NOTE:

In the page enumeration, page numbers 66 and 67 have been omitted.
DETERMINATION OF THE CHARACTERISTICS OF A LINEAR NETWORK FROM INPUT AND OUTPUT RECORDS UNDER NORMAL OPERATION IN THE PRESENCE OF NOISE

by

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After explaining what a dynamic response is, and how it is described, methods that have been suggested for the determination of the dynamic response of systems from normal operating data are reviewed. All are found to employ arbitrary constraint conditions which

(i) are difficult to support as physically significant,
(ii) are unsuitable for generalisation,
(iii) cannot be tested for compatibility, and
(iv) require the derivation of functions which cannot yet be adequately estimated from experimental records.

Objections (i), (iii) and (iv) above make these methods suspect.

Objections (i), (ii) and (iii) have been removed in this investigation by the adoption of 'least-squares' constraints, similar to, but more widely applicable than, those used by Wiener (1949) for the synthesis of filter and prediction operators. Since mathematicians, and others, are currently attempting to remove objection (iv) (which has occurred as a difficulty of many recent investigations) this has been considered as a subsidiary, and temporary, difficulty only.

Least squares methods are developed for the
selection of optimum linear, time-invariant, descriptions of n-port system dynamics. Examples are included demonstrating the application of these methods, and showing that linear dependence of input variables (such as occur in linear passive closed loop systems) present special difficulties. These difficulties are studied in detail, and methods of resolving them are suggested.

A final chapter is devoted to a study of finite memory systems. The methods there developed are not subject to objection (iv) above. These methods, like those of the previous chapters, reveal that mathematical solutions of little or no value may result if the mathematical results are extremely sensitive to small changes (errors) in observed and estimated quantities. In such cases it is found that the data (not the method) are at fault. Such data are unsuitable for deriving the required information, and must be supplemented by the introduction of test disturbances before a realistic description of the dynamics can be obtained.
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LIST OF PRINCIPAL SYMBOLS

\( f(k), f(t), f_e(t), \text{etc.} \) = impulse response of a linear, time-invariant, passive system from which data was collected.

\( \mathcal{L}_{IS}(\tau, \tau) \) = weighting function of a linear system.

\( \mathcal{L}(t), \mathcal{R} \), \( \mathcal{R}_i(t) \), etc. = weighting functions of linear time-invariant systems.

\( n(t), n_r(t), \text{etc.} \) = contribution to the output variable \( y(t) \), \( x_r(t) \), etc. of internal active disturbances of active systems.

\( x(t), x_r(t), x_{rs}(t) \) = input variables.

\( y(t), y_e(t) \) = output variable.

\( z(t) \) = an approximation to \( y(t) \).

\( m \) = total number of input variables to an \( n \)-port.

\( n-m \) = total number of output variables of an \( n \)-port.

\( E(j\omega) \) = Fourier Transform of \( \zeta(t) \).

\( F, F_i, \text{etc.} \) = Functional operators.

\( F(j\omega), F_i(j\omega), \text{etc.} \) = Fourier Transforms of \( f(t), f_i(t), \text{etc.} \).

\( G_{xx}(\omega) \) = Power spectra of input variables.

\( G_{xy}(\omega) \) = Cross power spectra of input and output

\( H(j\omega), H_i(j\omega) \) = Fourier Transforms of \( h(t), h_i(t), \text{etc.} \).

\( X(j\omega), X_r(j\omega), X_{rs}(j\omega) \) = Fourier Transforms of \( x(t), x_r(t), \text{etc.} \).

\( Y(j\omega) \) = Fourier Transform of \( y(t) \).
\( \varepsilon(t) \) = error of an approximation = \( y(t) - z(t) \).

\( \overline{\varepsilon^2(t)} \) = mean square value of \( \varepsilon(t) \).

\( \eta \) = efficiency of a data fitting approximation.

\( \phi_x(t) \) = Correlation function of input variables.

\( \phi_{xy}(t) \) = cross-correlation function of input and output variables.

\( E \) = vector composed of \( n \) elements \( E(jw) \) determined over a series of trials.

\( G_{xx} \) = matrix of input spectral density functions.

\( G_{xy} \) = column vector of cross spectral density functions.

\( H \) = column vector of transfer functions.

\( X \) = matrix of input quantities.

\( Y \) = column vector of output quantities.
1.1. INTRODUCTION.

To demonstrate at the outset the general need for, and the wide application of, the results of system dynamic studies, consider the following selection of problems:

(a) The prediction of the motion of a mechanical linkage resulting from the application of a postulated set of forces.

(b) The prediction of the current flow in one branch of an electrical network resulting from the application of specified electromotive forces.

(c) The prediction of heat flow in a heat transfer problem when specified temperature variations occur in a system of thermal conductors.

(d) The prediction of the changes in chemical purity of a distillate in a chemical process resulting from specified fluctuations in flow rate, temperature, etc.

(e) The prediction of the effect on the standard of living of a community resulting from changes in external conditions of trade, internal economic controls, etc.
(f) The prediction of the effects of changes of environment on the sense of social responsibility of the individual.

(g) The prediction of animal behaviour resulting from specified command stimuli.

(h) The making of policy decisions in industrial management.

These, and many similar examples drawn from equally diverse interests, are all aspects of the same general problem. These are all examples requiring the prediction of the dynamic response of a system. Each problem enumerated requires that the behaviour of a system shall be determined when the conditions that govern the behaviour are changing.

1.2. CAUSE, EFFECT, AND SYSTEM.

All system dynamic studies involve three constituent parts. These are (a) the effects, the prediction of which are required, (b) the causes, which produce these effects and (c) the system through which the causes and effects are related. The problem in each of the examples given above is the determination of a 'cause-effect' relationship. It must be emphasised, however, that there is no clearly defined division of these problems into the constituent parts 'cause', 'effect' and 'system'.
Table 1.1 indicates, for the above examples, a possible division, but the choice adopted is not a unique one, neither can it be made so without more precise definitions of these terms than are associated with present usage.

<table>
<thead>
<tr>
<th>EXAMPLE</th>
<th>CAUSE</th>
<th>SYSTEM</th>
<th>EFFECT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>MECHANICAL FORCES</td>
<td>MECHANICAL LINKAGE</td>
<td>MOTION</td>
</tr>
<tr>
<td>(b)</td>
<td>E.M.F.'s</td>
<td>ELECTRICAL NETWORK</td>
<td>CURRENT</td>
</tr>
<tr>
<td>(c)</td>
<td>TEMP. VARIATIONS</td>
<td>THERMAL CONDUCTORS</td>
<td>HEAT FLOW</td>
</tr>
<tr>
<td>(d)</td>
<td>TEMP. OR FLOW RATE VARIATIONS</td>
<td>DISTILLATION PLANT</td>
<td>CHEMICAL PURITY</td>
</tr>
<tr>
<td>(e)</td>
<td>VARIATIONS IN EXTERNAL CONDITIONS OF TRADE AND/OR INTERNAL ECONOMIC CONTROLS</td>
<td>THE COMMUNITY</td>
<td>STANDARD OF LIVING</td>
</tr>
<tr>
<td>(f)</td>
<td>ENVIRONMENTAL CHANGE</td>
<td>THE INDIVIDUAL</td>
<td>SENSE OF SOCIAL RESPONSIBILITY</td>
</tr>
<tr>
<td>(g)</td>
<td>COMMAND STIMULI</td>
<td>ANIMAL</td>
<td>OBSERVED BEHAVIOUR PATTERN</td>
</tr>
<tr>
<td>(h)</td>
<td>CONSUMER DEMAND DATA, PROFIT AND LOSS ACCOUNT, ETC.</td>
<td>INDUSTRY</td>
<td>MANAGEMENT DECISION.</td>
</tr>
</tbody>
</table>

**TABLE 1.1**

To make this point, consider example (g). Taking the human animal, he has contact with his environment via
the senses (visual, aural, etc.). Through all these senses command stimuli may be applied. The effect of these are observed by changes in behaviour that result.

In a given investigation, only a particular behaviour pattern resulting from a particular command stimulus may be of interest. To take a specific example, one might be interested in a manual response to a visual stimulus. In this case, fig.1.1 indicates the probable division of the problem into its three constituent parts that would be adopted. In this case some of the causes which can modify the observed effect are included as disturbances (noise) in the system. Such a system is called an 'active' one.

Figure 1.2 gives an alternative division of the same problem into its constituent parts in quite a different way. This might be of value, for example, in studying the effect of background noise on a machine operator who is required to make some manual adjustment as a result of a visual observation.

It is obvious that the result of system dynamic studies must depend upon the way this division is carried out. Without a precise definition of how a problem shall be divided into its constituent parts 'cause', 'effect', and 'system', there can be no unique description of the 'system'.

12.
1.3. THE NEED FOR SYSTEM DYNAMIC STUDIES

Information obtained from system dynamic studies is used normally to assist in making "policy decisions". Thus it was common to all the problems given as examples that they required the prediction of system responses (the effects) to a postulated set of stimuli (the causes). The need for such prediction is to help assess the desirability, or otherwise, of introducing certain "correcting influences" aimed at improving the behaviour of the system.

The concept of "degrees of desirability", implicit in such a use of the information, suggests that an optimum condition, describing an ideal state of system behaviour, has been defined, a priori. The study of system dynamics makes it possible to decide, in advance, whether a proposed "correcting influence" may be expected to have the desired effect (of bringing the actual and desired behaviour more nearly into agreement), or not. It may
be pointed out that system dynamic studies cannot help in obtaining the optimum code of behaviour. This must be derived from different considerations.

From what has been said about the use to which information obtained from system dynamic studies is to be put, it must be concluded that it is essential to all studies directed towards the control of system behaviour. Whether the system be physical, economic, social, biological, industrial, or some other, the essentials of control are:

(a) observe the present behaviour and compare with the previously postulated desired behaviour;

(b) use the information so obtained to introduce a correcting influence, so making the observed behaviour more desirable.

Figure 1:3 shows the essentials of a very simple control problem involving one 'effect' and one 'cause'. A comparison of observed and desired behaviour leads to a decision on the type of correcting influence to apply.

FIG. 1:3. A SIMPLE CONTROL SYSTEM.
In this, as in more difficult control problems, difficulties of description arise. The question of the division of a problem into 'cause', 'effect' and 'system' has already been discussed. Here it is of interest to ask whether the 'correcting influence' is to be considered as an effect of the initial cause, or whether one considers that the observed behaviour of the system is a consequence of two distinct causes, (a) the initial cause (b) the correcting influence? If the latter point of view is adopted, every control system must contain at least one component in which the effect is the result of the simultaneous applications of two causes. The study of system dynamics must thus include multi-variable as well as single variable systems.

1.4 A MATHEMATICAL MODEL.

1.4.1 The approach adopted to system dynamic studies is to seek to establish a mathematical model of the problem. In this model, mathematical operations are performed on a given set of variables to produce a new set of variables. The given set of variables represent the previously defined 'causes', while the variables produced as a consequence of the operations performed represent the desired effects. The functional dependence between these two sets of variables represents
the constraints imposed by the system in producing the observed effects as a consequence of the observed causes.

The simplest problem, that involving a single cause and a single effect, may be represented by the model indicated by fig.1:4.

![Diagram of a two-port system](image)

**FIG. 1:4. A TWO PORT PROBLEM**

Such systems will be referred to as 'two-ports'. One, the input port, is the point at which the cause is applied, while the other, the output port, is the point at which the resulting effect is observed.

As has already been noted, not all system dynamic studies can be described in this way. A much more general class of problem involves the study of a number of different effects resulting from the simultaneous application of a number of independent causes. Problems of this type are characterised by \( n \)-ports, as in fig.1:5. Here \((n-m)\) different effects resulting from the application of \(m\) causes are studied.
1.4.2. The need for system dynamic studies discussed in section 1.2 above require predictions of possible future conditions from observed past behaviour, when the causes of the observed behaviour change with time. It follows that the study is of quantities which vary with time, and the mathematical representation found most suitable are functions of a single independent variable $t (= \text{time})$. The value $f(t_i)$ of this function for any time $t = t_i$ is a direct measure of the observed phenomenon at the corresponding time.

The types of function necessary to describe these observed phenomena may be divided into two general classes. Firstly, there are random time variations: (see Laning and Battin, 1956, for a description of these). Examples are afforded by such time variations as thermal noise in electrical circuits, Brownian motion, variations of natural phenomena such as barometric pressure, temperature, rainfall, etc., speech waveforms in
communication equipment, consumer demand variations in economic systems and so on. These are by far the most common type of variation encountered in practice, and are characterised by the absence of any analytic description.

The second class includes all specific time variations, analytically defined. Examples of this class are cisoidal functions, step functions, impulse functions and so on. These have assumed importance in the testing of system dynamic responses rather than because of their frequent occurrence in practical problems. It is an important property of the analytic functions given above that, for linear systems, the response to many arbitrary inputs \( x(t) \) can be computed if the response to any one of these functions is known. It is this fact which makes the study of the response of linear systems to test variations of these forms of value.

1.4.3. When seeking a mathematical model of a system, it is necessary to know that the system possesses, or may be reasonably assumed to possess, certain very general properties, by which the particular system may be catalogued, as a member of a particular class of system. The choice of attributes used for classification is arbitrary. Typical sub-divisions with the attributes that characterise them are given later in this chapter.
1.5. **ANALYSIS AND SYNTHESIS PROBLEMS IN SYSTEM DYNAMIC STUDIES.**

A few examples of system dynamic studies can be found in which the present state of knowledge is such that a mathematical model can be constructed from known laws governing the behaviour of the constituent parts of the system. Examples, mainly from the problems of physics and engineering, spring to mind. Thus Newton's Laws of Motion permit a mathematical description of simple mechanical linkages to be formulated.

In those situations in which known laws can be used to describe

(a) the behaviour of the constituent parts, and
(b) the constraints imposed by the interdependence of these parts,

the determination of the mathematical model of the system, and the prediction therefrom of the behaviour for any postulated set of causes, is a problem in system analysis. In these situations

(a) the mathematical description of the system, and
(b) a postulated set of stimuli,

are known, and the problem is to predict the responses that result from the application to the system of the postulated stimuli.

In the majority of problems arising in practice,
insufficient knowledge is available of

(a) the fundamental laws governing the behaviour
    of the constituent parts, and

(b) the influence of one part upon another through
    the interconnections which make up the system.

It is not possible in such cases to proceed, by
deductive reasoning, from a set of known laws to a
mathematical description of the response. Further, in
many situations of interest (e.g. chemical and industrial
systems, and biological, economic, and social studies),
the complexity is such that it seems unlikely that any
foreseeable advance in the study of system dynamics will
be made in this way.

For situations such as these, it is necessary,
in the first instance, to **synthesize** a mathematical model
of the whole system from experimentally acquired
information about the past behaviour, and then to use
this model, once synthesised, to predict the behaviour
when subjected to the postulated set of stimuli. It may
be observed that this synthesis procedure is precisely
the one by which the fundamental laws of the physical
sciences were first developed. The difference here is
one of degree only, the procedure being applied to systems
of far greater complexity than were those studied in, for
example, the experimental derivation of Ohm's Law.
1.6. DEFINING THE PROBLEM.

1.6.1. The present investigation is concerned with the synthesis of mathematical models describing the dynamics of systems, about which the only available information is a collection of observed responses and stimuli.

The approach to this problem is a data-fitting approach. Where it is known that the system from which the data was collected possess certain general attributes which make it a member of a certain sub-class, a search is made from among all the members of that sub-class for the one which fits the measured data.

When no information is available to suggest the sub-class amongst which to search, it is necessary to postulate a sub-class and to seek among the members of that sub-class for the one which gives a 'best' fit to the given data. If this best fit is a good approximation, and remains invariant with changes of data fitted, the assumption as to the sub-class is considered justified. If the fit is poor, and does not remain invariant with changes of input data, then it is necessary to extend the sub-class. Obviously, in problems of this type, it is advisable to keep the sub-class as general as possible, so that the search for a best operator shall range over as wide a class as possible.
In some situations it is convenient to postulate a sub-class to which the system is known not to belong, and to find, from that sub-class a useful approximation to the system dynamics. Such a case arises in the linearisation of non-linear problems. When data is known to have been collected from a non-linear system, it is often required, for simplicity, that a 'best linear approximation' to the non-linear system shall be found. One would then postulate a linear sub-class, and select therefrom that member which best approximates the given data (with 'best' suitably defined). It may be emphasised here that the best linear approximation to a non-linear system is dependent upon the input data, and hence, to be of greatest value, any linearisation of non-linear system dynamics should be performed from normal operating data rather than from any specific test input as is often the case at present.

The problem to be investigated may now be formally stated as follows:

"Given m recorded causes \( x_1(t), x_2(t), \ldots, x_m(t) \), and \((n-m)\) effects \( y_1(t), y_2(t), \ldots, y_{n-m}(t) \), determine the functional dependence of the dependent variables \( y_1(t) \) on the independent variables \( x_s(t) \). This functional dependence is to be defined as the description (mathematical model) of the system dynamics."
1.6.2. The most general functional dependence is given by

\[ y_r(t) = F_r[x_1, x_2, \ldots, x_m, t], \quad r = 1, 2, \ldots (n-m). \]  

(1.1)

Equation (1.1) is simply a statement that a functional dependence exists between \( x_s(t) \) and \( y_r(t) \). The inclusion in (1.1) of an independent functional dependence of \( y_r(t) \) on the variable \( t \) recognizes that \( y_r(t) \) may be functionally dependent on other, unrecorded independent variables (such as occurs, for example, in the determination of the dynamics of a system with noise disturbances).

System dynamics are said to be linear if the 'effects' may be expressed as a linear superposition of the causes. The functional dependence for the linear sub-class of (1.1) is thus defined by

\[ y_r(t) = \sum_{s=1}^{m} F_{rs}[x_s] + \eta_r(t), \quad r = 1, 2, \ldots (n-m) \]  

(1.2)

where, by linear superposition (see Appendix 7.1) \( F_{rs}[x_r] \) is of the general form

\[ F_{rs}[x_s] = \int_{-\infty}^{+\infty} \mathcal{K}_{rs}(\xi, \tau) x_s(\xi) d\tau \]  

(1.3)
Combining (1.2) and (1.3), linear system dynamics are defined by the general functional dependence relationship

\[ \chi_r(t) = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} h_{rs}(\tau, \tau') \chi_r(\tau) d\tau + n_r(t) \quad (1.4) \]

It is with linear system dynamics, defined by the functional dependence relation (1.4) that the investigation is concerned.

Further attributes exist by which sub-classes of the linear class (1.4) may, with convenience, be defined. Sub-classes of particular interest here are

1. the linear, **time-invariant** sub-class, for which
   \[ h_{rs}(t, \tau) = h_{rs}(t - \tau) \quad (1.5) \]
2. the linear **passive** sub-class, for which
   \[ n_r(t) = 0 \quad (1.6) \]
3. the linear, **causal** sub-class for which
   \[ h_{rs}(t, \tau) = 0 \text{ for } \tau > t \quad (1.7) \]
4. the linear causal sub-class of **memory time** \( T \), for which
   \[ h_{rs}(t, \tau) = 0 \text{ for } \tau > t \quad (1.7) \]
   and \[ h_{rs}(t, \tau) = 0 \text{ for } \tau \leq (t - T) \quad (1.8) \]
Obviously further sub-division of these sub-classes is possible by a combination of these constraints. Thus, for example, the functional dependence

\[ y(t) = \sum_{s=1}^{m} \int_{-\infty}^{+\infty} h_{ss}(t-\tau) x_s(\tau) \, d\tau, \quad \tau = \tau, \ldots, (n-m) \]

with \( h_{ss}(t-\tau) = 0 \) for \( \tau > t \)
and \( h_{is}(t-\tau) = 0 \) for \( \tau \leq (t-\tau) \)

defines a linear, time-invariant, passive, causal, n-port having a memory time \( T \). This is a sub-class of each of the classes defined by equations (1.4) to (1.8).

The determination of the functional dependence for the special case

\[ m = (n-m) = 1 \quad (1.10) \]

represents the simplest n-port problem, that of the 2-port. This problem has received considerable attention elsewhere (Chang et alia, 1956, Florentin et alia, 1959, Goodman et alia, 1956, Laning and Battin, 1956, and Westcott, 1956), and is also treated at length here.

For future convenience, the defining equations for the various sub-classes of linear 2-ports are enumerated here:

(i) linear 2-ports are defined by the functional dependence

\[ y(t) = \int_{-\infty}^{+\infty} h(t, \tau) x(\tau) \, d\tau + n(t) \quad (1.11) \]
(ii) linear time-invariant 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t-\tau) x(\tau) \, d\tau + \eta(t) \] (1.12)

(iii) linear, passive, 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t, \tau) x(\tau) \, d\tau \] (1.13)

(iv) linear, time-invariant, passive 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t-\tau) x(\tau) \, d\tau \] (1.14)

(v) linear, causal, 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t, \tau) x(\tau) + \eta(t) \]
\[ \mathcal{H}(t, \tau) = 0 \text{ for } \tau > t \] (1.15)

(vi) linear, time-invariant, causal 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t-\tau) x(\tau) \, d\tau + \eta(t) \]
\[ \mathcal{H}(t-\tau) = 0 \text{ for } \tau > t \] (1.16)

(vii) linear, time-invariant, passive, causal 2-ports are defined by
\[ y(t) = \int_{-\infty}^{\infty} \mathcal{H}(t-\tau) x(\tau) \, d\tau \]
\[ \mathcal{H}(t-\tau) = 0 \text{ for } \tau > t \] (1.17)
(viii) linear, time-invariant passive causal

2-ports having a memory time T are defined by

\[ y(t) = \int_{-\infty}^{\infty} h(t-\tau) x(\tau) \, d\tau \]

\[ h(t-\tau) = 0 \text{ for } \tau > t \]

\[ h(t-\tau) = 0 \text{ for } \tau \leq (t-T) \]

\[ (1.18) \]
CHAPTER TWO

A CRITICAL SURVEY OF EXISTING TECHNIQUES

2.1 INTRODUCTION.

Considerable thought has been given to the determination of system dynamic behaviour from data collected during normal operation (Chang et alia 1956, Florentin et alia, 1959, Goodman et alia, 1956, Henderson, 1958, Reswick, 1955, Westcott, 1956 (a), (b).)

In these investigations, the application has been to the determination of the dynamic response of physical systems such as chemical plant, rolling mills, heat exchangers, etc.

The situations studied have been those for which no adequate analysis technique exists, and exhaustive experimental investigations of the plant were excluded by economic or other considerations. In order adequately to predict the response of such systems when subjected to proposed control stimuli, it was necessary to synthesise a model from the available measured data. From this model (which may be mathematical or physical) the response to any proposed stimuli can be predicted, and a decision made on the desirability or otherwise of the proposal.

It has been a factor of the studies mentioned that it has not been known with certainty to which
sub-class the system belonged from which the data was collected, and, in every case, a 'best' member of a postulated sub-class has been sought. The characteristic property of these investigations is that they have all led to an underdetermined set of equations (j equations in k unknowns where j < k). Such equations may be either
(a) mutually inconsistent, and have no solution, or
(b) mutually consistent, and have many solutions.

The choice of a 'best' solution to such an under determined set of equations requires the introduction of additional equations of constraint to define the meaning of the term "a best solution" in this context.

2.2 CATEGORIES OF DATA FITTING PROBLEMS

Data fitting problems fall naturally into two classes.

In one class is the type of problem in which data is supposedly recorded over all time. Such situations are characterised by the type of data indicated in fig 2:1 in which the stimuli are assumed to be zero for
\[-\infty \leq \tau \leq \tau_A \quad \text{and} \quad \tau_B \leq \tau \leq +\infty,\] and are recorded for \(\tau_A \leq \tau \leq \tau_B\). Likewise, the responses to these stimuli are as indicated in fig.2:1 and must be recorded from quiescent conditions before the application of the stimulus until quiescent conditions are observed again.
after the removal of the stimulus. These conditions may be associated with systems having a finite memory (or, conceivably, a finite anticipation).

![Diagram](image)

**FIG. 2:1 THE FIRST CLASS OF DATA FITTING PROBLEM.**

The other class of problem consists of situations in which only a sample of the stimulus and response data are recorded during a time interval $t_A \leq \tau \leq t_B$ as indicated in fig. 2:2. No information is available about the nature of the stimulus and response data outside this interval, but there is good reason to suppose that it cannot be assumed identically zero as in the previous case.

These classes can conveniently be subdivided again into two. In the first sub-class are two port data fitting problems - i.e. they involved only one response and one stimulus.
In the second sub-class are n-port data fitting problems - i.e. they involve m responses to n-m stimuli where n > m. The 2 port problem is really a special case of this class, but is here treated separately for clarity of presentation, and the results subsequently generalised. The previous contributors mentioned have concentrated upon 2-port data fitting problems in which the data to be fitted was of the type indicated in fig. 2:2.

Goodman et alia (1956) set the pattern that has largely been followed by others investigating these problems. In Goodman's paper, an attempt was made to fit the data by a 'best' operator chosen from the linear,
active, time-invariant, causal, class.

It is of value to study critically the methods suggested in Goodman's original paper (1956), and also those proposed subsequently. It is hoped that such a study will reveal the limitations and weaknesses of these methods and lead to an alternative approach capable of generalisation to n-port problems.

2.3 GOODMAN'S METHOD FOR OPEN LOOP PROBLEMS

For reasons that will emerge, Goodman (1956) found it necessary to develop separate methods for
(a) open loop, and
(b) closed loop problems,
according to whether the data was collected from a physical system with feedback from output to input, or not.

In the case of open loop problems, Goodman postulated that the system from which the data had been collected was a linear, time-invariant, causal one. This was not stated formally, but must be inferred from the form of physical system depicted (fig.2:3),

FIG.2:3. GOODMAN'S OPEN LOOP PROBLEM.
and his initial equation defining the permitted class of operator, which was

$$y(t) = \int_{-\infty}^{\infty} \mathcal{E}(\tau) x(t-\tau) d\tau + n(t)$$  \hspace{1cm} (2.1)

and may be compared with (1.16) from which it can be derived by a change of variable. This equation implies

(i) linearity through the superposition integral (see (1.4));

(ii) an active system through the presence of \( n(t) \) in (2.1), (see (1.6));

(iii) a time-invariant system through the form of the weighting function \( \mathcal{E}(\tau) \) in the superposition integral (see (1.5));

(iv) a causal system through the range

\( 0 \leq \tau \leq +\infty \) of integration in the superposition integral (see (1.7)). This range of integration requires that \( y(t) \) is influenced only by past and present values of the stimulus \( x(t) \), not by future values.

It should be pointed out here that equation (2.1) is underdetermined, being a single equation in two unknowns \( n(t) \) and \( h(\tau) \). This single equation has an infinity of possible solutions since any \( h(\tau) \) may be arbitrarily selected from members of the permitted class and
substituted into (2.1). The integral may then be calculated and a corresponding function \( n(t) \) deduced.

In order to define the 'best' solution of the infinity of possible solutions, Goodman arbitrarily introduced the constraint equation

\[
\mathcal{Q}_{\infty}(t_i) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t-t_i) n(t) \, dt = 0 \quad (2.2)
\]

for all \( t_i \).

Multiplying (2.1) throughout by \( x(t-t_i) \), and averaging, he obtained

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t-t_i) y(t) \, dt = \int_{0}^{\infty} \mathcal{R}(\nu) \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t-t_i) x(t-\nu) \, dt \right\} \, d\nu \\
+ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t-t_i) n(t) \, dt \quad (2.3)
\]

or, from (2.2) and (2.3)

\[
\mathcal{Q}_{xy}(t_i) = \int_{0}^{\infty} \mathcal{R}(\nu) \mathcal{Q}_x(t_i-\nu) \, d\nu \quad (2.4)
\]

Equation (2.4) is the integral equation defining the 'best \( h(\nu) \)' according to Goodman's definition of 'best'. The problem may then be considered solved if a function \( h(\nu) \) can be found having the properties

(a) that it satisfies equation (2.4);
(b) that it is a causal $h(\tau)$ - i.e.

$$h(\tau) = 0 \text{ for } -\infty \leq \tau \leq 0$$

With data of the type used by Goodman, a solution of equation (2.4) cannot be found. Instead, an approximate solution of a modified equation, of the form

$$Q_{xy}(t_i) = \sum_{j=1}^{N} h_j Q_{xx}(t_i - t_j T)$$

(2.5)

for an undefined range of values of $t$, was sought (see Goodman, 1956).

The reasons for the difficulties with which Goodman was faced in finding a solution of (2.4) will be discussed in detail later (section 2.6, 2.7, and Chapter 5).

It suffices for the moment to make the observation, from (2.5) that the form of solution sought excludes all members of the class linear, active, time-invariant, causal (the class originally permitted by equation (2.1) except those having a finite memory time less than, or equal to, some previously defined maximum memory time $T$).

2.4 EXISTING TECHNIQUES FOR CLOSED LOOP PROBLEMS

When the data is known to have been collected from a closed loop system, several constraint conditions defining a 'best' solution to the data fitting problem have been suggested.
2.4.1. Goodman (1956) postulated, for the closed loop problem he considered, that the recorded data $x(t)$ and $y(t)$ should be considered to have been collected from a linear, active, time invariant, causal, closed loop system reducible to the form indicated by fig. 2.4, for which the defining equations are

\begin{align}
  y(t) &= n_i(t) + \int_{-\infty}^{\infty} H_l(\tau) x(t-\tau) \, d\tau \tag{2.6} \\
  x(t) &= n_i(t) + \int_{-\infty}^{\infty} H_l(\tau) y(t-\tau) \, d\tau \tag{2.7}
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{fig2.4.png}
\caption{GOODMAN'S CLOSED LOOP PROBLEM.}
\end{figure}

As in the open loop problem, these equations are underdetermined, and have an infinity of possible solutions from members of the permitted class.

Goodman stated that, because of the feedback path from output to input, in the closed loop problem, constraints defining a best solution of (2.6) and (2.7) similar to those used in the open loop problem must be
excluded. The reasons for this statement were not adequately explained, but can be supported (Woodrow, 1959).

Equations (2.6) and (2.7) contain four unknown parameters $n_1(t)$, $n_2(t)$, $h_1(\tau)$ and $h_2(\tau)$, together with two recorded parameters $x(t)$ and $y(t)$. It is thus necessary to introduce two additional equations of constraint to define a best solution.

It should be emphasised here that each of equations (2.6) and (2.7) contains two different unknowns. Equation (2.6) contains $n_1(t)$ and $h_1(\tau)$, while equation (2.7) contains $n_2(t)$ and $h_2(\tau)$.

If the constraints imposed defining a 'best' solution restrains an unknown in one equation without reference to the unknowns in the other equation (called here a type A constraint) each of equations (2.6) and (2.7) defines a separate optimisation process.

If, on the other hand, the constraints relate the unknowns in one of the equations (2.6) and (2.7) to the unknowns in the other (called here a type B constraint), this imposes a simultaneous constraint on both optimisation problems.

Of the two, the type A constraint is the easier to handle, since it enables the problem to be reduced to two separate problems whereas the type B constraint does
not have this orthogonal property. Unless there are
good reasons (e.g. on physical grounds) to suppose that
type B constraints make a significant contribution to
the solution, it seems pointless to introduce them
with the added difficulties they introduce.

2.4.2. Goodman introduced into his solution of
the closed loop problem the constraint conditions

\[ \Phi_{\eta x}(t_i) = \lim_{t \to \infty} \frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} \eta_i(t, t_i) \eta_2(t) \, dt = 0 \]  

(2.8)

and

\[ \Phi_{xx}(t_i) = \lim_{t \to \infty} \frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} x_i(t, t_i) \eta(t) \, dt = 0 \]  

(2.9)

for \( t_i > A \)

These are type B constraints because (2.8) simultaneously
places a constraint upon both (2.6) and (2.7).

In his original paper, Goodman contented himself
with the derivation of the equation for a best \( \hat{h}_i(t) \)
only, obtaining from (2.6) and (2.9) the integral
equation

\[ \Phi_{xy}(t_i) = \int_{-\infty}^{\infty} \hat{h}_i(t) \Phi_{xx}(t_i, t) \, dt \], \text{ for } t_i > A \]

(This solution can be derived from (2.6) and (2.9) in
exactly the same way that (2.4) was derived from (2.1)
and (2.2).
2.4.3. An alternative procedure for the fitting of data collected from closed loop systems reducible to the arrangement of fig. 2.4 has been suggested by Westcott (1956(a)).

This method postulated that in addition to $x(t)$ and $y(t)$, the data $Z_2(t)$ (see fig. 2.4) should be recorded, and the single constraint introduced that

$$\lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} \eta(t) \eta_{2}(t) \, dt = 0 \quad (2.10)$$

For this situation, the equations to be solved are

$$y(t) = \eta_{2}(t) + \int_{0}^{\infty} \mathcal{H}_2(\tau) \, x(t-\tau) \, d\tau \quad (2.6)$$

and

$$Z_2(t) = \int_{0}^{\infty} \mathcal{H}_2(\tau) \, y(t-\tau) \, d\tau \quad (2.11)$$

together with the constraint condition (2.10).

In this situation, by multiplying (2.6) throughout by $\eta_{2}(t+\xi)$

$$\eta_{2}(t+\xi) - x(t+\xi) = Z_2(t+\xi)$$

and taking a time average, the integral equation defining the required operator $h_{i}(\tau)$ becomes

$$\varphi_{yx}(t_{i}) - \varphi_{yZ_2}(t_{i}) = \int_{0}^{\infty} \mathcal{H}_i(\tau) \varphi_{x}(t_{i} - \tau) \, d\tau$$

$$- \int_{0}^{\infty} \mathcal{H}_i(\tau) \varphi_{xZ_2}(t_{i} - \tau) \, d\tau \quad (2.12)$$

Equation (2.11) is the corresponding integral equation.
defining the impulse function \( h_z(\tau) \). If required, the corresponding correlation equation

\[
\phi_{z,y}(\xi) = \int_0^\infty h_2(\tau) \phi_y(\xi - \tau) \, d\tau
\]  

(2.13)

may be taken as the defining equation for the function \( h_z(\tau) \).

2.4.4. An alternative procedure for the same problem (Woodrow, 1958) suggested that \( n_x(t) \) should be made available as recorded data, and that the constraint condition (2.10) should be applied.

With these conditions, equation (2.7) is an integral equation to be solved for \( h_z(\tau) \) while equations (2.6) and (2.10) combine to give

\[
\phi_{n_z,y}(\xi) = \int_0^\infty h(\tau) \phi_{n_x}(\xi - \tau) \, d\tau
\]  

(2.14)

as the integral equation to be solved for \( h_t(\tau) \).

As was shown (Woodrow, 1958) equation (2.14) and equation (2.12) are the same. Equation (2.14) has the practical advantage over (2.12) that it involves the determination of only two correlation functions, while (2.12) requires that four correlation functions be computed.

2.4.5. It was further suggested (Woodrow, 1958) that, should the condition of the problem prevent the
recording of the data \( n(t) \), but be such that

\[
n(t) = \eta(t) + s(t)
\]

(2.15)

where \( n(t) \) = contribution to \( x(t) \) due to disturbances in the feedback path and

\( s(t) \) = disturbances introduced into the loop at this point, which may be recorded, then \( s(t) \) can be recorded and used with the constraint equations

\[
\phi_{\eta}(t) = \phi_{s}(t) = 0
\]

(2.16)

For such conditions it follows from (2.6), (2.7), (2.15) and (2.16) that

\[
\phi_{sy}(t) = \int_{-\infty}^{\infty} h_s(t,\tau) \phi_{sx}(t,\tau) d\tau
\]

(2.17)

and

\[
\phi_{sx}(t) = \phi_{s}(t) + \int_{0}^{\infty} h_s(t,\tau) \phi_{sy}(t,\tau) d\tau
\]

(2.18)

are the integral equations to be solved for \( h_s(t,\tau) \) and \( h_s(t,\tau) \) respectively.

2.5 GENERAL OBSERVATIONS ON EXISTING METHODS.

2.5.1. In the preceding pages a number of constraints which have been suggested in the literature have been discussed. While it is possible to compare the relative merits of these methods from a point of view of mathematical convenience, nothing can be said about the physical significance of any one method of data
fitting as compared with any other.

In every case, an inadequate amount of available information has resulted in

(a) the formulation of under-determined sets of equations, and

(b) the introduction of arbitrary constraint equations, (chosen largely for mathematical convenience) in an attempt to make a solution possible.

In such a situation, only two courses are open. One must either

(a) make more information available to weaken, or remove, the indeterminacy, or

(b) accept the limitation on the data, and satisfy oneself with an approximate solution based upon intuitively chosen conditions of constraint.

The conditions of the problems studied must be assumed to exclude the possibility of making sufficient information available to completely define the system dynamics, and one is forced to accept a 'best' approximation.

This being the case, there exists a need for a general set of constraint conditions to define the best approximation which are both physically significant and mathematically convenient. Further, in order to unify
the concepts, and generalise the results, the constraint conditions adopted should be equally applicable to data collected from either open or closed loop systems.

Finally, any method that is adopted for two-port problems of the type so far studied should be capable of generalisation to include n-port problems of the type mentioned in chapter one (section 1.3) above.

From the examples given, it is apparent that there are, as yet, no general underlying principles governing the selection of the conditions of constraint, each problem being considered in isolation. Such an approach does no more than provide particular approximations to particular problems. It cannot produce a data fitting theory which is of general application in all system dynamic studies.

2.5.2. A second observation which may be made from the examples studied above is that, in each case, the constraint conditions introduced have been statistical ones involving the concept of uncorrelated variables (as, for example, equation (2.2) for the open loop problem, and equations (2.8), (2.10) and (2.16) for closed loop problems).

These methods have finally reduced to the solution of integral equations of the general form

\[
\phi(t) = \int_{-\infty}^{\infty} h(\tau) \phi(t-\tau) \, d\tau
\] (2.19)
where $\mathcal{H}(\tau) = 0$, for $-\infty \leq \tau \leq 0$

Two important assumptions are implied by the use of these methods.

Firstly, it is implied that equation (2.19) has a solution. In other words, it is assumed that, for the given data, there exists, in the postulated class of system operators, at least one which satisfies the arbitrarily imposed conditions of constraint.

Secondly, it is assumed that the correlation functions $\phi_1(t)$ and $\phi_2(t)$ in (2.19), or good approximations thereto, can be made available from the measured data $x(t)$ and $y(t)$, recorded in some interval $t_A \leq t \leq t_B$ only.

It will be necessary to consider these assumptions in detail later (sections 2.6 and 2.7), but, for the moment, it is sufficient to anticipate the results of this survey, and to say that both assumptions are suspect.

When seeking alternative constraint conditions (which are physically significant, mathematically convenient, and of general application) attempts must be made to choose conditions which weaken the dependence of the results upon the validity of these assumptions.

2.6 THE SOLUTION OF AN INTEGRAL EQUATION.

Turn now to the question, mentioned earlier, of whether the integral equation (2.19) defining the 'best'
approximation to the given data has a unique solution.

At least two arbitrary assumptions were made in the derivation of this equation. A permitted class of operator (linear, active, time-invariant causal in the problems studied) was first postulated, and subsequently the selection of one particular member of this class, was attempted. Other postulates were introduced defining the property (or properties) which the particular member of the permitted class sought was required to possess.

Since the constraints defining the best member of the permitted class were imposed arbitrarily, it is conceivable that, for the given data, either (a) no member, or (b) many members, of the permitted class of operator exist which have the properties postulated by the conditions of constraint introduced. Situation (a) above occurs if the assumptions postulating the class of operator are incompatible, for the data given, with the postulates defining the 'best' member of the class.

If this is the situation, then the integral equation (2.19) can have no solution. Constraint conditions cannot be chosen arbitrarily, but must, in fact, be chosen with considerable care, so as to define
attributes which are possessed by at least one (and preferably only one) member of the postulated permitted class.

It is obviously of interest to ask whether, in a given problem, a proposed set of constraint conditions do define a member of the postulated class of system operators. The answer to this question is to be found in the resulting equation defining the member selected. If this equation has no solution, then the constraint conditions do not make a choice possible - i.e. the assumptions as to the properties of the class excludes all operators having the particular property ascribed to the member required.

Ignore, for the moment, the additional complication, present here, of the actual determination of the correlation functions \( \varphi_1(t) \) and \( \varphi_2(t) \) from samples of recorded data, and assume that these functions can be made available.

Suppose, initially, that (2.19) has a solution. Take Fourier Transforms of both sides, giving

\[
\int_{-\infty}^{+\infty} \varphi_1(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(t) \varphi_2(t-r) e^{-j\omega t} dr dt
\]

or

\[
G_1(j\omega) = H(j\omega) G_2(j\omega)
\]
where $G_1(j\omega) = \text{Fourier Transform of } \phi_1(t)$

$G_2(j\omega) = \text{Fourier Transform of } \phi_2(t)$

$H(j\omega) = \text{Fourier Transform of } h(t).$

From Lerch's Theorem (Pipes, 1958, page 555), on the uniqueness of the Fourier Transform, it follows that, provided $G(j\omega),$ $G_2(j\omega),$ and $H(j\omega)$ exist, then $H(j\omega)$ is given uniquely by

$$H(j\omega) = \frac{G(j\omega)}{G_2(j\omega)} \quad (2.21)$$

Taking the inverse Fourier transform gives

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{G_1(j\omega)}{G_2(j\omega)} \right\} e^{j\omega t} d\omega. \quad (2.22)$$

Hence, if a causal solution to (2.19) exists, then that solution is given by (2.22) which must, therefore, define a causal $h(t).$

A test of whether or not a given constraint conditions may be imposed is thus to be found in the form of $h(t)$ given by (2.22). If this is a causal $h(t),$ then the constraint conditions do define a member of the class linear active, time-invariant, causal.

If, on the other hand, (2.22) does not define a causal $h(t),$ then it is necessary to extend the permitted
class (by the removal of the constraint 'causal') in order to allow these constraints to define the desired member.

It is not possible, in practice, to carry out this procedure to assess whether proposed constraint conditions define a member of the postulated class, because of the practical difficulty of isolating this problem from the additional problem of finding the correlation functions \( \varphi_1(t), \varphi_2(t) \), and the corresponding spectral density functions \( G_1(j\omega) \) and \( G_2(j\omega) \) from the recorded data.

The question of the compatibility of the assumptions

(a) as to the class of operator defining the system from which the data was collected, and

(b) the properties of the member of that class which best describe the data,

has never been raised in the literature.

As has already been stated (see equation (2.5)), the practical difficulty of finding \( \varphi_1(t), \varphi_2(t), G_1(j\omega) \) and \( G_2(j\omega) \) has compelled those seeking to solve equations of the same general form as (2.19) to make two unsubstantiated assumptions, namely

(a) that the integral equation (2.19) has a solution, and
(b) that a good approximation to this solution is given by any approximate solution to the modified equation

\[ \phi(t_i) = \sum_{z=1}^{N} h_z \phi_2(t_i - \frac{z}{N}) \]  \hspace{1cm} (2.23)

These assumptions obscure this problem, but, because of them, one can say nothing about the significance of any of the proposed data fitting procedures. One is unable to say whether, for given data, the constraints imposed are compatible or incompatible.

2.7 THE DETERMINATION OF CORRELATION AND SPECTRAL DENSITY FUNCTIONS.

Turn now to the second assumption, (mentioned in section 2.5.2.), that a good estimate of correlation (or spectral density) functions of the data \( x(t) \) and \( y(t) \) may be made available from such data as that depicted in fig.2:2.

The constraints currently adopted in the formulations of the problems discussed have all been found to require that an integral equation of the form (2.19) be solved for \( h(t) \). An analytic solution of this type of integral equation requires that Fourier Transforms be used (see Coales, 1959, and Laning and Battin, 1956, pp.280-283). Further, it is essential
if the compatibility of the several assumptions employed are to be tested, that spectral density estimates be made available. Hence the proposed constraints cannot be considered to define a useful description of the system dynamics unless realistic estimates of spectral density functions $G_1(j\omega)$ and $G_2(j\omega)$ can be made available from experimental data.

Wiener (1949, page 55) pointed out that "in all practical cases the correlation coefficient of a message is not completely determined by its own past. If it were so determined, then at no period in the message would it be possible to introduce new information". This statement makes it quite clear that the entire past history of the data is insufficient to make a precise calculation of the correlation coefficients possible. The whole life history of the system for $-\infty \leq t \leq +\infty$ is required.

When only a part of the past history is known, to ask for correlation and spectral density functions is to ask more than available information can provide. The constraints introduced in the formulation of these problems have failed to observe the restriction imposed upon any method of finding a realistic description of the system dynamics, by the properties of the recorded data (that it is only defined for a finite, past, time interval).
Two quite distinct approaches to this dilemma seem possible. Firstly, one may accept constraint conditions in the optimisation problem which take no account of the restriction imposed on the problem by the finite duration of the recorded data, and then seek with the aid of statistical sampling theory an estimate of the required properties of the whole population, from the corresponding properties of a representative sample of that population. Such an approach requires the assumption of certain general statistical properties for the data of which a sample is available. For example, Laning and Battin (1956, p.161, section 4.3) have studied data which are samples of stationary, ergodic, Gaussian, random processes, while Fuller (1958, sections 5 and 6) studied stationary, ergodic, processes having both Gaussian and Poisson distributions). The weaknesses of this approach are

(a) the lack of a procedure by which the hypotheses regarding the statistical properties assumed for the data may be readily tested;

(b) the difficulty of making an a priori decision on the size of sample necessary to be representative of the whole population;
(c) as a consequence of (b), the extreme difficulty of fitting confidence limits to the estimates of the statistical properties of the whole population so derived.

Considerable effort is being expended by numerous statisticians (e.g. Lomnicki and Zaremba, 1957, Blackman and Tukey, 1958, Bartlett, 1955, Grenander and Rosenblatt, 1957, Fuller, 1958) on this problem of deriving realistic estimates of power spectra from finite samples of recorded data. The difficulties are considerable, but it seems reasonable to hope that the outcome of this concentrated attack on the problem will be a satisfactory technique for the estimation of spectral density data from samples of finite time duration. The application of the results developed in the next two chapters must await the realisation of this hope.

The alternative approach, by which this difficulty may be circumvented, is to look again at the constraint conditions imposed upon the optimisation problem, and to ensure that the constraints are realistic in the sense that they recognize the finite time duration of the recorded data. Such constraints remove the need to estimate spectral density functions, since these no longer arise in the description of the dynamics. This approach is considered in chapter 5.
3.1. INTRODUCTION

The discussions of the previous chapter have revealed the need for the introduction into data fitting problems of constraint conditions which are physically significant, mathematically convenient, equally applicable to open and closed loop problems, and capable of generalisation to n-port problems of which the 2-port becomes a special case. Further, it has been observed that, associated with the concept of physical significance, there is the condition that the constraint defining the best member of a given class of operator shall be a realistic constraint (in the sense that a member of the permitted class shall exist which satisfies the given constraint).

In this chapter a data fitting procedure is developed for, and applied to, 2-port problems. This procedure is extended in the next chapter to include n-port data fitting problems. In chapter 5 consideration is given to the necessary modifications to these ideas.
if the limitations placed upon any solution by the nature of the available data are to be respected.

3.2 ON CHOOSING THE PERMITTED CLASS.

In the methods described in chapter 2, a best operator was selected from the class linear, active, time-invariant, causal. The methods of this chapter are developed initially for the class linear, passive, time-invariant. Attempts are made later to extend this class.

The class here chosen differs in two respects from those used in the earlier work. Firstly a passive, rather than an active, class is required. This gives recognition to the fact that active problems, with a sufficiently highly developed measuring technique, may be treated as multivariable passive systems. However, in attempting to generalise the permitted class later, the possibility of replacing the passive constraint by an active one will be considered.

