A THEORETICAL STUDY OF ALPHA-PARTICLE
SCATTERING AND REACTIONS WITH
HEAVY NUCLEI

by

M. J. RHOADES-BROWN

A Thesis submitted to the Faculty of Mathematical
and Physical Sciences of the University of Surrey
for the Degree of Doctor of Philosophy

SEPTEMBER 1977
TO MY WIFE

LYDIA
ABSTRACT

This thesis is concerned for the main part with the elastic scattering of α-particles on heavy nuclei and nuclear reactions in which an alpha particle is transferred.

Various $^{208}\text{Pb}$ potentials are calculated microscopically using a single folding model in which the target nuclear density is constructed from shell model wavefunctions. Using these potentials it is shown that it is possible to reproduce elastic scattering data at medium energies. The problems of applying these potentials to more complex nuclear reactions is discussed.

The alpha decay of various Polonium isotopes and the α-transfer reaction $^{208}\text{Pb} \ (O^{16}C^{12}) \ P_{0}^{212}$ are then analysed using perturbation theory. By expanding the initial A=4 system in a cluster model representation and taking the difference between the full Hamiltonian of the system and a model Hamiltonian as the perturbation which causes alpha decay it is shown that the inconsistencies which existed in previous theories had been greatly reduced. Applying this technique to the ground state decay of polonium isotopes as well as the decay of the $18^+$ isomeric state $^{212}_{121}\text{Po}$ it is shown that the α-Pb spectroscopic factor is now much larger than the most complicated shell model calculations. This magnitude is confirmed in the analysis of the α-transfer reaction and leads to the conclusion that alpha decay and alpha transfer reactions can be considered, to a good approximation, as one step processes in which a real alpha particle in its ground state is transferred.
From the α-decay formalism a new energy-lifetime relation is derived which fits the appropriate experimental data of even Polonium isotopes, a range covering fourteen orders of magnitude.

The problem of antisymmetrisation for an alpha particle in orbit around $^{208}$Pb is investigated using a modified Resonating Group Method. A formalism is developed for the non-local kernels which uses a local density approximation. It is shown that the one particle exchange component is the dominant term and that exchange effects are not very important in the analysis of α-decay and α-transfer reactions to heavy systems. The non-local interaction kernel between an alpha particle and $^{208}$Pb is also examined and the expected dominant term is investigated.

Further improvements to the theory presented is given together with possible suggestions for extensions of the work.
ACKNOWLEDGEMENTS

I would like to thank:

My supervisor Professor D.F. Jackson for all her invaluable guidance and help throughout the course of this work.

Dr. R.C. Johnson and all members of the Nuclear Physics Group for making my stay at Surrey such a friendly and informative one.

Professor J. Blair, Professor M. Kawai, Dr. F.D. Santos and Dr. W.S. Pong for their enlightening discussions and interest shown at various stages of the work.

The Science Research Council for financial support in the form of a studentship.

Mrs. M.G. Shannon for her accurate typing of a difficult script.
## CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER 1  INTRODUCTION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 2  ALPHA PARTICLE SCATTERING NEAR THE COULOMB BARRIER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Introduction</td>
<td>6</td>
</tr>
<tr>
<td>2.2 The Microscopic Optical Model</td>
<td>7</td>
</tr>
<tr>
<td>2.3 Microscopic Analysis</td>
<td>11</td>
</tr>
<tr>
<td>2.3.1 Angular Distribution and Reaction Cross Sections</td>
<td>11</td>
</tr>
<tr>
<td>2.3.2 Reflection Coefficients</td>
<td>12</td>
</tr>
<tr>
<td>2.3.3 Critical Angular Momentum and Radii</td>
<td>14</td>
</tr>
<tr>
<td>2.4 Discussion</td>
<td>15</td>
</tr>
<tr>
<td>Tables</td>
<td>17</td>
</tr>
<tr>
<td>Figures</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 3  ALPHA DECAY</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Introduction</td>
<td>26</td>
</tr>
<tr>
<td>3.1.1 Statement of the Problem</td>
<td>26</td>
</tr>
<tr>
<td>3.1.2 Problems in Alpha Decay</td>
<td>28</td>
</tr>
<tr>
<td>3.2 Time Dependent Theory of Decaying States</td>
<td>31</td>
</tr>
<tr>
<td>3.2.1 Definitions and Notation</td>
<td>31</td>
</tr>
<tr>
<td>3.2.2 Derivation of the Decay Width</td>
<td>33</td>
</tr>
<tr>
<td>3.2.3 Choice of the Hamiltonians K and K'</td>
<td>38</td>
</tr>
<tr>
<td>3.2.4 Creation of a Bound Initial State</td>
<td>39</td>
</tr>
<tr>
<td>3.3 Evaluation of Decay Width Matrix Element</td>
<td>41</td>
</tr>
<tr>
<td>3.3.1 Excitation of the Residual Nucleas</td>
<td>41</td>
</tr>
<tr>
<td>3.3.2 Excitation of the Alpha Particle</td>
<td>45</td>
</tr>
<tr>
<td>3.4 Calculations of the $0^+ \rightarrow 0^+$ Transitions</td>
<td>45</td>
</tr>
<tr>
<td>3.4.1 Cluster Formalism</td>
<td>45</td>
</tr>
<tr>
<td>3.4.2 Results for the Decay Width</td>
<td>50</td>
</tr>
<tr>
<td>3.5 Calculations for the Decay of $^{212m}_{\text{Po}}$</td>
<td>51</td>
</tr>
<tr>
<td>3.5.1 Extended Cluster Formalism</td>
<td>52</td>
</tr>
<tr>
<td>3.5.2 Cluster Parentage Coefficients</td>
<td>54</td>
</tr>
<tr>
<td>3.5.3 Results for the Isomeric Decay Width</td>
<td>55</td>
</tr>
</tbody>
</table>
3.6 Relationship Between Alpha-Particle Energies and Decay Lifetimes 57

3.7 Summary and Conclusions 59

Tables 62

Figures 68

CHAPTER 4 ALPHA TRANSFER TO ALPHA DECAYING NUCLEI 75

4.1 Introduction 75

4.2 Expression of the Matrix Element 77
   4.2.1 The Form of \( W_f \) 81
   4.2.2 Reduction of \( T \)-matrix 84

4.3 Partial Wave Analysis 86
   4.3.1 Core Excitation 86
   4.3.2 No Core Excitation 89

4.4 Selection Rules 91

4.5 Optical and Bound State Potential Parameters 92

4.6 Results and Discussion 94
   4.6.1 Local Momenta 94
   4.6.2 Angular Momentum Mismatch 94
   4.6.3 Angular Distributions 95
   4.6.4 Spectroscopic Factors 97

4.7 Conclusions 99

Figures 101

CHAPTER 5 EXCHANGE EFFECTS IN THE ALPHA-NUCLEAR CLUSTER MODEL 109

5.1 Introduction 109

5.2 Resonating Group Method 110
   5.2.1 Alternative Approach and the Orthogonal Condition Model 116

5.3 The Normalisation Kernel 120
   5.3.1 Alpha Particle Density Matrix 121
   5.3.2 Density Matrix for \(^{208}\text{Pb}\) 123
   5.3.3 Spin Dependence of Density Matrices 126
   5.3.4 One Particle Exchange Kernel 127
   5.3.5 Two Particle Exchange Kernel 128
   5.3.6 Three Particle Exchange Kernel 132
   5.3.7 Four Particle Exchange Kernel 135
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4 Numerical Technique</td>
<td>136</td>
</tr>
<tr>
<td>5.5 Results and Discussion</td>
<td>138</td>
</tr>
<tr>
<td>Figures</td>
<td>142</td>
</tr>
<tr>
<td>CHAPTER 6 SUMMARY AND CONCLUSIONS</td>
<td>147</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>151</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>157</td>
</tr>
<tr>
<td>APPENDIX C</td>
<td>162</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>165</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

The elastic and inelastic scattering of alpha particles has proved to be a very useful tool for investigating the structure of nuclei (Mo69). Indeed Gamow's (Ga28) qualitative explanation of $\alpha$-decay in terms of quantum mechanical tunnelling through a potential barrier combined with early studies of $\alpha$-particle scattering from nuclei led to the first estimates of the size of the nuclei (Ru29).

There is at present considerable interest in the clustering of nucleons and since alpha particles are the most tightly bound nuclear cluster most of the current discussion concentrates on them. The shell model provides a way of calculating the wave functions of all the nucleons in a nucleus and from the overlap of these and an appropriate parametrized alpha particle wavefunction Harada (Ha62) has shown there is a strong probability of finding alpha particles in Polonium at the nuclear surface. The elastic scattering of alpha particles by some nuclei shows anomalously high cross-sections in the backward direction and it has been suggested that these could be due to interactions with alpha clusters in the nucleus (Ga69).

The alpha decay reaction is another topic closely associated with the formation of four particle clusters in heavy nuclei. Traditionally the reaction mechanism of alpha decay was analysed using $R$-matrix theory and for the intrinsic structure problem the shell model was used. However, it has always been difficult to
reproduce the experimental values for the decay half life. This topic is of current interest because of the possible existence of superheavy nuclei. An early comprehensive study of the alpha decay half life of these systems (FI72) has concentrated on the accurate calculation of the alpha decay energies but used empirical estimates for the effects associated with the intrinsic structure and reaction mechanism. A further interesting aspect of alpha decay is the experimental detection of a parity violating alpha transition from the $2^-$ state at 8.87 MeV to the $0^+$ ground state in $^{16}$O. The current theory of parity violating alpha decay has been reviewed by Blin-Stoyle (BL73) and is explained by parity impurities in the $2^-$ state from nearly $2^+$ states at 6.9 MeV, 9.8 MeV and 11.5 MeV. Although this decay mechanism is an ideal reaction to study the weak interaction there are problems because of the sensitivity of $R$-matrix theory to the channel radius $R_c$ i.e. the reaction mechanism is not good enough.

With the advent of heavy ion accelerators there has been considerable interest in the alpha transfer reaction itself as well as its relation to alpha decay (DA76). The concept of transferring an alpha cluster between nuclear systems is a promising reaction for studying new aspects of nuclear correlations, eg. "quartet states", if the parentage of the alpha clusters is relatively large. A problem in reactions involving $\alpha$-clusters is the form of the wave function describing the relative motion of the intrinsic nuclei. Indeed in a series of papers on alpha decay Fliessbach (FL75, FL176, FL276) has suggested that a correction of the relative motion wave-function because of inclusion of exchange effects between the alpha
particle and the core drastically effects the alpha decay rates in nuclei.

In the following Chapters the alpha-\(^{208}\text{Pb}\) interaction is first investigated and this interaction is then used in more complex reactions involving \(\alpha\)-clusters in nuclei.

In Chapter 2 the analysis of low energy elastic \(\alpha\)-particle scattering from \(^{208}\text{Pb}\) is briefly discussed. The real part of the \(\alpha\)-nuclear potential is calculated using a single folding model. In this calculation the form of the alpha-nucleus interaction is dependent on the alpha nucleon interaction averaged over the target nuclear density, thus simply taking into account the presence of other nucleons. The alpha-nucleon interaction was taken from analysis of proton alpha elastic scattering, making the assumption that the proton-alpha and neutron-alpha interactions are the same. A detailed description of the nature of alpha nucleus scattering is not given for this has been reviewed many times elsewhere (BE70,H071,JA70). However, an account of fitting to available data and the nature of the S-matrix for this scattering process is given.

In Chapter 3 the problems associated with the traditional alpha decay theories are outlined. A new formalism is then developed to reduce these problems and incorporate the microscopic real potentials developed in Chapter 2. This technique, which includes representing the initial \(A+4\) nucleon system as a summation over alpha particle and residual nuclear states, is then applied to the ground state transitions of various Polonium isotopes as well as the transition to excited
states in $^{208}\text{Pb}$ from the $18^+$ isomeric state of $^{212}\text{Po}$. The various possible decay modes to $^{208}\text{Pb}$ are shown in Figure A.

\begin{center}
\begin{tabular}{c|c|c}
$E$(MeV) & $E$ (MeV) \\
\hline
$18^+$ & 11.65 \\
$4^-$ & 3.745 & \textbf{2\%} \\
$5^-$ & 3.198 & \textbf{1\%} \\
$3^-$ & 2.615 & \textbf{97\%} \\
$0^+$ & 0 & \textbf{100\%} \\
$212\text{Po}$ & & \\
\end{tabular}
\end{center}

\textbf{FIGURE A}

Energy levels of $^{208}\text{Pb}$ and two possible decay modes of $^{212}\text{Po}$

The alpha decay energies and lifetimes of the Polonium isotopes are well established experimentally and have been the subject of a number of theoretical papers. In this way they represent an ideal decay mode for the testing of modified alpha decay theories and comparison with previous results. Indeed because of the large range of experimental data available it was possible to develop a new energy lifetime relation for alpha decay which will be shown to fit the experimental data over fourteen orders of magnitude.
In Chapter 4 a distorted wave born approximation (D.W.B.A.) formalism is developed for the heavy ion transfer reaction $^{208}_{\text{Pb}} (^{16}_{\text{C}}^{12}_{\text{C}})^{212}_{\text{Po}}$. This formalism, which uses the potentials generated in Chapter 2 and the alpha decay technique of Chapter 3, assumes a real $\alpha$-particle in its ground state transferred from $^{16}_{\text{O}}$ to $^{208}_{\text{Pb}}$. The relatively large fractional parentage coefficient for the alpha particle in $^{16}_{\text{O}}$ suggests this is a valid assumption. By calculating the differential cross section for this reaction comparison is made between the spectroscopic factor, or measure of alpha cluster probability, in $^{212}_{\text{Po}}$ deduced from this reaction and the alpha decay work of Chapter 3.

In Chapter 5 the modification of the $\alpha$-particle and $^{208}_{\text{Pb}}$ core relative motion wavefunction is considered when the identical nature of the A+4 nucleons is taken into account. This is examined within the context of a modified Resonating Group method. A formalism is developed to calculate the complicated normalisation kernel which arises in this method and is based on a density matrix representation of the two cluster systems. The effect on the relative motion wavefunction between these clusters is considered and a discussion of the implications to alpha-decay, alpha transfer and of Fliessbach's technique and conclusions are also given.

Finally in Chapter 6 a review of the results and conclusions is given with suggestions for possible extensions of the work presented.
2.1 Introduction

The interaction of alpha particles with nuclei presents some special features for, in the first place, the alpha particle is spinless so that there are no complications of spin dependent effects to consider and because it is a tightly bound structure break up effects are likely to be small. At certain energies the alpha particle is also strongly absorbed by nuclei so that semiclassical and diffraction models may be used with some success to describe many features of the scattering process. These models have been reviewed by Hodgson (H071). Generally the experimental data for elastic $\alpha$-nuclear scattering has been successfully analysed in terms of the phenomenological optical model.

More recently, attempts have been made to understand the real part of the $\alpha$-nucleus interaction potential in terms of the microscopic folding model. Morgan and Jackson (M069) concluded that the full microscopic model is capable of producing cross sections for elastic and inelastic scattering of medium energy $\alpha$-particles on $^{42}$Ca and $^{50}$Ti which are comparable in terms of quality of fit to those produced by the generalised optical model while Singh et al (SI75) showed they could describe elastic $\alpha$-nucleus scattering data of a wide range of bombarding energy (20-160 MeV) and target mass ($A = 24-90$) by using an energy dependent folding technique.
More specifically, in the most recent study of α-particle scattering from heavy nuclei Barnett and Lilley (BA174) analysed scattering from $^{208}$Pb and $^{209}$Bi in the energy range 17–22 MeV and suggested that one set of parameters for the phenomenological optical potential for $^{208}$Pb (Their Set A, See Table I) should be particularly suitable for calculations on α-decay of heavy nuclei.

In this chapter the construction of the real part of the microscopic optical potential for the α-$^{208}$Pb interaction will be developed together with an analysis of the elastic scattering cross section of α's on $^{208}$Pb at 22 MeV. These microscopic potentials are compared with the phenomenological Set A mentioned above and their applicability for α-decay calculations is discussed.

2.2 The Microscopic Optical Model

The local optical potential $V(R)$ for an alpha particle scattering from a spin zero nucleus is given by

$$V(R) = V_c(R) - U(R) - iW(R) \quad (2.1)$$

where $R$ is the distance of the centre of masses and the Coulomb potential is given by

$$V_c(R) = \begin{cases} \frac{Z_1 Z_2 e^2}{2R_c} \left(3 - \frac{R^2}{R_c^2}\right) & R \leq R_c \\ \frac{Z_1 Z_2 e^2}{R} & R > R_c \end{cases} \quad (2.2)$$
In this analysis the imaginary potential \( W(R) \) was treated phenomenologically.

The real nuclear potential \( U(R) \) can be calculated microscopically in a number of ways. If the density distributions \( \rho(r) \) for the two ions are assumed to be unperturbed by the scattering process the real potential is simply the expectation value of a nucleon-nucleon interaction \( v \) averaged over the two densities, ie.

\[
U(R) = \int dr \int dr_1 \rho(r) \rho(r_1) v(r_{12})
\]

(2.3)

where \( r \) and \( r_1 \) are nucleon coordinates in the two nuclear systems and \( r_{12} \) is given by \( r - r_1 \). This is called a double folding procedure.

If just one of the integrations in (2.3) is considered a nucleon nucleus potential is obtained and the effective real potential \( U(R) \) can be calculated using a single folding model. Thus, for \( \alpha \)-particle scattering the potential is

\[
U(R) = \int \rho(\xi) V_{\alpha n}(|R - \xi|) d\xi
\]

(2.4)

where in the calculation considered here \( \rho(\xi) \) is the \( ^{208}\text{Pb} \) density and \( V_{\alpha n} \) is an \( \alpha \)-nucleon interaction.

Because of the success in describing \( \alpha \)-particle scattering with a microscopic potential (Mo69) the potential \( U(R) \) was calculated using (2.4) where, for \( V_{\alpha n} \), a Gaussian form was chosen with parameters;
The difference between these parameters is that for H', the depth has been increased by a factor of 1.25; this yields a potential \( V_{\alpha n} \) which gives quite good agreement with the \( \alpha \)-nucleon angular distributions and excellent agreement with the S-wave phase shifts (BA171). This Gaussian form for \( V_{\alpha n} \) can be derived if a nucleon-nucleon interaction of the form

\[
V(r_{12}) = (V_{00} + V_{01} \cdot I_1 \cdot I_2 + (V_{10} + V_{11} \cdot I_1 \cdot I_2) \cdot s_1 \cdot s_2) \exp(-\gamma r_{12}^2)
\]  

(2.5)

where \( s \) and \( I \) represent the spin and isospin components respectively of a nucleon, is folded over a Gaussian \( \alpha \)-particle wave function (BE70).

A Wood-Saxon form for \( V_{\alpha n} \) given by Mailandt et al (MA73) was also used. The parameters are

\[
\begin{align*}
(V) & \quad V = 42.5 \text{ MeV} \quad a = 0.34 \text{ fm} \quad R = 1.43 - 0.0009E \text{ fm} \\
(M) & \quad V = 42.5 \text{ MeV} \quad a = 0.34 \text{ fm} \quad R = 1.43 - 0.0009E \text{ fm}
\end{align*}
\]

where \( E \) is the lab. energy for the nucleon-\( \alpha \) scattering. This interaction gives good agreement with the nucleon-\( \alpha \) phase shifts over a range of energies.

The nuclear density \( \rho(r) \) was constructed using an independent particle shell model, solving a radial Schrödinger equation with a potential of the form
\[ V(r) = V_c(r) - V_N f_N(r) - V_{so} \left( \frac{\hbar^2}{m_n c^2} \right)^2 \frac{1}{r} \frac{d}{dr} f_{so}(r) \]  

(2.6)

where \( f(r) \) is of a Wood-Saxon form

\[ f(r) = \left[ 1 + \exp \left( \frac{r-R}{a} \right) \right]^{-1} \]  

(2.7)

and

\[ R_N = r_N (A - 1)^{1/3} \]

\[ R_{so} = r_{so} (A - 1)^{1/3} \]

The Neutron and Proton density distributions

\[ \rho(r) = \sum_{n\ell j} \left| \psi_{n\ell j}(r) \right|^2 \]  

(2.8)

were calculated with parameters corresponding to the experimental separation energies of the least bound proton (BA69) and neutron (ZA69, BA271) levels and normalised to \( A \), the nucleon number. These parameters give a R.M.S. radii of 5.44 fm and 5.43 fm for the proton and neutron densities respectively, where the corresponding densities are plotted in Figure 1.

In Figure 2 a comparison is made between the total real potential for the three microscopic calculations for \( ^{208}\text{Pb} \) and the phenomenological potential of Barnett and Lilley. The microscopic potentials are much deeper than the phenomenological potential in the interior region but the potential derived from parameters \( H' \) agrees very closely with the phenomenological potential in the surface region between 8 and 10 fm as can also be seen from Figure 3.
A consistent feature of these potentials is that they reproduce the position and height of the Coulomb Barrier. These values are very close to the experimental results of Goldring et al (G070) as can be seen from Table V of Chapter 3.

2.3 Microscopic Analysis

2.3.1 Angular Distribution and Reaction Cross Sections

Using a standard procedure of partial wave analysis and the real potentials shown in Figure 2, the angular distribution of the ratio of the elastic cross section to the Rutherford cross section was calculated for 22 MeV α-particle scattering on $^{208}$Pb. Barnett and Lilley obtained equally good fits to their data with a Wood-Saxon volume absorption $f(R,R_I,a_I)$ or with a surface absorption of derivative form $4a_I f'(R,R_I,a_I)$. In fact the surface absorption allows more Coulomb-Nuclear interference at small angles (J76) but there are no experimental points in this angular region.

In Figure 4 the results obtained for the angular distribution using the three microscopic potentials with the same surface absorption behaviour as the phenomenological potential are compared with the angular distribution using Set A parameters. Figure 5 shows a similar calculation using the phenomenological imaginary potential of volume form. In both calculations potential H gives a much better fit while $H'$, despite being very close to the Barnett and Lilley potential in the surface region, gives very poor agreement. The agreement obtained with potential M is seen to be improved using volume absorption.
A further calculation was carried out in which the three parameters in the imaginary potential only were allowed to vary freely in order to minimise the quantity

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{\sigma_{TH}(\theta_i) - \sigma_{exp}(\theta_i)}{\Delta\sigma(\theta_i)} \right)^2$$  \hspace{1cm} (2.9)$$

where \(\sigma_{TH}(\theta_i)\) is the theoretical cross section at angle \(\theta_i\); \(\sigma_{exp}(\theta_i)\) is the experimental cross section at \(\theta_i\) and \(\Delta\sigma(\theta_i)\) takes into account the experimental uncertainty at \(\theta_i\). Good agreement with the data was obtained with the real microscopic potentials when both volume and surface absorption potentials were used with the parameters given in Table I.

In Figure 3 the imaginary potential for the H and H' potential are also plotted and it is seen that for the deeper H' real potential the imaginary potential required to fit the data is peaked further out in the surface, as would be expected.

2.3.2 Reflection Coefficients

Figure 6 shows the real and imaginary parts of the reflection coefficients \(\eta_L\) for the best fit potentials with surface absorption. The behaviour of the low partial waves immediately shows this is not a strong absorption situation (AU61). Austern (AU61) has attributed the fluctuation of \(\eta_L\) with L to interference between the reflection from the angular momentum barrier and reflection from the potential surface. Using the W.K.B. approximation he writes

$$\eta_L = \eta_L^{\text{BARRIER}} + \eta_L^{\text{SURFACE}}$$ \hspace{1cm} (2.10)$$
where

\[ \eta_{L}^{\text{BARRIER}} = - e^{-2iS_{L}(r_{L})} \]

\[ \eta_{L}^{\text{SURFACE}} = \frac{1}{r_{L}} \int_{r_{L}}^{\infty} \left( e^{-2iS_{L}(r')} - e^{-4iS_{L}(r_{L})} e^{2iS_{L}(r')} \right) \]

\[ \times \frac{k^{'(r')}}{k_{L}(r')} \, dr' \]

\[ S_{L}(r) = kr + \int_{\infty}^{r} [k(x) - k \, dx] \]

\[ k(r) = \left( k^{2} - \frac{\kappa(\ell+1)}{r^{2}} - \frac{2\mu}{\hbar^{2}} V(r) \right)^{\frac{1}{2}} \tag{2.11} \]

and \( r_{L} \) is the radius at which \( \text{Re}k_{L}(r) = 0 \).

The relative contributions to \( \eta_{L} \) have recently been analysed by Gelbke (GE75) for a strong absorption scattering situation. A Coulomb potential was included in this analysis using equations (2.10) and (2.11). It was found that fluctuations in \( \eta_{L} \) will disappear if either \( \eta_{L}^{\text{BARRIER}} \) or \( \eta_{L}^{\text{SURFACE}} \) is reduced.

In the strong absorption situation it is thought that reflections from the surface are small and for low partial waves the reflections at the barrier take place well inside the nucleus so that the absorptive part of the optical potential causes substantial attenuation. For a weak absorption situation \( \eta_{L}^{\text{BARRIER}} \) is larger because the ingoing and reflected waves are less attenuated. Strong reflection from the surface will occur if the real potential has a sharp surface, if the wavelength is comparable or greater than the surface thickness, and if the phase averaging is upset. In the present case we have a relatively weak absorption situation and the wavelength is 3.1 fm.
which is not small in comparison with the surface thickness. It is therefore not surprising that \( \eta_{\text{BARRIER}} \) and \( \eta_{\text{SURFACE}} \) can interfere to give the odd-even staggering effect.

2.3.3 Critical Angular Momentum and Radii

Because of the ambiguities in the optical potential of strongly absorbed particles it is necessary to introduce more fundamental size parameters for the interpretation of elastic scattering. One of these strong absorption radii \( R_i \) is calculated through the relation

\[
kR = \gamma \pm \left[ \gamma^2 + L(L+1) \right]^{\frac{1}{2}}
\]

where \( \gamma \) is the usual Coulomb parameter

\[
\gamma = \frac{Z_1 Z_2 e^2}{\hbar v} = \frac{\mu Z_1 Z_2 e^2}{\hbar^2 k}
\]

and the angular momentum is set equal to \( L \) through \( \Gamma(\eta L) = 0.5 \).

However because of the weak absorption and staggered nature of the phase shifts it is not possible to calculate this quantity consistently. For scattering from heavy nuclei Blair (BL54) has defined the quarter point angle \( \theta_c \) such that \( d\sigma/d\sigma_R = 0.25 \) and this is related to a radius through the expressions \( \theta_c = 2\tan^{-1}(\gamma/L_c) \) and (2.12). However from the cross sections this point is not clearly defined either.

As already stated the barrier radius is well determined and appears to be a significant radius parameter for this scattering process.
2.4 Discussion

The results show that it is possible to fit $\alpha$-particle elastic scattering from $^{208}\text{Pb}$ at 22 MeV very well using a microscopic real potential derived from a single folding model. All the microscopic potentials gave good agreement with the data when the imaginary potential was allowed to vary. This result implies that when attempts are made at these energies to study the microscopic structure of the optical potential it is always necessary to give information on the way the imaginary potential is treated.

If in the problem of $\alpha$-decay it is assumed that the imaginary potential is negligible for the very low $\alpha$-decay energies ($\approx$ 8 Mev) then because of the different shapes of the various microscopic real potentials it is obvious that no one real potential is necessarily the correct one for $\alpha$-decay. This is because of the inseparable problem of the combined shape and magnitude of the total complex potential to take account of absorption effects at these higher energies.

The problem can be seen more clearly by calculating the well known W.K.B. expression for the penetrability $P$ through a barrier for the various real potentials (Table II) where

$$P = \exp \left\{ -2 \frac{(\frac{\mu}{\hbar})^{\frac{1}{2}}}{2} \int_{r_1}^{r_2} (V(R) - T_\alpha)^{\frac{1}{2}} dR \right\}$$  \hspace{1cm} (2.14)

and $r_1$, $r_2$ are the inner and outer turning points respectively. $T_\alpha$ is the kinetic energy of the emitted $\alpha$-particle

It can be seen that the potential $H$ which fits the elastic
scattering data with the same imaginary potential as the Barnett and Lilley real potential yields a value differing by 25% from the Barnett and Lilley result while potential M yields a value differing by a factor of two. These differences arise from the different widths of the barrier as can be seen from Figure 2.
<table>
<thead>
<tr>
<th>Type</th>
<th>Parameters</th>
<th>Type</th>
<th>Parameters</th>
<th>σ Reaction (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phenomenological</td>
<td></td>
<td>Surface</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BL (Set A)</td>
<td>$V_R = 96.44$ MeV</td>
<td>BL (Set A)</td>
<td>$V_I = 32.0$ MeV</td>
<td>219.8</td>
</tr>
<tr>
<td></td>
<td>$R_R = 1.376$ fm</td>
<td></td>
<td>$R_I = 1.216$ fm</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_R = 0.625$ fm</td>
<td></td>
<td>$a_I = 0.42$ fm</td>
<td></td>
</tr>
<tr>
<td>Microscopic</td>
<td>$H$</td>
<td>Surface</td>
<td>$V_I = 31.3$ MeV</td>
<td>207.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BL (Set A)</td>
<td>$R_I = 1.383$ fm</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_I = 0.402$ fm</td>
<td></td>
</tr>
<tr>
<td>Phenomenological</td>
<td></td>
<td>Volume</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BL (Set A)</td>
<td>$V_R = 96.44$ MeV</td>
<td>BL (Set A)</td>
<td>$V_I = 32.0$ MeV</td>
<td>228.1</td>
</tr>
<tr>
<td></td>
<td>$R_R = 1.376$ fm</td>
<td></td>
<td>$R_I = 1.216$ fm</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_R = 0.625$ fm</td>
<td></td>
<td>$a_I = 0.42$ fm</td>
<td></td>
</tr>
<tr>
<td>Microscopic</td>
<td>$H$</td>
<td>Volume</td>
<td>$V_I = 9.53$ MeV</td>
<td>209.8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BL (Set A)</td>
<td>$R_I = 1.564$ fm</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_I = 0.324$ fm</td>
<td></td>
</tr>
<tr>
<td>Microscopic</td>
<td>$H'$</td>
<td>Volume</td>
<td></td>
<td>224.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$V_I = 55.2$ MeV</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$R_I = 0.533$ fm</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_I = 0.164$ fm</td>
<td></td>
</tr>
<tr>
<td>Microscopic</td>
<td>$M$</td>
<td>Volume</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE II

Values of Penetrability Using WKB Analysis for $^{212}$Po Decay

<table>
<thead>
<tr>
<th>Potential</th>
<th>Inner Turning Point (fm)</th>
<th>Penetrability</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>9.20</td>
<td>$5.13 \times 10^{-14}$</td>
</tr>
<tr>
<td>BL H'</td>
<td>9.11</td>
<td>$2.72 \times 10^{-14}$</td>
</tr>
<tr>
<td>H</td>
<td>8.90</td>
<td>$2.16 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS (Chapter 2)

Figure 1  The proton density distribution \( \rho_p(r) \), the neutron density distribution \( \rho_n(r) \) and the total density distribution calculated using the shell model technique outlined in the text.

Figure 2  Comparison of the real potentials \( \text{Re}V(R) = U(R) + V_c(R) \) for the phenomenological potential of Barnett and Lilley (BL) and the three microscopic potentials. Also shown are the surface imaginary part of the BL potential and the surface imaginary potential which gives the best fit with the \( H' \) real potential.

Figure 3  Comparison of the real nuclear potentials \( U(R) \) in the surface region.

Figure 4  Comparison of the angular distributions given by the three real microscopic potentials and the surface imaginary part of the BL potential.

Figure 5  Same as Figure 4 but with the volume imaginary part of the BL potential.

Figure 6  The reflection coefficients corresponding to the best fits for the microscopic potentials with surface absorption.
FIGURE 1
FIGURE 2
E = 22 MeV

FIGURE 4
FIGURE 5
FIGURE 6
3.1 Introduction

3.1.1 Statement of the Problem

Alpha decay, the process in which a nucleus in its ground or an excited state breaks up spontaneously into an α-particle and a residual nucleus, is the oldest observed reaction in nuclear physics. In the earliest attempt to understand this process Gamow (GA28) gave a qualitative explanation in terms of quantum mechanical tunnelling of the α-particle through the potential barrier, formed by the sum of the real attractive nuclear interaction and the Coulomb repulsion. But, as shown at the end of the last chapter, this technique introduces large ambiguities due to the choice of the total real potential taken from higher energy scattering calculations.

With the development of nuclear reaction theories subsequent theoretical formalism was based on R-matrix theory (LA58). The first in a long series of papers on alpha decay, based on this reaction theory, and trying to understand the formation of the alpha particle in terms of a microscopic shell model description of the parent nucleus, was introduced by Mang (MA60). Subsequent analysis, using this technique, was based on a more accurate description of the formation of the alpha particle, by using varying degrees of configuration mixing (HA61, HA62) and applying the theory to the α-decay of deformed nuclei (SA63). The results for the relative decay rates using these models are in rather
good agreement with the experimental values but the absolute decay rates are too small by between a factor of 10 to 1000. More recent theories of alpha decay of heavy nuclei (KA70, HA68) have been based on the unified theory of nuclear reactions developed by Feshbach (FE73) in which the $\alpha$-width is expressed as a scattering resonance for the system consisting of an alpha particle and the residual nucleus.

The problems associated with the prediction of absolute $\alpha$-decay rates can be stated as follows.

(i) In the R-matrix theory the penetrability is extremely sensitive to the choice of the arbitrary channel radius.

(ii) The phenomenological optical potential for $\alpha$-particle scattering from nuclei at medium energies shows ambiguities. In R-matrix theory (SC68) and in other methods (HA68) these ambiguities cause substantial variation in the calculated widths.

(iii) The initial state is normally treated in the oscillator shell model while the final state is represented by an $\alpha$-particle moving in the $\alpha$-nucleus potential. This can lead to problems of nonorthogonality, and it is difficult to show convincingly how the shell model state evolves into the final system of an $\alpha$-particle and residual nucleus.

In this Chapter an alpha-decay formalism is developed which does not involve an arbitrary channel radius and uses the microscopic
optical potentials developed from low energy α-particle scattering, the properties of which were described in Chapter 2. This formalism has been applied to the decays of the Polonium isotopes listed in Table III. With this formalism the problems (i) and (ii) mentioned above have been considerably reduced and, in addition, because a cluster model approach was used, it will be shown that this technique contributes substantially to the resolution of problem (iii) and yields satisfactory quantitative results.

3.1.2 Problems in α-Decay

In the remainder of the introduction the problems outlined above are expanded to give greater insight into the inconsistencies that existed in alpha decay.

In R-matrix theory the configuration space is separated into two regions, in the external region defined by \( r > R_c \), the total wavefunction can be expressed as a sum of channel functions of simple product form while the wavefunction in the internal region \( U_E(r) \) is replaced by a complete set of eigenfunctions \( U_\lambda(r) \) with real eigenvalues \( E_\lambda \).

\[
U_E(r) = \sum_\lambda A_\lambda U_\lambda(r) \quad 0 \leq r \leq R_c \quad (3.1)
\]

Using the boundary condition

\[
\frac{dU_\lambda(R_c)}{dr} = 0 \quad (3.2)
\]

and applying Greens theorem to the radial Schrödinger equation, then using the orthogonality properties of \( U_\lambda(r) \) it can be readily deduced
that
\[ U_E(r) = \frac{\hbar^2}{2M} \sum_{\lambda} \frac{U_{\lambda}(R_c) U_{\lambda}(r)}{E_\lambda - E} \left( \frac{dU_E}{dr} \right)_{R_c} \] (3.3)
\[ = G(r, R_c) (R_c dU_E(r)/dr) R_c \] (3.4)

The R-matrix is defined as \( G(R_c, R_c) \) and is related to the reduced width amplitude \( \gamma^2_{\lambda} \) through
\[ R = \sum_{\lambda} \frac{\gamma^2_{\lambda}}{E_\lambda - E} \] (3.5)
where
\[ \gamma_{\lambda} = \left( \frac{\hbar^2}{2MR_c^2} \right)^{\frac{1}{2}} U_{\lambda}(R_c) \] (3.6)

Equation (3.4) shows that the R-matrix is directly related to the reciprocal of the logarithmic derivative at \( R_c \). Applying outgoing wave boundary conditions at this point and comparing the phase shifts with the Briet-Wigner form (JA70) the width of a narrow resonance \( \Gamma \), in the single channel representation, is given by
\[ \Gamma \sim 2P\gamma^2_{\lambda} \] (3.7)

where \( P \) is called penetrability and is given by
\[ P = \frac{kR_c}{F^2(R_c) + G^2(R_c)} \] (3.8)

The functions \( F \) and \( G \) correspond to the usual regular and irregular Coulomb functions.
The one body widths are related to the many body widths through the spectroscopic factor $S$, ie.

$$\gamma^2 = s^2$$

(3.9a)

$$\Gamma_a = s \Gamma_{ob}$$

(3.9b)

It is well known that variation if the channel radius $R_c$ can lead to an order of magnitude variation in the penetrability. Calculations by Scherk and Vogt (SC68) for the decay of $^{212}$Po to $^{208}$Pb using the $\alpha$-particle optical potential for $^{208}$Pb due to Igo (IG59) showed that the penetrability increases by a factor of $\sim 10$ as $R_c$ increases from 9 fm to 10 fm. Strictly, $R_c$ should be sufficiently large so that there is no coupling between different channels in the external region but values of $R_c$ well inside the inner turning point have been used (RA67). Values of the penetrability derived from (3.8) are plotted in Figure 7 for various Polonium isotopes. The W.K.B. results, for potential $H$, are also shown plotted at the position of the inner turning point.

In alpha decay the reduced width amplitude $\gamma_\lambda$ is given by

$$\gamma_\lambda = \left\{ \frac{\hbar^2}{2 MR_c} \right\}^{1/2} \int dx_k dx_d R^d \phi^* \psi^\alpha (x_\alpha) \sum_m (Lm|JM-m)$$

$$Y^m_L(R) \psi_{J-M-m} (x_{\alpha})$$

(3.10)

where $\phi^{JM}$ and $\phi^{JM-m}$ are the wave functions of the parent and the
daughter nucleas respectively and \( \phi_i \chi^M_{L} (\hat{R}) \) are the internal wave function and the angular part of the center of mass motion of an \( \alpha\)-particle. However, if the parent nucleas is described by the single particle model then inconsistencies are introduced through the amount of configuration mixing present in the structure of the initial nucleas. Indeed a communication with Arima (AR75) showed that a shell model calculation for \(^{212}\)Po with admixture of 89 proton levels and 83 neutron levels yields a ground state spectroscopic factor of \(1.2 \times 10^{-3}\) while the lowest configuration of four particles outside the \(^{208}\)Pb core, ie, \((1h_{9/2})^2(2g_{9/2})^2\) yields \(1.98 \times 10^{-5}\). These results were also sensitive to the single particle potentials used.

The sensitivity of previously calculated theoretical decay rates to ambiguities in the optical potential can be seen in Table IV. All these results were calculated with a radius independent reaction theory and a constant amount of configuration mixing. They emphasize the need to remove the uncertainties in decay rates arising from potential ambiguities.

Further, the experimental values given in Table III immediately suggest that a numerical solution of a radial Schrödinger equation at these energies with complex energy and potentials in order to determine the width would be a very difficult procedure.

### 3.2 Time Dependent Theory of Decaying States

#### 3.2.1 Definitions and Notation

The total wavefunction \( \Psi \) for the \( A+4 \) system is a solution of the
The Hamiltonian can be written as

\[ H = H_\alpha + H_A + V_{\alpha A} + T_R + V_c \]  \hspace{1cm} (3.12)

where \( T_R \) is the kinetic energy operator for the motion of the \( \alpha \)-particle relative to the residual nucleus, \( V_c \) is the Coulomb potential and \( V_{\alpha A} \) is a many body operator representing the total nuclear interaction between the nucleons in the \( \alpha \)-particle and those in the residual nucleus. Choosing a set of relative coordinates of the form

\[ R = R_\alpha - R_A \]  \hspace{1cm} (3.13a)

\[ \xi_i = \xi_i - R_A \]  \hspace{1cm} (3.13b)

\[ \xi_k = \xi_{A+k} - R_\alpha \]  \hspace{1cm} (3.13c)

where

\[ R_\alpha = \frac{1}{4} \sum_{k=1}^{4} \xi_{A+k} \] \hspace{1cm} \[ R_A = \frac{1}{A} \sum_{i=1}^{A} \xi_i \]  \hspace{1cm} (3.14)

the interaction \( V_{\alpha A} \) can now be expressed as

\[ V_{\alpha A} = \sum_{i=1}^{A} \sum_{k=1}^{4} V_{nn} (\xi_{A+k} - \xi_i) = \sum_{i=1}^{A} \sum_{k=1}^{4} V_{nn} (\xi_k + R - \xi_i) \]  \hspace{1cm} (3.15)

The alpha particle has spin zero and therefore the \( \alpha \)-decay selection rules are very simple. The angular momentum of the \( \alpha \)-particle is
restricted to the interval

$$|j-I'| \leq \ell \leq j + I'$$ \hspace{1cm} (3.16)

where J and I' are the spins of the initial and final nuclear states respectively. The \(\ell\)-value is further restricted by parity conservation and is even if the parities of the initial and final nuclear states are identical and odd if the parities are different.

