define a dual pair structure on a suitable subset of $\text{M}_{2n}(\mathbb{R})$. In anticipation, we here introduce the notion of *mutually transitive actions*. We then explain how mutual transitivity allow us to view reduced spaces of one action as coadjoint orbits of the other. Finally, we indicate the relation of mutual transitivity to the more standard notion of a dual pair.

A fuller treatment of dual pairs and related concepts can be found in [LM87, Section IV.7], [OR04, Chapter 11], and [BW12]. We also refer to [SV19] for a more thorough discussion of mutual transitivity.

3.1 Mutually transitive actions

Let (N, Ω) be a symplectic manifold, and let $\Phi_1 : G_1 \times N \rightarrow N$ and $\Phi_2 : G_2 \times N \rightarrow N$ be symplectic actions. We assume N , G_1 , and G_2 are all finite-dimensional.

Definition 3.1. We say the actions Φ_1, Φ_2 are *mutually transitive* if the following three properties hold:

- Φ_1 and Φ_2 commute,
- Φ_1 and Φ_2 are Hamiltonian actions, with corresponding equivariant momentum maps J_1 and J_2 ,
- each level set of J_1 is a G_2 -orbit and vice versa, i.e., for any $x \in N$,

$$J_1^{-1}(J_1(x)) = G_2 \cdot x \quad \text{and} \quad J_2^{-1}(J_2(x)) = G_1 \cdot x.$$

Denoting the coadjoint orbit in \mathfrak{g}_i^* through μ_i by \mathcal{O}_{μ_i} , we then have

Proposition 3.2. *Let Φ_1, Φ_2 be mutually transitive actions. Then for all $x \in N$,*

$$J_1^{-1}(\mathcal{O}_{J_1(x)}) = J_2^{-1}(\mathcal{O}_{J_2(x)}).$$

Proof.

$$\begin{aligned} J_1^{-1}(\mathcal{O}_{J_1(x)}) &= G_1 \cdot J_1^{-1}(J_1(x)) && \text{since } J_1 \text{ is } G_1\text{-equivariant} \\ &= G_1 \cdot (G_2 \cdot x) && \text{since } \Phi_2 \text{ is transitive on the fibres of } J_1 \\ &= G_2 \cdot (G_1 \cdot x) && \text{since the actions } \Phi_1 \text{ and } \Phi_2 \text{ commute} \\ &= G_2 \cdot J_2^{-1}(J_2(x)) && \text{since } \Phi_1 \text{ is transitive on the fibres of } J_2 \\ &= J_2^{-1}(\mathcal{O}_{J_2(x)}) && \text{since } J_2 \text{ is } G_2\text{-equivariant.} \end{aligned}$$

□

Corollary 3.3 ([BW12, Theorem 2.8(i)]). *There exists a one-to-one correspondence between coadjoint orbits in $J_1(N)$ and $J_2(N)$ given by*

$$\mathcal{O}_{\mu_1} \mapsto J_2(J_1^{-1}(\mathcal{O}_{\mu_1})) = J_2(J_1^{-1}(\mu_1))$$

or equivalently

$$\mathcal{O}_{J_1(x)} \mapsto \mathcal{O}_{J_2(x)} \quad \text{for } x \in N.$$

3.2 The relation between coadjoint orbits and reduced spaces

From now on we assume that all group actions Φ_i are both free and proper (see [SV19] for a discussion when these conditions do not hold).

For $\mu_1 \in \mathfrak{g}_1^*$, let $(G_1)_{\mu_1}$ denote the coadjoint stabiliser of μ_1 , and N_{μ_1} the quotient space $J_1^{-1}(\mu_1)/(G_1)_{\mu_1}$ at μ_1 . It is well known that N_{μ_1} may be given a smooth structure making $\sigma_1 : J_1^{-1}(\mu_1) \rightarrow N_{\mu_1}$ a submersion, and a symplectic structure $\Omega_{N_{\mu_1}}$ satisfying

$$i_{\mu_1}^* \Omega = \sigma_1^* \Omega_{N_{\mu_1}}, \quad (1)$$

where $i_{\mu_1} : J_1^{-1}(\mu_1) \rightarrow N$ is the inclusion (see for example [OR04, Theorem 6.1.1 (i)]). The pair $(N_{\mu_1}, \Omega_{N_{\mu_1}})$ is called the *reduced space* or *Marsden-Weinstein-Meyer quotient*.

For mutually transitive actions, the reduced space under one action is symplectomorphic to a coadjoint orbit of the other action. To prove this result, we first give the following lemma:

Lemma 3.4. *The smooth map $J_2 : N \rightarrow \mathfrak{g}_2^*$ restricts to a smooth submersion $J_2 : J_1^{-1}(\mu_1) \rightarrow \mathcal{O}_{\mu_2}$ satisfying*

$$i_{\mu_1}^* \Omega = J_2^* \Omega_{\mathcal{O}_{\mu_2}}^+, \quad (2)$$

where $\mathcal{O}_{\mu_2} \subset \mathfrak{g}_2^*$ is a coadjoint orbit, and $\Omega_{\mathcal{O}_{\mu_2}}^+$ is the positive KKS symplectic form on \mathcal{O}_{μ_2} .

Proof. Since Φ_1 is free, the momentum map J_1 is a submersion [OR04, Corollary 4.5.13], and hence the level set $J_1^{-1}(\mu_1)$ is an embedded submanifold of N [OR04, Section 1.1.13]. Since $J_1^{-1}(\mu_1)$ is a G_2 -orbit, and J_2 is G_2 -equivariant, the image $J_2(J_1^{-1}(\mu_1))$ equals some coadjoint orbit $\mathcal{O}_{\mu_2} \subset \mathfrak{g}_2^*$, which is an initial submanifold [OR04, Proposition 2.3.12 (i)]. Hence $J_2 : N \rightarrow \mathfrak{g}_2^*$ restricts to a smooth map $J_2 : J_1^{-1}(\mu_1) \rightarrow \mathcal{O}_{\mu_2}$, and by G_2 -equivariance this restriction is a submersion.

Taking $\xi, \zeta \in \mathfrak{g}_2$, $x \in J_1^{-1}(\mu_1)$, and using the G_2 -equivariance of J_2 , we have

$$\begin{aligned} \Omega_x(\xi \cdot x, \zeta \cdot x) &= d_x \langle J_2, \xi \rangle (\zeta \cdot x) = \zeta \cdot x \langle J_2, \xi \rangle = \langle -\text{ad}_\zeta^* J_2(x), \xi \rangle \\ &= \langle J_2(x), [\xi, \zeta] \rangle = (\Omega_{\mathcal{O}_{\mu_2}}^+)_{J_2(x)}(-\text{ad}_\xi^* J_2(x), -\text{ad}_\zeta^* J_2(x)). \end{aligned}$$

Again using equivariance of J_2 , plus the fact that $J_1^{-1}(\mu_1)$ is a G_2 -orbit, gives (2). \square

Proposition 3.5 ([BW12, Theorem 2.8(iii)]). *Let Φ_1, Φ_2 be mutually transitive actions on N . Then any reduced space under the G_1 -action is symplectomorphic*

to a coadjoint orbit in $J_2(N) \subset \mathfrak{g}_2^*$, and similarly with 1 and 2 interchanged. Explicitly, for $x \in N$,

$$N_{J_1(x)} \simeq \mathcal{O}_{J_2(x)}, \quad N_{J_2(x)} \simeq \mathcal{O}_{J_1(x)},$$

via a resp. G_2 - and G_1 -equivariant symplectomorphism.

Proof. Choose $x \in N$, and let $\mu_i = J_i(x)$, $i = 1, 2$. By equivariance of J_1 and J_2 and the mutual transitivity property, it is not difficult to show that for any $y \in J_1^{-1}(\mu_1)$,

$$(G_1)_{J_1(y)} \cdot y = (G_2)_{J_2(y)} \cdot y = G_1 \cdot y \cap G_2 \cdot y.$$

Hence the fibres of $\sigma_1 : J_1^{-1}(\mu_1) \rightarrow N_{\mu_1}$ and $J_2 : J_1^{-1}(\mu_1) \rightarrow \mathcal{O}_{\mu_2}$ agree, and we get a bijection $\check{J}_2 : N_{\mu_1} \rightarrow \mathcal{O}_{\mu_2}$ making the following diagram commute

$$\begin{array}{ccc} & J_1^{-1}(\mu_1) & \\ \sigma_1 \swarrow & & \searrow J_2 \\ N_{\mu_1} & \xrightarrow{\check{J}_2} & \mathcal{O}_{\mu_2} \end{array} \quad (3)$$

Since σ_1 and J_2 are submersions, \check{J}_2 is a diffeomorphism. Equations (1) and (2), and commutativity of (3), implies that

$$\sigma_1^* \Omega_{N_{\mu_1}} = J_2^* \Omega_{\mathcal{O}_{\mu_2}^+} = \sigma_1^* \check{J}_2^* \Omega_{\mathcal{O}_{\mu_2}^+}.$$

Since σ_1 is a submersion, it follows that

$$\Omega_{N_{\mu_1}} = \check{J}_2^* \Omega_{\mathcal{O}_{\mu_2}^+},$$

i.e., \check{J}_2 is a symplectomorphism. □

Remark. The map \check{J}_2 has a simple interpretation: since the G_1 - and G_2 -actions commute, the G_2 -action drops to the reduced space $N_{J_1(x)}$. Using $(i^{\mu_1})^* \Omega = (\sigma_1)^* \Omega_{N_{\mu_1}}$, we see that \check{J}_2 is simply the momentum map for this reduced action, and is itself equivariant.

3.3 The relation to dual pairs

In this subsection we make contact with the notion of dual pair (in the sense of Weinstein [Wei83]).