The second difference is the dropping of the 'causal' constraint from the permitted class. Causality is frequently considered fundamental to physical systems. If the data in a data-fitting problem has been collected from a certain causal system, it is quite unnecessary to introduce this constraint into the problem. The data
itself favours a causal solution. Since the class linear, passive, time invariant causal is a sub-class of the class linear, passive, time-invariant, it is sufficient to seek a best operator from the enlarged class. If the constraint by which the best operator is selected is a physically significant constraint, then it must lead to a causal operator in those situations where the data originated in a system of the causal sub-class.

Two advantages result from a consideration of the larger class. Firstly, it is inconvenient mathematically to carry the causal constraint through the processes by which the defining equation for the optimum operator of the permitted class is obtained. Secondly, it is desirable, as a general principle, to keep the permitted class as unrestricted as available mathematical techniques permit. The less restricted the class of operator permitted, the wider is the choice, and hence the larger the class of data for which an acceptable fit is possible.

Further, bearing in mind what has been said in section 1.2 about the uses of the information obtained from system dynamic studies, it is conceivable that situations may arise in which a non-causal description
of system dynamics would be more acceptable (if it more accurately described the behaviour of the system) than a causal operator.

However, in recognition of the need, in some cases, to produce physical (as opposed to mathematical) models approximating the behaviour of systems from which data has been collected, the effect of introducing a causal constraint will be considered for those situations in which the best linear, passive, time-invariant, operator may be a non-causal one.

3.3 ON CHOOSING THE 'BEST' OPERATOR OF THE PERMITTED CLASS

Emphasis has already been given to the need to develop a general procedure which, while being mathematically convenient, is also physically significant. It is in the choice of the constraint by which the 'best' operator is defined that this is imposed.

A general constraint already widely applied in data fitting problems of all kinds, and more recently applied to the allied problems of the synthesis of optimum linear filter and predictor operators (Wiener 1949, Laning and Battin 1956, Bode and Shannon 1950) is the minimum mean square error criterion. This criterion requires that one chooses as the 'best' operator that member of the permitted class which makes
the mean square error of the approximation least. This is quite a different form of constraint from those discussed in the previous chapter, and has the important property (which those constraints did not have) that it always defines members of the permitted class. Without further study, it is not possible to say whether, for all data, this criterion necessarily defines a unique solution. It is conceivable that quite different approximations shall produce errors, the mean square values of which are the same. The question of uniqueness remains, for the moment, to be answered.

If the data to be fitted was collected from a system, the dynamics of which makes it a member of the permitted class, then there exists, in the permitted class, at least one member which fits the data exactly - i.e. which makes the error of the approximation zero. Since mean square values are necessarily positive, the least value that a mean square error can assume is zero. If it can be shown that a minimum mean square error constraint does lead to a unique solution, then it follows from what has been said above that this unique solution must describe the system dynamics exactly provided the system is, in fact, a member of the permitted class. This emphasises the need to keep the permitted class as general as possible.
If mathematical techniques were available by which a minimum mean square error solution could be developed for a class of operator which was so general as to include all systems from which data might be collected, then one could, in principle at least, predict with complete confidence from the mathematical model so synthesised. The making of policy decisions would then be reduced to a routine computation.

3.4 A LINEAR, TIME-IN Variant, PASSIVE, 2-PORT DATA FITTING PROBLEM.

Figure 3:1 (a) indicates the situation for a simple 2-port problem. The observed 'cause' \( x(t) \) and 'effect' \( y(t) \) are related by a system the dynamics of which is assumed to be linear, passive, time-invariant. Figure 3:1 (b) gives the mathematical description of the problem.

Denoting the impulse response (weighting function) of the linear passive approximation by \( h(t) \), the quantity \( z(t) \) of fig.3:1 (b) is given by either

\[
x(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau \quad (3.1)
\]

or

\[
x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau
\]
Further

\[ y(t) = \epsilon(t) + z(t) \]

\[ = \epsilon(t) + \int_{\infty}^{\infty} h(\tau) x(t-\tau) d\tau \]  \hspace{1cm} (3.2)

where \( \epsilon(t) \) is the error resulting from approximating the actual system dynamics by the linear passive time-invariant operator \( h(t) \).

![Diagram](a)

![Diagram](b)

**FIG. 3.1. THE 2-PORT PROBLEM.**
It is in the form of equation (3.1) that the restriction of the permitted class to linear passive time-invariant operators occurs. The restriction

(i) linear, is necessary to permit the superposition integral;

(ii) passive, is necessary to permit the form of \( z(t) \). (An additional term \( n(t) \) would be added to the right hand side if the permitted class were enlarged to include linear active systems).

(iii) time invariant is necessary to permit the given form of weighting function

(This would be replaced by \( \hat{h}(\xi, \tau) \) if the class were enlarged to include time-variant systems).

It may be observed at this point that, had the class been contracted so as only to include linear, passive, time-invariant, causal operators, this would leave (3.1) and (3.2) unchanged, but would introduce the further condition

\[
\hat{h}(\tau) = 0 \quad \text{for} \quad \tau < 0
\]

(3.3)

Since this constraint is not employed here, this condition is not considered.
3.5 ON APPLYING A MINIMUM MEAN SQUARE ERROR CONSTRAINT.

The next step is to choose that operator (or those operators) \( h(t) \), from the whole permitted class of operators, which minimises \( \varepsilon(t) \) where

\[
\varepsilon(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon(t) \, dt
\]  

(3.4)

Now, from (3.2) it follows that

\[
\varepsilon(t) = \gamma(t) - \int_{-\infty}^{\infty} h(t) \times (t-t) \, dt
\]  

(3.5)

and

\[
\varepsilon(t-t) = \gamma(t-t) - \int_{-\infty}^{\infty} h(t) \times (t-t, t) \, dt
\]  

(3.6)

Multiplying (3.5) and (3.6) and taking a time average of the product

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon(t) \varepsilon(t-t) \, dt = \]

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left\{ \gamma(t) - \int_{-\infty}^{\infty} h(t) \times (t-t) \, dt \right\} \left\{ \gamma(t-t) - \int_{-\infty}^{\infty} h(t) \times (t-t, t) \, dt \right\} \, dt
\]

or

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left\{ \gamma(t) \int_{-\infty}^{\infty} h(t) \times (t-t, t) \, dt \right\} \left\{ \gamma(t-t) \int_{-\infty}^{\infty} h(t) \times (t-t, t) \, dt \right\} \, dt
\]  

(3.7)

where

\[
\phi_{\varepsilon\varepsilon}(t) = \phi_{yy}(t) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t) \times h(t) \times \phi_{xy}(t, t + \tau) \, d\tau
\]

\[
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t) \times h(t) \times \phi_{xx}(t, t + \tau, t) \, d\tau \, d\tau
\]

(3.8)

is, by definition, the autocorrelation.
function of the error, and the other correlation functions are similarly defined.

Taking Fourier Transforms of both sides of (3.7)

\[ G_{ee}(\omega) = G_{yy}(\omega) - H(\omega)G_{xy}(\omega) - H(\omega)G^*_{xy}(\omega) \]
\[ + H^*(\omega)H(\omega)G_{xx}(\omega) \]  

where

\[ G_{ee}(\omega) = \int_{-\infty}^{\infty} \varrho_{ee}(t)e^{-j\omega t} dt \]  

and * denotes a complex conjugate, as, for example, in

\[ H^*(\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t} dt \]  

Now, from (3.10), taking inverse transforms

\[ \varrho_{ee}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ee}(\omega) e^{j\omega t} d\omega \]  

From (3.4) and (3.8) it follows that

\[ \overline{\varepsilon^2(t)} = \varrho_{ee}(\omega) \]  

and hence, from (3.12) and (3.13)

\[ \overline{\varepsilon^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ee}(\omega) d\omega \]  

etc.,
From (3.14) and (3.9)

\[
\overline{\varepsilon^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ G_{yy}(\omega) + H^*(\omega) H(\omega) G_{xx}(\omega) \right\} d\omega
\]

\[
- \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ H(\omega) G_{xy}(\omega) + H(\omega) G_{xy}^*(\omega) \right\} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{xx}(\omega) \left\{ \frac{G_{yy}(\omega)}{G_{xx}(\omega)} - \frac{G_{xy}(\omega) G_{xy}^*(\omega)}{G_{xx}(\omega) G_{xx}(\omega)} \right\} d\omega \quad \text{\{3.15\}}
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{xx}(\omega) \left\{ H(\omega) - \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \right\} \left\{ H^*(\omega) - \frac{G_{xy}^*(\omega)}{G_{xx}(\omega)} \right\} d\omega \quad \text{\{3.17\}}
\]

Since

\[
\left\{ H(\omega) - \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \right\} \left\{ H^*(\omega) - \frac{G_{xy}^*(\omega)}{G_{xx}(\omega)} \right\} = \left| H(\omega) - \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \right|^2
\]

the last term in (3.15) is necessarily positive or zero, and \( \overline{\varepsilon^2} \) will be least if \( H \) is chosen to make this term as small as possible (i.e. zero).

The best linear passive time-invariant operator in a minimum mean square error sense is therefore one having a frequency response \( H(\omega) \) given by

\[
H(\omega) = \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \quad \text{\{3.16\}}
\]

writing (3.16) in the form

\[
G_{xy}(\omega) = H(\omega) G_{xx}(\omega)
\]
and taking inverse transforms gives

$$\phi_{xy}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) \cdot G_{xx}(\omega) \cdot e^{j\omega \xi} \, d\omega$$

But \(H(w) \cdot G_{xx}(w)\) is the Fourier Transform of a convolution integral i.e.

$$\int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} h(\tau) \phi_{x\ast}(\xi - \tau) \, d\tau \right\} e^{-j\omega \xi} \, d\xi = H(\omega) \cdot G_{xx}(\omega)$$

Hence

$$\phi_{xy}(\xi) = \int_{-\infty}^{+\infty} h(\tau) \cdot \phi_{x\ast}(\xi - \tau) \, d\tau$$ \hspace{1cm} (3.17)

is the integral equation giving the time domain description of the optimum impulse response \(h(\tau)\) corresponding to the optimum frequency response function \(H(w)\).

If spectral density data is available, the function \(h(\tau)\) satisfying (3.17) follows directly from (3.16) thus

$$h(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) \cdot e^{j\omega \xi} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \right\} \cdot e^{j\omega \xi} \, d\omega$$ \hspace{1cm} (3.18)

Practical difficulties may arise in deriving good estimates of power spectra (because the limitations placed upon available data referred to previously, but
not considered in this optimisation procedure, only permit reasonable approximations to correlation functions to be derived for a finite range of $t$.

The present trend in such situations is to prefer (e.g. Florentin 1959, Reswick 1955) an approximate solution to the time domain description (3.17) not involving Fourier Transforms of the convolution integral. It is not possible to say in general whether such a procedure provides a better description of the system dynamics than could be obtained from approximations to the spectral density functions and the use of (3.16) and (3.18). An obvious test of the relative merits of the two procedures for the solution of any particular problem is to derive the mean square errors associated with each of the solutions obtained and find which is least.

The essential problems are not considered to be in the comparison of the relative merits of two different approximate solutions of the optimisation equation. The real problems are either

(i) to introduce constraints into the optimisation procedure by which the limitations of the data are respected, or

(ii) the development of techniques for the
derivation of realistic estimates of spectral density functions from samples of data, as discussed in section 2.7.

It should be observed that, for those problems for which equation (3.16) defines an \( H(\omega) \), it follows, from Lerch's Theorem (Pipes, 1958, p. 555) that the \( H(\omega) \) so defined is a unique solution to the optimisation problem, i.e., there is only one \( H(\omega) \) in the permitted class which minimises the mean square error.

3.6 THE CONDITIONS FOR AN EXACT SOLUTION.

If the data fitted had been collected from a system which was a member of the class linear, passive, time-invariant, then a member of that class exists which fits the data exactly, and, from what has been said above, it follows that a minimum mean square error optimisation procedure must lead to that member. It is of interest, therefore, to develop a test which can be applied to data to establish whether an exact fit can be found from members of the linear, passive, time-invariant class.

From equations (3.9) and (3.16) it follows that the error associated with the optimum operator \( H(\omega) \) has a spectral density function \( G_{e e}(\omega) \) given by

\[
G_{e e}(\omega) = G_{y y}(\omega) - \frac{G_{x y}(\omega) G_{y x}^*(\omega)}{G_{x x}(\omega)}
\] (3.28)
From (3.14) and the property of spectral density functions (Laning and Battin, p.126) that $G_{ee}(\omega)$ is a real valued, non-negative function of $\omega$, it follows that $\overline{e}(t)$ can only be zero if $G_{ee}(\omega)$ is identically zero, i.e. if

$$G_{xx}G_{yy} = G_{xy}G_{xy}$$

(3.29)

It should be emphasised here that while data collected from a linear passive, time-invariant system satisfies this condition, the converse has not been shown to be true - i.e. it does not follow that data which satisfies this test was collected from a linear, passive, time-invariant, system. As a test of the validity of the postulate that the data originated in a system of this type, this result must be used with caution.

In situations in which an exact fit to the data is not possible, it follows that the system dynamics does not belong to the linear, passive, time-invariant, class. The solution given by (3.16) and (3.18) represents an approximation to the system dynamics in the sense that the linear passive time-invariant system defined by these equations gives, for the input $x(t)$, an output $z(t)$ which is nearer (in the least squares sense) to the output $y(t)$ of the actual system than can be obtained from any other member of the same class. If the character of $x(t)$ is
changed, some other member of the class must be used to approximate the actual system. In short, the solution ceases to be invariant under changes of data (which is the essential property of a true description of system dynamics).

However, in the absence of an alternative procedure, one might accept such an approximation to the system dynamics and base policy decisions upon it, neglecting the effect of changes in the character of \( x(t) \). In such instances, equation (3.28) provides a check on the closeness (and hence the usefulness) of the approximation. If the closeness of the approximation is poor, the policy decisions based thereon are certainly of doubtful value, and it becomes essential to search for a better approximation from an enlarged class of operator.

A measure of the success of the approximation \( \mathcal{N} \) could be

\[
\mathcal{N} = \frac{\text{mean square value of the output of the 'best' approximation}}{\text{mean square value of the output data } y(t) \text{ being approximated}} \quad (3.30)
\]

Now \( Z^2(t) \) the mean square value of the output \( z(t) \) of the best linear passive time-invariant approximation in the 2-port problem is given by

\[
Z^2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{xx}(\omega) \, d\omega \quad (3.31)
\]
But \( z(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau \)

and hence

\[
\varphi_{zz}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) \varphi_{xx}(\tau_1 - \tau_2 + t) \, d\tau_1 \, d\tau_2
\]

Taking Fourier Transforms

\[
G_{zz}(\omega) = H(\omega) H^*(\omega) G_{xx}(\omega)
\]  
(3.32)

\[
= \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \frac{G_{xy}^*(\omega)}{G_{xx}(\omega)} G_{xx}(\omega)
\]  
from \(3.16\)

\[
G_{zz}(\omega) = \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \frac{G_{xy}^*(\omega)}{G_{xx}(\omega)} = G_{xx}(\omega)
\]  
\(3.33\)

and, from \(3.28\) and \(3.33\)

\[
G_{zz}(\omega) = G_{yy}(\omega) - G_{ee}(\omega)
\]  
\(3.34\)

Finally, from \(3.31\) and \(3.34\)

\[
\bar{Z}^2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ G_{yy}(\omega) - G_{ee}(\omega) \} \, d\omega
\]  
\(3.35\)

From \(3.30\) and \(3.35\)

\[
\eta = \frac{\bar{Z}^2(t)}{\bar{Y}^2(t)} = \frac{1}{\int_{-\infty}^{\infty} G_{ee}(\omega) \, d\omega} - \frac{\int_{-\infty}^{\infty} G_{yy}(\omega) \, d\omega}{\int_{-\infty}^{\infty} G_{yy}(\omega) \, d\omega}
\]  
\(3.36\)

Since \(G_{ee}(\omega)\) and \(G_{yy}(\omega)\) are both even functions of \(\omega\), the
range of integration may be contracted to positive \( w \) only.

Alternatively

\[
\eta = \frac{\mathbb{E}^2(t)}{\mathbb{Y}(t)} \quad , \text{by definition}
\]

\[
= 1 - \left\{ \frac{\mathbb{E}^2(t)}{\mathbb{Y}(t)} \right\} \quad , \text{from (3.35) and Parsival's Theorem.}
\]

This quantity is unity if the approximation is exact (\( \mathbb{E}^2(t) = 0 \)), and becomes fractional as the closeness of the approximation deteriorates. The worst possible approximation (when \( \mathbb{E}^2(t) = \mathbb{Y}(t) \)) is signified by the value \( \eta = 0 \).

Values of \( \eta \) which define acceptable approximations can only be decided by experience.

3.7 EXAMPLES OF THE APPLICATION OF A MINIMUM MEAN SQUARE ERROR CONSTRAINT TO TWO PORT DATA FITTING PROBLEMS.

This section is devoted to the discussion of selected examples of the application of the results derived above.

3.7.1. As a first example, suppose the data \( x(t) \) and \( y(t) \) originated in a linear, passive, time-invariant, causal, system, having an impulse response \( f(t) \). Then

\[
y(t) = \int_{-\infty}^{+\infty} f(\tau) \times (t - \tau) d\tau
\]

and \( \int_{-\infty}^{0} f(\tau) \quad \tau \leq 0 \) is the exact expression relating \( x(t) \) and \( y(t) \).
The best, linear, passive, time-invariant approximation to \( f(t) \) is to be sought from

\[
y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau + \epsilon(t) \quad (3.2)
\]

The solution of (3.2) given by

\[
h(\tau) = f(\tau) \quad (3.38)
\]

will fit the data exactly - i.e., if \( h(\tau) \) is chosen to satisfy (3.38), the corresponding value of \( \epsilon(t) \) is given by

\[
\epsilon(t) = 0 \quad (3.39)
\]

Since (3.39) defines the least possible value for equation (3.38) must define a minimum mean square error solution. Further, since the minimum mean square error procedure leads to a unique solution (3.38) represents the solution so obtained.

The same result may be established formally as follows. From (3.37), forming first an expression for the cross correlation function \( \phi_{xy}(\xi) \), and then taking the Fourier Transform to obtain a spectral density relationship, the results

\[
\phi_{xy}(\xi) = \int_{-\infty}^{\infty} f(\omega) \phi_{xx}(\xi-\omega) d\omega
\]

\[
G_{xy}(\omega) = F(\omega) G_{xx}(\omega) \quad (3.40)
\]

are obtained.
From (3.40)

\[
\frac{G_{xy}(\omega)}{G_{xx}(\omega)} = F(\omega) \tag{3.41}
\]

But, from (3.16), the \( h(\tau) \) in (3.2) which yields the minimum mean square error is one having the Fourier Transform \( H(\omega) \) where

\[
\frac{G_{xy}(\omega)}{G_{xx}(\omega)} = H(\omega) \tag{3.16}
\]

Comparing (3.41) and (3.16)

\[
F(\omega) = H(\omega) \tag{3.42}
\]

Taking inverse transforms of (3.42)

\[
h(t) = f(t) \tag{3.43}
\]

From (3.2), (3.37) and (3.43)

\[
y(t) = \int_{-\infty}^{+\infty} f(\xi) x(t - \tau) d\tau + \epsilon(t)
\]

\[= y(t) + \epsilon(t) \]

\[\therefore \quad 0 = \epsilon(t) \tag{3.44}
\]

which establishes the result.

The constraint \( f(\tau) = 0 \) for \( \tau \leq 0 \) places restraints on the positions of the singularities of \( F(\omega) \) in the complex frequency plane. These same restraints are transferred, through the data, to \( H(\omega) \) which, in turn, represents a causal \( h(t) \).
This example demonstrates that, with data collected from linear, passive, time-invariant, causal systems, the data favours a causal solution, and the introduction of a causal constraint into an optimisation procedure is unnecessary.

3.7.2. As a second example, suppose the data originated in a linear active time-invariant system, but this information is withheld, and a linear passive time-invariant system is postulated.

In this instance, the actual expression relating $x(t)$ and $y(t)$ is

$$y(t) = \int_{-\infty}^{\infty} f(\tau) x(t-\tau) d\tau + n(t) \quad (3.45)$$

Forming the cross correlation function $\phi_{xy}(\zeta)$ gives

$$\phi_{xy}(\zeta) = \int_{-\infty}^{\infty} f(\tau) \phi_{xx}(\zeta-\tau) d\tau + \phi_{xx}(\zeta) \quad (3.46)$$

Taking Fourier Transforms of both sides of (3.46) gives

$$G_{xy}(\omega) = F(\omega) G_{xx}(\omega) + G_{x}(\omega)$$

or

$$\frac{G_{xy}(\omega)}{G_{xx}(\omega)} = F(\omega) + \frac{G_{x}(\omega)}{G_{xx}(\omega)} \quad (3.47)$$

From (3.16) and (3.47)

$$H(\omega) = F(\omega) + \frac{G_{x}(\omega)}{G_{xx}(\omega)} \quad (3.48)$$
If $\mathcal{C}_{\text{x,n}}(\omega) = 0$ in (3.48) - i.e. if $x(t)$ and $n(t)$ in (3.45) are uncorrelated, then

$$H(w) = F(w)$$

(3.49)

and by taking inverse transforms of (3.49)

$$h(t) = f(t)$$

(3.50)

The error $e(t)$ in (4.2) is not zero in this case, however, but is given, from (3.2), (3.45) and (3.50) by

$$e(t) = n(t)$$

The existence of $e(t)$, with a consequent fractional value for the success of the approximation $\mathcal{N}$, indicates the fact that the postulated permitted class was incorrect.

If the data were represented by the best linear, passive, time-invariant operator $h(t)$ together with an active source generating $n(t)$ so as to establish an exact fit to the data as in fig. 3:2, this combination happens, in this case, to be a correct description of the system dynamics, although the postulated class was an incorrect one. This result depends entirely upon the orthogonal property of $x(t)$ and $n(t)$.  

![Diagram](image.png)

**FIG. 3:2.**
3.7.3. If, in the previous example, either $x(t)$ or $n(t)$ changed so that $\phi_{x_n}(\xi)$ is no longer zero, a better (in the sense that $\mathcal{H}$ is nearer unity) linear, passive, time-invariant, approximation than $f(t)$ can be found.

If $\phi_{x_n}(\xi) \neq 0$, it follows from the inverse transform of (3.48) that the best linear passive time invariant operator $h(t)$ is given by

$$h(t) = f(t) + f'_1(t)$$  \hspace{1cm} (3.51)

where

$$f'_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{G_{x_n}(\omega)}{G_{x_n}(\omega)} \right\} e^{j\omega t} \, d\omega$$  \hspace{1cm} (3.52)

The best linear, time-invariant, passive approximation $h(t)$ to the linear time-invariant, active system of fig. 3:2(a) is then that of fig. 3:3.

Now the error spectral density $G_{\epsilon e}(\omega)$, associated with the best linear passive time-invariant approximation is given by (3.28)
In the same way that equation (3.9) was derived from (3.5) it can be shown from (3.45) that

\[ G_{yy}(\omega) = G_{nn}(\omega) + F(\omega) F^*(\omega) G_{xx}(\omega) \]
\[ + F(\omega) G_{xx}^*(\omega) + F^*(\omega) G_{nn}(\omega) \]  \hspace{1cm} (3.53)

From (3.47) it follows that

\[ G_{xy}(\omega) = F(\omega) G_{xx}(\omega) + G_{xn}(\omega) \]  \hspace{1cm} (3.54)

and

\[ G_{yx}^*(\omega) = F^*(\omega) G_{xn}(\omega) + G_{xn}^*(\omega) \]  \hspace{1cm} (3.55)

From (3.28), (3.53), (3.54) and (3.55) it follows that

\[ G_{ee}(\omega) = G_{nn}(\omega) - \left\{ G_{xn}(\omega) G_{xn}^*(\omega) \right\} / G_{xx}(\omega) \]  \hspace{1cm} (3.56)

It follows from (3.56) that the mean square error of the best linear passive approximation will be zero if the data has the property that

\[ G_{nn}(\omega) G_{xx}(\omega) = G_{xn}(\omega) G_{xn}^*(\omega) \]  \hspace{1cm} (3.57)

This example illustrates the statement already made that the mean square error may be zero, and hence \( \mathcal{N} \) may be unity even when the system from which the data was collected is not a member of the postulated class. It is apparent that the dynamics of the system of fig.3:3 will not continue to give an exact fit to the data.
collected from the system of fig. 3:2(a) if the
character of either x(t) or n(t) changes. In other
words, the result obtained here is not an exact
description of the system dynamics even though η = 1,
since the description of the system dynamics does not
remain invariant under arbitrary changes in the input data.

Another point of interest arises in this example
if it is supposed that the f(t) of (3.45) is a causal
operator. (This would be the case for any data collected
from physical systems). In case 3.7.2., the best
operator *H(t) from the linear passive time invariant class
is also necessarily causal because *H(t) = f(t) in that
problem.

In this case, however, while the operator f(t) of
fig. 3.2 (a) is assumed a causal one, there is no guarantee
that the operator f(t) of (3.52) is also causal. Causal
and non-causal operators f(t) can both occur depending
upon the singularities of

\[ \frac{G_x(\omega)}{G_{xx}(\omega)} \]

This difficulty does not arise because the
constraint causal has been omitted from the postulated
class linear, passive, time-invariant. It arises instead
because the permitted class is not sufficiently general.
The permitted class should be extended to include linear
active time-invariant operators if the dynamics of systems of this type are to be adequately described. The possibility of applying a minimum mean square error constraint to operators of this class is considered later in this chapter.

3.7.4. As a final example of the application of a minimum mean square error constraint to 2-port problems, consider the closed loop problem of fig. 3.4(a), in which a linear passive time-invariant description of the dynamics of systems A and B are to be found.

Figure 3.4(b) denotes the linear passive time invariant operators describing the dynamics, together with the corresponding errors of the description. From fig. 3.4(b) it follows that

\[ y_1(t) = \xi_1(t) + \int_{-\infty}^{\infty} h_1(\tau) x_1(t - \tau) \, d\tau \]  
(3.58)

and

\[ y_2(t) = \xi_2(t) + \int_{-\infty}^{\infty} h_2(\tau) x_2(t - \tau) \, d\tau \]  
(3.59)

where

\[ x_1(t) = y_1(t) \]  
(3.60)

and

\[ x_2(t) = y_2(t) + n(t) \]  
(3.61)

Assuming that \( n(t) \), \( y_1(t) \) and \( y_2(t) \) are all recorded data, equations (3.58) and (3.59) are of exactly the same form as (3.2) and may be treated as two independent 2-port data
FIG 3:4
fitting problems, the solutions of which are either

\[ H_1(\omega) = \frac{G_{x,y_1}(\omega)}{G_{x,x_i}(\omega)} \]

\[ = \frac{G_{x,y_1}(\omega) + G_{n,y}(\omega)}{G_{y,y_1}(\omega) + G_{n,y_2}(\omega) + G_{n,n}(\omega)} \]  \hspace{1cm} (3.62)

and

\[ H_2(\omega) = \frac{G_{y,y_1}(\omega)}{G_{x,y_1}(\omega)} \]  \hspace{1cm} (3.63)

if power spectral density data can be made available, or the solutions of the integral equations

\[ \phi_{y_1,y_i}(\tau_i) + \phi_{n,y_i}(\tau_i) = \int_{-\infty}^{\tau_i} \phi_i(\tau) \left\{ \phi_{y_1,y_2}(\tau_i-\tau) + \phi_{n,n}(\tau_i-\tau) + \phi_{y_2,n}(\tau_i-\tau) \right\} d\tau \]  \hspace{1cm} (3.64)

and

\[ \phi_{y_1,y_2}(\tau_i) = \int_{-\infty}^{\tau_i} \phi_i(\tau) \phi_{y_1,y_2}(\tau_i-\tau) d\tau \]  \hspace{1cm} (3.65)

for \( \phi_i(\tau) \) and \( \phi_i(\tau) \) if use of correlation data is preferred.

This closed loop problem has reduced to two independent 2-port problems similar to open loop problems because the dynamics of one important element in the loop was assumed known. This element, having 2 inputs \( n(t) \) and \( y_i(t) \), and one output, \( x(t) \), is a 3-port, for which the dynamics is assumed in formulating (3.61).

In general, it is not possible to postulate, a priori, the dynamics of every element in a closed loop problem which is of greater complexity than a 2-port.
If the dynamics of the 3-port is not known, fig. 3:4(a) would need to be replaced by fig. 3:5.

This situation involves a 3-port, and, before a solution is possible, consideration must be given to the problem of the optimisation of n-ports. This is delayed until Chapter 4.

3.8 ON INTRODUCING A CAUSAL CONSTRAINT.

Before proceeding to a study of n-port data fitting problems, the question of the introduction of a causal constraint to 2-port problems will be considered.

It has already been observed that, provided the data originated in a linear, passive, time-invariant, causal system, then it is quite legitimate to drop the causal constraint, and to search among members of the enlarged, linear, passive, time-invariant class. It is in the nature of the data to favour the causal sub-class when making the choice of an optimum operator via a minimum mean
square error optimisation procedure. Of examples 3.7.1., 3.7.2., and 3.7.3., the only linear time-invariant 2-port problem for which a non-causal approximation to data collected from linear time-invariant causal systems is obtained is that of seeking a linear passive time-invariant operator when the data originated in a linear active time-invariant system. The difficulty arises here, not because the constraint 'causal' was dropped, but because the class linear, passive, time-invariant, postulated as the class to which the system belonged, was not sufficiently general. Enlarging the permitted class to include linear, active, time-invariant, systems (of which the class linear, passive, time-invariant becomes a sub-class) is considered the correct approach to this problem, and emphasises again the need to continually enlarge the class of permitted operators in data fitting problems, so as to encompass data of more diverse origin. The introduction of additional constraints, such as a 'causal' one into the definition of the permitted class is to contract the permitted class, the very opposite of what is required.

However, as was mentioned in section 1.6, the need to solve a rather different problem does arise for which it is necessary to contract the permitted class of operator in this way. The problem is that of selecting a best
approximation to the dynamics of a given system from a class of system operators which does not include that to which the system producing the data actually belonged. When attempting to determine the system dynamics from operating data, the method adopted should be to continually expand the permitted class of operator with the object of finding a description of the dynamics which involves a mean square error of zero for any \( x(t) \).

In the problem of finding an approximation to the dynamics of a given system from among members of a specified class of system, one is not interested in finding a precise description of the system dynamics. Interest is confined, from the outset, to a particular class of operator irrespective of whether the data belongs to dynamics of this class or not. This is the situation that arises in such problems as:

(a) finding a linear passive, time-invariant, causal, approximation to the dynamics of linear active systems (this is not to be confused with the problem of finding the description of the system dynamics which involves a search among linear, active, time-invariant, operators);

(b) the linearisation of non-linear problems, (which normally requires the determination of a linear, passive, time-invariant, causal approximation to non-linear system dynamics.)
time-invariant, may result in the selection of an operator from the causal sub-class. In such problems the need for causality adds no new difficulties.

Suppose, however, that in a particular problem a selection from the linear, passive, time-invariant class has yielded a non-causal operator as that one which minimises $\overline{\varepsilon^2(t)}$, while the conditions of the problem require a best causal approximation. The system of fig. 3:1(b) is thus supposed to have been developed for the given $x(t)$ and $y(t)$, and has the property that $h(\tau)$ is not zero for negative values of $\tau$ (the property of a non-causal approximation). It is possible to proceed from this non-causal operator to the desired causal one. The method is indicated by fig. 3:6, in which

\[ \epsilon(t) = \text{error of the best linear passive time invariant operator.} \]

\[ z(t) = \text{output of the best linear passive time invariant operator } h(t) \text{ for an input } x(t). \]

\[ \epsilon_c(t) = \text{error in approximating the data } x(t) \text{ and } z(t) \text{ by a linear, passive, time-invariant, causal approximation } h_c(t). \]
From fig. 3.6

$$z(t) = e_i(t) + \int_{-\infty}^{+\infty} h_c(\tau) x(t-\tau) d\tau$$

(3.66)

where

$$h_c(\tau) = 0 \text{ for } \tau < 0$$

Also

$$y(t) = e(t) + e_i(t) + \int_{-\infty}^{+\infty} h_c(\tau) x(t-\tau) d\tau$$

(3.67)

The required operator $h_c(\tau)$ is that one which minimises

$$\left[ e(t) + e_i(t) \right]^2.$$ But

$$\left[ e(t) + e_i(t) \right]^2 = e^2(t) + 2e(t)e_i(t) + e_i^2(t)$$

(3.68)

But, from (3.66)

$$e_i(t) = z(t) - \int_{-\infty}^{+\infty} h_c(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} \left[ h_c(\tau) - h_c(\tau) \right] x(t-\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} \left[ h_c(\tau) - h_c(\tau) \right] \phi_{e}(\tau) d\tau$$

(3.69)

It can be shown (Woodrow 1959), that

$$\phi_{e}(\tau) = 0$$

(3.23)

for the linear, passive, time-invariant, approximation, and hence, from (3.68), (3.69) and (3.23), it follows that

$$\left[ e(t) + e_i(t) \right]^2 = e^2(t) + e_i^2(t)$$

(3.70)
Two approaches to the optimisation problem are now possible.

(a) \( z(t) \) may be calculated from the linear, passive, time-invariant, approximation, i.e.

\[
z(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau
\]

where

\[
h(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \frac{G_{xy}(\omega)}{G_{xx}(\omega)} \right\} e^{j\omega\tau} d\omega
\]

With \( z(t) \) and \( x(t) \) as available data, \( h_c(\tau) \) may be selected to minimise \( \epsilon^2(\tau) \) in (3.66). It follows from (3.70) that this also minimises the total mean square error \( \left[ \epsilon(t) + \epsilon_c(t) \right]^2 \), and is thus the required \( h_c(\tau) \).

(b) From equation (3.67) it is possible, as an alternative, to select \( h_c(\tau) \) to minimise \( \left[ \epsilon(t) + \epsilon_c(t) \right]^2 \) directly. This method does not require an appeal to a best linear, passive, time-invariant approximation, but confines attention, from the outset, to the causal subclass.

In 2-port problems, it is probable that method (b) is the more direct. The first method has the advantage that it makes it clear the increase in the error, resulting from the addition of a causal constraint to the problem. Method (a) may also find applications when introducing causal constraints into n-port problems (see Chapter 4) where it will be considered again.
In either case, the minimisation procedure is the same, and follows well established techniques (Wiener, 1949).

3.9 EXTENDING THE PERMITTED CLASS

In the last section it was suggested that the approach to the difficulties of example 3.7.3. should be to enlarge the permitted class to include linear active time invariant operators rather than to contract to exclude all operators except linear, passive, time-invariant, causal ones. To enlarge the permitted class is to recognize the existence, in the linear active time-invariant class, of an operator which can reduce the mean square error of the approximation to zero for every input \( x(t) \). If it can be established that an optimisation process, employing the concept of a minimum mean square error as the optimising criterion, leads to a unique solution for this class of operator, then it follows that this operator is the one obtained by the procedure. As in passive data fitting problems, it may be stated here that, if the data originated in a linear, active, time-invariant, causal system – i.e. in a particular sub-class of the permitted class, then the data will require that the optimisation procedure shall favour that sub-class.

Consider the choice of a best linear, active, time-invariant operator for the system of fig. 3:7.
The characteristic equation of the linear, active, time-invariant class is

\[ z(t) = n(t) + \int_{-\infty}^{t} h(\tau) x(t-\tau) d\tau \]  \hspace{1cm} (3.71)

where \( n(t) \) = equivalent disturbance referred to output terminals = value of \( z(t) \) when \( x(t) = 0 \)

\( h(\tau) \) = the time invariant operator relating \( x(t) \) to \( z(t) \).

From fig. 3:7 and (3.71) it follows that

\[ \varepsilon(t) = y(t) - \left[ n(t) + \int_{-\infty}^{t} h(\tau) x(t-\tau) d\tau \right] \]  \hspace{1cm} (3.72)

Study of (3.72) reveals that an optimisation procedure in which \( h(t) \) and \( n(t) \) may both be selected arbitrarily to minimise \( \varepsilon(t)^2 \) can lead to no useful result. Thus, if \( h(t) \) is arbitrarily selected, and the integral determined, a value of \( n(t) \) could then be chosen, in
association with the given h(t), which reduces the right hand side of (3.72) to zero. An infinity of combinations of h(t) and n(t) thus exist which will minimise $\epsilon^2(t)$. It is apparent that the direct application of a minimum mean square error constraint of this type to (3.72) yields no useful result.

However, if some additional constraint describing certain statistical properties of n(t) can be introduced into the problem, useful results are obtained.

To proceed with the optimisation, form $\Phi_\epsilon(\epsilon)$ from (3.72), and take Fourier Transforms of the resulting equation to obtain an expression relating the spectral density functions of the data to the frequency response function H(ω) of the system. The steps follow those of section 3.5 exactly, except that the function y(t) of that section is now replaced by [ y(t) - n(t) ]. This requires the following modifications to (3.9):

(a) $G_{yy}$ is replaced by $(G_{yy} + G_{nn} - G_{ny} - G_{yn})$

(b) $G_{xy}$ is replaced by $(G_{xy} - G_{xn})$

giving

$$G_{\epsilon\epsilon} = (G_{yy} + G_{nn} - G_{ny} - G_{yn} - \frac{(G_{xy} - G_{xn})(G_{yx} - G_{nx})}{G_{xx}}) + \frac{G_{xy} - G_{xn}}{G_{xx}}(H - \frac{G_{xy}}{G_{xx}})(H^* - \frac{G_{xy}^*}{G_{xx}})$$

(3.73)
Since \( \bar{\varepsilon}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{\varepsilon\varepsilon}(\omega) d\omega \), the value of \( H \) which makes \( \bar{\varepsilon}^2 \) least is

\[
H = \frac{G_{xy} - G_{xn}}{G_{xx}} \tag{3.74}
\]

In order to solve (3.74), it is necessary that an estimate of \( G_{xn}(\omega) \) be available, in addition to the quantities \( G_{xy} \) and \( G_{xx} \) which are also required by the linear passive, time-invariant data fitting problem.

Assuming an estimate of \( G_{xn} \) can be made available, and hence a best operator \( H \) derived from (3.74), it can easily be demonstrated that the appropriate choice for \( n(t) \) is \( n_i(t) \) where

\[
n_i(t) = y(t) - \int_{-\infty}^{t+\infty} h(\tau) x(t-\tau) d\tau \tag{3.75}
\]

because

(a) this choice has the required spectral density function \( G_{xn} \), and

(b) this choice reduces \( \bar{\varepsilon}^2 \) to zero. To show these results, form \( G_{xn}(t_i) \) from (3.75) and take Fourier Transforms to give

\[
G_{xn} = G_{xy} - HG_{xx} \tag{3.76}
\]

\[
= G_{xy} - \frac{G_{xy} - G_{xn}}{G_{xx}} \cdot G_{xx}
\]

\[
= G_{xn}
\]
From (3.72) and (3.75) it follows immediately that this choice of \( n(t) \) reduces to zero, and hence minimises

\[ f(t) \]

If \( G_{xn} \) had been available in examples 3.7.2 and 3.7.3, and the optimisation procedure here outlined applied to those problems, the solution

\[ h(t) = f(t) \]
\[ n(t) = n(t) \]

is obtained. In other words, if linear, active, time-invariant operators are permitted for these examples and an appropriate constraint is imposed defining the cross spectral density function \( G_{xn} \), then a description of the system dynamics results which remains invariant under changes of input data \( x(t) \).

Comparison of (3.16) and (3.74) reveals that the best linear time-invariant passive approximation (3.16) of linear time-invariant active dynamics remains sensibly invariant under changes of data only so long as

\[ G_{xy} > G_{xn} \]

If this inequality is not satisfied, the best linear time-invariant passive approximation

(i) does not remain invariant under changes in the statistical properties of \( x(t) \);
(ii) gives a biased estimate of the system dynamics,
as can be seen from (3.74) and (3.16) by writing (3.74) as

\[ H_{\text{active}} = H_{\text{passive}} - \frac{G_{xn}}{G_{xx}} \]

Repeating the estimation R times

\[ \frac{1}{R} \sum_{r=1}^{R} (H_{\text{active}})_r = \frac{1}{R} \sum_{r=1}^{R} (H_{\text{passive}})_r - \frac{1}{R} \sum_{r=1}^{R} \frac{(G_{xn})_r}{(G_{xx})_r} \]

Now, provided the statistical properties of \( x(t) \) and \( n(t) \) remain invariant throughout the R trials, \( (G_{xn})_r \) and \( (G_{xx})_r \) are sample estimates of the true functions \( G_{xn} \) and \( G_{xx} \).

Let \( (G_{xn})_r = G_{xn} + \varepsilon_{rn} \)

and \( (G_{xx})_r = G_{xx} + \varepsilon_{rs} \)

where \( \varepsilon_{rn} \) and \( \varepsilon_{rs} \) are the errors in the sample estimates of \( G_{xn} \) and \( G_{xx} \) derived from the \( r^{th} \) sample of data.

Then

\[ \overline{H}_{\text{active}} = \overline{H}_{\text{passive}} - \frac{G_{xn}}{G_{xx}} \left[ \frac{1}{R} \sum_{r=1}^{R} 1 + \frac{\varepsilon_{rn}}{G_{xn}} - \frac{\varepsilon_{rs}}{G_{xx}} \right] \]

where \( \overline{H}_{\text{active}} \) and \( \overline{H}_{\text{passive}} \) are the active and passive approximations averaged over a large number, \( R \), of trials. Thus the best passive approximation is a biased estimate of the dynamics of active systems if the signal and noise are correlated. The bias is negligible only so long as \( G_{xy} \) is large compared with \( G_{xn} \).
CHAPTER FOUR

A MINIMUM MEAN SQUARE ERROR CONSTRAINT FOR n-PORT DATA FITTING

4.1 INTRODUCTION.

The techniques developed in the last chapter are extended in this chapter to include n-port data fitting problems where n is greater than two.

It has already been observed (example 3.7.4. of the previous chapter) that closed loop problems always include, in their description, at least one n-port having n > 2. But it is not only in closed loop problems that such situations are encountered. In any situation in which:

(a) (n-1) output responses are of interest, but each of these is a function of the same input variable,

(b) a single output response is of interest, but this is influenced by (n-1) different causes (input variables),

(c) (n-m) different output variables are of interest, and each of these is influenced by m different input variables where n > m > 1, and n > 2, the methods of Chapter three cannot be applied directly.

The three situations described above are arranged in increasing order of complexity, and they will be studied
in that order.

Situation (a) is described mathematically by the general description

\[ y_r(t) = F_r[ x, t ] \quad r = 1, 2, \ldots, (n-1) \] (4.1)

Each of the output variables \( y_r(t) \) is derived by some functional operator, \( F_r \), operating upon the single input variable \( x(t) \). The functional operators \( F_r \) are the mathematical description of the system dynamics which this investigation seeks.

Situation (b) above is described mathematically by the general description

\[ y(t) = F[ x_1, x_2, \ldots, x_{n-1}t ] \] (4.2)

Again the investigation seeks the operator \( F \) which relates the input variables \( x_1(t), x_2(t), \ldots, x_{n-1}(t) \) to the output variable \( y(t) \).

The most general of the three, that of (c) above, is described by

\[ y_r(t) = F_r[ x_1, x_2, \ldots, x_n, t ] \quad r = 1, 2, \ldots, (n-m) \] (4.3)

Situations (a) and (b) may be considered special cases of (c) for which (a) \( m = 1 \) and (b) \( m = (n-1) \), respectively. Likewise, the 2-port of the previous chapter is a special case of (4.3) for which \( n = 2 \) and \( m = 1 \).
While recognizing that (4.3) represents the most general problem, it is proposed to build up to the general solution via a consideration of the special cases of (a) and (b) above. Anticipating the results of this treatment, the problems of Type (a) may be reduced to \((n-1)\) separate 2-port problems for which the previous results may be directly applied. Type (b) problems cannot be reduced to a number of separate 2-port problems, and hence it is for these that an extension of the methods of the previous chapter are required. The Type (c) n-port can be reduced to \((n-m)\) separate problems of Type (b) in the same way that problems of Type (a) are reduced to a number of separate 2-port problems.

As in the 2-port case, it is to be assumed initially that the data originated in a linear, passive, time-invariant system, and the functional operators \(F_r\) are restricted, in the first instance, to operators of this class. Attempts are made at the end of the chapter to generalise the permitted class of operator \(F_r\), so as to include data originating in classes originally excluded. The introduction of a causal constraint when a linear, passive, time-invariant, approximation to system dynamics is required is also discussed.

The best operator of the linear, passive, time-invariant, class is defined, as before, as that one which
minimises the mean square error. No attempt is made in this chapter to include any additional constraint which may be placed upon the problem by limitations of the recorded data. Consideration of these additional complications is delayed until the next chapter. As in chapter three, the methods here assume that correlation and spectral density estimates can be made available.

4.2 THE TYPE (a) n-PORT

Consider, first of all, the problem posed by (a) above and described by equation (4.1). This equation defines a set of (n-1) equations, each one of which defines one functional operator $F_r$. This situation is represented by the flow diagram of fig. 4:1.

![Flow diagram of a type (a) n-port](image)

**FIG. 4:1. A TYPE (a) n-PORT.**

The restriction of $F_r$ to linear, passive, time-
invariant operators imposes the condition on (4.1) that

\[ y_i(t) = \int_{-\infty}^{t} h_i(\tau) \cdot x(t-\tau) \, d\tau + \epsilon_i(t) \]  

(4.4)

where \( \epsilon_i(t) \) is the error arising as a result of approximating \( F_r \) by the linear, passive, time-invariant operator \( h_i(t) \). Rearranging (4.4) gives

\[ \epsilon_i(t) = y_i(t) - \int_{-\infty}^{t} h_i(\tau) \cdot x(t-\tau) \, d\tau \]  

(4.5)

The 'best' operator of the permitted class has already been defined as that one which minimises the mean square error \( \overline{\epsilon^2_i(t)} \) where

\[ \overline{\epsilon^2_i(t)} = \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} [\epsilon_i(t)]^2 \, dt \]

The value of \( \overline{\epsilon^2_i} \) can be reduced to zero if \( y_i(t) \) and \( x(t) \) are actually related by a functional operator \( F_r \) which is linear passive time-invariant. Provided the optimisation procedure leads to a unique solution of the minimisation problem, and the data belongs to the permitted class (in this case linear, passive, time-invariant), it follows that this optimisation leads to an exact description of the system dynamics.

The flow diagram of fig. 4.2 represents the situation of equations (4.4) and (4.5).
Since each of the equations (4.5) for \( r = 1, 2, \ldots, (n-1) \) is of exactly the same form as equation (3.2), and since the operators \( h_r(t) \) of (4.5) are to be chosen to minimise \( \bar{\xi}_r^2(t) \), in the same way as \( h(t) \) was chosen in (3.2), it follows that the 2-port optimisation may be applied to (4.5). By equation (3.18), the required choice of operators for the Type (a) \( n \)-port is given by

\[
H_r(\omega) = \frac{G_{xy}(\omega)}{G_{xx}(\omega)}
\]

or

\[
h_r(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_r(\omega) e^{j\omega t} d\omega
\]

This demonstrates the result, already stated, that Type (a) \( n \)-port data fitting problems reduce to \( (n-1) \) separate 2-port data fitting problems.
4.3 THE TYPE (b) n-PORT.

