THE HIGH ENERGY THEORY
OF THE OPTICAL POTENTIAL

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by

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ABSTRACT

The Watson multiple scattering theory is used to generate the first and second order optical potentials for nucleon-nucleus elastic scattering at high energies. A method is developed for treating exactly the non-locality of the second order potential and the scattering from the exact calculation is compared to that from a local approximation for the second order potential. The results are significantly different both for the angular distributions and the polarisation.

Simple methods are used to include in the theory the finite range of the two body interaction and off-two-body-energy-shell effects. The inclusion of these phenomena are seen to improve the agreement with the data. Neither the exact nor the approximate calculation gives very good fits to the data and some reasons for this are considered.

Some comparison is made with the work of other authors, both in terms of the formal theory and the resulting calculations. Differences due to choosing different two body phase shifts are seen to be comparable to the errors due to approximating the second order potential.
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THE DEVELOPMENT OF THE THEORY

"What experience and history teach is this: that people never have learned anything from history, or acted on principles deduced from it."

- Hegel: The Philosophy of History

1.1 Introduction

Ever since the very existence of the atomic nucleus and the fact that its constituents are protons and neutrons were revealed by collision experiments,\(^{(1,2,3)}\) the mathematical description of collision processes has played a dominant rôle in nuclear physics. On the basis of the idea of wave-particle duality, provided by the theory of quantum mechanics, the wave motion required for "looking" at nuclei has been found in the nuclear and sub-nuclear particles themselves. Nuclei and fundamental particles are extremely useful probes of nuclei.

The notion of describing the nucleon-nucleus interaction in terms of a complex potential, of which the imaginary or absorptive part represents processes which are not otherwise included, was introduced in the 1930's by Breit et.al.\(^{(4,5)}\) The long mean free path enjoyed by the incident nucleon, as predicted by the potential model, was contradicted by the compound nucleus model of Bohr\(^{(6)}\), which seemed to indicate that the nucleus, as well as the future of the optical model, was "black". However, the success of complex potentials in fitting the scattering data led to the famous "Model for Nuclear Reactions with Neutrons" by Feshbach et.al.\(^{(7)}\) As a result of the theory presented in that paper, the model is understood to describe the "gross structure" or "average"
scattering. This averaging occurs physically at higher energies (typically $E > 1$ MeV), since the compound states are too numerous and too short lived to be distinguished in a scattering experiment. The connection between the optical model and the compound nucleus is dealt with in the book of G. Brown(8).

Thus, with the optical model fairly firmly established in the early 1950's, the detailed nature of the optical potential became the object of many studies. It continues today to be the object of many studies, including the present effort. The attack on the problem can be loosely divided into two flanks: the phenomenological and theoretical approaches.

1.2 Phenomenological Methods

The phenomenological approach is not the main concern of this work although the results of its use are directly relevant. The method of assuming a form for the potential and varying its parameters until a fit to the scattering data is obtained is well known. The mathematical techniques, together with the results obtained during the first flush of success of the model, are described in the book of Hodgson(9). More recent analyses are summarised in the book by Green, Sawada and Saxon(10). In an early analysis Woods and Saxon(11) used the Fermi shape for their potential; this form is now almost universally employed and the strong desire of physicists to commemorate important advances has thus resulted in three names for one function! It has three parameters (depth, radius, diffuseness).

The optical potential with real and absorptive central terms, and real and absorptive spin-orbit terms has a minimum
of six parameters if Saxon-Woods shape is used. Of course, if one allows each term to have different radius and diffuseness as well as depth, the number of parameters is increased. If a spin-spin term is added, as suggested by Feshbach\(^{(12)}\), together with the symmetry term of Lane\(^{(13)}\), then the number of parameters can be as large as nineteen. With such a battery of parameters, all varying with energy, a somewhat cynical view would be that the optical model is merely a neat way of storing the experimental data. This would surely be a case of knowing the price and not the value of the model. In practice, fits over a wide range of nuclei and energies can be obtained with only four parameters (the potential depths), since the diffuseness remains nearly constant and the radius goes as the cube root of the mass number.

Even more encouraging has been the work of Green\(^{(14)}\), Wyatt\(^{(15)}\) et. al., and Perey and Buck\(^{(16)}\) on phenomenological non-local potentials. These authors have shown that, by using a non-local potential, energy independent parameters can be found which fit the scattering data (up to 24 MeV). Perey and Buck give an expression for an equivalent local potential which displays the usual energy dependence. Essentially, the non-local potential represents a velocity dependence of a definite form – an improvement over the local potentials, where the energy variation of the parameters is not predictable. Other complications can occur with local potentials which fit the data, such as the necessity for peculiar shapes. An example is Elton's\(^{(17)}\) work on proton-iron scattering at 180 MeV, which is represented by a potential with a repulsive central region.
Another important point concerning phenomenological potentials is their non-uniqueness. This is due both to different parameterizations and to the fact that, for a given energy and target, the available experimental data does not suffice to determine a unique set of parameters. As an example of the former reason, the shape of the absorptive central potential is an interesting case. Surface or volume absorption can be made to fit the data, but volume absorption is favoured above an energy of about 50 MeV, while surface absorption is used for energies lower than that, since with this arrangement less energy variation of the parameters is required.\(^\text{(18)}\) Attempts to justify these choices usually state that, at lower energies, the incoming nucleon more easily excites the nucleons in the surface of the target, whilst collisions in the centre are reduced by the Pauli exclusion principle. Such arguments are extremely tenuous. Surface nucleons may be more easily excited, but there are fewer of them in a given volume. The Pauli exclusion principle might reduce collisions in the centre, yet surely the effect should also be observable in the real central part, since elastic scattering should also be inhibited. There is theoretical support for the arguments, based both on nuclear matter calculations\(^\text{(19,20)}\) and on the Thomas-Fermi model\(^\text{(21)}\). There is also theoretical dissent, based on similar models\(^\text{(22,23)}\). Such controversies serve as a warning not to take detailed shapes of potentials too seriously when considering theoretical derivations; it is the scattering obtained from a potential which is important.
1.3 The Theoretical Approach

This thesis is mainly concerned with the theoretical optical potential for high energy nucleon-nucleus elastic scattering, where "high energy" means incident nucleon kinetic energies greater than about 100 MeV. However, much of the theoretical formalism is applicable to other projectiles, so that publications of theories for different projectiles are also mentioned in this section. Also, some low energy theories are considered.

The theoretical approach to the optical potential grew from the fact that a potential model predicts a long mean free path for the projectile inside the nucleus. In 1947, Serber\(^{(24)}\) reasoned that, since the mean free path of a nucleon in a nucleus is at least of the same magnitude as the radius of the nucleus, the scattering should be describable in terms of a few collisions with individual target nucleons. A formulation of multiple scattering theory had already been made by Foldy.\(^{(25)}\) By analogy with the properties of light propagation, Fernbach et.al.\(^{(26)}\) developed these ideas into a theory of a nucleus represented by a medium of complex refractive index, thus giving the model its rather strange name.

The next step was to consider the structure of the nucleus, in so far as it affected the model, and this work was begun by Lax et.al.\(^{(27,28)}\). It was continued in a veritable deluge of papers by Watson and collaborators. First\(^{(29)}\) the multiple scattering formalism for the pion-nucleus system was developed, then\(^{(30)}\) an optical potential was identified. The work was extended to nucleon-nucleus
scattering and a most important step was the relating of the optical potential to free two nucleon scattering data, with estimates of the error introduced. It was shown that study of the scattering from a nucleus could thus show up the properties of the two body force. The potential obtained from these theories was put into a form allowing calculations, including treatment of correction terms, for both pion and nucleon scattering. The whole subject is reviewed in depth in Goldberger and Watson's book.

Contemporaneously with these developments, many other authors developed somewhat different formal theories, notably the unified theory of Feshbach, which is capable of dealing with resonance scattering, and the semi-classical theory of Glauber, which has been remarkably successful at high energies (for example, the analysis of Bassel and Wilkin). Closer to the Watson approach was the work of Kerman, McManus and Thaler (written as K.M.T. in any subsequent reference) and of Rodberg; these authors essentially make different multiple scattering expansions for the optical potential and their work is reviewed in a straightforward way in the book by Rodberg and Thaler.

Thus, by about 1960, the main features of the theory had been made clear; the optical potential takes the form of a multiple scattering series. The first term, called the first order potential, involves the nuclear density, the second term (second order potential) involves the nuclear two nucleon correlation function and so on. It was hoped that the series was rapidly convergent, but it was soon realised that the second order term, at least, was important. Unfortunately,
after the first term, the series becomes extremely complicated and, as recently as 1970, Feshbach and Hüfner\textsuperscript{(46)} could report no "convincing method for evaluating the scattering" due to the second order term. An attempt to find and use such a method is presented in this thesis.

Before the work done with the theory in the high energy domain is considered, mention must be made of some low energy scattering theories. The relating of nucleon-nucleus scattering to free nucleon-nucleon scattering is a high energy method based on Chew's\textsuperscript{(47)} impulse approximation. At lower energies, it might be more helpful to expand the multiple scattering series in terms of the bound two-nucleon scattering. This latter has been extensively studied by Brueckner (a summary, together with the references, is given in Preston's book\textsuperscript{(48)}). The very first derived optical potential for nucleon-nucleus scattering was obtained by Fay\textsuperscript{(49)} from an effective two nucleon interaction. More recently, authors have either used nuclear matter theory or the unified Feshbach theory. Examples of the former are the papers by Shaw\textsuperscript{(19)}, Tang et al.\textsuperscript{(50)}, Dabrowski and Sawicki\textsuperscript{(51)}, Terasawa\textsuperscript{(52)}, Young\textsuperscript{(53)}, Kidwai and Rook\textsuperscript{(54)} and Azziz\textsuperscript{(55,56)}. The Feshbach theory has been used by, amongst others, Ripka\textsuperscript{(57)}, Ahmad and Rhaman Khan\textsuperscript{(58)}, Passatore\textsuperscript{(59,60)} and Gross\textsuperscript{(61)}. It should be noted that use of the Brueckner effective interaction leads to a real optical potential (in first order) so that the connection with the high energy theory (giving a first order complex potential) is not straight-forward. Some work on this problem is presented in the Kidwai and Rook paper\textsuperscript{(54)}.
1.4 Use of the High Energy Theory

The complications in the multiple scattering series have meant that, although the subject is a fine exercise for theoreticians, calculations using the theory do not lie too thickly on the ground. The earliest calculations\(^{(62, 63)}\) were restricted to the first order potential used in Born approximation, which Bethe\(^{(64)}\) has shown to be fairly good for small angle scattering polarization in this theory. The basic problem with the second order potential is that it is non-local in a nasty sort of way, involving a propagator and the correlation function. One method to deal with this is to attempt to localise it, using the methods of Johnston and Watson\(^{(36, 37)}\), which involve the eikonal approximation on the propagator and simple models for the correlation function. McDonald and Hull\(^{(65)}\) used this method to calculate nucleon-nucleus scattering over an energy range of 100 to 300 MeV and a fair selection of nuclei with moderate success. The second order potential, as calculated by them, did not necessarily improve fits to the data, but was certainly found to be non-negligible. McDonald and Hull introduced the idea of using the Watson theory to distinguish between various sets of two-nucleon scattering phase shifts, which has been followed in the calculations mentioned below. In view of the uncertainties concerning the approximations in the large second order potential, this may well have been an over-ambitious use of the theory.

A similar, but somewhat less complete, set of calculations was made by Tatischeff and Willis\(^{(66, 67)}\) using the K.M.T. formalism. Again, fits to data are reasonable rather than
good. A different approach is that of Chalmers and Saperstein, who, in a series of three papers \((68, 69, 70)\), examine the scattering of nucleons from light nuclei at 142 MeV, 210 MeV and 310 MeV. Their calculations are carried out in momentum space, avoiding the approximations of forcing the second order potential into being local. Their second two papers \((69, 70)\) are probably the best calculations using the Watson theory which are available to date. Their results suggest that the Pauli exclusion principle is the most important source of correlations in the second order term. However, their method of calculating these second order effects is still dependent on the Fermi gas model of the nucleus, which is not appropriate to the very light nuclei considered. As with the other authors, the polarizations prove difficult to fit.

Another method of attempting to deal with the second order term is to use an equivalent local approximation as given by Mulligan \((71)\). Such a calculation is Kujawski's \((72)\), concerning scattering of nucleons by light nuclei at 630 MeV and 1 GeV. Even at such high energies, the method has only qualitative success and hardly that for the polarizations.

The simplification of non-local-potentials if a separable form is used, has led to the separable two body interaction of Yamaguchi \((73)\) being invoked for use in the multiple scattering problem. Early authors \((74, 75)\) contented themselves with calculating the optical potential using only the s-state two body separable interaction. In a recent paper \((76)\), Foldy and Walecka generalize this work by using a sum of separable potentials generated by a set of fixed scatterers. The
expressions for the optical potential are still complicated but the theory shows, in a particularly clear manner, just what types of scattering are included in the multiple scattering series.

Feshbach and Hüfner\(^{(46)}\) have studied the problem of using the second order optical potential. They suggest a method involving an infinite set of coupled equations which under certain circumstances might be reduced to just two. The method developed independently in this thesis also involves coupled equations. It will be shown that, given the shell model of the nucleus, in which a closed shell nucleus is represented by a single Slater determinant, the correlations derived from the Pauli principle result in a finite number of coupled equations for calculation of the scattering wave function. The number of such equations is small enough for an exact numerical calculation to be made for the case of light nuclei.
THE FORMAL THEORY

"Well, the Tale is now told, from first to last. Here we all are, and here is the [problem]. But we have not yet come any nearer to our purpose. What shall we do with it?" – Tolkien: The Lord of the Rings.

2.1 The (N+1) Body Problem

The problem under consideration is that of the elastic scattering of a high energy nucleon from a nucleus consisting of N nucleons. If only two body forces are assumed to be acting, then the complete (N+1) body problem can be defined in terms of the following operators:

\[ H_T = \text{Complete target nucleus Hamiltonian} \]
\[ K = \text{Projectile Kinetic Energy Operator} \]
\[ v_i = \text{Interaction between projectile and target nucleon (i).} \]

Thus nuclear nucleons are distinguished by a subscript (e.g. the position of target nucleon (i) will be denoted by \( r_i \)), whereas the projectile is distinguished by no subscript (e.g. the position of the projectile will be denoted by \( r \)). The complete Schrödinger equation is:

\[ (E - K - H_T) |\Psi\rangle = \Sigma_{i=1}^{N} v_i |\Psi\rangle \quad (1) \]

where \( E \) is the total energy of the (N+1) particle system and \( |\Psi\rangle \) is the complete solution, which should be antisymmetric under exchange of any two of the (N+1) nucleons, if isospin
formalism is adopted. However, the antisymmetry with respect to the incident particle is neglected for the moment.

It is convenient to consider equation (1) in the centre of mass frame.* This involves factorizing out the centre of mass motion, so that in equation (1), E becomes the total energy in the C.M. frame, $K$ is the relative kinetic energy operator for the projectile and $H_T$ is the internal target Hamiltonian. The result of this procedure (which just expresses conservation of linear momentum) is that of the $(N+1)$ nucleons, the spatial coordinates of $N$ are independent but once these are fixed, the spatial coordinates of the final nucleon are also fixed. This fact is ignored in what follows but, as pointed out in the appendix of Foldy and Walecka\(^{(76)}\), the effects are small, especially for large $N$.

Working in the $(N+1)$ C.M. frame, the solution to equation (1) with the correct boundary conditions for scattering can be written:

$$|\psi_k> = |\hat{\phi}_k> + \frac{1}{E-K-H_T+i\epsilon} \sum_i v_i |\psi_k>$$

where:

$$|\hat{\phi}_k> = |k,0>$$

which represents an antisymmetrized nuclear ground state and a free particle with relative wave number $k$ (i.e. momentum is $\hbar k$). The transition amplitude for elastic scattering is then:

$$T_{el} = \langle \hat{\phi}_k | \sum_i v_i |\psi_k> \quad |k| = |k'|$$

* Referred to as C.M. frame hereafter.
2.2 Derivation of the Optical Potential

The optical potential, \( U \), is the one body potential which, when used in a one body Schrödinger equation, will give the same scattering amplitude \( T_{el} \) as the full solution given by (3). The derivation presented here is essentially the Watson method, using the uncluttered notation of Rodberg\(^{(43,44,45)}\).

Some definitions:

The following notation will be used throughout this work:

(i) States denoted by round brackets are \( N \)-particle target nucleus states. i.e. They are eigenstates of the target internal Hamiltonian \( H_T \), normalized to unity:

\[
H_T |n\rangle = E_n |n\rangle \quad (n|n) = 1
\]  

(4)

For convenience, the ground state is taken to have zero energy:

\[
H_T |0\rangle = 0 \quad (0|0) = 1
\]  

(5)

The physical states \( |n\rangle \) are antisymmetrized in the \( N \) target particles.

(ii) States denoted by Dirac brackets and a capital letter are \( (N+1) \) particle states. e.g. \( |\Psi\rangle \) in equation (1).

(iii) States denoted by Dirac brackets and a small letter are one particle states. e.g. A free particle
with wave number \( k \) is represented by \( |k> \).

(iv) The following operators are defined by:

\[
H = H_T + K \\
G = \frac{1}{E-H_T-K+i\epsilon} = \frac{1}{E-H+i\epsilon}.
\]

**Neglect of Antisymmetry**

The derivation starts from the full solution, equation (2):

\[
|\psi_k> = |\xi_k> + \frac{1}{E-H+i\epsilon} \sum_i v_i |\psi_k>.
\]

The only approximation made at this stage is the neglect of the identity of the projectile with the \( N \) target nucleons. This problem will be considered later. With this neglect, the initial state becomes a simple product:

\[
|\xi_k> = |0>|k>.
\]  

**Equivalent One Body Problem**

This can be found in the following manner. First, define a 'coherent' wave function \( |X_k> \)

\[
|X_k> = |0>(0|\psi_k>.
\]

and a single particle wave function \( |\psi_k> \)

\[
|\psi_k> = (0|\psi_k>.
\]

where the product in equations (9), (10) implies summation over all target coordinates, so that \( |\psi_k> \) is a function only
of the projectile coordinates and \( |0\rangle\langle 0| \) projects onto the
ground state of the nucleus. \( |\psi_k\rangle \) can be found from equation
(2), since:

\[
|0\rangle\langle \psi_k| = |0\rangle\langle \tilde{\psi}_k| + (0| \sum_i \frac{1}{E-H+i\epsilon} v_i | \psi_k\rangle
\]

i.e.

\[
|\psi_k\rangle = |k\rangle + \frac{1}{E-K+i\epsilon} (0| \sum_i v_i | \psi_k\rangle
\]

(11)

where equations (5), (7), (8) and (10) have been used.

If it is possible to find an operator \( \Omega \), such that:

\[
\Omega|X_k\rangle = \Omega|0\rangle\langle 0| \psi_k\rangle = |\psi_k\rangle
\]

(12)

then (11) becomes:

\[
|\psi_k\rangle = |k\rangle + \frac{1}{E-K+i\epsilon} (0| \sum_i v_i \Omega|0\rangle | \psi_k\rangle.
\]

(13)

The required optical potential is thus given by:

\[
U = (0| \sum_i v_i \Omega|0\rangle
\]

(14)

and the equivalent one body problem by:

\[
|\psi_k\rangle = |k\rangle + \frac{1}{E-K+i\epsilon} U| \psi_k\rangle
\]

(15)

which is just the scattering solution to the Schrödinger
equation with potential \( U \). The elastic scattering is
described by:

\[
T_{el} = \langle k' | U | \psi_k\rangle.
\]

(16)

That this is the same matrix element as given by equation (3)
is shown by using (8), (12) and (14) in (3):

\[ T_{el} = \langle \xi, \sum_{i} v_i | \psi > \]
\[ = \langle \xi, (0| \sum_{i} v_i \Omega |0) | \psi > \]
\[ = \langle \xi | U | \psi > . \]

Finally, the operator \( \Omega \) is required. Write equation (2) for \( |\psi_k> \) in terms of \( |X_k> \) by using (7) and (12):

\[ |\psi_k> = |\xi_k> + G \sum_{i} v_i \Omega |X_k> . \quad (17) \]

An expression for \( |X_k> \) is obtained by multiplying \( |\psi_k> \) by \( |0> \). Hence multiply equation (15) by \( |0> \) and use (5) with (7):

\[ |X_k> = |\xi_k> + GU |X_k> . \quad (18) \]

Direct comparison of (17) and (18) leads to:

\[ |\psi_k> = (1 + G(\sum_{i} v_i \Omega - U)) |X_k> . \]

Comparing this last result with equation (12), gives:

\[ \Omega = 1 + G(\sum_{i} v_i \Omega - U) . \quad (19) \]

Equations (14) and (19) give the formal solution to the optical potential. The following points are worth mentioning:

(i) The expression for \( \Omega \) shows that \( U \) is a potential containing the effects of multiple scattering.
(ii) The first approximation to $U$ would be found by using:
\[ \Omega = 1. \text{ Therefore } U = (0|\Sigma v_i|0) \]
Such a potential would be real.