3.2.2 Derivation of the Decay Width

In this section a formula for the width of a decaying state is developed, based on the time dependent analysis given by Goldberger and Watson (GO64,Chapter8). This formula does not involve an arbitrary channel radii, and therefore avoids the problem of the radius dependent width associated with R-matrix theory.

The time dependent wavefunction for the A+4 system can be written as

$$\psi(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE e^{-iEt/\hbar} G^+(E) \psi_0$$ \hspace{1cm} (3.17)

where \(G^+(E)\) is the Greens function

$$G^+(E) = (E-H+i\epsilon)^{-1}$$ \hspace{1cm} (3.18)

In this expression \(H\) is the total Hamiltonian and \(\psi_0\) is the wavefunction of the initial state at time \(t=0\). The wavefunction can always be expanded in the form
\psi(t) = \psi_o J_{oo}(t) + \sum_b \psi_b^* J_{bo}(t) \tag{3.19}

where \psi_b^* represents the final states of the A+4 system and \( J_{oo}(t) \), \( J_{bo}(t) \) are the probability amplitudes of finding the initial system in state \( \psi_o \) or \( \psi_b^* \) respectively at time \( t \), ie.

\[ J_{oo}(t) = \langle \psi_o | \psi(t) \rangle = \frac{1}{2\pi i} \int dE e^{-iEt/\hbar} \langle \psi_o | G^+(E) | \psi_o \rangle \tag{3.20} \]

\[ J_{bo}(t) = \langle \psi_b^* | \psi(t) \rangle = \frac{1}{2\pi i} \int dE e^{-iEt/\hbar} \langle \psi_b^* | G^+(E) | \psi_o \rangle \tag{3.21} \]

In the problem of alpha decay the initial state is described as one of a set of discrete eigenstates of an Hamiltonian \( K \) and the difference \( (H-K) \) was interpreted as giving rise to the decay process. The final states were taken to be eigenstates of a Hamiltonian \( K' \) and if \( K \neq K' \), as is usually the case, some nonorthogonality is introduced into the description of the process. These points are covered in more detail in section 3.2.4.

Following the analysis of Goldberger and Watson, consider the introduction of a level shift operator \( R \) where

\[ R = (H-K)F \tag{3.22} \]

\[ F = 1 + (E-K)^{-1} Q (H-K) F \tag{3.23} \]

and \( Q \) is a projection operator which projects off the ground state. The matrix elements of \( R \) are
\[ R_0(E) = \langle \psi_o | (H-K) | F' \psi_o \rangle \]  
(3.24)

\[ R_{bo}(E) = \langle \psi_b | (H-K) | F' \psi_o \rangle \]  
(3.25)

and
\[ R^+(E) = \lim_{\epsilon \to 0} R(E+i\epsilon) \]  
(3.26)

The operator \( R \) is called the "level shift operator" because its diagonal elements represent the shift between the unperturbed energy eigenvalues of the Hamiltonian \( K \) and the perturbed energy \( E \). Thus when the full Hamiltonian \( H \) operates on a decaying state the imaginary part of \( R \) is directly related to the width of this state. Indeed the full perturbation expansion can be obtained by iterating equation (3.22). The matrix elements of \( R \) are related to the Greens functions through the relations

\[ \langle \psi_o | G^+(E) | \psi_o \rangle = \frac{1}{E - E_o - R_o(E)} \]  
(3.27)

\[ \langle \psi_b | G^+(E) | \psi_o \rangle = \frac{1}{E - E_b - R_{bo}(E)} \frac{1}{E - E_o - R_o(E)} \]  
(3.28)

Using (3.27) and (3.28) and writing,

\[ R(E) = D(E) - i\Gamma(E) = D(E) - \frac{i\Gamma}{2} \]  
(3.29)

where \( \Gamma \) is the decay width, the probability amplitudes (3.20) and (3.21) may be readily evaluated by the residue theorem to give

\[ |J_{oo}(t)|^2 = e^{-\Gamma t/\hbar} \to 0 \]  
(3.30)

\[ t \to \infty \]
\[ J_{bo}(t) = \frac{e^{-E_b t/\hbar} R_{bo}(E_b)}{E_b - E_0 - R(E_0)} - \frac{e^{-i(E_0 + D(E_0)) t/\hbar} -i t/2\hbar R_{bo}(E_0)}{E_b - E_0 - R(E_0)} \]  

(3.31)

\[ R_{bo} e^{i E_b t/\hbar} = \frac{R_{bo}}{E_b - E_0 - R(E_0)} \left( 1 - \exp[i(E_b - E_0 - D(E_0) + \frac{i t}{\hbar})] \right) \]  

(3.32)

where the approximations \( R_{bo}(E_b) \sim R_{bo}(E_0) \) and \( R(E_b) \sim R(E_0) \) have been used. Thus

\[ |J_{bo}(t)|^2 \to \int d\rho(E_b) \frac{|R_{bo}(E_b)|^2}{[E_b - E_0 - D(E_0)]^2 + \Gamma^2/4} \]  

(3.33)

where \( d\rho(E_b) \) is the density of states per unit energy interval. The formula for the width of the resonance may be extracted by using the unitarity properties of the level shift operator. Writing the formal solution of (3.22) and (3.23) as

\[ R = (H-K) + (H-K)Q \frac{1}{E-H} Q(H-K) \]  

(3.34)

where \( H_a = K + Q(H-K)Q \)  

(3.35)

Then using the unitarity properties of \( R \)

\[ R(E \pm i\eta) - R^+(E \pm i\eta) = 2i\Pi(E) \]  

(3.36)

\[ = (H-K)Q \left( \frac{1}{E \pm i\eta - H_a} - \frac{1}{E \mp i\eta - H_a} \right) Q(H-K) \]  

(3.37)

\[ = \mp 2i\eta R^+ \frac{Q}{(E \pm i\eta - K)(E \mp i\eta - K)} R \]  

(3.38)
Now when $\eta$ is small and positive (GO64, Chapter 5)

\[
\frac{2i\eta}{(E-k)^2 + \eta^2} = 2\pi i \delta(E-k) \tag{3.39}
\]

Thus

\[
\Gamma_b = 2\pi \int |R_{b0}(E_b)|^2 \delta(E_b) \tag{3.40}
\]

where

\[
\Gamma = \sum_b \Gamma_b \tag{3.41}
\]

and the width is to be evaluated at $E_b = E_o + R_{b0}(E_o)$. The expression (3.40) for the width can be rewritten using (3.22) to give

\[
\Gamma_b = 2\pi \int \delta(E_b) \left| \langle \psi_b^* | (H-K) | \psi_o \rangle \right| d\rho(E_b)
+ \left| \langle \psi_b^* | (H-K) \frac{Q}{E-K} | \psi_o \rangle \right|^2 \tag{3.42}
\]

The second term represents a process in which the nucleus is excited to a virtual intermediate state and then decays.

The above formula for the width can be derived in many different ways including the reaction theory due to MacDonald (MA64), in which a T-matrix formulism is used. The formalism developed by Feshbach (FE73) gives a formula for the width which is extremely satisfactory for the decay of an excited state by nucleon or $\alpha$-particle emission but it does not give a simple prescription for the type of decay process which arises as the result of tunnelling through a barrier without excitation. An account of these reaction theories, their inter relations, and their relevance to alpha decay has been given.
3.2.3 Choice of the Hamiltonians $K$ and $K'$

In the previous papers on alpha decay based on the formula (3.40) there seems to arise problems in the choice of the unperturbed Hamiltonian and the form of the wavefunction $\Psi_0$.

Kadmenski and Kalechits (KA70) give as their choice for the unperturbed Hamiltonian

$$K = K' = H_A + H_\alpha + T_R + \frac{2Ze^2}{R}$$

and consequently the states $\Psi_0$ and $\Psi_b$ (by0) form a complete set of eigen-states. However this Hamiltonian will not give any bound states for the relative motion because there is no potential barrier and the whole of the nuclear interaction is contained in the residual interaction. Hence, at $t=0$, the nucleus is not in a bound state or in a resonant state. In practice they take the wavefunction $\Psi_0$ to be a shell model state constructed from single particle wavefunction of oscillator form. This means that the nonorthogonality of the states $\Psi_0$ and $\Psi_b$ is reintroduced through the use of a different (shell model) Hamiltonian which is not related to the operators $H$ and $K$ defined in their paper. Their value for the decay width of $^{210}\text{Po}$ is listed in Table IV.

Harada and Rauscher (HA69) also take $\Psi_0$ to be a shell model state but draw attention to the problem of nonorthogonality. However, they choose a perturbation of the form
where \( U_N \) is the nuclear part of the phenomenological optical potential for the \( \alpha \)-particle. In practice \( V_{\alpha A} \) is approximated by the sum of real parts of the nucleon-nucleus potentials averaged over the ground state wavefunctions of the alpha particle and residual nucleus, called \( \hat{U}_N \). Now using Feshbach's formula for the generalised optical potential (FE73)

\[
U_N = \langle 0 | V_{\alpha A} | 0 \rangle + \langle 0 | V_{\alpha A} Q \frac{1}{(E-QHQ)} QV_{\alpha A} | 0 \rangle \quad (3.45a)
\]

\[
= \hat{U}_N + \Delta \hat{U}_N \quad (3.45b)
\]

Thus the difference \( \hat{U}_N - U_N \) corresponds in principle to the second order term. The Coulomb barrier and potential appear to play no special role in the formalism which could equally well apply to nucleon emission.

3.2.4 Creation of a Bound Initial State

Even for long lived naturally radioactive nuclei the true initial state is not a stationary state which persists until acted on by the interaction \( (H-K) \) but a resonant state with a complex energy. To reduce the problem of nonorthogonality and tackle the problem of ambiguities in the \( \alpha \)-nucleus optical potential the initial state was transformed into a bound state by introducing the model Hamiltonian

\[
K = H_A + H_\alpha + T_R + U_R \quad (3.46)
\]
where \( r_b \) is the barrier radius, i.e. the position of the top of the barrier formed by the potential \( U_N + V_c \). The required bound state for the \( \alpha \)-particle now has the single-particle energy

\[
E_\alpha = T_\alpha - U_R(r_b) - \frac{\hbar^2}{2\mu} \frac{L(L+1)}{r_b^2}
\] (3.48)

and the interaction is

\[
H - \mathcal{K} = V_{\alpha A} - U_N \quad R \leq r_b
\] (3.49)\[
= V_{\alpha A} + V_c - U_R(r_b) \quad R > r_b
\]

The physical interpretation of this expression is that the transition represents the lowering of the constant potential \( U_R(r_b) \) to allow the \( \alpha \)-particle to escape, as shown in Figure 8. This approach is likely to be a good approximation for the long lived decays in which penetration through the barrier is the dominant aspect of the process. In this case the wavefunction for the final state must be obtained from the Hamiltonian

\[
K' = H_A + H_\alpha + T_R + U_N + V_c
\] (3.50)

This leads to nonorthogonality again but, since \( K-K' \) is zero in the
region $R < r_b$ this should not be a significant effect.

The experimental position and height of the Coulomb barrier, as shown in Chapter 2, can be very well reproduced using phenomenological potentials or a microscopic folding model. The consistency in reproducing this parameter and the process in which the initial state is made into a bound state is used in the next section to help reduce ambiguities in calculations of alpha decay rates.

A technique of making the resonant state into a bound state has been used by Garside et al (GA65) for neutron reactions at low energy.

3.3 Evaluation of Decay Width Matrix Element

3.3.1 Excitation of the Residual Nucleas

In section 3.1.1 the difficulties associated with the formation of an $\alpha$-cluster from an independent shell model approach were outlined. Because of the uncertainties associated with the shell model approach and the formation of the $\alpha$-cluster in heavy nuclei a different approach was taken in which the $A+4$ many body wavefunction for the initial state was written in the form of a cluster model expansion.

$$\psi_{o}^{JM_{J}} = \sum_{IMLM} C_{L}^{J}(I)(LMIM|JM_{J})\phi_{A}^{IM}(\eta)\phi_{A}(\xi)\chi_{A}^{-LM}(R)$$  

(3.51)

where $\eta, \xi, R$ were defined in section 3.2.1 and $C_{L}^{J}(I)$ is the fractional parentage coefficient. The excitation of the alpha particle and the role of the antisymmetrisation operator are neglected at present. The
relative wavefunction $\chi_{\alpha}$ was expanded as

$$\chi_{\alpha}^{\text{I}M} = R^{-1}u_{\text{NL}}(R)Y_{L}^{m}(\hat{R})$$  \hspace{1cm} (3.52)

so that the normalisation is given by

$$\int |\chi_{\alpha}^{\text{I}M}|^2 dR = \int_{0}^{\infty} |u_{\text{NL}}|^2 dR = 1$$ \hspace{1cm} (3.53)

The wavefunction for the final state must be coupled to the same channel spin, to give

$$\psi_{b}^{JM,J'} = \sum \left( \xi \xi' M' \right) \phi_{\alpha}^{I'M'}(\eta) \phi_{\alpha}(\xi) \chi_{\alpha}^{\xi'\xi'}(R)$$ \hspace{1cm} (3.54)

with

$$\chi_{\alpha}^{\xi'\xi'} = A_{\alpha} R^{-1} u_{\alpha}(R) Y_{\xi}^{\nu}(\hat{R}) Y_{\xi}^{\nu*}(\hat{\xi})$$ \hspace{1cm} (3.55)

To evaluate the alpha decay widths only the leading term in equation (3.42) will be used. Thus using (3.49) for the perturbing interaction the matrix element to be evaluated is

$$X(JM_{J} \rightarrow I'M') = \langle \psi_{b}^{JM} | V_{ab} - U | \psi_{o}^{JM} \rangle$$ \hspace{1cm} (3.56)

where the potential $U(R)$ connects only those terms with $I=I'$. Because the alpha particle is not excited, in the present formalism, the many body interaction $V_{ab}$ can be folded with the $\alpha$-particle to give

$$\langle \phi_{\alpha}(\xi) | V_{ab} (\eta \xi \hat{R}) | \phi_{\alpha}(\xi) \rangle = V_{a\eta}(\bar{R}\eta)$$ \hspace{1cm} (3.57)
Making use of a multipole expansion of the interaction potential and the Wigner-Eckart theorem (BR68) the interaction, between nuclear states, is given by (JA70)

\[
\langle I'M' | V_{\text{an}} (R) | IM \rangle = \sum_{kq} \langle I'M' | V_{kq} | IM \rangle \left[ i^{k} Y_{k}^{q}(\hat{R}) \right]^* \tag{3.58}
\]

\[
= \sum_{kq} (4\pi)^{\frac{1}{2}} (\text{IM}_q \text{ I'M'}) V_{k}^{\text{II'}_1} (R) [i^{k} Y_{k}^{q}(\hat{R})]^* \tag{3.59}
\]

Combining all the equations of this section the matrix element \(X\) becomes

\[
X(JM_j \text{ I'M'}) = \sum_{\text{IM}_L m l \text{ kq} \text{ M'}_j} C^{J}_{L}(1) (-1)^{q} (-i)^{k} A_{k} Y_{l}^{\ast} Y_{l}^{q} (R) (\text{Lo}k_{o} | z_{o}) \frac{R_{k}}{x}
\]

\[
\times (Lm_{k-q} | E_{q}) (Lm_{I'M} | J_{M} J) (E_{q}I'M' | J_{M} J) (\text{IM}_q | I'M')
\]

\[
\times \int_{0}^{\infty} u_{l} (R) [V_{k}^{\text{II'}_1} (R) - \delta_{k_{o}U(R)}]_{\nu_{NL}} dR \tag{3.60}
\]

where \(x = (2x+1)^{\frac{1}{2}}\) and standard relations between Gablesh-Gordon coefficients as well as the relation

\[
\int Y_{l_{1}}^{m_{1}}(\Omega)Y_{l_{2}}^{m_{2}}(\Omega)Y_{l_{3}}^{m_{3}}(\Omega) d\Omega = \frac{\hat{\lambda}_{1} \hat{\lambda}_{2}}{\hat{\lambda}_{3} (4\pi)^{\frac{1}{2}}} \left( \frac{\hat{\lambda} l_{1} l_{2} l_{3} o}{\hat{\lambda} l_{1} m_{1} l_{2} m_{2} l_{3} m_{3}} (-1)^{m_{3}} \right) \tag{3.61}
\]

have been used (BR68). Using the relation between \(3_j\) symbols and the Racah coefficient (BR68), ie.
\[ W(k\ell j|I' L) = \sum_{qM M'} (2\ell + 1)(-1)^{I-L-I'-q+M_j} \]

\[
\begin{bmatrix}
 k & I & I' \\
 q & M & M'
\end{bmatrix}
\begin{bmatrix}
 J & \ell & I' \\
 -M_j & V & M'
\end{bmatrix}
\begin{bmatrix}
 I & J & L \\
 M & -M_j & m
\end{bmatrix}
\begin{bmatrix}
 \ell & k & L \\
 \nu & q & -m
\end{bmatrix}
\]

(3.62)

then (3.60) becomes

\[ X(JM_J+I'M') \]

\[ = \sum_{I\ell V L k} C^I_{L}(1)(-1)^{I-I'\ell} (-i)^{k_A} \hat{y}^u_{L}(\hat{L}) (\text{Loko}|\ell_0) \frac{\hat{J}_{L,\ell}^{2}\hat{f}_{L}^{2}}{q^2} \]

\[ \times W(k\ell J|I'L) \int_0^\infty U_k(R)[V^II'_{k}(R)-\delta_{k_0} U(R)]u^*_N(R) dR \quad (3.63) \]

When \( I=I' \) and corresponds to the ground state of the residual nucleus then (3.59) becomes

\[ V^II_o(R) = \int \rho_A(\eta) V_{an}(\eta) d\eta \equiv \hat{U}_N \quad (3.64) \]

This expression is just the single folding model which was used in Chapter 2 to describe the \( \alpha \)-nucleus interaction. Thus the formalism enables the \( \alpha \)-nuclear potential to be treated by microscopic means. The other terms \( V^II_k \) are the potentials which would appear in a coupled channels description of excitation of several levels in the residual nucleus. The potentials \( V_k \) can be parametrized directly in terms of the derivatives of the optical potential (JAC70) ie.

\[ V_k(R) = -i^{k}[4\pi(2k+1)]^{-\frac{1}{2}}(E_k R) \frac{d\hat{U}_N}{dR} \quad k > 0 \quad (3.65) \]
where $\beta_{k0}$ is the transition strength, which is taken experimentally, and may be related to the rotational or vibrational model, as appropriate.

3.3.2 Excitation of the \(\alpha\)-Particle

The excitation of the \(\alpha\)-particle, which is not included in the subsequent analysis, can be easily handled by the previous formalism. Equation (3.51) becomes

$$\psi^{JM}_{O} = \sum_{Lm\mu} c_{L}(j)(Lm\mu|JM_{J})\psi_{\alpha\alpha}(\xi, \eta) x_{\alpha}(R)$$  \hspace{1cm} (3.66)

where

$$\psi^{j}_{\alpha\alpha}(\xi, \eta) = \sum_{j_{\alpha} I M} (j_{\alpha} I M|j_{\mu})\phi^{IM}_{\alpha}(\eta)\phi^{j}_{\alpha}(\xi)$$  \hspace{1cm} (3.67)

Further, the \(\alpha\)-nucleon interaction $V_{\alpha n}(Rn)$ must be replaced by the following.

$$V^{j}_{\alpha n}(\xi, \eta) = \langle \phi_{\alpha}(\xi)|V_{\alpha}(\xi, \eta, R)|j_{\alpha}\rangle$$  \hspace{1cm} (3.68)

3.4 Calculations for the \(0^{+} \rightarrow 0^{+}\) Transitions

3.4.1 Cluster Formalism

In this section the ground state to ground state transitions of the Polonium isotopes listed in Table III are calculated using the lowest order formula for the width, equation (3.56). Only the contribution to the decay width matrix element from the first term in the cluster expansion for the initial nucleus was taken into
account. This term with L-I=0 represents a real α-particle moving relative to the residual nucleus. This approximation has recently been successfully applied to nuclear structure problems in light nuclei (BU177). With these conditions, (3.60) simplifies considerably to give an expression for the width of the form.

\[ \Gamma = [C^0(0)]^2 \Gamma_{ob} \quad (3.69) \]

where the one body width is

\[ \Gamma_{ob} = 2\pi \int d\rho (E_B) |<\chi^+_A|\{H(1-K)|\overline{\chi}_A>|^2 \quad (3.70) \]

and the absence of subscripts indicates ground nuclear states.

Using equation (3.63), the well known expression for the density of states, (LE66), and the angular momentum selection rule for α-decay, equation (3.16), this equation can be reduced to one of the form

\[ \Gamma = 2\pi \int \frac{1}{(2\pi)^3} k^2 \frac{dk}{dE} d\Omega |\int_0^{\infty} u_o [V^{oo}_{o}-U] \tilde{u}_{no} dR|^2 \quad (3.71) \]

Thus \([C^0(0)]^2\) has the role of the spectroscopic factor defined by equation (3.9b) and represents the probability of finding in the initial system a component which resembles a free α-particle and the residual nucleus in its ground state. In equation (3.71) \(V^{oo}_{o}\) if given by (3.64) may be directly equated to the microscopic folding potentials analysed in Chapter 2. The potential U is related to the model Hamiltonian (3.46).
To apply the concept of making the initial relative motion wavefunction into a bound state the values of the barrier height and its position \( r_b \) were calculated for each isotope using the method given in Section 2.2. The results are given in Table IV and are in good agreement with the experimentally deduced values (G070). In calculations on the decay of \(^{212}\text{Po}\) the Wood-Saxon bound state potential due to Davies et al (DA76) was also used and is denoted DLF.

The quantum numbers \( NL \) of the bound state function alpha particle wave function were generated by the oscillator rule

\[
2(N-1)+L = \sum_{i=1}^{4} [2(n_i-1)+l_i] \quad (3.72)
\]

where \( n_i, l_i \) are the quantum numbers for the two protons and two neutrons in the least bound single particle levels in the \( A+4 \) system. When configuration mixing is taken into account higher values of \( N \) are also possible. These values, for the lowest \( s \)-state in \(^{212}\text{Po}\), are given in Table VI, where the admixture coefficients are taken from the work of Glendenning and Harada (GL65) and terms with coefficients less than \( \pm 0.1 \) have been omitted. The effect on the decay width for different values of \( N \) will be considered later.

To obtain a bound state for each isotope at the correct energy \( E^\alpha \), defined by equation (3.48), the potential may have to be varied. The original model potential defined by (3.47) was therefore modified to be

\[
U_R(R) = g[\hat{U}^N_N(R)+V_c(R)] + (1-g)U_R(r_b) \quad R \leq r_b
\]

\[
= U_R(r_b) \quad R > r_b \quad (3.73)
\]
where the parameter $g$ was allowed to vary until a bound $s$-state with the required $N$ was obtained at the correct energy $E^*$. This procedure, which is different to the standard bound state technique in which, say, the depth of the real nuclear potential is varied, leaves the barrier height and position unchanged.

The potentials obtained for $^{208}$Pb, after completion of the search procedure are plotted in Figure 9. It can be seen that the differences between the microscopic potentials has been largely eliminated but these potentials differ from the Woods-Saxon forms because of the shape of the latter in the nuclear interior. The parameter $g$ took the values 0.4-0.6 for the microscopic potentials but was 1.54 for the phenomenological potential (BL). Using the bound state potential due to Davies et al (DA76) a value for $g$ of unity was obtained. This suggests the two different techniques (See Chapter 4) are comparable. The bound state wavefunctions obtained using potentials H and D.L.F. are plotted in Figure 10.

With this modification the scattering wavefunction is now generated in the potential

$$U(R) = U_R(R) \quad R \leq r_b$$

$$= \hat{U}_N(R) + V_c(R) \quad R > r_b$$

$$U(R) = U_R(R) \quad R \leq r_b$$

$$= \hat{U}_N(R) + V_c(R) \quad R > r_b$$

which has the correct behaviour at large distances and the barrier height and position consistent with elastic scattering data.
To analyse the boundary conditions and normalisation of the scattering wavefunction consider expanding in the form,

\[ \chi^+_\alpha(R) = (2\pi)^{-3/2} \frac{dE}{dk}^{1/2} \sum_{\ell\nu} \int_{\mathbb{R}} i^\ell e^{i\sigma^\ell} u^\ell(kR) Y^\ell \nu(k) Y^\nu (\hat{R}) \]

\[ = \sum_{\ell\nu} i^\ell e^{i\sigma^\ell} \frac{u^\ell(kR)}{R} Y^\ell \nu (\hat{R}) Y^\nu (\hat{R}) \]

where \[ a = \frac{2}{\pi} \frac{dE}{dk}^{1/2} \quad \sigma^\ell = u^\ell / kR \]

and \( \sigma^\ell \) is the Coulomb phase shift.

The radial function \( u^\ell \) is, of course, a solution of a radial Schrödinger equation which obeys the following boundary conditions,

\[ u^\ell = 0 \quad \text{at} \quad R = 0 \]

\[ u^\ell = \frac{a}{2} \left[ (F^\ell +iG^\ell) + e^{i\delta^\ell}(F^\ell -iG^\ell) \right] \quad \text{at large} \ R \]

where \( F^\ell, \ G^\ell \) are regular and irregular Coulomb functions. Hence using (3.75) the wavefunctions are normalised so that (LE66)

\[ \langle \chi^+_\alpha(E) | \chi^+_\alpha(E) \rangle = \delta(E-E') \delta(k-k') \]

and the density of states is now given by \( d\Omega_k \) (LE66). Comparing (3.55) with (3.75) then \( \Lambda^\ell = i^\ell e^{i\sigma^\ell} \) and equation (3.71) for the one body width reduces to
3.4.2 Results for the Decay Width

The results for the decay widths of ground state to ground state transitions of Polonium isotopes are given in Table VII.

The results for $^{212}\text{Po}$ using the microscopic potentials $H$, $H'$ and $M$ show that, to a very large degree, the uncertainties associated with the choice of the optical potential have been eliminated. This can be seen more clearly by comparing these results with those in Table IV. The markedly different result obtained from the phenomenological potential (BL) is due to the need to lower this potential to make it contain a $12s$ bound state. This means that a very substantial difference is introduced between the phenomenological potential and the microscopic potentials in the surface region, as can be seen from Figure 9, and this leads to a substantial difference between the bound state wavefunctions. The DLF potential is also a phenomenological Woods-Saxon potential but it is chosen to give a $12s$ bound state at the required energy and consequently gives a value for the width which is closer to the values given by the microscopic potential. The results were also found to be sensitive to the barrier height (See Table V) because this leads to slight differences in the binding energy defined by Equation (3.48). This effect has been verified by fixing the barrier for the $\alpha + ^{208}\text{Pb}$ system at $20.4$ MeV, which yields the results given in Table VIII (RH76).

The value of the spectroscopic factors for all three isotopes
are at least an order of magnitude bigger, than the complicated shell model calculation of Arima (AR75) (outlined in Section 3.12). This seems to confirm the difficulties encountered in the independent particle formalism with the formation of a four particle cluster. Further these values are consistent with nuclear structure considerations, in particular the very low value for $^{210}$Po reflects the need to break the closed neutron shell.

The effect on the decay width for different principal quantum numbers (see Table VI) was investigated using the potential for $^{208}$Pb + α derived from the H parameters. The results are given in Table IX which shows that uncertainty in N of one unit leads to about the same variation in $\Gamma$ as that arising from the different choices for the parameters of the α-nucleon interaction. The full H potential supports a 15s state at the given α-particle energy.

3.5 Calculations for the Decay of $^{212m}$Po

In order to apply the alpha decay theory beyond ground state transitions the decay of the isomeric state $^{212m}$Po was investigated.

In the alpha decay of this isomeric state, which lies at 2.85 MeV above the ground state, 97% of the decays go to the ground state of $^{208}$Pb, 1% to the first 3$^-$ state at 2.62 MeV, and 2% to the 5$^-$ state at 3.20 MeV. According to the shell model calculations of Glendenning and Harada (GL65) the spin of this state is 18$^+$ and the principal configurations of the extra core nucleons are $(1h_{9/2})_p^8$
The decay parameters for this isomeric state are indicated in Table III.

Rauscher et al (RA67), using the shell-model with configuration mixing and R-matrix theory, found that the theory underestimated the relative decays to the \( 0^+ \) and \( 3^- \) and \( 5^- \) states for channel radius 9.5 fm. Once again the results were very sensitive to the value of the channel radius and the amount of configuration mixing used. Indeed a complicated coupled channels calculation was carried out to try and eliminate the discrepancy between theory and experiment but, of course, once again the results were sensitive to the value of the channel radius.

3.5.1 Extended Cluster Formalism

To calculate the decay widths to the various states in \(^{208}\text{Pb}\), equation (3.60) was used. Neglecting, once again, the excitation of the alpha particle the initial isomeric \( 18^+ \) state was approximated by expanding over the lowest \( 0^+ \), \( 3^- \) and \( 5^- \) states of the residual nucleus. This expansion leads to the transitions indicated in Figure 11.

Neglecting coupling between the final states of the residual nucleus the decay widths were calculated in three separate ways, with increasing complexity. Initially only the ground state term in the expansion were retained and the decay widths to the \( 0^+ \), \( 3^- \) and \( 5^- \) states in \(^{208}\text{Pb}\) were calculated. Indeed the \( 18^+ \to 0^+ \) transition can be calculated using a slightly modified equation (3.80) with the correctly evaluated barrier height, ie.
\[ \Gamma = 2\pi c_{18}^{18}(0)^2 |A_{18}| \int_{r_b}^{\infty} u_{18}^{18}[\tilde{u}_N(R) + V_c(R) - U_k(R)] \tilde{u}_3^{18} dR |^2 \] (3.81)

where the selection rule (3.16) and the oscillator rule (3.72) with appropriate four nucleon configuration (GL65) has been used. The normalisation of the scattering wavefunction was, of course, consistent with (3.75). These results are given in Table X and are denoted as Set A. Returning to Figure 1(b), the widths corresponding to the transitions represented by vertical arrows were next calculated using the coefficients \( B_{18}^{18}(I) = C_{18}^{18}(I)/C_{18}^{18}(0) \), obtained from perturbation theory (see Section 3.5.2). These results are given in Table X as Set B. Finally these two different calculations were combined, allowing for interference effects, to give Set C in Table X. In this way all the transitions represented by double arrows in Figure 11 have been taken into account and the quantity denoted by \( \Gamma_{ob} \) is the width calculated from (3.60) divided by \( [C_{18}^{18}(0)]^2 \).

In most of the transitions it was required to calculate \( \gamma_{kk}^{II} \) through (3.65). For these calculations the collective strength parameters for the vibrational states in lead were taken as (AL67)

\[ \beta_{3}^{R_o} = 0.81 \quad \beta_{5}^{R_o} = 0.48 \]

where the uncertainty on these parameters is 10%. These parameters were taken from fits to inelastic alpha particle scattering from \(^{208}\text{Pb}\) using a Woods-Saxon potential. Thus there could be additional uncertainties in relating \( R_o \) to the microscopic potentials of Chapter 2.

The derivative \( d\tilde{U}_N/dR \) for the H potential peaks at 6.4 fm.
3.5.2 Cluster Parentage Coefficients

To calculate the Sets B and C mentioned in the preceding section it was necessary to calculate the coefficient \( B^1_L(I) = C^1_L(I)/C^1_L(0) \).

Neglecting initially angular momentum coupling, equation (3.51) for the initial state can be written as

\[
\psi_o = C_o \left[ \phi_{\alpha' A} \chi_{\alpha A} + \sum_{n_\alpha n_A > 0} B^n_{n_\alpha n_A} \phi_{A}^{n_\alpha A} \psi_{\alpha A} \chi_{\alpha A} \right] \quad (3.82)
\]

where \( B^n_{n_\alpha n_A} = C^n_{n_\alpha n_A} \). Now using first order perturbation theory \( (G064) \), where \( V_{\alpha A} \) is defined as the perturbing interaction, the coefficients \( B^n_{n_\alpha n_A} \) are given by

\[
(E_o - E_n) B^n_{n_\alpha n_A} = \langle \phi_{A}^{n_\alpha A} \psi_{\alpha A} \chi_{\alpha A} | V_{\alpha A} | \phi_{A}^{n_\alpha A} \psi_{\alpha A} \chi_{\alpha A} \rangle \quad (3.83)
\]

On inserting angular momentum coupling and using the same technique employed to derive equation (3.60), then (3.83) reduces to

\[
(E_I - E_{I'}) B^J_L(I') = \sum_{M m M' m'} (-1)^k (-1)^q (L o k | L' o 0) (L m k - q | L' m')
\]

\[
\times (L' m' I' M' | J M_j) (L m M | J M_j) (I M k q | I' M')
\]

\[
\times \int \int \int_{L'} \int_{L'} \int_{L'} u_{N' L'} u_{k} u_{N L} dR \quad (3.84)
\]

where \( V_{k}^{II'} \) is given by (3.65).

When \( I = M = 0 \) this equation can be used with the collective
strength parameters mentioned earlier to evaluate $B^J_L(I')$ through the simplified relation

$$(E_o - E_{I'})B^J_L(I') = \int_{0}^{\infty} \sum_{\text{NL}} u_{N',L'}^* v_{L'} v_{I'}^* u_{NL} \, dR \quad (3.85)$$

In using the oscillator rule to determine $N'$ one quantum of excitation of the core was added to the right hand side of (3.72). This is consistent with the wavefunctions used by Rauscher et al (RA67) to describe these collective states by combinations of shell model configurations.

The values obtained for $B^18_L(I')$ are given in Table XI. Using the orthogonality conditions of the wavefunctions in equation (3.82) a normalisation relation of the form

$$C^2_o = 1/(1 + \sum_{n} B^2_n) \quad (3.86)$$

arises. Using the values of $B_n$ in Table XI then $C^2_o = 0.25$, a value consistent with a spectroscopic factor of 0.128 obtained with (3.81) for the $18^+ \to 0^+$ transition (see Set A, Table IX), considering the crudeness of the calculation.

3.5.3 Results for the Isomeric Decay Width

The decay widths for the various sets of calculations outlined in Section 3.5.1 are shown in Table X. These indicate that for Set A the relative width for the $3^-$ state is about correct but the relative width for the $5^-$ state is too small by a factor of $\sim 16$. This difference, however, could be corrected for by increasing $\beta_J R_o$ by a
factor of 4. For Set B the relative widths for both the $3^-$ and $5^-$ states are too small by a factor of only 2 but, as can be seen, the ratio is in excellent agreement with experiment. Thus, in this case, relatively small changes in the coefficients $B_{L}^{18}(I)$ would yield exact agreement with the data. This suggests that what is important in the decay of the isomeric state is not the collective excitation of the residual nucleus $^{208}$Pb but rather the cluster structure of the decaying state $^{212m}$Po. Further support for this comes from analysing the ratio of the $^{212m}$Po $\rightarrow$ $^{208}$Pb (g.s.) width using only the ground state cluster expansion with the width $^{212}$Po (g.s.) $\rightarrow$ $^{208}$Pb (g.s.) taken from Table VII. The value of this ratio is in very good agreement with experiment (~ a factor of 0.9) whereas, in contrast, Glendenning and Harada (GL65) using R-matrix theory and the shell model with configuration mixing give a ratio greater than experiment by a factor of 45 for channel radius of 9.5 fm and of 18 for a channel radius of 9.0 fm.

Thus it seems, despite the simplicity of the calculation for the coefficients in the cluster expansion, and the limited nature of the form of the initial state wavefunction, that the basic properties of the relative decay widths can be understood using a cluster model expansion; in contrast to the most complex R-matrix calculations with shell model wavefunctions.

This conclusion is supported in calculations of Rauscher et al (RA67); who used a similar technique, the theory gave a relative width for the $3^-$ state which was too small by a factor of 250 and a relative width for the $5^-$ state which was too small by a factor of
14. Both results were obtained with channel radius of 9.0 fm.

When the two modes of transition were put together to give Set C it was found that the interference effects changed the widths so that the relative width of the $3^-$ state is too large by a factor of 7 while the $5^-$ width is too small by a factor of 2. However, this calculation depends sensitively on the values of $B_L^{18}(I)$ which also depend, in turn on the parameters $R_o$. Finally it was found that the most important contribution to each transition comes from the term with the lowest value of $I$ due mainly to the effect of the angular momentum barrier.

3.6 Relationship Between Alpha-Particle Energies and Decay Lifetimes

An important feature of α-decay is the dependence of the decay half life on the decay energy. Of the empirical formulae previously given (MA69, and references therein) surely the most famous is that due to Geiger and Nuttall which relates the logarithm of the decay constant $\lambda$ to the logarithm of $V_\alpha$, the velocity of the emitted α-particle.

In this section a relation is derived between these two parameters by making approximations on the form of the decay width for ground-state to ground state transitions, equation (3.80). In order to obtain an analytical relation between half life and decay energy the nuclear potential beyond the barrier radius was neglected so that (3.49) becomes
Using a simplified W.K.B. approximation for the scattering wavefunction in the region of the potential barrier and assuming the bound state wavefunction can be accurately described, in this region, by a decaying exponential then (3.80) becomes simply

\[
\Gamma_{ob} \propto \int_{r_b}^{r_t} e^{\beta R} \left( \frac{2Ze^2}{R} - \frac{2Ze^2}{r_b} \right) e^{-\gamma R} dR \quad (3.88)
\]

where

\[
\gamma^2 = \frac{2\mu}{\hbar^2} |E_{\alpha\alpha}| = \frac{2\mu}{\hbar^2} [U_R(r_b) - T_\alpha]
\]

\[
\beta^2(R) = \frac{2\mu}{\hbar^2} [U_{\alpha\alpha}(R) - T_\alpha]; \quad \beta^2(r_b) = \gamma^2; \quad \beta^2(r_t) = 0 \quad (3.89)
\]

\(r_t\) is the outer turning point \(= \frac{2Ze^2}{T_\alpha}\) and contributions to \(\Gamma_{ob}\) beyond \(r_t\) have been neglected. Integrating by parts and neglecting terms containing \(1/R^n\) and \((d\beta/dr)^n\) it was readily shown that

\[
\log_e T_\alpha (A(T_\alpha) + \log B(T_\alpha)) \quad (3.90)
\]

where, with \(x = \frac{2Ze^2}{r_b}\)

\[
A(T_\alpha) = \gamma r_t = \frac{2Ze^2}{T_\alpha} \left( \frac{2\mu x}{\hbar^2} \right) \left( 1 - \frac{T_\alpha}{x} \right) \quad (3.91)
\]

\[
B(T_\alpha) = \left( \gamma r_b \right)^2 \left( 1 - \frac{r_b}{r_t} \right)^{-2} = r_b^2 \left( \frac{2\mu x}{\hbar^2} \right) \left( 1 - \frac{T_\alpha}{x} \right)^{-1} \quad (3.92)
\]

Although this dependence on \(T_\alpha\) does not resemble any of the previous empirical formulae it can be seen in Figure 12 that
equation (3.90) does represent the behaviour of the observed half lives of the even isotopes of polonium which cover fourteen orders of magnitude, despite the extreme approximations made to derive it.

This suggests equation (3.60) should yield the correct lifetime energy relationship in a wide range of circumstances.

3.7 Summary and Conclusions

In this Chapter a method was developed for calculating the alpha decay widths of heavy nuclei which did not contain any arbitrary radii or phenomenological potentials. Indeed there is only one adjustable parameter, namely the scaling factor g, which changes the depth of the bound state potential to give a bound state with the correct number of modes. It was shown that this method leads to consistent and acceptable results for the one-body widths of ground state transitions in the even polonium isotopes and for the branching ratios for the decay of the isomeric state $^{212m}$Po. In this way the ambiguities associated with the phenomenological optical potential and the uncertainties due to the choice of the channel radius in R-matrix theory were overcome.