Definition 3.6 ([Bla01, Appendix E]). Let N be a symplectic manifold, and P_1, P_2 Poisson manifolds. A pair of Poisson maps

$$P_1 \xleftarrow{J_1} N \xrightarrow{J_2} P_2$$

is called a *full dual pair* if J_1, J_2 are submersions, and

$$(\ker T J_1)^\Omega = \ker T J_2.$$

Proposition 3.7. *If Φ_1, Φ_2 are mutually transitive and free, then their momentum maps J_1, J_2 form a full dual pair.*

Proof. Let $x \in N$. The identity $G_1 \cdot x = J_2^{-1}(J_2(x))$ implies that

$$\mathfrak{g}_1 \cdot x = T_x(J_2^{-1}(J_2(x))).$$

A standard result says that the former equals $(\ker T_x J_1)^\Omega$. Also, as in Lemma 3.4, freeness of Φ_1, Φ_2 implies that J_1, J_2 are submersions. Then $J_2^{-1}(J_2(x))$ is an embedded submanifold, and so

$$T_x(J_2^{-1}(J_2(x))) = \ker T_x J_2.$$

The result follows. \square

4 The dual pair structure on $\mathrm{GL}(2n, \mathbb{R}) \subset \mathrm{M}_{2n}(\mathbb{R})$

The subset $\mathrm{GL}(2n, \mathbb{R})$ of $\mathrm{M}_{2n}(\mathbb{R})$ is open, and hence Ω restricts to a non-degenerate form on $\mathrm{GL}(2n, \mathbb{R})$. Denote the restriction of Ω and the momentum maps to $\mathrm{GL}(2n, \mathbb{R})$ by the same symbols for simplicity.

With this restriction, we can demonstrate that the left $\mathrm{Sp}(2n, \mathbb{R})$ - and right $\mathrm{O}(2n)$ -actions are mutually transitive (Definition 3.1). For this, it is more convenient to work with the equivalent double fibration

$$\begin{array}{ccc} & \mathrm{GL}(2n, \mathbb{R}) & \\ \swarrow \mathfrak{j}_{\mathrm{Sp}} & & \searrow \mathfrak{j}_{\mathrm{O}} \\ \mathfrak{sp}(2n, \mathbb{R}) & & \mathfrak{o}(2n) \end{array}$$

Proposition 4.1. (i) *The left $\mathrm{Sp}(2n, \mathbb{R})$ -action acts transitively on the fibres of $\mathfrak{j}_{\mathrm{O}}$.*

(ii) *The right $\mathrm{O}(2n)$ -action acts transitively on the fibres of $\mathfrak{j}_{\mathrm{Sp}}$.*

Proof. (i) Using Proposition 2.2, we have that for any $E, E' \in \mathrm{GL}(2n, \mathbb{R})$,

$$\begin{aligned} \mathfrak{j}_{\mathrm{O}}(E') &= \mathfrak{j}_{\mathrm{O}}(E) \\ \iff -(E')^\top \mathbb{J} E' &= -E^\top \mathbb{J} E \\ \iff (E' E^{-1})^\top \mathbb{J} (E' E^{-1}) &= \mathbb{J} \\ \iff E' &= S E \quad \text{for some } S \in \mathrm{Sp}(2n, \mathbb{R}). \end{aligned}$$

(ii) Similar. \square

Proposition 4.2. *The left $\mathrm{Sp}(2n, \mathbb{R})$ -action and right $\mathrm{O}(2n)$ -action are free and proper.*

Proof. We prove the result for the $\mathrm{Sp}(2n, \mathbb{R})$ -action only; the $\mathrm{O}(2n)$ case is similar. Freeness is trivial. To prove properness: suppose we have two convergent sequences (E_k) and $(S_k \cdot E_k)$ in $\mathrm{GL}(2n, \mathbb{R})$ with $E_k \rightarrow E$ and $S_k E_k \rightarrow E$. Then $S_k = S_k E_k E_k^{-1} \rightarrow E E^{-1} = I$. \square

Remark. For any $E \in M_{2n}(\mathbb{R})$, it remains true that $E \cdot O(2n) \subset j_{\text{Sp}}^{-1}(j_{\text{Sp}}(E))$ and $\text{Sp}(2n, \mathbb{R}) \cdot E \subset j_{\text{O}}^{-1}(j_{\text{O}}(E))$. However restriction to $\text{GL}(2n, \mathbb{R}) \subset M_{2n}(\mathbb{R})$ is necessary to get equality in the second inclusion. For example

$$j_{\text{O}}^{-1}(0) = \{E \in M_{2n}(\mathbb{R}) \mid \text{span } E \text{ is an isotropic subspace of } \mathbb{R}^{2n}\},$$

and this is not simply an orbit of $\text{Sp}(2n, \mathbb{R})$ (since for example the $\text{Sp}(2n, \mathbb{R})$ -action preserves the dimension of $\text{span } E$). It is however possible to prove that $E \cdot O(2n) = j_{\text{Sp}}^{-1}(j_{\text{Sp}}(E))$ for all $E \in M_{2n}(\mathbb{R})$.

With the restriction from $M_{2n}(\mathbb{R})$ to $\text{GL}(2n, \mathbb{R})$, the momentum map j_{Sp} acquires a nice interpretation. As before, let E_a denote the a th column of E , so $\{E_1, \dots, E_{2n}\}$ defines a basis of \mathbb{R}^{2n} . Let $g(E)$ denote a metric on \mathbb{R}^{2n} with respect to which the basis $\{E_a\}$ is orthonormal. Then any basis related to E by the right $O(2n)$ -action defines the same metric, i.e.,

$$g(E \cdot O) = g(E) \quad \text{for all } O \in O(2n),$$

and in fact the set of metrics on \mathbb{R}^{2n} is in 1-1 correspondence with the $O(2n)$ -orbits of $\text{GL}(2n, \mathbb{R})$. We then have the following result:

Proposition 4.3. *The metric $g(E)$ and canonical symplectic form $\omega_{\mathbb{R}^{2n}}$ are related by*

$$\omega_{\mathbb{R}^{2n}}(u, v) = g(E)(j_{\text{Sp}}(E)u, v).$$

Proof. If $\{s_a\}$ is the standard basis of \mathbb{R}^{2n} (so $E_a = \sum_{b=1}^{2n} E_{ba}s_b$), the definition of $g(E)$ implies that

$$g(E)(s_a, s_b) = [(EE^\top)^{-1}]_{ab}.$$

From this, it follows that

$$\begin{aligned} g(E)(j_{\text{Sp}}(E)u, v) &= (j_{\text{Sp}}(E)u)^\top (EE^\top)^{-1}v = u^\top (-EE^\top \mathbb{J})^\top (EE^\top)^{-1}v \\ &= u^\top \mathbb{J}v = \omega_{\mathbb{R}^{2n}}(u, v). \end{aligned}$$

□

Inverting this identity to give

$$g(E)(u, v) = \omega_{\mathbb{R}^{2n}}((j_{\text{Sp}}(E))^{-1}u, v),$$

we see that $j_{\text{Sp}}(E)$ defines the metric $g(E)$, and so essentially corresponds to it. In the next section we will consider this correspondence in the particular case that the ordered basis (E_a) forms a *symplectic* frame of $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.

5 Reduction

Following Ohsawa [Ohs15], we consider in this section the Marsden-Weinstein quotient for the right $O(2n)$ -action at a particular value of momentum. Using the correspondence between reduced spaces and coadjoint orbits provided by the dual pair structure, we will give a geometrical characterisation of this space. Using this characterisation, we will explain how to naturally introduce a global coordinate chart on the reduced space, reproducing Ohsawa's description of the space as the Siegel upper half plane.

5.1 Reduction through a general point

It will be more convenient to work with the Lie-algebra valued momentum maps j_O and j_{Sp} . For $\xi \in \mathfrak{g}$, now let G_ξ denote the *adjoint* stabiliser of ξ , and $\mathcal{O}_\xi \subset \mathfrak{g}$ the *adjoint* orbit through ξ . Writing $N = \text{GL}(2n, \mathbb{R})$, the reduced space through $E \in \text{GL}(2n, \mathbb{R})$ is

$$N_{j_O(E)} = \frac{j_O^{-1}(j_O(E))}{\text{O}(2n)_{j_O(E)}}.$$

Using the trace form to transfer the Poisson structure from $\mathfrak{sp}(2n, \mathbb{R})^*$ to $\mathfrak{sp}(2n, \mathbb{R})$, Proposition 3.5 tells us there exists a $\text{Sp}(2n, \mathbb{R})$ -equivariant symplectomorphism

$$\check{j}_{Sp} : N_{j_O(E)} \rightarrow \mathcal{O}_{j_{Sp}(E)}.$$

5.2 Reduction through the identity

We consider the reduced space for the left $\text{O}(2n)$ -action through $I = I_{2n} \in \text{GL}(2n, \mathbb{R})$. From Proposition 2.2(ii) we see that the level set of j_O through I is the set

$$j_O^{-1}(j_O(I)) = \{E \in \text{GL}(2n, \mathbb{R}) \mid -E^T \mathbb{J} E = -\mathbb{J}\} = \text{Sp}(2n, \mathbb{R}),$$

which corresponds to the set of *symplectic* frames of $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. Meanwhile,

$$\text{O}(2n)_{j_O(I)} = \{O \in \text{O}(2n) \mid \text{Ad}_O(-\mathbb{J}) = -\mathbb{J}\} = \text{Sp}(2n, \mathbb{R}) \cap \text{O}(2n) \simeq \text{U}(n),$$

and we see that the reduced space $N_{j_O(I)}$ is isomorphic to $\text{Sp}(2n, \mathbb{R})/\text{U}(n)$, as originally obtained by Ohsawa [Ohs15].

