Stochastic finite element method for vibroacoustic loads prediction

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To my family.
Abstract

Reliable and efficient vibroacoustic loads prediction is often critical in structural design, yet it remains a challenging task for many applications. Spacecraft structures are characterised by extensive use of composite materials, complex connections between components and various non-trivial geometrical features. Accurate treatment necessitates the construction of highly detailed numerical models, traditionally employing deterministic representations. Simultaneously, the broadband acoustic excitation due to the diffuse sound field experienced during launch requires modelling the fluid domain and solving the resulting elasto-acoustic interaction at multiple frequencies.

To alleviate the computational demand implications for large problem sizes, substructuring and reduction techniques for the structural domain are commonplace, component mode synthesis (CMS) being a framework widely adopted in the aerospace industry. Nevertheless, despite ongoing research, the topic still presents a range of difficulties when a universal, robust method of accounting for model uncertainties is sought.

In this study, two CMS-based approaches are proposed and evaluated. Firstly, the Craig-Bampton stochastic method (CBSM) is improved via a set of modifications enhancing its efficiency, and subsequently adapted for use in a vibroacoustic setting. Optimal perturbation levels and scope of validity of the technique are established against a probabilistic structural analysis (PSA) simulation for a spacecraft structure.

Secondly, a novel stochastic finite element method (FEM) is presented. The underlying mathematical foundation is derived so that uncertainty can naturally be controlled at the subsystem level, in partitions of the corresponding condensed mass and stiffness matrices. This decomposition-based approach ensures that realisations of the random matrices have key properties such as positive-(semi)definiteness strictly preserved, guaranteeing complete robustness. The method is validated with a spacecraft test case, comparing its predictions against PSA, the improved CBSM, and experimental data. A coupling scheme with a hierarchical matrix accelerated boundary element method is formulated, resulting in the construction of a complete stochastic vibroacoustic solver.

**Keywords:** Structural dynamics, Stochastic matrices, Finite element method, Vibroacoustics, Component mode synthesis
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Abbreviations

ABC  absorbing boundary condition
ACA  adaptive cross approximaton
BC   boundary condition
BE   boundary element
BEM  boundary element method
CAD  computer-aided design
CB   Craig-Bampton
CBIE conventional boundary integral equation
CBSM Craig-Bampton stochastic method
CLF  coupling loss factor
CMS  component mode synthesis
COV  coefficient of variation
DLF  damping loss factor
DOF  degree of freedom
DS   dynamic substructuring
DSF  diffuse sound field
DSFEM decomposition-based stochastic finite element method
FE   finite element
FEM  finite element method
FMM  fast multipole method
GEP  generalised eigenvalue problem
GMRES generalized minimum residual
I/O  input/output
IEM  infinite element method
LHS  left-hand side
LV   launch vehicle
MC   Monte Carlo
MCS  Monte Carlo simulation
MPC  multi-point constraint
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<tr>
<td>MVP</td>
<td>matrix-vector product</td>
</tr>
<tr>
<td>NDBIE</td>
<td>normal derivative boundary integral equation</td>
</tr>
<tr>
<td>NF</td>
<td>natural frequency</td>
</tr>
<tr>
<td>NSM</td>
<td>non-structural mass</td>
</tr>
<tr>
<td>PD</td>
<td>positive definite</td>
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<tr>
<td>PDF</td>
<td>probability density function</td>
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<tr>
<td>PML</td>
<td>perfectly matched layer</td>
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<tr>
<td>PSA</td>
<td>probabilistic structural analysis</td>
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<tr>
<td>PSD</td>
<td>positive semi-definite</td>
</tr>
<tr>
<td>RHS</td>
<td>right-hand side</td>
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<tr>
<td>RMS</td>
<td>root mean square</td>
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<tr>
<td>SEA</td>
<td>statistical energy analysis</td>
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<tr>
<td>SFEM</td>
<td>stochastic finite element method</td>
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<tr>
<td>SPC</td>
<td>single point constraint</td>
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<tr>
<td>SPL</td>
<td>sound pressure level</td>
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<td>SSC</td>
<td>Surrey Space Centre</td>
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<td>SSTL</td>
<td>Surrey Satellite Technology Limited</td>
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<tr>
<td>SVD</td>
<td>singular value decomposition</td>
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<td>TBL</td>
<td>turbulent boundary layer</td>
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Symbols

(,) elements of a matrix, also n-tuple e.g. \((a_1, \ldots, a_n)\)

\{\cdot\} set, e.g. \(\{5, 3, k, \ldots\}\)

\langle \cdot, \cdot \rangle inner product

\lfloor \cdot \rfloor floor function

\abs{\cdot} absolute value

\norm{\cdot} vector norm

\norm{\cdot}_2 spectral norm, equal to \(\sigma_{\text{max}}(.)\)

\norm{\cdot}_F Frobenius norm

\(A_{m \times n}\) matrix with \(m\) rows and \(n\) columns

\(A^+\) pseudoinverse

\(A^{+\ast}\) shorthand for \((A^+)\ast\); similarly \(A^{+T} = (A^+)^T\)

\(A^g\) generalised inverse

\(A^\ast\) conjugate transpose

\(A^{-\ast}\) shorthand for \((A^\ast)^{-1}\); similarly \(A^{-T} = (A^T)^{-1}\)

\(A^T\) matrix transpose

\(A / X\) the (possibly generalised) Schur complement of \(X\) in \(A\)

\(\lambda_i(A)\) \(i\)-th eigenvalue of \(A\), where \(\lambda_1 \geq \cdots \geq \lambda_n\)

\(\sigma_i(A)\) \(i\)-th singular value of \(A_{m \times n}\), where \(\sigma_1 \geq \cdots \geq \sigma_{\min\{m, n\}}\)

cov\((X, Y)\) covariance of the random variables \((X, Y)\)

\(\dim(U)\) the dimension of a vector subspace \(U\)

E\((X)\) mathematical expectation for a random variable \(X\)

Im\(\{X\}\) imaginary part of a complex value \(X\)

ker\((A)\) the nullspace of a matrix \(A\); also kernel

range\((A)\) the range of a matrix \(A\); also image

rank\((A)\) the rank of a matrix \(A\)

\(r(X, Y)\) linear correlation coefficient between \(X\) and \(Y\)

span\(\{V\}\) the set of all linear combinations \(\{\sum_{i=1}^n \alpha_i v_i : n \in \mathbb{N}, \alpha_i \in \mathbb{C}, v_i \in V\}\)

var\((X)\) variance of the random variable \(X\)

\(\cap\) set intersection; \(A \cap B\): all elements in both \(A\) and \(B\)
Symbols

∪ set union; $A \cup B$: all elements in either $A$, $B$, or both
⊆ subset; $A \subseteq B$ if for any $x \in A$, $x \in B$
⇔ if and only if
⇒ therefore
:= equals by definition
∃ there exists (at least one); negated by $\not\exists$
∃! there exists one and only one
∀ for all
◦ element-wise product
∈ is member of, belongs to; negated by $\notin$
$\nabla^2$ Laplace operator
∅ the empty set
$\mathbb{C}$ set of complex numbers
$\mathbb{N}$ set of natural numbers
$\mathbb{R}$ set of real numbers
$\succ$ positive definite matrix: $A \succ 0$
$\succeq$ positive semi-definite matrix: $A \succeq 0$
$\bar{x}$ complex conjugate of $x$
$\bar{x}$ mean value of a random variable $x$
$\bar{x}$ random variable $x$
$\mu_x$ mean of a random variable $x$
$\sigma_x$ standard deviation of a random variable $x$
Definitions

*-congruent for $A, B \in \mathbb{C}^{n \times n}$ if $\exists$ invertible $S : S^*BS = A$

diagonalisable $A_{n \times n}$, similar to a diagonal matrix, i.e. $\exists B : B^{-1}AB$ is diagonal

Frobenius norm the norm $\|A\|_F = \sqrt{\text{trace}(A^*A)}$

generalised inverse $A^g$, such that $AA^gA = A$

Gramian $G \in \mathbb{C}^{n \times n} : G = A^*A \succeq 0, \forall A \in \mathbb{C}^{m \times n}$

Hermitian $A \in \mathbb{C}^{n \times n} : A = (A^T) := A^*$

inertia the triplet $\text{In}(A) = (p, n, z)$ of number of positive, negative and zero $\lambda_i(A)$

invertible $A_{n \times n}$, such that $\exists B : AB = BA = I$; also nonsingular, nondegenerate

normal $U \in \mathbb{C}^{n \times n} : U^*U = UU^*$

nullspace $\ker(A) := \{x : Ax = 0\}$; also kernel

orthogonal $U \in \mathbb{R}^{n \times n} : U^TU = UU^T = I$

orthonormal orthogonal unit vectors, i.e. $u, v : \langle u, v \rangle = 0, \|u\| = \|v\| = 1$

positive definite $A_{n \times n} : \lambda_i(A) > 0$ for all $i = \{1, \ldots, n\}$

principal minor the determinant of a matrix block: $\det(A_{i:j,i:j})$

pseudoinverse a special case of $A^g$, denoted $A^+$

range $\text{range}(A_{m \times n}) := \text{span}\{a_1, \ldots, a_n\}$ for columns $a_i$; also image, column space

rank the dimension of $\text{range}(A)$, denoted $\text{rank}(A)$

similar for $A, B \in \mathbb{C}^{n \times n} : A \sim B \iff \exists P : P^{-1}BP = A$

spectral norm the norm $\|A\|_2 = \sigma_{\text{max}}(A)$

trace the sum of the diagonal elements, $\text{trace}(A_{n \times n}) = \sum_{i=1}^n A_{i,i}$

unitary $U \in \mathbb{C}^{n \times n} : U^*U = UU^* = I$
Chapter 1

Introduction

1.1 Industrial motivation

The harshness of the launch environment spacecraft are subjected to is a key driving factor in their design. Ensuring that the structural integrity and functionality of the payload are not endangered during this initial stage of the mission demands the availability of numerical models that enable reliable loads prediction prior to the test campaign. Aeroacoustic excitations induced by the launch vehicle (LV) are most prominent during take-off and transonic flight, and estimating their effect proves particularly challenging. The resultant fluid-structure interaction, broadband in nature, is arguably yet to be solved sufficiently well over the full frequency range of interest by a single method. In the wider structural dynamics context, accounting for model uncertainties in a reliable way, without imposing a burden of excessive computational resource demand, has historically been problematic as well. The scope of this thesis is confined between these two topics.

The work presented hereafter was enabled by the support and cooperation of Surrey Satellite Technology Limited (SSTL). The company has a long-standing history of conducting research in collaboration with the Surrey Space Centre (SSC). The inception of this project is owed to the Craig-Bampton stochastic method originally developed by Remedia et al. [1]–[3] for estimating the effects of uncertainty on microvibration levels. Expanding the formulation and capabilities of the original CBSM, in order to enhance its efficiency and accommodate low- to mid-frequency vibroacoustic analysis, was targeted. Following integration with an acoustic domain representation, the new technique was expected to supersede the weakly stationary random process approach, discussed in Section 2.5.1, that was used by SSTL at the time. An investigation of the method’s limitations and determination of optimal perturbation parameters was intended with the aid of spacecraft models and test data supplied by SSTL. The developed Improved CBSM and corresponding software were to be integrated into the company’s design process and utilised in future SSC research.

In the process of fulfilling the aforesaid objectives, the inherent limitations of the CBSM and its improved version were better understood. Initially, allowing the injection of uncertainty in specific local partitions of the component mass matrices was prioritised.
1.2 Scientific context

It was a means of making the technique suitable for a wider range of models occurring in practical use. Over time, the idea of developing a completely new, more universal stochastic [FEM] was conceived. From a strictly industrial perspective, efficiency and robustness were paramount. Strong emphasis was also placed on its ability to facilitate dynamic analyses not restricted to acoustic loads. At a more fundamental level, natural integrability within the [CMS] framework was requested. Furthermore, the possibility to unrestrictedly control uncertainty magnitude at a model scale required by the user, i.e. typically at a CMS subsystem level, was strongly preferable. Streamlined practical use procedures, manifested in a limited number of variables requiring independent modelling or control, was also coveted by SSTL. Finally, usability of the new method in combination with different existing acoustic solvers was seen as an advantage for vibroacoustic analysis.

The work contained in the ensuing chapters spans both described methods. At present, they are being utilised in ongoing doctorates at [SSC] and included in SSTL’s software toolkit for various spacecraft design and verification purposes.

1.2 Scientific context

The content of this thesis crosses a broad range of topics in structural mechanics, uncertainty modelling and acoustics. From the beginning and throughout the evolution of this research, three key scientific directions have been identified, and are expansively covered in the following chapters.

Firstly, the evolution of the [CBSM]. The continuous refinement of finite element (FE) representations in the space industry means existing parametric Monte Carlo (MC)-type methods would benefit from reduced computational complexity, even when applied to condensed models. To that effect, a reformulation of the original technique, aimed at efficiency improvements, was required. Exploiting the specific matrix structure occurring due to the Craig-Bampton (CB) reduction was seen as a potentially viable path to achieving that. Moreover, extending the CBSM’s capability to cover distributed excitations, particularly ones occurring due to vibroacoustic interaction, was a primary objective. Finally, studying the effect of perturbation levels under different conditions, against an established technique, such as [PSA] in order to find optimal parameters and global usability limitations of the scheme, was planned.

Subsequently, the development of a novel stochastic FE method that allows the robust construction of random mass and stiffness matrices for general structural dynamics problems was targeted. This was expected to include the derivation of a mathematical framework enabling the preservation of algebraic matrix properties inherently important to the underlying physical problem. Stemming from various practical considerations, the proposal of a construction allowing the generation of random FE matrices on a $2 \times 2$ block partitioning level was envisioned. The extension of the concept to more complex cases, such as generic CMS-derived model matrices was another logically sound aim. Naturally, upon the successful completion of the aforementioned tasks, practical investigation of the properties of random matrices obtained via the new method would be needed. Verification of the anticipated behaviour would ensue. The ultimate test
would be the validation of the new stochastic FEM against the improved CBSM, PSA simulation and test data for vibroacoustic analysis of a real spacecraft structure.

Finally, subject to the completion of the first two sets of tasks, the integration of the new stochastic FEM within a fully coupled vibroacoustic solver, utilising hierarchical matrix ($\mathcal{H}$-matrix) accelerated boundary element method (BEM) treatment for the fluid domain, was to be attempted. While this was not a chief objective, it was devised with large-scale complex problems in mind. It was asserted that some practical improvements could be investigated for the $\mathcal{H}$-matrix implementation. These were to be focused around enhancements permitting the construction of minimal-overhead software. As a last potential goal, the initial validation of the convergence behaviour of the so-obtained coupled FEM-BEM was to be assessed. The plan included the employment of iterative solvers, and was to be executed through the use of large-scale problems with simple geometries.

1.3 Novel contributions

Numerical methods for problems in structural dynamics are subject to continuous development, due to the rapidly growing complexity of models that need to be analysed. This issue is particularly relevant to the aerospace industry, which constantly pushes the boundaries of computational analysis refinement, and therefore demands. Component mode synthesis is a frequently employed technique for problem size reduction in the context of structures. In recent years, it has also gradually started to gain popularity as a platform for uncertainty analysis and simulation - another topic of rising importance, inciting vast amounts of research activity. However, clear limitations still exist. A sufficiently good integration of model order reduction and ability to handle variability is yet to be exhibited by a single approach.

The key advancements of the state of the art stemming from the current work are centred exactly around the aforesaid issue. The contributions largely correspond to the pursuit of the research objectives exposed in Section 1.2. More specifically:

- The Craig-Bampton stochastic method’s efficiency is enhanced by various mathematical and procedural structure improvements. In addition, a wider understanding of the CBSM is gained through an extensive parametric survey. Recommendations are given on the conditions under which the CBSM is expected to show near-optimal prediction accuracy against PSA, while practical limitations are also outlined. Applicability to distributed excitations is addressed.

- A complete mathematical framework is derived for a new class of blockwise random Hermitian matrices. The formulation is rigorously constructed from the outset with nonparametric stochastic FEM in mind as the target application. Fundamental properties, such as rank and nonnegative definiteness of matrix realisations, are strictly guaranteed to be preserved. Unlike existing techniques, the method allows individual treatment of off-diagonal mass and stiffness matrix partitions to be carried out.
1.4 Thesis outline

The novel factorisation based stochastic FEM is studied via both artificially generated matrices and a real spacecraft example. The projected excellent performance is verified. A coupling scheme with hierarchical matrix BEM is delineated to form a complete, efficient low- to mid-frequency vibroacoustic solver. Due to the generality of their underlying algebraic formulation, the new stochastic matrices are seen as highly versatile and not restricted in usability to FEM numerical analysis. In fact, the enveloping complex Hermitian case for matrices not subjected to any particular structure requirements is derived. The resultant insights can be used directly to construct methods for restoring nonnegative definiteness of perturbed matrices, occurring in various fields.

1.4 Thesis outline

The thesis is organised in a total of six chapters. Excluding the current introductory one, content is ascribed to the remaining five so that they form, as much as reasonable, self-contained reflections of the research carried out within this project.

In Chapter 2 the survey of existing literature on a broad range of topics concerning this thesis is contained. Three principal logical partitions can be identified, incrementally tending in scope towards contemporary vibroacoustic techniques for complex, uncertainty-containing structures. Initially, attention is drawn to structural dynamic analysis principles. Thorough investigation into the state of the art in CMS methods, as well as uncertainty modelling approaches, is laid out. Subsequently, the focus is shifted towards the modelling of acoustic environments, predominantly in relation to space applications. Combining the previous two topics, elasto-acoustic analysis methods are reviewed, especially in the context of low- to mid-frequency problems. Generally, the chapter is intended to be as representative as possible of the whole field of structure-acoustic interaction, viewed from a structural response estimation perspective.

Chapter 3 encompasses a range of discussions on the Craig-Bampton stochastic method. An introduction to the main tenets is provided, followed by the derivation of a more efficient reformulation of the original technique. A simple benchmark example is used to demonstrate the method. A realistic spacecraft test case is subsequently employed to carry out a far-reaching parametric survey on the CBSM perturbation values. A comparison baseline is established via a conventional PSA approach. Based on the findings, recommendations are given on the optimal CBSM application conditions. The chapter is concluded by a computational cost juxtaposition against the PSA and unmodified CBSM, along with a study of the scope limitations of the method.

Chapter 4 is devoted to the introduction of a new class of stochastic matrices. The underpinning mathematical concepts are first reviewed. Spectral and singular value decompositions, pseudoinverses and generalised Schur complementation shape the main array of principles used subsequently within the chapter. A set of requirements, rooted in FEM-based considerations, is construed for the desired properties of the random matrices. The remainder of the chapter is entirely concentrated on the construction of a suitable algebraic platform for the generation of the required partition-wise random
matrices, with several important claims being formulated and proven. Various relevant aspects are covered, from basic principles, to the effective definition and realisation sampling of stochastic mass and stiffness matrices, arising from domain decomposition strategies in FEM problems.

The practical implementation aspects of the new stochastic FEM are studied in Chapter 5. Initially, the method is applied to artificially generated dense matrices, in order to study the entry-wise statistical behaviour of the resultant sampled random matrices. Their eigenvalue distributions are also covered and deliberated. The ensuing section deals with the technique’s application to a real spacecraft structure, condensed with the Craig-Bampton reduction. Comparison is made against test data, PSA and the improved CBSM, for the case of vibroacoustic simulation. Computational requirements are established against the other two numerical methods as well. At the end of the chapter, a coupling scheme with a hierarchical matrix BEM is outlined as a potential high-performance vibroacoustic solver, capable of the estimation of response variability due to the presence of structural uncertainty.

Finally, Chapter 6 summarises the presented research outputs. Some ideas on potential future work are given and the conclusions of the thesis are drawn.
Chapter 2

Literature survey

2.1 Introduction

The topic of vibroacoustic numerical simulation lies at the intersection of structural
dynamics, acoustics and applied mathematics in computational engineering. As such,
this is a field with a rich history, yet innovation is continuously brought about by strong
practical and academic interest. Nevertheless, the complexity and therefore demands
of engineering problems have been growing at a matching rate, requiring improvements
of existing procedures or the development of entirely new ones.

As a classic example, low-frequency fluid-structure interactions are traditionally solved
by established deterministic techniques, such as the finite element method and the
boundary element method. Their primary advantage is that arbitrary geometries and
mechanical properties can be represented. However, considering higher frequencies
and therefore lower wavelengths, proportionally smaller elements are needed. The
consequent increase in problem size often renders the available computing resources
inadequate, both in terms of time and memory requirements. In addition, deterministic
methods are prone to suffering from large sensitivity in cases of heavily refined meshes,
compromising the quality of the solution. Due to the prevalence of FEM and BEM,
there is a clear motivation to remedy their shortcomings and extend their validity into
the mid-frequency range.

For the structural domain, the first challenge discussed is the alleviation of the comput-
ing cost of FEM. Strong emphasis is placed on component mode synthesis, which is the
main tool for this purpose, especially in the aerospace industry. The common concepts
underpinning CMS are briefly reintroduced and key advancements in the state of the
art are reviewed in Section 2.2. For example, a lot of recent research has been centred
around improving the accuracy or efficiency of existing techniques, such as the Craig-
Bampton method. Simultaneously, CMS has been recognised by many authors as an
environment suitable for stochastic analysis, giving rise to some prominent numerical
schemes. This line of research is clearly of notable importance to this thesis, which for
the most part shares the same scientific direction.

In general, FEM predictions deviate from real behaviour for reasons beyond the afore-
mentioned sensitivity at higher frequency. Modelling inaccuracy, uncertain mechanical
and geometric characteristics and greater modal density, leading to more chaotic behaviour, complement each other to produce systems better suited to statistical, rather than deterministic descriptions. In this regard, the next objective is the development of methods representing the structure in stochastic terms within the FE analysis framework. The progress and existing developments in this field are reflected in Section 2.3 of the current chapter.

While structures are almost inevitably modelled with FE, the acoustic domain presents its own set of peculiarities and difficulties. A survey on the numerical schemes available is given in Section 2.4, compiled with particular attention to the relevant context of aeroacoustic excitations on spacecraft.

Low- to mid-frequency acoustic medium representations utilising finite elements need special treatment when unbounded domains are of interest, in order to cater for an infinite space which evidently cannot be discretised with the standard FEM. The boundary element method is often a meaningful alternative, but its implementation and use are arguably not free of predicaments. Many aspects of BEM’s efficient and robust application necessitate much more involved mathematical theory. In spite of this, the positive aspects of BEM have fuelled its rapid evolution in the last few decades, and it is currently an established technique routinely incorporated in both pure wave propagation and fluid-structure interaction analyses. At the high end of the frequency spectrum, the dominance of energy-based methods is consolidating, mostly through the wide adoption of statistical energy analysis (SEA).

Finally, research directly concerning vibroacoustic analysis is examined in Section 2.5. Integration of two or more schemes yielding mixed formulations like coupled FEM-BEM and FEM-SEA has gradually become the norm in the field. Motivation is clearly grounded in the attempt to extract the benefits of each constituent approach. Contemporary formulations even envelop advanced hybridisation of statistical and deterministic descriptions within the same model. The pursuit of efficiency and validity in the mid-frequency range has also spawned entirely new ideas, such as ‘thermal’ and wave based methods, briefly explained in Section 2.5.5. An entirely different viewpoint involves considering idealised geometries, such as flat plates or cone-cylinder-cone structures, commonly encountered in the aerospace industry. Their relative simplicity allows the exercise of custom built analytical models which often offer more realistic results and may be integrable within a standard element-based method frame.

Overall, the current chapter aims to provide a concise overview of the diverse research area of elasto-acoustic numerical simulation, especially for the case of aerospace structures. Its organisation is such that the efficient treatment of time-harmonic problems in structural dynamics is initially introduced, followed by the relevant theoretical and practical principles in modelling the fluid space. Lastly, vibroacoustic analysis methods are outlined. The chapter is concluded with a summary of the reviewed research.
2.2 Model condensation and substructuring

2.2.1 Dynamic substructuring

Core principles

Domain decomposition is a term under which a number of concepts in numerical analysis are unified. In fact, the phrase may be understood in its broad mathematical sense, where it refers to solving a boundary value problem by splitting it into several smaller ones, represented by their corresponding subdomains. A solution then involves considering the subdomains separately, whereas their interdependence is described by a top-level 'coarse' problem, containing only a small number of variables per subdomain. Visualising the benefits of such an approach is easy: it lends itself to parallel computation, and has the potential to be more efficient in cases where the global problem is of a higher than linear complexity order, which very often is the case.

Dynamic substructuring (DS) is a special case of domain decomposition in the context of FEM in structural dynamics. Apart from gaining computational parallelism, analysing parts of a mechanical structure independently brings additional dividends. Local dynamic behaviour becomes more apparent, allowing analysis times to be shortened in terms of engineering effort as well. Explicitly having separate FE models on a component basis enables sharing and combining parts from different projects. Finally, DS opens up the possibility of constructing reduced order models via component mode synthesis, a powerful technique discussed in Section 2.2.2.

Let us consider a structural domain Ω divided into \( N_s \) non-overlapping parts, denoted \( \Omega^{(1)}, \ldots, \Omega^{(N_s)} \), as depicted on Figure 2.1. The dynamic equilibrium equation for a substructure \( k \), corresponding to \( \Omega^{(k)} \) is given by:

\[
M^{(k)}\ddot{u}^{(k)} + C^{(k)}\dot{u}^{(k)} + K^{(k)}u^{(k)} = f^{(k)} + g^{(k)}
\]  

(2.1)

Here, \( M, C, K \), have the traditional meaning of mass, damping and stiffness FE matrices, \( u \) is the displacement vector, \( f \) the external force vector, \( g \) is the vector of connecting forces between components and the superscript \( k = \{1, \ldots, N_s\} \) is the subsystem number.

Figure 2.1: Illustration of substructuring: (a) partitioning of Ω, (b) definition of component interface connections
The additional term \( g \) is necessary in order to provide constraints ensuring displacement continuity at the interface. Assembling the global governing equation for \( \Omega \) from \( (2.1) \) yields

\[
M_d \ddot{u}_d + C_d \dot{u}_d + K_d u_d = f_d + g_d
\]

(2.2)

where subscript \( d \) refers to a 'dual' assembly mode of DS. This concept is clarified in the following paragraphs. Explicitly, the terms of \( (2.2) \) are given by

\[
M_d = \text{diag} \begin{pmatrix} M^{(1)}, \ldots, M^{(N_s)} \end{pmatrix}, \quad C_d = \text{diag} \begin{pmatrix} C^{(1)}, \ldots, C^{(N_s)} \end{pmatrix}, \quad K_d = \text{diag} \begin{pmatrix} K^{(1)}, \ldots, K^{(N_s)} \end{pmatrix},
\]

(2.3)

\[
u_d = \begin{pmatrix} u_d^{(1)} \\ \vdots \\ u_d^{(N_s)} \end{pmatrix}, \quad f_d = \begin{pmatrix} f_d^{(1)} \\ \vdots \\ f_d^{(N_s)} \end{pmatrix}, \quad g_d = \begin{pmatrix} g_d^{(1)} \\ \vdots \\ g_d^{(N_s)} \end{pmatrix}
\]

The condition enforcing interface compatibility can be written as

\[
\mathcal{B} u_d = 0
\]

(2.4)

where \( \mathcal{B} \) is a signed Boolean matrix. In other words, its entries take values of \{−1, 0, 1\}. \( \mathcal{B} \) operates on the interface, guaranteeing that any matching pair of connecting DOFs between two substructures have equal displacements. For instance, if the \( i \)-th and \( j \)-th degree of freedom of components \( k_1 \) and \( k_2 \), respectively, are connected, \( u_{k1}^i = u_{k2}^j \) must hold. In general, multi-point constraints (MPCs) and non-conforming meshes between parts result in \( \mathcal{B} \) that is no longer Boolean, but \( (2.4) \) remains valid. Now, we can write the boundary equilibrium as

\[
\mathcal{L}^T g_d = 0
\]

(2.5)

where \( \mathcal{L} \) is a Boolean localisation matrix, mapping substructural connection DOFs to the global dual set corresponding to \( u_d \). The physical interpretation of \( (2.5) \) is that the joining forces between a DOF pair have a null sum, that is \( g_{k1}^i + g_{k2}^j = 0 \). Further details on the construction of \( \mathcal{B} \) and \( \mathcal{L} \) are provided in Chapter 3. It is worth pointing out that \( \mathcal{L} \), \( \mathcal{B} \) are not explicitly used in practice. Rather, they mathematically represent the extraction, reordering and summation of interface degrees of freedom from the full dual set. In programming terms this data is already available at the stage when the full system FE model has been partitioned. At this point, \( \Omega \) is fully described by the combination of \( (2.2) \), \( (2.4) \) and \( (2.5) \):

\[
\begin{cases}
M_d \ddot{u}_d + C_d \dot{u}_d + K_d u_d = f_d + g_d \\
\mathcal{B} u_d = 0 \\
\mathcal{L}^T g_d = 0
\end{cases}
\]

(2.6)

Equation \( (2.6) \) is valid for substructuring of arbitrary complexity, number of components and interconnections.
2.2. Model condensation and substructuring

An overview of the topic of DS with classification of existing techniques can be found in [4], whereas the underlying mathematical theory is explored in detail in [5]. Some interesting modern developments include algebraic substructuring, introduced by Bennighof and Lehoucq [6]. It involves automatically partitioning the global system matrices, potentially in a hierarchical manner. The domain decomposition need not correspond to separation of distinct physical components. The method is primarily intended for use within CMS, with remarkable efficiency being reported in some recent papers, for instance [7] and [8].

Primal assembly

The traditional approach in dynamic substructuring is to define a unique degree of freedom set by eliminating duplicate interface DOFs. Since this corresponds to the global FE model as if it had not been subjected to any partitioning, a notation dropping the subscripts $d$ is adopted throughout this thesis. Then the dual DOFs are related to the unique ones through

$$u_d = \mathcal{L} u$$  \hspace{1cm} (2.7)

thus $\mathcal{B}\mathcal{L} u = 0$ from (2.4). Substituting (2.7) into (2.2) and premultiplying by $\mathcal{L}^T$, the global equation of motion assumes the classic form

$$M\ddot{u} + C\dot{u} + Ku = f$$  \hspace{1cm} (2.8)

where $g$ has been cancelled out and

$$M = \mathcal{L}^T M_d \mathcal{L}, \quad C = \mathcal{L}^T C_d \mathcal{L}, \quad K = \mathcal{L}^T K_d \mathcal{L}, \quad f = \mathcal{L}^T f_d$$  \hspace{1cm} (2.9)

Dual assembly

The second basic pathway to enforcing boundary compatibility is to keep the equilibrium equation in its dual form. The motivation behind it is that (2.2) is intrinsically suitable to processing on parallel computer architectures on a subpart basis. The formulation is arrived at by defining an a priori interface equilibrium for the dually assembled system. This is attained by specifying the assembling force

$$g_d = -\mathcal{B}^T \mu$$  \hspace{1cm} (2.10)

by the Lagrange multipliers $\mu$, physically representing force intensity. Then

$$\mathcal{L}^T g_d = -\mathcal{L}^T \mathcal{B}^T \mu = 0$$  \hspace{1cm} (2.11)

The dually assembled equivalent to (2.8) for $\Omega$ then reads

$$\begin{pmatrix} M_d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{u}_d \\ \mu \end{pmatrix} + \begin{pmatrix} C_d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u}_d \\ \mu \end{pmatrix} + \begin{pmatrix} K_d & \mathcal{B}^T \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} u_d \\ \mu \end{pmatrix} = \begin{pmatrix} f_d \\ 0 \end{pmatrix}$$  \hspace{1cm} (2.12)

The advancement of dual formulations has spawned a whole class of solvers known as ‘finite elements tearing and interconnecting’ [4, 5].
2.2. Model condensation and substructuring

Analysis in the frequency domain

An equivalent version of the preceding statements can straightforwardly be made for the important case of frequency domain solution. A Fourier transform of the general DS equation (2.6) results in

\[
\begin{cases}
Z_d(\omega)u_d(\omega) = f_d(\omega) + g_d(\omega) \\
Bu_d(\omega) = 0 \\
L^Tg_d(\omega) = 0
\end{cases}
\]  

(2.13)

where \( Z_d(\omega) = -\omega^2M_d + i\omega C_d + K_d \) is the dynamic stiffness, \( u_d, f_d \) and \( g_d \) being complex amplitudes of the harmonic response, whereas \( i \) and \( \omega \) are the imaginary unit and frequency, respectively. The primal assembly is then written as

\[ Zu = f \quad \text{where} \quad Z = \mathcal{L}^TZ_d\mathcal{L}, \quad f = \mathcal{L}^Tf_d \]  

(2.14)

and its dual counterpart is found using the same reasoning as before:

\[
\begin{pmatrix} Z_d & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_d \\ \mu \end{pmatrix} = \begin{pmatrix} f_d \\ 0 \end{pmatrix}
\]  

(2.15)

2.2.2 Component mode synthesis

General framework

Over half a century of research and practical use have proven dynamic substructuring as an environment especially well-suited to accommodating model order reduction methods. The latter involve projecting the original problem onto a suitable basis, which can be truncated to a small number of generalised degrees of freedom, resulting in a numerical model of a diminished size. In DS, the procedure is applied separately to each component. Historically, the substructures’ actual eigenvectors were used as a reduction basis, giving rise to the name component mode synthesis. However, at present ‘mode’ takes on a far more expansive meaning in the context of CMS, and can be any vector of the reduction space.

To obtain a condensed system representation, some reduction matrix, facilitating the transformation of the physical degrees of freedom \( u_d \) to modal ones \( \xi_d \), is initially constructed:

\[ u_d \approx T_d\xi_d, \quad T_d = \text{diag}(T_d^{(1)}, \ldots, T_d^{(N_s)}) \]  

(2.16)

The approximation arises because in practice \( T_d \) is a truncated basis for the modal space, otherwise the desired reduction in problem size would not be realised. In generalised coordinates, the exact interface compatibility condition becomes

\[ B_r\xi_d = 0, \quad \mathcal{B} = \mathcal{B}^T \]  

(2.17)

Pertaining to the generic treatment presented here, the assembly operator \( \mathcal{L}_r \) may no longer be Boolean, unlike the analogous localisation matrix in physical coordinates. It abides to \( \xi = \mathcal{L}_r^T\xi_d \), and therefore

\[ \mathcal{B}_r\mathcal{L}_r\xi = 0 \quad \Rightarrow \quad \mathcal{L}_r = \text{ker}(\mathcal{B}_r) \]  

(2.18)
The nullspace \( \text{ker}(B_r) \) of \( B_r \) may need to be extracted explicitly in some cases \[9\]. At this stage, utilising the above definitions, writing the reduced versions of the coupled dynamic substructuring equations is trivial. It is done by simply substituting all quantities with their modal counterparts:

\[
\begin{align*}
A_{d,r} &= T_d^T A_d T_d \\
v_{d,r} &= T_d^T v_d
\end{align*}
\]

and

\[
\begin{align*}
A_r &= \mathcal{L}_r^T A_{d,r} \mathcal{L}_r \\
v_r &= \mathcal{L}_r^T v_{d,r}
\end{align*}
\]

(2.19)

where \( A \) is a placeholder for any of the matrices \( M, C, K \) or \( Z \) and \( v \) stands for any of the vectors \( f \) or \( g \).

**Common reduction methods**

Multiple definitions of the substructure modal bases have been invented since the early days of CMS. Broadly speaking, a division between fixed and free interface component modes can be employed in order to classify them. The original method of Hurty [10] was the first CMS formulation and constructed the transformation matrix by considering fixed interface modes, rigid body modes and constraint modes. Application was cumbersome as partitioning of the interface DOF set was demanded. At about the same time the Guyan reduction [11] was introduced. In essence it neglects inertial effects and statically condenses the mass and stiffness matrices to the boundary set.

Combining the ideas of Hurty and Guyan, the Craig-Bampton method [12] emerged, and remains the most universally adopted CMS technique to date. The reduction basis \( T_r \) is composed of fixed interface and constraint modes. Each constraint mode is a vector describing the static deformation of the component occurring due to a unit displacement applied at an interface DOF, while the remaining boundary degrees of freedom are kept fixed. Explicitly

\[
T_r = \begin{pmatrix}
I_B & 0_i \\
-K_I^{-1}K_{IB} & \Phi_I
\end{pmatrix}
\]

(2.20)

where subscripts \( I \) and \( B \) refer to internal and boundary degrees of freedom and \( \Phi_I \) contains the truncated set of fixed interface eigenvectors. Note that the connector DOFs in the reduced model coincide with their original physical coordinates. For convenience, the boundary set is therefore often augmented by DOFs not belonging to the actual substructure joints. They may directly be used for excitation input or output collection.

The specifics of the Craig-Bampton method are conferred at length in Chapter 3, to the content of which it is central.

The CB reduction was followed by several methods that incorporated attachment modes in the definition of the transformation matrix, in other words, component responses to unit boundary loads. The internal substructure dynamics were described through free interface modes. This led to the inception of the classic MacNeal [13], Rubin [14] and Craig and Chang [15] CMS approaches in the 70s. Later takes on the idea also exist in literature, for example [16].
2.2. Model condensation and substructuring

Further improvements

Many of the aforementioned standard techniques have been revisited by different authors in the pursuit of improved accuracy and performance. Rixen [17] devised a dual CB method. The paper showed that approximately two orders of magnitude decrease in the relative error of the low-frequency eigenvalues against the baseline non-reduced model could be gained. The comparison was made with respect to the similarly performing MacNeal and standard CB methods. This could be attributed to the incorporation of a weak interface compatibility, which permits the boundary displacement continuity to be broken within small limits. Use of nonconforming boundary meshes is also possible. Generally, such formulations pose the benefit of avoiding the definition of an overly stiff interface [9].

Several publications that have been the subject of significant academic attention focused on modernising the original [18] (expanding the idea from [19]) and dual CB methods [20]. In both situations, the authors proposed an augmented transformation matrix including high-order effects of residual modes, i.e. ones normally truncated in the selection of a condensation basis. In either case a noteworthy accuracy gain was observed in comparison to the traditional formulations. The main drawback is that penalty is incurred in terms of time required for building the CMS model, due to the evaluation of extra terms demanded for the updated reduction matrices.

A peculiar consequence of the procedure involved in [20] is that spurious modes having negative associated eigenvalues are shifted towards higher frequencies. These erroneous eigenvalues are a known problem for weak interface compatibility, especially if an insufficient reduction basis has been defined, which causes them to appear at lower frequencies. They signify unphysical behaviour and could become an obstacle to the correct approximation of the full FE model. An alternative treatment was suggested by Rixen [21]. The enhanced CB method was successfully used to conduct coupled loads analysis for satellites in [22]. Additional modifications to the scheme were reported in [23], where it was used in conjunction with algebraic substructuring and interface reduction. Test cases indicated impressive runtime performance with respect to the standard and enhanced CB, supplemented by greater accuracy of the condensation.

When referring to interface reduction, the application of condensation techniques to the $B$ set degrees of freedom is meant. Under unfavourable circumstances, even CMS finite element models could grow to excessive dimensions. This may happen in the presence of finely meshed inter-component joints, or inversely, physically large substructure connection boundaries, the approximation of which necessitates the retention of many DOFs in the $B$ set.

Various methods have been explored, traceable back to Craig and Chang [15]. Current procedures addressing the problem can involve a diverse range of reduction strategies. Examples include MPCs [24], singular value decomposition (SVD) [24], Legendre polynomials [25], or preliminary substructure analysis with coarse meshes [26]. More traditional means of reducing the interface are rooted in performing eigenvalue analysis on the boundary DOF set before or after component assembly. A review for CB-type approaches is given in [9]. Overall, different aspects of the topic are covered with a good degree of completeness in [9], [25] and [27].
In other relevant articles, Kim et al. [28] discussed existing component mode selection methods and proposed a new one, which was rigorously derived and tested by means of several example structures. Holzwarth and Eberhard [29] derived a CB-like method based on input-output ansatz functions originating from control theory and serving to replace eigenmodes in the CMS. They also reviewed other ‘exotic’ reduction bases, such as Krylov vectors, SVD-derived ones and amalgamations of both.

2.3 Uncertainty modelling in structural dynamics

2.3.1 Overview and classification of techniques

Modelling mechanical systems can be viewed as the generation of a mathematical idealisation of a collection of variables describing physical and geometric properties and their relation to a set of responses, e.g. displacement, stress and so on. Regardless of the model’s level of precision, a deterministic description inherently provides only a partial insight into the real world behaviour that could be observed. Factors such as manufacturing tolerances, non-ideal materials and approximations involved in the construction of the numerical representation intrinsically carry a degree of uncertainty. In structural dynamics, such effects are aggravated for an analysis of high frequency modes when element-based methods are used.

The comparatively new field of stochastic structural mechanics studies the modelling and quantification of uncertainty. The state of the art has been exhaustively studied and characterised in a number of publications [30]–[35]. These works have been chronologically ordered herein, and albeit not completely uniform in focus, together they form a complete description of the pivotal advances in the sphere. Due to the sheer volume of research conducted in the past three decades, even the classification of current methods is not a trivial task. A logical division is available in [31], and will generally be conformed to in this section. The fundamental theories can be ascribed to three main categories:

- Strategies aiming to evaluate the first two statistical moments of the desired response variables, in other words the mean and the variance. Perturbation methods are a prominent example.
- Reliability approaches, viewing the uncertainty problem as a means of extracting the probability of failure under some predefined conditions. Normally this is governed by the occurrence of rare events, thus the tails of the PDFs of the response quantities are of interest.
- Methods based on various flavours of the stochastic finite element method (SFEM), whose objective is the calculation of the global probabilistic nature of the aforesaid response variables, when viewing them as random processes.

It is worth emphasising that this division is subjective and quite blurred in reality. Many of the methods are strongly interconnected or share the same basic principles. For instance, given the response variance of a structure is known to a reasonable degree of accuracy, reliability can easily be ascertained. A further remark to be made is that
2.3. Uncertainty modelling in structural dynamics

Uncertainty at high frequencies is tractable with completely different methods, discussed in Section 2.5.4. Here, the discussion is limited to FEM.