(iii) The presence of $U$ on the right hand side of (19) results in a self consistency problem. Physically, it is there to remove certain types of double counting inherent in the other terms. This will be considered in section (2.3).

(iv) Use of (14) and (19) results in a form of Born series for $U$. This will be slowly converging and truncation after a few terms would be a poor approximation to the complete series. A re-summation is necessary to obtain fast convergence. The bewildering plethora of different objects possessing the title of optical potential stems from this re-summation. The point will be pursued further in section (2.4).

**Re-summation of the multiple scattering series**

The result obtained so far is given by (14)
\[ U = (0|\Sigma v_i \Omega|0) . \]

From (19) it can be seen that $U$ contains a sub-series of the form:
\[
(\langle 0 | \sum_i v_i | 0 \rangle + \langle 0 | \sum_i v_i G v_i | 0 \rangle + \langle 0 | \sum_i v_i G v_i G v_i | 0 \rangle + \ldots).
\]

\[
= \langle 0 | \sum_i (v_i + v_i G v_i + v_i G v_i G v_i + \ldots) | 0 \rangle
\]

\[
= \langle 0 | \sum_i t_i | 0 \rangle \quad \text{where} \quad t_i = v_i + v_i G t_i .
\]

It is to be emphasised that \( t_i \) is not meant to be the exact \( t \)-matrix for the scattering of the projectile and target particle \((i)\) inside the nucleus. It contains no allowance for the exclusion principle. Nor does it take into account all effects due to particle \((i)\), since this particle also contributes to the propagator \( G \) (which appears in other terms than \( t_i \)) and to \( U \), which appears on the right hand side of (19). \( t_i \) is a device for re-summing the series, which can be done as follows. Defining a new operator \( \Omega_i \) such that:

\[
t_i \Omega_i = v_i \Omega
\]

one has:

\[
v_i \Omega_i + v_i G t_i \Omega_i = v_i + v_i G \sum_j \Omega_j \Gamma_j - v_i G U
\]

\[
= v_i + v_i G \sum_j \Omega_j \Gamma_j - v_i G U .
\]

Collecting terms:

\[
v_i (\Omega_i - 1) = v_i (G \sum_{j \neq i} \Omega_j \Gamma_j - G U),
\]

therefore

\[
\Omega_i = 1 + G (\sum_{j \neq i} \Omega_j \Gamma_j - U)
\]

and

\[
U = \langle 0 | \sum_i t_i \Omega_i | 0 \rangle .
\]

Equations (21) and (22) represent the formal multiple scattering series solution for the optical potential; they
are an exact solution if the projectile is not a nucleon – the problem of antisymmetrizing with respect to an incoming nucleon has yet to be considered.

2.3 Truncation of the Series and the Meaning of its Terms

The exact solution (22) is an infinite series and must be truncated for practical calculations. This is carried out by taking successive approximations to \( \Omega_i \) (21).

1st Order Optical Potential

This is obtained by using:

\[
\Omega_i^{(1)} = 1. \quad \text{Therefore} \quad U^{(1)} = (0|\Sigma \ t_i|0). \quad (23)
\]

Since \( t_i \) is complex, \( U^{(1)} \) will be complex.

2nd Order Optical Potential

The next term in \( \Omega_i \) is found by putting \( \Omega_i^{(1)} \) and \( U^{(1)} \) into the right hand side of (21):

\[
\Omega^{(2)} = G\left( \sum_{j \neq i} t_j - U^{(1)} \right). 
\]

Therefore

\[
U^{(2)} = (0|\Sigma \sum_{i,j \neq i} t_i G t_j|0) - (0|\sum_i t_i G U^{(1)}|0). \quad (24)
\]

The second term above can be simplified as \( U^{(1)} \) is an operator only on projectile coordinates.

Therefore

\[
(0|\sum_i t_i G U^{(1)}|0) = (0|\sum_i t_i G|0)U^{(1)}. 
\]

Further, if a free particle Green's function is defined by:
\[ g = \frac{1}{E - \Omega - i\epsilon} \]

then using equations (5) and (6):

\[ t_i^G |0\rangle = t_i |0\rangle g. \]

The second order potential is then:

\[ U(2) = \sum_{i \neq i} \sum (0|t_i G t_j|0) - \sum_{i} (0|t_i|0) g U(1). \]  

(25)

To show the structure of \( U(2) \) more clearly, define the projection operator for projecting off the nuclear ground state:

\[ Q = 1 - |0\rangle\langle 0| \]  

(26)

and write \( U(1) \) in full in equation (25). Then part of the second term can be accommodated into the first

\[ U(2) = \sum_{i \neq i} \sum (0|t_i G t_j|0) - \sum_{i} (0|t_i|0) g(0|t_i|0). \]  

(27)

Double Counting and its Cancellation

The second term in \( U(2) \) is present to remove some double counting resulting from \( U(1) \); it is often referred to as a \((1/N)\) term as there is only one summation over the target nucleons whereas there is a double summation in the first term. To show this double counting, write \( U(1) \) in a shorthand form:

\[ U(1) = \sum_i (0|t_i|0) = \sum_i (t_i). \]

In a Born scattering series this gives:
The second term contains an object of the form:

$$T(1) = \sum_{t_1} + \sum_{j,t_j} g(t_j) + \ldots \ldots$$ \hspace{1cm} (28)

However, the first term of (28) is:

$$\sum_{t_1} g(t_1) = \sum_{v_1} g(v_1) + \ldots$$ \hspace{1cm} (29)

and this already contains everything which appears in (29), since \(G\) becomes \(g\) for unexcited intermediate states, as shown when dealing with \(U^{(2)}\). Thus the Born series with \(U^{(1)}\) alone counts (29) twice. However, the second term of \(U^{(2)}\) is precisely minus the object (29) and so cancels it when the scattering from \(U^{(1)} + U^{(2)}\) is calculated. In the first term of \(U^{(2)}\), the only intermediate states are excited ones and the summation is over different target nucleons just so as to avoid further double counting. The first few terms of the Born series (as far as first order in the propagator) for \(U^{(1)} + U^{(2)}\) are shown in figure (1).

Fig. (1) shows clearly that, apart from the \((1/N)\) correction term, \(U^{(2)}\) involves the two particle nuclear correlation function. The projectile excites the nucleus by collision with a nucleon, then de-excites it by collision with a different nucleon. This can only occur if the motion of the two nucleons is correlated in some manner. In fact, the series is an expansion in the target correlations (apart from the \((1/N)\) corrections). The third order potential contains effects due to nuclear three body correlations.
\[ T(1) = u(1) + u(1)g + \ldots \text{ higher terms in } u(1). \]

This term is spurious

\[ T(2) = u(2) + \ldots \text{ higher terms in } u(1) \text{ and } u(2). \]

Excited Intermediate State

Cancels the spurious term in \( T(1) \)

Fig. 1: Scattering from \( u(1) \) and \( u(2) \)
2.4 The Non-Uniqueness of the Optical Potential

Since the elastic scattering amplitude is given by the T matrix of equation (16):

\[ T_{el} = \langle k' | U | \psi_k \rangle \]

it follows that the same result can be obtained from a potential \((U+W)\) if:

\[ \langle k' | W | \psi_k \rangle = 0 . \quad (30) \]

This ambiguity is one reason why several sets of phenomenological parameters can be found to fit given scattering data. What is the effect on the theory presented here? Equation (22) gives the theoretical potential as:

\[ U = (0| \Sigma t_i \Omega_i | 0) . \]

The ambiguity allowed by (30) means that any new series, say:

\[ W = (0| \Sigma t'_i \Omega'_i | 0) \]

can be added to \(U\), so long as (30) is satisfied. Since \(U\) and \((U+W)\) give identical results, this might seem unimportant. However, when it is remembered that any series solution for \(U\) is to be truncated for practical purposes, the ambiguity could be calamitous. Put another way, the series must be fast converging, which is why the re-ordering discussed in the previous section is important. The re-ordering chosen is found to be quite quickly converging, since it is an expansion in the nuclear correlations, which are a reasonably small perturbing effect, for the simple reason that the
independent particle model of the nucleus is a good approximation to the real thing.

The differences between various theories can largely be traced to different re-orderings of the series. These are effected by choosing to re-sum expression (14) in terms of different t-matrices. The t-matrix used in this work is that given in equation (20):

\[ t_i = v_i + v_i \frac{1}{E - H + i\epsilon} t_i. \]

In other works a t-matrix with a projection operator in the second term is often used in order to make the series more quickly convergent. For example, Goldberger and Watson\(^{38}\) choose:

\[ t_{G.W.} = v_i + v_i \frac{Q}{E - H - i\epsilon} t_{G.W.} \]

where Q is given by equation (26) and is the projector for projecting off the nuclear ground state. K.M.T.\(^{42}\) choose:

\[ t_{K.M.T.} = v + v \frac{A}{E - H + i\epsilon} t_{K.M.T.} \]

where A projects onto antisymmetrized target nuclear states and so allows subscripts on v and t to be dropped.

These new t-matrices result in new series operators, i.e. modifications of \( \Omega_i \) (equation (21)). Such modifications are somewhat self defeating, since the only way of dealing with them seems to be that of finding the effects of the projector by a perturbation expansion of the new t-matrix in terms of the old one. This usually leads to extra terms in the second order optical potential, so that the series is not,
in effect, any more quickly converging. One special case of
a projector in the t-matrix is the Brueckner G-matrix, for
which the projector is the Pauli projection operator and the
denominator does not possess the boundary condition of
outgoing waves, since the G-matrix applies to two particles
embedded in nuclear matter. Hence the G-matrix is more
relevant to scattering from large nuclei at low energies.

Another possible modification of the theory concerns the
treatment of the \((1/N)\) correction terms, such as the second
term of equation (27). Goldberger and Watson\(^{(38)}\) show that
they can be absorbed by redefining t-matrix as already
mentioned. With their t-matrix (31) the optical potential
becomes:

\[
U = (0| t^{G.W.}_i \Omega^{G.W.}_i |0)
\]

where:

\[
\Omega^{G.W.}_i = 1 + GQ \sum_{j \neq i} t^{G.W.}_{ij} \Omega^{G.W.}_j .
\]

By comparison with equation (22), it can be seen that \(U\) has
been removed from the definition of \(\Omega_i\) by this definition of
the series. However, the \((1/N)\) terms reappear if \(t^{G.W.}_j\) is
expanded. Another method of dealing with these terms is
used by Francis and Watson\(^{(30)}\) and K.M.T.\(^{(42)}\). In their
methods, a modified Schrödinger equation is to be solved and
this results in a simplification of \(U\). An application of
this method to the potential \(U\) of equations (21) and (22) is
given in the appendix. Such an arrangement is not adopted
in this work, since what is calculated is then an optical
potential in the usual sense and the \((1/N)\) corrections are
useful for cancellation of terms arising from the rest of the
potential.
THE FORMAL APPROXIMATIONS

"They love not poison that do poison need."
- Shakespeare: Richard II

3.1 Requirement of Approximations

The multiple scattering theory developed in the previous chapter is merely a formal manipulation of the problem and, elegant as it is, does not really bite into the problem of actually calculating the scattering from a nucleus. Such a theory requires 'formal' approximations to bring it to a recognizable numerical problem.

The complete solution of equation (22) is an infinite series. Not only this, every term is an \((N+1)\) body problem, since each term contains expressions in \(t_1\) which is an \((N+1)\) body operator. The aim of the approximations presented here is to separate the scattering problem from the nuclear structure problem, since methods for dealing with each of these are well known. The advantage of a formal theory is that approximations can be removed as new computing techniques become available. The Watson theory makes successive approximations to the exact answer. In contrast, the Glauber\(^{40}\) theory starts with an all embracing approximation and then works back towards the exact answer. A comparison of the two approaches has been made by Remler\(^{77}\).

3.2 The Multiple Scattering Approximation

The infinite series of equation (22) must be truncated, as mentioned in the last chapter. As was indicated there,
the n-th order potential is generated by the n-particle correlation function. By stopping at the second order potential, three particle correlations (and higher) are being ignored. This is basically the approximation to be tested in this work. By experience it is found that the second order potential causes changes in the scattering observables of typically 25% at 90 MeV. The error incurred by neglecting the third order potential is thus expected to be typically 6% at this energy, which is typical of the errors introduced in approximating \( U(1) \) and \( U(2) \). The higher order potentials become less important with increasing energy, as shown in chapter V. Thus, the potential to be considered is:

\[
U = U(1) + U(2)
\]

where \( U(1) \) is given by equation (23) and \( U(2) \) by equation (27).

3.3 The Impulse Approximation

As used here, the impulse approximation consists of three steps for simplifying the matrix elements of \( t_i \) which appear in both \( U(1) \) and \( U(2) \). These three approximations will be considered in turn.

(i) The t-matrix appearing in \( U \) is given by equation (20):

\[
t_i = v_i + v_i \frac{1}{E - H + i\epsilon} t_i.
\]

This is approximated by the t-matrix for scattering of the projectile from a free nucleon (i):

\[
t_i^f = v_i + v_i \frac{1}{E - K - K_i + i\epsilon} t_i^f.
\]
This is the impulse approximation considered by Chew\(^{(47)}\), Chew and Wick\(^{(78)}\), and Chew and Goldberger\(^{(79)}\). The error associated with this approximation is given by\(^{(38)}\):

\[
t_i \approx t_i^f (1 + kM\frac{B_{AV}}{E})
\]

where \(k\) is the wave number of the incident particle and \(M\) is the forward scattering amplitude due to \(t_i^f\). \(B_{AV}\) is the average binding energy per nucleon. This estimate suggests that the error could be as much as 20\% for carbon at \(E = 90\) MeV but drops to 7\% at \(E = 300\) MeV. These are large errors, and it would seem that the impulse approximation requires further investigation. However, as is evident from the work done using this theory (reviewed in Chapter I), it seems to have become accepted that (for light nuclei, at least) the impulse approximation is reasonable for energies as low as about 100 MeV. This assumption is made here for the purposes of simplifying the expressions for \(U\) - the possibility of making some simple correction for the error involved is considered in chapter VI.

It is worth remarking that the term "impulse approximation" (I.A.) has been used to describe a number of different things. Early workers\(^{(47,78,79)}\) tended to include the multiple scattering approximation with I.A. More recently\(^{(42,69)}\), the name "I.A. correction" has been applied to corrections due only to projectors appearing in \(t_i\) (Chapter II) and not to corrections for the neglect of nuclear effects in the propagator in \(t_i\).
(ii) The matrix elements required are those appearing in \( U^{(1)} \) and \( U^{(2)} \):

\[
U^{(1)} = \sum_{i=1}^{N} (0|t_i^f|0)
\]

The nuclear ground states are antisymmetrized, so this becomes:

\[
U^{(1)} = N(0|t^f_1|0) = N(0|t^f_1|0)
\]

where \( t^f_1 \) is \( t^f_1 \) averaged over the ground state spin and isospin variables of the nucleus, so that only spatial effects have then to be considered in \( (0|t^f_1|0) \). Consider the matrix elements of \( U^{(1)} \) in momentum space:

\[
<k'|U^{(1)}|k> \propto <k'|N(0|t^f_1|0)|k> = <t>
\]  

which is taken to define \( <t> \). \( k' \) and \( k \) are nucleon-nucleus relative momenta. The momentum transfer is defined by:

\[
q = k' - k
\]

Stage (ii) of the impulse approximation consists of neglecting the initial momentum of nucleon (1) in the \( t \)-matrix.

K.M.T. show that this, together with the conservation of two body momentum implied by \( t^f_1 \) leads to:

\[
<t> = <k_0|t^f_1|k_0> P(q)
\]  

where \( k_0 \) is the initial relative momentum of the projectile and nucleon (1):

\[
k_0 = \frac{N+1}{2N} k
\]
and $k'_o$ is the final value of this quantity:

$$k'_o = k_o + q$$  \hspace{1cm} (33c)$$

$P(q)$ is the Fourier transform of the single particle nuclear density function (normalised to unity). The energy at which $t^f_{11}$ is to be evaluated is fixed by the energy of the nucleon-nucleus system (see equation (32)).

Stages (i) and (ii) of the impulse approximation can be written formally as:

$$\langle k'|0|t^f_{11}|0 \rangle k > \approx \langle 0 | e^{-i q \cdot r} t^f_{11}(k'_o, k_o) |0 \rangle$$  \hspace{1cm} (35)$$

where $t^f_{11}(k'_o, k_o)$ is an operator only in spin space. Reverting to the notation of equations (33), $U^{(1)}$ becomes:

$$\langle k'|U^{(1)}|k \rangle = \langle k'_o|t^f_{11}|k_o \rangle p(q)$$  \hspace{1cm} (36)$$

where, for convenience, the density function $\rho(r)$ is renormalised:

$$\int \rho(r) dr = N \quad \rho(r) = NP(r).$$

This separation of the nuclear structure problem from the two body scattering problem results from the neglect of the Fermi motion of the nuclear nucleons. The error due to this approximation has been estimated at less than 3% for scattering at $E = 100$ MeV. The Fermi motion is of the order of 40 MeV, a sizeable fraction of the incident energy.
However, the error in neglecting it is small because the optical potential involves the ground state average of $t^f_1$. The momenta of the nuclear nucleons are isotropic and so the average momentum of a struck particle in the nucleus is zero.

(iii) There is still one more stage in the impulse approximation if nucleon-nucleus scattering is to be related to nucleon-nucleon scattering. In the two body $t$-matrix in equation (36), $k_0$ and $k'_0$ can take on any values because $k$ and $k'$ can have any values. These values are not restricted by two-body energy conservation. Hence $t^f_1$ is required off the two body energy shell. However, if the energy variation of $t^f_1$ is assumed to be weak, the off-shell elements can be replaced by on-shell ones. Then $t(k'_0, k_0)$ becomes, for a given incident energy, just a function of scattering angle and is defined only on the two body energy shell. It is then related to the two body phase shifts. The corrections due to this approximation have been considered by Mulligan (71), Reading and Mackellar (80) and Reading (81). To obtain these corrections, a model of the two body interaction is required. Reading and Mackellar (80) point out that their calculations are very model dependent and that their semi-heuristic techniques tend to confuse other corrections with the one required.

The three stages of the impulse approximation give:

\[ \langle k' | U^{(1)} | k \rangle = - \frac{4\pi\hbar^2}{m} \overline{M}(k'_0, k_0) \rho(q) \]  

(37)

where $\overline{M}$ is the spin averaged two body scattering amplitude.
evaluated at the incident energy under consideration. The "ultimate" approximation is to use the fact that $p(q)$ is a rapidly falling function of $q$, so that only $M(0)$ is required i.e. the forward scattering amplitude:

$$<k' | U_0^{(1)} | k> \approx -\frac{4\pi \hbar^2}{m} M(0) p(q).$$ \hspace{1cm} (37a)

As will be shown later, this is not a good approximation for light nuclei, where $p(q)$ does not fall rapidly enough.

To illustrate the nature of the impulse approximation, it is necessary to consider certain scattering events. Stages (i) and (ii) result in equation (36), which gives the scattering from a "free" particle whose presence is governed by the nuclear density and in which $(N+1)$ body kinematics are employed. The struck nucleon recoils quasi-freely. This is the weak binding situation illustrated in figure (2). The opposite case, tight binding, is illustrated in figure (3). In this case, the nucleus as a whole feels the effects of the collision and the impulse approximation is not valid.

To illustrate stage (iii), consider the Born approximation for scattering from the potential $U^{(1)}$. If $p$ is the lab. incident nucleon momentum, the relevant quantities are:

$$k = \frac{N}{N+1} \frac{p}{p} \quad k' = k + q$$

$$T \approx <k' | U^{(1)} | k> \quad |k'| = |k|.$$ 

The last equation expresses the fact that elastic scattering from a nucleus governs the kinematics. However, stage (iii)
Scattering Events in the Lab. Frame

Fig. 2: Weak Binding

Fig. 3: Tight Binding

Off Shell Effects

(N+1) Body Kinematics

2 Body Kinematics

Fig. 4: Scattering in 1st Born Approximation
uses two body elastic scattering in evaluating $U^{(1)}$, which means that:

$$|k'_o| = |k_o| \quad q_o = k'_o - k_o .$$

As shown in figure (4), the difference between the shells corresponds to a rotation of $q$ to give $q_o$, and instead of calculating $t^f_1$ on the nucleon-nucleus energy shell:

$$< k_o + q | t^f_1 | k_o >$$

it is calculated on the two nucleon energy shell:

$$< k_o + q_o | t^f_1 | k_o >$$

where $|q| = |q_o|$ but $q \neq q_o$.

Fig. (4) shows that these off shell effects are zero for $q = q_o = 0$ and small for small angles.