5.3 Geometric interpretation of the adjoint orbit $\mathcal{O}_{j_{Sp}(I)}$

As explained above, there is a $\text{Sp}(2n, \mathbb{R})$ -equivariant map

$$\check{j}_{Sp} : N_{j_O(I)} \rightarrow \mathcal{O}_{j_{Sp}(I)}.$$

We now give an alternative geometric characterisation of $\mathcal{O}_{j_{Sp}(I)}$ in terms of complex structures on \mathbb{R}^{2n} .

Definition 5.1. Let $j : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a complex structure on \mathbb{R}^{2n} , i.e., an \mathbb{R} -linear map satisfying $j^2 = -I$. The map j is called $\omega_{\mathbb{R}^{2n}}$ -*compatible* if it satisfies the two properties

- (i) j is symplectic (i.e., $j \in \text{Sp}(2n, \mathbb{R})$), and
- (ii) the bilinear form $g_j(u, v) := \omega_{\mathbb{R}^{2n}}(u, jv)$ is positive-definite.

Denote the space of $\omega_{\mathbb{R}^{2n}}$ -compatible complex structures on \mathbb{R}^{2n} by $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.

Proposition 5.2. *The adjoint orbit $\mathcal{O}_{j_{Sp}(I)} \subset \mathfrak{sp}(2n, \mathbb{R})$ is precisely the set $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.*

Proof. An arbitrary element of $\zeta \in \mathcal{O}_{j_{Sp}(I)}$ is of the form $\zeta = \text{Ad}_S[j_{Sp}(I)] = \text{Ad}_S[-\mathbb{J}]$. Since $(-\mathbb{J})^2 = -I$ we have $\zeta^2 = -I$, and since $-\mathbb{J} \in \text{Sp}(2n, \mathbb{R})$ we

have $\zeta = \text{Ad}_S(-\mathbb{J}) \in \text{Sp}(2n, \mathbb{R})$. Finally, writing $\zeta = -SS^\top \mathbb{J}$, we see that for $u \in \mathbb{R}^{2n} - \{0\}$,

$$g_\zeta(u, u) = \omega_{\mathbb{R}^{2n}}(u, \zeta u) = u^\top \mathbb{J}(-SS^\top \mathbb{J})u = u^\top (S^\top \mathbb{J})^\top (S^\top \mathbb{J})u > 0.$$

So we have that $\mathcal{O}_{\text{j}_{\text{Sp}}(I_{2n})} \subset \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.

Conversely, suppose $j \in \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. Then

$$g_j(ju, jv) = \omega_{\mathbb{R}^{2n}}(ju, j^2v) = \omega_{\mathbb{R}^{2n}}(u, jv) = g_j(u, v),$$

the second identity following since j is symplectic. Hence j is orthogonal with respect to the metric g_j . A standard theorem guarantees the existence of a real orthonormal basis $\{u_\alpha, v_\alpha \mid \alpha = 1, \dots, n\}$ of \mathbb{R}^{2n} such that $j(u_\alpha - iv_\alpha) = e^{i\theta_\alpha}(u_\alpha - iv_\alpha)$. The identity $j^2 = -I$ implies that $e^{i\theta_\alpha} = \pm i$ for all α , and by switching u_α and v_α if necessary we can choose $e^{i\theta_\alpha} = i$ for all α . So

$$ju_\alpha = v_\alpha, \quad jv_\alpha = -u_\alpha.$$

It can be verified that $(u_1, \dots, u_n, v_1, \dots, v_n)$ forms a symplectic basis for $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. For example

$$\omega_{\mathbb{R}^{2n}}(u_\alpha, v_\beta) = \omega_{\mathbb{R}^{2n}}(u_\alpha, ju_\beta) = g_j(u_\alpha, u_\beta) = \delta_{\alpha\beta}$$

by orthonormality. Taking $E = \begin{bmatrix} u_1 & \dots & u_n & v_1 & \dots & v_n \end{bmatrix} \in \text{Sp}(2n, \mathbb{R})$, the above relations can be written $jE = -E\mathbb{J}$, which implies

$$j = -E\mathbb{J}E^{-1} = \text{Ad}_E[-\mathbb{J}] = \text{Ad}_E[\text{j}_{\text{Sp}}(I)].$$

Thus $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \subset \mathcal{O}_{\text{j}_{\text{Sp}}(I)}$. □

Adapting diagram (3) to our case, we obtain

$$\begin{array}{ccc} & \text{j}_O^{-1}(\text{j}_O(I)) = \text{Sp}(2n, \mathbb{R}) & \\ \sigma_O \swarrow & & \searrow \text{j}_{\text{Sp}} \\ N_{\text{j}_O(I)} & \xrightarrow{\check{\text{j}}_{\text{Sp}}} & \mathcal{O}_{\text{j}_{\text{Sp}}(I)} = \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \end{array} \quad (4)$$

with $\check{\text{j}}_{\text{Sp}}$ an $\text{Sp}(2n, \mathbb{R})$ -equivariant symplectomorphism. An equivalent diagram was recently obtained independently by Ohsawa and Tronci [OT17].

6 The upper half plane coordinates

As demonstrated in the previous section, the adjoint orbit $\mathcal{O}_{\text{j}_{\text{Sp}}(I)}$ can be naturally viewed as the space of $\omega_{\mathbb{R}^{2n}}$ -compatible complex structures $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. It is a standard result that $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ has another natural interpretation, that of the negative Lagrangian Grassmannian $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$. In this section, we outline this correspondence, and explain how it can be used to introduce a coordinate system on $\mathcal{O}_{\text{j}_{\text{Sp}}(I)}$.

6.1 The negative Lagrangian Grassmannian

Consider the complexification $(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^{2n}$ of \mathbb{R}^{2n} . Real-linear functions on \mathbb{R}^{2n} extend to complex-linear functions on \mathbb{C}^{2n} in the obvious way, and we will generally use the same symbol to denote both a function and its complex extension.

Define a sesquilinear form $s : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}$ by

$$s(w, z) := -i\omega_{\mathbb{R}^{2n}}(w, \bar{z}).$$

Definition 6.1. The *negative Lagrangian Grassmannian* $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ is the set of (complex) Lagrangian subspaces of \mathbb{C}^{2n} on which s restricts to a negative-definite form,

$$\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}) := \{\Gamma \text{ a subspace of } \mathbb{C}^{2n} \mid \Gamma \text{ is Lagrangian, } s|_{\Gamma} < 0\}.$$

Proposition 6.2 ([Sat80, Chapter II, Lemma 7.1]). *There is a bijection between $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ and $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$.*

Proof. (Sketch) For any $j \in \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$, j symplectic implies that it is diagonalisable, and $j^2 = -I$ implies that its eigenvalues are $\pm i$. The $-i$ -eigenspace is simply

$$\Gamma^j = \frac{1}{2}(I + ij)\mathbb{R}^{2n} \subset \mathbb{C}^{2n}.$$

It is straightforward to verify that $\Gamma^j \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$, and so we obtain a map

$$\Gamma^{\cdot} : \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \rightarrow \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}).$$

Conversely, suppose $\Gamma \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$. For any $w \in \Gamma - \{0\}$, the negative-definite condition implies that $-i\omega(w, \bar{w}) < 0$. Since Γ is Lagrangian, it follows that $\bar{w} \notin \Gamma$. Hence $\Gamma \cap \bar{\Gamma} = \{0\}$, and $\mathbb{C}^{2n} = \Gamma \oplus \bar{\Gamma}$. Define the complex-linear map $j_{\Gamma} : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ by

$$j_{\Gamma}(w) = \begin{cases} -iw & w \in \Gamma \\ iw & w \in \bar{\Gamma} \end{cases}.$$

Since it commutes with complex commutation, j_{Γ} restricts to a real-linear map $j_{\Gamma} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. It is straightforward to verify that $j_{\Gamma} \in \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$, and so we obtain a map

$$j : \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}) \rightarrow \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}).$$

It may be checked that $\Gamma^{\cdot} : \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \rightarrow \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ and $j : \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}) \rightarrow \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ are inverses of each other. \square

Following Proposition 6.2, we may extend the commutative diagram (4) to

$$\begin{array}{ccccc} & & \mathfrak{j}_{\mathbb{O}}^{-1}(\mathfrak{j}_{\mathbb{O}}(I)) = \mathrm{Sp}(2n, \mathbb{R}) & & \\ & \swarrow \sigma_{\mathbb{O}} & & \searrow \Gamma^{\mathfrak{j}_{\mathrm{Sp}}(\cdot)} & \\ & N_{\mathfrak{j}_{\mathbb{O}}(I)} & \xrightarrow{\mathfrak{j}_{\mathrm{Sp}}} & \mathcal{O}_{\mathfrak{j}_{\mathrm{Sp}}(I)} = \mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) & \xrightarrow{\Gamma^{\cdot}} & \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}) \end{array} \quad (5)$$

We note that for $S \in \mathrm{Sp}(2n, \mathbb{R})$,

$$\Gamma^{\mathrm{Ad}_S j} = \frac{1}{2}(I + i \mathrm{Ad}_S j) \mathbb{R}^{2n} = \frac{1}{2}(I + i \mathrm{Ad}_S j) S \mathbb{R}^{2n} = S \left(\frac{1}{2}(I + ij) \mathbb{R}^{2n} \right) = S \Gamma^j,$$

where we have used the invertibility of S to write $\mathbb{R}^{2n} = S \mathbb{R}^{2n}$ in the second equality. Hence Γ is $\mathrm{Sp}(2n, \mathbb{R})$ -equivariant with respect to the natural left action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$.