Finally, it ought to be pointed out that the remainder of this section is focused on probabilistic methods. Prominent emerging non-probabilistic techniques, namely the interval FEM and its extension through fuzzy analysis [36] are covered in great depth in [32], [37]. Their case is argued by many researchers who perceive the lack of objective information on the statistical properties of structural uncertainties to render probabilistic simulation unjustifiable or unrealistic. Interval FEM prompts the analyst to define individual upper and lower bounds within which each uncertainty is contained. Fuzzy analysis extends this methodology by defining a level of membership that expresses to what extent a given value is in the range of possible input values. The actual solution is carried out through a succession of interval FEM analyses.

2.3.2 PSA, SFEM and reliability methods

Random variables

There is little doubt in the historical prevalence of perturbation methods to construct the statistical relationships linking the input and output variables of a system. In these approaches, a Taylor series expansion is applied to the mathematical operator describing this relation. Typically truncation to first or second order terms is employed. Perturbation theory is efficient for the evaluation of the first two statistical moments - it requires only the knowledge of the mean and variance of the input parameters. However, it becomes inaccurate when major parameters have a sufficiently large coefficient of variation (COV) normally around 0.15, according to [33]. This would not be unexpected with space structures [38]. Note that COV is defined as the ratio between the standard deviation and mean, i.e. $\text{COV} = \sigma / \mu$. The degree of nonlinearity present also negatively affects the applicability of perturbation schemes [38].

Within a more general setting, permitting model parameters of significant scatter, a Monte Carlo-based approach is the traditionally assumed path [39]. Realisations of the stochastic structure are explicitly generated, each one resulting in a deterministic solution case. Upon collection of a sufficiently large sample of the population, the response variability is estimated by direct statistical manipulation. This procedure requires some knowledge of the distributions of the random model properties.

The conventional probabilistic structural analysis dictates that they are modelled as independent random variables. Explicitly, the probability $P(a \leq X \leq b)$ of a random quantity $X$ in an interval $[a, b]$, and its mean $E(X)$, are given by its PDF $f_X$ as

$$P(a \leq X \leq b) = \int_a^b f_X(x) \, dx \quad (2.21)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx \quad (2.22)$$

where $E$ is formally the mathematical expectation operator. The $n^{th}$ central moment is given by

$$m_n = \int_{-\infty}^{\infty} (x - E(X))^n f_X \, dx \quad (2.23)$$
with the variance being $m_2$ or $\text{var}(X)$. The covariance $\text{cov}(X, Y)$ is

$$\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

(2.24)

and must be zero if $X$ and $Y$ are independent.

The standard PSA is advantageous primarily due to its straightforward implementation. Upon performing a direct Monte Carlo simulation (MCS), the procedure tends to be quite representative of the real uncertain structure and is thus the de facto verification technique for more advanced schemes. Such use is plentiful in literature, for instance in [40]–[42] in the framework of spacecraft design and analysis. Prior to these contributions, Esnault and Klein [38] compiled a guideline for the COV of various loads, materials and basic geometries encountered in space applications.

Among the intensely debated topics in PSA central place is taken by methods targeting reduction of the required number of samples for MCS. Normally, for each random variable, such as Young’s modulus or shell thickness, an appropriate PDF is assigned. Sampling of the global problem is done by taking uncorrelated realisations of all random variables. For this unoptimised procedure, required population sample sizes commonly encountered in literature are in the vicinity of 500 to 2000 realisations ([34], [40], [42], [43]). Sampling techniques obviating the use of such naïve approaches are summarised in [32] and include:

- Importance: more samples are generated in critical regions defined a priori. A widely adopted approach.
- Adaptive: after each simulation, the sampling distribution is modified based on the preceding results.
- Directional: uniformly distributed sample direction vectors are defined. Failure probability is the mean of the conditional probabilities obtained along each vector. Importance and adaptive sampling can be applied to each direction.
- Latin hypercube: the sampling space is divided into equi-probable subsets. In other words, the range of a random variable is split into a fixed number of non-overlapping, equally probable intervals.

**Random fields**

Many researchers have recognised modelling of the parameter space as a set of uncorrelated random variables as an oversimplification. It is unlikely that a physical quantity governed by some form of uncertainty would deviate from its mean value uniformly over its domain. For example, geometrical variation of a component due to limitations of manufacturing processes would likely be distributed, rather than fixed for the whole part. Modelling spatial variability of parameters gives rise to random fields, and respectively stochastic processes, if temporal effects are also included. This idea is the foundation of the stochastic finite element method.

Comprehensive overviews of the SFEM and its underlying principles are given in [44]–[46]. In essence, random fields are expressed in terms of field variables of the form...
2.3. Uncertainty modelling in structural dynamics

$v(x, \theta)$, with the arguments representing spatial variation and probabilistic behaviour, respectively. Generally, $E(v)$, $\text{var}(v)$, along with the corresponding spatial dependency of the field variable

$$\text{cov}(x_1, x_2) = E\{(v(x_1, \theta) - E[v(x_1, \theta)])(v(x_2, \theta) - E[v(x_2, \theta)])\} \quad (2.25)$$

fully define the stochastic field. Special treatment is usually required for the representation of (2.25). However, a more challenging aspect of SFEM is that a discrete computational domain has to be specified for the random fields over the application geometry. This translates into the necessity to manage an additional mesh in the FE model, which is one of the method’s drawbacks. Note that both perturbation and MCS approaches can be employed with the stochastic FEM in likeness to PSA.

In spite of SFEM’s academic popularity, deployment in commercial software remains currently limited, although work is being done in this direction \[47\]. Specialised SFEM programs also exist, and are discussed by Stefanou \[34\], whereas a MATLAB implementation was laid out in \[31\] and is freely accessible. Some practical applications have been published, such as design optimisation of shell structures \[48\].

### Reliability methods

A resurgence of approaching problems in stochastic mechanics from the point of view of structural reliability has been observed in the past two decades. In brief, the goal is to estimate the failure probability of the system, rather than the response confidence bands. It can be shown that direct Monte Carlo simulation demands a sample of the population of order $1/p_F$, where $p_F$ is the failure probability. Clearly, even the computational capabilities of today would be insufficient to cope with solving circa $10^6 \sim 10^8$ realisations of a complex problem in reasonable time.

The so-called variance reduction methods prioritise the estimation of failure probability $p_F$, which is much smaller than that of the system response quantities and hence the direct MCS. Two recent advancements have enabled that. Line sampling, introduced in \[49\], incorporates the somewhat abstract idea of identifying important directions towards high failure probability regions in the input parameter space. Realisations are then evaluated along this direction from randomly selected starting points, and the intersection with the failure regions are found. In \[50\] the procedure was shown to require only a few hundred samples for certain problems, contrasting with $\sim 10^7$ for conventional MCS. Application to spacecraft ensued shortly after, proving the robustness of the method \[40\], which even under the most unfavourable conditions performs comparably to the direct MCS.

The second strategy is called subset simulation \[51\]. Calculation of $p_F$ is achieved by expressing it as the product of a set of conditional failure probabilities, facilitated through intermediate failure events. The original problem is then reduced to the evaluation of the conditional probabilities, which can be made sufficiently large upon appropriate modelling. To that end, a Markov chain MCS based on the Metropolis-Hastings algorithm is employed. The method was found to have a performance similar to line sampling for a satellite structure \[41\], with both schemes requiring only a few hundred realisations for $p_F$ of order $10^{-6}$.
2.3.3 Modal approaches

Several techniques have been devised with the aim of lowering the computational intensity of MCS for probabilistic FEM on a per realisation basis. Early attempts circumvented the problem mostly by exploiting the advent of parallel computing \[52\]. In this article, the authors employed a finite element tearing and interconnecting type of domain decomposition to make the global problem amenable to concurrent processing.

Further insights on the topic were provided slightly later by Székely and Schuëller \[53\]. However, a more valuable contribution was the use of the subspace iteration method to hasten the computation of each Monte Carlo realisation. Using the set of eigenpairs extracted from the \(i\)-th sample as a starting solution for the next one, a speed-up of a factor of 3 was attained. An increase to 4 was possible, depending on whether a specific sample sorting procedure was applied. The idea was subsequently expanded with the incorporation of the subspace iteration-based method within CMS, along with some additional performance enhancements \[54\].

An interesting publication by Mace and Shorter \[55\] emerged at about the same time, and could be recognised as one of the first works to fully embrace component mode synthesis as a framework for uncertainty quantification. The suggested concept was to avoid perturbing properties in the physical space and instead directly inject variability into the modal parameters of the CMS subsystems. More expressly, the condensed component mass and stiffness are given by

\[
\begin{align*}
M_r^{(k)} &= \begin{pmatrix} M_{bb}^{(k)} & M_{bm}^{(k)} \\ M_{mb}^{(k)} & I \end{pmatrix}, & K_r^{(k)} &= \begin{pmatrix} K_{bb}^{(k)} & 0 \\ 0 & \Lambda_{mm}^{(k)} \end{pmatrix}
\end{align*}
\]

(2.26)

where subscript \(m\) corresponds to the modal representation of the \(I\)-set, i.e. the non-boundary DOFs. Lower case \(b\) refers to the reduced \(B\)-set matrices. \(\Lambda_{mm}^{(k)} = \text{diag} (\omega_1^{(k)^2}, \ldots, \omega_{m_k}^{(k)^2})\) contains the \(m_k\) fixed interface eigenvalues of subsystem \(k\). The authors defined a random diagonal matrix \(\tilde{\Lambda}_{mm}^{(k)}\) by modelling the respective \(I\)-set natural frequencies as random variables.

Assuming the perturbations are contained within some relatively narrow limits, the well-known results of perturbation theory could be utilised to efficiently propagate the random local parameters to the system-level eigensolutions \[56\]. In turn, they are cheaply superimposed to infer the final response of each MCS realisation. Indeed, the first order approximation expressions for the rate of change of the \(i\)-th global eigenvalue \(\lambda_i\) of the structure with respect to some design variable \(\nu\) is given by

\[
\frac{\partial \lambda_i}{\partial \nu} = \phi_i^T \left( \frac{\partial K}{\partial \nu} - \lambda_i \frac{\partial M}{\partial \nu} \right) \phi_i
\]

(2.27)

where \(\phi_i\) is the corresponding eigenvector. Note \(i\) is a generic counter and is unrelated to the \(i\)-set. For the sensitivity of \(\phi_i\) with respect to \(\nu\), it is possible to write

\[
\frac{\partial \phi_i}{\partial \nu} = -\frac{1}{2} \left( \phi_i^T \frac{\partial M}{\partial \nu} \phi_i \right) + \sum_{j=1,j\neq i}^N \frac{1}{\lambda_i - \lambda_j} \phi_j^T \left( \frac{\partial K}{\partial \nu} - \lambda_j \frac{\partial M}{\partial \nu} \right) \phi_i \phi_j
\]

(2.28)
where $N$ is the global system’s size. Invoking the definition of $\bar{\Lambda}^{(k)}_{nm}$ translates into a vast simplification of (2.27) and (2.28) for each realisation:

$$
\delta \lambda_i = \sum_s \phi_i^T \phi_i \delta \lambda^{(k)}_s, \quad \delta \phi_i = \sum_s \sum_{j \neq i} \phi_i^T \phi_i \phi_j \frac{\delta \lambda^{(k)}_s}{\lambda_i - \lambda_j}
$$

(2.29)

Here, $\delta \lambda^{(k)}_s$ is the perturbation imposed to $\omega^{(k)}_s$. By means of a simple non-uniform rod example, the preceding method was demonstrated to yield a good approximation to MCS of the unreduced structure. An extended discussion of the approach, within a more far-reaching context, was laid out in [57]. In a related article, Van den Nieuwenhof and Coyette [58] derived an efficient mixed formulation, employing a first order approximation for the eigenpairs in conjunction with Monte Carlo SFEM, while also proposing treatment for uncertain geometric features.

Recently, Remedia et al. [1]–[3] revisited the idea from [55], introducing the Craig-Bampton stochastic method. In essence, uncertainty is ascribed to the same modal stiffness parameters of the subsystems. The CBSM was formulated in the context of estimating responses induced by the action of multiple microvibration sources on spacecraft. Of course, it is applicable to more general structural dynamics problems, as they pertain to the same governing principles and equations. From a stochastic analysis perspective, the CBSM differs from [55] in that a direct MCS was used as its foundation instead of propagation of perturbations.

Attention is drawn to the fact that the authors validated the CBSM through test cases of substantial complexity, comprised of real spacecraft finite element models. Juxtaposition of experimental data, MCS of the full physical model and CBSM predictions indicated a good correlation for a COV of 10% in $\bar{\Lambda}^{(k)}_{nm}$. An immediate consequence is that the accuracy of (2.29) might be impaired, or its use invalidated altogether, for sufficiently complex realistic structures. The underlying assumption of small perturbations to the modal parameters would likely prove inadequate to envelop the actual behaviour variability.

Additional investigations into the CBSM are the main subject matter of Chapter 3. The preceding argument is extended by studying the acceptable randomisation levels for the case of a spacecraft FE model exposed to a distributed load. To that end, the original methodology is reformulated to permit such excitations. In addition, multiple steps are taken to significantly enhance its computational efficiency without compromising prediction accuracy.

### 2.3.4 Nonparametric probabilistic approach

Constructing the stochastic structural representation without an explicit a priori knowledge or modelling of random variables or processes poses an obvious appeal. Such a methodology has the potential to save valuable time from design efforts and data collection through test campaigns. The nonparametric probabilistic approach of Soize [59]–[61] is the most universally acclaimed technique developed in relation to this concept. It has seen substantial academic interest and has recently been integrated in commercial structural analysis software, e.g. Nastran.
Two underlying principles are central to the method. Firstly, random matrix theory, which was introduced in the 30s in mathematical statistics and has later been the subject of numerous studies by physicists and mathematicians. A random matrix is essentially a matrix-valued stochastic variable. A key concept is the study of ensembles, that is, sets of random matrices with specific statistical properties. Secondly, the maximum entropy principle, which arises from information theory. It could be interpreted as a method of building a stochastic model of a system within the constraints of the available information, such that 'entropy', or in mechanical terms, uncertainty, is maximised.

In the nonparametric probabilistic approach, new random matrix ensembles are defined with the intention of satisfying algebraic requirements originating from mathematical properties of structural mass and stiffness matrices. Expressly, for a known matrix $A$, which here is a placeholder for $M, C, K$, the technique prescribes the construction of a random matrix $\tilde{G}_A$ from the aforesaid new ensemble. Realisations of $\tilde{A}$ are explicitly obtained by sampling $\tilde{G}_A$, using the relation

$$\tilde{A} = L_A^T \tilde{G}_A L_A, \quad A = L_A^T L_A$$

The known deterministic $A$ is in fact precisely the mean of $\tilde{A}$. The dispersion of the stochastic matrix $\tilde{G}_A$ is controllable via a single positive real parameter $\delta_A$, independent of the dimension of the problem $N$. Statistical moments of the uncertain structure can then be computed by a standard MCS. In addition, note that $A$ is not restricted to physical space discretisations and may also represent some reduced counterparts of the system matrices. An updated version of the nonparametric approach was also recently published [62]. It allows modelling errors and system parameter uncertainties to be considered separately.

The original technique has been extensively validated in various settings. For example, Capiez-Lernout et al. [43] compared the method against a traditional parametric Monte Carlo simulation for the low-frequency random response of a satellite in two configurations: free and coupled to the launcher. In [63], comparison is done in a more strictly mathematical sense with the new random matrix ensembles being evaluated against classic ones, such as the Gaussian orthogonal ensemble. A conceptually related nonparametric method was demonstrated in [64], where modelling of uncertain boundaries in the DS framework was emphasised.

### 2.4 Acoustic space modelling

#### 2.4.1 Spacecraft aeroacoustic excitations

**Description of the launch environment**

In the context of the spacecraft launch environment, the acoustic excitation defining the limiting vibroacoustic responses can be classified into two main categories - diffuse sound field (DSF) and turbulent boundary layer (TBL) as recognised in [65]. The former one occurs at lift-off, when the high velocity plumes generate acoustic waves.
that are in turn reflected from the launch pad and finally reach the LV again. Since sound is a pressure fluctuation in a medium, these waves induce vibration of the LV’s surface. Part of the energy is dissipated within the LV’s structure, some is transmitted into the fairing as acoustic noise, and some in the form of random vibrations through the spacecraft-LV interface. The actual engine operation also contributes to the structure-borne noise emitted into the fairing.

The turbulent boundary layer on the other hand is at the peak of its prominence during transonic flight, and to a lesser extent when the LV reaches supersonic speeds. As the name suggests, the noise source is due to turbulence in the boundary layer of the flow around the launcher. Non-smoothness of the external shape of the vehicle leads to detached aerodynamic flow. Local shockwave formation and termination at transonic speeds measurably aggravates this effect, promoting violent flow separations. The resultant TBL detachments manifest themselves as acoustic noise in the payload cavity.

Typically, the expected maximum noise levels are provided by launcher manuals in terms of sound pressure level (SPL) given by

\[
SPL = 10 \log_{10} \frac{p^2}{p_{ref}^2}
\]  

(2.31)

where \( p_{ref} = 2 \times 10^{-5} \text{ Pa} \) and \( p \) is the root mean square (RMS) value of the actual pressure. Equivalently, in terms of power spectral density (PSD):

\[
PSD_k = \frac{p_k^2}{\Delta f_k} = \frac{p_{ref}^2}{\Delta f_k} \frac{SPL_k}{10^{\frac{SPL_k}{10}}} \]  

(2.32)

The division is normally made into one or one third octave bands, numbered herein by the subscript \( k \), with a central frequency of \( f_k \) and width of \( \Delta f_k \).

The sound field due to the launch environment varies with flight stage, but is overall broadband and random in nature. As such, it is amenable to statistical modelling as a product of a spatial correlation function and a spectral density one. The former is normally obtained in a semi-empirical manner and contains the distribution of the excitation in space, whereas the latter represents its intensity in the frequency domain. Such excitations are considered weakly stationary random processes from a mathematical viewpoint.

The diffuse sound field

The principal characteristic of the DSF, also known as a reverberant field, is a sound pressure level that is spatially and temporally invariant. It is formed by sound waves travelling in all directions, which is equivalent to a large set of independent random noise sources. The lack of correlation between them means interference does not produce any regular features in the pressure field. Consequently, it is completely homogeneous and isotropic.

The characterisation of DSFs is an important concept, and the fundamental theory can be found in a multitude of works, such as [66] and [67]. A common mathematical
idealisation is called a pure-tone diffuse sound field. It is based on the assumption that infinitely many plane waves of random amplitude, incidence and directivity are summed to form the DSF. The resultant standard correlation function is expressed as the cardinal sine:

\[ f_c(r, \omega) = \frac{\sin(kr)}{kr} = \text{sinc}(kr) \] (2.33)

where \( r = |p - q| \) is the distance separating the points \( p \) and \( q \), while \( \omega \) and \( k = \omega/c \) bear the usual meanings of frequency and acoustic wavenumber, respectively. An extended derivation of the spatial-temporal correlation, which is not restricted to use in pure-tone or band-limited reverberant fields, has been presented by Rafaely \[68\].

Since the DSF is homogeneous, it holds that its cross-power spectral density is defined by a reference spectrum \( S_0 \), constant at any point, as

\[ S_{pq}(\omega) = S_0(\omega) f_c(r, \omega) \] (2.34)

**Turbulent boundary layer**

Unlike reverberant fields, the excitations induced by detached flow along the launch vehicle are inhomogeneous processes. The PSD is related to the geometric mean of the spatially variable auto-spectral density. More specifically, it is obtained from

\[ S_{pq}(\omega) = \sqrt{S_p(\omega) S_q(\omega)} f_c(p, q, \omega) \] (2.35)

Different definitions of the correlation function exist for turbulent boundary layer excitations. Most commonly the pressure field induced by the TBL is expressed by the Corcos model \[69\]:

\[ f_c(p, q, \omega) = e^{-|r_x/L_x} e^{-|r_y/L_y} e^{-i\omega r_y/U_c} \] (2.36)

Here, subscripts \( x \) and \( y \) denote the Cartesian coordinates in the spanwise and streamwise directions, respectively, while \( r_x \) and \( r_y \) are the distances between \( p \) and \( q \) in these directions. \( L_x, L_y \) are the corresponding correlation lengths, \( U_c \) is the convection velocity and \( i \) has the usual meaning of the imaginary unit, as harmonic time-dependence is followed in the model. In addition, it is normally assumed that

\[ L_x = \frac{\alpha_x U_c}{\omega}, \quad L_y = \frac{\alpha_y U_c}{\omega} \] (2.37)

with \( \alpha_x \) and \( \alpha_y \) being constants. It is worth pointing out that unlike (2.33), the Corcos model is an empirical result. It suggests the correlation structure is varying independently in the streamwise and spanwise directions. Alternative modelling formulations are also available for the TBL, for example by Cockburn and Robertson \[65\]. The more recent research by Goody \[70\] is extensible to evaluating pressure fluctuation auto-spectra along shaped bodies, as opposed to only flat structures.
2.4.2 FEM for exterior problems

Modelling with finite elements

Similarly to representing structures where the finite element method is accepted as the default approach, FE descriptions of the acoustic space are widely adopted. The main benefits against competing approaches, such as BEM, can be found in FEM’s long history of active use and development. Currently, algorithms dealing with the sparse matrices arising in FEM are highly optimised and arguably more straightforward to manipulate in general [71]. An extensive introduction to the numerical aspects of FEM implementation for sound propagation problems can be found in [72]. Another excellent resource is the recent book by Atalla and Sgard [73], essentially covering all critical aspects of the broad topic of acoustic simulation with deterministic schemes.

![Diagram of interior and exterior acoustic problems](image)

Figure 2.2: Interior and exterior acoustic problems. Γ denotes the fluid-structure boundary and \( n \) the surface normal.

Similarly to solids, finite element representations of fluids are obtained by discretisation with 3-D elements. However, a clear distinction is made on the basis of whether the acoustic space is physically contained within the structure, or encloses it. The two cases are referred to as interior and exterior problems, respectively, and are illustrated on Figure 2.2. Special treatment by means of imposing some form of artificial boundary is needed for the latter one, as it calls for the filling of an infinite space with finite elements. Due to the relevance of exterior problems to vibroacoustic modelling for space applications, this particular case will be discussed in more detail.

Regardless of the problem configuration, in a homogeneous and isotropic medium the propagation of waves is governed by the Helmholtz equation

\[
\nabla^2 p(x) + k^2 p(x) = 0
\]

where \( \nabla^2 \) is the Laplace operator, \( p(x) \) is the pressure at a point \( x \) in space and \( k \) is the wavenumber. For an external acoustic medium, a crucial requirement that must be fulfilled is the Sommerfeld radiation condition, stating that waves infinitely far away...
from the boundary $\Gamma$ are outgoing. It is written as

$$\lim_{|x| \to +\infty} |x|^{(d-1)/2} = \left( \frac{\partial p(x)}{\partial |x|} - ikp(x) \right)$$

(2.39)

where $d = 1, 2, 3$ is the space dimension. A well-posed mathematical description dictates that an appropriate boundary condition (BC) should be imposed on the defined artificial boundary. This proves a challenging task, particularly if a general solution is sought that is expected to be numerically stable, efficient, accurate in its representation of the underlying physics and also valid for a range of media, geometries and wave types. Harari [74] reports the main approaches developed in response to this issue.

Absorbing boundary conditions, PMLs and IEMs

In the pursuit of applying FEM to unbounded domains, several methods have gained widespread approval over the decades. Absorbing boundary conditions (ABCs) were the first to emerge, with significant breakthroughs made in the late 70s. Broadly speaking, the aim of ABCs is to define a specific condition on the outward fluid boundary which forms the edge of the FE space, such that spurious reflections are eliminated. Among the most well-known ones are the ABCs proposed by Bayliss and Turkel [75], with many others to follow or improve on the concept.

In brief, classic ABCs often yield satisfactory results, but this assertion is strongly affected by the specifics of the application [72]. Notwithstanding, they have enabled the definition of the contemporary high-order absorbing boundary conditions summarised in [76]. The principal advantage of the newer formulations is that solution accuracy up to any desired order can be required without causing an ill-conditioning of the problem.
The concept of absorbing layers was conceived chronologically in parallel to absorbing boundary conditions. Like ABCs, an artificial boundary is defined, but the principal difference is that it is a narrow region of finite extent. The equations of motion undergo certain modifications within that space, devised so that the outgoing waves can be absorbed.

The absorbing layer technique paved the way for the seminal work of Berenger [77], in which the so-called perfectly matched layer (PML) is proposed. A PML is designed to have completely zero reflectivity with respect to any plane wave at the artificial boundary $\Gamma_R$, depicted on Figure 2.3. In addition, the acoustic solution decays exponentially within the PML region, with waves potentially undergoing multiple reflections between its inner and outer bounds. Waves reaching the fluid-structure interface $\Gamma$ tend to be of insignificant strength. The extension of PMLs, originally addressing problems in electromagnetism, to acoustics is owed to Turkel and Yefet [78].

An approach rather different to the previously described ones also emerged at the same time. Infinite element methods (IEMs) make use of ‘elements’ represented by semi-infinite prisms with associated shape functions attempting to mimic the behaviour of the solution in the far-field. Modern versions trace their foundations to the works of Burnett [79] and Astley et al. [80]. Such IEMs are constructed in separate, usually spheroidal, coordinate systems and employ shape functions that automatically satisfy the Sommerfeld radiation condition. Overall, present-day IEMs may be considered a viable alternative to boundary element methods for exterior problems. For an in-depth analysis the reader is referred to the papers of Tsynkov [81] and later Thompson [82], who also cover the related techniques detailed above.

### 2.4.3 Boundary element methods

#### Fundamentals of BEM

In the field of computational acoustics, boundary element methods have been used for decades and can be viewed as a long-standing rival of FEM. As suggested by its name, the BEM is also a deterministic technique. Its primary asset is that no spatial discretisation is required. Instead, only a surface mesh on the interface of the fluid medium with the solid body needs to be constructed, which makes BEM instantly appealing for resolving exterior problems. In addition, the Sommerfeld radiation condition (2.39) is inherently satisfied, constituting a further advantage over FEM.

Obtaining and utilising the BEM formulation for the Helmholtz equation (2.38) is extensively described in literature. For instance, the book by Ciskowski and Brebbia [83] is entirely dedicated to the subject of boundary elements (BEs) in acoustics. More recent works, such as [72] and [73], also provide an excellent overview. The fundamental developments and features of numerous formulations have been reported in the review by Yu et al. [84]. Thus, the relevant theory is only compactly summarised herein.

Let the domain of the wave propagation medium be denoted $\Omega$ and $\Gamma$ be its boundary. A Dirichlet BC prescribes a known pressure $p(x)$ $\forall x \in \Gamma$, whereas for a Neumann BC
the fluid normal surface velocity $v_n^f(x)$ is available on $\Gamma$ and follows the relationship

$$q(x) = \frac{\partial p(x)}{\partial n(x)} = i\omega v_n^f(x)$$

(2.40)

where $n(x)$ is the unit surface normal at $x$, $\rho$ is the fluid density and $\omega$ the angular frequency. A Robin BC is any linear combination of the other two types. The governing equation (2.38) can be reformatted into the conventional boundary integral equation (CBIE):

$$c(x)p(x) + \int_{\Gamma} \frac{\partial G(x,y)}{\partial n(y)} p(y) d\Gamma(y) = \int_{\Gamma} G(x,y)q(y) d\Gamma(y) + p_{inc}(x)$$

(2.41)

Here, $x$ and $y$ are receiver and source points, respectively. Additionally, $c(x)$ takes values depending on where $x$ is located. For exterior problems it is defined as $c(x) = 1$ for a point in the fluid region, but not on $\Gamma$, $c(x) = 1/2$ for $x \in \Gamma$ if the surface is smooth, and 0 in the interior. The term $p_{inc}(x)$ describes the pressure due to an incident wave and does not appear for pure acoustic radiation problems. Finally, the free-space Green’s function in 3-D space, for $e^{-i\omega t}$ time-dependence, is

$$G(x,y) = \frac{e^{ikr}}{4\pi r}$$

(2.42)

with $r = |y - x|$ being the distance between the points. Discretisation of (2.41) may be done by a collocation, Galerkin or Nyström method. Considering the first case, which tends to be the most straightforward to implement and enjoys notable popularity ([85], [86]), we can set

$$H_{ij} := \delta_{ij}c(x_i) + \int_{\Delta \Gamma_j} \frac{\partial G(x_i,y)}{\partial n(y)} \phi_j(y) d\Gamma(y)$$

(2.43a)

$$G_{ij} := \int_{\Delta \Gamma_j} G(x_i,y)\phi_j(y) d\Gamma(y)$$

(2.43b)

where subscript $ij$ denotes element position in the matrix, $\delta_{ij}$ is the Kronecker delta, with values of 1 at $i = j$ and zero otherwise, $\Delta \Gamma_j$ is the part of the boundary represented by the surface of the $j$-th element, and $\phi_j$ are the interpolation, or shape, functions. Then the matrix form of the CBIE is written in terms of the discrete forms of the single-layer and double-layer potential operators $G$ and $H$ as

$$Hp = Gq + p_{inc}$$

(2.44)

Provided a BC is specified over $\Gamma$, i.e. known pressure (Dirichlet), normal velocity (Neumann) or a Robin condition, (2.44) is amenable to solution via standard linear system approaches [83]. Note that for collocation, the system matrices are in general full, complex and non-symmetric, whereas Galerkin BEM yields better conditioned, symmetric systems, but requires the evaluation of double instead of single surface integrals, which is typically more challenging and computationally expensive [87]. The matrix denseness in particular proves perhaps the most prohibitive shortcoming of BEM for the solution of large problems. Its circumvention is a fascinating topic that is briefly addressed later in this literature survey.
2.4. Acoustic space modelling

Treatment of irregular frequencies

A crucial point of concern is raised with the use of the classic CBIE for the resolution of exterior domains. It fails to provide unique solutions at characteristic frequencies associated with the interior Dirichlet problem. This issue is recognised entirely as a mathematical drawback and has no attached physical meaning. Two vastly different schemes have been proposed in the late 60s and early 70s, and have since been accepted as the de facto approaches.

Initially, the combined Helmholtz integral equation formulation was proposed by Schenck [88]. Additional constraints are defined on interior points, causing an overdetermined linear system. While the non-uniqueness problem is resolved at low frequencies, this is not necessarily the case in other ranges. Furthermore, no consensus has been made on how to define the constraint points in an arbitrary setting.

The second solution, demonstrated by Burton and Miller [89], requires taking the normal derivative of the CBIE, thus obtaining the normal derivative boundary integral equation (NDBIE). In the original paper, a proof was presented, demonstrating that a linear combination of these two equations yields a unique solution at any frequency, provided that the coupling constant $\alpha$ has a nonzero complex part.

Starting from (2.41), the NDBIE can be written explicitly:

$$c(x) \frac{\partial p(x)}{\partial n(x)} + \int_{\Gamma} \frac{\partial^2 G(x, y)}{\partial n(y) \partial n(x)} p(y) d\Gamma(y) = \int_{\Gamma} \frac{\partial G(x, y)}{\partial n(x)} q(y) d\Gamma(y) + \frac{\partial p_{\text{inc}}(x)}{\partial n(x)}$$

(2.45)

The discrete forms of the hypersingular operator and the adjoint double-layer potential, $E$ and $F$ respectively, can be defined as

$$E_{ij} := \int_{\Delta \Gamma_j} \frac{\partial^2 G(x_i, y)}{\partial n(y) \partial n(x_i)} \phi_j(y) d\Gamma(y)$$

(2.46a)

$$F_{ij} := -\delta_{ij} c(x_i) + \int_{\Delta \Gamma_j} \frac{\partial G(x_i, y)}{\partial n(x_i)} \phi_j(y) d\Gamma(y)$$

(2.46b)

and the matrix form of (2.45) follows analogously to (2.44):

$$Ep = Fq + q_{\text{inc}}$$

(2.47)

Finally, taking a linear combination of (2.44) and (2.47), the discrete Burton-Miller equation is arrived at:

$$(H + \alpha E)p - p_{\text{inc}} = (G + \alpha F)q + \alpha q_{\text{inc}}$$

(2.48)

A typical value for the coupling parameter is $\alpha = i/k$, but care must be exercised how it is selected, on the basis of the assumed time-harmonic dependence [90]. A survey on its effect on (2.48) is available in [91]. A noteworthy issue with schemes stemming from the NDBIE is the presence of $O(1/r^2)$ and $O(1/r^3)$ terms in the integral kernels, resulting in
what is known as strongly-singular and hypersingular integrals, respectively. Standard methods, such as Gauss-Legendre quadratures over the elements cannot handle the problem, and a lot of research has been conducted to mitigate this difficulty. Scuderi [92] proposed a solution for collocation and subsequently Galerkin BEM [87], based on various nonlinear variable transformations to smooth the singularity, or move it away from the integration domain. Matsumoto et al. [85] derived a closed-form expression for all integrands in the classic Burton-Miller equation for piecewise-constant triangular elements, a popular choice in BEM. Their work was later refined by Wu et al. [93]. A promising very recent formulation is found in [86], showing a non-singular version of the governing equation for linear shape function triangles. Yet another notable recent enhancement is owed to Li and Huang [94]. The authors incorporated a singularity subtraction technique to regularise the strongly-singular and hypersingular operators, so that only weak singularities remain.

Fast BEMs

Inherently, the influence coefficient matrices produced by BEM are fully populated, regardless of the employed governing equation form and discretisation scheme. The natural consequence is a storage requirement, with respect to the total number of elements \( N \), of \( O(N^2) \) and \( O(N^3) \) linear system solution complexity. The latter can be sharpened to \( O(N^2p) \), where \( p \ll N \), by substituting traditional direct procedures with an iterative method that progressively approximates the solution in a Krylov subspace, and \( p \) is the number of iterations performed to convergence. The most commonly embraced one for a broad class of matrices, including BEM-derived ones, is the generalized minimum residual (GMRES) method [95]. Nonetheless, this still fares unfavourably compared to the asymptotically much lower cost of FEM.

Successful implementation of Krylov methods relies to a great extent on computing matrix-vector products (MVPs) efficiently. This is the cost associated with avoiding cubic complexity operations, such as LU decompositions or inversions. A breakthrough in the late 80s enabled matrix-vector multiplication with much lower complexity, ranging from log-linear \( O(N \log(N)) \) to linear. In the seminal paper of Greengard and Rokhlin [96], a technique called fast multipole method (FMM) was introduced. It was used to effectively construct data-sparse coefficient matrices for N-body interaction problems. Boundary integral equations are very similar to interacting particles in that the constituent non-local operators are discretised via approximations on finite sets of elements and points. Thus the FMM is intrinsically suitable for boundary value problem applications.

The FMM functions by exploiting smoothness properties of the underlying integrals and uses multipole expansions of the kernels to approximate far-field interactions. In other words, interaction between clusters of points, sufficiently well separated geometrically, is described by the aforesaid approach. In physical terms, the effect of an ensemble of objects in the far-field may be expressed as a single, combined influence at the observation cluster. Algebraically this is equivalent to a low-rank matrix partition, as per Figure 2.4. On the other hand, the near-field is only amenable to traditional treatments and is constructed numerically by standard BEM quadratures.
Modern FMMs make use of multilevel hierarchical partitioning of the physical space, with some achieving the optimal $O(N)$ complexity for matrix-vector multiplications. Noteworthy developments include the one by Shen and Liu, where a low-frequency FMM BEM for Helmholtz problems was based on the Burton-Miller equation. Later, Gunerov and Duraiswami successfully defined a broadband version of the algorithm. Their improvement solved one of the main limitations of the fast multipole method, namely the need to carefully tailor the selected multipole expansions to the solution frequency of interest.

![Diagram of matrix block](image)

Figure 2.4: Matrix block $t \times s$ corresponding to the far-field interaction between the subdomains $\Omega_t$ and $\Omega_s$.

Exploiting the idea of hierarchically-defined interactions further, several other fast BEM methods were created, such as the panel clustering approach. However, it also depended on the explicit knowledge of the integral kernels and their expansions. The mosaic-skeleton approach of Tyrtyshnikov paved the way for purely algebraic schemes for the compression of large BEM matrices. A particularly influential observation was that low-rank matrix block approximations could be built by calculating only a few of the original matrix’s entries. The idea was refined by Bebendorf and the resulting technique was called the adaptive cross approximation (ACA). It requires only the extraction of specific rows and columns of a matrix block in order to build a low-rank representation. Further developments of ACA and related work are available in many subsequent papers, for example [103-105].

Subdivision of a matrix into a hierarchical block tree, together with the evaluation of low-rank approximations of the blocks, formed the basis of the $\mathcal{H}$-matrix framework ($\mathcal{H}$ stands for 'hierarchical'). Mathematically, it can be thought of as an algebraic generalisation of the FMMs that shares a close relation to panel clustering. Also introduced by Hackbusch, $\mathcal{H}$-matrices have proven to be among the most well-received fast BEM methods. Special formatted arithmetic has been developed, enabling matrix-vector and matrix-matrix operations, as well as various decompositions, inversions and other traditional notions of linear algebra, to be transferred to the $\mathcal{H}$-matrix framework. Exhaustive descriptions can be found in [106, 107] and in a book by Bebendorf, entirely dedicated to this subject [108].

Enhancements to the $\mathcal{H}$-matrix approach include a possible reduction of storage requirements and complexity of MVPs from $O(N \log^a(N))$ to $O(N)$ [109]. The so-called $\mathcal{H}^2$-matrices form a special $\mathcal{H}$-matrix subset, and are conceptually very close to FMMs.
In [108] and [110], an alternative scheme was devised to achieve the same storage performance, by using Chebyshev polynomial recompression for ACA. A popular system for the block structure refinement of already constructed hierarchical matrices is ‘coarsening’, presented by Grasedyck [99]. An example is depicted on Figure 2.5. The rationale behind it is that the initial and often geometry-based partitioning tends to yield sub-optimal compression. A block agglomeration strategy is detailed in response to the problem. It has been shown to measurably decrease the Η-matrix memory footprint and MVP calculation time for numerous problems.

Finally, the wavelet Galerkin scheme is a method principally different to the FMM, panel-clustering and their algebraic generalisations, i.e. mosaic-skeleton and Η-matrices. The use of wavelet basis functions permits the representation of the integral kernels on various resolution levels. This feature, together with some other special properties, allows the Galerkin BEM coefficient matrices to be constructed with primarily zero or nearly-zero entries that can be truncated. The compression pattern is very different to the multilevel block structure of the other fast BEM methods. An overview is available in [111], and the mathematical principles are covered in [112].

For realistic problems, the wavelet BEM proves challenging to implement due to the very involved quadrature definitions [113]. Nevertheless, it exhibits vast performance benefits over FMMs and ACA-based Η-matrices, according to the study published in [114]. A comparison between the latter has been reported by Brunner et al. [115] for a case of surface ships, i.e. half-space acoustic problems. The authors concluded that the choice of time-wise best performing technique is dictated by the number of GMRES iterations that are needed to attain solution convergence. With increasing mesh size, the FMM gained a significant lead in terms of memory consumption, a result later affirmed in [114].
2.5 Vibroacoustic analysis

2.5.1 FEM-based methods

Coupled elasto-acoustic FEM-FEM

In the comparatively simple case of a sufficiently rigid structure, the presence of the fluid does not significantly modify its modes of vibration and dynamic response. A stationary body can be specified purely as an incident field scatterer and the resulting problem is entirely a fluid one, with a Neumann $BC$ $\forall x \in \Gamma : v_n^i(x) = 0$. Alternatively, a vibrating solid on which the acoustic medium has only a marginal effect represents a radiation problem. Both settings are subject to efficient treatment by deterministic methods. For radiation, one need first compute the normal surface velocity at the interface via standard FE analysis, and then use it to impose a Neumann boundary condition for the acoustic problem. In brief, such uncoupled problems lend themselves to an uncomplicated sequential solution for the two domains.

Unfortunately, such a procedure is commonly not acceptable. Aerospace applications are characterised by lightweight, thin structures and high-energy broadband excitations. This combination dictates the need to simultaneously consider both physical phenomena. In this coupled context, a traditional procedure of describing structure-acoustic interactions involves modelling both problems with finite elements (116, 117). The general form of the governing equation can be expressed as:

$$
\begin{bmatrix}
K_s + i\omega C_s - \omega^2 M_s & T_{sf} \\
\rho \omega^2 T_{sf}^T & K_f + i\omega C_f - \omega^2 M_f
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
= 
\begin{bmatrix}
f_s \\
f_f
\end{bmatrix}
$$

(2.49)

Here, $u$ and $p$ indicate nodal displacement and pressure. $K$, $C$ and $M$ have the usual meaning of stiffness, damping and mass, with subscripts $s$ and $f$ denoting structural and fluid quantities, respectively. The corresponding nodal loads are given by $f_s$ and $f_f$, while $T_{sf}$ is a coupling matrix mapping the structural to the acoustic DOFs. Finally, $Z$ is the global dynamic stiffness matrix.

Note that (2.49) is dependent on $\omega$, and a separate solution is needed at each frequency point. Of course, the same applies to BEM equations, such as (2.48). Various procedures exist for solving (2.49). Without delving into excessive detail on their execution, the most common strategies will be summarised next, particularly for the relevant aeroacoustic loads introduced in Section 2.4.1.

General solution strategy

Denoting the total nodal load vector by $x$, the corresponding output $y$ is given by

$$
y = H x
$$

(2.50)

where $H = Z^{-1}$ is the input-output transfer function. As the matrix $H$ tends to be very large, solutions using direct inversion of $Z$, or decomposition-based variants thereof,
are not normally viable. Modal procedures, on the contrary, successfully mitigate that problem. Ideally, $H$ would be symmetric or Hermitian. Then powerful algorithms, such as the Lanczos method, can be employed to rapidly solve (2.49). Otherwise, more general routines exist which can still make use of the sparsity of $Z$. The reader is referred to the book by Golub and Van Loan [118] for an expansive resource that covers the relevant topics in linear algebra.

