EFFICIENT MATRIX RANDOMISATION METHODOLOGY FOR REDUCED SPACECRAFT MODELS IN STOCHASTIC FEM-BEM VIBROACOUSTIC PROBLEMS

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ABSTRACT

In this paper, the mathematical framework for a computationally efficient stochastic finite element method (FEM) is outlined. It is devised for a range of applications in structural dynamics, where uncertainties need to be reliably dealt with in the context of reduced model formulations. It allows random mass and stiffness matrices to be robustly generated at the subsystem level in component mode synthesis (CMS) applications. The technique is validated for the particularly challenging case of mid-frequency FEM-FEM vibroacoustic analysis of a spacecraft structure. Results are compared against both test data and full parametric Monte-Carlo simulation. Finally, the method's applicability to coupled vibroacoustic problems utilising hierarchical matrix boundary element method (BEM) acoustic formulations is evaluated.

1. INTRODUCTION

The accurate and robust numerical representation of the dynamics of complex structures in the mid-frequency range has traditionally been a challenging discipline. This frequency band is characterised by the need for fine domain discretisation when classic element-based techniques are used, while statistical methods may not be fully applicable. Accounting for the inherent model uncertainties, such as structural parameters, or accuracy of the numerical representation, calls for some form of stochastic formulation. This commonly translates into solving multiple instances of a problem, each having the same complexity as the original one. Vibroacoustic analysis, in particular, further aggravates the problem, due to the additional issue of modelling the acoustic domain and solving the coupled fluid-structure interaction.

In practice, structural FEM representations are often sufficiently detailed to yield model sizes reaching millions of degrees of freedom (DOFs). Unsurprisingly, a variety of methods have been developed, such that the non-deterministic behaviour can be quantified and analysed within reasonable timeframes. The reader is referred to [1, 2, 3, 4] for some of the more contemporary works on the topic, all using reduced-order models to achieve good efficiency.

Component mode synthesis is a widely used tool for producing reduced dynamic models of structures, that it is naturally suitable for treating uncertainties at the component level [5]. In CMS, substructures are reduced separately by suitable projecting physical to modal coordinates via suitably chosen basis functions. Some interface DOFs are retained for subsequent reassembly of the original full model into a much more compact version, yielding a partitioning of the global mass, stiffness and damping matrices into two types of blocks, containing the component modal representation and the interface, respectively. Arguably, the most widespread CMS approach is the Craig-Bampton (CB) method [6], along with its recent enhanced variants [7, 8].

In this article, a decomposition-based stochastic method that defines the random mass and stiffness matrices by exploiting the particular block structure of the global CMS matrices is presented. Its development draws inspiration from the works of Remedia et al. [9, 10] and also Shorter and Mace [11], which utilise perturbation of substructures’ natural frequencies to obtain the global random matrices. A validation example for the method is subsequently provided, comparing vibroacoustic simulation FEM-FEM results with test data for the NovaSAR spacecraft, designed and built by Surrey Satellite Technology Limited (SSTL).

In the field of vibroacoustic analysis, the advent fast boundary element techniques for the solution of wave scattering and radiation problems has made it feasible to carry out coupled FEM-BEM simulations for highly complex structures. Due to some inherent advantages of BEM, such as handling of infinite domains and the fact no spatial discretisation of the fluid domain is needed, its investigation for use in a wider range of practical problems is justified. Therefore, at the end of this paper, the potential use of the proposed technique as a part of an efficient stochastic FEM-BEM tool is laid out. The intrinsically high construction and storage requirements of the acoustic matrices is mitigated by employing hierarchical matrices [12, 13] (referred to as $\mathcal{H}$-matrices) to compress the discrete representations of the boundary integrals.