6.2 Coordinates on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$

In order to introduce coordinates on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$, we first introduce the two complex Lagrangian subspaces of \mathbb{C}^{2n}

$$\Gamma_1 = \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_n\} \quad \text{and} \quad \Gamma_2 = \mathrm{span}_{\mathbb{C}}\{e_{n+1}, \dots, e_{2n}\},$$

where $\{e_1, \dots, e_{2n}\}$ is the canonical basis of \mathbb{R}^{2n} . Note that $\mathbb{C}^{2n} = \Gamma_1 \oplus \Gamma_2$, and denote the direct sum projections by $p_i : \mathbb{C}^{2n} \rightarrow \Gamma_i$. We use the following lemma.

Lemma 6.3. *Let $\Gamma \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$. Then the projection $p_i : \mathbb{C}^{2n} \rightarrow \Gamma_i$ restricts to a bijection on Γ , for $i = 1, 2$.*

Proof. Since all three are Lagrangian, $\Gamma, \Gamma_1, \Gamma_2$ all have complex dimension n . So we just need to prove that $p_i|_{\Gamma} : \Gamma \rightarrow \Gamma_i$ is injective.

Suppose $p_1(w) = 0$ for $w \in \Gamma$. Then $w \in \Gamma_2$, and so $\bar{w} \in \bar{\Gamma}_2 = \Gamma_2$. Since Γ_2 is Lagrangian, we have that $-i\omega_{\mathbb{R}^{2n}}(w, \bar{w}) = 0$, i.e., $s(w, w) = 0$. But since $s|_{\Gamma} < 0$, this implies that $w = 0$.

The same argument shows that $p_2|_{\Gamma} : \Gamma \rightarrow \Gamma_2$ is a bijection. \square

In light of Lemma 6.3, for any $\Gamma \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ we may define a map $\widehat{W}_{\Gamma} : \Gamma_1 \rightarrow \Gamma_2$ by

$$\widehat{W}_{\Gamma} = (p_2|_{\Gamma}) \circ (p_1|_{\Gamma})^{-1}.$$

Then for any $w \in \Gamma$,

$$w = (p_1 + p_2)|_{\Gamma}(w) = (I + \widehat{W}_{\Gamma})((p_1|_{\Gamma})(w)),$$

demonstrating that

$$\Gamma = (I + \widehat{W}_{\Gamma})\Gamma_1. \tag{6}$$

It follows from identity (6) that the map $\Gamma \mapsto \widehat{W}_{\Gamma}$ is injective. We now use this representation of $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ as linear maps to introduce coordinates on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$. The following lemma is straightforward to verify.

Lemma 6.4 ([Sat80, Chapter II, Lemma 7.2]). *Let $\widehat{W} : \Gamma_1 \rightarrow \Gamma_2$ be a linear map, and define the matrix $W \in M_n(\mathbb{C})$ by*

$$\widehat{W}(e_{\alpha}) = \sum_{\beta=1}^n W_{\beta\alpha} e_{n+\beta} \quad \text{for } \alpha = 1, \dots, n.$$

Then

$$(I + \widehat{W})\Gamma_1 \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}) \iff W^{\top} = W \text{ and } \mathrm{Im} W > 0.$$

The set

$$\Sigma_n = \{W \in M_n(\mathbb{C}) \mid W^{\top} = W, \mathrm{Im} W > 0\}$$

arising in Lemma 6.4 is traditionally referred to as the *Siegel upper half plane*. It has a well-known symplectic structure. In subsection 6.5 we show this structure is the pushforward of the usual Kostant-Kirillov-Souriau symplectic structure on $\mathcal{O}_{\mathrm{isp}}(I)$ under the bijection Γ of diagram (5).

6.3 The action of $\mathrm{Sp}(2n, \mathbb{R})$ in upper half plane coordinates

There is a natural action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$, given by $(S, \Gamma) \mapsto S\Gamma$. We wish to express this action in terms of upper half plane coordinates, i.e., to express $W_{S\Gamma}$ in terms of S and W_Γ .

Since $\Gamma_1 = \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_n\}$, we see from equation (6) that a basis for $\Gamma \in \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ is given by

$$e'_\alpha := (I + \widehat{W}_\Gamma)e_\alpha = e_\alpha + \sum_{\beta=1}^n (W_\Gamma)_{\beta\alpha} e_{n+\beta}, \quad \alpha = 1, \dots, n.$$

It follows that $S\Gamma$ has basis

$$e''_\alpha = Se'_\alpha = Se_\alpha + \sum_{\beta=1}^n (W_\Gamma)_{\beta\alpha} Se_{n+\beta}, \quad \alpha = 1, \dots, n.$$

Writing $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, this becomes

$$e''_\alpha = \sum_{\gamma=1}^n [(A + BW_\Gamma)_{\gamma\alpha} e_\gamma + (C + DW_\Gamma)_{\gamma\alpha} e_{n+\gamma}], \quad \alpha = 1, \dots, n.$$

Now recall that the coordinates $W_{S\Gamma}$ of $S\Gamma$ are simply the matrix elements of $\widehat{W}_{S\Gamma} = (p_2|_{S\Gamma}) \circ (p_1|_{S\Gamma})^{-1} : \Gamma_1 \rightarrow \Gamma_2$. We have

$$\begin{aligned} p_1(e''_\alpha) &= \sum_{\gamma=1}^n (A + BW_\Gamma)_{\gamma\alpha} e_\gamma \\ \implies (p_1|_{S\Gamma})^{-1}(e_\delta) &= \sum_{\alpha=1}^n [(A + BW_\Gamma)^{-1}]_{\alpha\delta} e''_\alpha \\ \implies \widehat{W}_{S\Gamma}(e_\delta) &= (p_2|_{S\Gamma})((p_1|_{S\Gamma})^{-1}(e_\delta)) = \sum_{\alpha=1}^n [(A + BW_\Gamma)^{-1}]_{\alpha\delta} (p_2(e''_\alpha)) \\ &= \sum_{\gamma=1}^n [(C + DW_\Gamma)(A + BW_\Gamma)^{-1}]_{\gamma\delta} e_{n+\gamma}. \end{aligned}$$

It follows that for $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$,

$$W_{S\Gamma} = (C + DW_\Gamma)(A + BW_\Gamma)^{-1}. \quad (7)$$

6.4 The projection in upper half plane coordinates

Referring again to diagram (5), we have demonstrated the existence of an $\mathrm{Sp}(2n, \mathbb{R})$ -equivariant map

$$\Gamma^{\mathrm{j}_{\mathrm{Sp}}(\cdot)} : \mathrm{j}_O^{-1}(\mathrm{j}_O(I)) \rightarrow \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}}).$$

The set $\mathrm{j}_O^{-1}(\mathrm{j}_O(I)) \subset \mathrm{GL}(2n, \mathbb{R})$ is simply $\mathrm{Sp}(2n, \mathbb{R})$, and we have shown that the Siegel upper half plane provides a global coordinate chart on $\mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$. Using the results of the previous section, we can derive an expression for the projection $\Gamma^{\mathrm{j}_{\mathrm{Sp}}(\cdot)}$ in terms of the coordinates on its domain and range manifolds.

First note that $\Gamma^{\text{jSp}(I)} = \frac{1}{2}(I + i\text{jSp}(I))\mathbb{R}^{2n} = \frac{1}{2}(I - i\mathbb{J})\mathbb{R}^{2n}$ has a basis

$$\{e_\alpha + ie_{n+\alpha} \mid \alpha = 1, \dots, n\}.$$

It follows easily from equation (6) that

$$W_{\Gamma^{\text{jSp}(I)}} = iI \in M_n(\mathbb{C}).$$

By equivariance of $\Gamma^{\text{jSp}(\cdot)}$, we have that $\Gamma^{\text{jSp}(S)} = S\Gamma^{\text{jSp}(I)}$ for any $S \in \text{jSp}^{-1}(\text{jSp}(I))$, and so $W_{\Gamma^{\text{jSp}(S)}} = W_{S\Gamma^{\text{jSp}(I)}}$. Writing $S = \begin{bmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{bmatrix}$ for the coordinates on $\text{jSp}^{-1}(\text{jSp}(I))$, and using equation (7), this becomes

$$W_{\Gamma^{\text{jSp}(S)}} = (P_1 + P_2 W_{\Gamma^{\text{jSp}(I)}})(Q_1 + Q_2 W_{\Gamma^{\text{jSp}(I)}})^{-1} = (P_1 + iP_2)(Q_1 + iQ_2)^{-1}.$$

In summary, the projection written in (global) coordinates is

$$\Gamma^{\text{jSp}(\cdot)} : \begin{bmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{bmatrix} \mapsto (P_1 + iP_2)(Q_1 + iQ_2)^{-1}. \quad (8)$$

6.5 Upper half plane coordinates on $\mathcal{O}_{\text{jSp}(I)}$

Using $\Gamma : \mathcal{O}_{\text{jO}(I)} \rightarrow \mathcal{G}_-(\mathbb{C}^{2n}, \omega_{\mathbb{R}^{2n}})$ to pull coordinates back to $\mathcal{O}_{\text{jO}(I)}$, we may instead interpret (8) as the coordinate expression for the momentum map $\text{jSp} : \text{jO}^{-1}(\text{jO}(I)) \rightarrow \mathcal{O}_{\text{jSp}(I)}$

$$\text{jSp} : \begin{bmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{bmatrix} \mapsto (P_1 + iP_2)(Q_1 + iQ_2)^{-1}. \quad (9)$$

This is in agreement with [Ohs15, Equation (21)], which was obtained by a method involving the Iwasawa decomposition of $\text{Sp}(2n, \mathbb{R})$.