The values derived for the spectroscopic factors were in the range 0.10–0.15 and the calculation of the cluster parentage coefficients for the isomeric state supports this magnitude. The shell model yields much smaller values for this, depending on the amount of configuration mixing used in the calculation. These
results suggest that the very complicated many body structure of
the initial state and the formation of the \( \alpha \)-particle are more ac-
curately represented by a cluster model expansion. Indeed it appears
that for these heavy nuclei the shell model, even with a substantial
amount of configuration mixing, is far from adequate and any attempt
to compensate in R-matrix theory leads to abnormally low values of
the channel radius.

All the calculations reported in this Chapter have been restricted
to the expected dominant terms in the cluster model expansion but the
formalism will allow the study of many additional effects. Because
the ambiguities in the reaction mechanism itself have been considerably
reduced it will now be possible to study the formation of four particle
clusters in nuclear matter in greater detail. Also with the current
interest in superheavy nuclei it should be possible, using an extension
of this formalism, to obtain a more realistic estimate of their \( \alpha \)-decay
lifetimes. Alpha decay in light nuclei is also of current interest
with the measurement of a parity violating alpha decay width in \( ^{16}O \)
(NE74). This decay has been analysed using extended perturbation
theory and a parity violating potential by Apayi (AP76) but he points
out a more thorough calculation needs to be carried out.

Further improvements of the results for the decay widths can
be achieved through more consistent data on the magnitude of the
Coulomb barrier and including more terms in the initial cluster
expansion. A Coulomb excitation term should probably be added to
equation (3.65) since in low energy alpha particle scattering this
gives an important contribution to the excitation (AU70). In the
analysis of $^{212}_{\text{Po}}$ the inclusion of the low lying $2^+$ and $4^+$ states would be an obvious improvement. In equation (3.51) the identical nature of nucleons and their effect on the alpha decay rates, through the application of an antisymmetrisation operator has been neglected. However, in Chapter 5, the effects of such an operator is considered.
### TABLE III

Decay Parameters for the Polonium Isotopes

<table>
<thead>
<tr>
<th>Decay</th>
<th>$T_\frac{1}{2}$ (sec)</th>
<th>$\Gamma_{\text{exp}}$ (MeV)</th>
<th>$\Gamma_\alpha$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{210}\text{Po} \rightarrow ^{206}\text{Pb}$ (g.s.)</td>
<td>$1.73 \times 10^{-7}$</td>
<td>$3.82 \times 10^{-29}$</td>
<td>$5.30$</td>
</tr>
<tr>
<td>$^{212}\text{Po} \rightarrow ^{208}\text{Pb}$ (g.s.)</td>
<td>$3.04 \times 10^{-7}$</td>
<td>$1.52 \times 10^{-15}$</td>
<td>$8.80$</td>
</tr>
<tr>
<td>$^{214}\text{Po} \rightarrow ^{210}\text{Pb}$ (g.s.)</td>
<td>$1.60 \times 10^{-4}$</td>
<td>$2.85 \times 10^{-18}$</td>
<td>$7.70$</td>
</tr>
<tr>
<td>$^{216}\text{Po} \rightarrow ^{212}\text{Pb}$ (g.s.)</td>
<td>$1.58 \times 10^{-1}$</td>
<td>$2.89 \times 10^{-21}$</td>
<td>$6.81$</td>
</tr>
<tr>
<td>$^{208}\text{Pb}$ (g.s.)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{212}\text{Po} \rightarrow ^{208}\text{Pb}$ (3⁻)</td>
<td>$4.50 \times 10^{-1}$</td>
<td>$1.01 \times 10^{-25}$</td>
<td>$9.03$</td>
</tr>
<tr>
<td>$^{208}\text{Pb}$ (5⁻)</td>
<td></td>
<td>$2.02 \times 10^{-25}$</td>
<td>$8.45$</td>
</tr>
</tbody>
</table>

### TABLE IV

Predictions for α-decay widths of $^{210,212}\text{Po}$ (in MeV)

<table>
<thead>
<tr>
<th></th>
<th>$^{210}\text{Pb}$</th>
<th>$^{212}\text{Pb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Gamma$</td>
<td>$\Gamma_{\text{exp}}/\Gamma_{\text{calc}}$</td>
</tr>
<tr>
<td>Kadmenskii &amp; Kalechits</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$1.14 \times 10^{-29}$</td>
<td>$2.68$</td>
</tr>
<tr>
<td>B</td>
<td>$3.5 \times 10^{-32}$</td>
<td>$1.1 \times 10^3$</td>
</tr>
<tr>
<td>C</td>
<td>$6.5 \times 10^{-27}$</td>
<td>$5.8 \times 10^{-2}$</td>
</tr>
<tr>
<td>D</td>
<td>$2.3 \times 10^{-26}$</td>
<td>$1.6 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$3.8 \times 10^{-30}$</td>
<td>$1.0 \times 10^1$</td>
</tr>
</tbody>
</table>
**TABLE V**

Calculated Barrier Heights and Positions

<table>
<thead>
<tr>
<th></th>
<th>( r_b ) (fm)</th>
<th>( V(r_b) ) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ^{206}_{\text{Pb}} )</td>
<td>H 10.8</td>
<td>20.61</td>
</tr>
<tr>
<td></td>
<td>H' 11.0</td>
<td>20.34</td>
</tr>
<tr>
<td>( ^{208}_{\text{Pb}} )</td>
<td>H 10.8</td>
<td>20.60</td>
</tr>
<tr>
<td></td>
<td>H' 11.0</td>
<td>20.33</td>
</tr>
<tr>
<td></td>
<td>M 11.0</td>
<td>20.56</td>
</tr>
<tr>
<td></td>
<td>BL 10.9</td>
<td>20.49</td>
</tr>
<tr>
<td>( ^{210}_{\text{Pb}} )</td>
<td>H 10.8</td>
<td>20.54</td>
</tr>
<tr>
<td></td>
<td>H' 11.0</td>
<td>20.27</td>
</tr>
<tr>
<td>( ^{212}_{\text{Pb}} )</td>
<td>H 10.9</td>
<td>20.48</td>
</tr>
<tr>
<td></td>
<td>H' 11.0</td>
<td>20.21</td>
</tr>
</tbody>
</table>
### TABLE VI

Four-Nucleon Configurations in $^{212}$Po

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Coefficient</th>
<th>$2(N-1)+L$</th>
<th>Lowest s-State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(J_p=0, J_n=0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1h_9/2)^2 (2g_9/2)^2$</td>
<td>0.6682</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>$(1h_9/2)^2 (1i_{11/2})^2$</td>
<td>0.3900</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>$(1h_9/2)^2 (1j_{15/2})^2$</td>
<td>-0.2386</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>$(1h_9/2)^2 (3d_5/2)^2$</td>
<td>0.1067</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>$(2f_7/2)^2 (2g_9/2)^2$</td>
<td>0.2305</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>$(2f_7/2)^2 (1i_{11/2})^2$</td>
<td>0.1064</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>$(1i_{13/2})^2 (1i_{11/2})^2$</td>
<td>-0.1665</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>$(1i_{13/2})^2 (2g_9/2)^2$</td>
<td>-0.2030</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>$(1i_{13/2})^2 (1j_{15/2})^2$</td>
<td>0.1026</td>
<td>26</td>
<td>14</td>
</tr>
<tr>
<td>$(J_p=2, J_n=2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1h_9/2)^2 (2g_9/2)^2$</td>
<td>0.2385</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>Isotope</td>
<td>Potential</td>
<td>$\Gamma_{ob}$ (MeV)</td>
<td>$S_\alpha = \Gamma_{exp}/\Gamma_{ob}$</td>
</tr>
<tr>
<td>---------</td>
<td>-----------</td>
<td>---------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>210\text{Po}</td>
<td>H</td>
<td>$1.30 \times 10^{-27}$</td>
<td>0.029</td>
</tr>
<tr>
<td>210\text{Po}</td>
<td>H'</td>
<td>$1.64 \times 10^{-27}$</td>
<td>0.023</td>
</tr>
<tr>
<td>210\text{Po}</td>
<td>M</td>
<td>$1.02 \times 10^{-14}$</td>
<td>0.149</td>
</tr>
<tr>
<td>210\text{Po}</td>
<td>BL</td>
<td>$1.12 \times 10^{-14}$</td>
<td>0.136</td>
</tr>
<tr>
<td>210\text{Po}</td>
<td>D.L.F.</td>
<td>$1.46 \times 10^{-14}$</td>
<td>0.104</td>
</tr>
<tr>
<td>212\text{Po}</td>
<td>H</td>
<td>$1.45 \times 10^{-13}$</td>
<td>0.010</td>
</tr>
<tr>
<td>214\text{Po}</td>
<td>H</td>
<td>$3.84 \times 10^{-14}$</td>
<td>0.040</td>
</tr>
<tr>
<td>214\text{Po}</td>
<td>H'</td>
<td>$5.40 \times 10^{-17}$</td>
<td>0.053</td>
</tr>
<tr>
<td>214\text{Po}</td>
<td>H'</td>
<td>$6.80 \times 10^{-17}$</td>
<td>0.042</td>
</tr>
</tbody>
</table>
TABLE VIII

Decay Widths for the L=0 Transition in $^{212}$Po
Calculated with a Barrier of 20.4 MeV

<table>
<thead>
<tr>
<th>Potential</th>
<th>$\Gamma_{ob}$ (MeV)</th>
<th>$S_\alpha=T_{exp}/\Gamma_{ob}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>$1.27 \times 10^{-14}$</td>
<td>0.120</td>
</tr>
<tr>
<td>H'</td>
<td>$1.37 \times 10^{-14}$</td>
<td>0.111</td>
</tr>
<tr>
<td>M</td>
<td>$1.32 \times 10^{-14}$</td>
<td>0.115</td>
</tr>
</tbody>
</table>

TABLE IX

One-Body Width for Various Relative s-States in $^{212}$Po,
Calculated for Fixed Binding Energy $\varepsilon_\alpha$ in the H Potential

<table>
<thead>
<tr>
<th>Principal Quantum Number N</th>
<th>Scaling Factor $g$</th>
<th>$\Gamma_{ob}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.67</td>
<td>$1.02 \times 10^{-14}$</td>
</tr>
<tr>
<td>13</td>
<td>0.77</td>
<td>$1.38 \times 10^{-14}$</td>
</tr>
<tr>
<td>14</td>
<td>0.87</td>
<td>$1.87 \times 10^{-14}$</td>
</tr>
<tr>
<td>15</td>
<td>0.98</td>
<td>$2.48 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
TABLE X

Widths for Decay of $^{212}$Po to States of $^{208}$Pb (in MeV)

<table>
<thead>
<tr>
<th></th>
<th>Experiment</th>
<th>Theory $T_{ob}$</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Set A</td>
<td>Set B</td>
<td>Set C</td>
<td></td>
</tr>
<tr>
<td>$^{212}$m$\text{Po} \rightarrow ^{208}\text{Pb}(0^+)$</td>
<td>$9.84 \times 10^{-24}$</td>
<td>$7.67 \times 10^{-23}$</td>
<td>$7.67 \times 10^{-23}$</td>
<td>$7.67 \times 10^{-23}$</td>
<td></td>
</tr>
<tr>
<td>$^{212}$m$\text{Po} \rightarrow ^{208}\text{Pb}(3^-)$</td>
<td>$1.01 \times 10^{-25}$</td>
<td>$1.00 \times 10^{-24}$</td>
<td>$4.11 \times 10^{-25}$</td>
<td>$5.49 \times 10^{-24}$</td>
<td></td>
</tr>
<tr>
<td>$^{212}$m$\text{Po} \rightarrow ^{208}\text{Pb}(5^-)$</td>
<td>$2.02 \times 10^{-25}$</td>
<td>$9.81 \times 10^{-26}$</td>
<td>$7.32 \times 10^{-25}$</td>
<td>$7.02 \times 10^{-25}$</td>
<td></td>
</tr>
<tr>
<td>Ratio $0^+/3^-$</td>
<td>97.0</td>
<td>76.7</td>
<td>186.6</td>
<td>14.97</td>
<td></td>
</tr>
<tr>
<td>Ratio $0^+/5^-$</td>
<td>48.5</td>
<td>781.8</td>
<td>104.8</td>
<td>109.3</td>
<td></td>
</tr>
<tr>
<td>Ratio $3^-/5^-$</td>
<td>0.5</td>
<td>10.2</td>
<td>0.56</td>
<td>7.8</td>
<td></td>
</tr>
</tbody>
</table>

TABLE XI

Values of the Leading B Coefficients

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{15}^{18}$ (3^-)</td>
<td>0.97</td>
</tr>
<tr>
<td>$B_{17}^{18}$ (3^-)</td>
<td>-1.14</td>
</tr>
<tr>
<td>$B_{19}^{18}$ (3^-)</td>
<td>0.54</td>
</tr>
<tr>
<td>$B_{13}^{18}$ (5^-)</td>
<td>0.22</td>
</tr>
<tr>
<td>$B_{15}^{18}$ (5^-)</td>
<td>-0.31</td>
</tr>
<tr>
<td>$B_{17}^{18}$ (5^-)</td>
<td>0.43</td>
</tr>
<tr>
<td>$B_{19}^{18}$ (5^-)</td>
<td>-0.20</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS (Chapter 3)

Figure 7  Variation of the penetrability, defined by equation (3.8), as a function of the channel radius $R_c$. Crosses mark the value obtained using the W.K.B. approximation.

Figure 8  Sketch of the behaviour of the bound state potential $U_R$ for $L = 0$.

Figure 9  The bound state potentials $U_R$ and for $^{212}_{\text{Po}}$.

Figure 10  Comparison of the bound state wavefunctions using potentials H and D.L.F.

Figure 11  Diagram to illustrate the transitions contributing to the decay of $^{212m}_{\text{Po}}$.

Figure 12  Plot of $\log_{10} T_1$ for the experimental half-lives of even polonium isotopes against $A(Ta)$ (dashed line) and against $A(Ta) + \log_e B(Ta)$ (full line). A value of $r_b = 10.8$ fm predicted by potential H has been used in the calculation.
FIGURE 7

Penetrability
(R-Matrix)

\[ 10^{-10} \]

\[ 10^{-11} \]

\[ 10^{-12} \]

\[ 10^{-13} \]

\[ 10^{-14} \]

\[ 10^{-15} \]

\[ 10^{-16} \]

\[ 10^{-17} \]

\[ 10^{-18} \]

\[ 10^{-19} \]

\[ 10^{-20} \]

\[ R_C (\text{fm}) \]

212\text{Po}

\times \text{W.K.B.}

214\text{Po}

\times \text{W.K.B.}

216\text{Po}

\times \text{W.K.B.}
\[ U_R(R) \text{(MeV)} \]

\[ R \text{(fm)} \]

\[ r_b \]

FIGURE 9
FIGURE 10
\[ \{ \sigma_{d}, \alpha \} \text{^{(5)}} \] + \[ \Sigma \text{^{(3)}} \text{^{(1)}} \] + \[ \Sigma \text{^{(18)}} \text{^{(5)}} \alpha + \text{^{(208)}} \text{^{(3)}} \alpha \] + \[ \text{^{(208)}} \text{^{(5)}} \alpha \] + \[ \text{^{(208)}} \text{^{(0)}} \text{^{(3)}} \alpha \] + \[ \text{^{(208)}} \text{^{(5)}} \alpha \]

**Initial state of ** \( \text{^{212m}Po} \)

**Final state of ** \( \text{^{208}Pb} \)

**FIGURE 11**
FIGURE 12
4.1 Introduction

The alpha transfer reaction is an important spectroscopic tool for the study of α-clustering and certain aspects of the collective behaviour of nuclei (MI77).

In the recent work of Davies et al (DA76) a comparison was made between the spectroscopic factors deduced from the alpha transfer reaction and alpha decay in the lead region. In this analysis the α-decay rates were calculated using R-matrix theory and an analysis of the \((0^+_1,^{12}C)\) reaction at 93 MeV on lead isotopes and bismuth was carried out using a full finite range Distorted Wave Born Approximation code (DWBA). The close agreement for the spectroscopic factors obtained in this work was interpreted as evidence that an α-particle in its ground state was transferred in a one-step process. These conclusions support the use of the cluster model representation for the Polonium isotopes given in Chapter 3. Further evidence for the close correspondence between alpha transfer and alpha decay is given in the work of Milder et al (MI77) who used the \((d, ^6Li)\) alpha pickup reaction on rare-earth nuclei at 35 MeV. Here good agreement between measured and calculated angular distributions was obtained using a one-step, zero-range DWBA theory with alpha cluster wave functions. Once again R-matrix theory was used for α-decay calculations.

The α-decay calculations used in these papers was not based on
a shell model description of the parent nucleus, outlined in the last Chapter. A ground state cluster model was used and the radial function \( \psi_G(r) \) for the parent system (target plus alpha) was calculated at the resonance energy. For this system the R-matrix reduced alpha-particle width is given by (DA76)

\[
\gamma_\alpha^2 = \frac{n^2 a}{2m} S_\alpha |\psi_G(a)|^2
\]

(4.1)

where \( m \) is the reduced mass and \( S_\alpha \) is related to the spectroscopic factor. In practice \( \psi_G(r) \) is treated as a 'quasi-bound' state and is normalised to unity in an analogous way to equation (3.53).

However because \( \psi_G(r) \) is a resonant state, which means scattering state boundary conditions, uncertainties are introduced because of the oscillatory nature of \( \psi_G(r) \) for large \( r \) and the finite numerical value of the integration limit. This wavefunction \( \psi_G(r) \) was used, as a bound state, in the DWBA calculations.

In this Chapter a formalism is developed to calculate the \( \alpha \)-transfer cross section for the reaction \( {}^{208}_{\text{Pb}} ({}^{16}_{\text{O}}, {}^{12}_{\text{C}}) {}^{212}_{\text{Po}} \) at 93 MeV. The reaction was treated as a one step process in which a real alpha particle in its ground state was transferred. This formalism was based on the technique of introducing a model Hamiltonian which makes the \( \alpha \)-particle resonant state into a bound state and so avoids the problems of normalisation outlined above. Because the relative wavefunction is modified at large distances and, since this region is likely to be important in heavy ion transfer reactions, an expression for the DWBA matrix element was used which takes account
of the difference between the model wavefunction and the true wavefunction. To calculate the transition matrix a zero range code was developed, extending the alpha decay work, and incorporating the Coulomb function subroutine especially developed by Barnet et al (BA74) for heavy ion reactions.

The spectroscopic factors deduced from this analysis are compared with those of Chapter 3, deduced from alpha decay, and the calculations of Davies et al (DA76).

4.2 Expression of the Matrix Element

The transfer reaction can be represented as

\[ a + A + b + B \]  \tag{4.2}

where

\[ a = b + \alpha \quad B = A + \alpha \]  \tag{4.3}

and, in this case, B is the alpha decaying nucleas \(^{212}\)Po. To be consistent with the analysis of Chapter 3 the systems a and B were treated in a cluster model representation. The exact Hamiltonians for the initial and final states are given by

\[ H_i = H_\alpha + H_{ab} + T_{ab} + V + V_{Coul} + V_{Coul} + V_{Coul} + V_{Coul} + V_{Coul} + V_{Coul} \]  \tag{4.4}

\[ H_f = H_\alpha + H_{A} + T_{A} + V + V_{Coul} + V_{Coul} + V_{Coul} + V_{Coul} + V_{Coul} \]  \tag{4.5}
where $H_k$ is the internal Hamiltonian for the cluster $k$, $V_{jk}$ is the total nuclear interaction between the clusters $j$ and $k$, and $V_{jk}^{\text{COUL}}$ is the Coulomb interaction between the same pair. Denoting the optical potential for scattering of cluster $j$ from cluster $k$ as $U_{jk}$ and the total potential appropriate to the bound or resonant state as $\bar{U}_{jk}$, then the model Hamiltonians for the initial and final states are given as

\[ K_i = H + H_{\alpha b} + T_{\alpha b} + \bar{U}_{\alpha b} + H_{A} + T_{A} + U_{A} + U_{\alpha A} + U_{\alpha A} \tag{4.6} \]

\[ K_f = H + H_{\alpha A} + T_{\alpha A} + \bar{U}_{\alpha A} + H_{b} + T_{b} + U_{b} + U_{b B} + U_{b B} \tag{4.7} \]

The perturbation for the prior form of the matrix element is given by

\[ \Delta V_i = H_i - K_i \]

\[ = (V_{ab} + V_{ab}^{\text{COUL}} - \bar{U}_{ab}) + (V_{aA} - U_{aA}) + (V_{A} - U_{A}^{\text{COUL}}) \tag{4.8} \]

\[ = (V_{ab} + V_{ab}^{\text{COUL}} - \bar{U}_{ab}) + V_{aA} + (V_{aA} - U_{aA}) + (V_{aA} - U_{aA}^{\text{COUL}}) \tag{4.9} \]

and similarly the perturbation for the post form of the matrix element is given by

\[ \Delta V_f = H_f - K_f \]

\[ = (V_{aA} + V_{aA}^{\text{COUL}} - \bar{U}_{aA}) + (V_{bB} - U_{bB}) + (V_{bB}^{\text{COUL}} - U_{bB}) \tag{4.10} \]
Thus using the post form of the matrix element a term arises \( \Delta V_1 \) which can be equated with the perturbation which brings about \( \alpha \)-decay, as in the analysis of Chapter 3 (Equation 3.49). This term now occurs because of the difference between the model description of the final state of nucleus B and the true representation for this state. Both \( \Delta V_1 \) and \( \Delta V_3 \) can cause core excitation, the former through the interaction of the \( \alpha \)-particle with nucleus A and the latter through the interaction of particle b with the core A. \( \Delta V_2 \) is the normal dominant term of the post DWBA perturbation. (Can be equated to \( V_{np} \) for p,d reactions.)

The nuclear wavefunctions for the systems B and a were treated in a cluster model representation. In this way system B \( (\Phi^212 \) in this case) can be treated using the cluster model expansion as before, i.e. equation (3.51), but now 'dashes' are introduced to indicate a final state ie.

\[
\psi^J_M_B = \sum_{L M_{I'M'}} C^J_L(I')(LmI'M'|JM_J)\phi_\alpha^I'M'(\xi)\phi_A^I'M'(\eta)\chi^L_{\alpha A}(R) \tag{4.13}
\]

In a similar manner the wavefunction of the incident composite particle can be written as

\[
\phi_a^{S_{a\mu a}} = \sum_{S_{b\mu b}} C_{S a}(S_b) \langle S_{a\mu b} | S_{a\mu a} \rangle \phi_\alpha^S_{ab}(\xi) \phi_b^{S_{b\mu b}}(\lambda) \chi_{ab}^{S_{b\mu b}}(\xi) \tag{4.14}
\]
where $\lambda$ represents the internal coordinates of b and $\mathbf{r}$ is a relative coordinate.

In both (4.13) and (4.14) the antisymmetrication operator has been omitted and it is assumed that the $\alpha$-particle remains in its ground state (omission of superscripts again implies a ground state term).

The transition matrix for a rearrangement collision is given by, in the post form (RO67)

$$T_{fi} = \langle \psi_f^+ \mid \Delta V_f \mid \psi_i^+ \rangle$$ (4.15)

where $\psi_i^+$ is the full initial state vector of the system. The bra of the above matrix element is calculated using the model Hamiltonian $K_f$. It can be seen from equation (4.10) that $K_f$ does not contribute directly to the rearrangement process but contributes only to the formation of the final bound state and elastic scattering of b from B. It is the difference between the full Hamiltonian and the model Hamiltonian which causes the rearrangement process.

Approximating $\psi_i^+$ by distorted waves in the initial channel (DWBA) equation (4.15) becomes, by neglecting spin coupling at present;

$$T_{fi} = \sum \sum C_{S \sigma}^{\alpha \beta} C_{\sigma n}^{\alpha \beta} \langle \phi\beta \phi\alpha A | nA^{-n} \phi\beta \phi\beta B \chi\beta B | \Delta V_f | \phi\beta \phi\alpha A | -S \phi\beta \phi\beta A \chi\beta A \rangle$$

$$= \sum \sum C_{S \sigma}^{\alpha \beta} C_{\sigma n}^{\alpha \beta} \langle \chi\alpha A \beta B | \chi\beta B | \Delta V_f | \chi\alpha A \beta B | \chi\beta A \rangle$$ (4.16)
where $\chi^-$ and $\chi^+$ are the distorted waves for the outgoing and incoming channel respectively and

$$W_f = \langle \phi_a \phi_b \phi_A | \Delta V_f | \phi_a \phi_b \phi_A \rangle$$  \hspace{1cm} (4.17)

In (4.16), (4.17) and in subsequent analysis it will be assumed that $\phi_b$ remains in its ground state.

In figure 13 the definitions of the coordinates used in this analysis are given. They are related by the equations

$$r_f = yR - r \hspace{1cm} r_i = R - xR$$  \hspace{1cm} (4.18)

where

$$y = \frac{M_A}{M_b} = \frac{A}{A+4} \hspace{1cm} x = \frac{M_b}{M_a} = \frac{b}{b+4}$$  \hspace{1cm} (4.19)

Because the model bound state for the $\alpha + ^{208}Pb$ system depends on the coordinate $R$, and this wavefunction has already been calculated for $\alpha$-decay, the transition matrix element was evaluated by integrating over the coordinates $R$ and $r$ ie.

$$T_{fi} = J \sum c_{x}^{ab} c_{n}^{aA} \int \int$$

$$\chi^*_{bb}(k_{f},yR-r)\chi^{n*}_{aa}(k_{i},R-xR)drdR$$

where the Jacobian $J = \frac{3(r_{i}r_{f})}{3(RR)}$ is unity for this transformation.

4.2.1 The Form of $W_f$

To examine the potential terms $W_f(RR)$ appearing in (4.20) consider expanding $W_f$ into three parts.
\[ W_f = W_1 + W_2 + W_3 \]  
(4.21)

where the individual components correspond to the three parts of the interaction i.e. in terms of coordinates defined earlier

\[ \Delta V_1 = V_{\alpha A}(\xi, \eta_R) + V_{\alpha A}^{\text{Coul}}(R) - \bar{U}_{\alpha A}(R) \]  
(4.22)

\[ \Delta V_2 = V_{ab}(\xi \lambda \tau) \]  
(4.23)

\[ \Delta V_3 = V_{bA}(R-\tau, \eta \lambda) - U_{bb}(r_f) \sim V_{bA}(\xi_f, \lambda \eta) - U_{bb}(r_f) \]  
(4.24)

Now using the same technique that was used in section (3.31) and neglecting the possibility of excitation of particle b, the Wigner-Eckart theorem (BR68) gives

\[ W_1 = \sum_{kq} (4\pi)^{1/2} (IMkq|I'M') \]

\[ \left[ i \frac{kq}{r} (\hat{r}) \right] \left[ V_{\alpha A}^{\text{Coul}}(R) + \delta_{k \alpha} V_{\alpha A}^{\text{Coul}} - \delta_{k \beta} \bar{U}_{\alpha A}(R) \right] \]  
(4.25)

\[ W_2 = \sum_{n\nu} (4\pi)^{1/2} (S_{b\nu}^m b, n\nu|S_{b\nu}^m b) \left[ i \frac{n\nu}{r} (\hat{r}) \right] \left[ V_{bA}^{\text{Coul}}(r_f) - \delta_{k \nu} U_{bb}(r_f) \right] \]  
(4.26)

\[ W_3 = \sum_{kq} (4\pi)^{1/2} (IMkq|I'M') \]

\[ \left[ i \frac{kq}{r} (\hat{r}) \right] \left[ V_{bA}^{\text{Coul}}(r_f) - \delta_{k \nu} U_{bb}(r_f) \right] \]  
(4.27)

where the term \( W_4 \) corresponding to \( \Delta V_4 \) has been neglected for the moment. When \( S_b = 0 \), \( W_2 \) reduces to
where $U^{(r)}_{ab}(r)$ is the bound state potential for the initial composite system. Neglecting the problem of Coulomb excitation ($n=0, k=0$) the interaction term $\Delta V_4$ is given by equation (4.11) as

$$\Delta V_4 = U^{\text{COUL}}_{ba} + U^{\text{COUL}}_{ab} - U^{\text{COUL}}_{bb}$$ (4.29)

where

$$U^{\text{COUL}}_{ba} = Z_b Z_a e^2 f(|R-f|, R_{ba})$$

$$U^{\text{COUL}}_{ab} = Z_a Z_b e^2 f(r, R_{ba})$$

$$U^{\text{COUL}}_{bb} = Z_b (Z_a + Z_b) e^2 f(r, R_{bb})$$

The function $f(x, R_c)$ denotes the usual radial dependence of the Coulomb interaction, given by equation (2.2). Thus on neglecting the difference between the parameters $R_{ba}$ and $R_{bb}$ $\Delta V_4$ becomes

$$\Delta V_4 = \delta_{n0} \delta_{l0} Z_b Z_a e^2 f(r, R_{ba}) - \delta_{k0} Z_b Z_a e^2 f(r, R_{bb})$$ (4.33)

where these terms add on to $W_2$ and $W_3$ respectively.

Further, equating the potential $\bar{U}_{\alpha A}$ with the model potential defined in Chapter 3 (equation 3.47) then for $k=0$ $W_1$ becomes

$$\frac{\nabla II^0}{\bar{U}_{\alpha A}} - U^{\text{COUL}}_{\alpha A} - \bar{U}_{\alpha A} = 0 \quad \text{for } R \leq r_b$$

$$= \hat{U}_{\alpha A}(R) + V^{\text{COUL}}_{\alpha A}(R) - \bar{U}_{\alpha A}(R_b) \quad \text{for } R \geq r_b$$ (4.34)
where $\hat{U}^{\alpha A}_{\alpha A}(R)$ is the microscopic folded $\alpha$-nuclear potential given by equation (3.64) and calculated in Chapter 2. This is the same as the perturbing interaction used in the alpha-decay calculations of section (3.4).

Equations (4.25) and (4.27) show that even for a $0^+ \rightarrow 0^+$ transition there are contributions from terms with $L = 1 \neq 0$. These terms will enter in the potential terms $W_1$ and $W_3$ through $k \neq 0$ values and will add coherently to those with $k = 0$.

### 4.2.2 Reduction of T-matrix

Even after assuming the transition matrix may be approximated by the DWBA expression, equation (4.16), there still arises a problem in the evaluation of the six dimensional integral because of the relation between the coordinates involved.

Consider writing the distorted waves in the form

$$\chi^{+}_{\alpha A}(k_i, R-x) = e^{-ixR} \chi^{+}_{\alpha A}(k_i, R) = e^{-ixR} \chi^{+}_{\alpha A}(k_i, R)$$  \hspace{1cm} (4.35)

and

$$\chi^{-}_{\beta B}(k_f, y) = e^{-iyR} \chi^{-}_{\beta B}(k_f, R) = e^{-iyR} \chi^{-}_{\beta B}(k_f, R)$$  \hspace{1cm} (4.36)

where $\hat{k}_i$ and $\hat{k}_f$ are operators that operate either on $\chi^{+}_{\alpha A}(k_i, R)$ or $\chi^{-}_{\beta B}(k_f, R)$, as the case may be. These exact equations may be reduced by replacing $\hat{k}_i$ and $\hat{k}_f$ by the appropriate local momenta, ie.

$$\chi^{+}_{\alpha A}(k_i, R-x) = e^{-ixq_i} \chi^{+}_{\alpha A}(k_i, R)$$  \hspace{1cm} (4.37)
These expressions for the momenta are valid if $\mathbf{r} \cdot \nabla [U \mp U^\text{COUL}]$ and $\mathbf{x} \cdot \nabla [U \mp U^\text{COUL}]$ as well as higher order terms in the exponential expansion can be neglected (BR74). In this approximation, because $q_i$ and $q_f$ are functions of $R$, the integrand in (4.16) does not reduce to two independent functions of $R$ and $r$.

The six dimensional integral may be separated into two independent three dimensional integrations if the local momenta $q_i$ and $q_f$ are approximated by the asymptotic momenta $k_i$ and $k_f$. In this case the product of the distorted waves becomes

$$\chi_{bB}^{-\ast}(k_f, yR) = e^{iQ \cdot \mathbf{x}} \chi_{aA}^{\ast}(k_i, yR)$$

(4.38)

where

$$q_i^2(R) = \frac{2\mu_{aA}}{\hbar^2} [E_i - \text{Re} U_{aA}(R) - U_{aA}^\text{COUL}(R)]$$

(4.39)

$$q_f^2(R) = \frac{2\mu_{bB}}{\hbar^2} [E_f - \text{Re} U_{bB}(R) - U_{bB}^\text{COUL}(R)]$$

(4.40)

and

$$Q_E = q_f(R) - xq_i(R).$$

These expressions for the momenta are valid if $\mathbf{r} \cdot \nabla [U \mp U^\text{COUL}]$ and $\mathbf{x} \cdot \nabla [U \mp U^\text{COUL}]$ as well as higher order terms in the exponential expansion can be neglected (BR74). In this approximation, because $q_i$ and $q_f$ are functions of $R$, the integrand in (4.16) does not reduce to two independent functions of $R$ and $r$.

The six dimensional integral may be separated into two independent three dimensional integrations if the local momenta $q_i$ and $q_f$ are approximated by the asymptotic momenta $k_i$ and $k_f$. In this case the product of the distorted waves becomes

$$\chi_{bB}^{-\ast} \chi_{aA}^{\ast} = e^{iQ \cdot \mathbf{x}} \chi_{bB}^{-\ast}(k_f, yR) \chi_{aA}^{\ast}(k_i, yR)$$

(4.41)

where

$$Q = k_f - xk_i$$

(4.42)

In these alpha transfer calculations the approximation corresponding to equation (4.41) was used together with an effective zero range calculation in which the dependence of the distorted waves on the coordinate $r$ was neglected. This is equivalent to the condition
\[ V_{ab}(r) = D_0 \delta(r) \]  

(4.43)

In both of these approximations equation (4.20) factorizes into two independent integrations and the vector diagram in Figure 13 greatly simplifies.

In the DWBA analysis of single nucleon transfer on light nuclei finite range improvements to (4.43) are well known (BU64). However, as will be shown later, there are more basic problems for this heavy ion reaction. These arise because of inconsistancies in the distorted wave optical potentials.

4.3 Partial Wave Analysis

4.3.1 Core Excitation

The unpolarized cross section for a transition from state I in nucleus A to state J in nucleus B is given by (RO67)

\[ \frac{d\sigma}{d\Omega}(I+J) = \frac{\mu_{AB}^*}{(2\pi\hbar)^2} \frac{k_f}{k_i} \frac{N}{(2I+1)(2S_{\alpha}+1)} \sum_{M_f M_j} |T_{fi}|^2 \]  

(4.44)

where \( \mu_{AA} \) and \( \mu_{BB} \) are the reduced masses in the appropriate channels and the summation variables are related to equations (4.13) and (4.14) \( N \) is an antisymmetrization factor (see Section 4.6.4).

To evaluate the transition matrix \( T_{fi} \) the relative motion wavefunction appearing in (4.13) and the distorted waves in the initial and final channels are expanded in the form;
\[ x_{\lambda m}(R) = u_{NL}(R)Y_{\lambda}^{m*}(\hat{R})/R \]  

\[ H_{\lambda}^{+}(k_{1}R) = \sum_{\lambda} i^{2} [4\pi(2\lambda+1)]^{\frac{3}{2}} e^{i\sigma_{\lambda}} u_{\lambda}(k_{1}R)Y_{\lambda}^{\sigma}(\hat{R})/(k_{1}R) \]  

\[ \chi_{\lambda}^{-\lambda}(k_{f}R) = 4\pi \sum_{\lambda'} i^{-\lambda'} e^{i\sigma_{\lambda'}} u_{\lambda'}(k_{f}yR) \]  

\[ Y_{\lambda}^{m'}(\hat{R})Y_{\lambda}^{m'}(\hat{k}_{f})/(k_{f}yR) \]  

where it is being assumed that the distorted waves can be written in the form of equations (4.37) and (4.38). Hence on using the general relation (BR68)

\[ Y_{\lambda_{1}}^{m_{1}}(\Omega)Y_{\lambda_{2}}^{m_{2}}(\Omega) = \sum_{LM} \frac{\hat{\lambda}_{1} \hat{\lambda}_{2}}{(4\pi)^{\frac{1}{2}}} \]  

\[ (\hat{\lambda}_{1} \hat{\lambda}_{2} \Omega \mid L_{0})(\hat{\lambda}_{1} \hat{\lambda}_{2} \Omega \mid LM)Y_{L}^{M}(\Omega) \]  

and the forms for \( W_{f} \) given in section (4.2.1); the equation (4.16) or the T-matrix takes the form

\[ T_{fi} = \sum_{LmL'm'_{s}S_{b_{s}}} C_{I}^{J}(I')C_{s}^{S_{a}}(S_{b}) (LmI'M'_{s}M_{s}) (S_{s}S_{s}m_{s}m_{s}) \]  

\[ \times (IMkq|I'M') \{ g_{S_{s}}^{m_{s}}(Q)F_{Lm}^{II'k}(k_{i}k_{f}) \]  

\[ + h_{S_{s}}^{m_{s}}(Q)L_{m}(k_{i}k_{f}) \delta_{II'} \delta_{k_{o}} + J_{S_{s}Lm}^{II'k}(k_{i}k_{f}) \} \]  

where
\[ F_{Lm}^{II'}(k_{\perp}, k_{\perp}) = \sum_{\ell, I, p} 4\pi \ell' \ell - \ell' - k e^{i(\ell \alpha, \ell' \alpha')} \frac{\ell' \ell' \ell' \ell'}{p^2} (-1)^{k+m+q} \]
\[ \times Y_{\ell}^{m-q}(k_{\perp}) (\ell \alpha \ell' m+q | p m+q) (\ell \alpha \ell' o | po) (Lm_{kq}| p m+q) \]
\[ \times (Lm_{ko}|po) \int (k_{i} k_{f} y R) u_{\ell}^{f}(k_{f} y R) u_{\ell}^{p}(k_{i} R) u_{NL}(R) \]
\[ \times [V_{\alpha \alpha}(R) + \delta_{ko} V_{Coul}(R) - \delta_{ko} \bar{U}_{\alpha \alpha}(R)] dR \quad (4.50) \]

\[ g_{Sm_{s}}(q) = \int e^{iQ \cdot r} r^{Sm_{s}}(r) dr \quad (4.51) \]

\[ h_{Sm_{s}}(q) = \int e^{iQ \cdot r} r^{Sm_{s}}(r) [U_{ab}(r) + U_{Coul}(r)] dr \quad (4.52) \]

The term \( J_{Lm}^{II'}(k_{i}, k_{f}) \) in expression (4.49) is identical to equation (4.50) but does not contain the potential term, in square brackets for the integration over \( R \). \( J_{Sm_{s}Lm}^{II'} \) is the matrix element of \( W_{3} \).

Because of the coordinate problem the only simple way of evaluating this term, for the general case of core excitation, is to use the zero range approximation when it reduces to

\[ J_{Sm_{s}Lm}^{II'} = g_{Sm_{s}Lm}^{II'}(k_{i}, k_{f}) \quad (4.53) \]

where, once again, \( J_{Sm_{s}Lm}^{II'} \) is exactly the same in form as \( J_{Lm}^{II'} \) except that the radial integral contains the factor

\[ [V_{ba}(R) - \delta_{ko} U_{ba}(R) - \delta_{ko} \bar{U}_{ba}(R) e^{2f(R, R_{c})} \] \[ (4.54) \]

instead of the quantity in brackets in equation (4.50).
Summing over \( m_a \) and \( m_b \) and using the orthogonality properties of Glebsh–Gordon coefficients (BR68) the differential cross section for the reaction is given by

\[
\frac{d\sigma}{d\Omega} (I+J) = \frac{N}{(2\pi)^2} \sum_{S_n M M_J} |c_{S_n}^{S_b}(S_b)|^2 
\]

\[
\times \sum_{LmI'} \sum_{M'i'k} C_{L}^{J}(I') (LmI'M'|JM_J)(IMq|I'M') \{g_{Sm_s}(Q) F_{Lm}^{I'I'}(k_{I'I'}k_{f}) \\
+ h_{Sm_s}(Q) I_{Lm}(k_{I'I'}k_{f}) \delta_{I'I'}k_{o} + h_{Sm_s}(Q) I_{Lm}(k_{I'I'}k_{f}) \}^2 \] (4.55)

To be consistent with the numerical factors in equation (4.44) the radial distorted waves in (4.46) and (4.47) were matched onto Coulomb functions, as in equation (3.78), but in this case \( a/4\pi = 1 \). Thus for the alpha transfer calculation the distorted waves were normalised to \( \delta(k-k') \). Using the definition of the density of states in Chapter 3 and the normalisation (equation (3.79)), both techniques are, of course, identical (LE66).