3.4 The Closure Approximation

To evaluate the second order potential, as given by equation (25):

$$U^{(2)} = \sum_i \sum_j \langle 0 | t_i \frac{1}{E-H+i\xi} t_j | 0 \rangle - \sum_i \langle 0 | t_i | 0 \rangle \frac{1}{E-K+i\xi} U^{(1)}$$

an approximation in addition to those required for the calculation of $U^{(1)}$ is employed. The first term of $U^{(2)}$ contains the $(N+1)$ particle propagator. Consider expanding this propagator in a complete set of nuclear states:
\begin{align*}
(0 | t_i \frac{1}{E-H+i\varepsilon} t_j | 0) &= (0 | t_i \frac{1}{E-K_{HT}+i\varepsilon} t_j | 0) \\
&= \sum n (0 | t_i \frac{1}{E-K-E_N+i\varepsilon} (n!) t_j | 0).
\end{align*}

The closure approximation replaces $E_N$ in the denominator by some average excitation energy $\overline{E_N}$. Then the sum over the intermediate states just gives unity:

\begin{align*}
(0 | t_i \frac{1}{E-H+i\varepsilon} t_j | 0) \approx (0 | t_i \frac{1}{E-K-E_N+i\varepsilon} t_j | 0).
\end{align*}

This step can only be justified if the values of $E_N$ giving non-negligible contributions to $U^{(2)}$ all lie near some fixed value $\overline{E_N}$. In fact, this is expected to be true for the following reason (38, 72). Excitation is caused by collision of projectile and particle (j) but de-excitation has to be effected by collision with particle (i). Since these are different particles, the excitation must be small, for if particle (j) recoiled with high energy, a subsequent collision with particle (i) could not remove this excitation. Thus the scattering is taken to be quasi-elastic and $\overline{E_N}$ is just the recoil energy of the nucleus as a whole. If the value of $E$ appearing in $U^{(2)}$ is taken as the relative projectile-nucleus energy and the reduced mass of the projectile is used, then $\overline{E_N}$ can be put to zero. The result is:

\begin{align*}
U^{(2)} &= \sum_i \sum_{j \neq i} (0 | t_i^f \frac{1}{E-K+i\varepsilon} t_j^{f*} | 0) - \sum_i (0 | t_i^f | 0) \frac{1}{E-K+i\varepsilon} U^{(1)}
\end{align*}
where the impulse approximation has been introduced. This expression can be re-written if $U^{(1)}$ is expressed in terms of $t^f_1$ and the formal expression for the impulse approximation, equation (35), is used:

$$\langle k' | U^{(2)} | k \rangle = \int \frac{dk''}{E-K(k'')+i\epsilon} \left[ \Sigma \Sigma (0|e^{-iq'}t_{i}^{f}(k'',k_{0})t_{j}^{f}(k_{0},k_{0})|0) - \Sigma (0|e^{-iq}t_{i}^{f}(k_{0},k_{0})|0)(0|e^{-iq}t_{j}^{f}(k_{0},k_{0})|0) \right]$$

(38)

where:

$$q' = k' - k'' = k_{0} - k_{0} \quad q = k'' - k = k_{0} - k_{0} \quad (38a)$$

The detailed nature of $U^{(2)}$ will be considered in the remaining chapters. The essential structure of $U^{(2)}$ is seen by using the forward scattering approximation and carrying out the spin averages for the $t^f_1$. Then the exponential factors result in the fourier transform of the two body and one body nuclear density functions:

$$\langle k' | U^{(2)} | k \rangle \approx \int \frac{dk''}{E-K(k'')+i\epsilon} \left[ \rho^{(2)}(q',q)t_{1}^{f}t_{2}^{f} - \rho(q')\rho(q)t_{1}^{f}t_{2}^{f} \right]$$

(39)

where $\rho^{(2)}(r,r')$ is the two body density normalised to $N(N-1)$. $t_{1}^{f}t_{2}^{f}$ is the spin average of the product of $t_{1}^{f}$ and $t_{2}^{f}$ (forward scattering) and $t_{1}^{f}t_{2}^{f}$ is the product of the spin averages of
Apart from \( \frac{1}{N} \) corrections, the square bracket in (39) is of the form of a binary average of \( t^f_1 \) multiplied by the Fourier transform of the nuclear two-particle correlation function, since:

\[
C(r, r') = P^{(2)}(r, r') - P(r)P(r')
\]

where \( C \) is the correlation function and the capital "P" form of the density is normalised to unity.

Therefore \( P^{(2)}(r, r') = \frac{1}{N(N-1)} \rho^{(2)}(r, r') \).

3.5 The Antisymmetrization Approximation

Having developed the theory neglecting the antisymmetrization required for the projectile, it is now necessary to take its identity into account, at least approximately. Takeda and Watson\(^{(31)}\) have shown formally that this is done by using properly antisymmetrized two-body t-matrices, which are just the free scattering amplitudes found by experiment, if the impulse approximation is employed. The reasoning is as follows. The identity of the \( N \) target nucleons is taken into account by using properly antisymmetrized nuclear ground states \( |0 \rangle \) in the expressions for \( U(1) \) and \( U(2) \). The identity of the two nucleons directly involved in each of the multiple collisions is included by antisymmetrizing the two-body t-matrices. The exchange processes which are neglected in this approximation are what Takeda and Watson\(^{(31)}\) call "target exchange" effects. In a collision between the projectile and target nucleon (1) an
exchange with particle (2) takes place and the projectile remains in the nucleus while particle (2) is ejected. Such effects are small at high energy since they involve the overlap of a free, high energy particle wave function with a bound state wave function.

The error associated with this approximation was estimated by Takeda and Watson\(^{(31)}\) to be about 3% at 100 MeV incident energy and decreasing with increasing energy. An estimate made by Sawicki\(^{(82)}\) makes the error 0.5% at 80 MeV. Thus, this approximation is probably very good at the energies considered here.

The antisymmetrization adopted here does take into account the fact that certain momenta (or more exactly certain momentum states) are prohibited for the projectile in the intermediate states. This is the effect estimated by Goldberger, using the Fermi gas model\(^{(82)}\); it would be incorrect to attempt to make a separate correction for this effect. To see how this restriction of scattering in intermediate states is included, consider the form of \(U^{(2)}\).

It has been shown that (equation (39)):

\[
U^{(2)} \propto \sum_i \sum_{j \neq i} t^f_{ij} C(i,j) t^f_{ij}
\]

where \(C\) is the correlation function for nucleons (i) and (j) and \(t^f_{ij}\) is the two-body t-matrix, now antisymmetrized. The effect of the \(C(i,j)\) derived from the Pauli principle is to restrict the possible momenta (or possible positions) of nucleon (i), given the presence of nucleon (j). However, exchange of the projectile for nucleon (i) or (j) is now
possible via $t_i^f$ and $t_j^f$. Hence, the "projectile" now feels the effects of the correlation function. The state of nucleon (i) or (j) then restricts the possible states of the "projectile" in two ways. First, the recoil allowed to (i) or (j) is restricted by $C(i,j)$. Secondly, the exchange due to $t_i^f$ and $t_j^f$ means that the "projectile" is subjected to the effects of $C(i,j)$, which affects the exchange processes which are possible.

3.6 The Nonrelativistic Approximation

Nonrelativistic quantum mechanics is used throughout this work for the following reasons. The problem of what rôle a potential plays in relativistic theories is somewhat ambiguous and is not considered here; Elton\(^{(83)}\) has examined this question. Also, at the energies under study here (100 - 300 MeV), the kinematic corrections due to relativity are small. Using the results of the appendix of K.M.T.\(^{(42)}\), the correction to $U^{(1)}$ at 100 MeV incident energy for nucleon-carbon scattering is an increase of about 2.5%. This rises to about 7% for an energy of 300 MeV. Thus it is only for energies at the upper end of the range that this error approaches those already accepted. In any case, the main emphasis in subsequent chapters is consideration of methods of treating the non-locality of the optical potential. So long as consistent approximations of other effects are maintained, the results of non-locality will be unambiguously displayed, in spite of approximations.
THE OPTICAL POTENTIAL IN CONFIGURATION SPACE

"I could be bounded in a nut-shell,
and count myself a king of infinite
space ...."  — Shakespeare : Hamlet

4.1 Procedure for the Calculation of the Potential

In the previous chapter, the formal theory was subjected to successive approximations until a calculable expression was obtained. In this chapter, the reverse procedure is adopted. The simplest expression for the potential is considered and successive refinements dealt with. It is the effect of these refinements and the problems involved in calculating them that is of interest, since the basic calculations have been studied before\(^{42,37,65}\). In this chapter, the first order potential, \(U^{(1)}\), is dealt with in detail. The second order potential, \(U^{(2)}\), is considered in outline, details being reserved for Chapter V.

4.2 The "Ultimate" Approximation for \(U^{(1)}\)

This expression was given in Chapter III in equation (37a)

\[
U^{(1)}_0(q) = \langle k' | U^{(1)}_0 | k \rangle = -\left(\frac{4\pi \hbar^2}{m}\right) M(o) \rho(q) \tag{40}
\]

where \(m\) is the nucleon mass.

It can be seen from the derivation given in Chapter III that the above expression for \(U^{(1)}\) implies:

\[
\rho(q) = \frac{1}{(2\pi)^3} \int e^{-i q \cdot r} \rho(r) \, dr \tag{41}
\]
The transform of (40) to coordinate space is trivial and gives a local potential:

\[ \langle \Sigma | U_0^{(1)} | r' \rangle = -\left(\frac{4\pi \hbar^2}{m}\right) M(0) \rho(r) \delta(r-r') \]  

so that:

\[ U_0^{(1)}(r) = -\left(\frac{4\pi \hbar^2}{m}\right) M(0) \rho(r) \]  

(42a)

As can be seen, all that is required is the spin average of the forward scattering amplitude and the nuclear density function. These will be considered in turn.

The density function required in the above potential is normalised to the nuclear mass number N:

\[ \rho(r) = \sum_{i=1}^{N} \delta(r-r_i) \]  

(43)

If the ground state \( |0\rangle \) is a single Slater determinant of single particle states \( \mu \), then:

\[ \rho(r) = \sum_{\mu} |<\mu| r|0 \rangle|^2 \]  

(43a)

In general, the states \( |\mu\rangle \) must be found in a "realistic" way**, such as by generating them in a Saxon-Woods potential well, including a spin orbit component. The parameters are fixed by requiring that the single particle binding energies be in accordance with experimental separation energies and that the charge density should fit elastic electron scattering. Such calculations are described by Elton and

* The spin orbit term complicates this a little — this is considered below.

** The nuclei considered are assumed to be spherical.
However, as discussed by Jackson and Murugesu, for carbon and oxygen, the "realistically" generated density can be approximated quite accurately by a density generated by harmonic oscillator wavefunctions. The algebra of these functions is dealt with in many books, including that of von Buttlar. The spin-orbit force is ignored, as is the difference between neutron and proton potential wells. The only radial wavefunctions required are for \( n = 1 \) with \( \ell = 0,1 \).

\[
R_0(r) = 2\left(\frac{1}{\pi a^6}\right)^{1/4} e^{-r^2/2a^2}
\]

\[
R_1(r) = 2\left(\frac{4}{9\pi a^{10}}\right)^{1/4} r e^{-r^2/2a^2}.
\]

These give a density function as follows:

\[
\rho(r) = 4\left(\frac{1}{\pi a^2}\right)^{3/2} \left(1 + \frac{r^2}{a^2}\right) e^{-r^2/a^2}
\]

where \( b = 4/3 \) for carbon and \( b = 2 \) for oxygen. The only free parameter is the radius parameter \( a \), which is related to the mean square radius by:

\[
a^2 = \frac{4 + 6b}{6 + 15b} \langle r^2 \rangle.
\]

The mean square radius found from electron scattering is that of the charge distribution. The density appearing in (42a) is the matter distribution. As discussed by Slalina and McManus, the radii of these distributions are related by:

\[
\langle r^2 \rangle_{\text{ch.}} = \langle r^2 \rangle_{\text{m}} + \langle r^2 \rangle_{\text{p}}
\]
where $<r^2>_p$ represents the finite size of the proton as "seen" by an electron:

$$<r^2>_p \approx 0.64 \text{ fermi}.$$  

The value of $<r^2>_{ch}$. is obtained from electron scattering and the results of various analyses are found in the report of Hofstadter and Collard\(^{(88)}\). The values used here are:

Carbon $<r^2>_{ch}^{\text{C}} = 2.41 \text{ f.}$; Oxygen $<r^2>_{ch}^{\text{O}} = 2.71 \text{ f.}$

A comparison between a "realistic" calculation of $\rho(r)$, using the Elton-Swift\(^{(84)}\) potential, and a harmonic oscillator potential of the same radius is shown in Fig. (5). As can be seen, the two cases only diverge in the extreme tail, where the harmonic oscillator results are too small, as expected. This divergence will only affect the small angle scattering and such effects will be submerged in the Coulomb scattering for protons. A far more serious effect on the tail of $U^{(1)}$ is discussed in section 4.3.

The two body scattering amplitude is also required for the potential of equation (42a). As an operator in the spins $\sigma$, $\sigma_i$ and isospins $I$, $I_i$ of the projectile and target nucleon (i), $M_i$ may be written in the Stapp\(^{(89)}\) parameterization:

$$M_i = [A_o + C_o (\sigma + \sigma_i) \cdot \vec{\eta} + B_o \sigma \cdot n \sigma_i \cdot n$$
$$+ \frac{1}{2} (G_{o} + H_{o}) \sigma \cdot \vec{m} \sigma_i \cdot \vec{m} + \frac{1}{2} (G_{o} - H_{o}) \sigma \cdot \vec{e} \sigma_i \cdot \vec{e}]$$
$$+ [A_t + C_t (\sigma + \sigma_i) \cdot \vec{\eta} + B_t \sigma \cdot n \sigma_i \cdot \vec{\eta}$$
$$+ \frac{1}{2} (G_{t} + H_{t}) \sigma \cdot \vec{m} \sigma_i \cdot \vec{m} + \frac{1}{2} (G_{t} - H_{t}) \sigma \cdot \vec{e} \sigma_i \cdot \vec{e}] \tau \cdot \tau_i ,$$
where, in terms of relative momenta, the unit vectors are:

\[ \vec{\gamma} = \frac{\vec{k}_o \wedge \vec{k}'_o}{|\vec{k}_o \wedge \vec{k}'_o|}, \quad \vec{\eta} = \frac{\vec{k}_o + \vec{k}'_o}{|\vec{k}_o + \vec{k}'_o|}, \quad \vec{\chi} = \frac{\vec{k}_o - \vec{k}'_o}{|\vec{k}_o - \vec{k}'_o|}. \]

The coefficients are functions of \( k_o \) and \( k'_o \); this momentum dependence will be discussed in the next section. For the moment only forward scattering is considered. The coefficients can be expressed in terms of the scattering for a given isospin state:

\[ A_o = \frac{1}{4}(3A_3 + A_1), \quad A_1 = \frac{1}{4}(A_3 - A_1). \]

\( A_3 \) is the coefficient for the isospin triplet state and \( A_1 \) corresponds to the singlet state.

The first order potential (40) requires an average of \( M_i \) over all nuclear nucleons (\( i \)). In a spin-zero, isospin-zero nucleus (e.g. \( C^{12}, O^{16} \)) terms in \( g_i \) and \( i_i \) average to zero. Thus:

\[ \bar{\mathcal{M}} = A_o + C_o \cdot \frac{\vec{k}_o \wedge \vec{k}'_o}{|\vec{k}_o \wedge \vec{k}'_o|}. \quad (47a) \]

Since \( \bar{\mathcal{M}} \) is an invariant, the above can be expressed in terms of momenta in the \((N+1)\) body frame:

\[ C_o \frac{\vec{k}_o \wedge \vec{k}'_o}{|\vec{k}_o \wedge \vec{k}'_o|} = C_o \frac{\vec{k} \wedge \vec{k}'}{|\vec{k} \wedge \vec{k}'|}. \]

Also, by ignoring off shell effects (see fig. 4) one has:
Therefore: \( \frac{k}{k_o} \frac{\kappa}{\kappa_o} = \frac{k}{k_o} \frac{k}{k_o} \frac{k}{k_o} \).

For elastic two body scattering: \( |k_o \times k_o'| = k_o^2 \sin \theta_o \).

Hence: \( C_o \times \bar{n} = \frac{C_o}{k_o k \sin \theta_o} \times \bar{n} \times k \times k' \).

For convenience one can define \( \tilde{C}_o \) by:

\[
\tilde{C}_o = \frac{C_o}{k_o k \sin \theta_o} .
\]

The forward scattering approximation consists here of retaining \( g \times k \times k' \) but evaluating \( \tilde{C}_o \) for forward scattering.

Then: \( \tilde{C}_o(0) = \frac{1}{k_o} \tilde{C}_o(0) \quad \tilde{C}_o(0) = \left. \frac{C_o(\theta_o)}{\sin \theta_o} \right|_{\theta_o=0} \).

The potential then is:

\[
U_o^{(1)}(q) = -\left(\frac{4\pi n^2}{m}\right) [A_o(0) + \tilde{C}_o(0) g \times k \times k'] \rho(q) .
\]

The algebra of the Fourier transform of \( U^{(1)}(q) \) has been presented by Riesenfeld and Watson (32). The result is:

\[
U_o^{(1)}(r) = -\left(\frac{4\pi n^2}{m}\right) \left[ A_o(0) + i\tilde{C}_o(0) g \times \frac{1}{r} \frac{d}{dr} \right] \rho(r) , \quad (48)
\]
where $\ell$ is the orbital angular momentum of the projectile.

The potential of equation (48) will be the purely nuclear potential and will be the same for proton and neutron scattering from $S = 0, T = 0$ nuclei, if the two body scattering coefficients of equation (47) are those calculated from the two body nuclear force. Such coefficients were calculated by K.M.T. $^{(42)}$ from the Gammel-Thaler $^{(90)}$ phase shifts and these values are used throughout this work. Many other phase shift sets exist which fit the two body data and, as described in chapter I, other authors have attempted to use nucleon-nucleus scattering to distinguish between such sets. In this work the aim is to compare different calculations with the same two body data, so any reasonable set will suffice. The K.M.T. calculations of two body data were chosen purely on grounds of convenience.

The Coulomb effects in proton-nucleus scattering are included by taking the coulomb potential to be that due to a uniform charge distribution of the same mean square radius as the nuclear charge distribution.

\[
U_{\text{Coul}}(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{zZe^2}{r} & r > R_c \\ \frac{1}{4\pi\varepsilon_0} \frac{zZe^2}{2R_c} \left(3 - \frac{r^2}{R_c^2}\right) & r < R_c \end{cases}
\]  

(49)

where:

\[
z = 1 \quad Z = \text{Nuclear charge}
\]

\[
R_c^2 = \frac{5}{2} \langle r^2 \rangle_{\text{ch}}.
\]
The neglect of the angular variation of the scattering coefficients in $U_o^{(1)}$ corresponds to the use of a local, delta-function two body $t$-matrix in coordinate space. This use of a zero range force is only justified in the case of large nuclei where the true range of this force is much less than the nuclear radius. The resulting potential, $U_o^{(1)}$, is energy dependent only in its depth - its shape is energy independent.

4.3 The Angular Variation of the Two Body Amplitudes

To maintain the local nature of $U^{(1)}$, and yet include the angular variation of $M$ in equation (37), $M$ must be considered to be a function of momentum transfer $q$ only. How far is this justifiable? In fig. (6) the central term, $A_0$, as calculated by K.M.T., is plotted against two body centre of mass scattering angle, $\theta_0$. Since $\theta_0$ is very simply related to $q$ for elastic scattering:

$$q = 2k_0 \sin \frac{\theta_0}{2}$$

(50)

it is tempting to fit $A_0(\theta_0)$ as a function of $q$. The data stops, of course, at $q = 2k_0$, corresponding to $\theta_0 = 180^0$. The peak at backward scattering angles in the real part of $A_0$ leads to an immediate pitfall. If $A_0$ is a function of $q$ only, then the central part of $U^{(1)}$ becomes:

$$\left( - \frac{m}{4\pi\hbar^2} \right) U_c^{(1)}(r) = \int e^{iq \cdot \vec{r}} A_0(q) \rho(q) dq$$.