For $\eta \in \mathcal{O}_{\text{jSp}(I)}$, let $W_\eta \in \Sigma_n$ denote the corresponding coordinate. Then equation (7) becomes

$$W_{\text{Ad}_S \eta} = (C + DW_\eta)(A + BW_\eta)^{-1}. \quad (10)$$

It was shown by Siegel (see for example [Sie73, Chapter 6, Section 3]) that Σ_n is a Kähler manifold, with Hermitian form $H_{\Sigma_n} = \text{Tr}(\mathcal{B}^{-1}d\bar{W} \otimes \mathcal{B}^{-1}dW)$ and symplectic form $\Omega_{\Sigma_n} = \text{Im} H^{\Sigma_n}$, i.e.,

$$\Omega_{\Sigma_n} = \text{Tr}(\mathcal{B}^{-1}d\mathcal{A} \otimes \mathcal{B}^{-1}d\mathcal{B} - \mathcal{B}^{-1}d\mathcal{B} \otimes \mathcal{B}^{-1}d\mathcal{A}).$$

(We remind that $\mathcal{A} := \text{Re} W$, $\mathcal{B} := \text{Im} W$.) Siegel also proves that the forms H_{Σ_n} , and so Ω_{Σ_n} , are invariant under pullback by the $\text{Sp}(2n, \mathbb{R})$ -action $W \mapsto (C + DW)(A + BW)^{-1}$.

Correspondingly, the standard Kostant-Kirillov-Souriau symplectic form $\Omega_{\mathcal{O}}^+$ on the adjoint orbit $\mathcal{O}_{\text{jSp}(I)} = \mathcal{O}_{-\mathbb{J}}$, given by

$$(\Omega_{\mathcal{O}}^+)_\eta(\text{ad}_\xi \eta, \text{ad}_\zeta \eta) = \langle \eta, [\xi, \zeta] \rangle,$$

for $\eta \in \mathcal{O}_{-\mathbb{J}}$, $\xi, \zeta \in \mathfrak{sp}(2n, \mathbb{R})$, is invariant under the adjoint $\text{Sp}(2n, \mathbb{R})$ -action.

Hence, if we can show that Ω_{Σ_n} and $\Omega_{\mathcal{O}}^+$ are proportional at $-\mathbb{J}$ (corresponding to coordinate $W = iI$), then they are proportional on the entire adjoint orbit $\mathcal{O}_{-\mathbb{J}}$.

To see this is the case, let $\xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \in \mathfrak{sp}(2n, \mathbb{R})$. From (10) we deduce

$$\left. \frac{d}{dt} \right|_{t=0} W_{\text{Ad}_{\exp t\xi}(-\mathbb{J})} = (\xi_{21} + i\xi_{22}) - i(\xi_{11} + i\xi_{12}),$$

implying that

$$\text{ad}_\xi(-\mathbb{J}) = (\xi_{12} + \xi_{21}) \left. \frac{\partial}{\partial \mathcal{A}} \right|_{iI} + (\xi_{22} - \xi_{11}) \left. \frac{\partial}{\partial \mathcal{B}} \right|_{iI}. \quad (11)$$

Note that $\xi^\top \mathbb{J} + \mathbb{J} \xi = 0 \implies \xi_{12}^\top = \xi_{12}$, $\xi_{21}^\top = \xi_{21}$, $\xi_{22}^\top = -\xi_{11}$, and so both coefficient matrices in (11) are symmetric.

We have that

$$\begin{aligned} (\Omega_{\mathcal{O}}^+)_{-\mathbb{J}}(\text{ad}_\xi(-\mathbb{J}), \text{ad}_\zeta(-\mathbb{J})) &= \langle\langle -\mathbb{J}, [\xi, \zeta] \rangle\rangle \\ &= \frac{1}{2} \text{Tr}(-(\xi_{21}\zeta_{11} + \xi_{22}\zeta_{21}) + (\xi_{11}\zeta_{12} + \xi_{12}\zeta_{22}) - (\xi \leftrightarrow \zeta)). \end{aligned} \quad (12)$$

Meanwhile, at $W = iI$ the symplectic form on Σ_n simplifies to

$$(\Omega_{\Sigma_n})_{iI} = \text{Tr}(d\mathcal{A} \otimes d\mathcal{B} - d\mathcal{B} \otimes d\mathcal{A})_{iI},$$

and so (11) gives

$$(\Omega_{\Sigma_n})_{iI}(\text{ad}_\xi(-\mathbb{J}), \text{ad}_\zeta(-\mathbb{J})) = \text{Tr}((\xi_{12} + \xi_{21})(\zeta_{22} - \zeta_{11}) - (\xi_{22} - \xi_{11})(\zeta_{12} + \zeta_{21})).$$

Comparison with (12) shows that $(\Omega_{\mathcal{O}}^+)_{-\mathbb{J}} = \frac{1}{2}(\Omega_{\Sigma_n})_{iI}$, and therefore

$$\Omega_{\mathcal{O}}^+ = \frac{1}{2}\Omega_{\Sigma_n}.$$

7 Dynamics

In this section we finally come to the main point of the previous constructions, namely to describe the dynamics of semiclassical Gaussian wave packets in terms of Hamiltonian dynamics on the frame bundle of $T^*\mathbb{R}^n$, and its symplectic reduction.

7.1 The Gaussian wave packet ansatz

Consider the time-dependent Schrödinger equation on \mathbb{R}^n ,

$$i\hbar \frac{\partial \psi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \Delta \psi(x, t) + V(x) \psi(x, t), \quad (13)$$

where the potential $V(x)$ is at most *quadratic* in x . Following [Lub08], we recall here the Gaussian wave packet ansatz, in both the Heller [Hel76] and Hagedorn [Hag80] parametrisations.

In Heller's parametrisation, the wavefunction is written

$$\psi(x, t) = \left(\frac{\det \mathcal{B}(t)}{(\pi \hbar)^n} \right)^{\frac{1}{4}} \exp \left\{ \frac{i}{\hbar} \left[\frac{1}{2} (x - q(t))^\top (\mathcal{A}(t) + i\mathcal{B}(t)) (x - q(t)) + p(t)^\top (x - q(t)) + \phi(t) \right] \right\}, \quad (14)$$

with $(q, p) \in T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, ϕ a real phase, and $\mathcal{A} + i\mathcal{B} \in \Sigma_n$ with \mathcal{A}, \mathcal{B} real. Substitution of the ansatz into equation (13) shows that it produces a solution provided the following equations are satisfied

$$\begin{aligned} \dot{q} &= \frac{p}{m}, & \dot{p} &= -D_q V, \\ \dot{\mathcal{A}} &= -\frac{1}{m} (\mathcal{A}^2 - \mathcal{B}^2) - D_q^2 V, & \dot{\mathcal{B}} &= -\frac{1}{m} (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}), \\ \dot{\phi} &= \frac{p^2}{2m} - V(q) - \frac{\hbar}{2m} \text{Tr } \mathcal{B}. \end{aligned} \quad (15)$$

Here, D^2V denotes the Hessian of V .

In Hagedorn's parametrisation, the wavefunction is

$$\psi(x, t) = \frac{1}{(\pi \hbar)^{\frac{n}{4}}} \frac{1}{(\det Q(t))^{\frac{1}{2}}} \exp \left\{ \frac{i}{\hbar} \left[\frac{1}{2} (x - q(t))^\top P(t) Q(t)^{-1} (x - q(t)) + p(t)^\top (x - q(t)) + S(t) \right] \right\}, \quad (16)$$

for some appropriate choice of the square root in the denominator, with $(q, p) \in \mathbb{R}^{2n}$, S a real phase, and $Q, P \in M_n(\mathbb{C})$ satisfying

$$Q^\top P - P^\top Q = 0 \quad \text{and} \quad Q^\dagger P - P^\dagger Q = 2iI_n.$$

Substitution into equation (13) yields

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -D_q V, \quad \dot{Q} = \frac{P}{m}, \quad \dot{P} = -D_q^2 V Q, \quad \dot{S} = \frac{p^2}{2m} - V(q). \quad (17)$$

The two parametrisations (14) and (16) are related by

$$W = PQ^{-1}, \quad S = \phi - \frac{\hbar}{2} \arg(\det Q). \quad (18)$$

7.2 Description as Hamiltonian dynamics on the frame bundle

Given a symplectic manifold (M, ω) , recall that the a *frame* at $z \in M$ is an ordered basis of $T_z M$, and the *frame bundle* $\mathcal{F}(M)$ consists of all such ordered bases as z ranges over M . $\mathcal{F}(M)$ is naturally a right principal $\text{GL}(2n, \mathbb{R})$ -bundle, with group action

$$(v_1, \dots, v_{2n}) \cdot E = \left(\sum_{a=1}^{2n} v_a E_{a1}, \sum_{a=1}^{2n} v_a E_{a2}, \dots, \sum_{a=1}^{2n} v_a E_{a,2n} \right). \quad (19)$$

We say that a frame (v_a) at $z \in M$ is symplectic if

$$\omega_z(v_a, v_b) = \mathbb{J}_{ab} \quad a, b = 1, \dots, 2n,$$

and a local section $\mathbf{s} : U \subset M \rightarrow \mathcal{F}(M)$ of the frame bundle is symplectic if $\mathbf{s}(z)$ is a symplectic frame for each $z \in U$.