4.3.2 No Core Excitation

When core excitation is neglected (k=0 only) then equation (4.49) greatly simplifies. If, for this condition, the term containing \( W_3 \) is neglected then the transition matrix becomes

\[
T_{fi} = \sum_{LmI'M'J} C_{L}^{J}(I') C_{S}^{S_a}(S_b) (LmI'M'|JM_J) (S_m S_a m_a) \\
\times \{g_{Sm_s}(Q) F_{Lm}^{I'I'}(k_{I'I'}k_{f}) + h_{Sm_s}(Q) I_{Lm}(k_{I'I'}k_{f}) \} \] (4.56)
where, in this approximation;

\[
F_{LM}^{II^*}(k_{f},k_{p}) = \sum_{\ell \ell'} 4\pi i^{\ell-\ell'} e^{i(\sigma \xi + \xi^* \sigma') \frac{2\ell^*}{L}} (-1)^{m_{e} - m_{f}} (k_{f}) \\
\times (\ell \ell' m | Lm) \times (\ell \ell' o | Lo) \\
\times \int_{r_b}^{\infty} (k_{f} k_{R})^{-1} u_{\ell}^{*}(k_{f} yR) u_{\ell'}(k_{i} R) [u_{\ell}^{*}\alpha(R) + v_{\ell'}^{\text{Coul}}(R) - \bar{u}_{\ell'}^{\alpha}(R)] \\
\times u_{NL}(R) dR \quad (4.57)
\]

ie. the model form of the interaction has been assumed.

\[
I_{LM}(k_{fi}) \text{ is now given by}
\]

\[
I_{LM}(k_{fi}) = \sum_{\ell \ell'} 4\pi i^{\ell-\ell'} e^{i(\sigma \xi + \xi^* \sigma') \frac{2\ell^*}{L}} (-1)^{m_{e} - m_{f}} (k_{f}) (\ell \ell' m | Lm) \\
\times (\ell \ell' o | Lo) \\
\times \int_{0}^{\infty} (k_{f} k_{R})^{-1} u_{\ell}^{*}(k_{f} yR) u_{\ell'}(k_{i} R) u_{NL}(R) dR \quad (4.58)
\]

By neglecting the dependence of the matrix element on $k$ it is possible to sum over $m_{a}, m_{b}, M$ and $M_{J}$, in which case

\[
\frac{d \sigma}{d \Omega} (I \rightarrow J) = \frac{u_{\alpha} \bar{u}_{\text{b}}}{(2\pi \hbar^2)^{2}} \frac{k_{f}}{k_{i}} (2I+1)N \sum_{S_{Lm}} \frac{|c_{L}^{j}(I) c_{S}^{S_{Lm}}|^{2}}{(2L+1)(2S+1)} \\
\times |e_{S_{Lm}} F_{LM} + h_{S_{Lm}} I_{LM}|^2 \quad (4.59)
\]
In the numerical calculations of the differential cross section the asymptotic form of the local momentum Q was taken as well as the zero range limit. With these approximations the integrations \( g_{sm} \) and \( h_{sm} \) are independent of the coordinate \( R \).

### 4.4 Selection Rules

The selection rules for the \( \alpha \)-transfer reaction can be seen from equations (4.49) and (4.50) to be

\[
S + S_b = S_a \quad (4.60a)
\]

\[
L + I' = J \quad I + k = I' \quad (4.60b)
\]

\[
L + k = p = \lambda + \lambda' \quad L + \lambda + \lambda' = \text{even} \quad (4.60c)
\]

where \( L \) is the orbital angular momentum of the \( \alpha \)-particle in the system \( B = A+\alpha \), \( S \) is the orbital angular momentum of the \( \alpha \)-particle in the system \( a = b+\alpha \), \( I \) is the spin of the target \( A \), and \( J \) is the spin of the residual nucleus \( B \). Once again, excitation of the \( \alpha \)-particle from its ground state is neglected. When \( k=0 \) equations (4.60b) and (4.60c) reduce to

\[
L = J - I = \lambda + \lambda' \quad L + \lambda + \lambda' = \text{even} \quad (4.61)
\]

In the calculations of the alpha-transfer reaction \( ^{208}Pb(0^{16},c^{12}) \)
the quantum numbers I, J and s are equal to zero.

Neglecting core excitation (k=0) and using the result of Chapter 3 that the alpha particle in $^{212}_{\text{Po}}$ is in an s-state, then $\ell=\ell'$, i.e. the angular momentum transfer is zero.

4.5 Optical and Bound State Potential Parameters

In the calculations of the distorted waves two different sets of optical potentials were used. The potential used by Davies et al (DA76), in his analysis of the $^{208}_{\text{Pb}}(0^+_1)^{12}_{\text{C}}^{212}_{\text{Po}}$ reaction, is referred to as the standard potential S. This potential gives a good fit to elastic scattering of $^{16}_{\text{O}}$ ions from $^{208}_{\text{Pb}}$ at 104 MeV (BE72). The parameters are

\[ V_0 = 40 \text{ MeV} \quad W_0 = 15 \text{ MeV} \quad a = 0.45 \text{ fm} \]

\[ R_0 = r_o\left(\frac{A_1^{1/3} + A_2^{1/3}}{2}\right) \quad r_o = 1.31 \text{ fm} \]

The parameters due to Vandenbosch and Webb (VA75) were also used and are denoted as set V

\[ V_0 = 40 \text{ MeV} \quad a_R = 0.49 \text{ fm} \quad r_R = 1.31 \text{ fm} \]

\[ W_0 = 60 \text{ MeV} \quad a_I = 0.36 \text{ fm} \quad r_I = r_R + 0.14 \text{ fm} \]

These parameters were obtained by fitting elastic scattering of $^{16}_{\text{O}}$ ions from $^{208}_{\text{Pb}}$ at 87 MeV. The same parameters were used for the
incoming and outgoing channels, with appropriate values for \( A_1 \) and \( A_2 \). These two sets are plotted, for the incoming channel in Figure 14.

To generate the model bound state of the alpha particle in \( ^{212}_{Po} \), the microscopic potential \( H \) was used, as described in the alpha decay work of Chapter 3. Using the oscillator rule, (equation 3.72), the bound state of an \( \alpha \)-particle in \( 0^{16} \) was taken to be a 3S state. This bound state wavefunction was generated in a Wood-Saxon potential using the radial parameters due to DeVries (DE73) and denoted as Set A.

\[
R_o = 1.25 \left( 12^{1/3} + \frac{1}{4}^{1/3} \right) \text{ fm} \quad a = 0.65 \text{ fm}
\]

The depth of this potential was varied to generate a 3S wavefunction with the correct separation energy of 7.16 MeV (DE73). To be consistent with the calculations of Davies et al (DA76) the following parameters were also used (Set B)

\[
R_o = 1.25 \left( 12^{1/3} \right) \text{ fm} \quad a = 0.65 \text{ fm}
\]

where, according to Blair et al (BL74), this latter choice improves agreement between approximate and exact treatment of recoil because of the smaller range of the initial bound state potential.
4.6 Results and Discussion

4.6.1 Local Momenta

Taking the angle between the local momenta \( q_i \) and \( q_f \) to be equal to the center of mass scattering angle \( \theta \) the quantities \( Q_E = |q_f - q_i| \) and \( Q = |k_f - k_i| \) were calculated. The resulting values are plotted in Figure 15. It was found that the important contributions to the matrix element came from the barrier region (= 12 fm) and it is evident, in this region, \( Q_E \ll Q \). It was found the transforms \( g(Q) \) and \( h(Q) \) fall very rapidly with increasing \( Q \) and this, together with Figure 15 suggest the use of an effective WKB approximation with the asymptotic momenta \( Q \) will be a very poor approximation. The quantity \( Q_E \) is non zero because the center of mass energies in the initial and final channels, ie. 86.4 MeV and 69.7 MeV respectively, are above the corresponding Coulomb Barriers.

Because \( Q_E \) has a minimum in the barrier region and the initial bound alpha particle cluster has a large separation energy of 7.16 MeV the numerically simplified zero-range calculation seems to be a very useful tool for this reaction.

4.6.2 Angular Momentum Mismatch

The angular momentum transfer in the transition to the ground state of \( ^{212}\text{Pb} \) is zero. However, one quantity that affects the magnitudes of the radial integrals in equation (4.57) and (4.58) is the change of linear momentum \( k_1 \cdot \overrightarrow{y} \cdot k_f \). The value of these radial integrals is dominated by the overlap of the radial distorted waves.
This overlap is reduced if the momenta \( k_i \) and \( y_k \) are very different, for then the distorted waves oscillate at very different rates and the overlap averages to a small value (AU70). To yield large cross sections reactions must provide good momentum matching, i.e. \( k_i \approx y_k \).

In Figure 16 the angular dependent quantity \( k_E (=|y_q - q_i|) \) is plotted as a function of \( R \). At the angles covered by the experimental data, which are \( \theta \sim 60^\circ-120^\circ \), \( k_E \) is not zero and hence the kinematic momentum transfer \( k ER \) is not zero. It is obvious that there is a condition of angular momentum mismatch for this reaction and thus it would not be expected to find sharp localization in angular momentum space or in configuration space. This conclusion is confirmed by the graph shown in Figure 17 of the quantities \( |F_0| \) and \( |I_0| \) calculated from equations (4.57) and (4.58) with \( L=0 \) and \( \xi=\xi' \). The spread in \( \xi \) is \( \Delta \xi \sim 20 \). However, these functions behave smoothly with \( \xi \) and there is no significant contribution from small \( \xi \) or very large \( \xi \). An \( \xi \) max of 100 was used in the analysis. It was found the main contributions in configuration space came from the region \( R \sim 8.5 \text{ fm} \) to 14.5 fm.

4.6.3 Angular Distributions

Figure 18 shows the angular distribution obtained in zero-range approximation (normalised to the peak in the cross section), using Set S for the optical parameters and Set B for the bound state parameters. Although \( |F_0| \) is larger than \( |I_0| \), \( g_0 \) is very much smaller than \( h_0 \) and hence the term \( h_0 I_0 \) gives the maximum contribution to the cross section. This is consistent with treating the correction due
to the use of model wavefunctions by means of perturbation theory. Nevertheless, when the coherent sum of the two terms is taken the peak of the cross section increases by a factor of \( \sim 1.7 \). The absolute magnitude of the cross section emphasizes the angular momentum mismatch.

Figure 19 shows the result of using Set V for the optical potential and the same bound state parameters. The importance of the correction term is slightly increased and the shape of the angular distribution is changed. However, it was found that the absolute value of the cross section is 31 times smaller than the one shown in Figure 18. A calculation with the optical parameters S with the alternative bound state parameters A only altered the numerical values of \( g_0 \) and \( h_0 \) and gave, of course, a cross section of identical shape but the magnitude was increased by a factor of 1.44.

The characteristic bell shaped cross section in heavy ion transfer reactions can be understood using semi-classical arguments (SC73). For this reaction the wavelength of relative motion is small (\( \approx 0.8 \text{ fm} \)) and thus this localisation of the wave packet leads to the concept of a well defined classical orbit with a peak in the cross section at \( \theta_c \) where the nuclei suffer a grasing collision. For larger angles the nuclei overlap and absorption reduces the transfer cross section, while for smaller angles the nuclei do not come into contact so that the transfer probability is small. For a Rutherford orbit
where $R_1 + R_2$ are the sum of the nuclear radii and $E$ the center of mass energy. To get the peak at 78° center of mass angle it is required that $R_1 + R_2 = 14$ fm.

4.6.4 Spectroscopic Factors

It can be seen from equation (4.59) that in order to deduce the spectroscopic factor for $^{212}$Po $\rightarrow ^{208}$Pb + $\alpha$ it is necessary to include the antisymmetrisation factor $N$ and have a reliable value for the coefficient $|C|^2$ of the $^{12+C} + \alpha$ system.

In the calculations of Davies et al (DA76) a value of 0.55 for $|C|^2$ was used. This was obtained by DeVries (DE73) by fitting the data for the $^{12+C} (^{16}_O, ^{12}_C)$ reaction. Other fits to data give $|C|^2 = 0.23$ for the $^{16}_O(d, ^6Li)$ reaction (GU71) and 0.295 for the $^{16}_O (^3He, Be)$ reaction (DE69). The theoretical shell model calculation due to Kurath (KU73) gives $|C|^2 = 0.79$ while a cluster model calculation by Coelho (CO73) obtained the values $|C|^2 = 0.28-0.36$.

The antisymmetrisation factor, which multiplies the matrix element and takes partial account of the indistinguishability of particles, is given by $N^\frac{1}{2} = (N_1/N_2)^\frac{1}{2} \times X$ (AU70). Here $X$ is the number of ways each term in the wavefunction for the initial system $a+A$ can be converted to a term in the final $b+B$ system and $N_1, N_2$ are the number of nucleon permutations resulting in mutually orthogonal product functions in the initial and final states, respectively.

In the treatment of $\alpha$-decay of $^{212}$Po the $^{208}$Pb nucleus was taken to
be a closed core. This gives $N_2=1$ and $N_1=X$. Choosing Kurath's prescription for the $^{12}_{C} + \alpha$ spectroscopic factor, which already includes the factor $X$ mentioned above, the $\alpha$-particle is removed from the p shell in $^{16}_{0}$. Thus the cross section must be multiplied by $N=225$.

In Table XII the results of the spectroscopic factors for the $^{212}_{Po} \rightarrow ^{208}_{Pb} + \alpha$ system are given. Davies et al (DA76) obtained a

| Table XII |
| Values of the coefficient $|C|^2$ for $^{212}_{Po}$ obtained with $M=225$ and $|C|^2=0.79$ for $^{160}_{Po}$. The symbols $S$, $A$, etc, refer to the potentials defined in sections 6.1 and 6.2. |

| $S+A$ | $S+B$ | $V+A$ | $V+B$ |
| $|C|^2$ | 0.0048 | 0.0069 | 0.149 | 0.214 |

value of 0.0073 for the potential set of $S+B$ but used the experimental value of 0.55 (DE73) for the $^{16}_{0}$ spectroscopic factor. Also some discrepancy is to be expected between the two calculations because of their different treatment of the $^{212}_{Po}$ wavefunction and because they used a finite range code.

The large variation produced by the different potentials makes it difficult to draw conclusions about the spectroscopic factor. However it can be seen that for the potential Set $S$ this calculation is consistent with Davies' results, even taking into account the different aspects of the calculations. Agreement with the $\alpha$-decay results of Chapter 3.
favours the use of potential Set V.

4.7 Conclusions

In this Chapter a formalism was developed for a transfer reaction in which the final state of the transferred particle was not a bound state. Using this technique it is possible to obtain good shape agreement with the data under consideration by using only a zero-range approximation. A full finite range calculation can be expected to change the absolute magnitude of the cross section by some factor but there seems to be little justification, at the moment, for a major computational effort in this respect. This is because of the different results obtained with the two sets of distorted wave optical potentials, the large experimental errors in the differential cross section, and the possibility of core excitation which may also be expected to change the magnitude of the cross section.

It can be seen from Figure 14 that the different values obtained for the spectroscopic factors in this reaction can be largely attributed to the different range and magnitude of the imaginary potential used in the generation of the distorted waves. A communication with Vandembosh (VA77) has shown that his data covers a much wider angular range than the earlier work of Becchetti (BE72) and indeed to fit his elastic scattering data of $^16\text{O}$ on $^{208}\text{Pb}$ at 87 MeV it was necessary to adjust Becchetti's (BE72) original optical potential parameters. The energy dependence of the imaginary part of the optical potential for this reaction has been shown in a recent publication due to Videbaek
et al (VI77). In view of these results it would be interesting if Davies could repeat his calculation using the potential due to Vandenbosh and Webb (VA75) which has been fitted to a large angular range and at an incident energy very near the one of interest for this reaction.

Despite the uncertainties outlined above it is clear that the spectroscopic factors obtained in this treatment of α-decay are not inconsistent with α-transfer calculations and it seems the two processes can be treated with a similar formation. Both the work of Davies, who used a cluster model formalism, and the calculations reported here give spectroscopic factors at least an order of magnitude larger than shell model calculations.

In conclusion it seems that it is possible to give a complete description of α-decaying states of heavy nuclei in a cluster model representation without the use of arbitrary radii and recourse to R-matrix theory.
Figure 13  Definitions of coordinates for the $\alpha$-transfer reaction.

Figure 14  Comparison of the two optical potentials used in the $\alpha$-transfer reaction for angular momentum value $\ell = 0$.

Figure 15  The effective local momentum $Q_E(R) = |q_f - xq_i|$ for large $R$ and the angular range of interest, calculated with potential set S. The asymptotic momentum $Q(\theta)$ is plotted at the value of $R$ where $Q_E$ is a minimum to reflect the maximum difference between the two quantities.

Figure 16  Plot of the kinematic momentum transfer $K_E(R) = |yq_i - q_f|$ for large $R$ and the angular range of interest, for potential set S.

Figure 17  Comparison of $|F_\ell|$ and $|I_\ell|$ as functions of incident angular momentum $\ell$, calculated for potential set S.

Figure 18  The angular distributions for the $\alpha$-transfer reaction obtained using potential parameters $S$ and $^{16}O$ bound state parameters $B$. The peak value of the calculated cross section is normalised to the experimental data. The two lower curves show the contributions from the two parts of the matrix element normalised by the same factor.

Figure 19  The same quantities as shown in Figure 18 but calculated with potential set V.
MODULUS OF $\text{Fo}(t)$ AND $\text{Io}(t)$
ARBITRARY UNITS

FIGURE 17
\[
\frac{d\sigma}{d\Omega} (\mu b/sr) \quad \begin{array}{c}
10^0 \\
10^{-1} \\
10^{-2} \\
10^{-3} \\
10^{-4}
\end{array}
\]

\[\begin{align*}
208_{\text{Pb}}^{16}O, C & \rightarrow^{212}_{\text{Po}(gs)} \\
93 \text{MeV}
\end{align*}\]

\text{FIGURE 19}
5.1 Introduction

In the analysis of previous Chapters on alpha decay and alpha transfer it was assumed that the Polonium isotopes can be simply represented by a product of wave functions for Pb and for the alpha particle and the appropriate wave function of relative motion between them. The relatively large values obtained for the spectroscopic factors were seen to be consistent with this approximation. However, in this analysis the exclusion principle was only taken into account explicitly by determining the lowest allowed state of relative motion through the oscillator rule, equation (3.72).

There have been a number of papers investigating the effects of the Pauli principle. Pong and Austern (P075), who calculated a local equivalent for the Pauli correction term in the deuteron nucleus optical potential, showed that the real part of this potential was decreased by 10% when exchange effects were included. In the work of Perkin et al (PE75) the decrease in the binding energy of composite particles inserted into nuclear matter was shown to have a large effect on the momentum dependent optical potential. In this analysis the internal wavefunction of the deuteron, say, was described by the equation

\[ \Psi = \phi - \frac{Q}{T-\varepsilon} v\Psi \]

(5.1)

where \( T \) is the mutual kinetic energy of the nucleon pair, \( \varepsilon \) is the internal binding energy to be determined, and \( Q \) is an operator which...
projects out states forbidden by the Pauli principle. The interaction \( v \) was taken to be of separable form and a Thomas–Fermi model was used to describe nuclear matter (DE74).

Another approach to the problem of antisymmetrisation is the technique known as the Resonating Group Method (R.G.M.). In this theory a correct treatment of antisymmetrisation between two clusters leads to a modified relative motion wavefunction. The internal structure of the projectiles is unchanged. The difficulty with this method is the calculation of a normalisation and interaction kernel arising in an integro-differential equation for the relative motion function.

In this Chapter the effects of antisymmetrisation on the cluster expansion for Polonium, equation (3.51), are considered within the context of a modified Resonating Group formalism. After an introduction to the R.G.M. a technique of calculating the normalisation kernel is given which uses an approximation for a density matrix description of the two nuclear clusters. This kernel is used in a calculation based on a reinterpreted R.G.M. due to Buck et al (BU277) to calculate the modified relative motion function. A discussion of Fliessbach's technique of including antisymmetrisation effects in alpha decay (FL75) and the form of the non-local interaction kernel are also given.

5.2 Resonating Group Method

Introducing an antisymmetrisation operator \( \hat{A} \) into the cluster model expansion, equation (3.51) becomes
where, in this notation, the sum over $n$ implies summation over all appropriate quantum numbers and $\phi_\alpha(\eta)$, $\phi_\alpha(\xi)$ are fully antisymmetrised wavefunctions. The operator $\hat{A}$ is given by (ME61)

$$\hat{A} = 1 + \sum_{\nu=1}^{4} \binom{4}{\nu} (-1)^\nu P_{1+A+\nu} \cdots P_{\nu, A+\nu}$$

where $P_{i, A+i}$ is the permutation operator that exchanges nucleons $i$ and $A+i$. Separating $\psi_o$ into direct and exchange components equation (5.2) becomes

$$\psi_o = \sum_n [\phi_d(n) + \sum_{\nu=1}^{4} \sum_S (-1)^\nu \phi_{EX, \nu}(n)]$$

$$= \sum_n [\phi_d(n) + \phi_{EX}(n)]$$

where the sum over $S$ denotes all the different functions arising from a given $\nu$-exchange.

By inserting equation (5.5) into the Hill-Wheeler equation (WH37)

$$\int \phi_A^*(\eta)\phi_A^*(\xi)(E-E)\psi_o d\eta d\xi = 0$$

the full R.G.M. equation is derived as a set of coupled integro-differential equation for $\chi_\alpha(R)$ i.e.

In the following equation $T(R)$ is the kinetic energy operator $-\hbar^2 R^2/2m$, $\epsilon_\alpha = E - E_A - E_\alpha$, and the previous definition of the optical
The exchange integrals are normally represented as

\[ \int \phi_A(n) \phi_A^*(\xi) \phi_{EX}(n) \, d\eta \, d\xi = - \int K_n(RR') \chi_n^*(R') \, dR' \]  

(5.9)

\[ \int \phi_A(n) \phi_A^*(\xi) V_{\alpha A} \phi_{EX}(n) \, d\eta \, d\xi = - \int G_n(RR') \chi_A^*(R') \, dR' \]  

(5.10)

where \( K_n(RR') \) is the normalisation operator introduced by Feshbach and whose properties have been outlined for nucleon-nucleus scattering (FE73).

In their analysis of alpha decay of light nuclei Arima and Yoshida (AR74) derived equation (5.7) for the single channel case. On truncating equation (5.2) to the ground state term, as in
Chapter 3, this form may be deduced from equation (5.7), ie.

\[ [T(R)+V_c(R)+\hat{U}_N(R)]\chi_\alpha = \varepsilon_\alpha (1-\hat{K})\chi_\alpha +\hat{K}[T(R')+V_c(R')][\chi_\alpha +\delta\chi_\alpha \ (5.11) \]

In all the subsequent analysis on exchange effects it will be assumed that equation (5.2) may be represented by a single channel approximation. Equation (5.11) may be represented in a more convenient form using the notation due to Buck et al (BU277). Writing equation (5.2) in the equivalent form (for convenience the centre of mass vector is retained as \( R \))

\[ |\psi> = \int dR\hat{A}|\phi_A(\eta)\phi_\alpha(\xi)\delta(R-R')\chi_\alpha(R) \ (5.12) \]

then equation (5.11) can be written as (BU277)

\[ \int dR'H_{c}(R'R')\chi_\alpha(R') = \varepsilon_\alpha \int dR'A(RR')\chi_\alpha(R') \ (5.13) \]

or

\[ H_c|\chi> = \varepsilon_\alpha A|\chi> \ (5.14) \]

where \( H_c(RR') = \langle \phi_\alpha A | (T+V)A |\phi_\alpha A \delta(R-R') \rangle \) \ (5.15)

\[ A(RR') = \langle \phi_\alpha A | \hat{A} |\phi_\alpha A \delta(R-R') \rangle \] \ (5.16)

\[ = \delta(R-R') - K(RR') \] \ (5.17)

The delta function corresponds to the direct term.
From equations (5.12), (5.16) and (5.17) it is evident that the normalisation condition for $|\psi\rangle$ is given by (BU77, FL75, FE73)

$$<\psi|\psi> = <\tilde{\chi}_\alpha|1-K|\tilde{\chi}_\alpha> = \delta_{ij} \quad (5.18)$$

where the condition $\hat{A}^2 = \hat{A}$ has been used (ME61).

The R.G.M. has been applied successfully in a series of papers by Tang et al (TA71) to describe light ion scattering. More recently Friedrich (FR74) and Ando et al (AN75) have applied equation (5.11) to the problem of scattering of light and medium heavy nuclei. In the latter paper a calculation for $^0_{16} - ^0_{16}$ scattering showed that the inner parts of the R.G.M. relative wave function were considerably damped, irrespective of angular momentum quantum number. The wavefunction in the surface region was also slightly modified. This was said to be due to the strong non-locality of the exchange terms.

The role of exchange effects in alpha decay has been examined by Fliessbach (FL75, FL76, FL276) using Mangs R-matrix theory of $\alpha$-decay. He represented the final $A+4$ system by equation (5.12) and in order to take account of the normalisation condition, equation (5.18) a new scattering function $\Omega(R)$ was introduced such that

$$\Omega(R) = (1-K)^{1/2} \chi_{\alpha}(R) \quad (5.19)$$

ie. equation (5.12) becomes

$$|\psi> = \int dR \Omega(R) |\hat{A}(1-K)^{-1/2} \phi_{\alpha} \phi_{A}\delta(r-R)> \quad (5.20)$$
In this way Mang's definition (MA60) of the reduced width amplitude $G(R)$, i.e. $\langle \phi_{A+4} |\delta (r-R) \phi_\alpha |\phi_A \rangle$, is now modified to give

$$G_N(R) = (1-K)^{-\frac{1}{2}} G(R)$$

(5.21)

where in Mang's theory a new spectroscopic factor is now given by

$$S = \int dR |G_N(R)|^2$$

(5.22)

Using a shell model description of the initial state it was found that the spectroscopic factor defined by eqn. (5.22) was two orders of magnitude larger than Mang's result. However Fliessbach's work contains a number of uncertainties. By using a shell model description of the initial state and Mang's theory his results are still sensitive to the old problems outlined in section (3.1.2).

Further, he estimates the channel radius $R_c$ by examining the effective $\alpha$-nucleus potential, which is derived from $G_N(R)$, by using the radial Schrödinger equation. This analysis gives a value of $R_c$ of 6.1-7.0 fm, well inside the inner turning point values calculated in Chapter 2. Although the new radial wave function $\Omega_E(R)$ obeys the more conventional form of delta function normalisation there seems to be no reason to assume that making a new definition of the relative motion wavefunction through equation (5.19) implies that this function is now a solution of radial Schrödinger equation with local potentials only. In practice however, it is assumed that $\Omega_E(R)$ may be approximated by the W.K.B. method in which the $\alpha$-nuclear local potential can be taken from Igo's work (IG59).
Thus apart from the problems of R-matrix theory and the value used for \( R \), there also seems to be an ambiguity between the introduction of an antisymmetrisation operator in the cluster expansion and calculating \( \Omega_E(R) \) as if it was the solution of a conventional radial Schrodingers equation with a local potential.

5.2.1 Alternative Approach and the Orthogonal Condition Model

One technique of approximating the complicated R.G.M. equation arises through examining the properties of the normalisation Kernel \( K \). It was pointed out by Saito (SA69) that if the normalisation operator \( K \) was expanded in terms of its eigenstates and eigenvalues, ie.

\[
K = \sum_{n\ell m} |g_{n\ell m}^\lambda|^2
\]

(5.22)

where the \( \lambda \) on the eigenstates indicate it is associated with a certain eigenvalue, \( n \) is the principal quantum number, \( \ell \) the angular momentum quantum number and \( m \) its \( z \) component, then for \( \alpha-\alpha \) scattering the values obtained for \( \lambda_{n\ell} \) made it possible to approximate eqn. (5.11) so that a calculation of the complicated non-local interaction kernel can be avoided. This is called the Orthogonal Condition Model (O.C.M.) and arises from the concept of redundant states in the R.G.M. equation.

The properties of \( K \) have been stated by Feshbach (FE73). \( K \) is an Hermitean, positive definite operator and the operator \( (1-K) \) is also positive definite. Becuase of these properties the quantity \( \lambda_{n\ell} \) defined in equation (5.23) can only take the values \( 0 \leq \lambda_{n\ell} \leq 1 \).
The redundant states of the R.G.M., if they exist, are defined as those \( |g^\lambda\rangle \) for which \( \lambda = 1 \). This can be seen more clearly if the one body states are constructed with such a \( |g^\lambda\rangle \). In this case equation (5.18) becomes:

\[
<g^\lambda | \mathbf{H} | g^\lambda > = 0 \quad (5.24)
\]

From equation (5.12) this corresponds to

\[
\int d\mathbf{R} \phi_\alpha(\xi) \phi_A(n) \delta(\xi-R) \mathbf{H} g^\lambda(R) = 0 \quad (5.25)
\]

Hence the relative motion states corresponding to the eigenvalue \( \lambda = 1 \) are forbidden by the Pauli principle. For the case of nucleon scattering from a nucleus described by a Slater determinant, \( \lambda \) takes the value one for all \( n \) and \( m \) (FE73). In this case \( K \) becomes a summation over appropriate occupied single particle levels. For the interaction of two composite particles this simple situation no longer applies but it is always possible to project out the \( |g^\lambda\rangle \) by the following operator (SA69, BU277).

\[
\Lambda = 1 - \sum_{\lambda=1}^{\lambda} |g^\lambda\rangle <g^\lambda| \quad (5.26)
\]

and so produce a modified R.G.M. equation for \(|u\rangle\) where \(|u\rangle = \Lambda |\chi_\alpha\rangle\)

\[
\Lambda(T(R)+V_c(R)+\hat{U}_N(R)-\xi_\alpha)u = \Lambda(G-\xi_\alpha K + KT(R')+KV_c(R'))u \quad (5.27)
\]

In the case of \( \alpha-\alpha \) scattering it was found (SA69) that for the
non-redundant solutions the values of \(\lambda\) in equation (5.23) were very small. In this situation the right hand side of equation (5.27) reduces considerably and indeed it seems reasonable to neglect the right hand side of this equation if the quantity \(\Delta G\) is also small. This approximation gives the O.C.M. equation, ie.

\[
\lambda(T(R)+V_c(R)+\hat{U}_N(R)-\varepsilon_\alpha)u' = 0 \tag{5.28}
\]

The usual O.C.M. technique is to use the separable form of the expansion appearing in equation (5.26). A more general expansion is to replace equation (5.26) by \(\lambda = (1-K)\). In this way any values of \(\lambda\) between 0 and 1, corresponding to semiredundant states, are also included. Using a partial wave expansion for the bound state \(|u\rangle\) and the kernel \(K\), equation (5.28) reduces to an integro-differential equation of the form

\[
(k^2 + \frac{d^2}{dR^2} - V_T(R))u(R) = 0
\]

\[
= \int_0^\infty K'_0(RR')\left(-\frac{d^2}{dR'^2} - V_T(R')\right)R'u(R')dR' \tag{5.29}
\]

where \(k\) is the relative wave number, \(K'_0(RR') = RK(R'R')R\), and

\[
V_T = \frac{2U_c}{\hbar^2} (V_c + \hat{U}_N). \quad \text{It has been assumed in equation (5.29) that } \ell = 0.
\]

Other work on the R.G.M. (AN75) has suggested that for heavier clusters the conditions outlined above for the eigenvalues in \(\alpha-\alpha\) scattering are not present and so this casts doubt on the O.C.M. in these situations. For the examples of \(\alpha+0^{16}\) and \(0^{16}+0^{16}\) the eigenvalues \(\lambda_{n\ell}\) varied smoothly from 0 to 1 (AN75). However a
recent publication due to Buck et al (BU277) has shown that an equation of the same form as the O.C.M. may be derived if $H_c$, of equation 5.15, is approximated by

$$H_c = \hat{A}^{\frac{1}{2}} [T+V] \hat{A}^{\frac{1}{2}}$$  \hspace{1cm} (5.30)$$

where $V$ is an effective potential which may contain some nonlocality. In this approximation of the R.G.M. it is suggested that even if there are many partially redundant states ($\lambda \neq 0$ or 1) the O.C.M. form may still be valid, but it must be interpreted as an equation for $\hat{A}^{\frac{1}{2}}$ times the R.G.M. function $|\chi\rangle$

$$|u\rangle = \hat{A}^{\frac{1}{2}} |\chi\rangle$$  \hspace{1cm} (5.31)$$

where $|u\rangle$ is now the solution of (5.27).

Using this reinterpretation of the O.C.M. it was found that the results using eqn. (5.27) and a full resonating group calculation were comparable in many situations. Indeed it seems exchange effects are present in the O.C.M. and are accounted for by the redefinition of the wave function $|u\rangle = \hat{A}^{\frac{1}{2}} |\chi\rangle$.

In view of the results of this reinterpreted version of the O.C.M. (BU277) it was decided to examine the role of exchange effects in the $\alpha + ^{208}\text{Pb}$ system by calculating the modified relative motion function using equation (5.28). For the expansion of $^{212}\text{Po}$, represented by eqn (5.2), the solution of (5.28) corresponds, of course, to a modified bound state wave-function. Although this approach avoids a calculation
of the complex non-local interaction kernel the normalisation kernel K still has to be computed.

5.3 The Normalisation Kernel

The v particle exchange normalisation kernel is given as, from equation (5.17) and (5.3)

\[ K(RR') = \sum_{v=1}^{4} \binom{4}{v} (-1)^v \langle \phi_\alpha^A \delta(r-R') | \sum_{P_{1,5} \cdots P_{4+\nu}} | \phi_\alpha^A \delta(r-R') \rangle \]

where, in subsequent analysis the individual v particle exchange kernel \( K_v(RR') \) will be examined, ie.

\[ K(RR') = \sum_{v=1}^{4} K_v(RR') \quad (5.33) \]

It is well known that in the situation that the two nuclear clusters are described by Harmonic Oscillator wave functions (DE74) the normalisation kernel can be worked out analytically (SA69, AN75, MA75). However because of the asymptotic behaviour of oscillator wave functions and the sensitivity of alpha decay and alpha transfer to the nuclear surface region a different approach was taken. The first step in this analysis is given in Appendix A. This work represents a vector analysis of equation (5.32) and expresses the normalisation kernel in terms of density matrices of the two nuclear clusters, see equation (A.21).
In general the local one particle density matrix can be expressed in terms of the Pauli spin and isospin matrices (B069), ie.

\[
\rho_{\text{rst}} = \frac{1}{4} \rho_{\text{oo}}(r) + \frac{1}{2} \sum_{i=x,y,z} \sigma_i \rho_{\text{lo},1}(r)
+ \frac{1}{2} \sum_{j=x,y,z} \tau_j \rho_{\text{ol},1}(r)
+ \sum_{i,j} \sigma_i \tau_j \rho_{\text{11},ij}(r)
\]

(5.34)

where \( \rho_{\text{oo}}(r) \) is a scalar in spin space and isospin space while \( \rho_{\text{lo}} \) is a vector in spin space and a scalar in isospin space, and so on. However in all the following analysis a separate neutron-proton formalism will be used and only the first term in equation (5.34) is considered. From the definition of \( \rho_{\text{oo}}(r) \) this means that in calculating the kernels from equation (A.21) the trace must be taken over the spin space coordinates \( \sigma \) as well as integration over the space variables already indicated. It was also assumed that \(^{208}\text{Pb}\) can be taken as an infinitely heavy nucleus. This corresponds to a modified weighting factor in equation (A.21) and the coordinates (A.26)-(A.29).

5.3.1 Alpha Particle Density Matrix

In the neutron-proton formalism the ground state alpha particle wave-function can be expressed in two equivalent forms as (SA69, HA68)
\[ \phi_a(\xi_1, \xi_2, \xi_3) = \left( \frac{1}{2} \right)^{3/2} (\frac{a}{\pi})^{9/4} \exp\left( -\frac{a}{2} \frac{1}{2} \xi_1^2 + \frac{2}{3} \xi_2^2 + \frac{3}{4} \xi_3^2 \right) \]

\[ \times [x_{1/2}(\sigma_{p1})x_{1/2}(\sigma_{p2})]_{oo} \times [x_{1/2}(\sigma_{n1})x_{1/2}(\sigma_{n2})]_{oo} \]  

(5.35)

or

\[ \phi_a(\xi_1', \xi_2', \xi_3') = \left( \frac{1}{2} \right)^{3/2} (\frac{a}{\pi})^{9/4} \exp\left( -\frac{a}{2} \frac{1}{2} \xi_1'^2 + \frac{2}{3} \xi_2'^2 + \frac{3}{4} \xi_3'^2 \right) \]

\[ \times [x_{1/2}(\sigma_{p1})x_{1/2}(\sigma_{p2})]_{oo} \times [x_{1/2}(\sigma_{n1})x_{1/2}(\sigma_{n1})]_{oo} \]  

(5.36)

where \( a = 0.535 \text{ fm}^{-2} \) and \([x_{1/2}(\sigma)x_{1/2}](\sigma)]_{oo} \) are the singlet spin space wavefunctions. The two sets of coordinates are defined by

\[ \begin{align*}
\xi_3 &= \xi_2 - \xi_1 \\
\xi_3' &= \xi_3 - \xi_4 \\
\xi_2 &= \xi_3 - \frac{1}{2} (\xi_1 + \xi_2) \\
\xi_2' &= \frac{1}{2} (\xi_1 + \xi_2 - \xi_3 - \xi_4) \\
\xi_1 &= r_4 - \frac{1}{3} (\xi_1 + \xi_2 + \xi_3) \\
\xi_1' &= \xi_1 - \xi_2
\end{align*} \]  

(5.37)

where \( \xi_1 - \xi_4 \) are nucleon coordinates. From equation (A.2) it can be seen that equation (5.36) is the correct form for even particle exchanges and equation (5.35) for odd particle exchanges. Hence on using (A.26) and (A.28) the spatial part of the density matrix is given by (from equation A.12);

**One Particle Exchange**

\[ \rho(\xi_1, \xi_1', \xi_1 - \xi_1) = \left( \frac{3}{4} \right)^{3/2} (\frac{a}{\pi})^{3/2} \exp\left( -\frac{a}{3} (\xi_1 - \xi_1')^2 - 3\alpha (R - R')^2 \right) \]  

(5.38)
Two Particle Exchange

\[ \rho(S_1, S_2; I_1, I_2) = \left( \frac{1}{2} \right)^{3/2} \left( \frac{a}{\pi} \right)^{3/2} \exp\left( -\frac{a}{4}(\xi_1^2 + n_1^2) \right) \]

\[ - a(y - R + R')^2 \]

\[ \times \exp(-a(R - R')^2) \]  

(5.39)

Three Particle Exchange

\[ \rho_\alpha(S_1, S_2; I_1, I_2, I_3; n_1, n_2, n_3) = \left( \frac{1}{2} \right)^{3/2} \left( \frac{a}{\pi} \right)^{4/2} \]

\[ \times \exp\left( -\frac{a}{4}(\xi_1^2 + n_1^2) - \frac{a}{3}(\xi_2^2 + n_2^2) \right) \]

\[ \times \exp(-3(y - R + R')^2 - \frac{1}{3}(R - R')^2) \]  

(5.40)

Four Particle Exchange

\[ \rho_\alpha(S_1, S_2; I_1, I_2, I_3; n_1, n_2, n_3) = \left( \frac{1}{2} \right)^{3/2} \left( \frac{a}{\pi} \right)^{9/2} \]

\[ \times \exp\left( -\frac{a}{4}(\xi_1^2 + n_1^2) - \frac{a}{2}(\xi_2^2 + n_2^2) \right) \]

\[ - \frac{a}{4}(\xi_3^2 + n_3^2) \]  

(5.41)

5.3.2 Density Matrix for \(^{208}\text{Pb}\)

In order to be consistent with the previous descriptions of \(^{208}\text{Pb}\) it was once again assumed that this nucleus can be represented
by a Slater determinant (B069) of single particle wavefunctions \( \psi_j(r) \), calculated in the potentials outlined in Section (2.2). Using this model the one particle exchange density matrix can be written down straight away as

\[
\rho_{PB}(\gamma_i \eta_j) = \frac{1}{B} \sum_i \psi_i(\gamma_i) \psi_i(\eta_j)
\]

where \( i \) is a sum over all occupied quantum levels. Equation (5.42) can be put into a more manageable form using the density matrix expansion of Negele and Vautherin (NE72) i.e.

\[
B\rho_{PB}(\gamma_i \eta_j) = \frac{3}{Sk_f} j_1(Sk_f) \rho(R) + \frac{35}{2Sk_f^3} j_3(Sk_f)
\]

\[
\times \{ \frac{1}{4} V^2 \rho(R) - \tau(R) + \frac{3}{5} k^2_f \rho(R) \}
\]

(5.43)

where \( \tau(R) \) is the kinetic energy density

\[
\tau(R) = \sum_i |V\psi_i(R)|^2
\]

(5.44)

and

\[
R = \frac{\gamma_1 + \eta_1}{2}, \quad S = \gamma_1 - \eta_1
\]

(5.45)

In this expression \( k_f \) is the Fermi momentum (DE74) and the second term in curly brackets represents surface corrections to the Thomas-Fermi Model (DE74). \( \rho(R) \) is the nuclear density the calculation of which was outlined in Section (2.2). Equation (5.43) actually represents an average of \( \rho_{PB}(\gamma_i \eta_j) \) over the directions of \( S \) and its validity as a density matrix expansion has been studied by Sprung et al (SP75).
On replacing equation (5.44) by the modified Thomas-Fermi expression (B076)

\[ \tau(R) = \frac{3}{5} k F^2(R) + \frac{1}{36} \left( \frac{(\nabla \rho)^2}{\rho} + \frac{1}{3} \nabla^2 \rho \right) \]  \hspace{1cm} (5.45)

the second term in equation (5.42) can be reduced to terms involving derivatives of the nuclear density only. Hence by using equation (5.43) and (5.45) the complicated shell model structure of $^{208}$Pb may be reflected simply in the density matrix by terms involving the nuclear density only. The construction of this quantity was outlined in Chapter 2.