Several works in which symmetrisation of $Z$ is explicitly targeted also exist in literature. In [119], the authors devised such an approach and an apparent good agreement was indicated for a simple one dimensional problem against an analytical solution. More recently, Ding and Chen [120] exercised a similar approach for modelling thin-walled acoustic cavities. Again, reasonably good correlation to closed-form solutions was shown for simple problems.

**Weakly stationary random process excitation**

In a general vibroacoustic setting, it could be expected that the fluid domain requires some form of explicit modelling, similarly to the structural one. However, this is not necessarily the case with acoustic fields amenable to treatment as weakly stationary random processes, such as those outlined in Section 2.4.1. In fact, application of the DSF or TBL directly as a distributed random excitation to a standard structural FE model presents an opportunity for a simplified, streamlined analysis.

Indeed, for a weak coupling scenario, Séon and Roy [121] suggested such a scheme in which only the elastic FE matrix is built and no acoustic space discretisation is done. The authors claimed acceptable agreement of their model’s predictions compared to experimental results, the case study being the design of a re-entry vehicle. Nevertheless, they did recognise the importance of the coupling strength on the applicability of such techniques. At this stage it must be pointed out that the procedures described next have been derived with a full elasto-acoustic FEM model in mind, as per (2.49). However, they remain completely unaltered and valid for an elastic-only dynamic stiffness $Z$, as was the case with their deployment in [121].

To apply a distributed random load, a nodal pressure cross-spectral density matrix $S_P$ is initially constructed. In essence, it is a discrete version of $S_{pq}(\omega)$, introduced in equations (2.34) and (2.35) for the cases of DSF and TBL, respectively. More general theory on the numerical synthesis of correlated random pressure fields can be found in [122]. Entries of $S_P$ assume non-zero values only when the corresponding pair of nodes is on the fluid-structure interface. Through the coupling matrix $T_{sf}$,

$$S_x = T_{sf} S_P T_{sf}^T$$

conversion of $S_P$ to a nodal loads matrix $S_x$ is facilitated. Then the matrix $S_y$ of PSD outputs is given by

$$S_y = \overline{H} S_x H^T = \overline{H} T_{sf} S_P T_{sf}^T H^T$$

where $\overline{H}$ is the complex conjugate of the transfer function.
Efficient evaluation of $S_y$

For realistic problems, direct evaluation of $S_y$ may be undesirable or even impossible on acceptable timescales. Two conventional paths of dealing with this obstacle can be identified. The first one involves a preliminary dynamic reduction, representing (2.49) in modal space on a truncated basis of eigenvectors. The method follows the standard logic of performing modal condensations to reduce computing costs, as described in Section 2.2.2. Following some analytical steps shown in [117], equation (2.52) can be rewritten in terms of modal quantities, indicated by a subscript $\Phi$:

$$S_{\Phi y} = H_{\Phi} T_{\Phi y} S_{\Phi} T_{\Phi y}^T H_{\Phi}^T$$  \hspace{1cm} (2.53)

The modal receptance and PSD output matrices $H_{\Phi}$ and $S_y$ are of size $r \times r$, where $n$ is the number of physical degrees of freedom, $r$ the number of eigenvectors in the basis $\Phi$ and $r \ll n$. Furthermore, $H_{\Phi}$ is usually diagonal.

Another option is an evaluation of $S_y$ in physical coordinates via a ‘pseudo load-cases’ approach, proposed by Coyette et al. [117], and later refined in [123]. Taking a decomposition of the form

$$S_P \approx Q_{sp} D_{sp} Q_{sp}^T$$  \hspace{1cm} (2.54)

where the diagonal matrix $D_{sp}$ contains only the $r$ dominant eigenvalues of $S_P$. Substituting (2.54) into (2.52), the approximation

$$S_y \approx (HT_{sf} Q_{sp}) D_{sp} (HT_{sf} Q_{sp})^T = XD_{sp} X^T$$  \hspace{1cm} (2.55)

is obtained. This decomposition can be interpreted as presenting the random process algebraically as a set of $r$ uncorrelated load cases associated with the columns of $Q_{sp}$.

A slightly different method involves starting by factorising $S_x$ instead of $S_P$. The computation cost of $X$ is determined by $r$, and tends to rise for turbulent boundary layer excitations and at higher frequencies. To avoid this setback, Coyette and Meerbergen [123] devised a procedure that involves a direct partial decomposition of $S_y$ and demonstrated that its efficiency surpasses the pseudo load-case method, since lower ranks $r$ are required irrespectively of the solution frequency.

Even more recent takes on the problem are outlined in [124] and [125]. An algebraic sampling procedure was proposed for the generation of realisations of the random process determined by $S_P$. A Cholesky decomposition $S_P = L_{sp} L_{sp}^T$ is taken instead of (2.54). Samples $S_{sp}^k$ are extracted from

$$S_{sp}^k = L_{sp} \zeta, \quad C_k = e^{i\theta_k}, \quad \theta_k \in [0, 2\pi]$$  \hspace{1cm} (2.56)

where $\zeta$ is a vector of random angles with $k$-th entry $\theta_k$. The overall response $y$ is built by statistical manipulation of a sufficiently large pool of realisations.

In addition, both papers covered the idea of direct sampling, in which the diffuse sound field is approximated by a set of discrete plane waves with random phase angles. Each of them produces a blocked pressure excitation along the fluid-structure interface, and the final response is also obtained via statistical manipulation of the associated load cases. An updated spatial correlation function for the DSF has been derived, in order to aid the selection of appropriate phases for the plane waves.
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2.5.2 Coupled FEM-BEM

Standard formulation

Considering the intrinsic suitability of the boundary element method for exterior problems, the fact that coupled FEM-BEM methods are commonplace in vibroacoustics is unsurprising. Logically, the statements from Section 2.5.1 pertain to this approach in the case of a negligible elasto-acoustic interaction, and a sequential solution is possible. Within the broader context, a coupled FEM-BEM equation similar to (2.49) can be derived. Before proceeding with a condensed introduction to the relevant theory, it is worth noting that a brief further discussion on the topic is presented in Chapter 5. For the Burton-Miller formulation (2.48), let

\[ \hat{H} = H + \alpha E, \quad \hat{G} = G + \alpha F \]  (2.57)

and consider the typical Robin boundary condition, as introduced in Section 2.4.3, given in terms of the admittance \( \beta \), i.e. the inverse of the acoustic impedance:

\[ v_f^j(x) - v_s^s(x) = \beta(x)p(x) \]  (2.58)

Invoking the definition of the fluid-structure coupling matrix which maps BE pressure to FE forces, the total load on the structure can be expressed as \( f_s + T_{sf}p \). The second component indicates the excitation contribution due to the fluid. Substituting this result and (2.58) into (2.40) translates into

\[ q = i\omega \rho \beta \circ p + \omega^2 \rho T_{sf}^T u \]  (2.59)

where \( \circ \) is the element-wise (Hadamard) product, thus \( \beta \circ p \) simply represents the vector \( \beta(x_k)p(x_k) \), with \( x_k \) indicating the location of the \( k \)-th degree of freedom. To enhance notation clarity, the diagonal matrix \( Y = i\omega \rho I(\beta \circ p) \) is defined. The coupled FEM-BEM equation can then be written:

\[ \begin{pmatrix} K_s + i\omega C_s - \omega^2 M_s & -T_{sf} \\ -\rho\omega^2 \hat{G} T_{sf}^T & \hat{H} - \hat{G} Y \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f_s \\ p_{inc} + \alpha q_{inc} \end{pmatrix} \]  (2.60)

Note that ordinarily deployed simplifications include a constant admittance \( \beta(x) = \beta \).

The particular case of a 'sound-hard' boundary is obtained with \( \beta = 0 \), corresponding to \( v_f^j = v_s^s \), and causing the term \( \hat{G} Y \) to vanish. This condition is the de facto approach for vibroacoustic simulations in the space industry, but it also finds wide applicability in many other problems. Another useful remark is that the CBIE can be used instead of the Burton-Miller combined integral equation. Firstly, \( \hat{G}, \hat{H} \) are respectively substituted for \( G, H \) in (2.60). Then, by omitting the incident flux term \( \alpha q_{inc} \) on the right-hand side (RHS) the desired reformulation is acquired.

The CBIE version of the coupling scheme has seen extensive utilisation as the backbone of various research in the field. For example, in [126], [127] it was used to devise a procedure for the assessment of individual modal contribution to radiated power for exterior problems with sound-hard boundaries. Chen et al. [128] considered the admittance BC scenario and demonstrated a technique for parametrisation of shell geometries, enabling
sensitivity analysis of acoustic properties with respect to the structural shape. An interesting method was very recently proposed by Liu et al. [129] for the structural-acoustic simulation of thin shells with an admittance BC. It exhibits the ability to handle highly complex geometries, but its main advantage is the adoption of isogeometric discretisation. Isogeometric analysis is an emerging approach aiming to unify computer-aided design (CAD) and analysis methods under a single workspace, by enabling the direct use of high-order CAD-derived geometry representations in element-based solvers.

Ultimately, there exist numerous publications on FE-BE coupling and their review is far overreaching the scope of this introduction. A comprehensive survey can be found in [130]. A more compact overview is available in [131], where in addition, alternative coupling schemes to the one presented herein were investigated. Some newer, non-standard coupling systems have also been demonstrated, e.g. in [132] and [133]. The latter exploits a prominent recent technique called edge-smoothed FEM [134], which promises enhanced stiffness matrix accuracy against standard FEM elements, a straightforward implementation and no associated computing cost penalty.

Solution strategy

For a large portion of practical applications, the direct solution of the coupled elasto-acoustic equation would incur an inordinately long runtime. The coefficient matrix on the left-hand side (LHS) of (2.60) is clearly unsymmetrical and the partitions corresponding to BEM operators are dense, rendering sparse linear solvers unusable. Overall, the effect of large problem sizes is further aggravated by the poor conditioning of the FE-BE equation.

An adaptation more amenable to computation is obtained by elimination of the displacement unknowns and solving for the pressure. The procedure involves employing Schur complementation, which is discussed in greater detail in Chapter 4 as well as in [129], [131], [135]. For now, it is sufficient to state the final equation

$$\begin{align*}
\left[\hat{H} - \hat{G}(Y + Y_C)\right]p &= \hat{G}q_s + p_{inc} + \alpha q_{inc} \\
\text{(2.61)}
\end{align*}$$

where the following set of simplifications have been incorporated

$$\begin{align*}
q_s &= \rho \omega^2 T s f Z_s^{-1} f_s \\
Y_C &= \rho \omega^2 T s f^T Z_s^{-1} T s f \\
Z_s &= K_s + i \omega C_s - \omega^2 M_s \\
\text{(2.62a, b, c)}
\end{align*}$$

In other words, $Z_s$ is the structural dynamic stiffness matrix, whereas $q_s$ is a vector accounting for acoustic velocity contributions from the structure. The densely populated matrix $Y_C$ can be understood as a global admittance condition over the surface of the elastic solid domain. Solving (2.61) enables the calculation of the displacement field via (2.50). Explicitly, the relationship reads

$$u = Z_s^{-1} (f_s + T s f p)$$

$$\text{(2.63)}$$

Close examination of (2.61) and (2.62) immediately reveals two key issues. Namely, the presence of $Z_s^{-1}$ and matrix products involving it, as well as efficiently handling
the dense matrices $\hat{H}, \hat{G}$. The former can be addressed in the standard manner, that is, by performing a modal reduction [129], [135]. Alternatively, the condensation of the elastic domain matrices can be done by projection on a Krylov basis, as suggested by Peters et al. [127]. An incomplete Crout lower/upper factorisation is used in [136] in order to construct $Z_s^{-1}$.

Treatment of the BEM operator matrices is generally more challenging. Contemporary implementations make use of fast BEM techniques, with $H$-matrices and fast multipole methods being especially well-received. An equation akin to (2.61) is then solved iteratively, usually by GMRES through evaluating matrix-vector products with quasi-linear complexity. Examples of such works are accessible in [128], [131], [136], [137], among many others.

An interesting approach aimed at degrading the coupled FEM-BEM domain interaction to an uncoupled one consists of performing solutions for each subspace sequentially and updating the coupling boundary condition at each iteration. The idea can be traced back to the article of Lin et al. [138]. Formulations for elastostatics [139], transient elastodynamics [140] and fluid-structure problems incorporating FEM non-linearities [141] exist. However, iterative coupling appears to have never been used in an industrial vibroacoustic context, or validated for large models of arbitrary geometrical topology. Guaranteeing convergence might also be troublesome - a survey on the underpinning conditions was done by El-Gebeily et al. [142] and later in [143].

**Related developments**

The previously exposed considerations in composing and resolving elasto-acoustic problems within the FEM-BEM framework are certainly not exhaustive. While they should be representative of the typically assumed approaches, several related studies posing theoretical and practical interest can be identified, and deserve being mentioned.

Firstly, Wilkes [144] showed it is possible to construct a dual fast multipole BEM. That is, the elastic body in contact with the acoustic medium is also modelled with a FMM-accelerated BEM, instead of finite elements. In a subsequent work, Wilkes and Duncan [145] demonstrated the method is applicable to solids with irregular geometrical features. Nevertheless, the authors acknowledged the current limitation that solely piecewise-constant triangular elements are permitted and a broadband FMM is yet to be incorporated. In addition, it seems unclear how the scheme fares against a more traditional combination of FEM and fast BEM.

A particularly interesting technique was depicted by Zerbib et al. [146]. In the article, a so-called ‘adaptive absorbing boundary condition’ was proposed in order to yield a hybridisation of FEM and fast multipole BEM. This is achieved by the introduction of a (possibly thin) volumetric FEM mesh around the elastic structure, similarly to the FE treatment of exterior problems discussed in Section 2.4.2. In contrast to the typical conditions imposed on the artificial boundary like PMLs or infinite elements, BEM has been used. Although the so-obtained domain decomposition leads to an additional FE computational space, the merit of the scheme is a vastly sped up GMRES convergence. This claim was supported in the article via an example of wave scattering from a
submarine. Performance gains for the new method of at least an order of magnitude in terms of solver iterations and total computing time were established against FMM-only acoustic representations with collocation and Galerkin BEM.

In conclusion to this topic, attention is drawn to the publications of Harari and Hughes [147], Atalla and Bernhard [148], as well as Bolejko and Dobrucki [116]. A thorough cost comparison of FEM and BEM is presented for acoustic and vibroacoustic problems, providing an indication on which type of modelling is more appropriate for the fluid medium. While contemporary accelerated BE approaches are not discussed, the analyses remain valid for the respective near-field BEM calculations, which use standard quadrature rules.

2.5.3 Statistical energy analysis

Fundamental principles of SEA

The inadequacy of element-based models to capture the high-frequency physics of dynamical problems has led to the development of a number of predominantly non-deterministic methods. In this section, statistical energy analysis is briefly introduced as the dominant approach adopted for solutions in the high-frequency domain. Some key developments and modern implementations, expanding the SEA framework to lower frequencies and wider range of problems, are then discussed. For completeness, newer schemes targeted at covering the problematic mid-frequency range are subsequently mentioned.

![Diagram of energy exchange between three SEA subsystems](image_url)

Figure 2.6: A schematic representation of the energy exchange between three SEA subsystems

The initial derivation of the SEA principles can be traced to the early 60s, motivated by the demand to analyse complex aerospace structures subjected to distributed random excitations. The observations made suggested an analogy between the laws of thermodynamics and vibroacoustic systems. That is, under appropriate conditions, the exchange of power between two coupled oscillators with one degree of freedom is
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proportional to their energy difference. The governing principles were established by Lyon and DeJong [149], who are the original contributors to the creation of the SEA method. This classic book serves as a comprehensive resource on SEA, containing the relevant theoretical descriptions and also reflecting practical knowledge of its applicability gained since its inception.

Extension of the basic idea to complex systems involves a decomposition into several coupled substructures and performing a power balance between them, as depicted on Figure 2.6. The following annotation is defined: \( P_{i,\text{in}} \) and \( P_{i,\text{diss}} \) represent the power injected to and dissipated by the \( i \)-th subsystem, respectively. \( P_{ij} \) is the power exchanged between subsystems \( i \) and \( j \). Writing the equilibrium equation is facilitated by expressing the various powers in terms of the time-averaged total energies \( E_i \) and modal densities \( n_i \) of the considered subsystems. To that end, several constants, typically referred to as the 'SEA parameters' can be identified. Firstly, the **damping loss factor (DLF)** \( \eta_i \) is given by:

\[
P_{i,\text{diss}} = \omega \eta_i E_i \tag{2.64}
\]

Physically, \( \eta_i \) characterises the combined dissipative effect of structural damping mechanisms and acoustic radiation into the surrounding fluid, if such is present. In addition, under the assumption of proportionality between the modal energy difference and exchanged power:

\[
P_{ij} = \omega (\eta_{ij} E_i - \eta_{ji} E_j) \tag{2.65}
\]

where \( \eta_{ij} \) is known as the **coupling loss factor (CLF)**. The reciprocal relationship linking \( \eta_{ij} \) and \( \eta_{ji} \) is

\[
n_i \eta_{ij} = n_j \eta_{ji} \tag{2.66}
\]

Combining equations (2.64)-(2.66), the SEA equation for \( k \) coupled subsystems can be written in matrix form:

\[
\begin{pmatrix}
\eta_1 + \sum_{j \neq 1}^{k} \eta_j & -\eta_{21} & \cdots & -\eta_{k1} \\
-\eta_{12} & \eta_2 + \sum_{j \neq 2}^{k} \eta_{2j} & \cdots & -\eta_{k2} \\
\vdots & \vdots & \ddots & \vdots \\
-\eta_{1k} & -\eta_{2k} & \cdots & \eta_k + \sum_{j \neq k}^{k} \eta_{kj}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_k
\end{pmatrix} =
\begin{pmatrix}
\frac{P_{1,\text{inj}}}{\omega} \\
\frac{P_{2,\text{inj}}}{\omega} \\
\vdots \\
\frac{P_{k,\text{inj}}}{\omega}
\end{pmatrix} \tag{2.67}
\]

Due to the reciprocity relationship, provided \( n_i = n_j \), the coefficient matrix on the **LHS** of (2.67) is symmetric. Moreover, it is inherently much smaller than FEM or BEM derived descriptions of the global problem. Thus SEA is appealing precisely because it only needs top-level 'discretisation', while simultaneously benefiting from the intrinsic parametric uncertainty of modes at high frequencies.

**Estimation of SEA parameters and scope of validity**

In contrast to the stated pros, the construction of (2.67) implies the knowledge of all modal densities, DLFs and CLFs. Obtaining them is usually not a trivial task and represents the main obstacle to the practical application of SEA.
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The modal density is the estimated number of modes per frequency band. Its presence in the law of reciprocity means miscalculation of the $n_i$ terms gives rise to a notable disparity between the real and estimated coupling loss factors. An analytical computation approach for the modal densities involves considering the natural frequencies of the subsystems in a simply supported state, and is generally assumed to be reliable at high frequency. Examples for composite structures can be found in [150].

In general, typical wave-context approaches employed in the evaluation of the SEA coefficients rely on analytical solutions for (semi-)infinite systems with high modal overlap, and are therefore limited to relatively simple problems. Note that a 'wave approach' to SEA approximately means viewing a complex system as an ensemble of propagating wavetypes, as opposed to a modal description. In addition, 'overlap' refers to the diffuseness of the vibrational field of each subsystem (spatial modal overlap) and the number of modes that exist within the frequency band of application (spectral modal overlap).

In complex systems, either FE modelling (e.g. [151], [152]) or experimental measurement data for similar problems [153] can be used to find the SEA parameters. In both situations, the power injected method may be used. It involves applying a rain on the roof excitation to the system - essentially a set of arbitrary forces the combination of which reproduces white noise. The importance of this method is seen primarily in the frequency domain, where it is more accurate than wave approaches. More details on techniques applicable to finding CLFs and DLFs are available in [154].

Apart from the intricacy of extracting the SEA parameter values for complex systems, the practical usability of the method is hindered by a set of underlying hypotheses that have to be satisfied:

- The studied system is not in a transient regime. Excitations are random, stationary and uncorrelated in a statistical sense.
- Each of the subsystems the model is partitioned into is only weakly dissipating. The coupling between any pair of them is weak, or stated equivalently, no global modes exist.
- Coupling between subsystem pairs is carried out by mass, stiffness and gyroscopic effects, therefore damping is small and losses in the coupling are ignored.
- The subsystems are reverberant, thus modal overlap is high, guaranteeing the correctness of a statistical representation.
- The behaviour of several grouped modes is amenable to description as an averaged mode. Additionally, the mean total energy in a frequency band is only affected by the constituent modes in that band, which also contribute to the entirety of the energy transmission to other subsystems.

In summary, the preceding requirements mostly translate into restrictions on the lower bound of the applicable frequency and the permitted characteristics of junctions between subsystems.
2.5.4 Hybrid deterministic-statistical approaches

Notable developments

At present, the firmly established status of SEA as the principal tool in high-frequency vibroacoustic analysis is undisputed. This popularity has instigated continuous research on two main fronts of improvement. Firstly, the extension of SEA’s potential to mid-frequency analysis, especially via coupling to, or hybridisation with FEM and/or BEM. Secondly, the automated retrieval of the modal coefficients, often in the context of the aforementioned multi-method schemes.

In [155], an interesting deterministic-statistical framework was proposed for the modelling of complex elasto-acoustic problems. The model DOFs are divided into ‘global’ and ‘local’ sets, the former being solved with traditional element-based techniques, and inclusion of the effects of the local DOFs is done in a manner akin to fuzzy structure theory, briefly outlined in Section 2.3.1. The short wavelength set is subjected to treatment by SEA, but with additional power input from the global set. A simple rod test case indicated good broadband predictions.

Later, Shorter and Langley [154] showed a conceptually different deterministic-statistical hybridisation, deriving the fundamental principles without explicit reference to SEA. The authors recognised the wave approach to SEA as a subset of their formulation. An important feature was the ability of flexibly incorporating local element-based submodels, for example at junctions, mitigating the necessity of building a full model deterministic representation.

The next appreciable advancement in the field was enabled by [156] and [157]. The demonstrated ‘diffuse field reciprocity relation’ postulates that the cross-spectral density matrix of the forces exerted by a vibrational or acoustic field can be expressed in terms of the wave field energy and the direct field dynamic stiffness matrix $D_{dir}$ of the boundary. The exact meaning of $D_{dir}$ is easily explained as the dynamic stiffness corresponding to the interface DOFs, in the absence of reflections from waves propagating within the host subsystem. The reverberant loading $f_{rev}$ is the difference between the actual boundary force and the one due to $D_{dir}$, that is

$$D_{dir}u = f + f_{rev}$$ (2.68)

since $Du = f$. The governing equations of motion for the mixed FE-SEA can be then written as

$$D_{tot}u = f + \sum_{j} f_{rev}^{(j)}$$ (2.69a)

$$D_{tot} = D_d + \sum_{j} D_{dir}^{(j)}$$ (2.69b)

where $j$ is a subsystem number, $D_d$ is the dynamic stiffness matrix arising from the FE model of the deterministic portion of the system, $D_{tot}$ is $D_d$ when augmented by the direct field dynamic stiffness of the SEA subsystems. The hybrid FE-SEA of [151] then makes use of the diffuse field reciprocity

$$S_{ff,rev}^{(j)} = E[f_{rev}^{(j)} f_{rev}^{(j)*}] = \left(\frac{4E_j}{\omega \pi n_j}\right) \text{Im}\{D_{dir}^{(j)}\}$$ (2.70)
where $S_{ff,\text{rev}}^{(j)}$ is the cross-spectral matrix of the reverberant force, $E$ is the mathematical expectation, $E_j$ and $n_j$ are average energy and modal density. Im and the power * denote imaginary part and conjugate transpose, respectively. The authors showed that with the hybrid FE-SEA, not only the mean response, but also ensemble variance of the statistical components’ responses can be ascertained. This was accomplished based on the tenets initially outlined for general built-up structures in [158] and later [159] and [160].

Further development of the preceding idea is found in [161], where the usability of the reciprocity relation was extended to the case of coupling over 'large' domain boundaries, such as fluid-structure contact surfaces encountered in vibroacoustic problems. Note that in general, these hybrid methods do not restrict modelling element-based subsystems to FE only, and permit the automatic calculation of SEA parameters.

Considering the former discussion, it is evident that deterministic-statistical approaches provide a solid framework for the vibroacoustic analysis of systems with delineated mid-frequency behaviour features. Randomness of some components is implicitly accounted for by SEA without demanding MC simulations. However, some subsystems might possess a degree of randomness insufficient to justify a probabilistic description, while invalidating the assumed deterministic model. A possible solution has been explored by Cicirello and Langley [162], who incorporated a parametric uncertainty model into the FE subsystems. A later extension by the same authors [163] dealt with lowering the computational demand of the original method.

### Relevance to aerospace structures

Mixed deterministic-statistical approaches are naturally well-equipped for vibroacoustic simulation of aerospace structures. The described hybrid FE-SEA has been recently applied to an aircraft model [164], affirming good correlation against Monte Carlo simulation and enhanced low-frequency correctness against pure SEA. Larko and Cotoni [165] suggested such hybrid models exhibit competitive performance in terms of solution accuracy against acoustic BEM, based on a comparison for a spacecraft antenna.

Within the context of the DSF excitation of the launch environment, Roibás et al. [166] investigated what was the optimal combination of FEM, BEM and SEA. Nine different mixed formulations were employed for a spacecraft structure, based on a band partitioning of the frequency range between 0 Hz and 10 kHz. Conversely, the construction of multi-technique procedures based on experimental results was discussed in [153]. Likewise, in [167], finite and boundary element predictions were used to derive realistic inputs to a hybrid model for non-trivial satellite equipment panel configurations.

### 2.5.5 Alternative high- and mid-frequency techniques

During the last few decades, the gap between low- and high-frequency analysis has been narrowed by the advent of sophisticated numerical methods. Stochastic extensions to deterministic approaches and multi-method formulations, relying on domain coupling or statistical-deterministic hybrid descriptions, constitute the majority of this progress.
Within these categories, several alternatives to FEM/BEM/SEA have emerged. An exhaustive and insightful review was compiled by Desmet \cite{168}, and only several key ideas will be mentioned here.

A number of authors have pursued the concept of deriving deterministic representations more efficient than the traditional ones, such as FEM. The strategy usually revolves around a Trefftz approach, which uses wave basis functions to describe the system’s response. The rationale is that functions exactly satisfying the governing differential equations provide better solution approximation than the polynomial interpolations used by the classic methods. The variational theory of complex rays \cite{169} and the wave based method \cite{170} fall within that category. The latter has received a lot of attention since its introduction by Desmet \cite{171}, and seems to exhibit all the desirable characteristics of a ’true’ mid-frequency approach. In other words, computational efficiency, invariance with geometrical complexity and ability to handle non-negligible fluid-structure coupling.

The wave based scheme was successfully used by Vergote et al. \cite{172} as a replacement for SEA in a hybrid deterministic-statistical context. It was applied as a strictly deterministic vibroacoustic tool coupled to a finite element model in \cite{173}. The authors argued the case of using FE for the structural domain in order to exploit its geometric versatility, whereas the wave approach proved suitable for the fluid space, particularly in the presence of relatively simply-shaped acoustic cavities.

Energy flow analysis embodies another principally different idea that spawned a class of ’thermal’ methods. The name originates from the founding assumption that vibrational energy obeys the steady-state heat flow equation for solids. The energy FEM facilitates the numerical implementation of the principle \cite{174}. Analysis can therefore be carried out by existing thermal solvers, which is advantageous in terms of convenience. However, the main appeal of the energy FEM actually lies in its ability to supply more finely detailed information on the spatial distribution of energy within subsystems, as opposed to SEA, which does not model internal dissipation and conduction. In fact, the fundamental equations may be thought of as a generalisation of the basic SEA axioms.

Since the energy FEM is complementary to standard low-frequency finite element descriptions, already available models are used to provide insight into the system behaviour at higher frequencies. Irregular geometries, strong coupling and damping are allowed. For instance, Mace and Shorter \cite{175} demonstrated how energy flow models can be obtained from FEM. In the article implementation in a CMS context was shown. A formal introduction to energy FEM can be found in \cite{72} together with several examples of varying complexity.

Overall, a number of new techniques for assessing the dynamic behaviour of systems in the elusive mid-frequency regime are presently under development. The ones discussed in this section display marked potential, but the list is not exhaustive. It should be pointed out that many traditional approaches share a ’bottom-up’ philosophy, attempting the modification of current low-frequency methods. Such reduction and/or uncertainty-based schemes have been explored in Section 2.3.
2.5.6 Geometry-specific formulations

To ensure completeness of the current literature survey, the importance of advances made in the semi-analytical modelling of vibroacoustic interactions has to be acknowledged. In fact, it would often be unreasonable to attempt the validation of a new ‘global/universal’ numerical model without an in-depth understanding of the physical principles governing the basic constituents of a problem. In aerospace elasto-acoustic applications, thin planar, cylindrical and conical geometries are vastly prevalent at the full model scale.

Thin plates, usually built from layered composite materials, have arguably received the most attention. In [176], a detailed study on the sound radiation of unbaffled homogeneous flexible plates was presented. An analytical model was backed by comparisons against BEM and experimental data, showing excellent correlation at low frequencies. A later work demonstrated a low-frequency tool for estimating diffuse field transmission loss of double-walled sound barriers [177], where the reverberant loading was modelled in the typical fashion of a sum of uncorrelated plane waves. Foin et al. [178] introduced a method for vibroacoustic analysis of multilayer plates, which enables the construction of more compact mass and stiffness matrices and exhibited excellent correlation against physical test data.

In a slightly more recent article [179], flat sandwich composite panels are discussed, whereas in [180] the same authors started from a wave-based approach in order to build a vibroacoustic prediction tool within the SEA framework, which was intended for curved laminates, potentially containing thick layers. Following a parametric study, it was concluded that the transmission loss of such laminates can be controlled by utilising carefully selected orthotropic arrangements and ply thicknesses. Finally, Xin and Lu [181] extended the aforementioned research by devising an analytical description considering the influence of the imposed BC type on the sound transmission properties of finite double panel configurations coupled via an air cavity. Differences between the behaviour of simply supported and clamped BCs were highlighted for cases of oblique sound incidence.

With regard to more complex geometries, Ghinet et al. [182] employed the so-called thin layered method to diffuse field propagation in infinite cylindrical composite structures. The aforementioned technique is founded on discretisation of the structure in the sense of lamination. Harmonic wave propagation is assumed to govern the motion within each sublayer. In general, with such formulations, a complex polynomial eigenvalue problem is obtained.

Cotoni et al. [152] showed an interesting formulation for SEA subsystems in which FEM, CMS and periodic structure theory are combined. The objective was to allow the calculation of SEA parameters for very general structural panels, such as isogrid fairings and composite fuselages containing various stringers and ribs. Important quantities, such as modal density and radiation efficiency, transmission loss and the elasto-acoustic vibroacoustic response vector were obtained. Insight on a modal approach to arriving at the same predictions is also available in the paper.

In the recently published work of Chronopoulos et al. [183], cone-cylinder-cone shell arrangements were analysed. Low- to mid-frequency behaviour was described by an
2.6 Summary

In this chapter, a top-level overview on the fundamentals of modern numerical vibroacoustic analysis was laid out. Four principal subtopics were identified, allowing the convoluted problem of robustly and efficiently simulating coupled elasto-acoustic interaction to be broken down into separate sets of techniques and developments. Throughout their incremental presentation within the chapter, strong emphasis was placed on the specifics related to aerospace structures.

In Section 2.2 the concept of model order reduction and domain decomposition schemes for finite element structural representations was explored. As a natural extension to physical space dynamic substructuring methods, component mode synthesis was viewed...
as a powerful tool enabling them to be applied more efficiently in the context of complex structures. In particular, the Craig-Bampton reduction and its recently refined versions were introduced, due to their widespread use within the aerospace industry, but also other fields of engineering. Several interesting works were identified, for example the enhanced CB method, permitting quicker generation of more compact and physically accurate reduced system matrices.

Subsequently, methods for uncertainty treatment in structural dynamics were studied. The typical philosophy embedded in such approaches is that low-frequency predictions can be complemented with mid-frequency ones via the construction of some form of a stochastic version of the original FE model. The standard PSA was deemed generally reliable, but often cumbersome in implementation and expensive computationally. The second issue was addressed by newer means of uncertainty modelling, such as the nonparametric probabilistic approach or the CBSM. Both could be used with reduced built-up structures, but were found to be either excessively mathematically involved, or somewhat restricted in scope.

Section 2.4 dealt primarily with the deterministic methods available for modelling acoustic media. The DSF and TBL aeroacoustic excitations generated by the launch vehicle were described in a mathematical sense, due to their importance as design driving factors for spacecraft. Usually their application to a FEM or BEM model demands the definition of an unbounded acoustic medium. For this reason, concepts enabling the use of acoustic FEM in an exterior problem context were detailed. Various types of absorbing boundary conditions were discussed, with IEM and PML being highly capable modern schemes. While BEM is naturally suitable for exterior problems, it is intrinsically computationally heavy as well. Hierarchical approximation methods were found to successfully overcome this hindrance.

The chapter was concluded with Section 2.5 dedicated to coupled elasto-acoustic analysis. The concepts of the preceding sections were assembled to yield a discussion on some classic low-frequency methods, such as FEM-FEM and FEM-BEM. Furthermore, SEA was recognised as the dominant methodology for high-frequency problems and its underlying theory was displayed in a condensed form. Bridging the gap between low- and high-frequency analysis was found to have initiated substantial research efforts in the hybridisation of deterministic and statistical methods, most commonly FEM and SEA. Nevertheless, it was acknowledged that mid-frequency simulation was not yet free of predicaments. Promising works aimed at addressing the issue with completely new techniques, such as the energy FEM or the wave based method, were mentioned. However, their current state of maturity appeared insufficient for tackling problems of arbitrary complexity.
Chapter 3

The improved Craig-Bampton stochastic method

3.1 Definition of the method

3.1.1 Background

Usually complex satellite assemblies are built-up of several discrete components. They are represented by separate finite element models, all of which are eventually assembled into a full FE system for final analyses. The substructuring approach allows each component to be handled more easily and efficiently in terms of both engineering effort and numerical simulation. In addition, it provides flexibility to the design process, permitting quicker modification or substitution of subsystems, such as on-board instrumentation, without necessitating modification of the full model every time. The principles of dynamic substructuring have been conferred at greater length in Section 2.2.1.

In the context of acoustic loading, the satellite substructures can essentially be divided in three categories, which are schematically illustrated on Figure 3.1. Firstly, parts that are directly exposed to the pressure field can be identified. Next, components that may be sensitive to the transmitted loads arising due to that acoustic excitation, such as optical instruments or electronics boards. Finally, the remaining, or 'residual' structure, which is also likely to be comprised of multiple components itself. The latter also affects the dynamic interaction between the rest of the subsystems, as well as the transmission of air-borne vibrations through the spacecraft. Some components may belong to several of these categories simultaneously. A remark should be made that in a CMS context, residual structure commonly bears the meaning of 'non-reduced'. In the following sections, the former definition shall be implied instead.

By today’s standards, performing low-frequency vibroacoustic simulations of the fully assembled model in physical coordinates, often encompassing several million degrees of freedom, is not an excessively intensive computing task. In fact, historically it has been bottlenecked by the modelling of the fluid domain rather than the structure, but major advances in numerical methods have enabled the pressure field acting on the satellite to
3.1. Definition of the method

be obtained relatively quickly. A thorough discussion has been supplied in Section 2.4. Throughout the current thesis, it is proposed that the acoustic load is computed only in the nominal (deterministic) solution, and random realisations of the uncertain structure are sampled once that has been accomplished. The initial elasto-acoustic problem may be treated as either coupled or uncoupled to the structural FE model, and could be resolved by any suitable technique. This approach allows the independent handling of the structure within the framework of a stochastic vibroacoustic implementation, while only a single acoustic solution is required as a prerequisite.

This separation of the structural and acoustic analyses allows the efficient and unrestricted use of model order reduction schemes. The most commonly used one in industry is the classic Craig-Bampton reduction. When paired with a parametric uncertainty modelling technique, applied directly to the condensed (fixed interface) component stiffness matrices, it lays the foundation of a very efficient, yet straightforward to implement scheme, namely the Craig-Bampton stochastic method. The remainder of the current chapter is dedicated to various aspects of the CBSM. They include efficiency improvements, particularities regarding its use in conjunction with distributed excitations and parametric studies aimed at providing a broader understanding of the method’s performance and usability restrictions.

3.1.2 Craig-Bampton reduction

Domain decomposition

In the present section, the principles underpinning the original CBSM are explained. The full theory of the underlying Craig-Bampton reduction is readily available in literature, which has been extensively reviewed and referenced in Section 2.2 from the
broader point of view of dynamic substructuring and CMS. Therefore only the key equations, relevant to the CBSM, shall be presented. The starting point for deriving the method is the dynamic equation of motion of the unconstrained structure, which is restated here, along with other results of the aforementioned section, for the sake of clarity:

$$M \ddot{u} + C \dot{u} + Ku = f$$

(3.1)

with the symbols taking on the standard meanings related to DS. Subdivision into $N_s$ non-overlapping parts in the traditional DS manner yields a partitioning of the global degrees of freedom into the boundary set $B$, possibly augmented by DOFs not belonging to inter-component joints, and the internal elastic ones $I$:

$$\begin{pmatrix} M_{BB} & M_{BI} \\ M_{IB} & M_{II} \end{pmatrix} \begin{pmatrix} \ddot{u}_B \\ \dot{u}_I \end{pmatrix} + \begin{pmatrix} C_{BB} & C_{BI} \\ C_{IB} & C_{II} \end{pmatrix} \begin{pmatrix} \dot{u}_B \\ \dot{u}_I \end{pmatrix} + \begin{pmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{pmatrix} \begin{pmatrix} u_B \\ u_I \end{pmatrix} = \begin{pmatrix} f_B \\ f_I \end{pmatrix}$$

(3.2)

Employing the previously defined notation, the $k$-th substructural displacement vector is written as

$$u^{(k)} = \begin{pmatrix} u^{(k)}_B \\ u^{(k)}_I \end{pmatrix} = \begin{pmatrix} \mathcal{L}^{(k)}_B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_B \\ u_I \end{pmatrix}$$

(3.3)

or in other words $\mathcal{L}^{(k)}_B u_B = u^{(k)}_B$, with $\mathcal{L}^{(k)}_B$ being a localisation matrix mapping the $k$-th local boundary onto the full system $B$-set. The terms in the global equilibrium equation (3.2) are recast into

$$M = \begin{pmatrix} \sum_{k=1}^{N_s} \mathcal{L}^{(k)T}_B M^{(k)}_{BB} \mathcal{L}^{(k)}_B & \mathcal{L}^{(1)T}_B M^{(1)}_{BI} & \cdots & \mathcal{L}^{(N_s)T}_B M^{(N_s)}_{BI} \\ M^{(1)}_{IB} \mathcal{L}^{(1)}_B & M^{(1)}_{II} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ M^{(N_s)}_{IB} \mathcal{L}^{(N_s)}_B & M^{(N_s)}_{II} \end{pmatrix}, \quad u = \begin{pmatrix} u_B \\ u^{(1)}_I \\ \vdots \\ u^{(N_s)}_I \end{pmatrix}$$

(3.4)

and if present, the matrix $C$ assumes the same form as its $M$ and $K$ counterparts, being a linear combination thereof in Rayleigh damping. Of course, (3.4) depicts the governing equation of motion as a primal assembly of its constituent subdomains.

**Projection on generalised coordinates**

The concept of the Craig-Bampton reduction for component mode synthesis is to project (3.4) onto a mixed physical-modal space basis. For each component, the local CB coordinates are linked to the original ones of the substructure via the transformation matrix $T_r^{(k)}$:

$$\begin{pmatrix} u^{(k)}_B \\ u^{(k)}_I \end{pmatrix} = T_r^{(k)} \begin{pmatrix} u^{(k)}_B \\ u^{(k)}_I \end{pmatrix}, \quad T_r^{(k)} := \begin{pmatrix} I_B & 0_B \\ \Phi^{(k)}_B & \Phi^{(k)}_I \end{pmatrix}$$

(3.5)
3.1. Definition of the method

where \( u_B = u_b \), but the lower case \( b \) is intentionally used to signify relation to the interface of the condensed subsystem. As in Section 2.2.2, subscript \( m \) corresponds to the modal representation of the \( I \) set. The construction of \( T_r^{(k)} \) involves the separate computation of its bottom blocks. Firstly, \( \Phi_I^{(k)} \) is the constraint mode matrix. Each of its constituent vectors expresses the subsystem’s deformation imposed by a unit displacement at a boundary DOF, with the remainder of the \( B \) set kept motionless. On the other hand, \( \Phi_B^{(k)} \) essentially is a truncated eigenbasis of size \( m_k \) for the subsystem under a fixed interface condition. Explicitly, they are obtained from

\[
\Phi_B^{(k)} = -K_B^{(k)} \Phi_I^{(k)} \Phi_B^{(k)T} u_I^{(k)} = 0
\]

where \( \phi_{i,j}^{(k)} \) and \( \omega_{i,j}^{(k)} \) are the \( j \)-th structural mode and corresponding natural frequency \( \text{(NF)} \), respectively, of the \( k \)-th part with a fixed boundary. The Craig-Bampton equation for the component is arrived at by pre- and post-multiplication of the substructural equivalent of (3.2), obtained through simply assigning all quantities a superscript \( (k) \), by \( T_r^{(k)} \) and \( T_I^{(k)} \), respectively. Following some algebraic manipulation \[12\],

\[
\begin{pmatrix}
M_{bb}^{(k)} & M_{bm}^{(k)} \\
M_{mb}^{(k)} & I
\end{pmatrix}
\begin{pmatrix}
\ddot{u}_b^{(k)} \\
\ddot{u}_m^{(k)}
\end{pmatrix}
+
\begin{pmatrix}
K_{bb}^{(k)} & 0 \\
0 & \Lambda_{mm}^{(k)}
\end{pmatrix}
\begin{pmatrix}
u_b^{(k)} \\
u_m^{(k)}
\end{pmatrix}
=
\begin{pmatrix}
f_B^{(k)} + \Phi_B^{(k)T} f_I^{(k)} \\
\Phi_I^{(k)T} f_I^{(k)}
\end{pmatrix}
\]

with \( \Lambda_{mm}^{(k)} = \text{diag}(\omega_1^{(k)}^2, \ldots, \omega_{m_k}^{(k)}^2) \). Furthermore, observe that the \( bb \) submatrices actually differ from the non-reduced interface ones, even though they correspond to the same degree of freedom set and still represent the inter-component connections. Note that (3.7) obeys the common assumption of mass normalised modes.