If the function of figure (6) is represented by a falling
Fig. 6: Two Body Amplitude $\tilde{A}_o$ as function of Scattering Angle

"Direct"

$$f(\mathbf{k}_o' - \mathbf{k}_o)$$

"Exchange"

$$f(\mathbf{k}_o' + \mathbf{k}_o)$$

Fig. 7: Scattering Events in the C.M. Frame
function of \( q \) and a delta function at \( q = 2k_0 \):

\[
A_0(q) = a(q) + b \delta(q-2k_0)
\]

then \( U_0^{(1)}(r) \) becomes:

\[
\left( -\frac{m}{4\pi\hbar^2} \right) U_0^{(1)}(r) = V_a(r) + \frac{2k_0b}{r} \sin(2k_0r)\rho(q=2k_0)
\]

where \( V_a(r) \) is a "sensible" potential due to the falling function \( a(q) \). The second term in the above is a slowly falling, oscillatory function of \( r \) and this behaviour is confirmed by calculations with "realistic" peaks in \( A_0 \) around \( q = 2k_0 \). This second term would be negligible if \( \rho(q=2k_0) \) were negligible, as is the case in heavy nuclei. However, for \( E = 90 \text{ MeV}, 2k_0 = 2.08 \text{ fm}^{-1} \) and for \( {^1}C_{12} \) and \( {^1}O_{16} \), \( \rho(q=2.08) \) is still about 2% of \( \rho(q=0) \).

This peculiar behaviour of \( U^{(1)} \) indicates that something is amiss with the above treatment of \( A_0(\theta_0) \). The peak at backward angles is, of course, an exchange effect, which results in the following characteristic behaviour for the two body scattering amplitude:

\[
M(k_0',k_0) = \alpha f_1(k_0'-k_0) + \beta f_2(k_0'+k_0).
\]  

Both terms result in scattering through an angle \( \theta_0 \); they are illustrated in fig. (7). The exchange term is present for two reasons:

(i) In n-n or p-p scattering, the particles are experimentally indistinguishable, so that scattering
through $\theta_0$ and $\pi-\theta_0$ are indistinguishable.

(ii) The two nucleon force itself contains strong exchange components, in particular the Majorana \cite{91} space-exchange force and the Heisenberg \cite{92} charge-exchange force. Such forces result in strong backward angle scattering in n-p experiments as well as in the n-n and p-p cases.

The exchange term results in a non-local potential. This problem has been studied by Slalina and McManus \cite{87}. They find that the equivalent local potential changes the potential obtained from direct forces but very little at high energies. In view of this, the procedure adopted to deal with the angular variation of $\bar{M}(k'_0,k_0)$ is that adopted by Goldberger \cite{82}. The functions $f_1$ and $f_2$ of equation (51) tend to be large only when their arguments are close to zero, so that for small angle scattering only $f_1(k'_0-k_0)$ should be important. Then, exactly as suggested by Goldberger, the most practical procedure is to fit the experimentally observed nucleon-nucleon angular distribution with a function of $q$ in the range $0 < \theta_0 < \pi/2$ and use this function to represent $\bar{M}(q)$ for all values of $q$. Equation (40) is then replaced by:

$$u^{(1)}(q) = -\frac{4\pi \hbar^2}{2} \bar{M}(q)\rho(q)$$

where:

$$\bar{M}(q) = A_0(q) + C_0(q)\sigma \cdot \vec{n}.$$

In fact, the functions to be fitted from two body data are
A\(_o\)(q) and the quantity \(v\)\(_o\)(q) of equation (47b) which is independent of the nucleus under consideration - unlike the quantity \(v\)\(_o\)(q). The first order potential then becomes:

\[
U^{(1)}(r) = U_c^{(1)}(r) + \frac{1}{r} \frac{dU^{(1)}_{so}}{dr} \hat{q} \cdot \hat{\ell} 
\]  

(52)

where:

\[
U_c^{(1)}(r) = -\frac{4\pi \hbar^2}{m} \int dq \, e^{iq \cdot r} A\(_o\)(q) \rho(q) 
\]  

(52a)

\[
U^{(1)}_{so}(r) = -\frac{4\pi \hbar^2}{m} i \int dq \, e^{iq \cdot r} \tilde{v}\(_o\)(q) \rho(q) . 
\]  

(52b)

An important consequence of equations (52) is that the mean square radius of the optical potential is given by the sum of the mean square radii of \(\rho\) and the relevant coefficient of \(\tilde{M}\).

\[
\langle r^2 \rangle_u = \langle r^2 \rangle_{\rho} + \langle r^2 \rangle_{M} . 
\]

This follows from the relation between \(\langle r^2 \rangle_u\) and \(U(q)\):

\[
\langle r^2 \rangle_u = \left[ - \frac{\nabla^2 U(q)}{\hat{\rho} U(q)} \right]_{q=0} 
\]

which means that if for small \(q\), \(U(q)\) can be written:

\[
U(q) = \alpha[1 + \beta q^2 + \ldots \ldots] 
\]

(53)

then:

\[
\langle r^2 \rangle_u = -6\beta . 
\]

The above shows that a function of finite mean square radius
cannot have a term in $q$ in an expansion for small values of $q$
(and from (53), for non zero $\langle r^2 \rangle$, $\beta$ must be non-zero.)

The real and imaginary parts of the functions $A_0(q)$ and
$\hat{C}_0(q)$ were separately fitted at the energies $E = 90, 156,$
310 MeV by functions of the form:

$$
\begin{align*}
A_0(q) & = a(1+bq^2)e^{-cq^2} \\
\hat{C}_0(q) & = b(1+bq^2)e^{-cq^2}
\end{align*}
$$

where $b$ and $c$ are parameters which were varied to give a
least-squares fit. The parameter $a$ was fixed by demanding
that $A_0(q=0)$ be correctly fitted. This is an important
restriction as the presence of $\rho(q)$ restricts the function
$U(q)$ to values close to $q = 0$. The choice of the function
of equation (54) makes it possible to fit a scattering
coefficient which takes negative values as well as positive
values.

In table (I) are shown the parameters obtained from the
fitting procedure. The contribution to the mean square
radius is also tabulated:

$$
\langle r^2 \rangle_M = 6(c-b) \quad \text{fermi}^2.
$$

Typical fits are shown in figures (8) and (9) (for the case
of 90 MeV). The comparison of $U(1)$ and $U_0(1)$ is made in
figures (10) and (11) (for nucleon–$^{12}\text{C}$ scattering at 90 MeV).
The scattering from $U(1)$ and $U_0(1)$ is plotted in figures (12)
and (13).

These calculations show the importance of the angular
variation of the two body scattering amplitudes. The results
essentially confirm the conclusions of McDonald and Hull (65). The central part of the potential is made considerably more diffuse — its tail is much extended and its central region somewhat reduced — by including the two body angular variation. This increase in diffuseness is so large that the spin–orbit potential no longer peaks in the surface region, although its value in this region is not much changed. Also, because of the removal of the central depression in $U_{so}$, the spin–orbit potential no longer changes sign near the origin. This difference has little effect on the scattering observables, as near the origin, the centrifugal force dominates the spin–orbit force. As expected, the small angle scattering is increased while the large angle scattering is reduced — the "rotation" of the differential cross section noted by McDonald and Hull. Unless otherwise indicated, the potential $U^{(1)}$ of equations (52) will be employed subsequently as the first order potential.
TABLE I

The parameters for the fitted two body scattering amplitudes.

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>(a) (fermi)</th>
<th>(b) (fermi(^2))</th>
<th>(c) (fermi(^2))</th>
<th>(\langle r^2 \rangle_M) (fermi(^2))</th>
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<td>(E = 90) MeV</td>
<td>0.59250</td>
<td>0.28043</td>
<td>0.90676</td>
<td>3.758</td>
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<td>0.182</td>
</tr>
</tbody>
</table>
Fig. 8: Fits Obtained to Two Body Amplitude $\tilde{A}_o$. 
Fig. 9: Fits Obtained to Two-Body Amplitude $C_\Omega$.
Comparison of Potentials from Zero Range and Finite Range

Fig. 10: The Central Potentials

Fig. 11: The Spin-Orbit Potentials
4.4 Other Causes of Non-locality in $U^{(1)}$

$U^{(1)}$ is non-local because the matrix elements of the t-operator appearing in $U^{(1)}$ are non-local. Even after impulse approximation has been made, $U^{(1)}$ is non-local, as was shown above, due to the exchange nature of the two body force. There are other causes. The direct two body force is probably non-local — one way of dealing with the Jastrow hard core is to use a non-local two body force as shown by Herndon et al. (93) and Darewych and Green (94). Even if the two body potential was purely a direct, local object, $U^{(1)}$ requires the evaluation of the two body t-matrix off the two body energy shell and, as shown by Mulligan (71), this results in a non-local $U^{(1)}$.

By utilising equations (52) for $U^{(1)}$, all forms of non-locality are being ignored. To put this another way, the assumption being made is that the various operators involved have a far stronger dependence on $(k'-k)$ than they have on $(k'+k)$. This is probably reasonable for small angles of scattering, where a change in scattering angle has a large effect on $(k'-k)$ but only a small effect on $(k'+k)$.

4.5 The Non-locality of $U^{(2)}$

$U^{(2)}$ in momentum space is given by equation (39) of chapter III. The configuration space form of $U^{(2)}$ is given by:

$$
<r|U^{(2)}|r'> = \int \int \frac{dk' dk e^{i(k'-k)\cdot r} e^{-ik\cdot r'}}{(2\pi)^3} <k'|U^{(2)}|k'> .
$$

Ignoring complications caused by the spin-orbit force, this
Fourier transformation can be carried out to give:

\[
\langle \mathbf{r} | U^{(2)} | \mathbf{r}' \rangle = \langle \mathbf{r} | \frac{1}{E-K+i\epsilon} | \mathbf{r}' \rangle \sum_{\mathbf{r}_1, \mathbf{r}_2} \rho^{(2)}(\mathbf{r}, \mathbf{r}') t_1^f t_2^f - \rho(\mathbf{r}) \rho(\mathbf{r}') t_1^f t_2^f \]

(55)

where the same normalisation has been maintained for \( \rho(\mathbf{r}) \) and \( t^f \) as is used in the case of \( U^{(1)} \). This is defined by equations (40) and (41). The other expressions are:

\[
\rho^{(2)}(\mathbf{q}', \mathbf{q}) = \frac{1}{(2\pi)^6} \int d\mathbf{q}' d\mathbf{q} e^{-i\mathbf{q}' \cdot \mathbf{r}} e^{-i\mathbf{q} \cdot \mathbf{r}'} \rho^{(2)}(\mathbf{r}, \mathbf{r}') \]

(56)

\[
\langle \mathbf{r} | \frac{1}{E-K+i\epsilon} | \mathbf{r}' \rangle = \int \frac{d\mathbf{k}''}{(2\pi)^3} \frac{1}{E-K(k'')+i\epsilon} \frac{e^{i\mathbf{k}'' \cdot (\mathbf{r} - \mathbf{r}')}}{E-K(k'')+i\epsilon}
\]

\[
= -\frac{2\mu}{4\pi^2} \frac{e^{ik'' \cdot (\mathbf{r} - \mathbf{r}')}}{4\pi^2} ,
\]

(57)

where \( \mu \) is the nucleon-nucleus reduced mass and \( k''^2 = 2\mu E/\hbar^2 \).

The potential of equation (55) is highly non-local and only the most violent approximations can make it local. (This "violence" is considered in chapter V). However, if \( \rho^{(2)}(\mathbf{r}, \mathbf{r}') \) is generated by shell model wavefunctions, the following structure is obtained:

\[
\rho^{(2)}(\mathbf{r}, \mathbf{r}') = \sum_{\alpha, \beta} \varphi_{\alpha}(\mathbf{r}) \varphi_{\beta}(\mathbf{r}') ,
\]

(58)

where both \( \alpha \) and \( \beta \) involve two sets of single-particle quantum numbers (as yet unspecified). Since the term involving the one-particle densities can also be absorbed
into the sum (58), \( U^{(2)} \) can be written:

\[
\langle r | U^{(2)} | r' \rangle = \sum_{\gamma, \delta} \Gamma_{\gamma, \delta} \phi_{\gamma}(r) \langle r | \frac{1}{E-K+i\epsilon} | r' \rangle \phi_{\delta}(r'), \tag{59}
\]

where the \( \Gamma_{\gamma, \delta} \) represents the product of t-matrices relevant to the term in \( \phi_{\gamma} \cdot \phi_{\delta} \). Now, in a Schrödinger equation involving the local \( U^{(1)} \) and the non-local \( U^{(2)} \), one has:

\[
[E-K-U^{(1)}(r)] \langle r | \psi_k \rangle = \int dr' \langle r | U^{(2)} | r' \rangle \langle r' | \psi_k \rangle 
= \sum_{\gamma, \delta} \Gamma_{\gamma, \delta} \phi_{\delta}(r) \int dr' \langle r | \frac{1}{E-K+i\epsilon} | r' \rangle \phi_{\delta}(r') \langle r' | \psi_k \rangle. \]

The fact that \( U^{(2)} \) is a sum of separable terms multiplied by the propagator means that \( \langle r | \psi_k \rangle \) can be calculated exactly.

A subsidiary set of functions \( \langle r | \chi_\delta \rangle \) is defined as follows:

\[
\langle r | \chi_\delta \rangle = \int dr' \langle r | \frac{1}{E-K+i\epsilon} | r' \rangle \phi_{\delta}(r') \langle r' | \psi_k \rangle. \tag{60}
\]

The propagator in the above function is just the free particle Green's function, which satisfies:

\[
(E-K) \langle r | \frac{1}{E-K+i\epsilon} | r' \rangle = \delta(r-r'). \tag{61}
\]

Hence, one obtains coupled equations for \( \langle r | \psi_k \rangle \) and \( \langle r | \chi_\delta \rangle \)

\[
[E-K-U^{(1)}(r)] \langle r | \psi_k \rangle = \sum_{\gamma, \delta} \Gamma_{\gamma, \delta} \phi_{\delta}(r) \langle r | \chi_\delta \rangle 
[E-K] \langle r | \chi_\delta \rangle = \phi_{\delta}(r) \langle r | \psi_k \rangle. \tag{62}
\]
Equations (62) are the formal statement of the method used to calculate the scattering from the first and second order optical potentials. Only if the number of subsidiary functions $\langle \Xi | \chi_5 \rangle$ is small (typically less than ten) is the method practicable. Fortunately, for light nuclei with closed shells, this is the case.
"The [scientist] of today is not a romantic animal... but a quiet, grave man, busied in charts, exact in sums, occupied in trivial detail..."

- Bagehot: The English Constitution

5.1 Connection between $U^{(2)}$ and the Two Body Density Function

This chapter gives an account of the details involved in employing $U^{(2)}$. The goal of the process is a differential radial equation for the wavefunction of a particle scattered by the optical potential. The algebra of $U^{(2)}$ is complicated both by its non-locality and by the fact that it involves spin averages over a product of two body amplitudes, which are themselves complicated operators. The notation used here is that of Chalmers and Saperstein.\(^{(69)}\)

After making the approximations of chapter III, including the forward scattering approximation for the two body $t$-matrix, $U^{(2)}$ can be written:

$$
\langle k' | U^{(2)} | k \rangle = \int dk'' g(k'') \left\{ \sum_i \sum_i (0| e^{-i q' \cdot \Sigma_{i} e^{-i q \cdot \Sigma_{j} t_{i} t_{j}}} | 0) 
- \sum_i \sum_j (0| e^{-i q' \cdot \Sigma_{i} t_{i}} | 0)} (0| e^{-i q \cdot \Sigma_{j} t_{j}} | 0) \right\}
$$

where:

$$g(k'') = \frac{1}{E - K(k'') + i\epsilon}$$

$$q' = k' - k'' \quad q = k'' - k$$

and $t$-matrices are operators only in spin space. To simplify
the spin averages, it is usual to assume that there are only two distinct two-body density functions - for space symmetric and space antisymmetric states, denoted $\rho_{s}^{(2)}$ and $\rho_{a}^{(2)}$. Then, using the methods of Lax and Feshbach as developed by Fowler and Watson, one has:

$$<k'|U^{(2)}|k> = \int dk'' \sum_{ij} \left[ \frac{\rho_{s}^{(2)}(q_1,q_2)+\rho_{a}^{(2)}(q_1,q_2)}{2N(N-1)} \left( \sum_{i\neq j} (0|t_{ij}|0) \delta(q_1-q_2) - \delta(q_1+q_2) \right) + \frac{\rho_{s}^{(2)}(q_1,q_2)-\rho_{a}^{(2)}(q_1,q_2)}{2N(N-1)} \sum_{i \neq j} (0|t_{ij}|P_{ij}|0) \right]$$

where $P_{ij}$ is the space exchange operator, and:

$$\rho_{s,a}^{(2)}(r,r') = \sum_{ij} \delta(r-r_i)\delta(r-r_j)\frac{1}{2}P_{ij}|0\rangle \langle 0| \tag{64a}$$

In the Chalmers and Saperstein notation, the spin averages of $t_i$ and $t_j$ are, for $T = 0, S = 0$ nuclei:

$$\bar{X} = \sum_{ij \neq i} (0|t_{ij}|0) = \left(\frac{4\pi\hbar^2}{m}\right)^2 \left[ \frac{N^2\alpha - N\beta}{4N}\right] \tag{65a}$$

$$\bar{Y} = \sum_{ij \neq i} (0|t_{ij}|P_{ij}|0) = \left(\frac{4\pi\hbar^2}{m}\right)^2 \left[ \frac{4N\alpha - \frac{N^2}{4\beta}}{4N}\right] \tag{65b}$$

$$\bar{Z} = \sum_{ij} (0|t_{i}|0)(0|t_{j}|0) = \left(\frac{4\pi\hbar^2}{m}\right)^2 \left[ \frac{N^2\alpha}{4N}\right] \tag{65c}$$

where, in terms of the scattering coefficients of equation...
(47) one has:

\[
\alpha = A_o^2 + C_o^2 + 2A_o C_o \mathbf{g} \cdot \mathbf{n}
\]

\[
\beta = [A_o^2 + 2C_o^2 + B_o^2 + \frac{1}{2}(G_o^2 + H_o^2) + 2C_o(A_o + B_o) \mathbf{g} \cdot \mathbf{n}]
+ 3[A_t^2 + 2C_t^2 + B_t^2 + \frac{1}{2}(G_t^2 + H_t^2) + 2C_t(A_t + B_t) \mathbf{g} \cdot \mathbf{n}].
\]

Johnston (37) has shown that for small angles:

\[
G_o + H_o \approx 2B_o \quad G_t + H_t \approx 2B_t.
\]

Also, \( C_o \) is zero at zero angle, so that, except in the \( \mathbf{g} \cdot \mathbf{n} \) terms, it can be ignored. Eliminating \( G_o \) and \( G_t \) from \( \alpha \) and \( \beta \) gives:

\[
\alpha = A_o^2 + 2A_o C_o \mathbf{g} \cdot \mathbf{n} \quad \text{(66a)}
\]

\[
\beta = [A_o^2 + 3B^2 - 2B_o H_o + H_o^2 + 2(A_o + B_o)C_o \mathbf{g} \cdot \mathbf{n}]
+ 3[A_t^2 + 3B_t^2 - 2B_t H_t + H_t^2 + 2(A_t + B_t)C_t \mathbf{g} \cdot \mathbf{n}] \quad \text{(66b)}
\]

A useful modification of this notation is:

\[
\tilde{X} = X_o + X_s \mathbf{g} \cdot \mathbf{n} \quad \alpha = \alpha_o + \alpha_s \mathbf{g} \cdot \mathbf{n} \quad \text{etc.} \quad \text{(67)}
\]

The coefficients \( \alpha_o, \alpha_s \) etc. can be identified from equations (66). As previously mentioned, the scattering coefficients are taken from K.M.T. (42) (they are also listed by Johnston (37)). The averages given by equations (66) are:
It is also convenient to define a "direct" and a "space exchange" two body density as follows:

\[ \rho_{D}^{(2)} = \rho_{s}^{(2)} + \rho_{a}^{(2)} \quad \rho_{SE}^{(2)} = \rho_{s}^{(2)} - \rho_{a}^{(2)} \]

so that:

\[ \rho_{D}^{(2)}(\mathbf{r}, \mathbf{r}') = \sum_{i \neq j} \left( 016(\mathbf{r}-\mathbf{r}_i)\delta(\mathbf{r}'-\mathbf{r}_j)10 \right) \]

\[ \rho_{SE}^{(2)}(\mathbf{r}, \mathbf{r}') = \sum_{i \neq j} \left( 016(\mathbf{r}-\mathbf{r}_i)\delta(\mathbf{r}'-\mathbf{r}_j)P_{ij}10 \right) \]

In terms of these quantities, \( U^{(2)} \) becomes:

\[ \langle k' | U^{(2)} | k \rangle = \int d\mathbf{k}'' g(k'') \left\{ \frac{\rho_{D}^{(2)}(q,q)}{N(N-1)} \frac{X}{2} + \frac{\rho_{SE}^{(2)}(q,q)}{N(N-1)} \frac{Y}{2} \right\} \]

\[ - \frac{\rho(q')\rho(q)}{N^2} \frac{Z}{2} \].

*The meaning of the hat on \( \hat{\alpha}_s \) and \( \hat{\beta}_s \) is the same as for \( \hat{\alpha}_o \) in chapter IV: equation (47b).
5.2 The Local Approximation for $U^{(2)}$

The local approximation is presented here for purposes of comparison with the results obtained from the coupled equations method. Also, as the spin-orbit potential correction arising from $U^{(2)}$ is small compared to the spin-orbit potential arising from $U^{(1)}$ and since it is not readily amenable to the coupled equations method, it will be dealt with in the manner presented here.