Consider now the case when $M = T^*\mathbb{R}^n$ and $\omega = \sum_{\alpha=1}^n dq^\alpha \wedge dp_\alpha$ is the usual cotangent bundle symplectic form. We will write the natural coordinates on $T^*\mathbb{R}^n$ as either (z_1, \dots, z_{2n}) or $(q^1, \dots, q^n, p_1, \dots, p_n)$, depending on situation. The bundle $\mathcal{F}(T^*\mathbb{R}^n)$ has a *global* symplectic frame $\mathbf{s} : T^*\mathbb{R}^n \rightarrow \mathcal{F}(T^*\mathbb{R}^n)$, given by

$$\mathbf{s}(z) := \left(\frac{\partial}{\partial q^1} \Big|_z, \dots, \frac{\partial}{\partial q^n} \Big|_z, \frac{\partial}{\partial p_1} \Big|_z, \dots, \frac{\partial}{\partial p_n} \Big|_z \right) = \left(\frac{\partial}{\partial z_1} \Big|_z, \dots, \frac{\partial}{\partial z_{2n}} \Big|_z \right),$$

which induces a global trivialisation $\Lambda : T^*\mathbb{R}^n \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow \mathcal{F}(T^*\mathbb{R}^n)$ of the frame bundle, given by

$$\Lambda(z, E) = \mathbf{s}(z) \cdot E = \left(\sum_{a=1}^{2n} E_{a1} \frac{\partial}{\partial z_a} \Big|_z, \dots, \sum_{a=1}^{2n} E_{a,2n} \frac{\partial}{\partial z_a} \Big|_z \right).$$

We define the \hbar -diagonal lifted symplectic form Ω^h on $T^*\mathbb{R}^n \times \mathrm{GL}(2n, \mathbb{R})$ by

$$\Omega_{(z,E)}^h((v_z, V_E), (w_z, W_E)) := \omega_z(v_z, w_z) + \frac{\hbar}{2} \Omega_E(V_E, W_E). \quad (20)$$

By analogy with [Ohs15], for any Hamiltonian $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$ we define the lifted Hamiltonian on the frame bundle $H^h : T^*\mathbb{R} \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow \mathbb{R}$ to be

$$H^h(z, E) := H(z) + \frac{\hbar}{4} \mathrm{Tr}(E^\top D_z^2 H E), \quad (21)$$

where $D_z^2 H$ denotes the Hessian of H evaluated at z .

We can give a more geometric interpretation to (21) as follows: for any $E \in \mathrm{GL}(2n, \mathbb{R})$, let $g(E)$ denote the metric on $T^*\mathbb{R}^n$ with respect to which the global frame $\Lambda(z, E) = \mathbf{s}(z) \cdot E$ is orthonormal, i.e., the vector fields

$$e_a(z) = \sum_{b=1}^{2n} E_{ba} \frac{\partial}{\partial z_b}, \quad a = 1, \dots, 2n,$$

form an orthonormal basis at each point. Explicitly

$$g(E) \left(\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_b} \right) = [(EE^\top)^{-1}]_{ab}. \quad (22)$$

Then the Laplacian $\Delta_{g(E)} : C^\infty(T^*\mathbb{R}^n) \rightarrow C^\infty(T^*\mathbb{R}^n)$ associated with the metric $g(E)$ is given for $H \in C^\infty(T^*\mathbb{R}^n)$ by

$$\Delta_{g(E)} H = \sum_{a=1}^{2n} e_a(e_a H) = \sum_{a,b,c=1}^{2n} E_{ba} E_{ca} \frac{\partial^2 H}{\partial z_b \partial z_c} = \mathrm{Tr}(E^\top D^2 H E),$$

and so (21) can be alternatively written as

$$H^h(z, E) = H(z) + \frac{\hbar}{4} (\Delta_{g(E)} H)(z). \quad (23)$$

To find the Hamiltonian vector field corresponding to H^{\hbar} , note that

$$d_{(z,E)}H^{\hbar} = (d_1)_z \left(H + \frac{\hbar}{4} \Delta_{g(E)} H \right) + (d_2)_E \left(\frac{\hbar}{4} \Delta_{g(\cdot)} H(z) \right),$$

where d_1, d_2 denote the exterior derivatives in $T^*\mathbb{R}^n$ and $\text{GL}(2n, \mathbb{R})$ respectively. The first term can be written

$$i_{X_{H + \frac{\hbar}{4} \Delta_{g(E)} H}} \omega.$$

The second term satisfies

$$\begin{aligned} (d_2)_E \left(\frac{\hbar}{4} \Delta_{g(\cdot)} H(z) \right) (V_E) &= \frac{\hbar}{4} \text{Tr} (V^\top D_z^2 H E + E^\top D_z^2 H V) \\ &= \frac{\hbar}{2} \text{Tr} (E^\top D_z^2 H V) \\ &= \frac{\hbar}{2} \Omega_E ((\mathbb{J} D_z^2 H) \cdot E, V_E) \\ &= \frac{\hbar}{2} (i_{(\mathbb{J} D_z^2 H) \cdot E} \Omega_E) (V_E). \end{aligned}$$

Here we have used the identity $\zeta \cdot E = (\zeta E)_E$. Overall

$$d_{(z,E)}H^{\hbar} = i_{(X_{H + \frac{\hbar}{4} \Delta_{g(E)} H}(z), (\mathbb{J} D_z^2 H) \cdot E)} \Omega_{(z,E)}^{\hbar},$$

i.e.,

$$X_{H^{\hbar}}(z, E) = \left(X_{H + \frac{\hbar}{4} \Delta_{g(E)} H}(z), (\mathbb{J} D_z^2 H) \cdot E \right). \quad (24)$$

In the situation where $H : T^*\mathbb{R}^n \rightarrow \mathbb{R}$ is at most quadratic in the coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$, the Hessian $D_z^2 H$, and hence $\Delta_{g(E)} H$, will be constant, and the Hamiltonian vector field $X_{H^{\hbar}}$ simplifies to

$$X_{H^{\hbar}}(z, E) = \left(X_H(z), (\mathbb{J} D_z^2 H) \cdot E \right) = (\mathbb{J} D_z H, (\mathbb{J} D_z^2 H) \cdot E)$$

In this case, motion on the frame bundle consists of classical motion on the base space $T^*\mathbb{R}^n$, and the corresponding induced linearised motion on the frame. This picture of semiclassical wave mechanics has much in common with the *nearby orbit approximation* discussed by Littlejohn [Lit86, Section 7]—semiclassical motion consists of the classical motion, plus motion of a frame moving along the classical trajectory, and describing to first order the classical flow relative to it.

In particular, taking the standard Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

and writing $E = \begin{bmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{bmatrix}$ yields the first four equations of the Hagedorn equations (17).

In the general case (24) where H is not quadratic, there is an additional correction term in the motion on the base space, leading to a deviation from strictly classical motion. This deviation has recently been proposed as a means of introducing quantum tunnelling into semiclassical quantum mechanics [OL13].

7.3 $O(2n)$ -invariance and conservation of symplectic frame

The right $GL(2n, \mathbb{R})$ -action (19) on the bundle $\mathcal{F}(T^*\mathbb{R}^n) \simeq T^*\mathbb{R}^n \times GL(2n, \mathbb{R})$ restricts to a right $O(2n)$ -action, and the lifted Hamiltonian H^{\hbar} is easily seen to be invariant under this restricted action. By Noether's theorem, the corresponding momentum map $j_O \circ \pi_2 : T^*\mathbb{R}^n \times GL(2n, \mathbb{R}) \rightarrow \mathfrak{o}(2n)$ is conserved under the dynamical evolution corresponding to H^{\hbar} , i.e.,

$$j_O(E) = -E^\top \mathbb{J} E$$

is conserved. In particular, the previously considered level set

$$j_O^{-1}(j_O(I)) = \{E \in GL(2n, \mathbb{R}) \mid -E^\top \mathbb{J} E = -\mathbb{J}\}$$

simply corresponds to the collection of *symplectic* frames at a point, and conservation of j_O says that such a frame remains symplectic under dynamical evolution.

7.4 Description in the reduced space

Since we have a Hamiltonian H^{\hbar} that is invariant under the right $O(2n)$ -action, the dynamics on $\mathcal{F}(T^*\mathbb{R}^n)$ will drop to the symplectic quotient. Taking the symplectic quotient at momentum $j_O(I) = -\mathbb{J}$, and using the notation of diagram (5), gives

$$\frac{(j_O \circ \pi_2)^{-1}(j_O(I))}{O(2n)_{j_O(I)}} = T^*\mathbb{R}^n \times \frac{j_O^{-1}(j_O(I))}{O(2n)_{j_O(I)}} = T^*\mathbb{R}^n \times N_{j_O(I)},$$

and we have the corresponding extension of diagram (4)

$$\begin{array}{ccc} & T^*\mathbb{R}^n \times j_O^{-1}(j_O(I)) & \\ \text{id} \times \sigma_O \swarrow & & \searrow \text{id} \times j_{\text{Sp}} \\ T^*\mathbb{R}^n \times N_{j_O(I)} & \xrightarrow{\text{id} \times \check{j}_{\text{Sp}}} & T^*\mathbb{R}^n \times \mathcal{O}_{j_{\text{Sp}}(I)}. \end{array}$$

The map $\text{id} \times \check{j}_{\text{Sp}}$ is a symplectomorphism, and so we can equivalently describe the reduced dynamics on $T^*\mathbb{R}^n \times \mathcal{O}_{j_{\text{Sp}}(I)}$. Rewriting equation (21) as

$$H^{\hbar}(z, E) = H(z) + \frac{\hbar}{4} \text{Tr}(-EE^\top \mathbb{J}^2 D_z^2 H) = H(z) + \frac{\hbar}{4} \text{Tr}(j_{\text{Sp}}(E) \mathbb{J} D_z^2 H)$$

we see that

$$H^{\hbar} = \check{H}^{\hbar} \circ (\text{id} \times j_{\text{Sp}}),$$

where $\check{H}^{\hbar} : T^*\mathbb{R}^n \times \mathcal{O}_{j_{\text{Sp}}(I)} \rightarrow \mathbb{R}$ is the reduced Hamiltonian

$$\check{H}^{\hbar}(z, \zeta) := H(z) + \frac{\hbar}{4} \text{Tr}(\zeta \mathbb{J} D_z^2 H). \quad (25)$$