By assuming the ground state of $^{208}$Pb can be represented by a Slater determinant the density matrices for higher order exchange processes can be simply expressed in terms of the one particle density matrix, i.e. for two particle exchange, say,

\[ \rho_{Pb}(y_1 \pi_1; y_2 \pi_2) = \binom{B}{2}^{-1} \frac{1}{2} \]

\[ (\rho_{Pb}(y_1 \pi_1) \rho_{Pb}(y_2 \pi_2) - \rho_{Pb}(y_1 \pi_2) \rho_{Pb}(y_2 \pi_1)) \]  \hspace{1cm} (5.47)

Hence in this way it will be shown that the higher order particle exchanges can be expressed in terms of integrals of the same form as those which arise in lower order exchange mechanisms, thus reducing the large computational effort required in exchange calculations.
5.3.3 Spin Dependence of Density Matrices

To include the spin dependence for one proton exchange, say, equation (5.38) must be multiplied by the factor $\Lambda_{oo}(\sigma^1 P_1) / \sqrt{2}$ where

$$\frac{1}{\sqrt{2}} \Lambda_{oo}(\sigma^1 P_1) = \frac{1}{\sqrt{2}} \sum_m (\frac{1}{2} m \pm m|oo)(-1)^{\frac{1}{2} m} \chi^{\frac{1}{2} m}(\sigma^1 P_1) \chi^{\frac{1}{2} m}(\sigma^1 P_1)$$  \hspace{1cm} (5.48)

and $\sigma^1 P_1$ is the spin space coordinate of the exchanged particle. This term may be readily deduced by considering the singlet spin structure of the alpha particle and summing over unexchanged nucleon spin coordinates. Similarly, of course, equation (5.42) must be multiplied by $\Lambda_{oo}(\sigma^1 P_1) / \sqrt{2}$. As explained in Section (5.3.1) only the trace over the spin space variables is of interest in this analysis. This can be calculated from the relation.

$$\text{Tr}(\Lambda_{j m_j}(\sigma^1) \Lambda_{j' m'_j}(\sigma^1)) = (-1)^{m_j} \delta_{jj'} \delta_{m_j m'_j}$$  \hspace{1cm} (5.49)

Using the same notation the following identities will be used in higher order exchange formalisms, i.e.

$$[\chi^{\frac{1}{2}}(\sigma_1) \chi^{\frac{1}{2}}(\sigma_2)]_{oo} [\chi^{\frac{1}{2}}(\sigma'_1) \chi^{\frac{1}{2}}(\sigma'_2)]_{oo}^* =$$

$$= \sum_{j_1 m_1} \frac{1}{2} (-1)^{m_1-j_1} \Lambda_{j_1 m_1}(\sigma_1 \sigma'_1) \Lambda_{j_1 m_1}(\sigma_2 \sigma'_2)$$  \hspace{1cm} (5.50)
\[
\left[ \chi^1 (\sigma_1) \chi^1 (\sigma_2) \right]_{oo} \left[ \chi^1 (\sigma'_1) \chi^1 (\sigma'_2) \right]_{oo}^* = \]

\[
= - \sum_{j_1 m_1} \frac{1}{2} (-1)^{m_1-j_1} \Lambda_{j_1 m_1} (\sigma_1 \sigma'_2) \Lambda_{j_1 m_1} (\sigma_2 \sigma'_1) (5.51)
\]

\[
\sum_n \left[ \chi^1 (\sigma_1) \chi^1 (\sigma_2) \right]_{1n} \left[ \chi^1 (\sigma'_1) \chi^1 (\sigma'_2) \right]_{1n}^* = \]

\[
= \frac{3}{2} \Lambda_{oo} (\sigma_1 \sigma'_1) \Lambda_{oo} (\sigma_2 \sigma'_2)
\]

\[
+ \frac{1}{2} \sum_{m_1} (-1)^{m_1} \Lambda_{1m_1} (\sigma_1 \sigma'_1) \Lambda_{1-m_1} (\sigma_2 \sigma'_2) (5.52)
\]

5.3.4 One Particle Exchange Kernel

From equations (A.21) and (5.3) the one particle exchange kernel may be expressed as, for one neutron exchange say

\[
K^1_{n}(RR') = - \binom{2}{1} \binom{N}{1} \frac{8}{3} \frac{3}{3} \int d \eta \rho_a (\xi_1, \xi_1, -X) \rho_{\beta} (\eta_{1, \eta_{1}, X}) (5.53)
\]

Using the equations (5.38), (5.42) and (5.48) this becomes

\[
K^1_{n}(RR') = - \binom{2}{1} \binom{N}{1} \frac{8}{3} \frac{3}{3} \frac{3a}{4n} \frac{3}{2} \exp(-3a(R-R'))
\]

\[
\times \int d \eta \exp(-\frac{a}{3}(\eta-R+R'))^2 \frac{1}{N} \rho_p (\eta)(5.54)
\]

\[
\times \frac{\text{Tr}}{2} (\Lambda_{oo} (\sigma \sigma'_1) \Lambda_{oo} (\sigma'_1 \sigma))
\]

where \(2\xi_1 = \eta + 4(R-R')\) and \(2\xi'_1 = \eta - 4(R-R').\)
It can be seen from equation (5.54) that the factor \( \binom{N}{1} \) simply cancels and thus the one neutron exchange and one proton exchange kernels can be added where

\[
\rho_{PB}(rr') = \rho_{PB}^n(rr'_1) + \rho_{PB}^p(rr'_1)
\]  

(5.55)

Thus using equations (5.49), (5.43) and making the substitution

\[
h(r) = \frac{1}{4} \nabla^2 \rho(r) - \tau(r) + \frac{3}{5} k_f^2 \rho(r)
\]  

(5.56)

equation (5.54) becomes for the total one particle exchange kernel

\[
K^1_1(RR') = (-4)^3 \binom{3}{n}^{3/2} \exp(-3\alpha|\mathbf{R}-\mathbf{R}'|^2)
\]

\[
\times \int dy \exp(-\frac{\alpha}{3} (y-R+R'))^2)
\]

\[
\times \left[ \frac{3j_1(Sk_f)}{Sk_f} \rho(y) + \frac{35}{2Sk_f^3} j_3(Sk_f)h(y) \right]
\]  

(5.57)

where \( S = 4(R-R') \) and the Fermi momentum \( k_f \) is related to the nuclear density by (DE74)

\[
k_f(R+R') = \left( \frac{3\pi^2}{2} \right)^{1/3} \rho^{1/3} \left( \frac{R+R'}{2} \right)
\]  

(5.58)

5.3.5 Two Particle Exchange Kernel

In the neutron-proton formalism there are three separate two particle exchange mechanisms, ie. \( lplp, llin \) and \( lpin \). Considering,
for the present, only lpln exchange the kernel is given by equation (A.21) as
\[
K_{np}^{(2)}(RR') = 2^3 (\frac{2}{3})^2 (\frac{N}{1}) (\frac{Z}{1}) \int dy \, d\xi \, d\eta \, d\rho 
\times \rho_a (\xi_1, \eta_1, \xi_2, \eta_2 - x) \rho_b (n_1, \xi_1, n_2, \eta_2 + x) \tag{5.59}
\]
Substituting in equation (5.39) and recognising that no symmetry requirements need be introduced for the lpln density matrices in $^{208}_{\text{Pb}}$ this becomes
\[
K_{np}^{(2)}(RR') = 2^3 (\frac{2}{3})^2 (\frac{N}{1}) (\frac{Z}{1}) \int dy \, d\xi \, d\eta \, d\rho 
\times \rho_n (\xi_1, \eta_1, \xi_2, \eta_2) \rho_p (\xi_2, \eta_2) \times \frac{1}{N \times Z}
\]
\[
\frac{1}{4} \text{Tr} (A_{oo} (\sigma_{n1} \sigma_{n1}^I) A_{oo} (\sigma_{p1} \sigma_{p1}^I))
\]
\[
A_{oo} (\sigma_{n1} \sigma_{n1}^I) A_{oo} (\sigma_{p1} \sigma_{p1}^I)) \tag{5.60}
\]
where $\xi_1 \xi_1' \xi_2 \xi_2'$ are nucleon coordinates, at present.

Using equation (5.49) and the density matrix expansions for $\rho^n$ and $\rho^p$ this equation can be reduced to a more convenient form by changing the integration variables to $p$ and $q$ where
\[ \mathbf{p} = \mathbf{r}_1 + \mathbf{r}_1' \quad \mathbf{q} = \mathbf{r}_1 - \mathbf{r}_1' \]

e.i. \[ 2\mathbf{y} - \mathbf{p} = \mathbf{r}_2 + \mathbf{r}_2' \quad 2\mathbf{x} - \mathbf{q} = \mathbf{r}_2 - \mathbf{r}_2' \]  \hspace{1cm} (5.61)

In terms of these coordinates equation (5.60) becomes

\[
K^{(2)}_{np} (\mathbf{R}\mathbf{R}') = \left( \frac{1}{2} \right)^{3/2} \frac{3}{\pi} \exp \left( -\alpha |\mathbf{R}-\mathbf{R}'|^2 \right) \sum_{ij=1}^{2} G_{ij}(\mathbf{R}\mathbf{R}';\mathbf{x}) \\
\times \int dy \exp \left( -\alpha (y-R+R')^2 \right) H_{ij}^{np}(y) \]  \hspace{1cm} (5.62)

where in this new notation

\[
G_{ij}(\mathbf{R}\mathbf{R}';\mathbf{x}) = \int dq \exp \left( -\frac{\alpha}{2} (q-x)^2 \right) g_{ij}^{kf}(q) g_{ij}^{kf}(2x-q) \]  \hspace{1cm} (5.63)

\[
H_{ij}^{np}(y) = \int dp \exp \left( -\frac{\alpha}{2} (p-y)^2 \right) h_{ij}^{np}(p) h_{ij}^{np}(\frac{2y-p}{2}) \]  \hspace{1cm} (5.64)

\[
g_{ij}^{kf}(q) = 3 j_1(qk_f)/qk_f \]  \hspace{1cm} (5.65)

\[
g_{ij}^{kf}(q) = \frac{35}{2qk_f^3} j_3(qk_f) \]  \hspace{1cm} (5.66)

\[
h_{ij}(p) = \rho(p) \]  \hspace{1cm} (5.67)

\[
h_2(p) = \frac{1}{4} \nabla^2 \rho(p) - \tau(p) + \frac{3}{5} k_f^2 \rho(p) \]  \hspace{1cm} (5.68)

The \[ \mathbf{R}\mathbf{R}' \] dependence in \[ G_{ij}^{np} \] is due to the form of \[ k_f \] from equation (5.58). The remaining two proton and two neutron exchange kernels have a similar form to equation (5.60) but now, because of
the Slater determinant representation for $^{208}$Pb, the two particle exchange density matrix takes the form of equation (5.47). Recognising that the two protons in the alpha particle are coupled to zero spin the two proton normalisation kernel takes the form:

$$K_{pp}(RR') = 2^3 \binom{2}{2} \binom{2}{2} \int d\Sigma \cdot d\xi_1 \ d\eta_1 \ \left(\frac{3}{2}\right)^{3/2} \left(\frac{3}{n}\right)^3$$

$$\times \exp\left(-\alpha |R-R'|^2\right) \exp\left(-\frac{\alpha}{4} (\xi_1^2 + \eta_1^2) - \alpha (y-R+R')^2\right)$$

$$\times \left(\binom{2}{2}^{-1} \frac{1}{2} \text{Tr}\left[\chi_{p_1}^1 a_{p_2}^1 \chi_{p_2}^1 \right]_\infty \right)$$

$$\times \left[ \chi_{p_1}^1 \chi_{p_2}^1 \right]_\infty$$

$$\times \left[ \rho_p (r_1 r_1')^p (r_2 r_2')^p \right] \frac{\Lambda_{oo}(\sigma_1 \sigma_1')}{\sqrt{2}} \frac{\Lambda_{oo}(\sigma_2 \sigma_2')}{\sqrt{2}}$$

$$- \rho_p (r_1 r_1')^p (r_2 r_2')^p \frac{\Lambda_{oo}(\sigma_1 \sigma_1')}{\sqrt{2}} \frac{\Lambda_{oo}(\sigma_2 \sigma_2')}{\sqrt{2}}$$

$$\left(5.69\right)$$

Using equations (5.50), (5.51) and (5.49) this reduces to

$$K_{pp}(RR') = \left(\frac{1}{2}\right)^{3/2} \left(\frac{3}{n}\right)^3 \int d\Sigma \ d\xi_1 \ d\eta_1$$

$$\times \exp\left(-\frac{\alpha}{4} (\xi_1^2 + \eta_1^2) \right) \exp\left(-\alpha (y-R+R')^2\right)$$

$$\times \left[ \rho_p (r_1 r_1')^p (r_2 r_2')^p \right] \left(5.70\right)$$

If, as in equation (5.61), the substitution $r_1 + r_2 = p'$ and $r_2 - r_2' = q'$ is made it is easily deduced that the evaluation of the
second term in the square brackets is mathematically equivalent to
the first, and so the two terms may be added. The two neutron exchange
kernel is, of course, given by equation (5.70) with the appropriate
neutron density functions. Exactly the same techniques can be used
for equation (5.70) as in the 1pln exchange; hence the total two
particle kernel is given by

$$K^{(2)}(RR') = \frac{1}{\sqrt{8}} \left( \frac{5}{n} \right)^3 \exp(-\alpha|R-R'|^2) \sum_{ij=1}^2 C_{ij}(R+R';X)$$

$$\times \int dy \exp(-\alpha(y-R+R')^2)$$

$$\times \left[ H_{ij}^{nn}(y) + H_{ij}^{pp}(y) + H_{ij}^{np}(y) \right]$$

(5.71)

5.3.6 Three Particle Exchange Kernel

For the $\alpha + ^{208}$Pb system there are only two possible three particle
exchange mechanisms. There are the (1plp) + 1n and (1nln) + lp exchanges.
The general three particle exchange normalism kernel takes the form

$$K^{(3)}_{nnp}(RR') = \left( \frac{2}{2} \right) \left( \frac{2}{1} \right) \left( \frac{N}{Z} \right) \left( \frac{Z}{3} \right)^3 \int dy \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2$$

$$\times \rho_\alpha(\xi_1 \eta_1; \xi_2 \eta_2; \xi_3 \eta_3 - X)$$

$$\times \rho_\beta(\eta_1 \xi_1; \eta_2 \xi_2; \eta_3 \xi_3 + X)$$

(5.72)

for the (1lnln) + lp exchange component, say.

This equation can be reduced using the same techniques as in
Section (5.3.5) except that in this case \( \rho_\beta \) is given by

\[
\begin{align*}
\nonumber \left[ \rho_n (\xi_1 \xi_1') \rho_n (\xi_2 \xi_2') \right] &= \Lambda_{oo} (\sigma_1 \sigma_1') \Lambda_{oo} (\sigma_2 \sigma_2') \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} - \\
\nonumber \rho_n (\xi_1 \xi_1') \rho_n (\xi_2 \xi_1) \Lambda_{oo} (\sigma_1 \sigma_2') \Lambda_{oo} (\sigma_2 \sigma_1') \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \]  \\
\nonumber \times \rho_p (\xi_3 \xi_3') \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \Lambda_{oo} (\sigma_3 \sigma_3') \quad (5.73) \\
\end{align*}
\]

and the spin component of the alpha particle density matrix is given by

\[
\rho^{(3)}_\alpha (\text{spin}) = [x^{1/2} (\sigma_1) x^{1/2} (\sigma_2)]_{oo} [x^{1/2} (\sigma_1') x^{1/2} (\sigma_2')]_{oo} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \Lambda_{oo} (\sigma_3 \sigma_3') \quad (5.74)
\]

Hence taking the trace over the spin variables equation (5.72) becomes

\[
K^{(3)}_{nnp} (RR') = \left( \frac{2}{3} \right)^3 \left( \frac{a}{n} \right)^{4/2} \int d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
\times \exp \left( -3a(\xi - R + R')^2 \right) \exp \left( - \frac{a}{4} (\xi_1^2 + \eta_1^2) \right) \\
\nonumber - \frac{a}{3} (\xi_2^2 + \eta_2^2) - \frac{a}{3} (R - R')^2 \rho_p (\xi_3 \xi_3') \\
\nonumber \times \left[ \rho_n (\xi_1 \xi_1') \rho_n (\xi_2 \xi_2') \rho_n (\xi_1 \xi_3') \rho_n (\xi_2 \xi_3') \right] \quad (5.75)
\]

After using the density matrix expansion, equation (5.43), and relating the nucleon coordinates \( r_1 - r_1' \), \( r_1 + r_1' \) etc. to the integration variables this equation takes on a seemingly impossible form. However
by making the following change of integration variables it was found possible to reduce equation (5.75) to terms defined in the two particle exchange analysis; i.e. equations (5.63) and (5.64). Using this technique results generated in the course of the two particle exchange analysis can be used in the calculation of higher order exchange processes and the number of integrations required is considerably reduced. Thus making the substitutions

$$p_n = r_n + r'_n \quad q_n = r_n - r'_n \quad n = 1,2,3 \quad (5.76)$$

and redefining the appropriate terms in the alpha particle density matrix the three particle exchange normalisation kernel can be written as

$$K^{(3)}(RR') = \left(\frac{2}{3}\right)^3 \left(\frac{a}{\pi}\right)^{9/2} \exp\left(-\frac{a}{3} |R-R'|^2\right) \times \int d\gamma \exp\left(-3a(\gamma-R+R')^2\right) \sum_{ijk} \int dp_3$$

$$\times \exp\left(-\frac{3}{8} \alpha (p_3-\gamma)^2\right) \{H_{iij}^{nn}(3\gamma-p_3)h^p_k(p_3)$$

$$+ H_{ij}^{pp}(3\gamma-p_3)h^n_k(p_3)\} \times \int dq_3 \exp\left(-\frac{3}{8} \alpha (q_3-\gamma)^2\right) G_{ij}(R+R';3\gamma-q_3) g_k(q_3)$$

where all the terms are defined in equations (5.63) to (5.68). In this expression the $(lplp) + 1n$ component has been added to equation (5.75) and simply contains the appropriate density functions.
Equation (5.77) involves integration over three vector coordinates only. This is because $H_{ij}$ and $G_{ij}$ themselves have been reduced to integrations over $p_1$ and $q_1$. As stated before these functions have already been calculated in the two particle exchange analysis though $H_{ij}$ and $G_{ij}$ are now defined for a different argument from equations (5.63) and (5.64).

### 5.3.7 Four Particle Exchange Kernel

From equation (A.21) the four particle exchange normalisation kernel is given as

$$K^{(4)}(RR') = \binom{2}{2} \binom{2}{2} \binom{N}{2} \int d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3$$

$$\times \rho_\alpha(\xi_1;\xi_2;\xi_3;\eta_1;\eta_2;\eta_3) \rho_\beta(\eta_1;\eta_2;\eta_3)$$

where for four particle exchange there is, of course, only one possible exchange component. In this expression $\rho_\beta$ is simply given by a product of terms identical to those used in equation (5.69) while the spatial part of $\rho_\alpha$ is given by equation (5.41). The spin component of $\rho_\alpha$ is represented by a product of the singlet wave functions with appropriate spin-space coordinates, ie.

$$[x^{1/2}(\sigma_{p_1})x^{1/2}(\sigma_{p_2})][x^{1/2}(\sigma_{n_1})x^{1/2}(\sigma_{n_2})]_{oo}$$

$$\times \frac{1}{16} [x^{1/2}(\sigma_{p_1})x^{1/2}(\sigma_{p_2})]_{oo}$$

and taking the trace over the spin space coordinates simply introduces a factor of $1/16$ into the general expression.
Once again the number of integrations to be carried out can be considerably reduced by using the same technique as in Section (5.3.6) and reducing the four particle exchange kernel to terms involving functions calculated in the two particle exchange analysis. Hence reducing the product form of $\rho_p$ by equation (5.43) and integrating over $\xi_1 \xi_3$ and $n_1 n_3$ equation (5.78) becomes

$$K^{(4)}(R,R') = \frac{1}{16} \left( \frac{a}{\pi} \right)^3 \left( \frac{a}{\pi} \right)^{9/2} \int d\xi_2 d\eta_2 \exp\left( -\frac{a}{2} (\xi_2^2 + \eta_2^2) \right)$$

$$\sum_{i,j,k,l} H_{ij}^{pp}(R+R', 2X - \frac{\xi_1 + \xi_3}{2}) G_{kl}(R+R', 2X - \frac{\xi_1 + \xi_3}{2})$$

$$H_{ij}^{pp}(2R+R' - \frac{\xi_1 - \xi_3}{2}) G_{kl}(2R+R' - \frac{\xi_1 - \xi_3}{2})$$

$$\text{and}$$

$$H_{ij}^{nn}(2R+R' - \frac{\xi_1 - \xi_3}{2}) H_{kl}^{pp}(2R+R' - \frac{\xi_1 - \xi_3}{2})$$

where $\xi_1 \ldots \xi_4$ and $\xi'_1 \ldots \xi'_4$ are nucleon coordinates and are related to $\xi_2$ and $\eta_2$ through equations (A.2) and (A.3). Making the substitution $2p = \xi_2 + \eta_2$ and $Q = \xi_2 - \eta_2$ equation (5.80) reduces to

$$K^{(4)}(R,R') = \frac{1}{16} \left( \frac{a}{\pi} \right)^3 \left( \frac{a}{\pi} \right)^{9/2} \int dP dQ \exp\left( -\frac{a}{4} Q^2 - \alpha P^2 \right)$$

$$\sum_{i,j,k,l} H_{ij}^{nn}(R+R', 2X - \frac{\xi_1 + \xi_3}{2}) H_{kl}^{pp}(R+R', 2X - \frac{\xi_1 + \xi_3}{2}) G_{ij}(R+R', R-R' - \frac{\xi_1 - \xi_3}{2}) G_{kl}(R+R', R-R' + \frac{\xi_1 - \xi_3}{2})$$

$$\text{(5.81)}$$

5.4 Numerical Technique

It was shown in the formalism of the last section that the initially complicated normalisation kernel can be reduced to terms based on equations (5.63) and (5.64). Indeed the three and four particle exchange kernels are expressed in terms of integrals of a
similar form to these equations. From a computational point of view this means only one general code is needed. Equation (5.63) or (5.64) was reduced to a more convenient form for computing by using the general identity (BR68)

\[
e^{-\gamma(a-b)^2} = \sum_{\ell m} 4\pi i^{-\ell} e^{-\gamma(\ell^2 + b^2)} j_{\ell}(2i\gamma ab) Y^m_\ell(\Omega_\ell) Y^m_\ell(\Omega_b) \quad (5.82)
\]

and a Lagrange interpolation routine was introduced so that \( G_{ij} \) and \( H_{ij} \) can be calculated at different values of their argument. The results of the previous section give the various components of the normalisation kernel as functions of \(|R+R'|\) and \(|R-R'|\). To reduce this quantity to the scalar \( K'_o(RR') \) one further integration is needed over the angle \( \hat{R} \cdot \hat{R}' \), for all values of \( R \) and \( R' \) of interest.

In order to study the eigenvectors and eigenvalues of \( K(RR') \) it was required to solve the equation:

\[
\int K(RR') e^{\theta(R')} dR' = \lambda \theta e^{\theta(R)} \quad (5.83)
\]

This was achieved by rewriting equation (5.83), in matrix form, as \( KWg = \lambda g \), where \( W \) is a Simpsons rule weighting factor. This matrix equation was then solved using well established computer library codes.

In order to solve equation (5.29) a code developed by Pong (P077) for \( \alpha-\alpha \) scattering was modified for the numerical kernels calculated in the \( \alpha-^{208}\text{Pb} \) case and the special form of the bound state potential in this situation. A discussion of the numerical method used to solve this equation is given in Appendix C.
5.5 Results and Discussion

In Figure 20 the diagonal elements of the normalisation kernel components are plotted as functions of the centre of mass distance \( R \). In order to compare the solution of equation (5.29) with the wavefunction of the \( \alpha \)-particle bound in a relative s-state used in the alpha decay and alpha transfer reactions only the component \( K_\ell (RR') \), where \( \ell = 0 \), was calculated. It can immediately be seen that the one particle exchange kernel has the largest magnitude and longest range and that the relative importance of the higher order exchange kernels becomes progressively less important. This is due to two effects; the number of different ways each exchange process can occur falls off rapidly with each higher order exchange component and because of the formal relation between the two, three and four particle exchanges the higher order exchange kernels depend on ever increasing powers of the local density matrices. In these calculations it was found that the surface correction term in equation (5.43) was very small when applied to two particle exchange and thus was neglected for higher order exchange components. This reduces the number of summations in equations (5.77) and (5.81) considerably.

In Figure 21 the results of the diagonalisation process, equation (5.83), are given. The eigenvalues \( 1 - \lambda_{no} \) are plotted as a function of the principal quantum number, or number of numerically generated nodes in the eigenvectors \( |g^\lambda> \). It can be seen from this Figure that there is a relatively smooth behaviour in the eigenvalues with values between 0 and 1 which correspond to semi-redundant states. In particular the \( N = 12-15 \) states, the form of the relative motion
functions used in the $0^+ \rightarrow 0^+$ transitions, lie within this semi-redundant region. Thus some effect on these states, due to exchange effects, can be expected.

In figures 22 and 23 the normalised bound state solution of equation (5.29) is compared with the equivalent wave function used in previous alpha decay and alpha transfer reactions. This solution was calculated using the nuclear potential $H$ outlined in Chapter 2. It can be seen that the effect of the non-locality is to reduce the amplitude of the wave function at small $R$ and because of the normalisation condition the amplitude in the surface region is consequently increased. This is consistent with previous calculations (AN75, FR74).

In the analysis of the $0^+ \rightarrow 0^+$ transitions of Polonium isotopes in Chapter 3 the contributions to the matrix elements for values of $R$ less than the Coulomb barrier radius are zero. Assuming that any modification to the decay width will come from the difference in tail of the bound state wave functions the $^{212}\text{Po}$ decay width is increased by a factor of 6.5. This approximation assumes that the new non-local terms arising in the perturbing interaction of the decay width matrix element can be neglected. A similar conclusion can be drawn for the alpha transfer reaction because the major contributions to the matrix element occur for values of $R$ corresponding to the tail of the bound state wave function. Also, the reduction of the wave function for small separation distances implies that for the more complicated situation of alpha decay to excited states of the daughter nucleus, in which all $R$ values are used, exchange effects will modify the widths and appropriate cluster expansion coefficients.
Assuming that exchange effects will have an analogous effect on the relative motion wave-function used in R-matrix theory, Figure (22) suggests that for the traditional values of the channel radius of \( R_c \geq 9.0 \text{ fm} \) there will be a slight modification to the width. Although for values of \( R \) less than this the effect will be more significant it is difficult to see how Fliessbach (FL74, FL76) arrived at his modification factor of 100. The relatively small change in the R-matrix value of the width for \( R_c = 9.0 \text{ fm} \) gives further evidence that the main problem in earlier descriptions of alpha decay was the shell model structure of the initial \( A+4 \) system.

An interesting extension of the normalisation kernel analysis is given in Appendix B in which the interaction kernel \( G(\mathbf{R} \mathbf{R}') \), defined in equation (5.10) is analysed. Figure (25) is a schematic representation of the various components of the interaction kernel. The solid line represents a nucleon in the alpha particle and similarly the dotted line represents a nucleon in \(^{208}\text{Pb}\). The interaction between two nucleons is represented by the vertical line and the exchange of nucleons is represented by the appropriate lines crossing. For each of these diagrams there are, of course, four possible exchange processes making sixteen components of \( G(\mathbf{R} \mathbf{R}') \). An examination of previous work on the non-local interaction between alpha particles and \(^{208}\text{Pb}\) (EI76, HA77) shows that they have essentially calculated the one particle exchange component to type II only. Although it is not immediately obvious whether types I and III can be completely neglected the results from the calculations of the normalisation kernel, shown in Figure 20, indicate that higher order exchange processes, above the one particle exchange, can be neglected.
For one particle exchange only the form of the interaction kernel corresponding to type II reduces to

\[ V_{nn}(\mathbf{R}'-\mathbf{R})\rho_{\alpha}(\xi_1;\xi_1-\mathbf{R}'-\mathbf{R})\rho_{\beta}(\eta_1;\eta_1+\mathbf{R}'-\mathbf{R}) \]  \hspace{1cm} (5.84)

where equations (B-6), (B-9) and (A-21) have been used.

Thus for a given nucleon nucleon interaction this type of interaction has essentially already been calculated although, of course, it is still in a non-local form and it will be necessary to incorporate the spin dependent terms of the nucleon-nucleon interaction.
FIGURE CAPTIONS (Chapter 5)

Figure 20    Comparison of the diagonal elements of the normalisation kernel \( K_{\nu}(RR') \) for each of the individual exchange components. The total diagonal element of the full kernel is also shown (dashed line).

Figure 21    Graph of \( 1 - \lambda_{\text{NO}} \) against principal quantum number for the eigenvalues and eigenvectors defined in equation (5.83). The semi-redundent states correspond to those values of \( 1 - \lambda_{\text{NO}} \) between 0 and 1.

Figure 22    Comparison of bound state wavefunction used in analysis of Chapter 3 with the solution of the modified Resonating Group Equation (5.29).

Figure 23    Comparison of wavefunctions shown in Figure 22 beyond the nuclear surface region.
FIGURE 20

\[ |K_\nu (RR)| \]
(Arbitrary Units)

- \( |K_{RR}| \)
- \( |K_{RR}| \) (dashed line)

1 Particle Exchange
2 Particle Exchange
3 Particle Exchange
4 Particle Exchange

R (fm)
FIGURE 22

- U(R) (Arbitrary Units)
- Local Potential Solution
- Modified Resonating Group Solution

R (fm)
FIGURE 23

- Local Potential Solution
- Modified Resonating Group Solution
CHAPTER 6

SUMMARY AND CONCLUSIONS

In this Chapter the main results and conclusions presented at the end of previous chapters are drawn together to give overall conclusions together with suggestions for possible extensions of the work presented.

The work of Chapter 2 on $\alpha-^{208}_{\text{Pb}}$ scattering at an energy just above the Coulomb barrier has shown that it is possible to fit the available experimental data by using a simple complex optical potential where the real part of the alpha nucleus interaction is constructed using a single folding model. By extrapolating these potentials to lower energies and assuming that the imaginary potential is negligible at these energies the alpha decay widths, in the simple W.K.B. model, were shown to be different for each microscopic potential because of the different widths of the potential barriers.

In Chapter 3 a formalism for the alpha-decay width was developed in which the initial $A+4$ system was represented as an expansion over alpha-particle and nuclear states. Taking the lowest order term in this expansion and using this formalism for the $0^+ \rightarrow 0^+$ transitions of Polonium it was found that to a large extent the problems associated with R-matrix theory of alpha-decay had been eliminated and the results yielded spectroscopic factors about 100 times larger than the traditional shell model values. Indeed there is only one adjustable parameter namely the scaling factor $g$ of the potential inside the
Coulomb barrier radius. Table IX shows that the results for these transitions are not very sensitive to the scaling factor, or depth of the potential well, but as shown in Table VII and VIII the alpha-decay width, in this model, is sensitive to the height of the Coulomb barrier, an experimentally determined parameter. By approximating the expression for the ground state decay width it was shown that it is possible to derive an equation which can relate the half lives and decay energies of even Polonium isotopes, a range spanning fourteen orders of magnitude. The analysis of the decay of $^{212\text{m}}$Po indicated that the emphasis of theoretical investigation should not be the excitation of the residual nucleus $^{208}\text{Pb}$ but rather the cluster structure of the parent nucleus. Using the normalisation relation, equation (3.86), it was shown that the ground state spectroscopic factor devised in this analysis was within a factor of two of that deduced for the $0^+ \rightarrow 0^+$ transition, a good agreement considering the approximations made in the analysis of $^{212\text{m}}$Po decay.

A natural extension of this work is the study of alpha-decay of strongly deformed nuclei. Apart from the structure considerations it would now no longer be reasonable to consider an isotropic transition matrix. It now seems possible, using the alpha-decay code, to make a reasonable prediction of the alpha-decay half lives of superheavy nuclei, especially as the shell structure (M076) and decay energies (FI72, BU76) of these nuclei have been reported. Indeed because of the success of the relation between alpha particle energies and decay lifetimes, equation (3.90), it doesn't seem necessary to repeat the calculation for every possible superheavy system especially if values for the predicted Coulomb barriers are
The overall implication of the results of Chapter 3 suggest that it is more accurate to represent the initial decaying state as an alpha particle in orbit around $^{208}$Pb then to describe the initial $A+4$ system by an independent shell model from which an alpha cluster evolves.

Extending the alpha decay work to the alpha transfer reaction $^{208}$Pb$(O^{16},C^{12})P^{212}$ it was found that this reaction gave a similar spectroscopic factor to the $0^+ \rightarrow 0^+$ transitions when potential Set V was used. It was also found possible to get good shape agreement with the data using a zero range approximation. Although the magnitude of the $\alpha^{208}$Pb spectroscopic factor suggest that other nuclear states may be important, the results imply it is reasonable to assume a cluster model for $^{212}$Po and $^{16}O$ and that a real $\alpha$-particle in its ground state is transferred in the reaction. It would be interesting if the analysis could be repeated by modifying a standard finite range code or alternatively changing the form of the coordinates of interest in the $T$-matrix and expressing the $\alpha-C^{12}$ bound state wavefunction as an expression of Hankel functions (SA73). In this way the calculation retains the structure of the D.W.B.A. zero range code. Also another interesting alpha transfer reaction to study the $\alpha^{208}$Pb spectroscopic factor is the $^{208}$Pb($Li^{6})P^{212}$ reaction.

In Chapter 5 it was shown that the one particle exchange component in the $\alpha^{208}$Pb system is the dominant exchange mode. This is a promising result for the analysis of other many particle systems for
this component is the simplest both from a mathematical and numerical point of view, especially when a local density matrix approximation is used. It was shown by using a modified R.G.M. equation that exchange effects in alpha decay and alpha transfer reactions are not as important as Fliessbach's work suggests, mainly because of the importance of large radii in these reactions. Any process sensitive to small radii, however, should include effects from exchange processes. However it seems from the form of this wavefunction that it may be possible to simulate exchange effects by the introduction of a phenomenological hard core potential. If in some interaction of two heavy systems exchange effects may be considered to be important the analysis of Chapter 5 shows that the one particle exchange component will be dominant and that the formalism may be easily extended to include two local density approximations for each shell model system. An interesting extension of this work would be to analyse the solution of the modified R.G.M. equation using a separable form for the normalisation kernel and neglecting the semiredundant states. This then could be compared with the results presented here and perhaps a full R.G.M. calculation. By investigating the connection between the non-local interaction kernel and the normalisation kernel it seems that it is now possible to study the effect of the non-locality on α-Pb scattering, or, because of the relatively simple form of the one particle exchange component, to derive a local energy dependent equivalent for this potential.
APPENDIX A

VECTOR ANALYSIS OF n PARTICLE EXCHANGE NORMALISATION KERNEL

The normalisation Kernel $K(RR')$ for a general $n$ particle exchange process is given by

$$<\delta(S-R')\phi_\alpha(1..A)\phi_\beta(A+1..A+B)|\Pi_1,A+1\cdot\Pi_{n,A+n}|$$

$$\phi_\alpha(1..A)\phi_\beta(A+1..A+B)\delta(S-R)> \quad (A1)$$

where $\phi_\alpha$, $\phi_\beta$ are antisymmetrised nuclear wave functions and $\Pi_{i,A+i}$ is the permutation operator that exchanges nucleon $i$ with nucleon $A+i$. In the following analysis due to Pong (P076) the coordinates to be used are expressed in terms of the nucleon coordinates $\xi_i$, ie.