Turning now to the type of problem posed by equation (4.2), the flow diagram of fig. 4:3 corresponds to this situation.

\[ f = \text{F}_{x_1} + \text{F}_{x_2} + \ldots + \text{F}_{x_n} \]

**FIG. 4:3. A GENERAL TYPE (b) n-PORT**

Imposing a linear, time-invariant, passive constraint on the operator \( F \) requires that

\[ \text{F}\{ x_1(t), x_2(t), \ldots, x_n(t) \} \]

\[ = \text{F}\{ x_1 \} + \text{F}_2\{ x_2 \} + \ldots + \text{F}_{n-1}\{ x_{n-1} \} \]  \hspace{1cm} (4.7)

The flow diagram of fig. 4:4 corresponds to the situation for linear Type (b) n-ports, where \( F_1, F_2, \ldots, F_{n-1} \) are linear operators.
Approximating the linear operator $F_r \{ x_r \}$ in (4.7) by the linear, passive, time-invariant, operator having an impulse response $h_r(t)$ gives

$$F_r \{ x_r \} = \int_{-\infty}^{\infty} h_r(\tau) x_r(t-\tau) d\tau + \epsilon_i(t) \quad (4.8)$$

where $\epsilon_i(t)$ is the error of the approximation.

Introducing (4.7) and (4.8) into (4.2) gives

$$y(t) = \sum_{r=1}^{n-1} \left[ \int_{-\infty}^{+\infty} h_r(\tau) x_r(t-\tau) d\tau + \epsilon_i(t) \right]$$

$$= \epsilon(t) + \sum_{r=1}^{n-1} \int_{-\infty}^{+\infty} h_r(\tau) x_r(t-\tau) d\tau \quad (4.9)$$

where

$$\epsilon(t) = \sum_{r=1}^{n-1} \epsilon_r(t)$$

is the total error of the approximation.
The flow diagram corresponding to (4.9) is that of fig. 4:5.

FIG. 4:5. A LINEAR, PASSIVE, TIME-ININVARIANT, APPROXIMATION TO A TYPE (b) n-PORT PROBLEM.

The problem here is to simultaneously select the (n-1) operators $h_x(t)$ to minimise $\overline{\epsilon^2(t)}$ in (4.9). This is quite different from the Type (a) optimisation which required (n-1) consecutive selections of (n-1) operators. No amount of manipulation can reduce this problem to one of Type (a), because the functional dependence involves $m$ independent variables where $m > 1$.

The consideration of the actual optimisation procedure for (4.9) will be delayed until after a consideration of the Type (c) n-port data fitting problem.

4.4 THE TYPE (c) n-PORT.

As with equation (4.1) and Type (a) n-ports, so in this case equation (4.3) defines a number of separate
n-port data fitting problems. Each of the Type (c) problems is one of Type (b) as is indicated by the flow diagram of fig. 4:6.

FIG. 4:6. A GENERAL TYPE (c) n-PORT.

If a linear, passive, time-invariant constraint is imposed upon the functional operators \( F_1, F_2, \ldots, F_{n-m} \), then each of the \( F_r \) of fig. 4:6 may be reduced, in turn, to the form of fig. 4:5.

It is to be concluded, therefore, that provided an optimisation procedure can be developed from equation (4.9) for the Type (b) n-port, then the more general Type (c) problem may be solved by \((n-m)\) applications.
of that result to each of the \((n-m)\) operators \(F_r\) of fig. 4.6. The Type (b) \(n\)-port stands in the same relation to the Type (c) \(n\)-port that the 2-port stands in relation to the Type (a) \(n\)-port.

4.5 THE OPTIMISATION OF TYPE (b) \(n\)-PORTS.

4.5.1. Having established that the general \(n\)-port requires \((n-m)\) successive applications of a Type (b) optimisation procedure, it remains to select the \(m (=n-1)\) operators \(h_r(t)\) in the equation

\[
\epsilon(t) = \gamma(t) - \sum_{r=1}^{m} \int_{-\infty}^{+\infty} h_r(\xi) x_r(t-\xi) d\xi
\]

which collectively minimise \(\bar{\epsilon}^2(t)\), where

\[
\bar{\epsilon}^2(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\epsilon(t)]^2 dt
\]

From (4.10)

\[
\bar{\epsilon}^2(t) = \gamma^2(t) - 2 \sum_{r=1}^{m} \int_{-\infty}^{+\infty} h_r(\xi) \gamma(t) x_r(t-\xi) d\xi + \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_r(\xi) h_s(\xi') x_r(t-\xi) x_s(t-\xi') d\xi d\xi'
\]

\[
\bar{\epsilon}^2(t) = \gamma^2(t) - 2 \sum_{r=1}^{m} \int_{-\infty}^{+\infty} h_r(\xi) \phi_{x,y}(t-\xi) d\xi + \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_r(\xi) h_s(\xi') \phi_{x,y}(t-\xi) \phi_{x,y}(t-\xi') d\xi d\xi' (4.11)
\]

If a set of functions \(h_r(t)\) can be chosen which make \(\bar{\epsilon}^2(t)\) a minimum, changing from \(h_r(t)\) to

\[
\{ h_r(t) + k_r p_r(t) \}
\]

where
\[ p_r(t) = \text{an arbitrary function of } t, \]
\[ k_r = \text{an arbitrary small scalar multiplier}, \]

increases \( \varepsilon_2^2(t) \) (to \( \varepsilon_2^2(t) \) say). Then

\[
\varepsilon_2^2(t) = y^2(t) - 2 \sum_{r=1}^{m} \int_{-\infty}^{+\infty} \left[ h_x(\tau) + k_r P_r(\tau) \right] \varphi_{x,y}(\tau) \, d\tau
\]

\[
+ \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ h_x(\tau) + k_r P_r(\tau) \right] \left[ h_s(\tau) + k_s P_s(\tau) \right] \varphi_{x,x_s}(\tau,\tau_s) \, d\tau \, d\tau_s
\]

(4.12)

\[
\frac{\partial \varepsilon_2^2}{\partial k_q} = -2 \int_{-\infty}^{+\infty} P_q(\tau) \varphi_{x,y}(\tau) \, d\tau
\]

\[
+ \sum_{r=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_q(\tau) \left[ h_x(\tau) + k_r P_r(\tau) \right] \varphi_{x,x_r}(\tau,\tau_r) \, d\tau \, d\tau_r
\]

\[
+ \sum_{r=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_q(\tau) \left[ h_s(\tau) + k_s P_s(\tau) \right] \varphi_{x,x_s}(\tau,\tau_s) \, d\tau \, d\tau_s
\]

(4.13)

Now if all the scalar multipliers \( k_r \) are made to tend to zero together, \( \varepsilon_2^2 \) tends to \( \varepsilon_1^2 \), and since, by definition, \( \varepsilon_1^2 \) is a minimum \( \frac{\partial \varepsilon_1^2}{\partial k_q} = 0 \) for

\( q = 1, 2, \ldots, m \). Therefore

\[
0 = 2 \int_{-\infty}^{+\infty} P_q(\tau) \varphi_{x,y}(\tau) \, d\tau
\]

\[
- \sum_{r=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_q(\tau) h_x(\tau) \varphi_{x,x_r}(\tau,\tau_r) \, d\tau \, d\tau_r
\]

\[
- \sum_{r=1}^{m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_q(\tau) h_s(\tau) \varphi_{x,x_s}(\tau,\tau_s) \, d\tau \, d\tau_s
\]

(4.14)

Since it is a property of correlation functions that

\[
\varphi_{x,x_r}(\tau,\tau_r) = \varphi_{x,x_r}(\tau-\tau_r)
\]

(4.15)
the last two terms of (4.14) are identical. Hence (4.14) may be written

\[ \int_{-\infty}^{\infty} p(x) \left\{ q^0(x) - \sum_{r=1}^{m} \int_{-\infty}^{\infty} h_r(x) q^0(x-r_x) \, dx \right\} \, dx = 0 \]  

Equation (4.16) is true for \( p(x) \), arbitrary, and hence

\[ q^0(x) = \sum_{r=1}^{m} \int_{-\infty}^{\infty} h_r(x) q^0(x-r_x) \, dx, \]  

(4.17)
is the condition required to ensure that the selected \( h_r(t) \) make \( \overline{\varepsilon^2(t)} \) a stationary value.

Giving \( q \) the values 1, 2, ..., \( m \) in succession yields a set of \( m \) simultaneous integral equations to be solved for the \( m \) operators \( h_r(t) \).

To solve (4.17) take Fourier Transforms of both sides to obtain

\[ G_{xqy}^{(q)}(\omega) = \sum_{r=1}^{m} H_r(\omega) \ G_{xqy}^{(r)}(\omega), \quad q = 1, 2, ..., m. \]  

(4.18)

This step reduces the set of simultaneous integral equations (4.17) to the set of simultaneous linear algebraic equations (4.18).

In the notation of matrix algebra, the set of equations (4.18) are written

\[ G_{xqy} = G_{xqy}^H \]  

(4.19)
where

\[ G_{xy} \] is a column vector \( \{ G_{x_1 y}^{(\omega)}, G_{x_2 y}^{(\omega)}, \ldots, G_{x_m y}^{(\omega)} \} \)

\[ \mathbf{H} \] is a column vector \( \{ H_1^{(\omega)}, H_2^{(\omega)}, \ldots, H_m^{(\omega)} \} \)

and \( G_{xx} \) is the \( m \times m \) Hermitian matrix

\[
\begin{bmatrix}
G_{11}^{(\omega)}, & G_{12}^{(\omega)}, & \ldots, & G_{1m}^{(\omega)} \\
G_{21}^{(\omega)}, & G_{22}^{(\omega)}, & \ldots, & G_{2m}^{(\omega)} \\
\vdots & \vdots & \ddots & \vdots \\
G_{m1}^{(\omega)}, & G_{m2}^{(\omega)}, & \ldots, & G_{mm}^{(\omega)}
\end{bmatrix}
\]

Provided \( G_{xx} \) is non-singular, (4.19) has a solution, and that solution is given uniquely by

\[
\mathbf{H} = G_{xx}^{-1} \cdot G_{xy} \quad (4.20)
\]

The determination of the elements \( H_r^{(\omega)} \) of the vector \( \mathbf{H} \) from (4.20) is particularly simple in the case when the input variables \( x_i(t) \) and \( x_s(t) \) are uncorrelated. Then \( G_{x_i x_s}^{(\omega)} = 0 \) for \( i \neq s \), and the Hermitian matrix \( G_{xx} \) reduces to a diagonal matrix, in which form the inverse is easily obtained.

If \( G_{x_i x_s}^{(\omega)} = 0 \), the elements \( H_r^{(\omega)} \) are given by

\[
H_r^{(\omega)} = \frac{G_{x_i y}^{(\omega)}}{G_{x_i x_r}^{(\omega)}} \]

In general, this property of lack of correlation between the various inputs will not obtain, and the simultaneous
set (4.20) must be solved.

4.5.2. It still remains to establish that the stationary value defined by (4.20) is a true minimum. To this end, derive, from (4.10), the expression

\[ \varphi_{\text{ee}}(\xi) = \varphi_{\text{yy}}(\xi) - \sum_{r=1}^{m} \int_{-\infty}^{+\infty} h_{+}(r) \varphi_{\text{xy}}(\xi + r) \, dr \]

\[ - \sum_{r=1}^{m} \int_{-\infty}^{+\infty} h_{-}(r) \varphi_{\text{yx}}(\xi - r) \, dr \]

\[ + \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{-\infty}^{+\infty} h_{r}(r) h_{s}(s) \varphi_{\text{xx}}(\xi - r + s) \, dr \, ds \]

and take Fourier Transforms of both sides to obtain

\[ G_{\text{ee}}(\omega) = G_{\text{yy}}(\omega) - \sum_{r=1}^{m} \left\{ H_{r}(\omega) G_{\text{xy}}(\omega) + H_{r}(\omega) G_{\text{yx}}^{*}(\omega) \right\} \]

\[ + \sum_{r=1}^{m} \sum_{s=1}^{m} H_{r}(\omega) H_{s}(\omega) G_{\text{xx}}^{*}(\omega) \]

\[ (4.21) \]

In the notation of matrix algebra (4.21) is written

\[ G_{\text{ee}} = G_{\text{yy}} - H' G_{\text{xy}} - G_{\text{yx}} H + H' G_{\text{xx}} H \]

\[ (4.22) \]

where \( G_{\text{xx}}, G_{\text{xy}} \) and \( H \) have the same meaning as in (4.19) and the superscript prime (as in \( H' \) and \( G_{\text{xy}}' \)) denotes 'complex conjugate transpose'. The matrices \( G_{\text{ee}} \) and \( G_{\text{yy}} \) are each \( 1 \times 1 \) matrices, being \( G_{\text{ee}}(\omega) \) and \( G_{\text{yy}}(\omega) \) respectively.
Now from the matrix identity

\[
\begin{align*}
\left[ H' - G_{xy} \left[ G_{xx} \right]^{-1} \right] G_{xx} \left\{ H - \left[ G_{xx} \right]^{-1} G_{xy} \right\} \\
\equiv \quad H' G_{xx} H - G_{xy} \left[ G_{xx} \right]^{-1} G_{xx} H - H' G_{xx} \left[ G_{xx} \right]^{-1} G_{xy} \\
+ \quad G_{xy} \left[ G_{xx} \right]^{-1} G_{xx} \left[ G_{xx} \right]^{-1} G_{xy}
\end{align*}
\]

(4.23)

it follows, for \( G_{xx} \) Hermitian - i.e. for

\[
G_{xx}^\prime = G_{xx}
\]

that

\[
\begin{align*}
\left[ H' - G_{xy} \left[ G_{xx} \right]^{-1} \right] G_{xx} \left\{ H - \left[ G_{xx} \right]^{-1} G_{xy} \right\} \\
\equiv \quad H' G_{xx} H - G_{xy} H - H' G_{xy} + G_{xy} \left[ G_{xx} \right]^{-1} G_{xy}
\end{align*}
\]

(4.24)

since \( G_{xx} \left[ G_{xx} \right]^{-1} = \left[ G_{xx} \right]^\prime G_{xx} = I \), the unit matrix.

From (4.22) and (4.24) it follows that

\[
G_{ee} = G_{yy} - G_{xy} \left[ G_{xx} \right]^{-1} G_{xy} \\
+ \left[ H' - G_{xy} \left[ G_{xx} \right]^{-1} \right] G_{xx} \left\{ H - \left[ G_{xx} \right]^{-1} G_{xy} \right\}
\]

(4.25)

Denoting the column vector

\[
\left[ H - \left[ G_{xx} \right]^{-1} G_{xy} \right]
\]

by \( Y \)

(4.26)

equation (4.25) may be written

\[
G_{ee} = G_{yy} - G_{xy} \left[ G_{xx} \right]^{-1} G_{xy} + Y G_{xx} Y
\]

(4.27)
The mean square error \( \bar{e}^2(t) \) is given by

\[
\bar{e}^2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{\xi\xi}(\omega) \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ \begin{array}{c} G_{\xi} \\ G_{\eta} \end{array} \right]_{\text{opt}} \, d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{xX} \, Y \, d\omega \quad (4.28)
\]

where

\[
\left[ G_{\xi\eta} \right]_{\text{opt}} = G_{\xi\xi} - G_{\xi\eta} \left[ G_{xx} \right]^{-1} G_{\eta\eta}
\]

(4.29)

The stationary value defined by the \( m \) simultaneous integral equations (4.17), the solution of which is given by (4.20) provided \( G_{XX} \) is non-singular, defines a minimum if \( G_{XX} \) is such that no vector \( Y \) can be found which makes the term

\[
\int_{-\infty}^{+\infty} Y' G_{XX} Y \, d\omega
\]

in equation (4.28) negative.

4.5.3. Before attempting to establish the constraints, if any, that this requirement imposes upon the data in the general case, it is of interest to consider the special case which arises when the inputs are all uncorrelated so that \( G_{xX} = 0 \) if \( t \neq s \). In this case, \( G_{XX} \) becomes the diagonal matrix

\[
\begin{bmatrix}
G_{xX_1} & & \\
& G_{xX_1} & \\
& & \ddots
\end{bmatrix}
\]

\[
\begin{bmatrix}
G_{xX_1} & \cdots & G_{xX_1} \\
\vdots & \ddots & \vdots \\
G_{xX_1} & \cdots & G_{xX_m}
\end{bmatrix}
\]
If \( \underline{\gamma} \) = \{ \gamma_1, \gamma_2, \ldots, \gamma_m \}

\[
\underline{\gamma}' G_{\underline{x}, \underline{\gamma}} \underline{\gamma} = |\gamma_1|^2 G_{x_1 x_1} + |\gamma_2|^2 G_{x_2 x_2} + \ldots + |\gamma_m|^2 G_{x_m x_m}
\]  \hspace{1cm} (4.30)

From (4.30) it follows that

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{\gamma}' G_{\underline{\gamma}, \underline{\gamma}} \underline{\gamma} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\gamma_1|^2 G_{x_1 x_1} + \ldots + |\gamma_m|^2 G_{x_m x_m} d\omega > |\gamma|_{\min}^2 x_1^2 + \ldots + |\gamma|_{\min}^2 x_m^2
\]  \hspace{1cm} (4.31)

The right hand side of (4.31) is always positive or zero, and \( \varepsilon^2(t) \) in (4.28) is least when

\[
|\gamma_1|^2 = |\gamma_2|^2 = \ldots = |\gamma_m|^2 = 0
\]

which, from (4.26) is the case when

\[
H = \left[ G_{\underline{xx}} \right]^{-1} G_{\underline{x}, \underline{\gamma}}
\]  \hspace{1cm} (4.20)

It may therefore be concluded that, if the input variables \( x_r(t) \) and \( x_s(t) \) are uncorrelated, then (4.20) defines a minimum mean square error solution for all values of \( m \). The question of \( G_{xx} \) being non-singular does not arise so long as each input \( x_r(t) \) yields a non zero spectral density function \( G_{x_r x_r}(\omega) \).

4.5.4. The general result may be developed by arguments similar to those of the orthogonal case of 4.5.3. above.
It may be shown (Ferrar, 1951, page 147, Theorem 36) to be the result of a complete eigenvalue analysis of a Hermitian matrix that the matrix $G_{xx}$ may be expressed as the product of three matrices — i.e.

$$G_{xx} = \Lambda' \Lambda \Lambda$$

where $\Lambda$ is a diagonal matrix having the $m$ real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $G_{xx}$ as diagonal elements, while the columns of $\Lambda$ are the corresponding eigenvectors.

Hence

$$Y' G_{xx} Y = Y' \Lambda' \Lambda \Lambda Y$$

$$= Z' \Lambda Z$$

where $Z$ is the column vector defined by

$$A \cdot Y = Z \quad (4.32)$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} Y' G_{xx} Y d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z' \Lambda Z d\omega$$

$$= \sum_{r=1}^{m} \frac{1}{2\pi} \int_{\rho}^{+\infty} |Z_r|^2 \lambda_r(\omega) d\omega \quad (4.33)$$

The right hand side of (4.33) is certainly positive if all the eigenvalues $\lambda_r$ are positive. Further, if any one of the $\lambda_r$ is negative for any finite range of values of $\omega$, then a vector $Y$ can be constructed which makes the left hand side of (4.33) negative. The stationary value of $H$ defined by (4.20) is not a minimum in this case.
It may be concluded, therefore, that in those problems in which cross-correlation of some of the input variables is observed, it is necessary for all \( m \) eigenvalues of the Hermitian matrix \( G_{xx} \) to be real and positive if (4.20) is to define a minimum. Such matrices are termed positive definite, Hermitian (see Aitken, p.37).

It is felt probable, although no proof has yet been established, that any non-singular \( G_{xx} \) matrix of spectral density functions is of this form. The Hermitian property of \( G_{xx} \) is sufficient to ensure that all the eigenvalues are real (Aitken, p.73), but not all Hermitian matrices are positive definite, so that this property, if observed generally, must be a consequence of the method of derivation of the elements \( G_{xrxs} \) of \( G_{xx} \).

4.6 THE APPLICATION OF OPTIMUM n-PORT THEORY TO A PASSIVE CLOSED LOOP DATA FITTING PROBLEM.

4.6.1. As an example of the application of these concepts, consider again the closed loop problem discussed in section 3.7.4. In that section it was assumed, through the use of equation (3.61), that the dynamics of the 3-port element (a summing device) was known. This assumption reduced the problem of the determination of the dynamics of the components of the closed loop of fig. 3:4 to two separate 2-port optimisation problems. It was pointed out
(see fig. 3:5), however, that this procedure was not always possible.

Consider again the problem posed by fig. 3:5, redrawn for convenience in fig. 4:7. (A slight change of notation has been introduced to bring the notation here into line with that used in the development of the n-port data fitting theory earlier in this chapter).

![Diagram](image)

**FIG. 4:7**

Using a linear, time-invariant, passive description of the 3-port as in fig. 4:5, the linear, time-invariant, passive approximation of fig. 4:8, to the dynamics of the closed loop of fig. 4:7, results.
The determination of the best linear, time-invariant, passive, approximation to the dynamics of system B (i.e. the operator \( h(t) \)) which minimises \( \bar{\varepsilon}^2 \) is a straightforward 2-port problem in this example. The recorded input variable for this element is \( y(t) \), while \( x_2(t) \) is the recorded output variable.

To determine the best linear, time-invariant, passive approximation to the dynamics of system A requires the choice of the two operators \( h_1(t) \) and \( h_2(t) \) which together minimise \( \bar{\varepsilon}^2 \) when

\[
\varepsilon(t) = y(t) - \sum_{\tau_1}^{\tau_2} \int_{-\infty}^{\infty} h_{\tau}(\tau) x_{\tau}(\tau - \tau) d\tau
\]  

(4.34)
From (4.20) it follows that this choice is given by

\[
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} = 
\begin{bmatrix}
G_{\dot{x}_1 \dot{x}_1} & G_{\dot{x}_1 \dot{x}_2} \\
G_{\dot{x}_2 \dot{x}_1} & G_{\dot{x}_2 \dot{x}_2}
\end{bmatrix}
\begin{bmatrix}
G_{x_1 y} \\
G_{x_2 y}
\end{bmatrix}
\]

provided

\[
\begin{bmatrix}
G_{\dot{x}_1 \dot{x}_1} & G_{\dot{x}_1 \dot{x}_2} \\
G_{\dot{x}_2 \dot{x}_1} & G_{\dot{x}_2 \dot{x}_2}
\end{bmatrix}
\]

is non-singular.

The question of the determination of the singularity of (4.36) is of special interest in the determination of the dynamics of n-ports in closed loops.

4.6.2. Suppose, for example, that the linear time-invariant system of fig. 4.7, in which \(x_1(t), x_2(t)\) and \(y(t)\) originated, was actually a linear, time-invariant, passive, causal one, as in fig. 4.9.

With no information other than the recorded variables \(x_1(t), x_2(t)\) and \(y(t)\), it is required to estimate the dynamics of the system of fig. 4.9.

Since the actual system dynamics belong to a sub-class of the class from which the optimum operators (4.35) were chosen, it is to be expected that the optimisation procedure should lead to an exact description of the dynamics, but this is not the case.
The determination of $h(t)$ the best approximation to $f(t)$ is obtained by a straightforward application of equation (3.16) for 2-port problems, giving

$$H(\omega) = \frac{G_{y2}(\omega)}{G_{yy}(\omega)}$$  \hspace{1cm} (4.37)

From fig. 4.9 and the defining equation (1.17) of linear, time-invariant, passive, causal, 2-port, operators, it follows that

$$x_2(t) = \int_{-\infty}^{+\infty} f(\tau) y(t-\tau) d\tau$$  \hspace{1cm} (4.38)

where $f(\tau) = 0$ for $\tau < 0$
Multiplying (4.38) throughout by \( y(t-t') \), taking the time average of the product to obtain the correlation equation corresponding to (4.37), and finally taking the Fourier Transform of this to obtain the spectral density equation yields

\[
G_{yx}(\omega) = F(\omega) G_{yy}(\omega)
\]  
(4.39)

From (4.37) and (4.39) it follows that

\[
H(\omega) = F(\omega)
\]  
(4.40)

or, taking inverse transforms

\[
h(t) = f(t)
\]

It follows, therefore, that the best linear, time-invariant, passive operator describing the linear, time-invariant, passive, causal, dynamics of system B of fig. 4:9 provides an exact description.

4.6.3. Before attempting to determine an approximation to the dynamics of system A of fig. 4:9, by the solution of (4.35), the data must be tested to ensure that (4.36) is satisfied.

For the given data, it follows from fig. 4:9 that

\[
y(t) = \int_{-\infty}^{\infty} f_1(\tau) x_1(t-\tau) d\tau + \int_{-\infty}^{\infty} f_2(\tau) x_2(t-\tau) d\tau
\]  
(4.41)

where \( f_1(t) = f_2(t) = 0 \) for \( t < 0 \)
From equations (4.38) and (4.41), forming the appropriate spectral density relationships, it follows that

\[ G_{x_2y} = F_1 G_{x_2y} \] \hspace{1cm} (4.42)

\[ G_{x_1x_2} = F_2 G_{x_1y} \] \hspace{1cm} (4.43)

\[ G_{x_1y} = F_1 G_{x_1x_1} + F_2 G_{x_1x_2} \] \hspace{1cm} (4.44)

and

\[ G_{x_2y} = F_1 G_{x_2x_1} + F_2 G_{x_2x_2} \] \hspace{1cm} (4.45)

Using (4.42) and (4.43) to eliminate \( G_{x_1y} \) and \( G_{x_2y} \) from (4.44) and (4.45) gives

\[ G_{x_1x_2} (1 - FF_2) = FF_1 G_{x_1x_1} \] \hspace{1cm} (4.46)

\[ G_{x_2x_2} (1 - FF_2) = FF_1 G_{x_2x_1} \] \hspace{1cm} (4.47)

From (4.46) and (4.47) it follows that the available data has the property that

\[ G_{x_1x_1} G_{x_2x_2} = G_{x_1x_2} G_{x_2x_1} \] \hspace{1cm} (4.48)

Equation (4.48) is the condition that the matrix (4.36) shall be singular. The optimisation procedure fails to define a unique pair of operators \( H_1(\omega) \) and \( H_2(\omega) \) for this data. The method leaves the question of the best approximation to the dynamics of system A of fig. 4.9 unanswered.
One is compelled to conclude that normal operating data collected from linear, time-invariant, passive, closed loops are unsuitable for the determination of the dynamics of the elements of the loop.

The explanation for this is to be found in the linear dependence of the input variable $x(t)$ on $x(t)$. It follows from (4.43) and (4.46) that

$$G_{x,y} = \frac{E_1}{1 - FF_2} G_{x,x}, \quad (4.49)$$

and

$$G_{x,x_2} = \frac{FF_1}{1 - FF_2} G_{x,x}, \quad (4.50)$$

Denoting the overall frequency response function of the closed loop by $Y$, where

$$Y \equiv Y_a + (Y - Y_a) = \frac{E_1}{1 - FF_2}, \quad \text{and}.$$

$Y_a$ is any arbitrary linear, passive, time-invariant causal transfer function.

Let

$$Y - Y_a = \frac{FF_1}{1 - FF_2} Y_b$$

then

$$Y_b = \frac{1 - FF_2}{FF_1} (Y - Y_a), \quad (4.51)$$

which is also a linear, passive, time-invariant, transfer function.

The overall frequency response $Y$ may thus be produced by any arrangement as Fig. 4:10(a) to which the above calculation applies.
Any pair of frequency response functions \( Y_a \) and \( Y_b \) satisfying (4.51), and connected as in fig.4.10(b) will exactly fit the data \( x_1(t), x_2(t), \) and \( y(t) \) collected from system A of fig.4.9. Since \( Y_a \) may be selected arbitrarily, there exists an infinity of 'best' operators, in a minimum mean square error sense, to fit the given data. The optimisation procedure does not favour any one of these, and an underdetermined set of equations results. This indeterminacy can only be removed if the linear dependence of \( x_2(t) \) on \( x_1(t) \) is destroyed, so that at no point in fig. 4:10(a) can the variable \( x_2(t) \) be produced by a linear, time-invariant, passive operation on \( x_1(t) \).
4.7 ACTIVE DISTURBANCES IN CLOSED LOOPS.

4.7.1. Since data from linear, time-invariant, passive, closed loops do not permit a description of the system dynamics, it is necessary to introduce disturbances into the loop to destroy the linear dependence of \( x_2(t) \) on \( x_1(t) \). Suppose a disturbance is associated with the 3-port of fig. 4:9, modifying the linear time-invariant system of fig. 4:7 from which the data was collected to that of fig. 4:11.

The presence of \( n(t) \) modifies (4.41), but not (4.38). Thus

\[
\begin{align*}
\dot{x}_2(t) &= \int_{-\infty}^{+\infty} f(\tau) \, y(t-\tau) \, d\tau \\
\text{with } f(t) &= \begin{cases} 0 & \text{for } t < 0 \\ \end{cases} \\
\text{(4.38)}
\end{align*}
\]
and

\[ y(t) = n(t) + \int_{-\infty}^{+\infty} f_1(\tau) x_1(t-\tau) d\tau \]

\[ + \int_{-\infty}^{+\infty} f_2(\tau) x_2(t-\tau) d\tau \]

with

\[ f_1(t) = f_2(t) = 0 \quad \text{for} \quad t < 0 \] \hspace{1cm} \text{(4.52)}

If a solution of (4.35) is attempted for this data, one is seeking to approximate the dynamics of a linear, time-invariant, active 3-port by a linear, time-invariant, passive approximation. The result of such a procedure cannot provide an exact description of the system dynamics, even when an exact fit to the given data can be obtained.

Before attempting to solve (4.35) it is again necessary to test the data for the validity of condition (4.36). From (4.38), (4.52) and arguments similar to those used in section 4.6.3. to derive equations (4.46) and (4.47), it follows that

\[ G_{x_1 x_2} (1 - FF_2) = FF_1 G_{x_1 x_1} + FG_{x_1 n} \] \hspace{1cm} \text{(4.53)}

and

\[ G_{x_2 x_1} (1 - FF_2) = FF_1 G_{x_2 x_1} + FG_{x_2 n} \] \hspace{1cm} \text{(4.54)}

From (4.53) and (4.54)

\[ F_1 (G_{x_1 x_1} G_{x_2 x_2} - G_{x_1 x_2} G_{x_2 x_1}) = G_{x_1 x_2} G_{x_1 n} - G_{x_1 n} G_{x_2 x_1} \] \hspace{1cm} \text{(4.55)}

and hence (4.36) is satisfied provided

\[ G_{x_1 x_2} G_{x_1 n} - G_{x_1 n} G_{x_2 x_1} \neq 0 \] \hspace{1cm} \text{(4.56)}
From (4.53) and (4.54)
\[ G_{x_2x_2} = \frac{FF_1}{1-FF_2} G_{x_2x_1} G_{x_1n} + \frac{F}{1-FF_2} G_{x_2n} G_{x_1n} \] (4.57)
\[ G_{x_2x_n} = \frac{FF_1}{1-FF_2} G_{x_2x_1} G_{x_1n} + \frac{F}{1-FF_2} G_{x_2n} G_{x_1n} \] (4.58)
\[ G_{x_2x_2} G_{x_2x_n} = G_{x_2x_1} G_{x_2x_1} + G_{x_2x_1} G_{x_2x_1} \]
\[ = \frac{FF_1}{1-FF_2} \left( G_{x_2x_1} G_{x_1n} - G_{x_2x_1} G_{x_2n} \right) \] (4.59)

But, from (4.58) and (4.54)
\[ G_{nx_2} = F G_{ny} \] (4.60)
\[ G_{ny} = G_{nn} + F_1 G_{nx_1} + F_2 G_{nx_2} \] (4.61)

Also, from (4.53)
\[ G_{x_2x_2} = \frac{FF_1}{1-FF_2} G_{x_2x_1} + \frac{F}{1-FF_2} G_{x_2n} \] (4.63)

From (4.55), (4.59), (4.62), (4.63) and the property of spectral density functions that
\[ G_{rs} = G_{sr}^* \]

it follows that
\[ G_{x_2x_2} G_{x_2x_2} - G_{x_2x_1} G_{x_1x_1} = \left| \frac{F}{1-FF_2} \right|^2 \left( G_{x_2x_1} G_{x_1n} - G_{x_2x_1} G_{x_2n} \right) \] (4.64)

From (4.64) it follows that (4.36) is satisfied provided
\[ G_{x_1x_1} G_{nn} - G_{x_1n} G_{nx_1} \neq 0 \] (4.65)
Assuming the disturbance \( n(t) \) satisfies (4.65), with a given \( x_i(t) \), a solution of (4.35) is possible. It remains to determine the best linear time-invariant passive approximation to the dynamics of the active system of fig. 4.11.

From equation (4.35), the best linear, time-invariant passive approximation to the dynamics of system A of fig. 4.11 is defined by the pair of operators \( H_1(\omega) \) and \( H_2(\omega) \) where

\[
H_1(\omega) = \left\{ \frac{G_{x,y} G_{x,x_1} - G_{x,x_1} G_{x,y}}{G_{x,x_1} G_{x,x_2} - G_{x,x_1} G_{x,x_2}} \right\} \text{ (4.66)}
\]

and

\[
H_2(\omega) = \left\{ \frac{G_{x,y} G_{x,x_1} - G_{x,x_1} G_{x,y}}{G_{x,x_1} G_{x,x_2} - G_{x,x_1} G_{x,x_2}} \right\} \text{ (4.67)}
\]

From (4.38), it has already been established that

\[
G_{x_2,x_2} = F G_{x,y} \text{ (4.42)}
\]

\[
G_{x,x_2} = F G_{x,y} \text{ (4.43)}
\]

These equations are still applicable, since the disturbance introduced into the loop in fig. 4.11 left the dynamics of system B unchanged.

From (4.66) and (4.42) and (4.43)

\[
H_1(\omega) = F \left\{ \frac{G_{x,y} G_{x,y} - G_{x,y} G_{x,y}}{G_{x,x_1} G_{x,x_2} - G_{x,x_1} G_{x,x_2}} \right\} \text{ (4.68)}
\]

\[
= F \times 0
\]
From (4.67), (4.42) and (4.43)

\[ H_2(\omega) = \frac{1}{F} \left\{ \frac{G_{xx_1} G_{x_2 x_2} - G_{x_2 x_1} G_{x_1 x_1}}{G_{x_2 x_1} G_{x_2 x_2} - G_{x_1 x_1} G_{x_1 x_2}} \right\} \]

(4.69)

Figure 4.12 gives a comparison of (a) the actual dynamics of the closed loop with (b) the best linear, time-invariant passive approximation.

It can readily be shown that the operators of fig. 4.12(b), when operating upon the input variables \( x_1(t) \) and \( x_2(t) \) produce the output variable \( y(t) \) exactly.
i.e. the error of the approximation is zero. The procedure has produced an exact fit to the data, but the lack of exactness in the description of the dynamics of the system of fig. 4.12(a) requires no comment.

Anticipating the results of section 4.8, it may be stated that the discrepancy arises because the class of operator permitted in the optimisation procedure is not sufficiently general. It excludes operators defining the dynamics of linear, time-invariant, active systems to which the data actually belong.

Two important conclusions are to be drawn from these results, namely:

(a) a search for the best linear time-invariant, passive approximation to the dynamics of an active n-port, in a closed loop data fitting problem, leads, inevitably, to a regenerative description of the system dynamics, in which the input disturbance $x_1(t)$ is isolated from the loop; and

(b) it cannot be assumed that an optimisation procedure which leads to an exact fit for the data, necessarily defines an exact description of the system dynamics.

4.7.2. Since the introduction of disturbances into the 3-port cannot be considered to yield a satisfactory
description of the dynamics of system A from (4.35), consider the effect of introducing the disturbance into system B of fig. 4:9, while leaving the dynamics of system A, linear, time-invariant passive. Instead of fig. 4:11, the loop of fig. 4:9 is now to be modified to that of fig. 4:13

\[
\dot{x}_1(t) = f_1(t) + \sum
\]

\[
\dot{x}_2(t) = f_2(t) + \sum n(t)
\]

\[
f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \end{cases}
\]

\[
\gamma(t) = \int_{-\infty}^{\infty} f_1(t) x_1(t-\tau) d\tau + \int_{-\infty}^{\infty} f_2(t) x_2(t-\tau) d\tau
\]

with \( f_1(t) = f_2(t) = 0 \) for \( t < 0 \)

The actual relationships of the data are now given by

\[
\dot{x}_2(t) = \eta(t) + \int_{-\infty}^{\infty} f(t) y(t-\tau) d\tau
\]

with (4.70)

(4.71)
In this case the dynamics of the 3-port (system A) are linear time-invariant passive causal, and the optimisation procedure, by which (4.35) was developed, permits a choice from operators of this class.

Provided (4.36) is satisfied, which can again be shown to be the case if

$$\mathcal{O} \neq G_{x_1x_1} G_{nn} - G_{x_1n} G_{nx_1}$$

equation (4.35) defines a unique pair of operators $H_1(\omega)$ and $H_2(\omega)$.

To establish the relationships of the best approximations $H_1(\omega)$ and $H_2(\omega)$ to the actual dynamics $F_1(\omega)$ and $F_2(\omega)$, the following pair of spectral density equations may be derived from (4.71)

$$G_{x_1y} = F_1 G_{x_1x_1} + F_2 G_{x_1x_2} \quad (4.72)$$

$$G_{x_1y} = F_1 G_{x_1x_1} + F_2 G_{x_1x_2} \quad (4.73)$$

Solving (4.72) and (4.73) for $F_1$ and $F_2$ yields

$$F_1(\omega) = \frac{\{G_{x_1x_1} G_{x_1y} - G_{x_1x_2} G_{x_2y}\}}{\{G_{x_1x_1} G_{x_1x_2} - G_{x_1x_2} G_{x_2x_1}\}}(4.74)$$

$$= H_1(\omega)$$
from (4.66), and

\[
F_z(\omega) = \left\{ G_{x, x_1} G_{x, y} - G_{x_2, x_1} G_{x, y} \right\} \bigg/ \left\{ G_{x, x_1} G_{x, x_2} - G_{x_2, x_1} G_{x, x_2} \right\}
\]

from (4.67).

Equations (4.74) and (4.75) demonstrate that the combination of an active 2-port with a passive 3-port in closed loops of the type shown in fig. 4.7 allow an exact description of the dynamics of the 3-port to be obtained from a solution of (4.35). This is to be compared with the previous observations;

(a) that the combination of a passive 2-port with a passive 3-port provides data \(x_1(t), x_2(t)\) and \(y(t)\) which does not lead to any solution for (4.35), while

(b) the combination of a passive 2-port with an active 3-port provides data which can define a solution to (4.35), but the solution is of no physical significance.

If it is required to determine the dynamics of the 2-port, this requires data collected before the introduction of a disturbance into this part of the loop; otherwise one is compelled to approximate active dynamics by passive
operators (assuming the optimisation procedure adopted only permits passive operators as in the development of (4.16)).

4.7.3. Finally, consider the case when the linear time invariant systems A and B of fig. 4.7 are both linear time-invariant active, then the relationships of the data become

\[ x_2(t) = n_2(t) + \int_{-\infty}^{+\infty} f(\tau) y(t-\tau) d\tau \]  \hspace{1cm} (4.76)

with \( f(t) = 0 \) for \( t < 0 \)

and

\[ y(t) = n(t) + \int_{-\infty}^{+\infty} \int_{0}^{\infty} f_2(\tau) x_2(t-\tau) d\tau + \int_{-\infty}^{+\infty} f_1(\tau) x_1(t-\tau) d\tau \]  \hspace{1cm} (4.77)

with \( f_1(t) = f_2(t) = 0 \) for \( t < 0 \)

Assuming the disturbances are such that (4.36) is satisfied, it follows from (4.76) that

\[ G_{yx_2} = G_{yn_2} + \Gamma G_{yy} \]

or

\[ \frac{G_{yx_2}}{G_{yy}} = \Gamma + \frac{G_{yn_2}}{G_{yy}} \]  \hspace{1cm} (4.78)

or

\[ \frac{1}{G_{yy}} = \Gamma + \frac{G_{yn_2}}{G_{yy}} \]

from (4.37).
Also, from (4.77)

\[ G_{x,y} = G_{x,n_1} + F_1 G_{x,x_1} + F_2 G_{x,x_2} \]  
(4.79),

and

\[ G_{x_2,y} = G_{x_2,n_1} + F_1 G_{x_2,x_1} + F_2 G_{x_2,x_2} \]  
(4.80)

Solving (4.79) and (4.80) gives

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \begin{bmatrix}
G_{x,x_1} & G_{x,x_2}
\end{bmatrix}^{-1} \begin{bmatrix}
G_{x,y} - G_{x,n_1}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} - \begin{bmatrix}
G_{x,x_1} & G_{x,x_2}
\end{bmatrix}^{-1} \begin{bmatrix}
G_{x,n_1}
\end{bmatrix}
\]
(4.81)

from (5.35)

This emphasises the terms which are important in their contribution to the error resulting from the approximation of linear, time-invariant, active dynamics by linear, time-invariant passive operators. Thus, for \( H \) to be a good approximation to \( F \),

\[ G_{y,x_2} \gg G_{y,n_2} \]

and for \( H_1 \) and \( H_2 \) to be good approximations to \( F_1 \) and \( F_2 \),

\[ G_{x,y} \gg G_{x,n_1} \]

\[ G_{x_1,y} \gg G_{x_2,n_1} \]
It is apparent, from (4.81), that any problem in which $G_{xx}$ is 'nearly singular' can lead to considerable error, even when $G_{x_n}$ and $G_{x_i n}$ are themselves small.

It may be concluded, from (4.78) and (4.81) that a need exists to extend the optimisation procedure leading to (4.35), so as to permit a choice from both active, and passive, linear, time-invariant operators when systems subject to disturbances require to be investigated.

4.8 ON CHOOSING LINEAR, TIME-INVARIANT, ACTIVE OPERATORS.

The results established in sections 4.7.1. and 4.7.3. reveal the dangers associated with linear time-invariant, passive approximations to the dynamics of linear-time-invariant, active systems. The possibility of developing an optimisation procedure permitting a selection from active, time-invariant, linear, as well as from passive, time-invariant, linear system operators needs to be studied. The method will be developed here for the 3-port system discussed in section 4.8, and subsequently extended to include all Type-(b) n-ports.

Consider then the problem of selecting linear, time-invariant active operators to define the dynamics described by the given data $x_1(t)$, $x_2(t)$ and $y(t)$. The problem is to seek functions $h_1(t)$, $h_2(t)$ and $n(t)$ in the equation

$$
\varepsilon(t) = y(t) - \left\{ \sum_{r=1}^{2} \int_{-\infty}^{\infty} h_r(r) x_r(t-r) \, dr + n(t) \right\}
$$

(4.82)
which collectively minimise $\bar{\varepsilon}^2$. The term $n(t)$ in (4.28) is the contribution to $y(t)$ due to the internal disturbances of the active approximation.

As in the 2-port case (see section 3.9) an optimisation procedure which permits complete freedom of choice of $n(t)$, as well as of $h_1(t)$ and $h_2(t)$, provides no useful results because an infinite number of operators then exist which can reduce $\bar{\varepsilon}^2$ in (4.82) to zero.

If the optimisation procedure only permits freedom of choice of $h_1(t)$ and $h_2(t)$, while $n(t)$ is subjected to some previously determined constraint, useful results can be developed.

The optimisation procedure in this case is the same as that leading, for the general n-port to (4.20), or, for the 3-port, to (4.35). In fact, the result can be quoted directly by replacing $y(t)$ in the passive optimisation by $y(t) - n(t)$ for the active optimisation. Thus, from (4.35), the 'best' operators $h_1(t)$ and $h_2(t)$ which, in association with a previously defined $n(t)$, minimise $\bar{\varepsilon}^2$ in (4.82), are given by

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} G_{x_1x_1} & G_{x_1x_2} \\ G_{x_2x_1} & G_{x_2x_2} \end{bmatrix}^{-1} \begin{bmatrix} G_{x_1y} - G_{x_1n} \\ G_{x_2y} - G_{x_2n} \end{bmatrix}$$

(4.83)

If the constraint imposed on $n(t)$ makes estimates of $G_{x_1n}$ and $G_{x_2n}$ available, (4.83) may be solved for
$H_1(\omega)$ and $H_2(\omega)$. The passive case of (4.35) is a special case of (4.83) in which $n(t) = 0$, when $G_{x_1n}$ and $G_{x_2n}$ do not exist.

The result for an active n-port follows directly from (4.20) in the same way, and is defined by the matrix equation

$$H = G_{xx}^{-1} \left[ G_{xy} - G_{xn} \right]$$

(4.84)

where $G_{xn}$ is the column vector $\{G_{x_1n}, G_{x_2n}, \ldots, G_{x_nn}\}$

Consider the application of this result to the linear time-invariant active closed loop problem of section 4.7.3. Equations (4.79) and (4.80) may be rearranged to give

$$\begin{bmatrix} G_{x_1x_1} & G_{x_1x_2} \\ G_{x_2x_1} & G_{x_2x_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} G_{xy} - G_{x_1n} \\ G_{x_2y} - G_{x_2n} \end{bmatrix}$$

or

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} G_{x_1x_1} & G_{x_1x_2} \end{bmatrix}^{-1} \begin{bmatrix} G_{xy} - G_{x_1n} \\ G_{x_2y} - G_{x_2n} \end{bmatrix}$$

(4.85)

provided $G_{xx}$ is non-singular.

If estimates of $G_{x_1n}$ and $G_{x_2n}$ can be made available, and the spectral densities $G_{x_1n}$ and $G_{x_2n}$ of the best linear, time-invariant, active approximation
are constrained to have the same properties, then it follows from (4.83) and (4.85) that

\[
\begin{align*}
H_1(\omega) &= F_1(\omega) \\
H_2(\omega) &= F_2(\omega)
\end{align*}
\]

Provided the conditions permit estimates of the cross spectral density functions $G_{x_n}(\omega)$ to be made, realistic estimates of the dynamics of linear, time-invariant, active systems may be obtained.

4.9 A VARIABLE LOAD PARAMETER PROBLEM.

As a second example of the application of an n-port optimisation procedure, consider the determination of the dynamics of systems which are themselves linear, time-invariant, passive, but which are subject to load disturbances during normal operation. More specifically, consider the electrical network of fig. 4:14 when the load impedance $Z_L$ is time variant.

\[ \mathbf{V}_i = \mathbf{x}_i(t) \]

\[ \mathbf{V}_z = \mathbf{y}(t) \]

\[ \mathbf{Z}_L \]

\[ \mathbf{I}_z = \mathbf{x}_z(t) \]

**FIG. 4:14.**
The problem here is to determine the system dynamics from observed values of \( V_1 \), \( V_2 \) and \( i_z \).

The system to the left of the load terminals in fig. 4:14 may be replaced by an equivalent Thevenin generator in series with the Thevenin impedance. The Thevenin generator defines the open circuit voltage transfer ratio, while the Thevenin impedance is the output impedance of the network. Fig. 4:14 then reduces to the arrangement of fig. 4:15.

\[
\begin{align*}
i_z &= x_z(t) \\
V_1(t) &= \int_{-\infty}^{+\infty} g_1(\tau) x_1(t-\tau) \, d\tau
\end{align*}
\] (4.86)

Then \( g_1(t) \) = response observed at terminals 1, 1, when these terminals are on open circuit, and \( x_1(t) \) is a unit impulse. The Fourier Transform of \( g_1(t) \) is thus the open circuit voltage transfer ratio.

Now let the voltage drop \( V(t) \) across the impedance due to the current \( i_2 \) (= \( x_z(t) \)) be denoted by

\[
V(t) = \int_{-\infty}^{+\infty} g_2(\tau) x_2(t-\tau) \, d\tau
\]
Then
\[ g_2(t) = \text{voltage across } Z_T \text{ when } x_2(t) \text{ is a unit impulse.} \]
The Fourier Transform of \( g_2(t) \) is thus the impedance \( Z_T \) itself, and is therefore the output impedance of the network.