It should be stressed that the reduced matrix \( C_r^{(k)} \) would normally have the same form as its stiffness counterpart in (3.7) if structural damping were present. However, damping does not affect the local modal properties, which conform to (3.6b). In this work, the Craig-Bampton damping matrix is not derived as an explicit condensation of an existing structural \( C \) available in physical space. Instead, modal damping is directly applied at the response computation step, after all subsystems have been reduced, synthesised, and the resultant generalised eigenvalue problem (GEP) has been solved.

Synthesis of components

At the stage when all components have been specified in the form prescribed by (3.7), the reconstruction of the original problem in global CB coordinates can commence. One of the chief advantages of fixed interface methods, such as CB, is that the \( B \) set DOFs are retained as generalised coordinates after the reduction has been completed. An important implication is that the reduced primal assembly operators remain Boolean, unlike the unspecific case (2.18) conferred in Section 2.2.2. It will demonstrated that the global CB localisation matrix \( L_r \) can straightforwardly be expressed in terms of the known \( L_B^{(k)} \). For conciseness, the arguments shall be constricted to the mass matrices,
as the cases of damping and stiffness share a completely identical logic. Firstly, the
dually assembled mass, split into local $B$ and $I$ sets, is

$$M_d = \begin{pmatrix}
M_{BB}^{(1)} & \cdots & M_{BI}^{(1)} \\
\vdots & \ddots & \vdots \\
M_{IB}^{(1)} & \cdots & M_{II}^{(1)}
\end{pmatrix}
\begin{pmatrix}
\cdots & \cdots & M_{BB}^{(N_s)} \\
M_{BI}^{(N_s)} & \cdots & M_{BI}^{(N_s)} \\
M_{IB}^{(N_s)} & \cdots & M_{II}^{(N_s)}
\end{pmatrix}
$$

(3.8)

with all entries not falling on the three block diagonals being null matrices. The re-
spective primal assembly operator may be explicitly constructed as

$$L = \begin{pmatrix}
L_B^{(1)} & J_{BI} \\
\vdots & \ddots & \vdots \\
L_I^{(N_s)} & J_{BI}
\end{pmatrix},
J_{BI} = \begin{pmatrix}
I_{BI}^{(1)} \\
\vdots \\
I_{BI}^{(N_s)}
\end{pmatrix}
$$

(3.9)

where $I_{II}$ is an identity matrix of size equal to the number of $I$ set DOFs. All $I_{BI}^{(k)}$
are rectangular with ones on the main diagonal, zeros elsewhere and dimensions of
$b_k \times i_k$, i.e. the same as the respective $M_{BI}^{(k)}$. Also, $J_{BI}$ has a size of $\sum_{l=1}^{N_s} b_l \times \sum_{l=1}^{N_s} i_l$, 
whereas for $0_{IB}$ it is equal to that of $M_{IB}$. It is easy to verify directly that the product
$L^T M_d L$ is precisely $M$, as given in (3.4). More generally $u_d = L u$, thus $L$ satisfies
the primal assembly equation (2.9). To translate the former definitions into the CMS
of Craig-Bampton components, it is only required to notice

$$M_{d,r} = \begin{pmatrix}
M_{bb}^{(1)} & \cdots & M_{bm}^{(1)} \\
\vdots & \ddots & \vdots \\
M_{mb}^{(1)} & \cdots & M_{mm}^{(N_s)}
\end{pmatrix}
\begin{pmatrix}
\cdots & \cdots & M_{bb}^{(N_s)} \\
M_{bm}^{(N_s)} & \cdots & M_{bm}^{(N_s)} \\
M_{mb}^{(N_s)} & \cdots & I_{mm}
\end{pmatrix}
$$

(3.10)

thereby the desired CB localisation matrix is simply:

$$L_r = \begin{pmatrix}
L_B^{(1)} \\
\vdots \\
L_I^{(N_s)}
\end{pmatrix},
J_{bm} = \begin{pmatrix}
I_{bm}^{(1)} \\
\vdots \\
I_{bm}^{(N_s)}
\end{pmatrix}
$$

(3.11)

The dimensions of the submatrices are defined in the same way as their physical space
equivalents from (3.9). Overall, as was originally stated, the reassembly stage of the
CMS procedure is in fact reasonably straightforward. It adheres to precisely the same principles and form as (3.4), and

\[
\mathbf{u}_r^{(k)} = \begin{pmatrix} \mathbf{u}_b^{(k)} \\ \mathbf{u}_m^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_B^{(k)} & 0 \\ 0 & \mathbf{I}_{mm}^{(k)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_b^{(k)} \\ \mathbf{u}_m^{(k)} \end{pmatrix}
\]

is the exact analogue of (3.3) in the generalised coordinate space.

### 3.1.3 The original CBSM

The Craig-Bampton stochastic method has been recently proposed by Remedia et al. [1]–[3] for the treatment of structural uncertainty in spacecraft. The fundamental application for the technique that was demonstrated in these publications was the modelling of onboard microvibration experienced by satellites. Conceptually, the CBSM is rooted in the earlier work of Mace and Shorter [55] in the sense that perturbations are imposed on the modal partitions of the CB component matrices, hence the naming of the method. The substructural equation of motion is obtained from (3.7) by writing

\[
\begin{pmatrix} \mathbf{M}_{bb}^{(k)} & \mathbf{M}_{bm}^{(k)} \\ \mathbf{M}_{mb}^{(k)} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_b^{(k)} \\ \ddot{\mathbf{u}}_m^{(k)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{\tilde{C}}_{mm}^{(k)} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_b^{(k)} \\ \ddot{\mathbf{u}}_m^{(k)} \end{pmatrix} + \\
\begin{pmatrix} \mathbf{K}_{bb}^{(k)} & 0 \\ 0 & \mathbf{\tilde{\Lambda}}_{mm}^{(k)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_b^{(k)} \\ \mathbf{u}_m^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_B^{(k)} + \Phi_B^{(k)T} \mathbf{f}_i^{(k)} \\ \mathbf{\Phi}_1^{(k)T} \mathbf{f}_i^{(k)} \end{pmatrix}
\]

where the accented \( \tilde{x} \) denotes a random quantity \( x \). The modal stiffness is given by

\[
\mathbf{\tilde{\Lambda}}_{mm}^{(k)} = \text{diag} \left( \tilde{\omega}_1^{(k)^2}, \ldots, \tilde{\omega}_{mk}^{(k)^2} \right)
\]

with the diagonal entries being the constrained interface natural frequencies, modelled as random variables. The boundary terms in the damping matrix are assumed negligible. The pragmatic reasoning underpinning this simplification is that these quantities tend to be non-standard and not amenable to experimental verification. Pointedly,

\[
\mathbf{\tilde{C}}_{mm}^{(k)} = 2J \mathbf{\tilde{\Lambda}}_{mm}^{(k)} \mathbf{\zeta}_m^{(k)}
\]

is a diagonal matrix of modal damping values of the constrained boundary subsystem \( k \). The vector \( \mathbf{\zeta}_m^{(k)} \) contains the equivalent viscous damping of the fixed interface modes, defined as the ratio of actual to critical damping. This approach is widely adopted in the aerospace industry and preliminary dynamic analyses often resort to approximations, such as a constant \( \zeta = 0.01 \). Defining a variable damping between modes may be based on engineering judgement, or be a refinement resultant from post-test model correlation activities. Nevertheless, as has been mentioned in the Section 3.1.2 in all applications shown in the remainder of this thesis, non-random modal damping \( \mathbf{C}_r = \mathbf{I} \mathbf{\zeta} \) is applied to the global generalised coordinate modes of the synthesised structure.

Returning to the CBSM, the traditional use of the Craig-Bampton reduction involves localised excitations. As a matter of convenience, all the point load input locations are affixed to the \( B \) set. Then the RHS of (3.13) is simplified and only the \( \mathbf{f}_B^{(k)} \) force
term remains. Provided that the boundary DOF set is construed to also encompass
the nodes desired for response output, the transformation matrices $\Phi_B^{(k)}$ and $\Phi_I^{(k)}$ are
never required after the reduction stage.

Once all the subsystem condensations have been performed, the respective stochastic
$m$-set NFs are sampled directly prior to reassembly, which can be done as previously
explained. A Monte Carlo simulation of the random reduced model is carried out, as
opposed to the uncertainty propagation approach used in [55]. Statistical manipulation
of the results yields the mean and variance of the dynamic response.

A benefit of the CBSM is that the perturbation intensity can be selected so as to
represent the level of uncertainty in each subsystem. In the original CBSM papers,
validation of the technique for various SSTL spacecraft has indicated very good agree-
ment against PSA-derived results and test data. The computational time has been
reduced by around two orders of magnitude with respect to the full parametric MCS.
Overall, the technique appears as a promising platform for rapidly performing Monte
Carlo analysis on complex built-up structures.

3.1.4 A more efficient formulation

Treatment of the generalised eigenvalue problem

In this section, an enhanced formulation of the CBSM is outlined, which allows a more
efficient solution of each stochastic realisation of the non-deterministic structure. To
this end, it is worth briefly introducing the generalised eigenvalue problem. Firstly, the
ordinary eigenvalue problem is given by

$$ (A - \lambda_i I) v_i = 0, \quad i = \{1, 2, \ldots, n\} $$

(3.16)

where $\lambda_i$ and $v_i$ are the eigenvalues and corresponding eigenvectors of a square matrix $A$. In a manner akin to (3.16), in the GEP, a set of scalars $\lambda_i$ and corresponding vectors $x_i$ are sought, such that:

$$ (A - \lambda_i B) x_i = 0, \quad i = \{1, 2, \ldots, n\} $$

(3.17)

Analogously to the ordinary eigenvalue problem, $\lambda_i$ and $x_i$ are known, respectively, as
the generalised eigenvalues and eigenvectors of the pair $(A, B)$. In FEM, $A$ and $B$
respectively correspond to the mass and stiffness matrices, whereas $\lambda_i$ and $x_i$ are the
square of the $i$-th natural frequency and the $i$-th mode shape vector of the structure.
The generic notation employed herein is used in order to avoid ambiguity with regard
to already defined symbols.

If $A$ and $B$ are symmetric, the GEP has characteristics similar to the ordinary eigen-
value problem. Additionally, if at least one of the matrices is positive definite (PD) i.e.
has only strictly positive eigenvalues, it is therefore also nonsingular, and the GEP can
be readily reduced to the form of (3.16) by:

$$ (B^{-1} A) x_i = \lambda_i x_i $$

(3.18a)

$$ (A^{-1} B) x_i = \frac{1}{\lambda_i} x_i $$

(3.18b)
3.1. Definition of the method

It is evident that both cases can be treated in exactly the same manner, and here we shall consider the second one. Note that $A^{-1}B$ is generally not symmetric, which is undesirable from a computational standpoint when solving (3.16). After rewriting in matrix form, (3.18b) takes the equivalent, and sometimes more convenient shape

$$ (A^{-1}B)X = X \Lambda^{-1} $$

(3.19)

where $\Lambda^{-1}$ is simply $\text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$. Recall that for a symmetric positive definite matrix $D$, there exists a unique lower triangular $L_D$, such that $D = L_D L_D^T$. This is known as the Cholesky factorisation of $D$. Within the remainder of this thesis, a subscript coinciding with the name of a matrix should normally be interpreted as a factor of the latter, although not necessarily Cholesky. Taking $A = L_A L_A^T$, equation (3.19) can be rewritten into

$$ (L_A L_A^T)^{-1}BX = X \Lambda^{-1} $$

$$ L_A^T L_A^{-1}B (L_A^{-1} L_A^T)X = X \Lambda^{-1} $$

(3.20)

or, concisely, setting $S := L_A^{-1}B L_A^{-T}$, $\Psi_S := L_A^T X$ and $\Lambda_S := \Lambda^{-1}$:

$$ S \Psi_S = \Psi_S \Lambda_S $$

(3.21)

The GEP is therefore reduced to the ordinary eigenvalue problem, but now with $S$ being symmetrical. The latter is amenable to solution schemes of superior efficiency. The reader is referred to the classic books [95], [118], [185] and [186] for a complete review of the respective numerical methods, the last three being particularly relevant to FEM computer treatments.

Note that a selected number of dominant eigenvectors can be extracted from (3.17) through the Lanczos algorithm. This is the conventional approach adopted for FEM, due to its pronounced efficiency for sparse linear systems, and commercial solvers are normally based on this method. Comprehensive descriptions of the topic are available in the aforesaid references. After (3.21) is solved, the required eigenvectors $x_i$ can be recovered by computing

$$ x_i = L_A^{-T} \psi_{S,i} $$

(3.22)

where $\psi_{S,i}$ is the $i$-th eigenvector of (3.21), and respectively column of $\Psi_S$.

The improved CBSM

In a purely finite element analysis sense, the procedure delineated above is applicable if the model under consideration is sufficiently constrained with SPCs and MPCs. Non-singularity of the matrices translates into an absence of rigid body modes. In practice, the structure is always specified to fulfil that, unless a preliminary modal analysis is being performed in a free-free condition. Moreover, as subsequently demonstrated in Section 3.2.4, a direct application of the Lanczos method is arguably not an adequate strategy for the rapid solution of the GEP of $(\tilde{K}_r, M_r)$, arising at each CBSM realisation. The rationale is that the off-diagonal blocks of $M_r$ are intrinsically non-sparse, impairing the Lanczos method’s execution quickness.
In light of these arguments, it is justifiable to assume a reduction of the GEP to an ordinary symmetric problem could be beneficial. The logic is supplemented by the fact that the particular block structure of the $\tilde{K}_r$ matrix can be exploited in order to eliminate some excessive computation done when transforming the CB stochastic method GEP into the form prescribed by (3.21). Let the latter be expressed as

$$ S_r \Psi_{Sr} = \Psi_{Sr} \Lambda_{Sr} $$

and letting $L_{K_r}$ be the Cholesky factor of $K_r,$

$$ S_r = L_{K_r}^{-1} M_r L_{K_r}^{-T} $$

The random matrices corresponding to the quantities in (3.23) and (3.24) and any subsequent equations shall abide to the formerly defined annotation style, e.g. $\tilde{\Psi}_{Sr}, \tilde{S}_r, \tilde{L}_{Kr}$ and so on. Now, in order to substantiate the original claim, consider the primal assembly of the random stiffness matrix by using (3.10) and (3.11):

$$ \tilde{K}_r = \begin{pmatrix} \sum_{k=1}^{N_s} L_b^{(k)} T_k^{(k)} & K_{bb} & 0 & \ldots & 0 \\ 0 & \Lambda_{mm}^{(1)} & 0 \\ \vdots & \ddots & \ddots \\ 0 & 0 & \Lambda_{mm}^{(N_s)} \end{pmatrix} = \begin{pmatrix} K_{bb} & 0 \\ 0 & \Lambda_{mm} \end{pmatrix} $$

Then the Cholesky factor of $\tilde{K}_r$ and its inverse transpose become

$$ \tilde{L}_{K_r} = \begin{pmatrix} L_{K_{bb}} & 0 \\ 0 & \Lambda_{mm}^{1/2} \end{pmatrix}, \quad \tilde{L}_{K_r}^{-T} = \begin{pmatrix} L_{K_{bb}}^{-T} & 0 \\ 0 & \Lambda_{mm}^{-1/2} \end{pmatrix} $$

Verification requires simply noticing that $\tilde{L}_{K_r} \tilde{L}_{K_r}^{-T} = \tilde{K}_r$ and $\tilde{L}_{K_r} \tilde{L}_{K_r}^{-1} = I$ hold, whereas $\tilde{L}_{K_r}$ is indeed lower triangular. Therefore, as $K_{bb}$ is kept constant in the CBSM, so is its Cholesky factor, which does not need to be recalculated at each realisation of the method. Once $\tilde{\omega}_m$ has been generated at each stochastic instance, $\tilde{L}_{K_r}, \tilde{L}_{K_r}^{-T}$ can be directly acquired with effectively no extra computation.

At this stage, a significant portion of the cost of constructing (3.24), respectively (3.23), has been alleviated. Now, the calculation of the costly double matrix-matrix product in (3.24) can be addressed via the following simplification

$$ \tilde{L}_{K_r}^{-1} M_r \tilde{L}_{K_r}^{-T} = \begin{pmatrix} L_{K_{bb}}^{-1} & 0 \\ 0 & \Lambda_{mm}^{-1/2} \end{pmatrix} \begin{pmatrix} M_{bb} & M_{bm} \\ M_{tb} & I \end{pmatrix} \begin{pmatrix} L_{K_{bb}}^{-T} & 0 \\ 0 & \Lambda_{mm}^{-1/2} \end{pmatrix} $$

$$ = \begin{pmatrix} L_{K_{bb}}^{-1} M_{bb} & L_{K_{bb}}^{-1} M_{bm} \\ \Lambda_{mm}^{-1/2} M_{tb} & \Lambda_{mm}^{-1/2} \end{pmatrix} \begin{pmatrix} L_{K_{bb}}^{-T} & 0 \\ 0 & \Lambda_{mm}^{-1/2} \end{pmatrix} $$

$$ = \begin{pmatrix} L_{K_{bb}}^{-1} M_{bb} L_{K_{bb}}^{-T} & L_{K_{bb}}^{-1} M_{bm} \Lambda_{mm}^{-1/2} \\ \Lambda_{mm}^{-1/2} M_{tb} L_{K_{bb}}^{-T} & \Lambda_{mm}^{-1/2} \end{pmatrix} $$

Clearly, top left block $S_{bb}$ of the RHS in (3.27) is independent of the random natural frequencies of the substructures. Coincidentally, it also represents the only partition of
that involves the product of three dense matrices, but can instead be precomputed once and then stored. In fact, the majority of the non-sparse matrix operations are invariant with respect to \( \bar{\Lambda}_{mm} \). For the off-diagonal partitions, we can write

\[
\tilde{S}_{bm} = L_{\bar{\Lambda}_{mm}}^{-1} M_{bm} \bar{\Lambda}_{mm}^{-1/2} = \left( \bar{\Lambda}_{mm}^{-1/2} M_{bm} L_{K_{bb}}^{-T} \right)^T = \left( \bar{\Lambda}_{mm}^{-1/2} P^T \right)^T = \tilde{S}_{mb}^T
\]

(3.28)

Then, with the help of the preceding assertions, it is evident that generating realisations of \( \tilde{S}_r \) only demands a single initial evaluation of \( S_{bb} \) and \( P \) and then storing them for subsequent use. \( \bar{\Lambda}_{mm}^{-1} \) is obtained trivially, and so is \( \bar{\Lambda}_{mm}^{-1/2} P \), which represents a scaling of the columns of \( P \) by the diagonal elements of \( \bar{\Lambda}_{mm}^{-1/2} \), viz. \( \bar{\omega}_{m,i}^{-1} \). Thus, the cubic complexity of the evaluation of \( \tilde{S}_r \) is lowered to quadratic, owed entirely to the \( M_{bm} \) term. In this way, the strategy for efficiently constructing the ordinary eigenvalue problem for the CBSM instances is completed. The eigenmodes of the structure, say, \( \bar{\Phi}_r \), are extracted from the relationship (3.22), i.e.:

\[
\bar{\phi}_{r,i} = \bar{L}_{K_r}^{-T} \bar{\psi}_{Sr,i}
\]

(3.29)

A final observation could be called upon to further aid the process outlined above. Namely, the use of (3.25) and its respective mass matrix equivalent are made redundant by (3.27), and it is sufficient to reassemble the boundary submatrices only once.

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**Figure 3.2:** Process diagram of the original CBSM loop. Reduced subsystem matrices and loads obtained in the same way as for the optimised method shown on Figure 3.3
3.1. Definition of the method

Figure 3.3: Process diagram of the improved CBSM for distributed load applications

To appreciate the differences between the original method and the enhanced formulation presented in the current section, a schematic of the original CBSM loop is given on Figure 3.2. Note the reduced component matrices and loads are obtained in the same way as discussed in Section 3.1.2 and respectively Figure 3.3.

The full process of splitting a model into subsystems, applying the Craig-Bampton reduction, and the addition of the proposed modification to the CBSM, is illustrated
on Figure 3.3. The diagram demonstrates the relatively straightforward case of only having two condensed components and a single unreduced one, but it can easily be seen how the process is generalised. When the improved CBSM is used for boundary-only non-distributed excitations, the load reduction steps are omitted. The latter are discussed in the next section. Observe that the proposed method only involves an implicit CMS model update through the sampled random natural frequencies, rather than an actual assembly of components. Hence, for its practical applications laid out in Section 3.2 custom code has been written to reflect this procedural change.

3.1.5 Extension to distributed loads

Software implementation specifics

Several practical considerations need to be taken into account when the described method is intended for application with acoustic pressure. Degrees of freedom that are to be retained in physical coordinates during the reduction are normally selected so that they span the whole set of connections between the subsystems, along with force input and response output locations. However, this approach is infeasible when dealing with vibroacoustic excitations, because in that case translational forces are exerted on most of the model’s nodes. Clearly, retaining all of them in the \( B \)-set would defeat the purpose of the reduction, yielding condensed matrices of size comparable to the original ones.

Consequently, the computation of the force terms in the RHS of (3.7) dictates the need to output and store the transformation matrices for all the subsystems that require load reduction. In practice, it is important that this is achieved at the component reduction stage, as opposed to separately reacquiring them after the latter has taken place, which would evidently be inefficient. More importantly, the signs of the eigenvectors in \( \Phi^{(k)}_I \) are not fixed. They have to satisfy the scaling \( \Phi^{(k)}_I M^{(k)}_B \Phi^{(k)}_I = I \), which only restricts their absolute values. It follows that the column signs of the modal participation factor matrices \( M^{(k)}_m \) and the elements of the load vectors \( f^{(k)}_m = \Phi^{(k)}_I f^{(k)}_I \) are also not fixed, and are determined when (3.6b) is solved. Therefore ensuring (3.7) is consistent demands the same transformation matrix to be used on the LHS and RHS.

As an effect of the explicit availability of the transformation matrices after the CMS has been completed, conversion of the \( m \) set solution for any component back to physical space becomes trivial. Using (3.5), it is given by

\[
\mathbf{u}^{(k)}_I = \Phi^{(k)}_B \mathbf{u}^{(k)}_B + \Phi^{(k)}_I \mathbf{u}^{(k)}_m \tag{3.30}
\]

Applying pressure excitations

In elasto-acoustic problems, the distributed surface load is normally specified as an elemental pressure vector \( \mathbf{p} \), rather than a nodal force one. A fluid-structure coupling matrix needs to be constructed in order to facilitate the conversion, as discussed in Section 2.5. A general process for non-conforming meshes is shown in [129], for example.
Figure 3.4: Conversion between element pressure and nodal loads

Here, the treatment will be restricted to the commonly encountered case of matching surface discretisations. Indeed, elements defined with constant interpolation functions and solution points internal to their domain, as depicted on Figure 3.4, are widespread. They arise especially often in collocation BEM. Before proceeding, let the degree of freedom ordering in the global FE model be specified as

\[(T_1^1, T_2^1, T_3^1, R_1^1, R_2^1, R_3^1, T_1^2, \ldots, R_2^s, R_3^s)\]

where \(T_i^j\) and \(R_i^j\) respectively denote the \(i\)-th translational and rotational degree of freedom of node \(j\), while \(s\) is the total number of nodes. Under those conditions, the following construction can be made:

\[
T_{sf} := \begin{pmatrix}
A_{e,1}^1 n_1^1 e_1 & A_{e,2}^1 n_1^1 e_1 & \ldots & A_{e,m}^1 n_1^1 e_1 \\
A_{e,1}^1 n_2^1 e_2 & A_{e,2}^1 n_2^1 e_2 & \ldots & A_{e,m}^1 n_2^1 e_2 \\
A_{e,1}^1 n_3^1 e_3 & A_{e,2}^1 n_3^1 e_3 & \ldots & A_{e,m}^1 n_3^1 e_3 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{e,1}^2 n_1^2 e_1 & A_{e,2}^2 n_1^2 e_1 & \ldots & A_{e,m}^2 n_1^2 e_1
\end{pmatrix}
\]

(3.31)

Here, the orthogonal unit vectors \((e_1, e_2, e_3)\) define the global coordinate system and \((n_1^i e_1, n_2^i e_2, n_3^i e_3)\) is the unit surface normal at element \(i\), whose corresponding pressure \(p_i\) acts on node \(j\) through the effective nodal area \(A_{e,i}^j\). Additionally, the total number of elements is \(m\), hence \(T_{sf}\) has a size of \(6s \times m\) prior to the definition of any single- or multi-point constraints. The zero rows correspond to the rotational nodal DOFs. The matrix is sparse, since the coefficient \(A_{e,i}^j = 0\) when node \(j\) is not in contact with element \(i\), or more formally, when it is not in the domain of the support of the interpolation function of that element.
3.1. Definition of the method

The evaluation of the entries of $T_{sf}$ depends on the degree to which the fluid and structural meshes are conforming. Broadly speaking, $A_{e,i}^j$ can be calculated from an entirely geometrical perspective if piecewise constant shape functions are used for the acoustic pressure and flux. Only the splitting of the element into appropriate non-overlapping regions is required. It can be achieved, for instance, by connecting its geometric centre to the midpoints of its edges. In a more general setting, let $\Delta \Gamma_i$ be the part of the acoustic boundary $\Gamma$ corresponding to element $i$. Then

$$A_{e,i}^j = \int_{\Delta \Gamma_i^j} \phi_i(x) d\Gamma(x)$$

(3.32)

where $\Delta \Gamma_i^j$ is the area enclosed by the geometric partitioning of the element that acts on node $j$, $\phi_i$ is the shape function and $x$ is spatial location. For a constant $\phi_i(x)$, equation (3.32) indeed simply represents an area calculation.

The complete load reduction process

Taking into account the aforementioned considerations, the practical procedure for obtaining the total load $f_r$ in hybrid CB coordinates is depicted on Figure 3.5. The reduction performed on each subsystem is standard, with the exception that all the transformation matrices are kept for later use, viz. Figure 3.3. It is assumed that the pressure fields $p^{(k)}$ are known, and may be computed with any appropriate technique discussed in Section 2.4. The actual conversion to FE forces is done by $f^{(k)} = T_{sf}^{(k)} p$. Partitioning of $T_{sf}$ is done row-wise, as it is ordered to match the DOF set of the displacement coordinate $u$.

Figure 3.5: Generation of the CB reduced nodal load from acoustic pressure input
3.2 Performance and behaviour

3.2.1 Benchmark example

The example presented in this subsection has been designed to examine the benefits of the proposed method. This is accomplished by first establishing a baseline solution for comparison by running a frequency response analysis on an unperturbed model in physical coordinates. The test structure is then subjected to both a direct Monte Carlo simulation in full physical coordinates and a CBSM one, utilising the mixed modal-physical space representation. Both sets of results are compared against the nominal solution.

![Figure 3.6: CBSM benchmark example - subsystem definition and boundary set nodes, including SPCs, interface DOFs and output locations](image)

The selected model is shown on Figure 3.6: a flat plate of size $1 \text{ m} \times 0.9 \text{ m}$, made up of 3 subsystems. All of the latter have identical material properties, with Young’s modulus $E = 7.3 \times 10^{10} \text{ N mm}^{-2}$, Poisson’s ratio $\nu = 0.33$ and a density $\rho = 2800 \text{ kg m}^{-3}$. The plates have a distributed non-structural mass (NSM) of $2.5 \text{ kg m}^{-2}$, $1.5 \text{ kg m}^{-2}$ and $2.2 \text{ kg m}^{-2}$, respectively. The set of enforced displacement constraints, along with the nodes forming the boundary DOF set have also been indicated on Figure 3.6.

A non-uniform pressure load derived from a random combination of 10 incident plane waves has been applied to one side of the plate, as shown on Figure 3.7. The resultant complex load vector was kept constant over the range of solution frequencies. Since the CB modal bases were chosen so that the condensed model is representative up to $500 \text{ Hz}$, the excitation spanned the range of $1 \text{ Hz} - 500 \text{ Hz}$. The full Monte Carlo simulation’s uncorrelated random variables were modelled with normal distributions. They were defined in terms of the COV as explained in Section 2.3.2. The mean and standard deviation for a finite set of realisations are given by the conventional formulae corresponding to the mathematical expectation \ref{2.22} and the square root of the second
3.2. Performance and behaviour

Figure 3.7: Distribution of pressure applied to the benchmark model

central moment (2.23):

$$\mu_X = \frac{1}{N} \sum_{j=1}^{N} X_j, \quad \sigma_X = \sqrt{\frac{1}{N} \sum_{j=1}^{N} (X_j - \mu_X)^2}$$ (3.33)

Here, $N$ is the total number of samples and $X_j$ is the $j$-th realisation of the random variable $X$.

The perturbations were set at 5% for the Young’s modulus, 3% for density, Poisson’s ratio and shell thickness and 8% for NSM. These COVs were chosen in a way as to reflect the expected variance of the physical quantities, namely, non-structural mass is much more likely to vary within a wider band than material density. Conditional truncation of the distributions was implemented as a precaution, in order to avoid inappropriate values, such as excessively thin shells. However, due to the relatively small COV values, such occurrences were extremely unlikely. The NFs in the CBSM simulation were modelled with a uniform distribution of ±10% limits, having a standard deviation equivalent to 5.77% COV. For both stochastic methods, the final prediction $\tilde{x}$, represented by a confidence band of $k \sigma$, is given by

$$\tilde{x} = \mu_x \pm k \sigma_x$$ (3.34)

Results for the displacement responses of two different nodes can be found on Figure 3.8 and Figure 3.9. It should be noted that in the former case, only the mean ($k = 0$) prediction was taken. In both cases it can be observed that the displacement obtained from the CB stochastic method is in-line with the final direct MCS prediction, especially in the range of 100 Hz to 500 Hz. As modal density increases, the two statistical results nearly converge, indicating that applying the CBSM’s natural frequency randomisation becomes viable in low- to mid-frequency, yielding results very similar to the parametric Monte Carlo. Of course, this was the original motivation for extending the CBSM to distributed load applications. The observation is enforced by the results presented on Figure 3.9, on which the two techniques are seen to provide nearly identical solutions over approximately 420 Hz.
3.2. Performance and behaviour

Figure 3.8: Comparison of CBSM and MCS against the nominal solution (Node 193)

Figure 3.9: Comparison of CBSM and MCS $\mu + 3\sigma$ predictions (Node 254)

As expected, both the Monte Carlo and Craig-Bampton solutions exhibit a more deterministic behaviour at lower frequencies, and the few modes dominating the response can be clearly identified. This also translates into a narrower solution confidence band, agreeing with the expected physical behaviour of actual structures.

3.2.2 Realistic spacecraft structure

In order to demonstrate the methodology outlined in the previous sections with a more realistic example, a test case was set-up, based on the SSTL300 spacecraft, designed and built by [SSTL]. Since the structural qualification model provided by the company did not include the exact flight payload and instrumentation, a mock-up pressure field was applied in a manner similar to that explained for the benchmark example. The spacecraft is shown on Figure 3.10.
3.2. Performance and behaviour

Figure 3.10: Improved CBSM realistic test case: the SSTL300 spacecraft structure

The surface load was generated by simulating a diffuse sound field composed of 50 plane wave sources evenly distributed around a unit sphere whose centre coincides with the approximate geometric centre of the model. Firstly, sources are modelled as mutually repelling particles, constrained to the surface of the sphere. The particles are initially randomly distributed, and their motion and position is iteratively updated, until criteria for the minimum distance between any pair of points is fulfilled. This approach is closely related to the Riesz $s$-energy of measures, but that idea is only used herein to ensure a unique, yet smooth spatial source distribution at every frequency. Once a satisfactory allocation of the sources is obtained, their phases and magnitudes are randomised to emulate a diffuse field.

Applying the aforesaid procedure, an incident field was evaluated at 100 frequency points, logarithmically spaced between 1 Hz and 1500 Hz. Since test data was not available, the incident field was used directly as structural excitation, whereas normally the acoustic domain would be properly accounted for. Nevertheless, the principal aim of this section is to assess the improved CBSM with distributed excitations against the well-established PSA technique for a realistic finite element model, rather than to test the quality of the model itself.

The Craig-Bampton reduction of the spacecraft involved dividing it into 10 subsystems, 9 of which were its main external panels. Each of the panels was subjected to acoustic excitation, and no loads were directly applied to the internal structure. The reduction and reassembly of the acoustic excitations was performed as per the procedures explained in Section 3.1.5. The full implementations of the improved CBSM and the parametric MCS were programmed and executed in Matlab and Fortran 95. The latter
served more intensive computational and input/output tasks. The structural solver employed throughout was MSC Nastran.

All component condensations included local modes corresponding to NFs of up to $2.5\,\text{kHz}$. The final CB model, representative of the full FE up to $1.5\,\text{kHz}$, consisted of 1569 degrees of freedom, 1029 of which were modal. In contrast, the full physical representation of the spacecraft had a total of $243\,738$ DOFs. The condensed model was expected to be a good approximation of the type of CB matrices that would occur in practice, perhaps barring the fact more non-interface DOFs could normally be assigned to the $B$ set at points where physical sensor locations would exist.

The comparison baselines for the CBSM were the nominal solution and PSA facilitated through a direct Monte Carlo simulation. The MCS's uncorrelated random variables were defined in accordance with the values given in Table 3.1. A detailed prescription of suggested COVs for parameter modelling can readily be found in the literature reviewed and referenced in Section 2.3.2. Especially relevant articles directly addressing this topic are [38], [40]–[42]. A remark should be made that some of the values presented in Table 3.1 differ from the work of the authors cited above, thus results are not directly comparable. The decision was supported by SSTL input of knowledge of the dynamic behaviour of the SSTL300 spacecraft, gained through various physical tests. Investigating uncertainty in the damping was not within the scope of this work, therefore a frequency-invariant modal value of $1\%$ was utilised. Detailed studies of damping level's effect can be found in papers such as [40] and [41].
Table 3.1: Assumed coefficients of variation for the SSTL300 parametric model’s random properties

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<th>Type</th>
<th>Property</th>
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<th>St. deviation</th>
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<tr>
<td></td>
<td>Poisson’s ratio</td>
<td>ν</td>
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<td>Shear modulus</td>
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<td></td>
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</tbody>
</table>

*Not applicable to the test case, or is described as a function of already defined values, e.g. \( G = \frac{E}{2(1+\nu)} \)

### 3.2.3 Optimal perturbation PDF and variance

**CBSM performance assessment criterion**

In order to build a wider understanding of how the improved Craig-Bampton stochastic method behaves when applied to structures of such complexity, a detailed study on the influence of the NF perturbation level was conducted. Furthermore, the investigation was carried out over three different underlying distributions for the entries in \( \tilde{\omega}_m \), namely uniform, normal and lognormal. The criterion selected for similarity comparison between the proposed method and the PSA was the Pearson correlation coefficient \( r(X,Y) \), defined by:

\[
r(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}, \quad \text{cov}(X,Y) = \frac{1}{N} \sum_{j=1}^{N} (X_j - \mu_X)(Y_j - \mu_Y)
\]  

The sets \( X \) and \( Y \) of size \( N \) represent sampled populations of the underlying random variables, while \( \text{cov}(X,Y) \) is the discrete equivalent of the general definition given in
The value \( r(X, Y) \) is also known as the linear correlation coefficient and originates from the Cauchy-Bunyakovsky-Schwartz inequality. It falls within the closed interval \([-1, 1]\), where \( r(X, Y) = \pm 1 \) indicates a total positive or negative linear correlation between \( X \) and \( Y \).

To guarantee the reliability of the obtained data, a mean correlation coefficient was calculated from a set of the responses of 100 boundary and 100 modal degrees of freedom scattered throughout the model, over the full frequency range under consideration. For example, for a specified perturbation level and distribution, 500 realisations of the improved CBSM were run and responses from 1 Hz to 1500 Hz were collected for the aforementioned DOFs. The correlation for the mean CBSM response against the parametric Monte Carlo one was computed for each of these DOFs. Finally, the average of the produced 200 values for the Pearson correlation coefficient was taken. The process may be formalised through the following equation:

\[
\bar{r}_\mu = \frac{1}{N_d} \sum_j N_s \sum_j r(\bar{x}_j^{MCS}, \bar{x}_j^{CBSM}), \quad \bar{x}_j = (x_j(\omega_1), \ldots, x_j(\omega_{N_s}))
\]

Here, \( \bar{x}_j \) is the mean response vector for the \( j \)-th DOF, with entries corresponding to the different solution frequencies \( \omega_s \) up to the last one, indexed \( N_s \). \( N_d \) is the total number of DOFs comprising the correlation dataset. An identical process was followed for the corresponding standard deviations, i.e. \( \bar{r}_\sigma \) was evaluated from \( r(\sigma_x^{MCS}, \sigma_x^{CBSM}) \). The so-obtained linear correlation coefficients \( \bar{r}_\mu \) and \( \bar{r}_\sigma \) were devised to serve as the primary indicators reflecting the quality of the improved CBSM solution, as they capture its overall closeness to the direct MCS predictions.

**Parametric survey**

An extensive investigation of the effects of varying the COV and the underlying distribution of \( \tilde{\omega}_m \) was intended as a principal output of this work. To this end, three PDFs were assessed for modelling the random subsystem NFs. Uniform distribution is characterised by a constant probability density within an interval \([a, b]\), with a mean equal to \((a + b)/2\). It is straightforwardly reproduced by scaling and shifting a sample generated by a (pseudo)random number generator that yields numbers evenly spread on the interval \([0, 1]\). Such tools are readily available in most programming languages. Either uniform or Gaussian distributions were used in the original papers introducing the CBSM, but the latter did not directly compare them, or examine the effect of their variance on the reliability of the collected responses. In the ensuing discussion, these PDFs were also juxtaposed against a lognormal distribution. The three types are depicted on Figure 3.12.

Lognormal distributions often arise naturally in stochastic FEM problems, as deliberated in [158], [160], [162], [163] and also [40]–[43], in relation to modal parameters of aerospace structures. Furthermore, they do not need to be artificially truncated, as opposed to Gaussian, since negative parameter values cannot occur. In essence, if \( X \) is a stochastic variable, the lognormal PDF is such that \( \log(X) \) is normally distributed. In order to acquire meaningful assertions, the PDFs were defined to have a mean of 1 and
3.2. Performance and behaviour

Figure 3.12: PDFs of uniform, Gaussian and lognormal distributions with the same mean $\mu = 1$ and $\sigma = \sqrt{3}/6$

equal variances, given in terms of equivalent uniform randomisation limits. Explicitly, the interdependencies can be written as

\[
\text{var}(X_u) = \frac{(b - a)^2}{12} \quad (3.37a)
\]

\[
\text{var}(X_n) = \log \left( \frac{\text{var}(X_n)}{X_n^2} + 1 \right) \quad (3.37b)
\]

\[
X_{ln} = \log \left( \frac{X_n^2}{\sqrt{\text{var}(X_n)} + X_n} \right) \quad (3.37c)
\]

where subscripts $u$, $n$ and $ln$ respectively indicate the uniform, normal and lognormal distributions, $a$, $b$ are the uniform distribution lower and upper limits and $X$ bears the standard meaning of average when used in a statistical context.

Using the preceding arguments, analysis of the SSTL300 spacecraft, as introduced in Section 3.2.2 was repeated over randomisations ranging from $\pm 8\%$ to $\pm 50\%$ uniform distribution limits, at increments of $2\%$. All generated distributions were used to scale the original $\omega_m$ in order to yield $\tilde{\omega}_m$, in other words, $\tilde{\omega}_m = X \circ \omega_m$. The full collected dataset is summarised on Figure 3.13, on which the global correlations $r_\mu$, $r_\sigma$ of (3.36) are plotted. Conversion to equivalent COV is obtained by dividing the uniform perturbation level by $\sqrt{3}$, as can be calculated from (3.37a) for the case of interest, i.e. distributions normalised to have a mean of 1.

**Principal findings**

Overall, all the tested distributions led to very reliable response estimation at levels higher than approximately $\pm 14\%$. As expected, the best correlation for the uniform
distribution was reached at lower natural frequency randomisation, owing to the substantially higher probability density near the edges of the distribution. On the other hand, the Gaussian case yielded the lowest correlation with respect to the physical space MCS. Note that at the high extreme of the perturbation range, the equivalent coefficient of variation is 28.9%, corresponding to a confidence band of $\pm 3.46\sigma$. Physically, all random NFs must remain positive. The probability of a negative one being produced at this COV is $\approx 0.025\%$, which is non-negligible and forces distribution truncation in many of the improved CBSM realisations.
3.2. Performance and behaviour

Figure 3.15: Acceleration response magnitude at different nodes: (a) B set, interface between two subsystems - node 1187, (b) B set, sensor location, not used for CMS reassembly - node 701124, (c) I set, recovered from the CBSM solution - node 1100884
The lognormal distribution intrinsically does not suffer from such issues. It provides the highest correlation for both the mean and standard deviation between the CBSM and the PSA-based simulation, with \( r_\mu \approx 0.97 \) and \( r_\sigma \approx 0.955 \), respectively. This was achieved over a relatively wide band of \( \pm 34\% \sim \pm 44\% \) randomisation limits. The optimal level for the Gaussian PDF occurred between \( \pm 32\% \) and \( \pm 40\% \), yielding approximately 1% lower global correlation coefficients. The uniform distribution evidently posed the least desirable characteristics, producing slightly smaller \( r_\mu, r_\sigma \) values than the normal one, while necessitating greater COV to be applied.