Inside Nucleus $\alpha$

$$R_A = \sum_{i=1}^A \frac{r_i}{A}$$

$$\xi_1 = \xi_{n-1} - \xi_n$$

$$\xi_K = \xi_{n-K} - \sum_{i=0}^{K-1} \frac{r_{n-1}}{K}$$

$$\xi_n = \frac{1}{n} \sum_{i=1}^n \xi_i - \frac{1}{(A-n)} \sum_{i=n+1}^A \xi_i$$

$$\xi_{n+1} = \xi_{n+1} - \sum_{i=n+2}^{A} \frac{\xi_i}{(A-n-1)}$$

and similarly Inside Nucleus $\beta$
A vector diagram of this system is given in Figure 24, where \( \xi_1 \ldots \xi_{n-1} \) and \( n_1 \ldots n_{n-1} \) are the coordinates of the exchanged particles. In terms of these coordinates (A1) becomes

\[
\int dS \prod_{i=1}^{A-1} d\xi_i \prod_{j=1}^{B-1} dn_j \phi_\alpha(\xi_1 \ldots \xi_{A-1})^* \phi_\beta(n_1 \ldots n_{B-1})
\]

\[
\times \phi_\alpha(\xi_1 \ldots \xi_n; \xi_{n+1} \ldots \xi_{A-1}) \times \phi_\beta(n_1 \ldots n_n; n_{n+1} \ldots n_{B-1})
\]

(A4)

where the dashed coordinates represent exchanged particles, i.e.

\[
\xi_i = n_i \quad i = 1, n - 1
\]

(A5)

\[
n_i = \xi_i \quad i = 1, n - 1
\]

(A6)

\[
\xi_n = \xi_n - \frac{1}{n} \left( \sum_{i=1}^{n} \xi_i - \sum_{i=1}^{A} \xi_i \right) = \xi_n \chi
\]

(A7)
\[ n' = n + \frac{1}{n} \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_{A+i} \right) = n + \frac{X}{n} \quad (A8) \]

and
\[ S = \frac{A}{n} \sum_{i=1}^{A} \frac{x_i}{A} - \frac{B}{n} \sum_{i=A+1}^{A+n} \frac{x_i}{B} \quad (A9) \]

The Jacobian for the transformation \( x_1 \cdots x_A \rightarrow R_{A-n} \cdots R_{n-1} \) is unity. Equation (A4) can be reduced by relating the relative centre of mass vectors before and after exchange, i.e.
\[ S' = S - \left( \frac{1}{A} + \frac{1}{B} \right) \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_{A+i} \right) \quad (A10) \]
\[ = S - \left( \frac{1}{A} + \frac{1}{B} \right) nX \quad (A11) \]

Thus on integrating over \( S \) and using the following definition of the density matrix
\[
\int d\xi_{n+1} \cdots d\xi_{A-1} \phi^* (\xi_1 \cdots \xi_n; \xi_{n+1} \cdots \xi_{A-1}) \phi^* (\xi_1' \cdots \xi_n'; \xi_{n+1} \cdots \xi_{A-1}) \\
= \rho_\alpha (\xi_1 \cdots \xi_n; \xi_{n+1} \cdots \xi_{A-1}) \\
\rho_\beta (\xi_1' \cdots \xi_n'; \xi_{n+1} \cdots \xi_{A-1}) \quad (A12)
\]
equation (A4) becomes
\[
\int \prod_{i=1}^{n-1} d\xi_i \prod_{j=1}^{n-1} d\eta_j \rho_\alpha (\xi_1 \cdots \xi_{n-1} \eta_{n-1} \cdots \eta_n; \xi_n \eta_n - X) \\
\rho_\beta (\eta_1 \xi_1 \cdots \eta_{n-1} \xi_{n-1}; \eta_n \eta_n + X) \times \delta^{('R-R-(1/\text{A}+1/\text{B})nX)} \quad (A13)
\]
Choosing
\[ X = \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \sum_{i=1}^{n} x_{A+i} \quad (A14) \]
Equation (A13) can be further reduced by relating $\xi_n$ and $\eta_n$ to $X$ and $Y$. From equations (A2) and (A3)

\begin{align*}
  A & \frac{R_A}{A} + (A-n)\xi_n = \frac{A}{n} \sum_{i=1}^{A} \xi_i \tag{A16} \\
  B & \frac{R_B}{B} + (B-n)\eta_n = \frac{B}{n} \sum_{i=1}^{B} \xi_{A+i} \tag{A17}
\end{align*}

On adding and subtracting (A16) and (A17) it can be readily deduced that

\begin{align*}
  X &= R' + \frac{(A-n)}{A} \xi_n - \frac{(B-n)}{B} \eta_n \tag{A18} \\
  Y &= \frac{(B-A)}{(A+B)} R' + \frac{(A-n)}{A} \xi_n - \frac{(B-n)}{B} \eta_n \tag{A19}
\end{align*}

Thus on integrating over $X$, where the Jacobian of the transform is given by

\begin{equation}
  \frac{\partial(x, y)}{\partial(x, y)} = \left( \frac{AB}{2(A-n)(B-n)} \right)^2 \tag{A20}
\end{equation}

equation (A13) becomes

\begin{equation}
  \left[ \frac{A^2 B^2}{2n(A-n)(B-n)(A+B)} \right]^2 \int dy \int_{i=1}^{n-1} \frac{d\xi_i d\eta_i}{n} \\
  \times \rho_\alpha(\xi_1, \eta_1, \ldots, \xi_n; \eta_n, X) \times \rho_\beta(\xi_1, \eta_1, \ldots, \eta_n, n + X) \tag{A21}
\end{equation}

where from equations (A18), (A19) and the relation $X = (R' - R) \frac{AB}{n(A+B)}$
\[ \xi_n = \frac{A}{2(A-n)} \left[ \chi + \frac{B}{(A+B)} \left( \frac{A}{n} - 2 \right) R' - \frac{A}{n} R \right] \]  \hspace{1cm} (A22)

\[ \eta_n = \frac{B}{2(B-n)} \left[ \chi + \frac{A}{(A+B)} \left( \frac{B}{n} R - \left( \frac{B}{n} - 2 \right) R' \right) \right] \]  \hspace{1cm} (A23)

\[ \xi_n^{-X} = \frac{A}{2(A-n)} \left[ \chi - \frac{B}{(A+B)} \left( \frac{A}{n} R' - \left( \frac{A}{n} - 2 \right) R \right) \right] \]  \hspace{1cm} (A24)

\[ \eta_n^{-X} = \frac{B}{2(B-n)} \left[ \chi + \frac{A}{(A+B)} \left( \frac{B}{n} R' - \left( \frac{B}{n} - 2 \right) R \right) \right] \]  \hspace{1cm} (A25)

In the limit \( B \to \infty \), (A22) - (A25) become

\[ \xi_n = \frac{A}{2(A-n)} \left[ \chi + \left( \frac{A}{n} - 2 \right) R' - \frac{A}{n} R \right] \]  \hspace{1cm} (A26)

\[ \eta_n = \frac{1}{2} \left[ \chi + \frac{A}{n} (R-R') \right] \]  \hspace{1cm} (A27)

\[ \xi_n^{-X} = \frac{B}{2(A-n)} \left[ \chi - \left( \frac{A}{n} R' - \left( \frac{A}{n} - 2 \right) R \right) \right] \]  \hspace{1cm} (A28)

\[ \eta_n^{-X} = \frac{1}{2} \left[ \chi + \frac{A}{n} (R'-R) \right] \]  \hspace{1cm} (A29)
VECTOR DIAGRAM FOR n PARTICLE EXCHANGE

Exchanged Particles

Origin

Nucleus A

Nucleus B

FIGURE 24
APPENDIX B

THE n PARTICLE EXCHANGE INTERACTION KERNEL

Recognising that the interaction $V_{\alpha A}$ represents a summation over various nucleon-nucleon interactions, i.e. $V_{i, A+j}$ the full interaction kernel $G(RR')$ can be written for a general n particle exchange process as

$$G(RR') = \langle \delta(S-R) \phi_\alpha (1..A) \phi_\beta (A+1..A+B) \rangle$$

$$\times \sum_{i,j} V_{i, A+j} \hat{A} | \phi_\alpha (1..A) \phi_\beta (A+1..A+B) \delta(S-R') > \quad (B-1)$$

where $\hat{A}$ is the antisymmetrisation operator defined by equation (5.3).

In Figure 25 the various types of interaction kernels are represented in the graphical form of Kamimura and Matsuse (KA74). In this figure the overlapping lines signify exchanged particles. The different types can be classified as follows

**Type 1:** The particles with labels $i$ and $A+j$ are not exchanged, denoted $G_{I}(RR')$.

**Type 2:** The particles with labels $i$ and $A+j$ are both exchanged, denoted $G_{II}(RR')$.

**Type 3:** Only one of the particles labelled $i$ or $A+j$ are exchanged, denoted $G_{III}(RR')$.

Hence equation (B1) can be expressed for the exchange terms as
The structure of the interaction kernel is analogous to the normalisation kernel outlined in Appendix A and the same vector technique and definition of the density matrices can be applied. The vector coordinates of $V_{ij}$ can be deduced immediately from Figure 24 and equations (A22)-(A25) for the various types of exchange mechanisms outlined above. Using equation (5.3) and the results of Appendix A equation (B1) can be readily evaluated for the different types of interaction kernel, ie.

Type 1

$$G_{I}(RR') = \sum_{n=1}^{4} (-1)^{n} c_{n}^{I} \int \prod_{i=1}^{n} d^{n} \rho_{\alpha} \left(\xi_{1}^{n-1}, \cdots, \xi_{n-1}^{n}, \xi_{n}^{n+1} \right) \rho_{\beta} \left(\eta_{1}^{n-1}, \cdots, \eta_{n}^{n}, \eta_{n+1}^{n+1} \right) \times \prod_{i=1}^{n} d^{n} \xi_{i}^{n+1} \xi_{i+1}^{n+1} \rho_{\gamma} \left(\gamma_{1}^{n}, \cdots, \gamma_{n}^{n+1}, \gamma_{n+1}^{n+1} \right)$$

\[ (B-3) \]

where $c_{n}^{I}$ contains the numerical factors of equation (5.3) and (A.21) as well as the appropriate weighting factors for this particular case, ie.

$$c_{n}^{I} = \begin{vmatrix} A \end{vmatrix} \begin{vmatrix} B \end{vmatrix} \begin{vmatrix} A-n \end{vmatrix} \begin{vmatrix} B-n \end{vmatrix} \frac{A^2 B^2}{2n(A-n)(B-n)(A+n)}$$

\[ (B-4) \]

From Figure 24 $Z_{I}$ is given by

$$Z_{I} = \frac{(A-n-1)}{(A-n)} \xi_{n+1}^{n} - \frac{(B-n-1)}{(B-n)} \eta_{n+1}^{n} + \eta_{n}^{n} - \xi_{n}^{n} + \xi$$

\[ (B-5) \]
Type 2

\[ G_{II}(RR') = \sum_{n=1}^{4} (-1)^n C_{n}^{II} \int_{i=1}^{n-1} dy \pi d\eta_{i} d\xi_{i} V(Z_{II}) \]

\[ \times \rho_{\alpha}(\xi_{n} n_{n} n_{n} n_{n} \xi_{n} n_{n} \xi_{n} n_{n}) \rho_{\beta}(\eta_{n} n_{n} n_{n} n_{n} + X) \]

(B-6)

where in an analogous way to the analysis of type 1.

\[ C_{n}^{II} = \binom{A}{n} \binom{B}{n} n^2 \left[ \frac{A^2 B^2}{2n(A-n)(B-n)(A+B)} \right]^3 \]

and

\[ Z_{II} = \frac{(n-1)}{n} \xi_{n-1} - \frac{(n-1)}{n} \eta_{n-1} + X \]

\[ = \frac{(n-1)}{n} \xi_{n-1} - \frac{(n-1)}{n} \eta_{n-1} + \frac{AB}{(A+B)n} R' - R \]  

(B-9)

Type 3

\[ G_{III}(RR') = \sum_{n=1}^{4} (-1)^n C_{n}^{III} \int_{i=1}^{n-1} dy \pi d\xi_{i} d\eta_{i} d\xi_{n+1} V(Z_{III}) \]

\[ \times \rho_{\alpha}(\xi_{1} n_{1} \cdots \xi_{n} n_{n} \xi_{n} n_{n} \xi_{n} n_{n} \xi_{n} n_{n} \xi_{n} n_{n} + X) \]

\[ + C_{n}^{III} \int_{i=1}^{n-1} dy \pi d\xi_{i} d\eta_{i} d\xi_{n+1} V(Z_{III}) \rho_{\alpha}(\xi_{1} n_{1} \cdots \xi_{n} n_{n} \xi_{n} n_{n} \xi_{n} n_{n} + X) \]

\[ \times \rho_{\beta}(\eta_{n} n_{n} n_{n} n_{n} + X) \]

(B-10)

For this particular component

\[ C_{n}^{III} = \binom{A}{n} \binom{B}{n} \left[ \frac{A^2 B^2}{2n(A-n)(B-n)(A+B)} \right]^3 n(B-n) \]  

(B-11)
\[ C_{n2}^{III} = \binom{A}{n} \binom{B}{n} \left[ \frac{A^2B^2}{2n(A-n)(B-n)(A+B)} \right]^3 n(A-n) \]  \hspace{1cm} (B-12)

and

\[ Z_{III}^{1} = \frac{(n-1)}{n} \frac{\xi_{n-1}}{\xi} - \frac{(B-n-1)}{(B-n)} \frac{\eta_{n+1}}{\xi} + \eta_{n} + \xi \]  \hspace{1cm} (B-13)

\[ Z_{III}^{2} = \frac{(n-1)}{n} \frac{\eta_{n-1}}{\xi} - \frac{(A-n-1)}{(A-n)} \frac{\xi_{n+1}}{\xi} + \xi_{n} \]  \hspace{1cm} (B-14)
DIAGRAMS OF PARTICLE EXCHANGES CORRESPONDING TO VARIOUS PARTS OF $G(R, R')$

Direct Interaction

$V(R) \delta(R-R')$

Type 1

$G_I (R, R')$

Type 2

$G_{II} (R, R')$

Type 3

$G_{III} (R, R')$

FIGURE 25
APPENDIX C

THE NUMERICAL SOLUTION OF THE MODIFIED RESONATING GROUP EQUATION

The radial form of the modified resonating group equation is given, from equation (5.29) as:

\[
\left(\frac{d^2}{dR^2} + V_T(R) + k^2\right)\phi(R) = \int_0^\infty K'_o(R') \left(\frac{d^2}{dR'^2} + V_T(R')\right)\phi(R')dR' \quad (C-1)
\]

where as defined in Chapter 5, \( V_T(R) = 2\mu/h^2(\tau_c(R) + \mu_{N}(R)) \), \( K'_o(R'R) = Rk_o(R'R') \) and \( k \) is the \( ^{208}\text{Pb} \) relative wave number. The numerical solution of this equation has been given by Pong (P077). By this method part of the right hand side of this equation was put into a more manageable form by twice integrating by parts to obtain the result

\[
\int_0^\infty K'_o(R'R) \frac{d^2}{dR'^2} \phi(R')dR' = \int_0^\infty \left\{ \frac{d^2}{dR'^2} K'_o(R'R') \right\} \phi(R')dR' \quad (C-2)
\]

where the property of \( K'_o(R'R') \) and \( \phi(R') \) at the origin have been used.

Introducing the notation

\[
\hat{L} = \frac{d^2}{dR^2} + V_T(R) + k^2
\]

the first step in solving equation (C-1) was to calculate the solution of the homogeneous equation \( \hat{L}f_o = 0 \) where \( f_o(R) \) was subject to the usual bound state boundary conditions. This was achieved using standard techniques in which the second order differential equation was replaced by the recurrence relation...
\[ \phi(R+h)\left\{ 1-h^2V(R+h)/12 \right\} = h^2V(R)\phi(R) + \]
\[ 2\phi(R)\left\{ 1-h^2V(R)/12 \right\} - \phi(R-h)\left\{ 1-\frac{h^2}{12}V(R-h) \right\} \quad (C-4) \]

where \( h \) is a step length, typically 0.1 fm, and \( V(R) = -V_T(R) - k^2 \).

From this solution it is possible to construct the function \( F_0(R) \) where
\[ F_0(R) = \int_{-\infty}^{\infty} K'(RR')f_0(R')dR' \equiv \hat{K}'f_0 \quad (C-5) \]

and thus to set up the following iterations, ie.

\[ \hat{L}g_n = F_{n-1} \quad (C-6) \]
\[ G_n = \hat{K}'g_n \]

and
\[ F_n = \alpha G_n + \beta F_{n-1} \quad (C-8) \]

where \( \alpha \) and \( \beta \) are arbitrary constants such that \( \alpha + \beta = 1 \). By substituting equation (C-8) into equation (C-6) the iteration proceeds until the equality \( G_n - F_{n-1} = H_n = \gamma F_0 \) is met, or more formally the iteration stops when
\[ \left| \left| H_n - \frac{F_0}{\langle F_0 | F_0 \rangle} \right| F_0 \right|^2 < \varepsilon \quad (C-9) \]

where \( \varepsilon \) is a small constant.
When this condition is met equation (C-6) becomes

\[ \hat{L}_n = \hat{F}_{n-1} = C_n - \gamma F_o \]  \hspace{1cm} (C-10)

Thus from equations (C-5) and (C-6) this becomes

\[ \hat{L}(g_n - \gamma f_o) = \hat{K}'(g_n - \gamma f_o) \]  \hspace{1cm} (C-11)

and the solution is finally given by

\[ \phi(R) = g_n(R) - \gamma f_o(R) \]  \hspace{1cm} (C-12)
REFERENCES


BL54 J. Blair, Phys. Rev. 95 (1954) 1218.


BO76 O. Bohigas, X. Campi, H. Krivine and J. Treiner, Phys. Lett. 64 (1976) 381.


FE73 H. FESHBACH, "Reaction dynamics" (1973) Gordon and Breach.


GA28 G. GAMOW, Z Physik 51 (1928) 204.


GO64 M. GOLDBERGER and K. WATSON, "Collision theory" (1964) Wiley.


MA64  W.M. MACDONALD, Nucl. Phys. 54 (1964) 393.


PO75  W.S. PONG and N. AUSTERN, Ann. of Phys. 93 (1975) 369.
<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s) and Title</th>
</tr>
</thead>
</table>


ALPHA-PARTICLE SCATTERING NEAR THE COULOMB BARRIER

DAPHNE F. JACKSON and M. RHOADES-BROWN
Department of Physics, University of Surrey, Guildford, UK
Received 18 March 1976

Abstract: The elastic scattering of 22 MeV α-particles from 208Pb has been fitted using microscopic optical potentials constructed from a simple folding model. It is shown that several different microscopic potentials give agreement with the data provided that the imaginary part of the potential is suitably adjusted. It is shown also that the relation between the real and imaginary parts of the potential in the surface region is not unimportant, and this casts doubt on the use of the ingoing wave boundary condition model at these energies. The reflection coefficients show an odd-even staggering effect which is interpreted in terms of interference between reflection at the angular momentum barrier and reflection at the potential surface. Some comments are made on the suitability of certain potentials for calculations of α-decay rates.

1. Introduction

Previous work on α-particle scattering from heavy nuclei at energies comparable with the height of the Coulomb barrier 1-4) has shown that the barrier radius \( r_b \) and the height of the barrier \( V(r_b) \) are well-determined. The calculated cross sections were found to be insensitive to the behaviour of the real part of the optical potential in the interior region and to the shape of the imaginary part of the potential 3). These calculations have also shown 2, 3) that the potential in the barrier region can be satisfactorily reproduced by means of a folding model.

In the most extensive of these studies, Barnett and Lilley 3) have measured and analysed α-particle scattering from 208Pb and 209Bi in the energy range 17-22 MeV. They suggest that one set of parameters for the phenomenological optical potential for 208Pb (set A) should be particularly suitable for calculations on α-decay of heavy nuclei. Since we are at present engaged in microscopic studies of α-decay, we have compared our microscopic potentials with the one proposed by Barnett and Lilley and this has led to somewhat unexpected results concerning the role of the imaginary part of the potential in scattering at these energies.

2. The microscopic optical potential

We write the α-particle optical potential as

\[ V(R) = V_C(r) + U(R) + iW(R), \]

where the Coulomb potential \( V_C \) and the imaginary nuclear potential \( W(R) \) will be
Fig. 1. Comparison of the real potentials $\text{Re}V(R) = U(R) + V_c(R)$ for the phenomenological potential of Barnett and Lilley (BL) and the three microscopic potentials. Also shown are the surface imaginary part of the BL potential and the surface imaginary potential which gives the best fit with the $H'$ real potential.

Fig. 2. Comparison of the real nuclear potentials $U(R)$ in the surface region.
treated phenomenologically. We calculate the real nuclear potential microscopically using the single folding procedure, i.e.

\[ U(R) = \int \rho(r) V_{\text{an}}(|R-r|) \, dr, \]

where \( \rho(r) \) is the nuclear density and \( V_{\text{an}} \) is the \( \alpha \)-nucleon interaction. The nuclear density for \( ^{208}\text{Pb} \) was constructed from single-particle wave functions generated in the neutron bound state potential of Zaidi and Darmodjo \(^5\), as modified by Batty \(^2\), and the proton potential of Batty and Greenlees \(^6\).

For \( V_{\text{an}} \), we have used a Gaussian form with parameters

- (H): \( V_0 = 43 \text{ MeV}, \quad K = 0.526 \text{ fm}^{-1} \),
- (H'): \( V_0 = 53.75 \text{ MeV}, \quad K = 0.526 \text{ fm}^{-1} \).

The difference between these parameters is that for H' the depth has been increased by a factor of 1.25; this yields a potential \( V_{\text{an}} \) which gives quite good agreement with the nucleon-\( \alpha \) angular distributions and excellent agreement with the s-wave phase shifts \(^7\). The corresponding folded nuclear potentials for Pb and Bi give good agreement with results for the barrier heights and positions \(^2\). We have also used the Woods-Saxon form for \( V_{\text{an}} \) given by Mailandt \( \text{et al.} \) \(^8\)

\[ (M) \quad V = 42.5 \text{ MeV}, \quad a = 0.34 \text{ fm}, \quad R = 1.43 - 0.0009E \text{ fm}, \]

where \( E \) is the lab energy for the nucleon \( \alpha \)-scattering. This interaction gives good agreement with the nucleon-\( \alpha \) scattering over a range of energies.

In fig. 1 we compare the total real potential \( U(R) + V_C(R) \) for the three microscopic calculations for \( ^{208}\text{Pb} \) with the phenomenological potential (set A) of Barnett and Lilley. The microscopic potentials are much deeper than the phenomenological potential in the interior region but the potential derived from parameters H' agrees very closely with the phenomenological potential in the surface region between 8 and 10 fm, as can be seen also from fig. 2. (The halfway radius of the real phenomenological potential is 8.15 fm.) The nuclear potential derived from parameters M crosses the phenomenological potential at \( \approx 10.2 \text{ fm} \), while the potential derived from parameters H crosses the phenomenological potential at \( \approx 7.3 \text{ fm} \) and remains

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values of the barrier radius ( r_b ) and the barrier height ( V(r_b) )</td>
</tr>
<tr>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>( V(r_b) ) (MeV)</td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>( \text{Goldring et al. }^\text{a}) )</td>
</tr>
<tr>
<td>Barnett and Lilley (set A) (^b)</td>
</tr>
<tr>
<td>H (^c)</td>
</tr>
<tr>
<td>H' (^c)</td>
</tr>
<tr>
<td>M (^c)</td>
</tr>
</tbody>
</table>

\(^a\) Ref. \(^1\). \(^b\) Ref. \(^3\). \(^c\) Ref. \(^2\).
smaller thereafter. All the potentials shown give values of \( r_b \) and \( V(r_b) \) which are very close to the results of Goldring et al.\(^1\), as can be seen from table 1.

From this comparison of the real potentials, we conclude that, if the low energy scattering depends only on \( r_b \) and \( V(r_b) \), all three microscopic potentials should give similar results to the phenomenological potential; on the other hand, if there is any sensitivity to the shape of the real potential in the surface region, then potential H' may be preferred.

### 3. Calculations and Results

#### 3.1. Angular Distributions and Reaction Cross Sections

We have calculated the angular distribution of the ratio of the elastic cross section to the Rutherford cross section for 22 MeV \( \alpha \)-particle scattering from \( ^{208}\text{Pb} \) using the real potentials shown in fig. 1.

Barnett and Lilley \(^3\) obtained equally good fits to their data with volume absorption or with surface absorption of derivative form \( 4a_l f'(R, R_l, a_l) \). In fact, the surface absorption allows more Coulomb-nuclear interference at small angles, as shown in fig. 3, but there are no experimental points in this angular region.

In fig. 4 we show the results obtained using the three real microscopic potentials with the same surface absorption as for the phenomenological potential. No variation of parameters has been allowed. In each case we compare with the cross section given by the Barnett and Lilley phenomenological potential with surface absorption; it can be seen that potential H gives by far the closest agreement while potential H' gives very poor agreement. In fig. 5, the angular distributions obtained with the three microscopic real potentials and the phenomenological volume absorption are compared, and we again use the angular distribution given by the Barnett and Lilley potential with surface absorption as a reference curve. It can be seen that the agreement obtained with potential M is improved but the result for potential H is still substantially better than that for potential H'.

A further set of calculations was carried out using the microscopic real potential H' and searching on the parameters of the imaginary potential. Good agreement was obtained with both volume and surface absorption with the parameters given in table 2.

Good agreement with the data and the reference curve obtained from the Barnett and Lilley potential was obtained for potential M and a volume absorption term whose parameters are given in table 2. A search with this potential starting from surface absorption produced an imaginary potential which tended closely to a volume shape. We were not able to find a parameter set, for volume absorption, with more conventional values of both \( R_q \) and \( a_l \) which gave a reasonable value of \( \sigma_R \).

The surface imaginary potential for the Barnett and Lilley potential, which also gave agreement for the microscopic potential H, is plotted in fig. 1. Also plotted in this figure is the surface imaginary potential obtained by searching with the potential
Fig. 3. Fits to the data using the phenomenological BL potential with volume and surface absorption.

Fig. 4. Comparison of the angular distributions given by the three real microscopic potentials and the surface imaginary part of the BL potential.
H'. The peak of the surface term has moved out from \( \approx 7.2 \) fm where the BL and \( ^{5}H \) potentials cross to \( \approx 8.2 \) fm where the BL and H' potentials coincide.

The calculated reaction cross sections are also given in table 2. It can be seen that potential M yields a value closest to the BL value, potential H yields somewhat lower values and potential H' somewhat higher values.

3.2. REFLECTION COEFFICIENTS

Fig. 6 shows the real and imaginary parts of the reflection coefficients \( \eta_{L} \) for the Barnett and Lilley potentials with surface and volume absorption. With surface absorption there is an odd-even staggering effect in \( \eta_{L} \) which does not occur for volume absorption. Fig. 7 shows the reflection coefficients corresponding to the angular distributions shown in fig. 4 while fig. 8 shows the reflection coefficients for the best fits with surface absorption. Although the stagger in \( \text{Re}\eta_{L} \) is out of phase with \( \text{Im}\eta_{L} \), the behaviour of \( |\eta_{L}| \) is still not smooth and \( \arg\eta_{L} \) shows a very pronounced odd-even staggering for the low partial waves. The reflection coefficients corresponding to
Table 2

<table>
<thead>
<tr>
<th></th>
<th>$U(R)$ parameters</th>
<th>$W(R)$ parameters</th>
<th>$\sigma$ reaction (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>phenomenological</strong></td>
<td>$V_R = 96.44 \text{ MeV}$</td>
<td>$V_I = 32.0 \text{ MeV}$</td>
<td></td>
</tr>
<tr>
<td>BL (set A)</td>
<td>$R_R = 1.376 \text{ fm}$</td>
<td>$R_I = 1.216 \text{ fm}$</td>
<td>192.8</td>
</tr>
<tr>
<td></td>
<td>$A_R = 0.625 \text{ fm}$</td>
<td>$a_I = 0.42 \text{ fm}$</td>
<td></td>
</tr>
<tr>
<td><strong>microscopic</strong></td>
<td>$H$</td>
<td>surface</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$V_I = 32.0 \text{ MeV}$</td>
<td>as above</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R_I = 1.216 \text{ fm}$</td>
<td>$a_I = 0.42 \text{ fm}$</td>
<td></td>
</tr>
<tr>
<td><strong>phenomenological</strong></td>
<td>$V_R = 96.44 \text{ MeV}$</td>
<td>$V_I = 55.2 \text{ MeV}$</td>
<td></td>
</tr>
<tr>
<td>BL (set A)</td>
<td>$R_R = 1.376 \text{ fm}$</td>
<td>$R_I = 0.533 \text{ fm}$</td>
<td>213.3</td>
</tr>
<tr>
<td></td>
<td>$A_R = 0.625 \text{ fm}$</td>
<td>$a_I = 0.164 \text{ fm}$</td>
<td></td>
</tr>
<tr>
<td><strong>microscopic</strong></td>
<td>$H'$</td>
<td>volume</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$V_I = 9.53 \text{ MeV}$</td>
<td>as above</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R_I = 1.564 \text{ fm}$</td>
<td>$a_I = 0.324 \text{ fm}$</td>
<td>224.5</td>
</tr>
<tr>
<td><strong>microscopic</strong></td>
<td>$M$</td>
<td>volume</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$V_I = 55.2 \text{ MeV}$</td>
<td>as above</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$R_I = 0.533 \text{ fm}$</td>
<td>$a_I = 0.164 \text{ fm}$</td>
<td>213.3</td>
</tr>
</tbody>
</table>

the angular distributions given by the microscopic potentials with volume absorption shown in fig. 5 lie on smooth curves but the best fits with volume absorption for potentials $M$ and $H'$ show slight odd-even staggering.

It appears from fig. 7 that the microscopic potential $H'$ with the Barnett and Lilley surface absorption gives too much absorption for the low partial waves, and moving the imaginary potential out corrects this. In all cases, the behaviour of the reflection coefficients and angular distributions indicates that the scattering lies in the region between non-diffractive scattering and Fresnel scattering 9).

Austern 10) has attributed the fluctuation of $\eta_L$ with $L$ to interference between the reflection from the angular momentum barrier and reflection from the potential surface. Using the WKB approximation, he writes

$$\eta_L \approx \eta_L^{\text{barrier}} + \eta_L^{\text{surface}},$$

where

$$\eta_L^{\text{barrier}} = -e^{-2iS_L(r_L)}$$

$$\eta_L^{\text{surface}} = \frac{1}{2i} \int_{r_L}^{r} \left[ e^{-2iS_L(r')} - e^{-4iS_L(r_L)} e^{+2iS_L(r')} \right] \frac{k_L(r')}{k_L(r)} \, dr',$$

$$S_L(r) = kr + \int_{r_0}^{r} [k(x) - k] \, dx,$$

$$k(r) = |k|^2 \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2} V(r) \frac{1}{r^4}.$$

\[\text{\hspace{1cm}}\]
Fig. 6. The reflection coefficients for the BL potential with surface and volume absorption.

Fig. 7. The reflection coefficients corresponding to the angular distributions shown in fig. 4.
and $r_L$ is the radius at which $\text{Re} k_L(r) = 0$. These formulae can be generalised to include the Coulomb effects more accurately\(^\text{11}\): Fluctuations in $\eta_L$ will disappear if either $\eta^\text{barrier}_L$ or $\eta^\text{surface}_L$ is reduced.

In the strong absorption situation it is thought that reflections from the surface are small and for low partial waves the reflections at the barrier take place well inside the nucleus so that the absorptive part of the potential causes substantial attenuation. This attenuation is primarily a volume effect. For a weak absorption situation, $\eta^\text{barrier}_L$ is larger because the ingoing and reflected waves are less attenuated. Strong reflection from the surface will occur if the real potential has a sharp surface, if the wavelength is comparable or greater than the surface thickness, and if the phase averaging is upset. In the present case we have a relatively weak absorption situation and the wavelength is 3.1 fm which is not small in comparison with the surface thickness. It is therefore not surprising that $\eta^\text{barrier}_L$ and $\eta^\text{surface}_L$ can interfere to give the odd-even staggering effect, particularly for surface absorption. However, the odd-even staggering does not appear to be necessary to fit the available data.

3.3 CRITICAL ANGULAR MOMENTA AND RADI

It is not possible to use the calculated reflection coefficients to define accurately the
critical angular momentum $L_\pm$ through the relation $\text{Re} \eta_L(L_\pm) = 0.5$, and so to obtain the corresponding radius $R_\pm$. For scattering from heavy nuclei Blair \cite{12} has defined the "quarter-point" angle $\theta_c$ such that $d\sigma/d\sigma_R = 0.25$, and this is related to a radius $R_c$ through the expressions

$$\theta_c = 2\tan^{-1}(n/L_c), \quad kR_c = n + (n^2 + L_c^2)^{1/2},$$

where $n$ is the Coulomb parameter. However, it can be seen from figs. 3–5 that the quarter point is not defined except for the cross section calculated using the $H'$ potential. If we take $\theta_c \approx 180^\circ$ or $L_c = 0$, both methods yield approximately $kR_c \approx 2n$.

As we have already shown, the barrier radius is well-determined and appears to be the only significant radius parameter for this scattering process.

4. Discussion

We have shown that it is possible to fit $\alpha$-particle scattering from $^{208}\text{Pb}$ at 22 MeV very well using a microscopic real potential derived from a simple folding model. When we took the imaginary potential to be given from the phenomenological analysis by Barnett and Lilley \cite{3} our prediction from fig. 1, namely that the microscopic potential $H'$ which most closely reproduced the real phenomenological potential would give the best fit, proved to be incorrect. On the other hand, all the microscopic potentials gave good agreement with the data when the imaginary potential was allowed to vary. This result implies that when attempts are made at these energies to study the microscopic structure of the optical potential it is necessary to give information on the way the imaginary potential is treated. It also suggests that the ratio of the real to the imaginary potential in the surface region is a significant quantity, as has been noted in heavy ion scattering \cite{13}. More precise measurements of reaction cross sections and of angular distributions in the Coulomb-nuclear interference region would serve to determine the shape and magnitude of the imaginary potential more clearly.

In their work on $\alpha$-particle scattering at energies near the Coulomb barrier, Eisen et al. \cite{4,14} have used the ingoing wave boundary (IWB) condition model with a real potential. However, the IWB model is valid in a strong absorption situation but not

<table>
<thead>
<tr>
<th>Potential</th>
<th>Inner turning point (fm)</th>
<th>Penetrability</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>9.20</td>
<td>$5.13 \times 10^{-14}$</td>
</tr>
<tr>
<td>BL $H'$</td>
<td>9.11</td>
<td>$2.72 \times 10^{-14}$</td>
</tr>
<tr>
<td>$H$</td>
<td>8.90</td>
<td>$2.16 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
when there is a weak absorption and interference between ingoing and outgoing waves\textsuperscript{15}, as is the case for scattering at these energies from heavy nuclei. We do not agree with the conclusion of Eisen \textit{et al.}\textsuperscript{14} that the study of elastic scattering near the Coulomb barrier is free from complications associated with the imaginary potential.

We have also examined the suggestion by Barnett and Lilley that their real potential is particularly suitable for \(a\)-decay calculations. Values of the penetrability for the decay of \(^{212}\)Po calculated in WKB approx using the real potentials shown in fig. 1 are given in table 3. It can be seen that the potential \(H\) which fits the elastic scattering data with the same imaginary potential as the BL real potential yields a value differing by 25\% from the BL result while potential M yields a value differing by a factor of two. These differences arise from the different widths of the barrier.

\textbf{References}

2) C. J. Batty and E. Friedman, Phys. Lett. \textbf{34B} (1971) 1
6) C. J. Batty and G. W. Greenlees, Nucl. Phys. \textbf{A133} (1969) 673
7) C. J. Batty, E. Friedman and D. F. Jackson, Nucl. Phys. \textbf{A175} (1971) 1
12) J. S. Blair, Phys. Rev. \textbf{95} (1954) 1218
14) Y. Eisen, B. Day and E. Friedman, Phys. Lett. \textbf{56B} (1975) 313
Theories of Alpha-Decay

DAPHNE F. JACKSON AND M. RHODES-BROWN

Department of Physics, University of Surrey, Guildford GU2 5XH, England

Received August 24, 1976

The derivation of the width for $\alpha$-decay is examined with particular emphasis on methods which do not involve arbitrary channel radii. A new method of treating the initial decaying states is introduced and the use of ambiguous phenomenological potentials is avoided. This method yields consistent and acceptable quantitative results for the g.s. to g.s. (ground state) transitions in even polonium isotopes and for the branching ratio for the decay of the isomeric state $^{212m}$Po.

CONTENTS

1. Introduction
   1.1. Statement of the problem
   1.2. $R$-matrix theory
   1.3. Sensitivity to ambiguities in the optical potential
   1.4. Other formulae for the width
   1.5. Definitions and notation

2. Time-Dependent Theory of Decaying States
   2.1. Derivation of an expression for the width
   2.2. Choice of the Hamiltonians $K$ and $K'$
      2.2.1. Method of Kadomskii and Kalechits
      2.2.2. Method of Harada and Rauscher
      2.2.3. Method of Furman, Holan, Kadomskii and Stratan
      2.2.4. Creation of an initial bound state
      2.2.5. Method of Schlitter

3. Unified Reaction Theory
   3.1. Feshbach's projection operator formalism
      3.1.1. Derivation of an expression for the width
      3.1.2. Relation to other theories
   3.2. Macdonald's formalism
      3.2.1. Derivation of an expression for the width
      3.2.2. Relation to other theories
   3.3. Excitation of the residual nucleus
   3.4. Excitation of the $\alpha$-particle

4. Calculations for $0^+ \rightarrow 0^+$ Transitions
   4.1. Cluster formalism
   4.2. Construction of the microscopic potentials
   4.3. The functions $\chi_a$ and $\chi_a^+$
   4.4. Results for the decay widths

151
5. Calculations for the Decay of $^{212}\alpha\text{Po}$
   5.1. Structure of the isomeric state
   5.2. Decay to the g.s. of $^{208}\text{Pb}$
   5.3. Decay to the lowest $3^-$ and $5^-$ states in $^{208}\text{Pb}$

6. Antisymmetrization
   6.1. The effect of the exclusion principle
   6.2. Cluster formalism
   6.3. Discussion

7. Summary and Conclusions

Appendix. Estimates of the cluster parentage coefficients

Acknowledgments

1. INTRODUCTION

1.1. Statement of the Problem

The study of $\alpha$-decay in heavy nuclei has a long history. A qualitative explanation of experimental data is readily given in terms of barrier penetration but it has proved more difficult to obtain consistent quantitative results for the absolute values of decay rates in heavy nuclei [1, 2]. Interest in $\alpha$-decay of light nuclei has developed recently in connection with studies of four-particle correlations [3], and of nonconservation of parity [4]. Lifetimes of superheavy nuclei are also of interest.

The theoretical formalism for $\alpha$-decay and, in particular, expressions for the decay width can be developed by means of the perturbation theory of decaying states or in terms of resonance theory for the scattering of the $\alpha$-particle by the residual nucleus. Several authors have displayed the connection between these methods, usually in terms of $R$-matrix theory [1, 5, 6].

The problems associated with the prediction of absolute decay rates can be stated as follows: (i) in $R$-matrix theory the penetrability is extremely sensitive to the choice of the arbitrary channel radius [7, 8, 9, 10]. (ii) The phenomenological optical potential for $\alpha$-particle scattering from nuclei at medium energies shows ambiguities. In $R$-matrix theory [8, 11] and in other methods [12], these ambiguities cause substantial variation in the calculated widths. (iii) The initial state is normally treated in the oscillator shell model while the final state is represented by an $\alpha$-particle moving in the $\alpha$-nucleus potential. This can lead to problems of nonorthogonality, and it is difficult to show convincingly how the shell model state evolves into the final system of an $\alpha$-particle and residual nucleus.

In this paper we wish particularly to circumvent problems (i) and (ii) and we therefore build our formalism on those methods which do not involve an arbitrary channel radius, and also construct the optical potentials microscopically. We also work as far as possible within the cluster model, and will be able to show that this contributes substantially to the resolution of problem (iii) and yields satisfactory quantitative
results. For examples to illustrate our method we use the decays of the polonium isotopes listed in Table I.

**Table I**

Decay Parameters for the Polonium Isotopes

<table>
<thead>
<tr>
<th>Decay</th>
<th>$T_{1/2}$ (sec)</th>
<th>$T_{\text{exp}}$ (MeV)</th>
<th>$T_{\text{n}}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{210}\text{Po} \rightarrow ^{206}\text{Pb}(\text{g.s.})$</td>
<td>$1.19 \times 10^7$</td>
<td>$3.82 \times 10^{-23}$</td>
<td>$5.30$</td>
</tr>
<tr>
<td>$^{212}\text{Po} \rightarrow ^{208}\text{Pb}(\text{g.s.})$</td>
<td>$3.04 \times 10^{-7}$</td>
<td>$1.52 \times 10^{-15}$</td>
<td>$8.80$</td>
</tr>
<tr>
<td>$^{214}\text{Po} \rightarrow ^{210}\text{Pb}(\text{g.s.})$</td>
<td>$1.64 \times 10^{-4}$</td>
<td>$2.85 \times 10^{-18}$</td>
<td>$7.70$</td>
</tr>
<tr>
<td>$^{214}\text{Po} \rightarrow ^{212}\text{Pb}(\text{g.s.})$</td>
<td>$1.58 \times 10^{-1}$</td>
<td>$2.89 \times 10^{-21}$</td>
<td>$6.81$</td>
</tr>
<tr>
<td>$^{212}\text{Po} \rightarrow ^{208}\text{Pb}(\text{g.s.})$</td>
<td>$4.5 \times 10^1$</td>
<td>$9.84 \times 10^{-31}$</td>
<td>$11.65$</td>
</tr>
<tr>
<td>$^{208}\text{Pb}(3^-)$</td>
<td>$9.84 \times 10^{-25}$</td>
<td>$9.03$</td>
<td></td>
</tr>
<tr>
<td>$^{208}\text{Pb}(5^-)$</td>
<td>$2.02 \times 10^{-26}$</td>
<td>$8.45$</td>
<td></td>
</tr>
</tbody>
</table>

### 1.2. R-Matrix Theory

In R-matrix theory the configuration space is separated into two regions; in the external region defined by $R > R_c$, the total wavefunction can be expressed as a sum of channel functions of simple product form while the wavefunction in the internal region is replaced by a complete set of eigenfunctions $X_\alpha$ with the real eigenvalues $E_\alpha$. In the single channel approximation, the width of a narrow resonance is given by

$$
\Gamma = 2P\gamma_\alpha^2
$$

where $\gamma_\alpha^2$ is the reduced width determined by the overlap of $X_\alpha$ and the channel function at the channel surface, and $P$ is the penetrability

$$
P = \frac{kR_c}{F^2(R_c) + G^2(R_c)}.
$$

The functions $F$ and $G$ are those solutions of the one-body Schrödinger equation containing the Coulomb potential and the nuclear optical potential whose asymptotic behavior corresponds to the usual Coulomb functions $F$ and $G$. The $R$-matrix is

$$
R = \sum_\alpha \frac{\gamma_\alpha^2}{E_\alpha - E}.
$$

The one-body widths are related to the many-body widths through the spectroscopic factor $S_a$, i.e.

$$
\gamma_a^2 = S_a\gamma_{ab}^2,
$$

$$
\Gamma_a = S_a\Gamma_{ab}.
$$
The experimental width $\Gamma_r$ is related to the mean lifetime $\tau = 1/\lambda$ by the relation

$$\Gamma_r = \frac{\hbar}{\tau} = \frac{0.658 \times 10^{-15} \text{ eV}}{\tau (\text{sec})}.$$  

It is well-known that variation of the channel radius $R_c$ can lead to an order of magnitude variation in the penetrability. Calculations by Scherk and Vogt [8] for the decay of $^{212}\text{Po}$ to $^{208}\text{Pb}$ using the $\alpha$-particle optical potential for $^{208}\text{Pb}$ due to Igo [13] show that the penetrability increases by a factor of $\sim 10$ as $R_c$ increases from 9 fm to 10 fm. Strictly, $R_c$ should be sufficiently large so that there is no coupling between different channels in the external region but values of $R_c$ well inside the inner turning point have been used [7, 9]. Values of the penetrability derived from the microscopic optical potentials calculated in this work (see Section 4.2) are plotted in Figure 1. The WKB results are also shown, and are plotted at the position of the inner turning point. In agreement with earlier work [11] we find that the WKB approximation gives an underestimate of the penetrability.