In fact, from the expression (22) for the metric $g(E)$, we see that

$$g(E) \left(\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_b} \right) = [(j_{\text{Sp}}(E) \mathbb{J})^{-1}]_{ab},$$

and so the metric $g(E)$ drops to a metric $\check{g}(\zeta)$ on $T^*\mathbb{R}^n$, depending on $\zeta \in \mathfrak{sp}(2n, \mathbb{R})$,

$$\check{g}(\zeta) \left(\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_b} \right) = [(\zeta \mathbb{J})^{-1}]_{ab}.$$

Then the reduced Hamiltonian (25) can be written, analogously to (23), as

$$\check{H}^{\hbar}(z, \zeta) := H(z) + \frac{\hbar}{4} (\Delta_{\check{g}(\zeta)} H)(z). \quad (26)$$

Since $H^{\hbar} = \check{H}^{\hbar} \circ (\text{id} \times \text{j}_{\text{Sp}})$, and $\text{id} \times \text{j}_{\text{Sp}}$ is a Poisson map, the Hamiltonian vector fields

$$X_{H^{\hbar}}(z, E) = \left(X_{H + \frac{\hbar}{4} \Delta_{g(E)} H}(z), (\mathbb{J} D_z^2 H) \cdot E \right)$$

and

$$X_{\check{H}^{\hbar}}(z, \zeta) = \left(X_{H + \frac{\hbar}{4} \Delta_{\check{g}(\zeta)} H}(z), \text{ad}_{\mathbb{J} D_z^2 H} \zeta \right)$$

are $(\text{id} \times \text{j}_{\text{Sp}})$ -related,

$$T(\text{id} \times \text{j}_{\text{Sp}}) \circ X_{H^{\hbar}} = X_{\check{H}^{\hbar}} \circ (\text{id} \times \text{j}_{\text{Sp}}).$$

Again specialising to quadratic Hamiltonians, these $(\text{id} \times \text{j}_{\text{Sp}})$ -related vector fields are

$$X_{H^{\hbar}}(z, E) = (X_H(z), (\mathbb{J} D_z^2 H) \cdot E)$$

and

$$X_{\check{H}^{\hbar}}(z, \zeta) = (X_H(z), \text{ad}_{\mathbb{J} D_z^2 H} \zeta).$$

The unreduced dynamics, generated by $X_{H^{\hbar}}$ and reproducing the the Hagedorn equations (17), will drop to the reduced dynamics generated by $X_{\check{H}^{\hbar}}$, and these will reproduce the Heller equations (15), by virtue of the relationship (9) expressing j_{Sp} in coordinates on $\text{j}_O^{-1}(\text{j}_O(I))$ and $\mathcal{O}_{\text{j}_{\text{Sp}}}(I)$.

8 Generalisation to other symplectic manifolds

Now that we have outlined the construction of a Hamiltonian system on the frame bundle of $T^*\mathbb{R}^n$, an obvious question is whether this construction can be extended to more general symplectic manifolds. In this section we suggest a generalisation by employing the results of Cordero and de León [CdL83]. Essentially they show that the frame bundle can be provided with a symplectic structure if it has a trivialisaton compatible with the symplectic structure, encoded in the form of a symplectic connection. Their construction may be seen as the symplectic analogue of the Sasaki-Mok metric on the frame bundle of a Riemannian manifold [Mok78].

8.1 The almost-symplectic form on the frame bundle

Let (M, ω) be an arbitrary $2n$ -dimensional symplectic manifold, and let $\pi : \mathcal{F}(M) \rightarrow M$ denote its frame bundle, which is a principal right $\text{GL}(2n, \mathbb{R})$ -bundle in the usual way. For a frame $e = (e_1, \dots, e_{2n}) \in \mathcal{F}(M)$, and $\xi \in \mathfrak{gl}(2n, \mathbb{R})$, define

$$e \cdot \xi := \left. \frac{d}{ds} \right|_{s=0} e \cdot \exp(s\xi) \in V_e \mathcal{F}(M) \subset T_e \mathcal{F}(M),$$

where V_e denotes the vertical space at $e \in \mathcal{F}(M)$. Given a connection 1-form $A \in \Omega^2(\mathcal{F}(M), \mathfrak{gl}(2n, \mathbb{R}))$, Cordero and de León [CdL83] define two natural lifts of the symplectic 2-form ω to the frame bundle $\mathcal{F}(M)$:

- the vertical lift $\omega^{\text{ver}} = \pi^*\omega$;
- the horizontal lift ω^{hor} , given by

$$\omega_e^{\text{hor}}(X_e, Y_e) := \left(\bigoplus_{a=1}^{2n} \omega \right)_e (e \cdot A_e(X_e), e \cdot A_e(Y_e)).$$

In the definition of the horizontal lift we have used the canonical isomorphism $\bigoplus_{a=1}^{2n} T_z M \simeq V_e \mathcal{F}(M)$ (for $z = \pi(e)$) given by

$$(v_1, \dots, v_{2n}) \in \bigoplus_{a=1}^{2n} T_z M \longleftrightarrow \left. \frac{d}{ds} \right|_{s=0} (e_1 + sv_1, \dots, e_{2n} + sv_{2n}) \in V_e \mathcal{F}(M),$$

and here $e \cdot A_e(X_e)$ is the infinitesimal generator at $e \in \mathcal{F}(M)$ corresponding to $A_e(X_e) \in \mathfrak{gl}(2n, \mathbb{R})$. We then define the \hbar -diagonal lift ω^{\hbar} of the symplectic form ω by

$$\omega^{\hbar} := \omega^{\text{ver}} + \frac{\hbar}{2} \omega^{\text{hor}}.$$

This is essentially the diagonal lift defined in [CdL83], up to a factor of $\frac{\hbar}{2}$.

Denote the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ to $\mathcal{F}(M)$ by X^{hor} , and the infinitesimal generator on $\mathcal{F}(M)$ corresponding to $\xi \in \mathfrak{gl}(2n, \mathbb{R})$ by $\xi_{\mathcal{F}(M)}$. Then it is straightforward to verify

- $\omega^{\hbar}(X^{\text{hor}}, Y^{\text{hor}}) = \omega(X, Y) \circ \pi$,
- $\omega^{\hbar}(Z^{\text{hor}}, \xi_{\mathcal{F}(M)}) = 0$, and
- $\omega^{\hbar}(\xi_{\mathcal{F}(M)}, \zeta_{\mathcal{F}(M)})(e) = \frac{\hbar}{2} \sum_{a=1}^{2n} \omega_z((e \cdot \xi)_a, (e \cdot \zeta)_a) = \frac{\hbar}{2} \sum_{a,b,c=1}^{2n} \omega_z(e_b, e_c) \xi_{ba} \zeta_{ca}$.

Using the non-degeneracy of ω to infer that $\omega_z(e_b, e_c)$ is an invertible matrix in the last identity, it follows that the 2-form ω^{\hbar} is non-degenerate, and so defines an almost-symplectic form on $\mathcal{F}(M)$.

8.2 Conditions for the lifted form to be closed

In order for the almost-symplectic form ω^{\hbar} to be a symplectic form, it must satisfy the additional condition $d\omega^{\hbar} = 0$. The conditions for this to be the case are computed in [CdL83].

Proposition 8.1 ([CdL83, Proposition 6.1]).

- (i) $d\omega^{\hbar}(X^{\text{hor}}, Y^{\text{hor}}, Z^{\text{hor}}) = (d\omega(X, Y, Z)) \circ \pi$;
- (ii) $d\omega^{\hbar}(X^{\text{hor}}, Y^{\text{hor}}, \xi_{\mathcal{F}(M)}) = \frac{\hbar}{2} \omega^{\text{ver}}(F^A(X^{\text{hor}}, Y^{\text{hor}})_{\mathcal{F}(M)}, \xi_{\mathcal{F}(M)})$;
- (iii) $d\omega^{\hbar}(X^{\text{hor}}, \xi_{\mathcal{F}(M)}, \zeta_{\mathcal{F}(M)}) = \frac{\hbar}{2} (\nabla_X^A \omega)^{\text{ver}}(\xi_{\mathcal{F}(M)}, \zeta_{\mathcal{F}(M)})$;
- (iv) $d\omega^{\hbar}(\xi_{\mathcal{F}(M)}, \zeta_{\mathcal{F}(M)}, \chi_{\mathcal{F}(M)}) = 0$.

Here, $F^A \in \Omega^2(\mathcal{F}(M), \mathfrak{gl}(2n, \mathbb{R}))$ denotes the curvature 2-form of A , and ∇^A denotes the covariant derivative associated with A .

Corollary 8.2.

$$d\omega^{\mathfrak{h}} = 0 \iff d\omega = 0, \quad F^A = 0, \quad \nabla^A \omega = 0,$$

i.e., $\omega^{\mathfrak{h}}$ is closed if and only if ω is closed and A is a flat, symplectic (i.e. ω -preserving) connection.