In brief, the lognormal distribution appeared the most suitable for modelling the fixed interface natural frequencies of the CB subsystems. However, all three probability density functions led to mean linear correlation of over 0.9 for both the mean and standard deviation of the solutions, above circa \( \pm 18\% \) level. The latter corresponds to COV = 0.104, which is very close to the magnitude suggested by the authors of the original CBSM, i.e. COV = 0.1. Nevertheless, the ideal coefficient of variation, ranging from 0.15 to 0.26, gives noticeably enhanced predictions. This is easily visualised and affirmed by the results plotted on Figure 3.15. Three distinct cases are depicted: a boundary and a non-boundary \( B \) set node, as well as the solution recovered from an internal elastic \( I \) set degree of freedom, using (3.30). The \( \pm 36\% \) line (20.8% COV) represents the optimal case, whereas the remaining curves serve as a basis for comparison of how the solution degrades when deviating from this ideal value. The spatial locations of the nodes in the actual spacecraft model are indicated on Figure 3.14.

Generally, the improvement of greatest significance observed by increasing the applied COV is a better correlation of the low-frequency dynamics with respect to the MCS. The connection interface node’s response on Figure 3.15(a) is a minor exception, showing substantial gains in \( r_\mu, r_\sigma \) well into the mid-frequency range. Combining the preceding arguments, it is easy to see why applying a lognormal distribution with a relatively large variance for \( \tilde{\omega}_m \) is beneficial. In light of this conclusion, it can also be deduced that a first order perturbation approach, as per Section 2.3.3, would not be practically viable to accurately propagate the variability of \( \tilde{\omega}_m \) into the final solution. Indeed, the method of Mace and Shorter [55] only employed a COV of 0.02, yet discrepancies against the full modal solution were visible in the supplied results.

### 3.2.4 Convergence and computational cost

#### CBSM solution convergence

In terms of how a suitable number of realisations was selected to build the correlation datasets of Figure 3.13 and Figure 3.17, 2000 instances were initially run for both the full and condensed model stochastic simulations. This exact choice was influenced by a combination of factors. Firstly, a sample of this size is considered to be at the high end of the spectrum for direct Monte Carlo simulations when only the first two statistical moments are of interest. Further details have been supplied in Section 2.3.2. Secondly, even samples containing as few as 100 stochastic instances have been shown to yield tractable results in practice, for example in the original CBSM articles. This was the case for tests ranging from a 1-D rod to a real satellite model. The number and type of random variables present did not appear to affect this result.
As a conservative scenario, the improved CBSM case employed ±50% NF randomisation with uniform distribution, expected to show poor convergence due to the extreme perturbation limits. The standard definition of the physical space MCS, as per Section 3.2.2, was used. The average solution relative error for $\mu$ and $\sigma$ was evaluated over all DOFs in terms of the number of completed solutions, for the full solution frequency range. The procedure pertains to the same principles as the construction of $r_{\mu}$ and $r_{\sigma}$. The main findings can be seen on Figure 3.16.

![Figure 3.16: Maximum relative error of the CBSM and MCS with respect to solution based on 2000 MCS realisations. Convergence criterion set to 1% relative error.](image)

A convergence criterion of 1% maximum relative error was deemed a reasonable target. The condition was met at approximately 300 realisations for the mean and 400 for the standard deviation. From the observed trends, it is additionally apparent that the CBSM’s convergence rate is very similar to that of the MCS, regardless of whether degrees of freedom in physical or modal space representations are being compared. On the basis of this study, 500 realisations were considered as a reliable choice for executing CBSM-based analyses, providing some margin of safety over the number necessitated by the already conservatively defined test case.

**Computational requirements against PSA**

It is worth briefly discussing computational time of the CBSM against the baseline parametric MCS. The former would be expected to hold an advantage, in view of the sizeable reduction in the problem size owing to the CMS representation of the model. To quantify the performance of the method, two metrics were extracted from the executed simulations, with results summarised in Table 3.2. Solution time was comprised of various tasks, such as constructing or reassembling the mass and stiffness matrices, solving the GEP and recovering the required final outputs by modal superposition. Overhead was attributed to input/output (I/O) operations, the code facilitating the PSA and the original/improved CBSM implementations, as well as other activities unrelated to the principal computations.
Table 3.2: Computational time per realisation, 100 frequency points

<table>
<thead>
<tr>
<th>Implementation</th>
<th>PSA Nastran</th>
<th>Original Nastran</th>
<th>Original Matlab</th>
<th>Improved Matlab</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
<td>297 s</td>
<td>11.9 s</td>
<td>12.5 s</td>
<td>0.6 s</td>
</tr>
<tr>
<td>Overhead</td>
<td>436 s</td>
<td>4.7 s</td>
<td>0.4 s</td>
<td>&lt;0.1 s</td>
</tr>
<tr>
<td>Total</td>
<td>733 s</td>
<td>16.6 s</td>
<td>12.9 s</td>
<td>0.7 s</td>
</tr>
</tbody>
</table>

* A custom code avoiding excessive I/O
* GEP reduction to an ordinary eigenvalue problem, no explicit reassembly

The results affirm that in reality, the replacement of a PSA-based stochastic analysis with more efficient approaches can be easily justified from a speed of execution standpoint. The original CBSM accelerated the solution for 500 realisations from almost 102 h to about 2 h 18 min and 1 h 47 min, respectively, for two different deployments. The latter was further reduced by a factor of 18.4 with the improved CBSM, in other words when the transformation of the GEP to an ordinary eigenvalue problem is done. Its solution cost was dominated by solving the latter via a generic non-sparse algorithm, and the lack of explicit reassembly at each realisation significantly lessened the performance loss incurred due to excessive I/O operations.

It must be pointed out that the parametric MCS and the first rendition of the original CBSM were carried out in Nastran, while the second one and the improved CBSM were implemented entirely in Matlab, using mass and stiffness system matrices initially constructed in Nastran. Therefore the stated overhead times are not necessarily analogous, as different forms of I/O operations were facilitated.

In terms of the fundamental solution times, all methods call highly optimised subroutines using multi-core parallelism. Indeed, recreating Nastran’s Lanczos iterative solution in Matlab resulted in an average runtime of 12.5 s for the original CBSM with a similar number of calculated MVPs. This signifies the inadequacy of sparse iterative solvers to cope with relatively dense matrices, such as the ones occurring in CMS. Finally, observe that the reduced SSTL300 satellite problem is quite small at 1569 DOFs. It is expected that for larger condensed systems, a gain in efficiency of the same order would be exhibited by the improved CBSM, since the algorithmic complexity remains identical, i.e. a cubic scaling.

Taking into account the promising results demonstrated on Figure 3.15(c) for solution recovery of non-boundary DOFs, it is also possible to assert that the nodes whose physical responses are required do not have to be identified prior to performing the model condensation, therefore do not need to be ascribed to the $B$ set. This also enables the efficient storage of the complete numerical solution as a combination of physical and modal responses along with a set of transformation matrices. For the particular SSTL300 spacecraft case investigated here, the storage requirements are presented in Table 3.3. Note that this capability enabled the conduction of the parametric survey presented in Section 3.2.3, which consolidated data acquired from tens of thousands of CBSM executions.
### 3.2. Performance and behaviour

#### Table 3.3: Solution vector storage requirements for a full stochastic simulation

<table>
<thead>
<tr>
<th></th>
<th>Per realisation(^a)</th>
<th>Transformation</th>
<th>500 realisations</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBSM</td>
<td>2.4 MB</td>
<td>1382.3 MB</td>
<td>2.52 GB</td>
</tr>
<tr>
<td>PSA</td>
<td>371.9 MB</td>
<td>181.6 GB</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) For all DOFs over 100 frequency points

Generally, the Craig-Bampton stochastic method would allow saving the full results dataset in a sufficiently compact form even for much larger FE models and at more frequency points. This advantage further supplements the drastically reduced computing cost, especially pronounced for the improved CBSM variants.

#### 3.2.5 Scope of validity

The findings outlined in the prior sections delineated the merits of the improved CBSM for analysis of complex structures exposed to distributed loads. Nonetheless, the conclusions drawn were in part based on the assumption that the spacecraft CMS model was representative of how such structures would be treated in an industrial engineering environment. In particular, the optimality of a certain probability distribution and variance for the description of the uncertain substructural NFs, conferred in Section 3.2.3, demands some further exploration. It is feasible to suspect that the overall stiffness of the CMS interface might affect the predictions of the CBSM.

#### Table 3.4: Number of degrees of freedom per subsystem for reduction 1 and 2 of the SSTL300 model

<table>
<thead>
<tr>
<th>Subsystem</th>
<th>Reduction 1</th>
<th>Reduction 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>modal</td>
<td>boundary</td>
</tr>
<tr>
<td>1</td>
<td>92</td>
<td>114</td>
</tr>
<tr>
<td>2</td>
<td>97</td>
<td>114</td>
</tr>
<tr>
<td>3</td>
<td>118</td>
<td>186</td>
</tr>
<tr>
<td>4</td>
<td>79</td>
<td>72</td>
</tr>
<tr>
<td>5</td>
<td>98</td>
<td>114</td>
</tr>
<tr>
<td>6</td>
<td>37</td>
<td>24</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>30</td>
</tr>
<tr>
<td>8</td>
<td>37</td>
<td>24</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>359</td>
<td>366</td>
</tr>
<tr>
<td>Total</td>
<td>1029</td>
<td>1074</td>
</tr>
<tr>
<td>CB reassembled</td>
<td>1029</td>
<td>540</td>
</tr>
</tbody>
</table>

To that end, a second Craig-Bampton reduction of the SSTL300 satellite was performed. Table 3.4 contains the details of it, as well as the initial condensation. The division into components remained unaltered, but multiple arbitrarily selected \( B \) set points...
were introduced for each subsystem. The aim was to replicate the behaviour that the structure would have exhibited in the presence of either an oversized boundary or a very large count of sensor locations. The effect was intentionally exaggerated by transferring a number of \( I \) set points to the \( B \) set that would be conceivably unrealistic in practice. In this case, the \( B \) set amounted to 3.27% of the overall 243738 physical degrees of freedom, whereas this value was only 0.47% for the original CMS.

The analysis carried out followed the tenets of Section 3.2.3, although a complete parametric study of the same extent was deemed unnecessary. A lognormal PDF was used to model \( \omega_m \) and again 500 realisations were extracted at each COV step, which had between \( \pm 10\% \) and \( \pm 70\% \) equivalent uniform distribution limits. Increments of 10\% were assigned. The responses obtained from the improved CBSM were gathered and statistically processed, so that the indicators \( \tau_\mu \) and \( \tau_\sigma \) could be evaluated. The key results are summarised on Figure 3.17. For comparison purposes, the CBSM-PSA correlations from the optimum perturbation for reduction 1, namely 36\% lognormal, have also been included.

![Figure 3.17: Coefficients \( \tau_\mu \) and \( \tau_\sigma \) for two separate condensations of the SSTL300 satellite. NFs modelled with a lognormal distribution.](image)

Examination of Figure 3.17 immediately reveals the detrimental effect of the excessively large \( B \) set on the quality of the CBSM solution. The highest achieved correlation was \( \tau_\mu \approx 0.9 \) and \( \tau_\sigma \approx 0.88 \), which closely resembles the lognormal distribution results for reduction 1 at 12\% \( \sim \) 14\% randomisation, i.e. at a COV level of sub 0.08. This kind of correlation might still mean the response variability had been successfully predicted to an acceptable accuracy, as demonstrated on Figure 3.15 for the original reduction. Notwithstanding, unphysical behaviour can be expected when such perturbation magnitudes are being applied, as can be inferred from Figure 3.18 which shows the results for the same nodes, obtained with the second configuration. While being quite extreme, the \( \pm 70\% \) value stood out as clearly insufficient for the CBSM to achieve its maximum closeness to the direct Monte Carlo simulation. In addition, further increases in the coefficient of variation were not tested, as distortion of the response near structural resonances was starting to become apparent.
3.2. Performance and behaviour

Figure 3.18: Acceleration response magnitude at different nodes for reduction 2: (a) B set, interface between two subsystems - node 1187, (b) B set, sensor location, not used for CMS reassembly - node 701124, (c) I set, recovered from the CBSM solution - node 1100884
3.3. Summary

The low-frequency CBSM solution predominantly exhibited severe nodal acceleration underprediction. The range below 300 Hz-400 Hz was nearly unaffected by the presence of the random natural frequencies. Broadly speaking, the mid- to high-frequency response variability of the parametric MCS was still reasonably well approximated by the CBSM, especially for the nodes that experienced larger accelerations in absolute terms, such as 1100884 and 701124. The subsystem connector node 1187 could be argued as a counterexample. Only peaks near heavily excited global modes were close to the MCS $\mu + 3\sigma$ curve. They were, however, unexpectedly prominent as well, protruding upwards on the chart to what would correspond to at least a $\mu + 5\sigma$ PSA estimate line. These should be very low likelihood occurrences, and were certainly far less pronounced for the first modal condensation of the spacecraft.

From a finite element modelling perspective, this study might be somewhat unrealistic. An interface definition that engulfs over 3% of the physical space DOFs is not unreasonable, but the fact the corresponding nodes were randomly distributed within the subsystems might have made the CMS representation overly constrained. In a sense, a definition like the second reduction can be thought of as removing model stiffness from the fixed interface component DOF sets, while further constraining the remaining internal elastic DOFs of the subsystems. Under such circumstances, the Craig-Bampton reduction itself demands extended modal bases to properly replicate the original structure’s dynamics, which is reflected in Table 3.4. The participation of the constraint modes becomes more prevalent in terms of the global behaviour, but their effect is not manifested in the modal DOF partition of the mass and stiffness CB matrices.

In light of the former arguments, the results observed in the current section could logically be expected. The presented test case, albeit extreme, indicates the limitations of the CBSM under the effect of specific factors. To remedy this issue and enable stochastic vibroacoustic analysis of completely general FE models, a novel decomposition-based method has been developed and is presented in the succeeding chapter. Nevertheless, the improved CBSM is certainly a viable platform for rapid dynamic analysis of uncertain structures. It is easily implemented and, under appropriate circumstances, highly robust and reliable, due to the wide band of permitted COVs that give acceptable results from low to relatively high solution frequencies.

3.3 Summary

In the present chapter, the Craig-Bampton stochastic method was thoroughly studied. The main objectives were to expand the CBSM’s capabilities in terms of efficiency and ability to handle acoustic loads, as well as to gain deeper understanding of its general limitations. Section 3.1 primarily dealt with its theoretical foundations. A set of enhancements to the CBSM’s formulation were proposed. An altered CBSM process structure obviated explicit reassembly of the subsystems containing random component natural frequencies. This included an efficient recasting of the GEP, solved at each realisation, to an ordinary eigenvalue problem, achieved by exploiting the structure of the CB matrices. Load reduction and application to the CMS model were deliberated alongside the recovery of non-boundary DOF responses.
Section 3.2 was split into five essential subtopics, centred around practical implementation matters. Firstly, the improved CBSM was compared against parametric MCS for a simple benchmark example. Then, the set-up of a high-complexity test case based on the SSTL300 spacecraft was shown, with direct Monte Carlo PSA forming a baseline prediction of its response variability. Section 3.2.3 introduced the indicators $\tau_{\mu}$, $\tau_{\sigma}$, based on the Pearson correlation coefficient, as a means of appraising the global proximity of the improved CBSM solution to the full MCS one. A comprehensive parameter study was conducted and $\tau_{\mu}$, $\tau_{\sigma}$ were evaluated over a broad spectrum of coefficients of variation of the random substructural NFs, spanning the range of ±8% to ±50% equivalent uniform distribution limits. The lognormal distribution was deemed the most suitable PDF for the NFs, with respect to Gaussian and uniform. The ±34% to ±44% level yielded nearly perfect values of $\tau_{\mu}$, $\tau_{\sigma}$. In comparison to the PSA, the computational cost of the new method was around three orders of magnitude smaller. A crucial factor for this was the transition from sparse system Lanczos solvers to general matrix ones. Finally, the scope of validity of the CBSM was investigated in the context of a second reduction of the satellite, characterised by a substantially larger interface DOF set. It was concluded that the CBSM is tolerant to wide COV ranges of the random natural frequencies for typical models. However, engineering judgement must be exercised to determine whether the method is suitable for the analysis of specific non-standard CMS representations on a case by case basis. Examples would include models containing non-local subsystem interfaces or very large number of sensor locations, both resulting in a significant expansion of the boundary DOF set.
Chapter 4

Block decomposition-based stochastic Hermitian matrices

4.1 Introduction

Uncertainty modelling in computational mechanics is presently a highly active field of research, incited by the demand of numerical methods’ development to remain on par with the advent of ever faster computers. Expansive review of the topic has been provided in Section 2.3 and therefore only the motivation behind the work contained within the current chapter is elucidated here.

In Chapter 3 the Craig-Bampton stochastic method has been explored in great detail. Upon the implementation of a set of suggested modifications and improvements, the technique was deemed reliable and efficient for solving low- to mid-frequency distributed excitation problems in structural dynamics in the presence of uncertainty. Nevertheless, limitations of its range of applicability were also discovered, and in part attributed to the fact that only the modal coordinate subpartition of the reduced stiffness matrix was responsible for encompassing the entire variability of the structure.

On the other hand, several competing techniques that also avoid the tedious use of traditional PSA are in existence, such as the nonparametric probabilistic approach, delineated in Section 2.3.4. However, it is intrinsically primed to generate random realisations of the original model matrices that do not have their sparsity preserved. Furthermore, while convenient from an engineering perspective, the reduction of the structural randomness to a single dispersion-controlling parameter might be seen as restrictive. In the context of CMS, which is extremely relevant to aerospace structures and other high performance applications, it would be undesirable to produce full matrices that negate the present blockwise sparsity pattern. Some rare publications try to explicitly inject uncertainty through a random matrix model on a subsystem level, e.g. [64], where the boundary stiffness $K_{bb}^{(k)}$ submatrices of Craig-Bampton models are considered for the purpose. However, a definitive lack of stochastic techniques that are capable of dealing with off-diagonal matrix blocks individually is observed. Even in the paper referenced above, the fact that the anti-diagonal partitions $K_{bm}^{(k)}$, $K_{mb}^{(k)}$ equal the null matrix is of paramount importance.
The reason for the above claim will be made apparent in the ensuing sections. In this chapter, a novel approach to the mathematical description of finite element model nondeterminism is presented, as partially introduced recently by Yotov et al. [187, 188]. Indeed, the particular issue of robustly and unrestrictedly generating random blocks of Hermitian matrices is addressed in a completely generic setting, which essentially can be thought of as a superset of linear time-harmonic problems originating in FEM. Unlike existing schemes, the presented framework is entirely founded on principles of linear algebra and can flexibly accommodate different statistical representations of the random variables controlling the stochastic matrix blocks. As such, it lends itself to straightforward software deployment and optimisation. Vitally, block structures, such as the ones transpiring in CMS or automatic algebraic substructuring, are inherently well-suited to manipulation with the novel technique. Independent treatment on a component level is naturally permitted and simultaneously the structure of the matrices is not distorted.

For the purpose of completeness and improved readability of the subsequent sections, a concise reintroduction to a number of fundamental algebraic concepts, encountered and used throughout the rest of this chapter, is presented in Section 4.2. The complete set of symbols and mathematical notation used can be found in the glossary supplied at the beginning of this work. A remark should be made that whenever deemed appropriate, concrete equations or definitions already explained in the preceding chapters might be restated to ensure a smoother flow of arguments. Finally, it is pointed out that a more mathematically oriented presentation style is adopted, with theorems, corollaries and other logical structures encapsulated in individual enumerated blocks. Simultaneously, intermediate results might intentionally be unnumbered for the sake of clarity.

### 4.2 Mathematical apparatus and prerequisites

#### 4.2.1 Spectral decomposition

A complex square matrix $A$ of size $n \times n$ will be denoted $A \in \mathbb{C}^{n \times n}$. Sometimes, $A_{n \times n}$ will also be used for improved clarity, especially when indicating the size of matrix partitions in a block matrix. Finally, for the zero and identity matrices, respectively $0$, $I$, the shorthand $0_n$ is taken to be the same as $0_{n \times n}$. Now, recall the standard eigenvalue problem (3.16):

$$(A - \lambda_i I)v_i = 0, \quad i = \{1, 2, \ldots, n\}$$

(4.1)

where $\lambda_i$ and $v_i$ are the eigenvalues and corresponding eigenvectors of $A$. Defining the matrices $Q = (v_1, \ldots, v_n)$, $\Lambda = \text{diag}(\lambda_i)$, which is equivalent to $\Lambda_{i,j} = \lambda_i$ for $i = j$ and $0$ otherwise, (4.1) can be rewritten as $AQ = QA$. Premultiplying by $Q^{-1}$ yields the factorisation

$$A = Q\Lambda Q^{-1}$$

(4.2)

which is known as the eigendecomposition. Of course, $A$ has to have $n$ linearly independent (not necessarily distinct) eigenvectors. More broadly speaking, it has to be **diagonalisable**. Without loss of generality, $\|v_i\| = 1$ will be assumed in any further use,
4.2. Mathematical apparatus and prerequisites

although the eigenvectors need not be normalised in general. Before proceeding, recall the following special cases, depending on the type of $A$:

1. if $A$ is normal, $Q$ is unitary, therefore $QAQ^{-1} = Q\Lambda Q^*$
2. if $A$ is also Hermitian, all eigenvalues $\lambda_i$ are real
3. if $A$ is additionally real, therefore symmetric, both of the above clearly hold, and $Q \in \mathbb{R} \Rightarrow Q^* = Q^T$. In other words, $Q$ is orthogonal
4. if $A$ is unitary, $\lambda_i$ have values on the complex unit circle, i.e. $|\lambda_i| = 1$

Now, it is worth mentioning that (4.2) is also known as the spectral decomposition, due to the following theorem:

**Theorem 4.2.1 (Spectral theorem).** If $A$ is a Hermitian matrix, there exists an orthonormal basis of its eigenvectors: $Q = (v_1, \ldots, v_n)$, and all of its eigenvalues $\lambda_i(A)$ are real

The first statement immediately results in (4.2). Finally, whenever applicable, i.e. for Hermitian matrices, let the eigenvalues be sorted in a weakly decreasing order, so the $n$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

4.2.2 Singular value decomposition

A continuation of the previously outlined spectral decomposition concept is the so-called singular value decomposition (SVD). It is the underpinning for the proposed decomposition-based stochastic finite element method (DSFEM) intended predominantly for the robust and efficient generation of random CMS mass and stiffness matrices. Moreover the SVD is extensively utilised in various stages of the hierarchical matrix compression of BEM matrices, explained in slightly more detail in Chapter 5.

Here, a definition of the SVD, together with its properties relevant to the proceeding derivations is outlined. For an exhaustive description of its theory and applications, along with other topics covering all fundamental aspects of linear algebra, the reader is referred to [118], [189] and [185].

The singular value decomposition can be viewed as a generalisation of the eigendecomposition of a positive semi-definite (PSD) matrix. However, it can be applied to any matrix $A \in \mathbb{C}^{m \times n}$, and is given by

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^*$$

where $U$ and $V$ are complex or real unitary matrices, i.e. such that $U^*U = UU^* = I$.

In the latter special case, they are orthogonal. The columns of $U$ and $V$ are respectively the left and right singular vectors of $A$. The matrix $\Sigma$ is rectangular diagonal with $r = \min\{m, n\}$ nonnegative values on the main diagonal. For example, for $m \leq n$ it has the form

$$\Sigma_{m \times n} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \begin{pmatrix} 0 \\ m \times m \end{pmatrix} \begin{pmatrix} 0 \\ m \times n - m \end{pmatrix}$$
The unique values \( \sigma_i \) are called singular values of \( A \), and are conventionally sorted in a weakly descending order. Analogously to the spectral decomposition, we shall use \( \sigma_i(A) \) for the \( i \)-th singular value of \( A \), where \( \sigma_1 \geq \cdots \geq \sigma_r \). The connection of the SVD to the spectral decomposition is rather simple to visualise. Taking the matrix \( AA^* \) into consideration and using (4.3),

\[
AA^* = U \Sigma V^* (U \Sigma V^*)^* \\
= U \Sigma V^* V \Sigma^* U^* \\
= U (\Sigma \Sigma^*) U^*
\]

which is indeed the eigendecomposition of \( AA^* \). The latter, being a Gramian matrix, is of course positive semi-definite. Analogously, \( A^* A = V^* (\Sigma^* \Sigma) V \). Therefore the left and right singular vectors of \( A \) are actually the orthonormal eigenvectors of \( AA^* \) and \( A^* A \), respectively. As \( \Sigma^* \Sigma = \Sigma \Sigma^* = \text{diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0) \), the relation \( \sigma_i(A) = \sqrt{\lambda_i(\Sigma \Sigma^*)} \) instantly follows. A noteworthy remark is that unlike the eigendecomposition, the SVD is normally not unique. It is, however, in the trivial case \( m = n \), when the two factorisations coincide.

An advantage of the SVD is that it naturally encompasses an explicit characterisation of the range and nullspace of a matrix. For a graphical illustration, let the factorisation of \( A \) from (4.3) be further split as specified below, with \( \Sigma_r \) being the diagonal matrix containing the \( r \) nonzero singular values of \( A \):

\[
\begin{pmatrix}
  u_1 & \cdots & u_r \\
  U_2
\end{pmatrix}
\begin{pmatrix}
  \Sigma_r & 0 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  V_1 & v_{r+1} & \cdots & v_m
\end{pmatrix}^*
\]

The left singular vectors corresponding to the \( \Sigma_r \) form a basis for the range of \( A \), that is \( \text{range}(A) = \text{span}\{u_1, \ldots, u_r\} \) and \( \text{rank}(A) = r \). Furthermore, the right singular vectors for which \( \sigma_i(A) = 0 \), form a basis for the nullspace \( \text{ker}(A) = \text{span}\{v_{r+1}, \ldots, v_m\} \).

Similarly, \( U_2 \) and \( V_1 \) are bases for \( \text{ker}(A^*) \) and \( \text{range}(A^*) \), respectively. Therefore the SVD provides an explicit representation of all four fundamental subspaces of \( A \).

In addition to the former remark, a key result is that the best possible low-rank approximation to the original matrix in both the Frobenius and spectral norms can be obtained from the singular value decomposition. This is known as the Eckart-Young-Mirsky theorem. To rephrase the claim in a slightly more formal manner, let \( S \) be a matrix having the same size as \( A \), and \( \text{rank}(S) = k \leq r \). Define the truncated SVD of \( A \), containing its \( k \) largest singular values and their corresponding singular vectors, as

\[
A_k := \sum_{i=1}^{k} \sigma_i(A) u_i v_i^* = U_k \Sigma_k V_k^* \approx A
\]

then for any \( S \) as defined, it holds that

\[
\|A - A_k\|_F \leq \|A - S\|_F \quad (4.8a)
\]
\[
\|A - A_k\|_2 \leq \|A - S\|_2 \quad (4.8b)
\]

which, in turn, has several major implications. The one of considerable interest that will be mentioned here is in fact the obvious - the generation of minimum possible rank
approximants of predetermined accuracy to some known matrices. It is an application instrumental to the successful construction of efficient $\mathcal{H}$-matrices, which in turn enable the full $\mathcal{H}$-matrix accelerated DSFEM-BEM method for vibroacoustic simulations of uncertain structures, proposed in Chapter 5.

4.2.3 Positive (semi-)definiteness of matrices

Schur complements

At this stage, the concept of Schur complements is briefly introduced, largely for the purpose of deriving a set of necessary and sufficient conditions for establishing the positive (semi-)definiteness of a block matrix based on its partitions. These conditions are fundamental to the logic of the developed stochastic matrix construction algorithm, introduced in Section 4.3. Moreover, Schur complementation is used for efficiently computing the structural response of coupled FEM-BEM systems in the context of vibroacoustic simulations, viz. Section 2.5.2. For an in-depth review of the topic, including detailed proofs and numerous further properties and applications of the Schur complement, the reader is referred to the specialised book by Zhang [190].

Now, let $M$ be an $n \times n$ matrix, partitioned in blocks $A$, $B$, $C$, $D$, and $r + q = n$:

$$M = \begin{pmatrix} A_{r \times r} & B_{r \times q} \\ C_{q \times r} & D_{q \times q} \end{pmatrix} \quad (4.9)$$

**Definition 4.2.2 (Schur complement).** Consider the block partitioning of $M$ as in (4.9). Then

$$M/A := D - CA^{-1}B \quad (4.10a)$$

$$M/D := A - BD^{-1}C \quad (4.10b)$$

are the Schur complements of blocks $A$ and $D$ in $M$, respectively.

Clearly, the existence of $M/D$ requires that $D$ is nonsingular. The same argument applies to the complement of block $A$. Using Schur complementation, several important properties can be written in terms of the submatrices. For instance, the determinant and rank of $M$ may be expressed as

$$\det(M) = \det(D) \det(M/D) \quad (4.11)$$

$$\text{rank}(M) = \text{rank}(D) + \text{rank}(M/D) \quad (4.12)$$

In the context of checking for positive (semi-)definiteness of matrices, (4.11) immediately yields Sylvester’s criterion. The latter states that a real symmetric matrix $M \in \mathbb{R}^{n \times n}$ is PD if and only if all of its leading principal minors are positive, e.g. \( \forall i \in \{1, \ldots, n\} : \det(M_{1:i,1:i}) > 0 \). However, the Schur complement can be used to obtain a more general condition in terms of the matrix partitions. Let $M$ be Hermitian such that

$$M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \quad (4.13)$$
In addition, let us define the expression \( X > 0 \) to mean that a Hermitian matrix \( X \) is strictly positive definite, and \( X \geq 0 \) to mean that \( X \) is positive semi-definite. Analogously, the symbolic representations for negative definite and negative semi-definite matrices are defined by inverting the sign of the former 'inequalities'. Now, the following set of necessary and sufficient conditions for \( M > 0 \) can be written in terms of either diagonal block and its Schur complement:

\[
M > 0 \iff \begin{cases} 
A > 0 \\
M/A = D - B^*A^{-1}B > 0
\end{cases}
\]  
\[(4.14a)\]

\[
M > 0 \iff \begin{cases} 
D > 0 \\
M/D = A - BD^{-1}B^* > 0
\end{cases}
\]  
\[(4.14b)\]

In a similar manner, \( M \geq 0 \) is satisfied when a diagonal block is positive definite, if and only if its Schur complement is nonnegative definite:

\[
A > 0 \Rightarrow M \geq 0 \iff M/A \geq 0
\]  
\[(4.15a)\]

\[
D > 0 \Rightarrow M \geq 0 \iff M/D \geq 0
\]  
\[(4.15b)\]

**Generalised inverses and Schur complements**

When the block whose complement in \( M \) is taken is not invertible, or not even square, it is still possible to form a similar set of conditions. However, the concept of generalised inverses needs to be introduced first.

**Definition 4.2.3** (generalised inverse). For a given matrix \( A_{m \times n} \), a generalised inverse \( A^g \) is an \( n \times m \) matrix, such that \( AA^gA = A \).

Normally, \( A^g \) needs not be unique, and in the trivial case when \( m = n \) and \( A \) is non-singular, \( A^g \) coincides with the ordinary (and unique) inverse \( A^{-1} \). Indeed, excluding the aforementioned special case, there are infinitely many distinct generalised inverses of \( A \). With the previous definition, the generalised Schur complement [191] is specified in the same way as in **Definition 4.2.2**, but using some generalised inverse instead of the ordinary one. Subsequently, it can be shown that the conditions for positive (semi-)definiteness equivalent to (4.15a) and (4.15b) are:

\[
M \geq 0 \iff \begin{cases} 
A \geq 0 \\
D - B^*A^gB \geq 0 \\
(I - AA^g)B = 0
\end{cases} \iff \begin{cases} 
D \geq 0 \\
A - BD^gB^* \geq 0 \\
(I - DD^g)B^* = 0
\end{cases}
\]  
\[(4.16)\]

The result is due to the work of Albert [192], later enveloped as a subset of the scope of [190]. For simplicity, consider the first case, involving \( M/A \), as it is identical to that of \( M/D \). The last condition implies that \( AA^g \) acts on \( B \) as an identity matrix. It is possible to show that this is the case if and only if \( \text{range}(B) \subseteq \text{range}(A) \). For the intents and purposes of this thesis, the concept of \( A^g \) will be restricted to the pseudoinverse also frequently referred to as a Moore-Penrose inverse, introduced in [193]. This is arguably the most widely known and used type of generalised inverse, for a number of reasons. It will be denoted using the symbol \( A^+ \).
**Definition 4.2.4** (pseudoinverse). A unique generalised inverse, for which all of the following statements are valid:

\[
\begin{align*}
AA^gA &= A \\
A^gAA^g &= A^g \\
(-AA^g)^* &= AA^g \\
(A^gA)^* &= A^gA
\end{align*}
\]

Adding the second condition defines a so-called reflexive generalised inverse. Furthermore, \(A^+\) must fulfil the second pair of requirements, which guarantee its uniqueness. In fact, if \(USV^*\) is a singular value decomposition of \(A\), the pseudoinverse is given by

\[
A^+ = V\Sigma^+U^* \tag{4.18}
\]

where \(\Sigma^+ = \text{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0)\) and \(r = \text{rank}(A)\). This clearly demonstrates its strong connection to least squares, along with other concepts related to the SVD. Moreover, a few supplementary important properties can be extracted.

**Corollary 4.2.5.** For \(AA^+\) to act as an identity matrix on \(B\):

\[
\text{range}(B) \subseteq \text{range}(A) \Leftrightarrow AA^+B = B
\]

**Proof.** Using (4.18),

\[
AA^+ = USV^*V\Sigma^+U^* = U\Sigma\Sigma^+U^*
\]

and additionally, let \(k = \text{rank}(A)\). Clearly, \(k \leq r\), where \(r\) is the size of \(A\), otherwise \(AA^+ = I_r\). Then \(\Sigma\) is a diagonal matrix which, together with its pseudoinverse, can be written as \(\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)\), and \(\Sigma^+ = \text{diag}(\sigma_1^{-1}, \ldots, \sigma_k^{-1}, 0, \ldots, 0)\), respectively. Thus

\[
\Sigma\Sigma^+ = \begin{pmatrix}
I_k & 0 \\
0 & 0_{r-k}
\end{pmatrix}
\]

with the matrices \(AA^+, A^+A\) naturally being Hermitian:

\[
\begin{align*}
AA^+ &= U \begin{pmatrix} I_k & 0 \\ 0 & 0_{r-k} \end{pmatrix} U^* \tag{4.19a} \\
A^+A &= V \begin{pmatrix} I_k & 0 \\ 0 & 0_{r-k} \end{pmatrix} V^* \tag{4.19b}
\end{align*}
\]

They are called orthogonal projections, by virtue of the additional property

\[
(\cdot AA^+)^2 = AA^+, \quad (\cdot A^+A)^2 = A^+A
\]

which, in turn, is directly acquired from (4.19). This means that \(AA^+\) acts as an identity matrix on any \(B \subseteq \text{range}(AA^+)\). Expressly, from **Definition 4.2.4** combined with the fact that for any vector \(y = Ax\), \(y \in \text{range}(A)\),

\[
AA^+y = AA^+Ax = Ax = y \tag{4.20}
\]
Now, letting \((u_1, \ldots, u_k)\) be the left singular vectors of \(A\), corresponding to its nonzero singular values, and noting that (4.19a) actually provides an SVD of \(AA^+\), we deduce:

\[
\text{range}(AA^+) = \text{span}\{u_1, \ldots, u_k\} = \text{range}(A) \tag{4.21}
\]

Thereby the image of \(AA^+\) indeed coincides with the column space of \(A\), but then combining (4.20) and (4.21), Corollary 4.2.5 is evident. Finally, note a similar result can be obtained via the application of precisely the same set of arguments for the nullspace of \(A\) and \(\ker(A^+A)\).

The implications of Corollary 4.2.5 will be shown to be particularly useful for the proposed approach to constructing random matrices for the DSFEM technique, as it establishes precisely under what circumstances (4.16) can be applied. Finally, the following crucial remark should be made: if \(M\) is not necessarily Hermitian, as in (4.9), given \(\text{range}(B) \subseteq \text{range}(A)\) and \(\text{range}(C^*) \subseteq \text{range}(A^*)\) hold, (4.12) is also valid. If \(M\) is Hermitian, trivially, the second condition is equivalent to the first one. Otherwise, \(\text{rank}(M) > \text{rank}(D) + \text{rank}(M/D)\) is generally possible. For thorough analysis, one can consult [190].

Weyl eigenvalue inequality

Having conferred some necessary and sufficient conditions for \(M > 0\) and \(M \succeq 0\) in terms of its block partitions, it would be beneficial to cover the topic of positive definiteness of a matrix formed as a sum of Hermitian matrices, for the sake of completeness of the present section. A commonly asked question in linear algebra is, for Hermitian \(P, Q \in \mathbb{C}^{n \times n}\), to what degree do the eigenvalues of \(P + Q\), denoted \(\lambda_1(P + Q), \ldots, \lambda_n(P + Q)\), are constrained by the eigenvalues \(\lambda_1(P), \ldots, \lambda_n(P)\) and \(\lambda_1(Q), \ldots, \lambda_n(Q)\) of the original matrices. The problem was proposed in the early 20th century by Herman Weyl, and, as a matter of fact, is highly non-trivial. Without further information, its full solutions relies on the so called Horn conjecture. The latter had long stood as an open problem in mathematics, until it was proven about two decades ago by Knutson and Tao [194].

In engineering and physical sciences, the problem often arises naturally when dealing with random matrices. In such situations, \(Q\) is often a perturbation matrix that is, in some sense, small with respect to \(P\). Such a formulation lends itself to a more simplistic treatment. In the case of interest for this work, which is indeed related to generating a stochastic matrix with specific properties, based on a known, possibly mean one, \(Q\) cannot necessarily be assumed to be small. On the contrary, \(P\) and \(Q\) would typically have very close eigenvalues. Regardless, some well-known results that do not completely describe \(\lambda_i(P + Q)\) prove to be a useful tool for the analysis of such problems on many occasions.

The Weyl, Lidskii and Ky Fan eigenvalue inequalities ([195], [196]) are perhaps the most famous ones directly concerning the previously described problem. They can be derived from the property of the trace i.e. \(\text{trace}(A) = \sum_{i=1}^n \lambda_i(A)\), the Courant-Fischer min-max theorem and consideration of partial traces of \(A\). Here, we present only the dual Weyl inequality, which states:
Theorem 4.2.6 (Weyl eigenvalue inequality). The inequalities
\[
\begin{align*}
\lambda_{i+j-1}(P + Q) &\leq \lambda_i(P) + \lambda_j(Q), \quad i + j \leq n + 1 \quad (4.22a) \\
\lambda_{i+j-n}(P + Q) &\geq \lambda_i(P) + \lambda_j(Q), \quad i + j \geq n - 1 \quad (4.22b)
\end{align*}
\]
hold for any pair of indices \(1 \leq i, j \leq n\).

Using the preceding theorem, by selecting appropriate index pairs \(i, j\), the following key consequence can be immediately established:

Corollary 4.2.7. For all indices \(1 \leq k \leq n\)
\[
P \succeq 0 \Rightarrow \lambda_k(P + Q) \geq \lambda_k(Q) \quad (4.23)
\]

Proof. Indeed, starting from the second inequality, only a simple observation is required. Rearranging \((4.22b)\) and setting \(i = n, k = i + j - n = j\):
\[
\lambda_k(P + Q) - \lambda_k(Q) \geq \lambda_n(P) \geq 0
\]
which instantly yields \((4.23)\). \(\square\)

At this point an important remark should be made - if \(P\) is not known to be positive semi-definite a priori, the statement is not valid with certainty. Hence \(\lambda_k(P + Q) \geq \lambda_k(Q)\) is necessary, but not sufficient to guarantee \(P + Q \succeq 0\). Further to the eigenvalue inequalities, \((4.22a)\) can be extended for the singular values of nonsquare matrices and their sum. Namely, let \(P, Q \in \mathbb{C}^{m \times n}\), and \(r = \min\{m, n\}\). For all pairs of indices \(1 \leq i, j \leq r\), satisfying \(i + j \leq r + 1\):
\[
\sigma_{i+j-1}(P + Q) \leq \sigma_i(P) + \sigma_j(Q) \quad (4.24)
\]
where the singular values of \(P, Q\) and their sum are the eigenvalues of the Hermitian matrices \(P', Q'\) and \(R'\), defined as
\[
P' = PP^*, \quad Q' = QQ^*, \quad R' = (P + Q)(P + Q)^*
\]

In fact, it is possible to take \(S' = P' + Q'\) instead, and observe that the Weyl inequality holds for for \(P', Q'\), and \(S'\), as all of them are Hermitian. Then using the fact that \((\sigma_k(A))^2 = \lambda_k(AA^*)\), with the same restrictions for \(i, j\), both \((4.22a)\) and \((4.22b)\) are clearly valid for the corresponding singular values, provided the original matrices are PSD. Otherwise, \((4.22b)\) in particular has no direct equivalent in terms of the singular values of the original matrices. For example, consider the case \(P = I, Q = -I, P + Q = 0\), which straightforwardly leads to:
\[
\sigma_{i+j-n}(P + Q) = 0 \geq \sigma_i(P) + \sigma_j(Q) = 2
\]
which is obviously a contradiction. On the other hand, it is easy to verify that the inequality prescribed by \((4.24)\) is indeed satisfied.
4.2.4 Generalised eigenvalue problem for Hermitian matrices

Before progressing with the derivation and construction of the actual DSFEM algorithm, a final assertion needs to be made in order to demonstrate the relevance of the ideas summarised in Section 4.2.3 to numerical model representations arising in FEM. To that effect, it is first and foremost shown why the conditions $M \succeq 0$, $K \succeq 0$ are of paramount importance in practice. To achieve that, the focus is drawn once again towards the generalised eigenvalue problem, underpinning the FEM modal solution procedure.