2. THE STOCHASTIC FEM METHOD

2.1 Algebraic formulation

Firstly, let us define the following notation. Conjugate transpose and pseudo-inverse, respectively, are denoted $A^\dagger, A^\ast$. For compound operations, shorthand versions are used, i.e. $A^{-1}$ is the same as $(A^\ast)^{-1}$. In the context of singular value decompositions (SVDs), $\sigma_i(A)$ is the $i$-th singular value of $A$, with the standard ordering
\(\sigma_1 \geq \cdots \geq \sigma_n\). Similarly, \(\lambda(K, M)\) are the generalised eigenvalues of the pencil \((K, M)\), ordered in an ascending manner. \(\tilde{A}\) is used for random variables, as in \(\tilde{x}\). Now, consider the standard generalised eigenvalue problem (GEP):

\[
(K - \lambda_i M)\phi_i = 0, \quad i = \{1, \ldots, n\}
\]

(1)

where \(\phi_i\) is the \(i\)-th structural mode, and \(K, M\) are the discrete stiffness and mass, typically both real in FEM. It is possible to show that unless \(K, M \succeq 0\), i.e. they are at least positive semi-definite, the GEP has negative eigenvalues. In practical terms, this gives rise to complex natural frequencies of the structure, since they are given by \(\omega_i^2 = \lambda_i(K, M)\). Correspondingly, \(\phi_i \in \mathbb{R}\) is no longer true. To avoid this, for any realisation of the stochastic matrices \(\tilde{K}, \tilde{M}\) that we aim to construct, the positive-(semi-)definiteness of the original matrices must be strictly preserved.

Now, consider any \(A \times B\) block partitioning, and its random counterpart \(\tilde{G}\):

\[
G = \begin{pmatrix}
A & B \\
B^* & D
\end{pmatrix}, \quad \tilde{G} := \begin{pmatrix}
\tilde{A} & \tilde{B} \\
\tilde{B}^* & \tilde{D}
\end{pmatrix}
\]

(2)

Since all CMS component matrices must be at least positive semi-definite and symmetric, exactly like the global ones, \(G\) in (2) is used to represent any \(M\), or \(K\). Now, it is possible to derive the following condition for \(G \succeq 0\), entirely based on properties of \(G\)’s submatrices:

**Theorem 1.** Let \(G\) be a Hermitian matrix with a 2 \times 2 block partitioning \((\begin{pmatrix} A & B \\ B^* & D \end{pmatrix})\). Then \(G\) is positive semi-definite if and only if

(a) all singular values \(\sigma_i(L_A^+ BL_B^{-1}) \leq 1\), where \(A = L_A L_A^*\) and \(D = L_D L_D^*\)

(b) \(\text{range}(B) \subseteq \text{range}(A)\)

(c) \(\text{range}(B^*) \subseteq \text{range}(D)\)

Broadly speaking, a proof of **Theorem 1** can be constructed calling upon the generalised Schur complement condition, i.e. \(G \succeq 0\) if and only if

\[
A \succeq 0, \quad D - B^* A^+ B \succeq 0, \quad (I - AA^+) B = 0
\]

(3)

along with the observation that for a Hermitian \(T\), a necessary, but not sufficient condition for \(STS^* \succeq 0\) is \(\text{range}(V_n) \cap \text{range}(S^*) = \{0\}\), where \(V_n\) is the matrix of eigenvectors of \(T\) with negative associated eigenvalues.

Attention is drawn to the fact \(L_A\) and \(L_D\) may be any suitable factors of the diagonal blocks. For a strictly positive definite block, a Cholesky decomposition can be taken. In the general case, when \(A\) or \(D\) may be singular, one can use the eigendecomposition/SVD:

\[
A = Q_A \Sigma_A Q_A^*, \quad L_A := Q_A \Sigma_A^{1/2}
\]

(4)