Re-writing equation (69) in terms of the capital "P" forms of the density distributions, one has:

$$
\begin{align*}
\mathcal{P} &= 1 \mathcal{N} \\
\mathcal{P}_s^{(2)} &= \frac{1}{\mathcal{N}(\mathcal{N}-1)} \mathcal{P}_s^{(2)} \\
\mathcal{P}_a^{(2)} &= \frac{1}{\mathcal{N}(\mathcal{N}-1)} \mathcal{P}_a^{(2)}
\end{align*}
$$

and using the explicit Fourier transform expressions given by equations (41) and (56) gives:

$$
<k' | U^{(2)} | k> = \frac{1}{(2\pi)^6} \int dk'' \mathcal{g}(k'') \int d\mathbf{r} \int d\mathbf{r}' \quad e^{-i\mathbf{q} \cdot \mathbf{r}} e^{-i\mathbf{q} \cdot \mathbf{r}'} .
$$

$$
\begin{align*}
\left\{ \frac{\mathcal{P}_s^{(2)}(\mathbf{r},\mathbf{r}') + \mathcal{P}_a^{(2)}(\mathbf{r},\mathbf{r}')} 2 \right\} \\
+ \frac{\mathcal{P}_s^{(2)}(\mathbf{r},\mathbf{r}') - \mathcal{P}_a^{(2)}(\mathbf{r},\mathbf{r}')}{2} \cdot \mathcal{P}(\mathbf{r})\mathcal{P}(\mathbf{r}') \end{align*}
$$

The first step in making the local approximation consists of expressing the two body density functions in the form:

$$
\mathcal{P}_{s,a}^{(2)}(\mathbf{r},\mathbf{r}') \sim \mathcal{P}(\mathbf{r})\mathcal{P}(\mathbf{r}')[1 + \mathcal{G}_{s,a}(\mathbf{r}-\mathbf{r}')] .
$$

(70)
This step is only possible if the functions $G_s$ and $G_a$ can be found. $G_s$ is the space-symmetric correlation function and $G_a$ the space-antisymmetric one. Since calculations are to be done in the "independent particle model", only Pauli correlations are considered. These arise because the ground state wave function is not a simple product of single particle states, but a Slater determinant of these states. Only if a simple model of the nucleus is chosen, can an expression of the form (70) be found analytically. Almost invariably the Fermi gas model is used. For a light nucleus, Chalmers and Saperstein (69) find:

$$G_s = \frac{1}{4N+1} G', \quad G_a = -G'$$

$$G'(\mathbf{x}) = \frac{1}{4N(4N-1)} \sum_i \sum_{j \neq i} e^{-i(k_i-k_j) \cdot \mathbf{x}}$$

where:

$$\mathbf{x} = \mathbf{r} - \mathbf{r}' .$$

The factors of $\frac{1}{4N}$ come from the fact that there are four nucleons per Fermi state. The summation is treated by adding and subtracting the $(i=j)$ terms:

$$\sum_i \sum_{j \neq i} e^{-i(k_i-k_j) \cdot \mathbf{x}} = \sum_i \sum_j e^{-i(k_i-k_j) \cdot \mathbf{x}} - \frac{4}{4N} . \tag{71}$$

The double sum is approximated by an integral of the form:

$$\sum_i \sum_j e^{-i(k_i-k_j) \cdot \mathbf{x}} \approx \left(\frac{4}{4N}\right)^2 \left[ \frac{1}{4\pi^2k_F^2} \int_{k<k_F} \frac{dk}{k} e^{-ik \cdot \mathbf{x}} \right]$$

$$= \left(\frac{4}{4N}\right)^2 \left[ \frac{4}{k_F^2} \frac{(k_F x)^2}{2} \right] = \left(\frac{4}{4N}\right)^2 G_F(x) .$$
Putting this in (70) gives:

\[ G'(x) = \frac{\frac{1}{4} N G_F(x) - 1}{\frac{1}{4} N - 1} \]  \hspace{1cm} (72)

That these are reasonable results for light nuclei can be seen by considering He\(^4\) where all the nucleons are in the same spatial state. Also (\(\frac{1}{4} N - 1\)) is zero, so the He\(^4\) limit has to be taken carefully. Since all the \(k_i\) are equal, the sum in equation (71) just gives:

\[ \sum_i \sum_j e^{-i(k_i - k_j) \cdot x} = \frac{(iN)^2}{4N} - \frac{1}{4N} \]

which means: \(G' = 1\)

and therefore: \(G_S = 0\) \hspace{0.5cm} \(G_a = -1\).

Equations (70) then show that:

\[ P_s^{(2)}(\bar{r}, \bar{r}') = P(r)P(r') \hspace{1cm} P_a^{(2)}(\bar{r}, \bar{r}') = 0 \]

which is the correct result for He\(^4\) where any two nucleons are in a spatially symmetric state. Of course, for other nuclei, the integration over plane waves carried out to give \(G_F(x)\) means that the expressions are only rough estimates of the \(G_S\) and \(G_a\) which would be obtained with realistic wave functions.

If the Fermi gas results are used in the \(P_s^{(2)}\) of equation (70) and these are put into (69a) together with the full expressions (65) for \(\bar{X}, \bar{Y}, \bar{Z}\), a remarkable simplification of the bracket of (69a) occurs:
\[ \left\{ \frac{P_s(\mathbf{r}, \mathbf{r}') + P_a(\mathbf{r}, \mathbf{r}')} {2} \right\} \frac{1}{x} + \left\{ \frac{P_s(\mathbf{r}, \mathbf{r}') - P_a(\mathbf{r}, \mathbf{r}')} {2} \right\} \frac{1}{1 - P(r)P(r')Z} \]

\[ = - \frac{1}{2} N^2 \left( \frac{4\pi \hbar^2}{m} \right)^2 \beta P(r) P(r') G_F(1r-1r') \]

with the result that \( U^{(2)} \) becomes:

\[ \langle k' | U^{(2)} | k \rangle = - \frac{1}{2} N^2 \left( \frac{4\pi \hbar^2}{m} \right)^2 \beta \frac{1}{(2\pi)^2} \frac{1}{2} \int dk'' g(k'') \int dr \int dr' e^{-i\mathbf{q} \cdot \mathbf{r}'} P(r) P(r') G_F(1r-1r') \]

\[ = - \frac{1}{2} N^2 \left( \frac{4\pi \hbar^2}{m} \right)^2 \beta I(k', k) \quad (73) \]

which defines the integral expression \( I \). Writing out the free Green's function and \( q, q' \) in full from (63):

\[ I(k', k) = \frac{2\mu}{\hbar^2} \int \frac{dk''}{(2\pi)^3} \int \frac{dr}{(2\pi)^3} \int \frac{dr'}{(2\pi)^3} \frac{e^{i\mathbf{k}'' \cdot (\mathbf{r} - \mathbf{r}')}}{k''^2 - k'^2 + i\epsilon} P(r) P(r') G_F(1r-1r') e^{-i\mathbf{k}' \cdot \mathbf{r}} e^{i\mathbf{k} \cdot \mathbf{r}'} \quad (73a) \]

To make \( I \) a function only of \( q \), the following approximations have to be made (33):

(i) Assume that \( G_F(1r-1r') \) is a short range function so that only positions satisfying \( r - r' \approx 0 \) need to be considered. Then replace \( P(r)P(r') \) by \( \left[ P(\frac{r + r'}{2}) \right]^2 \) and change the variables of integration in the equation for \( I \) to:
Chen, if a function \( F(q) \) is defined by:

\[
F(q) = \frac{1}{(2\pi)^3} \int dR \ e^{-i\mathbf{q} \cdot \mathbf{R}} \mathcal{F}^2(R)
\]

the expression for \( I \) becomes:

\[
I(k', k) = -\frac{2\mu}{4\pi \hbar^2} F(q) \int dx \ \frac{e^{-\frac{i\mathbf{k}_E \cdot \mathbf{x}}{x}}}{x} e^{i\mathbf{K} \cdot \mathbf{x}} G_F(x)
\]

where the integral over \( k'' \) in (73a) has given the usual configuration space form of the Green's function.

(ii) The remaining integral is treated in the eikonal approximation. This implies that only forward scattering is important so that \( K \) is replaced by \( k_E \) (a dangerous approximation, since it is a potential, rather than a scattering amplitude, which is being calculated). The remaining integral then is:

\[
\int dx \ \frac{e^{-\frac{i\mathbf{k}_E \cdot \mathbf{x}}{x}}}{x} e^{i\mathbf{k}_E \cdot \mathbf{x}} G_F(x) = 2\pi \int_0^\infty dx \ e^{\frac{ik_E x}{x}} \frac{e^{\frac{ik_E x}{x}} - e^{-\frac{ik_E x}{x}}}{ik_E} G_F(x)
\]

\[
\approx -\frac{2\pi}{ik_E} \int_0^\infty dx \ G_F(x)
\]

where the eikonal approximation of taking only the stationary value in the integrand has been used. A fermi correlation
length can be defined by:

\[ R_F = \int dx G_R(x) = \int_0^\infty dx \left[ \frac{j_1(k_Fx)}{k_Fx} \right]^2. \]

The integration can be done analytically (95):

\[ R_F = \frac{3\pi}{5k_F} \quad \text{i.e. } R_F \approx 1.38 \text{ fermi} \quad (75) \]

where the value obtained is that given by using \( k_F = 1.36 \text{ fermi}^{-1} \). Collecting these results into \( I \), one has:

\[ I(k', k) = \frac{\mu R_F}{i\hbar^2 k_E} F(q) \]

so that the local approximation to \( U(2) \) is, from (75):

\[ \langle k' | U(2) | k \rangle = \frac{\mu R_F}{i\hbar^2 k_E} \left( \frac{4\pi^2}{m} \right)^2 (i\beta)^4 N^2 R_F F(q). \quad (76) \]

To transform to configuration space, one splits \( \beta \) into its spin independent part, \( \beta_0 \), and its spin dependent part, \( \beta_s \alpha \), as in equation (67). The transform of the potential \( U(2) \) then goes through as described for \( U(1) \) in chapter IV (in the forward scattering case). The result is:

\[ U(2)(r) = \frac{\mu}{4\hbar^2 k_E} \left( \frac{4\pi^2}{m} \right)^2 R_F \left[ i\beta_0 \rho^2(r) - 2\beta_s \beta \rho(r) \frac{d\rho(r)}{dr} \right]. \quad (77) \]

where:

\[ \tilde{\beta}_s = \frac{1}{kk_0} \frac{\beta_s(\theta_0)}{\sin \theta_0} \bigg|_{\theta_0=0} = \frac{1}{kk_0} \beta_s. \]
and the renormalised density has been used, together with the relation:

$$\frac{1}{r} \frac{d}{dr} \rho^2(r) = \frac{1}{r} \frac{d\rho^2}{d\rho} \frac{d\rho}{dr} = \frac{2}{r} \rho \frac{d\rho}{dr}. $$

Since a factor $\rho(r)$ can be factorised out of $U^{(2)}(r)$ and in view of the approximations made to reach (77), this factor of $\rho(r)$ is often replaced by the Fermi gas density:

$$\rho(r) \rightarrow \rho_0 = 4 \frac{k_F^3}{6\pi^2} \text{ and } \frac{1}{r} \frac{d\rho^2}{d\rho} \rightarrow \frac{\rho_0}{r} \frac{d\rho_0}{dr}$$

giving a potential dependent on the density function $\rho(r)$ in a familiar way:

$$U^{(2)}(r) = \frac{\mu}{\hbar^2 k_F^2} \left( \frac{4\hbar^2}{m^2} \right) \rho_0 R_F \left[ i\beta_o - \tilde{\beta}_s g \cdot \ell \frac{1}{r} \frac{d}{dr} \right] \rho(r). \quad \text{(77a)}$$

The expression (77a) is the second order potential as found by Johnston(37) and in a different way by Glauber(40), and as used by McDonald and Hull(65).

Table II shows that for 90 MeV and 156 MeV, $\beta_o$ has positive real and imaginary parts, so that the depth of the real part of the potential is increased while the depth of the imaginary part is decreased. At 310 MeV, both real and imaginary parts are decreased. Also, since $\tilde{\beta}_s$ is proportional to $\hat{\beta}_s$, table II shows that the correction to the real spin-orbit potential (which in the first order potential is the dominant part) tends to be very small and it fluctuates in sign. The imaginary part is increased in
depth. In the approximation of equation (77a), \( U^{(2)}(r) \) has the same geometry as \( U^{(1)}(r) \) calculated in the forward scattering or "ultimate" approximation. Actual numerical values will be discussed in chapter VI.

5.3 The Independent Particle Model for \( U^{(2)} \)

Only the central part of \( U^{(2)} \) will be treated by the coupled equations method, so that, starting from equations (68) and (69) for \( U^{(2)} \) in momentum space, the Fourier transform to configuration space can be easily accomplished. Using equations (56) and (57), the result is

\[
\langle r | U^{(2)} | r' \rangle = \left\{ \frac{\rho_d^{(2)}(r, r')}{N(N-1)} \frac{\kappa_0}{2} + \frac{\rho_{SE}^{(2)}(r, r')}{N(N-1)} \frac{\kappa_0}{2} \right.
\]

\[
- \frac{\rho(r)\rho(r')}{N^2} \frac{\kappa_0}{Z_0} \right\} g(|r-r'|) \quad (78)
\]

where the densities are defined by equations (68) and the two body scattering data is included in \( \kappa_0, \kappa_0, \kappa_0 \) as defined in equations (65), the subscript indicating that only the central terms are included. The Green's function is defined by equations (57):

\[
g(|r-r'|) = \langle r | \frac{1}{E - \kappa_0 i\epsilon} | r' \rangle = -\frac{2\mu}{4\pi\hbar^2} e^{\frac{i\kappa_E |r-r'|}{\hbar}}.
\]

The density functions are to be found from the independent particle model of the nucleus, where the ground state wave function is represented by a Slater determinant of single particle states. Such states are represented by
a set of quantum numbers $\mu$. For a single-particle state of
definite total angular momentum $j, m$ one has:

$$\mu = n, \ell, j, m, m_{\tau}$$

and if particle (1) is in the state $\mu$, the result is:

$$\mu(1) = \Sigma \frac{R_{\ell j}(r_{1})Y_{\ell}(r_{2})}{m_{s}}\chi_{m_{s}}(\sigma_{1})\chi_{m_{\tau}}(\tau_{1}).$$

The dependence on quantum number $n$ has been dropped as the
nuclei considered only have $n = 1$ states. Also, Coulomb
forces have been neglected, so that $R_{\ell j}$ has no isospin
dependence. If $\mu$ and $\nu$ are defined by:

$$\mu = \ell, j, m, m_{\tau} \quad \nu = \ell', j', m', m'_{\tau}$$

the required density functions are:

$$\rho_{D}^{(2)}(x; x') = \Sigma \Sigma \{<\mu(1)\nu(2)|\delta(x-x')\delta(r_{1}-r_{2})|\mu(1)\nu(2)>$$

$$- <\mu(1)\nu(2)|\delta(x-x_{1})\delta(r_{1}'-r_{2})|\mu(2)\nu(1)>\}$$

$$\rho_{SE}^{(2)}(x; x') = \Sigma \Sigma \{<\mu(1)\nu(2)|\delta(x-x_{1})\delta(r_{1}'-r_{2})P_{12}|\mu(1)\nu(2)>$$

$$- <\mu(1)\nu(2)|\delta(x-x_{1})\delta(r_{1}'-r_{2})P_{12}|\mu(2)\nu(1)>\}$$

where $P_{12}$ is the space exchange operator for particles (1)
and (2), and the summations go over all occupied states.
For a closed $j$-shell nucleus, for example $^{\text{12}}$C, the results
are:
\[ \rho_{D}(2)(\mathbf{r}, \mathbf{r}') = \rho(\mathbf{r})\rho(\mathbf{r}') - \frac{1}{2}\varphi_{\text{ex}}(\mathbf{r}, \mathbf{r}') \]

\[ \rho_{\text{SE}}^{(2)}(\mathbf{r}, \mathbf{r}') = \varphi_{\text{ex}}(\mathbf{r}, \mathbf{r}') - \frac{1}{2}\rho(\mathbf{r})\rho(\mathbf{r}') \]

where:

\[ \rho(\mathbf{r})\rho(\mathbf{r}') = 4\sum_{\ell j} \sum_{\ell' j'} \frac{(2j+1)(2j'+1)}{16\pi^2} R_{\ell j}(\mathbf{r})R_{\ell' j'}(\mathbf{r}') \]

\[ \varphi_{\text{ex}}(\mathbf{r}, \mathbf{r}') = 2\sum_{\ell j} \sum_{\ell' j'} \frac{(2l+1)(2l'+1)(2j+1)(2j'+1)}{4\pi} \]

\[ \begin{pmatrix} \ell & \ell' & I \end{pmatrix} \begin{pmatrix} \ell & j & \frac{1}{2} \end{pmatrix}^{2} \]

\[ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j & l' & L \end{pmatrix} \]

\[ R_{\ell j}(\mathbf{r})R_{\ell' j'}(\mathbf{r}')R_{\ell j}(\mathbf{r})R_{\ell' j'}(\mathbf{r})Y_{\ell}^{M}(\mathbf{r})Y_{L}^{M}(\mathbf{r}') \]

\[ \rho_{D}^{(2)}(\mathbf{r}, \mathbf{r}') = \rho(\mathbf{r})\rho(\mathbf{r}') - \frac{1}{2}\varphi_{\text{ex}}(\mathbf{r}, \mathbf{r}') \]

\[ \rho_{\text{SE}}^{(2)}(\mathbf{r}, \mathbf{r}') = \varphi_{\text{ex}}(\mathbf{r}, \mathbf{r}') - \frac{1}{2}\rho(\mathbf{r})\rho(\mathbf{r}') \]

L and M are "dummy quantum numbers" introduced to combine spherical harmonic functions\(^{(96)}\). The summation over L, M is automatically restricted by the allowed values of \( \ell j \), \( \ell' j' \). For a closed \( \ell \)-shell nucleus, for example \( 0^{16} \), the above results simplify if the \( j \)-dependence of \( R_{\ell j} \) is ignored:
where:

\[ p(r) p(r') = 16 \sum_{\ell \ell'} L \frac{(2l+1)(2l'+1)}{16\pi} R_\ell^2(r) R_{\ell'}^2(r') \]

\[ \xi_{ex}(r, r') = 16 \sum_{\ell \ell' M} \frac{(2l+1)(2l'+1)}{4\pi} (\ell, \ell', L)^2 \]

\[ R_{\ell j}(r) R_{\ell' j'}(r') Y_L^M(\hat{r}) Y_L^M(\hat{r}') \]

As indicated in chapter 17, the neglect of Coulomb and g.\ell \rangle forces in deriving density functions for light nuclei is an acceptable approximation. These forces become important in heavier nuclei where a large number of protons and high \ell values are encountered. The expressions (79) or (80) are now used in the second order potential (78). The function \( U^{(2)} \) thus found is put into a Schrödinger equation to find the scattering wave function.

5.4 Schrödinger Equation with Non-local \( U^{(2)} \)

All the quantities required for \( U^{(1)} \) and \( U^{(2)} \) have been defined in chapters IV and V so that the one body problem (15) can now be defined precisely. As the potentials are used in configuration space, the vector \( \mathbf{k} \) will be taken to define the initial projectile-nucleus relative momentum:

\[ \hbar \mathbf{k} = \mu \mathbf{v} \]

\[ E = \frac{\hbar^2 k^2}{2\mu} \]

\[ \mu = \frac{N}{N+1} m \]

where \( \mathbf{v} \) is the lab. frame projectile initial velocity and \( \mu \)
is the projectile-nucleus reduced mass. The one body problem can be written:

\[ (E - K - U(1)) | \psi_k \rangle = U(2) | \psi_k \rangle \]  

(E - K - U(1)) is the local potential of chapter IV and U(2) the non-local potential found in the previous section. For the present, the spin-orbit of U(1) and U(2) will be ignored, so that | \psi_k \rangle can be expanded in terms of spherical harmonics, without reference to spin:

\[ <r | \psi_k > = \sum_{p,q} \frac{u^q_p(r)}{r} \frac{v^q_p(r)}{r} \]  

Using (82) in (81) and projecting onto a particular partial wave p, q gives:

\[ \frac{1}{r} \left[ x + \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} - \frac{\mu (p+1)}{2\mu^2} - U(1)(r) \right] u^q_p(r) \]

\[ = \int dx y^q_p(x) \int dx' <r | U(2) | x' > <x' | \psi_k > \]

\[ = \int dx y^q_p(x) \int dx' g(r-r') \left[ \frac{\rho_D(x, x')}{N(N-1)} \frac{1}{2} + \frac{\rho_{SE}(x, x')}{N(N-1)} \frac{1}{2} \right. \]

\[ - \frac{\rho_S(x, x')}{N^2} \frac{1}{2} \right\} <r' | \psi_k > \]

where the expression (78) for U(2) has been substituted. To obtain the right hand side of the above equation explicitly, the partial wave expansion for <r' | \psi_k > given by (82) is substituted. In addition the expansion for g(x, x') is required.  