While real, symmetric mass and stiffness matrices $M$, $K$, are of interest for the applications of the proposed method, covering the more general Hermitian case is fairly uncomplicated, and shall be laid out here. As per Section 3.1.4, recall the main idea of the GEP: a set of scalars $\lambda_i$ and corresponding vectors $\phi_i$ are sought, satisfying

$$(K - \lambda_i M)\phi_i = 0, \quad i = \{1, \ldots, n\}$$

Strictly speaking, $K - \lambda_i M$ is called a linear matrix pencil. In order to maintain self-consistent notation, let $\lambda_i(K, M)$ denote the $i$-th generalised eigenvalue. This annotation will be used whenever the shorter version $\lambda_i$ causes ambiguity. Note that a weakly increasing ordering is used, i.e. $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If at least one of $(K, M)$ is strictly PD, the pair is said to form a definite Hermitian pencil, and the GEP can be reformulated into the ordinary eigenvalue problem, viz. Section 3.1.4. Again, let us consider the case of invertible $K$, equivalent to (3.18b).

The generalised eigenvalues $\lambda_i$ are real, but unlike the ordinary eigenvectors, $\phi_i$ are not orthogonal in the usual sense. They are, nevertheless, distinct when $M \succ 0$ and are instead called $M$-orthogonal, since $\phi_i M \phi_j = 0$ for $i \neq j$ and 1 otherwise. In fact, they diagonalise $K$ as well, and

$$\Phi^* M \Phi = I$$
$$\Phi^* K \Phi = \Lambda$$

Naturally, $M, K \in \mathbb{R}$ results in $\Phi \in \mathbb{R}$. Before examining how perturbing the initial matrices affects the generalised eigenvalues of the Hermitian pencil they form, a couple of key concepts are first formally introduced.

**Definition 4.2.8** (Cholesky decomposition). For a Hermitian matrix $A$, there exists a unique lower triangular $L_A$, such that $A = L_A L_A^*$, which is known as the Cholesky factorisation of $A$.

**Definition 4.2.9** (Inertia of Hermitian matrices). The number of positive, negative and zero eigenvalues of a matrix $A$, indicated, respectively, by the triplet $\text{In}(A) := (p(A), n(A), z(A))$.

The first concept has already been utilised in the case of symmetric matrices within the context of the improved Craig-Bampton stochastic method. At this stage it is feasible to ascertain that if the initial positive definiteness of $M$ is not preserved upon applying some modification, the negative eigenvalues $\lambda_i(M)$ induce an equal number of negative $\lambda_i(K, M)$. 
4.2. Mathematical apparatus and prerequisites

Proof. Pertaining to the analytical steps undertaken in (3.19) and (3.20), with the exception that $K$ is now Hermitian, (4.25) is recast into

$$(L_K^{-1}ML_K^{-*})(L_K^*\Phi) = (L_K^*\Phi)\Lambda^{-1}$$

or, compactly, $C\Phi_C = \Phi_C\Lambda_C$. The original claim can be reformulated to prove that $\ln(C) = \ln(\Lambda_M)$, where $Q_M\Lambda_MQ_M^*$ is the spectral decomposition of $M$. Recall Sylvester’s law of inertia, which states that

$$\ln(A) = \ln(B) \iff \exists S : A = S^*BS$$

where $S$ is invertible, and $A, B$ are called *-congruent. Then set $G := Q_M^*L_K^{-*}$ and observe:

$$C = L_K^{-1}ML_K^{-*} = (L_K^{-1}Q_M)\Lambda_M(Q_M^*L_K^{-*}) = G^*\Lambda_MG$$

Thereby $\ln(C) = \ln(\Lambda_M) = \ln(\Lambda)$ is evident, with the minor addition that $\lambda_i(C) = 1/\lambda_i$, i.e. their signs are equal. \hfill \Box

If $M$ is singular, any infinite values arising on the diagonal of $\Lambda_C$ correspond to zero generalised eigenvalues of the original problem. In a physical sense, they usually signify rigid body modes of the structure, or in other words, motions occurring when the latter is not fully constrained. As an immediate consequence of the aforesaid proof, if $M, K \in \mathbb{R}$ and the mass matrix becomes indefinite due to an improperly prescribed perturbation, unphysical behaviour is obtained when solving the GEP. In practical terms, this gives rise to some complex natural frequencies of the structure, since they are related to the generalised eigenvalues by $\omega_i^2 = \lambda_i(K, M)$. For their corresponding generalised eigenvectors, $\phi_i \in \mathbb{R}$ is no longer true.

Remark. All of the preceding conclusions can similarly be reached for the equivalent case of $M \succ 0$ and indefinite $K$. Furthermore, in the general setting when neither $K$ nor $M$ is positive definite, the GEP is solved by the generalised Schur decomposition (also known as QZ factorisation):

$$M = QU_MZ^*, \quad K = QU_KZ^*$$

where $Q, Z$ are unitary, $U_M, U_K$ are upper triangular, and the ratio of their $i$-th diagonal values produces the corresponding generalised eigenvalue, namely $u_{Mi}/u_{Ki} = \lambda_i(K, M)$. If required, the eigenvectors of the pencil $U_K - \lambda U_M$ are computed by back substitution. If $K$ and $M$ are simultaneously singular and share a common nullspace, any value can be regarded as a generalised eigenvalue of the pencil, which is called singular. In general, the Cholesky factorisation method presented above would typically be a preferred approach for a definite Hermitian pencil, subject to appropriate conditioning of the latter.
4.3 Derivation of the DSFEM framework

4.3.1 Desirable properties and behaviour

In Section 4.2, the main algebraic constructs related to the ensuing definition of the block decomposition-based random Hermitian matrices have been summarised. The viewpoint of generalised eigenvalue problems originating in FEM has been extended in the sense that the discussion was not restricted to real matrices. For the majority of the current section, the deliberation will pertain to the same level of generality. Elevated attention was drawn towards conditions of nonnegative definiteness. Based on the key identified factors, a list of properties desirable to be exhibited by a class of random Hermitian matrices intended for FEM-like applications has been composed. It serves to bridge the gap between the anterior exposition of selected mathematical theory and the succeeding array of claims, proofs and miscellaneous algebraic constructions. The following set of qualities is required for the method that will be proposed:

1. Robustness. The technique, referred to as decomposition-based stochastic finite element method, should be rigorously defined so that physical accuracy of the objects described by the constituent random matrices is not distorted. Due to the applications intended for this thesis, references to the method will imply the subset of FEM representations, not general Hermitian pencils.

2. Preservation of positive (semi-)definiteness. The inability to do so is an issue sometimes encountered in literature, for instance with the dual Craig-Bampton method, outlined in Section 2.2.2. As has been shown, this requirement is instrumental to the reliable definition of random FEM models that do not have erroneous negative eigenvalues. To an extent, it is a subset of the previous specification of robustness.

3. Intrinsic capability to allow the individual construction of stochastic off-diagonal matrix blocks. This has been identified as one of the main limitations of existing techniques to be naturally suitable to the treatment of DS and CMS models.

4. Inherent suitability for integration within CMS-based uncertainty analysis. This condition is intimately related to the previous requirement, as will be demonstrated in Section 4.4.3.

5. Efficiency. The sampling of the random matrices, i.e. mass and stiffness, ought to demand computer runtime of order at most similar to the solution of the corresponding GEP. In CMS terms, a reasonable benchmark would be similarity to the computational time of a realisation of the improved CBSM.

6. Preservation of the rank of the original matrices. This is an extension of the imposed PSD condition. Mass and stiffness matrices of structures that are not in some way disjointed have a rank deficiency of at most 6.

7. Optionally, provide a means of preservation of the nullspace of the original matrices. This is an exacting specification for stochastic mass and stiffness produced by the new scheme to mimic the conditions under which rigid body motion takes place. Therefore it reflects a physical characteristic of the structure.
4.3.2 A PSD condition for partitioned Hermitian matrices

In this subsection we shall develop the core mathematical framework laying the foundations for the novel DSFEM approach. It is constructed in a way as to guarantee the fulfillment of the specified requirements. It will be shown that the underlying apparatus accommodates the generation of random matrices not necessarily restricted in scope to the ones arising due to the component mode synthesis of real, symmetric mass and stiffness discrete representations. For instance, the fundamental assumptions and derivations do not make use of the fact that $M, K \in \mathbb{R}$, so the enveloping Hermitian case is valid. In addition, particularities, such as the component modal mass of Craig-Bampton models being the identity matrix, need not be obeyed.

In order to refrain from defining and dealing with superfluous notation, or repeating equivalent equations, let us consider a generic $2 \times 2$ block partitioning of a Hermitian matrix, without specifying the order of its diagonal blocks, namely:

$$G = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \text{ or } G = \begin{pmatrix} D & B^* \\ B & A \end{pmatrix} \quad (4.27)$$

If, for instance, the first definition from the above equation is being used, and the Schur complements $G/A$ or $G/D$ are of interest, observe that by either reordering the blocks of $G$, or equivalently, selecting the second definition, treatment of the two cases remains completely identical. Thereby provided no special restrictions are known that make differentiation between $A$ and $D$ obligatory, without loss of generality, the first definition shall be used.

For the intents and purposes of constructing a random matrix $\tilde{G} \succeq 0$, in some sense based on a known, possibly mean, PSD matrix $G_0$, let us define

$$\tilde{G} := \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^* & \tilde{D} \end{pmatrix} \quad (4.28)$$

and take $G$, partitioned as explained, to be a realisation of $\tilde{G}$. Note that specifics, such as measures of 'closeness' of $\tilde{G}$ to $G_0$, will be dealt with later. Clearly, $G$ must be at least positive semi-definite. Establishing a result that explicitly provides sufficient conditions for the partitions of $G$, such that they satisfy (4.28), is vital. To this end, we first re-derive a classic result, viz. Property 4.3.1, then extend it by formulating and proving the related Proposition 4.3.2. Both are subsequently used in the proof of a theorem instrumental to the definition of the new stochastic method.

**Property 4.3.1.** For a Hermitian matrix $T$,

(a) if $T \succeq 0$, then $STS^* \succeq 0$ for any $S$

(b) if $S$ is nonsingular, then $STS^* \succeq 0 \iff T \succeq 0$, with the equivalent claim for strict positive definiteness following trivially

**Proof.** If $T$ is PSD, we can write $STS^* = (SL_T)(SL_T)^*$ which is Gramian thus PSD. On the other hand, to prove (b) assume $T$ has at least one negative eigenvalue and
\(STS^* \succeq 0\). Take the spectral decomposition \(T = Q_T \Lambda_T Q_T^*\) and let \(G := S^{-*}Q_T\). Then
\[
G^*STS^*G = Q_T(S^{-*}S^*)^*Q_T \Lambda_T Q_T^*(S^*S^{-*})Q_T
= Q_TQ_T^* = \Lambda_T
\]
so \(STS^*\) is \(\ast\)-congruent to a diagonal matrix with at least one negative eigenvalue and cannot be PSD - a contradiction with the assumption.

**Proposition 4.3.2.** For a Hermitian \(T\), a necessary, but not sufficient condition for \(STS^* \succeq 0\) is \(\text{range}(V_n) \cap \text{range}(S^*) = \{0\}\), where \(V_n\) is the matrix of eigenvectors of \(T\) with negative associated eigenvalues.

**Proof.** Here, a slightly more involved approach is called for. Assume \(\text{range}(V_n)\) and \(\text{range}(S^*)\) have a nontrivial intersection, i.e. there exists a vector \(y \neq 0\), which is in the image of both \(V_n\) and \(S^*\). Then \(y\) can be expressed as a linear combination of some \(k\) columns of \(V_n:\)
\[
y = \sum_{i=1}^{k} c_i v_i
\]
where \(v_i : \lambda_i(T) < 0\) are eigenvectors of \(T\), and \(c_i\) are nonzero constants. Now, let
\[
\hat{y} := S^{++}y = S^{++} \sum_{i=1}^{k} v_i c_i
\]
Then, as \(y \in \text{range}(S^*)\), and recalling Corollary 4.2.5, we can write:
\[
\hat{y}^*STS^*\hat{y} = y^*S^+STS^*S^{++}y = y^*Ty
\]
Moreover, the latter can be rearranged into:
\[
y^*T \sum_{r=1}^{k} v_r c_r = y^* \sum_{r=1}^{k} T v_r c_r
= (\sum_{p=1}^{k} c_p^* v_p^*) \left( \sum_{r=1}^{k} \lambda_r(T) v_r c_r \right)
= \sum_{p=1}^{k} \sum_{r=1}^{k} c_p^* v_p^* \lambda_r(T) v_r c_r
= \sum_{r=1}^{k} c_r^* \lambda_r(T) c_r < 0
\]
The last line is obtained from the fact \(v_i\) are orthonormal, thus \(v_p^* v_r = 1\) if \(p = r\) and 0 otherwise. The final assertion is owed to \(c_r^* c_r > 0\) for any \(c_r \neq 0\), and that by our definition, \(\lambda_r(T) < 0\). However, this is a contradiction, because for \(A \in \mathbb{C}^{n \times n}\)
\[
A \succeq 0 \iff x^* A x \geq 0 \ \forall x \in \mathbb{C}^n
\]
and we have shown \(\hat{y}^*STS^*\hat{y} < 0\). Thereby the intersection \(\text{range}(V_n) \cap \text{range}(S^*)\) must indeed be the null vector. Finally, inspect the following simple case:
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Example 4.3.3. Consider the matrices $T$ and $S$, where their spectral and singular value decompositions, $Q_T \Lambda_T Q_T^*$ and $U_S \Sigma_S W_S^*$, respectively, are specified as below:

$$T = I \Lambda_T I^* = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}, \quad U_S = (1), \quad \Sigma_S = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_S = \begin{pmatrix} 3/5 & 4/5 \\ 0 & 0 \end{pmatrix}$$

and $STS^* = (-2)$, which is evidently not PSD but

$$\text{range}(V_n) \cap \text{range}(S^*) = \text{span}\left\{\frac{3}{5} \right\} \cap \text{span}\left\{1\right\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $V_n$ is defined as in Proposition 4.3.2. The preceding construction clearly provides a counterexample to the assumption that the originally stated condition is sufficient for $STS^* \succeq 0$. This completes the proof.

Theorem 4.3.4. Let $G$ be a Hermitian matrix with a $2 \times 2$ block partitioning $(A \ B; B^* \ D)$. Then $G$ is positive semi-definite if and only if

(a) all singular values $\sigma_i((L_A^+ B L_D^+)^*) \leq 1$, where $A = L_A L_A^*$ and $D = L_D L_D^*$

(b) $\text{range}(B) \subseteq \text{range}(A)$

(c) $\text{range}(B^*) \subseteq \text{range}(D)$

Proof. Notice that $A$ and $D$ are implicitly required to be positive semi-definite, since they are both Gramian (as they must have factorisations of the form $XX^*$), and therefore so are their pseudoinverses. Keeping this in mind, set $H := B^* A^+ B$, which is PSD due to Property 4.3.1(a). The generalised Schur complement condition from (4.16) takes the form

$$G \succeq 0 \iff \begin{cases} A \succeq 0, \\ D - B^* A^+ B \succeq 0, \\ (I - AA^+) B = 0 \end{cases}$$

(4.29)

The first condition is already satisfied, whereas the last one is equivalent to Theorem 4.3.4(b) as a consequence of Corollary 4.2.5. Therefore only $G/A \succeq 0$ remains to be examined. For a strictly PD, say, $A$, $L_A$ may directly be taken as its Cholesky factor. However, in order to treat the general case when the blocks could be singular, let

$$L_A := Q_A \Sigma_A^{1/2} X_A^*, \quad L_D := Q_D \Sigma_D^{1/2} X_D^*$$

(4.30)

where $L_A L_A^* = Q_A \Sigma_A Q_A^*$ is the SVD of $A$, $X_A^*$ is any unitary matrix, and $L_D$ is obtained analogously. Invoking Corollary 4.2.5, we can deduce

$$H = (L_D L_D^*) H \iff \text{range}(H) \subseteq \text{range}(L_D)$$

$$H = H (L_D^+ L_D^*) \iff \text{range}(H^*) \subseteq \text{range}(L_D^*)$$

$$=$$
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Regarding the premultiplication by $L_D D_L$, let $\text{rank}(D) = r$. Since $Q_D \Sigma_D^{1/2} X_D^*$ is an SVD of $L_D$, after denoting the orthonormal vectors of $Q_D$ by $q_D$, make the observations:

\[
\text{range}(L_D) = \text{span}\{q_D_1, \ldots, q_D_r\} = \text{range}(D)
\]
\[
\text{range}(H) = \text{range}(B^* A^+ B) \subseteq \text{range}(B^*)
\]
\[
\text{range}(B^*) \subseteq \text{range}(D)
\]

the last of which is given. Combining all of them, $\text{range}(H) \subseteq \text{range}(L_D)$ is affirmed.

Ensuring the equality between $H$ and $HL_D^* L_D^+$ can be dealt with in a comparable manner, or simply by using the fact $H$ and $D$ are Hermitian. If any different definition of the factor $L_D$ was selected, the same arguments as above could be made, after taking an SVD of $L_D$ first. Returning to the original task, proceed by inspecting the Schur complement of $A$ in $G$:

\[
G/A = D - B^* A^+ B
= L_D L_D^* - H
= L_D (I - L_D^* H L_D^*) L_D^*
= L_D H' L_D^*
\]

Then, recalling that $(FF^*)^+ = F^+ F^+$, (an easily verifiable characteristic of the pseudoinverse):

\[
H' = I - L_D^* H L_D^+
= I - L_D^* B^* L_A^+ L_D^* B L_D^*
= I - (L_D^* B^* L_A^+)(L_D^* B^* L_A^+)^*
= I - WW^*
\]

Where the matrix $W$ and its singular value decomposition are defined as

\[
W := L_D^* B^* L_A^+ = U_w \Sigma_w V_w^*
\]

Since $U_w, V_w$ are unitary and $\Sigma_w \in \mathbb{R}$ is diagonal, we can write

\[
H' = I - (U_w \Sigma_w V_w^*)(U_w \Sigma_w V_w^*)^*
= U_w U_w^* - U_w \Sigma_w \Sigma_w^T U_w^*
= U_w (I - \Sigma_w \Sigma_w^T) U_w^*
\]

which is actually the eigendecomposition of $H'$. Then $H' \succeq 0 \iff \Sigma_{wii} \leq 1$, but $W$ has the same singular values as $W^* = L_D^* B L_D^*$. Then, from Property 4.3.1(b) if $\sigma_i(W) < 1 \Rightarrow H' \succeq 0$, and therefore $G/A \succeq 0$.

It is only left to show that the Schur complement of $A$, thus $G$, cannot be positive semi-definite if $W$ has any $\sigma_i(W) > 1$. Indeed, letting $u_i$ be the $i$-th column of $U_w$, assume there is at least one such singular value, and its corresponding left singular
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Vector is \( u_k \). Clearly, as \( 1 - \Sigma^2 \leq 0 \), and \( U_w \) is the left singular vector matrix of \( L_D^*B^*L_A^{-*} \):

\[
\begin{align*}
  u_k \in \text{span}\{u_i : \Sigma_{Wi} \neq 0\} = \text{range}(W) \subseteq \text{range}(L_D^+) = \text{range}(L_D^*)
\end{align*}
\]

However, \( H^t \) is obviously Hermitian, and Proposition 4.3.2 postulates that the positive semi-definiteness of \( L_D^*H^tL_D^* \) necessitates

\[
\begin{align*}
  \text{span}\{u_i : 1 - \Sigma^2 \leq 0\} \cap \text{range}(L_D^*) = 0
\end{align*}
\]

which is, of course, a contradiction with the assumption, since we have specifically defined \( u_k \) to be a singular vector of \( W \) for which \( \Sigma_{Wi} > 1 \), and have shown that \( u_k \in \text{range}(L_D^*) \).

Corollary 4.3.5. Let \( G \) be Hermitian matrix with a 2 \( \times \) 2 block partitioning \((A \quad B \quad B^* \quad D)\). Then \( G \) is positive definite if and only if \( \sigma_1(L_A^{-1}B^*D^{-1}) < 1 \), where \( A = L_A^*L_A \), \( D = L_D^*L_D \).

Proof. Since Theorem 4.3.4 (b) and (c) are automatically satisfied, and the diagonal submatrices \( A, D \) of \( G \) must be positive definite (as they are Gramian and nonsingular), we can directly arrive at the concluding argument of the previous proof:

\[
\begin{align*}
  G/A = L_D^*H^tL_D^* = L_D^*U_w(I - \Sigma_W \Sigma_W^t)U_W^*L_D^*
\end{align*}
\]

By using either Property 4.3.1(b) or \(*\)-congruence, \( G/A > 0 \iff \Sigma_{Wi} < 1 \) is inferred. \( \square \)

Theorem 4.3.4 and its corollary will later enable the complete DSFEM. Note that intuitively, it may appear that combining requirements (b) and (c) in Theorem 4.3.4 could be excessively restrictive. After all, the Schur complement of either diagonal block of \( G \) being PSD is sufficient for the positivity of \( G \), and the latter only imposes a condition on either range(\( B \)), or the row space, range(\( B^* \)). The remaining condition becomes specified implicitly. However, it is actually necessary to constrain both the row and column spaces of \( B \), as Theorem 4.3.4 is not directly equivalent to the Schur complement criteria. From a similar viewpoint, the explicit specification of both \( A, D > 0 \) in Corollary 4.3.5 may be questioned, but it is also necessary. To substantiate these claims through a visual illustration, explore the following simple example:

Example 4.3.6. Consider the matrices

\[
G_1 = \begin{pmatrix}
1 & 0 & -\sqrt{2}i \\
0 & 0 & 0 \\
\sqrt{2}i & 0 & 0
\end{pmatrix}, \quad G_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -\sqrt{2}i \\
0 & \sqrt{2}i & 1
\end{pmatrix}
\]

where, for either of them, \( A \) is the top left 2 \( \times \) 2 block, and \( D \) is the 1 \( \times \) 1 remaining diagonal block. The matrix \( G_1 \) has eigenvalues \((-1, 0, 2)\), even though \( \sigma(L_A^+B^*D^{-1}) = 0 \). Of course, range(\( B^* \)) \subseteq range(\( D \)) is not true in this case. It can be further verified that \( G_1/D \) is positive definite, but \( G_1/A \) is not.
For $G_2$, on the other hand, $\sigma(L_A^+B L_D^{+*}) = 0$, and the rowspace of $B$ is contained in that of $D$. However, $G_2$ still has eigenvalues $(-1, 1, 2)$. In this case, the range of $B$ is clearly not a subset of that of $A$.

Finally, in the context of Corollary 4.3.5, make the observation that $\sigma(L_A^+B L_D^{+*}) < 1$, with only one diagonal block of $G$ prescribed as strictly PD does not guarantee the other diagonal submatrix is also PD. Indeed, assume only $D \succ 0$ was required: $G_2$ is a clear counterexample. If, instead, only $A \succ 0$ was enforced, repartition $G_1$ so that $A = (1)$ is the top left $1 \times 1$ block, and $D$ is bottom right $2 \times 2$ null matrix. Obviously, neither $D$ nor $G_1$ are positive definite. In fact, the latter even is indefinite.

4.3.3 Strategy for constructing random blocks

Off-diagonal partitions

In light of the prior derivations, a desirable condition for positive (semi-)definiteness of the block matrix $G$ has been acquired. In fact, Theorem 4.3.4 proves very convenient to enable the creation of a method that encompasses all the properties outlined in Section 4.3.1. To achieve that, we firstly claim:

Proposition 4.3.7. Any positive semi-definite matrix with the same diagonal blocks $A$, $D$ as the original $G$ can robustly be generated as a realisation of $\tilde{G}$, by applying a suitable SVD perturbation to block $B$ (implicitly, via $L_A^+B L_D^{+*}$), provided the conditions of Theorem 4.3.4 are met.

Proof. To simplify the notation, $G$ (with no subscript) has been taken as the original matrix, on which $\tilde{G}$ is based. Let:

$$A = Q_A \Sigma_A Q_A^*, \quad D = Q_D \Sigma_D Q_D^* \quad (4.31)$$

be singular value decompositions of $A$, $D$, and $Z$ and its SVD be:

$$Z := L_A^+ B L_D^{+*} = U_Z \Sigma_Z V_Z^* \quad (4.32)$$

which is simply $W^*$ from the proof of Theorem 4.3.4. Then $B$ can be expressed in terms of $Z$, pre and postmultiplied by the factors $L_A$ and $L_D$:

$$L_A Z L_D^{+*} = L_A^+ B L_D^{+*} L_D^{+*} = B \quad (4.33)$$

The second part of the equality is only possible to ascertain since Theorem 4.3.4(b) and (c) are fulfilled. Temporarily, let us assume $A$, $D$ are either fixed, or their perturbation is of no interest. Now, the earlier representation of $B$ will be used for ‘implicitly’ defining $\tilde{B}$.

Construction 4.3.8 (Indirect generation of $\tilde{B}$). Let $\tilde{R}_A$, $\tilde{R}_D$ be stochastic unitary matrices of the same size as $A$ and $D$, respectively. Additionally, let $\tilde{\Sigma}_Z \in \mathbb{R}$ be a random diagonal matrix of the size and rank of $\Sigma_Z$, with elements not exceeding unity. Then we can write

$$\tilde{Z}' := U_Z \tilde{\Sigma}_Z V_Z^* \quad (4.34a)$$

$$\tilde{Z} := \tilde{R}_A \tilde{Z}' \tilde{R}_D^* = (\tilde{R}_A U_Z) \tilde{\Sigma}_Z (\tilde{R}_D V_Z)^* \quad (4.34b)$$

$$\tilde{B} := L_A \tilde{Z} L_D^{+*} = L_A \tilde{R}_A \tilde{Z}' \tilde{R}_D^* L_D^{+*} \quad (4.34c)$$
Using Construction 4.3.8, make the observation that (4.34b) explicitly provides an SVD of \( \tilde{Z} \) in terms of a singular value decomposition of \( Z \). By virtue of the definitions in Construction 4.3.8, the validity of condition (a) of Theorem 4.3.4 is naturally preserved upon substituting \( B \) with its random counterpart. Remember that \( A \) and \( D \) were intentionally kept fixed, thus the transition from \( Z \) to \( \tilde{Z} \) does not affect their factors \( L_A \) and \( L_D \), and is contained entirely in \( \tilde{B} \). The validation of the remaining two requirements of Theorem 4.3.4 is rather trivial:

\[
\text{range}(\tilde{B}) = \text{range}(L_A \tilde{Z} L_D^*) \subseteq \text{range}(L_A) = \text{range}(A)
\]

\[
\text{range}(\tilde{B}^*) = \text{range}(L_D \tilde{Z}^* L_A^*) \subseteq \text{range}(L_D) = \text{range}(D)
\]

A couple of essential properties of defining \( \tilde{B} \) as per the above way are immediately evident. Firstly, if no further restrictions are imposed on \( \tilde{R}_A, \tilde{R}_D \) and \( \tilde{\Sigma}_Z \), the domain of \( \tilde{B} \) is precisely \( \{B : G \succeq 0\} \), i.e. the ensemble of all matrices \( B \) for which \( \tilde{G} \) with constant diagonal blocks is positive semi-definite.

More formally, assume there exists some \( B_1 \) that is not a possible realisation of \( \tilde{B} \), and yet satisfies the prescriptions of Theorem 4.3.4. Then there must exist an SVD of the corresponding \( Z_1 \) that is not in the domain of \( \tilde{Z} \), which implies either \( \Sigma_{Z_1} \) is not in the domain of \( \tilde{\Sigma}_Z \), or that \( \tilde{R}_A : R_{A_1} U_z = U_{z_1} \). All of the preceding statements are obvious contradictions, the last one because we can always select \( R_{A_1} = U_{z_1} U_z^{-1} \), as the RHS forms a unitary matrix. Identical argumentation can also be used for the case of \( R_{D_1}, V_z, V_{z_1} \). With this, the proposition is proven.

The direct implication of Proposition 4.3.7 is that the elusive requirement (3) of Section 4.3.1 can be fully satisfied. In addition, the fact that any possible PSD matrix, having the same diagonal blocks as the original deterministic one, can be sampled from \( \tilde{G} \), means there exist no restrictions on the outcomes of the physical quantity or process described by \( G \). This translates into Construction 4.3.8 being ‘non-restrictive’, in the sense that there are no physical uncertainties that cannot be reproduced. In contrast, the CBSM cannot replicate variability in the interface between subsystems. Overall, existing techniques are unable to individually control the relation between two DOF sets, i.e. the \( B \) block, linking DOFs corresponding to \( A \), to those of \( D \).

**Diagonal blocks**

What remains to be demonstrated is that the diagonal partitions are also amenable to such treatment. Since \( A \) and \( D \) are already PSD matrices, it is possible to use existing techniques to model \( \tilde{A}, \tilde{D} \), like the nonparametric probabilistic approach, described in Section 2.3.4. One of its benefits is that it promises to preserve the rank and nullspace of the random square blocks. Another basic idea that achieves the same is a direct eigenvalue modification of the form

\[
\tilde{A} = Q_A \tilde{\Sigma}_A Q_A^*, \quad \tilde{D} = Q_D \tilde{\Sigma}_D Q_D^*
\]

following the notation adopted in (4.31) for the factorisations of \( A \) and \( D \). Restricting the subsequent comments to \( \tilde{A} \) for brevity, we can require

\[
\text{rank}(\tilde{A}) = \text{rank}(\tilde{\Sigma}_A) = \text{rank}(A)
\]
from the same physical constraints as the ones imposed on the whole matrix \( \tilde{G} \). Furthermore, preserving the rank of \( A \) also keeps its range and nullspace unchanged, as depicted in (4.6). Then the two image inclusion conditions of Theorem 4.3.4 automatically remain satisfied upon substitution of \( A \) with a realisation of \( \tilde{A} \) in \( G \). However, this does not guarantee that part (a) is still valid. Indeed, retracing the steps of the proof of the theorem, we can write:

\[
\frac{\tilde{G}}{\tilde{\mathbf{A}}} = \tilde{D} - \mathbf{B}^* \tilde{\mathbf{A}}^+ \mathbf{B}
\]

\[
= \tilde{L}_D^* (\mathbf{I} - (\tilde{L}_D^+ \mathbf{B}^* \tilde{L}_A^+)^*) \tilde{\mathbf{L}}^*_D
\]

Now, we can expand the term \( \tilde{\mathbf{L}}_D^+ \mathbf{B}^* \tilde{\mathbf{L}}_A^+ \) using the factorisations defined in (4.30), and accounting for the fact the eigenvalue matrices of the diagonal blocks are random, due to (4.35):

\[
\tilde{\mathbf{L}}_D^+ \mathbf{B}^* \tilde{\mathbf{L}}_A^+ = X_D \Sigma_D^{1/2} Q_D^* B \Sigma_A^{1/2} X_A^* \tilde{J}^*
\]

and as \( X_A, X_D \) are unitary

\[
\sigma_i(\tilde{\mathbf{L}}_D^+ \mathbf{B}^* \tilde{\mathbf{L}}_A^+) = \sigma_i(\tilde{\mathbf{J}})
\]

but from Theorem 4.3.4 (a), for \( \tilde{G} \) to be positive semi-definite, \( \sigma_i(\tilde{\mathbf{J}}) \leq 1 \) \( \forall i \). This does not hold in general:

**Example 4.3.9.** Consider the matrix

\[
G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

with blocks \( A = B = D = 1 \), eigenvalues \((2, 0)\) and \( Q_A = \Sigma_A = \cdots = J = 1 \). We can freely select a realisation \( A_1 = \Sigma_A \) such that \( 0 < A_1 < 1 \), which conforms to (4.35). However, \( \tilde{G} \) then becomes indefinite. Explicitly, keeping \( D \) fixed and calling \( \tilde{J}_1 \) the corresponding realisation of \( \tilde{J} \), we obtain \( \tilde{J}_1 = A_1^{-1} > 1 \) for the above choice of \( A_1 \).

Irrespectively of that, we can still use the idea of Construction 4.3.8 in order to remedy the issue that arises. Indeed, since the crucial range inclusion conditions of Theorem 4.3.4 are not broken, we can make the following assertion:

**Construction 4.3.10.** (Generation of \( \tilde{\mathbf{B}} \) from sampled \( \tilde{\mathbf{A}}, \tilde{\mathbf{D}} \).) Let \( A_s \) and \( D_s \) be realisations of \( A \) and \( D \), as specified by (4.35). We can then compute the respective

\[
Z_s = L_{A_s}^+ B L_{D_s}^{+*} = U_{Z_s} \Sigma_{Z_s} V_{Z_s}^*
\]

where, as displayed, \( \Sigma_{Z_s} \) may have diagonal values exceeding unity. Define

\[
\tilde{Z}_s := U_{Z_s} \tilde{\Sigma}_{Z_s} V_{Z_s}^*
\]

\[
\tilde{Z} := \tilde{R}_A \tilde{Z}_s \tilde{R}_D^* = (\tilde{R}_A U_{Z_s}) \tilde{\Sigma}_{Z_s} (\tilde{R}_D V_{Z_s})^*
\]

\[
\tilde{B} := L_{A_s} \tilde{Z} L_{D_s}^* = L_{A_s} \tilde{R}_A \tilde{Z}_s \tilde{R}_D^* L_{D_s}^*
\]

in the same way as in (4.34).
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The only principal difference with the case when the diagonal submatrices were kept fixed is that now $\tilde{\Sigma}_Zs$ might be impossible to remain static, and could incur a potentially larger change of its singular values, in order to guarantee all of the latter still have an upper bound of 1. Observe that Construction 4.3.10 is invariant with respect to how $\tilde{A}$ and $\tilde{D}$ have been defined in order to retrieve the realisations $A_s$ and $D_s$. Without loss of generality, we could have assumed

$$\tilde{A} = \tilde{Q}_A \Sigma_A \tilde{Q}_A^*, \quad \tilde{D} = \tilde{Q}_D \Sigma_D \tilde{Q}_D^*$$

instead of (4.35), as long as $\tilde{Q}_A$, $\tilde{Q}_D$ are such that the image and kernel of the non-stochastic original matrices are preserved. As explained, this should be the case if they are extracted by the nonparametric probabilistic approach. Otherwise, barring further restrictions, (4.39) describes all possible pairs of matrices whose dimensions are equal to those of $A$ and $D$. Not only would the preceding analyses be invalidated, since $\text{range}(B) \subseteq \text{range}(A)$ and $\text{range}(B^*) \subseteq \text{range}(D)$ are no longer certain, but the physical significance of $A$ and $D$ would be violated.

**Sampling $\tilde{G}$**

At this stage, it is feasible to construct a holistic process for the computation of realisations of the full random Hermitian matrix $\tilde{G}$, with all blocks being stochastic.

**Algorithm 1** Sampling $\tilde{G}$ based on $G$

1: procedure SAMPLEMATRIX($A, B, D, N_s$) \hspace{1cm} ▷ Input: blocks of $G$, $N_s$
2: \hspace{1cm} $A_s \leftarrow$ SAMPLEBLOCK($A$)
3: \hspace{1cm} $D_s \leftarrow$ SAMPLEBLOCK($D$)
4: \hspace{1cm} $L_{A_s} \leftarrow$ DECOMPOSE($A_s$) \hspace{1cm} ▷ Any $L_{A_s}: L_{A_s}L_{A_s}^* = A_s$
5: \hspace{1cm} $L_{D_s} \leftarrow$ DECOMPOSE($D_s$) \hspace{1cm} ▷ Any $L_{D_s}: L_{D_s}L_{D_s}^* = D_s$
6: \hspace{1cm} $Z_s \leftarrow L_{A_s}^*B L_{D_s}^*$
7: \hspace{1cm} $U_{Zs}, \Sigma_{Zs}, V_{Zs} \leftarrow$ SVD($Z_s$)
8: \hspace{1cm} $k = 1$
9: \hspace{1cm} while $k \leq N_s$ do
10: \hspace{2cm} $R_{A_k} \leftarrow$ SAMPLEUNITARY($n$) \hspace{1cm} ▷ Same size as $A$
11: \hspace{2cm} $R_{D_k} \leftarrow$ SAMPLEUNITARY($m$) \hspace{1cm} ▷ Same size as $D$
12: \hspace{2cm} $\Sigma_{Zs,k} \leftarrow$ SAMPLEDIAG($\Sigma_{Zs}$) \hspace{1cm} ▷ $\sigma_1(\Sigma_{Zs,k}) \leq 1$
13: \hspace{2cm} $Z_{s,k} \leftarrow R_{A_k} U_{Zs} \Sigma_{Zs,k} V_{Zs}^* R_{D_k}^*$
14: \hspace{2cm} $B_k \leftarrow L_{A_s} Z_{s,k} L_{D_s}^*$
15: \hspace{2cm} $k \leftarrow k + 1$
16: \hspace{1cm} end while
17: return $A_s, D_s, B_1, \ldots, B_{N_s}$
18: end procedure

In Algorithm 1 the sampling routines represent the functions that would be used in practice to obtain realisations of the corresponding stochastic matrices. As already discussed, the ones operating on the blocks $A$ and $D$ may be specified based on either (4.35) or (4.39), depending on the chosen underlying method.
4.3. Derivation of the DSFEM framework

The loop from 1 to $N_s$ is intentionally defined to demonstrate the idea of extracting multiple realisations of $\tilde{G}$ per single one of the diagonal partitions. This corresponds to creating $N_s$ instances of $\tilde{G}$, acquired via Construction 4.3.8, with $\tilde{A}$ and $\tilde{D}$ fixed to $A_s$ and $D_s$, respectively. The resultant sample of $\tilde{G}$ is the set

$$\left\{ G_{s,i} = \begin{pmatrix} A_s & B_i \\ B_i^* & D_s \end{pmatrix} : \ i = 1, \ldots, N_s \right\}$$

The purpose of such an approach is to avoid the unknown and expectedly large computational cost of sampling the diagonal blocks with, say, the nonparametric model of uncertainty. Even in the case of the eigenvalue modification of (4.35) being used, calculation of the factors $L_A$, $L_D$ and their pseudoinverses is still a costly operation that is preferably averted. Moreover, lines 6 and 7 of Algorithm 1 represent computationally heavy operations which need not be repeated until $A_s$ and $D_s$ change (e.g. to the next realisation $A_{s+1}$ and $D_{s+1}$).

4.3.4 Definition of the stochastic unitary matrices

Modelling the singular values of $\tilde{Z}$

What remains to be discussed regarding Algorithm 1 is how an appropriate selection of statistical models for $\tilde{R}_A$, $\tilde{R}_D$ and $\tilde{\Sigma}_Z$ can be made. The last one is arguably the least complicated, due to the exacting requirement $\sigma_i(\tilde{\Sigma}_Z) \leq 1 \forall i$ and the fact that $\tilde{\Sigma}_Z$ is diagonal. Provided no further information is available on the properties of $G$, the only indirect criterion for some form of proximity of a realisation $G_s$ to $G$ would be the relation between $Z_s$ and $Z$. One possibility is to choose

$$\sum_{i=1}^{r_Z} \sigma_i(\tilde{Z}) \approx \sum_{i=1}^{r_Z} \sigma_i(Z) = \sum_{i=1}^{r_Z} \Sigma_{Zii}$$

(4.40)

where $r_Z = \text{rank}(Z)$. Given a not excessively large coefficient of variation for $\sigma_i(\tilde{Z})$, the spectral norm $\|\tilde{Z}\|_2 = \sigma_{\max}(\tilde{Z})$ is approximately preserved, as the norms are invariant with respect to unitary transformations, viz. $\tilde{R}_A$ and $\tilde{R}_D$. On the other hand, the Frobenius norm is given by

$$\|\tilde{Z}\|_F = \sqrt{\text{trace}(\tilde{Z}^* \tilde{Z})} = \sum_{i=1}^{r_Z} \sigma_i^2(\tilde{Z})$$

and should also stay relatively close to $\|Z\|_F$. Overall, (4.40) is based on the intuitive notion that $\tilde{Z}$, with singular values somewhat constrained by those $Z$, would exhibit some measure of closeness to $Z$. As this is certainly not a strict statement, supporting numerical validation results are presented in Chapter 5.

Appropriate choice of $\tilde{R}_A$ and $\tilde{R}_D$

Now, the interesting case of building populations for $\tilde{R}_A$ and $\tilde{R}_D$, in a manner suitable for the purposes of stochastic FEM analysis, has to be addressed. Two considerations
of paramount importance can be identified straight away. Firstly, based on the definitions so far, no information about the original matrix $G$ is embodied in $\tilde{R}_A$ and $\tilde{R}_D$. Therefore they have to be specified in a way as to allow control of the dispersion of the anti-diagonal block $\tilde{B}$. Secondly, they participate in a matrix-matrix product required to evaluate realisations of $\tilde{B}$, that is to say (4.38c). It would be highly advantageous for the efficiency of the complete DSFEM approach if a sparse representation was a possible option for the random unitary matrices.

A slightly less general and more FEM-oriented approach shall henceforth be adopted. In particular, let $\tilde{R}_A, \tilde{R}_D \in \mathbb{R}$ and so be orthogonal. For the purpose of fulfilling the aforesaid requirements, two definitions are firstly introduced.

**Definition 4.3.11** (Rotation matrix). An orthogonal $R \in \mathbb{R}^{n \times n}$: $\det(R) = 1$

**Definition 4.3.12** (Givens rotation). A rotation in the plane spanned by two axes.

The principal idea is to use Givens rotations as a building block for the stochastic unitary matrices, now restricted to the subset of orthogonal ones. It is partly rooted in the geometric meaning of performing rotations in Euclidean space. Indeed, for such a matrix $R^{n \times n}$ and some vector $u \in \mathbb{R}^n$, the linear transformation $\hat{u} = Ru$ can be thought of as either representing $u$ in a basis rotated by $R$, or alternatively, rotating $u$ with respect to the original basis. In 3-D space the notion is quite intuitive and can easily be explained by a rotation about an axis, whose magnitude can be characterised by some spatial angle $\theta$.