From fig. 4.15
\[ \gamma(t) = \nu_T(t) - \nu(t) \]
\[ = \int_{-\infty}^{+\infty} g_1(\tau) x_1(t-\tau) d\tau - \int_{-\infty}^{+\infty} g_2(\tau) x_2(t-\tau) d\tau \tag{4.87} \]

The problem now reduces to the determination of \( g_1(t) \) and \( g_2(t) \) given \( x_1(t), x_2(t) \) and \( y(t) \), which is a 3-port data fitting problem.

If a pair of operators \( h_1(t) \) and \( h_2(t) \) are chosen from the relation
\[ y(t) = \varepsilon(t) + \sum_{i=1}^{2} \int_{-\infty}^{+\infty} h_i(\tau) x_i(t-\tau) d\tau \]
to satisfy the constraint that they jointly minimise \( \overline{\varepsilon} \), then \( h_1(t) \) is the required dynamics approximating \( g_1(t) \) (and \( H_1(\omega) \) is the open circuit voltage transfer ratio of the system), while \(-h_2(t)\) is the required approximation to \( g_2(t) \) (and \(-H_2(\omega)\) is the output impedance of the network).

It is again observed in this problem that, if \( x_2(t) \) is linearly dependent on \( x_1(t) \), the resulting \( G_{xx} \) matrix is again singular, and no solution is possible.
This linear dependence is observed, in practice, if the network is linear, time-invariant, passive, and the load impedance is also linear, time-invariant, passive. It is not possible to determine both the open circuit voltage transfer ratio and the output impedance from measurements made with a linear time-invariant load impedance.

Since the problem here involves time-variant loads, the question of the singularity of the $G_{xx}$ matrix does not generally arise.

With the open circuit voltage transfer ratio, and the output impedance found, the voltage transfer ratio for any load impedance may be predicted for any $x_i(t)$, provided the source impedance of $x_i(t)$ remains invariant.

4.10 ON INTRODUCING A CAUSAL CONSTRAINT TO n-PORTS.

The same need may arise with n-ports, as with 2-ports, to approximate the dynamics of systems of one class with operators from another, as, for example, in the linearisation of non-linear dynamics. The class linear, time-invariant, passive, causal, may be a desirable one, in such situations.

It has already been observed (e.g. in section 4.7.2) that when the dynamics of the system from which data are collected is linear, time-invariant, passive, causal, the causal condition may be dropped. A search
among all linear, time-invariant, passive, operators
(causal and non causal) for that one which minimises the
mean square error of the approximation to the data leads
to a causal choice.

The need to approximate the dynamics of systems
which are not linear, time-invariant, passive, causal, by
operators of this class arises whenever lumped constant,
linear, passive, physical, analogues approximating the
behaviour of the actual system are required. It should
be emphasised again that it does not follow that such
physical analogues provide better policy decisions than
are obtained from mathematical models chosen from a more
general class of operator. The problem of finding
operators defining the dynamics of systems from given
input and output variables should not be confused with
the problem of finding an optimum design of a physical
system for a certain task (e.g. filtering or prediction).
The need for causality in this case is an expression of
the need to mechanise the mathematical operations defining
an optimum design. In the determination of system
dynamics, input and output variables are supplied, and a
set of mathematical operators defining the dynamics are
required. No problem of mechanisation arises unless one
prefers to use physical, rather than mathematical,
descriptions of the dynamics.
Although it is felt that the causal constraint belongs more to the province of selecting optimum transfer functions for the design of physical systems for particular tasks than to system dynamic studies, the need to approximate the dynamics of systems of one class by operators of another is sufficiently common that the problem is considered here without apology.

If, in an $n$-port data fitting problem the method developed above leads to a set of causal operators $h_r(t)$, there is obviously no need to introduce causality into the optimisation procedure. The data is such that no non-causal, linear, time-invariant, passive, operators can be found which are associated with a smaller mean square error than is associated with the causal choice.

On the other hand, if the data permits the choice of a set of non-causal operators which reduce the mean square error to a smaller value than is associated with the best causal choice then

(a) the system dynamics is certainly not linear, time-invariant, passive, causal, and

(b) the dynamics can be more accurately approximated (in the minimum mean square error sense) by non-causal than by causal operators.

It is in finding the best causal approximation to dynamics of this class that a special constraint is necessary.
As with 2-port (section 3.8), the causal constraint may be introduced into the mathematics from the outset. If this constraint is imposed upon the optimisation procedure of section 4.5, the mathematics leading to (4.16) may be formally repeated. The difference between the two cases is that the causal constraint imposes upon \( P_q(\tau) \) in (4.16) the condition that

\[
P_q(\tau) = 0 \quad \text{for } \tau < 0
\]

This modifies (5.17) by the addition of a corresponding constraint condition. The set of integral equations defining the optimum n-port, linear, time-invariant, passive, causal operators are thus

\[
\varphi_{x_q y} (\tau) = \sum_{r=1}^{m} \int_{0}^{\infty} h_r(\tau) \varphi_{x_q x_r} (\tau - \tau_r) d\tau_r, \quad \tau > 0
\]

for

\[
q = 1, 2, \ldots, m
\]

Equations (4.88) cannot be reduced to a set of linear algebraic equations by taking Fourier Transforms because they are not applicable for all \( \tau \) in the range \(-\infty \leq \tau \leq +\infty\) as were (4.17). No simple result corresponding to (4.18) has been found. A solution has been developed (Wiener 1949, Chap.IV, section 4.1, pp.105-109).
A less exact solution than Wiener's, but one which may be easier to apply, can be developed using the ideas of section 3.8 above. Thus, suppose in a particular problem, $s$ of the $m$ operator $H_{x}(\omega)$ obtained from a solution of (4.20) are non-causal, while $(m-s)$ are causal as indicated in fig. 4:16.

Replace each of the non-causal operators by a causal approximation plus an associated error as in fig. 4:17.
The error $E(t)$ of the causal approximation is

given by

$$E(t) = e(t) + \sum_{r=1}^{m} e_r(t)$$

($m$-s of the terms $e_r(t)$ in (4.89) may be zero, but this does not violate the argument).

Hence

$$E^2(t) = e^2(t) + \sum_{r=1}^{m} e(t) e_r(t) + \left[ \sum_{r=1}^{m} e_r(t) \right]^2$$

Now, from fig. 4.17 and fig. 4.16

$$e_r(t) = z_r(t) - \int_{-\infty}^{+\infty} h_{rc}(\tau) x_r(t-\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} h_{rc}(\tau) x_r(t-\tau) d\tau - \int_{0}^{+\infty} h_{rc}(\tau) x_r(t-\tau) d\tau$$

$$\frac{e(t) e_r(t)}{\overline{e(t) e_r(t)}} = \int_{-\infty}^{+\infty} h_{rc}(\tau) \overline{\chi_r(\zeta)} d\tau - \int_{0}^{+\infty} h_{rc}(\tau) \overline{\chi_r(\zeta)} d\tau$$

$$= 0$$

(4.91)
since $\varepsilon(t)$ can be shown to be uncorrelated with every $x_r(t)$.

From (4.90) and (4.91)

$$
\overline{E^2(\varepsilon)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E^2(\varepsilon) \, dt
$$

$$
= \overline{\varepsilon^2} + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{r=1}^{m} \sum_{s=1}^{n} \varepsilon_r(t) \varepsilon_s(t) \, dt
$$

$$
= \overline{\varepsilon^2} + \sum_{r=1}^{m} \sum_{s=1}^{n} \phi(\omega) \delta_{rs}
$$

(4.92)

If it is assumed that the cross correlation between the errors $\varepsilon_r(t)$ and $\varepsilon_s(t)$ is small compared to their auto-correlations, then

$$
\phi(\omega) = \begin{cases} 0 & \text{for } r \neq s \\ \overline{\varepsilon^2(\varepsilon)} & \text{for } r = s \end{cases}
$$

when (4.92) becomes

$$
\overline{E^2(\varepsilon)} = \overline{\varepsilon^2} + \sum_{r=1}^{m} \overline{\varepsilon_r^2(\varepsilon)}
$$

(4.93)

The quantity $\overline{E^2(\varepsilon)}$ in (4.93) will be minimised if each $\overline{\varepsilon_r^2(\varepsilon)}$ is separately minimised.

The assumption that $\phi_{\varepsilon_r\varepsilon_s}(\omega)$ may be set equal to zero is correct if $x_r(t)$ and $x_s(t)$ are uncorrelated; otherwise it is an approximation. However, it is quite legitimate to accept the approximation, and to check the values of $\phi_{\varepsilon_r\varepsilon_s}(\omega)$ from the solution obtained.
Provided
\[ \sum_{\tau \neq s} \phi_{\tau s}(\omega) \ll \left\{ \varepsilon_{\tau}^{2} + \sum_{\tau = 1}^{m} \varepsilon_{\tau}^{2} \right\} \]
(the superscript 'dash' on the summation sign indicates \( \tau \neq s \)), then the assumption is justified.

Minimisation of each \( \varepsilon_{\tau}^{2}(\ell) \) separately requires the choice of the causal operator \( h_{rc}(t) \) which minimise \( \varepsilon_{\tau}^{2}(\ell) \) in the equation

\[ \varepsilon_{\tau}(\ell) = Z_{r}(\ell) = \int_{0}^{\infty} h_{r}(\tau) x_{r}(t-\tau) d\tau \]

This is exactly the same as the 2-port causal optimisation procedure described in 3.8, and leads to the integral equation

\[ \phi(\tau) = \int_{0}^{\infty} h_{r}(\tau) \phi_{x \tau}(\tau-\tau_{c}) d\tau_{c} \]

or the corresponding frequency domain solution.
CHAPTER FIVE

LEAST SQUARES OPTIMISATION WITHOUT SPECTRAL DENSITY ESTIMATES

5.1 INTRODUCTION.

5.1.1. In section 2.7 of chapter 2, it was stated that the methods to be developed in chapters 3 and 4 (for application to system dynamics studies when the recorded data does not define \( x_r(t) \) and \( y(t) \) for all \( t \)) required that realistic estimates of the power spectra derived from \( x_r(t) \) and \( y(t) \) be available. Considerable effort is being expended by statisticians and others (e.g. Bartlett 1955, Blackman and Tukey 1957, Fuller 1958, Grenander and Rosenblatt 1957, Lomnicki and Zaremba 1957) in attempts to derive such estimates from finite samples of data. While there is every hope that suitable methods of estimation of these quantities will eventually emerge, methods for the derivation of optimum system dynamic operators which do not require such estimates are of immediate interest.

In assuming that realistic estimates of power spectra may be made available, there is implied the additional assumption that the data possesses power spectra. This is not true of all types of data.
Attention will be given in this chapter to the determination of optimum system dynamic operators when

(i) \( x_r(t) \) and \( y(t) \) are supposed defined for all time, but in such a way (e.g. as in fig. 2:1) that the data do not possess power spectra,

(ii) \( x_r(t) \) and \( y(t) \) are supposed defined during some finite time interval only, and the optimising procedure is constrained in such a way that the limitations in the time duration of the data are respected.

5.1.2. As an example of a problem of the first type, consider applying the methods by which equation (4.20) was derived to data of the type indicated in fig. 2:1. It becomes obvious, early in the procedure, that

\[
\bar{\varepsilon^2(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon^2(t) \, dt = 0
\]

whatever choice of linear, time-invariant, passive, causal, finite memory, \( h_r(t) \) is made. It is necessary, therefore, to use a modified constraint condition.

More generally, if the experimental data \( x_r(t) \) may be supposed to define a set of causes \( x_r(t) \) and the corresponding effect \( y(t) \) over the whole time interval

* Only type-(b) n-ports of fig. 4:3 are treated in this chapter. It follows from the discussions of Chapter 4 that more general n-ports may be treated by repeated applications of a type b n-port procedure.
\( -\infty \leq t \leq +\infty \), and \( x_r(t) \) and \( y(t) \) possess Fourier Transforms \( X_r(\omega) \) and \( Y(\omega) \), then the error function \( \varepsilon(t) \) in the equation

\[
\varepsilon(t) = \gamma(t) - \sum_{r=0}^{\infty} \int_{-\infty}^{+\infty} h_r(\tau) x_r(t-\tau) d\tau
\]

also possesses a Fourier Transform, \( E(\omega) \), for every operator \( h_r(\tau) \) which is Fourier Transformable.

It is well known (Titchmarsh, 1937.), that a necessary condition for the existence of \( E(\omega) \) is that

\[
\int_{-\infty}^{+\infty} \varepsilon(t)^2 dt
\]

is finite.

But, any \( \varepsilon(t) \) which satisfies this condition requires that

\[
0 = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \varepsilon(t) dt
\]

By similar considerations to the above, one is led to the conclusion that, if \( x_r(t) \) and \( y(t) \) possess Fourier Transforms, they cannot also be considered to possess spectral density functions, and hence equation (4.20) becomes meaningless for such data.

5.1.3. In the second class of problem, in which \( x_r(t) \) and \( y(t) \) are supposed defined during some finite time interval only, and are undefined outside this interval, it is not possible, on the evidence of the available data, to decide whether the data possess Fourier
Transforms, or power spectra. This depends entirely upon assumptions made about the properties of the data in the undefined region.

In using the minimum mean square error constraint of chapters 3 and 4, it is assumed (by the use of a stationary ergodic hypothesis) that the properties of $x_r(t)$ and $y(t)$ in the undefined region are such that the data does possess spectral density functions. The need to derive estimates of these quantities arises directly from the use of this minimum mean square constraint. In formulating such a constraint, the limitations placed upon the problem by the nature and amount of experimental data available is ignored, and the subsequent dilemmas created.

The gap between the mathematical requirements, and the experimental information available, is bridged by an assumption. Thus it is assumed that

(i) $x_r(t)$ and $y(t)$ are samples of stationary ergodic random processes,

(ii) the mean value of the sample correlation and spectral density functions, computed over an ensemble of such samples gives a good approximation to the 'true' correlation and spectral density functions, and
(iii) that the variance of sample correlation and spectral density functions is small, and hence the probability high that such quantities, computed from a single sample, are good approximations to the mean value computed over the whole ensemble.

Before such assumptions can be accepted with confidence, it is necessary to

a) devise tests to verify the assumptions, and

b) find whether the optimum dynamics so obtained remains sensibly invariant under small changes in the values of correlation or spectral density functions (these changes being the difference between the functions computed from the samples and the corresponding functions computed from the whole ensemble.)

No test has yet been developed by which to check the assumptions as to the properties of the data in those time intervals in which \( x(t) \) and \( y(t) \) are unrecorded. It is difficult to see how such verification can be found, for, as has been pointed out (Wiener, 1949, p. 55), "the correlation coefficient of a message is not completely determined by its own past".

If the need for correlation and/or spectral density functions of the data is to be avoided, the
optimisation procedure must be reformulated, and in such a way that the mathematical operations required recognize the limitations imposed upon these problems by the fact that the recorded variables do not define \( x(t) \) and \( y(t) \) for all values of \( t \) in the interval \(-\infty \leq t \leq +\infty\).

5.2 FOURIER TRANSFORMABLE DATA

5.2.1. Turning first to the problem of section 5.1.2. above in which the conditions of the experiment permit the assumption that \( x(t) \) and \( y(t) \) are zero outside some region as in fig. 2:1. Such data possess Fourier Transforms, but not spectral density functions.

If the constraint condition

\[
\lim_{t \to \infty} \frac{1}{2\pi} \int_{-\infty}^{t} \varepsilon^2(t) \, dt = a \text{ minimum} \quad (5.2)
\]

used in the derivation of (4.20) is replaced by the modified constraint condition

\[
\int_{-\infty}^{\infty} \varepsilon^2(t) \, dt = a \text{ minimum} \quad (5.3),
\]

the optimisation procedure seeks to minimise the 'total error energy', instead of the error power. In any situation in which the total error energy, as defined by (5.3) is finite, the error power of (5.2) is necessarily zero. In these situations the methods of the previous
chapters fail. Those methods are applicable to situations in which members of the permitted sub-class of system operators all provide an error signal $\epsilon(t)$ which may be supposed to possess a finite 'error power', and hence an infinite 'error energy'. A sub-class including Fourier Transformable system operators $h_r(\tau)$, when operating upon Fourier Transformable data $x_r(t)$ and $y(t)$ does not lead to such a situation.

In situations such as the present one, in which the total error energy is finite, the error power of (5.2) is zero, irrespective of the magnitude of the error energy, and the use of (5.2) as an optimising constraint inevitably leads to an optimising problem having no meaningful solution. However, if one seeks a set of mathematical operators from among the members of a linear, time-invariant, passive class of system operators to satisfy the constraint (5.3), the optimising problem can be successfully solved.

5.2.2. Consider the application of the constraint (5.3) to select the $m$ optimum operators $h_r(\tau)$ from the linear, time-invariant, passive, sub-class defined by (5.1), when the variables $\epsilon(t)$, $x_r(t)$, $y(t)$ and $h_r(t)$ are all Fourier transformable.
From (5.1), taking Fourier Transforms, it follows that

\[ E(\omega) = Y(\omega) - \sum_{r=1}^{m} H_r(\omega) X_r(\omega) \quad (5.4) \]

Also

\[
\int_{-\infty}^{\infty} \epsilon^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon(t) \left\{ \int_{-\infty}^{\infty} E(\omega) e^{i\omega t} d\omega \right\} dt
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) \left\{ \int_{-\infty}^{\infty} \epsilon(t) e^{i\omega t} dt \right\} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) E^*(\omega) d\omega \quad (5.5)
\]

so that the constraint (5.3) is equivalent to the constraint

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) E^*(\omega) d\omega = a \text{ minimum} \quad (5.6)
\]

Writing (5.4) in matrix form

\[
E = Y - XH
\]

where

- \( E = 1 \times 1 \) 'vector'
- \( Y = 1 \times 1 \) 'vector'
- \( X = 1 \times m \) matrix
- \( H = \text{column vector} \)
In the same notation
\[
E(\omega) = E^*(\omega) = E' \mathcal{E}
\]

\[
= \begin{bmatrix} Y' - H'X' \end{bmatrix} \begin{bmatrix} Y - XH \end{bmatrix}
\]

(5.8)

where superscript \( ' \) denotes complex conjugate transpose.

From (5.5) and (5.8)

\[
\int_{-\infty}^{\infty} e^2(t) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \begin{bmatrix} Y' - H'X' \end{bmatrix} \begin{bmatrix} Y - XH \end{bmatrix} d\omega
\]

(5.9)

Since the left hand side of (5.9) is non-negative, its least value is zero. Any vector \( \mathcal{H} \) which satisfies the equation

\[
Y - XH = 0
\]

(5.10)

thus satisfies (5.3) and represents a solution of the type sought.

If \( m = 1 \), equation (5.10) has the unique solution:

\[
\mathcal{H}_1(\omega) = \frac{Y_1(\omega)}{X_1(\omega)}
\]

(5.11)

If \( m > 1 \), an infinity of vectors \( \mathcal{H} \) can be found to satisfy (5.10), \((m-1)\) of the elements of \( \mathcal{H} \) being arbitrarily selected, and the remaining one chosen to satisfy (5.10). Thus for the case \( m = 2 \), for example,
(5.10) may be written

\[ \mathcal{H}_1(\omega) = \left\{ \mathcal{Y}(\omega) - \mathcal{X}_1(\omega) \mathcal{H}_2(\omega) \right\} / \mathcal{X}_1(\omega) \]  

(5.12)

The operator \( \mathcal{H}_2(\omega) \) in (5.12) may be selected arbitrarily, and \( \mathcal{H}_1(\omega) \) found from (5.12) to provide a vector \( \mathcal{H} \) which satisfies (5.10).

It follows, as a general conclusion that data of this type, collected during a single experiment, is adequate to provide a unique solution to the optimisation problem only in the case of 2-port system operators \( m = 1 \). A linear, time-invariant, passive operator can always be found to reduce the error energy to zero in this case, irrespective of the class to which the system dynamics actually belongs. For \( m > 1 \) data from a single experiment is insufficient to provide a unique description of the system dynamics, and it becomes necessary to repeat the experiment a number of times with different input variables \( x_r(t) \).

5.2.3. Instead of a single experiment, consider the case when \( N \) separate trials are conducted. Let

\[ x_{sr}(t) = \text{input variable recorded during trial } s \text{ at input terminal } r. \]

\[ y_s(t) = \text{output variable recorded during trial } s. \]

\[ \epsilon_s(t) = \text{the error resulting when approximating the data collected during trial } s \text{ by any set of linear, time-invariant, passive operators.} \]
Then

\[ \varepsilon_s (t) = \gamma_s (t) - \sum_{r=1}^{3} \int_{-\infty}^{+\infty} h_r (\tau) x_{s_r} (t-\tau) \, d\tau \]

(5.13)

One may now seek to minimise either

(a) the total error energy \[ \varepsilon \] = \[ \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \varepsilon_s^2 (t) \, dt \]  (5.14)

or

(b) the mean error energy \[ \varepsilon \] = \[ \frac{1}{N} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \varepsilon_s^2 (t) \, dt \]  (5.15)

Either constraint leads to the same set of equations defining the optimum operators. Using (5.15), it is possible to write

\[ \frac{1}{2} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \varepsilon_s^2 (t) \, dt = \frac{1}{2\pi N} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} E(\omega) E^*(\omega) \, d\omega \]

(5.16)

From (5.13)

\[ \sum_{s=1}^{N} \{ \gamma_s^* (\omega) - \sum_{r=1}^{3} X_{s_r}^* (\omega) H_r^* (\omega) \} \{ \gamma_s (\omega) - \sum_{r=1}^{3} X_{s_r} (\omega) H_r (\omega) \} = \sum_{s=1}^{N} E_s (\omega) E^*_s (\omega) \]

(5.17)

where * denotes 'complex conjugate'.

Writing (5.17) in matrix notation

\[ \mathbf{E}^T \mathbf{E} = \{ \mathbf{Y}^T - \mathbf{H}^T \mathbf{X}^T \} \{ \mathbf{Y} - \mathbf{X} \mathbf{H} \} \]

(5.18)
where

\[ \mathbf{E} = \text{column vector} \{ E_1(\omega), E_2(\omega), \ldots, E_N(\omega) \} \]

\[ \mathbf{Y} = \text{column vector} \{ Y_1(\omega), Y_2(\omega), \ldots, Y_N(\omega) \} \]

\[ \mathbf{X} = N \times m \text{ matrix} \quad \begin{bmatrix} \mathcal{X}_{ss'}(\omega) \end{bmatrix} \]

\[ \mathbf{H} = \text{column vector} \{ H_1(\omega), H_2(\omega), \ldots, H_m(\omega) \} \]

and the superscript \(^*\) indicates complex conjugate transpose.

From (5.16), (5.17) and (5.18)

\[
\frac{1}{2^{nN}} \int_{-\infty}^{+\infty} \left\{ Y' - \mathbf{H}' \mathbf{X}' \right\} \left\{ Y - \mathbf{X} \mathbf{H} \right\} d\omega = \frac{1}{N} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \mathcal{E}_s(\xi) d\xi \tag{5.19}
\]

Again, if a vector \( \mathbf{H} \) can be found which satisfies the matrix equation

\[ \mathbf{Y} = \mathbf{X} \mathbf{H} \tag{5.20} \]

such an \( \mathbf{H} \) vector reduces the quantity

\[
\frac{1}{N} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \mathcal{E}_s(\xi) d\xi
\]

to zero, which is the least possible value.

There is a fundamental difference between equations (5.10) and (5.20). Equation (5.10) is a single equation in \( m \) unknowns, whereas (5.20) defines a set of \( N \) simultaneous equations in \( m \) unknowns. Whereas (5.10) is always underdetermined for \( m > 1 \), a number of possibilities occur with (5.20).
(a) The set may be inconsistent (a test for this is given by Aitken, 1951, p.70), and have no solution. In this case, no vector \( \mathbf{H} \) can be found which reduces the quantity

\[
\frac{1}{N} \sum_{s=1}^{N} \int_{-\infty}^{+\infty} \varepsilon_s^2(t) \, dt
\]

to zero, and the vector \( \mathbf{H} \) which reduces this quantity to a minimum (not zero) must be found.

(b) The \( N \) non-homogeneous equations (5.20) may be consistent but underdetermined (i.e. the rank of the matrix \( \mathbf{X} \) in (5.20) may be less than \( m \), when some of the elements of \( \mathbf{H} \) may be arbitrarily assigned, giving an infinity of vectors \( \mathbf{H} \) satisfying (5.20)). If equations (5.20) are consistent, but \( N < m \), then, because the rank of an \( N \times m \) matrix cannot be greater than the lesser of \( N \) and \( m \), (5.20) defines a consistent under-determined set. This is the situation that arose when the method of section 5.2.2, was applied to systems having \( m \geq 2 \).

(c) The \( N \) non-homogeneous equations (5.20) may be consistent and \( \mathbf{X} \) of rank \( m \) (if \( N \geq m \)) when they possess a unique solution.

If, instead of attempting to solve (5.20), a set of equations are derived from (5.19) defining the
vector $\mathbf{H}$ which minimises the expression

$$\frac{1}{N} \sum_{s=1}^{N} \int_{-\infty}^{\infty} \epsilon_s^2(t) \, dt$$

these equations have one of the properties

(a) that they possess a unique solution, but one for which the minimum value of the error energy is not zero,

(b) that they have no unique solution,

(c) that they have a unique solution which is associated with a zero value of total error energy,

corresponding to the three conditions which may arise when a solution of (5.20) is attempted.

5.2.4. The equations defining the vector $\mathbf{H}$ which minimises the total error energy are derived from (5.16) and (5.18) and the matrix identity

$$\begin{align*}
\left\{ \mathbf{H}' \mathbf{x}' - \mathbf{Y}' \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \right\} - \left\{ \mathbf{x} \mathbf{H}' - \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \mathbf{Y} \right\} \\
\equiv \left\{ \mathbf{Y}' - \mathbf{H}' \mathbf{x}' \right\} \left\{ \mathbf{Y} - \mathbf{x} \mathbf{H} \right\} \\
- \left\{ \mathbf{Y}' - \mathbf{Y}' \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \right\} \left\{ \mathbf{Y} - \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \mathbf{Y} \right\}
\end{align*}$$

From (5.21) and (5.18)

$$\begin{align*}
\mathbf{E}' \mathbf{E} & = \left\{ \mathbf{Y}' - \mathbf{Y}' \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \right\} \left\{ \mathbf{Y} - \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \mathbf{Y} \right\} \\
& \quad + \left\{ \mathbf{H}' \mathbf{x}' - \mathbf{Y}' \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \right\} \left\{ \mathbf{x} \mathbf{H}' - \mathbf{x} \left[ \mathbf{x}' \mathbf{x} \right]^{-1} \mathbf{x}' \mathbf{Y} \right\}
\end{align*}$$

(5.22)
\[ \frac{1}{2\pi N} \int_{-\infty}^{+\infty} E' E d\omega = \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_i Z_i d\omega \]
\[ + \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_2 Z_2 d\omega \]  
(5.23)

where \( Z_i \) = column vector  \{ Y - X [X'X]^{-1} X' Y \} 
and \( Z_2 \) = column vector  \{ X H - X [X'X]^{-1} X' Y \} 

Now, both \( \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_i Z_i d\omega \) = a non-negative quantity, and 
\[ \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_2 Z_2 d\omega \] = a non-negative quantity.

Hence
\[ \frac{1}{2\pi N} \int_{-\infty}^{+\infty} E' E d\omega \geq \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_i Z_i d\omega \]  
(5.24)

If \( H \) is chosen to make \( Z_2 \) a zero vector
\[ \frac{1}{2\pi N} \int_{-\infty}^{+\infty} E' E d\omega = \frac{1}{2\pi N} \int_{-\infty}^{+\infty} Z'_i Z_i d\omega \]
which is the least value it can assume.

The optimum vector \( H \) is thus given by
\[ O = X \left\{ H - [X'X]^{-1} X' Y \right\} \]
for all arbitrary \( X \) matrices. Hence
\[ H = [X'X]^{-1} X' Y \]  
(5.25)

defines the vector \( H \) which reduces \( \frac{1}{2\pi N} \int_{-\infty}^{+\infty} \varepsilon^2(t) dt \) to a minimum.
5.2.5. Provided $X'X$ is a non-singular matrix, (5.25) defines a unique $H$, while a singular $[X'X]$ possesses no inverse. This constraint on $[X'X]$ is to be compared with the condition $G_{xx}$ is non-singular in (4.19) and (4.20). As an example, consider the determination of the dynamics of the passive 3-port of fig. 5:1, from data of the type here defined. This problem has already been considered, (fig. 4:9, and section 4.6.2.) with reference to data $x_r(t)$ and $y(t)$ which was assumed to possess spectral density functions. Because of the linear dependence of $x_2(t)$ on $x_1(t)$, it was found in section 4.6.2. that $G_{xx}$ was a singular matrix, and it is to be suspected here that $X'X$ may be singular for the same reason.

FIG. 5:1.
To show that this is the case set

\[ X_{s_2}(\omega) = A(\omega) X_{s_1}(\omega) \]  \hspace{1cm} (5.26)

in \( X \), where

\[ A(\omega) = \text{open loop transfer function} \]

If \( N = 2 \), i.e. if it is assumed that only 2 trials are conducted, then

\[
\begin{pmatrix}
X' & X \\
\end{pmatrix}
= \begin{pmatrix}
X^*_n & X^*_\omega \\
X^*_\omega & X^*_\omega \\
\end{pmatrix}
\begin{pmatrix}
X_n & X_{12} \\
X_{12} & X_{22} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
X^*_n & X^*_\omega \\
A^*X^*_n & A^*X^*_\omega \\
\end{pmatrix}
\begin{pmatrix}
X_n & AX_n \\
X_{12} & A X_{12} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
|X_n|^2 + |X_{12}|^2, & A \{ |X_n|^2 + |X_{12}|^2 \} \\
A^* \{ |X_n|^2 + |X_{12}|^2 \}, & A A^* \{ |X_n|^2 + |X_{12}|^2 \}
\end{pmatrix}
\]

And hence

\[
\|X'X\| = |A|^2 \left[ |X_n|^2 + |X_{12}|^2 \right] - |A|^2 \left[ |X_n|^2 + |X_{12}|^2 \right]^2
\]

\[
= 0
\]

which establishes that \( X'X \) is a singular matrix.

Increasing the value of \( N \) beyond 2 does nothing to destroy the linear dependence of the variables, and hence cannot yield a non-singular matrix, as can be seen
by substituting (5.26) into the general 3-port X matrix

\[
\begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22} \\
\vdots & \vdots \\
X_{N1} & X_{N2}
\end{bmatrix}
\]

when \(X'X\) is given by

\[
X'X = \begin{bmatrix}
\sum_{s=1}^{N} X_{s1} X_{s1}^* & A \sum_{s=1}^{N} X_{s1} X_{s1}^* \\
A^* \sum_{s=1}^{N} X_{s1} X_{s1}^* & A A^* \sum_{s=1}^{N} X_{s1} X_{s1}^*
\end{bmatrix}
\]

which is singular for all \(N\).

Suppose, however, an attempt is made to destroy the linear dependence of \(x_2(t)\) on \(x_1(t)\)

(a) by changing the controller dynamics between each trial. Equation (5.30) is then modified to

\[
X_{s2}(\omega) = A_s(\omega) X_{s1}(\omega)
\]

(5.27)

Substituting the dependence (5.27) into the \(X\) matrix gives

\[
\|X'X\| = \frac{1}{2} \sum_{r=1}^{N} \sum_{s=1}^{N} \|X_r\|^2 \|X_s\|^2 |A_r - A_s|^2
\]

\[\neq 0\]

since \(A_r \neq A_s\), \(r \neq s\)

by the conditions of the experiment.
Changing the controller setting between trials has the desired effect, therefore, since $\mathbf{X}'\mathbf{X}$ becomes non-singular, making an optimisation possible.

(b) by leaving the setting of the controller dynamics unchanged for all trials, but injecting independent disturbances (which need not be recorded) into the loop in addition to the recorded variable $x_i(t)$

Then

$$x_2(\omega) = \left[ A(\omega) x_2(\omega) + x_3(\omega) \right]$$

This situation was discussed in some detail in section 4.6. Just as in that study the usefulness of the results obtained depended upon the position within the loop that the disturbances were introduced, so the same observation may be made here. If the disturbances arise within the 3-port (i.e. the system dynamics are made linear, time-invariant, active, by the introduction of the disturbances), and yet a linear, time-invariant, passive sub-class is postulated for the selection of the optimum operator, the result obtained (compare equations (4.68) and (4.69)), are

$$H_1(\omega) = 0$$
$$H_2(\omega) = \frac{1}{F(\omega)}$$

where $F(\omega) = \text{transfer function of the controller.}$
If the disturbance originates in the controller, however, a realistic description of the dynamics of the 3-port system is obtained.

5.3. SHORT DURATION SAMPLES OF OPERATING DATA.

5.3.1. It is proposed to return now to the second type of problem mentioned in section 5.1.1, namely that in which \( x_r(t) \) and \( y(t) \) are defined in some finite interval of \( t \), and undefined outside this interval. It was for this type of data that the results of chapters 3 and 4 were developed, but without imposing limitations upon the mathematical processes corresponding to the limitations on the experimental information. It is the purpose of this section to look again at these procedures, and to modify them in such a way that the mathematical demands on experimental information can be satisfied.

5.3.2. Consider first the 2-port data fitting problem of fig. 3.1. In chapter 3, optimum operators were selected by a solution of the pair of equations

\[
\epsilon(t) = y(t) - \int_{-\infty}^{t} h(\tau) x(t - \tau) d\tau \tag{3.2}
\]

and

\[
\overline{\epsilon^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \epsilon^2(t) dt = \text{a minimum} \tag{3.4}
\]

Because \( x_r(t) \) and \( y(t) \) are not defined over the
whole range \(-\infty \leq t \leq +\infty\), the value of \(\bar{\varepsilon}^2\) defined by (3.4) cannot be determined for any \(h(t)\) of the class defined in (3.2). It is in the use of the optimising constraint (3.4), with its implication that \(\varepsilon(t)\) is defined in the whole interval \(-\infty \leq t \leq +\infty\) for every linear, time-invariant passive operator \(h(t)\), that the necessity for correlation and spectral density estimates is introduced into the mathematics.

A constraint condition, which is more realistic than (3.4) in that it recognizes the limitations of the available data, is obtained if one seeks that operator, from a given class of operators, which minimizes the mean square error over that range of \(t\) for which \(\varepsilon(t)\) is defined by the available data for every member of the permitted class. Since the range of \(t\) involved in such an optimization is finite, both the total error energy, defined by
\[
\int_0^\infty \varepsilon^2(t) \, dt
\]
and the error power
\[
\frac{1}{T} \int_0^T \varepsilon^2(t) \, dt
\]
are finite simultaneously. Optimisation procedures can be developed in this case, which seeks to minimize either of these quantities, and, whichever optimization procedure is adopted, the same description of the dynamics results.
It is immediately apparent that the constraint condition suggested above places a restriction upon the class of operators which may be permitted. For example, those operators \( h(t) \) of the linear time-invariant, passive class permitted in (3,2), which exist for all \( t \) in the interval \(-\infty \leq \tau \leq +\infty \) must be excluded if every operator of the permitted class is to define \( \varepsilon(t) \) over some finite range of \( t \). In other words, finite samples of data restricts the class of operator (from amongst which an optimum operator may be sought) more severely than was supposed when it was assumed that estimates of spectral density functions may be made available.

The arguments of this section develop a procedure for the selection of optimum operators from a linear, time-invariant, passive, causal, finite-memory, class, having a memory time less than or equal to some pre-assigned value.

Assume that
(a) \( x(t) \) and \( y(t) \) are recorded only for values of \( t \) in the interval \( 0 \leq \ell \leq \ell, \) and
(b) the preassigned memory time of the permitted class is \( T \) (which must be less than \( t, \) if \( \varepsilon(t) \) is to be defined in any range of \( t \).
Equations (3.2) and (3.4) are then to be replaced by the equations

$$
\varepsilon(t) = \gamma(t) - \int_0^\infty h(\tau) x(t - \tau) \, d\tau $

where

$$
h(\tau) = 0 \quad \text{for} -\infty \leq \tau \leq 0, \quad T \leq \tau \leq +\infty$

and

$$
\varepsilon^2 = \frac{1}{T - t} \int_t^T \varepsilon^2(\tau) \, d\tau = \text{a minimum}
$$

Equation (5.28) defines the error resulting when approximating the dynamics of the system from which the data was collected by a particular operator $h(t)$ chosen from a class containing only linear, time-invariant, passive, causal operators having a memory time less than or equal to $T$.

The constraint (5.29), when imposed upon $\varepsilon(t)$ of (5.28) is the mathematical statement of the modified minimum mean square error constraint proposed above.

A simultaneous solution of (5.28) and (5.29) requires the selection of that particular linear, time-invariant, passive, causal, finite memory operator which best defines the system dynamics.

Because the method of solution requires considerable
numerical computation, it is convenient to record 
x(t) and y(t) as time series, so that numerical
integration methods may be employed. With such data, 
h(t) is only defined at the sample points. This is
allowed for by restricting still further the permitted
class of operator. It is to be assumed here that the
optimum operators h(t) is to be selected from that
sub-class of the linear, time-invariant, passive, causal,
finit-memory class defined by

\[
    h(t) = \frac{N}{\pi} \sum_{r=1}^{N} h_r \delta \left( t - \frac{T}{N} \right) \tag{5.30}
\]

where \( \delta \left( t - \frac{T}{N} \right) = 1 \) if \( t - \left( t - \frac{1}{2} \right) T_N \leq t \leq \left( t + \frac{1}{2} \right) T_N \)
\( = 0 \) outside this range.

From (5.28) and (5.30)

\[
    \epsilon(t) = y(t) - \int_{\infty}^T \sum_{r=1}^{N} h_r \delta \left( \tau - \frac{T}{N} \right) \times (t - \tau) d\tau
\]

\[
    = y(t) - \sum_{r=1}^{N} h_r \times (t - \frac{T}{N}) \tag{5.31}
\]

where \( \epsilon(t) \), \( y(t) \) and \( x(t) \) are time series at
interval \( \frac{T}{N} \).

The significance of operators of the class
defined by (5.30) is indicated in fig. 5:2. Any linear,
time-invariant, passive, causal operator having a memory time less than or equal to $T$ may be approximated to any desired degree of accuracy by a member of the sub-class defined by (5.30) if $N$ is made sufficiently large.

It cannot be overemphasised that the search for a 'best' operator $h(t)$ (i.e. a best set of coefficients $h_r$ in (5.31) is restricted to a selection from operators of a very small sub-class of all possible linear operators, and can only yield an exact description of the dynamics in those problems in which the dynamics belongs to this sub-class. When applying the procedure to obtain a description of the dynamics of systems in which disturbances were present (e.g. active, or time-variant systems), an approximation to the correct description of the dynamics is all that is possible,
because the choice of active or time variant operators is not permitted by (5.31). With finite samples of data, it is difficult to see how one can do better than to approximate the dynamics of more general systems by the best approximation from operators permitted by (5.31) until realistic estimates of various correlation and spectral density functions become available, when the methods of the previous chapters may be applied.

5.3.3. To minimise the mean square error, equation (5.31) is written in finite difference form

$$\epsilon_q = y_q - \sum_{r=1}^{N} h_r x_{q-r} \tag{5.32}$$

where

$$\epsilon_q = \text{value of } \epsilon(t) \text{ at the data point } t = T(i + \frac{q}{N}) \tag{5.33}$$

$$y_q = \text{value of } y(t) \text{ at the data point } t = T(i + \frac{q}{N}) \tag{5.34}$$

$$x_{q-r} = \text{value of } x(t) \text{ at the data point } t = T(i + \frac{(q-r)}{N}) \tag{5.35}$$

Now

$$\overline{\epsilon^2} = \frac{1}{\ell - r} \int_{-r}^{\ell} \epsilon^2(\ell) \, d\ell$$

$$= \frac{1}{M(\frac{r}{N})} \sum_{q=1}^{M} \epsilon_q^2$$

where

$$\ell_i = (1 + \frac{M_i}{N}) T$$

Hence

$$\overline{\epsilon^2} = \frac{1}{M} \sum_{q=1}^{M} \epsilon_q^2$$

$$= \text{a minimum} \tag{5.36}$$

is the time series equivalent of (5.29).
Giving $q$ the values $1, 2, \ldots, M$ in (5.32) yields $M$ simultaneous equations in the $N$ unknowns $h_r$ and the $M$ unknown error values $\epsilon_q$.

This set of simultaneous equations may be written as a single matrix equation

$$
\xi = \gamma - Xh
$$

(5.37)

where

- $\xi$ is a $M \times 1$ column vector of errors.
- $\gamma$ is a $M \times 1$ column vector of output function values.
- $h$ is a $N \times 1$ column vector defining the operator $h(t)$ of (5.30).
- $X$ is a $M \times N$ matrix of input function values.

Now

$$
\frac{1}{M} \sum_{q=1}^{M} \epsilon_q^2 = \frac{1}{M} \xi^T \xi
$$

is a minimum by (5.36),

where $\xi^T$ is transpose of $\xi$.

The optimisation problem now reduces to the choice of a vector $h$ in (5.37) which minimises the length of the vector $\xi$.

Since

$$
\frac{1}{M} \xi^T \xi = \frac{1}{M} \left\{ \gamma - h \bar{x} \right\} \left\{ \gamma - Xh \right\}
$$

(5.38)

from (5.37), it can be readily derived from the
matrix identity

$$\frac{1}{\bar{M}} \{ \bar{y} - \bar{x} \bar{x} \} \{ y - x h \}$$

$$\equiv \frac{1}{\bar{M}} \{ \bar{y} \bar{x} - \bar{y} \bar{x} [ \bar{x} \bar{x} ]^{-1} \bar{x} \} \{ x h - x [ \bar{x} \bar{x} ]^{-1} \bar{y} \}$$

$$+ \frac{1}{\bar{M}} \{ \bar{y} \bar{x} - \bar{y} \bar{x} [ \bar{x} \bar{x} ]^{-1} \bar{x} \} \{ y - x [ \bar{x} \bar{x} ]^{-1} \bar{y} \} \quad (5.39)$$

that the vector $h$ which minimises $\frac{1}{\bar{M}} \bar{y} \bar{x}$ is given by

$$h = \left[ \frac{1}{\bar{M}} \bar{x} \bar{x} \right]^{-1} \left[ \frac{1}{\bar{M}} \bar{x} \bar{y} \right] \quad (5.40).$$

The derivation of (5.40) from (5.38) and (5.39) follow the same argument as was used in the derivation of (5.25) from (5.18) and (5.21), and hence will not be repeated here.

Again the property, $[ \bar{x} \bar{x} ]$ is non-singular, is required if (5.40) is to define a vector $h$. A necessary (although not a sufficient) condition for $[ \bar{x} \bar{x} ]$ to be non-singular is that

$$\bar{M} \geq N.$$ 

(see Ferrat 1941, p.110, Theorem 33), from which it is concluded that the duration of the records must be at least twice the memory time $T$ of the system. If the duration of records is exactly twice the memory time, (5.37) is not overdetermined, and the solution cannot be considered a least squares optimisation. The more the duration of the records exceeds twice the system memory
time, the more the resulting equations (5.37) are over-
determined, and the greater is the probability that
the least squares procedure has minimised the effect
upon the solution of extraneous errors.

5.3.4. A certain similarity exists between the
solution (5.40), and the corresponding solution
(Wiener, 1949, p. 132) in terms of the correlation
estimates of the discrete data. To illustrate the
similarity, and to reveal the differences that exist,
it is of value to compare the elements of \( \frac{1}{M} \tilde{X} \tilde{X} \) and
\( \frac{1}{M} \tilde{X} \tilde{Y} \) with the corresponding expressions for the cross-
and auto-correlation functions.

Denoting the \( k \)th element of the column vector
\( \left[ \frac{1}{M} \tilde{X} \right] \) by \( \varphi_{xy}(k) \), this is given by

\[
\varphi_{xy}(k) = \frac{1}{M} \sum_{s=1}^{M} \tilde{x}_{s-k} y_s
\] (5.41)

Similarly, denoting the element in row \( j \) and column \( k \)
of the matrix \( \left[ \frac{1}{M} \tilde{X} \tilde{X} \right] \) by \( \varphi_{xx}(j,k) \), then

\[
\varphi_{xx}(j,k) = \frac{1}{M} \sum_{s=1}^{M} \tilde{x}_{s-j} \tilde{x}_{s-k}
\] (5.42)

The essential differences between (5.41) and
(5.42), and the results given by Wiener (1949, 132-134) are

(i) The averaging process defining \( \varphi_{xy}(k) \) and
\( \varphi_{xx}(j,k) \) is performed over a finite set of
data \( M \), instead of over an infinite set, and
(ii) There is no reason to suppose that \( \phi_{xx}(j, k) \) in (5.41) depends on \( j \) and \( k \) only through the difference \((j-k)\) as is assumed in a correlation formulation. A dependence of \( \phi_{xx}(j, k) \) on \( j \) and \( k \) only through \((j-k)\) is a consequence of the assumptions

(a) that \( x(t) \) is a stationary time-series,

(b) that \( M \) is sufficiently large that the sample correlation is a realistic estimate of the true correlation function.

The function \( \phi_{xx}(j, k) \) may be expected to be of the appropriate form when these hypotheses are valid for the given sample of data \( x(t) \). Unlike previous methods, however, equation (5.40) does not require the validity of these hypotheses for a successful solution.

5.3.5. With the vector \( h \), given by (5.40) defining the system dynamics, it is possible to proceed to a corresponding frequency domain description of the linear, passive, time-invariant, causal, finite memory system used to approximate the data.

Since

\[ H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt \]

it follows, from (5.30), that for operators \( h(t) \) here
considered

\[
H(\omega) = \int_{-\infty}^{+\infty} \frac{N}{\tau} \sum_{r=1}^{N} h_r \delta (t - r T_N) e^{-j\omega t} dt
\]

= \sum_{r=1}^{N} h_r \int_{(r-\frac{1}{2}) T_N}^{(r+\frac{1}{2}) T_N} e^{-j\omega t} dt \quad (5.43)

Provided \( \frac{T_N}{\tau} \) is sufficiently small that, for the whole range of values of \( \omega \) of interest, \( e^{-j\omega t} \) is sensibly constant over every range

\((r - \frac{1}{2}) T_N \leq t \leq (r + \frac{1}{2}) T_N\), (5.43) may be approximated by

\[
H(\omega) \approx \sum_{r=1}^{N} h_r e^{-j\omega T_N} \quad (5.44)
\]

To find the range of values of \( \omega \) for a given\( \frac{\tau}{T_N} \) for which (5.44) is a reasonable approximation to (5.43) integrate (5.43) to give

\[
H(\omega) = \frac{N}{\tau} \sum_{r=1}^{N} h_r \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_{(r-\frac{1}{2}) T_N}^{(r+\frac{1}{2}) T_N}
\]

= \frac{N}{\tau} \sum_{r=1}^{N} h_r e^{-j\omega T_N} \left\{ \frac{2 \sin(\omega T_N/2N)}{\omega T_N/2N} \right\}

\[
H(\omega) = \sum_{r=1}^{N} h_r e^{-j\omega T_N} \left\{ \frac{\sin(\omega T_N/2N)}{\omega T_N/2N} \right\} \quad (5.45)
\]

Now

\[
\frac{\sin(\omega T_N/2N)}{(\omega T_N/2N)} = 1 - \frac{1}{6} \left( \frac{\omega T_N}{2N} \right)^2 + \frac{1}{120} \left( \frac{\omega T_N}{2N} \right)^4 \ldots
\]
and differs from unity by less than 5% so long as

\[ \left( \frac{\omega T}{2N} \right) \leq \frac{1}{2} \]  

(5.46)

If \( H(\omega) \) is of interest only for values of \( \omega \) in this range, equation (5.44) may be used to determine \( H(\omega) \) from the optimum vector \( h^{*} \). When \( H(\omega) \) is required for values of \( \omega \) for which (5.46) is not satisfied, a correction, given by (5.45) must be applied to the value of \( H(\omega) \) found from (5.44).