The density terms, which are also substituted explicitly, are given in the previous section. The Schrödinger equation for scattering from $0^\pm$ can be written in terms of the following quantities:

(i) Three coefficients derived from two body scattering:

\[
g(\mathbf{r}-\mathbf{r}') = \frac{2\mu}{\hbar^2} \sum_{r,s} g_r(r,r') Y_r^s(\mathbf{r}) Y_r^{s^*}(\mathbf{r}').
\]

\[
K_0 = \left\{ \frac{\mathbf{Y}(\mathbf{r})}{2} - \frac{\mathbf{X}(\mathbf{r})}{2} \right\} \frac{N(N-1)}{N^2} + \frac{Z}{N^2}
\]

\[
K_1 = \left( \frac{\mathbf{X}(\mathbf{r})}{2} \right) \frac{2}{N(N-1)}
\]

\[
K_2 = \left( \frac{\mathbf{Y}(\mathbf{r})}{2} \right) \frac{4}{N(N-1)}
\]

(ii) Two geometry factors:

\[
A_1(\ell\ell'jj;pr) = \frac{4(2\ell+1)(2\ell'+1)(2j+1)(2j'+1)}{L} \sum \langle 0\ell'0100;\ell\ell'jj;pr \rangle^2 \langle 0\ell'0100;\ell\ell'jj;pr \rangle^2
\]

\[
A_2(\ell\ell'jj;pr) = \frac{4(2j+1)(2j'+1)}{L} \sum \langle 0\ell'0100;\ell\ell'jj;pr \rangle^2 \langle 0\ell'0100;\ell\ell'jj;pr \rangle^2.
\]

Using these expressions, the radial equation is:
\[ \frac{1}{r} \left[ E + \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} - \frac{p(p+1)\hbar^2}{2\mu r^2} - U^{(1)}(r) \right] u_p(r) \]

\[ = K_0 \cdot \frac{2\mu k}{\hbar^2} \rho(r) \int dr' r' g_p(r,r') \rho(r') u_p(r') \]

\[ + \frac{2\mu k}{\hbar^2} \sum_{\ell \ell' \ell''} \frac{1}{16\pi^2} \sum_{jj'} \left\{ \frac{K_{A1}(\ell \ell' jj'pr) - K_{A2}(\ell \ell' jj'pr)}{2} \right\} R_{\ell j}(r) R_{\ell' j'}(r) \int dr' r' g_r(r,r') R_{\ell j}(r') R_{\ell' j'}(r') u_p(r') . \quad (84) \]

There is no real dependence on the magnetic substate \( q \) in \( U_p^d(r) \), so reference to \( q \) has been dropped. This is equivalent to cancelling a common factor \( Y^q(\vec{r}^*) \) from the radial Schrödinger equation.

In the case of \( 0^{16} \), the situation is a little simpler.

One requires:

(i) Two coefficients derived from two body scattering:

\[ H_0 = \left\{ \frac{1}{N(N-1)} \left( \frac{1}{2} \frac{\bar{X}_O}{2} - \frac{\bar{Z}_O}{2} \right) + \frac{\bar{Z}_O}{N^2} \right\} = K_0 \]

\[ H_1 = \left\{ \frac{4}{N(N-1)} \left( \frac{\bar{X}_O}{2} - 16 \frac{\bar{Y}_O}{2} \right) \right\} = 2K_1 - 4K_2 . \]

(ii) One geometry factor:

\[ B(\ell \ell'pr) = \sum_{L \ell} (2\ell+1)(2\ell'+1) <\ell0\ell'0|L0>^2 <L0p0|r0>^2 . \]

The radial equation for scattering from \( 0^{16} \) then is:
\[
\frac{1}{r} \left[ E + \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} \frac{\rho(p+1)\hbar^2}{2\mu r^2} - U^{(1)}(r) \right] u_p(r)
\]

\[
= \frac{2\mu k}{\hbar^2} H_0 \rho(r) \int dr' r' \rho(r') g_p(r,r') u_p(r')
\]

\[
+ \frac{2\mu k}{\hbar^2} H_1 \sum_{r \ell \ell'} \frac{B(\ell \ell' pr)}{16\pi^2} R_\ell(r) R_{\ell'}(r) \int dr' r'
\]

\[
R_\ell(r') R_{\ell'}(r') g_r(r,r') u_p(r'). \tag{85}
\]

5.5 Coupled Equations for Calculations with Non-local \(U^{(2)}\)

Equations (84) and (85) can be reduced to sets of equivalent coupled radial equations. In the case of \(c^{12}\), the labelling of functions by both \(\ell, \ell'\) and \(j, j'\) is superfluous, since for a given \(\ell\) value, there is only one \(j\)-value. The \(j, j'\) labels will be dropped. To deal with equations (84), the following coupling potentials are defined:

\[
V^{(0)}(r) = K_o^\frac{1}{2} \rho(r)
\]

\[
V_{\ell, \ell'}^{p, r}(r) = \frac{\{K_1 A_1 (\ell \ell' jj'pr) - K_2 A_2 (\ell \ell' jj'pr)\}^\frac{1}{2}}{4\pi} R_{\ell j}(r) R_{\ell' j'}(r)
\]

\(\tag{86}\)

That the functions \(V\) do have units of energy is easily shown. The units are as follows:

\[
K^\frac{1}{2} = [t\text{-matrix}] = [\text{Energy} \cdot \text{Length}^3]
\]

\[
\rho = R_{\ell j}^2 = [\text{Length}]^{-3}.
\]

Therefore: \(V = [\text{Energy}]\).
In terms of the potentials (85), equation (84) becomes:

\[
\frac{1}{r} \left[ E + \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{p(p+1)\hbar^2}{2\mu r^2} - u(1)(r) \right] u_p(r) 
= \frac{2\mu k}{\hbar^2} V^{(0)}(r) \int dr' r' \varepsilon_p(r, r') V^{(0)}(r') u_p(r') 
\]

(87)

A set of auxiliary functions is now defined:

\[
\frac{\varepsilon_p^{\ell', \ell'}(r)}{r} = \frac{2\mu k}{\hbar^2} \int dr' r' \varepsilon_p(r, r') V^{\ell', \ell'}(r') u_p(r'). 
\]

(88)

Finally, the special properties of the Green's function are used:

\[
\left[ E + \frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} \right] \frac{2\mu k}{\hbar^2} \varepsilon_p^{\ell}(r, r') = -\frac{\delta(r-r')}{rr'} 
\]

(89)

where \( \frac{d^2}{dr^2} \cdot r \) means first multiply by \( r \), then carry out the differentiation. By substituting (88) into (87) and using (89) on (88) a set of coupled equations is found. After dividing through each equation by \( E \) and using:

\[
E = \frac{\hbar^2 k^2}{2\mu} \quad x = kr 
\]

these equations can be written:
where the auxiliary functions satisfy:

\[
\begin{align*}
\left[ 1 - \frac{d^2}{dx^2} - \frac{2(p+1)}{x^2} \right] v^{(o)}(x) &= - \frac{V^{(o)}(x)}{E} u_p(x) \\
\left[ 1 + \frac{d^2}{dx^2} - \frac{2(r+1)}{x^2} \right] v^{(o)'}(x) &= - \frac{V^{(o)'}(x)}{E} u_p(x) \\
\end{align*}
\]

In the approximation that the difference in the radial wavefunctions for \( l \frac{3}{2} \) and \( l \frac{1}{2} \) states is ignored, equations (90) also describe the scattering from \( 0^{16} \). However, in this case, the coupling potentials are modified to contain the geometry factors relevant to \( 0^{16} \). From equation (85) it can be seen that, for \( 0^{16} \):

\[
v^{(o)}(r) = H_0^{\frac{1}{2}} \rho(r) = K_0^{\frac{1}{2}} \rho(r) \\
V^{(o)'}_{\ell', \ell'}(r) = \left( \frac{H_0 B(\ell \ell' pr)}{4\pi} \right)^{\frac{1}{2}} R_\ell(r) R_{\ell'}(r). \\
\]

As can be seen from equations (90b), the number of coupled equations is defined by the number of different functions \( V(x) \). This is governed by the possible values of the quantum numbers \( \ell, \ell', r \) for a given value of \( p \), which are restricted by the ground state properties of the nucleus and
the geometry factors $A_1$, $A_2$ or $B$. Within the approximations mentioned above, scattering from $O^{12}$ and $O^{16}$ is governed by the same number of coupled equations. The possible values of the quantum numbers appearing in $A_1$, $A_2$ and $B$ are set out in table III.

The table shows clearly that the symmetrical cases $\ell = 0$, $\ell' = 1$ and $\ell = 1$, $\ell' = 0$ give rise to identical auxiliary functions. Thus for $p > 2$, the number of coupling potentials is seven:

$$V(0) \text{ and } V_{p,r}^{\ell,\ell'} = V_{p,p}^{0,0}, \left\{ V_{p,p-1}^{1,0}, V_{p,p}^{0,1}, V_{p,p+1}^{0,1}, V_{p,p-2}^{1,1}, V_{p,p+1}^{1,1}, V_{p,p+2}^{1,1} \right\}$$

where the bracketed potentials are equal. Thus the number of coupled equations is eight. This number is reduced to seven for $p = 1$, since $r = p-2$ is then not allowed. For $p = 0$, the number of equations is six, as neither $r = p-2$ nor $r = p-1$ is allowed.
The possible values of the quantum numbers allowed by the geometry factors.

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<tr>
<th>$p = 0$</th>
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<th>$\ell'$</th>
<th>$L$</th>
<th>$r$</th>
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<td>0</td>
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<table>
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<th>$\ell$</th>
<th>$\ell'$</th>
<th>$L$</th>
<th>$r$</th>
</tr>
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<th>$\ell'$</th>
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<th>$r$</th>
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<td>0</td>
<td>$p$</td>
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<td></td>
<td>$p+2$</td>
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5.6 Inclusion of Spin-Orbit Potentials with Non-local $U^{(2)}$

The spin-orbit potentials which will be used in conjunction with the first and second order central potentials are given by equations (52) and (77). These combine to give a potential of the form $W(r)g.l$. The scattering wave function is expanded in terms of states of definite total angular momentum in which basis the operator $g.l$ is diagonal (45). As the nucleon has spin $\frac{1}{2}$, for a given orbital angular momentum $p$, there are only two possible values for total angular momentum, $(p+\frac{1}{2})$ and $(p-\frac{1}{2})$. The corresponding values of $g.l$ are $+p$ and $-(p+1)$. Thus, for a given value of $p$ there are two radial wavefunctions $u^+_p(r)$ and $u^-_p(r)$. The resulting coupled equations are very little different from those of equations (90) but the new ones are listed here for completeness. The Coulomb potential is also included:

$$\begin{align*}
\left[1 + \frac{d^2}{dx^2} - \frac{p(p+1)}{x^2} \right] & \frac{U^{(1)}(x)}{E} - \frac{U_{COUL}(x)}{E} \left[ \begin{array}{c}
\frac{+p}{E} \\
\frac{- (p+1)}{E}
\end{array} \right] \frac{W(x)}{E} \right] u^\pm_p(x) \\
= \frac{V^{(o)}(x)}{E} v^{(o)\pm}(x) + \sum \sum \frac{V_{p,l}^{l,l'}(x)}{E} v_{p,l,l'}^{l,l'}(x)
\end{align*}$$

(92a)

The auxiliary functions are:

$$\begin{align*}
\left[1 + \frac{d^2}{dx^2} - \frac{p(p+1)}{x^2} \right] v^{(o)\pm}(x) &= - \frac{V^{(o)}(x)}{E} u^\pm_p(x) \\
\left[1 + \frac{d^2}{dx^2} - \frac{r(r+1)}{x^2} \right] v_{p,l,l'}^{l,l'}(x) &= - \frac{V_{p,l,l'}^{l,l'}(x)}{E} u^\pm_p(x)
\end{align*}$$

(92b)

where $U^{(1)}$ is the central first order potential and $W$ is the
combined first and second order spin-orbit potentials. The solutions of equations (92) and the results obtained from them are considered in the next chapter.
VI

COMPUTATION AND RESULTS

"Damn it! The fellow's a mere pounder after all."
- Wellington. (Of Napoleon at Waterloo)

6.1 Elastic Scattering from a Local Complex Potential

The partial wave method for evaluating the scattering from a local potential is given in almost any book on quantum mechanics, for example the books by Rodberg (45), Messiah (96) or Jackson (97). The nuclear potentials of equations (52) and (77) lead to a radial Schrödinger equation for proton-nucleus scattering of the form:

\[
\left[ 1 + \frac{d^2}{dx^2} - \frac{p(p+1)}{x^2} - \frac{U(x)}{E} - \frac{U_{\text{Coul}}(x)}{E} - \left\{ \frac{+p}{(p+1)} \right\} \frac{W(x)}{E} \right] u_{p}^{\pm}(x) = 0
\]

where \( U \) is the combined first and second order central potentials, \( W \) is the combined spin-orbit potentials and:

\[
x = kr \quad k = \frac{\sqrt{2\mu E}}{h} \quad \mu = \frac{N}{N+1} m.
\]

At sufficiently large distances from the origin, only \( U_{\text{Coul}} \) of the potentials is non-negligible and the solutions \( u_{p}^{\pm} \) become linear combinations of the regular and irregular Coulomb functions \( F_{p} \) and \( G_{p} \). In terms of incoming and outgoing waves these solutions are:

\[
u_{p}^{\pm}(x) \overset{x \rightarrow \infty}{\longrightarrow} [G_{p}(x) - iF_{p}(x)] - s_{p}^{\pm}[G_{p}(x) + iF_{p}(x)]. \quad (94)
\]
All the information required to calculate elastic scattering is in the scattering matrix \( S \); overall normalization is not required. With complex potentials one has:

\[
S_p^± = e^{2i\delta_p} , \quad |S_p^±| < 1 .
\]

The last result is obtained because the \( \delta_p^± \) are complex, and are obtained from potentials with an absorptive part. The expressions for \( S_p^± \) and the asymptotic form of \( u_p^± \) show that the short range potentials have both shifted the phase of and partially absorbed the outgoing pure Coulomb wave.

The observables are related to the \( S_p^± \) via the following relations for the asymptotic region:

\[
\frac{U_{\text{COUL}}(x)}{E} = \frac{1}{4\pi\varepsilon_0} \frac{zZe^2}{r} \frac{2\mu}{\hbar^2k^2} = \frac{2\eta}{x} \\
\eta = \frac{1}{4\pi\varepsilon_0} \frac{\mu zZe^2}{\hbar^2k} , \quad z = 1 , \quad Z = \text{Target Charge}.
\]

For very large values of \( x \) (\( x \gg [p(p+1) + \eta^2] \)), the Coulomb functions have asymptotic forms as follows:

\[
F_p(x) \xrightarrow{x \to \infty} \sin(x - \eta \log_e 2x - \frac{1}{2}p\pi + \sigma_p) \\
G_p(x) \xrightarrow{x \to \infty} \cos(x - \eta \log_e 2x - \frac{1}{2}p\pi + \sigma_p)
\]

where \( \sigma_p \) is the Coulomb phase shift. The partial waves are summed to give the scattering amplitudes:
A(θ) = \frac{1}{2\pi i} \sum_{p=0}^{\infty} [(p+1)S_p^+ + pS_p^- - (2p+1)]e^{2i\sigma_p} \cos p\theta

B(θ) = \frac{1}{2\pi i} \sum_{p=0}^{\infty} [S_p^+ - S_p^-]e^{2i\sigma_p}' \cos p\theta

where \( f_c(θ) \) is the Coulomb scattering amplitude:

\[ f_c(θ) = -\frac{n}{2\pi} \frac{1}{\sin^2(θ/2)} \exp[-2i\eta \log_e (\sin θ/2) + 2iσ_0]. \]

In terms of these amplitudes one has the differential cross section:

\[ \frac{dσ}{dΩ}(θ) = |A(θ)|^2 + |B(θ)|^2 \]  \hspace{1cm} (95a)

the polarisation:

\[ P(θ) = \frac{2\text{Im}[A(θ)B^*(θ)]}{dσ/dΩ(θ)} \frac{\hat{n}}{\eta}; \quad \hat{n} = \frac{k\hat{A}k^*}{|k\hat{A}k^*|} \]  \hspace{1cm} (95b)

and the absorption cross section:

\[ q_A = \frac{π}{k^2} \sum_{p=0}^{\infty} [(p+1)(1 - 1S_p^+ 2 + p(1 - 1S_p^- 2)]. \]  \hspace{1cm} (95c)

In the case of neutron scattering, \( z, U_{\text{COUL}}, η, q_p \) and \( f_c \) are zero. The Coulomb functions revert to the spherical Bessel functions (multiplied by their arguments):

\[ F_p(x) \rightarrow x j_p(x) \quad G_p(x) \rightarrow x n_p(x) \]

which define the signs of \( j_p \) and \( n_p \). These are such that:

\[ j_0(x) = \frac{\sin x}{x} \quad n_0(x) = \frac{\cos x}{x}. \]
The formulae (95) are then valid for neutron scattering, and in addition one can define a total elastic scattering cross section:

\[ \sigma_E = \int d\Omega \frac{dg(\theta)}{d\Omega} = \frac{\pi}{k^2} \sum_{p=0}^{\infty} [(p+1)|1 - S_p^+|^2 + p(1 - S_p^-)^2] \quad (96a) \]

and a total cross section:

\[ \sigma_T = \sigma_E + \sigma_A. \quad (96b) \]

To obtain the \( S_p^\pm \) values, the equation (93) for \( u_p^\pm \) must be numerically integrated from the origin \( x = 0 \), where \( u_p^\pm(0) = 0 \), to a point \( x = x_M \), the matching point. \( x_M \) must be sufficiently large so that the nuclear potentials are negligible and the solution \( u_p^\pm(x) \) is well represented by the asymptotic form (94). By matching the numerical solution to the asymptotic solution at two points e.g. \( x = x_M \) and \( x = x_M + h \), where \( h \) is the integration step length, the \( S_p^\pm \) can be determined. Two points are required as the overall normalisation is arbitrary.

The computational methods for calculation of Coulomb and Legendre functions together with integration procedures for the Schrödinger equation are reviewed in an article by Melkanoff, Sawada and Raynal. Suffice it here to say that the usual methods of calculating standard functions were employed, and the integration procedure adopted was the Fox-Goodwin method. This method applies to second order differential equations with the first derivative absent i.e. equations of the form (93) which can be written:

\[ y''(x) = f(x)y(x) \quad (97) \]
where \( y(x) \) represents the particular radial wave function \( u_p^+(x) \) under consideration, \( y''(x) \) is its second derivative with respect to \( x \) and \( f(x) \) represents the remaining terms of (93). The Fox-Goodwin solution for integration of (97) is:

\[
(1 - \frac{\hbar^2}{12f_{i+1}})Y_{i+1} = (2 + \frac{5}{6}\hbar^2 f_i)Y_i - (1 - \frac{\hbar^2}{12f_{i-1}})Y_{i-1}
\]

where, if \( Y_{i-1} \) is the value of \( y(x) \) at \( x = x_o \), \( Y_i \) is the value at \( x_o + h \) etc. The procedure can be written more compactly as:

\[
Y_{i+1} = F_i Y_i - Y_{i-1}
\]

where:

\[
Y_i = (1 - \frac{\hbar^2}{12f_i})Y_i \quad F_i = \frac{(2 + \frac{5}{6}\hbar^2 f_i)}{(1 - \frac{\hbar^2}{12f_i})}
\]

Since \( y(o) = 0 \) and the normalisation is arbitrary, the Fox-Goodwin method requires no starting series for values of \( y(x) \) near the origin. The initial conditions are taken to be:

\[
y_o = 0 \text{ (origin)} \quad y_1 = \text{constant (arbitrary)}
\]

giving \( Y_0 = 0 \text{ (origin)} \quad Y_1 = \text{constant (arbitrary)} \).

These values are incorrect for the partial wave \( p = 1 \), since with potentials which are finite at the origin, the solution near the origin in this case is:

\[
y(x) = a_0 x^2 + a_1 x^4 + \ldots
\]

Hence \( y''(0) = f_0 y_0 = 2a_0 \) and \( Y_0 = -\frac{1}{6}a_0 \hbar^2 \).
This means that, in the $p = 1$ case, $y_1$ is not arbitrary. However, since $a_1 x^4$ is of the order $10^{-3} a_0 x^2$ for $x = h \approx 0.1$, it is sufficient to take:

$$y(x) = a_0 x^2 \quad (a_0 \text{ arbitrary})$$

to define the starting conditions for the $p = 1$ partial wave.