Obtaining an equivalent statement for the strict case \(G \succ 0\) is also a consequence of **Theorem 1**. The latter corresponds to the absence of rigid body modes in \(M\) or \(K\), i.e. fully constrained model. Nevertheless, the general case explicitly enables the treatment of matrices of the free-free boundary condition FEM model. Thus, an advantage of the proposed method is that random \(M, K\) can be produced irrespective of the selection of model constraints, or lack of such. To build the stochastic blocks of \(\tilde{G}\), initially consider \(B\). Taking an SVD of \(A\), as per Eq. (4), and an equivalent representation for \(D\), and defining \(Z\) and its SVD:

\[
Z := L_A^* B L_B^{-1} = U_z \Sigma_z V_z^*
\]

(5)

Now, let \(\tilde{R}_A, \tilde{R}_B\) be stochastic unitary matrices of the same size as \(A\) and \(D\), respectively. Additionally, take \(\Sigma_z \in \mathbb{R}\) to be a random diagonal matrix of the size and rank of \(\Sigma_z\), with elements not exceeding unity. Then

\[
\tilde{Z} := U_z \Sigma_z V_z^*, \quad \tilde{B} := L_z \Sigma_z L_z^* = L_z \tilde{R}_A^* \tilde{R}_A L_z^*
\]

(6)

Swapping \(B\) with its random counterpart naturally preserves the validity of condition (a) of **Theorem 1** provided that \(A\) and \(D\) are kept fixed, the transition from \(Z\) to \(\tilde{Z}\) is contained entirely in \(\tilde{B}\). Barring further restrictions on \(\tilde{R}_A, \tilde{R}_B\) and \(\Sigma_z\), the domain of \(\tilde{B}\) is precisely \(\{B : G \succ 0\}\), i.e. the set of all matrices \(B\) for which \(\tilde{G}\) with constant diagonal blocks is positive semi-definite. The construction of \(\tilde{B}\) is hence ‘uncoupled’ from that of \(\tilde{A}, \tilde{D}\). Regarding the latter blocks, a perturbation of the same form as in Eq. (6b) may be applied, thus only the more involved case of \(\tilde{D}\) is discussed here. Realisations of \(\tilde{G}\) involve sequentially computing instances of \(\tilde{A}, \tilde{D}\), then \(\tilde{B}\).

### 2.2 Selection of appropriate perturbation matrices \(R_i\) and overall computing cost

A suitable selection of the random matrices \(\tilde{R}_A, \tilde{R}_B\) is paramount. Ideally, they should be sparse, owing to the matrix multiplications operations in Eq. (5c), and be easily definable so that \(\text{range}(B), \text{range}(B^*)\) are preserved. Matrices compounded of Givens rotations with random angles of specified probability density satisfy all of these conditions:

\[
\tilde{R}_i = \Pi_i \tilde{J}_i \Pi_i^T, \quad \tilde{J}_i = \begin{pmatrix}
\tilde{P}_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \tilde{P}_k
\end{pmatrix}
\]

(7)

\[
\tilde{P}_k = \begin{pmatrix}
\cos \tilde{\theta}_k & -\sin \tilde{\theta}_k \\
\sin \tilde{\theta}_k & \cos \tilde{\theta}_k
\end{pmatrix}
\]
Table 1: Operations for generating a realisation of a random submatrix (for $q \geq r$)

<table>
<thead>
<tr>
<th>Perturbation type</th>
<th>$A_{r \times r}$</th>
<th>$D_{q \times q}$</th>
<th>$B_{r \times q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only singular values $\sigma_i(.)$</td>
<td>$r^3 + r^2$</td>
<td>$q^3 + q^2$</td>
<td>$q^2r + r^2$</td>
</tr>
<tr>
<td>$R_i$ and constant $\sigma_i(.)$</td>
<td>$r^3 + 2r^2$</td>
<td>$q^3 + 2q^2$</td>
<td>$q^2r + 2r^2$</td>
</tr>
<tr>
<td>$R_i$ and random $\sigma_i(.)$</td>
<td>$2r^3 + 3r^2$</td>
<td>$2q^3 + 3q^2$</td>
<td>$q^2r + 2q^2 + 3r^2$</td>
</tr>
</tbody>
</table>