Therefore it stands to reason to expect that a 'small rotation' $\tilde{R}_A$ applied to $\tilde{Z}'$, as per (4.38b), would trigger a limited change in the properties of $\tilde{Z}$ as a linear transformation acting on some vector. That would, respectively, translate into a 'small change' in $B$ with respect to $\tilde{B}$, since it is the only term modifying $\tilde{Z}$ once $A$ and $D$ have been sampled. The same logic can be exercised for the postmultiplication of $\tilde{Z}'$ by $\tilde{R}_D$. In a slightly more formal and abstract language, the rotations in $n$-space form the special orthogonal group $SO(n)$ of distance-preserving transformations that also preserve a fixed point, i.e. the origin. One of the key properties of rotations is that they can be compounded, namely, the product of rotation matrices is a rotation matrix. There always exists a suitable basis $Q$, under which

$$QRQ^T = \begin{pmatrix} R_1 & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & R_k \end{pmatrix} \pm 1 \pm 1$$ (4.41)$$

is a block diagonal matrix and $R_1, \ldots, R_k$ are $2 \times 2$ plane rotations, written as

$$R_i = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \quad i = \{1, \ldots, k\}$$ (4.42)$$

for angles of rotation $\theta_i \in [0; 2\pi]$. We will use this concept to model the desired $\tilde{R}_A$ and $\tilde{R}_D$. Again, constricting the comments to $\tilde{R}_A$ for conciseness, the strategy is to define
4.3. Derivation of the DSFEM framework

Each realisation $\mathbf{R}_{Ak}$ as compounded of $d = \lfloor \frac{n}{2} \rfloor$ non-overlapping Givens rotations. Explicitly, we can write

$$
\mathbf{R}_{Ak} = \prod_{i=1}^{d} \mathbf{E}_{k,i}, \quad \mathbf{E}_{k,i} = \begin{pmatrix}
1 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
\sin \theta_{k,i} & \cdots & \cdots & \cos \theta_{k,i} \\
\sin \theta_{k,i} & \cdots & \cdots & \cos \theta_{k,i} \\
\vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & 1
\end{pmatrix}
$$

(4.43)

where $n$ is the size of $\mathbf{A}$ and respectively $\tilde{\mathbf{R}}_A$. The subscript $k,i$ is used to prevent ambiguity and to signify the fact that $\mathbf{E}_{k,i}$ specifically refers to $\mathbf{R}_{Ak}$.

Now it is only left to make the observation that (4.43) can be acquired without having to perform the chain of matrix multiplications. This is because the underlying Givens rotations have been required to act on $\lfloor \frac{n}{2} \rfloor$ mutually orthogonal 2-D planes. In other words, precisely one pair of values $(\cos \theta_{k,i}, \pm \sin \theta_{k,i})$ is present on any row or column of $\mathbf{R}_{Ak}$. If $n$ is odd, there is a single exception, viz. some diagonal value equal to unity.

The following structure can then be employed in order to supply the sampling function demanded by Algorithm 1 at lines 10, 11:

Construction 4.3.13 (Sampling $\tilde{\mathbf{R}}_A, \tilde{\mathbf{R}}_D$). Let $\tilde{\mathbf{R}}$ be a random orthogonal matrix of size $n$, realisations of which are required by Algorithm 1. Then it is given by

$$
\tilde{\mathbf{R}} := \tilde{\Pi} \tilde{\mathbf{Y}} \tilde{\Pi}^T, \quad \tilde{\mathbf{Y}} := \begin{pmatrix}
\tilde{\mathbf{P}}_1 \\
\begin{array}{c}
\cdots \\
\vdots \\
\cdots \\
\end{array} \\
\tilde{\mathbf{P}}_d \\
1
\end{pmatrix}, \quad \tilde{\mathbf{P}}_i = \begin{pmatrix}
\cos \bar{\theta}_i & -\sin \bar{\theta}_i \\
\sin \bar{\theta}_i & \cos \bar{\theta}_i
\end{pmatrix}
$$

(4.44)

with $d = \lfloor \frac{n}{2} \rfloor$ and $\tilde{\Pi}$ being a random permutation matrix. In addition, all of the stochastic angle variables obey $\mathbb{E}(\bar{\theta}_i) = 0$.

The condition imposed on the mean of $\bar{\theta}_i$ stems from the direct assertion

$$
\lim_{\theta_i \to 0} \tilde{\mathbf{R}} = \mathbf{I} \Rightarrow \tilde{\mathbf{Z}} = \mathbf{Z} \Rightarrow \tilde{\mathbf{B}} = \mathbf{B}
$$

which also supports the original idea that a 'small rotation' applied to $\tilde{\mathbf{Z}}'$ would not yield a disproportionately magnified change in $\tilde{\mathbf{B}}$. Hence, the $\tilde{\mathbf{B}}$ controllability requirement has been translated into the specification of $\text{var}(\theta)$. This claim is not formally proven within this work. However, supporting numerical results are provided in the following chapter. Finally, Construction 4.3.13 obviously conforms to the requirement for sparsity of $\tilde{\mathbf{R}}_A$ and $\tilde{\mathbf{R}}_D$. As a matter of fact, this is accomplished with the near-optimal $n + \lfloor \frac{n}{2} \rfloor$ elements for a realisation of an orthogonal matrix of dimension $n$. 


4.4 Further considerations

4.4.1 Preservation of rank and kernel

A rank formula for matrix products

In this section, two important aspects of the formerly described random Hermitian matrices are analysed. Firstly, the rank preservation for realisations of $\tilde{G}$ with respect to the original matrix is addressed. This issue is rooted in the underlying physical properties of finite element mass and stiffness matrices, as recognised in requirement \[6\] of Section 4.3.1. Subsequently, the nullspace of the stochastic matrix is commented on, as per the optionally specified condition \[7\]. To that end, a formula for the rank of matrix products, helpful for the ensuing arguments, is firstly proven.

**Proposition 4.4.1.** For $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times q}$ it holds that

$$\text{rank}(AB) = \text{rank}(B) - \dim(\ker(A) \cap \text{range}(B))$$

**Proof.** Let $\{b_1, \ldots, b_s\}$ denote a basis for $\ker(A) \cap \text{range}(B)$. In addition, it can be extended to a full basis for $\text{range}(B)$ by $\{w_1, \ldots, w_t\}$, i.e.

$$\text{span}\{b_1, \ldots, b_s, w_1, \ldots, w_t\} = \text{range}(B)$$

Then the original statement can be proven by showing $\{Aw_1, \ldots, Aw_t\}$ is a basis for $\text{range}(AB)$. Firstly, assume $\{Aw_1, \ldots, Aw_t\}$ are linearly dependent. Then there exist scalars $c_1, \ldots, c_t$ for which

$$0 = \sum_{i=1}^{t} c_i Aw_i = A \left( \sum_{i=1}^{t} c_i w_i \right) = Av \Rightarrow v \in \ker(A)$$

Therefore

$$v \in \ker(A) \cap \text{range}(B)$$

but then it can be written as a linear combination of the vectors $b_i$, which is a contradiction, as $\{b_1, \ldots, b_s, w_1, \ldots, w_t\}$ are linearly independent. Now, let $y = ABx \in \text{range}(AB)$:

$$y = A(Bx) = A \left( \sum_{i=1}^{s} d_i b_i + \sum_{i=1}^{t} f_i w_i \right) = A \sum_{i=1}^{t} f_i w_i = \sum_{i=1}^{t} f_i(Aw_i)$$

since $d_i b_i \in \ker(A)$. But then $\{Aw_1, \ldots, Aw_t\}$ are both linearly independent and spanning $\text{range}(AB)$, therefore are a basis for it.

**Rank of $\tilde{G}$**

Now, let us shift the focus back to the study of the properties of $\tilde{G}$. The plan for carrying out rank analysis of its realisations is centred around the use of Proposition 4.4.1 along
with equation (4.12). The latter provides a means of estimating the rank of $\tilde{G}$ in terms of the Schur complements of either of its diagonal submatrices. As an interesting fact, due to Haynsworth's work, it is known that not only the rank, but also the inertia of Hermitian matrices is additive on the Schur complement, given the block, whose complement is taken, is invertible. Since $\tilde{G}$ has already been defined to be strictly nonnegative definite, the statement degrades to the rank condition. Furthermore, we are interested in the case when $A$ and $D$ might simultaneously be singular.

For the sake of simplification, it shall be assumed that $\tilde{G}$ is being treated under the condition of fixed diagonal blocks. As has been demonstrated in Construction 4.3.10, this is equivalent to the more general situation, where realisations of $A$ and $D$ would be sampled, and then any further treatment of $\tilde{B}$ would proceed as if they were constant. The case of $\tilde{G}/A$ will be shown henceforth. Firstly, we have

$$G/A = L_D(I - Z^*Z)L_D^*$$  \hspace{1cm} (4.45)

and taking an SVD of $J$:

$$Z = X_A(\Sigma_A^{1/2}Q_A^*BQ_D\Sigma_D^{1/2})X_D^* = X_AJX_D^*$$  \hspace{1cm} (4.46)

Combining (4.34) and (4.46), or alternatively, using properties of unitary transformations, we ascertain $\Sigma_J = \Sigma_Z$ and $\tilde{\Sigma}_J = \tilde{\Sigma}_Z$. Then it holds:

$$\tilde{Z} = R_AX_AJX_D^*$$  \hspace{1cm} (4.47)

At this stage, the following important observation is necessary

$$\tilde{Z} = L_AL_D^* \Leftrightarrow \begin{cases} L_AL_D^*\tilde{Z} = \tilde{Z} \\ \tilde{Z}L_D^*L_D^* = \tilde{Z} \end{cases}$$ \hspace{1cm} (4.48)

Before continuing, make the ensuing simplification of $\tilde{Z}^*\tilde{Z}$ and expansion of $I$, viz.:

$$\tilde{Z}^*\tilde{Z} = \tilde{R}_DX_DX_D^*\tilde{\Sigma}_Z\tilde{\Sigma}_Z^*V_J^*X_D^*R_D^*$$

$$I = (\tilde{R}_DX_DV_J)(\tilde{R}_DX_DV_J)^*$$

Then, if and only if (4.48) is true, we can substitute (4.47) into (4.45) directly, and with the aid of the last two results:

$$G/A = L_D(I - \tilde{Z}^*\tilde{Z})L_D^*$$

$$= L_D(\tilde{R}_DX_DV_J(I - \tilde{\Sigma}_Z\tilde{\Sigma}_Z^*)V_D^*\tilde{R}_D^*)L_D^*$$  \hspace{1cm} (4.49)

Clearly, (4.49) is invariant with respect to the choice of $\tilde{R}_A$. From considerations of symmetry, the same is true for the other Schur complement, $G/D$, with regard to $\tilde{R}_D$. However, from

$$\operatorname{rank}(G) = \operatorname{rank}(D) + \operatorname{rank}(G/D) = \operatorname{rank}(A) + \operatorname{rank}(G/A)$$  \hspace{1cm} (4.50)
and the diagonal submatrices being fixed, it transpires that \( \text{rank}(\tilde{G}) \) is not dependent on the choice of the stochastic unitary \( \tilde{R}_A \) and \( \tilde{R}_D \). Indeed, simply let \( \tilde{R}_A = I \). Then, by (4.50). The rank of \( \text{rank}(\tilde{G}/A) \) must be fixed regardless of \( \tilde{R}_D \), and vice versa. Now, what remains to be seen is exactly how \( \text{rank}(\tilde{G}) \) varies with \( \tilde{\Sigma}_*^*Z \). To this end, set \( \tilde{S}\tilde{S}^* = I - \tilde{\Sigma}_*^*\tilde{\Sigma}_z \), transforming (4.49) into

\[
\tilde{G}/A = \left( L_D\tilde{R}_D X_D V_J \tilde{S} \right) \left( L_D\tilde{R}_D X_D V_J \tilde{S} \right)^* = \tilde{N}\tilde{N}^*
\]

hence we have:

\[
\text{rank}(\tilde{G}/A) = \text{rank}(\tilde{N}\tilde{N}^*) = \text{rank}(\tilde{N})
\]

Additionally, from (4.30), \( \tilde{N} \) can be further expanded into

\[
\tilde{N} = Q_D \Sigma_D^{1/2} X_D^* \tilde{R}_D X_D V_J \tilde{S}
\]

and since \( Q_D \) is unitary,

\[
\text{rank}(\tilde{N}) = \text{rank}(\Sigma_D^{1/2} X_D^* \tilde{R}_D X_D V_J \tilde{S})
\]

Invoking Proposition 4.4.1 twice,

\[
\text{rank}(\Sigma_D^{1/2} \tilde{T}\tilde{S}) = \text{rank}(\tilde{S}) - \dim(\ker(\Sigma_D^{1/2} \tilde{T}) \cap \text{range}(\tilde{S}))
= \text{rank}(\tilde{T}\tilde{S}) - \dim(\ker(\Sigma_D^{1/2} \tilde{T}) \cap \text{range}(\tilde{T}\tilde{S}))
\]

where \( \text{rank}(\tilde{S}) = \text{rank}(\tilde{T}\tilde{S}) \), due to \( \tilde{T} \) being unitary, so:

\[
\dim(\ker(\Sigma_D^{1/2} \tilde{T}) \cap \text{range}(\tilde{S})) = \dim(\ker(\Sigma_D^{1/2} \tilde{T}) \cap \text{range}(\tilde{T}\tilde{S}))
\]

If \( D \) is invertible, the LHS and RHS of (4.52) are evidently equal to zero, for example from \( \ker(\Sigma_D^{1/2}) = 0 \). This immediately gives

\[
\text{rank}(\Sigma_D^{1/2} \tilde{T}\tilde{S}) = \text{rank}(\tilde{S}) = \text{rank}(\tilde{N}) = \text{rank}(\tilde{G})
\]

meaning that if at least one of the blocks \( A, D \) is nonsingular, the rank of \( \tilde{G} \) is determined only by \( \text{rank}(\tilde{S}) = \text{rank}(I - \tilde{\Sigma}_*^*\tilde{\Sigma}_z) \). The latter has a deficiency exactly equal to the number of diagonal elements \( \tilde{\Sigma}_z = 1 \). Therefore, if the number of singular values of \( \tilde{Z} \) equal to unity is preserved with respect to those of \( Z \), the rank of the Schur complement is unchanged, and then so is the rank of \( \tilde{G} \).

Attention is drawn to the fact that in CMS problems, practically at least one of the mass or stiffness diagonal block submatrices is nonsingular, being a truncated modal representation. Therefore, even without further analysis on the current matter, (4.48) can be guaranteed by setting one of \( \tilde{R}_A, \tilde{R}_D \) to \( I \). The ‘ignored’ random unitary matrix must be \( \tilde{R}_A \) for invertible \( D \), and \( \tilde{R}_D \) for nonsingular \( A \), respectively. Examine the first case more closely, by letting \( U_j^1 \) correspond to the nonzero singular values in \( \tilde{\Sigma}_z \).

Then it obviously holds that

\[
\text{range}(\tilde{Z}) = \text{range}(\tilde{R}_A X_A U_j^1) = \text{range}(X_A U_j^1) = \text{range}(Z) \subseteq \text{range}(A)
\]
and range($\tilde{Z}^*$) $\subseteq$ range($D$) is instantly valid due to the full rank of $D$. With these image inclusions fulfilled, we have explicitly shown, in the case for nondegenerate $D$, that rank($\tilde{G}$) is not influenced by $\tilde{R}_D$, so the latter can now be constructed without any restrictions.

The rank analyses demanded when $\tilde{Z}$ does not abide to the conditions of (4.48) is more complicated and outside the scope of this work. The same can be stated for the case when both diagonal partitions of $G$ are simultaneously degenerate. This does not invalidate the positive (semi-)definite nature of the realisations of $\tilde{G}$, it merely implies controllability of their rank may not be certain. Strictly speaking, as long as the random unitary matrices can be built so that the equalities

$$\text{range}(\tilde{R}_A X_A U_1^J) = \text{range}(X_A U_1^J)$$
$$\text{range}(\tilde{R}_D X_D V_1^J) = \text{range}(X_D V_1^J)$$

are concurrently valid, the nonsingularity of $A$ or $D$ need not be required for (4.51) to hold. Here, $V_1^J$ is analogous to $U_1^J$, and both conditions can clearly be treated in the same way. Overall, the approach delineated below can be taken for addressing FEM-derived problems with complete robustness regarding strict rank preservation:

- **Both $A^{-1}$ and $D^{-1}$ exist.** Equation (4.48) is automatically satisfied and the assertions of (4.50) and (4.53) can be used directly, while no constraints are placed on the random unitary matrices. The number of $i$ for which $\sigma_i(Z) = 1$ equals the rank deficiency of $\tilde{Z}$.

- **Either $A$ or $D$ is singular.** Define the corresponding $\tilde{R}_A$ or $\tilde{R}_D$ as the identity matrix and proceed as per the above case.

- **Both $A$ and $D$ are degenerate.** Specify both $\tilde{R}_A$ or $\tilde{R}_D$ as identity matrices and inject variability in $\tilde{B}$ only through $\tilde{\Sigma}_Z$.

Finally, it is stressed that for future work on the topic presented in this chapter, Proposition 4.4.1 can be employed to derive requirements of the form of (4.52) for $\tilde{R}_A$, $\tilde{R}_D$ and $\tilde{\Sigma}_Z$, in order for $\tilde{G}$ to exhibit rank consistency with respect to $G$. This would be of interest from a purely mathematical perspective, although the theoretical platform exposed in the current chapter is sufficient for the successful definition of blockwise random FEM matrices that abide to the stipulations of Section 4.3.1.

### Nullspace of $\tilde{G}$

A logical extension to the preceding rank analyses of $\tilde{G}$ would be the consideration of ker($\tilde{G}$) with respect to ker($G$). Strict preservation of the nullspace would be a desirable quality of the produced random Hermitian matrices, due to its underlying physical meaning, elucidated in Section 4.3.1. The successful derivation of exacting conditions under which this occurs has not been achieved within this thesis, and remains an open question. Nevertheless, two potential approaches to studying the matter are outlined below. Firstly, we may write

$$\tilde{G} z = 0 \iff \begin{cases} A x + \bar{B} y = 0 \\ \bar{B}^* x + D y = 0 \end{cases}$$

(4.54)
where \( z \) is partitioned into \( x \) and \( y \) conformally with \( G \). When \( A \) and \( D \) are nonsingular, the linear system (4.54) can be recast into the equivalent form

\[
\begin{cases}
(A - \tilde{B}D^+\tilde{B}^*)x = 0 \\
(D - \tilde{B}^*A^+\tilde{B})y = 0
\end{cases}
\]  

(4.55)

Therefore ensuring that the set of all solutions \( z \) of (4.54) is exactly the nullspace of \( \tilde{G} \) is identical to guaranteeing that \( \ker(\tilde{G}/D) = \ker(G/D) \) and \( \ker(\tilde{G}/A) = \ker(G/A) \) hold simultaneously. An alternative approach would encompass the fact any \( z \in \ker(\tilde{G}) \) must be in \( \ker(G) \) and vice versa. Then

\[
\tilde{G}z = 0 \iff \begin{cases}
Ax + By = 0 \\
Bx + Dy = 0 \\
Ax + \tilde{B}y = 0 \\
\tilde{B}^*x + Dy = 0
\end{cases} \implies \begin{cases}
(B - \tilde{B})y = 0 \\
(B^* - \tilde{B}^*)x = 0
\end{cases}
\]  

(4.56)

However, with both of the preceding ideas, the issue was not pursued further and is yet to be addressed.

### 4.4.2 Regularisation strategy for off-diagonal blocks

#### Condition of linear systems

One of the potential drawbacks of the exposed technique for generating random PSD Hermitian matrices is the requirement to explicitly perform operations with pseudoinverses, i.e. \( L_A^+ \) and \( L_D^+ \). The latter become ordinary inverses if their corresponding blocks are strictly positive definite. In either situation, the calculations are prone to effects of poor conditioning of \( A \) and/or \( D \) with respect to inversion, since computer arithmetic is not exact. To illustrate the issue, the concept of conditioning is first briefly introduced. Thorough treatments are found in texts such as [118] and [197], which is specifically dedicated to this topic. To begin with, consider the generic linear system

\[
Cx = b
\]

where \( C \in \mathbb{C}^{n \times n} \) has a singular value decomposition \( U_C \Sigma_C V_C^* \). Then it holds that the best approximation (or exact) solution \( \hat{x} \) is

\[
\hat{x} = C^+b = V_C \Sigma_C^+ U_C^* b = \sum_{i=1}^{r_C} \frac{u_i^*b}{\sigma_i(C)} v_i
\]

where \( r_C = \text{rank}(C) \) and \( u_i, v_i \) are the left and right singular vectors of \( C \). It is evident that a relatively small change in \( C \) or \( b \) might cause a large difference in \( \hat{x} \) when \( \sigma_i(C) \) is small. The ratio of the relative error in \( x \) to that in \( b \) is a measure of the sensitivity of the system. It is possible to show that the maximum value is, for some norm \( ||.|| \) and invertible \( C \), proportional to

\[
\kappa(C) = ||C^{-1}||||C||, \quad \kappa_2(C) = \frac{\sigma_{\text{max}}(C)}{\sigma_{\text{min}}(C)}
\]

(4.57)
The value $\kappa$ is called condition number, and the one based on the spectral norm is commonly employed. For a general $C$, the usual error estimate is

$$\|x - \hat{x}\| \leq \|C^+\| \|C\hat{x} - b\|$$

which might be extremely pessimistic, even if $\hat{x}$ is a good approximation to $x$ that is not overly polluted by noise.

**Tikhonov regularisation**

One of the typical schemes for improving the conditioning of the system is an SVD truncation, as per (4.7), e.g. treating all singular values below some multiple of the machine epsilon as zero. It can easily be inferred from (4.57) that this reduces $\kappa_2(C)$. Still, that treatment would not be ideal if many of the singular values of $C$ are approximately zero, which tends to occur with FEM matrices. The other standard approach is the Tikhonov regularisation, also known as ridge regression by statisticians’ naming conventions. The idea is to solve the modified problem

$$\hat{x} = (C^*C + h^2 R^* R)^{-1} C^* b$$

with the normally taken path being to add a small factor of the identity to the pseudo-inverse, that is, $R = I$:

$$\hat{x} = (C^*C + h^2 I)^{-1} C^* b$$

(4.58)

Note $C^*C$ is Hermitian even for rectangular $C$, and $C^*C + h^2 I$ has its eigenvalues on the interval $[h^2, h^2 + \|C\|^2_2]$, therefore the condition of the linear system is raised as $h$ increases. The principal difficulty is exercising a choice of $h$ that balances the decrease in noise against the modification of the original problem.

**Relevance to the computation of $\tilde{B}$**

As has been extensively deliberated in Section 4.3.3, the anti-diagonal blocks of the random matrix $\tilde{G}$ are explicitly given by the expression $\tilde{B} = L_{A_s}Z_s L_{D_s}^{*}$, which relies on the computation of the unperturbed

$$Z_s = L_{A_s}^+ B L_{D_s}^{++}$$

(4.59)

for some realisation $s$ of $A$ and $D$. To avoid the direct computation of the pseudoinverses, we can, instead, set

$$B L_{D_s}^{++} = \tilde{X}$$

(4.60)

Taking the transpose of the equality, we can arrive at

$$L_{D_s}^{+} B^* = \tilde{X}^* \Rightarrow L_{D_s} \tilde{X}^* = B^*$$

(4.61)

This is equivalent to solving a linear system for each pair of identically indexed columns of $\tilde{X}^*$ and $B^*$. Then Tikhonov regularisation (4.58) can be used, and adhering to an
assumption that all column pairs are treated with the same value of the constant \( h \), the sequence of operations below is acquired:

\[
\begin{align*}
\widetilde{X}^* &= (L_{D_s}^* L_{D_a} + h_D^2 I)^{-1} L_{D_s}^* B^* \quad (4.62a) \\
Z &= (L_{A_s}^* L_{A_a} + h_A^2 I)^{-1} L_{A_s}^* \widetilde{X}_s \quad (4.62b)
\end{align*}
\]

The second one is inferred from \( Z = L_{A_s}^* \widetilde{X}_s \), obtained by substituting (4.60) in (4.59). Then the same logic as before is applied, but this time for the columns \( \widetilde{x}_i \) of \( \widetilde{X} \) and respectively \( \widetilde{x}_i \) of \( \tilde{X} \).

**Remark.** The factors \( L_{A_s}, L_{D_s} \) can still be interpreted as implicit inversion of \( A \) and \( D \), especially in the cases of Cholesky or LDL decompositions being used to define them. Nonetheless, competing random matrix techniques also rely on the existence of similar factorisations for the mass, stiffness and damping matrices, hence would be subject to the same level of numerical precision degradation. Additionally, the condition number of such factors is not necessarily overly large, being the square root of that of the original matrix. This can be seen by examining (4.30) and (4.57).

### 4.4.3 Dynamic substructuring and CMS

#### Dual assembly

It is worth briefly commenting on the suitability of the previously defined blockwise stochastic matrices for the uncertainty modelling of built-up structures. It is crucial to clarify that whether condensed, i.e. CMS, or physical coordinate representations are being used, bears no relevance to the DSFEM. It is a purely algebraic construct, unaffected by the origin of the underlying matrix blocks. The Tikhonov regularisation strategy for the extraction of \( Z \) is also optional. In light of the rank analysis carried out in Section 4.4.1, it is immediately asserted that dually assembled structures do not experience an introduction of rank deficiency upon modelling the substructural mass and stiffness with the suggested random matrix model. Limiting the comments to the masses, the system matrix has the block diagonal form

\[
\widetilde{M}_d = \text{diag} \left( \widetilde{M}^{(1)}, \ldots, \widetilde{M}^{(N_c)} \right)
\]

for \( N_c \), total number of parts, as reflected in detail in Section 2.2. For the first component, the Schur complement in \( \widetilde{M}_d \) can be expressed as

\[
\widetilde{M}_d / \widetilde{M}^{(1)} = \text{diag} \left( \widetilde{M}^{(2)}, \ldots, \widetilde{M}^{(N_c)} \right)
\]

therefore

\[
\text{rank}(\widetilde{M}_d) = \text{rank}(\widetilde{M}_d^{(1)}) + \text{rank}(\widetilde{M}_d / \widetilde{M}^{(1)})
\]

from (4.12). Applying the process to the complement \( \widetilde{M}_d / \widetilde{M}^{(1)} / \widetilde{M}^{(2)} \) and so on, by mathematical induction it follows that rank deficiency is simply the sum of the ranks of its constituent blocks. Since for each of them rank is preserved with respect to the original components, rank deficiency is not introduced in the global matrix. How the individual \( \widetilde{M}_d^{(k)} \) are partitioned internally in order to employ Algorithm 1 is irrelevant to the claim. Whether or not a recalculation of the nominal Lagrange multipliers facilitating the reassembly is necessary is not investigated within this thesis.
Primal assembly

In contrast, analysis for the primal assembly is somewhat more demanding. A relatively straightforward strategy involves beginning by performing the synthesis of the DS model, and subsequently setting

\[
\tilde{G} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^T & \tilde{D} \end{pmatrix} = \begin{pmatrix} \tilde{M}_{BB} & \tilde{M}_{IB} \\ \tilde{M}_{IB}^T & \tilde{M}_{II} \end{pmatrix} 
\]

where the partitioning is conformal, and the case of reduced matrices is entirely analogous. Obviously, Algorithm 1 can be utilised straight away, with the sampling of the orthogonal matrices \( \tilde{R}^* \) facilitated by Construction 4.3.13. A supplementary remark can be made, concerning the pre-sampling of \( \tilde{M}_{BB} \) and \( \tilde{M}_{II} \), before a realisation of \( \tilde{M}_{IB} \) is extracted. For the former, it holds

\[
\tilde{M}_{II} = \text{diag} \left( \tilde{M}_{II}^{(1)}, \ldots, \tilde{M}_{II}^{(N_c)} \right)
\]

since the internal elastic degrees of freedom are non-overlapping by definition. Then we can explicitly write the eigendecomposition of \( \tilde{M}_{II} = Q_{MII} \Lambda_{MII} Q_{MII}^T \) in terms of the spectral factorisations of the individual non-boundary component matrices, i.e.

\[
\tilde{M}_{II} = \begin{pmatrix} Q_{MII}^{(1)} \Lambda_{MII}^{(1)} Q_{MII}^{(1)T} \\ \vdots \\ Q_{MII}^{(N_c)} \Lambda_{MII}^{(N_c)} Q_{MII}^{(N_c)T} \end{pmatrix} = \begin{pmatrix} Q_{MII}^{(1)} \\ \vdots \\ Q_{MII}^{(N_c)} \end{pmatrix} \begin{pmatrix} \Lambda_{MII}^{(1)} \\ \vdots \\ \Lambda_{MII}^{(N_c)} \end{pmatrix} \begin{pmatrix} Q_{MII}^{(1)T} \\ \vdots \\ Q_{MII}^{(N_c)T} \end{pmatrix} \tag{4.63}
\]

which is simply a case of multiplying block diagonal matrices. Moreover, the columns of \( \text{diag} \left( \tilde{Q}_{MII}^{(1)}, \ldots, \tilde{Q}_{II}^{(N_c)} \right) \) on the RHS are clearly linearly independent unit vectors, that are easily verified to be orthogonal. Then this is indeed the spectral decomposition of \( \tilde{M}_{II} \). Consequently, realisations of \( \tilde{M}_{II} \) can be calculated on a per-block basis with any of the methods suitable for Construction 4.3.10 discussed in Section 4.3.3. For instance, this can be attained by injecting variability only through \( \tilde{\Lambda}_{MII}^{(k)} \), or alternatively, by implicitly using the more general equation (4.39), stemming from the nonparametric model of uncertainty or similar techniques. Either scenario preserves the blockwise sparsity, image and kernel of \( \tilde{M}_{II} \), which could be expected to drastically surpass \( \tilde{M}_{BB} \) in size for general domain decompositions of structures.

The preceding conclusion demonstrates one of the merits of the novel DSFEM approach. The anticipated efficiency gains against methods generating dense \( \tilde{M} \) are enhanced by the fact that (4.63) permits the factor \( L_{MII} \), and the respective Moore-Penrose inverse \( L_{MII}^+ \), to also be extracted in an uncoupled fashion, i.e. on a subpartition basis, employing the same arguments as for (4.63). The complete process structure is summarised in Algorithm 2. Notice that it is virtually identical to Algorithm 1 with the only difference being the partition-wise treatment of block \( D \) and its factor \( L_D \).
4.4. Further considerations

Algorithm 2 Sampling of primal assemblies by DSFEM

1: procedure SAMPLEPRIMAL($G, N_s$) \hfill ▷ Placeholder for $M, C, K$
2: \quad $A \leftarrow G_{BB}$ \hfill ▷ Boundary submatrix
3: \quad $B \leftarrow G_{BI}$ \hfill ▷ Off-diagonal block
4: \quad for $k \leftarrow 1 : N_c$ do \hfill ▷ Substructure
5: \quad \quad $D_s^{(k)} \leftarrow$ SAMPLEBLOCK($G_{BB}^{(k)}$)
6: \quad \quad $L_s^{(k)} \leftarrow$ DECOMPOSE($D_s^{(k)}$) \hfill ▷ Any $L_{Da}^{(k)} : L_{Da}^{(k)} L_{Da}^{(k)T} = D_s^{(k)}$
7: \quad end for
8: \quad $D_s \leftarrow \text{DIAG}(D_s^{(1)}, \ldots, D_s^{(N_c)})$
9: \quad $L_{Da} \leftarrow \text{DIAG}(L_{Da}^{(1)}, \ldots, L_{Da}^{(N_c)})$
10: \quad $A_s \leftarrow$ SAMPLEBLOCK($A$)
11: \quad $L_{As} \leftarrow$ DECOMPOSE($A_s$) \hfill ▷ Any $L_{As} : L_{As} L_{As}^T = A_s$
12: \quad $Z_s \leftarrow L_s^T A_s B L_{Da}^T$ \hfill ▷ Optional: regularisation
13: \quad $U_{zs}, \Sigma_{zs}, V_{zs} \leftarrow$ SVD($Z_s$)
14: \quad for $k \leftarrow 1 : N_s$ do
15: \quad \quad $R_{Ak} \leftarrow$ SAMPLEUNITARY($n$) \hfill ▷ Same size as $A$
16: \quad \quad $R_{Dk} \leftarrow$ SAMPLEUNITARY($m$) \hfill ▷ Same size as $D$
17: \quad \quad $\Sigma_{zs,k} \leftarrow$ SAMPLEDIAG($\Sigma_{zs}$) \hfill ▷ $\sigma_1(\Sigma_{zs}) \leq 1$
18: \quad \quad $Z_{sk} \leftarrow R_{Ak} U_{zs} \Sigma_{zs,k} V_{zs}^T R_{Dk}^T$
19: \quad \quad $B_k \leftarrow L_{As} Z_{sk} L_{Da}^T$
20: \quad end for
21: \quad return $A_s, D_s, B_1, \ldots, B_{N_s}$
22: end procedure

A more aggressive sparsity preserving technique could also be defined for the decomposition-based stochastic approach. Indeed, Algorithm 3 demonstrates the concept, which extends the aforesaid one by also requiring the boundary DOF component matrices to be sampled before reassembly of the global structure. This method also does not impair the sparsity of the interface matrices. However, it is not known if strict preservation of the rank of the global matrix remains unaltered. Numerical test results presented in Section 5.2 indicate that it does.

Algorithm 3 Sampling of primal assemblies by DSFEM, version II

1: procedure SAMPLEPRIMAL($G$) \hfill ▷ Placeholder for $M, C, K$
2: \quad for $k \leftarrow 1 : N_c$ do \hfill ▷ Substructures
3: \quad \quad $A_s^{(k)}, D_s^{(k)}, B_s^{(k)} \leftarrow$ SAMPLEREGULARIZATION($G_{BB}^{(k)}, G_{BI}^{(k)}, G_{II}^{(k)}, 1$) \hfill ▷ Algorithm 1
4: \quad end for
5: \quad $A \leftarrow$ REASSEMBLE($A_s^{(1)}, \ldots, A_s^{(N_c)}$)
6: \quad $B \leftarrow$ REASSEMBLE($B_s^{(1)}, \ldots, B_s^{(N_c)}$)
7: \quad $D \leftarrow$ REASSEMBLE($D_s^{(1)}, \ldots, D_s^{(N_c)}$)
8: \quad return $A_s, D_s, B_s$
9: end procedure

In order to visually illustrate the differences between Algorithm 2 and Algorithm 3 an example case of the primal assembly of a mass matrix for a 3-component system
4.4. Further considerations

is depicted on Figure 4.1. In Algorithm 2, the whole $M_{BI}$ segment is treated as a single block, and therefore its realisations become dense, as opposed to their original blockwise sparse structure. Realisations of $\tilde{M}_{BB}$ and $\tilde{M}_{II}$ are constructed a priori, by sampling the subsystem blocks $\tilde{M}_{BB}^{(k)}$, $\tilde{M}_{II}^{(k)}$ with any appropriate technique, as per Section 4.3.3. By contrast, Algorithm 3 samples $\tilde{M}_{BI}^{(k)}$ individually as well, thus preserving the sparsity of not only the diagonal, but also the off-diagonal blocks of the global matrix. It is stressed that stiffness and damping matrices are treated in a completely identical manner. Furthermore, even though the subscripts $B$, $I$, indicative of a physical space representation, have been used in this example, reduced matrices have the same general form as has been deliberated in Section 3.1.2. It is restated that the DSFEM is a completely general technique and the derivations, proofs and algorithms shown within this chapter remain valid regardless of whether DS, CMS, or in fact any non-FEM $2 \times 2$ partitioned random nonnegative definite Hermitian matrix is being sampled.

Ultimately, the new approach proposed in this chapter obviates the usual hindrances to the application of a classic divide and conquer strategy to uncertain built-up structures. As an exception, the Craig-Bampton stochastic method also achieves that, but it is easily seen that it is a trivial subset of the DSFEM, obtained when $\tilde{D} = \tilde{A}_D$, $\tilde{A} = A$ and $\tilde{B} = B$. Note that with the new decomposition-based technique, component-wise treatment is done in a robust manner from the point of view of mathematically exact preservation of positive (semi-)definiteness and rank. Both are guaranteed when Algorithm 2 is used. Therefore all requirements specified as mandatory in Section 4.3.1 are inherently fulfilled by the DSFEM, based on the series of proofs and derivations shown in Section 4.3 and Section 4.4. The only exception is Algorithm 3 which has not been proven to be strictly rank-preserving. This property is only validated experimentally in Section 5.2 Nonetheless, Algorithm 2 can always be used instead.
4.5 Summary

The current chapter was devoted to establishing the mathematical framework of a novel approach, called decomposition-based stochastic FEM. At first, Section 4.2 dealt primarily with the introduction of the relevant algebraic apparatus, subsequently used for the derivation of the proposed technique. Strong emphasis was placed on conditions for the positive (semi-)definiteness of Hermitian matrices. In Section 4.2.4 it was shown that failure to preserve this condition, in the context of random FEM matrices, inevitably results in the introduction of spurious negative generalised eigenvalues. Singular value decompositions, pseudoinverses and the generalised Schur complement formed the core mathematical tools required for the remainder of the chapter.

In Section 4.3 a detailed derivation was exposed for a class of random Hermitian matrices based on a $2 \times 2$ block partitioning, following a set of critical requirements identified at the beginning of the section. Key claims were proposed and proven, so that a platform permitting the definition of the global stochastic matrix on a per-block basis could be constructed. A chief advantage of the new method is the fact that it inherently allows anti-diagonal partitions to be unrestrictedly and robustly defined as stochastic submatrices. To the best of the author’s knowledge, this has not been achieved in preceding works in the field, which can be recognised as problematic when uncertain built-up structures comprise the applications of interest. The full process facilitating the definition of the global random matrix was given in an algorithmic format. Subprocedures, such as the one in Section 4.3.4 for the sampling of sparse stochastic orthogonal matrices instrumental to the definition of the DSFEM, were deliberated, and particular strategies were suggested. In Section 4.4, several further topics, such as the preservation of the rank and kernel of the random matrix, were investigated. The use of a chain of Tikhonov regularisations was outlined as a means of avoiding the explicit computation of pseudoinverses. Finally, algorithms for the efficient implementation of the full stochastic approach in the context of dynamic substructuring and CMS were presented. As a concluding remark, it is stressed that the complete DSFEM technique, along with the underlying formulation of the blockwise random Hermitian matrices, encompassing all the relevant propositions, proofs, theorems, constructions, algorithms and miscellaneous analyses, are novel and have been developed entirely within this research work.
Chapter 5

The decomposition-based stochastic FEM

5.1 Properties of the random matrices

In the present chapter, the practical aspects of the blockwise random Hermitian matrices are investigated. The current section deals with the deliberation of the statistical distributions arising for their elements and eigenvalues. In particular, a special version of Construction 4.3.8 that does not employ Construction 4.3.13 has been implemented for sampling of the stochastic unitary matrices $\tilde{R}_A$ and $\tilde{R}_D$. The rationale is that a certain measure of closeness to the identity matrix has intentionally been affixed to the latter, in order for them to satisfy a dispersion controllability requirement. Instead, a QR algorithm was used here to yield well-distributed random unitary matrices, that is to say, with eigenvalues uniformly spread on the unit circle. The QR factorisation owes its name to the fact it represents a matrix as

$$ C = QR $$

where $Q$ is unitary and $R$ is upper triangular. It is well-known that by producing $C$ element-wise, with a standard random number generator, and requiring the diagonal of $R$ to be positive, the aforesaid uniform distribution of the eigenvalues of $Q$ is acquired. Indeed, this procedure was employed for the sampling of $\tilde{R}_A$, $\tilde{R}_D$. Additionally, the singular values $\sigma_i(\tilde{Z})$ were allowed to vary uniformly on the interval $[0, 1]$, with no other constraints. As has been proven in Construction 4.3.8 the unification of these two constructs is equivalent to allowing any possible realisation of the random matrix of interest (under the condition of fixed diagonal partitions), as long as it remains PSD. Hence, the applied process was as follows

1. A matrix $C \in \mathbb{R}^{200 \times 200}$ with random entries on the interval $[-1, 1]$ was generated.
2. $C = QR$ was computed, and a Hermitian matrix $G$ was defined by $G = RR^*$. 
3. $G$ was partitioned into blocks $A_{50 \times 50}$, $B_{50 \times 150}$ and $D_{150 \times 150}$.
4. The matrices $L_A$, $L_D$ were taken as the Cholesky factors of $A$ and $D$. 

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5.1. Properties of the random matrices

(5) Matrix $Z$ was obtained from (4.32) and its SVD $U_Z \Sigma_Z V_Z^*$ was calculated.

(6) Realisations of $\tilde{R}_A, \tilde{R}_D, \tilde{\Sigma}_Z$ were extracted, as explained.

(7) Realisations of $\tilde{Z}, \tilde{B}, \tilde{G}$ were sequentially computed, the last one being $G_i$.

(8) The eigenvalues of $G_i$ were evaluated and stored together with $G_i$ itself.

Steps (6)-(8) were repeated $10^5$ times to form the final population sample of the so-obtained matrix $\tilde{G}$. Two metrics were deemed of interest - the distribution of the elements of $\tilde{B}$, and that of the eigenvalues of $\tilde{G}$. Examples of the results are shown on Figure 5.1 and Figure 5.2 respectively. They were based on an automated distribution fitting code.