Then

$$f_0 y_0 = 2a_0 \quad y_1 = a_0 h^2$$

$$y_0 = -\frac{1}{6} a_0 h^2 \quad y_1 = \left[1 - \frac{1}{12} h^2 f(h)\right] a_0 h^2.$$

The somewhat approximate treatment of the $p = 1$ partial wave starting conditions is adopted to avoid having to develop a starting series (which becomes an extremely complicated matter in the coupled equations case) and is further justified by the fact that the $p = 1$ case is only one of a number of partial waves. The number of values of $p$ which must be taken into account is governed by the fact that for large values of $p$, $S_p^\pm$ become unity and these partial waves do not contribute to $A(\theta)$ or $B(\theta)$.

i.e. As $p \to \infty \quad S_p^\pm \to 1$.

At some value of $p$, say, $p_{\text{max}}$, $S_p^\pm$ has become sufficiently close to unity that the error in neglecting all partial waves $p > p_{\text{max}}$ is acceptable. The relation of the choice of integration parameters to the resulting accuracy of the calculated observables is discussed in the reviews of computing techniques\(^{(98, 100, 101)}\). To obtain $A(\theta)$ and $B(\theta)$ with computing errors of less than 1%, typical values are:
(i) Step length (in variable $x = kr$), $h = 0.2$

(ii) For $^{12}C$ or $^{16}O$, $p_{\text{max}}$ was taken as:

- $E = 90$ MeV, $p_{\text{max}} = 14$;
- $E = 156$ MeV, $p_{\text{max}} = 17$;
- $E = 310$ MeV, $p_{\text{max}} = 22$

(iii) The cut off radius $x_M$ is the value of $x$ for which all nuclear potentials have dropped to less than $10^{-3}$ of their values at $x = 0$.

6.2 Elastic Scattering from the Coupled Equations

The radial equations for $u_p^\pm(x)$ when the non-local second order potential is treated by means of coupled equations are given by formulae (92). The expressions presented in the previous section concerning the scattering observables and their relation to the $S_p^\pm$ remain valid but the $S_p^\pm$ become far more difficult to obtain. The methods required for the solution of the coupled equations have been summarised by Buck et al. (102).

Boundary conditions must now be specified for both the scattering wave function $u_p^\pm$ and the auxiliary functions $v_r^\pm$ (where for simplicity, only the orbital angular momentum subscript has been retained). As before, $u_p^\pm$ consists of incoming wave and modified outgoing wave in the asymptotic region:

$$u_p^\pm(x) \xrightarrow{x \to \infty} \left[ G_p(x) - iF_p(x) \right] - S_p^\pm \left[ G_p(x) + iF_p(x) \right]. \quad (99a)$$

The auxiliary functions have no incoming part and so consist only of a phase shifted, partially absorbed outgoing wave:
The quantities $T_r^\pm$ satisfy $|T_r^\pm| < 1$ and are only calculated because their values are required to find the $S_p^\pm$.

The problem with coupled equations is that, apart from the complication of the numerical integration procedure (see below), there are many independent solutions. For $n$ coupled, second-order differential equations there are $n$ independent, regular solutions for each of the $n$ functions involved. Since the boundary conditions of equations (99) are asymptotic, all $n$ solutions for each function must be generated and the correct linear combination to satisfy the asymptotic boundary conditions found. By matching at two points in the asymptotic region ($x_M$ and $x_M + h$), the $S_p^\pm$ and $T_r^\pm$ can be expressed in terms of the coupling coefficients and Coulomb functions, and $n$ algebraic equations for the $n$ unknown coupling coefficients set up. Thus the matching results in an $(n \times n)$ matrix inversion problem.

To apply the Fox–Goodwin method to equations (92), these are written in the form:

$$y''(x) = f(x)y(x) + \sum_{j=1}^{n-1} g(j)(x)z(j)(x)$$

$$z''(j)(x) = h(j)(x)z(j)(x) + k(j)(x)y(x)$$

where $y(x)$ represents a particular partial wave $u_p^\pm(x)$ and $z(j)(x)$ are the auxiliary functions coupled to $y(x)$. Each individual equation in the set (100) is treated as an inhomogeneous equation of the form:
\[ y''(x) = f(x)y(x) + b(x) \]

where \( b(x) \) contains the coupling potentials and auxiliary functions. The Fox–Goodwin method of relating the value of \( y \) at \( x_{i+1} \) to its values at \( x_i \) and \( x_{i-1} \) is easily generalized to inhomogeneous equations:

\[
Y_{i+1} = F_i Y_i - Y_{i-1} + B_i
\]

where \( B_i = \frac{h^2 b_i}{1 - \frac{h^2}{i^2} f_i} \)

and \( F_i, Y_i \) were defined by equations (98). Unfortunately, the functions \( y \) and \( z^{(j)} \) have to be identified at each step and this involves matrix inversion which normally makes Fox–Goodwin too cumbersome for coupled equations. However, the particularly simple coupling of equations (100), where each auxiliary function \( z^{(j)} \) is coupled only to \( y \) (and not to the other \( z^{(j)} \)) redeems the situation.

The method of generating \( n \) independent sets of solutions is to use \( n \) independent starting conditions. The first set of solutions is obtained by using:

\[
y_0 = 0 \quad y_1 = \text{constant (arbitrary)}
\]

\[
z_0^{(j)} = 0 \quad z_1^{(j)} = 0 \quad (\text{all } j).
\]

The second set is obtained by using:

\[
y_0 = 0 \quad y_1 = 0
\]

\[
z_0^{(1)} = 0 \quad z_1^{(1)} = \text{constant (arbitrary)}
\]

\[
z_0^{(j \neq 1)} = 0 \quad z_1^{(j \neq 1)} = 0
\]
and so on. In terms of the quantities of equations (101) this corresponds to using, for each set of solutions:

\[ Y_0 = 0, \ Y_1 = 0 \]  
\[ \text{for n-1 functions} \]
\[ Y_0 = 0, \ Y_1 = \text{constant} \]  
\[ \text{for the n^\text{th} function} \]

If the function with non-zero starting conditions also happens to be a \( p = 1 \) partial wave, the procedure for relating \( Y_1 \) and \( Y_0 \) described earlier is adopted.

The program developed in connection with the present problem uses the methods outlined above. The following checks were made on its accuracy.

(i) If the potentials of equations (92) are all square wells, solutions for a pair of coupled equations can be obtained exactly without difficulty. The exact solutions can then be compared with the computed ones.

(ii) If the coupling potentials are real and symmetric (in the sense defined below), there is no loss of flux. To see this, consider a matrix formulation:

\[
\begin{pmatrix}
 k^2 + v^2 & 0 \\
 k^2 + v^2 & \ddots \\
 0 & \ddots & \\
 \end{pmatrix}
\begin{pmatrix}
 \phi_1 \\
 \phi_2 \\
 \vdots \\
 \end{pmatrix}
= 
\begin{pmatrix}
 v_{11} & v_{12} & \cdots \\
 v_{21} & v_{22} & \cdots \\
 \vdots & \vdots & \ddots \\
 \end{pmatrix}
\begin{pmatrix}
 \phi_1 \\
 \phi_2 \\
 \vdots \\
 \end{pmatrix}
\]

It is simple to show that, where \( j \) is the quantum mechanical current operator, \( \text{div} \ j = 0 \) if \( V_{12} = V_{21} \) etc. In this case, the \( S_p \) and \( T_r \) of equations (99) satisfy:

\[ |S_p|^2 + \sum_r |T_r|^2 = 1. \]
This is a useful check for any number of coupled equations.

(iii) Numerous checks were made to ensure that reasonable values of step length, matching radius and number of partial waves included were chosen.

6.3 Some Exploratory Calculations

The first calculations using the coupled equations were made for the following simplified model. The incident particle was taken to be a spinless, uncharged "nucleon", so that the free scattering between two such "nucleons" is described by just the amplitude \( A_0(\theta_0) \) — the first term of the true scattering amplitude, \( M(\theta_0) \), equation (47). The forward scattering approximation was made, so that the first order potential becomes:

\[
U_0^{(1)}(r) = -\left(\frac{\hbar^2}{m}\right)A_0(0)\rho(r).
\] (102)

For the second order potential, only one type of nuclear two body density function was considered, the "direct" function:

\[
\rho^{(2)}_D(\vec{r},\vec{r}') = \sum_i \sum_{j\neq i} \delta(\vec{r}-\vec{r}_i)\delta(\vec{r}-\vec{r}_j)\rho_0.
\]

The result is that the term involving the product \( \rho(r)\rho(r') \) in \( U^{(2)} \) is self-cancelling (this can be seen from equation (79) for \( \rho_D(\vec{r},\vec{r}') \) and equation (55) for \( U^{(2)} \)) and the number of coupled equations is reduced by one, as the coupling potential \( V^{(0)} \) no longer exists. Then, for either \( \text{O}^{16} \) or \( \text{C}^{12} \), the coupled equations are:
where:

\[ V_{\ell', \ell'}^{\ell, \ell'}(x) = - \left( \frac{4\pi^2}{m} \right) \frac{A_o(0)}{2\pi} \frac{D^{\ell, \ell'}(x)}{R_F(x)R_{\ell'}(x)}. \]

For \(C^1\): \(D^{\ell, \ell'}(x) = B^{\ell, \ell'}(x)\)

For \(C^2\): \(D^{\ell, \ell'}(x) \rightarrow \frac{1}{2} A_1^{\ell, \ell'}(x)\)

\(A_1^{\ell, \ell'}(x)\) and \(B^{\ell, \ell'}(x)\) were defined in chapter V.

The local approximation to the second order potential is obtained by the methods of chapter V, which are applied to the simplified model to give:

\[ U^{(2)}(x) = \frac{i\mu}{\hbar^2} \left( \frac{4\pi^2}{m} \right)^2 A_0^2(0) R_F^2 \rho^2(x). \]

This is the second order potential studied in some detail in the famous article by Glauber. Following his methods, \(R_F\) and one factor of \(\rho(x)\) are replaced by their Fermi gas values, which are given in terms of the Fermi momentum, \(k_F\):

\[ R_F = \frac{3\pi}{5k_F}, \quad \rho_0 = 4 \frac{k_F^3}{6\pi^2}. \]

Also, defining a complex number \(f\) by:

\[ f = \left( \frac{4\pi^2}{m} \right) A_0(0) = f_R + i f_I. \]
where \( f_R \) and \( f_I \) are real numbers, and splitting \( U^{(2)} \) into its real and imaginary parts, one has:

\[
\text{Re} [U^{(2)}(r)] = 4 \frac{\mu k_F^2}{\pi \hbar^2} f_I \text{Re} [U^{(1)}(r)]
\]

\[
\text{Im} [U^{(2)}(r)] = -2 \frac{\mu k_F^2}{\pi \hbar^2} \frac{(f_R^2 - f_I^2)}{f_I} \text{Im} [U^{(1)}(r)].
\]

(104)

Since \( f_I \) is positive throughout the 90 - 310 MeV energy range, \( U^{(2)} \) always increases the depth of the real part of the potential in this range. If \( f_R > f_I \), \( U^{(2)} \) decreases the imaginary part of the potential. This is true up to about 200 MeV, above which energy \( f_I > f_R \). Thus, in the lower part of the range, the sign of both real and imaginary parts of \( U^{(2)} \) agrees with the more complete theory of chapter V.

The first calculations using the coupled equations (103) were made at 90 MeV and compared with the results obtained from the local potential (104). Since some approximations beyond the localization of the potential were required to obtain the \( U^{(2)} \) of equations (103), i.e. the neglect of the \( \frac{1}{N} \) term and replacement of \( \rho(r) \) by \( \rho_o \), the following procedure was adopted. The concept of a fermi momentum \( k_F \) does not have much meaning in a light nucleus, so the value of \( k_F \) was adjusted so that the local approximation gave the same forward scattering as the full non-local calculation. In fact \( k_F \) had to be reduced (thus reducing \( U^{(2)} \)), since replacing a factor of \( \rho(r) \) by \( \rho_o \) much overestimates the tail of the potential, which largely governs small angle scattering.
The results of this calculation for an idealised "neutron", scattering from C\textsuperscript{12} at 90 MeV, are shown in figure (14). The curve labelled \( U_o^{(1)} \) is the scattering obtained from the first order potential, equation (102), that labelled \( U_L^{(2)} \) is obtained from the combined first and second order local potentials and the \( U^{(2)} \) curve represents the full coupled equations calculation. As mentioned above, curves \( U_L^{(2)} \) and \( U^{(2)} \) coincide at zero degrees. The amount by which they differ at finite angles indicates the differences that the local and non-local calculations produce in the angular distributions (differences not attributable to the overall normalization of the cross sections). The total cross sections are as follows (in milli-barns):

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_E )</th>
<th>( \sigma_A )</th>
<th>( \sigma_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_o^{(1)} )</td>
<td>252</td>
<td>300</td>
<td>552</td>
</tr>
<tr>
<td>( U_L^{(2)} )</td>
<td>307</td>
<td>296</td>
<td>604</td>
</tr>
<tr>
<td>( U^{(2)} )</td>
<td>318</td>
<td>282</td>
<td>600</td>
</tr>
</tbody>
</table>

A notable feature, which persists throughout the calculations, is that even when \( U_L^{(2)} \) has been adjusted to give the same forward elastic scattering as \( U^{(2)} \), the effects with the non-local potential are more violent than with the local potential. Thus \( \sigma_E \) is increased more and \( \sigma_A \) is decreased more with \( U^{(2)} \) than with \( U_L^{(2)} \). The fact that forward elastic scattering is equal for the two cases means that \( \sigma_T \) will be at least similar for both potentials, as it
Fig. 14
is related to forward scattering via the optical theorem:

\[ \sigma_T = \frac{4\pi}{k} \text{Im}[f(0)]. \]

The plot of the phase shift functions \( S_p \) in figure (15) shows that, for the lower partial waves, both the real and imaginary parts of \( S_p \) are numerically smaller for \( U_L^{(2)} \) than for \( U^{(2)} \). Thus the \( |S_p|^2 \), plotted in figure (16) are smaller for the local potential, representing greater absorption than with the non-local potential.

To investigate these effects a little further, the "trivially equivalent potential" to \( U^{(2)} \), as defined by Perey(103), was obtained. This equivalent local potential, \( U_{TE}^{(2)} \), is found by performing the complete calculation numerically and then evaluating the expression for the exactly equivalent local potential:

\[ U_{TE}^{(2)}(x) = \frac{1}{\langle \psi | \psi \rangle} \int \text{d}r' \langle \psi | U^{(2)}(r') \psi \rangle, \]

This potential is non-central so that \( U_{TE} \) is different for each partial wave. In terms of the coupled equations (103), the trivially equivalent potential for the \( p \)th partial wave is:

\[ U_{TE}^{(2)}(x) = \frac{1}{u_p(x)} \sum_{\ell, \ell'} \sum_r V_{p, r}^{\ell, \ell'}(x) v_{p, r}^{\ell, \ell'}(x). \]

In figures (17, 18) and (19) are plotted the functions \( U_{TE}^{(2)}(r) \) for the partial waves \( p = 0 \), \( p = 1 \) and \( p = 7 \) respectively. For comparison, the approximate local
Fig. 15: The Phase Shift Functions
Fig. 16: The Modulus of the Phase Shift Functions
potential is also plotted. The sharp peaks in $U_{TE}^{(2)}$ correspond to radii where the real or imaginary part of the partial wave goes through zero. Since these peaks are very narrow, they are not significant (the value of $2\pi/k$, the incident wavelength, is about 3 fermi at an energy of 90 MeV). What is significant is the overall shape of the potential. In all partial waves, the imaginary part of $U_L^{(2)}$ is much smaller than that of $U_{TE}^{(2)}$, explaining why the non-local calculation results in a far larger reduction in $\sigma_A$. For the lower partial waves, the real part of $U_L^{(2)}$ agrees roughly with the equivalent local potential, but the $p = 7$ figure shows that the real part of $U_{TE}^{(2)}$ changes sign for the higher partial waves. This drastic dependence on orbital angular momentum results in the differences in angular distribution (away from $\theta_{CM} = 0^\circ$), the approximate local potential being the same for all partial waves.

For the next group of calculations, the first order spin orbit potential is added and the finite range of the two nucleon interaction is folded into the first order potentials. Thus the first order potential is the complete theoretical one represented by equations (52). The second order potentials are still those obtained from idealised situation outlined above. The effects of the second order potential on the polarization can now be observed. The results for proton-carbon scattering at 90 MeV are shown in figures (20) and (21). Again, the full calculation using $U^{(2)}$ shows more violent deviation from that using $U^{(1)}$ only than does the approximate calculation with $U_L^{(2)}$. Although the idealised theory used here to obtain expressions for $U^{(2)}$ and
$U_L^{(2)}$ makes a comparison with data at this stage not too meaningful, one encouraging aspect is apparent. The flattening of the polarisation curve in the range $\theta_{\text{CM}} \sim 5^0 - 15^0$ found in the $U^{(2)}$ calculation is also observed in the data. Local second order potentials do not seem to be as successful in producing this effect.

6.4 Complete Calculations

In this section, all references to the first order potential will imply the use of the potentials of equations (52). The second order potentials used are the complete ones of chapter V. No attempt is made to adjust $U_L^{(2)}$ to give the same results as $U^{(2)}$ by adopting an unusual value for $k_F$. The value used is the accepted one (104).

$$k_F = 1.36 \text{ fm}^{-1}.$$ 

One improvement over the theory of chapter V is included. So far no attempt has been made to fold the finite range of the two nucleon interaction into the second order potential. To do so properly would be a lengthy procedure - it involves a knowledge of the $q$ dependence of all the two body amplitudes of equation (47). If the folding is not done, $U^{(2)}$ and $U_L^{(2)}$ will be too large near the origin and too small in the surface of the nucleus. The following simple procedure was adopted. The increase in the mean square radii for the first order potentials was found in chapter IV. The same increase in mean square radius can be obtained by suitably adjusting the harmonic oscillator parameter.
according to equation (46). This adjusted parameter was used in $U^{(2)}$ and $U_L^{(2)}$ and has the effect of "spreading" the second order potentials by roughly the correct amount but their shapes are not affected.

The results, again for $p$-$^{12}C$ at 90 MeV are shown in figures (22) and (23) with data points from Dickson and Salter(105) corrected according to Jarvis and Rose(106).

The most striking feature is that $U^{(2)}$ now produces much larger effects than in the idealised calculation reported above and also much larger effects than given by $U_L^{(2)}$. The former is easily understood. In the idealised calculation, the strength of the second order potential is governed by one complex number:

\[ A_0^{(2)}(0) = 0.153 + i0.526 \text{ (at 90 MeV)} \]

In the complete calculation the strength is governed by the numbers appearing in table II. The dominant number for the spin independent part of $U^{(2)}$ is $\beta_0$:

\[ \beta_0 = 0.923 + i0.874 \text{ (at 90 MeV)} \]

Hence the full spin average over the pair of two body t-matrices involved in $U^{(2)}$ gives a larger strength than does the idealised calculation. That $U^{(2)}$ gives larger effects than $U_L^{(2)}$ is a feature of the non-locality which has already been suggested by the earlier calculations.

Turning now to a comparison with the data, it can be seen that with $U^{(1)}$ alone, the theory gives a curve for the elastic cross section which is above the data. The second
order potentials merely make the situation worse. Correspondingly the polarisation using $U^{(1)}$ alone is not correct in shape while that from $U^{(2)}$ has the correct shape but is too low in comparison to the data. Such considerations seem to have led other authors\(^{65,66,68}\) to abandon the Gammel-Thaler two body phase shifts in favour of more recent sets. It is true that these authors do obtain better fits to the data (even with $U^{(1)}$ alone) with some of these recent two body data.

However, the failure of the Gammel-Thaler set in the present calculations does not automatically mean that these data can be ruled out. The theory used here rests on the impulse approximation which is certainly suspect at 90 MeV. A simple correction to the impulse approximation has been suggested by Mulligan\(^{71}\). Without going into details, it is noted here that his correction concerns the fact that in evaluating $\langle 01t_1^f10 \rangle$, off shell elements of $t_i^f$ are required, which result in a non-local first order potential. On calculating the equivalent local potential, Mulligan finds that his potential depth is reduced by a factor $a_M$ compared to the ordinary K.M.T. calculation. The values of $a_M$ at various energies are:

<table>
<thead>
<tr>
<th>$E$(MeV)</th>
<th>90</th>
<th>156</th>
<th>310</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_M$</td>
<td>0.68</td>
<td>0.81</td>
<td>0.91</td>
</tr>
</tbody>
</table>

The calculations depend on a model for the two nucleon interaction and the model chosen by Mulligan fits two body scattering at 90 and 156 MeV but not at 310 MeV — hence the
query against the value of $a_M$ at 310 MeV. Since it is uncertain, $a_M$ was put to unity for $E = 310$ MeV in the present calculation. The first order potential is reduced by a factor $a_M$, but the second order potential is reduced by $a_M^2$ as it essentially involves evaluating the matrix elements of a product of two $t$-matrices. Hence the Mulligan reduction factor is extremely important at $E = 90$ MeV.