where $k = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$, $n$ is the size of the perturbation rotation matrix, subscript $i$ refers to either $\tilde{R}_i$ or $\tilde{R}_{ij}$, and $\tilde{\Pi}_i$ is a permutation matrix. If $G$ is singular, $\tilde{\Pi}_i$ and $\tilde{J}_i$ can be built such that the image of the perturbed $A$, $B$ or $D$ remains unaltered. This way all the conditions of Theorem 1 are met. Observe that an a priori explicit representation of the range and nullspace of these blocks is available, since in Eq. (6b) a known SVD is pre- and post-multiplied by orthogonal matrices of the form in Eq. (7).

Sparse matrix operations are accounted for. Observe that the computational time would depend both on the chosen perturbation type, and whether the singular values of $A$, $D$, $Z$ are kept deterministic or not. Algorithmic complexity remains unchanged in either scenario. It should be pointed out that FLOPS estimates for the initial decompositions of the blocks of $G$, expressed in terms of their corresponding sizes, is provided in Table 1. Sparse matrix operations are accounted for. Observe that the computational time would depend both on the chosen perturbation type, and whether the singular values of $A$, $D$, $Z$ are kept deterministic or not. Algorithmic complexity remains unchanged in either scenario.

3. VALIDATION - SPACECRAFT STRUCTURE

To ascertain the viability of the method drafted in Section 2, SSTL’s NovaSAR spacecraft has been used as a realistic, high complexity test case. Reverberation chamber acceleration spectral density (ASD) acoustic test data was available for comparison at several sensor locations on the satellite.

In addition, a FEM-FEM vibroacoustic solution was provided for the unreduced structure, obtained with FFT Actran and MSC Nastran solvers for the fluid and structural domains, respectively. The diffuse sound field excitation was defined in conjunction with the physical test sound pressure levels. Consequently, it was possible to carry out a parametric full Monte-Carlo (FMC) simulation, in which uncertainties in the model are approximated as Gaussian random variables, and the spacecraft’s representation in physical coordinates is kept. In this case the latter had 411786 DOFs. The purpose of the FMC was to establish a second baseline for comparison for the stochastic CMS method, using an established numerical scheme. The values used for the uncertainties are given in Table 2, with NSM standing for non-structural mass, and $\mu$ being the mean. The selection of appropriate values had been thoroughly studied, and such investigations can be found in [14]. The total number of instances run for the full Monte-Carlo simulation was 200. It should be pointed out that the vibroacoustic coupling was not taken into account on each of them, and the initially computed nominal pressure field was reused instead.

Figure 1: SSTL NovaSAR spacecraft, with indicated nodes / test sensor locations used for comparison
**Table 2: Assumed distributions for modelling the variables in the parametric model**

<table>
<thead>
<tr>
<th>Type</th>
<th>Property</th>
<th>Symb.</th>
<th>St. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic mat.</td>
<td>Young’s modulus</td>
<td>$E$</td>
<td>0.08µ</td>
</tr>
<tr>
<td></td>
<td>Shear modulus</td>
<td>$G$</td>
<td>0.012µ</td>
</tr>
<tr>
<td></td>
<td>Density</td>
<td>$\rho$</td>
<td>0.04µ</td>
</tr>
<tr>
<td>Solid element</td>
<td>Property matrix</td>
<td>$G_{ij}$</td>
<td>0.12µ</td>
</tr>
<tr>
<td></td>
<td>Density</td>
<td>$\rho$</td>
<td>0.04µ</td>
</tr>
<tr>
<td>Beams, rods</td>
<td>Section dimension</td>
<td>$L$</td>
<td>0.05µ</td>
</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08µ</td>
</tr>
<tr>
<td></td>
<td>Ply thickness</td>
<td>$t_i$</td>
<td>0.05µ</td>
</tr>
<tr>
<td>Composites</td>
<td>Fibre orientation</td>
<td>$\Theta_i$</td>
<td>1.0°</td>
</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08µ*</td>
</tr>
<tr>
<td>Thin shell</td>
<td>Thickness</td>
<td>$t$</td>
<td>0.05µ</td>
</tr>
<tr>
<td></td>
<td>Non-structural mass</td>
<td>NSM</td>
<td>0.08µ</td>
</tr>
<tr>
<td>Spring</td>
<td>Stiffness</td>
<td>$K_i$</td>
<td>0.06µ</td>
</tr>
<tr>
<td>Point mass</td>
<td>Mass</td>
<td>$m$</td>
<td>0.05µ</td>
</tr>
<tr>
<td>Damping</td>
<td>Modal value</td>
<td>constant</td>
<td></td>
</tr>
</tbody>
</table>

