RADIAL WAVEFUNCTIONS
IN p-SHELL NUCLEI AND
THE (p,d) REACTION

by
I.S. TOWNER

A Thesis submitted to the Faculty of Science
of the University of London for the Degree
Doctor of Philosophy
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ABSTRACT.

A single-particle shell model has been used to give simultaneously fits to elastic electron scattering data and \((p,d)\) angular distributions for \(1p\)-shell nuclei. Radial wavefunctions have been numerically computed using both a Saxon-Wood and a finite oscillator potential. The numerical procedure is described in detail.

For the case of a Saxon-Wood potential, the arbitrary parameters have been determined by fitting the \(p\)-shell proton separation energy, and the elastic electron scattering data at forward angles. The longitudinal Coulomb term has been calculated, using Born approximation, and both \(C_0\) and \(C_2\) contributions were considered. Good fits were obtained for the \(p_{3/2}\) nuclei, \(\text{Li}^6, \text{Li}^7, \text{Be}^9, \text{B}^{11}\) and \(\text{C}^{12}\), but for the first three several ambiguous sets of parameters were found. The r.m.s. radius of these nuclei are listed.

The DWBA formalism for the \((p,d)\) reaction is briefly presented. The reaction \(\text{C}^{12}(p,d)\text{C}^{11}\) g.s. at a proton energy of 155 MeV has been used as a test for a high energy DWBA calculation. It is found that the local energy approximation gives a distinct improvement over the usual zero-range calculation. Furthermore it is shown that the usual procedure of taking the neutron parameters from the proton optical potential is inadequate; but the prescription of taking the parameters from fitting elastic electron scattering data was
quite successful. Detailed fits for the 155 MeV data of C^{12}(p,d) C^{11}, B^{10}(p,d) B^9 and Be^{9}(p,d) Be^8 are presented.

Also obtained were the experimental spectroscopic factors. Owing to uncertainties and ambiguities in the optical potentials they can only be determined to within an accuracy of 30%. Nevertheless they were compared with the theoretical predictions of a single-particle shell model. Qualitatively the experimental trends were reproduced but the quantitative agreement was poor. Two explanations are briefly considered, the possibility of inelastic scattering processes contributing to the (p,d) cross-section, or the possibility of improving on the simple shell model description by including admixtures of higher states in the ground state wavefunction. Also considered is the application of a single-particle rotational model, but none of these could improve the agreement.
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CHAPTER 1.
Introduction.

Stripping and pick-up reactions have been extensively used in recent years as a method of investigating nuclear spectroscopy. Spins and parities of the excited states in the final nucleus have been identified using the selection rule

$$\frac{1}{2} + J_i + J_f \geq \ell \geq \left| J_i - J_f - \frac{1}{2} \right|$$

(1.1)

where $J_i$ and $J_f$ are the initial and final nuclear spins and $\ell$ is the orbital angular momentum of the transferred particle. From the shape of the angular distributions for the outgoing particles, it is possible to identify the $\ell$ -value which is dominating in the reaction, thus if $J_i$ is known, certain limits can be put on the value of $J_f$.

Furthermore the parity change in the reaction is given by

$$\Delta \pi = (-)^\ell$$

(1.2)

Strictly speaking this is only correct in special cases, such as stripping from S-state projectiles like deuterons, however if the zero-range limit is used for the interaction then the parity change is necessarily given by $(-)^\ell$. Thus if the parity of the initial state is known, the parity of the
final state is determined.

However there is no single theory of direct reactions, nor in fact has the more fundamental question of defining a direct reaction been satisfactorily answered. Yet the basic concept is reasonably clear. If there is a good overlap between the wavefunctions in the incident and exit channels, so that the collision may take place with a minimum of re-arrangement of the constituent nucleons then the reaction may be said to be direct. With this concept in mind, the most natural description for the reaction is to use either the coupled-channels formalism or the distorted wave method. These and various modifications of these methods have been described and appraised in a recent review article by Satchler 2).

In this thesis, we shall describe the most widely used method, namely the distorted-wave Born Approximation (D.W.B.A). In particular we shall be concerned with the (p,d) reaction at the high proton energy of 156 MeV. This affects the calculation by demanding a large number of partial waves (≈ 30) in both the incident and final channels. However since the whole calculation was performed on a computer, this gives no inherent difficulty.

In the DWA, the cross-section for the collision, can formally be written as a product of two terms 3)

\[
\frac{d\sigma}{d\Omega} = S(\ell) \sigma(\ell, Q, \theta)
\]  

(1.3)
The first term is angle-independent, and is known as the spectroscopic factor. It is a measure of the probability that in the initial nuclear state, all but one of the nucleons will find themselves in an arrangement corresponding to the final state. It is clear that the spectroscopic factor depends only on the wavefunctions of the nuclear states involved, and consequently provides a useful basis for comparison between experiment and the predictions of various nuclear models.

However before such a comparison can be meaningful we must ensure that an adequate description of the reaction term $\sigma(\ell, Q, \Theta)$ has been obtained. Here $\Theta$ is the centre-of-mass scattering angle, and $Q$ the $Q$-value of the reaction. Since the reaction is normally assumed to proceed through the transfer of a single $\ell$-value, we see that the spectroscopic factor therefore manifests itself as a multiplicative constant in the cross-section. As such then, this is perhaps the most difficult quantity to determine, since it requires the experimentalist to produce an accurate absolute measure of the cross-section, and the theoretician to produce an accurate absolute account of the reaction mechanism. Consequently in our analysis of the reaction using the DWBA, we have taken care to stress the effect on absolute values, that various uncertainties in the theory have.

As a test of the DWBA description of the reaction mechanism, we examine in detail in Chapter 4, the $^{12}_C(p,d)$
The Shell model description of the nucleus $^{12}\text{C}$ is used, thus the spectroscopic factor is known to within the limits of the two extreme coupling schemes employed in the shell model, that is the j-j and the L-S coupling schemes. It will be shown that a satisfactory description can be obtained, provided that the local energy approximation is incorporated into the usual DWBA formalism. However the uncertainties and ambiguities in the optical model parameters used, appreciably affect the magnitude of the cross-sections; hence the spectroscopic factor can only be determined to within an accuracy of $30\%$. Nevertheless in Chapter 5, we make a comparison of these experimentally determined spectroscopic factors with the predictions of various nuclear models.

The nuclei under study in this programme are Be$^8$, B$^9$ and C$^{11}$. For such light nuclei in the p-shell, the natural description is in terms of the shell model, although some calculations using the rotational model have been made on the deformed Be$^8$ and B$^9$ nuclei. The predictions and limitations of the shell model have been extensively discussed elsewhere and will not be considered here. However the two basic assumptions are

(1) There exist single-particle orbits, each characterised by a radial quantum number $n$ and an orbital angular momentum $\ell$.

(2) A strong spin-orbit interaction depresses each
\( j = 1 + \frac{1}{2} \) level relative to the corresponding \( j = 1 - \frac{1}{2} \) level where \( j = 1 + s. \)

With these postulates we can write the nuclear shell-model Hamiltonian in the form

\[
H = \sum_{i=1}^{A} H_i + \sum_{i<j} V_{ij} + \alpha \sum_{i=1}^{A} s_i \cdot s_i
\]

(1.4)

where \( H_i \) is the single particle Hamiltonian for a particle in the central "shell-model potential" \( V(r) \). \( V_{ij} \) represents the residual two-body interaction, while the third term is a one-body spin-orbit potential. The two-body interaction parameters and the strength \( \alpha \) of the spin-orbit potential are adjusted so that the model Hamiltonian, \( H \), gives as good a description as it can of the observable properties of the nucleus. It is therefore to be expected that the appropriate values of the various parameters will vary with \( A, n \) and \( l. \)

If the parameter \( \alpha \) is so small that the spin-orbit interaction is negligible, then we have the limit of L-S coupling. Conversely if \( \alpha \) is large so that we can neglect the residual interaction, then we have the \( j-j \) coupling limit. In either of these two schemes, the model Hamiltonian \( H \) can be written directly as the sum of single-particle Hamiltonians, consequently the total wavefunction for the nucleus can be written as a product of single particle wavefunctions. More generally any linear combination of these products satisfies the Schrödinger equation for the model Hamiltonian \( H \). However since the nucleus is a system
of fermions, we need to construct that \( n \)-particle wavefunction which is antisymmetric with respect to the interchange of any two single-particle wavefunctions. This problem will not be considered here.

An improvement over the two extreme coupling schemes is the intermediate coupling approximation which takes into account both the residual interaction and the spin-orbit interaction. No experimental data has been found to date whose interpretation demands conclusively that \( V_{ij} \) be anything other than central, static and charge-independent.

Thus

\[
V_{ij} = (W + MP^\sigma_{ij} + BP^\sigma_{ij} + HP^\sigma_{ij} P^\sigma_{ij}) \psi(r_{ij})
\]

(1.5)

where

\[
P^\sigma = \frac{1}{2}(1 + \mathbf{S}_i \cdot \mathbf{S}_j)
\]

is the spin exchange operator.

Since a number of operators commute with \( H \), the wavefunctions are characterised by certain quantum numbers. Among these are the total angular momentum, \( J \), the parity \( \pi \), (and as long as Coulomb effects are small) the iso-spin \( T \). The procedure then is to write the nuclear wavefunction as a linear combination of either \( L-S \) or \( j-j \) wavefunctions each of specified \( J \pi \) and \( T \). The amplitudes of each state
in this linear combination are then determined by
diagonalising the Hamiltonian \( H \). The radial integrals
involved have been parameterised, i.e. let

\[
L = \int \psi_1^*(r_1) \psi_2^*(r_2) \nu(r_{12}) \psi_1(r_1) \psi_2(r_2) \, dr_1 \, dr_2
\]

\[
K = \int \psi_1^*(r_1) \psi_2^*(r_2) \nu(r_{12}) \psi_1(r_2) \psi_2(r_1) \, dr_1 \, dr_2
\]

then \( L, K \) and the strength of the spin-orbit interaction \( a \) are adjusted individually for each nucleus, such that the low
lying energy level schemes are reproduced. Calculations using
L-S basis wavefunctions have been performed by Inglis \(^7\) and
using j-j basis wavefunctions by Kuruth \(^8\). Both have
used the simplified exchange variant \( W = H = 0, M = 0.8, \)
\( B = 0.2 \). Recently Boyarkina \(^9\) has repeated the calculations
in the L-S basis using the Rosenfeld mixture \( W = 0.13, \)
\( M = 0.93, H = -0.26 \) and \( B = 0.46 \). Boyarkina's wavefunctions
have been used in this thesis.

Nevertheless, no matter which coupling scheme is employed
the nuclear wavefunction is written as a linear combination of
a product of single-particle wavefunctions. Each single-
particle wavefunction satisfies a Schrödinger equation of the
type

\[
\left[ -\frac{\hbar^2}{2M} \nabla^2 + \nu(r) \right] \psi = \varepsilon \psi
\]

(1.7)
In $j-j$ coupling the potential $V(r)$ will contain a spin-orbit term. In principle this potential $V(r)$ should be determined by a self-consistent Hartree-Fock type calculation. In practice however a phenomenological potential is chosen containing arbitrary parameters, which may be configuration dependent.

In Chapter 2, we consider a three-parameter and a four-parameter potential, and show how a single-particle wavefunction can be generated on a computer.

The problem remains of how best to choose the parameters of this phenomenological potential. Previous work has normally used an oscillator potential, since then an analytic expression for the wavefunction is obtainable. However as we shall show, reactions involving the transfer of a single particle must have a specific asymptotic behaviour, and hence a finite potential is required.

Since we are considering the single-particle shell model, we can write the nuclear density distribution as the square of the nuclear wavefunction $^{10}$,

$$\rho(r) = \sum_{i=1}^{A} |\psi_i|^2$$

(1.8)

Consequently, elastic electron scattering proceeding through the Coulomb interaction between the electron charge and the nuclear charge density distribution gives information on $\rho(r)$. Since the interaction is considered to be known, fitting
elastic electron scattering data is equivalent to determining $\rho(r)$ which in turn is related to the phenomenological potential $V(r)$. In Chapter 3, we determine parameters for $V(r)$ for $^{6}\text{Li}$, $^{7}\text{Li}$, $^{9}\text{Be}$, $^{11}\text{B}$ and $^{12}\text{C}$ according to this prescription.

Let us return to consider direct reactions involving the transfer of a single nucleon. We stated earlier that the formal expression for the cross-section can be factorized into a nuclear structure and a reaction term. This is only true when a single particle model is used to describe the nuclear wavefunctions. Consequently in this thesis we shall attempt to give a consistent single-particle shell model treatment for both the elastic electron scattering and the $(p,d)$ results.
CHAPTER 2.
Bound State Wavefunctions.

2.1 Introduction.

In the last chapter we briefly discussed the nuclear shell model, and indicated how a nuclear wavefunction is made up of a product of single particle wavefunctions. In this chapter, we look more closely at the problem of calculating a single particle wavefunction, in particular, with the advent of high-speed computers, we shall describe a technique which has been specifically developed for use on a computer. We shall assume that the single particle of reduced mass $M$, moves in a central potential, then its wavefunction is given by the Schrodinger Equation

$$\left[ -\frac{\hbar^2}{2M} \nabla^2 + V(r) \right] \psi(r, \theta, \phi) = E \psi$$

(2.1)

We shall define the zero of energy by requiring that the potential $V(r) \to 0$ as $r \to \infty$. Then for bound states we must have $E < 0$ and $\psi(r, \theta, \phi) \to 0$ as $r \to \infty$.

Now the Schrodinger equation is separable in the radial and angular co-ordinates giving

$$\psi_{n\ell m}(r, \theta, \phi) = \frac{1}{R} u_{n\ell}(r) Y_{\ell m}(\theta, \phi)$$

(2.2)

where $u_{n\ell}(r)$ satisfies the radial equation
with $U_{nl}(r) \to 0$ as $r \to \infty$. It will be noted that $U_{nl}(r)$ vanishes at $r=0$, and may or may not have other zeros. The integer $n$ is a quantum number which determines the number of nodes in the radial function. We shall use the convention where $n$ is the number of nodes including the one at the origin but excluding the one at infinity.

Now for every bound state $E_{nl} < 0$ there is a point $r_T$, the turning point for which

$$E_{ne} - V(r_T) - \frac{\hbar^2}{2M} \frac{l(l+1)}{r_T^2} = 0$$

(2.4)

If $l > 0$, then there is more than one such turning point. We shall denote by $r_T$ the turning point which is furthest from the origin, then it follows from consideration of the radial equation, that all the nodes of $U_{nl}(r)$, except the one at infinity are between $r=0$ and $r=r_T$. We shall return to this point in discussing the numerical procedure.

Now the most frequently used central potential is the infinite harmonic oscillator

$$V(r) = -V_0 + \frac{1}{2} k r^2$$

(2.5)
where $V_0$, the well depth, and $K$ are arbitrary parameters.

For this potential, the radial equation can be solved analytically

\[
\frac{1}{r} u_{n\ell}(r) = r R_{n\ell}(r)
\]

\[
= N (r/a)^{\ell} e^{-\frac{r}{2a} + \frac{3}{2}} \left[ F_1 \left( -[n-\ell]; \ell + \frac{3}{2}; \frac{r^2}{a^2} \right) \right]
\]

and the energy eigenvalues are given by

\[
E_{n\ell} = -V_0 + \left[ 2(n-\ell) + \ell + \frac{3}{2} \right] \hbar \omega
\]

where $N$ is a normalisation constant, the oscillator length parameter $a = (\hbar^2/K)^{\frac{1}{2}}$, and the energy interval $\hbar \omega = (\hbar^2 K/M)^{\frac{1}{2}}$. While oscillator wavefunctions are widely used, they suffer from being unrealistic in the asymptotic region. The oscillator potential $V(r) \to \infty$ as $r \to \infty$, this has the effect of causing $u_{n\ell}(r)$ to go to zero faster than $e^{-\alpha r}$, which is the asymptotic behaviour of a radial wavefunction generated from a finite well. The constant $\alpha$ is given by $\alpha^2 = \frac{2M}{\hbar^2} |E_{n\ell}|$. We may well expect this difference in asymptotic behaviour to be important in a direct reaction, such as $(p,d)$, where the angular distributions are sensitive to the nuclear surface.
One of the most common finite potentials in use is the Saxon-Wood potential

\[ V(r) = \frac{-V_0}{1 + e^{\left(\frac{r-R}{a}^A\right)}} \]  

(2.8)

where \( R = r_0^A \), the well-depth \( V_0 \), the diffuseness \( a_0 \) and the radius \( r_0 \) are all arbitrary parameters.

However, in order to have a parameter which will affect the 'tail' region without influencing the interior region, we propose a finite oscillator potential defined by

\[ V(r) = \begin{cases} 
-V_0 + \frac{1}{2} K r^2, & t < t_1 \\
(-V_0 + \frac{1}{2} K t_1^2) e^{-\sqrt{t-t_1}}, & t > t_1
\end{cases} \]  

(2.9)

This potential is algebraically the same as the 'smoothed finite potential' introduced by Jackson\(^{18}\), except that the condition that the derivative of the potential should be continuous, was relaxed. This gives the potential four arbitrary parameters, \( V_0 \), \( K \), \( r_1 \) and \( t_1 \) with the requirement that the tail parameter \( t_1 \) is completely independent of the interior. The aim is to see if a single potential can reproduce the separation energies of both the s- and the p-shell protons as well as fitting the elastic electron scattering data. This is considered in the next chapter.
Our problem now is to solve the radial equation (2.3), for potentials of the type (2.8) and (2.9). So far we have tacitly assumed that the eigenvalue of the radial equation is to be the energy $E_{nl}$. However with there being ever increasing data on separation energies and binding energies from ($p,2p$) experiments, we often want to reverse the procedure and given the energy $E_{nl}$, to find one of the arbitrary parameters of the potential $V(r)$. With this problem in mind, we now develop a more general numerical procedure.

2.2 Eigenvalue Problem.

We are required to solve a two-point eigenvalue problem with one boundary at infinity, and with the eigenvalue contained implicitly in a function. That is we must solve an equation of the type

$$\frac{d^2u}{dx^2} + \left[ F(\lambda, x) + g(x) \right] u = 0 \quad (2.10)$$

such that

$$u(0) = 0, \quad u \to 0 \quad \text{as} \quad x \to \infty$$

$$\int_{0}^{\infty} |u|^2 \, dx = 1. \quad (2.11)$$
These boundary conditions can only be satisfied for certain discrete values of $\lambda$, and the problem is to find these values and their corresponding eigenfunctions $u$. We will assume that we can write

$$F(\lambda, x) = \lambda f(\lambda, x)$$

Then the procedure we shall use follows that of Buck\textsuperscript{12} where the somewhat simpler problem of

$$F(\lambda, x) = \lambda f(x)$$

has been solved. Our method differs only in that a Taylor series expansion of $f(\lambda, x)$ has been used to determine the dependence of the function $f$ on the eigenvalue $\lambda$.

Other treatments of this problem have been given by Hartree\textsuperscript{13} and by Fox\textsuperscript{14} for the case when the function $f$ is a constant only; and has been used by Ridley\textsuperscript{15} for a problem in atomic spectroscopy.

The first stage in the calculation is to divide the range of integration into two regions, (1) the interior region, $0 \leq x \leq x_M$ and (2) the exterior region $x_M < x \leq x_E$, where $x_E$ is a suitably chosen large value of $x$ to represent infinity. $x_M$ is known as the matching radius. The procedure is then to use step-by-step integration methods forwards from $x=0$, and
backwards from \( x = x_0 \) and to "match in the middle".

Let the initial guess to \( \lambda \) be \( \lambda_0 \) such that

\[
\lambda = \omega \lambda_0
\]  

(2.13)

then (2.10) can be written

\[
u'' + \left[ \lambda_0 f_0 + g \right] u = \left[ \lambda_0 f_0 - \omega \lambda_0 f \right] u 
\]  

(2.14)

where we have added \( \lambda_0 f_0 u \) to both sides of the equation, and where \( f = f(\lambda, x) \) and \( f_0 = f(\lambda_0, x) \). Now we expand \( f \) in a Taylor series

\[
f(\lambda, x) = f_0 + (\lambda - \lambda_0) \frac{df_0}{d\lambda} + \ldots
\]  

(2.15)

then

\[
u'' + (\lambda_0 f_0 + g) u = (1 - \omega) \lambda_0 \left[ f_0 + \omega \lambda_0 \frac{df_0}{d\lambda} \ldots \right] u
\]  

(2.16)

Pre-multiplying all through by \( u \) and integrating over the range, we get

\[
\int_a^x u [ u'' + (\lambda_0 f_0 + g) u ] \, dx = (1 - \omega) \lambda_0 \int_a^x \left[ f_0 + \omega \lambda_0 \frac{df_0}{d\lambda} \ldots \right] u \, dx
\]  

(2.17)
where we have neglected terms of the order of \((1-o)^2\)

Now let \(u_0\) be a solution of

\[
u_0'' + (\lambda_0 f_0 + g) u_0 = 0
\]  

(2.18)

satisfying the boundary conditions (2.11), where \(u_0\) has been found by forward integration from \(x = 0\) to \(x_M\), denoted by \(u_{of}\),

and by backward integration from \(x_E\), denoted by \(u_{ob}\).

The size of one of these solutions, say \(u_{ob}\), is arbitrary

while the size of the other is fixed by the condition that \(u_0\) be continuous at \(x_M\). Hence \(u_0\) is defined as

\[
u_0 = \begin{cases} 
\frac{u_{ob}(x_M)}{u_{of}(x_M)} u_{of}(x), & x \leq x_M \\
\frac{u_{ob}(x)}{u_{of}(x_M)} u_{of}(x), & x > x_M 
\end{cases}
\]  

(2.19)

However the derivative \(u_0'\) will not necessarily be continuous at \(x = x_M\), in fact there will be a mismatch in the slopes, except when \(\lambda_0\) is the required eigenvalue \(\lambda\).

Thus the integral

\[
I = \int_0^{x_E} u_0 \left[ u_0'' + (\lambda_0 f_0 + g) u_0 \right] \, dx
\]  

(2.20)

vanishes identically everywhere except in the vicinity of \(x_M\),

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hence

\[ I = \int_{x_m^-}^{x_m^+} u_0 u_0'' \, dx + \int_{x_m^-}^{x_m^+} (\lambda_0 f_0 + g) u_0^2 \, dx \]

The second term vanishes, since \( u_0 \) and hence the integrand is continuous across \( x_m \). Hence

\[ I = \left[ u_0 u_0' \right]_{x_m^-}^{x_m^+} - \int_{x_m^-}^{x_m^+} (u_0')^2 \, dx \]

where we have integrated by parts. Again the second term vanishes since \( u_0' \) is not infinite at \( x_m \), so

\[ I = u_0(x_m) \left[ u_0'b - \frac{u_0b}{u_0f} u_0' \right]_{x=x_m} \quad (2.21) \]

\[ = \xi_0 , \quad \text{say} \]

Now we can find an approximate value \( \omega_0 \) for \( \omega \) from equation (2.17) using the function \( u_0 \) in place of \( u \).

Hence we get

\[ \xi_0 = (1 - \omega_0) \lambda_0 \left[ A_0 + \omega_0 \lambda_0 b_0 \right] \quad (2.22) \]

i.e. \( \omega_0 = 1 - \frac{\xi_0}{\lambda_0 A_0 + \omega_0 \lambda_0^2 b_0} \)

where

\[ A_0 = \int_0^{x_E} f_0 u_0^2 \, dx , \quad b_0 = \int_0^{x_E} \frac{\delta f_0}{\delta \lambda} u_0^2 \, dx \]
solving the quadratic for \( \omega_0 \) we get

\[
2 \lambda_0 b_0 \omega_0 = (\lambda_0 b_0 - A_0) + (A_0 - \lambda_0 b_0) \left[ 1 - \frac{4b_0 \gamma_0}{(A_0 + \lambda_0 b_0)^2} \right]^{1/2}
\]  \((2.23)\)

If however \( 4b_0 \gamma_0 > (A_0 + \lambda_0 b_0)^2 \), then our quadratic has complex roots, nevertheless we can find an approximate value for \( \omega_0 \), by replacing \( \omega_0 \) on the right hand side of \((2.22)\) by unity. Hence

\[
\omega_0 = 1 - \frac{\gamma_0}{\lambda_0 A_0 + \lambda_0^2 b_0}
\]  \((2.24)\)

Thus our iteration scheme is now clear. With a trial value for the eigenvalue, we integrate the equation \((2.10)\) numerically, and hence calculate the quantities \( \gamma_0, A_0 \) and \( B_0 \) defined by equations \((2.21)\) and \((2.22)\). Then we can calculate a correction to our initial guess for the eigenvalue from either \((2.23)\) or \((2.24)\), giving

\[
\lambda_1 = \omega_0 \lambda_0
\]  \((2.25)\)

This procedure is then repeated until the required accuracy is reached, hence

\[
\lambda = \omega n \omega_{n-1} \ldots \omega \omega_0 \lambda_0
\]  \((2.26)\)
Now at the $i^{th}$ stage of iteration, we have (2.23), that

$$2B \lambda_{i+1} = (\lambda_i B - A) + (A + \lambda_i B) \left[ 1 - \frac{4B \lambda_i}{(A + \lambda_i B)^2} \right]^{1/2}$$

(2.27)

where we have assumed that $A$ and $B$ are approximately independent of $i$. Expanding binomially, we get

$$\lambda_{i+1} - \lambda_i = -\frac{\lambda_i}{A + \lambda_i B}$$

(2.28)

Hence the condition for convergence

$$\left| \frac{\lambda_{i+1} - \lambda_i}{\lambda_i - \lambda_{i-1}} \right| < 1$$

becomes

$$\left| \frac{\lambda_i}{\lambda_{i-1}} \cdot \frac{A + \lambda_{i-1} B}{A + \lambda_i B} \right| < 1$$

(2.29)

or approximately

$$\left| \frac{\lambda_i}{\lambda_{i-1}} \right| < \left| \frac{\lambda_{i-1}}{\lambda_i} \right|$$

(2.30)

Hence we have deduced the rather obvious conclusion, that for convergence the mismatch in the slopes must be reduced at each stage. If at the $n^{th}$ stage, $\lambda_n \rightarrow 0$, then from the definition (2.21), we have

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that is, when we have converged upon an eigenvalue, our
eigenfunction automatically becomes continuous at \( x = x_M \).

Finally, we make a few remarks about the choice of the
matching radius \( x_M \). This we take to be the last zero of

\[
\lambda f(\lambda, x_M) + g(x_M) = 0 \tag{2.32}
\]

This is precisely what we called the turning point when we were
discussing the radial wave equation. This is the point where
the solution goes over from an oscillatory to an exponential
form, hence our step-by-step integration, forwards for
\( x < x_M \) and backwards for \( x > x_M \) is in just that direction
which keeps the unwanted solution to a minimum. For example
in the asymptotic region, the analytic solution gives an
exponentially increasing solution, as well as the required
decreasing solution. If we integrated step-by-step in the
forward direction for \( x > x_M \), then the computed solution may
well follow the exponentially increasing solution, thus
swamping the required solution. Thus by suitably choosing
our matching radius, we obtain greater numerical stability.
2.3 An Example.

The phenomenological potential for single-particle bound-state wavefunctions is normally written

\[ V(r) = V_c(r) + V_N(r) + V_{s.o.}(r) \hat{\mathbf{\sigma}} \cdot \hat{\mathbf{\sigma}} \]  

(2.33)

where the spin-orbit term is introduced so as to give the correct sequence of nuclear states in the shell model. It is usually taken to be of the Thomas form, that is

\[ V_{s.o.}(r) = \lambda \left( \frac{\hbar}{2mc} \right)^2 \frac{1}{r} \cdot \frac{dV_N(r)}{dr} \]

where the multiplying constant \( \lambda \) is of the order of \( 30 \times 10^7 \). \( V_N(r) \) is the nuclear potential, and \( V_c(r) \) is the Coulomb potential which is approximated to the potential due to a hard sphere of radius \( R_c \). That is

\[ V_c(r) = \begin{cases} \frac{2Z_b(Z_n-Z_b)e^2}{2R_c} \left( 3 - \frac{r^2}{R_c^2} \right), & r < R_c \\ \frac{Z_b(Z_n-Z_b)e^2}{r}, & r \geq R_c \end{cases} \]

where \( Z_b \) is the charge on the bound particle, and \( Z_n \) the charge of the nucleus.

By way of an example, let us calculate the tail parameter \( \mathcal{V} \) of the finite oscillator potential (2.9) corresponding to a given energy \( E_{nl} \). Exactly analogous techniques could be
used to find the diffuseness of the Saxon-Wood potential $a_0$.
For expediency we will drop the spin-orbit term, in principle,
however, its inclusion is straightforward.

It is more convenient not to find the tail parameter $\mathcal{V}$
directly, but instead to find the related quantity
$\lambda = e^{\mathcal{V} r_1}$. This enables one to make the factorisation (2.12)

$$f(\lambda, +) = \lambda f(\lambda, +)$$

more simply. Then the radial wave equation (2.3) is expressed in the form (2.10), when

$$f(\lambda, +) = -\frac{2M}{\hbar^2} (-\mathcal{V}_0 + \frac{1}{2} \kappa r_1^2)(\lambda)^{-\mathcal{V}/r_1}, \quad r > r_1$$

Note that the function $f(\lambda, r)$ has a finite discontinuity at $r = r_1$, however the error caused in the numerical procedure of taking the average value over the discontinuity is negligible.

Hence it is straightforward to calculate the correction to an initial guess of $\lambda_0$ using (2.22)

$$\omega_0 = 1 - \frac{\lambda_0}{\lambda_0 A_0 + \omega_0 \lambda_0^2 B_0}$$

and $A_0 = \int_0^{x_0} f_0 u_0^2 \, dr$ and $B_0 = \int_0^{x_0} \frac{\partial f_0}{\partial \lambda} u_0^2 \, dr$, where $u_0$ defined by (2.19) has been found by numerical integration.
However these integration methods require a value of $u_0$ and $u_0'$ at the starting points. For the backward integration from $r_E$, an asymptotic solution to the radial equation with $V_N(r) = 0$ can be found. In the case of bound neutrons $V_C(r) = 0$, we have

$$u(\tau) = \lim_{\tau \to \infty} e^{-\alpha \tau} \left[ 1 + \frac{\ell(\ell+1)}{2\alpha \tau} + \frac{\ell(\ell-1)(\ell+1)(\ell+2)}{2! \left(2\alpha \tau\right)^2} + \ldots \right]$$

while for bound protons the corresponding asymptotic wavefunction is

$$u(\tau) = \lim_{\tau \to \infty} e^{-\alpha \tau} \left[ 1 + \frac{\ell(\ell+1) - \kappa(\kappa+1)}{2\alpha \tau} + \frac{[\ell(\ell+1) - \kappa(\kappa+1)] [\ell(\ell+1) - (\kappa+1)(\kappa+2)]}{2! \left(2\alpha \tau\right)^2} + \ldots \right]$$

where $\kappa = Z_b (Z_n - Z_b) M_e^2 / \hbar^2$ and $\alpha^2 = \frac{2M}{\hbar^2} |E_{n1}|$.

For the forward integration, we have $u(0) = 0$ and $u(0)'$ can be arbitrarily chosen, since this affects the size of the forward solution only, and this will be matched to the backward solution at $r_E$.

In the case of the finite oscillator potential, there are four arbitrary parameters, consequently it may be required that two of these parameters be found simultaneously corresponding to two known energy eigenvalues. An example of this is for Li$^6$, for which the binding energies of 1p and 1s protons have been found from ($p,2p$) experiments.
By choosing the well-depth parameter $V_0$ to correspond to
the $1s$ binding energy and the tail parameter $\nu$ to the $l_p$
binding energy, the two eigenvalue problems become almost
independent. Thus rapid convergence was obtained, using
the decoupled procedure of finding $V_0$ with a trial $\nu$, and
then using this value of $V_0$ to find $\nu$. 
CHAPTER 3.
Electron Scattering.

3.1 Born Approximation Formalism

High energy electrons have one advantage over nuclear projectiles as a probe of nuclear structure. In the case of nuclear projectiles the precise form of the interaction between the projectile and the target nucleus is unknown, and consequently is frequently parameterised. Thus the interpretation of data in terms of nuclear structure is obscured by the uncertainties in the reaction mechanism. However, when electrons are used as projectiles, the interaction being electromagnetic, is well known and information on the structure of the nucleus is obtained directly. This has been illustrated by Hofstadter \(^{31}\) using elastically scattered electrons to measure the charge distributions of nuclei in their ground state.

In this chapter we shall calculate the cross-section for the scattering of electrons by nuclei in Born approximation. This is applicable when \( Z \propto \ll 1 \), where \( \propto = 1/137 \) is the fine structure constant, hence we expect this approximation to be accurate for light nuclei, particularly in the \( lp \) shell. A numerical comparison between the Born approximation and the partial wave analysis is made in paragraph 3.3.

Consider now an incident electron of momentum \( h\kappa \).
scattered by a nucleus, and coming off with $h\mathbf{k}_f$, then the momentum transfer $h\mathbf{q}$, and the scattering angle $\Theta$ are defined by the relations

$$a = k_1 - k_f$$

$$k_1 \cdot k_f = k_1 k_f \cos \Theta$$

As the electron scatters from the nucleus, its charge and current interact with the nuclear charge, current and magnetisation densities $\rho_N(r)_{f1}$, $\mathbf{j}_N(r)_{f1}$ and $\mathbf{\mu}_N(r)_{f1}$. The multipole matrix elements are defined as (82)

$$\langle f | M( C, \mu, q ) | i \rangle = \frac{(2\lambda+1)!!}{q^{\lambda+1} (\lambda+1) \lambda^{\lambda+1}} \int \mathbf{j}_\lambda(qr) Y_{\lambda\mu}(\hat{r}) \rho_N(r)_{f1} d^3r$$

$$\langle f | M( E, \mu, q ) | i \rangle$$

$$= \frac{(2\lambda+1)!!}{q^{\lambda+1} (\lambda+1) \lambda^{\lambda+1}} \int \left[ \nabla \mathbf{j}_\lambda(qr) Y_{\lambda\mu}(\hat{r}) \right] \cdot \left[ \frac{1}{c} \mathbf{j}_N(r)_{f1} + \nabla \times \mathbf{\mu}_N(r)_{f1} \right] d^3r$$

$$\langle f | M( M, \mu, q ) | i \rangle$$

$$= -i \frac{(2\lambda+1)!!}{q^{\lambda} (\lambda+1)} \int \left[ \mathbf{L} \mathbf{j}_\lambda(qr) Y_{\lambda\mu}(\hat{r}) \right] \cdot \left[ \frac{1}{c} \mathbf{j}_N(r)_{f1} + \nabla \times \mathbf{\mu}_N(r)_{f1} \right] d^3r$$

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The parity of the Coulomb and transverse electric multipole operators, $M(C \lambda)$ and $M(E \lambda)$, is $(-)^{\lambda}$, while the parity of the transverse magnetic multipole operator $M(M \lambda)$ is $(-)^{\lambda+1}$. If $J_i$ and $J_f$ are the initial and final nuclear spins, then using the Wigner-Eckhart theorem we have

$$\langle J^M_f | M(\lambda, \mu, q) | J^M_i \rangle$$

$$= (-)^{J_f - J_i} \frac{\langle \lambda \mu J^M_i | J^M_f \rangle}{(2J_f + 1)^{\lambda/2}} \langle J_f | M(\lambda, q) | J_i \rangle$$

The reduced matrix elements defined by (3.3) contain the angular momentum and parity selection rules for the nuclear transition

$$| J_f - J_i | \leq \lambda \leq J_f + J_i$$

$$\Pi_f + \Pi_i + \Pi_{\lambda} = \text{even}$$

The cross-section for the transition $J_i \rightarrow J_f$ is

$$d\sigma = \sum_{\lambda=0}^{\infty} d\sigma_{C\lambda} + \sum_{\lambda=1}^{\infty} d\sigma_{E\lambda} + \sum_{\lambda=1}^{\infty} d\sigma_{M\lambda}$$

$$d\sigma_{C\lambda} = \left( \frac{k_i}{k_f} \right)^2 \left( \frac{e}{\hbar c} \right)^2 \frac{4\pi}{[(2\lambda + 1)!]^2} \frac{2\lambda}{k_i^2}$$

$$B(C\lambda, q, J_f, J_i) V_L(\mathcal{O}) \ d\Omega$$

(3.5)
\[
\frac{d\sigma}{d\Omega}(E) = \left(\frac{k_f}{k_i}\right)^2 \left(\frac{e}{\hbar c}\right)^2 \frac{4\pi(\lambda+1)}{\lambda \left[(2\lambda+1)!!\right]^2} \frac{q^{2\lambda}}{k_i^2} \mathcal{B}(E, q, J_f, J_i) V_T(\Theta) d\Omega
\]

\[
\frac{d\sigma}{d\Omega}(M, \lambda) = \left(\frac{k_f}{k_i}\right)^2 \left(\frac{e}{\hbar c}\right)^2 \frac{4\pi(\lambda+1)}{\lambda \left[(2\lambda+1)!!\right]^2} \frac{q^{2\lambda}}{k_i^2} \mathcal{B}(M, \lambda, q, J_f, J_i) V_T(\Theta) d\Omega
\]

where

\[
\mathcal{B}(\lambda, q, J_f, J_i) = \frac{1}{2J_i+1} \left| \left\langle J_f \mid M(\lambda, q) \mid J_i \right\rangle \right|^2
\]

are nuclear form factors which in the limit \( q \to 0 \) become the reduced transition probabilities defined by Alder et al. \(^\text{83}\) The angular factors are given by

\[
V_L(\Theta) = \frac{4 \cos^2 \Theta/2}{\left[1 + \left(\frac{k_f}{k_i}\right)^2 - 2 \left(\frac{k_f}{k_i}\right) \cos \Theta \right]^2}
\]

\[
V_T(\Theta) = \frac{k_i^2 + k_f^2 + k_i k_f (1 - \cos \Theta)}{2k_f^2 (1 - \cos \Theta) \left[1 + \left(\frac{k_f}{k_i}\right)^2 - 2 \left(\frac{k_f}{k_i}\right) \cos \Theta \right]}
\]

Let us now specifically consider the elastic scattering of electrons of energy \( E \). Then

\[
k_i = k_f = k = \frac{E}{\hbar c}
\]

\[
q = \frac{2E}{\hbar c} \sin \Theta/2
\]

and hence the angular functions (3.6) become

\[
V_L(\Theta) = \frac{\cos^2 \Theta/2}{4 \sin^4 \Theta/2} \quad V_T(\Theta) = \frac{1 + \sin^2 \Theta/2}{8 \sin^4 \Theta/2}
\]

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Furthermore $J_f = J_i = J$ and there is no parity change, so the selection rules (3.4) become

$$0 \leq \lambda \leq 2J$$

$$\Delta \Pi = 0$$

(3.8)

This means we can have $E0, M1, E2, M3, \ldots$ multipoles and so on. Substituting in (3.5) the total elastic cross-section can be written

$$\frac{d\sigma}{dt} = \left(\frac{e^2 \alpha^2}{2\hbar^2}\right) \frac{\cos^2 \Theta/2}{\sin^4 \Theta/2} \left[ F_L^2(q) + \left(\frac{1}{2} + \tan^2 \Theta/2\right) F_T^2(q) \right]$$

(3.9)

where the longitudinal form factor $F_L(q)$ and transverse form factor $F_T(q)$ are given by

$$F_L^2(q) = \sum_{\lambda = 0}^{\lambda_{\text{even}}} \frac{4\pi q^{2\lambda}}{(2\lambda + 1)!!} \mathbb{B}(C \lambda, q)$$

(3.10)

$$F_T^2(q) = \sum_{\lambda = 1}^{\lambda_{\text{odd}}} \frac{4\pi (\lambda + 1)\lambda}{(2\lambda + 1)!!} q^{2\lambda} \mathbb{B}(M \lambda, q)$$

$$+ \sum_{\lambda = 2}^{\lambda_{\text{even}}} \frac{4\pi (\lambda + 1)\lambda}{(2\lambda + 1)!!} q^{2\lambda} \mathbb{B}(E \lambda, q)$$

(3.11)

The term preceding the square bracket is usually called the Mott cross-section for electron elastic scattering by a point unit charge

$$\sigma_{\text{Mott}} = \left(\frac{e^2}{2\hbar^2}\right) \frac{\cos^2 \Theta/2}{\sin^4 \Theta/2}$$

(3.11)
An estimate of the relative importance of charge and magnetic scattering has been given by Bishop. The Coulomb interaction depends on $Z\epsilon$ and the magnetic moment interaction on $q\mu$, thus the ratio of the two effects will be of the order

$$\frac{V_T(\theta) F_T^2(q)}{V_L(\theta) F_L^2(q)} \approx \left( \frac{1}{Z\epsilon} \tan^2 \theta/2 \right) \frac{q\mu}{Z\epsilon}$$

For $^{9}\text{Be}$ ($j = 3/2$) and $\mu/\mu_N = -1.18$, the above ratio becomes approximately $4\%$ for $q = 1 \text{fm}^{-1}$, $E = 300 \text{MeV}$ and hence $\theta \approx 40^\circ$. However this ratio increases with $\theta$ and at backward angles ($\theta \sim 180^\circ$), the magnetic affects dominate.

Nevertheless in the work reported here, we shall neglect the transverse form factor $F_T(q)$ and write the cross-section for elastic electron scattering as

$$\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} F_L^2(q)$$

(3.14)

where

$$F_L^2(q) = \sum_{\lambda = 0}^{\text{even}} \frac{4\pi q^2 \lambda}{(2\lambda + 1)!!} B(\lambda, q)$$

From the definition (3.5) and the inverse of the Wigner-Eckhart theorem (3.3) with $M_e = M_T = M_\theta$,
\[ B(C, q) = \frac{2\lambda + 1}{2J+1} \sum_{M} |\langle JM | M(C, q) | JM \rangle|^2 S_{\mu, 0} \]

\[ = \frac{2\lambda + 1}{2J+1} \frac{((2\lambda + 1)!!)^2}{q^{2\lambda}} \sum_{M} \left| \int j_{\lambda}(qr) Y_{\lambda 0}(\hat{r}) \rho(r) \, d^3r \right|^2 \quad (3.15) \]

having used the definition (3.2). Interchanging the order of summation,

\[ F_L^2(q) = \]

\[ \frac{1}{2J+1} \sum_{M} \left| \int \sum_{\lambda=0}^{\text{even}} \left[ 4\pi (2\lambda + 1) \right]^{3/2} j_{\lambda}(qr) Y_{\lambda 0}(\hat{r}) \rho(r) \, d^3r \right|^2 \quad (3.16) \]

If the z-axis is chosen along the direction \( q \), then using the expansion for a plane wave \( e^{iq \cdot r} \), (3.16) can be written

\[ F_L^2(q) = \frac{1}{2J+1} \sum_{M} \left| \int e^{iq \cdot r} \rho(r) \, d^3r \right|^2 \quad (3.17) \]

If it is assumed that the density distribution is spherically symmetric, then it cannot depend on magnetic quantum numbers, hence the summation over \( M \), merely gives \( (2J+1) \) times the integral. Thus

\[ F_L(q) = \int e^{iq \cdot r} \rho(r) \, d^3r \quad (3.18) \]
Carrying out the angular integrations, one arrives at the familiar expression

\[ F_L(q) = \frac{4\pi}{q} \int \sin qr \ \rho(r) \ r \ dr \quad (3.19) \]

If we normalise the density distribution, such that

\[ 4\pi \int \rho(r) \ r^2 \ dr = 1 \quad (3.20) \]

we see that in the limit \( q \to 0 \), \( F_L(q) \to 1 \). We shall use this normalisation for our form factor throughout.

Furthermore, if for small \( q \), one expands \( \sin qr = qr - \frac{1}{6}q^3r^3 \ldots \) then

\[ F_L(q) = 1 - \frac{1}{6} \langle r^2 \rangle q^2 \ldots \quad (3.21) \]

where \( \langle r^2 \rangle \) is the mean square radius of the density distribution defined by

\[ \langle r^2 \rangle = 4\pi \int \rho(r) \ r^2 \ dr \quad (3.22) \]

Let us summarise then. Experimentally the cross-section \( \frac{d\sigma}{d\Omega} \) is measured for elastically scattered electrons. By dividing by \( \sigma_{\text{Mott}} \) one obtains the form factor \( F_L^2(q) \). In principle now we can obtain the density distribution by merely taking the Fourier transform of (3.19), thus obtaining

\[ \rho(r) = \frac{1}{2\pi^2 r} \int_0^\infty \frac{F_L(q)}{q} \sin qr \ q \ dq \quad (3.23) \]
Hence the mean square radius \( \langle r^2 \rangle \) can be found from (3.22). In practice, however, the range of \( q \) experimentally obtained is not sufficient to allow the integral (3.23) to be reliably computed. Instead then one constructs \( \rho(r) \) from some nuclear model, which may therefore contain one or two arbitrary parameters, and which are thereby determined by fitting to the experimental \( F_L(q) \).

Lastly, when we are considering nucleon scattering by a nucleus, then if the incident particle is charged it moves in the combined Coulomb and nuclear field of the target nucleus. The potential due to the nuclear charge may be derived from the known distribution of charge in the nucleus \( \rho(r) \) by applying Poisson's equation. In practice however it is sufficient to take the Coulomb potential as that due to a uniformly charged sphere of radius \( R_c = r_c A^{1/3} \), with constant density inside the sphere and zero outside. Then one can relate \( R_c \) to the mean square radius giving

\[
R_c^2 = \frac{5}{3} \langle r^2 \rangle \\
\text{i.e. } r_c = \left[ \frac{5}{3} \langle r^2 \rangle \right]^{1/2} A^{-1/3} \tag{3.24}
\]

We shall determine \( r_c \) from electron scattering data by applying (3.24) and use it when considering nucleon scattering.
3.2 Single Particle Model.

In the single particle model, the nucleus is considered as a collection of nucleons, and the nuclear charge density operator is the sum of the charge density operator for the individual nucleons \(82\)

\[
\rho_N(\mathbf{r})_{\text{op}} = |e| \sum_{j=1}^{A} \delta_j \delta(\mathbf{r} - \mathbf{r}_j) \tag{3.25}
\]

Thus

\[
\rho_N(\mathbf{r})_{\text{fi}} = |e| \int \sum_{j=1}^{A} \delta(\mathbf{r} - \mathbf{r}_j) \psi^*(1..A) \psi_j(1..A) d^3r_1 \ldots d^3r_A \tag{3.26}
\]

where

\[
\delta_j = \frac{1}{2} \left( 1 + 2t_{jz} \right) \xi_p + \frac{1}{2} \left( 1 - 2t_{jz} \right) \xi_n
\]

\[
+ \frac{i}{2}, \text{proton} \quad \xi_p = 1
\]

\[
- \frac{i}{2}, \text{neutron} \quad \xi_n = 0
\]

For elastic electron scattering, then, the ground state charge density can be written

\[
\rho(\mathbf{r}) = |e| \left\langle JM \left| \sum_{j=1}^{A} \delta(\mathbf{r} - \mathbf{r}_j) \delta_j \right| JM \right\rangle \tag{3.27}
\]

Since the density operator has been written as a sum of single-particle operators, then

\[
\rho(\mathbf{r}) = Z |e| \left\langle JM \left| \delta(\mathbf{r} - \mathbf{r}_1) \right| JM \right\rangle \tag{3.28}
\]
Hence the normalisation becomes

$$\int \varphi(r) \, d^3r = Z |\varphi|$$

(3.29)

Now from (3.14) the longitudinal form factor

$$F_L^2(q) = \sum_{\lambda} \frac{4 \pi q^{2\lambda}}{(2\lambda + 1)!!}^2 \left\{ (C_{\lambda}, q) \right\}^2$$

$$= \frac{1}{2J+1} \sum_{\lambda} \frac{4 \pi q^{2\lambda}}{(2\lambda + 1)!!}^2 \left| \langle J | M(C_{\lambda}, q) | \| J \rangle \right|^2$$

(3.30)

The reduced matrix element can be calculated from the full matrix element for the substate $M=J$

$$\left| \langle J | M(C_{\lambda}, q) | \| J \rangle \right|^2 = \frac{2J+1}{\langle JJ \lambda \mu | JJ \rangle^2} \left| \langle JJ | M(C_{\lambda}, \mu, q) | JJ \rangle \right|^2$$

(3.31)

Hence from the definition (3.2) of the multipole matrix element, we have

$$F_L^2(q) = \sum_{\lambda} \left[ \int j_\lambda(qr) \, Y_{\lambda,0}(\hat{r}) \left\langle \varphi(r) \right\rangle_{M=J} \, d^3r \right]^2$$

$$= \sum_{\lambda} f_{\lambda}^2(q)$$

(3.32)

where we have introduced the factor $1/(Ze)^2$ so as to maintain our normalisation that $F_L(q) \to 1$ as $q \to 0$. This has been necessary due to the change in the normalisation of the density distribution (3.29). The quantity $\langle \varphi(\hat{r}) \rangle_{M=J}$ is the
expectation value of the density operator evaluated in the substate $M=J$.

Let us now consider the effect of a non-spherical charge distribution on electron scattering. Following Meyer-Berkhout, we expand $\langle \mathcal{F}(r) \rangle_{M=J}$ such that

$$\langle \mathcal{F}(r) \rangle_{M=J} = \rho_0(r) + \rho_2(r) Y_{2,0}(r) \quad (3.33)$$

Since no net charge is associated with $\rho_0$, we have the normalisation

$$4\pi \int \rho_0(r) r^2 \, dr = Z |e| \quad (3.34)$$

Now $\rho_0(r)$ is the spherically symmetric part of the density distribution. Thus from the definition (3.27), $\rho_0(r)$ will depend only on the radial part of the nuclear wavefunction, which in the single-particle model will be some product of single-particle wavefunctions. Thus for lp-shell nuclei, which have a proton configuration $(1s)^2 (1p)^Z-2$, we expect

$$\rho_0(r) = c_1 R_{10}^2(r) + c_2 R_{11}^2(r) \quad (3.35)$$

where $R_{10}(r)$ and $R_{11}(r)$ are the radial single-particle wavefunctions for 1s and 1p protons respectively. These are computed by the method given in Chapter 2. Let them be normalised such that

$$4\pi \int R_{10}^2(r) r^2 \, dr = \frac{2}{Z} \quad (3.36)$$

$$4\pi \int R_{11}^2(r) r^2 \, dr = \frac{(Z-2)}{Z}$$
then substituting (3.35) in (3.24), we get

\[ \frac{2}{Z} c_1 + \frac{Z-2}{Z} c_2 = Z \epsilon \]

One possible solution is that \( c_1 = c_2 = Z \epsilon \), then

\[ \rho_0(r) = Z \epsilon |r| R_{10}^2(r) + R_{11}^2(r) \]

and hence from (3.32)

\[ f_0(q) = \frac{4\pi}{Z \epsilon} \int j_0(qr) Y_{0,0}(\hat{\mathbf{r}}) \rho_0(r) \, d^3r \]

\[ = \frac{4\pi}{q} \int r \sin qr \left[ R_{10}^2 + R_{11}^2 \right] r^2 \, dr \]  \hspace{1cm} (3.38)

Now in the expansion (3.33) of the expectation value

\[ \langle \hat{\rho}(r) \rangle_{M=J} \], the term \( \rho_2(r) Y_{2,0} \) is a measure of the deformation of charge density. This can be related to the quadrupole moment, \( Q \), which is defined as the expectation value of \( \sqrt{\frac{16\pi}{5}} r^2 Y_{2,0}(\hat{\mathbf{r}}) \) in the substate \( M=J \),

\[ Q = \sqrt{\frac{16\pi}{5}} \langle JJ | r^2 Y_{2,0} | JJ \rangle \]

\[ = \sqrt{\frac{16\pi}{5}} \int \langle \hat{\rho}(r) \rangle_{M=J} r^2 Y_{2,0}(\hat{\mathbf{r}}) \, d^3r \]

\[ = \sqrt{\frac{16\pi}{5}} \int \rho_2(r) r^4 \, dr \]  \hspace{1cm} (3.39)

Now if the quadrupole moment arises from protons moving in the p-shell of an undeformed potential, then the radial dependence of \( \rho_2 \) is the same as that of the p-shell part of \( \rho_0 \).
Let us write

\[ \phi_2(r) = C R_{1l}^2(r) \quad (3.41) \]

then from (3.40) we have

\[ C = \left( \frac{5}{16\pi} \right)^{\frac{1}{2}} \frac{Q}{\langle r^2 \rangle_p} \quad (3.42) \]

where \( \langle r^2 \rangle_p = \int R_{1l}^2(r) r^4 \, dr \). Thus from (3.32)

\[ f_2(q) = \left( \frac{4\pi}{2l+1} \right)^{\frac{1}{2}} \int \frac{j_2(qr) Y_{2,0}}{\langle JJ20 | JJ \rangle} \langle \phi(r) \rangle_{M=J} \frac{d^3r}{r_1^2} \cdot (3.43) \]

where \( P_J \) is called the quadrupole projection factor and is given by

\[ P_J = \langle JJ20 | JJ \rangle^2 = \frac{J(2J-1)}{(J+1)(2J+3)} \quad (3.44) \]

Summarising then, in the single-particle model, the longitudinal form factor \( F_L(q) \) can be expressed in terms of radial single-particle wavefunctions. In particular, for \( l_p \)-shell nuclei we have

\[ |F_L^2(q)| = |f_0^2(q)| + |f_2^2(q)| \quad (3.45) \]

where \( f_0(q) \) and \( f_2(q) \) are given by (3.38) and (3.43) in...
terms of $R_{10}(r)$ and $R_{11}(r)$. These radial wavefunctions are generated from some finite potential and normalised according to (3.36). The cross-section for elastic electron scattering is then given by

$$\frac{d\sigma}{d\Omega} = c^2_{\text{Mott}} F_L^2(q)$$

(3.46)

Now there are two corrections which can be made to improve the shell model expression for $F_L(q)$. First, in the expression for the density distribution (3.28)

$$\varphi(r) = |e| \langle JM | \sum_j \delta(r-r_j) \xi_j | JM \rangle$$

we have assumed, by using a delta-function, that the proton can be represented by a point particle. Willek replaces the delta-functions by finite spatial density functions $\Delta(r-r_j)$, then the form factor which was

$$F_L(q) = \int e^{iQ \cdot r} \varphi(r) \, d^3r = |e| \sum_j e^{iQ \cdot r_j} \xi_j$$

now becomes

$$F_L'(q) = |e| \int e^{iQ \cdot r} \sum_j \xi_j \Delta(r-r_j) \, d^3r$$

$$= g(q) |e| \sum_j e^{iQ \cdot r_j} \xi_j$$

$$= g(q) F_L(q)$$

where the proton form factor is

$$g(q) = \int e^{iQ \cdot r'} \Delta(r') \, d^3r'$$

(3.47)
Thus the finite size of the proton is taken into account by multiplying the form factor $F_L(q)$ by the proton form factor $g(q)$. The experimental form factor of the proton's charge distribution can be fitted by a Gaussian

$$g(q) = e^{-a_p^2 q^2} \text{ with } a_p^2 = 0.43 \text{ fm}^2$$

(3.48)

Second, if shell model wavefunctions are used for the nucleon wavefunctions, a correction must be made for their lack of translational invariance due to the assumption of a fixed potential well for the nucleons to move in. This problem has been discussed by Barker and Tassie, who show that if oscillator wavefunctions are used, the effect of the centre of mass motion can also be factored out as a simple Gaussian, i.e. the form factor should be multiplied by

$$h(q) = e^{q^2 a^2 / 4A}$$

(3.49)

where $a$ is the oscillator length parameter

$$a = (\hbar / M_w)^{\frac{1}{2}}$$

(3.50)

This parameter can be found with fair accuracy for $lp$-nuclei, either from considerations of Coulomb energies as done by Sengupta, or from fitting elastic electron cross-sections.

Although in our work we are not using an infinite oscillator potential for shell model wavefunctions, we shall
nevertheless use the correction (3.49). For completeness then we give in Table 1., the oscillator parameter \( a \), taken from reference 85 and 88.

<table>
<thead>
<tr>
<th>Coulomb Energies</th>
<th>Electron Scattering</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li(^6)</td>
<td>1.82</td>
</tr>
<tr>
<td>Li(^7)</td>
<td>1.64</td>
</tr>
<tr>
<td>Be(^9)</td>
<td>1.78</td>
</tr>
<tr>
<td>B(^{11})</td>
<td>1.55</td>
</tr>
<tr>
<td>C(^{12})</td>
<td>1.62</td>
</tr>
</tbody>
</table>

Thus this correction for the centre of mass motion, and the finite proton size correction can be put together into one factor

\[
\Phi(q) = g(q) h(q) = \exp \left\{ -\frac{1}{4}q^2(a_p^2 - a^2/A) \right\}
\]

and hence the cross-section is given by

\[
\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} |\Phi(q)|^2 \left[ f_0^2(q) + f_2^2(q) \right]
\]

where \( f_0(q) \) and \( f_2(q) \) are given by (3.38) and (3.43) respectively.
3.3 Comparison with Experiment

Experimental data of elastic electron scattering on \( \text{Li}^{6, 89), 90}, \text{Li}^{7, 90}, \text{Be}^{9, 85}, \text{B}^{11, 85} \), and \( \text{Cl}^{12, 91), 92} \) is at present available for \( 1p_{3/2} \) nuclei. A comparison of the experimental longitudinal form factor, deduced from the cross-section using (3.46) is made with the theoretical value, obtained from equation (3.45) and corrected using (3.51). In this comparison, the radial wavefunctions have been obtained by solving the bound-state problem using a Saxon-Wood potential. A spin-orbit term has been included of strength \( \lambda = 30 \). The well-depth \( V_0 \) was chosen so that the binding energy for a \( 1p_{3/2} \) proton fitted the experimentally determined value from (p, 2p) experiments of Tyron\(^{17} \). The radius parameter \( r_0 \) and diffuseness \( a_0 \) were systematically varied until a good fit with experiment was obtained. The philosophy here is that we expect the Born approximation formalism to be suitably accurate for this fitting procedure to determine the parameters of our model density distribution, which in turn was constructed from a single-particle shell model.

A partial wave analysis programme has been written by Swift\(^{23} \) to calculate the \( EO \) contribution to the elastic electron scattering cross-section, starting with a model density distribution. In figure 1., we present a comparison of the longitudinal form factor \( F_L(q) \) for
Fig. 1. Comparison of Born Approx. and Partial Wave Analysis of elastic electron scattering on Li$^6$ & C$^{12}$
Li$^6$ and C$^{12}$ calculated using Born approximation and using partial wave analysis. Clearly the agreement is very good, the Born approximation only breaking down in the region of the first Born minimum.

It is also of interest to see how much the form factor $F_L(q)$ is affected by the correction $\tilde{g}(q)$ in equation (3.51). In figure 2, we plot $F_L(q)$ for Li$^6$ and C$^{12}$, with and without this correction. For small q-values it has little effect, but becomes increasingly more important for larger q-values. Since most difficulty in fitting the experimental results occurs at large q-values, this correction is important in the determination of 'best-fit' parameters.

All of the considered nuclei, except C$^{12}$, are known to have quadrupole moments. A recent compilation of nuclear moments has been made in the Nuclear Data Sheets\textsuperscript{94} series, consequently we shall take the quadrupole moment as a known quantity, and not treat it as an adjustable parameter. The values used in this calculation are given in Table 2.

<table>
<thead>
<tr>
<th>Quadrupole Moment, $Q$, in units $10^{-26}$ cm$^2$</th>
<th>$\frac{Q}{\sqrt{P^{3/2}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li$^6$</td>
<td>-0.08</td>
</tr>
<tr>
<td>Li$^7$</td>
<td>-4</td>
</tr>
<tr>
<td>Be$^9$</td>
<td>±3</td>
</tr>
<tr>
<td>B$^{11}$</td>
<td>±4</td>
</tr>
</tbody>
</table>
Fig. 2. Elastic Electron scattering on Li$^6$ and C$^{12}$ showing effect of the correction $Q(q)$
In figures 3 to 7 we present the theoretical and experimental form factors $F_L(q)$ plotted as a function of $q^2$. A search has been carried out over the parameter space spanned by $r_0$ and $a_0$. In the case of Li$^7$ and Be$^9$ a considerable ambiguity in the choice of parameters remains, and to a lesser extent also in Li$^6$. For B$^{11}$ and C$^{12}$ however a very precise determination of $r_0$ and $a_0$ is possible, since the experimental data passes beyond the first Born minimum. The parameter $r_0$ is very sensitive to the position of this minimum.

As an insert in the figures 3, 4, and 5, we have plotted the locus of the ambiguities in the parameters $r_0$ and $a_0$. In the case of Li$^7$, this is certainly due to the inadequate experimental data available. For Be$^9$, however, considerable difficulty was had in fitting the data, and the plotted best fit can only be described as reasonable. The difficulty is to account for the large q-values in the region of the Born minimum; this seemingly requires an unusually large diffuseness parameter. However, of the quadrupole moments listed in Table 2, Be$^9$ is the most uncertain. An investigation into the effect of increasing the quadrupole moment shows that improvement in the fit to the data can be obtained; but this still requires an unusually large diffuseness parameter.

Subject to the ambiguities listed above, we give in Table 3, the potential parameters which fit the $lp_{3/2}$
Fig. 3. Elastic Electron Scattering on Li$^6$
Experimental Values are taken from refs: 89 and 90. The ambiguities in the model parameters are plotted in the inset.

- 53 -
Fig. 4. Elastic Electron Scattering on Li$^7$
Experimental Values are taken from
Ref: 90.
Fig. 5. Elastic Electron Scattering on Be\textsuperscript{9}
Experimental values are taken from ref. 85. The contribution from \textsuperscript{10}B\textsubscript{0} and \textsuperscript{10}B\textsubscript{2} terms are shown by dashed lines.
Fig. 6. Elastic Electron Scattering on B''
Experimental values are taken from ref. 85
Fig. 7. Elastic Electron Scattering on C\textsuperscript{12}
Experimental values have been taken
from refs: 91 & 92
### Table 3

<table>
<thead>
<tr>
<th></th>
<th>$v_0$</th>
<th>$r_0$</th>
<th>$a_0$</th>
<th>$\lambda$</th>
<th>$r_p$</th>
<th>$E_s$</th>
<th>$r_0$</th>
<th>$\langle r^2 \rangle^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li$^6$</td>
<td>50</td>
<td>1.45</td>
<td>0.92</td>
<td>30</td>
<td>4.9</td>
<td>17.9</td>
<td>1.92</td>
<td>2.70 ± 0.1</td>
</tr>
<tr>
<td>Li$^7$</td>
<td>60</td>
<td>1.45</td>
<td>0.97</td>
<td>30</td>
<td>11.8</td>
<td>26.0</td>
<td>1.65</td>
<td>2.44</td>
</tr>
<tr>
<td>Be$^9$</td>
<td>76</td>
<td>1.20</td>
<td>1.26</td>
<td>30</td>
<td>16.4</td>
<td>30.7</td>
<td>1.57</td>
<td>2.53 ± 0.1</td>
</tr>
<tr>
<td>B$^{11}$</td>
<td>54</td>
<td>1.25</td>
<td>0.49</td>
<td>30</td>
<td>10.9</td>
<td>26.5</td>
<td>-</td>
<td>2.36 ± 0.1</td>
</tr>
<tr>
<td>C$^{12}$</td>
<td>54</td>
<td>1.32</td>
<td>0.45</td>
<td>30</td>
<td>14.7</td>
<td>29.5</td>
<td>1.36</td>
<td>2.42 ± 0.1</td>
</tr>
</tbody>
</table>
binding energy and the elastic electron scattering data. For completeness, we also give the 1s binding energy predicted by this potential, and the experimental value as given by Tyren\textsuperscript{17}). As can be seen from the table, there is no consistent trend in the parameters, which is probably due to the differing amounts of quadrupole $E2$ scattering contained in the longitudinal form factor. For the case of Be$^9$ and B$^{11}$, we plot the contribution to the form factor from quadrupole scattering in figures 5 and 6.

Elastic electron scattering data on $lp$ nuclei have previously been analysed by Meyer-Berkhout\textsuperscript{85}) using a variety of charge density functions. We list in Table 3, the mean value of the r.m.s. radius obtained from these functions; together with the r.m.s. radius deduced from this work. For the case of Be$^9$ and B$^{11}$ we have only included in the Meyer-Berkhout values, those functions which gave acceptable fits to the experimental data. As can be seen from the table, a satisfactory agreement is obtained between this work and that of Meyer-Berkhout.

It will be noted that for the two nuclei with little or no quadrupole scattering, Li$^6$ and Cl$^{12}$, that the predicted 1s binding energy is much less than the experimental value. This seems to indicate that the shell model single-particle potential is configuration dependent. This conclusion has been similarly reached by Elton and Swift\textsuperscript{95}) from considerations.
of electron scattering in medium mass nuclei. They have further shown that for a given nucleus, with the well-depth being made energy dependent, that a single non-local potential of the Perey and Buck type can be constructed for bound states.

To investigate this a little further, we have considered whether the finite oscillator potential, defined by equation (2.9), having four parameters can fit the binding energies and the electron scattering simultaneously. The parameters $a$ and $r_1$, $E_s$ and $E_p$ were varied subject only to the constraint that $E_s$ and $E_p$ should remain as close as possible to the experimental value for the binding energies. The best fit obtained for Li is shown in figure 8, using the parameters listed below

\[
V_0, \gamma, r_1, a, E_s, E_p
\]

\[
49, 0.75, 2.3, 1.7, 20.9, 5.5
\]

The well depth $V_0$ and the tail parameter $\gamma$ are determined from $E_s$ and $E_p$ as indicated in Chapter 2. Clearly the deviation of $E_s$ and $E_p$ from the experimental values of 22.7 and 4.9 is still significant, and is further evidence that the shell model potential is energy dependent.
Elastic Electron Scattering on Li$^6$

The model density distribution has been derived from the finite oscillator potential.
CHAPTER 4.

\((p,d)\) - reaction.

4.1 Introduction

We now consider a nuclear reaction involving the transfer of a single nucleon. The purpose of this is to use the single-particle wavefunctions generated in Chapter 2., and 'calibrated' in Chapter 3., and in this way to investigate one of the sources of uncertainty in the reaction theory. In particular we shall analyse the \((p,d)\) experiments of Radvanyi et al.\(^{19}\) at Orsay, where the synchrocyclotron produces a proton beam of 155 MeV.

At such a high energy, we are justified in using the direct reaction model, where the transition from the incident channel to the final channel is a one-step process, without the formation of any intermediate state. The cross-section predicted by such a theory does not show any sharp resonances; and this is indeed borne out by the angular distributions produced by Radvanyi.

We begin by briefly presenting the DWBA formalism for the \((p,d)\) reaction. Several approximations are made none of which will be justified or discussed in any detail. Instead, we take the view that we shall test the validity of the theory by comparing the cross-sections with the experimental values. A derivation of the DWBA expression for the transition amplitude can be found in Messiah\(^{20}\).
and its application to single-particle stripping or pick-up has been given by numerous authors\textsuperscript{1,21-27}\. The treatment here follows the work of Tobocman\textsuperscript{1} and Bassel, Drisko and Satchler\textsuperscript{21}.

4.2 \textbf{(p,d) Reaction}

Consider a target nucleus $A$, to comprise a neutron $n$, bound to a core $C$ of $(A-1)$ nucleons. The incoming proton $p$, then "picks-up" the neutron to form a deuteron $d$, leaving the residual core $C$ either in its ground state or an excited state. Diagrammatically

$$p + (n + C) \rightarrow (p + n) + C$$

(4.1)

The total Hamiltonian for the system can be written

$$H = H_p + H_n + H_C + V_{pn} + V_{pC} + V_{nC}$$

(4.2)

where $H_x$ contains the internal Hamiltonian and the kinetic energy term for particle $x$, and $V_{xy}$ is the interaction potential between particles $x$ and $y$.

Obviously there are many other ways in which the total Hamiltonian $H$ can be divided. For example in the incident channel, $\alpha$, \textsuperscript{63}
\[ H = H_p + (H_n + H_C + V_{nC}) + (V_{pn} + V_{pC}) \]
\[ = H_p + H_A + (V_{pn} + V_{pC}) \]
\[ = (h_p + h_A + T_\alpha) + (V_{pn} + V_{pC}) \]
\[ = H_\alpha + V_\alpha \quad (4.3) \]

where we have used the notation that \( h_\alpha \) represents the internal Hamiltonian for particle \( \alpha \), and \( T_\alpha = -\frac{\hbar^2}{2M_\alpha} \nabla^2 \) is the kinetic energy of relative motion of particles \( p \) and \( A \), in channel \( \alpha \). \( M_\alpha \) is the reduced mass.

Similarly in the final channel \( \beta \) we have

\[ H = (H_p + H_n + V_{pn}) + H_C + (V_{pC} + V_{nC}) \]
\[ = H_d + H_C + (V_{pC} + V_{nC}) \]
\[ = (h_d + h_C + T_\beta) + (V_{pC} + V_{nC}) \]
\[ = H_\beta + V_\beta \quad (4.4) \]

The eigenfunctions of \( H_\alpha \) and \( H_\beta \) are plane waves.

For example

\[ H_\alpha \Phi_i = E \Phi_i \]
\[ \Phi_i = \Phi_{\alpha i} e^{ik_i \cdot r_\alpha} \quad (4.5) \]

where \( \Phi_{\alpha i} \) is a product of the internal wavefunctions for the proton and the target nucleus \( A \). \( h_{ki} \) is the initial momentum and \( r_\alpha \) is the relative co-ordinate for the proton relative to the nucleus \( A \). The eigenfunction of the total Hamiltonian is
\[ H \psi_i(\pm) = E \psi_i(\pm) \]

where

\[ \psi_i(\pm) \xrightarrow{r \to \infty} \xi_i \exp \left[ i k_i r \right] \]

The superscript \((\pm)\) denotes the asymptotic behaviour, either an incoming or outgoing spherical wave. The scattering amplitude \(f_{i \alpha}(\Omega_i)\) is a function of \(\Omega_i\) the solid angle about the direction \(k_i\).

Now let us write

\[ V_{\alpha} = V_{\text{pn}} + V_{\text{nc}} \]

\[ = \overline{U}_{\text{pA}} + (V_{\text{pn}} + V_{\text{nc}} - \overline{U}_{\text{pA}}) \]  

(4.7)

where \(\overline{U}_{\text{pA}}\) is the optical model potential describing the elastic scattering of protons by the target nucleus \(A\).

Then the eigenfunctions of \(H_{\alpha} + \overline{U}_{\text{pA}}\) defined by

\[ (H_{\alpha} + \overline{U}_{\text{pA}}) \psi_i(\pm) = E \psi_i(\pm) \]  

(4.8)

are known as distorted waves. Analogous expressions to (4.5), (4.6) and (4.8) for the final channel eigenfunctions \(\Phi_f, \psi_f(\pm)\) and \(\psi_f(\pm)\) can be written down with

\[ V_{\Phi} = \overline{U}_{\text{dC}} + (V_{\text{pc}} + V_{\text{nc}} - \overline{U}_{\text{dC}}) \]

(4.9)
Then the 'post' form of the transition matrix element\(^{20}\) can be written exactly as

\[
T_{fi} = \langle \Phi_f \mid V_{nc} - V_{pn} + \bar{U}_{pA} \mid \psi_i^{(+)} \rangle \tag{4.10}
+ \langle \Phi_f^{(-)} \mid V_{pn} + V_{pC} - \bar{U}_{pA} \mid \psi_i^{(+)} \rangle
\]

The first term is identically zero, while the second term approximates to the well-known expression

\[
T_{fi} = \langle \psi_f^{(-)} \mid V_{pn} \mid \psi_i^{(+)} \rangle \tag{4.11}
\]

when the following two approximations are made.

1. The DWBA is used, which implies replacing the stationary wavefunction \(\Phi_f^{(-)}\) by the distorted wave \(\psi_f^{(-)}\), which is an eigenfunction of \(H_{\beta} + \bar{U}_{dC}\).

2. Assume a large overlap between \(V_{pC}\) and \(\bar{U}_{pA}\), i.e., \((V_{pC} - \bar{U}_{pA}) \sim 0\). The usual argument is that the elastic effects of \(V_{pC}\) are essentially included in \(\bar{U}_{pA}\), while the inelastic effects are small.

The resulting cross-section is\(^{20}\)

\[
\frac{d\sigma}{d\Omega_f} = \frac{m^2}{2\hbar^2 n^4} \frac{A(A-1)}{(A+1)^2} \frac{k_f}{k_i} S_{if} |T_{fi}|^2 \tag{4.12}
\]

where the symbol \(S_{if}\) denotes the operation of averaging over the initial nuclear states and summing over the final
nuclear states. $m$ is the nucleon mass, $k_F$ the deuteron momentum and $k_i$ the proton momentum.

Now if we assume that the two nuclei $p$ and $A$ in the initial channel and the two nuclei $d$ and $C$ in the final channel are made up of just nucleons, or of nucleons plus an inert core, then the transition matrix element must be modified to take into account the Pauli Exclusion Principle. Tanifuji\textsuperscript{29} has considered this problem and we quote his result

\begin{equation}
T_{fi} = \sqrt{A} \left( \langle \mathcal{A}_f \gamma_f^{(-)} | v_{pn} | \gamma_i^{(+)} \rangle \right) \tag{4.13}
\end{equation}

where $\mathcal{A}_f$ is the antisymmetrisation operator in the final channel, i.e. $\mathcal{A}_f = \sum_{n} (-)^{\sigma_n} p_n$ where $p_n$ produces a particular permutation among the nucleons, and $\sigma_n$ is the number of exchanges involved in $p_n$. The sum is over all distinct permutations. The number multiplying the matrix element in Tanifuji's article is $\sqrt{m}$, where $m$ is the number of neutrons available in the target. However we shall include iso-spin explicitly in all wavefunctions and hence we multiply our matrix element by $\sqrt{A}$. The procedure is to calculate the transition matrix element assuming the particles are distinguishable, then add the matrix elements for all the possible exchange processes each multiplied by the parity of exchange. For most of these terms the overlap between initial and final wavefunctions will be so small that the matrix element will be negligible. We, therefore, make a third approximation:
(3) Neglect all exchange terms, hence

\[ T_{fi} = \sqrt{A} \left\langle \psi_f(\cdot) \left| V_{pn} \right| \psi_i^{(+)} \right\rangle \quad (4.14) \]

The distorted wave \( \psi_f(\cdot) \) is a product wavefunction since \( H_{T} = h_d + h_c + T_{\beta} \). Hence

\[ \psi_f(\cdot) = \phi_{d}^{s_d \tau_d} \phi_{c}^{J_c \tau_c} \chi_f(\cdot) \quad (4.15) \]

where

\[ h_d \phi_{d} = E_d \phi_{d} \]

\[ h_c \phi_{c} = E_c \phi_{c} \quad (4.16) \]

\[ (T_{\beta} + \bar{T}_{d_c}) \chi_f(\cdot) = \left[ E - E_d - E_c \right] \chi_f(\cdot) \]

In the expression (4.15), the quantum numbers of spin and iso-spin have been written as superscripts and their projections as subscripts. However in (4.16) we have used the abbreviated notation of using a single Roman subscript to represent all the quantum numbers of the internal wavefunction.

Thus the transition matrix element

\[ T_{fi} = \sqrt{A} \left\langle \chi_f(\cdot)^{(\cdot)}( k_f, \xi_d) \phi_d( \xi_{pn} ) \phi_c( \xi ) \left| V_{pn} \right| \chi_i^{(+)}( k_i, \xi_{pnA} ) \phi_p \phi_A( \xi, \xi_{nC} ) \right\rangle \quad (4.17) \]
where

\[ \xi = \text{all co-ordinates of the residual nucleus } C. \]

\[ r_{xy} = r_x - r_y. \]

The first stage in the evaluation of (4.17) is to integrate over the co-ordinates \( \xi \)

\[ \psi(r_{nc}) = \left< \Phi_C(\xi) \right| \Phi_A(\xi, r_{nc}) \right> \quad (4.18) \]

This quantity \( \psi \) is called the overlap integral\(^{30}\) and provides the meeting point between nuclear structure and nuclear reaction theories. Quite obviously \( \Phi_A \) can be expanded in terms of the complete set of functions \( \Phi_C \).

\[ J_A^{J_A} T_A^{I_A} M_A^{I_A} M_{TA}^{I_A} = \sum_{jm} \tilde{\xi}(j) \Phi_C^{J_C T_C M_C^{I_C} M_{TC}^{I_C}}(\xi) \Phi^n_{0 \frac{1}{2}}(r_{nc}) \]

\[ (J_C^{M_C}JM_C^{M_{TC}} | J_A^{M_A} ) (T_C^{M_{TC}^{\frac{1}{2}}} | T_A^{M_{TA}} ) \quad (4.19) \]

where \( \tilde{\xi}(j) \) is the expansion coefficient, essentially a fractional parentage coefficient. We shall discuss this quantity in more detail in the next chapter. Then the overlap integral

\[ \psi(r_{nc}) = \sum_{jm} \tilde{\xi}(j) (J_C^{M_C}JM_C^{M_{TC}} | J_A^{M_A} ) (T_C^{M_{TC}^{\frac{1}{2}}} | T_A^{M_{TA}} ) \]

\[ \tilde{\xi}_n^{ \frac{1}{2} \frac{1}{2} } (r_{nc}) \quad (4.20) \]
where \( \mathcal{P}_{m}^{j, \frac{1}{2}} \) is some function of the transferred neutron's co-ordinates. Now a fourth approximation is to

(4) Replace \( \mathcal{P}_{m}^{j, \frac{1}{2}} \) by a single-particle neutron wavefunction \( \phi_{m}^{j, \frac{1}{2}} \), generated from some finite potential well. This can be done with little loss of accuracy, if we note that the overlap integral (4.21) satisfies a Schrodinger type equation. Following Bergrren\(^{30}\) we have

\[
\begin{align*}
\hat{h}_{A} \phi_{A} & = \varepsilon_{A} \phi_{A} \\
\hat{h}_{C} \phi_{C} & = \varepsilon_{C} \phi_{C}
\end{align*}
\]  

(4.21)

Then

\[
\begin{align*}
\langle \phi_{C} | h_{A} \phi_{A} \rangle & = \varepsilon_{A} \langle \phi_{C} | \phi_{A} \rangle \\
\langle \phi_{C} h_{C} | \phi_{A} \rangle & = \varepsilon_{C} \langle \phi_{C} | \phi_{A} \rangle
\end{align*}
\]

Assuming that the internal Hamiltonians, \( h \), are Hermitian, then on subtracting we get

\[
\begin{align*}
\langle \phi_{C} | h_{A} - h_{C} | \phi_{A} \rangle & = (\varepsilon_{A} - \varepsilon_{C}) \langle \phi_{C} | \phi_{A} \rangle
\end{align*}
\]  

(4.22)

Furthermore, the internal Hamiltonian of the \( A \)-particle system \( h_{A} \) can be written

\[
h_{A} = h_{C} + T_{nC} + V_{nC}
\]  

(4.23)

where \( h_{C} \) is the internal Hamiltonian of the \((A-1)\)-particle system, the core; and \( T_{nC} \) and \( V_{nC} \) are the kinetic
and potential energy terms of the $A^{th}$ particle, the transferred neutron relative to the core. Hence (4.22) becomes

$$\langle \Phi_C | T_{nc} + V_{nc} | \Phi_A \rangle = (\varepsilon_A - \varepsilon_C) \langle \Phi_C | \Phi_A \rangle$$

(4.24)

i.e. $$-\frac{\hbar^2}{2M_{nc}} \nabla^2 \langle \Phi_C | \Phi_A \rangle + \langle \Phi_C | V_{nc} | \Phi_A \rangle = (\varepsilon_A - \varepsilon_C) \langle \Phi_C | \Phi_A \rangle$$

where $M_{nc}$ is the reduced mass for the neutron-core system.

Thus the overlap integral $\langle \Phi_C | \Phi_A \rangle$, satisfies a Schrödinger-type equation (4.24). Asymptotically, $$\langle \Phi_C | V_{nc} | \Phi_A \rangle \overset{r_{nc} \to \infty}{\sim} 0$$

we have

$$\langle \Phi_C | \Phi_A \rangle \overset{r_{nc} \to \infty}{\sim} \frac{-\alpha r_{nc}}{r_{nc}}$$

(4.25)

where

$$\alpha^2 = \frac{2M_{nc}}{\hbar^2} (\varepsilon_A - \varepsilon_C)$$

Thus the radial dependence of the overlap integral is uniquely determined by the separation energy $(\varepsilon_A - \varepsilon_C)$ in the asymptotic region. Thus, using the fourth approximation of replacing the overlap integral by a single-particle wavefunction, is equivalent to writing

$$\langle \Phi_C | V_{nc} | \Phi_A \rangle \approx V_{nc} \langle \Phi_C | \Phi_A \rangle$$

(4.26)
If we use a single-particle model to describe the nuclear wavefunctions, then the relation (4.26) is exactly true. Consequently (4.24) reduces to a single-particle Schrödinger equation for the overlap integral

\[
\left(-\frac{\hbar^2}{2m_n} \nabla^2 + V_{nC}\right) \langle \phi_C | \phi_A \rangle = (\varepsilon_A - \varepsilon_C) \langle \phi_C | \phi_A \rangle.
\] (4.27)

This equation can now be solved numerically by the methods of Chapter 2, such that one of the parameters of $V_{nC}$ is adjusted so that the overlap integral has the correct asymptotic behaviour. The question now is, what is a suitable choice for the potential $V_{nC}$? We defer answering this question until paragraph 4.4, where we will investigate various procedures and give some numerical examples.

Using the expansion (4.20) and making the fourth approximation, the transition matrix element may now be written

\[
T_{fi} = \sqrt{A} \sum_{Jm} \mathcal{J}(j) \left(J_{C}M_{C}J_{m} \mid J_{A}M_{A}\right) \left(T_{C}M_{TC}2\tilde{J} \mid T_{A}M_{TA}\right)
\] (4.28)

\[
\langle X_{f}^{-}\left(k_{f},\pi_{dc}\right) \phi_{d}(\pi_{pn}) \mid V_{pn} \mid X_{i}^{+}\left(k_{i},\pi_{pa}\right) \phi_{n}(\pi_{nc}) \rangle
\]

A fifth approximation is to:

(5) Assume that all potentials are spin independent and central. Then all wavefunctions can be written as product functions of space, spin and iso-spin coordinates. That is
\[ \Phi_d(r_{pn}) = \sum_{d, \lambda_d} (\mathbf{l}_d \lambda_d \sigma_d | j_d m_d) \left[ i \mathbf{l}_d \mathbf{Y}_d^* \lambda_d \chi_d^* (r_{pn}) \right] \]

\[ \frac{\mathbf{u}_d}{r_{pn}} \chi^{s-d}_{\sigma_d} \zeta_{d}^{0} \]

\[ \Phi_p = \chi^{s-p}_{\sigma_p} \zeta_{p}^{0} \]

\[ \Phi_n(r_{nc}) = \sum_{\lambda, \sigma} (\mathbf{l}_n \sigma \lambda \lambda | j_m) \left[ i \mathbf{l}_n \mathbf{Y}_n^* \lambda_n \left( \frac{r_{nc}}{nc} \right) \right] \frac{\mathbf{u}_n}{r_{nc}} \chi^{s-d}_{\sigma_n} \zeta_{d}^{0} \]

Integrating over the spin co-ordinates gives

\[ \langle \chi^{s-d}_{\sigma_d} | \chi^{s-p}_{\sigma_p} \chi^{s}_{\sigma} \rangle = \langle \sigma_p, \sigma, \sigma | s_d, \sigma_d \rangle \]

\[ \left\langle \frac{1}{2}^{0}, \frac{1}{2}^{0}, \frac{1}{2}^{0} \right\rangle = \frac{1}{\sqrt{2}} \]

A further simplification can be made if it is assumed the deuteron ground state is a pure \( ^3S \) state. However, the deuteron is known to have a positive quadrupole moment, indicating that the ground state contains admixtures of a \( ^3D \) state. Nevertheless, the usual procedure is to make a sixth approximation:

(6) Neglect the D-state contribution to the deuteron wavefunction, then \( \mathbf{l}_d = 0 \) only, and

\[ \chi_d(r_{pn}) = i \mathbf{l}_d \mathbf{Y}_d^* \lambda_d \chi_d^* (r_{pn}) \frac{\mathbf{u}_d}{r_{pn}} = \frac{1}{(4\pi)^{1/2}} \frac{\mathbf{u}_0}{r_{pn}} \]

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Similarly defining
\[
\psi_n(r_{nC}) = \int \frac{Y_{\ell,j}(r_{nC})}{r_{nC}} \frac{\psi(x)}{x}\]
then substituting (4.29) to (4.31) back in (4.28)
\[
T_{\ell_1} = (\frac{\alpha}{2})^\frac{1}{2} \sum_{j,m} \langle j \mid J_{C} M_{C} j_{m} \mid J_{A} M_{A} \rangle (T_{M_{C} T_{C} \frac{1}{2} 2} \mid T_{A} M_{A} )
\]
\[
(s_{\kappa \sigma} s_{\kappa \sigma} | s_{\kappa \sigma} s_{\kappa \sigma} ) (2 \lambda \sigma \sigma | j_{m} ) \frac{E_{\lambda}}{E_{\lambda}}
\]
where
\[
E_{\lambda} = \langle X_{\kappa}^{(-)}(r_{\gamma}, r_{dC}) \mid V_{pn} \mid X_{i}^{(+)}(k_{i}, r_{pA}) \psi_{n}(r_{nC}) \rangle
\]
The last factor \( E_{\lambda} \) is just a six-dimensional integral over the variables \( r_{pn} \) and \( r_{nC} \). The vectors \( r_{dC} \) and \( r_{pA} \) are simply a linear combination of \( r_{pn} \) and \( r_{nC} \). From fig. 9, it is easy to show
\[
\begin{align*}
X_{dC} &= r_{nC} + \frac{1}{2} r_{pn} \\
X_{pA} &= r_{pn} + a r_{nC}
\end{align*}
\]
where \( a = 1 - 1/A \). Substituting back (4.32) into (4.12), one obtains after some Racah algebra
\[
\frac{d \sigma}{d \Omega_{T_{f}}(J_{A} T_{A} \rightarrow J_{C} T_{C})} = \frac{3m^{2}}{16 \pi^{2} n} \frac{A(A-1)}{(A+1)^{2}} \frac{k_{l}}{k_{i}} \frac{(T_{M_{C} T_{C} \frac{1}{2} 2} \mid T_{A} M_{A})^{2}}{S(1) \sum_{\lambda} \frac{E_{\lambda}}{E_{\lambda}}^{2}}
\]
Fig. 9. Vector diagram showing the relation between the four vectors $I_{pn}, I_{nc}, I_{dc}$ & $I_{pa}$
where the spin values \( s_d = 1 \), \( s_p = \frac{1}{2} \) and \( s = \frac{1}{2} \) have been included. It has been assumed that the final state is reached through the transfer of a single \( j \)-value. \( S(j) \) is called the spectroscopic factor\(^{31}\) and is defined as

\[
S(j) = A \left| \frac{d}{d\alpha} \right|^2
\]  

(4.36)

The cross-section for the inverse, deuteron stripping \((d,p)\) reaction is given by\(^{1}\)

\[
\frac{d\sigma}{d\Omega_f} (d,p) = \frac{2}{3} \frac{2J_A+1}{2J_C+1} \left( \frac{k_f}{k_i} \right)^2 \frac{d\sigma}{d\Omega_f} (p,d)
\]  

(4.37)

where \( k_i \) is the proton momentum, and \( k_f \) the deuteron momentum. The problem now remains of evaluating the six-dimensional integral \( B^f_\lambda \).

### 4.3 Calculation of \( B^f_\lambda \)

We have defined the six-dimensional integral \( B^f_\lambda \) in (4.33) to be

\[
B^f_\lambda = \left\langle X^{(-)}_{\lambda}(k_f, r_d) V (r_{pn}) X^{(+)}_{\lambda}(k_i, r_{pn}) \psi_n(r_{nc}) \right\rangle
\]

(4.38)

where we have defined a range function

\[
D(r_{pn}) = V_{pn}(r_{pn}) \psi_d(r_{pn})
\]

(4.39)

Now if the distorted waves are expanded in partial waves, all the angular integrations may be done analytically and
we are led to expressions involving double radial integrals. Even with the high speed computers available today, evaluating a large number of double integrals would require a vast amount of storage space and many hours of computing time. Nevertheless a formalism suitable for numerical computation has been given by Austern et al.\(^{32}\); and a computer code written for the IBM-7090 at Oak Ridge by Drisko and Satchler\(^{33}\). In the present work we have not undertaken such an ambitious project, instead we have taken one of the two alternative approximations which can be made to reduce \( B^\lambda \) to a three-dimensional integral. These two possible approximations are:

1. To use zero-range DWBA. This is based on the knowledge that we expect the interaction \( V_{pn} \) to be short-range, hence we write

\[
D(r_{pn}) = D_0 \delta (r_p - r_n) \tag{4.40}
\]

thus reducing \( B^\lambda \) to an integral over \( r_{nc} \).

2. To use plane-wave Born approximation, that is to replace the distorted waves \( \chi_f^{(-)} \) and \( \chi_i^{(+)} \) by plane waves \( \Phi_f \) and \( \Phi_i \). Then the integral \( B^\lambda \) reduces to a product of two three-dimensional integrals.

Let us first consider the zero-range DWBA. From the vector relations (4.34), we now have
\[ \Gamma_{dC} = \Gamma_{nC} \]
\[ \Gamma_{pA} = a \Gamma_{nC} \]  

\[ (4.41) \]

Then dropping the subscripts and writing \( r \) for \( \Gamma_{nC} \), the amplitude \( B^A \) becomes

\[ B^A = D_0 \int X_f(-\mathbf{k}_f, r) X_i(+)\mathbf{k}_i, \mathbf{a}_p) \psi_n(r) \, d^3r \]  

\[ (4.42) \]

The next stage is to make a partial wave expansion of the distorted waves, i.e.

\[ X_f(-\mathbf{k}_f, r) = 4\pi \sum_{n, \mathbf{n}, \mathbf{\mu}, \mathbf{n}', \mathbf{\mu}'} i^n f_n(-\mathbf{k}_f r) Y_{n', \mathbf{\mu}', \mathbf{\mu}'}(\mathbf{\hat{k}}_f) \]  

\[ X_i(\mathbf{k}_i, \mathbf{a}_p) = 4\pi \sum_{n, \mathbf{n}, \mathbf{\mu}, \mathbf{n}', \mathbf{\mu}'} i^n f_n(\mathbf{ak}_i r) Y_{n, \mathbf{\mu}}(\mathbf{\hat{k}}_i) Y_{n', \mathbf{\mu}'}(\mathbf{\hat{k}}_f) \]  

\[ (4.43) \]

where \( n, n' \) are orbital angular momentum quantum numbers and \( \mathbf{\mu}, \mathbf{\mu}' \) their projections; and where \( \mathbf{\hat{k}}_i \) and \( \mathbf{\hat{k}}_f \) represent the polar angles of \( \mathbf{k}_i \) and \( \mathbf{k}_f \) with respect to the z-axis. The function \( f_n(\mathbf{kr}) \) is a solution of the radial wave equation which satisfies the appropriate boundary conditions.

Hence the product of the two distorted waves given in the integral \((4.42)\) can be written

\[ X_f(-)^* X_i(+) = (4\pi)^2 \sum_{n, \mathbf{n}, \mathbf{\mu}, \mathbf{n}', \mathbf{\mu}'} i^{n-n'} f_n(\mathbf{ak}_i r) f_n(\mathbf{k}_f r) Y_{n, \mathbf{\mu}}(\mathbf{\hat{k}}_i) Y_{n', \mathbf{\mu}'}(\mathbf{\hat{k}}_f) \]  

\[ (4.44) \]
where we have used the reversibility or reciprocity theorem to derive the relation between the radial functions

\[ f_n(-)(kr) = f_n(+) (kr) \quad (4.45) \]

Having made the expansion (4.44) the integration over the polar angles \( \widehat{\Sigma} \) can be done explicitly

\[
\int Y_{n',\mu'}(\hat{\sigma}) Y_{n,\mu}^*(\hat{\sigma}) Y_{\lambda,\lambda}(\hat{\sigma}) \, d^2r
= (-)^{\mu' + \lambda} \left[ \frac{(2n+1)(2n'+1)}{4\pi(2\ell+1)} \right]^{\frac{1}{2}} (n|0\rangle \langle 0| n' - \mu' | \ell - \lambda) \]

Hence

\[
\beta_{\lambda} = D_0 (4\pi)^{3/2} \sum \sum (-)^{\mu' + \lambda} \left[ \frac{(2n+1)(2n'+1)}{(2\ell+1)} \right]^{\frac{1}{2}}
(n|0\rangle \langle 0| n' - \mu' | \ell - \lambda) Y_{n,\mu}(\hat{k}_1) Y_{n',\mu'}(\hat{k}_2) I_{nn'}(\mathbf{r})
\]

where \( I_{nn'}(\mathbf{r}) \) is the radial integral

\[
I_{nn'}(\mathbf{r}) = \int f_n^{(+)}(ak_1 r) f_{n'}^{(+)}(k_2 r) \, u_2(r) \, r \, dr \quad (4.48)
\]

and \( u_2(r) \) the neutron wavefunction defined in (4.31).

Now if we choose our z-axis along the direction \( k_1 \),
and let \( k_1 \) and \( k_2 \) define the x-z plane, then the angle
between these two vectors, \( \Theta \), will be the angle of scattering.

Hence

\[
\left( k_{-1,\mu} \right)^* = \left( \frac{2n+1}{4\pi} \right)^{\frac{1}{2}} \delta_{\mu,0}
\]
\[ Y_{n'}^{\mu'}(\hat{\mathbf{r}}) = \left( \frac{2n'+1}{4\pi} \right)^{\frac{1}{2}} \left[ \frac{(n'-\mu')!}{(n'+\mu')!} \right] \frac{1}{2} P_{n'}^{\mu'}(\cos \Theta) \quad (4.49) \]

where the \( P_{n'}^{\mu'}(\cos \Theta) \) are the associated Legendre functions.

Furthermore we have that \( \mu' = \lambda \), hence

\[ B_{\lambda}(\Theta) = (-)^{\lambda} \overline{B}_{-\lambda}(\Theta) \quad (4.50) \]

where

\[ \overline{B}_{\lambda}(\Theta) = D_0 (4\pi)^{\frac{1}{2}} \sum_{n,n'} (2\ell+1)^{\frac{1}{2}} \int_{n,n'} P_{n'}^{\lambda}(\cos \Theta) I_{nn'}(\ell) \]

\[ \quad \int_{n,n'} = \frac{1}{2^{\ell+1}} \left[ \frac{(n'-\ell)(n'+\ell+1)!}{(n'+\ell+1)!} \right] \left( n0n'0 \ell0 \right) \quad (4.51) \]

Now it follows from the properties of the Clebsch-Gordan coefficient \( (n0n'0 | \ell0) \) that \( n+\ell \geq n' \geq |n-\ell| \), and that \( n+n'+\ell \) must be even, so that the sum over \( n' \) is restricted to relatively few terms.

To complete the computation of \( B_{\lambda} \), we must make some estimate of the magnitude of \( D_0 \) and evaluate the radial integrals \( I_{nn'}(\ell) \). Following Bassel, Drisko, and Satchler \(^{21}\) we note that if \( V_{pn}(r_{pn}) \) is the potential which binds the deuteron, then the internal wavefunction \( \psi_d(r_{pn}) \) satisfies the Schrodinger equation

\[ \left( -\frac{\hbar^2}{2m_{pn}} \nabla^2 + V_{pn} \right) \psi_d = E_d \psi_d \]
where $M_{pn}$ is the reduced mass of the system and $\xi_d$ is the binding energy of the deuteron.

Defining $\alpha^2 = -\frac{2M_{pn}}{\hbar^2} \xi_d$, we have

$$\left(\nabla^2 - \alpha^2\right) \psi_d = \frac{2M_{pn}}{\hbar^2} D(r_{pn}) \tag{4.52}$$

However from the definition (4.40)

$$D_0 = \int V(r_{pn}) \psi_d(r_{pn}) \, d^3r_{pn}$$

$$= \frac{\hbar^2}{2M_{pn}} \int \left(\nabla^2 - \alpha^2\right) \psi_d(r_{pn}) \, d^3r_{pn}$$

$$= -\xi_d \int \psi_d(r_{pn}) \, d^3r_{pn} \tag{4.53}$$

where we have used (4.52). The term in $\nabla^2$ vanishes upon applying Green's theorem. Using the Hulthen form for the deuteron wavefunction

$$\psi_d(r) = \left[\frac{\alpha \beta (\alpha + \beta)}{2\pi (\alpha - \beta)^2}\right]^{\frac{1}{2}} \frac{e^{-\alpha r} - e^{-\beta r}}{r} \tag{4.54}$$

where $\beta > \alpha$, the integral (4.53) is simply evaluated to give\(^2\)

$$D_0^2 = \frac{8\pi \xi_d^2}{\alpha^3} \left(\frac{\alpha + \beta}{\beta}\right)^3$$

$$= 1.5 \times 10^4 \text{ MeV}^2 \text{ fermi}^3$$

using $\beta = 7\alpha$ and $\xi_d = -2.23 \text{ MeV}$.
Lastly, now, the radial integral \( I_{mn}(1) \) was defined as

\[
I_{mn}(1) = \int f_n^{(+)}(ak \cdot r) f_n^{(+)}(kr) u_{kn}(r) r \, dr
\]

where the function \( f_n^{(+)}(kr) \) is the radial part of the distorted wavefunction \( \chi^{(+)}(k, r) \) and is a solution of the equation

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \kappa^2 - \frac{2\gamma k}{r} - \frac{2M\chi}{\hbar^2} \bar{U}(r) - \frac{n(n+1)}{r^2} \right] f_n^{(+)}(kr) = 0
\]  

(4.56)

where \( \bar{U} \) is the complex optical potential describing the elastic scattering between the pair of particles \( \chi \), \( M_\chi \) being the reduced mass. \( \gamma \) is the Coulomb parameter

\[
\gamma = M_\chi z_1 z_2 e^2 / \hbar^2 k
\]

where \( z_1 \) and \( z_2 \) are the charges on the pair of particles, \( \chi \). At large radii, where \( \bar{U}(r) \) is negligible, \( f_n^{(+)}(kr) \) has the form (16)

\[
kr f_n^{(+)}(kr) = u_n^{(+)}(kr)
\]

\[
u_n^{(+)}(kr) = \frac{e^{i\sigma_n}}{2} \left[ (F_n + iG_n) + e^{2i\delta_n}(F_n - iG_n) \right]
\]  

(4.57)

where \( \sigma_n \) is the Coulomb phase shift, and \( F_n, G_n \) are the regular and irregular Coulomb functions respectively.

\( \delta_n \) is the phase shift introduced by the nuclear potential \( \bar{U}(r) \). Asymptotically
\[ F_n \sim \sin(x + \sigma_n) \quad G_n \sim \cos(x + \sigma_n) \]

\[ u_n(+) (kr) \sim -\frac{1}{2i} \left[ e^{-ix} - e^{2i(\delta_n + \sigma_n)} e^{ix} \right] \quad (4.58) \]

where \( x = (kr - \frac{\gamma}{2} \log 2kr + n\pi/2) \).

The radial integral \( I_{mn}(\mathbf{\ell}) \) has been integrated numerically using the computer programme for optical model analysis written by Buck et al. The phase shifts are determined and the solutions normalised by matching the numerical solution and its derivative to the asymptotic solution, and the correctly normalised solutions \( f_n(+) (kr) \) are used to calculate the radial integrals.

For completeness now, we briefly mention the plane-wave Born approximation (PWBA), since here we need make no assumption about the potential \( V_{pn}(r_{pn}) \); in other words we can treat the finite range of the interaction \( V_{pn} \) in an exact manner. In equation (4.38) for \( B_{\mathbf{\lambda}} \), we replace the distorted waves with plane waves

\[ \chi^{(+)}_{\mathbf{\ell}}(\mathbf{k}, \mathbf{r}, \mathbf{dC}) = e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{dC}} \]

\[ \chi^{(+)}_{\mathbf{1}}(\mathbf{k}, \mathbf{r}, \mathbf{pA}) = e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{pA}} \quad (4.59) \]

obtaining with the use of (4.34)

\[ B_{\mathbf{\lambda}}^\mathbf{\ell} = \int \int e^{-i\mathbf{k} \cdot (\mathbf{r}_{pn} + \mathbf{r}_{pn})} d^3 \mathbf{r}_{pn} \]

\[ e^{i\mathbf{k} \cdot (\mathbf{r}_{pn} + \alpha \mathbf{r}_{pn})} \mu_n(\mathbf{r}_{nc}) d^3 \mathbf{r}_{nc} d^3 \mathbf{r}_{pn} \]
Define
\[ a = k_p - ak_p \]
\[ K = k_d - \frac{2}{3}k_p \]  \hspace{1cm} (4.60)

then
\[ B_\lambda^2 = \int d^3r_{pn} e^{-iK \cdot r_{pn}} D^*(r_{pn}) \int d^3r_{nC} e^{i\alpha \cdot r_{nC}} \psi_n(r_{nC}) \]
\[ = G^*(K) F(q) \]  \hspace{1cm} (4.61)

where \( G(K) \) is the Fourier transform of the range function, and \( F(q) \) the Fourier transform of the neutron wavefunction. Thus the amplitude \( B_\lambda^2 \) factorizes into a product of two three-dimensional integrals, the first of which carries an averaging of the finite-range function \( D \). Finite-range effects therefore express themselves as a departure of the first factor from a simple constant value \( D_0 \), which it assumes in zero-range.

Now
\[ G(K) = \int e^{ik \cdot r} D(r) \, d^3r \]
\[ = \frac{\hbar}{2M_{pn}} \int e^{ik \cdot r} (\vec{\nabla}^2 - \alpha^2) \psi_d(r) \, d^3r \]  \hspace{1cm} (4.62)
\[ = - \frac{\hbar}{2M_{pn}} (K^2 + \alpha^2) \int e^{ik \cdot r} \psi_d(r) \, d^3r \]

where we have used (4.52).
Using the Hulthen wavefunction (4.54) for $\Psi_d(r)$, the integral is simply evaluated, giving

$$G(K) = D_0 \frac{\bar{\rho}^2}{\bar{\rho}^2 + \kappa^2}$$  \hspace{1cm} (4.63)$$

Thus in plane-wave theory the effect of the finite-range of the interaction $V_{pn}$ manifests itself in an angle-dependent factor $\bar{\rho}^2/(\bar{\rho}^2 + \kappa^2)$, where

$$\kappa^2 = k_i^2 + \frac{1}{4} k_f^2 - k_1 k_f \cos \Theta \hspace{1cm} (4.64)$$

Hence it would seem a logical step to introduce this plane-wave correction factor into the zero-range DWBA expression for $B^f$. In paragraph 4.4 we shall show the effect of this correction on $(p,d)$ cross-sections.

Recently much effort has been directed towards increasing the accuracy of zero-range calculations without causing much increase in either computer running time or programming difficulty. Two procedures have been suggested, the effective mass approximation\(^{36})^{37)}\) and the local energy approximation\(^{38)}\). These are equivalent to first order, and this first order term has been tested against the full finite-range calculations by Dickens et al\(^{39)}\), and a considerable improvement in the zero-range calculation was obtained.

To include this approximation in the normal zero-range formalism requires modifying the radial integral (4.48) to
\[ I_{nn'}(l) = \int [\wedge(r) \, f_n(+) \, \bar{\kappa}_{1r} \, f_{n'}(+) \, \kappa_{r} \, \bar{u}_e(r) \, r \, dr \] \quad (4.65) \]

where

\[ \wedge(r) = 1 - y(r) \]

\[ y(r) = \frac{\alpha^2}{\bar{\beta}^2 B_d} \left[ \bar{U}_{dC}(r) - \bar{U}_{pA}(ar) - \bar{V}_{nC}(r) - B_d \right] \] \quad (4.66)

\( \alpha \) and \( \beta \) are the Hulthen parameters of the deuteron wave-function \( (4.54) \), and \( B_d \) is the modulus of the deuteron binding energy, that is 2.23 MeV. Note that since the optical potentials are complex, the function \( y(r) \) is complex.

From the definition \( (4.66) \) of \( y(r) \), it can be seen that the correction is smallest when \( \bar{U}_{dC} = \bar{U}_{pA} + \bar{V}_{nC} + B_d \).

However, there is known to be considerable ambiguities in the possible deuteron optical potentials\(^{40}\), but there is some theoretical evidence\(^{41},^{42}\) to suggest that the deuteron optical potential should be approximately the sum of a proton and a neutron optical potential. This would suggest a deuteron potential of depth 100 MeV, and some extensive calculations\(^{43}\) on Ca\(^{40}\)(d,p) Ca\(^{41}\) seem to indicate that such a potential gives the best fit to the stripping cross-sections. Thus in the nuclear interior \( y(r) \) may be small due to cancellations in the potentials, but since the potentials have different shapes, \( y(r) \) will be significant in the important surface region.
4.4 Test for DWBA using $^1_2(p,d)^1_1$.

In the previous paragraphs of this chapter we have been concerned with the formalism of the DWBA. An expression has been derived for the angular distributions of deuterons in the $(p,d)$-reaction; equations (4.35) and (4.50), which has necessarily involved the making of a number of approximations. We have not however justified these approximations, instead we shall treat the end result as a model which is to be judged by direct comparison with experiment. This programme of work has been carried out by Lee et al.\textsuperscript{43)} for the reaction Ca$^{40}(d,p)$ Ca$^{41}$ for a deuteron energy range, 7 MeV to 12 MeV. However it is felt that a similar comparison should be made at much higher energies. There are two reasons for this. First, at high energies one may be tempted to use the WKB approximation for generating the distorted waves. If so, then it is necessary to have some yardstick for comparison purposes, and a detailed DWBA analysis could provide just this. Preliminary work of this kind has been reported by Jackson\textsuperscript{44)}, however uncertainties in the deuteron optical potential and in the form of the interaction obscured the interpretation of the results. Second, a new model for deuteron stripping has recently been proposed by Pearson and Butler\textsuperscript{45)} using the impulse approximation. The impulse approximation has had considerable success in interpreting high energy inelastic
scattering processes, and consequently it would be interesting to compare its application to rearrangement collisions such as deuteron stripping with a high energy DWBA calculation.

To test the conventional DWBA, we choose the experiment $^{12}(p,d)^{11}$ as performed by Radvanyi et al.\(^{19}\) for a proton energy of 155 MeV. There are three reasons for this choice. First, at such high energies there exists very little elastic proton and elastic deuteron scattering data necessary for fixing the optical model parameters. However for a carbon target, elastic proton data at 183 MeV\(^{46}\), and elastic deuteron data at 156 MeV\(^{47}\) is available. Second, as we shall later show the correct choice of parameters for the bound-state neutron wavefunction should be compatible with some shell model calculation. For carbon, we showed in Chapter 3 that a very precise determination of shell model parameters is possible using elastic electron scattering. Third, a simple shell model description for carbon enables us to theoretically estimate the spectroscopic factor $S$, which acts as a multiplicative constant in the expression for the cross-section (4.35). Using $j$-$j$ coupling model, $S = 8$, whereas in L-S coupling $S = 5^{1}/_{3}$. Consequently we can expect the deduced value of the spectroscopic factor $S$, derived from fitting the computed cross-section to the experimental cross-section to lie close to these limits.
4.4.1 Finite Range Effects.

We consider first the effect of making the zero-range approximation

\[ V_{pn}(r_{pn}) \psi_d(r_{pn}) = D_0 \delta(r_p - r_n) \]

and the two corrections put forward to improve upon this approximation. In Fig. 10, we show three curves for the \( ^{12}_p \rightarrow ^{11}d \) reaction. Curve A corresponds to the usual zero-range DWBA. Curve B is the zero-range cross-section multiplied by the angle-dependent plane-wave correction

\[ \text{i.e. } \left( \frac{\beta}{\beta^2 + K^2} \right)^2 \]

where \( \beta \) is the Hulthen deuteron wavefunction parameter and

\[ K^2 = \frac{l_k^2}{l_k^2 + k_1^2} - k_1 k_1 \cos \Theta \]

\( k_1 \) and \( k_1 \) are the deuteron and proton momenta respectively.

At a proton energy of 155 MeV, \( k_1 = 3.20 \) \( k_1 = 2.51 \) and hence the zero-range cross-section at \( 0^\circ \) is multiplied by 0.51. Thus in the angle range \( 0^\circ \) to \( 40^\circ \) considered here, the shape of the angular distribution is altered only slightly, but the magnitude is reduced by half. Curve C corresponds to the use of the local energy approximation (4.65). The improvement to the goodness of fit is remarkable.

Each of the theoretical curves have been normalised to the experimental points at \( 0^\circ \). We list over the spectroscopic
Fig. 10. Effect of the various approximations for the range of Vpn on the angular distribution.
We conclude therefore that the plane wave correction used at such high energies predicts incorrectly the absolute magnitude of the cross-section.

The success of the local energy approximation (LEA) in reproducing not only the experimental angular distribution but also the theoretical spectroscopic factor was unexpected. Previous calculations\cite{36} using this approximation at low energies concluded that the effect on the angular distribution was very small. Clearly this seems to indicate that finite range effects are much more important at higher energies; and that the approximate treatment of these effects through the LEA is justified.

The LEA has been used consistently throughout the rest of the work reported in this thesis.

4.4.2 Proton Optical Potential

The Uppsala data\cite{46} for elastic proton scattering on carbon at 183 MeV has been analysed using a conventional Optical Model procedure by Hodgson\cite{46} and by Satchler\cite{48}.

\begin{tabular}{|c|c|}
\hline Curve & Spectroscopic Factor \\
\hline A & 7.9 \\
B & 4.1 \\
C & 8.2 \\
\hline
\end{tabular}
Hodgson attempts to fit both the angular distributions and the polarisation data, but the fit can be described as only reasonable. Satchler in attempting to improve upon this published only detailed fits to the angular distribution; one can only assume that the corresponding polarisation fits were poor. Elton has subsequently suggested that in order to fit all the data simultaneously an optical potential whose real part is attractive in the nuclear surface, but just inside this becomes repulsive, should be used.

We have used the conventional optical model parameters deduced by Hodgson and Satchler but with the spin-orbit components put to zero. The Saxon-Wood potential is taken as

$$\overline{U}(r) = -\frac{V_0}{1 + e^x} - i \left( W_0 - 4\frac{d}{dx'} \right) \frac{1}{1 + e^{-x'}} + V_C$$

(4.67)

where

$$x = \frac{(r - r_A^{1/3})}{a_0}$$

$$x' = \frac{(r - r_A^{1/3})}{a_0'}$$

$V_C$ is the Coulomb potential from a uniform charge of radius $r_A^{1/3}$. Potential $A$ is the volume absorption potential of Hodgson, potential $B$ is the volume absorption potential of Satchler and potential $C$ the surface absorption potential of Satchler. The parameters are listed over
In Fig 11. we give the corresponding \((p,d)\) cross-sections for these three potentials. As can be seen they each give equally acceptable fits to the data. Again all three curves were normalised to the experimental data at \(0^\circ\). The most interesting feature however is the spectroscopic factors so deduced.

<table>
<thead>
<tr>
<th>Potential</th>
<th>Spectroscopic Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9.5</td>
</tr>
<tr>
<td>B</td>
<td>7.7</td>
</tr>
<tr>
<td>C</td>
<td>8.2</td>
</tr>
</tbody>
</table>

The mean of these three values is 8.5 with a deviation of approximately 10\%. The fact that the two Satchler potentials (B and C) give spectroscopic factors close to the \(j-j\) value, is probably not significant, since there are many other uncertainties. The important fact is, that it is possible to find several proton potentials which give acceptable fits to the elastic and the \((p,d)\) angular distributions and yet may vary by 10\% or more in the absolute magnitudes.
Fig. 11. Effect of the variation of the proton optical potential on the angular distribution.
4.4.3 Deuteron Optical Potential

The only data of elastic deuteron scattering on carbon at 156 MeV is that of Baldwin et al.\(^4\)\(^7\). The data is confined to about twelve points in the angular range \(0^\circ < \theta < 40^\circ\), with an absolute error of the order of 20%. No optical model potential to fit this data has been reported. We have carried out a limited parameter search with the potential (4.67) with a real well-depth \(V_0 = 100\) MeV. A reasonable fit was obtained with the parameters given below

\[
\begin{array}{ccccccccc}
V_0 & r_0 & a_0 & \bar{w}_0 & \bar{w}_D & r_0' & a_0' & r_C \\
D & 100 & 0.8 & 0.8 & - & 10 & 1.5 & 0.5 & 1.4
\end{array}
\]

An alternative procedure for finding the deuteron potential, starting from an optical potential for the interaction between each nucleon in the deuteron and the nucleus has been given by Watanabe\(^4\)\(^2\). The derived potential is given by

\[
V(R) = \int d^3r \, \psi_d^2(r) \left\{ \bar{U}_p \left( \frac{1}{R} | R + x | \right) + \bar{U}_n \left( \frac{1}{R} | R - x | \right) \right\}
\]

where \(R = (r_p + r_n)/2\), \(x = r_p - r_n\), \(\psi_d(r)\) is the internal deuteron wavefunction, and \(\bar{U}_p\) and \(\bar{U}_n\) are the proton and neutron optical potentials. Equation (4.68) can be simplified to give a more computable form\(^4\)\(^2\)

\[
V(R) = \frac{4\pi}{\bar{R}} \int_0^\infty x \left[ \bar{U}_p(x) + \bar{U}_n(x) \right] dx \int_{2R-x}^{2(R+x)} r \, \psi_d^2(r) dr
\]

To test this formula against the data of Baldwin at 156 MeV,
the Hulthen form is used for the deuteron wavefunction. If we assume that each nucleon in the deuteron has half the deuteron energy, then the potential parameters for $U_p$ and $U_n$ should be taken from elastic nucleon scattering from carbon at about 78 MeV. The closest approximation to this that could be found in the literature, was for neutron elastic scattering at 96 MeV performed by Salmon$^{50}$). The optical model fit to this data was performed by Hodgson$^{51}$) using a Saxon-Wood potential, the real well-depth was only 22.8 MeV. The best elastic proton data is that of Gerstein et al$^{52}$) at an energy of 90 MeV. This data has been analysed by Glassgold and Kellogg$^{53}$) using a Hill-Ford potential, the resulting real well-depth was 40 MeV.

Using these potentials, equation (4.69) was used to derive a deuteron optical potential. The three main qualitative features from this calculation were

(i) the potential depth was approximately the sum of the neutron and proton well depths.

(ii) the effective radius was about the same as that for the neutron and proton.

(iii) the diffuseness parameter was somewhat larger.

The numerically derived deuteron potential was approximated by a Saxon-Wood potential (4.67), and the resulting parameters are given over.
In fig 12, we show the cross-section for elastic deuteron scattering at 150 MeV from carbon as predicted by potentials D and E. One can see that the crudely derived potential E gives almost as good a fit as the phenomenologically chosen potential D.

However in fig 13, we show the resulting cross-section for the (p,d) reaction using the potentials D and E. Again both curves have been normalised to the experimental cross-sections at 0°, and the spectroscopic factors deduced.

<table>
<thead>
<tr>
<th>Potential</th>
<th>Spectroscopic Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>8.2</td>
</tr>
<tr>
<td>E</td>
<td>6.0</td>
</tr>
</tbody>
</table>

We are therefore forced to the conclusion that while potential E gives an acceptable fit to the elastic deuteron scattering data, it predicts a smaller magnitude and the wrong shape for the (p,d) cross-section. This agrees with the conclusions of Lee et al. 43, who also demonstrate that the most acceptable deuteron potential for reproducing stripping cross-sections is one 100 MeV deep.
Fig. 12. Angular distribution for elastic scattering of 150 MeV deuterons by C\textsuperscript{12}. The experimental values are taken from ref. 47 for 156 MeV deuterons on C\textsuperscript{12}. 

\[ \frac{d\sigma}{d\Omega} = \text{mb} \]

\[ \Theta_{\text{cm}} \]

\[ \text{Optical Model} \]

\[ \text{Exp. 107} \]
Fig. 13. Effect of variation of deuteron optical potential on angular distribution.
4.4.4 **Neutron Wavefunction.**

We showed in a previous paragraph, 4.2, that the approximating of the overlap integral by a single-particle wavefunction is equivalent to using a single-particle model for the nuclear wavefunctions. Consequently the choice of the neutron potential $V_{\text{nc}}$ should therefore be that central potential used in the single-particle model to construct the basis wavefunctions of the nuclear system. In theory, this should be found using a self-consistent approach of the Hartree-Fock type. In practice a phenomenological potential is used.

The usual procedure employed in stripping, see for example reference 43, is to take for the neutron potential, the real part of the proton optical potential. This implies that the central potential which describes the scattering states of a proton with the $A$-particle system can be extrapolated back to describe the negative energy states of a neutron bound to an $(A-1)$-particle system. This extrapolation may be reasonable for low energy scattering states, and for medium and heavy nuclei. Perhaps the apparent success of the procedure justifies this. However neither of these conditions apply in the work being reported here, and we shall further demonstrate the inadequateness of such a prescription.

A far better procedure is to attempt to find the shell
model central potential from other information. This is clearly a nuclear structure problem. A most useful technique is the analysis through elastic electron scattering, which attempts to find just this potential. However this and other nuclear structure calculations are designed, such that the bound state single-particle wavefunctions so generated are at their best in the nuclear interior. In direct reactions, however, the nuclear surface region is thought to be important. This may well be true at low energies, but at the high energies considered here, the nuclear interior is equally important. Nevertheless we still must not neglect the asymptotic behaviour of the single particle wavefunction, which is shown by equation (4.25) to be determined by the neutron separation energy.

Since the oscillator potential has had considerable success in shell-model calculations it might well be considered appropriate here. However it fails in the asymptotic region, oscillator wavefunctions at large radii go as $e^{-r^2/a^2}$, whereas the required behaviour is as $e^{-\alpha r}$.

The procedure we propose in this thesis is to find the phenomenological shell model potential by fitting elastic electron scattering as outlined in Chapter 3. However a little care must be used in considering which is the appropriate data. For the case $^{12}(p,d)\; ^{11}$ experiment, the neutron has a separation energy of 18.7 MeV, whereas the corresponding $^{12}(p,2p)\; ^{11}$ experiment, the proton separation energy is 16 MeV. The
difference can be attributed to the Coulomb energy. So the prescription is that the well-depth $V_0$ is found so that the energy eigenvalue for the neutron is 18.7 MeV, and $r_0$ and $a_0$ are found from fitting electron scattering on $^{12}$C. In fig. 14 curve (A) has been obtained in this way. Curve (B) has the neutron parameters taken from the proton optical potential and curve (C) has used an oscillator neutron wavefunction. The parameters used are listed below

<table>
<thead>
<tr>
<th>Saxon-Wood</th>
<th>Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>$r_0$</td>
</tr>
<tr>
<td>A</td>
<td>55.6</td>
</tr>
<tr>
<td>B</td>
<td>109.4</td>
</tr>
<tr>
<td>C</td>
<td>-</td>
</tr>
</tbody>
</table>

All three curves were normalised to the experimental data at $0^\circ$, and the spectroscopic factors so deduced are listed above.

The improvement of potential A over B in the fit is clearly demonstrated. Somewhat surprising however is the extremely good fit obtained with the oscillator. This is in part fortuitous in that the experimental separation energy is large so that the overlap integral will go rapidly to zero as $r$ goes to infinity. However the correctness of the shell model approach may well be reflected in this result.
Fig. 14. Effect of variation of the neutron wavefunction on the angular distribution.
Let us consider the \( \text{Be}^9(p,d)\text{Be}^8 \) reaction still at proton energies of 155 MeV. The neutron separation energy in this case is only 1.67 MeV. This gives computational difficulties, in that the neutron wavefunction with the correct asymptotic behaviour, has to be integrated out to approximately 30 fm before the radial integral is sufficiently accurate. This implies that the cross-section will be very sensitive to the shape of the neutron wavefunction. This statement is illustrated in fig.15 where four neutron wavefunctions have been considered, each one incidently being compatible with the electron scattering data on \( \text{Be}^9 \). We list the parameters used below together with the derived spectroscopic factors

<table>
<thead>
<tr>
<th></th>
<th>( V_0 )</th>
<th>( r_0 )</th>
<th>( a_0 )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>28.7</td>
<td>1.5</td>
<td>0.7</td>
<td>1.9</td>
</tr>
<tr>
<td>B</td>
<td>32.6</td>
<td>1.3</td>
<td>1.13</td>
<td>3.3</td>
</tr>
<tr>
<td>C</td>
<td>36.1</td>
<td>1.1</td>
<td>1.38</td>
<td>4.6</td>
</tr>
<tr>
<td>D</td>
<td>39.8</td>
<td>0.9</td>
<td>1.57</td>
<td>5.5</td>
</tr>
</tbody>
</table>

The theoretical spectroscopic factor deduced from a simple shell model calculation is in the range 0.5 to 0.7. We see that the shape fit is poor and the magnitudes wrong.

To see why this is, we note that the neutron is very loosely bound in \( \text{Be}^9 \), the separation energy being only 1.67 MeV. However the analysis of electron scattering
Fig. 13. Effect of variation of neutron wavefunction on Be$^9$ (p,d) angular distributions.
determines only information on the proton density
distribution which is theoretically correlated
to the shell model potential. Furthermore the
proton separation energy as determined from mass differences
is 17 MeV\(^5\). Clearly the neutron density distribution in
\(^9\)Be, no way resembles the proton density distribution, and
hence our prescription, which assumes that the proton density
distribution can be equated with the neutron density distribution,
can only be of use in nuclei with \(N = Z\).

However it might be argued that in \(^9\)Be, the proton
separation energy is \(-0.2\) MeV, and hence the proton density
distribution in \(^9\)Be may resemble the neutron density
distribution in \(^9\)Be. Unfortunately no experimental data
of electron scattering on \(^9\)Be is available. A crude
approximation to this might be to use the data on \(^11\)Be. In
fig.16, we show the \(^9\)Be\([p,d]\)^8Be cross-sections, curve A
takes the neutron parameters from electron scattering on \(^11\)Be,
curve B from the proton optical potential, and curve C uses
an oscillator for the neutron wavefunction. The parameters
used, together with the derived spectroscopic factors are
listed at the top of page 108.

Once again the inadequateness of the use of the proton
optical potential as the source of neutron parameters is clearly
demonstrated. Even more surprising is the goodness of fit
obtained using the oscillator, indicating that the nuclear
Fig. 16. Effect of further variation of the neutron parameters on $\text{Be}^9(p,d)\text{Be}^8$ angular distributions.
interior must be important for high energy reactions. Furthermore the agreement between the theoretical and experimental spectroscopic factors indicates the internal consistency of the single-particle shell model description of the nucleus.

4.4.5 Spin-orbit Effects.

To consider the effect of introducing a spin-orbit potential of the $L.S$ type into the optical potential, one calculation was made using the DWBA code of Macefield. To the optical potential (4.67) a further term

$$U_{s.o.} = \left( \frac{\hbar}{m_r c} \right)^2 (V_s + iW_s) \frac{1}{r} \frac{d}{dr} (e^r + 1)^{-1} L.S$$

was added. Proton potential $C$ and deuteron potential $D$ were used with the spin-orbit strengths

<table>
<thead>
<tr>
<th>$V_s$</th>
<th>$W_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deuteron</td>
<td>6</td>
</tr>
<tr>
<td>Proton</td>
<td>3.35</td>
</tr>
</tbody>
</table>
These deuteron parameters were chosen arbitrarily, whereas the proton parameters were taken from ref. 48.

Fig. 17 shows that the angular distribution is little affected by the introduction of spin-orbit coupling, but the magnitude was altered by 25%. This implies an even further uncertainty in the deduced spectroscopic factors.

If we assume there to be a 10% error in the spectroscopic factor due to uncertainties in the choice of the proton optical potential, a further 10% from the choice of the deuteron optical potential, and a further 25% from the neglect of spin-orbit coupling, then the absolute value of the spectroscopic factor determined from fitting the theory to the experiment will at best only be accurate to within 30%. A similar conclusion was reached by Lee et al. for much lower energies, where the optical potentials are much more precisely known.

4.4.6 \( \ell \)-dependence.

It is well-known that at lower energies, the shape of the forward peak in the angular distribution can determine the orbital angular momentum, \( \ell \), transferred in the reaction. It is for this reason that in recent years, direct reactions have provided such a powerful tool in identifying the spins and parities of nuclear energy levels excited by the reaction. As one proceeds towards higher incident energies, so the more
Effect of inclusion of spin-orbit term in the optical potentials on the (p,a) angular distribution. The LEA has not been included.
highly excited states became populated. This is particularly important for light nuclei, where it is hoped that the level spacing is greater than the possible energy resolution. Thus with incident protons of 155 MeV on Be\(^9\) it is possible to cause a rearrangement collision leaving a final nucleus Be\(^8\) in an excited state of 18.9 MeV. However the angular distribution of the outgoing deuterons is only measurable over a limited angular range — in this example only from 0° to 35°. The question, then, is it still possible to assign a unique \(\ell\)-value to the transition when only such a limited angular distribution is measurable?

We may further ask whether the \(j\)-dependence of the transferred neutron is detectable? In fig. 18 we show the computed angular distributions for \(^{12}_C(p,d)^{11}_C\) for angular momentum transfer \(\ell = 0,1,2\) and 3.

By including a spin–orbit term in the shell-model potential for the neutron wavefunction, the bound state wavefunction will then differ for \(j = \ell \pm \frac{1}{2}\). These differences have little effect on the angular distribution as can be seen from Fig. 18. However the shape of the forward peak is still sufficient to distinguish between different \(\ell\)-values.

Since \(^{12}_C\) ground state has spin 0\(^+\) the spin of the final state will be given by the \(j\)-value of the transferred neutron. It has been suggested\(^ {43}\) that it would be possible
Fig. 18. Effect on the \((p,d)\) angular distribution of variation in the \(l\)-value of the transition.
to distinguish between different \( j \)-values by considering the absolute magnitudes of the cross-sections. In table 4 we list the spectroscopic factors for different \( j \)-values, for the \( ^{12}\text{C}(p,d)^{11}\text{C} \) reaction. We also list the well-depth required in the neutron potential to give a binding energy equal to the separation energy, 18.721 MeV.

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( j^m )</th>
<th>( V_0 )</th>
<th>( S(j) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/2(^+)</td>
<td>37.8</td>
<td>45</td>
</tr>
<tr>
<td>1</td>
<td>3/2(^-)</td>
<td>56.5</td>
<td>8.2</td>
</tr>
<tr>
<td>1</td>
<td>1/2(^-)</td>
<td>66.8</td>
<td>8.1</td>
</tr>
<tr>
<td>2</td>
<td>5/2(^+)</td>
<td>79.9</td>
<td>4.4</td>
</tr>
<tr>
<td>2</td>
<td>3/2(^+)</td>
<td>109.9</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>7/2(^-)</td>
<td>107.3</td>
<td>5.3</td>
</tr>
<tr>
<td>3</td>
<td>5/2(^-)</td>
<td>174.4</td>
<td>8.3</td>
</tr>
</tbody>
</table>

For the \( f_{7/2} \) and \( f_{5/2} \) cross-sections, the difference in the spectroscopic factors is of the order of 50\%. However as is to be expected, the effect is smaller for smaller \( \ell \)-values, in this example \( \ell = 2 \) gives a difference of 15\%, and \( \ell = 1 \) less than 5\%. Therefore until the uncertainty in the correct choice of optical model parameters can be narrowed down, it will only be possible to distinguish between \( f_{7/2} \) and \( f_{5/2} \) by making a detailed comparison of spectroscopic factors with structure theory.
4.4.7 Localisation in Angular Momentum.

Recently a derivation of the DWBA expression for the transition matrix $T_{fi}$ has been given using the strong coupling approximation $^2$. It is argued that the failings of the DWBA in neglecting the coupling to other reaction channels can be better estimated in this way. Buck and Rook $^9$ have shown that the usual DWBA procedure of taking the optical model potentials from the corresponding elastic scattering data can be justified if a 'truncated unitarity relation' is satisfied. This relation is that the modulus of the S-matrix for the elastic scattering in the entrance channel should equal the modulus for the same in the exit channel, for the important partial waves. That is

$$|S_p| = |S_d|$$

(4.70)

In fig. 19 we plot the magnitude of the radial integrals $I_{n,n+1}$ and $I_{n,n-1}$ against the proton partial wave number $n$, for the $C^{12}(p,d)C^{11}$ reaction. Also plotted as a function of $n$ is the contribution to the $(p,d)$ cross-section from each partial wave, for $\theta = 10^\circ$. It is clear that the important partial waves are from $n = 4$ to $n = 11$. Furthermore the greater overlap occurs between the proton wavefunction $f_n$ and the deuteron wavefunction $f_{n+1}$, than between $f_n$ and $f_{n-1}$.
Fig. 19. Contribution to $(p,a)$ cross-section at $10^\circ$ and contribution to radial integrals, as a function of angular momentum.
Table 5.

| n  | $|S_p|$ | $|S_d|$ | $|S_p(n) - S_d(n+1)|$ |
|----|-------|-------|-----------------|
| 0  | 0.455 | 0.622 | 0.162           |
| 1  | 0.440 | 0.617 | 0.173           |
| 2  | 0.429 | 0.613 | 0.172           |
| 3  | 0.432 | 0.601 | 0.157           |
| 4  | 0.477 | 0.589 | 0.092           |
| 5  | 0.573 | 0.569 | 0.031           |
| 6  | 0.687 | 0.542 | 0.177           |
| 7  | 0.788 | 0.510 | 0.323           |
| 8  | 0.864 | 0.465 | 0.424           |
| 9  | 0.916 | 0.440 | 0.456           |
| 10 | 0.949 | 0.460 | 0.407           |
| 11 | 0.970 | 0.542 | 0.315           |
| 12 | 0.982 | 0.655 | 0.221           |
| 13 | 0.990 | 0.761 | 0.143           |
| 14 | 0.994 | 0.847 | 0.079           |
| 15 | 0.997 | 0.915 | 0.037           |
| 16 | 0.999 | 0.960 | 0.015           |

In Table 5 above, we list the S-matrix elements, $|S_p|$ and $|S_d|$, and the difference between $S_p(n)$ and $S_d(n+1)$. It is expected that for $4 \leq n \leq 11$, the latter column should be small. For the first part of this range of $n$, the condition is moderately well satisfied, but certainly not for $n \geq 7$. Yet, despite this, a well-fitted angular distribution and a reasonable spectroscopic factor has been obtained. This casts some doubt on the usefulness of the
truncated unitarity relation (4.70), and clearly much more work has to be done in this direction.

4.4.8 Localisation in Configuration Space.

We noted earlier the surprising success of the oscillator neutron wavefunction in giving as good a fit as the Saxon-Wood wavefunction to the (p,d) angular distributions; and further giving the correct absolute magnitude to the cross-section. From this we deduced that the nuclear interior must be important for these high energy reactions.

In fig. 20, the solid curve gives the contribution (expressed in arbitrary units) to the $C^{12}(p,d) C^{11}$ cross-section at $0^\circ$ as a function of the radial distance $r_{nc}$. The surprising feature is the strong contribution from the nuclear interior between 1 and 2 fermi, in addition to the usual surface peak between 3 and 4.5 fermi.

Also plotted is the bound state wavefunction $u(r)$ and the neutron potential $V(r)$ used in these calculations. Again the difference between the oscillator and the Saxon-Wood is small in the important regions, and the differing asymptotic behaviour only manifests itself at distances greater than 4.5 fermi. Clearly, therefore, we are justified in using oscillator wavefunctions under these high energy conditions.
Fig. 20. Solid curve gives the contribution to the $C_{12}(p,a)C''$ cross section at $0^\circ$ as a function of $r$. Dashed and dotted lines give neutron wavefunction $U(r)$ and neutron potential $V(r)$. 
4.5 Comparison with Experiment.

We now present the detailed fits to the Radvanyi\textsuperscript{19}) data for $^{12}\text{C}(p,d)^{11}\text{C}$, $^{10}\text{B}(p,d)^{9}\text{B}$ and $^{9}\text{Be}(p,d)^{8}\text{Be}$. The purpose here is to identify the $\ell$-value of the transition and to determine the experimental spectroscopic factor. We defer the discussion on the nuclear spectroscopy to the next chapter.

4.5.1 $^{12}\text{C}(p,d)^{11}\text{C}$.

Four angular distributions have been obtained corresponding to the excitation of the 0.00, 2.00, 4.81 and 6.9 MeV levels in $^{11}\text{C}$. In figs. 21 and 25, we show that the angular distributions for the first three levels are well fitted for $\ell = 1$. However in the case of the 6.9 MeV level, fig. 25 shows that it has not been possible to assign a unique $\ell$-value. This is most likely due to the admixture of other final states, probably the 6.35 and 6.49 MeV levels. The experimental spectroscopic factors are

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>Saxon-Wood</th>
<th>Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>$8.2 \pm 2.7$</td>
<td>$7.7 \pm 2.5$</td>
</tr>
<tr>
<td>2.00</td>
<td>$2.7 \pm 0.9$</td>
<td>$2.0 \pm 0.6$</td>
</tr>
<tr>
<td>4.81</td>
<td>$1.3 \pm 0.4$</td>
<td>$0.9 \pm 0.3$</td>
</tr>
</tbody>
</table>
Fig. 21. \( \ell = 1 \) angular distributions for \( C^{12}(p,\alpha)C^n \)
4.5.2 $^{10}\text{Be}(p,d)^9\text{Be}$.

Final states in $^9\text{Be}$ at 0.00 ; 2.4 ; 7.1 ; 9.6 ; 11.4 and 14.6 MeV have been identified. Each distribution has been fitted by an $\ell = 1$ DWBA curve. The fits are shown in figs. 22 and 23, and the spectroscopic factors obtained are listed below

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>Saxon-Wood</th>
<th>Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.8 ± 0.5</td>
<td>1.2 ± 0.4</td>
</tr>
<tr>
<td>2.4</td>
<td>1.9 ± 0.6</td>
<td>1.5 ± 0.5</td>
</tr>
<tr>
<td>7.1</td>
<td>1.2 ± 0.4</td>
<td>1.1 ± 0.4</td>
</tr>
<tr>
<td>9.6</td>
<td>1.4 ± 0.4</td>
<td>1.3 ± 0.4</td>
</tr>
<tr>
<td>11.4</td>
<td>2.0 ± 0.6</td>
<td>1.9 ± 0.6</td>
</tr>
<tr>
<td>14.6</td>
<td>0.7 ± 0.2</td>
<td>0.6 ± 0.2</td>
</tr>
</tbody>
</table>

4.5.3 $^{9}\text{Be}(p,d)^8\text{Be}$.

Final states in $^8\text{Be}$ at 0.00 ; 2.90 ; 11.4 ; 16.6 and 18.9 MeV have been identified. However Nuclear Data Sheets\textsuperscript{33)} show that for $^8\text{Be}$, levels at 16.62 ; 16.92 ; 17.64 ; 18.15 ; 18.9 ; 19.1 and 19.22 have all been identified. Consequently the two highly excited states in $^8\text{Be}$, probably contain admixtures to some of these other states in the angular distribution.

Nevertheless it has been possible to fit 0.00 ; 2.90 ; 16.6 and the 18.9 data with an $\ell = 1$ distribution. The 11.4 level is a very broad 4+ state. Transitions to this state are forbidden by a simple one-step process, which probably explains
Fig. 22. \( l = 1 \) angular distributions for \( B^{10}(p,d)B^9 \)
Fig. 23. $l=1$ angular distributions for $^8B(^p,d)^9B$
why a unique \( l \)-value cannot be attributed to this transition. The fits are shown in figs. 24 and 25, and the spectroscopic factors are listed below

<table>
<thead>
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<th>Energy (MeV)</th>
<th>Saxon-Wood</th>
<th>Oscillator</th>
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<tr>
<td>0.00</td>
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<td>0.5 ± 0.2</td>
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<tr>
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<td>1.2 ± 0.4</td>
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<tr>
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<td>1.3 ± 0.4</td>
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<tr>
<td>18.9</td>
<td>0.6 ± 0.2</td>
<td>0.5 ± 0.2</td>
</tr>
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</table>

The Saxon-Wood parameters for the optical potentials and the neutron bound-state potential used for these angular distributions are given in Table 9, on page 126a. The \( \text{B}^{10} \) proton potential is an interpolation of the values given in reference 48 for \( \text{Be}^9 \) and \( \text{C}^{12} \). The neutron well-depth \( V_0 \) quoted is that value which made the energy eigenvalue for the potential equal to the neutron separation energy for the ground-state transition. \( V_0 \) was adjusted for each excited state.

The oscillator neutron wavefunction is of the form given in equation (2.6). The length parameter \( a \) was taken from reference 88. The oscillator neutron potential is required in the calculation for the function \( \Lambda (r) \) used in the local energy approximation, equation (4.66). Again the well-depth \( V_0 \) was chosen so that the energy eigenvalue equalled the neutron separation energy.
Fig. 24. $l=1$ angular distributions for $\text{Be}^9(p,d)\text{Be}^8$
Fig. 25. Two angular distributions which could not be fitted by a unique $l$ value.
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<tr>
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<td><strong>Be$^{9}$</strong></td>
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<td>0.8</td>
<td>0.8</td>
<td>-</td>
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<td>1.59</td>
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<td>1.25</td>
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<td>-</td>
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</table>
CHAPTER 5.
Spectroscopic Factors.

5.1 Introduction

In the last chapter we presented the formalism for the \((p,d)\) reaction; in particular, in the assumption that the final state is reached through the transfer of a single \(j\)-value, the cross-section is given by (4.35)

\[
\frac{d\sigma}{d\Omega} = S(j) \sigma(Q, l, \phi)
\]  \hspace{1cm} (5.1)

where \(\sigma(Q, l, \phi)\) is the reaction factor depending on the \(Q\)-value for the reaction, the orbital angular momentum of the transferred neutron, \(l\), and the scattering angle \(\phi\)

\[
\sigma(Q, l, \phi) = \frac{3m^2}{16\pi^2h^4} \frac{A(A-1)}{(A+1)^2} \frac{k_F}{k_i} \frac{c^2}{2l+1} \sum_{\lambda} |B_{\lambda}(\phi)|^2
\]  \hspace{1cm} (5.2)

The \(Q\)-value of the reaction is defined from the energy conservation relation (4.16)

\[
Q = \mathcal{E}_A - \mathcal{E}_C - B_d
\]  \hspace{1cm} (5.3)

\[
= \hbar^2k_F^2/2M_N - \hbar^2k_i^2/2M_d
\]

where \(M_N = \frac{A}{A+1} m\) and \(M_d = \frac{2(A-1)}{A+1} m\), \(m\) being the nucleon mass, and \(B_d\) is the deuteron binding energy. \(C\) is the iso-
spin Clebsch-Gordan coefficient \((\mathbb{T}_{J L-J J} | T_{A M}^M_{A MTA})\).

The spectroscopic factor \(S(j)\) depends only on the wavefunctions of the nuclear states involved, and provides a useful basis for comparison between experiment and the predictions of nuclear models. It was defined in (4.36)

\[
S(j) = A |g(j)|^2
\]

where \(A\) is the mass number of the target nucleus, and \(g(j)\) was the expansion coefficient, when the target nuclear wavefunction was expanded in the complete set of \((A-1)\)-particle wavefunctions.

\[
\Phi_{M_{A MTA}}^{J_{A T A}} = \sum_j g(j) \Phi_{M_{C MTC}}^{J_{C T C}} \phi_{m_{\frac{1}{2}}}^{J_{\frac{1}{2}}} (J_{C M_{Cjm}} | J_{A M}) \quad (T_{C T C} \, | T_{A MTA})
\]

The development of a formalism for deriving the spectroscopic factor from nuclear models is given in a review article by Macfarlane and French\(^{31}\). We shall use their formulae to predict the spectroscopic factors for the \((p,d)\) experiments of Radványi et al\(^{19}\), and from the comparison derive spectroscopic information on the spins and parities of the excited states of the residual nuclei. We begin by briefly summarising the formulae we shall use.
5.2 Shell Model Expressions for $S(j)$

We first introduce the graphical notation used by Macfarlane and French.

The vector coupling of two commuting angular momentum operators $j_1$ and $j_2$ is represented diagrammatically by

$$\psi'(j_1j_2JM) = \begin{array}{c}
\text{Diagram}
\end{array}$$

The diagram not only represents the mode of coupling, but the wavefunction $\Psi$ itself as well. To couple $n$ commuting operators, the diagram

$$\begin{array}{c}
\text{Diagram}
\end{array}$$

is insufficient, since it only exhibits $(n+2)$ quantum numbers, whereas $2n$ is needed for a complete specification. The description is completed by specifying any $(n-2)$ non-intersecting diagonals. Naturally any two schemes for coupling the same $n$ angular momenta are related by a unitary transformation. Any such transformation can be compounded from three fundamental recoupling rules

$$a \begin{array}{c}
\text{Diagram}
\end{array} b = (-)^{a+b-c} b \begin{array}{c}
\text{Diagram}
\end{array} a$$

(5.6a)
where \( U(abcd;ef) \) is a normalised Racah coefficient. This is related to the 6j-symbol by

\[
U(abcd;ef) = (-)^{a+b+c+d} \frac{1}{\sqrt{(2e+1)(2f+1)}} \{a b e\} \{d c f\} \tag{5.7}
\]

Note that the projection quantum number \( M \) has been omitted from the diagrams, merely for convenience.

When a wavefunction is antisymmetric, this is indicated by a circular arc. For example, an antisymmetric state of the configuration \((n\ell)\) in an LST representation is written

Here the antisymmetric symbol is to apply to the whole wavefunction. A convenient "direct product" notation, which enables formal manipulations to be carried out independent of the particular representation used, is to write
Greek letters are used consistently to refer to such composite sets of quantum numbers.

The recoupling rules (5.6) are easily adapted, for example

\[
\sum_{\xi} U(\beta \gamma \xi \xi; \epsilon \tau) = \sum_{\xi} U(\beta \gamma \xi \xi; \epsilon \tau) = U(\beta \gamma \xi \xi; \epsilon \tau)
\]

provided that the Racah coefficient has the significance

\[
U(\beta \gamma \xi \xi; \epsilon \tau) = U(\beta \gamma \xi \xi; \epsilon \tau) = U(\beta \gamma \xi \xi; \epsilon \tau)
\]

in the case of an LS representation. Similarly

\[
(-)^{\beta} = (-)^{\beta^{L} + \beta^{S}}
\]

\[
(2\beta + 1) = (2L + 1)(2S + 1)
\]

Non-angular momentum quantum numbers (e.g. space symmetry) are written in Roman letters.
Now suppose it is required to construct the wavefunction for \( n \) equivalent particles of specified \( x \) and \( \beta \)

\[
\rho^n_{x, \beta}
\]

and that the wavefunctions

\[
\rho^{n-1}_{y, \xi}
\]

for \((n-1)\) equivalent particles is known. Then the functions

\[
\rho^{n-1}_{y, \xi}
\]

are antisymmetric in the first \((n-1)\) particles. However the required functions span a restricted subspace of the space spanned by these functions. Hence

\[
\rho^n_{x, \beta} = \sum_{y, \xi} \left( \rho^n_{x, \beta} \right) \rho^{n-1}_{y, \xi} \]

(5.9)

The expansion coefficients are called coefficients of fractional parentage (cfp). Orthonormality of the coefficients for different \( x \) yield the sum rule
Now let us recall the definition of the spectroscopic factor

\[ S(j) = A \left| \mathcal{g}(j) \right|^2 \]

where \( \mathcal{g}(j) \) is given by

\[ \mathcal{g}(j) = \begin{array}{c}
\langle j^n \mid t^n \rangle \\
J_A T_A \\
\langle j^{n-1} \mid s(n) \rangle \\
J_C T_C \\
\langle \varepsilon^{n-1} \mid t(n) \rangle \\
\end{array} \]

The necessary change of representation is obtained by using the recoupling rules \((5.6)\)
\[ f(n) \]

\[ J = y . y (n-l) \]

Hence

\[ S = A J (j) \times 3 (d) \]

\[ = A \sum_{z,z'} (-)^{z-z'} U(J_{C} A; z,j) U(J_{C} A; z'; j) g(z)^* g(z') \]

\[ = A \sum_{z,z'} (-)^{z-z'} \delta(z,z') g(z)^* g(z') \] \hspace{1cm} (5.13)

\[ = A \sum_{z} |g(z)|^2 \]

A single particle transition, which is sufficiently general for most practical uses is symbolised by

\[ \left\{ \begin{array}{c}
\alpha_2
\beta
\alpha_3
\end{array} \right\} \Gamma \]

\[ \rightarrow \left\{ \begin{array}{c}
\alpha_2'
\beta'
\alpha_3'
\end{array} \right\} \Gamma' \] \hspace{1cm} (5.14)

where \( \left\{ \begin{array}{c}
\dot{x}
\end{array} \right\}_\Gamma \) indicates vector coupling to a resultant \( \Gamma \).

The spectroscopic factor is

\[ S(\varphi_3) = (n_1 + n_2 + n_3) |g(\varphi_3)|^2 \] \hspace{1cm} (5.15)
\[ g(f_3) = \sum_{\sigma, \chi} \langle \rho_3^{n_3}; \sigma_3 | \rho_3^{n_3-1}; \sigma_3 \rangle \]

Manipulating the wavefunction on the left, and using (5.9) for the cfp.

where the antisymmetrisation operator \( A_{n_1+n_2+n_3} \) takes the three independently antisymmetric groups of \( n_1, n_2 \) and \( n_3 \) particles respectively and antisymmetrises over the space spanned by \( (n_1+n_2+n_3) \) particles. That is

\[
A_{n_1+n_2+n_3} = \left[ \frac{n_1! \ n_2! \ n_3!}{(n_1+n_2+n_3)!} \right] \sum_r (-)^r \ P_r
\]  

where \( P_r \) is the permutation operator, spanning all possible interchanges of the \( (n_1+n_2+n_3) \) particles.
Similarly the wavefunction on the right hand side of

\[(5.15)\]
can be written

\[\psi_{n_1+n_2+(n_3-1)}^{n_1+n_2+(n_3-1)} = \sum \Delta \]

By counting up the number of permutations from each side of
the overlap integral \[(5.15)\] which give a non-zero contribution,
the final result is obtained

\[S(\rho_3) = n_3 \left\langle \rho_3^{n_3} \alpha_3 \right| \rho_3^{n_3-1} \sigma_3 \rangle^2 \ u(\beta \sigma_3 \Gamma \rho_3; \xi \alpha_3)^2 \]

where the last factor is an abbreviated notation for a
product of Kronecker deltas.

Thus the spectroscopic factor

\[S(\rho_3) = n_3 \left\langle \rho_3^{n_3} \alpha_3 \right| \rho_3^{n_3-1} \sigma_3 \rangle^2 \ u(\beta \sigma_3 \Gamma \rho_3; \xi \alpha_3)^2 \]

Note that the inert groups of nucleons \([\rho_1^{n_1} \text{ and } \rho_2^{n_2}]\) affect
the spectroscopic factor only through their total quantum number $\mathbf{\ell}$. If it so happens that these groups are coupled to $\mathbf{\ell} = 0$ (for example, if both $\mathbf{\ell}_1$ and $\mathbf{\ell}_2$ are closed shells); then the Racah coefficient is unity and the spectroscopic factor reduces to

$$S(\mathbf{\ell}_3) = n_3 \langle \mathbf{\ell}_3 \alpha_3 \mid \mathbf{\ell}_3 \mathbf{\ell}_3^{n_3-1} \alpha_3 \rangle^2$$

(5.21)

If further we consider the particular case when the target nucleus has only one nucleon outside a closed shell configuration, i.e. $n_3 = 1$, the fractional parentage coefficient will be unity and the spectroscopic factor is

$$S(\mathbf{\ell}_3) = 1$$

(5.22)

This is a well known result. An example of the use of this fact is the stripping reaction $\text{Ca}^{40}(d,p)\text{Ca}^{41}$ which has recently been extensively studied as a test for DWBA calculations.

Let us consider the JJT coupling representation for $\text{l}p$ shell nuclei. For $A \leq 12$, the target nucleus can be represented by

![Diagram](image_url)
where we have assumed the $1s$ nucleons to form an inert closed-shell configuration. This, then, is the case of $\beta = 0$, hence

$$S = n \left \langle \frac{3}{2} J_A T_A \right \rangle \left \langle \frac{3}{2} J_C T_C \right \rangle^2$$

where $n$ is the number of nucleons in the $1p$-shell. The fractional parentage coefficient is a product of three terms: a spin-orbit cfp., an iso-spin cfp., and a weighting factor. For the $p$-shell, the spin-orbit cfp are tabulated by Edmonds and Flowers, but the iso-spin cfp. and the weighting factor must be taken from Jahn and van Wieringen. The appropriate factors appear under the ATOMIC fractional parentage coefficients.

For the LSJT coupling representation, the channel spin formalism is used. Thus the overlap integral can be written

$$\mathcal{J}(z) = \left \langle \ell^n L_A S_A t^n T_A \right \rangle \left \langle \ell^{n-1} L_C S_C \right \rangle \left \langle \ell^{n-1} T_C t(n) \right \rangle$$

Again using the parentage expansion for the left hand wavefunction, and using the recoupling rules, it is straightforward to show that $31)$
\[ f(z) = (-)^{L + C + L_A} \langle L_A S_A T_A | L_c S_C T_C \rangle \]

\[ U(L_c S_C T_C^2; J_A S_A) \]

where \( x \) denotes the orbital symmetry. The spectroscopic factor is then given by

\[ S = n \sum_z |f(z)|^2 \]  

For the \( p \)-shell nuclei, the fractional parentage coefficient is again a product of three factors: a charge-spin cfp., an orbital cfp., and a weighting factor. These are tabulated by Jahn and van Wieringen\(^{58}\), however the phases are incorrect in these tables, and for use in the formula (5.24) the cfp. of Jahn and van Wieringen must have the phases amended, as given by Elliott, Hope and Jahn\(^{59}\). These amendments must be made both horizontally and vertically in the original tables of cfp. This problem of the phases is unimportant for a pure LSJT coupled representation since the cfp. is squared; but in the case of intermediate coupling wavefunctions using an LS basis, the phases are very relevant. A full discussion of this point is to be found in Macfarlane and French\(^{31}\).

Equation (5.24) can be easily adapted for use with intermediate coupling wavefunctions. The target nucleus wavefunction is now written.
\( \mathcal{Q}^{A_A T_A} (\varepsilon^n) = \sum_{x_A L_A S_A} K^{x_A L_A S_A, A_A T_A} \mathcal{Q}^{x_A L_A S_A} (\varepsilon^n) \) (5.26)

and a similar expression for the residual nucleus wavefunction.

Then

\[
\mathcal{J} (z) = \sum_{x_A L_A S_A} K^{x_A L_A S_A, A_A T_A} K^{x_A L_A S_A, A_A T_A} \mathcal{J}^{x_C L_C S_C} (z) (5.27)
\]

where \( \mathcal{J} (z) \) in (5.27) is given by the expression for \( \mathcal{J} (z) \) in equation (5.24).

It is useful to regard (5.27) as a matrix equation,

\[
\mathcal{J} (z) = K_A (\beta_z)^\mu K_C^\dagger
\]

where \( K_A \) is a row vector, \( K_C^\dagger \) a column vector and \( (\beta_z)^\mu \) the rectangular matrix

\[
(\beta_z)^\mu = (-)^{L_C + L_A} \langle \ell^n; x_A L_A S_A T_A \mid \ell^{n-1}; x_C L_C S_C T_C \rangle
\]

\[
U(L_C J_A S_A; L_A z) U(L_C S_C z^2; J_C S_A)
\]

The rows and columns of these matrices are labelled by sets of quantum numbers

\[
\ell = (x_A L_A S_A T_A) \quad s = (x_C L_C S_C T_C)
\]

There is a separate matrix \( (\beta_z)^\mu \) for each pair of values
\( \mu = (J_A J_C) \). For every \( \mu \), \( r \) and \( s \) take all sets of values satisfying the triangular conditions \( \Delta (L_A S_A J_A) \) and \( \Delta (L_C S_C J_C) \) respectively.

Lastly for the case of JJT coupling, a very simple sum rule can be deduced when \( \beta = 0 \). From equation (5.21)

\[
\sum_{J_A^T_C} S = n \sum_{J_A^T_C} \left\langle \left( \frac{3}{2} \right)^{2n} J_A T_A \right| \left( \frac{3}{2} \right)^{2n-1} J_C^T_C \right\rangle^2
\]

\[
= n \tag{5.30}
\]

That is for a given target nucleus \( (J_A T_A) \), the sum of the spectroscopic factors for all accessible states in the residual nucleus is equal to the number of nucleons in the \( p_{3/2} \) shell of the target nucleus.

A sum rule for the case LSJT coupling cannot in general be written down, since it involves the mixing of 6j-symbols and ofp. However in certain cases an explicit summation can be made, for example \( C^{12} (p, d) C^{11} \).

In LSJT coupling, the ground state of \( C^{12} \) with \( J_A = 0, T_A = 0 \) is \( 11_S[44] \), where we have used the notation \( 2T+1, 2S+1 \)

Hence we have \( L_A = 0, S_A = 0 \). Further the vector equation defining the channel spin \( S \) is

\[
L + J_A = \mathbf{Z} = J_C + \frac{1}{2}
\]

- 141 -
thus for an $l = 1$ transition, we have $z = 1$ only, and $J_C = 3/2$ or $1/2$. Hence the spectroscopic factor is

$$
S = 8 \left\langle (p)^8 \left| {11S[44]} \right| (p)^7 \left| {2T_{1/2} + 1, 2S_{1/2} + 1} \right| L_C \left[ \frac{x_C}{2} \right] \right\rangle^2
$$

$$
U(1L_C^00; 0) \left( U(L_C S_{1/2}; J_C 0) \right)^2
$$

From the triangular condition on the first Wigner coefficient $\Delta (1L_C^0)$, we see that $L_C = 1$. Also the cfp. is non-zero only when $L_C = 1$, $S_C = \frac{1}{2}$ and $x_C = 43$, and then

$$
\left\langle {11S[44]} \left| {22P[43]} \right\rangle^2 = 1.
$$

Evaluating the Wigner coefficients, we obtain

$$
S = 8 \cdot \frac{2J_C + 1}{6}
$$

Summing over the two possible $J_C$ values

$$
\sum_{J_C} S = 8
$$

In general, however, such a simple result cannot be obtained.

5.3 Comparison with Experiment.

We now look specifically at the experiments of Radvanyi et al.\textsuperscript{19}, and attempt to identify spins and parities of various excited states in the residual nuclei. We shall further
compare the spectroscopic factor derived from fitting the
DWBA curve to the experimental angular distribution, with
values obtained from various nuclear models.

In using the DWBA, we have three selection rules to
help us in this aim. First the parity change in the
transition is given by

\[ \Delta \tau = (-)^{\ell} \]  \hspace{1cm} (5.31)

where \( \ell \) is the orbital angular momentum of the transferred
neutron. Since the parities of the initial and final nuclear
states are normally known, this rule usually limits the choice
of \( \ell \). Furthermore as we had seen in the last chapter, it
is nearly always possible to identify from the angular
distribution, a unique \( \ell \)-value. This is of course in the
spirit of a direct interaction mechanism, where we are
assuming the neutron is "picked-up" from a well defined
single particle orbital.

The second selection rule is that of angular momentum
conservation, which limits the possible final nuclear spins

\[ J_A + \frac{1}{2} \gtrsim J_C \gtrsim \left| J_A - |\ell - \frac{1}{2}| \right| \]  \hspace{1cm} (5.32)

If isotopic spin is a good quantum number, then we have a
third selection rule

\[ T_A + \frac{1}{2} \gtrsim T_C \gtrsim \left| T_A - \frac{1}{2} \right| \]  \hspace{1cm} (5.33)
Lastly, it should be remembered that due to the uncertainties of the theory (e.g. the exact choice of suitable optical model parameters), the spectroscopic factor deduced can at best only be accurate to within 30%.

5.3.1 \( C^{12}(p,d) C^{11} \).

In the experiment of Radvanyi, four excited states have been identified in the deuteron energy spectrum. These correspond to exciting the 0.00 ; 2.00 ; 4.81 and 6.90 MeV levels in \( C^{11} \). The angular distributions of the first three of these are all compatible with an assignment of \( \ell = 1 \). Since the ground state of \( C^{12} \) is \( J_A = 0^+, T_A = 0 \) the possible final nuclear spins are

\[
J_C = \frac{3}{2}^- \text{ or } \frac{1}{2}^- \quad T_C = \frac{1}{2}
\]

In fig. 26 we give the energy level diagram for \( C^{11} \). The right-hand column gives the levels which have been experimentally observed up to an excitation of 8 MeV. The left-hand column are the predicted energy levels from Boyarkina\(^9\) using intermediate coupling wavefunctions on an LS basis. The spins and parities of the low lying levels are well known from considerations of \( B^{11} \) energy level scheme, and from such experiments\(^60\) as \( B^{10}(d,n \gamma) C^{11*} \). Recently an assignment to the higher levels has been given by Roush et al\(^61\) from an experiment \( Be^{9}(He^{3},n \gamma) C^{11} \),
<table>
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<th>Energy Level</th>
<th>Spin</th>
<th>State Description</th>
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<td>7.50</td>
<td>1/2+, 3/2+</td>
<td></td>
</tr>
<tr>
<td>6.9</td>
<td>3/2</td>
<td></td>
</tr>
<tr>
<td>6.90</td>
<td>5/2+</td>
<td></td>
</tr>
<tr>
<td>6.49</td>
<td>7/2-</td>
<td></td>
</tr>
<tr>
<td>6.35</td>
<td>3/2+</td>
<td></td>
</tr>
<tr>
<td>5.7</td>
<td>7/2</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>5/2</td>
<td></td>
</tr>
<tr>
<td>4.81</td>
<td>3/2-</td>
<td></td>
</tr>
<tr>
<td>4.32</td>
<td>5/2-</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>1/2-</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>3/2</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>3/2-</td>
<td></td>
</tr>
</tbody>
</table>

Intermediate Coupling

Experimental

Fig. 26. Energy Level Diagram for Cl. All states have isotopic spin T=1/2.
where the angular correlations of the $\alpha$'s emitted in coincidence with the neutrons has been measured. Their calculations suggest that the 6.35 MeV level has a spin and parity $3/2^+$, the 6.90 MeV level $5/2^+$ and the 7.50 MeV either $3/2^+$ or $1/2^+$. Note all positive parity states.

From the j-selection rule, we have that the excitation of the 4.32 ($5/2^-$) and the 6.49 ($7/2^-$) MeV levels should be forbidden in the (p,d) reaction. This is confirmed by the experiment of Radvanyi, the excitation of the 4.32 level being only a few per cent of the 4.81 level.

### Table 6.

<table>
<thead>
<tr>
<th>Final State in 11C</th>
<th>Possible shell model configurations</th>
<th>Theoretical Spectroscopic Factor</th>
<th>DWBA fit to Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>JJT LS</td>
<td>jj LS IC</td>
<td>S-W R.O</td>
</tr>
<tr>
<td>0.00 v = 1, t = 1/2⁺</td>
<td>[22_p][43]</td>
<td>8 5.33 5.03</td>
<td>8.2±2.7 7.7±2.5</td>
</tr>
<tr>
<td>2.00 1/2⁻</td>
<td>[22_p][43]</td>
<td>- 2.67 1.02</td>
<td>2.7±0.9 2.0±0.6</td>
</tr>
<tr>
<td>4.81 3/2⁻</td>
<td>[22_d][43]</td>
<td>- 0.97</td>
<td>1.3±0.4 0.9±0.3</td>
</tr>
</tbody>
</table>

In table 6 above, we give the spectroscopic factors for the first three populated levels. In column two is the assumed configuration of the level for a pure JJT coupling scheme,
v is the seniority quantum number and t the reduced isotopic spin, that is, the isotopic spin of the configuration \((j)^v\). In column three, are the assumed configurations for a pure LSJT coupling scheme. Columns four, five and six give the spectroscopic factors calculated using formulae (5.21), (5.25) and (5.28) respectively. The amplitudes for the intermediate coupling calculations have been taken from tables prepared by Boyarkina. Columns seven and eight give the spectroscopic factors obtained at the end of Chapter 4, by fitting DWBA to experiment. Both Saxon-Wood (S-W) and oscillator (H.O.) values are given, the neutron parameters being taken from tables 3 and 1 respectively.

From table 6, then, only moderate agreement has been obtained between the intermediate coupling spectroscopic factors and the experimentally deduced values. There is however one criticism of the use of Boyarkina's amplitudes for the 4.81 MeV level, namely that Boyarkina predicts that this level should be at 6.9 MeV. However the position of the level depends critically on the choice of the parameters used in the intermediate coupling calculations. Boyarkina uses the Rosenfeld mixture for the exchange character of the nucleon-nucleon interaction; the other model parameters - the amplitude of the spin-orbit interaction a , and the radial integrals of the pairing interaction L and K - have
been chosen separately for each nucleus to obtain the best fit to energies of the lowest levels. A full discussion of the meaning of these parameters is given in review articles by Inglis \(^7\) and Kuruth \(^8\). For mass \(A = 11\) nuclei, Boyarkina chooses

\[ a = -5 \text{ MeV}, \quad K = -1.2 \text{ MeV}, \quad L/K = 6 \]

Kuruth \(^8\) shows that the position of the predicted energy level is critically dependent on \(a/K\), and changing the strength of spin-orbit interaction, \(a\) from \(-5\) MeV to \(-4\) MeV for fixed \(K\) is sufficient to bring the predicted level down to \(4.8\) MeV. The effect of such a variation in \(a\) on the amplitudes and hence on the spectroscopic factor is expected to be less than 10\%. An example of the variation of \(S\) with \(a\) is given by Macfarlane and French for \(^{12}\text{C}(d,p)\text{C}^{13}\).

The problem now, is the excitation of the 6.90 MeV level by the \((p,d)\) reaction. Roush predicts that the spin of this level should be \(5/2^+\), which implies the transition should have a characteristic \(l = 2\) angular distribution. However we have shown that the angular distribution for this level is not compatible with a unique \(l\)-value. Radvanyi, in a private communication, admits that the energy resolution for this state was poor, and there are probably admixtures from the 6.49 MeV level and even the 6.35 level. This would explain why a unique \(l\)-value could not be assigned to this transition.
However, there still remains the spectroscopic problem, that all these three levels are forbidden if a simple shell model description is invoked for the ground state of $^{12}$C, and the transition mechanism is taken to be direct. Recently an experiment by Pugh on $^{12}_C(p,d)^{11}_C$ with $E_p = 50$ MeV has been performed with better energy resolution. The states populated and their relative intensity at an angle of $20^\circ$ are listed below:

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$3/2^-$</th>
<th>Pugh</th>
<th>Radvanyi</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>1/2$^-$</td>
<td>0.23</td>
<td>0.3</td>
</tr>
<tr>
<td>4.32</td>
<td>5/2$^-$</td>
<td>0.13</td>
<td>-</td>
</tr>
<tr>
<td>4.81</td>
<td>3/2$^-$</td>
<td>0.15</td>
<td>0.4</td>
</tr>
<tr>
<td>6.35</td>
<td>3/2$^+$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6.49</td>
<td>7/2$^-$</td>
<td>0.13</td>
<td>-</td>
</tr>
<tr>
<td>6.90</td>
<td>5/2$^+$</td>
<td>-</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Clearly, the better energy resolution of Pugh indicates a rather more complicated description is required. There are two possibilities:

1. Suppose that there is some inelastic scattering whereby the target nucleus is first excited to the $2^+$ state at 4.435 MeV, and then a neutron is "picked-up" from this excited state.
The possible spin values $J_C$ for the residual nucleus assuming it still to be an $E=1$ transition are

$$J_C = 1/2^-, 3/2^-, 5/2^-, 7/2^-$$

Detailed analysis of how to treat such inelastic processes within the framework of direct reactions is given by Penny and Satchler. Studies of the inelastic scattering of protons on carbon have been carried out by several groups.

It is found that the cross-section for exciting the first 4.43 (2$^+$) level in $^{12}C$ is very strong and at larger angles even comparable with the elastic cross-section. For protons of energy $E_p = 185$ MeV, the experiments of Tyren show that for scattering angles between 20° and 40° the inelastic cross-section is comparable with the elastic. Thus it seems reasonable to expect such an inelastic process to contribute to the stripping cross-sections as measured by Radvanyi and Pugh. Furthermore the larger relative cross-sections for the excited states of $^{11}C$ in the experiments of Radvanyi at 150 MeV to the experiments of Pugh at 50 MeV is also consistent with the postulate of an inelastic process, since the relative probability of exciting the 4.43 level in $^{12}C$ is greater at 150 MeV than at 50 MeV.

(2) An alternative postulate is to assume that the mechanism is still direct; but that a more complicated description for
the ground state of $C^{12}$ should be used. Goswami and Pal have calculated the collective electric dipole, quadrupole and octupole states of $C^{12}$ assuming the presence of two hole, two particle ($2H - 2P$) pairs in the ground state.

An unperturbed hole-particle state is of the type
\[ \left| h^{-1}p^{1}\mu^{1}\rho^{1}\right\rangle , \]
where $h$ and $p$ denote the angular momenta of the hole and particle respectively, and $J$ and $T$ denote the total angular momentum and iso-spin with projections $M$ and $M_T$. Let $A^{+}_{\mu J\rho T} (h, p)$ denote the creation operator which operating on the closed shell ground state creates this state. If only the $2H - 2P$ terms in the ground state are considered, and further assume that the $2H - 2P$ states of $J=0$, $T=0$ are each formed by angular momentum coupling of two hole-particle pairs of equal $J$ and $T$, with opposite projection quantum numbers, then we can write for the ground state of $C^{12}$:

\[
\begin{align*}
\left| J_{0}\right\rangle &= N \left| 0 \right\rangle + \sum_{h \rho, p \rho'} (\epsilon^{J M + T - M_T} \left\{ 2\left[ J \right]\left[ T \right]\right\}^{-\frac{1}{2}}
\end{align*}
\]

\[
C_{h \rho, p \rho'} \left( A^{\mu J \rho T}_{\mu J \rho T} (h, p) A^{+ \mu J \rho T}_{\mu J \rho T} (h', p') \right) \left| 0 \right\rangle
\]

Here $\left| 0 \right\rangle$ denotes the closed shell state. The factors $(-)^{J M} \left[ J \right]^{-\frac{1}{2}}$ and $(-)^{T - M_T} \left[ T \right]^{-\frac{1}{2}}$ come from the Clebsch-Gordan coefficients, $\left[ x \right]$ is an abbreviation for $(2x+1)$, and $N$ is the normalisation constant.
\[
N = \left[ 1 + \sum_{J, T, h, p, h', p'} \left( C_{h, p, h', p'}^{J, T} \right)^2 \right]^{-\frac{1}{2}}
\]

(5.35)

Now we can rewrite the expression for the ground state of \( ^{12}\text{C} \) in the diagrammatic notation of Macfarlane and French.

\[
\left| \Psi_0 \right> = N \sum_{h, p, h', p'} \frac{C_{h, p, h', p'}^{J, T}}{\sqrt{2}} \left| s_{1/2}, 0, 0 \right> + \left| s_{3/2}, 0, 0 \right>
\]

(5.36)

Here the Clebsch-Gordan coefficients have been absorbed in the diagram. Further we have only drawn the diagram corresponding to the case when both holes are in the \( 1p_{3/2} \) shell and both particles have angular momentum \( p \). Clearly analogous diagrams can be drawn for the other cases corresponding to both holes in \( 1s_{1/2} \) shell, or the two holes being in different shells, or the two particles being in different shells. The sum over \( h, p, h', p' \) will include these other diagrams.

Now we assume that the ground state of \( ^{11}\text{C} \) corresponds to a \( p_{3/2}^{-} \) hole in the closed shell configuration \( |0> \), and the excited states all correspond to removing a particle \( p \) from the \( 2H - 2F \) states in the correlated ground state \( |\Psi_0> \).

For example
The spin and iso-spin of this state is $J_C = p, T_C = \frac{1}{2}$.

The overlap integral for the excited states is now

$$\mathcal{J}(p) = \sum_{J, T, M_J, M_T} C_{J, T, p, h} C_{h', p', J, T} \frac{\sqrt{2}}{\sqrt{A}} U(J, p; J, p') U(T, h, T, h')$$

This is simply evaluated using (5.19) giving

$$\mathcal{J}(p) = \sum_{J, T, h, h'} C_{J, T, p, h} C_{p, h', p'} \frac{\sqrt{2}}{\sqrt{A}} U(J, p; J, p') U(T, h, T, h')$$

The Wigner coefficients are unity, hence the spectroscopic factor is

$$S(p) = A |\mathcal{J}(p)|^2$$

$$= N^2 \sum_{J, T, h, h'} (C_{J, T, p, h})^2$$

$$= (2p+1) V_p$$

$$- 153$$
where $V_p$ is the occupational probability of the particle state $p$ in the correlated ground state.

$$V_p = \frac{1}{J!} \sum_{h, h', p', T} (2p+1)^{-1} \left( \frac{C_{h_p, h'_p}}{J! T} \right)^2$$

(5.40)

These occupational probabilities have been calculated by Goswami and Pal$^{67}$ and the corresponding spectroscopic factors are given in the table below.

<table>
<thead>
<tr>
<th>Particle state in $^{12}$C g.s.</th>
<th>$J^T$ for $^{11}$C of transition</th>
<th>$\ell$-value</th>
<th>Spectroscopic Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1f_{7/2}$</td>
<td>$7/2^-$</td>
<td>3</td>
<td>0.101</td>
</tr>
<tr>
<td>$1d_{5/2}$</td>
<td>$5/2^+$</td>
<td>2</td>
<td>0.101</td>
</tr>
<tr>
<td>$1d_{3/2}$</td>
<td>$3/2^+$</td>
<td>2</td>
<td>0.081</td>
</tr>
<tr>
<td>$2p_{3/2}$</td>
<td>$3/2^-$</td>
<td>1</td>
<td>0.007</td>
</tr>
<tr>
<td>$1f_{5/2}$</td>
<td>$5/2^-$</td>
<td>3</td>
<td>0.013</td>
</tr>
<tr>
<td>$1f_{7/2}$</td>
<td>$7/2^-$</td>
<td>3</td>
<td>0.101</td>
</tr>
<tr>
<td>$1d_{5/2}$</td>
<td>$5/2^+$</td>
<td>2</td>
<td>0.101</td>
</tr>
<tr>
<td>$1d_{3/2}$</td>
<td>$3/2^+$</td>
<td>2</td>
<td>0.081</td>
</tr>
</tbody>
</table>

We see that qualitatively this prescription allows all possible final states in $^{11}$C to be obtained through the transfer of a single nucleon with a unique $\ell$-value. Quantitatively, however, the prediction is that the $5/2^-$ and $3/2^-$ states at 4.32 MeV and 4.81 MeV are weakly excited, whereas the three levels at 6.32 MeV ($3/2^+$), 6.49 MeV ($7/2^-$) and 6.90 MeV ($5/2^+$) will all be comparatively strongly excited. This is clearly not the experimental situation.
5.3.2 \( \text{B}^{10}(p,d) \text{B}^{9} \).

The ground state of \( \text{B}^{9} \) and five excited states at 2.4, 7.1, 9.6, 11.4 and 14.6 MeV have all been populated in the \((p,d)\) reaction. Each state populated is compatible with an assignment of \( I = 1 \). Since the ground state of \( \text{B}^{10} \) is \( J_A = 3^+, T_A = 0 \), the possible final spins are

\[ J_C = \frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-, \frac{9}{2}^- , \quad T_C = \frac{1}{2} \]

In fig. 27 we give the experimental level scheme for states below 8 MeV for \( \text{B}^{9} \), together with the predictions of intermediate coupling calculations of Boyarkina. The \( \text{B}^{10}(\text{He}^3,\alpha) \text{B}^{9} \) experiment of Earwaker et al\(^{68}\), and the \( \text{Li}^7(\text{He}^3,n) \text{B}^{9} \) experiment of Duggan et al\(^{69}\), observe levels at 2.34, 2.81 and a weak level at 7.1 MeV as the only excited states in \( \text{B}^{9} \) of less than 8 MeV excitation. The spins of these states are probably \( \frac{5}{2}^-, \frac{1}{2}^-, \) and \( \frac{7}{2}^- \) respectively. However the \((p,n)\) experiments of Saji\(^{70}\) indicate the existence of levels at 4.1 and 4.9 MeV. By comparison with the intermediate coupling calculations these levels may have spins of \( \frac{3}{2} \) and \( \frac{5}{2} \). Nevertheless the existence of these levels does not contradict the experiments of Earwaker or Radvanyi, since as can be seen in Table 7 on page 157, the spectroscopic factors for these states are very small and consequently will not be detected in a single particle transition experiment. In Table 7, we give the spectroscopic
Fig. 27. Energy Level Diagram for the states of $^{9}$Be below 8 MeV.
Table 7.

<table>
<thead>
<tr>
<th>Final State in $^{9}$Be</th>
<th>Possible shell model configurations</th>
<th>Theoretical Spectroscopic Factor</th>
<th>DWBA fit to Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>J JT</td>
<td>LS</td>
<td>jj</td>
</tr>
<tr>
<td>0.00 3/2(^{-})</td>
<td>v=1</td>
<td>22p[41]</td>
<td>1.2</td>
</tr>
<tr>
<td>2.34 5/2(^{-})</td>
<td>v=3</td>
<td>22p[41]</td>
<td>1.37</td>
</tr>
<tr>
<td>2.81 1/2(^{-})</td>
<td>v=3</td>
<td>22p[41]</td>
<td>0</td>
</tr>
<tr>
<td>4.1 3/2(^{-})</td>
<td>--</td>
<td>22p[41]</td>
<td>--</td>
</tr>
<tr>
<td>4.9 5/2(^{-})</td>
<td>--</td>
<td>22p[41]</td>
<td>--</td>
</tr>
<tr>
<td>7.1 7/2(^{-})</td>
<td>v=3</td>
<td>22p[41]</td>
<td>3.43</td>
</tr>
</tbody>
</table>

factors of the excited states in $^{9}$Be below 8 MeV. A full description of this table which is identical in type to Table 6 is given on page 146. The Saxon-Wood neutron parameters used were those of $^{11}$Be in Table 3. The theoretical prediction that the ground state, the 2.34 and 7.1 MeV levels should be strongly excited in the $(p,d)$ reaction is qualitatively born out; but the quantitative agreement between theory and experiment is rather poor.
The situation above 8 MeV is rather complicated. The level at 9.6 MeV reported by Radvanyi is first evidence of this state. Recently, in a Li$^7$ (He$^3$,n) B$^9$ reaction, Dietrich reports new levels at 12.06, 14.01, 14.67 and 16.02 MeV. The level at 11.6 MeV seen by Radvanyi has also been reported by Fisher and Whaling with the pick-up B$^{10}$ (He$^3$,α) B$^9$ reaction. It has been suggested by Jackson, that the 11.6 level should have spin 9/2 and is a member of the K=3/2 rotation band built on the ground state. The 14.67 level has been the subject of much theoretical and experimental research. The corresponding level in Be$^9$ at 14.39 MeV has been identified as the first T=3/2 level by Lynch et al using the Li$^7$(He$^3$,p) Be$^9$ reaction. Dietrich suggests a T=3/2 assignment to the 14.6 state in B$^9$ due to its narrow width.

With the discovery that C$^9$ was stable, following the Cl$^2$(He$^3$,He$^6$) C$^9$ experiment of Cerny et al, the mass-9 multiplet of Li$^9$,Be$^9$,B$^9$ and C$^9$ of isotopic spin T=3/2 was completed, and the resulting agreement with the isobaric mass formula confirms the T=3/2 assignment. A summary of the experimental level scheme is given in fig. 28.

However, B$^{10}$ (p,d) B$^9$ experiment forbids the excitation of T=3/2 states from the isotopic spin selection rule (5.33). Thus, with the poor energy resolution of the Radvanyi data, it is plausible that the reported excitation of the 14.6 level may well include the excitation of the T=1/2 level at 14.01 MeV.
<table>
<thead>
<tr>
<th>Energy Level</th>
<th>Spectroscopic Factors</th>
<th>Energy Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L-S I.C</td>
<td></td>
</tr>
<tr>
<td>14.5 3/2</td>
<td>0.62 0.01</td>
<td>14.67 J=3/2 T=3/2</td>
</tr>
<tr>
<td>13.7 J=3/2 T=3/2</td>
<td>0 0</td>
<td>14.0</td>
</tr>
<tr>
<td>13.3 5/2</td>
<td>1.93 0.22</td>
<td></td>
</tr>
<tr>
<td>11.5 5/2</td>
<td>0.67 0.01</td>
<td>2.0</td>
</tr>
<tr>
<td>11.0 7/2</td>
<td>1.42 0.60</td>
<td>(9/2) 11.6</td>
</tr>
<tr>
<td>10.7 7/2</td>
<td>0.00 1.19</td>
<td></td>
</tr>
<tr>
<td>10.6 3/2</td>
<td>0.07 0.00</td>
<td></td>
</tr>
<tr>
<td>10.0 9/2</td>
<td>0.00 0.06</td>
<td></td>
</tr>
<tr>
<td>9.8 1/2</td>
<td>0 0</td>
<td>1.4 9.6</td>
</tr>
</tbody>
</table>

**Fig. 28.** Energy Level Diagram for states of B⁹ above 8 MeV. The theoretical and experimental spectroscopic factors are listed in the central columns.

- 159 -
Theoretically, the intermediate coupling calculations of Boyarkina predict eight $T=1/2$ levels between 9.8 and 14.6 MeV, and one $T=3/2$ state. In fig. 28 we give the theoretical spectroscopic factor for exciting these states, for both the pure L-S coupling scheme and the intermediate coupling calculations. It will be noticed that the largest spectroscopic factors are predicted for states of spin $5/2^-$ and $7/2^-$. Hence it is plausible to conclude that the levels observed by Radvanyi correspond to $J^\pi = 5/2^-$ or $7/2^-$, whereas the levels of spin $3/2$ and $9/2$ would not be appreciably excited.

5.3.3 Be$^9$(p,d) Be$^8$.

The ground state of Be$^8$ and four excited states at 2.9, 11.4, 16.6 and 18.9 MeV have been populated in the (p,d) reaction$^{19}$. With the exception of the 11.4 MeV level, the transitions to these states are compatible with an assignment of $\ell = 1$. For the 11.4 MeV ($J^\pi = 4^+$), it was not possible to assign a unique $\ell$-value to the transition and once again the possibility of competing inelastic channels should be considered. Since the ground state of Be$^9$ is $J_z = 3/2^-$, $T_A = 1/2$; the possible final spins for an $\ell = 1$ transition are

$$J_c = 0^+, 1^+, 2^+ \text{ or } 3^+; \quad T_c = 0 \text{ or } 1$$

The level scheme for Be$^8$ together with the theoretical
predictions of Boyarkina are given in fig. 29. Between 16.6 MeV and 19.3 MeV, seven levels are known, consequently it seems probable that the population to more that one final state has been measured in the angular distributions for the 16.6 and 18.9 levels reported by Radvanyi.

Furthermore the isotopic spin dependence of these excited states has been the subject of much work recently; and Marion has suggested that isotopic-spin $T$ is not a good quantum number in this region.

Nevertheless, in table 8, (page 163) we give the spectroscopic factors deduced using a simple shell model, with the configurations listed in columns 2 and 3. Again the qualitative agreement is good, with the general features of the theoretical predictions being reproduced experimentally. However the experimental value of $3.9 \pm 1.3$ for the 16.62 MeV level deserves some comment. It is the quantity $C^2 S$, which is determined by fitting the numerical cross-section to the experimental value, where $C$ is the iso-spin Clebsch-Gordan coefficient $\langle T_M^{1/2} | T_A^{TA} \rangle$ and $S$ the spectroscopic factor. This quantity $C^2 S$ was determined to be $1.3 \pm 0.4$, consequently in quoting the result of $3.9 \pm 1.3$ in table 8, we have made the assumption that the final state is pure $T = 1$, with $C^2 = 1/3$. 

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Intermediate Coupling Experiment

Fig. 29. Energy Level Diagram for Be\textsuperscript{8}
<table>
<thead>
<tr>
<th>Final State in Be 8</th>
<th>Possible shell model configurations</th>
<th>Theoretical Spectroscopic Factor</th>
<th>DWBA fit to Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>JJT</td>
<td>LS</td>
<td>jj</td>
</tr>
<tr>
<td>0.00 J=0^+, T=0</td>
<td>v = 0</td>
<td>11_S [4]</td>
<td>0.5</td>
</tr>
<tr>
<td>2.90 J=2^+, T=0</td>
<td>v = 2</td>
<td>11_D [4]</td>
<td>1.5</td>
</tr>
<tr>
<td>1.40 J=4^+, T=0</td>
<td>v = 4</td>
<td>11_G [4]</td>
<td>0</td>
</tr>
<tr>
<td>16.62 J=2^+, T=1</td>
<td>v = 2</td>
<td>33_P [31]</td>
<td>2.0</td>
</tr>
<tr>
<td>16.92 J=2^+, T=0</td>
<td>v = 2</td>
<td>13_P [31]</td>
<td>0</td>
</tr>
<tr>
<td>17.64 J=1^+, T=1</td>
<td>v = 4</td>
<td>33_P [31]</td>
<td>0.3</td>
</tr>
<tr>
<td>18.15 J=1^+, T=0</td>
<td>v = 4</td>
<td>13_P [31]</td>
<td>0.13</td>
</tr>
<tr>
<td>18.90 J=2, T=0</td>
<td>v = 4</td>
<td>13_D [31]</td>
<td>0.06</td>
</tr>
<tr>
<td>19.10 J=3, T=0</td>
<td>v = 4</td>
<td>13_D [31]</td>
<td>0.18</td>
</tr>
<tr>
<td>19.22 J=3^+, T=1</td>
<td>v = 2</td>
<td>33_D [31]</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>t = 0</td>
<td></td>
<td>163</td>
</tr>
</tbody>
</table>
5.4 Stripping in Deformed Nuclei.

Recently there have been many calculations\(^5\) using the Nilsson model\(^7\) for \(1p\)-shell nuclei. Both the level schemes of \(B\) and \(Be\) seem to indicate the presence of rotational bands of energy levels, consequently we shall look briefly into the formalism of stripping in deformed nuclei; with the aim of seeing whether the theoretical spectroscopic factors so obtained agree with the experimental values.

From (4.17) the transition amplitude in DWBA may be written

\[
T_{fi} = \sqrt{N} \left\langle X_d^{(-)} (k_i, \Sigma_{d0}) \phi_d (r_{pn}) \phi_{MC_{JK}}^J (\xi) | V_{pn} (r_{pn}) \right| \right.
\]

\[
\left. X_p^{(+)} (k_i, \Sigma_{pA}) \phi_p \phi_{MA_{KA}}^{J_A} (\xi, \Sigma_{nC}) \right\rangle
\]

(5.41)

where we have not included the iso-spin dependence hence the statistical factor outside the matrix element becomes \(\sqrt{N}\), where \(N\) is the neutron number for the target nucleus, instead of \(\sqrt{A}\).

Using now the rotational model, the symmetrised wavefunction for the final nucleus of spin \(J_C\) may be written\(^8\)

\[
\phi_{MC_{JK}}^J = \left[ \frac{2J_C + 1}{16 \pi^2 (1 + \xi_{KC,0})} \right]^{\frac{1}{2}} \phi_0 \left[ \phi_{MC_{JK}}^J (\theta) X_{KC} (i') \right.
\]

\[
\left. + (-)^{J_C - K_C} \phi_{MC_{-K_C}}^J (\theta) R_1 X_{KC} \right]
\]

(5.42)
where \( \phi_0 \) is the spinless core wavefunction of the vibrational ground state. \( \chi_{k C}^{i'} \) is the wavefunction of the particles outside the core, being in general a product of Nilsson's one-particle solutions; \( i' \) denotes all the co-ordinates of these particles in the body-fixed axes system. The \( \mathcal{D}_{JK}^J(\phi) \) are the usual rotational matrices depending on the Eulerian angles \( \phi \). Similarly the target nucleus wavefunction may be written

\[
\mathcal{D}^{J_A}_{M_A K_A} = \left[ \frac{2J_A+1}{16\pi^2(1+\delta_{K_A,0})} \right]^{1/2} \phi_0 \left[ \mathcal{D}^{J_A}_{M_A K_A}(\phi) \chi_{K_A}^{i',j'} \right] \\
+ (-)^{J_A-K_A} \mathcal{D}^{J_A}_{M_A',-K_A'}(\phi) R_{1}' \chi_{K_A}^{i',j'}
\]

where \( j' \) denotes all the co-ordinates of the transferred neutron in body-fixed axes. \( R_{1}' \) is a rotational operator, ensuring that the wavefunction is invariant with respect to a rotation of \( \Pi \) about an axis perpendicular to the symmetry axis.

Now both \( \chi_{K_A}^{i',j'} \) and \( \chi_{k C}^{i'} \) are antisymmetric in all active nucleons. Writing these as a product of one-particle functions we have

\[
\chi_{K_A} = A \chi_{n_1}^{(i')} \chi_{n_2}^{(2')} \ldots \chi_{n_n}^{(j')} \quad (5.44)
\]

where

\[
K_A = \Omega_1 + \Omega_2 + \ldots + \Omega_n
\]
and $A$ is the antisymmetrisation operator. Similarly

\[ \chi_{K_A}(i') = A \chi_{\Omega_1'}(1') \chi_{\Omega_2'}(2') \ldots \chi_{\Omega_{n-1}'}(n-1') \]  

with

\[ K_C = \Omega_1' + \Omega_2' + \ldots + \Omega_{n-1}' \]

Thus

\[ R_1 \chi_{K_A} = A R_1 \chi_{\Omega_1} \ldots \chi_{\Omega_n} \]

\[ = A (-)^{l_1 + \ldots + l_n} \chi_{-\Omega_1} \ldots (-)^{l_n} \chi_{-\Omega_n} \]

\[ = (-)^a A \chi_{-\Omega_1} \chi_{-\Omega_2} \ldots \chi_{-\Omega_n} \]

\[ = (-)^a \chi_{-K_A} \]  

(5.46)

where $a = n/2 - K_A + \sum_{i=1}^{n} l_i$

Similarly

\[ R_1 \chi_{K_C} = (-)^c \chi_{-K_C} \]  

(5.47)

where $c = n/2 - K_C + \sum_{i=1}^{n-1} l_i$. Furthermore now let us expand $\chi_{K_A}$ in terms of an $(n-1)$ particle system, i.e. make a fractional parentage type expansion. However the only good quantum number is the spin projection on the symmetry axis, hence the expansion becomes a direct product.

\[ \chi_{K_A}(i', \psi') = \frac{1}{\sqrt{n}} \mathfrak{g}(\Omega) \chi_{K_{A-\Omega}}(i') \chi_{\Omega}(\psi') \]  

(5.48)
where \( \chi_{K_A-N}(i') \) will be antisymmetric in the first (n-1) nucleons providing that we multiply by the normalisation factor \( n^{-\frac{1}{2}} \). The \( \mathcal{J}(\Omega) \) is essentially a fractional parentage coefficient for the expansion. Normally we will only be interested in cases where there are only one or two extra core nucleons, then \( \mathcal{J}(\Omega) = 1 \). Consequently

\[
\mathcal{R}_1 \chi_{K_A} = \frac{1}{\sqrt{n}} \mathcal{J}(\Omega) (-)^a \chi_{-K_A+N}(i') \chi_{-N}(y')
\]

Returning now to (5.41) for \( T_{fi} \), the first stage in the evaluation is to integrate over the final nuclear co-ordinates, i.e. \( i' \), \( \xi \) and \( \theta \). This involves evaluating the following four integrals

\[
\int \chi_{K_A}(i',y') \chi_{K_C}^*(i') \, di' = \frac{1}{\sqrt{n}} \mathcal{J}(\Omega) \chi_{\Omega}(y') \int \chi_{K_A-N}(i') \chi_{K_C}^*(i') \, di' = \frac{1}{\sqrt{n}} \mathcal{J}(\Omega) \chi_{\Omega}(y') \delta_{\Omega,K_A-K_C}
\]

\[
\int (\mathcal{R}_1 \chi_{K_A}) \chi_{K_C}^* \, di' = 0 \quad \text{since} \quad \Omega = K_A-K_C
\]

\[
\int \chi_{K_A} (\mathcal{R}_1 \chi_{K_C})^* \, di' = 0 \quad \text{(5.50)}
\]

\[
\int (\mathcal{R}_1 \chi_{K_A}) (\mathcal{R}_1 \chi_{K_C})^* \, di' = \frac{1}{\sqrt{n}} \mathcal{J}(\Omega) \chi_{-\Omega}(y') (-)^{a-c} \delta_{\Omega,K_A-K_C}
\]
Let us denote by $\ell$, 

$$\ell = \sum_{i=1}^{n+1} \ell_i - \sum_{i=1}^{n-1} \ell_i$$  \hspace{1cm} (5.51)

then $\ell$ is the orbital angular momentum of the transferred neutron. However there isn't, in general, a unique value of $\ell$ as can be seen from the definition. Then

$$(-)^{\ell-a-c} = (-)^{\ell+\frac{1}{2} - \Omega}$$  \hspace{1cm} (5.52)

As a result of the four integrals above, the overlap of the intrinsic wavefunctions results in the function $\chi_\Omega(\psi')$

This is a function of the space-fixed axes $\psi$ and the Eulerian angles $\Theta$, thus before the integration over $\Theta$ can be carried out, this neutron wavefunction must be first transformed into space-fixed axes. To achieve this, the wavefunction $\chi_\Omega$ must be expanded in eigenfunctions of the total angular momentum $j$,\hspace{0.5cm} (80)

i.e. $\chi_\Omega(\psi') = \sum_{j=m}^{n} c_{j,\Omega} \chi_j^{\Omega} (\psi')$

Then

$$\chi_\Omega(\psi') = \sum_{j=m}^{n} c_{j,\Omega} \mathcal{O}_m^{\Omega}(\Theta) \chi_j^{\Omega} (\psi)$$

$$\chi_{-\Omega}(\psi') = \sum_{j=m}^{n} c_{j,-\Omega} \mathcal{O}_m^{\Omega}(\Theta) \chi_j^{\Omega} (\psi)$$

Hence the integration over the final nuclear co-ordinates $i', \xi$ and $\Theta$ can now be made
\[ \int d\xi \, d\Omega \, \rho_{M_A K_A}^{J_A} \rho_{M_C K_C}^{J_C^*} = \frac{N_A N_C}{\sqrt{\pi}} \tilde{q}(\Omega) \frac{8\pi^2}{2J_A+1} \sum_{j \, m} (J_{C,M_C}^j m | J_{A,M_A}^j m) \chi_{j,m}^j (\Omega) \]

\[ = (J_{C,K_C}^j \Omega | J_{A,K_A}^j) c_j \Omega + (-)^d (J_{C,-K_C}^j - \Omega | J_{A,-K_A}^j) c_j - \Omega \]

where \[ d = J_A + J_C - K_C - \frac{1}{2} \pm - \Omega. \]

Furthermore using the relations

\[ (J_{C,K_C}^j \Omega | J_{A,K_A}^j) = (-)^{J_A + J_C - J} (J_{C,-K_C}^j - \Omega | J_{A,-K_A}^j) \]

\[ c_j \Omega = (-)^{l + \frac{3}{2} - j} c_j - \Omega \]  \hfill (5.57)

then from the phases involved it can be seen that the two terms in (5.55) are equal. Thus

\[ \int d\xi \, d\Omega \, \rho_{M_A K_A}^{J_A} \rho_{M_C K_C}^{J_C^*} = \frac{N_A N_C}{\sqrt{\pi}} \tilde{q}(\Omega) \frac{16\pi^2}{2J_A+1} \sum_{j \, m} (J_{C,M_C}^j m | J_{A,M_A}^j m) (J_{C,K_C}^j \Omega | J_{A,K_A}^j) \chi_{j,m}^j (\Omega) \]

\[ - 169 - c_j \Omega \chi_{j,m}^j (\Omega) \]
Comparing this with the result obtained in the spherical case, i.e. with equation (4.53)

\[
\langle \Phi_c^{J_c} | \Phi_A^{J_A} \rangle = \sum_j \sum_m j(j) (J_c M_c j m | J_A M_A) \Phi_m^{J_A} (n_0) \tag{5.59}
\]

we identify

\[
j(j) = \frac{N_{A} N_{C}}{\sqrt{\hbar}} \mathcal{J} (\Omega) \frac{16\pi^2}{2J_{A}+1} (J_{C} K_{C} j \Omega | J_{A} K_{A}) C_{j \Omega} \tag{5.60}
\]

Hence the spectroscopic factor

\[
S(j) = N \frac{\sum_j j(j)^2}{\sum_j j(j)^2} \mathcal{J} (\Omega) \frac{16\pi^2}{2J_{A}+1} (J_{C} K_{C} j \Omega | J_{A} K_{A}) C_{j \Omega} \tag{5.61}
\]

where

\[
\rho^2 = \frac{1}{(1 + \hat{\delta}_{K_A,0})(1 + \hat{\delta}_{K_C,0})}
\]

The coefficients \( C_{j \Omega} \) are simply related to the coefficient \( a_{\ell \Lambda} \), tabulated by Nilsson\(^7\) through the relation

\[
C_{j \Omega} \chi_{m}^{J_A}(\mathcal{Y}) = \sum_{\Lambda, \Sigma} (\ell A \frac{1}{2} \Sigma | j \Omega) a_{\ell \Lambda} \chi_{m}^{J_A}(\mathcal{Y}) \tag{5.62}
\]

The expression for the total cross-section is now derived in exactly the same way as in the spherical case. The result, without iso-spin, is

- 170 -
\[
\frac{d\sigma}{d\Omega} (J_A \rightarrow J_C) = \frac{3m^2}{8\pi^2n^4} \frac{\Lambda(\Lambda-1)}{(\Lambda+1)^2} \frac{k_P}{k_i} D_0 \sum_j S(j) \sum_{\Lambda} |B^{j\Lambda}|^2 \quad (5.63)
\]

where

\[
B^{j\Lambda} = \left< \chi_d(-)(k_P,z^) | \chi_p(+) (k_A,z^) \chi^j_m (\tau) \right> \quad (5.64)
\]

and \(\chi^j_m (\tau)\) is the intrinsic wavefunction of the transferred particle.

Lastly we note the following selection rules have been deduced

(i) \(\Omega = K_A - K_C\)
(ii) \(j > |\Omega|\)
(iii) \(|J_C - J_A| \leq j \leq J_C + J_A\)

Thus if either \(J_A\) or \(J_C\) is zero, only one \(j\)-value contributes.

We now look at a couple of specific examples and compare the experimental spectroscopic factor with that deduced using equation (5.61).

5.4.1 \(\text{Be}^9(p,d)\text{Be}^8\)

The ground state of \(\text{Be}^9(3/2^-)\) may be considered the first member of a \(K = 3/2\) rotation band, hence we have

\[
\sum_{M_A K_A} J_A \quad |J_A = 3/2, M_A, K_A = 3/2\rangle
\]

- 171 -
The ground-state \( (0^+) \) and first excited state \( (2^+) \) of Be\(^8\) are members of a \( K = 0 \) band, hence

\[
\begin{align*}
\Omega^J_{M_C K_C} (0^+) &= \left\{ J_C = 0, M_C = 0, K_C = 0 \right\} \\
\Omega^J_{M_C K_C} (2^+) &= \left\{ J_C = 2, M_C = 0, K_C = 0 \right\}
\end{align*}
\]

Furthermore we shall consider Be\(^8\) to be a closed shell in the Nilsson sense, i.e. there are no extra-core nucleons. Thus there is only one extra-core nucleon in Be\(^9\) and the parentage coefficient \( \Omega(\Omega) \) will be unity.

Now from the selection rules we have for the ground state transition

\[
\Omega = \frac{1}{2}, \quad j = \frac{3}{2}
\]

From Nilsson's tables and equation (5.62) we have that

\[
C_{\frac{3}{2} \frac{3}{2}} = 1, \text{ thus using (5.61)}
\]

\[
S_g = \frac{5}{2} \left( \frac{3}{2} \frac{3}{2} \frac{3}{2} - \frac{3}{2} \right) 0 0)^2
\]

\[
= 5/8
\]

For the first excited state, the selection rules indicate

\[
\Omega = \frac{3}{2}, \quad j = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}
\]

However the angular distribution is fitted with \( \ell = 1 \), thus limiting the choice to \( j = \frac{3}{2} \) only. Then
\[ S_1 = \frac{5}{2} \begin{pmatrix} 3/2 & 3/2 & 3/2 & -3/2 & 2 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}^2 = \frac{5}{8} \]

Thus we have the result, that if we are to believe a simple rotational model, the ground state and first excited state should be equally populated in the (p,d) reaction. Experimentally the ratio \( S_1/S_g \) is 2.4.

It might also be argued that the second excited state \((4^+)\), also being a member of the \(K = 0\) band, should have a non-zero spectroscopic factor using the rotational model. From the selection rules,

\[ O = 3/2, \quad j = 5/2, 7/2, 9/2, 11/2. \]

This implies \( l \)-values in the range 2 to 5. Even values are excluded from parity considerations. The principal quantum number \( N \) describing a Nilsson orbit is defined by the equation

\[ N = 2n + \ell \]

where \( n \geq 0 \). Thus for \( \ell = 3, N \geq 3 \); but the simple ground state configuration for \(^9\text{Be}\) in the Nilsson model comprises only \( N = 0 \) and \( N = 1 \) orbits. So if we assume that the picked-up neutron carries \( \ell = 3 \), this implies admixtures of orbits of energy \( 2\hbar \omega \) higher in the \(^9\text{Be}\) ground state. Since such admixtures are neglected by Nilsson, we shall neglect them from considerations here, and conclude that the \( 4^+ \) state is forbidden. Comparison of the theoretical and experimental
spectroscopic factors (without iso-spin) are given in the Table below. To be consistent with Nilsson model, the experimental values are those obtained using the oscillator.

<table>
<thead>
<tr>
<th>Rotational Model</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_e$</td>
<td>0.63</td>
</tr>
<tr>
<td>$S_0$</td>
<td>0.63</td>
</tr>
<tr>
<td>$S_0/S_e$</td>
<td>1</td>
</tr>
</tbody>
</table>

5.4.2 $^{10}\text{(p,d)}B^9$.

Consider the ground state of $^{10}B$ to be the first member of a $K = 3$ rotation band

$$\left\langle \frac{J_A}{M_A K_A} \right\rangle = \left\langle J_A = 3^+, M_A, K_A = 3 \right\rangle$$

The states 0.00 (3/2−), 2.34 (5/2−), 7.00 (7/2−) and 11.4 (9/2−) in $^{9}B$ are considered members of $K = 3/2$ rotation band. If again we consider $^{8}Be$ as a closed shell in the Nilsson sense, then in $^{10}B$ we have one extra-core proton and one extra-core neutron. Again therefore the parentage coefficient will be unity.

From the selection rules, and the identification of an $\Omega = 3/2$ transition, we have

$$\Omega = 3/2, \quad j = 3/2 \text{ only.}$$
Hence the spectroscopic factor for the state with spin \( J_C \)
in the \( K_C = 3/2 \) rotation band is

\[
S(J_C) = 5 \left( \begin{array}{c} 3 \ 3 \ 3/2 \ -3/2 \\ J_C \ 3/2 \end{array} \right)^2
\]

Thus the ratio of the excited state spectroscopic factors
to the ground state is merely given by the ratio of the square
of a Clebsch-Gordan coefficient. The results are given in
the table below

<table>
<thead>
<tr>
<th>Final State</th>
<th>Rotation in ( B^9 ) Model</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00 (3/2)</td>
<td>2.85</td>
<td>1.2 ( \pm ) 0.4</td>
</tr>
<tr>
<td>2.34 (5/2)</td>
<td>1.61</td>
<td>1.5 ( \pm ) 0.5</td>
</tr>
<tr>
<td>7.1 (7/2)</td>
<td>0.48</td>
<td>1.1 ( \pm ) 0.4</td>
</tr>
<tr>
<td>11.6 (9/2)</td>
<td>0.06</td>
<td>1.4 ( \pm ) 0.5</td>
</tr>
</tbody>
</table>

Clearly this simple rotational model approach gives no
improvement over the shell model in predicting the experimental
spectroscopic factors. It should be noted however, that the
value quoted for the experimental spectroscopic factor for the
9/2 state, almost certainly contains a sizable contribution from
the neighbouring 12.01 level as indicated in paragraph 5.3.2.

A possible improvement in the theory might be obtained by
including band mixing in the rotational model wavefunctions.
However, unless the coefficients for the mixing can be obtained
from other data, it is not considered worth while to determine
these coefficients by fitting the spectroscopic factors
exclusively.
CHAPTER 6.

Conclusions

The work of this thesis has been concerned with the construction and the use of radial wavefunctions for \( l_p \)-shell nuclei. The procedure has been to apply the simple 'single-particle shell model' to two very different reactions; elastic electron scattering, proceeding via an electromagnetic interaction and the high energy \( (p,d) \) reaction, proceeding via a nuclear interaction. As can be seen from the text, the formalism is such that both these reactions are extremely suitable for treatment through the single-particle model, and such a consistent treatment has not been made heretofore. The success however has only been moderate, for reasons we outline below.

The meeting point for the two reactions has been through the single-particle wavefunction. This was numerically computed as an eigenfunction of a Saxon-Wood potential using three arbitrary parameters. These parameters were chosen individually for each nucleus concerned, so that the separation energy of the \( p \)-shell proton, as determined from \( (p,2p) \) experiments\(^{17} \), and the elastic electron scattering data were fitted\(^{85,89-92} \). The dominant longitudinal Coulomb term in the electron scattering cross-section was calculated using Born approximation, and a comparison with a partial-wave analysis showed the approximation
to be very accurate for p-shell nuclei. Both $C_0$ and $C_2$ components were calculated and good fits to the experimental data were obtained for Li$^6$, Li$^7$, Be$^9$, B$^{11}$, and C$^{12}$. For the first three nuclei mentioned, several ambiguous sets of parameters were found. This was primarily due to insufficient data, more values in the region of, or beyond the first Born minimum being required. Without any further adjustment (save for the Coulomb difference between the neutron and proton separation energies) these model parameters were used for the neutron wavefunction in the (p,d) reaction.

It might be argued that high energy electrons give information primarily about the nuclear interior. Thus the single-particle wavefunction will only be accurate in this region. On the other hand the (p,d) reaction is thought to be primarily a surface reaction$^{22}$, the wavefunctions needing only to be accurate in the asymptotic region. While it is true, even at the high energies considered here (150 MeV), that the surface region is important, we showed in fig. 20 that a sizable contribution to the (p,d) cross-section came from the nuclear interior. Consequently the angular distribution becomes rather more sensitive to the form of the neutron wavefunction.

The nuclear interaction is approximated by a delta function in the (p,d) reaction, hence one is concerned with the overlap between the initial and final nuclear wavefunctions.
This is clearly a nuclear structure problem, and consequently the neutron wavefunction should be chosen accordingly. We show in figs. 14 and 16 that the usual procedure of taking the neutron Saxon-Wood parameters from the real part of the proton optical potential\textsuperscript{43) is quite unsuitable. Whereas the prescription offered here is quite successful.

However one must inject a note of caution. There are many other parameters concerned in the (p,d) formalism, namely those of the proton and deuteron wavefunctions. The usual procedure of taking these to be the elastic scattering wavefunctions as computed through some optical model potential has not been justified; but must be considered part of the model used to describe the reaction mechanism. Variation of these optical model parameters can produce wild fluctuations in the angular distribution\textsuperscript{43). The fact that without any adjustment of these parameters such good fits to the experimental data\textsuperscript{19) were obtained, indicates the suitability of the DWBA model for describing the reaction mechanism at these high energies of 150 MeV.

One further point here, is that in the DWBA treatment, the neutron-proton interaction $V_{\text{pn}}$ is usually taken to be a delta-function. The improvement upon this of using the 'local energy approximation',\textsuperscript{36-39) is shown in fig. 10 to be considerable, and surely indicates that finite range effects in the interaction are important at high energies.
A more stringent test of the single-particle model is through the spectroscopic factor. In the DWBA, this is an arbitrary multiplicative constant which is obtained by fitting the computed cross-section to the experimental value. In Chapter 5 we showed how this quantity could be theoretically predicted when using a single-particle model. We gave the values determined using pure j-j, pure L-S and intermediate coupling schemes.

However, no matter which coupling scheme is employed the qualitative predictions are frequently the same. For example the dominance of the $^{12}_{\text{(p,d)}} \ g.s. \ C_{11}^{\text{g.s.}}$ over transitions to any of the other excited states in $C_{11}^{\text{g.s.}}$ is clearly shown. \(B^9\) the observing of levels at 4.1 and 4.9 MeV in (p,n) reactions\(^7\), but their absence in the (p,d) reaction can be explained by the smallness of the spectroscopic factor.

However at higher excitation energies in the residual nucleus, the shell model description becomes poorer, and the energy levels become closer together, such that the experimental resolution may not be able to distinguish between two levels. Under these conditions a comparison becomes meaningless.

A further difficulty is that the experimental spectroscopic factor cannot be accurately determined. First, in using the DWBA it is possible to find several sets of optical potentials giving good fits to the elastic and (p,d) cross-sections and yet giving different spectroscopic factors\(^4\).
Second, it is required that the experimentalist should produce an accurate absolute measure of the cross-section. Consequently the derived spectroscopic factor can at best only be found to an accuracy of 30%.

Nevertheless there are considerable discrepancies between the theoretical and experimental spectroscopic factors. The qualitative trends are well predicted theoretically, but the quantitative agreement is very poor. We have investigated in detail two possible causes for the $^{12}\text{(p,d)}\, ^{11}$ reaction. First there is the possibility of inelastic scattering occurring in the reaction $^{64}$, and second there is the possibility of improving on the simple shell model description by including admixtures of higher states in the ground state wavefunction $^{67}$. We have considered both these second-order processes, and while both qualitatively can explain the observed final states in $^{11}$, the predictions of the latter seem to be quantitatively incorrect. However, only the latter prescription can predict positive parity final states in $^{11}$. Clearly there is a need for repeating the (p,d) experiments at high energies, greater than 50 MeV, necessary to obtain the higher excited states in $^{11}$, but with a much improved energy resolution.

In conclusion, then, we have been successful in using the single-particle model to obtain simultaneously fits to elastic electron scattering data and to the shape of the (p,d) angular
distributions. But we have been unsuccessful in obtaining a quantitative agreement between the experimental and theoretical single-particle spectroscopic factors.
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