Results for various projectiles, targets and incident energies, when the potentials are reduced according to Mulligan, are shown in the figures at the end of this chapter. Detailed conclusions are reserved for the next chapter but some general remarks are in place here. The best fits are at small angles (this is a small angle theory). The flattened shape of the $p-C^{12}$ 90 MeV polarisation mentioned earlier is well reproduced by $U(2)$. To the author's knowledge this has not been achieved in any other multiple scattering theory calculations.
$P(\theta) (p-C^{12}) 90$ MeV

$U^{(2)}$

$U^{(2)}_L$

$U^{(1)}$

Refs. (105, 106) 95 MeV

Ref. (107) 75 MeV

$\theta_{cm}$

Fig. 25
\[ \frac{d\sigma}{d\Omega} (p-C^\pi) 156 \text{ MeV} \]

- \( U_\alpha^{(2)} \)
- \( U_\alpha^{(2)} \)
- \( U_\alpha^{(3)} \)

- Ref. (104) 152 MeV
- Ref. (103) 16.3 MeV
- Ref. (105) 137 MeV

**Fig. 26**
\[ \frac{d\sigma}{d\Omega} (n-C^12) \] 156 MeV

Fig. 28
\frac{d\sigma}{d\Omega} (p-O^{16}) 156\text{ MeV}

Fig. 30
$P(0)(\rho-\sigma^0) 156$ MeV

$U^{(2)}$

$U^{(2)}$

$U^{(1)}$

$U^{(0)}$

Ref. (112) 158 MeV

Fig. 31
\[
\frac{d\sigma}{d\Omega} (p-C^12) 310 \text{ MeV}
\]

\[
\Delta \text{ Ref. (113) } 313 \text{ MeV}
\]

**Fig. 32**
Fig. 33
TABLE IV

Integrated Cross-sections from $U^{(1)}$, $U^{(2)}_L$ and $U^{(2)}$

The available data are collected in the review article by Barashenkov et al. (114) and the experimental numbers quoted are from that source.

<table>
<thead>
<tr>
<th>$E$ (MeV)</th>
<th>$U^{(1)}$</th>
<th>$U^{(2)}_L$</th>
<th>$U^{(2)}$</th>
<th>Expt. Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>287</td>
<td>268</td>
<td>219</td>
<td>$219 \pm 8 (77 \text{ MeV})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$233 \pm 7 (95 \text{ MeV})$</td>
</tr>
<tr>
<td>156</td>
<td>251</td>
<td>234</td>
<td>215</td>
<td>$223 \pm 6 (133 \text{ MeV})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$220 \pm 24 (134 \text{ MeV})$</td>
</tr>
<tr>
<td>310</td>
<td>232</td>
<td>228</td>
<td>239</td>
<td>$187 \pm 18 (305 \text{ MeV})$</td>
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<thead>
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<th>$E$ (MeV)</th>
<th>$U^{(1)}$</th>
<th>$U^{(2)}_L$</th>
<th>$U^{(2)}$</th>
<th>Expt. Values</th>
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<tr>
<td>156</td>
<td>254</td>
<td>236</td>
<td>215</td>
<td>$221 \pm 10 (140 \text{ MeV})$</td>
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<th>$E$ (MeV)</th>
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<th>$U^{(2)}_L$</th>
<th>$U^{(2)}$</th>
<th>Expt. Values</th>
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</thead>
<tbody>
<tr>
<td>156</td>
<td>124</td>
<td>129</td>
<td>155</td>
<td>$175 \pm 8 (90 \text{ MeV})$</td>
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<td></td>
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<td>$143 \pm 7 (270 \text{ MeV})$</td>
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<th>$U^{(2)}_L$</th>
<th>$U^{(2)}$</th>
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<tbody>
<tr>
<td>156</td>
<td>312</td>
<td>290</td>
<td>265</td>
</tr>
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</table>
"I seem to have been only on the sea shore, diverting myself in now and then finding a smoother pebble or prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

- Newton (from Brewster's Memoirs of Newton)

7.1 Effect of the Second Order Potential

The results presented at the end of chapter VI show that in the lower part of the energy range both the full non-local second order potential, $U^{(2)}$, and its approximate local form, $U_L^{(2)}$, increase the elastic scattering and reduce the absorption when added to the first order potential, $U^{(1)}$. The 310 MeV curves show that for higher energies $U^{(2)}$ and $U_L^{(2)}$ cause different changes in the observables. $U^{(2)}$ causes both the absorption and the small angle elastic scattering to increase. $U_L^{(2)}$ causes both of these quantities to decrease. However, at 310 MeV these changes are very small and do not significantly alter the fit to the experimental data.

MacDonald and Hull (65) and later Chalmers and Saperstein (69) using different two body data to those used in this work, found that $U_L^{(2)}$ and $U^{(2)}$ reduced the elastic cross section at small angles. These authors explained this on the basis of the fact that if $U^{(2)}$ is derived from Pauli correlations, then it is expected that the effect of the exclusion principle is to inhibit small angle scattering. However, Glauber (40) has explained that the exclusion principle is a two edged weapon when introduced into the multiple scattering theory. One effect is to inhibit small
angle scattering as explained by the Goldberger\textsuperscript{(82)} theory. In competition with this effect is the fact that the Pauli principle also has the result of keeping the nuclear nucleons further apart than otherwise would be expected, thus reducing the "shadow effect".

It is possible to think of the results of the present calculations in the following way. At the lower energies the Pauli principle, in giving rise to the second order potential, reduces the absorption of incident nucleons. The extra particles remaining in the elastic channel are scattered and increase the elastic scattering cross section. At higher energies the full $U^{(2)}$ calculation gives the following effects. The inhibition of absorption is dominated by the reduction in the eclipse effect so that absorption is actually increased. This reduction in the eclipse effect also allows greater small angle elastic scattering. The extra absorption is made at the expense of large angle scattering. As noted by Glauber\textsuperscript{(40)}, this change in the effects of $U^{(2)}$ at higher energies can be traced to changes in the two body data which occur around 200 MeV. Specifically, the imaginary parts of the two body amplitudes become dominant above 200 MeV, whereas the real parts were dominant at lower energies.

In terms of potential depths, the lower energy results show that $U^{(2)}_L$ increases the real part and reduces the imaginary part of the optical potential. This effect of the exclusion principle was predicted by Verlet and Gavoret\textsuperscript{(74)} and Dabrowski and Sobiczewski\textsuperscript{(75)} using simple theories.
The major difference between effects in the cross sections due to $U^{(2)}$ and $U_L^{(2)}$ is in the magnitude rather than the direction of the effect. $U^{(2)}$ causes changes of typically five times the size of those caused by $U_L^{(2)}$. The figures showing the "trivially equivalent local potential" to $U^{(2)}$ (figs. (17), (18), (19) of chapter VI) indicate that this large difference stems mostly from the imaginary part of the potential, $U_L^{(2)}$ being much too small. It seems that in reducing the second order potential to an approximate local form, most of the true nature of the potential is lost.

The fact that much is lost in reducing $U^{(2)}$ to $U_L^{(2)}$ is indicated by the polarization curves. $U_L^{(2)}$ and $U^{(1)}$ both contain only the nuclear density function for the dependence on nuclear properties. Thus they are very similar in shape; in fact, $U_L^{(2)}$ has the same shape as $U^{(1)}$. The result seems to be that $U_L^{(2)}$ causes little change in the shape of $P(\theta)$ but merely a displacement of the curve, either up or down. $U^{(2)}$ on the other hand can cause drastic changes in the shape of $P(\theta)$ which would seem to be due to the non-locality of $U^{(2)}$.

The trivially equivalent potential $U^{(2)}_{TE}$ displays the non-locality of $U^{(2)}$ by translating it to a dependence on $p$, the orbital angular momentum of the incident partial wave. The calculations of $U^{(2)}_{TE}$ described in chapter VI show that as $p$ increases, the magnitude of $U^{(2)}_{TE}$ decreases and the real part actually changes sign. These effects are most marked in partial waves corresponding to the surface region of the nucleus. This $p$ dependence can be understood in a qualitative way from the equations for $U^{(2)}_{TE}$ on page 104.
$U^{(2)}_{\text{TE}}$ depends on the overlap of a pair of bound state wave functions $R_\ell$ and $R_{\ell'}$, with the radial Green's function $g_r$ and the incident partial wave $u_p$. The quantum number $r$ is restricted by the possible values of $\ell$ and $\ell'$ according to: $p - \ell - \ell' < r < p + \ell + \ell'$ so that the bound state quantum numbers define the possible orbital angular momenta of the intermediate propagation. For high values of $p$, $r$ will also have only large values and $g_r$ will then have less overlap with $R_\ell$ and $R_{\ell'}$ than for small values of $p$ and $r$.

This important role of the occupied single particle states has been investigated in a paper by Beg (115). This paper is written throughout in "single particle language" and concerns the multiple scattering theory for the $\pi$-nucleus system. Except that Beg drops the $(1/N)$ term, his results very closely resemble the theory of chapter V. The intermediate propagation between the two scatterings represented by $U^{(2)}$ is restricted by the fact that both struck particles must start and finish in bound states. The results of this restriction fall out naturally in terms of the occupied orbitals $R_\ell$ and $R_{\ell'}$ in Beg's paper. In the present work $R_\ell$ and $R_{\ell'}$ are brought in via the single particle expressions for the correlation functions.

Another theoretical paper which has a significant overlap with this work is that by Feshbach et al. (46) already mentioned in chapter I. The method suggested in that paper for dealing with the correlations in the multiple scattering theory involved just two coupled equations. A calculation has recently been done by these authors for nucleon-nucleus scattering (118). However, their calculation is for an
incident energy of 995 MeV and in that energy region the correlations have very little effect.

7.2 The Results of the Theory and Experimental Data

The figures in chapter VI show the following general features. The complete calculations employing \( U(2) \) give elastic cross sections which are either good fits to the data or lie a little above the data. At the smaller angles \( U(2) \) gives either a good fit to the polarization or lies below the data. At 156 MeV, the \( P(\theta) \) found from \( U(2) \) tends to lie well below the data and this effect was also found by Chalmers and Saperstein\(^{69}\) in this energy region. These authors also found that the dip in \( P(\theta) \) around \( \theta = 30^\circ \) was an effect present only when \( U(2) \) was added to \( U(1) \). Figure (27) shows that when the Gammel-Thaler two body data is used, the reverse is true. The dip occurs with \( U(1) \) but not with the \( U(2) \) calculation.

The fact that there is a good deal of data for nucleon-carbon scattering allows the following general trend to be exhibited. For scattering at an energy \( E \), the calculations of elastic differential cross sections and to a lesser extend polarisations and absorption cross sections tend to agree better with the data for an energy of 20 to 30 MeV lower than \( E \). This fact was also noticed by MacDonald and Hull\(^{65}\) who therefore suggested that for scattering at an energy \( E \), the two body data to use is that for an energy of about \( (E + 30) \) MeV. This procedure allows for the fact that the two body collisions are taking place inside the nucleus and the struck particle possesses the energy of its Fermi motion.
Hence this represents another refinement on the impulse approximation.

A disturbing feature is that while the cross sections calculated with $U^{(2)}$ for carbon are quite good, the polarisations are rather poor, except at small angles. Indeed, in the case of $p-C^{12}$ at 156 MeV, the curve from $U^{(1)}$ alone is by far the best. Of course, the polarisation is much more sensitive to details of the potential than is the elastic cross section. Some possible explanations for the rather poor polarisations are considered below.

In general, the spin-orbit potential is less well treated in this theory than is the central potential. To obtain the usual $g.s.$ potential, it is necessary to make approximations beyond those involved in finding the central potential. Also, the second order spin-orbit potential has been treated in the same way in $U^{(2)}$ as for $U_L^{(2)}$, which means that the non-locality due to the propagator and correlation function has been ignored. By working in momentum space, Chalmers and Saperstein\(^\text{69}\) avoid these problems and their polarisations are better at around 150 MeV, but still are not good.

Another cause of discrepancy is the neglect of the angular variation of the two body data in the second order potential. Also, there will be a change in the shape of the potential due to off-shell effects. Only the depth is corrected for these effects by the Mulligan factor\(^\text{71}\). It might be necessary to include higher order terms in the multiple scattering series. At the lower energies $U^{(2)}$ has a large effect but higher order terms are unlikely to affect the results at 310 MeV as the effect from $U^{(2)}$ is already
very small. Better calculations of nuclear properties could improve the fits, although work by other authors\textsuperscript{(65, 69)} indicates that inclusion of short range correlations does not improve matters very much.

One likely possibility is that different sets of two body data may give better overall fits. Zohni and Hussein\textsuperscript{(116)} have shown that even the first order potentials can be quite different when computed from different sets of phase shifts. Other authors\textsuperscript{(65, 66, 69)} seem to have abandoned the Gammel-Thaler phase shifts in favour of more recent sets. The set which these authors seem to favour are the calculations done by the Yale group\textsuperscript{(117)}.

7.3 Summary

The conclusions from this work are summarised briefly here.

(i) For light nuclei, the angular variation of the two body data is important in determining the optical potential derived from the multiple scattering theory.

(ii) The second order optical potential is very important at 90 MeV but its effects are almost negligible at 310 MeV.

(iii) The non-locality of the second order potential is important and its effects are not well reproduced by the approximate local potential. The effects due to $U^{(2)}$ are both different in shape and larger in magnitude than the effects due to $U^{(2)}_L$, although the sign of the effects is usually the same for both potentials.
(iv) The impulse approximation is very suspect below 150 MeV. The off shell effects calculated by Mulligan (71) and the Fermi motion of the struck particle both become important in improving fits below this energy.

(v) In view of the large differences in the results from the local approximation and the more exact non-local calculation, there can be little justification in attempting to distinguish between sets of two body data on the basis of the former. This local approximation has been shown to be an inadequate representation of the true second order potential, so that if $U_L^{(2)}$ happens to give better fits to the data at certain energies and within certain angular ranges, this must be regarded as a fortunate circumstance rather than a vindication of the reduction of the potential to a local form. When the non-local calculation does not fit the data, the approximations inherent in the multiple scattering theory must be examined and some of the more likely sources of error were mentioned in the previous section.
APPENDIX

A.1 Modification Required of Formal Theory

The exact optical potential found in chapter II is given in equations (21) and (22):

\[ U = \sum_{i=1}^{N} (\Omega_i^1 \Omega_i^1 0) \quad (A1) \]

\[ \Omega_i = 1 + G \sum_{j \neq i} t_j \Omega_j - GU. \quad (A2) \]

The potential \((A1)\) satisfies the normal Schrödinger equation to give a transition operator \(T\):

\[ T = U + UgT. \quad (A3) \]

The propagators are:

\[ G = \frac{1}{E-K-H_T+i\epsilon} \quad G = \frac{1}{E-K+i\epsilon} \quad (A4) \]

An important relation is obtained between \(G\) and \(g\) by defining projection operators \(P\) and \(Q\)

\[ P = 10)(01) \quad Q = 1 - 10)(01) \quad (A5) \]

Then one has:

\[ P + Q = 1 \quad PG = GP = gP. \quad (A6) \]

The aim of this appendix is to modify equations \((A1)\), \((A2)\) and \((A3)\) in such a way that \(U\) does not appear on the right hand side of \((A2)\).
A.2 A Special Case

Consider two potentials $U$ and $U'$ such that:

$$ T = U + U gT = U' + U' g (1 - \frac{1}{N}) T. \quad (A7) $$

If (A7) is multiplied by $(1 - \frac{1}{N})$ one obtains

$$ T' = U'' + U'' g T' \quad (A8) $$

where:

$$ T' = (1 - \frac{1}{N}) T \quad U'' = (1 - \frac{1}{N}) U'. \quad (A9) $$

Thus, by using (A9) for a potential $U'$ which satisfies a peculiar Schrödinger equation (A7), one can solve in the normal way-(A8)-and merely invert (A9) to obtain the correct $T$. The relation between $U$ and $U'$ is easily obtained from (A7).

Solve for $U$:

$$ U = T \frac{1}{1 + g T} \quad (A10) $$

and for $U' - U$:

$$ (U' - U) = (U - U') g T + \frac{1}{N} U' g T. $$

Therefore $(U' - U)[1 + g T] = \frac{1}{N} U' g T$. Therefore $(U - U') = \frac{1}{N} U' g T - \frac{1}{1 + g T}$. Then use (A10) to simplify the above:

$$ (U' - U) = \frac{1}{N} U' g U. \quad (A11) $$
Thus, any two potentials \( U, U' \) satisfying (A11) will also satisfy (A7) and a potential \( U' \) can be treated easily as shown in (A8) and (A9).

A.3 Rearrangement of the Optical Potential

The methods of Francis and Watson\(^{(30)}\) can now be used to find an alternative expression for the potential given in (A1). Consider the expression for \( \Omega_1 \), equation (A2):

\[
\Omega_1 = 1 + G \sum_{j \neq i} t_j \Omega_j - G \sum_j (O_{1j} \Omega_j) \to 0
\]

where the full expression for \( U \) has been used on the right hand side. Operate with \( \Omega_1 \) on \( \to 0 \), remembering that \( G \) is diagonal in nuclear states

\[
\Omega_1 \to 0 = \to 0 + G \sum_j (P+Q)t_j \Omega_j \to 0 - G \sum_j t_j \Omega_j \to 0 .
\]

Combine the \( j \neq i \) terms in the above and use (A6):

\[
\Omega_1 \to 0 = \to 0 + G \sum_j (P+Q)t_j \Omega_j \to 0 - G \sum_j t_j \Omega_j \to 0 . \tag{A12}
\]

Because of the antisymmetry of \( \to 0 \), the last term in (A12) is independent of label \( i \). Hence, from (A1):

\[
(O_{1i} \Omega_i \to 0) = \frac{1}{N} U . \tag{A13}
\]

Similarly, using (A12), a quantity \( \Gamma \), independent of label \( i \),
is defined:

\[ \Gamma = (0|\Omega_{10}) = 1 - \frac{1}{N} gU \]

where use was made of:

\[ (0|Q = Q_{10}) = 0 . \]

Also useful are:

\[ Q^2 = Q , \quad P^2 = P . \]  \hspace{1cm} (A14)

Using (A12) and (A14), consider:

\[ Q\Omega_{10} = G \sum_{j \neq i} Q_{ij} \Omega_{10} \]

\[ = G \sum_{j \neq i} Q_{ij} (P + Q) \Omega_{10} \]

\[ = G \sum_{j \neq i} [Q_{ij} \Omega_{10} + Q_{ij} Q_{0j} \Omega_{10}] . \]

The above is of the form:

\[ Q\Omega_{10} = [w_i - 1] \Omega_{10} \Gamma \]  \hspace{1cm} (A15)

where \( w_i \) is determined by substitution:

\[ [w_i - 1] \Omega_{10} \Gamma = G \sum_{j \neq i} (Q_{ij} \Omega_{10} + Q_{ij} [w_j - 1] \Omega_{10} \Gamma) \]

\[ = G \sum_{j \neq i} Q_{ij} w_j \Omega_{10} \Gamma \]

which gives:

\[ w_i = 1 + G \sum_{j \neq i} Q_{ij} w_j . \]  \hspace{1cm} (A16)

Now define a new optical potential \( U' \):
\[ U' = \sum_i (O_{1i}w_{1i}10). \]  

(A17)

It remains to show that \( U' \) of (A17) and \( U \) of (A1) have the relationship (A11). From (A15):

\[ (O_{1i}Q_{1i}10) = (O_{1i}w_{1i}10)\Gamma - (O_{1i}10)\Gamma \]
\[ = (O_{1i}w_{1i}10)\Gamma - (O_{1i}P_{1i}10). \]

Therefore

\[ (O_{1i}Q_{1i}10) = (O_{1i}w_{1i}10)\Gamma. \]

Summing over (i); using the definition of \( \Gamma \):

\[ U = U'(1 - \frac{1}{N'}gU). \]

This gives:

\[ U' - U = \frac{1}{N}U'gU \]

which is exactly the form (A11). Hence \( U' \) can be treated by the methods of section A.2.

**A.4 Structure of \( U' \)**

Expand \( U' \) by using (A16) and (A17):

\[ U' = \sum_i (O_{1i}10) + \sum_i \sum_{ij\neq i} (O_{1i}G_{ij}10) + \ldots. \]

The advantage of employing \( U' \) is that it is in the form of an expansion in the nuclear correlations with no \((1/N)\) corrections. As can be seen, the intermediate states are
the excited states (due to the presence of Q) and de-excitation is caused by a collision with a different particle from the excited one — hence the correlation function is involved.
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