8.3 Consequences of the condition

Assuming the conditions in Corollary 8.2 hold, the frame bundle $\mathcal{F}(M)$ has a particularly simple description. Firstly, $F^A = 0$ implies that the horizontal distribution on $\mathcal{F}(M)$ is involutive, and hence integrable. If we further assume the absence of monodromy², this implies the existence of a global horizontal section $\mathbf{s} : M \rightarrow \mathcal{F}(M)$ through any point of $\mathcal{F}(M)$. Since the connection A is symplectic, the frames $\mathbf{s}(z) = (s_1(z), \dots, s_{2n}(z))$ are such that $\omega_z(s_a(z), s_b(z))$ is independent of z . Let us arrange for these frames to be symplectic, i.e.,

$$\omega_z(s_a(z), s_b(z)) = \mathbb{J}_{ab}.$$

Then $\mathcal{F}(M)$ is trivial, and we have a bundle morphism $\Lambda : M \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow \mathcal{F}(M)$ given by

$$\Lambda(z, E) = \mathbf{s}(z) \cdot E.$$

We use Λ to pull back the symplectic form $\omega^{\mathfrak{h}}$. Firstly,

$$\Omega^{\mathrm{ver}} := \Lambda^* \omega^{\mathrm{ver}} = (\pi \circ \Lambda)^* \omega = \pi_1^* \omega,$$

where $\pi_1 : M \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow M$ is projection onto the first factor. Also, defining $\Omega^{\mathrm{hor}} := \Lambda^* \omega^{\mathrm{hor}}$, and using

$$T_{(z, E)} \Lambda(v_z, V_E) = T_z \mathbf{s}(v_z) \cdot E + \mathbf{s}(z) \cdot V$$

we get

$$\begin{aligned} \Omega_{(z, E)}^{\mathrm{hor}}((v_z, V_E), (w_z, W_E)) &= \omega_{\mathbf{s}(z) \cdot E}^{\mathrm{hor}}(\mathbf{s}(z) \cdot V, \mathbf{s}(z) \cdot W) \\ &\quad \text{since } T_z \mathbf{s}(v_z) \cdot E \text{ is horizontal} \\ &= \sum_{a, b, c=1}^{2n} \omega_z((s_b(z), s_c(z))) V_{ba} W_{ca} \\ &= \mathrm{Tr}(V^{\top} \mathbb{J} W). \end{aligned}$$

Thus we have shown

$$\Lambda^* \omega^{\mathrm{hor}} =: \Omega^{\mathrm{hor}} = \pi_2^* \Omega,$$

where $\pi_2 : M \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow \mathrm{GL}(2n, \mathbb{R})$ is projection onto the second factor, and Ω is the symplectic 2-form on $\mathrm{GL}(2n, \mathbb{R})$ defined in Section 2.

²This is the case for example if M is simply connected—if not we can always go to universal cover.

Overall, we obtain the symplectic form $\Omega^{\hbar} := \Lambda^* \omega^{\hbar}$ on $M \times \mathrm{GL}(2n, \mathbb{R})$, given explicitly by

$$\Omega^{\hbar} = \Omega^{\mathrm{ver}} + \frac{\hbar}{2} \Omega^{\mathrm{ver}} = \pi_1^* \omega + \frac{\hbar}{2} \pi_2^* \Omega = \omega \oplus \frac{\hbar}{2} \Omega,$$

i.e.,

$$\begin{aligned} \Omega_{(z,E)}^{\hbar}((v_z, V_E), (w_z, W_E)) &= \omega_z(v_z, w_z) + \frac{\hbar}{2} \Omega_E(V_E, W_E) \\ &= \omega_z(v_z, w_z) + \frac{\hbar}{2} \mathrm{Tr}(V^{\top} \mathbb{J} W). \end{aligned} \quad (27)$$

We see that equation (27) agrees with equation (20), which corresponds to applying the above construction with A the obvious global flat connection on $T^* \mathbb{R}^n$, and global horizontal section

$$\mathbf{s}(z) = \left(\frac{\partial}{\partial q^1} \Big|_z, \dots, \frac{\partial}{\partial q^n} \Big|_z, \frac{\partial}{\partial p_1} \Big|_z, \dots, \frac{\partial}{\partial p_n} \Big|_z \right).$$

8.4 The frame bundle Hamiltonian

Again assuming the conditions in Corollary 8.2 hold, and given a Hamiltonian $H : M \rightarrow \mathbb{R}$, definition (23) of the lifted Hamiltonian carries over directly. To recap, a choice of connection $A \in \Omega^1(\mathcal{F}(M), \mathfrak{gl}(2n, \mathbb{R}))$ and global horizontal section $\mathbf{s} : M \rightarrow \mathcal{F}(M)$ induces a global bundle morphism $\Lambda(z, E) = \mathbf{s}(z) \cdot E$. For every $E \in \mathrm{GL}(2n, \mathbb{R})$, we define $g(E)$ to be the metric on M with respect to which the frame $\Lambda(z, E)$ is orthonormal. The lifted Hamiltonian $H^{\hbar} : M \times \mathrm{GL}(2n, \mathbb{R}) \rightarrow \mathbb{R}$ is then defined by

$$H^{\hbar}(z, E) := H(z) + \frac{\hbar}{4} (\Delta_{g(E)} H)(z).$$

As before, this may also be written

$$H^{\hbar}(z, E) := H(z) + \frac{\hbar}{4} \mathrm{Tr}(E^{\top} D_z^2 H E),$$

where now $D^2 H : M \rightarrow \mathfrak{gl}(2n, \mathbb{R})$ is given by

$$(D_z^2 H)_{ab} := s_a(z) (s_b H).$$

Note however that by contrast with equation (21), here $D^2 H$ is not necessarily symmetric, since s_b is also a function of z . Using a similar analysis to that in Section 7 gives the Hamiltonian vector field generated by H^{\hbar} as

$$X_{H^{\hbar}}(z, E) = \left(X_{H + \frac{\hbar}{4} \Delta_{g(E)}}(z), \frac{1}{2} \mathbb{J} (D_z^2 H + (D_z^2 H)^{\top}) \cdot E \right). \quad (28)$$

Symmetry of $D^2 H$ may be recovered by imposing a further condition on the connection A , viz. that it be *torsionless*. For then, applying the torsionless condition to the individual (covariantly constant) vectors of the global frame \mathbf{s} yields

$$0 = \nabla_{s_a}^A s_b - \nabla_{s_b}^A s_a - [s_a, s_b] \implies [s_a, s_b] = 0,$$

and it follows that

$$(D_z^2 H)_{ab} = s_a(z) (s_b H) = s_b(z) (s_a H) = (D_z^2 H)_{ba}.$$

Hence equation (28) reduces to equation (24).

From here on, the analysis is identical to Section 7. In particular, Λ intertwines the right $O(2n)$ -actions on the frame bundle $\mathcal{F}(M)$ and $M \times GL(2n, \mathbb{R})$

$$\Lambda(z, E) \cdot O = (\mathbf{s}(z) \cdot E) \cdot O = \mathbf{s}(z) \cdot (EO) = \Lambda(z, EO),$$

and the latter action has a momentum map $j_O : M \times GL(2n, \mathbb{R}) \rightarrow \mathfrak{o}(2n)$ as before. Also the (\mathbf{s} -dependent) left $Sp(2n, \mathbb{R})$ -action

$$S \cdot (\mathbf{s}(z) \cdot E) = \mathbf{s} \cdot (SE),$$

is intertwined with the left $Sp(2n, \mathbb{R})$ -action on $M \times GL(2n, \mathbb{R})$, and gives a momentum map $j_{Sp} : M \times GL(2n, \mathbb{R}) \rightarrow \mathfrak{sp}(2n, \mathbb{R})$ as before.

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References

- [Bla01] A. Blaom, *A geometric setting for Hamiltonian perturbation theory*, Mem. Amer. Math. Soc. **153** (2001), no. 727, xviii+112.
- [BW12] C. Balleier and T. Wurzbacher, *On the geometry and quantization of symplectic Howe pairs*, Math. Z. **271** (2012), no. 1-2, 577–591.
- [CdL83] L. A. Cordero and M. de León, *Lifts of tensor fields to the frame bundle*, Rend. Circ. Mat. Palermo (2) **32** (1983), no. 2, 236–271.
- [FL06] E. Faou and C. Lubich, *A Poisson integrator for Gaussian wavepacket dynamics*, Comput. Vis. Sci. **9** (2006), no. 2, 45–55.
- [Hag80] G. A. Hagedorn, *Semiclassical quantum mechanics. I. The $\hbar \rightarrow 0$ limit for coherent states*, Comm. Math. Phys. **71** (1980), no. 1, 77–93.
- [Hel76] E. J. Heller, *Time dependent variational approach to semiclassical dynamics*, J. Chem. Phys. **64** (1976), no. 1, 63–73.
- [Lit86] R. G. Littlejohn, *The semiclassical evolution of wave packets*, Phys. Rep. **138** (1986), no. 4-5, 193–291.
- [LM87] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics*, Mathematics and Its Applications, vol. 35, D. Reidel Publishing Company, Dordrecht, 1987.

- [Lub08] C. Lubich, *From quantum to classical molecular dynamics: Reduced models and numerical analysis*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2008.
- [Mok78] K. P. Mok, *On the differential geometry of frame bundles of Riemannian manifolds*, J. Reine Angew. Math. **302** (1978), 16–31.
- [MR99] J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry*, 2 ed., Texts in Applied Mathematics, vol. 17, Springer, New York, 1999.
- [Ohs15] T. Ohsawa, *The Siegel upper half space is a Marsden-Weinstein quotient: Symplectic reduction and Gaussian wave packets*, Lett. Math. Phys. **105** (2015), no. 9, 1301–1320.
- [OL13] T. Ohsawa and M. Leok, *Symplectic semiclassical wave packet dynamics*, J. Phys. A **46** (2013), no. 40, 405201, 28.
- [OR04] J.-P. Ortega and T. S. Ratiu, *Momentum maps and Hamiltonian reduction*, Progress in Mathematics, vol. 222, Birkhäuser, Boston, 2004.
- [OT17] T. Ohsawa and C. Tronci, *Geometry and dynamics of Gaussian wave packets and their Wigner transforms*, J. Math. Phys. **58** (2017), no. 9, 092105, 19.
- [Sat80] I. Satake, *Algebraic structures of symmetric domains*, Kanô Memorial Lectures, vol. 4, Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980.
- [Sie73] C. L. Siegel, *Topics in complex function theory. Vol. III*, John Wiley & Sons, Inc., New York, 1973.
- [SV19] P. Skerrett and C. Vizman, *Dual pairs for matrix groups*, J. Geom. Mech. **11** (2019), no. 2, 255–275.
- [Wei83] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.