![Penetrability versus $R_c$](https://example.com/penetrability_graph.png)

**Fig. 1.** Variation of the penetrability, defined by Eq. (1.2), as a function of the channel radius $R_c$. Crosses mark the value obtained using the WKB approximation.
THEORIES OF ALPHA-DECAY

For light nuclei, the deficiencies of the approximate formula (1.1) have been fully examined by Arima and Yoshida [14]. In more recent work on the polonium isotopes, DeVries et al. [15] replace the penetrability \( P \) by the expression \( k/R_e | \psi_G(R_e) |^2 \) where \( \psi_G(R_e) \) is a resonant state wavefunction with a fixed number of nodes in the interior region. They choose \( R_e \) to be the radius corresponding to the peak of the last maximum of \( \psi_G \), which gives \( R_e \sim 8.3-8.5 \) fm. This decreases the equivalent penetrability and increases the value of \( \gamma_a \) derived from Eq. (1.1).

1.3. Sensitivity to Ambiguities in the Optical Potential

The \( \alpha \)-particle optical potential determined by fitting low and medium energy scattering data has ambiguities in the parameters owing to the strong absorption features of the scattering process [13, 16]. Using optical potentials for \( ^{208}\text{Pb} \) whose depths varied between 50 MeV and 150 MeV, Scherk and Vogt [8] found widths for the \( ^{212}\text{Po} \) decay in the range \( (1.35-3.43) \times 10^{-15} \) MeV, while Bencze and Sandulescu [11] found that the penetrability for the \( ^{238}\text{Pu} \) decay increased by a factor of 5 when a phenomenological potential of depth 74 MeV was replaced by a calculated microscopic potential of depth 231 MeV. In the latter case the shape of the potential was also changed.

DeVries et al. [15] have removed much of the uncertainty in the phenomenological potential for decay of \( ^{212}\text{Po} \) to \( ^{208}\text{Pb} \) by using experimental data for \( \alpha \)-particle scattering from \( ^{212}\text{Pb} \) and \( ^{209}\text{Bi} \) in the energy range 17–22 MeV, and requiring a fit to these data and the correct number of nodes for the decaying state (see below, Section 4.3). However, their method of calculating the resonant state wavefunction \( \psi_G \) is sensitive to assumptions about the imaginary potential for scattering and the upper limit on the decay widths is not well determined.

In Section 2, we review methods developed by various authors to avoid the use of \( R \)-matrix theory and arbitrary radii. These methods do, nevertheless, involve use of phenomenological optical potentials. Table II shows the variation in the results obtained by Harada and Rauscher [12] using four different sets of parameters (A–D) for the optical potential. These results emphasise the need to remove the uncertainties arising from potential ambiguities.

### Table II

Predictions for \( \alpha \)-Decay Widths of \( ^{210,212}\text{Po} \) (in MeV)

<table>
<thead>
<tr>
<th>Reference</th>
<th>( ^{210}\text{Po} )</th>
<th>( ^{212}\text{Po} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Gamma )</td>
<td>( \Gamma_{\text{exp}}/\Gamma_{\text{cal}} )</td>
</tr>
<tr>
<td>Kadmenskii &amp; Kalechits [16]</td>
<td>( 1.43 \times 10^{-29} )</td>
<td>2.68</td>
</tr>
<tr>
<td>Harada &amp; Rauscher [10]</td>
<td>A ( 3.5 \times 10^{-28} )</td>
<td>1.1 \times 10^4</td>
</tr>
<tr>
<td></td>
<td>B ( 6.5 \times 10^{-27} )</td>
<td>5.8 \times 10^{-3}</td>
</tr>
<tr>
<td></td>
<td>C ( 2.3 \times 10^{-26} )</td>
<td>1.6 \times 10^{-8}</td>
</tr>
<tr>
<td></td>
<td>D ( 3.8 \times 10^{-25} )</td>
<td>1.0 \times 10^{-1}</td>
</tr>
</tbody>
</table>
1.4. Other Formulas for the Width

Within the framework of the optical model, the resonance width can be expressed as

\[ \Gamma \propto \int_{0}^{\infty} |u(r)|^2 W(r) \, dr \]  

(1.5a)

where \( u(r) \) is the optical model wavefunction and \( W(r) \) is the imaginary part of the optical potential. This formula is not very useful in the present context because \( W(r) \) is not well determined for low energy \( \alpha \)-particle scattering from heavy nuclei [15, 18].

The resonance width may also be written as [5]

\[ \Gamma = E/k \int_{0}^{B} |u(r)|^2 \, dr. \]  

(1.5b)

Here the upper limit \( B \) of the integral is arbitrary, except that it must be beyond the range of the nuclear potential. With this expression we still have to deal with the ambiguity in the real part of the optical potential, and the formalism does not describe excitation of a definite final state in the residual nucleus.

1.5. Definitions and Notation

The total wavefunction for the \( A + 4 \) system is a solution of the equation

\[(E - H)\Psi = 0.\]  

(1.6)

The Hamiltonian can be written as

\[ H = H_a + H_A + V_{aA} + T_R + V_C \]  

(1.7)

where \( T_R \) is the kinetic energy operator for the motion of the \( \alpha \)-particle relative to the residual nucleus, \( V_C \) is the Coulomb potential and \( V_{aA} \) is a many-body operator representing the total nuclear interaction between the nucleons in the \( \alpha \)-particle and those in the residual nucleus.

We choose a set of relative coordinates

\[ R = R_a - R_A \]  

(1.8a)

\[ \eta_i = r_i - R_A \]  

(1.8b)

\[ \xi_k = r_{A+k} - R_a \]  

(1.8c)

where

\[ R_a = \frac{1}{4} \sum_{k=1}^{4} r_{A+k}, \quad R_A = \frac{1}{A} \sum_{i=1}^{A} r_i. \]  

(1.9)

The \( \xi_k, \eta_i \) are not independent since

\[ \sum_{k=1}^{4} \xi_k = 0, \quad \sum_{i=1}^{A} \eta_i = 0. \]  

(1.10)
In terms of these coordinates, the interaction $V_{AA}$ can be written as

$$V_{AA} = \sum_{i=1}^{A} \sum_{k=1}^{\Lambda} V_{nn}(r_{A+k} - r_{i}) = \sum_{i=1}^{A} \sum_{k=1}^{\Lambda} V_{nn}(\xi_{k} + R - \eta_{i}). \quad (1.11)$$

The wavefunction for the initial state can be written in a cluster expansion as

$$\Psi_{0} = \sum_{n} C_{n}(n_{a}n_{A}) \Phi^{n_{a}}(\eta) \phi^{n_{A}}(\xi) \chi_{n}^{n}(R) \quad (1.12)$$

where the symbols $\xi$, $\eta$ represent the set of coordinates defined above, $\sum_{n}$ implies summation over all appropriate quantum numbers, and $\chi_{n}$ describes the relative motion of the four-particle system and the residual nucleus. At this stage we omit the antisymmetrization operator, but its role is considered explicitly in Section 6. The symbols $n_{a}$, $n_{A}$ represent all relevant quantum numbers for the four-particle system and the $A$-particle system, respectively, and $n_{a} = 0$, $n_{A} = 0$ implies the g.s. of each system. For simplicity we will frequently neglect terms involving excitation of the $\alpha$-particle, and whenever $n_{a} = 0$, $n_{A} = 0$ or $n = 0$ is intended the superscript will be omitted. The angular momentum coupling will be omitted except when explicitly needed.

The wavefunction in a given exit channel can be written in the same notation as

$$\Psi' \sim \Phi^{n_{a}'}(\eta) \phi^{n_{A}}(\xi) \chi_{n}^{n'}(R) \quad (1.13)$$

where $\chi_{n}^{n'}$ is a scattering state for the emitted $\alpha$-particle.

2. **Time Dependent Theory of Decaying States**

In this section we follow the description of the theory of decaying states given by Goldberger and Watson [19, Chap. 8].

2.1. **Derivation of an Expression for the Width**

The time-dependent wavefunction for the $A \rightarrow 4$ system can be written as

$$\Psi(t) = \frac{1}{2\pi i} \int dE \frac{e^{-iE}H}{E - H} \Psi_{0} \quad (2.1)$$

where $H$ is the total Hamiltonian and $\Psi_{0}$ is the wavefunction of the initial state at $t = 0$. The time dependent wavefunction can also be written as

$$\Psi(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE e^{-iE}H \left\{ \frac{1}{E - H + i\epsilon} - \frac{1}{E - H - i\epsilon} \right\} \Psi_{0} \quad (2.2)$$

where the second integral vanishes for $t > 0$. It can also be expanded in the form

$$\Psi(t) = \Psi_{0}J_{00}(t) + \sum_{b} \Psi_{b}J_{b0}(t) \quad (2.3)$$
where \( \Psi_0 \) represents the final states of the \( A + 4 \) system and

\[
J_{\theta_0}(t) = \frac{1}{2\pi i} \int dE \ e^{-it\hbar/\hbar} \langle \Psi_0 | G(E) | \Psi_0 \rangle 
\]

(2.4a)

\[
J_{\theta_0}(t) = \frac{1}{2\pi i} \int dE \ e^{-it\hbar/\hbar} \langle \Psi_0' | G(E) | \Psi_0 \rangle
\]

(2.4b)

with

\[
G(E) = \frac{1}{E - H}.
\]

(2.5)

We wish to describe the initial state as one of a set of discrete eigenstates of a Hamiltonian \( K \) and to treat the difference \( H - K \) as the interaction giving rise to the decay process. The final states are taken to be eigenstates of a Hamiltonian \( K' \) and if \( K \neq K' \), as is usually the case, some nonorthogonality is introduced into the description of the process. This is discussed in more detail in the following sections.

Following Goldberger and Watson, we introduce the operators

\[
R = (H - K)F
\]

(2.6)

\[
F\Psi_0 = \Psi_0 + \frac{1}{E - K} Q(H - K) F\Psi_0
\]

(2.7)

where \( Q \) is a projection operator which projects off the ground state. The matrix elements of \( R \) are

\[
R_0(E) = \langle \Psi_0 | (H - K) | F\Psi_0 \rangle
\]

(2.8)

\[
R_{\theta_0}(E) = \langle \Psi_0' | (H - K) | F\Psi_0 \rangle
\]

(2.9)

and

\[
R^\pm(E) = \lim_{\epsilon \to 0^\pm} R(E + i\epsilon).
\]

(2.10)

The analysis by Goldberger and Watson then gives

\[
J_{\theta_0}(t) \xrightarrow{t \to \infty} 0
\]

(2.11a)

\[
J_{\theta_0}(t) \xrightarrow{t \to \infty} \frac{R_{\theta_0}(E_0) e^{-iE_0/\hbar}}{(E_0 - E_0 - R_{\theta_0}(E_0))}
\]

(2.11b)

The probability for a transition to a definite final state at energy \( E_b \) is given by

\[
P_b = 2\pi \int \frac{|R_{\theta_0}(E_0)|^2 \ d\rho(E_0)}{[E_b - E_0 - \Re \ R_{\theta_0}(E_0)]^2 + \frac{1}{4} F^2}
\]

(2.12)
where \( dp(E_b) \) is the density of states per unit energy interval and

\[
\Gamma = \sum_b \Gamma_b
\]  

\[
\Gamma_b = 2\pi \int |R_{00}^+(E_b)|^2 dp(E_b).
\]  

The width \( \Gamma_b \) is to be evaluated at \( E_b = E_0 + R_0(E_0) \).

The expression (2.14) for the width can be rewritten using equation (2.6) to give

\[
\Gamma_b = 2\pi \int |\langle \Psi'_b | (H - K) | \Psi_0 \rangle|^2 dp(E_b)
\]  

which can be iterated using Eq. (2.7) to give

\[
\Gamma_b = 2\pi \int dp(E_b) |\langle \Psi'_b | (H - K) | \Psi_0 \rangle|^2
+ \langle \Psi'_b | (H - K) \frac{Q}{E - K} (H - K) | \Psi_0 \rangle \cdots |^2.
\]  

The second term represents a process in which the nucleus is excited to a virtual intermediate state and then decays.

2.2. Choice of the Hamiltonians \( K \) and \( K' \)

2.2.1. Method of Kadmenskii and Kalechits. Kadmenskii and Kalechits [20] have derived a similar expression for the width following the formalism of Goldberger and Watson [19]. They give as their choice for the unperturbed Hamiltonian

\[
K = K' = H_A + H_s + T_R + 2Ze^2/R
\]

and consequently the states \( \Psi_0, \Psi'_b \) \( (b \neq 0) \) form a complete set of eigenstates. It is evident, however, that this Hamiltonian will not give any bound states for the relative motion because there is no potential barrier and the whole of the nuclear interaction is contained in the residual interaction

\[
H - K = V_{66} + V_e - 2Ze^2/R.
\]

Hence, at \( t = 0 \) the nucleus is not in a bound state or in a resonant state.

In practice, Kadmenskii and Kalechits take the wavefunction \( \Psi_0 \) to be a shell model state constructed from single-particle wavefunctions of oscillator form. This means that the nonorthogonality of the states \( \Psi_0 \) and \( \Psi'_b \) is reintroduced through the use of a different (shell model) Hamiltonian which is not related to the operators \( H \) and \( K \) defined in their paper. They comment that in order to interpret the expression for the width, \( K \) must correspond to two spatially separated regions, internal and external, but this separation is not present in their formalism.

Their value obtained for the decay width of \(^{210}\)Po by the method is listed in Table II.
2.2.2. Method of Harada and Rauscher. Harada and Rauscher [12] also take $\Psi_0$ to be a shell model state and draw attention to the problem of nonorthogonality which arises. Their nucleon single-particle wavefunctions are generated in the potentials given by Blomquist and Wahlborn [21].

They proceed by time-dependent perturbation theory and obtain an expression for the width which in our notation is

\[ I_b = 2\pi |\langle \Psi'_0 | (H - K') | \Psi_0 \rangle|^2 \]  \hspace{1cm} (2.17)

with

\[ K' = H_A + H_a + U_N + V_a + T_R \]  \hspace{1cm} (2.18)
\[ H - K' = V_{aA} - U_N \]  \hspace{1cm} (2.19)

where $U_N$ is the nuclear part of the phenomenological optical potential for the $\alpha$-particle.

In order to examine the physical meaning of the expression (2.17), we consider the transition from the ground state of the decaying nucleus to the ground state of the residual nucleus and assume that the wavefunction for the initial state is represented reasonably well by a real $\alpha$-particle interacting with the residual nucleus in its ground state, i.e., we take a single term from the expansion (1.12).

The initial and final wavefunctions can then be written as

\[ \Psi_0 \sim \phi_a \phi_A \bar{\chi}_a \]  \hspace{1cm} (2.20a)
\[ \Psi'_0 \sim \phi_a \phi_A \chi_a^+ \]  \hspace{1cm} (2.20b)

where $\phi_a$ is the internal wavefunction of the $\alpha$-particle, $\bar{\chi}_a$ is the relative wavefunction in the initial state, $\chi_a^+$ is the relative wavefunction for the emitted $\alpha$-particle and $\phi_A$ is the ground state wavefunction of the residual nucleus. The expression (2.17) for the width reduces to

\[ I_b \propto |\langle \chi_a^+ | U_N - U_N | \chi_a \rangle|^2 \]  \hspace{1cm} (2.21)
\[ U_N = \langle \phi_a \phi_A | V_{aA} | \phi_a \phi_A \rangle = \langle 0 | V_{aA} | 0 \rangle. \]  \hspace{1cm} (2.22)

Now, Feshbach's formalism for the generalized optical potential [22] gives

\[ U_N = \langle 0 | V_{aA} | 0 \rangle + \langle 0 | V_{aA} \frac{1}{E - Q \overline{Q}^{\dagger} Q V_{aA}} | 0 \rangle \]  \hspace{1cm} (2.23a)
\[ = \bar{U}_N + \Delta \bar{U}_N \]  \hspace{1cm} (2.24b)

and hence the difference $\bar{U}_N - U_N$ corresponds in principle to the second order term $\Delta \bar{U}_N$. In practice, Harada and Rauscher take various phenomenological potentials for $U_N$ so that there is no simple connection between $\bar{U}_N$ and $U_N$.

In this approach, the width is that of a resonance in the system consisting of an $\alpha$-particle and the residual nucleus. The interaction is the residual nuclear interaction which couples the scattering channel to the resonant state of the decaying parent system. The Coulomb potential and the barrier appear to play no special role in the formalism which could apply equally well to nucleon emission. However, in the
calculations by Harada and Rauscher on decay of $^{212}$Po and $^{210}$Po the role of the Coulomb potential is crucial in determining the magnitude of $\chi^+ \chi^+ \chi^-$ in the interaction region.

Values obtained by Harada and Rauscher for the decay widths of $^{210}$Po and $^{212}$Po using four different alpha-nucleus optical potentials are given in Table II.

2.2.3. Method of Furman, Holan, Kadmenskii, and Stratan. Furman et al. [23] have developed the method of Kadmenskii and Kalechits [20] taking the same unperturbed Hamiltonian to give $\Psi_0$ and introducing a shell model wavefunction for $\Psi'_0$. They also replace the nuclear interaction $V_{aa}$ by the real part of the nucleon–nucleus optical potential summed over each nucleon in the $\alpha$-particle.

Since $V_{aa}$ is a many-body operator, replacing it by the one-body optical potential excludes the possibility of treating decay processes in which the residual nucleus is left in an excited state. The use of the phenomenological potential presents difficulties for the proper description of the finite size of the $\alpha$-particle, but Furman et al. [23] have given an approximate method involving expansion of the optical potential in a power series of the ratio of the $\alpha$-particle coordinates to the halfway radius of the potential.

2.2.4. Creation of a bound initial state. Even for long-lived naturally radioactive nuclei, the true initial state is not a stationary state which persists until acted on by the interaction $H - K$. We can make the initial state into a bound state, however, by introducing the approximate Hamiltonian

$$ K = H_A + H_\alpha + T_R + U_R $$

(2.24)

$$ U_R = U_N(R) + V_c(R), \quad R \leq r_b $$

(2.25)

$$ = U_R(r_b), \quad R > r_b $$

where $r_b$ is the barrier radius, i.e., the position of the top of the barrier formed by the potential $U_N + V_c$. The required bound state for the $\alpha$-particle has the single-particle energy.

$$ \epsilon_\alpha = T_\alpha - U_R(r_b) - \frac{\hbar^2}{2\mu_\alpha} \frac{L(L + 1)}{r_b^2}. $$

(2.26)

The interaction is now

$$ H - K = V_{aa} - U_N, \quad R \leq r_b $$

(2.27)

$$ = V_{aa} + V_c - U_R(r_b), \quad R > r_b $$

If we take $U_N = \bar{U}_N$, the width for the g.s. to g.s. transition is now given by

$$ \Gamma_{bb} \propto |\langle \chi^+_N | \bar{U}_N + V_c - U_R(r_b) | \chi^+_N \rangle |^2. $$

(2.28)

The physical interpretation of this expression is that the transition represents the lowering of the constant potential $U_R(r_b)$ to allow the $\alpha$-particle to escape, as shown in Fig. 2.
This approach is likely to be a good approximation for the long-lived decays in which penetration through the barrier is the dominant aspect of the process. In this case the wavefunction for the final state must be obtained from the Hamiltonian

\[ K' = H_A + H_a + T_R + U_N + V_c. \]  

(2.29)

This leads to nonorthogonality again but, since \( K - K' \) is zero in the region \( R < r_b \) where the bound state wavefunction for the \( \alpha \)-particle is large, this should not be a significant effect. This method has been applied to decays of the polonium isotopes in Section 4.

### 2.2.5. Method of Schlüter

Schlüter [24] also starts from the concept of decay of a bound state of a model Hamiltonian. This bound state \( \phi \) is transformed into the decaying state \( \psi \) according to the equation

\[ \psi = \phi + \frac{Q}{E - K + i\epsilon} (H - K) \psi. \]

(2.30)
He then shows that the total width is given by

\[ \Gamma = \sum_e \Gamma_e = 2\pi \sum_e |\langle \psi | (H - K) | \phi_e^c \rangle|^2 \] (2.31a)

\[ = 2\pi \sum_e |\langle \phi | H | \phi_e^c \rangle|^2 \] (2.31b)

where \( \phi_e^c \) is a continuum state of \( K \). The second form for \( \Gamma \) follows because \( K \) is chosen to be

\[ K = PHP + QHQ. \] (2.32)

The problem with this method then arises in the calculation of the channel state \( \phi_e^c \) which is a solution of

\[ (QHQ - E) \phi_e^c = 0 \] (2.33)

because \( P \phi_e^c = 0 \) and \( \phi_e^c \) is orthogonal to \( \phi \).

In an example of potential scattering of an \( \alpha \)-particle from \( ^{12}\text{C} \), three different approximate methods of choosing the initial bound state are examined. The results for the width vary by a factor of 2.

3. Unified Reaction Theory

3.1. Feshbach's Projection Operator Formalism

In this section we follow the unified theory of nuclear reactions developed by Feshbach [22] using projection operator techniques, and seek a formula for the width of a scattering resonance for the system consisting of an \( \alpha \)-particle and the residual nucleus.

3.1.1. Derivation of an expression for the width. We introduce the projection operators \( P \) and \( Q \), such that \( P\Psi \) represents all the open channels with a free \( \alpha \)-particle at infinity and \( Q = 1 - P \).

The Schrödinger equation

\[ (E - H) \Psi = 0 \] (3.1)

reduces to

\[ (E - H_{pp}) P\Psi = H_{po} Q\Psi \] (3.2a)

\[ (E - H_{oo}) Q\Psi = H_{op} P\Psi \] (3.2b)

where \( H_{po} = PHQ \), etc. We also introduce the wavefunction \( \psi^* \) which satisfies the equation

\[ (E - H_{pp}) \psi^* = 0 \] (3.3)
We neglect excited states of the $\alpha$-particle in order to write the projection operator $P$ in the form

$$P = \sum_{n'} | \phi_n \Phi_A^{n'} \rangle \langle \phi_n \Phi_A^{n'} |$$  \hspace{1cm} (3.4)$$

and define the scattering wavefunction for the $\alpha$-particle as (see Section 1.4)

$$\chi_{\alpha}^{n'+} = \langle \phi_n \Phi_A^{n'} | \psi^+ \rangle$$  \hspace{1cm} (3.5)$$

The $\chi_{\alpha}^{n'+}$ then obey coupled equations of the form

$$(\epsilon_{n'} - T_R - V_c - U_{n'n'}) \chi_{\alpha}^{n'+} = \sum_{m'n} U_{n'm'} \chi_{\alpha}^{m'+}$$  \hspace{1cm} (3.6)$$

where

$$U_{n'm'} = \langle \phi_n \Phi_A^{n'} | V_{A1} | \phi_m \Phi_A^{m'} \rangle$$  \hspace{1cm} (3.7)$$

Neglecting the coupling, we obtain, for the g.s. to g.s. transition,

$$(T_R + \hat{U}_N + V_c) \chi_{\alpha}^{n'+} = \epsilon_{\alpha} \chi_{\alpha}^{n'+}$$  \hspace{1cm} (3.8)$$

where $\hat{U}_N$ is the folded potential already defined in Eq. (2.22).

Eliminating $P\Psi$ from Eqs. (3.2) and using Eq. (3.3) we find

$$\left( E - H_{Q0} - H_{OP} \right) \frac{1}{E - H_{PP}} H_{PO} \right) \left. \Psi = H_{OP} \psi^+ \right|$$  \hspace{1cm} (3.9)$$

where

$$\frac{1}{E - H_{PP}} = \frac{\partial}{E - H_{PP}} - i\pi\delta(E - H_{PP}).$$  \hspace{1cm} (3.10)$$

For the single channel case it is easy to show that

$$\langle \psi^+ \mid H_{PO} \mid \Psi \rangle = \frac{\langle K \rangle}{1 + i\pi\langle K \rangle}$$  \hspace{1cm} (3.11)$$

where $\langle K \rangle$ is the reaction matrix defined as [22]

$$\langle K \rangle = \langle \psi^+ \mid H_{PO} \mid \frac{1}{E - \epsilon_{O0}} H_{OP} \mid \psi^+ \rangle$$  \hspace{1cm} (3.12)$$

with

$$E - \epsilon_{O0} = E - H_{O0} - H_{OP} \frac{\partial}{E - H_{PP}} H_{PO}$$  \hspace{1cm} (3.13)$$

Feshbach [22] introduces a set of eigenfunctions of the operator $\epsilon_{O0}$ so that

$$\frac{1}{E - H_{O0}} = \sum_{\lambda} \frac{1}{E - E_\lambda} \mid \Phi_\lambda \rangle \langle \Phi_\lambda \mid$$  \hspace{1cm} (3.14)$$
and obtains
\[ \langle K \rangle = \sum_{\lambda} \langle \psi^+ | H_{PQ} \frac{\Phi_\lambda \langle \Phi_\lambda | H_{OP} \psi^+ \rangle}{E - E_\lambda} \].
(3.15)

Thus \( \langle K \rangle \) is related to the \( R \)-matrix (see Section 1.2)
\[ R = \sum_{\lambda} \frac{\gamma_\lambda^2}{E - E_\lambda}. \]

The corresponding transition matrix element is
\[ T_{HH} = T_{HH}^{\text{pot}} + e^{i \eta_\lambda} \langle K \rangle \]
\[ = T_{HH}^{\text{pot}} + e^{i \eta_\lambda} \langle \psi^+ | H_{PQ} | Q \Psi' \rangle \]
\[ = T_{HH}^{\text{pot}} + e^{i \eta_\lambda} \langle \psi^+ | H_{PQ} \frac{1}{E - H_{QQ} - W_{QQ}} H_{OP} | \chi^+ \rangle \]
(3.16c)
where we have used Eq. (3.9) and have put
\[ W_{QQ} = H_{OP} \frac{1}{E - H_{PP}} H_{OP}. \]
(3.17)

If we assume that the space defined by \( Q \) contains only one resonant state, denoted by \( \Psi_0 \), we have
\[ Q = | \Psi_0 \rangle \langle \Psi_0 | \]
and
\[ \langle \Psi_0 | W_{QQ} | \Psi_0 \rangle = \langle \Psi_0 | H_{OP} \frac{\Psi}{E - H_{PP}} H_{PQ} | \Psi_0 \rangle \]
\[ = \Delta - \frac{1}{2} \eta \Gamma \]
(3.18)

Finally, we use Eq. (3.5) to expand \( \psi^+ \) in a complete set of eigenstates so that the partial width is given by
\[ \Gamma_b = 2\pi \sum_{n^\prime} |\langle \chi_{n^\prime}^{\ast} \phi_\alpha \Phi_{n^\prime}^{\ast} | H_{PQ} | \Psi_0 \rangle|^2 \]
(3.19a)
\[ = 2\pi \sum_{n^\prime} |\langle \Psi_0 | H_{OP} | \chi_{n^\prime}^{\ast} \phi_\alpha \Phi_{n^\prime}^{\ast} \rangle|^2 \]
(3.19b)
and the resonance energy is \( E_b = E_0 + \Delta \). For the g.s. to g.s. transition we have
\[ \Gamma_b = 2\pi |\langle \Psi_0 | H_{OP} | \chi_{n}^{\ast} \phi_\alpha \Phi_{n} \rangle|^2 \]
(3.20)
3.1.2. Relation to other theories. We have

\[ \langle \Psi_0 | QHP | \psi^+ \rangle = \langle \Psi_0 | H - PPHP | \psi^+ \rangle = \langle \Psi_0 | H - E | \psi^+ \rangle \]

(3.21)

where we have used Eq. (3.3) and the relation \( P\psi^+ = \psi^+ \). Hence the partial width (3.20) becomes

\[ \Gamma_b = 2\pi \left| \langle \Psi_0 | H - E | \chi_{n^+}^+ \phi \Phi_A \rangle \right|^2 \]

(3.22)

which is the same as the unantisymmetrized formula given by Mang [1] and by Fliessbach [25]. They note that if \( \psi_0 \) were an exact description of the \( A + 4 \) system it would satisfy the equation \( (E - H) \psi_0 = 0 \) but this does not hold for a model wavefunction. They argue that differences between the true wavefunction and the model wavefunction occur in the exterior region \( R > R_0 \) and hence

\[ \Gamma_b = 2\pi \left| \langle \Psi_0 | H - E | \chi_{n^+}^+ \phi \Phi_A \rangle_{R > R_0} \right|^2. \]

(3.23)

The radius \( R_0 \) is an arbitrary radius in the theory. We note, however, some similarity with our expression (2.28).

Using Eq. (3.8) the expression (3.22) can also be written as

\[ \Gamma_b = 2\pi \left| \langle \Psi_0 | V_{nA} - \hat{U}_N | \chi_{n^+}^+ \phi \Phi_A \rangle \right|^2. \]

(3.24)

Because we have neglected coupling to intermediate states, our optical potential has reduced to \( \hat{U}_N \) which is real. It follows from the definition (3.7) that Eq. (3.24) yields \( \Gamma = 0 \) for the g.s. to g.s. transition. Thus, this formalism, in which the width is determined by the residual interaction, is entirely satisfactory for the decay of an excited state by nucleon or \( \alpha \)-particle emission but it does not give a simple description, at least in lowest order, or the type of decay process which arises as the result of tunnelling through a barrier without excitation.

3.2. MacDonald's Formalism

3.2.1. Derivation of an expression for the width. MacDonald [26] has developed a unified theory for nucleon-nucleus scattering which pays particular attention to the structure of the compound states and allows these to be constructed in a shell model basis. In our case, these states will be cluster states of the \( \alpha \)-particle and the residual nucleus.

The transition matrix element is written in the form

\[ T_{H} = \langle \phi_T | U | \psi_i^+ \rangle + \langle \psi_i^- | T | \psi_i^+ \rangle \]

(3.25)

where \( \phi_T \) is a plane wave state and

\[ T = \nabla + \nabla(E - H + ie)^{-1} \nabla \]

(3.26a)

\[ \nabla = V_{nA} - U \]

(3.26b)

\[ H = H_A + H_a + T_R + V_{nA} + V_e = K + \nabla. \]

(3.26c)
Here \( U \) is an arbitrary real potential. MacDonald now introduces projection operators \( P_d, P_c \) such that \( P_d \) projects on to discrete bound states of \( K \). These discrete states are denoted by \( \{|n\rangle \} \). The compound state \( X_\mu \) can be written as

\[
X_\mu = \sum_n A_\mu^n |n\rangle
\]

where the \( A_\mu^n \) are coefficients for which

\[
(e_\mu - K + P_d \hat{T} P_d) X_\mu = 0.
\]

Here \( \hat{T} \) is the effective interaction

\[
\hat{T} = \hat{V} + \hat{V}(E - K + i\epsilon)^{-1} P_c \hat{T}
\]

and, since it is complex, the energy \( e_\mu = E_\mu - \frac{i}{2} \Gamma_\mu \) is also complex. MacDonald [26] proves the relation

\[
\text{Im} \hat{T} = -\pi \hat{T} P_c \delta(E - K) P_c \hat{T}^t
\]

so that the expression for the width becomes

\[
\Gamma_\mu = -2 \langle X_\mu | \text{Im} \hat{T} | X_\mu \rangle = 2\pi \int dE_b \rho(E_b) \delta(E - E_b) \langle X_\mu | \hat{T} | \Psi_b^+ \rangle^2
\]

(3.28)

where we have introduced the complete set of continuum states for \( K \). If there is one discrete compound state which we can identify with the initial state \( \Psi_0 \), then using the lowest order approximation \( \hat{T} = \hat{V} \), we obtain for the partial width the expression

\[
\Gamma_b = 2\pi \int dE_b \langle \Psi_0 | \hat{V} | \Phi_b^+ \rangle^2
\]

(3.29)

### 3.2.2. Relation to other theories.

In MacDonald’s theory the potential \( U \) is real and arbitrary. It may, therefore, be chosen to give a bound state instead of a resonant state, as we have done in Section 2.2.4 where \( U = U_R - V_C \).

For the g.s. to g.s. transition we have

\[
\Gamma_b \propto \langle \chi_0 | \hat{U}_N - U | \chi_0^+ \rangle^2
\]

where we have used the definition of \( \hat{U}_N \), Eq. (2.22). Thus, the expression for the width does not reduce to zero in this case. Formally, this result resembles quite closely that of Harada and Rauscher [12], although their treatment of \( V_{aa} - U \) is quite different in practice.

Dumitrescu and Kümmel [27] were also concerned with the structure of the compound state. They assume the validity of the shell model with effective residual interactions (SMERI), and obtain an expression for the total width of the form

\[
\Gamma_b = 2\pi \sum \langle \phi_e^s | V_{aa} | \lambda(E) \rangle^2
\]

(3.30)
where $\phi_{a}\xi$ is a continuum solution of

$$K = H_{A} + H_{a} + T_{R} + V_{a}.$$  

The many-body resonance function $\lambda(E)$ is a solution of

$$[E_{0} - H]\lambda(E) = 0, \quad E_{0} = E + (2i)^{-1} \Gamma_{a}(E).$$

For $\alpha$-decay, Dumitrescu and Kümmer [27] assume that

$$V_{aA} = \sum_{ij} V_{ij} \simeq \sum_{i=1}^{d} V_{i}^{\text{opt}}$$

where $V_{i}^{\text{opt}}$ is the real part of the nucleon-nucleus optical potential, and that

$$H \simeq \hat{H}_{\text{SMER1}} + \sum_{j} W_{j}$$

$$W_{j} = V_{j}^{\text{opt}} - U_{j}^{\text{sp}}$$

where $U_{j}^{\text{sp}}$ is the shell model potential. The difference between $H$ and $\hat{H}$ is included using the exponential operator technique, i.e.

$$| \lambda(E) \rangle = e^{\gamma} | \tilde{\lambda}(E) \rangle, \quad [E_{0} - \hat{H}] \tilde{\lambda}(E) = 0$$

so that the resonance character is generated by the operator $e^{\gamma}$.

We note that this model still contains uncertainties arising in the choice of the phenomenological optical potential, and that there is no formal connection between $V_{i}^{\text{opt}}$ and $U_{i}^{\text{sp}}$. Excitation of the residual nucleus is possible only if $V_{i}^{\text{opt}}$ is treated as a generalized optical potential containing collective coordinates. Numerical evaluation of the $\alpha$-decay width by this method would appear to be quite complicated.

3.3. Excitation of the Residual Nucleus

The expressions (3.19), (3.29) for the decay width allow the possibility of excitation of the residual nucleus. In this section we develop the formalism for this, still neglecting excitation of the $\alpha$-particle.

The wavefunction for the initial state can be written in the form

$$\Psi_{0}^{LM} = \sum_{IMlm} C_{L}(I)(LmLM | JMj) \Phi_{A}^{LM}(\eta) \phi_{a}(\xi) \chi_{a}^{LM}(R)$$

where $\eta, \xi, R$ are the internal coordinates defined in Section 1.4, and $C_{L}(I)$ is the fractional parentage coefficient. The relative wavefunction $\chi_{a}$ can be expanded as

$$\chi_{a}^{LM} = R^{-1} \bar{u}_{N}(R) Y_{LM}(\hat{R})$$
so that the normalization is
\[ \int | \chi_a^{LM} |^2 d^3 R = \int_0^\infty | \bar{u}_{NL} |^2 dR = 1 \] (3.35)

The wavefunction for the final state must be coupled to the same channel spin, to give
\[ \Psi_b^{J'M'} = \sum_{I'} (bI'M' | J'M) \Phi_A^{I'M'}(\xi) \phi_b(\xi) \chi_a^{LM}(R) \] (3.36)
with
\[ \chi_a^{LM} = A_i R^{-1} u_i(R) Y_i^\ell(R) Y_I^\ell(\hat{r}). \] (3.37)

We have to evaluate the matrix element
\[ X(J'M \rightarrow I'M') = \langle \Psi_b^{J'M'} | V \rightarrow U | \Psi_b^{J'M'} \rangle \] (3.38)
where the potential \( U(R) \) connects only those terms with \( I = I' \). Since we have assumed that the \( \alpha \)-particle is not excited, the many-body interaction \( V_{\alpha A} \) can be folded with the \( \alpha \)-particle density to give
\[ \langle \phi_b(\xi) | V_{\alpha A}(\eta, \xi, R) | \phi_b(\eta) \rangle = V_{\alpha}(R, \eta) \] (3.39)
and the matrix element of this interaction between states of nucleus \( A \) is given by
\[ \langle I'M' | V_{\alpha}(R, n) | IM \rangle = \sum_{kq} \langle IM' | V_{kq} | IM \rangle [Y^\ell(R)]^* \] (3.40)
\[ = \sum_{kq} (4\pi)^{1/2} (IMkq | I'M') V^\ell(R) \] (3.41)
Hence the matrix element \( X \) becomes
\[ X(J'M \rightarrow I'M') = \sum_{IMLM} C_{IMLM} (-1)^q (-i)^s A_l Y^\ell'(\hat{k}) \langle L0k0 | I \rangle \left[ \frac{(2L + 1)(2k + 1)}{2l + 1} \right]^{1/2} \] \[ \times (Lmk - q | \bar{\nu}) (LmIM | J'M)(l'\nu'M' | J'M) \] \[ \times \int_0^\infty u_i(R)[V^\ell(R) - \delta_{\bar{\nu}0} U(R)] \bar{u}_{NL}(R) dR. \] (3.41)

The effect of the integration over the density of states indicated in Eq. (3.32) is to give
\[ \int d\rho(\eta) | X |^2 = \int d\Omega_\delta Y^\ell(\hat{k}) Y_I^\ell(\hat{r}) = \delta_{I'I'} \delta_{\nu\nu'}. \] (3.42)

When \( I = I' \) corresponds to the g.s. of the residual nucleus we have
\[ V^\ell(\nu) = \int \rho_A(\eta) V_{\alpha}(R, \eta) d^3 \eta \equiv \delta_{\nu\nu'} \] (3.43)
where we have used the definition (2.22) of the optical potential $\hat{U}_N$ and $\rho_A$ is the g.s. density distribution for nucleus $A$. Similarly, when either $I$ or $I'$, but not both, corresponds to the g.s. of the residual nucleus we have

$$V_k^{II'}(R) = \int \rho_A^{II'}(\eta) V_{nn}(R, \eta) \, d^3\eta$$

(3.44)

where $\rho_A^{II'}$ is the transition density for nucleus $A$. The other terms $V_k^{II'}$ are the potentials which would appear in a couple channels description of excitation of several levels in the residual nucleus.

The transition density may be constructed microscopically using appropriate wavefunctions for nucleus $A$, or it may be parametrized in a macroscopic model, or it may be taken from fits to data if the g.s. of $A$ is stable or sufficiently long-lived to allow inelastic nucleon or electron scattering experiments to be carried out. In each case, the optical potential and the interaction potentials $V_k$ may be treated consistently with the same effective interaction $V_{nn}$ using the single-folding technique introduced into the analysis of elastic and inelastic $\alpha$-particle scattering [28]. Alternatively, the potentials $V_k$ may be parametrized directly in terms of the derivatives of the optical potential, i.e.,

$$V_k(R) = -i^k[4\pi(2k + 1)]^{-1/2} (\beta_\omega R_0) \frac{d\hat{U}_N}{dR}, \quad k > 0,$$

(3.45)

where $\beta_\omega R_0$ is the transition strength which may be related to the rotational or vibrational model, as appropriate.