The stochastic formulation of [Section 2](#) was applied to a Craig-Bampton reduction of the satellite, comprised of 3 subsystems. It had a total of 2136 DOFs, of which 1554 modal, and 582 interface ones. Gaussian distributions were assigned to the random singular values of each subsystem’s $K_{iq}, M_{iq}$ diagonal blocks, representing the component modal mass and stiffness. The distributions were defined as normalised with respect to the original singular values. Therefore they had $\mu = 1$, while $\sigma = 0.06\mu$ was specified. Similarly, the values of $\tilde{\theta}_k$, from Eq. (7), were also set to follow normal distributions with $\sigma = 0.06\pi$. However, the mean $\mu(\tilde{\theta}_k) = \pi$ was taken, since in the nominal model the rotation angles are zero, or in other words the matrices $R_i = I$. Again, 200 random model realisations were executed, which was found sufficient for result convergence to be attained. The results collected from the new stochastic CMS are plotted and compared against the other two data sets in **Figure 2** and **Figure 3**. The shown ASD response bands for either numeric scheme are mean solution $\pm 3\sigma$, equivalent to a 99.73% confidence interval. Generally, good agreement is established between the stochastic CMS, FMC and physical test data. While the former affects the low-frequency regime less than the FMC, the mid-frequency behaviour is remarkably closely matched. However, the reduced scheme also poses the advantage of completing in 139s in total, against 36h 45min for the full Monte-Carlo. Furthermore, while MSC Nastran was used for the FMC, the stochastic CMS was solved in Matlab, on the same machine, and the code was not fully optimised.

**Table 3: Comparison of RMS acceleration values**

<table>
<thead>
<tr>
<th>Node</th>
<th>Response</th>
<th>Stochastic CMS</th>
<th>Full MC</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>695897</td>
<td>$\mu$</td>
<td>3.91g</td>
<td>3.68g</td>
<td>5.46g</td>
</tr>
<tr>
<td>695897</td>
<td>$\mu + 3\sigma$</td>
<td>6.07g</td>
<td>6.45g</td>
<td></td>
</tr>
<tr>
<td>7923</td>
<td>$\mu$</td>
<td>13.02g</td>
<td>13.98g</td>
<td>8.31g</td>
</tr>
<tr>
<td>7923</td>
<td>$\mu + 3\sigma$</td>
<td>16.43g</td>
<td>18.36g</td>
<td></td>
</tr>
</tbody>
</table>

As an overall assessment of the response prediction’s reliability, the computed/measured RMS accelerations are compared in **Table 3**. The new method provided estimations slightly closer to the test data than the FMC, with greatly improved efficiency and ease of implementation.

![Figure 2: Acceleration spectral density for Node 695897 in the x-direction](image)
4. COUPLING WITH HIERARCHICAL MATRIX ACoustIC BEM

The classic BEM has prohibitive complexity for use in large scale problems, due to the fully populated acoustic matrices yielding $O(n^2)$ storage and $O(n^3)$ solution requirements. The latter can be reduced to $O(n^2p)$, where $p \ll n$ with the use of iterative solvers, like Generalised Minimum Residual (GMRES) but this is still not practical for large applications, such as spacecraft.