![Figure 5.1: PDF of the elements of a random $200 \times 200$ matrix, absolute value. Left: $\tilde{G}_{25,125}$, right: $\tilde{G}_{95,55}$](image1)

![Figure 5.2: PDF of the normalised eigenvalues of a random $200 \times 200$ matrix. Left: smallest, right: largest](image2)

In general, the elements of the stochastic anti-diagonal block appeared independent and following a Gaussian PDF very accurately. The normalised eigenvalues, on the other hand, were normally distributed only around the centre of the spectrum, otherwise slight shifts from unity were observed. The most extreme cases are demonstrated on Figure 5.2. Naturally, the sum of the eigenvalues of each realisation of $\tilde{G}$ is unaffected by $B$. The realisations of $\tilde{G}$ remained strictly positive (semi-)definite, and rank($\tilde{G}$) was determined by the number of $\sigma_i(\tilde{Z}) = 1$, as predicted in the previous chapter.
5.2 Validation: spacecraft vibroacoustic analysis

5.2.1 Model configuration

While providing interesting insights from an academic standpoint, Section 5.1 cannot serve to authenticate the suitability of the constructed stochastic matrices for realistic problem applications. Indeed, the arbitrarily generated matrices were not indicative of FEM representations occurring in practice, partly due to their density. In addition, specific tests on the sampling algorithms, devised for dynamic substructuring and CMS models, are needed. To this end, a complex vibroacoustic test case, based on the SSTL NovaSAR satellite, is presented in the following discussion. The spacecraft model is depicted on Figure 5.3.

Figure 5.3: DSFEM practical test case: the SSTL NovaSAR spacecraft

Five different sets of data, as is clarified in the current and next sections, were intended to form the comparison basis. The adopted approach closely resembles the thoroughly explained process in Section 3.2 of Chapter 3. In accordance to the procedures employed in the aforesaid study on the improved CBSM, only the deterministically calculated excitation was used for each realisation of each of the stochastic methods.

The load vector was supplied by SSTL, and was computed with commercial software, namely the finite element vibroacoustic solver module of MSC Actran. In total, 50 output frequencies between 20 Hz and 1000 Hz were required. The normal modes of the structure, used by Actran to perform the initial coupled elasto-acoustic analysis, were precomputed in MSC Nastran. The incident diffuse sound field, driving the fluid-structure interaction, was defined according to the reverberation chamber SPL levels achieved during the flight qualification test campaign for the spacecraft. Regarding the latter, sensor data was also provided by SSTL, given in terms of acceleration spectral density. The computed surface pressure field at 550 Hz is shown on Figure 5.4.
5.2. Validation: spacecraft vibroacoustic analysis

Figure 5.4: Applied acoustic pressure load at 550 Hz, magnitude, [Pa]

Table 5.1: Assumed coefficients of variation for the NovaSAR parametric model’s random properties

<table>
<thead>
<tr>
<th>Type</th>
<th>Property</th>
<th>Symbol</th>
<th>St. deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic material</td>
<td>Young’s modulus</td>
<td>$E$</td>
<td>0.08$\mu$</td>
</tr>
<tr>
<td></td>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.03$\mu$</td>
</tr>
<tr>
<td></td>
<td>Shear modulus</td>
<td>$G$</td>
<td>0.12$\mu$</td>
</tr>
<tr>
<td></td>
<td>Density</td>
<td>$\rho$</td>
<td>0.04$\mu$</td>
</tr>
<tr>
<td>Thin shell, orthotropic</td>
<td>Young’s modulus</td>
<td>$E_1, E_2$</td>
<td>0.08$\mu$</td>
</tr>
<tr>
<td></td>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.03$\mu$</td>
</tr>
<tr>
<td></td>
<td>Shear modulus</td>
<td>$G$</td>
<td>0.12$\mu$</td>
</tr>
<tr>
<td></td>
<td>Density</td>
<td>$\rho$</td>
<td>0.04$\mu$</td>
</tr>
<tr>
<td>Solid element</td>
<td>Property matrix</td>
<td>$G_{ij}$</td>
<td>0.12$\mu$</td>
</tr>
<tr>
<td></td>
<td>Density</td>
<td>$\rho$</td>
<td>0.04$\mu$</td>
</tr>
<tr>
<td>Simple beam</td>
<td>Section dimension</td>
<td>$L$</td>
<td>0.05$\mu$</td>
</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08$\mu$</td>
</tr>
<tr>
<td>Composite laminate</td>
<td>Ply thickness</td>
<td>$t_i$</td>
<td>0.05$\mu$</td>
</tr>
<tr>
<td></td>
<td>Fibre orientation</td>
<td>$\Theta_i$</td>
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</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08$\mu$</td>
</tr>
<tr>
<td>Thin shell</td>
<td>Thickness</td>
<td>$t$</td>
<td>0.05$\mu$</td>
</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08$\mu$</td>
</tr>
<tr>
<td>Spring</td>
<td>Stiffness</td>
<td>$k_i$</td>
<td>0.06$\mu$</td>
</tr>
<tr>
<td>Point mass</td>
<td>Mass</td>
<td>$m$</td>
<td>0.05$\mu$</td>
</tr>
<tr>
<td>Damping</td>
<td>Modal value</td>
<td>Constant</td>
<td>Not applicable to the test case, or is described as a function of already defined values, e.g. $G = \frac{E}{1+\nu}$</td>
</tr>
</tbody>
</table>
Using the so-obtained acoustic load data a conventional PSA simulation, relying on a direct Monte Carlo approach, was performed. Adhering to previously explained reasoning, 500 realisations were assumed as a number appropriate to guarantee satisfactory convergence of the mean and variance of the predicted responses. The particular random variable coefficients of variation used are specified in Table 5.1. Note that while similar, they are not identical to the ones found in Table 3.1 corresponding to the SSTL300 structure employed in the parametric survey of the CBSM. The reason is rooted predominantly in the differences in principal component connectors and nature of non-structural masses observed between the two satellites.

In addition to the sensor data and physical coordinate direct MCS solution, a CBSM simulation was performed. The underlying Craig-Bampton condensation involved the partitioning of the spacecraft into three components, with corresponding number of constituent degrees of freedom indicated in Table 5.2. Subsystem 1 and 2 refer, respectively, to the large $z$ direction solar panel, and the $-z$ four-panel array, seen on Figure 5.3. Component 3 was comprised of all remaining parts of the NovaSAR model. The eigenbasis for the CB reductions was selected to be representative up to at least 1500 Hz. In terms of the CBSM setup, the fixed interface natural frequencies were modelled with a lognormal distribution, having a COV = 0.16, as prescribed by the findings of Section 3.2.3. Note that numerically, the condensed NovaSAR model was almost identical to the SSTL300 one, as far as boundary, modal and total DOFs are concerned (cf. Table 3.4).

<table>
<thead>
<tr>
<th>Subsystem</th>
<th>CB Reduction</th>
<th>Full model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>modal</td>
<td>boundary</td>
</tr>
<tr>
<td>1</td>
<td>366</td>
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<tr>
<td>2</td>
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<td>3</td>
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<tr>
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<td>1320</td>
<td>420</td>
</tr>
<tr>
<td>Reassembled</td>
<td>1320</td>
<td>240</td>
</tr>
</tbody>
</table>

5.2.2 Set up of the DSFEM

The reduced spacecraft representation outlined in Section 5.2.1 was used to facilitate the implementation of the DSFEM. Prior to proceeding with the acquired results, a few noteworthy remarks should be made. Perhaps most importantly, the use of Algorithm 3 was selected for the sampling of the uncertain model. Secondly, Tikhonov regularisation for the computation of the matrix $\mathbf{Z}$ was deemed unnecessary in this situation. The decision was based on a simple test, involving the use of identity matrices in place of the random rotations $\tilde{\mathbf{R}}_A$ and $\tilde{\mathbf{R}}_D$ for each treated subsystem. No stochastic modelling was done for the singular values of $\mathbf{Z}$. Under exact arithmetic, this simply yields $\mathbf{B} = \mathbf{B}$. 
Expressly, a calculation of the type
\[
\tilde{B} = L_A U \Sigma_z V_z^* I L_D^*
\]
was performed for the evaluation of \(\tilde{B} \approx B\). The selected criteria for exactness were the standard relative errors
\[
\frac{\|B - \tilde{B}\|_2}{\|B\|_2}, \quad \frac{\|B - \tilde{B}\|_F}{\|B\|_F}
\]
in the spectral and Frobenius norms. It was found that for all of the three NovaSAR subcomponents, the relative errors were of order \(2.6e-15 \sim 4.1e-15\), which is close to the machine epsilon of \(2.2204e-16\), that is, in standard double precision mode.

The factors \(L_A\) and \(L_D\) were extracted from an LDL decomposition, which is in essence the same as Cholesky. The name of the factorisation comes from the way it is related to the latter, which is \(L \sqrt{D} = L_{\text{Chol}}\). The resultant factors are lower triangular, strictly having a main diagonal of ones, while \(D\) is a diagonal scaling matrix. In this specific test case, a block LDL version for indefinite matrices was implemented. The sole difference compared to the ordinary LDL is that the factors are triangular with respect to \(2 \times 2\) blocks. As a concluding note on this matter, all DSFEM deployments were entirely coded and executed in Matlab.

Two separate instances of the decomposition-based stochastic FEM were set up and run, for 500 realisations each. The nodes used for acceleration response collection are indicated on Figure 5.5. They coincide with the output extraction points for the PSA and CBSM, and sensor-derived experimental data was available for them.

The governing equation, according to which the stochastic models were constructed, assumed a form naturally very similar to that of (3.13) for the CBSM, as the same CMS representation was utilised. However, the principal difference was the addition of further random terms. All of the latter were related to the component mass matrices,
5.2. Validation: spacecraft vibroacoustic analysis

in contrast to the CB stochastic method, where masses remain deterministic. Explicitly written for the \( k \)-th substructure of the satellite, the equilibrium equation is

\[
\begin{pmatrix}
\hat{M}_{bb}^{(k)} & \hat{M}_{bm}^{(k)} & \hat{u}_b^{(k)} \\
\hat{M}_{mb}^{(k)} & \hat{M}_{mm}^{(k)} & \hat{u}_m^{(k)} \\
0 & \hat{\Lambda}_{mm}^{(k)} & \hat{u}_m^{(k)}
\end{pmatrix}
\begin{pmatrix}
\ddot{u}_b^{(k)} \\
\ddot{u}_m^{(k)}
\end{pmatrix}
+ \begin{pmatrix}
K_{bb}^{(k)} & 0 \\
0 & \hat{\Lambda}_{mm}^{(k)}
\end{pmatrix}
\begin{pmatrix}
\dot{u}_b^{(k)} \\
\dot{u}_m^{(k)}
\end{pmatrix}
= \begin{pmatrix}
f_b^{(k)} + \Phi_B^{(k)T} f_i^{(k)} \\
\Phi_I^{(k)T} f_i^{(k)}
\end{pmatrix}
\tag{5.1}
\]

where the modal mass \( \hat{M}_{mm}^{(k)} \) is diagonal, and has a mean value \( \mathrm{E}(\hat{M}_{mm}^{(k)}) = I \). The boundary DOF matrices \( \hat{M}_{bb}^{(k)} \) and \( \hat{K}_{bb}^{(k)} \) were left deterministic. Indeed, the principal novelty of the decomposition-based stochastic approach lies in the construction of the random off-diagonal blocks, namely \( \hat{M}_{mb}^{(k)} \), hence the purpose of this study was to isolate their effect on the system’s response. Finally, the aforementioned two setups, which shall be referred to as ‘DSFEM - A’ and ‘DSFEM - B’ were, respectively, as follows:

- Normally distributed diagonal \( \hat{\Lambda}_{mm}^{(k)} \), \( \hat{M}_{mm}^{(k)} \) and \( \hat{\Sigma}_Z \), with coefficients of variation of 0.08 for their constituent elements. In addition, the random angles \( \theta \), for the stochastic rotation matrices were also defined by Gaussian distributions, having a standard deviation of 0.08 radians, equivalent to 25°.

- A constant \( \hat{\Lambda}_{mm}^{(k)} = \Lambda_{mm}^{(k)} \). Normally distributed \( \hat{M}_{mm}^{(k)} \) and \( \hat{\Sigma}_Z \), with \( \text{COV} = 0.12 \). Random orthogonal matrices \( \hat{R}_A \) and \( \hat{R}_D \), specified by standard deviations of \( \theta \) equal to 0.12, or 37.4°.

5.2.3 Comparison against PSA, CBSM and test data

Principal findings

The main results of all stochastic simulations have been summarised on Figure 5.6 and Figure 5.7, for nodes 695897 and 7923, respectively. In both scenarios, three subgraphs have been plotted, each containing the \( \mu \pm 3\sigma \) confidence bands for the direct MCS, along with the physical sensor data curve. In addition, identical bands are shown for the CBSM, as well as the two previously explained setups of the proposed new method. These were intentionally presented on separate plots, for the sake of enhancing clarity. Moreover, Table 5.3 envelops the acceleration RMS values extracted from the data depicted on the figures, including the mean and \( \mu + 3\sigma \) predictions. They are given by the square root of the integral of the acceleration spectral density.

<table>
<thead>
<tr>
<th>Node</th>
<th>695897</th>
<th>7923</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response</td>
<td>( \mu )</td>
<td>( \mu + 3\sigma )</td>
</tr>
<tr>
<td>Direct MCS</td>
<td>1.92g</td>
<td>2.54g</td>
</tr>
<tr>
<td>CBSM</td>
<td>1.94g</td>
<td>2.54g</td>
</tr>
<tr>
<td>DSFEM - A</td>
<td>1.97g</td>
<td>2.55g</td>
</tr>
<tr>
<td>DSFEM - B</td>
<td>1.98g</td>
<td>2.56g</td>
</tr>
<tr>
<td>Test data</td>
<td>2.34g</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.6: Comparison against MCS for the response of node 695897, acceleration spectral density in direction $y$: (a) CBSM solution, (b) DSFEM - A, (c) DSFEM - B
5.2. Validation: spacecraft vibroacoustic analysis

Figure 5.7: Comparison against MCS for the response of node 7923, acceleration spectral density in direction $x$: (a) CBSM solution, (b) DSFEM - A, (c) DSFEM - B
The global correlations between the solutions acquired from the reduced order methods and the direct MCS are summarised in Table 5.4. As before, the coefficients $r_\mu$, $r_\sigma$ were defined by equation (3.36) and bear exactly the same meaning as in Chapter 3. Overall, the decomposition-based stochastic approach mimics the accuracy of the CBSM with respect to the full MCS remarkably well. Note that no DSFEM parametric study had been done a priori, so the results demonstrated herein do not necessarily represent the optimal scenario in terms of the random block definitions. The best achievable correlation by the new method might actually be greater than what is observed in Table 5.4. A detailed parametric survey on a similar test case would be beneficial as an extension to the findings of this chapter. It should be mentioned that here, the CBSM performed almost identically to the SSTL300 spacecraft case of Chapter 3 when the recommended coefficient of variation and PDF were used to model the random NFs.

<table>
<thead>
<tr>
<th></th>
<th>CBSM</th>
<th>DSFEM - A</th>
<th>DSFEM - B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_\mu$</td>
<td>0.965</td>
<td>0.959</td>
<td>0.955</td>
</tr>
<tr>
<td>$r_\sigma$</td>
<td>0.941</td>
<td>0.956</td>
<td>0.948</td>
</tr>
</tbody>
</table>

Several conclusions could be drawn in regard to the lack of strong conformity of the numerical methods' predictions to the reverberation chamber test data. First and foremost, this should not be interpreted as poor performance of the reduced order techniques, i.e., the DSFEM and CBSM. They do agree, clearly to a very high degree, with the PSA confidence bands. There is a substantially greater likelihood that the global discrepancy between experimental and computational data is owed to properties of the FE model itself. Alternatively, the specification of the nominal solution, obtained by Actran, might not have been as representative of the actual test as necessary. Some specific comments could be directed to the low-frequency behaviour observed on Figure 5.7. It could be expected that the acceleration in the range between 100 Hz and 250 Hz would be estimated quite accurately. However, for the simulations outlined here, effectively an uncoupled solution was sought. Indeed, as formerly conferred, the DSF pressure excitation was extracted by coupled FEM-FEM analysis, but subsequent applications did not take into account the effect of the altered modal behaviour of the random structure on the acoustic fluid.

Higher importance can be ascribed to the fact that, while not proven in a strict mathematical sense within this thesis, complete robustness in terms of the PSD matrix property was exhibited by Algorithm 3. Based on diagnostic data collected during the Matlab DSFEM runs, no spurious negative eigenvalues have occurred at any random realisation of the components. Furthermore, a similar statement can be phrased for the natural frequencies of the global structure, resultant from the solution of the GEP. None of them was complex, which would be the case if nonpositive generalised eigenvalues were triggered. In fact, the rank of the Craig-Bampton component mass matrices was also preserved, as well as that of the full reassembled mass.

Broadly speaking, the transition from the CBSM, through the first DSFEM, to the DSFEM - B version, did not exert any profound changes on the predicted responses.
It could be asserted that the last of the three aforementioned tests showed the widest confidence bands at higher frequency, with the corresponding opposite statement holding in the low-frequency range. Apart from this observation, it transpires that the DSFEM should have a much wider range of applicability than the CBSM. Indeed, a twofold lower COV for $\tilde{\Lambda}_{mm}(k)$ in the ‘A’ test replicated the CBSM’s $\mu \pm 3\sigma$ band very well. Interestingly, a slightly higher variance of $\tilde{\theta}$, combined with no randomness in $\Lambda_{mm}(k)$, yielded approximately the same outcome, viz. in DSFEM - B.

Finally, it must be stressed that the DSFEM has not, at this early stage of its maturity, been proven to produce random matrix distributions with means closely related to their ‘nominal’ deterministic counterparts. On the contrary, slight shift of the average response was noticed against the MCS-derived case. Therefore, while the results shown in this chapter appear promising, substantially more testing or alternatively, additional mathematical reasoning, is needed to fully validate the new method.

### Computational requirements

The last point of discussion concerned in the current section is related to the computational requirements of the decomposition-based stochastic approach. Recall item [5] of Section 4.3.1, which postulated the new method ought to pertain to computational time cost of the same order as the CBSM. The NovaSAR tests have affirmed this to be the case, bringing the topic to a completion. Indeed, the mean realisation time was approximately 0.9 s, against 0.7 s for the CBSM and 616 s for the PSA.

In a slightly more descriptive manner, it is possible to estimate the computational demands of the DSFEM explicitly, due to its purely algebraic nature. Indeed, Table 5.5 summarises the cost of generating a realisation of a random anti-diagonal partition $\tilde{B}$ of size $q \geq r$ in terms of floating point operations. The estimation is based on Construction 4.3.10, Construction 4.3.13, and the general case of dense factors of $A$ and $D$, obtained from (4.30), rather than a Cholesky-type factorisation.

<table>
<thead>
<tr>
<th>Generation of $\tilde{B}$</th>
<th>$A_{r \times r}$</th>
<th>$D_{q \times q}$</th>
<th>$B_{r \times q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only singular values $\sigma_i(\tilde{Z})$</td>
<td>$r^3 + r^2$</td>
<td>$q^3 + q^2$</td>
<td>$q^2 r + r^2$</td>
</tr>
<tr>
<td>$R_A, R_D$, constant $\sigma_i(\tilde{Z})$</td>
<td>$r^3 + 2r^2$</td>
<td>$q^3 + 2q^2$</td>
<td>$q^2 r + 2r^2$</td>
</tr>
<tr>
<td>$R_A, R_D$, random $\sigma_i(\tilde{Z})$</td>
<td>$2r^3 + 3r^2$</td>
<td>$2q^3 + 3q^2$</td>
<td>$q^2 r + 2q^2 + 3r^2$</td>
</tr>
</tbody>
</table>

Finally, note that with respect to the diagonal blocks, the first row of the table refers to the random eigenvalue case, i.e. equation (4.35). The last two lines are related to (4.39), if the bases $\tilde{Q}_A, \tilde{Q}_D$ were obtained by a multiplication of their non-random equivalents with a sparse unitary matrix. In any scenario, the complexity of the DSFEM sampling is inherently $O(N^3)$ due to the required matrix multiplications, but in domain decomposition problems $N$ is much smaller than the global system size.
5.3 Coupling with $\mathcal{H}$-matrix BEM

5.3.1 An introduction to the concept

Succeeding the spacecraft vibroacoustic validation case for the decomposition-based stochastic approach, one final topic will briefly be discussed. Namely, the coupling of the DSFEM to fast boundary element methods, which were thoroughly reviewed in Section 2.4.3. The concept that was initially envisaged involved the construction of a solver capable of simultaneously treating structural uncertainty and non-negligible elasto-acoustic interaction. The first condition necessitated some form of model order reduction for the structural domain, which in the case of the DSFEM can be effectively any existing technique, with CMS working especially well, based on the previously drawn conclusions and strict proofs.

The choice of BE method, on the other hand, is more varied, with a multitude of viable options, such as the FMM or panel clustering. The philosophy behind the selection of $\mathcal{H}$-matrices was based on the idea that an excessively large amount of computation would have to be repeated by an FMM-type approach if $N_f, N_s$ global solutions would have to be performed. Here, $N_f$ and $N_s$ denote the output frequencies and number of DSFEM realisations, respectively. Hierarchical matrices, on the other hand, are explicitly stored and generally once constructed, result in the fastest matrix-vector multiplication times. Perhaps the sole exception is the wavelet Galerkin scheme, which, however, is currently not completely mature and universally applicable.

The actual formulation that was planned involved a general equation of the form (2.60). However, for the case of space applications, the acoustic admittance $\beta$ is usually assumed zero, yielding a sound hard surface. This simplifies the Robin boundary condition (2.58) to

$$v_f^n(x) - v_s^n(x) = 0$$

which can be interpreted as enforced continuity between the fluid and structure normal velocities at points on the fluid-structure interface. The governing equation can be written as

$$\begin{pmatrix}
Z_s & -T_{sf} \\
-\rho\omega^2\mathcal{G}T_{sf}^T & \hat{H}
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix}
= 
\begin{pmatrix}
f_s \\
p_{inc} + \alpha q_{inc}
\end{pmatrix}$$

(5.3)

with $Z_s = K_s + i\omega C_s - \omega^2 M_s$ being the standard dynamic stiffness, and all remaining symbols also retaining their meanings defined in Section 2.5.2. The coupling matrix can explicitly be constructed using the process outlined in Section 3.1.5 of Chapter 3. A model order reduction can be applied to the structural domain, which, in the case of a built-up structure, would be facilitated by the coordinate transformation

$$\xi = \mathcal{L}_r^T T d \mathcal{L} \mathbf{u} = T \mathbf{u}$$

(5.4)

where $\mathcal{L}$ is the standard localisation matrix in physical space, and $\mathcal{L}_r^T$ is the one mapping dual reduced DOFs to the primally assembled ones. In the case of a Craig-Bampton reduction for component mode synthesis, the directly stated definitions of the
5.3. Coupling with \( \mathcal{H} \)-matrix BEM

Assembly operators from Section 3.1.1 can readily be used. Now, equation (5.3) can be recast in generalised coordinates as

\[
\begin{pmatrix}
\tilde{Z}_{r,s} & -\rho \omega^2 T^T G T_s^T \hat{H} \\
-T^T T_{sf} & \hat{H}
\end{pmatrix}
\begin{pmatrix}
\xi \\
p_{\text{inc}} + \alpha q_{\text{inc}}
\end{pmatrix} = \begin{pmatrix}
f_{r,s} \\
p_{\text{inc}}
\end{pmatrix}
\tag{5.5}
\]

with \( f_{r,s} = T^T f_s \). The deterministic \( Z_{r,s} \) is simply \( T^T Z_s T \), as per (2.19). The random dynamic stiffness \( \tilde{Z}_{r,s} \), in turn, is obtained by the DSFEM application on the mass, stiffness, and potentially damping matrices of which it is comprised. Solution of (5.5) for the acoustic pressure \( p \), for each frequency point and each realisation of \( \tilde{Z}_{r,s} \), continues with the standard Schur complementation procedure depicted in (2.61). The actual process is enabled by standard iterative solvers, and in the case of general matrices, such as here, the GMRES is the conventionally selected option.

5.3.2 Notes on the hierarchical matrix implementation

Hierarchical matrices are algebraic structures defined by two fundamental properties. Firstly, local partitions are either represented in standard element-wise form, or as an outer product. The first one is essentially a standard full submatrix, generated by classic BEM discretisation. The latter, on the other hand, is a concept of principal importance. For a matrix \( C \in \mathbb{C}^{m \times n} \), it is said that \( C \) is low rank if and only if

\[
\exists U \in \mathbb{C}^{m \times k}, V \in \mathbb{C}^{n \times k} : C = UV^*
\]

which is equivalent to

\[
C = \sum_{i=1}^{k} u_i v_i^*
\]

where \( u_i, v_i \) are the \( i \)-th columns of \( U \) and \( V \), respectively. Hence, only \( k(m + n) \) units of storage are needed for \( C \), instead of the standard \( mn \). In addition, MVPs can be computed more efficiently through:

\[
Cx = UV^*x = U(V^*x)
\]

This simple modification permits the reduction of the multiplication cost from \( 2mn \) to \( 2k(m + n) - k \) floating point operations, which can be significant for \( k \ll m, n \). In line with this observation, \( C \) is called a low rank matrix, from the point of view of approximations for BEM, if \( mn < k(m + n) \). All classic operations, such as addition, multiplication, SVD truncation approximations and so forth, can be expressed in terms of the outer product form, provided that it exists. The topic is far too broad for the scope of this work, which merely aims to introduce the idea of hierarchical matrices and show a simple test that was conducted in the context of the DSFEM. Excellent reviews on the topic, usually found in complete self-contained books, are available in many of the resources referenced in Section 2.4.3 for instance [108].

The second key notion, underpinning the FMMs and \( \mathcal{H} \)-matrix algebra, is called hierarchical partitioning. The concept is that a multi-level block structure, named hierarchical block-cluster tree, can be used to divide the matrix into non-overlapping
5.3. Coupling with $\mathcal{H}$-matrix BEM

rectangular partitions, based on the geometric distances between the physical sections of the acoustic boundary they represent. Recall that the idea was graphically illustrated on Figure 2.4. In principle, blocks that originate from parts of the structure separated by distance much larger than the size of the parts themselves, tend to be of very low rank, and need not be explicitly stored, or their entries computed. Rank-revealing methods, such as the adaptive cross approximation, are able to sample rows and columns of the submatrices and based on the ones computed so far, determine if an accurate enough outer product form was obtained for the unknown original block. Combined with hierarchical partitioning, these methods give rise to $\mathcal{H}$-matrices.

The topic is fascinating, but unfortunately, delving into further details would drastically expand the size of the current thesis, without contributing to its academic novelty. Due to this, suffice to say that an $\mathcal{H}$-matrix implementation was successfully carried out, employing the tenets outlined in [108] and [99]. The same applies to the discussion on BEM implementations. In this case, a plain collocation BEM with constant shape functions was devised. The integration of weakly- and strongly-singular terms, as well as the hypersingular operator, was facilitated through the closed-form analytical expressions found in [85]. Nearly-singular integration, necessary for points close to each other, but not lying on the same triangle, was done by the approach of Scuderi [92]. All of the $\mathcal{H}$-matrix construction and operation subroutines, as well as the underlying boundary element discretisation they call, were done in Matlab. Two basic examples of generated hierarchical BEM matrices can be seen on Figure 5.8.

![Figure 5.8](image)

Figure 5.8: Hierarchical matrix structure. Black indicates dense partitions, lighter colours indicate blocks of logarithmically lower rank. Left: 1-D rod, 1000 elements, right: 2-D circle, 5000 elements

5.3.3 Preliminary convergence properties

Several tests on elementary geometries were conducted with the purpose of assessing the convergence behaviour of the outlined DSFEM/$\mathcal{H}$-matrix BEM. The structures did not have any particular physical meaning, and were characterised by shell geometries of properties corresponding to the plate example used for the initial test on the improved
5.3. Coupling with $\mathcal{H}$-matrix BEM

CBSM, viz. Section 3.2.1. A non-preconditioned GMRES solver was used for the solution of the fluid-structure problem. The principal findings are summarised in Table 5.6 which shows the cases of a sphere with a 1 m diameter, comprised of 2 Craig-Bampton condensed hemispheres. In addition, a 1 m edge cube, built-up of 8 components, corresponding to each of its sides. The decomposition-based stochastic approach settings were identical to the the 'DSFEM-A' test for NovaSAR in Section 5.2.3.

Table 5.6: Coupled DSFEM-$\mathcal{H}$-matrix BEM iterations to convergence

<table>
<thead>
<tr>
<th></th>
<th>DSFEM-BEM</th>
<th>FEM-BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min max mean</td>
<td>mean</td>
</tr>
<tr>
<td>Cube</td>
<td>508 621 572</td>
<td>565</td>
</tr>
<tr>
<td>Sphere</td>
<td>471 589 541</td>
<td>532</td>
</tr>
</tbody>
</table>

Clearly, a very good agreement was attained between the deterministic FEM-BEM and the DSFEM-BEM coupled equation’s conditioning, as nearly insignificant difference in the number of iterations demanded arose. It is plausible that the cube necessitated slightly more computation by the iterative solver due to the presence of sharp edges. As an interesting remark, the actual $\mathcal{H}$-matrix structure of the sphere is displayed on Figure 5.9. It was defined to have approximately $3e+4$ surface elements, which would crudely correspond to the size of the NovaSAR model, had it been analysed with BEM. However, at the time the tests were performed, the stage of development of the boundary element codes did not permit the treatment of free-edge shell models.

Note that 13.4GB of storage would have been needed to store the BE system matrices of the sphere, whereas the $\mathcal{H}$-matrix compressed one only had a size of 262.3MB. The mean time for the crucially important MVP operation was 0.171 s, yielding a typical cost.
of 85 s per stochastic realisation per frequency point for the full equation (5.5). While good numerical stability was evident and performance of the GMRES was consistent, evaluating the coupled DSFEM-BEM solution at 50 discrete frequencies, i.e. as many as the NovaSAR test case used, would take 1 h 12 min per realisation, which would still be prohibitive if hundreds of realisations were demanded. Furthermore, it would be expected that a substantially larger frequency range would also be of interest, and would need to be discretised to a finer level. The coupled method could be viable, given a well-parallelised software implementation and the availability of a sufficiently powerful multi-processor machine.

Overall, the initial results showed that the DSFEM itself does not inherently cause inordinate execution times of coupled FEM-BEM analyses. However, the acoustic contribution to the fluid-structure interaction resolution might still render such applications inappropriate or unreasonable in engineering terms.

5.4 Summary

Within this chapter, various aspects of the practical behaviour of the previously defined blockwise random Hermitian matrices were investigated. Firstly, the discussion was centred around the element-wise and eigenvalue statistical distributions of arbitrary matrices of small size. It was demonstrated that, under the most nonrestrictive conditions permitted by Theorem 4.3.4, the entries of the random matrices generated by the proposed method, tend to be normally distributed. The same was not entirely true for the eigenvalues, normalised with respect to those of the nominal matrix. They exhibited a slight shift to either direction of unity, thus the obtained distributions for some of the smallest and largest random eigenvalues were skewed, while the rest were found to nearly follow a Gaussian curve.

Subsequently, a much more complex and involved test case was devised and thoroughly analysed in Section 5.2. The NovaSAR spacecraft was subjected to acoustic excitation, and two different applications of the DSFEM were compared against the CBSM, PSA and test campaign data. The reduced techniques were based on a three component Craig-Bampton condensation. Algorithm 3 was used for the definition of the random CB mass matrix for the DSFEM, in order to test whether it exhibited signs of instability or unphysical behaviour. However, no such occurrences were discovered. Overall, remarkably good agreement was seen between the two DSFEM options and the pure CBSM. The correlation coefficients \( \bar{r}_\mu, \bar{r}_\sigma \) were in the range of 0.94 to 0.97, which also indicated that all of the reduced random approaches behaved very similarly to the parametric MCS, as could be expected. Discrepancies between the numerical methods’ predictions and test data were present, likely due to properties of the FE model, or inaccuracy of the initial vibroacoustic solution. The computational time of the DSFEM was of the same order as the CBSM, with an added overhead per realisation not in excess of 30%. Finally, the chapter was concluded with the delineation of a reduced stochastic FEM-BEM solver, based on the coupling between DSFEM and H-matrices. Preliminary tests from the current state of its implementation showed convergence behaviour of iterative solvers to be in line with what could be expected in the absence of the DSFEM random matrix modelling.
Chapter 6

Conclusions & future work

6.1 Thesis summary

The primary subjects of this thesis have been element-based numerical methods for problems in structural dynamics and vibroacoustics, with strong emphasis placed on applications for the space industry. The bulk of the conducted research was centred around the development of highly efficient computational techniques for the uncertainty modelling of built-up structures, following an extensive study of the state of the art. The present section outlines the key points of this work in a condensed format.

The review of existing literature, contained in Chapter 2, was split into three main topics, representing the scientific fields relevant to the remainder of the thesis. The first segment introduced two areas of computational mechanics of paramount importance to the consequent chapters. Dynamic substructuring techniques, with CMS in particular, were identified as a principal means of treating complex finite element structural representations. On the other hand, the modelling of response variability of uncertain structures is another heavily researched subject matter, perceived by many researchers as a natural continuation of the currently highly evolved finite element solvers. However, it became apparent that the gap between model order reduction schemes and stochastic analysis ones has not completely been bridged. A discrepancy was observed between the availability of treatments for structural randomness in modal space, against the computational complexity of physical coordinate techniques. The former tended to be somewhat limited in scope, while the latter inevitably lead to inordinate execution runtime when the mid-frequency dynamics of large problems are evaluated.

The consequent sections of the state of the art survey were devoted to a range of subjects related to the modelling of structure-acoustic interactions. In line with the industrial motivation inciting this work, attention was initially drawn to the characterisation of aeroacoustic excitations experienced by spacecraft. Accordingly, the principles of FE and BE for the treatment of low- to mid-frequency exterior problems in linear acoustics were then demonstrated. The prominent fast boundary element methods were prioritised in the discussion. Consequently, the broad range of elasto-acoustic modelling formulations presently available was classified, from the point of view of aerospace structures analysis. It was established that hybridisation of SEA-based high-frequency
methods with classic FE approaches is among the most mature, yet actively developed medium frequency range techniques. Other schemes, such as the energy and wave FEM, appeared as viable alternatives, but are still in their infancy. Overall, a gap between the frequency ranges characterised by deterministic and stochastic behaviour remains. The use of none of the available numerical tools was deemed free of predicaments, especially in the presence of structural uncertainty.

In Chapter 3, the Craig-Bampton stochastic method was reintroduced, starting with its underlying theoretical foundations. A twofold improvement to the original solution process was suggested. Firstly, the generalised eigenvalue problem was efficiently reduced to an ordinary symmetric one by exploiting the matrix structure arising in the CB condensation. Consequently, only a CMS model update was necessary at each random structure realisation, as opposed to explicit mass and stiffness matrix reassembly. An array of numerical tests were conducted. Global correlation coefficients $\tau_\mu$ and $\tau_\sigma$ were devised to assess the improved CBSM’s statistical predictions against conventional PSA. It was shown that for a typical reduced spacecraft structure, subjected to a broadband distributed excitation, the CBSM could almost perfectly replicate the standard parametric Monte Carlo simulation results. The coefficients $\tau_\mu$ and $\tau_\sigma$ took near-optimal values of above 0.97 and 0.95, respectively, for the first two statistical moments of the solution. This was achieved by modelling the substructures’ fixed interface natural frequencies as lognormally distributed stochastic variables, with equivalent COV within the range of 0.15 $\sim$ 0.26. The efficiency gains of the improved CBSM, against the original method and the PSA, were observed to be in the vicinity of 1 and 3 orders of magnitude, respectively. The chapter was concluded by assessing limitations of the CBSM, demonstrating that while it is a viable method for mid-frequency dynamic analysis of built-up structures, caution must be exercised to ensure the conditions for its application are suitable.

In light of the aforesaid restrictions, Chapter 4 involved the adoption of a fundamentally different approach in the search for a stochastic matrix construction that could potentially exhibit universal applicability to random FE problems. The inability of existing techniques to individually deal with off-diagonal partitions of $2 \times 2$ random block matrices was initially recognised as their chief flaw. The rationale was rooted in the fact that nonempty anti-diagonal blocks naturally arise in domain decomposition FEM mass and stiffness representations. Methods intrinsically lacking the desired capability do not preserve the system level matrices’ sparse structure. To address this open problem, key mathematical concepts related to the properties of FEM matrices were initially reviewed. This included various factorisations, pseudoinverses and generalised Schur complements, in the context of their strong links to nonnegative definiteness, rank and kernel of Hermitian matrices.

The second part of the chapter employed the constructs mentioned above, to the effect of proposing, and consequently proving, a number of claims regarding matrices of the desired qualities. Crucially, a condition relating specific factorisations of the constituent blocks of a Hermitian matrix, along with two image inclusion specifications, to its positive (semi-)definiteness was shown, viz. Theorem 4.3.4. This finding was used to form a framework for the definition and sampling of random Hermitian matrices under stringent algebraic requirements, which guarantee matrix inertia preservation. The
latter was demonstrated via thorough analysis. The end of the chapter was marked by
the discussion of the numerical implementation aspects of the novel blockwise stochas-
tic matrices. A Tikhonov regularisation strategy was laid out, in order to the obviate
the explicit calculation of ill-posed products involving pseudoinverses. The suitabil-
ity of the suggested random matrices for DS and CMS was theoretically justified by
the presentation of concrete application process structures, namely Algorithm 2 and
Algorithm 3. Both propositions concerned the important case of primal assembly of
substructures, although the dual assembly was also commented on.

The final chapter, barring the present concluding one, dealt with the practical aspects of
the so-obtained decomposition-based stochastic FEM. Tests on artificially synthesised
small Hermitian matrices were run, in order to assess the element-wise statistical distri-
butions arising from the method’s application. Subsequently, a real spacecraft structure
test case was set up and comparisons were made against PSA and the CBSM, affirming
the new technique’s projected good computational performance and mathematically
robust behaviour. Finally, the coupling of the DSFEM with an $H$-matrix BEM was
delineated, and preliminary results regarding iterative solver convergence rates were
demonstrated to be in line with expectations.

6.2 State of the art advancements

It is believed that the present work enwraps several contributions to the current state
of the art in the field of numerical simulation of uncertain structures. They have
already been outlined in Section 1.3 but are also summarised herein for the sake of
completeness. In particular:

• The Craig-Bampton stochastic method, originally introduced for the purpose
of microvibration response prediction, has been reformulated to feature an im-
proved computational efficiency and the ability to handle distributed excitations.
Through an extensive parametric survey, the understanding of the scheme’s op-
timal parameters and scope of validity was broadened.

• A mathematical framework was rigorously defined for a new type of blockwise
random Hermitian matrices, intended for the representation of uncertain struc-
tures in FEM. The formulation permits individual treatment of anti-diagonal
block partitions, not allowed by other existing methods. Due to this, it is nat-
urally well-suited to the treatment of built-up structures, as the partition-wise
sparsity of the resultant reassembled matrices is preserved.

• The novel decomposition based stochastic FEM, arising from the aforesaid ma-
trices, has been implemented in software. Its expected excellent performance is
verified by several test cases, ranging in complexity from artificially generated
small Hermitian matrices to a real spacecraft model. Insights are gained on the
distributions of matrix elements and eigenvalues produced by the new approach.
Coupling with BEM for elasto-acoustic problems is outlined.
6.3 Prospective work

The present thesis gives rise to several areas of potential future work. Although it was studied in-depth, there are still a few aspects of the improved CBSM that could be investigated further, namely:

- The tests executed in the papers originally introducing the method could be repeated, using the perturbation values suggested in this work. It would be of practical interest to assess if, and to what extent, the solution accuracy in these tests would have been sharpened.

- The derivation of closed-form expressions for the variability induced by the modelling of the random natural frequencies in the CBSM might be attempted. This would avert the necessity to rely on experience, or the parametric survey presented in this work, in order to choose a representative coefficient of variation.

Additionally, several open questions remain with regard to the mathematical definition of the decomposition-based stochastic FEM. They are largely related to algebraic problems that have stayed unanswered in this study.

- A strict condition on the kernel preservation of random matrices induced by the method would be desirable. It is conjectured that the requirement would be of the form \( \text{range}(\tilde{Z}) = \text{range}(Z) \). It would be valuable to gain understanding of whether some of the usability restrictions on the random unitary matrices \( \tilde{R}_s \), with regard to the nondegenerateness of the corresponding diagonal blocks, could be relaxed or not. Also, to what extent the latter produce a mean random matrix coinciding with the original deterministic one.

- In relation to the previous point, rigorous analysis on the rank and nullspace behaviour of built-up structures in primal assembly, using the ’speculative’ Algorithm 3. While appearing completely robust in light of the satellite test case shown in Chapter 5, the exact mathematical implications of using this process, instead of Algorithm 2, are not known at present.

- Derivation of closed-form expressions for the dispersion of the global stochastic matrices, employing statistical principles. Two aspects would need to be considered, starting with the random singular values of the matrix \( \tilde{Z} \). Secondly, the effect of the variance of the random angles prescribing the compound Givens rotation matrices \( \tilde{R}_s \). Both would ideally be expressed in terms of \( E(\|B - \tilde{B}\|/\|B\|) \) and \( \text{var}(\|B - \tilde{B}\|/\|B\|) \) for the spectral and Frobenius norms.

- Thorough analytical or empirical study on the Tikhonov regularisation parameters that yield optimal accuracy for the computation of \( Z \). An assessment on the conditions, under which the employment of regularisations becomes beneficial for FEM problems, would be of significant practical value.

- Further empirical tests on the complete proposed stochastic FE method. For example, a parametric survey, resembling the one performed for the improved CBSM, might provide insights on how to choose appropriate variances for the random angles and singular values defining \( \tilde{Z} \).
6.4 Conclusion

The work presented within the body of this thesis has predominantly fulfilled the main aims outlined at the beginning of the research project. The principal advancement of the state of the art lies in the field of numerical simulation and prediction of variability in structures. A novel blockwise random Hermitian matrix definition was rigorously formulated from a mathematical perspective, in order to strictly obey a set of carefully prescribed algebraic requirements. The promising accuracy and computational performance of the resultant decomposition-based stochastic finite element method was practically supported through a number of examples, spanning the spectrum from simple synthetic tests to the vibroacoustic analysis of a real spacecraft structure. To conclude, it can be asserted that the pursuit of the fundamental academic objectives of this doctorate has successfully been brought to completion.
Bibliography


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