The right hand side of (5.45) may be calculated for any given value of \( \omega \) when \( h \) is known. The calculation may be generalised by the introduction of a matrix \( \Phi \) which, when premultiplying the given vector \( h \) gives a \( H \) vector for a specified set of values of \( \omega \). Such a matrix, if suitably derived, may be used to premultiply any \( h \) vector, irrespective of the memory time \( T \) of the system.

For example, from (5.45)

\[ H_{s} = \frac{\sin \pi s}{\pi s} \sum_{r=1}^{N} h_{r} e^{-j 2\pi s r}, \quad s = 0, 1, 2, \ldots, (m-1) \]

where

\[ H_{s} = \left[ H(\omega) \right]_{\omega = 2\pi s \left( \frac{S}{T} \right)} \]

Hence

\[ H = \Phi \; h \]
where \( \Phi = \begin{bmatrix} \Phi_{s\bar{s}} \end{bmatrix} = m \times N \) matrix

and \( \Phi_{s\bar{s}} = \frac{\sin \pi s}{\pi s} e^{-j2\pi s} \)

The \( \Phi \) matrix of table 5-1, when premultiplying a given 20 element \( h \) vector gives the corresponding \( H \) vector for the values

\[ \omega = 2\pi s \left( \frac{\gamma}{r} \right) \quad \text{where} \]

\[ s = 0(0.05) 0.5 \]

5.3.6. The procedure developed in the previous sections for the optimisation of 2-ports may be extended directly to n-ports in the same way that the 2-port procedures of chapter 3 were extended to give the n-port procedures of chapter 4.

Just as the time series equation (5.32) was developed from the 2-port continuous variable equation (3.2), so the equation

\[ e_q = \gamma_q - \sum_{s=1}^{m} \sum_{s=1}^{N} h_{s\bar{s}} x_s (q-t) , \quad q = 1, 2, \ldots, m (5.47) \]

may be developed from (4.9) as the time series description of the Type-(b) n-port.
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In equation (5.47)

\[ y_q = \text{value of } y(t) \text{ at } t = T(1 + \frac{q}{N}) \]

\[ x_s(q-r) = \text{value of } x_s(t) \text{ at } t = T(1 + \frac{(q-r)}{N}) \]

\[ h_{sr} = \text{value of } h_s(t) \text{ at } t = \frac{rT}{N} \]

\[ \epsilon_q = \text{value of } \epsilon(t) \text{ at } t = T(1 + \frac{q}{N}) \]

and the available records define \( x_s(t) \) and \( y(t) \) over the interval

\[ 0 \leq t \leq T(1 + \frac{M}{N}) \]

The \( M \) simultaneous equations (5.47) may be represented by the single matrix equation

\[ \mathbf{\epsilon} = \mathbf{y} - \sum_{q=1}^{M} \mathbf{X}_s \mathbf{h}_s \quad (5.48) \]

where

\[ \mathbf{y} = M \times 1 \text{ column vector of ordinates } y_q \]

\[ \mathbf{\epsilon} = M \times 1 \text{ column vector of ordinates } \epsilon_q \]

\[ \mathbf{h}_s = N \times 1 \text{ column vector of ordinates of } h_s(t) \]

\[ \mathbf{X}_s = M \times N \text{ matrix having the ordinate } x_s(q-r) \text{ for the element in row } q \text{ and column } r. \]

As in the 2-port problem

\[ \overline{\mathbf{\epsilon}^2} = \frac{1}{M} \sum_{q=1}^{M} \epsilon_q^2 \]

\[ = \frac{1}{M} \mathbf{\epsilon}^T \mathbf{\epsilon} \]
is to be minimised, by a suitable choice of vectors $h^s$.

The minimisation proceeds from (5.48) by arguments similar to those already used for the 2-port problem, and lead to the condition

$$
\frac{1}{M} \tilde{X}_p \tilde{Y} = \sum_{\tilde{s}_1} \frac{1}{M} \tilde{X}_p \tilde{X}_s h^s, \quad p = 1, 2, \ldots, m. \quad (5.49)
$$

Writing (5.49) as a matrix equation

$$
\begin{bmatrix}
\frac{1}{M} \tilde{X}_1 \tilde{X}_1, & \frac{1}{M} \tilde{X}_1 \tilde{X}_2, & \ldots, & \frac{1}{M} \tilde{X}_1 \tilde{X}_m \\
\frac{1}{M} \tilde{X}_2 \tilde{X}_1, & \frac{1}{M} \tilde{X}_2 \tilde{X}_2, & \ldots, & \frac{1}{M} \tilde{X}_2 \tilde{X}_m \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{M} \tilde{X}_m \tilde{X}_1, & \frac{1}{M} \tilde{X}_m \tilde{X}_2, & \ldots, & \frac{1}{M} \tilde{X}_m \tilde{X}_m
\end{bmatrix}
\begin{bmatrix}
h^1 \\
h^2 \\
\vdots \\
h^m
\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{M} \tilde{X}_1 \tilde{Y} \\
\frac{1}{M} \tilde{X}_2 \tilde{Y} \\
\vdots \\
\frac{1}{M} \tilde{X}_m \tilde{Y}
\end{bmatrix} \quad (5.50)
$$

Provided the partitioned matrix in (5.50) is non-singular, these $N_m \times M$ equations describe a unique set of ordinates $h^s$ describing the system dynamics.

The same result could have been obtained by treating (5.47) directly as a set of $M$ equations in $N_m$ unknowns $h^s$. The advantage of the partitioned form of matrix equation (5.50) is that the identity of the elements of the matrix, and their dependence on the
In equation (5.47)

\[ y_q = \text{value of } y(t) \text{ at } t = T(1 + \frac{q}{N}) \]
\[ x_s(q-r) = \text{value of } x_s(t) \text{ at } t = T\left(1 + \frac{(q-r)}{N}\right) \]
\[ h_{sr} = \text{value of } h_s(t) \text{ at } t = \frac{rT}{N} \]
\[ \epsilon_q = \text{value of } \epsilon(t) \text{ at } t = T(1 + \frac{q}{N}) \]

and the available records define \( x_s(t) \) and \( y(t) \) over the interval

\[ 0 \leq t \leq T\left(1 + \frac{M}{N}\right) \]

The \( M \) simultaneous equations (5.47) may be represented by the single matrix equation

\[ \epsilon = y - \sum_{s=1}^{M} X_s h_s \tag{5.48} \]

where

\[ y = M \times 1 \text{ column vector of ordinates } y_q \]
\[ \epsilon = M \times 1 \text{ column vector of ordinates } \epsilon_q \]
\[ h_s = N \times 1 \text{ column vector of ordinates of } h_s(t) \]
\[ X_s = M \times N \text{ matrix having the ordinate } x_s(q-r) \text{ for the element in row } q \text{ and column } r. \]

As in the 2-port problem

\[ \frac{\bar{\epsilon}^2}{M} = \frac{1}{M} \sum_{q=1}^{M} \bar{\epsilon}_q^2 \]

\[ = \frac{1}{M} \bar{\epsilon} \cdot \bar{\epsilon} \]
experiment may result in data which either

(a) render the optimisation problem insoluble, or

(b) provide a solution, but one which is of no value.

Before the results of an optimisation problem of
this sort can be used with confidence in the making of
policy decisions, two additional questions need
consideration, namely

(a) does the solution of the optimisation problem provide
    an adequate fit to the given data? And

(b) does the solution so obtained remain invariant under
    changes of data?

It has already been observed that, if the permitted
class of system operator is sufficiently general as to
include a sub-class to which the dynamics investigated belong,
then the answer to both these questions is in the affirmative.
It is seldom known in advance, however, whether the
postulated class is sufficiently general, and any attempt to
extend the permitted class beyond the linear time-invariant,
passive case requires experimental information (e.g. the
vector $G_{xn}$ in (4.84) for linear time-invariant active
operators) which may not be readily available. Under these
conditions, it is necessary to assume as general a class as
all available experimental information permits, to solve the
optimisation problem subject to this assumption, and then
to devise tests to reveal the suitability of the assumption. If the solution obtained from a particular experiment gives an efficiency of approximation \( \eta \) which is near unity, where

\[
\eta = 1 - \frac{\text{Mean square error}}{\text{Mean square value of observed output data}}
\]

then the optimisation may be said to give an adequate fit to the given data. The portion of the whole range of possible values of \( \eta (0 \leq \eta \leq 1) \) which represents an adequate approximation depends, amongst other things, on the use to which the information is to be put. No a priori range of acceptable values can be stated. It must be emphasised, however, that if \( \eta = 0.8 \), for example, then 80% of the actual output power (or energy) of the physical system investigated is accounted for by the operator approximating the dynamics. The remaining 20% may be considered a noise power associated with, but unaccounted for by, the approximate description of the dynamics. Hence if \( \eta = 0.8 \), the ratio

\[
\frac{\text{Signal Power}}{\text{Noise power}} = 4
\]

With such a ratio, the effect of the extraneous noise power, the effect of the noise power on the system behaviour can hardly be considered negligible. This suggests that
for \( \eta < 0.8 \) the usefulness of the results of an
optimisation of the type here discussed would be very
limited indeed.

Even in those situations in which the value of
obtained from a single trial is close to unity, it is
inadmissible to conclude that the permitted class adopted
is sufficiently general for the investigation in hand.
This was clearly revealed by the example discussed in
section 4.7.1. It is clearly inadequate in such a problem,
to find an optimum linear, time-invariant, passive operator,
to test the efficiency of such an approximation, and, finding
it unity, to accept the result as a useful description of
the system dynamics. It is essential to establish further
that the operator selected to define the system dynamics
shall remain sensibly invariant under arbitrary changes of
data, by repeating the optimisation a number of times with
different samples of operating data. Each success increases
confidence in the description of the system dynamics adopted.

If the normal operating data approximate stationary
ergodic time variables, changes from one sample of operating
data to the next will only produce small changes in the
elements of \( G_{xx} \) and \( G_{xy} \) (or \( X^T X \) and \( X^T Y \) in the problems of
this chapter). It is of interest to enquire if small
changes in the coefficients of a set of simultaneous equations
can produce profound changes in the solution. Such a situation must lead to a solution of the optimisation problem which is excessively sensitive to the sample of data fitted. Under these conditions the optimisation is critically dependent on the set of data selected, and the result of any single trial cannot be adopted as a useful description of the dynamics under investigation.

Sets of simultaneous equations having this property are common. Interpreted geometrically the problem is the determination of the co-ordinates of the point of intersection of \( n \) surfaces in \( n \)-dimensional space when two or more of the surfaces have similar direction cosines (are nearly parallel). The solution of this problem leads to a vector defining the point of intersection which is very sensitive to small changes in the direction cosines. The matrix description of this problem is characterised by a strongly skew angular matrix. If two or more of the surfaces are exactly parallel, there is no point of intersection, which is reflected in a singular matrix in the matrix formulation. This situation has no mathematical solution and even the related situation of a nearly singular matrix, while mathematically tractable, has no solution of physical significance because the excessive accuracy required in the coefficients is unattainable when these coefficients are the result of physical measurement.
Lanczos (1957 p.149-170) has given an excellent discourse of these difficulties, and concludes that "the critical quantity which decides the physical reliability of a strictly mathematical solution ... is the ratio of the largest to the smallest eigenvalue of the symmetrized matrix" (i.e. of $G_{xx}$ in the method of Chapter 4 or of $X'X$ in the methods of the present chapter). "It is the square root of this ratio which measures the magnification of the noise in the direction of the smallest eigenvalue". A test of the sensitivity of a solution upon the sample of data used is thus afforded by the ratio of these eigenvalues. Provided this ratio is not excessive, the solution of the data fitting problem may be expected to yield a physically useful description of the dynamics for the type of input data expected. What constitutes an "excessive value for this ratio can only be determined by experience, it depending upon the orders of magnitude of the changes in the elements of $G_{xx}$ (or $X'X$) resulting from changes in the sample of data employed. Certainly one can say, with Lanczos (p.169) that if this ratio is in excess of $10^4$ the result, while mathematically correct, is of little physical value. It may well happen that ratios considerably less than this are equally of doubtful value where the variance of the matrix elements computed from one sample to the next is appreciable.
5.5 CONCLUSIONS AND RECOMMENDATIONS.

Although the object of this investigation was to seek a suitable technique for the determination of the characteristics of linear networks from input and output records under normal operation in the presence of noise, much of the effort has been devoted to least squares procedures in which optimum linear time-invariant passive operators have been selected. The essential fact has emerged that the available recorded information is insufficient, in itself, to permit a least squares optimisation from the larger class containing active operators (to which systems subject to noise belong!). One is forced by the insufficiency of information to seek a passive approximation to active dynamics.

When additional information can be made available giving a quantitative measure of some suitable property of the noise associated with the system dynamics, it becomes possible to develop a constrained least squares optimisation procedure for the enlarged class including linear time-invariant active system operators, as well as the passive sub-class. This was demonstrated with spectral density data for a particular constraint ($G_{xn}$ a specified vector) in section 4.8. An identical analysis could be developed for the treatment of finite-memory problems by the methods of
thus a least square procedure for the selection of an optimum linear time-invariant, active, finite memory operator requires that the right hand side of (5.50) be modified to read

\[
\begin{bmatrix}
\frac{1}{M} \tilde{X}_1 y - \frac{1}{M} \tilde{X}_1 n \\
\frac{1}{M} \tilde{X}_2 y - \frac{1}{M} \tilde{X}_2 n \\
\vdots \\
\frac{1}{M} \tilde{X}_n y - \frac{1}{M} \tilde{X}_n n
\end{bmatrix} (5.51)
\]

If it was possible to place some realistic constraint defining vectors \( \tilde{X}_r \), this modified equation could be solved for the operators \( \tilde{h}_r \) (again assuming the square matrix on the left of (5.50) is non-singular).

It is conceivable that this particular constraint (or the corresponding one considered in section 4.8) is not, in all circumstances, the most suitable one. Work yet remains to be done to provide a catalogue of those effects of the noise disturbances which (a) can be readily observed, and (b) make a constrained least squares procedure feasible. An attempt to exploit knowledge of the autocorrelation function of the noise has, as yet, met with little success. It may be that such a constraint is not sufficiently strong.
to remove the indeterminacy from the problem.

Turning again to the passive approximation to active dynamics, it may be observed from the results of Chapter 4 (e.g. by comparison of (4.84) and (4.20)) that such an approximation can make a useful contribution to the making of 'policy decisions' in situations in which spectral density data is available if $G_{xn}$ is negligibly small, i.e. if each input $x_r(t)$ is sensibly uncorrelated with $n(t)$. For the finite memory problems of the present chapter a similar observation can be made. Thus comparison of the solution of (5.50) with the solution of the active optimisation (i.e. the right hand side of (5.50) replaced by (5.51)) reveals that the two solutions are sensibly the same provided the noise vector $n$ is sensibly orthogonal to every column vector in the matrices $X_1, X_2, \ldots, X_n$, when each $X_r n$ vector is negligibly small compared to the corresponding $X_r v$ vector. The interpretation of the expressions 'sensibly orthogonal' and 'sensibly uncorrelated' depends upon the conditioning of the equations to be solved. If the equations are ill-conditioned - i.e. the square matrix on the left hand side of the equation is nearly singular, the effect of the noise vectors $\tilde{X}_r n$ (and $G_{xn}$) upon the solution is profound even though these vectors may be small compared with the signal vectors $X_r v$ (and $G_{xy}$). One concludes, therefore, that input and output records alone
can only be expected to provide a useful description of the dynamics of systems in which noise is present in a very limited class of problem, and one must attempt to establish, in any given situation, that the experimental data can provide a physically useful solution. It still remains to develop suitable tests for this purpose. Short of producing estimates, where possible, of the noise vectors $X_{rn}$ (or $G_{xn}$), which then permits the development of an active description of the dynamics anyway — the only test which the author can envisage at present is to repeat the procedure a number of times, using different experimental data for each trial. If the solution remains invariant, within acceptable limits, over a number of trials, it may be assumed, confidently, that the noise contribution to the output is sufficiently small as to make the passive description a useful approximation. If the solutions do not remain invariant, the converse must be concluded, and information must be sought to make available a realistic estimate of the effect of the noise.

Any internal, active, element (i.e. noise) in a system, which can be experimentally observed, should be recorded and treated as an additional input variable, even though the dynamics relating this particular variable to the output may be of no immediate interest. In short, as many active elements as possible should be treated as input variables in a system dynamic study, while the system should
be confined, ideally, to include passive components only. In this way the effect of noise is minimised, and the usefulness of the resulting description of the dynamics maximised.

Work yet remains to be done to further extend the permitted class of system operators for which a least squares optimisation procedure exists. No attempt has been made here to remove the time invariant constraint. The removal of this constraint presents considerable difficulty because of the severely underdetermined nature of the resulting equations. At present it is not at all clear what is the essential experimental data which should be collected in order to permit the development of a least squares optimisation procedure for the selection of operators from a general linear class including both the time invariant and the time variant sub-classes. It is felt, however, that only by continually attempting to enlarge the permitted class of system operator in this way can (a) realistic estimates of the dynamics of complex physical and non-physical systems be made, and (b) decisions be made about desirable advances in measuring techniques in order to effect such estimates. Even within the confines of linear system dynamic studies much remains to be done.
6. REFERENCES

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(b) "The determination of process dynamics from normal disturbance records of a controlled process". Heidelberg Conference on Automatic Control, September.


1959: (a) "Data Fitting with Linear Transfer Functions". Jour.Electron. and Contr., 6, 454.
   (b) "On Finding a Best Linear Approximation to System Dynamics from Short Duration Samples of Operating Data". Jour.Electron. and Contr., 7, 176.

7.1 THE DERIVATION OF EQUATION (1.3)

By definition

\[ F_{rs}[x_s] = \text{contribution to the 'effect' } y_r(t) \text{ (observed at terminals } r), \text{ resulting from the 'cause' } x_s(t) \text{ (applied at terminals } s) \text{, when the functional dependence describing the system dynamics is a linear one.} \]

Now the continuous variable \( x_s(t) \) may be treated as the superposition of impulse functions. Thus one can write

\[ x_s(t) = \lim_{\varepsilon \to 0} \sum_{\tau = -\infty}^{+\infty} x_s(\tau) \delta(t-\tau) \delta\varepsilon \]  

(7.1)

where

\[ \delta(\tau) = \frac{1}{\varepsilon} \delta \varepsilon \text{ for } \tau \leq t \leq \tau + \delta \varepsilon \]

\[ = 0 \text{ outside this range} \]  

(7.2)

Denote the contribution to the output observed at terminal \( r \) at time \( t_1 \) by an applied input to terminal \( s \) of the form

\[ \lim_{\varepsilon \to 0} \int \delta(t-\tau) \delta \varepsilon \]

by \( h_{rs}(t_1, \tau) \)

Then, by superposition of the contributions, the
total contribution to \( y_s(t) \) by \( x_s(t) \) is given by

\[
F_{rs}[x_s] = \mathcal{L} \left[ \sum_{\tau=-\infty}^{\infty} h_{rs}(\tau, \tau) x_s(\tau) \delta(\tau - \tau) \right] = \int_{-\infty}^{+\infty} h_{rs}(\tau, \tau) x_s(\tau) \, d\tau \tag{7.3}
\]

On setting \( t_1 = t \), equation (1.3) follows.
A Closed Loop Data Fitting Problem†

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ABSTRACT

Study of data collected from a linear, passive, time-invariant, closed loop reveals that the linear dependence of the control variable upon the input variable makes data collected from such a system unsuitable for the determination of its dynamics.

It is shown that the introduction of a disturbance into the loop can destroy the linear dependence, but special care may still be necessary for the determination of the dynamics from the data.

It emerges that:

(a) to seek a linear, passive, time-invariant operator, to approximate the active element containing the disturbance, leads to a physically valueless solution of the optimization problem, while

(b) the insertion of the disturbance into the loop at a point external to the element studied or

(c) the choice of an optimum operator from a linear, active, time-invariant sub-class, makes a physically meaningful solution of the optimization problem possible.

§ 1. INTRODUCTION

In a previous paper (Woodrow 1959) a method for fitting a best linear, passive time-invariant, approximation to data collected from systems having more than one input variable, was developed. This paper applies these results to a closed loop data fitting problem, and shows that such data may possess properties which make special treatment necessary.

The example selected for study is the determination of the dynamics of the elements of closed loops which may be reduced to the general arrangement of fig. 1 (a). It is assumed that the data recorded for this study is $x_1(t)$, $x_2(t)$ and $y(t)$.

If, as is often assumed, the 3-port‡ is a summing device, then

$$z(t) = x_1(t) + x_2(t)$$

(1.1)

is also readily available. If the dynamics of the 3-port of fig. 1 (a) is not known, so that $z(t)$ cannot be developed from any known relationship similar to (1.1), the problem must be treated as in fig. 1 (b).

† Communicated by the Author.

‡ This term has been borrowed from passive network synthesis studies.

A general $n$-port is a system having $m$ input variables and $(n-m)$ output variables where $n$ and $m$ are integers and $n > m$. The $n$-port here has $m=2$ and $n=3$. 
In the case of fig. 1(a) the determination of the dynamics reduces to two separate 2-port optimization procedures. Using a minimum mean square error optimization, this problem has already been adequately treated (Woodrow 1959).

In the case of fig. 1(b), the determination of the system dynamics B is still a straightforward 2-port data fitting problem, \( y(t) \) being the observed input data, and \( x_2(t) \) the observed output data of this element. This presents no difficulty.

To determine the dynamics A, the suggestion (Woodrow 1959, p. 475) that the dynamics be described by the arrangement of fig. 2 is adopted.
The weighting functions \( h_1(t) \) and \( h_2(t) \) are to be chosen to minimize the mean square error \( e^2 \) in the equation
\[
e(t) = y(t) - \sum_{r=1}^{\infty} \int_{-\infty}^{+\infty} h_r(\tau)x_r(t-\tau) \, d\tau.
\] ............................ (1.2)

Fig. 2

The solution of such an optimization procedure requires (Woodrow 1959, Appendix IV) that the frequency response \( H_r(w) \) (the Fourier transform of \( h_r(t) \)) shall satisfy the equation
\[
\begin{bmatrix}
H_{1r} \\
H_{2r}
\end{bmatrix} =
\begin{bmatrix}
G_{x_r x_1} & G_{x_r x_2} \\
G_{s_r x_1} & G_{s_r x_2}
\end{bmatrix}^{-1}
\begin{bmatrix}
G_{x_r y} \\
G_{s_r y}
\end{bmatrix} \quad \cdots \cdots \quad (1.3)
\]

where \( G_{rs} \) denotes the spectral density function derived from \( r(t) \) and \( s(t) \).

Equation (1.3) has a unique solution provided the inverted matrix exists, i.e. provided the matrix
\[
\begin{bmatrix}
G_{x_r x_1} & G_{x_r x_2} \\
G_{s_r x_1} & G_{s_r x_2}
\end{bmatrix}
\] ............................ (1.4)
is non-singular. The question of the singularity of (1.4) is of special interest in closed loop problems.

\section*{§ 2. Power Spectra Relationships for Linear Passive Time-invariant Closed Loop Data}

Suppose \( x_1(t) \), \( x_2(t) \) and \( y(t) \) actually originated in a linear, passive, time-invariant system capable of representation by the circuit of fig. 3.

It is required to estimate the dynamics from the observed operating data \( x_1(t) \), \( x_2(t) \) and \( y(t) \) using a minimum mean square error optimization procedure.
The determination of the best approximation \( h(t) \) to the dynamics \( f(t) \) presents no problem. Using the optimization procedure already developed (Woodrow 1959, p. 466, eqn. (4.7)) for the value of \( h(t) \) which minimizes \( \varepsilon^2 \) in the equation

\[
e(t) = x(t) - \int_{-\infty}^{+\infty} h(\tau)y(t-\tau)\,d\tau,
\]

it is found that

\[
H(w) = \frac{G_{xx}}{G_{yy}},
\]

where \( H(w) \) is the Fourier transform of \( h(t) \).

It can readily be shown that, for this data

\[
H(w) = F(w),
\]

or

\[
h(t) = f(t).
\]

Before applying (1.3) to find \( H_1 \) and \( H_2 \), the best approximations to \( F_1 \) and \( F_2 \), it is necessary to test the data to establish that (1.4) is non-singular.

For the given data, it follows, from fig. 3, that

\[
x_2(t) = \int_{-\infty}^{+\infty} f(\tau)x_1(t-\tau)\,d\tau
\]

and

\[
y(t) = \int_{-\infty}^{+\infty} f_1(\tau)x_{12}(t-\tau)\,d\tau + \int_{-\infty}^{+\infty} f_2(\tau)x_2(t-\tau)\,d\tau
\]
A Closed Loop Data Fitting Problem

Forming the appropriate correlation functions from (2.2), and then taking Fourier transforms to establish spectral density relationships gives

\[ G_{x,y} = FG_{x,y}, \]  

\[ G_{x,y} = FG_{x,y}. \]  

(2.4)  

(2.5)

In the same way, (2.3) leads to the relations

\[ G_{x,y} = F_1 G_{x,x_1} + F_2 G_{x,x_2}, \]  

\[ G_{x,y} = F_1 G_{x,x_1} + F_2 G_{x,x_2}, \]  

(2.6)  

(2.7)

Using (2.4) and (2.5) to eliminate \( G_{x,y} \) and \( G_{x,y} \) from (2.6) and (2.7) gives

\[ G_{x,x_1}(1 - FF_2) = FF_1 G_{x,x_1}, \]  

\[ G_{x,x_1}(1 - FF_2) = FF_1 G_{x,x_1}. \]  

(2.8)

From the pair of eqns. (2.8) it follows that the available data has the property

\[ G_{x,x_1} G_{x,x_1} - G_{x,x_2} G_{x,x_2} = 0. \]  

(2.9)

Equation (2.9) is the condition that (1.4) shall be singular, and hence the optimization procedure fails, since no unique solution results, and both \( H_1(w) \) and \( H_2(w) \) are indeterminate.

Any attempt to solve (1.3) gives

\[ H_1(w) = (G_{x,y} G_{x,x_1} - G_{x,x_2} G_{x,y})/(G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2}), \]  

(2.10)

\[ H_2(w) = (G_{x,y} G_{x,x_1} - G_{x,x_2} G_{x,y})/(G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2}). \]  

(2.11)

From (2.4), (2.5) and (2.10)

\[ H_1 = (G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2})/F(G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2}). \]  

(2.12)

From (2.4), (2.5) and (2.11)

\[ H_2 = (G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2})/F(G_{x,x_1} G_{x,x_2} - G_{x,x_2} G_{x,x_2}). \]  

(2.13)

Equations (2.12) and (2.13) illustrate the indeterminacy of \( H_1(w) \) and \( H_2(w) \) when using the given data.

The conclusion to be drawn from this is that normal operating data collected from linear passive time invariant closed loops are unsuitable for the determination of the dynamics of the system. The explanation of this is not hard to find. The inputs \( x_1(t) \) and \( x_2(t) \) are not independent variables in this problem. \( x_2(t) \) is linearly dependent upon \( x_1(t) \) as is \( y(t) \).

From (2.5) and (2.6) it follows that

\[ G_{x,y} = \frac{F_1}{1 - FF_2} G_{x,x_1}, \]  

(2.14)

which establishes the linear dependence of \( y(t) \) on \( x_1(t) \).

From (2.8) and (2.9)

\[ G_{x,x_1} = \frac{F_1}{1 - FF_2} G_{x,x_1}, \]  

(2.15)

which establishes the linear dependence of \( x_2(t) \) on \( x_1(t) \).
Let \( Y = \frac{F_1}{1 - FF_2} \) = overall frequency response of the loop.

Write \( Y = Y_a + (Y - Y_a) \), where \( Y_a \) is arbitrary.

Write \( Y - Y_a = \frac{FF_1}{1 - FF_2} \cdot Y_b \)

or \( Y_b = \frac{1 - FF_2}{FF_1} (Y - Y_a) \) . . . . . . . . . . (2.16)
§ 3. A Linear Passive Approximation to Linear Active Dynamics

Since data from a linear passive closed loop does not define a unique solution, it is necessary to introduce a separate disturbance into the loop somewhere. In this section the disturbance is supposed associated with the 3-port of fig. 1 (b). This may be allowed for by adding an additional term \( n(t) \) to eqn. (2.3), so that the actual relations between \( x_1(t), x_2(t) \) and \( y(t) \) are now

\[
y(t) = n(t) + \int_{-\infty}^{+\infty} f_1(\tau)x_1(t-\tau) \, d\tau + \int_{-\infty}^{+\infty} f_2(\tau)x_2(t-\tau) \, d\tau \quad \text{(3.1)}
\]

and

\[
x_2(t) = \int_{-\infty}^{+\infty} f(\tau)y(t-\tau) \, d\tau. \quad \text{(3.2)}
\]

If a best linear passive time-invariant approximation to the 3-port is sought through eqn. (2.10), one is attempting to approximate dynamics of a linear, active, time-invariant system by a suitable linear passive time-invariant operator. The result of such a procedure can never yield an exact description of the system dynamics, although it may yield an exact fit to the given data. This particular example demonstrates this point.

Before applying (1.3) to find the solution to this problem, it is again necessary to test that (1.4) is non-singular. This is done by applying to (3.1) and (3.2) similar arguments to those leading from (2.2) and (2.3) to (2.9). In this case this gives

\[
G_{x_2} (1 - FF_{G}) = FF_{G} G_{x_2 x_1} + FG_{x_2 n} \quad \text{(3.3)}
\]

and

\[
G_{x_2} (1 - FF_{G}) = FF_{G} G_{x_2 x_1} + FG_{x_2 n} \quad \text{(3.4)}
\]

Dividing (3.3) by (3.4) gives

\[
G_{x_2} / G_{x_2 x_1} = (F_{G} G_{x_2 x_1} + G_{x_2 n}) / (F_{G} G_{x_2 x_1} + G_{x_2 n}), \quad \text{(3.5)}
\]

from which it follows that

\[
F_{G} (G_{x_2 x_1} - G_{x_2 n} G_{x_2 n}) = G_{x_2 x_1} G_{x_2 n} - G_{x_2 n} G_{x_2 n}, \quad \text{(3.6)}
\]

If, for example, \( x_1(t) \) and \( n(t) \) are uncorrelated, so that \( G_{x_1 n} \) becomes zero, the right-hand side of (3.6) cannot be zero since \( x_2(t) \) is correlated with both \( x_1(t) \) and \( n(t) \). Even when \( x_1(t) \) and \( n(t) \) are correlated, it is only under special circumstances, i.e. when \( G_{x_1 n} G_{n n} = G_{x_1 n} G_{n x_1} \), that the right-hand side of (3.6) reduces to zero.

It may be concluded that a solution to the data fitting problem is now possible, and it remains to find it.
From (3.2) it follows that

\[ G_{x_1 x_1} = F G_{x_1 y}, \quad \ldots \ldots \ldots \quad (3.7) \]

\[ G_{x_2 x_2} = F G_{x_2 y}, \quad \ldots \ldots \ldots \quad (3.8) \]

A comparison of (3.7) with (2.4) and (3.8) with (2.5) shows that eqns. (2.10) and (2.11) (which still define the best linear passive time-invariant approximation to the data) may be reduced, as in §2 to (2.12) and (2.13).
A Closed Loop Data Fitting Problem

But, in this case (1.4) is non-singular, and hence, from (2.12) and (2.13),

\[ H_1(w) = 0, \quad \ldots \ldots \ldots \ldots \quad (3.9) \]
\[ H_2(w) = 1/F(w). \quad \ldots \ldots \ldots \quad (3.10) \]

Figure 5 gives a comparison of (a) the actual system dynamics, with (b) the best linear passive time-invariant approximation.

It may readily be shown that the system of fig. 5 (b) produces the output variable \( y(t) \) exactly from the given input variables \( x_1(t) \) and \( x_2(t) \), i.e. the error of the approximation is zero. The procedure has thus led to an exact fit to the data, but the lack of exactness of the description of the dynamics of the system of fig. 5 (a) needs no comment. Anticipating the results of §5, it may be stated that this discrepancy arises because the class of operator permitted in the optimization procedure is not sufficiently general.

Two conclusions may be drawn from the results of this section, namely:

(a) A best linear passive approximation to an active n-port in a closed loop data fitting problem leads to a regenerative description of the system dynamics in which the input disturbance \( x_1(t) \) is isolated from the loop.

(b) It cannot be assumed that an optimization process which leads to an exact fit of the data necessarily gives an exact description of the system dynamics.

§ 4. Solving the Closed Loop Problem

The problem of the determination of the dynamics of the closed loop of fig. 1 (b) cannot yet be considered solved. It has so far been established that the data from an entirely passive loop can give no solution, while a best linear passive approximation to the active element leads to a solution, but one which is physically meaningless.

The remaining possibilities are (a) to situate the disturbance at some point in the loop outside the element for which a best linear passive time-invariant approximation is required, or (b) to generalize the optimization procedure to permit a choice of linear, active, time-invariant operators. These possibilities are considered in this section and in §5. Both are found to yield an exact description of the system dynamics.

Suppose a disturbance \( n(t) \) is introduced, this time, into the 2-port of fig. 1 (b), so that the actual relationships of the data are given, from appropriate modification of (2.2) and (2.3), by

\[ x_2(t) = n(t) + \int_{-\infty}^{+\infty} f(\tau)y(t-\tau)\,d\tau \quad \ldots \ldots \ldots \quad (4.1) \]

and

\[ y(t) = \int_{-\infty}^{+\infty} f_1(\tau)x_1(t-\tau)\,d\tau + \int_{-\infty}^{+\infty} f_2(\tau)x_2(t-\tau)\,d\tau. \quad (4.2) \]

The 3-port, the dynamics of which are required, is now a linear passive time-invariant one, and the optimization procedure requires a choice,
from the same class, of operators $h_1(\tau)$ and $h_2(\tau)$ to minimize $\bar{e}^2$ in the eqn. (1.2). Provided (1.4) is non-singular, which can again be shown to be the case, the optimization procedure leads to a solution.

From (1.3), it follows, as before, that $H_1$ and $H_2$ are given by (2.10) and (2.11). It remains only to find the relationship between $H_1$ and $F_1$ and between $H_2$ and $F_2$.

From (4.2) it follows that

$$G_{x_1y} = F_1 G_{x_2x_1} + F_2 G_{x_2x_2} \quad \ldots \ldots \quad (4.3)$$

and

$$G_{x_1y} = F_1 G_{x_2x_1} + F_2 G_{x_2x_2} \quad \ldots \ldots \quad (4.4)$$

Multiplying (4.3) by $G_{x_1x_1}$ and (4.4) by $G_{x_2x_1}$, subtracting, and comparing the result with (2.10), yields

$$H_1(w) = F_1(w). \quad \ldots \ldots \quad (4.5)$$

Similarly, multiplying (4.3) by $G_{x_1x_1}$ and (4.4) by $G_{x_2x_1}$, subtracting, and comparing the result with (2.11) yields

$$H_2(w) = F_2(w). \quad \ldots \ldots \quad (4.6)$$

Equations (4.5) and (4.6) show that, in this case, the presence of a disturbance in that portion of the loop external to the element studied, not only makes a solution possible, but also yields a correct physical description.

It must be concluded from this, that if a closed loop is suspected to be passive, a suitable disturbance must be injected into the 2-port when determining the dynamics of the 3-port. The dynamics of the 2-port, if required, should be calculated as in §1, from data collected before the disturbance is introduced.

§ 5. LINEAR ACTIVE TIME-IN Variant DATA FITTING OPERATORS

Finally, it is of interest to return to the problem of §3, in which it was supposed that the 3-port was an active one and the 2-port a passive element, as defined by (3.1) and (3.2). It follows from §3, that to seek a linear passive time-invariant approximation to the 3-port is of little value. Consider then, the possibility of fitting the data with a best set of operators from a linear active, time-invariant class.

With available data $x_1(t)$, $x_2(t)$ and $y(t)$ defined by (3.1) and (3.2), it is proposed to seek operators $\hat{h}_1(t)$ and $\hat{h}_2(t)$ in the equation

$$\epsilon(t) = y(t) - \left\{ \sum_{\tau=-\infty}^{2} \int_{-\infty}^{+\infty} h_\tau(t-\tau) d\tau + m(t) \right\} \ldots \ldots (5.1)$$

where $m(t)$ is the equivalent disturbance referred to the output terminals of the active system.

An optimization procedure which permits complete freedom of choice of $m(t)$ as well as $\hat{h}_1(t)$ and $\hat{h}_2(t)$ produces no useful results because an infinity of operators of the permitted class exists which can reduce the mean square error of the data fitting operation to zero.
If the optimization procedure only permits freedom of choice of $h_1(t)$ and $h_2(t)$, while $m(t)$ is subjected to a predetermined constraint, useful results can be developed.

In this case, the optimization procedure is the same as the linear, passive, time-invariant one by which (1.3) was developed. In fact, the result may be quoted directly, by replacing $y(t)$ in the passive optimization by

$$[y(t) - m(t)]$$

in the active optimization. By comparison with (1.3), the operator for the optimization of (5.1) with $m(t)$ constrained is given by

$$
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} = \begin{bmatrix}
G_{x_1,x_1} & G_{x_1,x_2} \\
G_{x_2,x_1} & G_{x_2,x_2}
\end{bmatrix}^{-1} \begin{bmatrix}
(G_{x_1,y} - G_{x_1,m}) \\
(G_{x_2,y} - G_{x_2,m})
\end{bmatrix}.
\tag{5.2}
$$

Provided an estimate of $G_{x_1,m}^2$ and $G_{x_2,m}^2$ may be made available, an optimization procedure is possible.

It may be emphasized that, in the closed loop problem of § 3, while it is physically possible that $G_{x_1,m}^2$ might, in some circumstances, be zero, the linear dependence of $x_2(t)$ on $y(t)$ makes it impossible, if $m(t)$ exists, for $G_{x_1,m}^2$ to be zero.

Assume, estimates of $G_{x_1,n}$ and $G_{x_2,n}$ were available in the case defined by (3.1) and (3.2), and the results (5.2) of an active optimization were used to find optimum values for $H_1$ and $H_2$ with the constraints

$$G_{x_1,m} = G_{x_1,n} \quad \text{and} \quad G_{x_2,m} = G_{x_2,n} \quad \ldots \ldots \quad (5.3)$$

imposed.

The expanded form of (5.2) gives

$$H_1 = \frac{[G_{x_1,y}G_{x_1,x_1} - G_{x_1,y}G_{x_1,x_2} + (G_{x_1,x_1}G_{x_2,m} - G_{x_1,x_2}G_{x_2,m})]}{[G_{x_2,x_1}G_{x_1,x_1} - G_{x_2,x_2}G_{x_2,x_1}]}.$$ \ldots \ldots \quad (5.4)

$$H_2 = \frac{[G_{x_2,y}G_{x_2,x_2} - G_{x_2,y}G_{x_2,x_1} + (G_{x_2,x_2}G_{x_1,m} - G_{x_2,x_1}G_{x_1,m})]}{[G_{x_2,x_1}G_{x_2,x_2} - G_{x_2,x_1}G_{x_2,x_1}]}.$$ \ldots \ldots \quad (5.5)

But it follows from (3.9) and (3.10) that

$$[G_{x_1,y}G_{x_1,x_1} - G_{x_1,y}G_{x_1,x_2}] / [G_{x_1,x_1}G_{x_2,x_1} - G_{x_1,x_2}G_{x_2,x_1}] = 0,$$ \ldots \ldots \quad (5.6)

$$[G_{x_2,y}G_{x_2,x_2} - G_{x_2,y}G_{x_2,x_1}] / [G_{x_2,x_2}G_{x_1,x_1} - G_{x_2,x_1}G_{x_1,x_1}] = 1/F.$$ \ldots \ldots \quad (5.7)

From (3.6), (5.3), (5.4) and (5.6) it follows that

$$H_1(w) = F_1(w).$$ \ldots \ldots \ldots \quad (5.8)

Multiplying (3.3) by $G_{x_1,x_1}$ and (3.4) by $G_{x_1,x_1}$ and taking the difference gives

$$[FF_2 - 1][G_{x_1,x_1}G_{x_2,x_2} - G_{x_1,x_1}G_{x_2,x_1} = F[G_{x_1,x_1}G_{x_2,m} - G_{x_1,x_2}G_{x_2,m}].$$ \ldots \ldots \quad (5.9)

From (5.3), (5.5), (5.7) and (5.9) it follows that

$$H_2(w) = F_2(w).$$ \ldots \ldots \ldots \quad (5.10)
Equations (5.8) and (5.10) demonstrate that the problem of §3 may be satisfactorily solved if the optimization procedure permits a choice of linear, active, time-invariant, operators. Comparison of (5.8) with (3.9), and (5.10) with (3.10) shows the advantage obtained by enlarging the permitted class of operator provided estimates of \( G_{x_m} \) and \( G_{x'm} \) may be made available.

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**REFERENCE**

A Further Note on Data Fitting with Linear Transfer Functions

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In a recent paper (Woodrow 1959) the question of data fitting with linear transfer functions was discussed for both single- and multi-variable problems. The purpose of this note is to elaborate on a result developed there.

In that paper it was proposed that a 'best' linear passive approximation to the given data should be sought. It was also proposed that 'best' should be interpreted as that one which minimizes the mean square error of the approximation.

The mathematical description of this error in the multi-variable problem was given as

$$e(t) = y(t) - \sum_{r=1}^{n} \int_{0}^{\infty} h_r(u) x_r(t-u) du.$$  \hspace{1cm} (5.26)

The corresponding expression for the single-variable case (eqn. (3.3), p. 463) follows from (5.26) by setting $n$ equal to unity.

In fact, this equation is not so general as to permit the choice of a best approximation from all linear passive operators. The choice is restricted to linear, passive, time-invariant, causal operators. It was from this sub-class of the class linear, passive, time-invariant that the best approximation was required.

In the variational processes by which the set of weighting functions $h_r(t)$ which minimize $\bar{e}^2(t)$ were chosen (Appendix III and §3.2), the constraint 'causal' was dropped, and a best operator selected from the enlarged class, linear, passive, time-invariant. Equations (3.9) and (III 6) define the best set of weighting functions for this class. Since the data fitted had been collected from a causal system (a general property of physical systems), it was supposed that it was in the nature of the data to favour the causal sub-class when fitting with linear passive time-invariant operators. In other words, it was assumed that, because of the causal property of the data, the best linear, passive, time-invariant operator would be a causal one.

Introducing this idea leads from (3.9) and (III 6) to (3.11) and (III 7). For all cases for which these equations have a solution, i.e. for all data for which the best linear, passive, time-invariant operator is a causal one,
that solution is the one required. It is convenient, because the resulting
integral equations are more easily solved, to exploit this property of causal
data (that it favours a causal solution), and to seek the solution from all
linear, passive, time-invariant operators rather than from the causal
sub-class only.

The validity of the postulate that causal data favour a causal approxima-
tion may be readily checked. In the single-variable problem, a Fourier
transform of (IV 7) gives
\[
\hat{h}_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y_1(j\omega) \exp(j\omega t) d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \{G_{x_1 y}(j\omega)/G_{x_1 x_1}(j\omega)\} \exp(j\omega t) d\omega.
\] (1)

If the postulate is correct, \(h_1(t)\) in (1) is a causal weighting function.
Similarly, Fourier transforms of each \(Y_r(j\omega)\) in (IV 6) check the multi-
variable case.

If, in a certain problem, the data are such that at least one of the \(h_r(t)\)
is found to be non-causal, and the conditions of the problem require a
causal approximation, a causal constraint must be re-introduced into the
optimization procedure.

Had the causal constraint been retained in the optimization procedure
of Appendix III, the only change in the calculation is the constraint
on \(p_r(u)\) that \(p_r(u) = 0\) for negative values of \(u\). This introduces the
le corresponding constraint on \(u\) into eqn. (III 7), giving
\[
\phi_{x_1 y}(u) = \sum_{s=1}^{n} \int_{0}^{+\infty} h_s(z) \phi_{x_s x_1}(u-z) dz \quad \text{for} \quad u > 0.
\] (2)

If it so happens that no correlation exists between the various inputs,
\(\phi_{x_r x_1}(u-z) = 0\) for \(r \neq s\), the \(n\) simultaneous integral equations obtained
by putting \(r = 1, 2, \ldots, n\) in (2) reduce to \(n\) independent equations each
of the form
\[
\phi_{x_r y}(u) = \int_{0}^{+\infty} h_r(z) \phi_{x_r x_r}(u-z) dz \quad \text{for} \quad u > 0.
\] (3)

These are of the same form as for the single-variable problem, and can be
solved by established techniques.

When this lack of correlation between the various inputs is not observed,
the \(n\) eqns. (2) must be solved as a simultaneous set. Because of the
constraint \(u > 0\), this is difficult.

A good approximation may be derived from the linear, passive, time-
invariant solution previously obtained. Thus, suppose a particular problem
is such that \(m\) of the \(n\) weighting functions in the solution of (III 7) are
found to be non-causal. Let \(h_s(t)\) operating on the input variable \(x_s(t)\)
(fig. 15) be one of these. Denote the contribution made to \(y(t)\) by this
branch by \(z_s(t)\). Then
\[
z_s(t) = \int_{-\infty}^{+\infty} h_s(\tau)x_s(t-\tau) d\tau.
\] (4)
Let the best linear, passive, time-invariant, causal operator be \( w_r(t) \), such that

\[
z_r(t) = \int_0^\infty w_r(\tau)x_r(t - \tau)\,d\tau + e_r(t)
\]  

(5)

where \( e_r(t) \) is the error of this causal approximation, i.e. is the addition to the error \( \varepsilon(t) \) of (5.26) which results from imposing a causal constraint on the operator \( h_r(t) \).

If \( w_r(t) \) is chosen to minimize \( e_r^2(t) \) the integral equation defining \( w_r(t) \) becomes

\[
\phi_{x_r x_r}(t) = \int_0^\infty w_r(t)\phi_{x_r x_r}(t - \tau)\,d\tau \quad \text{for} \quad t > 0.
\]  

(6)

Since \( z_r(t) \) can be made available by the solution of (4), eqn. (6) may be solved by established methods.

Each of the \( m \) non-causal weighting functions may be treated in this way.

Reference

On Finding a Best Linear Approximation to System Dynamics from Short Duration Samples of Operating Data†

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Abstract

A method for the determination of a linear passive system, which most nearly approximates (in the sense that it yields a minimum mean square error) the dynamic behaviour of a given physical system, is described. The method is applicable to situations in which

(a) the system investigated has $N$ input variables defining $M$ output variables;

(b) data is recorded at each of the input and output terminals during some finite time interval $0 \leq t \leq t_f$, which represents a short sample of a much longer past history.

Unlike methods already developed, the techniques used here do not require any appeal to statistical properties such as stationarity or ergodicity of the data, neither is it necessary to attempt to find a realistic estimate of correlation or spectral density functions.

It is suggested that the method described may help to decide whether, in a given trial, the best approximation may be expected to be sensitive to changes in the sample of operating data fitted.

§ 1. Introduction

Situations arise, in which it is required to assess transfer functions of physical systems, where the normal techniques for the determination of such information (sinusoidal, impulse or step response testing) cannot be used. Examples of such problems are to be found in

(a) the determination of the dynamics of continuous operating plant, and

(b) situations in which a linear passive approximation is required for a physical system which may be either linear active or non-linear. In this case the best approximation is a function of the input data, and hence should be deduced from normal operating records.

A paper by Goodman et al. (1956) set the pattern that has been followed in the analysis of problems of this type. The methods used in this and subsequent papers have been reviewed elsewhere (Woodrow 1959) and the criticism made that, with present techniques, no explanation is offered to suggest any good physical reason for accepting as ‘best’ the approximations derived. Woodrow (1959) suggested that the fitting of

† Communicated by the Author.
Determination of a Linear Passive System

recorded data by a best linear passive approximation, should incorporate, as a definition of 'best', a minimum mean square error criterion. This suggestion is accepted here, no other type of approximation being considered.

A second fundamental difficulty of existing methods, is that neither correlation nor spectral density functions (in terms of which 'best approximations' are at present described) can be adequately determined from a sample of finite time duration of a longer past history. It becomes necessary to assume statistical properties for the recorded variables. Present methods of solution of the data fitting problem (Florentin et al. 1959 and Westcott 1956) reduce to the solution of a Wiener Hopf type integral equation (see Laning and Battin 1956, p. 272, equn. 7.1-12 for a description of this equation) involving cross and auto-correlation functions of measured data. To solve an equation of this type requires (Lanning and Battin 1956, p. 291) that "the device is permitted to operate on the entire past history of the data". Wiener himself pointed out the even greater difficulty (Weiner 1949, p. 55) that "in all practical cases the auto-correlation coefficient of a message is not completely determined by its own past. If it were so determined, then at no period in the message would it be possible to introduce new information". When only a part of the past history is known, to ask for correlation and spectral density functions of the data is to ask more than the measurements can provide. The mathematical formulation of the problem has failed to recognize the necessary restriction, imposed by the finite time duration of the recorded data. As a result, the mathematical requirements and physical situation are incompatible. This fundamental contradiction between the mathematics and the physics is subsequently weakened, in present techniques, by the assumption that the recorded data are samples of stationary ergodic time series having the properties (i) that the mean of sample correlation functions computed over an ensemble of samples is a good approximation to the true correlation function and (ii) that the variance of the ensemble spread about the mean is small. If the data belongs to this class, then the probability is high that the correlation functions computed from single samples of data will give good approximations to the true correlation functions.

Before these assumptions can be used with confidence, it is essential

(a) to devise tests to establish that the data really does have the properties assumed for it, and

(b) assuming (a) established, to find whether the solution of the Wiener Hopf equation remains invariant under small changes in the correlation functions in a given trial (the small changes being the difference between the sample correlation functions of that trial and the true correlation functions).

No test has been found by which the assumptions about the data here described can be verified, and it is difficult to see how such verification can be found. It is of value to study the work that has been done to seek the cause of this dilemma, and to attempt to formulate the problem afresh, and
in such a way, that the mathematics is constrained (as the physical situation is constrained) to the use of samples of recorded data of finite time duration. This paper describes such a method.

§ 2. DATA FITTING FOR A TWO TERMINAL PAIR

2.1.

The simplest problem encountered in investigations of this type is that represented schematically by fig. 1.

![Figure 1](image)

Samples of the input variable $x(t)$ and the output variable $y(t)$ are collected, during some finite time interval $0 \leq t \leq t_f$, from a physical system having a single input variable. From these it is required to find a 'best' linear passive approximation to the dynamics of the physical system from which the data was collected.

The output $c(t)$ of any linear passive approximation is defined, for physically realizable systems, by the relation (Laning and Battin 1956, p. 184, eqn. 5.2-4)

$$c(t) = \int_0^\infty h(u) x(t - u) \, du \tag{2.1}$$

where $h(t)$ is the impulse response of the linear passive system chosen to approximate the data.

The difference $y(t) - c(t)$ between the desired output $y(t)$ and the actual output $c(t)$ of the linear passive approximation is the error $\epsilon(t)$ resulting from the approximation.

Hence

$$y(t) = \epsilon(t) + \int_0^\infty h(u) x(t - u) \, du \tag{2.2}$$

Of the infinity of possible solutions to (2.2) it is necessary to seek a 'best' one. A commonly accepted best solution used extensively by Wiener (1949, p. 131) and others requires the minimization of the quantity

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \epsilon^2(t) \, dt$$

and is referred to as the minimum mean square error criterion. This quantity, with its implied limits $-\infty \leq t \leq +\infty$ cannot be determined from (2.2) for any $h(t)$ if $x(t)$ and $y(t)$ are only defined in the interval $0 \leq t \leq t_f$. It is in adopting this criterion for the best fit to the data, implying the unreal condition that $x(t)$ and $y(t)$ are known for all time, that the difficulties partly originate.
In this paper, the best approximation to a finite sample of data is taken as that one which minimizes the mean square value of $e(t)$ over the range of $t$ for which $e(t)$ is defined by the available data.

It is immediately apparent from a study of (2.2) that, if $h(t)$ exists for all positive $t$, then input data must be available for the whole interval $-\infty \leq t \leq t_0$ if $e(t_0)$ is to be calculated for any value of $t_0$. Equation (2.2) and the suggested minimization criterion are still incompatible. The data does not define $e(t)$ for any value of $t$ if $h(t)$ may be any member of the class of functions defined by (2.1). How can this difficulty be overcome?

2.2

It is necessary at this point to return to the physical properties of linear passive systems. The important property here is that the observed output of a linear passive physical system resulting from an impulse type input ($= h(t)$ of eqn. (2.1)) is asymptotic to the zero line for large $t$. Hence $h(t)$ becomes indistinguishable from zero, to the accuracy to which physical measurements are possible, after some finite time $\tau$, which is referred to as the 'memory time' of the system. With finite data it is proposed to seek the 'best' linear passive operator from among that class of linear passive operators having a memory time less than, or equal to, some preassigned value $\tau$. This class of linear passive operator is a sub-class of the general class $h(t)$ defined by (2.1).

With this restriction eqn. (2.2) is replaced by

$$y(t) = e(t) + \int_{-\infty}^{\tau} h(u)x(t-u)\,du \quad \cdots \quad (2.3)$$

since $h(u) = 0$ for $y(t) = e(t) + \int_{-\infty}^{\tau} h(u)x(t-u)\,du$.

For systems of this class $e(t)$ is defined by the data for any time $t = t_0$ in the range $\tau \leq t_0 \leq t_1$.

Provided $\tau < t_1$ it is possible to ask that $h(u)$ shall be chosen in (2.3) to minimize $\overline{e^2(t)}$ where

$$\overline{e^2(t)} = \frac{1}{t_1-\tau} \int_{\tau}^{t_1} e^2(t)\,dt.$$

2.3

The minimization operation is most easily carried out if some slight additional restriction is placed upon the permitted class of operator. It is to be assumed with Goodman et al. (1956), Reswick (1955) and others, that the impulse response $h(t)$ to be determined shall be selected from a sub-class of the class having a memory time less than, or equal to, $\tau$ which is defined by the restriction

$$h(t) = \sum_{r=1}^{n} h_r \delta \left( t-r \frac{\tau}{n} \right) \text{ for } 0 \leq t \leq \tau$$

$$= 0 \quad \text{outside the range } 0 \leq t \leq \tau.$$  \quad (2.4)

where $\delta(t)$ is the Dirac delta function.

The significance of this form of impulse function is indicated in fig. 2.

Any member of the class having memory time less than, or equal to, $\tau$ can
be approximated to any desired accuracy by a function of this type if \( n \) is made sufficiently large.

Fig. 2

\[ h(t) \]

**actual weighting function**

**approximation in terms of**

\[ a \text{ sum of impulses} \]

\[ t \text{ memory time} \]

From (2.3) and (2.4)

\[
y(t) = \epsilon(t) + \int_0^\tau \sum_{r=1}^n h_r \delta(u - r(n)) x(t - u) \, du
\]

\[
= \epsilon(t) + \sum_{r=1}^n h_r x[t - r(n)]. \quad \ldots \ldots \ldots (2.5)
\]

It should be realized that the error function \( \epsilon(t) \) associated with (2.5) differs from the corresponding expression in (2.2) because of the restrictions placed upon the class of function from which it is permitted to seek a solution. In seeking a 'best' impulse response (i.e. a 'best' set of coefficients \( h_r \) in (2.5), the choice is restricted to a certain sub-class of that whole class which includes all possible physically realizable impulse response functions.

The restrictions imposed on the class of impulse function permitted are not considered severe. In fact, search of the literature reveals (e.g. Florentin* et al.* 1959, Reswick 1955) that disciples of the statistical school consider the problem adequately solved when a member of the sub-class here used has been found which *approximately* satisfies an *approximately formulated* Wiener Hopf equation (in the sense that the available data defines, at best, an approximation of doubtful accuracy to the correlation and spectral density data).

It would seem more logical to restrict the class of function from which a choice is to be made before the minimization procedure, and then to minimize the mean square error in such a way that the available data make such operations physically meaningful.

2.4

In order to solve (2.5) for a best fit to the available data, a number \( m \) of equations are set up for \( m \) different values of \( t \) in the range \( \tau \leq t \leq t_1 \), where the data is assumed to have been recorded in the interval \( 0 \leq t \leq t_1 \).
If the recorded data $x(t)$ and $y(t)$ are made available as a series of ordinates at time instants $t = k(\pi/n)$ where $k = 1, 2 \ldots (n + m)$ as indicated in table 1, the formulation of these $m$ equations is a simple operation.

Table 1

<table>
<thead>
<tr>
<th>$k$</th>
<th>$t=k(\pi/n)$</th>
<th>$y_k=y(t)$ for $t=(<a href="%5Cpi/n">k+n</a>)$</th>
<th>$x_k=x(t)$ for $t=(<a href="%5Cpi/n">k+n</a>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1($\pi/n$)</td>
<td>$y_1$</td>
<td>$x_{-(n-1)}$</td>
</tr>
<tr>
<td>2</td>
<td>2($\pi/n$)</td>
<td>$y_2$</td>
<td>$x_{-(n-2)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>window displaying one entry</td>
<td>window displaying $n$ entries</td>
</tr>
<tr>
<td>$n$</td>
<td>$(n\pi/n)$</td>
<td>$y_n$</td>
<td></td>
</tr>
<tr>
<td>$n+1$</td>
<td>$(n+1)(\pi/n)$</td>
<td>$y_{n+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{n+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{n+2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{n+3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{n+s-1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(n+s)(\pi/n)$</td>
<td>$y_{n+s}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{n+s}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_m$</td>
<td>$x_m$</td>
</tr>
</tbody>
</table>

From (2.5)

$$
\epsilon_s = y_s - \sum_{r=1}^{n} h_r x_{s-r}, \ldots \ldots (2.6)
$$

where

$$
\epsilon_s = \text{value of } \epsilon(t) \text{ at } t = (s+n)(\pi/n), \ldots \ldots (2.7)
$$

$$
y_s = \text{value of } y(t) \text{ at } t = (s+n)(\pi/n), \ldots \ldots (2.8)
$$

$$
x_s = \text{value of } x(t) \text{ at } t = (s+n)(\pi/n), \ldots \ldots (2.9)
$$

Giving $s$ the values $1, 2, \ldots m$ in succession yields $m$ simultaneous equations in the $n$ unknowns $h_r$ and the $m$ unknowns $\epsilon_s$.

This set may be written as the single matrix equation

$$
\epsilon = y - X \cdot h \ldots \ldots \ldots (2.10)
$$

where

$\epsilon = m \times 1$ column vector of errors,

$y = m \times 1$ column vector of output function values,

$h = n \times 1$ column vector defining impulse response,

$X = m \times n$ matrix of input function values.

†From (2.6) and table 1, row $s$ of the $X$ matrix of (2.10), $(x_{s-1}, x_{s-2}, \ldots x_{s-n})$ is displayed by the window in the last column of table 1, but read from the last entry upwards. The corresponding entry for the $y$ vector is that displayed by the window in column $y_k$. By moving a mask to display, in succession, the entries $y_1, y_2, \ldots y_m$ in the $y_k$ column, the successive rows of the $X$ matrix are the columns of entries displayed in the $x_k$ column window, reading upwards from the bottom.
The problem is now reduced to the choice of the vector $h$ in (2.10) which minimizes the length of the vector $e$.

A close correspondence exists between this minimization problem and the Wiener minimization problem. The latter is a limiting case of the above when (a) the data is assumed recorded for $-\infty < t < +\infty$ allowing $m$ to tend to infinity and (b) $\tau$ and $n$ both tend to infinity in such a way that $(\tau/n)$ tends to zero.

The same close correspondence exists between the operations of matrix algebra here used to solve the minimization problem for (2.10), and the corresponding operations of the calculus of variations used in the derivation of the Wiener Hopf integral equation (compare with Laning and Battin 1956, pp. 269 to 272). The form of solution obtained is likewise very similar.

The value $\bar{e}^2$ to be minimized is given by

$$\bar{e}^2 = (1/m) \sum_{i=1}^{m} e_i^2 = (1/m) e' \cdot e$$

where $e' = $ transpose of $e$.

But from (2.10)

$$(1/m) e' \cdot e = (1/m) (y' - h' \cdot X') (y - X \cdot h)$$

$$= (1/m) (y' \cdot y - h' \cdot X' \cdot y - y' \cdot X \cdot h + h' \cdot X' \cdot X \cdot h). \quad (2.11)$$

If a column vector $h$ exists which minimizes $\bar{e}^2$, then changing from this to a new column vector $(h + kb)$ where $k$ is an arbitrary small scalar multiplier and $b$ an arbitrary $n \times 1$ column vector increases the mean square error to $e_1^2$ where

$$e_1^2 = (1/m) \left[ y' \cdot y - (h' + kb') \cdot X' \cdot y - y' \cdot X (h + kb) + (h' + kb') \cdot X' \cdot X \cdot (h + kb) \right]$$

$$= (1/m) \left[ -b' \cdot X' \cdot y - y' \cdot X \cdot b + b' \cdot X' \cdot X \cdot h + h' \cdot X' \cdot X \cdot b \right]$$

$$= 0 \text{ since } e^2 = e_1^2 \bigg|_{k=0} \text{ is a minimum.} \quad \ldots \quad (2.12)$$

Now since $y' \cdot X \cdot b$ is the transpose of $b' \cdot X' \cdot y$,

$h' \cdot X' \cdot X \cdot b$ is the transpose of $b' \cdot X' \cdot X \cdot h$,

and each of these is a $1 \times 1$ matrix, having the property that the transpose is equal to the matrix itself, (2.12) reduces to

$$2b' [(1/m)X' \cdot X \cdot h - (1/m)X' \cdot y] = 0.$$ 

Since $b'$ is an arbitrary $1 \times n$ row vector this requires that

$$(1/m)X' \cdot X \cdot h = (1/m)X' \cdot y. \quad \ldots \quad (2.13)$$

Provided $X' \cdot X$, is a non-singular matrix the solution of (2.13) is

$$h = [(1/m)X' \cdot X]^{-1} \cdot [(1/m)X' \cdot y]. \quad \ldots \quad (2.14)$$
The proviso that \( X' \cdot X \) shall be a non-singular matrix is an important one, with important practical consequences which will be considered in detail later. A necessary, although not a sufficient, condition for non-singularity of \( X' \cdot X \) is that \( m \geq n \) (see Ferrat 1941, p. 110, Theorem 33)—i.e. at least as many equations must be formulated as there are elements in the vector.

Substitution into (2.4) of the ordinates given by the elements of the vector \( \mathbf{h} \) from eqn. (2.14) gives the impulse response \( h(t) \) of the linear passive approximation chosen as the best description of the data.

In the matrix eqn. (2.13), the vector \((1/m)X' \cdot \mathbf{y}\) occupies the position of the cross correlation function of the Wiener formulation, while the matrix \((1/m)X' \cdot X\) occupies the rôle of the auto-correlation function (compare (2.13) with eqn. (11), p. 134, Wiener 1949).

The correspondence that exists between these solutions is emphasized if the form of the general elements of the matrices \((1/m)X' \cdot X\) and \((1/m)X' \cdot \mathbf{y}\) are studied. The \( k \)th element \( \phi_{xy}(k) \) of the \( n \times 1 \) column vector \((1/m)X' \cdot \mathbf{y}\) is given by
\[
\phi_{xy}(k) = (1/m) \sum_{s=1}^{m} x_{s-k} \cdot y_{s}
\]
while the element in row \( j \) and column \( k \) of \((1/m)X' \cdot X\) is given by \( \phi_{xx}(j, k) \) where
\[
\phi_{xx}(j, k) = (1/m) \sum_{s=1}^{m} x_{s-j} x_{s-k}.
\]

These may be compared with Wiener's expressions (1949, p. 132) for cross and auto-correlation functions of discrete data.

The essential difference between the finite discrete data case here described and the infinite discrete data case studied by Wiener are

(i) that the averaging process describing \( \phi_{xy}(k) \) and \( \phi_{xx}(j, k) \) is performed over a finite set instead of an infinite set;

(ii) there is no guarantee that \( \phi_{xx}(j, k) \) is dependent only on \( j \) and \( k \) through their difference \( (j - k) \) as in Wiener's formulation. This dependence, in Wiener's solution, of \( \phi_{xx}(j, k) \) on \( j \) and \( k \) through the difference \( (j - k) \) only is a consequence of the stationarity hypothesis, (see Lanning and Battin 1956, p. 106) and \( \phi_{xx}(j, k) \) will be of the appropriate form only when this hypothesis is valid.

A further interesting comparison between the matrix and correlation formulation of the minimization problem follows if (2.10) is pre-multiplied by \( X' \) giving
\[
X' \cdot \mathbf{e} = X' \cdot \mathbf{y} - X' \cdot X \cdot \mathbf{h}.
\]
\[
= 0 \text{ from (2.13).} \quad \quad \quad \quad (2.15)
\]
Thus the error vector \( \mathbf{e} \) is orthogonal to each of the \( n \) row vectors of \( X' \) (which are the \( n \) column vectors of the original matrix \( X \)). This is to be compared with the property of the Wiener Hopf solution (see Woodrow
1959) that the input variable and the error of a minimum mean square error approximation are uncorrelated.

2.5

Having found the vector \( h \) from (2.14) defining the best linear passive approximation to the data as an impulse response function, it is possible to proceed directly to the frequency response function \( Y(j\omega) \) of that approximation.

It has been shown (Laning and Battin 1956, p. 193, eqn. 5.2-41), that

\[
Y(j\omega) = \int_{-\infty}^{+\infty} h(t) \exp(-j\omega t) dt
\] (2.16)

\[
= \sum_{r=1}^{n} h_r \exp\left[-j\left(\omega r \frac{\pi}{n}\right)\right]
\] (2.17) from (2.4) and (2.16).

Giving \( \omega \) any desired value, the right-hand side of (2.17) may be calculated since, for a given \( \tau \) and \( n \) the coefficients \( h_r \) are known. It is possible to generalize the result somewhat by the introduction of a matrix which has been previously constructed to transform a given \( h \) vector into a corresponding \( Y \) vector. The advantage of such a matrix is that it may be applied, once calculated to any \( h \) vector irrespective of the memory time \( \tau \) of the system investigated.

Denoting by \( Y_s \) the value of \( Y(j\omega) \) at \( \omega = s(2\pi n/\tau) \), it follows from (2.17) that

\[
Y_s = \sum_{r=1}^{n} h_r \exp(-jsr\pi).
\]

Giving \( s \) a set of \( m \) different values (i.e. choosing \( m \) values of \( \omega \) for which \( Y(j\omega) \) is to be computed) yields the set of equations

\[
Y = \Phi \cdot h
\]

where

- \( Y = m \times 1 \) column vector,
- \( h = n \times 1 \) column vector,
- \( \Phi = m \times n \) matrix transforming the \( h \) vector to the desired \( Y \) vector.

As an example, the \( \Phi \) matrix of table 2, when pre-multiplying a given 20 element \( h \) vector gives the corresponding \( Y \) vector for \( \omega = s(2\pi n/\tau) \) where \( s = 0(0.05)0.5 \).

§ 3. THE APPROXIMATION OF DYNAMIC SYSTEMS HAVING MULTIPLE INPUTS

3.1

Not all physical systems belong to the class for which the output is a function of a single input variable. A more general linear passive system is one having \( M \) separate output variables, each of which is a function of \( N \) input variables. It has been shown (Woodrow 1959) how such a system
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Determination of a Linear Passive System
Table 2 (cont.)—Values of Elements \( \phi_{jk} \) of Matrix \( \Phi \)

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may be reduced to $M$ simpler systems. Each of these has a single output variable which is a function of all $N$ input variables, and of the system dynamics, as in fig. 3.

For this system, the equation relating the best linear passive approximation to the system dynamics, the error of the approximation, and the recorded data, is taken (Woodrow 1959) as

$$y(t) = \epsilon(t) + \sum_{s=1}^{N} \int_{0}^{\infty} h_s(u) x_s(t-u) \, du$$  \hspace{1cm} (3.1)$$

where $h_s(u)$ = impulse response relating output $y(t)$ to input $x_s(t)$,

$$\epsilon(t) = \text{resulting error of the approximations } h_s(u).$$

Recognizing that physically realizable impulse response functions are zero, to the accuracy of data measurement, after some finite memory time $\tau$, (3.1) may be replaced by

$$y(t) = \epsilon(t) + \sum_{s=1}^{N} \int_{0}^{\tau} h_s(u) x_s(t-u) \, du.$$  \hspace{1cm} (3.2)$$

In (3.2) the search for the 'best' approximation to the data is restricted to consider only those members of the general class $h_s(t)$ having the property that $h_s(t) = 0$ for $t \geq \tau$, and $\epsilon(t)$ is the error associated with the choice from among members of this class.

Making the further restriction that $r = 1$ in (3.2) yields

$$y(t) = \epsilon(t) + \sum_{s=1}^{N} \int_{0}^{r} h_s(u) x_s(t-u) \, du.$$  \hspace{1cm} (3.3)$$

Assuming the ordinates of $y(t)$ and $x_s(t)$ are tabulated at intervals of $\tau/n$, then the following equations can be formed from (3.3),

$$y_q = \epsilon_q + \sum_{s=1}^{N} \sum_{r=1}^{n} h_{sr} x_s(q-r), \quad q = 1, 2, \ldots, m, \quad \hspace{1cm} (3.4)$$

where

$y_q =$ recorded ordinate $y[\tau + (q/\tau)] = y[(1 + (q/n))\tau]$,

$x_s(q-r) =$ recorded ordinate $x_s[1 + (q-r)/\tau]$,

and the duration of the available records are $\tau[1 + (m/n)]$. 
The \( m \) simultaneous equations (3.4) may be represented by the single matrix equation

\[
y = \varepsilon + \sum_{s=1}^{N} \mathbf{X}_s \cdot \mathbf{h}_s
\]

where

- \( y = m \times 1 \) column vector of the output ordinates \( y_q \)
- \( \varepsilon = m \times 1 \) column vector of the error ordinates \( \epsilon_q \)
- \( \mathbf{h}_s = n \times 1 \) column vector giving ordinates of \( h_s(t) \)
- \( \mathbf{X}_s = m \times n \) matrix of ordinate \( x_s(q-r) \) in row \( q \) column \( r \).

It is proposed to choose as the best set of \( \mathbf{h}_s \) in (3.5) that set which minimizes the length of the vector \( \varepsilon \). This is a minimum mean square error criterion, since the length of the vector \( \varepsilon = \sum_{q=1}^{m} \epsilon_q^2 = m \times \text{mean square error} \).

The minimization proceeds from (3.5) by similar arguments to those used in §2.4 for the case of systems with a single input variable, and leads to the formal solution

\[
\sum_{s=1}^{N} \frac{1}{m} \mathbf{X}_s' \cdot \mathbf{X}_s \cdot \mathbf{h}_s = \frac{1}{m} \mathbf{X}_s' \cdot \mathbf{y}, \quad p = 1, 2, \ldots N.
\]

Writing (3.6) in matrix form

\[
\begin{bmatrix}
(\frac{1}{m})\mathbf{X}_1' \cdot \mathbf{X}_1 & \cdots & (\frac{1}{m})\mathbf{X}_N' \cdot \mathbf{X}_N \\
\vdots & \ddots & \vdots \\
(\frac{1}{m})\mathbf{X}_N' \cdot \mathbf{X}_1 & \cdots & (\frac{1}{m})\mathbf{X}_N' \cdot \mathbf{X}_N
\end{bmatrix}
\begin{bmatrix}
\mathbf{h}_1 \\
\vdots \\
\mathbf{h}_N
\end{bmatrix}
= \begin{bmatrix}
(\frac{1}{m})\mathbf{X}_1' \cdot \mathbf{y} \\
\vdots \\
(\frac{1}{m})\mathbf{X}_N' \cdot \mathbf{y}
\end{bmatrix}.
\]

Provided the partitioned \( Nn \times Nn \) matrix in (3.7) is non-singular, it may be inverted to yield the vectors \( \mathbf{h}_1, \ldots, \mathbf{h}_N \) as the desired solution.

This same solution could have been obtained directly from a consideration of (3.4) as a set of \( m \) equations in the \( Nn \) unknowns \( h_{qpr} \) defining the \( n \) impulse functions, together with the \( m \) errors \( \epsilon_q \). The advantage of the partitioned matrix form (3.7) above is that the identity of the elements and their relationship to the observed data is preserved throughout the minimization operation.

As in the case of (2.13) the square matrix of (3.7) must be non-singular if (3.7) is to have a solution, and a necessary (although not a sufficient) condition for this is that \( m \geq Nn \).

3.2

Having found a set of impulse response functions which give a best fit to the available data, the equivalent frequency response functions corresponding to the impulse response functions \( \mathbf{h}_p \) may be found by \( N \) successive applications of the method of §2.5 to each of the impulse response functions obtained.
§ 4. Physical Significance of Results

Having found a linear passive approximation which is a best fit to the available data, it is necessary to consider two additional problems before the solution obtained can be used with confidence.

4.1

Does the approximation constitute an adequate fit to the data? It is essential to test that a linear passive system, from the sub-class of all linear passive systems here permitted, can be found which is an adequate approximation to the actual system investigated. If the fit obtained is poor (i.e. if the error of the approximation is large) it follows that the physical system cannot be approximated adequately by any member of the class of function here permitted. It then becomes necessary to search for an approximation from among members of a less restricted class of physical system.

It has been suggested (Woodrow 1959) that a suitable measure of the success of an approximation, \( \eta \), is

\[
\eta = 1 - \frac{\text{Mean square error}}{\text{Mean square value of output data}}
= 1 - \frac{\epsilon' \cdot \epsilon}{\gamma' \cdot \gamma} \quad \quad (4.1)
\]

in the discrete data problems here considered.

If the best linear passive system fits the data exactly, \( \eta = 1 \), and, as the success of the approximation deteriorates, \( \eta \) becomes fractional. In the worst possible case (when \( h(t) = 0 \), and the whole of the output is error), \( \eta \) becomes zero. Hence \( 0 \leq \eta \leq 1 \).

The portion of the whole range of possible values of \( \eta \) which represents an acceptable approximation will depend, amongst other things, on the use to which the information obtained is to be put. No a priori range of acceptable values can be given. Only experience can decide whether the figure of merit obtained in a given problem defines a useful approximation to the dynamics of the system investigated.

It should be pointed out, however, that if, for example, \( \eta = 0.8 \), then 80% of the actual output power of the physical system investigated is accounted for by the best linear passive approximation here found. The other 20% is associated with the error, which may be thought of as a noise power associated with the best linear passive approximation. With a ratio of signal power to noise power of four to one, the noise can hardly be considered negligible. From this argument it would seem that, for \( \eta < 0.8 \), the usefulness of the best linear passive approximation to the system dynamics would be limited.

4.2

Is the best approximation obtained invariant under changes of data? This question is, if anything, more important than the question of the
success of a given approximation. A value of $\eta$ near unity in any given trial may be purely fortuitous, and not indicative of a close approximation of the system dynamics at all. This may be easily demonstrated by a simple example.

In the physical system of fig. 4(a), $y(t) = v(t)$. If an input test signal $x(t) = v(t)$ is employed, then the best linear passive approximation to the input and output data is obtained from the system of fig. 4(b) for which $\eta = 1$ and mean square error is zero. But this is purely fortuitous depending upon the particular choice of $x(t)$. If the experiment had used an input $x(t)$ having the property that the $n$ column vectors of the resulting $X$ matrix are all orthogonal to the given $y$ vector (see eqn. (2.15)) the best approximation obtained would have been that of fig. 4(c) for which $\eta = 0$ and the whole of the output is error. Neither fig. 4(b) nor fig. 4(c) can be considered an adequate description of the system of fig. 4(a). In fact no adequate approximation to the system of fig. 4(a) can be found from the class defined by (2.4).

Fig. 4

![Diagram](image)

Obviously it would be quite inadmissible to assume that the physical system investigated was linear passive on the evidence that, in one trial, a linear passive system (that of fig. 4(b)) was found which gave a good fit to the data. It is essential to establish the further condition that the best approximation remains invariant under changes of sample of the operating data.

In order to establish that the best approximation is sensibly invariant under changes of data, it is necessary to repeat the analysis many times with different samples of input data. Each success increases the confidence in the best approximation to the data representing a good description of the physical system.

If, in a given situation, the assumptions of those applying statistical methods are reasonable (i.e. if the normal operating data approximates stationary ergodic time series for which (i) the mean value of correlation functions calculated over an ensemble of samples are close approximations to the true correlation functions and (ii) the variance of the ensemble spread about the mean is small), changes from one sample of operating data to another will only effect small changes in the values of elements of $X'X$ and $X'\cdot y$. It is therefore of interest to enquire if small changes in the known coefficients in a set of simultaneous equations can produce large changes in the solution. Such a situation can lead to a best linear passive
approximation to a given sample of operating data which is excessively sensitive to the sample of data used. Under these conditions, the best approximation is critically dependent upon the input data, and the result of any one trial cannot be considered a good description of the dynamics of the given system.

Sets of simultaneous equations having just this property are not uncommon. The same situation arises in the geometrical problem of the determination of the coordinates of the point of intersection of \( n \) surfaces in \( n \) dimensional space when two or more of the surfaces are nearly parallel (have the same direction cosines). The solution of this problem leads to a vector (set of coordinates) defining the point of intersection which is very sensitive to small changes in the direction cosines. The matrix description of the problem is characterized by the appearance of a strongly skew angular matrix.

If two or more surfaces in the geometrical problem are exactly parallel, there is no point of intersection. This is reflected in a set of simultaneous equations yielding a singular matrix. It has already been seen that the data fitting problem relies, likewise, for its success from a mathematical standpoint, upon the set of equations defining the best approximation to the data yielding a non-singular matrix. But if the matrix is nearly singular (corresponding to nearly parallel surfaces in the geometrical problem) the solution, while mathematically possible is of little physical significance because of the extreme sensitivity of the result on small changes in the value of the experimental data (compare small changes in imperfectly defined direction cosines in the geometrical problem).

Lanczos (1957, pp. 167–170) has given an excellent discourse on these difficulties, and concludes that “the critical quantity which decides the physical reliability of a strictly mathematical solution is not the determinant of the system, but the ratio of the largest to the smallest eigenvalue of the symmetrized matrix \( \mathbf{A}'\mathbf{A} \)” (read \( \mathbf{X}'\mathbf{X} \) in the problems of this paper). “It is the square root of this ratio which measures the magnification of the noise in the direction of the smallest eigenvalue”.

It would appear, therefore, that a test of sensitivity of the solution upon the sample of data used in the analysis is afforded by the ratio of these eigenvalues for the matrix \( \mathbf{X}'\mathbf{X} \). Provided this ratio is not excessive the solution of the data fitting problem may be expected to be a physically useful one for the type of input data expected. What constitutes an excessive value, for this eigenvalue ratio must be determined by experience, it depending upon the orders of magnitude of the changes in the value of the elements of \( \mathbf{X}'\mathbf{X} \) which may be expected from changes of data (or, what amounts to the same thing, how closely the normal operating data approximates the statistical properties assumed for it in other methods of analysis). Certainly one can say, with Lanczos, that if the eigenvalue ratio is in excess of \( 10^4 \) the result obtained is of doubtful value. It may well happen that ratios less than this are equally of doubtful value because of the uncertainty about the nature of the data.
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Data Fitting with Linear Transfer Functions†

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ABSTRACT

The problem of fitting a physically realizable linear passive system (defined either by an impulse response or a frequency response function) to a given set of input data and output data is discussed. It is shown that the problem leads to an under-determined set of equations, and that additional restraint conditions defining a 'best' approximation to the data must be introduced into any solution.

The various constraint conditions that have been proposed are surveyed, and a general criterion involving a minimization of mean square error is developed. Equations defining a minimum mean square error solution are derived, and solved, for a best linear passive approximation to data collected from systems having any number s of input terminal pairs.

The property of input and output data which produces a linear passive approximation, giving an exact fit, is deduced, and a criterion for the measure of success of an approximation is suggested, for situations in which an exact fit is not possible.

§ 1. INTRODUCTION

Considerable thought is at present being given to the development of techniques for the determination of system dynamics from performance records collected in normal operation (Chang et al. 1956, Florentin et al. 1959, Goodman et al. 1956, Reswick 1955, Westcott 1956 a, b, Woodrow 1958). In instances where application of the method is envisaged, little is known of the physical laws governing the dynamics of the process, and economic or other considerations preclude an exhaustive experimental investigation to determine such data.

The situation is thus one in which the only available information about process dynamics is contained in a collection of charts of measured variables. From these, one seeks to obtain as much information as the available records, and present methods of analysis, can provide.

It is necessary, in the first instance, to decide the form of the information to be extracted from the given data. Without exception, those attempting investigations of this type have sought a best linear passive description of the data, i.e. either a frequency response function or an impulse response function which, when operating upon the input variable \(x(t)\) of the process, produces an output variable \(c(t)\) which most nearly approximates, in some suitably defined sense, the actual output variable \(y(t)\) of the process.

† Communicated by the Author.
This leads to a description of the data as indicated in fig. 1.

In the literature (Goodman et al. 1956, Westcott 1956 a) the problem has been formulated somewhat differently. Thus, instead of asking for the best linear passive approximation to the data, and the associated error of the approximation, it has been the practice to assume that the records were taken from a linear active physical system. This assumption, in itself, does not define a solution to the problem, so that those using this approach are forced to assume further that some property may be ascribed to the active sources of the network. This leads to the formulation of the problem as in fig. 2, where $h(t)$ is the impulse response of the system operating on $x(t)$ and $n(t)$ is the equivalent active source referred to the output terminals. In other words, it is assumed that the principle of superposition may be applied to the process, when, with all the active sources made equal to zero, $y(t)$ would be equal to $c(t)$, while with $x(t)$ equal to zero, but with all the internal active sources operative, $y(t)$ would be equal to $n(t)$.

Comparison of figs. 1 and 2 show that they are identical, differing only in the interpretation. It seems more desirable to use the interpretation of fig. 1 than to use that of fig. 2, because this keeps constantly in mind the fact that a ‘best’ linear passive approximation to the given data is being sought, and that the solution obtained must depend upon the mathematical formulation of what is ‘best’. This may well vary from problem to problem, being influenced by the amount and form of any additional
knowledge available about the system, and also by the application to which the knowledge is to be put. The approach leading to fig. 2 obscures these facts in a number of plausible assumptions.

Search reveals that a number of criteria for the selection of the ‘best’ approximation to given data are in use, although nowhere are they clearly defined. It is hoped, in what follows, to state the criteria defining the ‘best’ linear passive approximations to data which are in use; to point out some of the difficulties which result from these choices; and to suggest a criterion which might have a general application.

\section{The ‘Best’ Approximation to Given Data}

\subsection{2.1}

In the literature of this subject (Goodman \textit{et al.} 1956, Westcott 1956 a, b, Woodrow 1958), a sharp distinction is drawn between data collected from open and closed loop physical systems, i.e. between those systems having an input variable \( x(t) \) which is not dependent upon the output variable \( y(t) \) of the process (open loop systems), and those having an input variable \( x(t) \) which is dependent on the output variable \( y(t) \) of the process (closed loop systems). This distinction is somewhat artificial, and arises because of a change of emphasis that occurs in the selection of the ‘best’ approximation in the two cases. To illustrate this point, it is useful to consider the open, and closed loop systems, which have already been investigated elsewhere, but in the light of the interpretation of fig. 1 rather than that of fig. 2.

\subsection{2.2}

As a first example, consider the situation of fig. 3 (a), when it is known that the input variable \( x(t) \) is in no way influenced by the output variable \( y(t) \), i.e. no feedback from output to input occurs. The problem is the determination of a linear passive system, the dynamic behaviour of which closely approximates that of the actual process. In other words, a linear differential equation (or some equivalent formulation) is sought, which, when operating upon an input variable \( x(t) \), produces an output variable which approaches \( y(t) \) as nearly as possible. It has been found convenient to seek an impulse response (weighting) function \( h(t) \) rather than a differential operator. This may be shown (Goldman 1949) to be an identical formulation, the differential operator being calculable if \( h(t) \) is known and vice versa.

In terms of this parameter, fig. 3 (b) defines the mathematical relation (Westcott 1956 a)

\[ y(t) = e(t) + \int_0^\infty h(u)x(t-u) \, du, \quad \ldots \ldots \quad (2.1) \]

where \( h(t) = \text{impulse response of the 'best' linear approximation to the data } x(t) \text{ and } y(t), \quad e(t) = \text{error of the best approximation.} \]

Equation (2.1) by itself does not define uniquely any function \( h(t) \), it being under-determined (a single equation in two unknowns). Thus in
(2.1) any $h(u)$ could be selected at will and, by solving the integral, a function $e(t)$ found to satisfy (2.1). In order to solve (2.1) an additional equation is required. This is the condition defining the meaning of 'best' in this context. This problem of under-determined sets of equations arises repeatedly in problems of this type, and it is this factor which makes it necessary to define and seek a 'best' solution to the problem. Stated mathematically, this search for a best solution, takes the form of a number of equations of constraint, which must be imposed in order to yield a soluble problem. It is in the formulation of these equations of constraint that the meaning of the phrase 'a best linear approximation' to the data is to be found.

The equation of constraint which has been used for open loop data (Westcott 1956 a) in the past is

$$\phi_{zt}(\tau) = \mathcal{L}_{T \to \infty} \int_{-T}^{T} \varepsilon(t)x(t-\tau)\,dt = 0. \quad (2.2)$$

With this equation of constraint, eqn. (2.1) may be solved for $h(t)$, but it cannot be emphasized too strongly that eqn. (2.2) has been introduced by 'inspired guesswork', based neither on experimental, nor theoretical, knowledge. Nobody has suggested, nor, it must be supposed, can suggest any good physical reason why this approximation to the data is more to be preferred than that obtained from any other equation of constraint.
Using (2.2) the elimination of \( e(t) \) from (2.1) to leave an equation in the single unknown \( h(t) \) follows established practice (Westcott 1956 a), and yields:

\[
\phi_{xy}(\tau) = \int_{0}^{\infty} h(u) \phi_{xx}(\tau-u) \, du \quad \ldots \ldots \quad (2.3)
\]

where

\[
\phi_{xy}(\tau) = \mathcal{L} \frac{1}{2T} \int_{-T}^{T} y(t) x(t-T) \, dt
\]

and

\[
\phi_{xx}(\tau) = \mathcal{L} \frac{1}{2T} \int_{-T}^{T} x(t) x(t-T) \, dt.
\]

It is not surprising, in view of the fact that the equation of constraint (2.2) was one involving statistical correlation, that eqn. (2.3) defines \( h(t) \) in terms of statistical correlation parameters.

Methods of solution of (2.3) for \( h(t) \) when \( \phi_{xy} \) and \( \phi_{xx} \) are available have already been described (Goodman et al. 1956, Reswick 1955).

2.3

As a second example, consider a situation in which the input variable \( x(t) \) of the process is known to be influenced by the output variable \( y(t) \) via an external feed-back path. This situation is of the type shown in fig. 4.

Fig. 4

It is now possible to seek the best linear passive approximation to the process, and likewise to the controller (box B of fig. 4). This leads, through fig. 5 to fig. 6. This is the closed-loop problem investigated in the literature (Goodman et al. 1956) but the emphasis here is different from that adopted there. In fig. 6 \( h(t) \) = 'best' linear passive approximation to the process dynamics, \( g(t) \) = 'best' linear passive approximation to controller dynamics, \( \epsilon(t) \) = error of the approximation to process dynamics, \( s(t) \) = sum of the error \( \epsilon(t) \) of the controller approximation and the portion \( d(t) \) of the input variable which is not influenced by \( y(t) \).
Data Fitting with Linear Transfer Functions

The equations relating system dynamics to the data are

\[ y(t) = e(t) + \int_{0}^{\infty} h(u)x(t-u) \, du, \quad \cdots \quad (2.4) \]
\[ x(t) = s(t) + \int_{0}^{\infty} g(u)y(t-u) \, du. \quad \cdots \quad (2.5) \]

With only \(x(t)\) and \(y(t)\) available as recorded data these equations contain four unknowns. The definition of the 'best' approximation requires in this case two conditions of constraint.

It is important to notice that each equation contains two different unknowns, \((2.4)\) containing only \(e\) and \(h\) while \((2.5)\) contains only \(s\) and \(g\).

If conditions of constraint are used which are similar to those used in the open loop case, i.e. if it is assumed that the 'best' linear passive approximations are those which make \(\phi_{xz}(\tau) = 0\) and \(\phi_{xv}(\tau) = 0\), then \((2.4)\) and \((2.5)\) reduce to two separate problems each of the open loop type, i.e.

\[ \phi_{xz}(\tau) = \int_{0}^{\infty} h(u)\phi_{xz}(\tau-u) \, du \quad \cdots \quad (2.6) \]
\[ \phi_{xv}(\tau) = \int_{0}^{\infty} g(u)\phi_{xv}(\tau-u) \, du. \quad \cdots \quad (2.7) \]

However, these have been considered unsuitable equations of constraint (Florentin et al. 1959, Goodman et al. 1956) and are not used for reasons which are not explained, and not immediately obvious. An explanation may be found (see Appendix I) supporting the rejection of this type of solution.

The equations of constraint which are used fall into one or other of two groups.
(a) Those which define a restraint placed upon an unknown in one of the eqns. (2.4) or (2.5) above, by virtue of the value of the unknowns in the other equation.

(b) Those which define a restraint placed upon an unknown in one of the eqns. (2.4) or (2.5) above, only through properties of the unknowns in that equation.

An example of a type (a) constraint is that the cross correlation of $e(t)$ and $s(t)$ shall be zero (assuming both $e(t)$ and $s(t)$ are unknown) while an example of a type (b) constraint is that $h(u)$ should be chosen to minimize the mean square value of $e(t)$. Of the two types, the type (b) constraint is the easier to handle, since it enables the solution of (2.4) and (2.5) to be split into two separate problems, whereas type (a) imposes a simultaneous restraint on the solution of both problems. Unless there are very good physical reasons for supposing that type (a) restraints really do make a significant contribution to the solution, it seems pointless to introduce them, with all the added complication of solution they entail.

2.4

A case in which type (a) restraint has been used in the literature (Goodman et al. 1956), imposed the equations of constraint defining the 'best' approximation to the data as follows:

$$
\phi_{es}(\tau) = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.8)
$$

and

$$
\phi_{xe}(\tau) = 0 \quad \text{for} \quad \tau > A. \quad \ldots \ldots \ldots \ldots \ldots (2.9)
$$

The solution of eqns. (2.4), (2.5), (2.8) and (2.9) is difficult, because of the equation of constraint (2.8), which simultaneously imposes restrictions on (2.4) and (2.5). Starting with these equations, the results obtained elsewhere (Goodman et al. 1956), follow (see Appendix II), but, what is difficult to see, is what special merit physically is attached to a solution of the underdetermined eqns. (2.4) and (2.5), which satisfy these constraint conditions.

2.5

An example where a type (b) restraint was imposed, but its possibilities not realized, and hence not exploited (Florentin et al. 1959), occurred in the assumption of the equations of constraint.

$$
s(t) = 0, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.10)
$$

$$
\phi_{xe}(\tau) = \text{a predetermined function.} \quad \ldots \ldots (2.11)
$$

In this case, eqns. (2.4) and (2.11) form a pair of equations, for the elimination of $e(t)$ from (2.4), to produce a single equation for the variable $h(t)$. Equations (2.5) and (2.10) form a second, quite separate pair of equations, for the formulation of an equation in the single unknown $g(t)$. Neither eqn. (2.10) nor (2.11) imposes a constraint simultaneously upon both (2.4) and (2.5).
The solution in this case could well proceed as follows:

From (2.4)

\[ \epsilon(t) = y(t) - \int_{0}^{\infty} h(u)x(t-u) du, \quad \ldots \ldots \quad (2.12) \]

\[ \epsilon(t-\tau) = y(t-\tau) - \int_{0}^{\infty} h(u)x(t-\tau-u) du. \quad \ldots \ldots \quad (2.13) \]

Multiplying (2.12) and (2.13), and taking a time average of the result yields:

\[ \phi_{e}(\tau) = \phi_{y}(\tau) - \int_{0}^{\infty} h(u)\{\phi_{x y}(u+\tau) + \phi_{x y}(u-\tau)\} du \]

\[ + \int_{0}^{\infty} \int_{0}^{\infty} h(u)h(z)\phi_{x z}(\tau+z-u) du dz. \quad \ldots \ldots \quad (2.14) \]

Equation (2.14) contains \( h(u) \) as the only unknown, and must be solved for this quantity.

Substitution of (2.10) into (2.5) yields an equation to be solved for \( g(t) \).

2.6

Another example of a type (b) constraint which has been suggested, (Woodrow 1958), is defined by the pair of constraint equations:

\[ s(t) = \text{a known function}, \quad \ldots \ldots \quad (2.15) \]

\[ \phi_{a}(\tau) = 0. \quad \ldots \ldots \quad (2.16) \]

With situations in which these conditions may be accepted, the solution of (2.4) and (2.5) is particularly simple. Equation (2.5) may be solved directly (\( g(u) \) is the only unknown in this case) while (2.4) may be manipulated (Woodrow 1958) to yield:

\[ \phi_{a y}(\tau) = \int_{0}^{\infty} h(u)\phi_{a x}(\tau-u) du. \quad \ldots \ldots \quad (2.17) \]

It may happen that \( s(t) \) cannot be recorded (because it contains, in part, the error of the approximation \( g(u) \) to the controller data). From fig. 6:

\[ s(t) = d(t) + \epsilon(t), \quad \ldots \ldots \quad (2.18) \]

where \( d(t) \) = portion of \( x(t) \) not influenced by \( y(t) \), and \( \epsilon(t) \) = error of the approximation \( g(u) \) to controller dynamics. If \( d(t) \) can be made available, as additional recorded data, this may be introduced into the problem, together with the equations of constraint:

\[ \phi_{a d}(\tau) = \phi_{e d}(\tau) = 0. \quad \ldots \ldots \quad (2.19) \]

From (2.4) and (2.19) it follows that

\[ \phi_{d y}(\tau) = \int_{0}^{\infty} h(u)\phi_{d x}(\tau-u) du. \quad \ldots \ldots \quad (2.20) \]

While from (2.5), (2.18) and (2.19)

\[ \phi_{d x}(\tau) = \phi_{d x}(\tau) + \int_{0}^{\infty} g(u)\phi_{d y}(\tau-u) du. \quad \ldots \ldots \quad (2.21) \]

Equations (2.20) and (2.21) lead to a solution when \( x(t) \), \( y(t) \) and \( d(t) \) are recorded data.
In the preceding sections, a number of possible equations of constraint which have been suggested in the literature, have been discussed. While it is possible to compare their relative merits from a point of view of mathematical convenience, nothing can be said about the physical significance of any one of these solutions, compared to any other one.

The fundamental problem in every case has been an inadequate amount of experimental information, leading inevitably to the formulation of sets of equations containing more unknowns than there are equations available. In such a situation one can only proceed by either (a) making more information available, or (b) accepting the limitation of the data, and confessing to being satisfied with an approximate result, based upon intuitively chosen equations of constraint.

There seems to exist a need for a plea: (a) to consider the problem with care, and ask oneself whether any useful additional experimental information might be collected, and whether the maximum use of the existing data is being made, (b) to keep the needs of mathematical convenience, and physical significance, in mind when selecting equations of constraint, and (c) to seek a general constraint condition which is both physically significant and mathematically convenient.

§ 3. A Minimum Mean Square Error Approximation to Given Data

3.1

A general constraint condition which can be supported on grounds of both physical significance, and mathematical convenience, is a minimum mean square error criterion. From a point of view of physical significance, it would be in order to pose the following problem: Suppose the data \( x(t) \) and \( y(t) \) were actually collected from a system, the dynamics of which were linear passive, having an impulse response \( W(t) \). In this case, the actual law relating \( x(t) \) and \( y(t) \) is the equation:

\[
y(t) = \int_{0}^{\infty} W(u)x(t-u) \, du. \tag{3.1}
\]

When presented with the data, without this knowledge of the system dynamics, one seeks a best linear passive approximation to the data \( h(t) \) through the equation

\[
y(t) = \int_{0}^{\infty} h(u)x(t-u) \, du + \epsilon(t). \tag{3.2}
\]

The problem is to devise a general constraint condition which (a) when imposed upon this problem ensures that the 'best' \( h(u) \) so defined is in fact \( W(u) \) (not all of the above criteria satisfy this), (b) when applied to problems in which the data cannot be exactly fitted with a linear passive system, shall yield a physically useful approximation to the data.

Such a criterion is the minimum mean square error criterion, i.e. a condition of constraint is imposed upon the solution of (3.2) requiring that
the value $e^2(t)$ associated with the 'best' linear passive approximation $h(u)$ shall have a minimum value.

Since the mean square error $e^2(t)$ is necessarily positive the minimum possible value it can assume is zero. Since $h(u) = W(u)$ gives $e^2(t) = 0$ in (3.2), this must represent a minimum mean square error solution satisfying (a) above.

In situations in which an exact fit to the data, with a linear passive system, is not possible, a minimum mean square error approximation gives a maximum signal/noise power description of the data, the signal power being that part of $y^2(t)$ defined by the approximation, the noise power being $e^2(t)$.

A final important consideration in favour of a minimum mean square error solution, is that it is amenable to mathematical formulation and analysis.

It seems desirable, therefore, to seek a solution to equations of the type (2.1) subject to the condition of constraint that the mean square value of $e^2(t)$ associated with the best linear passive description to the data shall be a minimum.

3.2

The equation of constraint for this condition may be derived as follows:

$$e(t) = y(t) - \int_{0}^{\infty} h(u)x(t-u) du. \quad . . . . \quad (3.3)$$

Since it is a property of the impulse response $h(t)$ of physically realizable networks that $h(t) = 0$ for $-\infty \leq t \leq 0$, the range of integration in (3.3) may be extended to $-\infty$ since

$$\int_{-\infty}^{0} h(u)x(t-u) du = 0.$$  

Consider

$$e(t) = y(t) - \int_{-\infty}^{+\infty} h(u)x(t-u) du, \quad . . . . \quad (3.4)$$

$$e^2(t) = y^2(t) - 2\int_{-\infty}^{+\infty} h(u)y(t)x(t-u) du$$

$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(u)h(z)x(t-u)x(t-z) du dz. \quad . . . . \quad (3.5)$$

Now mean square value of $e(t) = \bar{e}^2$ where

$$\bar{e}^2 = \mathcal{L} \frac{1}{2T} \int_{-T}^{T} e^2(t) dt.$$
Hence, integrating both sides of (3.5) between the appropriate limits, and dividing by the range of integration,

mean square error = \( e^2 \)

\[
= y^2 - 2 \int_{-\infty}^{+\infty} h(u)\varphi_{xy}(u) \, du + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(u)h(z)\varphi_{xz}(z-u) \, du \, dz,
\]

where

\[
\varphi_{xy}(\tau) = \mathcal{L} \left( \frac{1}{2T} \int_{-T}^{T} y(t)x(t-\tau) \, dt \right)
\]

and

\[
\varphi_{xz}(\tau) = \mathcal{L} \left( \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau) \, dt \right).
\]

Now, if \( h(u) \) is chosen to minimize the mean square value of \( e(t) \), any change in \( h(u) \) must result in an increase in \( e^2 \). Let \( h(u) \) be changed to \( \{h(u) + kp(u)\} \), and as a result, suppose \( e^2 \) changes to \( e'_1^2 \). Then

\[
\frac{d}{dk} \left| e^2 \right|_{k=0} = 0 \quad \text{(condition for turning value)}.
\]

Therefore

\[
0 = -2 \int_{-\infty}^{+\infty} p(u)\varphi_{xy}(u) \, du + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{p(u)h(z) + h(u)p(z)\}\varphi_{xz}(z-u) \, du \, dz
\]

or

\[
\int_{-\infty}^{+\infty} p(u) \left\{ \varphi_{xy}(u) - \int_{-\infty}^{+\infty} h(z)\varphi_{xz}(z-u) \, dz \right\} \, du = 0.
\]

(since \( \varphi_{xz}(z-u) = \varphi_{xz}(u-z) \): auto-correlation functions being even).

Since (3.9) must be valid for all functions \( p(u) \).

\[
\varphi_{xy}(u) = \int_{-\infty}^{+\infty} h(z)\varphi_{xz}(z-u) \, dz,
\]

or, noting again that \( h(z) \) is zero for \( -\infty < z < 0 \)

\[
\varphi_{xy}(u) = \int_{0}^{\infty} h(z)\varphi_{xz}(z-u) \, dz.
\]

Equation (3.11) has been derived elsewhere (Lanning and Battin 1956), but with a restraint condition \( u > 0 \), i.e. only to be applied for \( u + ve \).

This arises because, in that analysis, it was not appreciated that the lower limit of integration in (3.3) could be changed from 0 to \( -\infty \), because of the condition \( h(u) = 0 \) for \( -\infty < u < 0 \). Not to change this lower limit, is to leave \( h(u) \) undefined for \( u - ve \), whereas to change it, is to define \( h(u) \).
for all \( u \) positive and negative, but to define it as subject to the necessary restraint (for physical realizability), that \( h(u) = 0 \) for \( u \) negative.

Hence the integral equation defining the best linear passive approximation to the data in the minimum mean square error sense is

\[
\phi_{xy}(\tau) = \int_{0}^{\infty} h(u)\phi_{xx}(\tau-u)\,du, \quad \text{for all } \tau. \quad (3.12)
\]

Comparison of eqns. (3.12) and (2.3), shows that the equation for \( h(u) \) is the same in both cases, although derived from what appeared to be quite different criteria. This arises from the fact (see (4.4)), that it is a property of a minimum mean square error criterion, that the error \( e(t) \) is not linearly correlated with the input data \( x(t) \), which was the condition of constraint imposed in the derivation of (2.3).

The various criteria used previously and described above, which have been applied to closed-loop problems, have not led to a solution of the type indicated by (3.12), i.e. they do not include a minimization of a mean square error in their formulation. But the mean square error formulation of the problem defined above is in no way peculiar to data collected from open loop systems. No reference to whether the data originated in an open, or a closed, loop is made in the formulation of the problem, it being not relevant to the argument.

§ 4. CONDITIONS FOR AN EXACT LINEAR PASSIVE APPROXIMATION TO GIVEN DATA

Having formulated the integral eqn. (3.12), defining the best linear passive approximation, in the mean square error sense, to the given data, it is of some interest to establish those conditions which must be imposed upon given data, \( x(t) \) and \( y(t) \), in order that the mean square error, associated with the best linear passive approximation, shall be zero. Such conditions will obtain with data for which a linear passive network exists, which will produce an output \( y(t) \), when the input variable is \( x(t) \).

The determination of conditions for an exact linear passive interpretation of data follow from consideration of eqns. (3.2) and (3.12).

From (3.2), it is possible to systematically form the cross correlation functions \( \phi_{xy}(\tau) \) and \( \phi_{yx}(\tau) \), and the auto-correlation function \( \phi_{yy}(\tau) \).

These are given by

\[
\phi_{xy}(\tau) = \phi_{xx}(\tau) + \int_{0}^{\infty} h(u)\phi_{xx}(\tau-u)\,du, \quad \cdots \quad (4.1)
\]

\[
\phi_{yx}(\tau) = \phi_{yy}(\tau) + \int_{0}^{\infty} h(u)\phi_{yy}(\tau-u)\,du, \quad \cdots \quad (4.2)
\]

\[
\phi_{yy}(\tau) = \phi_{uu}(\tau) + \int_{0}^{\infty} h(u)\phi_{uu}(\tau-u)\,du. \quad \cdots \quad (4.3)
\]

From (3.12) and (4.1), it follows that

\[
\phi_{xx}(\tau) = 0, \quad \cdots \quad (4.4)
\]

i.e. it is a property of a minimum mean square error solution that the error is uncorrelated with the input data.
From (4.4) and (4.2), and the property of cross correlation functions
(James et al. 1947), that \( \phi_{x \alpha}(\tau) = \phi_{x \alpha}(-\tau) \), it follows that

\[
\phi_{\alpha \alpha}(\tau) = \phi_{\alpha \alpha}(\tau).
\]

From (4.3) and (4.5), and the above property of cross correlation functions,

\[
\phi_{\alpha \alpha}(\tau) = \phi_{\alpha \alpha}(\tau).
\]

it follows that

\[
\phi_{\alpha \alpha}(\tau) = \phi_{\alpha \alpha}(\tau) + \int_0^\infty h(u) \phi_{\alpha \alpha}(\tau - u) \, du
\]

since it is a property of auto-correlation functions that they are even.

Elimination of \( h(u) \) from (4.6) and (3.12), gives equations relating the
properties of the data, and of the error. This elimination is most easily
carried out if Fourier transforms are first taken, i.e. the relation is more
easily deduced in terms of power spectra than in terms of auto-correlation
data.

From (3.12)

\[
G_{\alpha \alpha}(\omega) = Y(j \omega) G_{\alpha \alpha}(\omega),
\]

and from (4.6)

\[
G_{\alpha \alpha}(\omega) = G_{\alpha \alpha}(\omega) + Y(j \omega) G_{\alpha \alpha}(\omega),
\]

where \( G_{\alpha \alpha}(\omega) = \int_{-\infty}^{+\infty} \phi_{\alpha \alpha}(\tau) \exp(-j \omega \tau) \, d\tau \)

cross spectral density of \( x(t) \)

and \( y(t) \), and the other spectral density functions are similarly defined, while

\[
Y(j \omega) = \int_{-\infty}^{+\infty} h(t) \exp(-j \omega t) \, dt \]

frequency response function of the

system approximating the data.

From (4.7) and (4.8), eliminating \( Y(j \omega) \) gives

\[
G_{\alpha \alpha}(\omega) = G_{\alpha \alpha}(\omega) + \frac{G_{\alpha \alpha} G_{\alpha \alpha}^*}{G_{\alpha \alpha}^2}
\]

where * denotes complex conjugate.

Formula (4.9) makes it possible, assuming power spectra of the data
can be made available, to check whether the approximation of the data,
by a linear passive system, may be expected to yield results that are of
value, by indicating the error of the approximation.

A suitable measure of the criterion of success (i.e. of the efficiency of
approximation \( \eta \)) might well be taken to be

mean square value of the output of the linear passive approximation
to the data

\[
\eta = \frac{\text{Mean square value of the output data } y(t) \text{ being approximated}}{	ext{Mean square value of the output of the linear passive approximation to the data}}
\]

Then

\[
\eta = \frac{\int_0^\infty (G_{\alpha \alpha}(\omega) - G_{\alpha \alpha}(\omega)) \, d\omega}{\int_0^\infty G_{\alpha \alpha}(\omega) \, d\omega}
\]

\[
= 1 - \frac{\int_0^\infty G_{\alpha \alpha}(\omega) \, d\omega}{\int_0^\infty G_{\alpha \alpha}(\omega) \, d\omega}.
\]
Data Fitting with Linear Transfer Functions

This efficiency is unity if the approximation is exact, i.e. if \( G_{ee}(\omega) = 0 \), and becomes fractional as the closeness of the approximation deteriorates. Values of \( \eta \) which define an adequately close approximation can only be gained by experience.

It follows from (4.9) that the condition that data shall yield an exact fit, with a linear passive system, i.e. that \( G_{ee} = 0 \), is that

\[
G_{yy}G_{xx} = |G_{xy}|^2 
\]

when, from (4.7)

\[
G_{yy}/G_{xx} = |Y|^2
\]

defines the modulus of the linear passive system, which fits the data, when an exact fit is possible.

§ 5. CONSIDERATION OF SPECIAL CASES

Suppose the records \( x(t) \) and \( y(t) \) originated in a linear passive system, having an impulse response \( W(t) \).

Then

\[
y(t) = \int_0^\infty W(u)x(t-u)\,du
\]

is the exact equation relating \( y(t) \) and \( x(t) \).

Form the cross correlation \( \phi_{xy}(\tau) \) from (5.1) giving

\[
\phi_{xy}(\tau) = \int_0^\infty W(u)\phi_{xx}(\tau-u)\,du.
\]

Comparing (5.2) with (3.12), the best linear passive approximation to the data, in the minimum mean square error sense, \( h(t) \), is given by \( h(t) = W(t) \). The error of the approximation in this case is obviously zero.

5.2

As a second example, suppose the records \( x(t) \) and \( y(t) \) were related by a linear active system, having a weighting function \( W(t) \), and an equivalent output disturbance \( d(t) \), i.e.

\[
y(t) = d(t) + \int_0^\infty W(u)x(t-u)\,du
\]

is the exact expression relating \( y(t) \) and \( x(t) \) to the system. From (5.3), forming the cross correlation function \( \phi_{xy}(\tau) \),

\[
\phi_{xy}(\tau) = \phi_{xd}(\tau) + \int_0^\infty W(u)\phi_{xx}(\tau-u)\,du.
\]

Comparing (5.4) and (3.12) it follows that (a) If \( x(t) \) and \( d(t) \) are uncorrelated, the best linear passive approximation to the data, in the minimum mean square error sense, is \( W(t) \), i.e.

\[
h(t) = W(t).
\]

(b) If \( \phi_{xd}(\tau) \neq 0 \), i.e. if the active sources \( d(t) \) are correlated with \( x(t) \), then a better approximation to the data, than that afforded by \( W(t) \), is possible.
Thus, if \( \phi_{xd}(\tau) \neq 0 \), the disturbance \( d(t) \) may be supposed to have originated from \( x(t) \), via a linear passive operator \( W_2(t) \), as in fig. 7, where

\[
\phi_{xd}(\tau) = \int_0^\infty W_2(u) \phi_{xx}(\tau - u) \, du
\]  

(5.6)

defines \( W_2(t) \), and \( \epsilon(t) \) has the property of a minimum mean square error approximation, that it is uncorrelated with \( x(t) \).

From (5.4) and (5.6)

\[
\phi_{xd}(\tau) = \int_0^\infty [W(u) + W_2(u)] \phi_{xx}(\tau - u) \, du.
\]  

(5.7)

Fig. 7

Comparing (5.7) and (3.12), the best linear passive approximation to the data, in the minimum mean square error sense, \( h(t) \), is given by

\[
h(t) = W(u) + W_2(u).
\]

That the mean square value of \( \epsilon(t) \), associated with this approximation, is less than the mean square value of \( d(t) \), is necessary from the form of solution, but may be separately demonstrated as follows:

From fig. 7

\[
d^2(t) = [c_1(t) + \epsilon(t)]^2
\]

\[= \epsilon^2(t) + c_1^2(t) + 2c_1(t)\epsilon(t).
\]  

(5.8)

Taking mean values of both sides of (5.8), and recognizing that the mean value of \( c_1(t)\epsilon(t) \) is zero, because of the lack of correlation of \( x(t) \) and \( \epsilon(t) \), it follows that

\[
d^2(t) = \epsilon^2(t) + c_1^2(t).
\]  

(5.9)

Since mean square values are necessarily positive, it follows from (5.9) that \( \overline{d^2} > \overline{\epsilon^2} \). Under certain conditions (those stated in (4.11)), it is possible for \( \epsilon(t) \) to be zero, although \( d(t) \neq 0 \).

5.3

As a third example of the application of a minimum mean square error criterion, consider the problem of fig. 4. (taken from Westcott 1956 b), in which neither the process, nor the controller, is restricted to be linear–passive, but a linear passive description of them is to be found having a minimum mean square error constraint. Thus fig. 4. may be replaced by fig. 8.
From fig. 8, it follows that
\[ y(t) = \varepsilon_x(t) + \int_0^\infty h_1(u)x(t-u) \, du, \quad \ldots \quad (5.10) \]
\[ z(t) = \varepsilon_y(t) + \int_0^\infty h_2(u)y(t-u) \, du. \quad \ldots \quad (5.11) \]

Fig. 8

Assuming with Westcott (1956 b) that \( x(t), y(t), z(t) \) are all recorded data, eqns. (5.10) and (5.11) are of the type considered for open-loop data, and each may be solved separately, by a minimum mean square error criterion, for the 'best' \( h_1(u) \) and the 'best' \( h_2(u) \), giving
\[ \phi_{xy}(\tau) = \int_0^\infty h_1(u)\phi_{xx}(\tau-u) \, du \quad \ldots \quad (5.12) \]
and
\[ \phi_{yz}(\tau) = \int_0^\infty h_2(u)\phi_{yy}(\tau-u) \, du, \quad \ldots \quad (5.13) \]
as the equations defining the best linear passive approximations to the process and controller dynamics respectively.

Not all cases of closed-loop systems are as simple as this one. In this case, each of the two eqns. (5.10) and (5.11) contained only two unknowns, an impulse response and the associated error. It is thus possible to consider these as two separate problems, each of the open-loop type, and consider each without cross reference to the other. Each \( h(t) \) is chosen to minimize its own \( \varepsilon(t) \). This is frequently not possible.

5.4

As an example of the difficulty that can arise in the application of a minimum mean square error criterion, consider the problem, studied elsewhere (Florentin et al. 1959), posed by the heat exchanger of fig. 9 and their corresponding flow diagram of fig. 10.
In terms of a best linear passive approximation to the system of fig. 10, there corresponds the system of fig. 11.

In fig. 11 $h_1(t) =$ 'best' linear passive approximation to the "water flow-rate to outlet temperature" dynamics, $\epsilon_1(t) =$ error of this best linear passive approximation, $h_2(t) =$ best linear passive approximation to the "valve position to temperature" dynamics, $\epsilon_2(t) =$ error of this best linear passive approximation, $h_3(t) =$ best linear passive approximation to the "temperature to valve position" dynamics, $\epsilon_3(t) =$ error of this approximation.
From fig. 11, the following equations result:

\[ n(t) = \epsilon_1(t) + \int_0^\infty h_1(u)x_1(t-u)\,du, \quad \ldots \quad (5.14) \]

\[ y(t) = n(t) + \epsilon_2(t) + \int_0^\infty h_2(u)x_2(t-u)\,du, \quad \ldots \quad (5.15) \]

\[ x_2(t) = \epsilon_3(t) + \int_0^\infty h_3(u)y(t-u)\,du. \quad \ldots \quad (5.16) \]

The difficulty arises, in this problem, because it was not found possible to record the data \( n(t) \). These equations are therefore more severely under-determined than those appearing in other problems. Equations (5.14) and (5.15) each have three unknown functions, one of which, \( n(t) \), is common to both.

Eliminating \( n(t) \) between (5.14) and (5.15) yields

\[ y(t) = \epsilon_1(t) + \epsilon_2(t) + \int_0^\infty h_1(u)x_1(t-u)\,du + \int_0^\infty h_2(u)x_2(t-u)\,du. \quad (5.17) \]

Equations (5.16) and (5.17) relate the measured records \( x_1(t), y(t) \) and \( x_2(t) \), to the dynamics being approximated, and the errors of the approximation.

In these two equations, the errors \( \epsilon_1(t) \) and \( \epsilon_2(t) \) are inextricably mixed, appearing only through their sum. To ask that the \( h(t) \) functions should each be separately chosen, to minimize its own error, is not legitimate with the amount of data available.

Had \( n(t) \) been available, the problem here would have been as previously, each equation containing one impulse response and its associated error as the unknowns, and each \( h(t) \) could have been separately chosen to minimize the corresponding \( \epsilon(t) \). But \( n(t) \) is not available, and the question is how to proceed.
The answer to this is to be found in the form of eqns. (5.16) and (5.17). It is legitimate, and reasonable, to ask of (5.16) that \( h_1(u) \) shall be chosen to minimize \( \varepsilon_1^2 \), because this equation is of the correct form for such an operation. But what of (5.17)? It is not legitimate to ask that \( h_1(u) \) be chosen to minimize \( \varepsilon_2^2 \), and \( h_2(u) \) to minimize \( \varepsilon_2^2 \) independently, but it is quite reasonable to require that \( h_1(u) \) and \( h_2(u) \) shall be adjusted simultaneously to minimize the mean value of \([\varepsilon_1(t) + \varepsilon_2(t)]^2\), i.e. to minimize the total mean square error at the terminal at which the output data was recorded.

If this operation is carried out, it follows (see Appendix III), that

\[
\phi_{x_1y}(u) = \int_0^\infty h_1(z)\phi_{x_1x}(u-z)\,dz + \int_0^\infty h_2(z)\phi_{x_2x}(u-z)\,dz, \tag{5.18}
\]

\[
\phi_{x_2y}(u) = \int_0^\infty h_1(z)\phi_{x_2x}(u-z)\,dz + \int_0^\infty h_2(z)\phi_{x_2x}(u-z)\,dz. \tag{5.19}
\]

These equations can be solved for the frequency response functions (Fourier transforms of \( h_1(z) \) and \( h_2(z) \)), giving (see Appendix IV):

\[
Y_1(j\omega) = \frac{G_{2x_1}G_{x_1x} - G_{x_1x}G_{x_2y}}{G_{x_1x}G_{x_2x} - G_{x_1x}G_{x_2x}}, \quad \ldots \quad (5.20)
\]

\[
Y_2(j\omega) = \frac{G_{2x_1}G_{x_1x} - G_{x_1x}G_{x_2y}}{G_{x_1x}G_{x_2x} - G_{x_1x}G_{x_2x}}. \quad \ldots \quad (5.21)
\]

It is of value to look more closely into the physical difference between the heat exchanger problem of § 5.4, and problems of the type indicated by fig. 3.

The difficulty arises mathematically because of the fact that the data \( n(t) \) is not available. Is the problem then really one of perfecting measuring techniques, or of perfecting data analysis techniques? Had \( n(t) \) been available, or had it been possible to make it available by perfecting new measuring techniques, the system of fig. 11, and the resulting mathematical description of the problem in eqns. (5.14) to (5.16), would be reduced to three separate problems, each of the type described by fig. 3.

Each of the mathematical equations (5.14), (5.15) and (5.16) could have been solved separately by the process of minimization of each of the error terms in turn, had \( n(t) \) been available. It is of some interest then to enquire why \( n(t) \) was not available, and whether a modified analysis technique is being sought, when the real search should be concentrated upon the perfection of measuring techniques. It is hoped to demonstrate, in what follows, that the problem really is a data analysis one, and not a ‘refinement of measurement’ one.

Turning first to fig. 3, systems of this type are characterized by the fact that they have two pairs of terminals, an input pair and an output pair. The electrical engineer refers to such systems (networks) as ‘two terminal pairs’. All physical systems are not of this type, the heat exchanger of
fig. 9 being one which is not. In the heat exchanger there are two input terminal pairs, one for cold water, and the other for steam. There are likewise two outputs, one the heated water, and the other the condensate. Such a system belongs to the general class of fig. 12.

![Fig. 12](image)

In the original problem, the dynamics relating the input data to output 2 were not discussed, but physical situations can easily be imagined, in which the dynamics relating the input to output 2 is of as great interest as the dynamics relating the input to output 1. (For example, input 2 might well be derived, after reheat, from output 2 in the heat exchanger problem, making an additional closed loop in the system under investigation.)

The system of fig. 12 may conveniently be described by that of fig. 13, where the dynamics relating the input to output 1, and the input to output 2, have been separated. Thinking of the mathematics for a moment, this is exactly what one would do there also. If simultaneous differential equations defining the outputs in terms of system dynamics and input data were available, these equations would be manipulated to give separate equations, one for each of the unknowns (the outputs, say), in terms of the known variable (the inputs and the system dynamics).

In passing from fig. 9 to fig. 10, it is assumed that the contents of box A of fig. 13 can be described by two independent two-terminal pairs, with a summation device which combines, through a simple summation operation, the outputs of the two separate two-terminal pairs as in fig. 10. Confession of failure of this assumption comes when it is stated that \( n(t) \) and \( c(t) \) cannot be measured, but \( y(t) \) can. The essential physical reason why \( n(t) \) and \( c(t) \) cannot be measured is because they have no physical existence except in very special cases, of which the heat exchanger is not one. In this situation, no amount of improvement of measurement technique can enable one to find these quantities. The problem is one of data analysis.

The question then arises, what is the best linear passive approximation to systems having more than one input terminal pair. In other words, when the output variable is a function of \( n \) measured input variables, instead of a function of a single measured input variable, as in fig. 3, how may the concept of a best linear passive approximation, in a minimum mean square error sense, be applied?

Consider the problem posed by fig. 14, in which an output variable \( y(t) \) is a function of \( n \) input variables \( x_1(t), x_2(t) \ldots x_n(t) \). It will be assumed
that records of data are taken at every terminal. If this is not the case, an extension of existing measuring techniques is called for, not of existing methods of data analysis.

It is proposed to define the linear passive approximation to such a system by fig. 15, or the corresponding mathematical equation

\[ y(t) = \sum_{r=1}^{n} \int_{0}^{\infty} h_r(u)x_r(t-u) \, du + \epsilon(t). \]  

Fig. 13

It will be noticed that both the two terminal pair of fig. 3, and the heat exchanger of fig. 11 (which can be redrawn as in fig. 15 with \( n = 2 \) and \( \epsilon = \epsilon_1 + \epsilon_2 \)), are special cases of this general problem for \( n = 1 \) and \( n = 2 \) respectively. In those cases the \( h(t) \) were chosen to minimize the mean square value of \( \epsilon(t) \). In this general case, the best linear passive approximation to the data will be defined as that set of values of \( h(t) \), which collectively minimize the mean square value of \( \epsilon(t) \).
A set of $n$ simultaneous integral equations defining these functions $h(t)$ can be calculated (see Appendix III), and are found to be of the form

$$
\phi_{sr}(u) = \sum_{s=1}^{n} \int_{0}^{\infty} h_s(z) \phi_{sr}(u-z) \, dz,
$$

$$
r = 1, 2, 3, \ldots n. \quad (5.27)
$$

In deriving this result, it is not necessary to resort to any information about the physics of the flow diagram external to the element itself, i.e. no reference is made to whether the element is part of a closed loop system or not, this not being relevant to the argument. Elements in a system differ, as far as this analysis is concerned, only in the value of $n$, the number of input terminal pairs, or, what is the same thing, the number of independent variables, which contribute to the output variable.

This supports the statement made earlier that the division into open-loop and closed-loop problems is an artificial one. The real division is a classification in terms of input terminal pairs at which data is, or can be, recorded with a sufficiently highly developed recording technique.

The solution of (5.27) is developed in Appendix IV.

5.6

To summarize, when seeking a minimum mean square error approximation to a physical system through recorded data collected during normal operation; (a) set up a flow diagram defining the dynamics, making sure that multiple input systems are shown as such, (b) set up the equations from the flow diagram defining the measured data in terms of a best linear passive approximation to the system dynamics, and the associated error; (c) solve for each element in the flow diagram in turn, using equation (IV 6), with the appropriate value for $n$ for the element.

This is exactly what was done in the heat exchanger problem of § 5.4, the heat exchanger itself was, for the purposes of that problem, an element having two input terminal pairs and one output terminal pair. Solving
the general equation (IV 6) for \( n = 2 \) yields (5.20) and (5.21) for \( Y_1 \) and \( Y_2 \).

The other element in the system controller of fig. 10 has one input terminal pair, and one output terminal pair, a solution of (IV 6), for \( n = 1 \), for this element gives the same description as taking the Fourier transform of a minimum mean square error solution of (5.16).

The method is quite general for any value of \( n \).

---

**APPENDIX I**

The Use of the Constraint Equation \( \phi_{vx}(\tau) = \phi_{vy}(\tau) = 0 \) in the Solution of the Problem of fig. 6

When the above restraint criteria are applied to eqns. (2.4) and (2.5), the impulse functions \( h(u) \) and \( g(u) \) are defined by

\[
\phi_{vx}(\tau) = \int_0^{\infty} h(u) \phi_{zx}(\tau - u) \, du, \quad \ldots \quad (11)
\]

\[
\phi_{xy}(\tau) = \int_0^{\infty} g(u) \phi_{yy}(\tau - u) \, du. \quad \ldots \quad (12)
\]

Equations (11) and (12) also define a minimum mean square error solution to the system of fig. 6, in which \( \epsilon(t) \) is looked upon as the error resulting from the approximation of input data \( x(t) \), and output data \( y(t) \), by a linear passive system having an impulse response \( h(t) \); and \( s(t) \) is looked upon as the error resulting from the approximation of input data \( y(t) \), and output data \( x(t) \), by a linear passive system having an impulse response \( g(t) \). It is not recognized in these constraint conditions that \( s(t) \) contains a contribution, due to a stimulus \( d(t) \), external to the loop. Hence what is being asked by these constraint conditions is a linear passive closed loop, with no external stimulus, which can itself generate stimuli \( x(t) \) and \( y(t) \) at appropriate points in the system. A closed-loop system capable of achieving this result is an oscillatory one (an unstable system), so it will not be surprising if such a solution results from these constraint equations.

To show that this is the solution obtained, take Fourier transforms of (11) and (12) giving

\[
G_{yx} = Y_1 \cdot G_{xx}, \quad \ldots \quad \ldots \quad (13)
\]

\[
G_{xy} = Y_2 \cdot G_{yy}, \quad \ldots \quad \ldots \quad (14)
\]

where \( Y_1 \) = frequency response equivalent of impulse response \( h(t) \), and \( Y_2 \) = frequency response equivalent of impulse response \( g(t) \). From (13) and (14) it follows that

\[
1 - Y_1 Y_2 = 1 - \frac{G_{yx} G_{xy}}{G_{xx} G_{yy}}. \quad \ldots \quad \ldots \quad (15)
\]
Since the solution required is a minimum mean square error one, it follows from (4.9), that
\[ G_{yy} = G_{xx} + \frac{G_{yx} G_{xy}}{G_{xx}} \]  \hspace{1cm} (I 6)
and
\[ G_{xx} = G_{ss} + \frac{G_{xy} G_{yx}}{G_{yy}} \]  \hspace{1cm} (I 7)
From (I 5), (I 6) and (I 7)
\[ 1 - Y_1 Y_2 = \frac{G_{yy}}{G_{yy}} = \frac{G_{ss}}{G_{xx}} \]  \hspace{1cm} (I 8)
Now a physically realizable solution to (I 8) can be found, which requires that \( G_{xx} = G_{ss} = 0 \), namely
\[ 1 - Y_1 Y_2 = 0 \]  \hspace{1cm} (I 9)
for this solution
\[ \bar{e}^2(t) = \int_{-\infty}^{+\infty} G_{ee}(\omega) d\omega = 0 \]  \hspace{1cm} (I 10)
and
\[ \bar{s}^2(t) = \int_{-\infty}^{+\infty} G_{ss}(\omega) d\omega = 0 \]  \hspace{1cm} (I 11)
Conditions (I 10) and (I 11) show that the solution (I 9) simultaneously reduces both \( \bar{e}^2(t) \) and \( \bar{s}^2(t) \) to the smallest value (zero) that a mean square value can assume, and hence is the minimum mean square error solution required by the given restraint condition.

But eqn. (I 9) is a statement of Nyquist’s stability criterion for a closed-loop system, and states that the system is unstable at every frequency having a component in \( G_{xx} \) and \( G_{yy} \). This is the type of solution the restraint criteria demanded. To look for such a solution when it is known that \( x(t) \) and \( y(t) \) result from an external stimulus \( d(t) \), and not from an unstable physical system with no external stimulus, is unwise. The essential problem is one of data recording (making \( d(t) \) available) and not one of data analysis. Had \( d(t) \) been available, a true minimum mean square error solution could have been derived.

**APPENDIX II**

*The Derivation of Goodman’s Result from the Constraint Equations*

From (2.4), forming an expression for the cross correlation function \( \phi_{xy}(\tau) \),
\[ \phi_{xy}(\tau) = \phi_{zz}(\tau) + \int_{0}^{\infty} h(u) \phi_{xx}(\tau - u) du. \]  \hspace{1cm} (II 1)
From (II 1), and the constraint condition (2.9),
\[ \phi_{xy}(\tau) = \int_{0}^{\infty} h(u) \phi_{xx}(\tau - u) du \text{ for } \tau > A. \]  \hspace{1cm} (II 2)
Equation (II 2) is the Goodman–Reswick equation for the determination of \( h(u) \). In their derivation (Goodman et al. 1956, Appendix I), both the constraint conditions (2.8) and (2.9) were used in obtaining this result. Equation (II 2) is, in fact, independent of the constraint condition (2.8).

Equation (2.8) makes it possible (though difficult) to solve (2.5) for \( g(u) \) also. This problem was not considered by Goodman and Reswick.

**APPENDIX III**

**A Minimum Mean Square Error Approximation to Systems with Multiple Inputs**

From (5.26), the equation defining the linear passive approximation of fig. 15, to a system having \( n \) input terminal pairs, and a single output terminal pair is

\[
y(t) = \sum_{r=1}^{n} \int_{0}^{\infty} h_r(u)x_r(t-u) \, du + \epsilon(t); \quad \ldots \ldots \quad (\text{III 1})
\]

\[
\therefore \quad \epsilon(t) = y(t) - \sum_{r=1}^{n} \int_{-\infty}^{\infty} h_r(u)x_r(t-u) \, du. \quad \ldots \ldots \quad (\text{III 2})
\]

The lower limit of the integral has been changed from 0 to \(-\infty\) in this expression, because \( h_r(t) = 0 \) for \(-\infty < t < 0\) for \( r = 1, 2, \ldots, n \).

Squaring both sides of eqn. (III 2) and taking time averages gives

\[
\bar{\epsilon}^2 = \bar{y}^2 - 2 \sum_{r=1}^{n} \int_{-\infty}^{+\infty} h_r(u)\phi_{x_r y}(u) \, du
\]

\[
+ \sum_{r=1}^{n} \sum_{s=1}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [h_r(u) + k_r p_r(u)][h_s(z) + k_s p_s(z)]\phi_{x_r x_s}(u-z) \, du \, dz. \quad (\text{III 3})
\]

If the set \( h_r(u) \) has been chosen to make \( \bar{\epsilon}^2 \) a minimum mean square error, change of \( h_r(u) \) to \( h_r(u) + k_r p_r(u) \), increases the mean square error to \( \bar{\epsilon}_1^2 \), where

\[
\bar{\epsilon}_1^2 = \bar{y}^2 - 2 \sum_{r=1}^{n} \int_{-\infty}^{+\infty} [h_r(u) + k_r p_r(u)]\phi_{x_r y}(u) \, du
\]

\[
+ \sum_{r=1}^{n} \sum_{s=1}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [h_r(u) + k_r p_r(u)][h_s(z) + k_s p_s(z)]\phi_{x_r x_s}(u-z) \, du \, dz.
\]

\[
\therefore \quad \frac{\partial \bar{\epsilon}_1^2}{\partial k_r} \bigg|_{k_r=0} = -2 \int_{-\infty}^{+\infty} p_r(u)\phi_{x_r y}(u) \, du + \sum_{r=1}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_r(u)h_s(z)\phi_{x_r x_s}(u-z)
\]

\[
\times \, du \, dz + \sum_{r=1}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_s(z)h_r(u)\phi_{x_r x_s}(u-z) \, du \, dz
\]

\[
= 0 \quad \text{since} \quad \bar{\epsilon}_1^2 \quad \text{is a minimum.}
\]

Since \( \phi_{x_r x_s}(t) = \phi_{x_s x_r}(-t) \) (James et al. 1947), the last two terms in this expression are identical.

\[
\therefore \quad \int_{-\infty}^{+\infty} p_r(u)[\phi_{x_r y}(u) - \sum_{s=1}^{n} \int_{-\infty}^{+\infty} h_s(z)\phi_{x_r x_s}(u-z) \, dz] \, du = 0. \quad (\text{III 6})
\]
Equation (III 6) is true for any arbitrary \( p_r(u) \). Hence

\[
\phi_{x,y}(u) = \sum_{s=1}^{n} \int_{0}^{\infty} h_s(z) \phi_{x_s,x_y}(u-z) \, dz. \quad (III 7)
\]

The lower limit of integration in (III 7) has been changed from \(-\infty\) to 0 since, for physically realizable systems, \( h_s(z) = 0 \) for \(-\infty < z < 0\) for all \( s \).

Giving \( r \) the values 1, 2 \ldots \( n \) in (III 7) gives \( n \) simultaneous equations to be solved for the \( n \) functions \( h_s(z) \).

Special cases of (III 7), of interest here, arise if \( n = 1 \) and 2.

(a) For \( n = 1 \), (III 7) reduces to

\[
\phi_{x_1,y}(u) = \int_{0}^{\infty} h_1(z) \phi_{x_1,x_1}(u-z) \, dz. \quad (III 8)
\]

Equation (III 7) is the Wiener-Hopf integral equation (3.12) already derived for systems having one input terminal pair and one output terminal pair.

(b) For \( n = 2 \), (III 7) becomes

\[
\begin{align*}
\phi_{x_1,y}(u) & = \int_{0}^{\infty} h_1(z) \phi_{x_1,x_1}(u-z) \, dz + \int_{0}^{\infty} h_2(z) \phi_{x_2,x_1}(u-z) \, dz \quad (III 9) \\
\phi_{x_2,y}(u) & = \int_{0}^{\infty} h_2(z) \phi_{x_2,x_2}(u-z) \, dz + \int_{0}^{\infty} h_1(z) \phi_{x_1,x_2}(u-z) \, dz. \quad (III 10)
\end{align*}
\]

These are eqns. (5.18) and (5.19) of the text used in connection with the heat exchanger problem of § 5.4.

**APPENDIX IV**

**Solution of the Simultaneous Integral Equations (III 7)**

The set of equations

\[
\phi_{x_r,y}(u) = \sum_{s=1}^{n} \int_{0}^{\infty} h_s(z) \phi_{x_s,x_y}(u-z) \, dz
\]

can be solved very simply if expressed in terms of power spectra rather than correlation data.

Taking Fourier transform of both sides

\[
G_{x,y} = \sum_{s=1}^{n} Y_s G_{x_s,x_y}, \quad r = 1, 2, \ldots n. \quad (IV 1)
\]

This is a set of simultaneous linear equations, and may be solved explicitly for \( Y_s \). Thus, in matrix form

\[
G_{x,y} = G_{x,x} Y, \quad . \ldots, \quad (IV 2)
\]

where \( G_{x,y} = n \times 1 \) column vector

\[
\begin{bmatrix}
G_{x_1,y} \\
G_{x_2,y} \\
\vdots \\
G_{x_n,y}
\end{bmatrix}
\]

. . . . (IV 3)
On Data Fitting with Linear Transfer Functions

\[ G_{xx} = n \times n \text{ matrix} \]
\[
\begin{bmatrix}
G_{x_1 x_1} & G_{x_1 x_2} & \cdots & G_{x_1 x_n} \\
G_{x_2 x_1} & G_{x_2 x_2} & \cdots & G_{x_2 x_n} \\
\vdots & \vdots & & \vdots \\
G_{x_n x_1} & G_{x_n x_2} & \cdots & G_{x_n x_n}
\end{bmatrix}
\]  

\[ (IV \ 4) \]

\[ Y = n \times 1 \text{ column vector} \]
\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{bmatrix}
\]

\[ (IV \ 5) \]

The solution of (IV 2) is
\[ Y = [G_{xx}]^{-1} G_{xy} \]  

\[ (IV \ 6) \]

giving all the \( Y_s \) values explicitly in terms of the spectral density properties of the data.

Again the special cases arise, which are of interest in the text:

(a) when \( n = 1 \)
\[ G_{x_1 y} = Y_1 G_{x_1 x_1}; \]  

\[ (IV \ 7) \]

(b) when \( n = 2 \)
\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix}
= 
\begin{bmatrix}
G_{x_1 x_1} & G_{x_1 x_2} \\
G_{x_2 x_1} & G_{x_2 x_2}
\end{bmatrix}^{-1}
\begin{bmatrix}
G_{x_1 y} \\
G_{x_2 y}
\end{bmatrix}
\]

\[ (IV \ 8) \]

which, when expanded, gives eqns. (5.20) and (5.21) of the text.

References


INSTRUMENTATION AND COMPUTATION
IN PROCESS DEVELOPMENT
AND
PLANT DESIGN
LONDON: 1959

(CONTRIBUTION TO DISCUSSION)
DISCUSSION: THE IMPROVEMENT OF PROCESS EFFICIENCY

provided by the makers—the programme for taking over the functions of the conventional controllers—because the makers would have acquired experience in writing such a programme. It should not be ambitious but should be based on the known dynamic behaviour of the measuring instruments and controls of the plant plus some of the predictable interactions which were known to occur when these systems were controlled by separate loops which could not be tied together because of the nature of the conventional controllers. The users would then operate on that crude programme in the first instance, just as management now operated on the human operator, by feeding in different kinds of set conditions. Later they would use the spare capacity of the machine for testing optimising programmes which would probably have to be of the statistical, evolutionary-operation type, working from one group of set conditions to another. Later still if they understood the process they could use the more sophisticated, dynamic optimisation kind of programme.

He thought that they must have that kind of package which paid for itself without years of research before there were too many confirmed sceptics who had installed a large general-purpose machine on a plant, which had generated a lot of work and proved very instructive, but which had never been intended as an economic proposition at the time, and which now seemed to prove conclusively that it never could be an economic proposition.

Mr. R. A. Woodrow said that he had been particularly interested in the correlation analysis of the heat exchanger.

His first comment concerned the type of solution which the authors sought. The system with which they were dealing was taken to be a linear active system as demonstrated by the form of equation (1) of the paper while the solution sought was a best linear passive approximation. That raised the question of what was the "best" linear passive description of the data.

Goodman and Reswick (Ref. 2 of the paper) had assumed that in the case of data taken from open loop systems such as in Fig. 1, the best linear passive description of the data was that one in which \( n(t) \) and \( x(t) \) were uncorrelated, while for data taken from closed loop systems such as Fig. 2 the best description of the data was assumed to be that in which \( y(t) \) and \( n(t) \) were uncorrelated.

In the present paper the authors suggested an alternative definition of the best linear passive approximation to the data defined as that quantity which minimised the mean square value of \( n(t) \) of Fig. 1. That was equivalent to the formulation of the problem: given the input variable \( x(t) \) and the output variable \( y(t) \) of a physical system, find a physically realisable linear passive operator which, when operating upon the input variable \( x(t) \) produces an output variable \( y(t) \) having the property that the operator minimises the mean square value of \( y(t) - y(t) \). That formulation took no account of the origin of the data, i.e., whether from an open or closed loop system—and could in certain circumstances lead to a quite different form of weighting function from that sought by Goodman and Reswick.

To demonstrate that consider the situation of Fig. 1, representing part of the closed loop system of Fig. 2. In that case the Goodman-Reswick method seeks to find a linear passive description of the data having the property that \( x(t) \) and \( n(t) \) are correlated in some particular fashion. But accepting the minimum mean square error criterion suggested in the present paper, the weighting function obtained was quite different. To appreciate that consider Fig. 1 with \( n(t) \) and \( x(t) \) correlated.

Seek a function \( h_1(t) \) such that

\[
n(t) = n_1(t) + \int_0^\infty h_1(u) x(t-u) \, du \quad (1)
\]

where \( n_1(t) \) is uncorrelated with \( x(t) \).

Multiplying throughout by \( x(t-T) \) and averaging

\[
\phi_{x,y}(T) = \int_0^\infty h_1(u) \phi_{x,y}(T-u) \, du + \phi_{y,y}(T) \quad (2)
\]

But for the overall system

\[
y(t) = n(t) + \int_0^\infty h(u) x(t-u) \, du \quad (3)
\]

which after multiplying by \( x(t-T) \) and averaging gives

\[
\phi_{x,y}(T) = \phi_{x,y}(T) + \int_0^\infty h(u) \phi_{x,y}(T-u) \, du \quad (4)
\]

Combining equations (2) and (4)

\[
\phi_{x,y}(T) = \int_0^\infty \{h(u) + h_1(u)\} \phi_{x,y}(T-u) \, du \quad (5)
\]

since \( \phi_{y,y}(T) = 0 \).

Equation (5) was the solution obtained by a minimum mean square error criterion for closed loop systems, and yields a weighting function \( h(u) + h_1(u) \) whereas the Goodman-Reswick formulation requires the determination of \( h(u) \).

Mr. Woodrow said that these calculations led him to wonder why the authors stated that "... these formulae cannot be applied directly in a feedback system ..." and why they then fell back on equation (5) of the present paper which assumed quite a different criterion for the best \( h(t) \) from that defined earlier in their paper. Could the authors make that point clear?

Mr. Woodrow continued that he had not attended the Dusseldorf Conference, neither had he access to Reference 7 which might account for his difficulty in interpreting the solution outlined in Fig. 7.
DISCUSSION: THE IMPROVEMENT OF PROCESS EFFICIENCY

The method described always led to a solution of the form:

\[ \phi_x(T) = \sum_{r} a_r \phi_x(T - rT) \]

which was equivalent to the assumption that the function h(t) of equation (2) of the paper was of the form

\[ h(u) = \sum_{r} a_r \delta(u - rT) \]

That must not be confused with the approximation

\[ h(u) = \sum_{r=1}^{n} a_r \delta(u - rT) \]

used in the method of solution of Reference 2, where nT was the memory time of the system and n was large (order 20 or more) to make T sufficiently small "so that the essential character of the response is retained" (Ref. 2 of paper).

In the present case the solution involved three terms only which suggested that the physical system could be reduced to a number of delays together with appropriate attenuators as in Fig. 3. Was it not an oversimplification accompanied by excessive inaccuracy to suppose that the weighting function could be approximated by as few as three delays?

![Diagram](image)

**Fig. 3.—Suggested system of a number of delays and appropriate attenuators**

Finally, he was puzzled about the question of lag windows and equations (11) and (7) of the text. These could be compared, but differed fundamentally in that equation (7) was exact if an ideal experiment of infinite time duration was assumed, whereas equation (11) involved the auto-correlation of a finite sample of x(t) (duration of 10 min in the case considered). Giving that the symbol \( \phi_x^*(t) \) to avoid confusion with the exact autocorrelation function of equation (7) and making the observation that the lag window D(r) was chosen to make the power spectral density functions of the sample [equation (11)] and of the complete record [equation (7)] closely the same it followed that

\[ \phi_x(t) = D(t) \phi_x^*(t) \]  \( \text{(6)} \)

The paper suggested that, with only \( \phi_x^*(t) \) as available experimental data it was possible to choose D(t) to satisfy equation (6). Could the authors say

(a) how, in general, D(t) may be selected,

(b) why it was not used in the time domain solution to improve the correlation records before attempting a solution to the integral equation (2)?

Mr. L. von Hamos, commenting on the paper by Florentin et al., said that some work had been done in his laboratories on the fundamental basic features of auto-correlation functions to find out about some of the problems which apparently existed. On the one side they had the theoretical mathematics, which gave complicated formulae for such information. On the other side there was the engineer who would like to have apparatus which gave a measure in a finite time of the statistical properties of the phenomena which were met, for example, in the chemical industry.