Hierarchical clustering techniques, such as $H$-matrices, make use of the underlying smoothness of the integral kernels in BEM, which enable low-rank approximations of matrix blocks in the form $M_{n \times m} = A_{n \times k} B_{m \times k}^T$, with $k \ll \min(n, m)$, and $k = \text{rank}(M)$ is small with respect to $m, n$. This representation allows $M$ to be take up $k(m + n)$, rather than $mn$ units of storage. Matrix-vector multiplications of the type $Mx$ can be computed in $2k(m + n) - k$, instead of $2mn$ operations. The latter directly translates into an efficient implementation of GMRES, which is based on computing a matrix-vector multiplication at each iteration.

An $H$-matrix is built upon a partitioning of the DOF index set in a so-called block-cluster tree. Leaves, i.e. submatrices, corresponding to sets of points that are geometrically close, thus represent near-field interactions, are called inadmissible. All the matrix entries in these partitions are explicitly computed using traditional BEM. However, low-rank approximations of the remaining blocks can be constructed by only evaluating a small number of matrix entries per block, for example by adaptive cross approximation - in essence, a rank-revealing LU decomposition. The overall storage and matrix-vector multiplication complexity of BEM is brought down to $O(n \log(n))$, and that of the GMRES to $O(pn \log(n))$, rendering solution of large problems feasible.

A coupled FEM-BEM solution was tested for a simple case of an elastic shell cube, with the plate forming each side representing a CMS subsystem. The underlying formulation was collocation Burton-Miller BEM with constant triangular elements and analytical evaluation of...
the singular integrals  [16]. The resulting BEM H-matrix structure is shown on Figure 4. It should be mentioned that further compression of the initially built H-matrix by ‘coarsening’  [17] had been done. It is based on agglomerating suitable H-matrix blocks into larger low-rank ones, whenever this operation leads to more efficient storage. The coupled system takes the traditional form

\[
\begin{pmatrix}
\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} & T_d \\
\omega^2 \rho G T_h & \mathbf{H}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u} \\
\mathbf{p}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{f}_s \\
\mathbf{p}_{inc}
\end{pmatrix}
\]

with \(\mathbf{M}, \mathbf{C}, \mathbf{K}\) being the standard FEM matrices, \(\mathbf{G}\) and \(\mathbf{H}\) the acoustic ones, and \(\mathbf{T}_d\) are coupling terms. It was solved with GMRES employing the standard Schur complement approach.

It was found that typically, 400-600 iterations were needed for the iterative solver to converge at each discrete frequency, following the application of the stochastic FEM. The matrix-vector multiplications required had a mean time cost of 0.171s, yielding a typical solution at each frequency point and each stochastic FEM realisation of 85.5s. Note that implementation was done entirely in Matlab, and the code was not fully optimised. While good numerical stability was observed, and convergence of the iterative solver was consistent, evaluating the coupled FEM-BEM solution at 490 discrete frequencies (as many as used in Section 3), results in over 11h 30min per realisation of the stochastic FEM. Clearly, accumulating hundreds of realisations, similarly to Section 3, would be impractical. If only a few frequency points are needed, the stochastic FEM-BEM can be solved within acceptable time. A similar statement is true if the fully coupled solution is required only once, or not of interest at all.

5. CONCLUSIONS

In summary, a novel stochastic method, applicable to Hermitian pencils arising in FEM has been introduced. Its inherent suitability to robustly and efficiently constructing random subsystem matrices for CMS reduced order models has been demonstrated in the context of mid-frequency vibroacoustic analysis of SSTL’s NovaSAR spacecraft. Comparison against test data, as well as a full parametric Monte-Carlo numerical simulation, showed reliable response predictions are attainable, at a greatly reduced computational cost. Finally, the suitability of the stochastic CMS for coupled vibroacoustic problems utilising H-matrix acoustic BEM has been briefly evaluated, by means of a simple test case of high mesh density.

REFERENCES

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