Equation (3.41) simplifies considerably in certain special cases.

(i) $J = M_f = 0$

$$X(00 \rightarrow I'M') = \sum_{L \in L} C_L^\eta(L)(-1)^{J_f+L}(-1)^J A_f Y_{I'M'}^*(\ell)(L000 | I'0)$$

$$\times [(2k + 1)/(2I' + 1)^{1/2}] \int_0^\infty u_f[V_k^{II'} - \delta_{00}u] \bar{u}_{NL} \, dR \quad (3.46)$$

(ii) $I' = M' = 0$

$$X(JM_f \rightarrow 00) = \sum_{L \in L} C_L^\eta(L)(-1)^J A_f Y_{II'}^*(\ell)(J000 | L0) \int_0^\infty u_f[V_k^{II} - \delta_{00}u] \bar{u}_{NL} \, dR$$

(3.47)

(iii) $J = M_f = 0, I' = M' = 0$

$$X(00 \rightarrow 00) = \sum_{L \in L} C_L^\eta(L) A_0 Y_0^*(\ell) \int_0^\infty u_0[V_k^{II} - \delta_{00}u] \bar{u}_{NL} \, dR$$

(3.48)

3.4. Excitation of the $\alpha$-Particle

The excited states of the four-particle system should be included in the expansion (3.33), so that the wavefunction becomes

$$\psi^{\mu'I\lambda}_0 = \sum_{L_{\eta\mu\nu}} C_L^\eta(j)(L_{\eta\mu\nu} | JM_f) \psi^{\mu'I\lambda}_\eta(A, \eta) \hat{X}^{L\eta}_\lambda(R)$$

(3.49)
THEORIES OF ALPHA-DECAY

\[ \psi_{\alpha \Lambda}(\xi, \eta) = \sum_{l_{\alpha}, m_{\alpha}} \frac{(j_{\alpha} \mu_{\alpha} l_{\alpha} m_{\alpha} | j_{\mu} \phi_{\alpha}^{J^{\mu}} \phi_{\alpha}^{J^{\mu}}(\xi))}{(3.50)} \]

When the Eq. (3.49) is inserted into the analysis of the previous section it is evident that the \(\alpha\)-nucleon interaction \(V_{\alpha}(R, \eta)\) must be replaced by

\[ V_{\alpha}^{J^{\mu}}(R, \eta) = \langle \phi_{\alpha}(\xi) | V_{\alpha}(\xi, \eta, R) | \phi_{\alpha}^{J^{\mu}}(\xi) \rangle. \]

Thus the inclusion of excitation of the \(\alpha\)-particle in the formalism of the previous section requires a double folding procedure starting with the nucleon–nucleon interaction \(V_{\alpha\Lambda}\). This procedure has also been introduced into the study of \(\alpha\)-particle scattering [29].

4. CALCULATIONS FOR \(0^+ \rightarrow 0^+\) TRANSITIONS

4.1. Cluster Formalism

In this section we calculate the widths for the g.s. to g.s. transitions in the polonium isotopes, whose parameters are listed in Table I. We use the lowest-order formula for the width developed in Section 2, Eq. (2.16) and in Section 3, Eq. (3.29), i.e.

\[ \Gamma_{\text{b}} = 2\pi \int dp(E_\beta) \langle \Psi_{\beta'} | (H - K) | \Psi_{\beta} \rangle^2 \]

(4.1)

where \(H-K\) is the perturbation and \(\rho(E_\beta)\) is the density of states.

We treat the initial system by the cluster expansion, represented by Eqs. (1.12) or (3.33). The transitions studied in this section are all \(0^+ \rightarrow 0^+\) transitions and the residual nucleus has 82 protons. In these preliminary calculations we therefore approximate the cluster expansion for the initial nucleus to the single term with \(L = I = 0\), which represents a real \(\alpha\)-particle moving relative to the residual nucleus in its ground state. Using Eq. (3.48), the expression for the width becomes

\[ \Gamma = [C_{0}^{n}(0)]^2 \Gamma_{\text{ob}} \]

(4.2)

where the one-body width is

\[ \Gamma_{\text{ob}} = 2\pi \int dp(E_\beta) \langle X_{\alpha}^+ \phi_{\alpha} \Phi_{\Lambda} | (H - K) | \Phi_{\alpha} \phi_{\alpha} \Phi_{\Lambda} \rangle^2 \]

(4.3a)

\[ = 2\pi A_0 \int_{0}^{\infty} u_{\alpha} V_{\alpha}^{00} - U \bar{u}_{\alpha}^{\Lambda N_0} \ dR \]

(4.3b)

Thus, \([C_{0}^{n}(0)]^2\) has the role of the spectroscopic factor defined by equation (1.4) and represents the probability of finding in the initial system a component which resembles a free \(\alpha\)-particle and the residual nucleus in its ground state.
Comparing the definition of the Hamiltonian $K$ given by Eq. (3.28) and by Eq. (2.24), we see that

$$V_0^0 = \hat{U}_N, \quad U = U_R - V_e$$

and hence

$$V_0^0 - U = \hat{U}_N + V_e - U_R \quad (4.4)$$

where $U_R$ is defined by Eq. (2.25) with $U_N = \hat{U}_N$, and $\hat{U}_N$ is our calculated optical potential, defined by Eq. (2.22).

4.2. Construction of the Microscopic Potentials

We have to construct the microscopic potential $\hat{U}_N$ defined by Eq. (2.22). The formula for $\hat{U}_N$ can be reduced to

$$\hat{U}_N(R) = \int \rho_A(r) V_{an}(R - r) \, d^3r \quad (4.5)$$

where $V_{an}$ is the $\alpha$-nucleon interaction and $\rho_A$ is the ground state density for the residual nucleus. In order to evaluate (4.5) we use the following forms for $V_{an}$[30, 31]:

- **Gaussian**
  - $H : V_0 = 43 \text{ MeV}, \quad K = 0.526 \text{ fm}^{-1}$
  - $H' : V_0 = 53.75 \text{ MeV}, \quad K = 0.526 \text{ fm}^{-1}$

- **Saxon-Woods**
  - $M : V_0 = 42.5 \text{ MeV}, \quad a = 0.34 \text{ fm}$
  - $R = (1.43 - 0.0009E) \text{ fm}$

where $E$ is the lab energy for the nucleon–alpha interaction. By using these forms for $V_{an}$ we take into account, at least approximately, the size and structure of the $\alpha$-particle in its ground state and exchange in the $n - \alpha$ system. The nuclear density distributions have been constructed from the single-particle model using the Batty-Greenlees potential [32] for the protons and the Zaidi-Darmodjo potential, as modified by Batty [33], for the neutrons. The depths of the potentials were varied to give the correct separation energy for the least bound proton or neutron in each lead isotope.

In the case of $^{212}$Pb this procedure cannot be used for the protons; consequently, we have used the proton distribution for $^{208}$Pb in this case.

The optical potentials derived by this method have been shown by several authors [18, 30, 31] to give good agreement with the elastic scattering data. The values of the barrier height $V(r_0)$ and its position $r_0$ for each isotope are given in Table III. The values obtained are in good agreement with those deduced from scattering data [34]. In calculations on the decay $^{212}$Po $\rightarrow^{208}$Pb we have also used the phenomenological optical potential of Barnett and Liley [35], which we denote by BL, and the potential of DeVries et al. [15], which we denote by DLF.
### THEORIES OF ALPHA-DECAY

#### Table III

<table>
<thead>
<tr>
<th></th>
<th>( r_b ) (fm)</th>
<th>( V(r_b) ) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{206}\text{Pb})</td>
<td>( H )</td>
<td>10.8</td>
</tr>
<tr>
<td></td>
<td>( H' )</td>
<td>11.0</td>
</tr>
<tr>
<td>(^{206}\text{Pb})</td>
<td>( H )</td>
<td>10.8</td>
</tr>
<tr>
<td></td>
<td>( H' )</td>
<td>11.0</td>
</tr>
<tr>
<td>(^{208}\text{Pb})</td>
<td>( M )</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>( BL )</td>
<td>10.9</td>
</tr>
<tr>
<td>(^{210}\text{Pb})</td>
<td>( H )</td>
<td>10.9</td>
</tr>
<tr>
<td></td>
<td>( H' )</td>
<td>11.0</td>
</tr>
</tbody>
</table>

4.3. The Functions \( \bar{\chi}_a \) and \( \chi_a^+ \)

The quantum numbers \( NL \) of the bound state function \( \bar{\chi}_a \) for the \( \alpha \)-particle were deduced from the oscillator rule

\[
2(N - 1) + L = \sum_{i=1}^{4} [2(n_i - 1) + l_i]
\]

where \( n_i, l_i \) are the quantum numbers for the two protons and two neutrons in the least bound single-particle levels in the \( A + 4 \) system. This procedure then yields

#### Table IV

<table>
<thead>
<tr>
<th>Configuration ((J_p = 0, J_n = 0))</th>
<th>Coefficient</th>
<th>( 2(N - 1) + L )</th>
<th>Lowest ( s )-state</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1h_{11/2})^2(2s_{1/2})^2)</td>
<td>0.6682</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>((1h_{11/2})^2(1i_{13/2})^2)</td>
<td>0.3900</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>((1h_{11/2})^2(1j_{15/2})^2)</td>
<td>-0.2386</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>((1h_{11/2})^2(3d_{5/2})^2)</td>
<td>0.1067</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>((2f_{7/2})^2(2s_{1/2})^2)</td>
<td>0.2305</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>((2f_{7/2})^2(1i_{13/2})^2)</td>
<td>0.1064</td>
<td>22</td>
<td>12</td>
</tr>
<tr>
<td>((1i_{15/2})^2(1i_{15/2})^2)</td>
<td>-0.1665</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>((1i_{15/2})^2(2s_{1/2})^2)</td>
<td>-0.2030</td>
<td>24</td>
<td>13</td>
</tr>
<tr>
<td>((1i_{15/2})^2(1j_{15/2})^2)</td>
<td>0.1026</td>
<td>26</td>
<td>14</td>
</tr>
<tr>
<td>((J_p = 2, J_n = 2))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1h_{11/2})^2(2s_{1/2})^2)</td>
<td>0.2385</td>
<td>22</td>
<td>12</td>
</tr>
</tbody>
</table>
$N = 12$ for the lowest relative $s$-state for $^{212-216}$Po and $N = 11$ for $^{210}$Po. When configuration mixing is taken into account higher values of $N$ are also possible; examples of possible values of $N$ for $^{212}$Po are given in Table IV, where the admixture coefficients are taken from the work of Glendenning and Harada [36] and terms with coefficients less than ±0.1 have been omitted.

In order to obtain a bound state for each isotope at the correct energy $\epsilon_\alpha$, defined by Eq. (2.26), we have to vary the potential. The potential $U_R$, defined originally be Eq. (2.25), is therefore modified to be

$$U_R(R) = g[U(R) + V_d(R)] + (1 - g) U(R_b), \quad R \leq R_b$$

$$= U(R_b), \quad R > R_b$$

(4.7)

and the parameter $g$ is allowed to vary until a bound $s$-state with the required $N$ is obtained at the correct value of $\epsilon_\alpha$. Figure 3 shows the potentials for $^{208}$Pb after com-

![Graph](image)

**Fig. 3.** The bound state potentials $U_R$ used for $^{212}$Po.
plication of the search procedure. It follows from the definition (4.7) that the barrier heights and positions are unchanged. The parameter \( g \) lies in the range 0.4–0.6 for the microscopic potentials but is 1.54 for the phenomenological potential (BL). The phenomenological potential, as we will see later, is quite different in shape from the microscopic potentials and this has a substantial effect on the gound state wavefunction.

The scattering wavefunction \( \chi_s^+ \) is generated in the potential

\[
U(R) = U_g(R), \quad R \leq r_b
\]
\[
= \hat{U}_N(R) + V_c(R), \quad R > r_b
\]

(4.8)

which has the correct behavior at large distances and the barrier height and position consistent with elastic scattering data.

The wavefunction can be expanded in the form

\[
\chi_s^+(R) = (2\pi)^{-3/2} k \left( \frac{dk}{dE} \right)^{1/2} 4\pi \sum_{J^P} i^J e^{iQ_J(kR)} Y_I^J(\hat{R}) Y_I^J(\hat{R})
\]

(4.9a)

\[
= \sum_{J^P} i^J e^{iQ_J} \frac{u_t(kR)}{R} Y_I^J(\hat{R}) Y_I^J(\hat{R})
\]

(4.9b)

where we have put

\[
a = \left( \frac{2}{\pi} \frac{dk}{dE} \right)^{1/2}, \quad a f_t = u_t/kR.
\]

(4.10)

The radial function \( u_t \) obeys the boundary conditions

\[
u_t = 0 \quad \text{at} \quad R = 0,
\]

(4.11a)

\[
u_t = (a/2)[(F_t + iG_t) + e^{2i\gamma}(F_t - iG_t)] \quad \text{at large } R,
\]

(4.11b)

where \( F_t, G_t \) are the usual regular and irregular Coulomb functions. Hence the wavefunctions are normalized so that

\[
\langle \chi_s(E) \mid \chi_s(E') \rangle = \delta(E - E') \delta(\hat{k} - \hat{k}')
\]

(4.12)

and the density of final states is \( d\Omega_k \). Comparing Eq. (3.37) with Eq. (4.9b), we find that the coefficient \( A_t \) is given by

\[
A_t = i^J e^{iQ_J}, \quad |A_t|^2 = 1
\]

(4.13)

and hence the one-body width for the \( L = 0 \) g.s. to g.s. transition reduces to

\[
\Gamma_{ob} = 2\pi \left| \int_{r_b}^\infty u_g(R)[\hat{U}_N(R) + V_c(R) - U_g(r_b)] \bar{u}_{n0}(R) dR \right|^2.
\]

(4.14)
4.4. Results for the Decay Width

Results for the decay widths are given in Table V. The values deduced for the spectroscopic factors are consistent with expectations from nuclear structure considerations, in particular the very low value for $^{210}$Po reflects the need to break the closed neutron shells. The results for $^{212}$Po are not included because they are subject to some doubt owing to the uncertainty in the proton distribution in $^{212}$Pb which was discussed in Section 4.2.

The results obtained with the microscopic potentials $H$, $H'$, and $M$ show that we have succeeded to a very large degree in eliminating the uncertainty associated with the choice of the optical potential, as can be seen by comparing the results in Table V with those in Table II. The remaining variation in the results is due to the slight differences in the barrier heights (see Table III) which leads to differences in the binding energy $e_b$ defined by Eq. (2.26). This effect has been verified by fixing the barrier for the $a + {^{208}}$Pb system at 20.4 MeV, which yields the results given in Table VI. The consistency of these results indicates that our method can give very precise and reliable results for the decay widths, but that is extremely sensitive to the barrier height $V(r_b)$.

### Table V

Calculated Values for Decay Widths and Spectroscopic Factors of $L = 0$ Decays of Po Isotopes

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Potential</th>
<th>$\Gamma_{ob}$ (MeV)</th>
<th>$S_a = \Gamma_{exp}/\Gamma_{ob}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{210}$Po</td>
<td>$H$</td>
<td>$1.30 \times 10^{-37}$</td>
<td>0.029</td>
</tr>
<tr>
<td></td>
<td>$H'$</td>
<td>$1.64 \times 10^{-37}$</td>
<td>0.023</td>
</tr>
<tr>
<td>$^{212}$Po</td>
<td>$H$</td>
<td>$1.02 \times 10^{-37}$</td>
<td>0.149</td>
</tr>
<tr>
<td></td>
<td>$H'$</td>
<td>$1.46 \times 10^{-37}$</td>
<td>0.104</td>
</tr>
<tr>
<td></td>
<td>$M$</td>
<td>$1.12 \times 10^{-37}$</td>
<td>0.136</td>
</tr>
<tr>
<td></td>
<td>$BL$</td>
<td>$1.45 \times 10^{-37}$</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>$D.L.F.$</td>
<td>$3.84 \times 10^{-37}$</td>
<td>0.040</td>
</tr>
<tr>
<td>$^{214}$Po</td>
<td>$H$</td>
<td>$5.40 \times 10^{-37}$</td>
<td>0.053</td>
</tr>
<tr>
<td></td>
<td>$H'$</td>
<td>$6.80 \times 10^{-37}$</td>
<td>0.042</td>
</tr>
</tbody>
</table>

### Table VI

Decay Widths for the $L = 0$ Transition in $^{211}$Po
Calculated with a Barrier of 20.4 MeV

<table>
<thead>
<tr>
<th>Potential</th>
<th>$\Gamma_{ob}$ (MeV)</th>
<th>$S_a = \Gamma_{exp}/\Gamma_{ob}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$1.27 \times 10^{-14}$</td>
<td>0.120</td>
</tr>
<tr>
<td>$H'$</td>
<td>$1.37 \times 10^{-14}$</td>
<td>0.111</td>
</tr>
<tr>
<td>$M$</td>
<td>$1.32 \times 10^{-14}$</td>
<td>0.115</td>
</tr>
</tbody>
</table>
The markedly different result obtained from the phenomenological potential \((BL)\) is due to the need to lower this potential to make it contain a 12s bound state, whereas we have to raise the microscopic potentials. This means that a very substantial difference is introduced between the phenomenological potential and the microscopic potentials in the surface region, as can be seen from Fig. 3, and this leads to a substantial difference between the bound state wavefunctions. The DLF potential is also a phenomenological Woods-Saxon potential but it has been chosen to give a 12s bound state at the required energy and consequently gives a value for the width which is closer to the values given by the microscopic potentials.

As can be seen from Table IV configuration mixing in the shell model corresponds to the introduction of states of relative motion in the cluster model having higher principal quantum numbers. We have investigated the effect of taking different values of \(N\) using the potential for \(208\text{Pb} + \alpha\) derived from the \(H\) parameters. The results are given in Table VII which shows that uncertainty in \(N\) of one unit leads to about the same variation in \(\Gamma\) as that arising from the different choices for the parameters of the \(\alpha\)-nucleon interaction. The full \(H\) potential supports a 15s state at the given \(\alpha\)-particle energy.

### Table VII

<table>
<thead>
<tr>
<th>Principal quantum number (N)</th>
<th>Scaling factor (g)</th>
<th>(\Gamma_{\text{ob}}) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.67</td>
<td>(1.02 \times 10^{-14})</td>
</tr>
<tr>
<td>13</td>
<td>0.77</td>
<td>(1.38 \times 10^{-14})</td>
</tr>
<tr>
<td>14</td>
<td>0.87</td>
<td>(1.87 \times 10^{-14})</td>
</tr>
<tr>
<td>15</td>
<td>0.98</td>
<td>(2.48 \times 10^{-14})</td>
</tr>
</tbody>
</table>

5. Calculations for the Decay of \(212m\text{Po}\)

5.1. Structure of the Isomeric State

The isomeric state \(212m\text{Po}\) lies at 2.85 MeV above the ground state. In the \(\alpha\)-decay of the state 97% of the decays go to the g.s. of \(208\text{Pb}\), 1% to the first 3\(^{-}\) state at 2.62 MeV, and 2% to the 5\(^{-}\) state at 3.20 MeV. According to the shell model calculations of Glendenning and Harada [36] the spin of the isomeric state is 18\(^{+}\) and the principal configurations of the extra core nucleons are \((1h_{9/2})_{j=8}(2g_{9/2}1h_{11/2})_{n=10}\) and \((1h_{9/2}2f_{7/2})_{j=8}(2g_{9/2}1h_{11/2})_{n=10}\). Using the oscillator rule, Eq. (4.6), these configurations yield \(N = 3, L = 18\) for the lowest allowed state of relative motion when the core is in its ground state with \(I = 0\).
5.2. Decay to the g.s. of $^{208}$Pb

An expression for the decay width may be obtained from Eq. (3.47). We consider first only the term with $I = 0$, i.e. the simplest cluster term, so that the width for the decay to the g.s. of the residual nucleus is given by

$$\Gamma = 2\pi \int d\rho(E_{0}) |X(JM_{J} \rightarrow 00)|^{2}$$

$$= 2\pi |C_{18}^{18}(0)|^{2} \left| \int_{0}^{\infty} u_{18} [V_{0}^{00} - U] \bar{u}_{2,18} dR \right|^{2}.$$ \hspace{1cm} (5.1)

Now, using Eqs. (4.4), (4.7), and (4.13) we have

$$\Gamma_{ob} = 2\pi \left| \int_{R_{p}}^{\infty} [u_{18} \bar{U}_{N}(R) + V_{c}(R) - U_{R}(R)] \bar{u}_{2,18} dR \right|^{2}.$$ \hspace{1cm} (5.2)

The expression for the one-body width has been evaluated using the microscopic potential $H$ with the correctly calculated barrier height. This yields a value of

$$\Gamma_{ob} = 7.67 \times 10^{-23} \text{ MeV}$$ \hspace{1cm} (5.3)

giving a spectroscopic factor

$$S_{o} = 0.128.$$ \hspace{1cm} (5.4)

which is very comparable with the values obtained for the g.s. to g.s. transition. Consequently, the ratio of our calculated one-body widths for the decay of $^{212}$Po and $^{212}$Po to the g.s. of $^{208}$Pb is in very close agreement with experiment. In contrast, the calculations of Glendenning and Harada [36] using $R$-matrix theory and the shell model with configuration mixing gave a ratio greater than experiment by a factor of 45 for a channel radius of 9.5 fm and of 18 for a channel radius of 9.0 fm.

5.3. Decay to the Lowest $3^{-}$ and $5^{-}$ States in $^{208}$Pb

Expressions for the widths of these decays can be obtained from Eq. (3.41). For simplicity, we include in the expansion for the $18^{+}$ state only the lowest $0^{+}$, $3^{-}$, and $5^{-}$ states of the residual nucleons. This leads to the transitions indicated schematically in Fig. 4. The vertical arrows represent transitions with $I = I'$ which have contributions from $k = 0$, as well as other values of $k$ if $I = I' \neq 0$. The other transitions have contributions from $k \neq 0$ through the excitation potential $V_{k}$, which we treat by means of Eq. (3.45).

We first neglected the second and third terms in the expansion of the cluster wavefunction for $^{212}$Po and calculated the decay widths to the $3^{-}$ and $5^{-}$ states in $^{208}$Pb. The ratios of these widths to the width for decay to the $0^{+}$ g.s. are then independent of the coefficient $C_{18}^{18}(0)$. These results are given in Table VIII and are denoted as set $A$. We next obtained the widths corresponding to the transitions represented by vertical arrows in Fig. 4 using the value of $B_{L}^{18}(I) = C_{L}^{18}(I)/C_{18}^{18}(0)$, calculated by perturbation
theory, as described in the Appendix. These results are given in Table VIII as set B. Finally, we combined these two sets of transitions so that we have taken into account all the transitions represented in Fig. 4 by double-line arrows and have allowed for interference effects. These results are given in Table VIII as set C. In each case the quantity denoted by $\Gamma_{ob}$ is the width calculated from Eq. (3.41) divided by $|C_{18}(0)|^2$.

**TABLE VIII**

Widths for Decay of $^{212}$Po to States of $^{208}$Pb (in MeV)

<table>
<thead>
<tr>
<th></th>
<th>Experiment</th>
<th>Theory $\Gamma_{ob}$</th>
<th>Set A</th>
<th>Set B</th>
<th>Set C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{212}$Po $\rightarrow$ $^{208}$Pb (0$^+$)</td>
<td>9.84 x $10^{-2}$</td>
<td>7.67 x $10^{-2}$</td>
<td>7.67 x $10^{-2}$</td>
<td>7.67 x $10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$^{212}$Po $\rightarrow$ $^{208}$Pb (3$^-$)</td>
<td>1.01 x $10^{-2}$</td>
<td>1.00 x $10^{-2}$</td>
<td>4.11 x $10^{-2}$</td>
<td>5.49 x $10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$^{212}$Po $\rightarrow$ $^{208}$Pb (5$^-$)</td>
<td>2.02 x $10^{-2}$</td>
<td>9.81 x $10^{-2}$</td>
<td>7.32 x $10^{-2}$</td>
<td>7.02 x $10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>Ratio 0$^+/3^-$</td>
<td>97.0</td>
<td>76.7</td>
<td>186.6</td>
<td>14.97</td>
<td></td>
</tr>
<tr>
<td>Ratio 0$^+/5^-$</td>
<td>48.5</td>
<td>781.8</td>
<td>104.8</td>
<td>109.3</td>
<td></td>
</tr>
<tr>
<td>Ratio 3$^-/5^-$</td>
<td>0.5</td>
<td>10.2</td>
<td>0.56</td>
<td>7.8</td>
<td></td>
</tr>
</tbody>
</table>

For these calculations, and for the calculations of the coefficients $B_L^{18}(I)$ we have taken the collective strength parameters to be [37]

$$\beta_2 R_0 = 0.81 \quad \beta_4 R_0 = 0.48.$$  

The uncertainty on these parameters is about 10%. They were obtained from DWBA fits to inelastic $\alpha$-particle scattering from $^{208}$Pb using the generalized optical model with Saxon–Woods potentials. This means that there may be an additional uncertainty in relating the $R_0$ parameter to our microscopic optical potential. The derivative $d\hat{U}_N/dR$ for the $H$ potential peaks at 6.4 fm.
We see from Table YIII that for set $A$ the relative width for the $3^-$ state is about right but the relative width for the $5^-$ state is too small by a factor of $\sim 16$. To correct this it would be necessary to increase $\beta_x R_0$ by a factor of 4. For set $B$ the relative widths for both the $3^-$ and the $5^-$ states are too small by a factor of 2 but the ratio is in excellent agreement with experiment. Thus, in this case, relatively small changes in the coefficients $B^{18}_L(I)$ would yield exact agreement with the data. This implies that what is important in the decay of the isomeric state is not the collective excitation of the residual nucleus $^{208}$Pb but rather the cluster structure of the decaying state $^{212}$Po.

Further support for this view comes from the result that the value of $[C_{18}^{18}(0)]^2$ deduced from the cluster perturbation calculation differs by only a factor of 2 from the value of $S_e$ obtained for the transition to the g.s., as can be seen by comparing Eqs. (A.10) and (5.4), despite the simplicity of our calculation of the coefficients in the cluster expansion. The difference could arise from small errors in the coefficients $B^{18}_L(I)$ due to the uncertainties in $\beta_x R_0$ and from neglect of terms in the cluster expansion involving the lowest $2^+$ and $4^+$ states in $^{208}$Pb, or from the neglect of the contributions to the g.s. transition from terms with $I \neq 0$, or from neglect of Coulomb excitation. Contributions from additional terms in the cluster expansion will reduce the value deduced for $[C_{18}^{18}(0)]^2$.

In set $B$ the most important contribution to each transition comes from the term with the lowest value of $l$ due mainly to the effect of the angular momentum barrier. When the two modes of transition are put together to give set $C$ we find that the interference changes the widths so that the relative width of the $3^-$ state is too large by a factor of 7 while the $5^-$ width is too small by a factor of 2. This cancellation depends sensitively on the value of $\beta_x R_0$ and on the values of the coefficients $B^{18}_L(I)$ which also depend on $\beta_x R_0$. For comparison we note that the calculation of Rauscher et al. [9] using the shell-model with configuration mixing and $R$-matrix theory gave a relative width for the $3^-$ state which was too small by a factor of 250 and a relative width for the $5^-$ state which was too small by a factor of 14. Both results were obtained with channel radius of 9 fm; it was found that the width for the $5^-$ state could be brought into agreement with experiment by reducing the channel radius to 7–5 fm but this still left the width for the $3^-$ state too small by a factor of 100.

6. Antisymmetrization

6.1. Effect of the Exclusion Principle

So far we have taken the exclusion principle explicitly into account only in the way we have determined the lowest allowed state of relative motion in the initial $A + 4$ system, using Eq. (4.6), although the use of effective interactions which fit elastic and inelastic scattering data should incorporate some exchange effects. However, a correct treatment of antisymmetrization leads to modification of the wavefunction describing the relative motion. Arima and Yoshida [14] take this into account approximately in the framework of $R$-matrix theory by renormalizing the interior wavefunction and
hence the reduced width but, because $R$-matrix theory is used, the normalization coefficient is a function of the channel radius. Fließbach [20] has developed the reaction theory due to Feshbach [22] to give a normalization coefficient $[N(R)]^{-2}$ which is very small inside the nucleus.

6.2. Cluster Formalism

In this section we introduce antisymmetrization into the cluster formalism we have previously used, following resonating-group techniques. We rewrite Eq. (1.12) introducing the operator which exchanges nucleons between the four-particle system and the $A$-particle system, and take $\phi_a$ and $\Phi_A$ to be fully antisymmetrized wavefunctions for these systems. Thus

$$\Psi_0 = \omega \sum_n C_n(n,\mu_a) \Phi_A^n(\eta) \phi_a^n(\xi) \bar{\chi}_a^n(R). \quad (6.1)$$

Following the method of Park, Schied and Greiner [38] we separate this wavefunction into direct terms of the form

$$\Phi_d(n) = C_n(n,\mu_a) \Phi_A^n \phi_a^n \bar{\chi}_a^n \quad (6.2)$$

and exchange terms $\Phi_{\text{ex}}(n)$ which correspond to exchange of $\nu$ particles, so that

$$\Psi_0 = \sum_n \left[ \Phi_d(n) + \sum_{\nu=1}^4 \sum_{S} (-1)^\nu \Phi_{\text{ex}_\nu}(n) \right] \quad (6.3a)$$

$$= \sum_n \left[ \Phi_d(n) + \Phi_{\text{ex}}(n) \right] \quad (6.3b)$$

where the sum over $S$ denotes all the different functions arising from a given $\nu$-exchange.

By inserting Eq. (6.3b) for $\Psi_0$ into the Hill–Wheeler equation

$$\int \Phi_A^* (\eta) \phi_a^* (\xi) (H - E) \Psi_0 \ d\eta \ d\xi = 0 \quad (6.4)$$

we obtain a set of coupled integrodifferential equations for $\bar{\chi}_a$

$$[T(R) + \bar{U}_0(R) + V_c(R) - \epsilon_a] \bar{\chi}_a(R)$$

$$= - \sum_{n=0}^{4} U_{nn} \bar{\chi}_a(R) + \epsilon_a \sum_n \int \Phi_A(\eta) \phi_a(\xi) \Phi_{\text{ex}}(n) \ d\eta \ d\xi$$

$$- [T(R') + V_c(R')] \sum_n \int \Phi_A(\eta) \phi_a(\xi) \Phi_{\text{ex}}(n) \ d\eta \ d\xi$$

$$- \sum_n \int \Phi_A(\eta) \phi_a(\xi) V_{\nu A} \Phi_{\text{ex}}(n) \ d\eta \ d\xi, \quad (6.5)$$
where we have used our previous definition, Eq. (2.22), of the optical potential $\tilde{U}_N$ and have put $\epsilon_a = E - E_A - E_n$, as before. The direct coupling potential is defined as

$$U_{on}^{n,A,n_a} = \langle \phi_d | V_{n,A} | \phi_d \rangle$$

which is identical with Eq. (3.7) when $n_a = 0$. Hence, if we omit the exchange terms and put $n_a = 0$, Eq. (6.5) reduces to Eq. (3.6).

The exchange integrals can be represented as

$$\int \Phi_A(\eta) \phi_a(\xi) \phi_{ex}(n) \, d\eta \, d\xi = - \int K_n(R, R') \tilde{x}_n(R') \, d^3R'$$

$$\int \Phi_A(\eta) \phi_a(\xi) V_{n} \phi_{ex}(n) \, d\eta \, d\xi = - \int G_n(R, R') \tilde{x}_n(R') \, d^3R'$$

where the $K_n(R, R')$ are matrix elements of the exchange operator $\tilde{K}$ introduced by Feshbach [22]. By neglecting the coupling to excited states we obtain the formal equation given by Arima and Yoshida [14]

$$[T(R) + V_c(R) + \tilde{U}_N(R)] \tilde{x}_a = \epsilon_a (1 - \tilde{K}) \chi_a + \tilde{K}[T(R') + V_c(R')] \tilde{x}_a + G \tilde{x}_a.$$  \hspace{1cm} (6.9)

The same set of equations apply to the scattering function $\chi_a^+$. The correct normalization of these functions is now [22, 25, 39]

$$\langle \chi_a(E) | 1 - \tilde{K} | \chi_a(E') \rangle = \delta(E - E')$$  \hspace{1cm} (6.10)

instead of Eq. (4.12). In order to take account of this change in normalization a new scattering function $\Omega(E)$ has been introduced such that [22, 25]

$$\Omega_a(E) = (1 - \tilde{K})^{1/2} \chi_a^+(E)$$  \hspace{1cm} (6.11)

and a new bound state function is defined as [25, 10]

$$G_a = (1 - \tilde{K})^{1/2} \tilde{x}_a.$$  \hspace{1cm} (6.12)

However, the operator $\tilde{K}$ is nonzero only in the interior region which does not contribute to the g.s. to g.s. transitions in our formalism.

6.3. Discussion

The normal procedure in resonating group calculations is to take the wavefunction to consist of only one antisymmetrized term from Eq. (6.1), namely the term with $n_a = n_A = 0$. This is a valid approximation for many problems such as nucleus–nucleus scattering at low energies but is less convincing for $\alpha$-decay from or to an excited state.

A study of the role of exchange in $\alpha$-decay is now in progress, although the evaluation of the functions $K_n$ and $G_n$ is extremely tedious for heavy nuclei. We note, however, that the low-lying states of relative motion excluded by the oscillator rule (4.6) have radial wavefunctions which fall rapidly at large distances and will therefore
have little influence on the α-decay widths for g.s. to g.s. transitions which, in our formalism, have no contribution from inside \( r_B \). On the other hand there is an overall renormalization arising from antisymmetrization and this may modify our values derived for the spectroscopic factors \( S_a \).

7. Summary and Conclusions

We have developed a method of calculating the widths for α-decay of heavy nuclei which does not contain any arbitrary radii or phenomenological potentials. Indeed, we have only one adjustable parameter, namely the scaling factor \( g \) which changes the depth of the bound state potential to give a bound state with the correct number of nodes. We have shown that this method leads to consistent and acceptable results for the one-body widths of g.s. transitions in the even polonium isotopes and for the branching ratio for the decay of the isomeric state \(^{212}\text{Po}\).

The values derived for spectroscopic factors are in the range 0.10–0.15 and our calculation of the cluster parentage coefficients for the isomeric state supports this magnitude. The shell model yields much smaller values than this. For example, a calculation for \(^{212}\text{Po}\) with admixture of 89 proton levels and 83 neutron levels yields \( 1.20 \times 10^{-3} \) while the lowest configuration \((1h_{9/2})^2(2g_{9/2})^2\) alone yields \( 1.98 \times 10^{-3} \) [40]. DeVries et al. [15] obtain results for α-decay widths using \( R \)-matrix theory with \( R_c \sim 8.3–8.5 \) fm which are \( 10^2–10^3 \) times larger than theoretical shell model values. Thus, it appears that for these heavy nuclei the shell model, even with substantial configuration mixing, is far from adequate and any attempt to compensate in \( R \)-matrix theory leads to abnormally low values of the channel radii.

There are many ways in which our calculation can be made more rigorous. A study of the effects of antisymmetrization is in progress but, as already pointed out, we do not expect the exchange kernel to have a significant effect on the one-body width for the g.s. to g.s. transitions although it could have a greater effect on the cluster parentage coefficients. In any case it would be more satisfying not to depend on the oscillator rule, Eq. (4.6), to determine the quantum numbers of the relative motion. More terms can be included in the cluster expansion for the initial state, the excitation of the residual nucleus for \( k \neq 0 \) can be treated microscopically but still within the cluster model, and the excitation of the α-particle can be included. All these modifications are feasible within the framework of our formalism.

APPENDIX: Estimates of the Cluster Parentage Coefficients

The general expression (1.12) for the wavefunction of the initial state can be written as

\[
\Psi_0 = C_0 \left[ \Phi_{\alpha} \Phi_{\alpha^*} \Phi_0 + \sum_{n_n a a > 0} B_n \Phi_n \Phi_n \Phi_n \Phi_n \right] \quad (A.1)
\]
where $B_n = C_n / C_0$. In first-order perturbation theory, the coefficients $B_n$ are given by

$$B_n(n_A, n_x) = \frac{\langle \Phi^n_{\alpha A} \Phi^n_{\alpha x} | V_{aA} | \Phi^n_{\alpha A} \Phi^n_{\alpha x} \rangle}{E_0 - E_n} \quad (A.2)$$

and the coefficients of terms with the $\alpha$-particle in its ground state are

$$B_n(n_A) = \langle \Phi^n_{\alpha A} \Phi^n_{\alpha x} | V_{aA} | \Phi^n_{\alpha A} \Phi^n_{\alpha x} \rangle. \quad (A.3)$$

Inserting the angular momentum coupling using Eq. (3.33) we find

$$(E_I - E_I') B_L^I(I') = \sum_{MmM'm'} \left[ (-1)^h (-1)^q (L0k0 | L'O')(Lmk - q | L'm') \times (N'l'MM') (LmIM | MJ)(IMq | I'M') \times \left[ \frac{(2L + 1)(2k + 1)}{(2L' + 1)} \right]^{1/2} \int_0^\infty \bar{u}_N' V^I_{h} \bar{u}_N dR \right] \quad (A.4)$$

where $V^I_{h}$ is defined as in Eq. (3.40). Equation (A.4) simplifies for certain special cases.

(i) $I = M = 0$

$$B_L^I(I') = \int (J0l0 | L0')(2I' + 1)\frac{1}{1} \int_0^\infty \bar{u}_N' V^I_{h} \bar{u}_N dR \quad (A.5)$$

(ii) $J = M_I = 0$

$$B_L^I(I') = \sum_k (-1)^k (-1)^{I+I'} (00k0 | I0')(2k + 1)\frac{1}{1} \int_0^\infty \bar{u}_N' V^I_{h} \bar{u}_N dR \quad (A.6)$$

(iii) $J = 0, I = 0$

$$B_L^I(I') = \int (2I' + 1)\frac{1}{1} \int_0^\infty \bar{u}_N' V^I_{h} \bar{u}_N dR. \quad (A.7)$$

If we now parametrize $V^I_{h}$ using equation (3.45) we have

$$i^I (2I' + 1)\frac{1}{1} \int_0^\infty \bar{u}_N' V^I_{h} \bar{u}_N dR = (-1)^{I+1} (B_{n}R_0)(4\pi)^{-1/2} \int_0^\infty \bar{u}_N' \frac{d\bar{u}_N}{dR} \bar{u}_N dR. \quad (A.8)$$

We have calculated the $B_L^I(I')$ for the $18^+$ state in $^{212}$Po taking $I' = 3, 5$ corresponding to the $3^-$ and $5^-$ states of the $^{208}$Pb core. In using the oscillator rule to determine the principal quantum number $N'$ we have added one quantum of excitation of the core to the right-hand side of Eq. (4.6). The values obtained for $B_L^I(I')$ are given in Table IX. From Eq. (A.1), the normalization condition becomes

$$C_0^I = \frac{1}{\left( 1 + \sum_n B_n^2 \right)}. \quad (A.9)$$
Theories of alpha-decay

Table IX

Values of the Leading $B$ Coefficients

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{11}^{3-}$</td>
<td>0.97</td>
</tr>
<tr>
<td>$B_{12}^{3-}$</td>
<td>-1.14</td>
</tr>
<tr>
<td>$B_{15}^{3-}$</td>
<td>0.54</td>
</tr>
<tr>
<td>$B_{13}^{5-}$</td>
<td>0.22</td>
</tr>
<tr>
<td>$B_{14}^{5-}$</td>
<td>0.22</td>
</tr>
<tr>
<td>$B_{15}^{5-}$</td>
<td>0.43</td>
</tr>
<tr>
<td>$B_{16}^{5-}$</td>
<td>-0.20</td>
</tr>
</tbody>
</table>

The values of the coefficients given in Table IX yield

$$C_\alpha^2 = [C_{16}^{16}(0)]^2 = 0.25$$  \hspace{1cm} (A.10)

Note added in proof. Since we completed this work our attention has been drawn to the work of Tobocman [41] on exchange effects in particle decay.

Acknowledgments

We are indebted to Professor A. Arima, Professor J. S. Blair, Dr. A. M. Lane and Professor G. H. Rawitscher for valuable comments and correspondence at various stages of this work, and to Dr. Won Sin Pong many helpful discussions and criticism of the manuscript. We are grateful to Mrs. Joan Hilton for help and advice in computing.

Mark Rhoades-Brown is supported by an SRC Research Studentship.

References

40. A. Arima and I. Tonozuka, private communication.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium