Closed form solution for the first natural frequency of offshore wind turbine jackets supported on multiple foundations incorporating soil-structure interaction

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Abstract

Offshore Wind Turbines (OWTs) are dynamically sensitive structures and as a result estimating the natural frequency of the whole system taking into effect the flexibility of the foundation is one of major design considerations. The natural frequency is necessary to predict the long-term performance as well as the fatigue life. Currently, jackets supported on multiple foundations (such as piles or suction caissons) are being considered to support WTG (Wind Turbine Generators) for deeper water developments. This paper presents a practical method to compute the natural frequency of a jacket supporting WTG by incorporating Soil-Structure-Interaction (SSI) based on closed form solutions. The formulation presented can be easily programmed in a spreadsheet type program and can serve as a convenient way to obtain natural frequency with least amount of input. The basis of this method is the Euler-Bernoulli beam theory where the foundations are idealized with a set of linear springs. In this method, a 3-Dimensional jacket is first converted into a two 2-Dimensional problem along the orthogonal planes of vibration which are essentially the principle axes of the foundation geometry. Subsequently, the jacket is converted into an equivalent beam representing its stiffness and a formulation is presented to find an equivalent beam for entire tower-jacket system. Using energy methods, an equivalent mass of the RNA (Rotor Nacelle Assembly)-tower-jacket system is also calculated and fixed base frequency of the jacket is estimated. To consider the flexibility effects of the foundation, a formulation for an equivalent rotational spring of the foundation is developed. A method to incorporate the mass of the transition piece is also presented. Finally, a step-by-step application of the methodology is presented by taking example problems from the literature which also serves the purpose of validation and verification.

Keywords:

Natural Frequency; Jacket Structure, Multiple Foundations, Soil-Structure Interaction
Nomenclature

<table>
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<tr>
<th>Parameter</th>
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<tbody>
<tr>
<td>(L_{\text{top}}) (m)</td>
<td>Top spacing of jacket leg chords</td>
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<tr>
<td>(L_{\text{bottom}}) (m)</td>
<td>Bottom spacing of jacket leg chords</td>
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<tr>
<td>(D_{\text{top}}) (m)</td>
<td>Tower top diameter</td>
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<tr>
<td>(D_{\text{bottom}}) (m)</td>
<td>Tower bottom diameter</td>
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<tr>
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<td>Tower wall thickness</td>
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<td>(A_{\text{c}}) (m(^2))</td>
<td>Area of jacket leg chords</td>
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<td>(h_{\text{J}}) (m)</td>
<td>Height of jacket</td>
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<td>(h_{\text{T}}) (m)</td>
<td>Height of tower</td>
</tr>
<tr>
<td>(m_{\text{i}}) (kg/m)</td>
<td>distributed mass of jacket</td>
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<tr>
<td>(m_{\text{T}}) (kg/m)</td>
<td>distributed mass of tower</td>
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<td>(m_{\text{eq}}) (kg/m)</td>
<td>equivalent distributed mass of tower-jacket system</td>
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<tr>
<td>(M_{\text{RNA}}) (kg)</td>
<td>Lumped mass of Rotor-Nacelle Assembly</td>
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<td>(M_{\text{TP}}) (kg)</td>
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</tr>
<tr>
<td>(E_{\text{IJ}}) (Nm(^2))</td>
<td>Jacket bending stiffness</td>
</tr>
<tr>
<td>(E_{\text{IT}}) (Nm(^2))</td>
<td>Tower bending stiffness</td>
</tr>
<tr>
<td>(E_{\text{IJ}}) (Nm(^2))</td>
<td>Tower-jacket system bending stiffness</td>
</tr>
<tr>
<td>(k_{\text{v}}) (N/m)</td>
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</tr>
<tr>
<td>(k_{\text{R}}) (NM/rad)</td>
<td>Rotational stiffness of the foundation</td>
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<tr>
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<tr>
<td>(C_{\text{f}})</td>
<td>Foundation flexibility parameter</td>
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<tr>
<td>(f_{\text{fb}}) (Hz)</td>
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<td>(f_{\text{0}}) (Hz)</td>
<td>Flexible natural frequency of the system</td>
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1.0 Introduction:

Jacket Supported Offshore Wind Turbines

Jackets or seabed frames supported on multiple shallow or deep foundations (see Figures 1 and 2), are currently being used to support WTG (Wind Turbine Generators) in deep waters typically ranging between 23m and 60m. Examples include Borkum Riffgrund 1 (Germany, water depth 23 to 29m), Alpha Ventus Offshore (Germany, water depth 28 to 30m), Aberdeen Offshore wind farm (Scotland, water depth 20 to 30m). The jackets are typically three or four legged and are supported on either deep foundations (piles) or shallow foundations (suction caissons). The height of the jackets currently in use is between 30 and 35 meters depending on water depth and wave height. However, it is expected that future offshore developments will see jacket heights up to 65m to support larger turbines (12MW to 20MW) in deeper waters.

Figure 1: Schematic of a jacket supported on shallow foundation together with modes of free vibration

Figure 2: Schematic of a jacket supported on deep foundations and modes of free vibrations
Dynamics plays an important role in the design of such structures as it is necessary to target the natural frequency of the whole system away from the excitation frequencies of 1P (rotor frequency), 2P/3P (blade passing frequency) and wave frequencies. The dynamic response on a fixed base jacket under different types of wave loads have been studied by (Wei, et al., 2017) where the dynamic amplification factors (DAFs) for regular and irregular waves is computed. The study shows that depending on the wave amplitude and period, the DAF may reach values of 1.2-1.3 which is significant given the magnitude of wave loads. Numerical studies by (Abhinav & Saha, 2015) shows the importance of incorporating the flexibility of the foundation in understanding the modes of vibration of the system and when predicting the structural response under numerous loads. In the study, the SSI effect was introduced through distributed springs along the depth of the foundation. (Abhinav & Saha, 2018) studied the effect of non-linearity of the ground profiles in loose sands, medium sands and dense sands and concluded that the effect of SSI becomes predominant in looser sands. Further details on different types of modes of vibrations as well as dynamic requirements are discussed in (Bhattacharya, et al., 2011) (Bhattacharya, et al., 2013) (Bhattacharya, et al., 2017). As shown in schematic Figures 1 and 2 and discussed in (Bhattacharya, et al., 2013) the modes of vibration for a jacket depends on the vertical stiffness of the foundation. Typically, jackets supported on piles will exhibit sway bending modes of undamped free vibration (see Figure 2) as the foundation is very stiff compared to the tower. On the other hand, a jacket supported on shallow foundation may have rocking modes of free vibration due to the relatively lower vertical stiffness of shallow foundations as also shown, see Figure 1. When considering the response to forced vibrations due to wind and wave loads, designers must check the effect of the misalignment between them which may vary up to 90° (Siedel, 2010).

Based on the above discussion, it is clear that estimating natural frequency of the wind turbine system is an important design requirement as it dictates the dynamic amplification factor of the response and the dynamic load factors. It is also imperative that designers need to optimise the jacket dimensions, jacket member stiffness and foundation stiffness to arrive at a configuration which will provide lower dynamic amplification factors. At the financial viability stage and before the business decision is made on the development of the wind farm, it is often not (economically) possible to carry out iterations of jacket configurations in a Finite Element (FE) program. FE software are necessary during the detailed design stage where understanding the specifics of the dynamics of the system is critical. Furthermore, it is also often necessary to verify high fidelity calculations carried out using software with some hand methods. Therefore, the aim and scope of the work are as follows:

1. Develop a closed form solution to estimate the natural frequency of a jacket by incorporating Soil-Structure-Interaction (SSI). This involves idealisation of the jacket as an equivalent beam and replace the foundation by a set of linear springs.

2. Verify and validate the method through examples available in the literature and through Finite Element Analysis.

This study is very similar to the earlier work carried out by the Surrey research group shown in (Arany, et al., 2016) whereby closed form solution is developed for monopile supported wind turbine by considering the SSI and Transition Piece (TP) stiffness. The proposed method can be easily coded in a spreadsheet type program and needs limited data about the wind turbine, ground condition, geometry of the jacket and the foundation. This work builds on previous research on monopile supported OWTs where the towers are modelled as Euler-Bernoulli Beams and supported on a set of independent spring which represent the foundations, see for example (Adhikari & Bhattacharya, 2011) (Adhikari & Bhattacharya, 2012) (Arany, et al., 2016). The work by (Arany, et al., 2015) compared the use of Timoshenko beam theory with Euler-Bernoulli beams and concluded the latter suffices with acceptable accuracy.
2.0 Methodology

The mechanical idealization adopted in this work is based on the study presented in (Bhattacharya, et al., 2013), (Bhattacharya, et al., 2017) where a 3-Dimensional system can be converted into a 2-Dimensional systems and the vibration across each axis can be studied separately. Further details of this conversion and the limitations are discussed in the next section. Figure 3 shows an overview of the whole methodology. The jacket and wind turbine towers are modelled as Euler-Bernoulli beam with a distributed mass. The foundations are modelled with a set of vertical elastic springs ($k_{v1}$ and $\alpha k_{v1}$) and the Rotor-Nacelle Assembly with a top lumped mass ($M_{RNA}$). Finally, a simplified spring-mass system is developed ($M_{eq}$ and $K_{eq}$).

![Figure 3: Mechanical idealization of jacket supported offshore wind turbines](image)

The calculation steps can be summarized as follows:

1. Convert the 3D vibration problem into a 2D vibration problem
2. Obtain the equivalent jacket bending stiffness $Ei_j$
3. Obtain the equivalent tower bending stiffness $Ei_t$
4. From steps 2 and 3, obtain a single value for the bending stiffness of the tower-jacket system $Ei_{T-J}$
5. Obtain the equivalent distributed mass $m_{eq}$ of the tower-jacket system
6. Obtain the equivalent rotational stiffness of the caisson or pile supports ($k_R$)
7. Obtain the natural frequency of the equivalent beam with one end spring hinged and carrying a lumped mass at the other free end.

It may be noted that damping is not required in the formulation (shown in Figure 3) as we are interested in natural undamped frequency. Damping is critical in terms of restricting fatigue damage and dynamic response. Further information on different types damping of OWT applications are stated in (Arany, et al., 2016). The next sections explain how each step can be obtained. To improve
readability of the paper, part of the mathematical derivations is provided in the appendices. Verification and validation of the method is also carried out.

2.1 Conversion from 3D vibration problem to a 2D vibration problem

For simplification purposes, a 3D offshore wind turbine can be modelled in 2D where the vibration across each axis can be studied separately. The vibration of such a complex system is a 3D problem where oscillations may occur over multiple coupled planes depending on the locations of the centre of mass and centre of stiffness. Under certain circumstances and practically for all real problems, a 3D vibration problem can be simplified into a 2D vibration problem, where vibrations in orthogonal planes may be uncoupled and studied separately. This is strictly applicable if the centre of mass coincides with centre of stiffness and can also be applied for different foundation arrangements. Even with asymmetric foundation arrangements (i.e. for example right-angled tripod), the locations of the centre of mass and the centre of stiffness will be within reasonable limits so that 2D idealization might still be applicable. Figures 4 to 6 explain the conversion for a 3-dimensional tetrapod and a tripod arrangement to a 2-dimensional problem.

As shown in Bhattacharya et al (2013, 2017), the wind turbine system will vibrate in two principle axes i.e. having the highest variance of moment of inertia. The foundation can be modelled as two springs connected by a rigid, whilst the superstructure (the jacket and wind turbine tower) can be modelled as an equivalent beam with a lumped mass at the tip representing the accelerating mass of the jacket, tower, and the Rotor-Nacelle Assembly. Thus, this two-dimensional mechanical model can be applied to both three legged or four legged jackets as shown in Figures 4 to 6. For four legged jackets, vibration can occur at X-X’ or Y-Y’ planes as shown in Figures 4. It may also be noted that a four-legged jacket may vibrate in the diagonal plane, see Figure 4(b). Similarly, for three legged jackets the rocking vibration modes with have three axes of symmetry as shown in Figures 5 and 6.

![Diagram of vibration of a rectangular base about orthogonal planes](image)

Figure 4: Vibration of a rectangular base about orthogonal planes
Figure 5: Vibration of a triangular base

Figure 6: Planes of Symmetry
2.2 Estimating equivalent beam stiffness of a jacket

In a simplified 2D idealization a truss may be modelled with an equivalent beam of a uniform cross-section using the parallel axis theorem as shown in Figure 7. A similar approach has been used by (MacLeod, 2005), (Giltner & Kassimali, 2000) in order to simplify finite element modelling of truss supported structures. Likewise, whilst studying the dynamic performance offshore oil and gas jackets, (Karsan, 1986) also idealizes the system as a beam where jacket legs represent the flanges and braces as the web. Hence, for the presented configuration, the moment of inertia is computed using Eq.1

\[
I_j = \frac{A_c d^2}{2} \quad \text{(Eq.1)}
\]

Where \( A_c \) is the cross sectional area of the leg and \( d \) is the spacing between the truss legs. For simplicity, Eq. 1 ignores the moment of inertia of a given member about its own centroidal axis and also assumes that the horizontal and diagonal members are connection members. Hence, they maintain the stability of the truss and are not included in the calculation of the moment of inertia. In reality, some shear stiffness is provided by the diagonal members where they transfer unbalanced axial forces from one leg to the other. Moreover, the braces also transfer wave, wind, and current loads to the jacket legs. If the ratio of the cross-sectional area of the braces to the cross-sectional area of the legs is low, then the jacket acts a Vierendeel frame where the deformations of the jacket would be large and plane sections would no longer remain plane, which in turn would contradict the Euler-Bernoulli beam theory. If the ratio of the areas is large, this will increase the lateral stiffness of the jacket and ensure that large distortions in shape would not occur. However, there is a threshold up to which increasing the brace cross-sectional area is beneficial to the lateral stiffness of the jacket. (Kumar, et al., 1985) showed that the lateral stiffness of a single storey truss started decreasing after a certain limit of the bracing cross-sectional area was exceeded. Based on the discussion above it is then assumed that the braces of the system are:

- Stiff enough such that deformations in the jacket legs are small and plane sections remain plane
- Are not beyond the optimum value
Clearly, the ignored shear stiffness will cause an underestimation of the natural frequency, however for the purpose of providing adequate preliminary jacket leg sizes and spacing, the additional contribution provided by the braces may be ignored. As it will be shown in section 3.0 this assumption is valid as the obtained results were comparable with Finite Element Analysis which includes the additional contribution of the shear stiffness of the braces.

As the spacing \( d \) between leg elements is variable (top and bottom chord spacing are different), a function was derived to obtain the equivalent cross-section of the truss

\[
L_{\text{bottom}} = m \cdot L_{\text{top}} \quad \text{(Eq.2)}
\]

\[
EI_{\text{j}} = EI_{\text{top}} \cdot f(m) \quad \text{(Eq.3)}
\]

Where \( f(m) = \frac{1}{3} \cdot \frac{m(m - 1)^3}{m^2 - 2m\ln(m) - 1} \) \quad \text{(Eq.4)}

The derivation for Eq.4 can be found in Appendix A

**3D to 2D conversion of the jacket structure**

In a similar manner to the method shown in Section 2.1, a jacket will vibrate about two diagonal planes. Consider a jacket with a square configuration as shown in Figures 8a and 8b, the bending stiffness in the \( x-x' \) direction can be obtained using Equation 5 to 7

\[
A_{\text{tot1}} = A_{\text{c}} + A_{\text{c}} = 2A_{\text{c}} \quad \text{(Eq.5)}
\]

\[
A_{\text{tot2}} = A_{\text{c}} + A_{\text{c}} = 2A_{\text{c}} \quad \text{(Eq.6)}
\]

Distance to Neutral Axis: \( L_{\text{top}}/2 \)

Moment of inertia in the \( x-x \) direction

---

**Figure 8a:** Stiffness of a jacket in \( x-x' \) bending direction

\[
A_{\text{tot1}} = A_{\text{c}} + A_{\text{c}} = 2A_{\text{c}}
\]

\[
A_{\text{tot2}} = A_{\text{c}} + A_{\text{c}} = 2A_{\text{c}}
\]
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\[ I_j = 2A_C\left(\frac{L_{\text{top}}}{2}\right)^2 + 2A_C\left(\frac{L_{\text{top}}}{2}\right)^2 = A_CL_{\text{top}}^2 \]  
*(Eq. 7)*

Similarly, the same analysis is repeated for the diagonal \(y-y'\) axis

![Diagram](image)

Figure 8b: Stiffness of a jacket in the diagonal \(y-y'\) bending direction

\[ A_{\text{tot}1} = A_C \]  
*(Eq. 8)*

\[ A_{\text{tot}2} = A_C \]  
*(Eq. 9)*

Distance to Neutral Axis: \(L_{\text{top}}\sqrt{2}/2\)

Moment of inertia in the diagonal direction:

\[ I_j = A_C\left(\frac{L_{\text{top}}\sqrt{2}}{2}\right)^2 + A_C\left(\frac{L_{\text{top}}\sqrt{2}}{2}\right)^2 = A_CL_{\text{top}}^2 \]  
*(Eq. 10)*

Comparison of Equations 7 and 10 suggests that the stiffness of a symmetric square configuration is the same about the different planes of symmetry. In reality, the diagonal members of a jacket will have a higher contribution to the stiffness in the \(x-x'\) direction. These results suggest that the natural frequencies of jackets with symmetric arrangements is expected to be similar in both axes due to their similar stiffness values.

### 2.3 Tower Bending Stiffness

Wind turbine towers are typically tapered tubular sections typically having varying thickness. Previous work by (Adhikari & Bhattacharya, 2011) (Arany, et al., 2016) (van der Tempel & Molenaar, 2002) (Zania, 2014) idealize the tower with a column of constant diameter and thickness as shown in Figure 9.
Following (Arany, et al., 2016) the average thickness of an OWT tower may be expressed as

\[ t_T = \frac{m_T}{\rho h_T D_T \pi} \quad \text{(Eq.11)} \]

Where \( D_T \) is the average tower diameter

\[ D_T = \frac{D_{\text{top}} + D_{\text{bottom}}}{2} \quad \text{(Eq.12)} \]

In a similar manner to the methodology presented in Section 2.2, the equivalent bending stiffness of a tapered tower can be calculated as follows

\[ I_T = \frac{\pi}{8} D^3 t_T \quad \text{(Eq.13)} \]

\[ EI_T = EI_{\text{top}} \cdot f(q) \quad \text{(Eq.14)} \]

Where \( f(q) = \frac{1}{3} \cdot \frac{2q^2(q-1)^3}{q^2(2lnq-3)+4q-1} \quad \text{(Eq.15)} \]

A full mathematical derivation is presented in Appendix B

### 2.4 Tower-Jacket Bending Stiffness

A single equivalent section for the tower-jacket system (Figure 10) may be obtained using Castigliano’s theorem of a linear elastic single degree of freedom structure and can be expressed as shown by Equations 16 to 20.
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![Figure 10: Equivalent Cross-Section of a tower-jacket system](image)

Say $\psi = \frac{h_j}{h_T}$  \hspace{1cm} (Eq.16)

And $\chi = \frac{E_T I_T}{E_J I_J}$  \hspace{1cm} (Eq.17)

$k_{T-J} = \frac{3E_T I_T}{h_T^3} \left( \frac{1}{1 + (1 + \psi)^3 \chi - \chi} \right)$  \hspace{1cm} (Eq.18)

Also $k_{T-J} = \frac{3E_T I_T I_{T-J}}{(h_T + h_J)^3}$  \hspace{1cm} (Eq.19)

Assuming both the jacket and the tower are composed of the same material and using Eq. 18 and Eq.19, the equivalent bending stiffness of the tower-jacket system is given by:

$EI_{T-J} = E_T I_T \left( \frac{1}{1 + (1 + \psi)^3 \chi - \chi} \right) \left( \frac{h_j + h_T}{h_T} \right)^3$  \hspace{1cm} (Eq.20)

Derivation of the above equation is given in Appendix D

2.5 Equivalent mass of a Tower-Jacket System

The masses of the jacket and the tower are assumed to be distributed constantly along their lengths as shown in Figure 11. Physically, due to the shape of the tower and the jacket the actual distributed mass of each component decreases along the height. However, in the simplified methodology, it is
assumed that the tower and the jacket have a constant mass distribution with depth given by \( m_T \) and \( m_J \) for the tower and jacket respectively in kg/m. It is shown later in the paper that this assumption is valid with acceptable levels of accuracy.

A method has been derived by (Moon & Hong, 2008) in order to obtain the equivalent distributed mass of a cantilever beam composed of 2 cross-sections, otherwise known as a stepped cantilever. It is assumed that the kinetic energy of the stepped cantilever and the uniform cross-section are equivalent. For this purpose, the first Eigen mode kinetic energy equation shown by (Hurty & Rubinstein, 1964) was used as shown in Eq.21

\[
KE = \int m(z)\phi^2 \, dz \tag{Eq.21}
\]

Where \( m(z) \) and \( \phi \) are the mass and Eigen mode function of a continuous cantilever system.

Equating the Kinetic energy of the tower-jacket system with the equivalent beam

\[
\int m(z)\phi^2 \, dz = \int m_{eq}\phi^2 \, dz \tag{Eq.22}
\]

\[
m_{eq} = \frac{\int m(z)\phi_i^2 \, dz}{\int \phi_i^2 \, dz} = \sum_{i=1}^{n} m_i \int_{z(i)}^{z(i-1)} \phi_i^2 \, dz + m_j \int_{0}^{h_j} \phi_j^2 \, dz + m_T \int_{h_j}^{h_j+h_T} \phi_i^2 \, dz \tag{Eq.23}
\]

The mode shape function and the integral of square the mode shape function may be evaluated using equations (Eq.24a) and (Eq.24b) respectively:

\[
\phi_i = \beta_i \left( \sin \frac{\lambda_i}{L} z - \sinh \frac{\lambda_i}{L} z \right) + \cosh \frac{\lambda_i}{L} z - \cos \frac{\lambda_i}{L} z \tag{Eq. 24a}
\]
\[ \int \phi_1^2 dz + \frac{1}{4\lambda_1} \sin \frac{2\lambda_1}{L} z - \frac{1}{2\lambda_1} \cos \frac{2\lambda_1}{L} z + \frac{1}{2\lambda_1} \sinh \frac{2\lambda_1}{L} z - \frac{1}{2\lambda_1} \cosh \frac{2\lambda_1}{L} z + \frac{2}{L} \sin \frac{\lambda_1}{T-J} z \sinh \frac{\lambda_1}{T-J} z - \frac{2}{L} \sin \frac{\lambda_1}{T-J} z \cosh \frac{\lambda_1}{T-J} z - \frac{1}{2\lambda_1} \sin \frac{\lambda_1}{T-J} z \sinh \frac{\lambda_1}{T-J} z = 0 \quad (Eq.24b) \]

\[ \frac{1}{\lambda_1} \sin \frac{\lambda_1}{L} z \cos \frac{\lambda_1}{L} z - \frac{1}{\lambda_1} \cos \frac{\lambda_1}{L} z \sin \frac{\lambda_1}{L} z - \frac{1}{\lambda_1} \sinh \frac{\lambda_1}{L} z \sinh \frac{\lambda_1}{L} z - \frac{1}{\lambda_1} \cosh \frac{\lambda_1}{L} z \cosh \frac{\lambda_1}{L} z = 0 \]

\[ L = h_J + h_T \]

Where \( \lambda_1 \) is the first root of the natural frequency equation, \( \beta_1 \) is the dimensionless natural frequency parameter of Euler-Bernoulli beam, and \( z \) is the height along the beam. For ready reference, the values of \( \lambda_1 \) and \( \beta_1 \) for a cantilever beam (fixed end, and not with an end rotational spring as the one shown in figure 3 (d)) can be obtained from Eqn 25a and 25b which are extracted from standard structural dynamics text books such as (Blevins, 1979).

\[ \lambda_1 = 1.8751 \quad (Eq.25a) \]

\[ \beta_1 = \frac{\cos \lambda_1 + \cosh \lambda_1}{\sin \lambda_1 + \sinh \lambda_1} \quad (Eq.25b) \]

Thus, all the components required to estimate the equivalent mass of the cross-section given in Eq. 23 can be carried out using simple computation.

As shown in Figure 3d, an equivalent rotational spring will be obtained for the foundation arrangement which is further elaborated upon in section 2.6. Hence, the procedure shown in Eq 22 and Eq 23 will be used to obtain the equivalent mass for a beam with an end rotational spring. This will allow for the comparison between the \( m_{eq} \) of a beam with a fixed end and a beam with a rotational spring at the end and check the differences in between. The work by (Chun, 1972) presents the function of the first mode shape of a beam which is supported by a rotational spring at one end and free at the other. This shown by Eq.26 where \( k_R \) is the stiffness of the rotational spring.

\[ \varphi_i = \beta_i \left( \frac{1}{\lambda_1} \right) \left( k_R \frac{L}{EI_T} \right) \left( \cos \frac{\lambda_1}{L} z \cos \frac{\lambda_1}{L} z \right) + \beta_i \left( \sin \frac{\lambda_1}{L} z \sinh \frac{\lambda_1}{L} z \right) + \beta_i \left( \sin \frac{\lambda_1}{L} z \cosh \frac{\lambda_1}{L} z \right) \]

Using Eq.23, \( m_{eq} \) was plotted for two \( m_J/m_T \) values and is shown in Figure 12. The two cases shown in the figure are for the two extremes cases:

(a) fixed end (having an infinite \( k_R \) i.e. jacket on deep foundations) shown in solid lines;

(b) low value of \( k_R \) (non-dimensional rotational stiffness of 1) shown with the dashed lines.

The mode shape function used for (a) is Eqn 24(a) and for (b) Eq. 26. From the figure two important points may be noted:

(a) If the tower height is more than 50% of the overall height of the structure, the ratio of \( m_{eq}/m_T \) is close to 1. This applies even if the mass of the jacket is twice that of the tower. As it is typical to have taller towers than jackets, the mass of the tower is likely to govern and any discrepancies in assuming the distributed mass of the jacket will not greatly affect the results.

(b) The degree of fixity of the support is insensitive to \( m_{eq} \) estimation. The plot shows closely matching results for both the fixed end (infinite \( k_R \)) and low rotational stiffness of the foundation. Due to the relative simplicity of the integral of the function, Eqn 24 i.e. the
cantilever function, is used in the solved example in section 3.0. The methodology of obtaining an equivalent rotational spring is explained in the following section.

![Graph](image)

Figure 12: Non-dimensional plot of $m_{eq}$ vs tower height

It may be noted that the methods presented above can also be applied for cases where the cross-sectional area of the jacket legs themselves also varies with height. Using the numerical derivations provided in the appendices, the integrals for stiffness and equivalent masses of the system may be considered at different intervals where the cross-section is expected to vary. However, in the solved example, for the sake of practicality, a single cross-section (with varying leg spacing) has been taken for the jacket members.

2.6 Equivalent rotational stiffness of a multiple supports.

The system is now effectively a beam with a uniform cross section supported on two vertical springs. A simpler method would be to obtain the equivalent rotational stiffness of a rigid support on multiple foundations as shown in Figures 13a and 13b and can also be solved using energy methods. As foundations of jackets and seabed frames are spaced wide apart, the vertical stiffness may not the similar and is therefore represented $k_v$ and $\alpha k_v$ as shown Figures 13a and 13b. Using the vertical equilibrium of the system and Castigliano’s theorem of strain energy, Equations 27 to 28 presents the results and the derivation is provided in Appendix D.
For two support condition, the centre of rotation can be obtained following equation 27.

\[
\mu = \frac{\alpha}{1 + \alpha}
\]  
(Eq.27)

\[
k_R = k_v L^2 \left[ \frac{\alpha}{1 + \alpha} \right]
\]  
(Eq.28)

As per Eq. 28, it is evident that the equivalent rotational stiffness of foundation on 2 supports is dependent on the spacing between the foundations and higher spacing between the vertical supports enhances rotational stiffness.

Similarly, as shown in Section 2.2.1, a 2D problem may have 3 springs as shown in Figures 5, 6 and 14. Similar methodology may be applied to accommodate vibration in the diagonal direction as given by Equations 29 to 30.

\[
\text{Figure 13a: Equivalent rotational stiffness of multi-support foundation}
\]

\[
\text{Figure 13b: Equivalent rotational stiffness in the diagonal direction}
\]
\[ \mu = \frac{\alpha + 0.5\nu}{1 + \alpha + \nu} \]  
(Eq.29)

\[ k_R = k_x L_1^2 \left[ \mu^2 + \alpha (1 - \mu)^2 + \nu (\zeta - \mu)^2 \right] \]  
(Eq.30)

In a similar manner to section 2.2.1, the rotational stiffness is calculated for both the x-x' and y-y' directions of a square configuration of the foundation base.

![Schematic of a square base](image)

**Figure 14:** Schematic of a square base

For plane x-x'

\[ k_1 = k_2 = k_3 = k_4 = k \]  
(Eq.31)

\[ k_{v1} = k_{v2} = 2k \]  
(Eq.32)

\[ \alpha = 1 \]  
(Eq.33)

\[ k_R = k_x L_1^2 \frac{\alpha}{1 + \alpha} = 2kL_2 \frac{1}{1 + 1} = kL_2 \]  
(Eq.34)

For vibrating about the diagonal axis y-y'

\[ k_{v1} = k_{v2} = k \]  
(Eq.35)

\[ \alpha = 1 \]  
(Eq.36)

\[ \nu = 2 \]  
(Eq.37)

\[ L_1 = L\sqrt{2} \]  
(Eq.38)

\[ \mu = \frac{1 + 0.5(2)}{1 + 1 + 2} = \frac{1}{2} \]  
(Eq.39)

\[ k_R = k_x L_1^2 \left[ \mu^2 + \alpha (1 - \mu)^2 + \nu (\zeta - \mu)^2 \right] = k \left( L\sqrt{2} \right)^2 \left[ 0.5^2 + (1 - 0.5)^2 + 2(0.5 - 0.5)^2 \right] = kL^2 \]  
(Eq.40)
Judging from Eq. 30 and Eq. 40 it is also evident that for a symmetric square configuration, both the jacket and foundation stiffness are the same and hence rocking about the x-x’ and diagonal y-y’ planes have the same natural frequency. Guidance in estimating the vertical stiffness of embedded deep and shallow foundations are provided in Appendix F.

It may be realized that this method assumes the presence of translational restraints in the lateral direction at foundation level and only the vertical stiffness is considered due to the load transfer mechanism. Even though embedded foundations exhibit a high lateral stiffness, some inherent flexibility is bound to be present in that direction which in turn would influence the value of the natural frequency. Ideally, this can be modelled by adding lateral springs (k_L) in addition to the vertical springs (k_v). Thus, after the selection of a certain foundation size using the proposed method (which only included vertical springs k_v) designers are encouraged to further refine structural models to include the lateral stiffness at the foundation level rather than a lateral restraint. Guidance on how to estimate the lateral stiffness of piles and shallow foundations is also presented in Appendix F.

2.7 Natural Frequency Estimation:
2.7.1 Continuous system solutions

The natural frequency of a beam with a distributed mass supported by a rotational restraint is expressed as:

$$f_0 = \frac{\lambda^2}{2\pi} \sqrt{\frac{EI}{mL^2}}$$  \hspace{1cm} (Eq. 41)

Where L is the span of the beam, m is the distributed mass and $\lambda$ is the root of the solution of the frequency equation and depends on the non-dimensional stiffness parameter as shown in equation Eq. 42 (Chun, 1972)

$$\frac{k_R L}{EI} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\cos \lambda \cdot \cosh \lambda} + 1 \right) - \tan \lambda + \tanh \lambda = 0$$  \hspace{1cm} (Eq. 42)

Hence, each root ($\lambda$) of Eq 42 can be achieved and substituted in Eq. 41 which allows designers to obtain the $n^{th}$ natural frequency of the system. Eq. 42 does not include the lumped mass at the end of the beam, which is the case of an offshore wind turbine system is the Rotor-Nacelle Assembly (RNA) mass and is significant and cannot be ignored. (Lee, 1973) provided the solution for the vibration of a uniform beam with one end spring hinged and carrying a lumped mass at the other free end. Expressed in terms of the parameters, the adjusted solution of the frequency equation is as follows:

$$\frac{k_R L}{EI} \left( \frac{M}{mL} \right) \left( \tan \lambda - \tanh \lambda \right) \left( \frac{1}{\lambda} \left( \frac{1}{\cos \lambda \cdot \cosh \lambda} + 1 \right) \right) + \tan \lambda - \tanh \lambda + 2\lambda \left( \frac{M}{mL} \right) \tan \lambda \cdot \tanh \lambda = 0$$  \hspace{1cm} (Eq. 43)

where $m$ is the distributed mass of the beam in kg/m and $M$ is the lumped mass at the beam end (in kg). From Equation 43, it is evident that the driving parameters in estimating the natural frequency of the system are the ratio of the lumped mass to the beam mass, and the ratio of spring stiffness to beam stiffness. Using any advanced mathematics software, one can find the roots of Equation 43 and
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obtain the natural frequency using Equation 41. Moreover, this method can enable the designer to obtain higher mode frequencies.

2.7.2 Equivalent SDOF system

As stated in section 2.7.1, using Eq.43 may require advanced mathematics software to find the roots, thus for simplification purposes, the problem is further broken into an equivalent mass-spring SDOF system as shown in Figure 15 where the natural frequency is expressed as in Eqn. 44. The equivalent system and equivalent lumped mass of the system are shown in Eq. 44 to 59 a full derivation is provided in Appendix E. $M_{eq}$ represents the total mass of the system lumped at the tip which is constituted of the beam mass (representing the tower and the jacket) and the RNA mass such that $M_{eq} = M_{eq-TJ} + M_{RNA}$. The mass of the RNA is already lumped at the tip and a method is presented to transform the beam distributed mass into a lumped at the tip ($M_{eq-TJ}$). On the other hand, $K_{eq}$ is a function of both the rotational stiffness of the foundation and the bending stiffness of the equivalent beam. Hence, $K_{eq}$ is a function of both $k_r$ and $EI_{TJ}$.

\[
f_0 = \frac{1}{2\pi} \sqrt{\frac{K_{eq}}{M_{eq}}},
\]

(Eq.44)

\[
h_{total} = h_T + h_J
\]

(Eq.45)

**Equivalent spring stiffness $K_{eq}$**

The deflection of a beam with end rotational restraint is estimated by Eq.46:
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\[ y(0) = \frac{1}{k_R} P \left( \frac{h_{\text{total}}}{h_{\text{total}}} \right)^2 + \frac{P \left( \frac{h_{\text{total}}}{3E_I} \right)^3}{3E_I} \]  
(Eq.46)

Say that \( \tau \) is the non-dimensional foundation stiffness given by

\[ \tau = \frac{k_R h_{\text{total}}}{E_I_{\text{T-J}}} \]  
(Eq.47)

Hence, \( \tau \) is a measure of how flexible the foundation is and is dependent on the vertical stiffness of the foundation and the spacing between the supports.

Hence, applying a unit load \( P \) will allow the computation of \( K_{\text{eq}} \)

\[ K_{\text{eq}} = \frac{1}{y(0)} = \frac{E_I_{\text{T-J}}}{\left( \frac{h_{\text{total}}}{3} \right) \left( \frac{1}{\tau} + \frac{1}{3} \right)} \]  
(Eq.48)

**Equivalent tower lumped mass \( M_{\text{eq-TJ}} \)**

In order to obtain the \( M_{\text{eq-TJ}} \) in kg, Eq.44 is equated to Eq. to 41 as shown in Eq.49. Effectively, this allows for the computation of the proportion of the beam distributed mass that can be lumped at the tip

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{K_{\text{eq}}}{M_{\text{eq-TJ}}}} = \frac{\lambda_1^2}{2\pi} \sqrt{\frac{E_I_{\text{T-J}}}{m_{\text{eq}} h_{\text{total}}^4}} \]  
(Eq.49)

Where \( \lambda_1 \) is the first root of the natural frequency equation shown in Eq.42

Through substituting Eq. 48 into 49

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{E_I_{\text{T-J}}}{M_{\text{eq-TJ}} \left( \frac{h_{\text{total}}}{3} \right) \left( \frac{1}{\tau} + \frac{1}{3} \right)}} = \frac{\lambda_1^2}{2\pi} \sqrt{\frac{E_I_{\text{T-J}}}{m_{\text{eq}} h_{\text{total}}^4}} \]  
(Eq.50)

Further algebraic simplification leads to Eq. 50 to 52

\[ M_{\text{eq-TJ}} \left( \frac{1}{\tau} + \frac{1}{3} \right) \left( \frac{1}{m_{\text{eq}} h_{\text{total}}} \right) = \frac{\lambda_1^4}{2\pi} \]  
(Eq.51)

\[ M_{\text{eq-TJ}} = \frac{\lambda_1^4}{2\pi} \left( \frac{1}{m_{\text{eq}} h_{\text{total}}} \right) \]  
(Eq.52)

Hence by replacing \( \frac{1}{\lambda_1^4 \left( \frac{1}{\tau} + \frac{1}{3} \right)} \) by \( \varepsilon \), Eq.52 may be expressed as:
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\[ M_{\text{eq-TJ}} = \varepsilon \left( m_{\text{eq}} h_{\text{total}} \right) \]  

(Eq.53)

Thus, \( \varepsilon \) represents the proportion of the Tower-Jacket mass (in kg) which can be lumped at the tip. The value of \( \varepsilon \) is a function of \( \lambda_i \) which in turn is a function of the fixity at the base as shown in Eqn 42. Hence, the proportion of the of the tower-jacket mass lumped at the tip depends on the value of \( k_b \). Considering a fixed base with rotational stiffness approaching infinity \( \tau \to \infty \) and \( \lambda_i = 1.8751 \), \( M_{\text{eq}} \) can be computed as shown in Eq.54

\[ M_{\text{eq-TJ}} = \frac{3}{1.8751} \varepsilon \left( m_{\text{eq}} h_{\text{total}} \right) = 0.243 \left( m_{\text{eq}} h_{\text{total}} \right) \]  

Hence \( \varepsilon = 0.243 \)  

(Eq.54)

Using the methodology above, value of \( \varepsilon \) was plotted against the non-dimensional rotational stiffness of the foundation as shown in Figure 16. The values of \( \lambda_i \) were obtained from finding the roots of Eq.42 for varying values of \( k_b \). Judging from the figure, it is evident that for practical values of non-dimensional rotational stiffness the value of \( \varepsilon \) is close to 0.243, thus it is safely assumed that the lumped mass of the tower \( M_{\text{eq-TJ}} \) is always 0.243 of \( m_{\text{eq}} h_{\text{total}} \) regardless of the value of \( k_b \). Hence, the total lumped mass of the system (beam and RNA) can be simply calculated as \( M_{\text{eq}} = 0.243 m_{\text{eq}} h_{\text{total}} + M_{\text{RNA}} \)

![Figure 16: variation of \( \varepsilon \) with non-dimensional rotational stiffness](image)

**Flexibility parameter \( C_J \)**

Finally, some further algebraic manipulation is performed to facilitate the understanding of the parameters influencing the natural frequency of the SSI of the system. A flexibility parameter \( C_J \) is introduced such that \( f_0 = C_J x f_b \) where \( C_J \) is a function of \( \tau \) which depends on rotational stiffness \( k_b \) of the foundation as shown in Equations 47. Hence, the flexible natural frequency is calculated as shown in Eq. 55-57
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\[ f_0 = C_J \times f_{ib} \]  \hspace{1cm} (Eq. 55)

\[ C_J = \frac{\tau}{\sqrt{\tau + 3}} \]  \hspace{1cm} (Eq. 56)

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{\tau}{\tau + 3}} \sqrt{\frac{3EI_{TJ}}{(0.243m_{eqh_{total}} + M_{RNA})(h_{total})^3}} \]  \hspace{1cm} (Eq. 57)

2.7.3 Transition Piece (MDOF System)

The work presented in sections 2.7.1 and 2.7.2 may be extended to include the effect of the transition piece mass on the natural frequency. Loads from the wind turbine tower are transferred to the jacket through the use of a transition piece (TP). The TP in this case is composed of a large concrete block with a considerable mass, see Figure 17 for a schematic. The TP can also be represented by a lumped mass \((M_{TP})\) located at \(h_j\) and its addition to the analysis introduces an additional degree of freedom which depending on the mass of the concrete block and the dimensions of the jacket may have an effect on the natural frequency of the system. There are numerous methods to incorporate the mass of the TP in the analysis of the natural frequency. For instance, (Liu & Huang, 1988) and (Wang, 2012) presented solutions for the natural frequency parameters \(\lambda_1\) and \(\beta_1\) for beams supporting multiple lumped masses at different locations.
There are other simpler methods for obtaining the natural frequency of a multi-degree of freedom system such as the method of influence coefficients which is summarized in Eq. 58-62 where the relation between the displacements and forces acting at different positions of the system can be evaluated using these coefficients. A full derivation leading to Eq. 62 can be found in (Chandrasekaran, 2016). Other methods may also be used such as Dunkerley’s method and the Rayleigh-Ritz method.

\[
M_{\text{eq}} = 0.243 m_{\text{eq h total}} + M_{\text{RNA}}
\]

\[
\begin{bmatrix}
U_{\text{eq}} M_{\text{eq}} - & 1 \\
4\pi^2 f_0^2 & U_{\text{TP}} M_{\text{TP}}
\end{bmatrix} \begin{bmatrix}
U_T \\
U_{\text{TP}}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(Eq.58)

Hence the roots of Eqn 59 provides the natural frequency of the system

\[
16\pi^4 M_{\text{eq}} M_{\text{TP}} \left( U_{\text{eq}} U_{\text{TP}} - U_{\text{eq i}} U_{\text{TP i}} \right) f_0^4 - 4\pi^2 \left( M_{\text{eq}} U_{\text{eq}} + M_{\text{TP}} U_{\text{TP}} \right) f_0^2 + 1 = 0
\]

(Eq.59)

Where
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$U_{eq}$: The displacement at the tower tip due to a unit force at that location ($h_{\text{total}}$)

$U_{TP}$: The displacement at the location of the TP due to a unit force at that location ($h_j$)

$U_{eq1}$: The displacement at the tower tip due to a unit force at the location of the TP ($h_j$)

$U_{TP1}$: The displacement at the location of the TP due to a unit force at the tower tip ($h_{\text{total}}$)

And maybe computed using eqns. 60-62

$$U_{eq} = \frac{h^2_{\text{total}}}{k_R} + \frac{h^3_{\text{total}}}{3EI_{TJ}}$$  \hspace{1cm} (Eq.60)

$$U_{TP} = \frac{h^2_j}{k_R} + \frac{2h^3_j}{6EI_{TJ}}$$  \hspace{1cm} (Eq.61)

$$U_{eq1} = U_{TP1} = \left(\frac{h^2_j}{k_R}\right)\left(\frac{h_{\text{total}}}{h_j}\right) + \frac{h^2_j \left(3h_{\text{total}} - h_j\right)}{6EI_{TJ}}$$  \hspace{1cm} (Eq.62)

It is important to note that this method assumes a rigid connection between the tower and transition piece. However, in reality a form of connection is required between the two such as a grouted connection which will add flexibility to the system and reduce the natural frequency. For this reason, designers are encouraged to check this using advanced finite element methods and ensure that the connection is stiff enough that it does not greatly influence the fundamental period.

3.0 Example 1: 4 legged jacket on deep foundations and shallow foundations

For the purpose of demonstration of the applied methodology a symmetrical four-legged jacket supporting a NREL 5 MW reference offshore wind turbine in deep waters is considered, see Figure 18. Details about the turbine can be found in (Jonkman, et al., 2009). The jacket dimensions are taken from (Alati, et al., 2015) where industry-standard software BLADED is used to obtain the fixed base frequency and SSI frequency of the system. The necessary dimensions of the jacket are summarized in Table 1.

<table>
<thead>
<tr>
<th>Table 1: Summary of input parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jacket</strong></td>
</tr>
<tr>
<td>$h_i$ (m)</td>
</tr>
<tr>
<td>$L_{\text{bottom}}$ (m)</td>
</tr>
<tr>
<td>$L_{\text{top}}$ (m)</td>
</tr>
<tr>
<td>Area of jacket leg $A_c$ (m$^2$)</td>
</tr>
<tr>
<td>$m_i$ including diagonals (kg/m)</td>
</tr>
<tr>
<td><strong>Tower</strong></td>
</tr>
<tr>
<td>$h_T$</td>
</tr>
<tr>
<td>$D_{\text{bottom}}$ (m)</td>
</tr>
<tr>
<td>$D_{\text{top}}$ (m)</td>
</tr>
<tr>
<td>$m_T$ (kg/m)</td>
</tr>
<tr>
<td><strong>RNA</strong></td>
</tr>
<tr>
<td>$M_{RNA}$ (kg)</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Transition Piece</th>
<th>$M_{TP}$ (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>666000</td>
</tr>
</tbody>
</table>

Note: It is assumed that lateral translational restraints are at the nodes on connecting the legs to the vertical springs.

The example will be solved by considering two cases: (a) ignoring the accelerating lumped mass of the Transition Piece (TP); (b) considering the accelerating mass of the TP. This was carried out to allow for some comparisons and understanding of the system in hand.

**Step 1: Obtain the jacket stiffness:**
For vibration along the main axis

\[ EI_J = EI_{top} \cdot f(m) \]

\[ I_{top} = \frac{A_L L_{top}^2}{2} = \frac{(2 \times 0.1281) \times 9.5^2}{2} = 11.56m^4 \]

\[ m = \frac{L_{bottom}}{L_{top}} = \frac{12}{9.5} = 1.26 \]

\[ f(m) = \frac{1}{3} \cdot \frac{m(m-1)^3}{m^2 - 2m\ln(m) - 1} = \frac{1}{3} \cdot \frac{1.26 \times (1.26 - 1)^3}{1.26^2 - 2 \times 1.26 \times \ln(1.26) - 1} = 1.425 \]

\[ EI_J = 2.1 \times 10^{11} \times 11.56 \times 1.425 = 3.46 \times 10^{12} Nm^2 \]

**Step 2: Obtain the tower Stiffness:**

\[ D_t = \frac{D_{top} + D_{bottom}}{2} = \frac{4 + 5.6}{2} = 4.8m \]

\[ t_t = \frac{m_t}{\rho \pi h_t D_t} = \frac{261100}{7850 \times \pi \times 70 \times 4.8} = 31.5mm \]

\[ I_{top} = \frac{\pi D_{top}^2}{8} t_t = \frac{\pi}{8} \times 4^3 \times 0.0315 = 0.7732m^4 \]

\[ q = \frac{D_{bottom}}{D_{top}} = \frac{5.6}{4.0} = 1.4 \]

\[ f(q) = \frac{1}{3} \cdot \frac{2q^2 (q - 1)^3}{q^2 (2\lnq - 3) + 4q - 1} = \frac{1}{3} \cdot \frac{2 \times 1.4^2 \times (1.4 - 1)^3}{2 \ln(1.4) - 3 + 4 \times 1.4 - 1} = 2.145 \]

\[ EI_T = EI_{top} \cdot f(q) \]

\[ EI_T = 2.1 \times 10^{11} \times 0.7732 \times 2.145 = 3.48 \times 10^{11} Nm^2 \]

**Step 3: Obtain the Tower-Jacket stiffness:**

\[ \psi = \frac{h_t}{h_J} = \frac{70}{70} = 1.0 \]

\[ \chi = \frac{E_J I_J}{E_t I_t} = \frac{3.48 \times 10^{11}}{3.47 \times 10^{12}} = 0.10 \]

\[ EI_{TJ} = E_J I_J \left( \frac{1}{1 + (1 + \psi) \chi - \chi} \right)^3 \left( \frac{h_J + h_t}{h_t} \right)^3 = 3.48 \times 10^{11} \times \left( \frac{1}{1 + (1 + 1) \times 0.1 - 0.1} \right)^3 \left( \frac{70 + 70}{70} \right) = 1.635 \times 10^{12} Nm^2 \]
**Step 4: Obtain the equivalent distributed mass:**

$L = h_1 + h_2 = 70 + 70$

$$\frac{L}{\lambda_1} = 70 + 70$$

$$\lambda_1 = 1.8751$$

$$\beta_1 = \frac{\cos \lambda_1 + \cosh \lambda_1}{\sin \lambda_1 + \sinh \lambda_1} = \frac{\cos 1.8751 + \cosh 1.8751}{\sin 1.8751 + \sinh 1.8751} = 0.7341$$

$$\int \varphi_1^2 z = 0 \rightarrow \beta_1 = \frac{1 + \beta_1^2}{2\lambda_1^2} \sin \lambda_1 \cos \lambda_1 - \frac{1 + \beta_1^2}{2\lambda_1^2} \cosh \lambda_1 \sinh \lambda_1 = -0.7341$$

$$\int \varphi_1^2 z = 70 \rightarrow \beta_1 = \frac{1 + \beta_1^2}{2\lambda_1^2} \sin \lambda_1 \cos \lambda_1 - \frac{1 + \beta_1^2}{2\lambda_1^2} \cosh \lambda_1 \sinh \lambda_1 = 47.52$$

$$\int \varphi_1^2 z = 140 \rightarrow \beta_1 = \frac{1 + \beta_1^2}{2\lambda_1^2} \sin \lambda_1 \cos \lambda_1 - \frac{1 + \beta_1^2}{2\lambda_1^2} \cosh \lambda_1 \sinh \lambda_1 = 165.71$$

$$\varphi_1 \mid_{z=0} = 0$$

$$\varphi_1 \mid_{z=70} = \frac{1}{2(0.0134)} + \frac{(-0.7341)^2}{2(0.0134)} = 47.52$$

$$\varphi_1 \mid_{z=140} = \frac{1}{2(0.0134)} + \frac{(-0.7341)^2}{2(0.0134)} - \frac{1}{2(0.0134)} \cos (0.0134 \times 70) \cdot \sin (0.0134 \times 70) = 165.71$$

$$\varphi_1 \mid_{z=140} = \frac{1}{2(0.0134)} + \frac{(-0.7341)^2}{2(0.0134)} - \frac{1}{2(0.0134)} \cos (0.0134 \times 140) \cdot \sin (0.0134 \times 140) = 1151$$

$$\int \varphi_1^2 z = 70 \rightarrow \int \varphi_1^2 z = 165.71 - 47.52 = 118.19$$

$$\int \varphi_1^2 z = 140 \rightarrow \int \varphi_1^2 z = 1151 - 165.71 = 985.4$$
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\[ h_y + h_j \int_0 \phi_i^2 z = 140 \int_0 \phi_i^2 z = 1151 - 47.52 = 1103.5 \]

\[ m_j \int_0^{h_j} \phi_i^2 dz + m_T \int_{h_j}^{h_f + h_j} \phi_i^2 dz = \frac{8.15(118.19) + 3.73(985.4)}{1103.5} = 4.20 \text{tons/m} \]

**Step 5: Calculate the fixed base natural frequency:**

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{K_{eq}}{(M_{eq} + M_{RNA})}} = \frac{1}{2\pi} \sqrt{\frac{\text{EI}_{T,J}}{(0.243m_{eq} h_{total} + M_{RNA})(h_{total})^3 \left(\frac{1}{\tau} + \frac{1}{3}\right)}} \]

\[ \tau \to \infty \]

\[ f_{fb} = \frac{1}{2\pi} \sqrt{\frac{3\text{EI}_{T,J}}{(0.243m_{eq} h_{total} + M_{RNA})(h_{total})^3}} = \frac{1}{2\pi} \sqrt{\frac{3 \times 1.635 \times 10^{12}}{(0.243(4200 \times 140) + 350000)(140)^3}} = 0.303 \text{Hz} \]

which is representative of the natural frequency if the jacket is supported on deep embedded piles.

**Step 6: Calculate C\(_J\) for a small value of stiffness of the springs:**

For the sake of completeness of the example, it is assumed that the jacket is supported on 4 suction caissons (4m diameter x 4 meter deep) resting on soft soils. Table 2 shows the properties of the foundation and the ground properties and the definition of the terms are given in Appendix F.

<table>
<thead>
<tr>
<th>Table 2: Foundation Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundation depth D (m)</td>
</tr>
<tr>
<td>Foundation radius R (m)</td>
</tr>
<tr>
<td>Depth to bed rock H (m)</td>
</tr>
<tr>
<td>Soil Shear Modulus G(_s) (MPa)</td>
</tr>
<tr>
<td>Soil Poisson's ratio (\nu_s)</td>
</tr>
</tbody>
</table>

Using Equation F.1 provided in Appendix F, a preliminary estimate of the vertical stiffness can be obtained:

\[ k = \frac{4G_s R}{1-\nu_s} \left(1 + 1.28 \frac{R}{H}\right) \left(1 + \frac{D}{2R}\right) \times \left[1 + \left(\frac{0.28D}{R}\right) \frac{D}{H} \frac{D}{H} \frac{H}{H}\right] = 93551087 \frac{\text{N}}{\text{m}} \]

\[ k_{v1} = 93551087 + 93551087 = 187102174 \frac{\text{N}}{\text{m}} \]

\[ k_{v2} = 93551087 + 93551087 = 187102174 \frac{\text{N}}{\text{m}} \]

\[ \alpha = 1 \]
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\[
k_R = k_L J^2 \left[ \frac{\alpha}{1 + \alpha} \right] = 187102174 \times 12^2 \times \left[ \frac{1}{1 + 1} \right] = 1.347 \times 10^{10} \text{Nm}
\]

\[
\tau = \frac{k_R h_{total}}{EIT_{-J}} = \frac{1.347 \times 10^{10} \times 140}{1.635 \times 10^{12}} = 1.153
\]

\[
C_j = \sqrt{\frac{\tau}{\tau + 3}} = \sqrt{\frac{1.153}{1.153 + 3}} = 0.53
\]

\[
f_0 = C_j \times f_{fb} = 0.53 \times 0.303 = 0.16 \text{Hz}
\]

As this is a symmetrical arrangement, the vibration in the diagonal plane will result in the same natural frequency. However, in reality due to ground variability, there may be two closely spaced frequencies.

**Additional step: Including lumped mass of the transition piece**

Using the influence coefficient method shown in section 2.7.3:

\[
M_{eq} = 0.243 m_{eq} h_{total} + M_{RNA} = 0.243 (4200 \times 140) + 350000 = 492884 \text{kg}
\]

\[
U_{eq} = \frac{h_{total}^2}{k_R} + \frac{h_{total}^2}{3EI_{T-J}} = \frac{140^2}{1.347 \times 10^{10}} + \frac{140^3}{3 \times 1.64 \times 10^{12}} = 2 \times 10^{-6} \text{m}
\]

\[
U_{TP} = \frac{h_j^2}{k_R} + \frac{2h_j^2}{6EI_{T-J}} = \frac{70^2}{1.347 \times 10^{10}} + \frac{2 \times 70^3}{6 \times 1.64 \times 10^{12}} = 4.33 \times 10^{-7} \text{m}
\]

\[
U_{eq} = U_{TP} = \left( \frac{h_j^2}{k_R} \right) + \frac{h_j^2 (3h_{total} - h_j)}{6EI_{T-J}} = \left( \frac{70^2}{1.347 \times 10^{10}} \right) \left( \frac{140}{70} \right) + \frac{70^2 (3 \times 140 - 70)}{6 \times 1.64 \times 10^{12}} = 9.0227 \times 10^{-7} \text{m}
\]

\[
16\pi^4 M_T M_{TP} \left( U_T U_{TP} - U_{TP} U_T \right) f_0^4 = 4\pi^2 \left( M_T U_T + M_{TP} U_{TP} \right) f_0^2 + 1 = 0
\]

\[26.55 f_0^4 - 50.31 f_0^2 + 1 = 0\]

\[a = f_0^2\]

\[26.55 a^2 - 50.31 a + 1 = 0\]

\[a = 0.020\]

\[f_0 = \sqrt{0.020} = 0.14 \text{Hz}\]

The same process was repeated with a fixed base \((k_k \to \infty)\) and the following was obtained

\[f_{fb} = 0.292 \text{Hz}\]

It is interesting to note that the mass of the transition piece did not greatly influence the natural frequency of the system. Therefore, for preliminary estimates it may be assumed that the transition piece mass need not be accurately estimated at an early stage of the design. However, if the transition
piece mass is considerably high (due to larger turbine installations for example or to resist a certain loading condition) one must always check the contribution of the transition piece to the natural frequency. As shown in Eq 59, a higher transition piece mass would mean that it will have a higher contribution to the quadratic equation which ultimately might alter the calculated frequency.

### 3.1 Comparison with Finite Element Analysis

A 3D finite element analysis was performed using the software package SAP2000 where a modal analysis was run to study the natural frequency of the undamped free vibration of the system. The jacket was constructed using beam elements with moment releases at the ends. The tower consisted of a non-prismatic section with a linear variation of the moment of inertia. As for the RNA it was modelled through a lumped mass at the tower top. The accelerating masses of the tower and the jacket are computed automatically by the software. The material model used for the jacket was linear elastic with a Young’s modulus for steel of 210 GPa and density of 7850 kg/m$^3$ and lateral restraints were applied at foundation level. The model was then meshed to 1200 beam elements and as the material model is linear elastic the analysis time required by the software was approximately half a minute. Typical deflected mode shape from the software output is shown in Figure 19 and 20. Figure 19 shows a sway-bending mode of vibration due to a fixed base at the supports, whilst Figure 20 shows rocking mode of vibration due to flexible linear elastic spring supports. It may be noted that in Figure 20 the natural frequency in the x-x’ (i.e. diagonal plane) were very close due to the fact that this is a symmetrical configuration. Table 3 shows a comparison between the results obtained using the proposed method and SAP2000

<table>
<thead>
<tr>
<th>Case:</th>
<th>SAP 2000 (Hz)</th>
<th>Proposed method (Hz)</th>
<th>(Alati, et al., 2015) BLADED (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Base (No Transition Piece)</td>
<td>0.312</td>
<td>0.303</td>
<td>-</td>
</tr>
<tr>
<td>Flexible Foundation (No transition piece)</td>
<td>0.156</td>
<td>0.16</td>
<td>-</td>
</tr>
<tr>
<td>Fixed Base (Including transition piece)</td>
<td>0.305</td>
<td>0.29</td>
<td>0.314-0.317</td>
</tr>
<tr>
<td>Flexible Foundation (Including transition piece)</td>
<td>0.141</td>
<td>0.14</td>
<td>-</td>
</tr>
</tbody>
</table>

As shown in table 3 the results obtained from SAP2000 match well with the proposed method and this validates the assumptions made about the mass distribution, stiffness idealizations, and equivalent rotational support. Similarly, the method also matches well with the fixed base natural frequency obtained through BLADED software provided in (Atali, et al., 2015) with slight discrepancies. These differences possibly arise firstly due to the numerical error of each software. Secondly, the RNA in BLADED is modelled with a higher geometrical and structural accuracy when compared to the lumped mass approach adopted in the proposed method and SAP2000. Thirdly, the foundation supports in BLADED are modelled using distributed springs along the length of the pile rather than lumped vertical springs. Finally, BLADED includes the additional stiffness of the jacket bracing.
Figure 19: Sway-bending mode of vibration due to a fixed base
Figure 20: Rocking mode of vibration
3.2: Solved Example 2: Asymmetric Triangle Case

The above methodology was repeated for an asymmetric triangle case, the value of \( L \) at the base is assumed to be 12m i.e. \( L=12m \) in Figure 21. Similarly, the same wind turbine is installed on this seabed frame. As this is an asymmetric configuration (Figure 21), the vibrations about the \( x-x' \) and \( y-y' \) planes are expected to be different as shown mathematically through Equations 63 to 68.

![Asymmetric Triangle Case](image)

Figure 21: (a) Plan view of asymmetric foundation base (b) Finite Element Model of the jacket

The vibration about the \( x-x' \) plane is studied first. This is a system which can be converted into 2D such that the jacket base is supported on two linear springs, where the stiffness of one spring is \( k \) and the stiffness of the other spring is \( 2k \). Hence using Eqs 27 and 28, the equivalent rotational stiffness of the foundation can be found as shown in Eq.63 and 64

\[
\mu = \frac{\alpha}{1+\alpha} = \frac{2}{1+2} = \frac{2}{3} \tag{Eq.63}
\]

\[
k_r = kL^2 \left[ A_c \left( \frac{\sqrt{2L}}{1+\alpha} \right) \right] = k \left( \frac{\sqrt{2L}}{2} \right)^2 \times \frac{2}{3} = \frac{kL^2}{3} \tag{Eq.64}
\]

Also using Eq.1, the stiffness of the jacket can be found as shown by Eq 65

\[
I_j = 2A_c \left( \frac{\sqrt{2L}}{6} \right)^2 + A_c \left( \frac{\sqrt{2L}}{3} \right)^2 = \frac{A_c L^2}{3} \tag{Eq.65}
\]

Similarly, the vibration about the \( y-y' \) plane can also be studied. In this case the jacket is assumed to be supported on 3 springs where Eqs. 29 and 30 may be used to obtain the rotational stiffness of the foundation. This is shown in Eq.66
\[ \mu = \frac{\alpha + 0.5\nu}{1 + \alpha + \nu} = \frac{1 + 0.5}{1 + 1 + 1} = 0.5 \quad \text{(Eq. 66)} \]

\[ k_R = k_L \left[ \mu^2 + \alpha (1 - \mu)^2 + \nu (\zeta - \mu)^2 \right] = k \left( \sqrt{2L} \right)^2 \left[ 0.5^2 + (1 - 0.5)^2 + (0.5 - 0.5)^2 \right] = kL^2 \quad \text{(Eq. 67)} \]

Finally, the stiffness of the jacket in the y-y' direction is shown in Eq. 68

\[ I_J = A_c \left( \frac{\sqrt{2L}}{2} \right)^2 + A_c \left( \frac{\sqrt{2L}}{2} \right)^2 = A_c L^2 \quad \text{(Eq. 68)} \]

Hence, comparing Eqs. 64 with 67 and Eq. 65 with 68, it is clear that the rotational stiffness of the foundation and jacket stiffness in the x-x' direction is different than that of y-y' direction. Therefore, 2 very closely spaced natural periods are expected and therefore two spectral peaks are expected. Research presented in (Bhattacharya, et al., 2011), (Bhattacharya, et al., 2013) showed 2 peaks in scaled tests on an asymmetric triangular sea bed frame. Table 4 summarizes the obtained results for both the proposed method and the Finite Element Analysis carried out in SAP 2000.

<table>
<thead>
<tr>
<th>Case:</th>
<th>SAP 2000 (Hz)</th>
<th>Proposed method (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Base (x-x' direction)</td>
<td>0.21</td>
<td>0.19</td>
</tr>
<tr>
<td>Fixed base (y-y' direction)</td>
<td>0.28</td>
<td>0.25</td>
</tr>
<tr>
<td>Fixed foundation (x-x' direction)</td>
<td>0.075</td>
<td>0.09</td>
</tr>
<tr>
<td>Flexible foundation (y-y' direction)</td>
<td>0.164</td>
<td>0.15</td>
</tr>
</tbody>
</table>

As shown in Table 4, the proposed method matches well with the FEM results with higher margin of errors when compared to a symmetric foundation. This is because in this case of an asymmetric configuration, the centre of mass does not match with the centre of stiffness of the foundation which results in slightly high margin of error. However, for the sake of preliminary sizing of the jacket members the method serves well.

3.3 Solved Example 3: Square Jacket on piles

This section takes an example of a pile supported jacket supporting a 4MW turbine proposed to be used in Fujian Pingtan Dalian island in the Chinese sea. The details of the turbine and the tower is given in Table 5. The jacket consists of a square base of 26m x 26m at the base and it tapers to 14m x 14m at the top. Each of the legs of the jacket is supported on 3m diameter piles 50m long. Table 6 and Figure 22 provides details of the jacket. This structure is analysed using SAP2000 software and also by using the proposed method. The same modelling assumptions shown in section 3.1 were applied with the exception that this model consisted of 900 frame elements. The results are shown in Table 7 and they are comparable.

<table>
<thead>
<tr>
<th>Case:</th>
<th>Tower Bottom Diameter (m)</th>
<th>Tower Top Diameter (m)</th>
<th>Tower Thickness (mm)</th>
<th>Tower Height (m)</th>
<th>Tower Mass (Tons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tower Bottom Diameter (m)</td>
<td>5.042</td>
<td>3.083</td>
<td>24-30</td>
<td>72</td>
<td>176</td>
</tr>
</tbody>
</table>

Table 5: Turbine and tower details
Table 6: Jacket configuration

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{\text{bottom}}$ (m)</td>
<td>26</td>
</tr>
<tr>
<td>$L_{\text{top}}$ (m)</td>
<td>14</td>
</tr>
<tr>
<td>Jacket Height (m)</td>
<td>40</td>
</tr>
<tr>
<td>Jacket Leg Section</td>
<td>Outer diameter 1600 mm, 32 mm wall thickness</td>
</tr>
<tr>
<td>Jacket Brace Section</td>
<td>Outer diameter 900 mm, 25 mm wall thickness (3 layers)</td>
</tr>
</tbody>
</table>

Figure 22: Schematic of example 3
4.0 Conclusion:

Obtaining natural frequency of a wind turbine system is one of the important design consideration. In this paper, a simple method has been presented to estimate the first natural frequency of a jacket supported offshore wind turbine considering the flexibility provided by the foundation. The input required are geometry of the tower and jacket, sizes of the legs of the jacket, RNA mass and vertical stiffness of the foundation. The methodology is based on converting a 3D vibration problem into a 2D problem and determining the equivalent bending stiffness of the superstructure. The methodology is validated using Finite Element solutions. Example problems are considered to show the applicability of the method.
Appendices:
Six appendices (Appendix A to F) are presented to explain the method.

Appendix A: Equivalent bending stiffness of a jacket

\[
I_j = \frac{A_c L^2}{2}
\]

\[
L_{\text{bottom}} = m \cdot L_{\text{top}}
\]

\[
L = \frac{L_{\text{top}}}{h_j} [h_j + (m-1)z]
\]

\[
I_j = \frac{A_c}{2} \left[ \left( \frac{L_{\text{top}}}{h_j} \right) \cdot (h_j + (m-1)z) \right]^2
\]

\[
I_j = I_{\text{top}} \left[ \left( \frac{1}{h_j} \right) \cdot (h_j + (m-1)z) \right]^2
\]

\[
I_j = I_{\text{top}} \left[ 1 + 2 \frac{(m-1)}{h_j} z + \left( \frac{m-1}{h_j} \right)^2 \right]
\]

\[
I_j = I_{\text{top}} \left[ 1 + \frac{(m-1)}{h_j} z \right]^2
\]

Assume \( a = \frac{m-1}{h_j} \)

\[
I_j = I_{\text{top}} (1 + az)^2 \tag{Equation A.1}
\]

The moment-curvature relation for section distant \( z \) from, the free end is given by

\[
EI \frac{d^2y}{dz^2} = Pz
\]

Substituting equation A.1

\[
EI_{\text{top}} (1 + az)^2 \frac{\partial^2 y}{\partial z^2} = Pz \quad \text{Which can be re-arranged to:}
\]

\[
\frac{d^2y}{dz^2} = \frac{Pz}{EI_{\text{top}} (1 + az)^2} \tag{Equation A.2}
\]

By integration the slope equation can be obtained:
\[
\frac{dy}{dz} = \frac{P}{EI_{\text{top}}a^2} \left( \ln|1 + az| + \frac{1}{1 + az} \right) + C_1 \]  
(Equation A.3)

\[
y = \frac{P}{EI_{\text{top}}a^3} \left[ (2 + az) \ln|1 + az| - az - 1 \right] + C_1z + C_2
\]  
(Equation A.4)

The boundary conditions for this problem are as follows:

At \( z = h_j \), \( \frac{dy}{dz} = 0 \)

At \( z = h_f \), \( y = 0 \)

This permutes the computation of the constants \( C_1 \) and \( C_2 \)

\[
C_1 = -\frac{P}{EI_{\text{top}}a^2} \left( \ln|1 + ah_j| + \frac{1}{1 + ah_j} \right)
\]  
(Equation A.5)

\[
C_2 = -\frac{P}{EI_{\text{top}}a^3} \left[ (2 + ah_j) \ln|1 + ah_j| - ah_j - 1 \right] + \frac{Ph_j}{EI_{\text{top}}a^2} \left( \ln|1 + ah_j| + \frac{1}{1 + ah_j} \right)
\]  
(Equation A.6)

Hence, the deflection at the free end is obtained by substituting \( z = 0 \) in Equation A.4

\[
y_{\text{tip}} = -\frac{P}{EI_{\text{top}}a^3} \left[ (2 + ah_j) \ln|1 + ah_j| - ah_j - 1 \right] + \frac{Ph_j}{EI_{\text{top}}a^2} \left( \ln|1 + ah_j| + \frac{1}{1 + ah_j} \right) - \frac{P}{EI_{\text{top}}a^3}
\]  
(Equation A.7)

By substituting equation A.1 in A.7

\[
y_{\text{tip}} = \frac{Ph_j^3}{EI_{\text{top}}} \left[ \frac{(1 + m \ln|m| - m)}{(m-1)^2} \right] + \frac{Ph_j^3}{EI_{\text{top}}} \left[ \frac{\ln|m| + \frac{1}{m}}{(m-1)^2} \right] - \frac{Ph_j^3}{EI_{\text{top}}(m-1)^3}
\]

Further algebraic simplification leads to

\[
y_{\text{tip}} = -\frac{Ph_j^3}{EI_{\text{top}}} \left[ \frac{m^2 - 2m \ln(m) - 1}{m(m-1)^2} \right]
\]  
(Equation A.8)

Which means for a unit displacement the equivalent bending stiffness of a tapered jacket \( EI_j \) maybe expressed as

\[
EI_j = EI_{\text{top}} \cdot f(m)
\]

Where \( f(m) = \frac{1}{3} \cdot \frac{m(m-1)^3}{m^2 - 2m \ln(m) - 1} \)  
(Equation A.9)

A quick check can be for a uniform jacket with \( L_{\text{top}} = L_{\text{bottom}} \)
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\[
\lim_{m \to 3} \left[ \frac{m(m-1)^3}{m^2 - 2mln(m)-1} \right] = 3 \quad \text{Which means } \quad EI_j = EI_{\text{top}}
\]

Appendix B: Equivalent bending stiffness of a tower

As offshore wind turbine towers have a varying tower diameter with a changing thickness, some expressions have been provided in literature to compute the equivalent tower thickness of a cross section. Following (Arany, et al., 2016) the average thickness of an OWT tower may be expressed as

\[
t_T = \frac{m_T}{\rho h_T D_T \pi}
\]

(Equation B.1)

Where \( D_T \) is the average tower diameter

\[
D_T = \frac{D_{\text{top}} + D_{\text{bottom}}}{2}
\]

(Equation B.2)

In a similar manner to the methodology presented in Appendix A the equivalent bending stiffness of a tapered tower can be calculated as follows

\[
I = \frac{\pi}{8} D^3 t_T
\]

\[
D_{\text{bottom}} = q \cdot D_{\text{top}}
\]

\[
D = \frac{D_{\text{top}}}{h_T} \left[ h_T + (q - 1)z \right]
\]

\[
I_T = \frac{\pi t_T}{8} \left[ \left( \frac{D_{\text{top}}}{h_T} \right) \cdot (h_T + (q - 1)z) \right]^3
\]

Similarly

\[
a = \frac{q - 1}{h_T}
\]

\[
I_T = I_{\text{top}} (1 + az)^3 \]

, using similar steps as equations A.2 to A.7, the following formulation is obtained for the equivalent bending stiffness of a tapered tower

\[
EI_T = EI_{\text{top}} \cdot f(q)
\]

Where \( f(q) = \frac{1}{3} \cdot \frac{2q^2(q - 1)^3}{q^2(2lnq - 3) + 4q - 1} \)

(Equation B.3)
Appendix C: Equivalent bending stiffness of a Tower-Jacket system

A single equivalent section may for the tower-jacket system may be obtained using Castigliano’s theorem of a linear elastic single degree of freedom structure can be expressed as:

\[ q = \frac{\partial U}{\partial Q} \]  
(Equation C.1)

Where \( U \) is the strain energy, \( Q \) is the force, and \( q \) is the generalised displacement. For the problem shown in figure 10,

\[ y(0) = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \int_{0}^{h_j} \left[ M(z)^2 \right] dz = \frac{\partial}{\partial P} \int_{0}^{h_j} \frac{F^2 z^2}{2EI} dz + \frac{\partial}{\partial P} \int_{h_j}^{h_j+h_j} \frac{F^2 z^2}{2EI} dz \]

\[ y(0) = \left[ \frac{Pz^3}{3EI} \right]_0^{h_j} + \left[ \frac{Pz^3}{3EI} \right]_{h_j}^{h_j+h_j} \]

\[ y(0) = \frac{Ph_j^3}{3EI} + \frac{P(h_j + h_j)^3}{3EI} - \frac{Ph_j^3}{3EI} \]  
(Equation C.2)

For a unit force \( P=1 \), the stiffness of the tower \( k_{T,J} \) tower-jacket system can be expressed as

\[ k_{T,J} = \frac{1}{y(0)} = \frac{1}{3EI} \left( \frac{h_j^3}{h_j} + \frac{(h_j + h_j)^3}{3EI} - \frac{h_j^3}{3EI} \right) \]  
(Equation C.3)

Say \( \psi = \frac{h_j}{h_j} \)  
(Equation C.4)

and \( \chi = \frac{E_JI_T}{E_JI_J} \)  
(Equation C.5)

Using algebraic simplification and substituting equations C.4 and C.5 into C.3

\[ k_{T,J} = \frac{3EI_J}{h_j} \left( \frac{1}{1 + (1 + \psi)^3} \right) \]  
(Equation C.7)

Also \( k_{T,J} = \frac{3EI_JI_{T,J}}{(h_j + h_j)^3} \)  
(Equation C.8)

Assuming both the jacket and the tower are composed of the same material and using equations C.7 and C.8, the equivalent bending stiffness of the tower-jacket system is

\[ EI_{T,J} = E_JI_T \left( \frac{1}{1 + (1 + \psi)^3} \right) \left( \frac{h_j + h_j}{h_j} \right)^3 \]  
(Equation C.9)
Appendix D: Equivalent rotational stiffness of the foundation

The vertical equilibrium of the system

\[ k_v (\mu L) \theta = \alpha k_v L(1 - \mu) \theta \]  
(Equation D.1)

Thus, \( \mu = \frac{\alpha}{1 + \alpha} \)  
(Equation D.2)

Using Castigliano’s theorem the equation of strain energy is as follows:

\[ U = -M\theta + \frac{1}{2} k_v[(\mu L)\theta]^2 + \frac{1}{2} \alpha k_v[L(1 - \mu)\theta]^2 \]  
(Equation D.3)

Differentiating with respect to \( \theta \) to obtain the maximum potential energy:

\[ \frac{\partial U}{\partial \theta} = -M + k_v \theta[(\mu L)]^2 + \alpha \theta[L(1 - \mu)]^2 = 0 \]  
(Equation D.4)

Further algebraic simplification leads to:

\[ \frac{M}{\theta} = k_v[(\mu L)]^2 + \alpha k_v[L(1 - \mu)]^2 \]  
(Equation D.5)

Where \( \frac{M}{\theta} \) is representative of the rotational stiffness of the system such that:

\[ k_R = \frac{M}{\theta} \]  
(Equation D.6)

Substituting equation D.6 into D.5

\[ k_R = k_v[(\mu L)]^2 + \alpha k_v[L(1 - \mu)]^2 \]  
(Equation D.7)

Further simplification of equation D.7:

\[ k_R = k_v L^2 [\mu^2 + \alpha (1 - \mu)^2] \]  
(Equation A5.8)

Substituting equation D.2 into D.8 leads to:

\[ k_R = k_v L^2 \left[ \frac{\alpha^2}{(1 + \alpha)^2} + \alpha - 2\alpha \left( \frac{\alpha}{1 + \alpha} \right) + \alpha - \frac{\alpha^2}{(1 + \alpha)^2} \right] \]  
(Equation D.9)

\[ k_R = k_v L^2 \left[ \frac{\alpha}{1 + \alpha} \right] \]  
(Equation D.10)

As per equation D.10, it is evident that the equivalent rotational stiffness of foundation on 2 supports is reliant on the spacing between the foundations. A higher spacing between the vertical supports enhances the rotational stiffness and avoids rocking vibrations discussed in

Similarly, as shown in section 2.2.1, a 2D problem may have 3 springs as shown in figure 14, hence the same methodology may be applied to accommodate vibration in the orthogonal direction.
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\[ k_v(\mu L)\theta = \alpha k_v L(1-\mu)\theta + \nu k_v L(\kappa - \mu)\theta \]  
(Equation D.11)

Equation D.11 may be simplified to

\[ \mu = \frac{\alpha + 0.5\nu}{1 + \alpha + \nu} \]  
(Equation D.12)

Using Castigliano’s theorem the equation of strain energy is as follows:

\[ U = -M\theta + \frac{1}{2} k_v \left( (\mu L)\theta \right)^2 + \frac{1}{2} \alpha k_v \left[ L(1-\mu)\theta \right]^2 + \frac{1}{2} \nu k_v \left[ L(\zeta - \mu)\theta \right]^2 \]  
(Equation D.13)

Differentiating with respect to \( \Theta \) to obtain the maximum potential energy and solving for \( K_R \):

\[ k_R = k_v L^2 \left[ \mu^2 + \alpha (1-\mu)^2 + \nu (\zeta - \mu)^2 \right] \]  
(Equation D.14)
Appendix E: Equivalent SDOF system

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{K_{eq}}{M_{eq}}} \]  
(Equation E.1)

\[ h_{total} = h_T + h_J \]  
(Equation E.2)

**Equivalent spring stiffness**

\[ y(0) = \frac{1}{k_R} P(h_{total})^2 + \frac{P(h_{total})^3}{3E_I T_J} \]  
(Equation E.3)

\[ \tau = \frac{k_R h_{total}}{E_I T_J} \]  
(Equation E.4)

\[ y(0) = \frac{P(h_{total})^3}{\tau \cdot E_I T_J} + \frac{P(h_{total})^3}{3E_I T_J} \]  
(Equation E.5)

\[ y(0) = \frac{P(h_{total})^3}{\tau \cdot E_I T_J} + \frac{P(h_{total})^3}{3E_I T_J} \]  
(Equation E.6)

\[ y(0) = \frac{P(h_{total})^3}{\tau \cdot E_I T_J} + \frac{P(h_{total})^3}{3E_I T_J} \]  
(Equation E.7)

\[ y(0) = \frac{P(h_{total})^3}{\tau \cdot E_I T_J} + \frac{P(h_{total})^3}{3E_I T_J} \]  
(Equation E.8)

\[ K_{eq} = \frac{1}{y(0)} = \frac{E_I T_J}{(h_{total})^3 \left( \frac{1}{\tau} + \frac{1}{3} \right)} \]  
(Equation E.9)
Appendix F: Guidance on estimating the vertical stiffness of foundations

Rigid Circular Embedded Footings:
The (DNV, 2002) provides guidance for rigid embedded shallow foundations over a bedrock layer and may be used as a preliminary estimate for suction caissons

\[ k_v = \frac{4G_R R}{1-\nu_s} \left( 1 + \frac{1.28 R}{H} \right) \left( 1 + \frac{D}{2R} \right) \times \left[ 1 + \left( \frac{0.85 - \frac{0.28 D}{R}}{1-\frac{D}{H}} \right) \frac{D}{H} \right] \]  
(Equation F.1)

\[ \frac{D}{R} < 2 \]
\[ \frac{D}{H} < \frac{1}{2} \]

Figure F1: Figure defining the terms in Equation F1

Vertical stiffness of piles
(Fleming, et al., 1992) suggested the following for embedded piles:

Shaft friction only**

\[ k_v = \frac{2\pi L_p G_s}{\zeta} \]  
(Equation F.2)

\( \zeta \) is between 3 and 5

LRFD guidelines for seismic design of bridge propose the following relation for vertical stiffness (Shama & El Naggar, 2015)

\[ k_v = 1.25 \frac{E_p A}{L_p} \]  
(Equation F.3)

In practice, t-z type of analysis or calibrated FEA (Finite Element Analysis) can be carried out to obtain the axial stiffness of the piles.
Lateral stiffness $k_L$ of piles

Following (Shadlou & Bhattacharya, 2016), the lateral stiffness of deep foundations can be computed as follows:

<table>
<thead>
<tr>
<th>Ground Profile/Pile Type</th>
<th>Rigid pile $k_L$</th>
<th>Flexible pile</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$DE_{50} f(\nu_s)$</td>
<td>$f_{\nu_s} E_{50} D_p$</td>
</tr>
<tr>
<td>Homogeneous</td>
<td>$3.2 \left( \frac{L}{D} \right)^{0.62}$</td>
<td>$1.45 \left( \frac{E_p}{E_{50}} \right)^{0.186} f_{\nu_s} E_{50} D_p$</td>
</tr>
<tr>
<td>Parabolic</td>
<td>$2.65 \left( \frac{L}{D} \right)^{1.07}$</td>
<td>$1.015 \left( \frac{E_p}{E_{50}} \right)^{0.27} f_{\nu_s} E_{50} D_p$</td>
</tr>
<tr>
<td>Linear</td>
<td>$2.35 \left( \frac{L}{D} \right)^{1.53}$</td>
<td>$0.79 \left( \frac{E_p}{E_{50}} \right)^{0.34} f_{\nu_s} E_{50} D_p$</td>
</tr>
</tbody>
</table>

Where

$L$: Pile Embedded length
$D$: Pile Diameter
$E_{50}$: Soil Young’s modulus at 1 diameter depth
$E_p$: Equivalent Young’s Modulus of the pile
$\nu_s$: Soil Poisson’s ratio

$f(\nu_s) = 1 + |\nu_s - 0.25|$

The readers are referred to the papers by Arany et al (2017) for a complete list of other types of formulations. Eurocode 8 (Part 5) also provides stiffness of flexible piles. Expert judgement is required for choosing the formulas. Conversely, one may carry out Beam on non-linear Winkler foundation model using state of the art $p$-$y$ curves to obtain the pile head stiffness values. The method to convert $p$-$y$ analysis to the pile head stiffness values is provided in Jalbi et al (2017).

Homogeneous soils are soils which have a constant stiffness with depth such as over-consolidated clays. On the other hand, a linear profile is typical for normally consolidated clays (or “Gibson Soil”) and parabolic behaviour can be used for sandy soils.

Lateral stiffness $k_L$ of suction caissons

Similarly, following (Jalbi, et al., 2018) the lateral stiffness of rigid suction caissons may be estimated using the following:

<table>
<thead>
<tr>
<th>Ground Profile/Pile Type</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$DE_{50} f(\nu_s)$</td>
</tr>
<tr>
<td>Homogeneous</td>
<td>$2.91 \left( \frac{L}{D} \right)^{0.56}$</td>
</tr>
<tr>
<td>Parablic</td>
<td>$2.7 \left( \frac{L}{D} \right)^{0.96}$</td>
</tr>
<tr>
<td>Linear</td>
<td>$2.53 \left( \frac{L}{D} \right)^{1.33}$</td>
</tr>
</tbody>
</table>
\[ f(\nu_s) = 1.1 \times \left( 0.096 \left( \frac{L}{D} \right) + 0.6 \right) \nu_s^2 - 0.7\nu_s + 1.06 \]
To appear in the Journal of *Soil Dynamics and Earthquake Engineering*

**References**


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