They found that the calculations were too complicated to be carried out by the methods of the pure mathematician. Moreover, if they found the mathematical formula, they had no time to obtain a numerical evaluation of it, even by the big machines such as the IBM 701. It took too much time. His approach was therefore different. He simulated the mathematics. Noise generators and filters were used to produce the noise of known characteristics—not the unknown noise which the authors had mentioned. He produced a well known noise and made experiments with it by analogue and digital computers. Thus he could calculate the estimates and the necessary window functions in a very short time. He suggested that they must obtain a more accurate idea about such things before they could investigate the very complicated dynamical processes by statistical means. The heat exchanger alone was very complicated. Some of its dynamical properties, such as resonance in the frequency response are not yet well understood. Results of A. Hempel (Oslo) and J. R. Jensen (Copenhagen) presented at a symposium in Bergen in December, 1958 show also discrepancies between theory and experiments.

One point to be noticed in the statistical technique, however, was that if the natural deviation of a control process were used (the natural noise), they were not sure to have the uniform spectral density of the input where there was a closed loop. Some of the spectrum was emphasised, where the frequencies were badly controlled, and that was not so where the frequencies were well controlled. There was no possibility of finding out the transfer functions in the frequency regions where there was perfect control.

It should be mentioned that the estimation of cross-correlation spectra has been treated by N. R. Goodman.

Mr. C. A. J. M. Van der Heudsen emphasised a comment made by Dr. Rosenbrock about the distillation column illustrated in Fig. 9 of the paper by Keating and Townend. The disadvantage of the scheme could be demonstrated by means of Fig. 19 of the same paper by assuming that a constant temperature in the top of the column meant a constant C content of the top product at constant pressure. That, of course, was not quite true, but the assumption might be made in the example to show qualitatively what happened if the feed composition changed, for instance, from T.B.P. 85 to T.B.P. 95. For an operation in which D/F equalled 11:54 and 1/C equals 0-2, Fig. 19 showed that the control scheme would try to change the internal reflux ratio from 12 to 70. In general it might be said that the control scheme tends to give very large changes of load in the column for quite normal disturbances in feed composition.

Mr. W. R. Burton wrote.

The paper by Coutie has given a lucid explanation of process optimisation by statistical procedure. It may be from consideration of space that Coutie has confined his review to consideration of process operating at steady levels of variables such as temperature, concentration, and pressure. This is characteristic of reactions where cost of materials, and consequently yields, dominate the cost of operation.

Many processes, however, involving expensive equipment and requiring product of given specification depend for their cost on throughput. This, in turn, may depend not on steady levels of operating characteristics but on the way in which the latter vary.
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An example may pinpoint the problem: in the reactions

\[ A + B \rightarrow AB \] \hspace{1cm} (6)
\[ A \rightarrow A^* \] \hspace{1cm} (7)
\[ A^* + B \rightarrow AB \] \hspace{1cm} (8)

the required product AB is formed by reactions (6) + (8) both of whose rates increase with temperature. Reaction (8), however, is much slower than reaction (6) because A* is an inactive species of A formed by reaction (2) which has also a positive temperature coefficient.

The time taken to reach a specified conversion to AB is thus markedly dependent on the temperature-time profile followed. In fact, in an actual plant case specifications could not be reached if the initial temperatures were too high.

Could Coutie outline how the optimum time-temperature path be most rapidly pointed out by statistical—and preferably evolutionary—procedures?

Mr. R. A. Woodrow wrote:

Assuming with Florentin et al. in the paper “Correlation Analysis of a Heat Exchanger” that Fig. 4a of their paper “is a reasonable compromise between mathematical simplicity and engineering reality”, I should like to ask why they have not exploited the mathematical simplicity of their assumption?

To make this point, let us formulate the equations relating the parameters defined in Fig. 4a. These are:

\[ n(t) = \int_0^\infty h_w(u) W(t-u) \, du \] \hspace{1cm} (9)
\[ T(t) = n(t) + \int_0^\infty h_w(u) x(t-u) \, du \] \hspace{1cm} (10)
\[ x(t) = \int_0^\infty g(u) T(t-u) \, du \] \hspace{1cm} (11)

These three equations contain four unknowns, and thus cannot be solved as they stand. But, in this problem, both open loop and closed loop data were collected, and these provide adequate information for a solution.

Denoting data collected from an open loop experiment \( [g(t) = 0] \) by a suffix o, and that collected from closed loop data \( [g(t) \text{ a function to be determined}] \) by the suffix c, the following equations result.

(a) Open loop test \( [g(t) = 0] \)

From equation (11):

\[ x(t) = 0 \] \hspace{1cm} (12)

From equations (10) and (12):

\[ T_c(t) = h_w(t) \] \hspace{1cm} (13)

From equations (13) and (9):

\[ T_o(t) = \int_0^\infty h_w(u) W_o(t-u) \, du \] \hspace{1cm} (14)

Equation (14) may be solved directly for \( h_w(u) \), this being the only unknown in the equation.

Turning to the closed loop test, and bearing in mind that, as a result of the previous test, \( h_w(t) \) is now known, it follows from equation (9) that

\[ n_o(t) = \int_0^\infty h_w(u) W_o(t-u) \, du \] \hspace{1cm} (15)

Equation (15) can be solved for \( n_o(t) \) from the known value of \( h_w(t) \) and the data \( W_o(t) \) collected during this test.

From equation (10)

\[ \{ T_c(t) - n_o(t) \} = \int_0^\infty h_w(u) x(t-u) \, du \] \hspace{1cm} (16)

Equation (16) may now be solved for \( h_w(t) \) the only unknown in this equation.

Finally, from equation (11)

\[ x(t) = \int_0^\infty g(u) T_c(t-u) \, du \] \hspace{1cm} (17)

which may be solved for \( g(u) \), the only unknown.

Equations (14), (16), and (17) are all of the same form, namely:

\[ f_o(t) = \int_0^\infty W(u) f_o(t-u) \, du \] \hspace{1cm} (18)

If it is assumed that:

\[ W(u) = \sum_{r=1}^n a_r \delta(t-r\frac{T}{n}) \] \hspace{1cm} (19)

where \( T \) is the memory time of the system, \( \delta(t) \) is the unit impulse function, and \( a_r \) are constants to be determined (see Ref. 2 of the paper for a detailed discussion of this assumption), then equation (18) reduces to:

\[ f_o(t) = \sum_{r=1}^n a_r f_0(t-r\frac{T}{n}) \] \hspace{1cm} (20)

Using equation (20) for \( n \) different values of \( t \) it is possible to formulate \( n \) different equations to solve for the parameters \( a_r \). These form a set of equations which may be represented by the single matrix equation

\[ F_1 a = f_o \] \hspace{1cm} (21)

where \( F_1 \) is an \( n \times n \) matrix, \( a \) an \( n \times 1 \) column vector and \( f_o \) is an \( n \times 1 \) column vector. The solution of equation (21) is \( a = F_1^{-1} f_o \).

Alternatively, one could do even better than this by using equation (20) for \( m \) different values of \( t \) where \( m > n \), producing an overdetermined set, and finding a best solution to this set in the minimum mean square error sense.

In this case \( F_1 \) would be an \( m \times n \) matrix where \( m > n \), \( a \) an \( n \times 1 \) column vector, and \( f_o \) an \( m \times 1 \) column vector.

The best solution of this set in the minimum mean square error sense is:

\[ a = (\tilde{F}_1 F)^{-1} \tilde{F}_1 f_o \] \hspace{1cm} (22)

where \( \tilde{F}_1 \) is the transpose of \( F_1 \) and \( (\tilde{F}_1 F)^{-1} \) is the inverse of the product of the \( F_1 \) matrix and its transpose.

Looking at Figs. 8 and 11 of the paper it appears that it was considered adequate to express the impulse responses as the sum of (9) in Fig. 8) and 6 (in Fig. 11) impulses. If \( n \) were of this order in the above matrix equations, the whole problem reduces to the solution of sets of simultaneous equations of order 10 or less—not nearly such a tedious task as the form of analysis here adopted.

Mr. Coutie replied that he agreed with Dr. Himsworth about the difficulties associated with using naturally occurring random plant variations when trying to derive information about the response surface. It was true that such variations
given to Mr. Dagnall's questions about continuous versus intermittent operations. Those things depended strictly on the exigencies of the moment and they did not think they had an important bearing on the research that they had reported.

The specifications of \( x^d, \Phi, \alpha \) as mentioned were essentially management decisions, depending on market conditions, particular requirements of particular customers, etc. Those matters were included in the paper to emphasise the important (but often overlooked) point: the mathematics of control problems could be solved only after some arbitrary assumptions had been made about how the system should (as distinct from could) behave. To wit, \( x^d \) represented (in a crudely abbreviated form) how much and what quality material and plant should produce. The matrix \( Q \) represented management's view as to how requirements of quality, quantity, cost, etc. should be compromised when they were (as usual) in conflict. The constant \( \alpha \) was a purely technical trick which was discussed in the more detailed expositions of their methods referred to in the bibliography.

A number of questions were raised concerning evolutionary operation and the possibility of obtaining information about process dynamics in a reasonable length of time without unduly upsetting the process. To date, methods of proposed ways for doing that were based on results from pure research as of about 1945. Considerable further progress had been made since then in statistical communication and detection theory, but little or no application had been made to process dynamics. To mention a specific point, the autocorrelation type of detection procedure suggested by Goodman and Reswick was not adequate when small signal-to-noise ratios were encountered (i.e., small process interference); in such cases, cross-correlation type of detection might be much more efficient, but could not be used without first having a fairly good model of the process itself. Needless to say, the possibility of identifying process dynamics was closely related to the quality of available measuring instruments. To be able to control, the instruments must transmit information about the process, but very little was known just how much information was needed to accomplish a given objective. Those were some of the unsolved or untouched problems which must be looked into more deeply before technological hopes could be translated into economic facts.

Perhaps the most urgent problem, however, was finding better means of representing dynamic systems. The linear representation used in the paper became poorer and poorer as the volume in the state space over which statistical averages of input-output data were taken increases. There were at present no acceptable methods of representing a nonlinear system; merely piecing together many linear systems obtained for different regions was too inefficient. That problem, too, had connections with information theory as shown by recent Russian work.

The authors concluded by saying that they did not disagree in substance with the various complaints which had been raised from the standpoint of the "practising" engineer. Rather, they tried to provide a small glimpse at the broad spectrum of research currently under way in many places and try to guess some significant aspects of the coming new technology.

Dr. Westcott replied that he agreed with Dr. Rosenbrock that it was very bad if frequencies were missing, but the same defects were found in the correlation function. The essential point was that the frequencies they obtained were those which the control received in normal circumstances and they were therefore the only frequencies in which they were interested.

The question of short records and whether errors were treated as noise posed the question as to what reliance could be placed on the records. That again was a compromise between accuracy and the time taken. What happened, in fact, was that they used a very inefficient process, continuing it so long as to make sure of the results.

Mr. Woodrow had pointed out many of the difficulties but he had slightly misunderstood one point: in their realisation of the transfer function they had dealt with the individual impulses and three impulses had been sufficient. In the particular case, and only that case, the three pulses had been accurate to \( \pm 2\% \). They could get very good results with a delay line synthesiser having as few as ten sections.

Referring to the lag window, Dr. Westcott said that as they were using short records and calculating the correlation function, as they continued with longer and longer delays the series went wild and was entirely unreliable. It thus was not satisfactory to transform the whole series which must be broken off somewhere. The question of the lag window was one of where and in what manner they broke it off and rejected inefficient data.

Messrs. Florentin, Hainsworth, Reswick, and Westcott later wrote:

Mr. Woodrow has raised the important question of the choice of the best solution for the transfer function. This is a general question which arises in any statistical work, and is a key problem in the development of the present method. In other applications several criteria are in use, e.g. maximum likelihood estimators, confidence intervals, least squares estimators, etc. Up to now it has proved very difficult to apply these criteria to regression problems involving several time series. The least squares estimator used in the paper is one of the few which has been worked out.

In the closed loop situation two equations are required to describe the system, one for the forward path, and one for the return path. To apply least squares error minimisation both equations must be taken into account, and this has not yet been done. It is possible, however, to deduce a relation between the population values of the correlation functions, this relation was used here with the hope that the records taken were long enough to make the errors small.

In choosing the window factor in spectral analysis, one makes a compromise between scatter in the magnitude of the results, and frequency resolution. In practice the choice will be influenced by the experience and judgment of the experimenter, and it is to be hoped that future use of the method will provide the experience which will be the best guide.

The window factor was introduced after a thorough analysis of the estimation properties of the power density spectrum. No analysis of the time domain approach of equivalent thoroughness has yet been carried out, and it is not known if an extra factor could be introduced which would improve the results.

Mr. Woodrow has raised the further point that this problem may be approached as a data-fitting problem, as opposed to a statistical analysis. This is an important philosophical distinction. Taking the statistical viewpoint, one assumes that the process is governed by a fixed set of parameters and probability distributions. This means that all variables in the system will be statistically fluctuating time series. An measured record will be one of many possible ones, and generally every repetition of the experiment will produce a
different result. One then wants to use the observations to calculate the best approximation to the underlying fixed parameters. There is here a basis for comparing different methods of using the observations to calculate the results, and a foundation for a discussion of the errors in the results.

In the data fitting approach one assumes fixed records with superposed experimental error. One then gets the best fit between the records, possibly minimising the effect of the experimental error, but disregarding the possibility of getting different results in a repetition of the experiment due to the records themselves being different. The error here can only relate to errors in measurement, and in the usual formulation of the method, auto-correlation in the error is not considered.

Often in engineering this distinction is of little practical importance. For instance, in the present example, in the open loop case one could include all the noise, regardless of its source, as experimental error. On applying the data fitting technique the same least squares equations would be obtained, but there is no longer any meaning to be attached to an ideal error-free (population) correlation function. There is no question of one method being better than the other, only the error-free (population) correlation function. This decision will vary from system to system.

In the authors' opinion, recognition of the statistical nature of the variables encountered in the typical plant parameter estimation problem will lead to a fuller understanding, and more fruitful use, of the experimental data.

Considering the first data fitting method proposed, no attempt is made to minimise the error, and in some circumstances it is excessively susceptible to instrumental error. On examining the second method for least squares error minimisation, it will be noticed that the coefficients in the "normal" equations are the covariance functions, i.e. non-normalised correlation functions. If a normalising factor is used it is interesting to note that the data fitting approach involves the same arithmetic as the statistical least squares estimator. It is also noteworthy that the more recent techniques of least squares data fitting (also discussed by Lanczos) use a Fourier analysis of the data followed by a smoothing procedure which is very similar to the window method.

The authors are pleased that Mr. Woodrow has raised these fundamental issues, because the present method will be of most use on large and complex plants; and where there are many aspects of plant behaviour to be considered, it is important that the underlying assumptions of the measuring technique should be put clearly before the experimenter.

Mr. Keating agreed with the need for simpler techniques and said that a more simple way of assessing controllability of plants was required especially in the design stage. He emphasised, however, that he was not suggesting that the method included in the paper should apply only to a particular plant. What he and his colleagues had hoped to achieve eventually was a technique which would be applicable to a wide variety of distillation plants. On the question of simpler techniques, he would point out that what was complicated today usually became simple tomorrow. He agreed that it was much easier to simulate mechanical control rather than process control because the process parameters were usually more difficult to define. However, he and his co-author intended to attempt simulation of a distillation process following on the present work.

The authors had not drawn as many general conclusions as they would have liked although they agree that controlling the top product rather than the reflux seemed to be the correct method. The reason might be that the reflux could be considered as having high inertia and consequently a large damping effect. It was dangerous to draw general conclusions too carelessly, however, because what might be good for one type of disturbance might be bad for another. He also pointed out that in the case of the present work, temperature was not sensitive, which was to be expected when dealing with close fractionation.

Dealing with the non-linearity of the system indicated in Fig. 4 on page 32, he pointed out that $D/F$ was rather sensitive at the higher values. From below a recovery of 70% to about 84% it was possible to increase $D/F$ and thereby increase recovery without an excessive increase in reflux. That was one conclusion and he had no doubt that many others could be drawn.

Mr. Broadhurst had asked whether the authors had any envisaged programme on the dynamics of a column. They agreed that any programme must be modified by current results but they hoped to obtain theoretical transient responses. That would be followed by some practical transient response measurement on the plant, and they were designing apparatus for that purpose. Any steady state practical data would naturally have to be obtained statistically and control charts were already in use in the particular refinery.

Replying to Mr. van der Heijden, Mr. Keating pointed out that unintentional feedstock changes of 75-85-95°C were not normal. The paper quoted them to give some clearly seen results. In fact, the plant was actually being controlled by volume control and that type of disturbance was not experienced. He also pointed out that constant temperature at point x in the column was not equivalent to the constant overhead purity at different feeds but in the particular case the difference was small. The maintenance of overhead purity by controlling the temperature in the column by regulating the heat load of the reboiler, that is with constant $D/F$ would result in a column loading which varied with feed composition, and might result in overloading the column.

Reference

CLOSED-LOOP DYNAMICS FROM NORMAL OPERATING RECORDS

By R. A. Woodrow, B.Sc., A.C.G.I., A.M.I.E.E.

SYNOPSIS

The application of correlation functions to the determination of system dynamics is discussed. General conditions are derived which are applicable when unrecorded disturbances modify observed records. In particular, it is shown how the source of the observed data can be chosen so that the theory developed for open-loop systems may be carried over directly to closed-loop systems. The determination of the frequency response of elements of single- and multiple-loop feedback systems, with disturbances originating in every element, is described. The results in the former case are compared with those obtained elsewhere by other arguments for a single-loop control system.

For situations in which correlated records obtained from normal operating data are unable to provide all the information required, a supplementary method is suggested.

INTRODUCTION

To predict the effect of a modification in the control of a physical system upon the overall performance requires an adequate assessment of the dynamic response of each element of the controlled process.

Two techniques are available—calculation and experimentation. To the calculator falls the task of applying fundamental physical and mathematical laws to the formulation and solution of the equations of constraint, defining each element of the system. The experimenter, presented with the same problem,

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prefer to measure the characteristics of the elements.

In complex systems, lack of adequate information about the physical mechanisms involved often makes the formulation of the defining equations difficult in the extreme. This, together with the mathematical difficulties of solving such equations once formulated, defeats the calculator in many situations in which the experimenter has a reasonable chance of success.

Currently, effort is being expended\textsuperscript{1}, \textsuperscript{2}, \textsuperscript{3}, \textsuperscript{4} on perfecting experimental methods for the measurement of system dynamics. Of particular interest are those\textsuperscript{5}, \textsuperscript{4} which require, as the available experimental data, no more than measured records taken at various points in the system, while normal operating conditions prevail. For the purpose of this discussion, the concept of normal operating conditions will be taken to imply that the observed records are arbitrary functions of time, and that uncontrolled and unrecorded disturbances (e.g. noise) may originate in any or all of the elements of which the system is composed.

**List of Symbols**

\[ \begin{align*}
  d(t), e(t), f(t), g(t), l(t) & \quad \text{Normal operating disturbances (functions of time)} \\
  d_s(t), d_p(t), d_l(t) & \quad \text{Disturbances generated within a system which are not amenable to experimental observation} \\
  Z(t) & \quad \text{An operating record which is linearly correlated with recorded fluctuations and uncorrelated with a specified unrecorded fluctuation} \\
  \delta(t) & \quad \text{Unit impulse function (Dirac function)} \\
  p_{rl}(t) & \quad \text{The output observed at terminals when the unit impulse } \delta(t) \text{ is applied at terminals } r \\
  \phi_{xy}(r) & \quad \text{Cross-correlation function of the recorded fluctuations } x(t) \text{ and } y(t) \\
  F(j\omega), G(j\omega) & \quad \text{Fourier transforms of certain cross-correlation functions} \\
  P(j\omega) & \quad \text{Complex frequency response to be determined from operating records}
\end{align*} \]

**THE UNCONTROLLED LINEAR PROCESS**

The problem here is the determination of the dynamic response of the uncontrolled process of Fig. 1 from the normal operating records \( f(t) \) and \( g(t) \), when an unrecorded disturbance \( d(t) \) is supposed to originate in the system. This has already been treated\textsuperscript{3}, \textsuperscript{4} but, for the sake of completeness, and to generalize the concepts to allow extension of the method to controlled processes, the arguments will be repeated here.

Application of the principle of superposition allows the contributions of \( f(t) \) and \( d(t) \) to the output record \( g(t) \) to be assessed separately.

Using the convolution integral relationship,\textsuperscript{5} the contribution to \( g(t) \) due to \( f(t) \) is

\[ g(t) = \int_{-\infty}^{\infty} p_{rl}(t) f(t-t_1) dt_1 \]

Hence, by the principle of superposition

\[ g(t) = \int_{-\infty}^{\infty} p_{rl}(t) f(t-t_1) dt_1 + \int_{-\infty}^{\infty} p_{rl}(t) d(t-t_2) dt_2 ... \] \text{(1)}

The problem posed here is the determination of \( p_{rl}(t) \) from equation (1), when \( g(t) \) and \( f(t) \) are observed random operating records.

Equation (1) cannot be solved directly because of the unknown contribution of the disturbance \( d(t) \) to the output record \( g(t) \). This difficulty may be overcome by application of a property of the correlation function of statistical mathematics.

For two functions of time \( x(t) \) and \( y(t) \), there exists a quantity \( \phi_{xy}(r) \) defined by

\[ \phi_{xy}(r) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t+y) y(t) dt \] \text{(2)}

The quantity \( \phi_{xy}(r) \) is called the cross-correlation function of the variables \( x(t) \) and \( y(t) \). It is a property of this function\textsuperscript{6} that \( \phi_{xy}(r) \) is identically zero, for all values of \( r \), if \( x(t) \) and \( y(t) \) are not linearly correlated.

Now assume a function \( Z(t) \) exists which is not linearly correlated with \( d(t) \) but is correlated with \( f(t) \). Such a function will obviously also be correlated with \( g(t) \) since \( g(t) \) is itself correlated with \( f(t) \). From equation (1) the cross-correlation of \( g(t) \) and \( Z(t) \) is given by

\[ \phi_{Zy}(r) = \int_{-\infty}^{\infty} p_{rl}(t) \phi_{xy}(r-t_1) dt_1 \] \text{(3)}

since, by the choice of \( Z(t) \), the cross-correlation of \( Z(t) \) and \( d(t) \) is identically zero, so the second integral of equation (1) reduces to zero after correlation.

In this particular situation, in the absence of any information to the contrary, it is reasonable to assume that \( f(t) \) and \( d(t) \) are uncorrelated, when \( f(t) \) is itself a function satisfying the requirements specified for \( Z(t) \). This is the case treated in the literature\textsuperscript{3}, \textsuperscript{4} It

\[ \begin{align*}
  \text{(a)} & \quad \text{Diagram of typical process} \\
  \text{(b)} & \quad \text{Diagram redrawn to show portion under test separated from remainder of loop}
\end{align*} \]
Again, by suitable choice of \( dp(t) \), correlation has removed the effect of the disturbances \( f(t) \). Any other \( Z(t) \) would necessitate three observed records, \( f(t) \), \( g(t) \), and \( Z(t) \), in order to produce the correlation functions of equation (3).

It is of interest to note that the function \( g(t) \) does not satisfy the conditions required for \( Z(t) \), so that a correlation equation derived from equation (1) by forming the auto-correlation of \( g(t) \) would not result in the removal of the disturbance contribution from the record. The equation that does result, namely

\[
\phi_{22}(r) = \int_{0}^{\infty} p_{32}(t) \phi_{2f}(r-t) \, dt + \int_{0}^{\infty} p_{3d}(t) \phi_{2d}(r-t) \, dt
\]

does not simplify the problem because \( g(t) \) and \( d(t) \) are correlated. This observation is significant in the development of techniques for the experimental determination of the dynamic characteristics of controlled processes.

**THE SINGLE-LOOP CONTROLLED PROCESS**

A typical controlled process with single-loop control is represented in Fig. 2a. It is supposed that the dynamic response of a portion of the process is to be investigated, and that a number of unrecorded disturbances originate within this portion of the process and also within the remainder of the control loop. These disturbances are supposed referred to equivalent single disturbances at convenient points, e.g. \( d_3(t) \) and \( d_4(t) \) in Fig. 2b.

It is required that normal operating records taken from such a system shall provide the required information. This raises the question of how many records must be collected and what are the properties of these records which will assure a solution to the problem.

Consider the portion of the process under test in Fig. 2b. By comparison with equation (1) for the uncontrolled process we have

\[
g(t) = \int_{0}^{\infty} p_{32}(t) f(t-t_1) \, dt_1 + \int_{0}^{\infty} p_{3d}(t) d_3(t-t_1) \, dt_1 \ldots (4)
\]

Again, assume a random function \( Z(t) \) is available which has the property that it is correlated with \( f(t) \) and uncorrelated with \( d_3(t) \). Then

\[
\phi_{22}(r) = \int_{0}^{\infty} p_{32}(t) \phi_{2f}(r-t) \, dt_1 \ldots (5)
\]

Again, by suitable choice of \( Z(t) \), the process of correlation has removed the effect of the disturbances \( d_3(t) \) from the correlated records.

In the case of the uncontrolled process, it was observed that \( f(t) \) was a function having the properties specified for \( Z(t) \). In the controlled process this is not the case. The contribution made by \( d_3(t) \) to the output record \( g(t) \) is fed back through the controller to contribute to \( f(t) \). The functions \( d_3(t) \) and \( f(t) \) are therefore linearly correlated. Any attempt to use \( f(t) \) leads to the same impasse which arose when attempting to use \( g(t) \) in the case of the uncontrolled process.

The situation is therefore one in which neither the record \( f(t) \), nor \( g(t) \), is suitable to fill the role required of \( Z(t) \). However, there exist in \( e(t) \), in the circuits of Fig. 2, just the properties required of \( Z(t) \), namely, that \( e(t) \) and \( d_3(t) \) may be assumed uncorrelated. Using a record of \( e(t) \) to take the place of the assumed function \( Z(t) \), equation (5) becomes:

\[
\phi_{22}(r) = \int_{0}^{\infty} p_{32}(t) \phi_{2f}(r-t) \, dt_1 \ldots (6)
\]

Measurement of \( e(t) \), \( g(t) \), and \( f(t) \) allows the required correlation functions \( \phi_{22} \) and \( \phi_{2f} \) of equation (6) to be determined. This may be shown (see Appendix I) to be the result derived elsewhere by other arguments, for the response of a controlled process.

**MULTIPLE-LOOP CONTROL OF INTERACTING PROCESSES**

More general situations can be envisaged than those discussed above. For example, consider Fig. 3 which might represent certain interacting processes subjected to multiple-loop control.

If it is assumed that unrecorded disturbances are generated within each of the eight elements indicated, it is possible to predict the response of any element from three operating records taken at suitable points in the system. Consider, as a typical case, the calculation of the response of element 3 of Fig. 3.

For this element, as in the previous cases considered,

\[
g(t) = \int_{0}^{\infty} p_{32}(t) f(t-t_1) \, dt_1 + \int_{0}^{\infty} p_{3d}(t) d_3(t-t_1) \, dt_1 \ldots (7)
\]

After correlation with a suitable function, \( Z(t) \), which is correlated with \( f(t) \) but not with \( d_3(t) \), equation (7) reduces to

\[
\phi_{22}(r) = \int_{0}^{\infty} p_{32}(t) \phi_{2f}(r-t) \, dt_1 \ldots (8)
\]

In this situation, either \( e_3(t) \) or \( e_4(t) \) is suitable to act as \( Z(t) \). Records of \( f(t) \), \( g(t) \), and either \( e_3(t) \) or \( e_4(t) \) enable \( \phi_{22} \) and \( \phi_{2f} \) to be found.

In a similar manner the response of any other element of Fig. 3 may be determined from observed input and output records of that element together with either \( e_3(t) \) or \( e_4(t) \).

The general conclusion may be drawn that, to find the response of any element in any closed-loop system, it is necessary to take (a) input and output records of that element, and (b) any one other record which is linearly correlated with these, but which may be...
It is convenient here to consider one of the essential functions performed by the controller, which is represented schematically by Fig. 5.

The measured output of the process is compared with a 'desired value' (also referred to as the 'set point'). The difference between the measured value of the output and the desired value constitutes an error signal which, after suitable modification in the controller, is used to correct the supply to the process in such a way as to reduce the difference between the measured output of the process and the desired output.

The desired value control is normally maintained steady. If, instead, small fluctuations about the steady position are caused to occur, much useful information can result. The magnitude of the fluctuation can be restricted to keep the output of the process within acceptable limits. The frequency spectrum, however, is under the control of the experimenter.

From a theoretical standpoint, such an operation is equivalent to an artificial increase in the disturbance \( d(t) \) of Fig. 2b. The essential difference is that the fluctuations of the setting of the desired value control \((= \bar{d}(t) \text{ say})\) are controllable and recordable, whereas the disturbances \( d(t) \) previously considered are not.

When the desired value is subjected to fluctuations, the arrangement of Fig. 2 is modified to that of Fig. 6. For this system

\[ e(t) = \int_{-\infty}^{\infty} p_u(t) f(t-t_c) \, dt + \int_{-\infty}^{\infty} p_u(t) d_p (t-t_c) \, dt \]

as before, and, after correlation with a function \( Z(t) \) correlated with \( g(t) \), but uncorrelated with \( d_p(t) \), this equation reduces to

\[ \phi_{zd}(\tau) = \int_{-\infty}^{\infty} p_u(t) \phi_{df}(t-\tau) \, dt \]

Since \( e(t) \) is supposed unsuitable to provide the desired information, the role of \( Z(t) \) may be filled by \( \bar{d}(t) \), when (11) becomes

\[ \phi_{zd}(\tau) = \int_{-\infty}^{\infty} p_u(t) \phi_{df}(t-\tau) \, dt \]

From the frequency domain definition of the problem

\[ P(j\omega) = \text{frequency response of the process} \]

\[ = \left\{ \int_{-\infty}^{\infty} \phi_{zd}(\tau) e^{-j\omega \tau} \, d\tau \right\} \left\{ \int_{-\infty}^{\infty} \phi_{df}(\tau) e^{-j\omega \tau} \, d\tau \right\} \]

By control of the frequency spectrum of \( d(t) \), equation (13) may be made determinate for those values of \( \omega \) for which a result is required.

The argument here formulated is in no way peculiar to a process subjected to single-loop control, but may be applied directly to multi-loop systems. The only

![Fig. 5—Desired-value setting of a controller](image)

![Fig. 6—Fig. 2b modified for 'desired-value setting'](image)
difference from the present situation is that the function \( Z(t) \) is replaced by \( d_i(t) \) because \( e_i(t) \) and \( e_d(t) \) are of unacceptable frequency content.

Assessment of Accuracy of Frequency Response

The application of correlation techniques and random disturbances to the problems of frequency-response measurement has not received unqualified support from those interested in making such measurements. The criticism levelled against the method is on the grounds of accuracy.

Bearing in mind that this method has been developed for use in situations which do not readily lend themselves to response determination by other methods, it is desirable that an assessment of probable accuracy of the final result should be attempted. Such an assessment would serve to establish confidence in the method, and in the results derived therefrom. It is unfortunate that, to date, no serious attempt has been made to compare accuracy of results determined by this method. Where results have been compared with those established by other measuring techniques, the agreement has left something to be desired. Possible reasons for inaccuracy are to be found in:

(i) Lack of validity of the assumption that certain disturbances to which the system is subject are uncorrelated.
(ii) The effects of non-linearity in the process—this is liable to introduce serious errors in the test results used for comparison, particularly if the test input required is so large as to violate quasi-linear approximations.
(iii) The normal operating data itself—for a process under closed-loop control this data may well validate a quasi-linear approximation, but at the same time may be so small and of such unsuitable frequency composition as to make for doubtful accuracy over part of the response curve \( P(j\omega) \).
(iv) The time duration of experimental records used—in practice, this may produce correlation records of very limited extent, instead of the infinite records required for the determination of the Fourier integrals of equation (10).

In the example quoted elsewhere\(^9\) a single result (a step response) obtained from operating records is compared with a similar result obtained by other methods (a measured step response). If the object of the experiment in each case is taken to be a determination of the frequency-response locus of the process when operating under quasi-linear conditions, both results are liable to error for the reasons indicated above. Such a comparison provides no useful data about the accuracy of either method.

An alternative technique is to carry out a set of similar experiments using normal operating records. For this purpose, random fluctuations of the desired value setting may be used as the record \( Z(t) \). The nature of this record (e.g. frequency spectrum and time duration) may be varied from one experiment to the next to ensure randomization of the errors. Work at present in hand is directed towards the collection and interpretation of such data.

APPENDIX I

In the case of the single-loop controlled process in which (a) the dynamic response of the whole process is to
LIMITATIONS ON REALIZABLE RESPONSE SHAPES FOR CERTAIN WIDE-BAND BANDPASS AMPLIFIER CIRCUITS

By

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LIMITATIONS ON REALIZABLE RESPONSE SHAPES FOR CERTAIN WIDE-BAND BANDPASS AMPLIFIER CIRCUITS

By R. A. WOODROW, B.Sc., Associate Member.

(1) INTRODUCTION

Two amplifier circuit configurations that have been recommended in the literature for use when wide-band bandpass properties are required are stagger-tuned amplifiers and chains of inverse-feedback pairs. The investigation from which this paper results attempted to unify and simplify the design procedures proposed for these circuits by developing simple graphical constructions to replace, as far as possible, procedures requiring algebraic manipulative ability and by devising a simple test which can be applied at the beginning of a design problem to verify that the specification may be realized with a given circuit.

It was not the primary object of this study to compare the relative merits of the two circuits for the experimental realization of a given specification, as this has been done elsewhere. Therefore no experimental results are offered. However, it so happens that, in the development of the test mentioned above, certain fundamental limitations of the circuits are disclosed. There thus emerges a criterion by which the merits of different circuit configurations could be compared.

(2) THE DESIGN PROCEDURE

The designer of a wide-band bandpass amplifier normally seeks to realize a design specification which requires that the amplifier shall

(a) Amplify a given band of frequencies defined by \( f_1 < f < f_2 \).

(b) Provide a certain specified passband gain with a permitted tolerance of \( \pm db \).

When proceeding from such a specification to a physical realization, two distinct problems occur, which may be called the 'approximation problem' and the 'realization problem'.

The solution of the approximation problem provides a suitable analytic approximation to the design specification, thereby defining the desired response shape. The approximation usually employed is either a Chebyshev or a Butterworth (maximally flat) polynomial. By an artifice developed here, it can be shown that the latter is a limiting case of the former, so that one solution of the approximation problem can be made to serve both types of approximation in common use.

Initially, the realization problem requires the development of a simple test to establish that the response shape given by the solution of the approximation problem can be obtained with a
certain circuit configuration; if it cannot, alternative circuits must be considered; if it can, the solution of the realization problem yields the required circuit.

The test of realizability has been reduced to a simple graphical construction defining the region(s) in a complex plane within which singularities of the desired response shape may be realized. A comparison with the positions of the singularities of the desired response shape in the same plane immediately reveals any unrealizable desired responses. The essential simplicity of this test is the result of a careful choice of the basic amplifier circuit (see Figs. 1 and 3). In each case this is chosen in such a way that it can be made to realize a complex conjugate pair of singularities. By cascading the appropriate number of basic units, the desired overall response shape is realized. In this way, the design of an n-stage amplifier can be reduced to repeated applications of the design procedure of the basic circuit.

(3) THE INVERSE-FEEDBACK PAIR

(3.1) The Gain Expression

As a first example of the application of this general procedure, suppose the amplifier of Fig. 1 is chosen as the basic amplifier circuit.

![Fig. 1.—The inverse-feedback pair.](image)

The voltage gain of this arrangement is given by

\[ G = \frac{g_m(g_m - g_f)}{z^2 + 2z\left(\frac{g + g_f}{\omega_m C}\right) + g^2 + 2gg_f + gfg_m} \]  

where \( \omega_m^2 = \omega_m \omega_f = 1/LC. \)

Factorizing the denominator of eqn. (1) gives

\[ G = \frac{g_m(g_m - g_f)}{(z - z_{p1})(z - z_{p2})} \]  

Provided that \( g_m > g_f, \) the poles \( z_{p1} \) and \( z_{p2} \) of the gain function \( G \) in eqn. (2) are complex conjugates. If \( g_m < g_f, \) both \( z_{p1} \) and \( z_{p2} \) are real and negative. This last situation is of little interest in the realization of Chebyshev approximations, and is not pursued.

Taking \( g_m > g_f, \) so that

\[ z_{p1} = x_p + jy_p \]  
\[ z_{p2} = x_p - jy_p \]  

it follows from eqns. (1), (2) and (3) that

\[ \omega_m C x_p = -(g + g_f) \]  
\[ (\omega_m C y_p)^2 = g_f(g_m - g_f) \]

(3.2) Regions of Realizable Singularities

It is essential, for a successful solution of the realization problem, that \( g \) and \( g_f \) be physically realizable conductances; i.e. they must be real, positive constants. This places certain restrictions on permitted values of \( z_{p1} \) and \( z_{p2}. \)

From eqn. (5) it follows that

\[ g_f = \frac{1}{2} \left\{ g_m \pm \sqrt{g_m^2 - (2\omega_m C y_p)^2} \right\} \]  

which shows that \( g_f \) is real provided that

\[ -\frac{g_m}{2\omega_m C} < y_p < \frac{g_m}{2\omega_m C} \]  

and that \( g_f \) is positive for all values of \( y_p \) for which it is real, whichever sign is associated with the square root.

From eqns. (4) and (6) it follows that

\[ g = -\frac{1}{2} \left\{ 2\omega_m C x_p + g_m \pm \sqrt{g_m^2 - (2\omega_m C y_p)^2} \right\} \]  

which shows that \( g \) is real if eqn. (7) is satisfied and is positive if

\[ 2\omega_m C x_p + g_m \pm \sqrt{g_m^2 - (2\omega_m C y_p)^2} < 0 \]

In the region where \( -g_m/2\omega_m C < x_p < 0, \) the quantity \( 2\omega_m C x_p + g_m \) is positive. In this region eqn. (9) can only be satisfied if the negative root sign is adopted, and then only if

\[ \sqrt{g_m^2 - (2\omega_m C y_p)^2} > 2\omega_m C x_p + g_m \]

which is the interior of the semicircle with centre \((-g_m/2\omega_m C, 0)\) and radius \( g_m/2\omega_m C\) in Fig. 2.

![Fig. 2.—Regions of physically realizable singularities.](image)
that response shape cannot be realized with a chain of inverse-feedback pairs.*

(3.3) The Constraint \( g > g_{\text{min}} \)

In practice, finite damping must be associated with the tuned load admittances caused by coil losses, valve damping, etc. This prevents the circuit designer from realizing values of \( g \) less than some specified minimum value, \( g_{\text{min}} \).

The effect of this practical limitation on the position of realizable singularities can be assessed by requiring that \( g \) in eqn. (8) shall satisfy the condition

\[
\frac{g}{g_{\text{min}}} \geq 1 \quad \ldots \quad (12)
\]

This constraint modifies eqn. (9) to

\[
2\omega_mC_p + g_m + 2g_{\text{min}} \pm \sqrt{(g_m - (2\omega_mC_p)^2)} < 0 \quad (13)
\]

Curve B of Fig. 2 shows the reduction in the region of realizable singularities necessary to satisfy eqns. (12) and (13).

(4) THE STAGGER-TUNED AMPLIFIER

(4.1) The Gain Expression

As an alternative, the stagger-tuned pair of Fig. 3 may be chosen as the basic amplifier element.

![Fig. 3.—The stagger-tuned pair.](image)

The voltage gain of this circuit is given by

\[
G = \frac{g_m^2}{\omega_1^2(\omega_2C)^2} \quad \ldots \quad (14)
\]

where \( \omega_1^2 = 1/L_1C \) and \( \omega_2^2 = 1/L_2C \).

If \( \omega_1, \omega_2, g_1 \) and \( g_2 \) are chosen to satisfy the conditions

\[
\omega_1^2\omega_2 = \omega_1\omega_2 = \omega_1^2 \quad \ldots \quad (15)
\]

and

\[
g_1\omega_1C = g_2\omega_2C = d_1 \quad \ldots \quad (16)
\]

then eqn. (14) can be manipulated to give

\[
G = \frac{(g_m^2}{\omega_m^2C^2} \quad \ldots \quad (17)
\]

where \( \mu_1 = \omega_1/\omega_m + \omega_m/\omega_1 = \omega_2/\omega_m + \omega_m/\omega_2 \). From eqn. (17),

\[
G = \frac{(g_m^2}{\omega_m^2C^2} \quad \ldots \quad (18)
\]

and provided that \( (d_1\mu_1)^2 < 4(d_1^2 + \mu_1^2 - 4) \), the poles \( z_{p1} \) and \( z_{p2} \) occur in complex conjugate pairs, as before.

For this circuit arrangement,

\[
z_{p1}z_{p2} = x_p^2 + y_p^2 = d_1^2 + \mu_1^2 - 4 \quad \ldots \quad (19)
\]

\[(z_{p1} + z_{p2}) = -2x_p = d_1\mu_1 \quad \ldots \quad (20)
\]

Eqs. (19) and (20) for the stagger-tuned amplifier correspond to eqns. (4) and (5) for the inverse-feedback pair.

(4.2) Regions of Realizable Singularities

The constraints imposed upon the positions of realizable singularities must again be considered. Since \( \omega_1, \omega_2, g_1 \) and \( g_2 \) must be positive real constants, it follows that \( d_1 \) is positive real, and \( \mu_1 \) is real and within the range \( 2 < \mu_1 < \infty \). In practice, the constraint that \( g_1 \) shall be a real positive constant is not sufficiently strong, because load circuits of infinite Q-factor are unattainable. This is allowed for by imposing a more severe constraint on \( d_1 \) than above, namely that \( d_1 \) is real and within the range \( d_{\text{min}} < d_1 < \infty \), where \( d_{\text{min}} \) is the minimum damping factor (=1/maximum Q-factor) attainable.

Eliminating \( d_1 \) from eqns. (19) and (20) and solving the resulting quadratic for \( \mu_1 \) gives

\[
\mu_1 = \left[ \frac{1}{4} \left( x_p^2 + y_p^2 + 4 \right) \pm \sqrt{\left( x_p^2 + y_p^2 + 4 \right)^2 - 16x_p^2} \right]^{1/2} \quad (21)
\]

But \( (x_p^2 + y_p^2 + 4)^2 - 16x_p^2 = (x_p^2 + y_p^2 - 4)^2 + 16y_p^2 \), which is positive for all values of \( x_p \) and \( y_p \). This shows that \( \mu_1 \) is real for all values of \( x_p \) and \( y_p \).

To satisfy the condition \( 2 < \mu_1 < \infty \), requires that

\[
0 < (x_p^2 + y_p^2 - 4) \pm \sqrt{(x_p^2 + y_p^2 - 4)^2 + 16y_p^2} \quad (22)
\]

Eqn. (22) can only be satisfied if the positive root is adopted and is always satisfied when this root is taken. Hence

\[
\mu_1 = \left[ \frac{1}{4} \left( x_p^2 + y_p^2 + 4 \right) + \sqrt{\left( x_p^2 + y_p^2 + 4 \right)^2 - 16x_p^2} \right]^{1/2} \quad (23)
\]

From eqns. (19) and (23), it follows that

\[
d_1 = \left[ \frac{1}{4} \left( x_p^2 + y_p^2 + 4 \right) - \sqrt{\left( x_p^2 + y_p^2 + 4 \right)^2 - 16x_p^2} \right]^{1/2} \quad (24)
\]

while from eqn. (20) it follows that only negative values of \( x_p \) are realizable, since both \( \mu_1 \) and \( d_1 \) are positive and real.

From eqn. (24) and the condition \( d_{\text{min}} < d_1 < \infty \) it follows finally that

\[
1 < \frac{x_p^2 - y_p^2}{d_{\text{min}}^2} < 4 - d_{\text{min}}^2 \quad \ldots \quad (25)
\]

which, for points in the left-half z-plane, defines a region bounded by a hyperbola such as is indicated in part by curve C of Fig. 2.*

The regions in the z-plane within which complex conjugate pairs of singularities are permitted for both an amplifier of inverse-feedback pairs and a stagger-tuned amplifier having been determined, it is necessary to turn to the approximation problem and to determine the position of the singularities in the z-plane corresponding to desired response shapes.

(5) THE APPROXIMATION PROBLEM

(5.1) General Remarks

The response shapes usually required are the Chebyshev or equal ripple response, and the Butterworth or maximally flat response. However, by a suitable choice of parameters, the

* The possibility of pairs of real roots situated on the negative axis of this region has not been investigated. It should not be assumed that arbitrary selections of pairs of negative real singularities in this region are admitted.
latter can be treated as a limiting case of the former, so that one analysis can be applied to both. This will be done here.

(5.2) The Chebyshev Response

A theorem due to Chebyshev, and a subsequent discussion of the theorem by Bernstein, shows that, if a polynomial of degree \( n \) in a real variable, \( x \), satisfies the condition that 
\[
-1 < f(x) < +1 \quad \text{for} \quad -1 < x < +1,
\]
then, for any value of \( x \) in the range \( |x| > 1 \), there is a maximum value which \( f(x) \) cannot exceed, and this maximum value is achieved if \( f(x) \) is a Chebyshev polynomial of the first kind and of order \( n \). Chebyshev polynomials therefore exhibit just the property required of a bandpass amplifier, namely the maximum attainable rate of cut-off outside the passband, and a maximum range within the passband having a gain which does not differ from the mid-band gain by more than \( 1 \). For this reason, the Chebyshev polynomial is considered the most suitable polynomial with which to approximate the design specification in situations in which the phase response is unimportant.

Since eqns. (11), (23) and (24) express the circuit parameters in terms of the positions of the singularities in the \( z \)-plane of the voltage transfer function, and since the constraints imposed upon the positions of these singularities are known, it is convenient to specify the desired response shape in terms of the positions of desired singularities in the plane of \( z \).

Define \( \omega_m = \omega_0 \omega_1 \) and \( 2nB = \omega_1 - \omega_0 \) and \( d = Bf_m \).

When \( \omega = \omega_1 \),
\[
z = x + jy = j(\omega_1 \omega_m - \omega_m \omega_0) = -jd
\]
When \( \omega = \omega_m \),
\[
z = x + jy = 0
\]
When \( \omega = \omega_0 \),
\[
z = x + jy = j(\omega_m \omega_0 - \omega_0 \omega_m) = +jd
\]
Consider next the expression
\[
\frac{G}{G_m} = \left[ \delta + e T_{2n}(\frac{y}{jkd}) \right]^{-1/2}
\] (26)
where \( |G_m| \) is the mid-band gain modulus,
\[
T_{2n}(\frac{y}{jkd}) = \cos [2n \cos (y/jkd)]
\]
is the Chebyshev polynomial of the first kind and of order \( 2n \) and \( \delta, e \) and \( \kappa \) are constants to be specified.

These constants can be chosen to relate eqn. (26) to the design specification through the following conditions:

When \( \omega = \omega_m \) and hence \( y = 0 \),
\[
|G| = |G_m| \quad \text{by definition of} \quad |G_m|
\]
When \( T_{2n}(y/jkd) = +1 \) and \( n \) is odd,
\[
20 \log_{10} \left| \frac{G}{G_m} \right| = -s
\]
(see Section 10.1).
When \( T_{2n}(y/jkd) = -1 \) and \( n \) is even,
\[
20 \log_{10} \left| \frac{G}{G_m} \right| = +s
\]
(see Section 10.1).

When \( \omega = \omega_0 \) or \( \omega_1 \) and hence \( y = \pm d \),
\[
|G/G_m| = |1/\sqrt{2}|
\]
from the specification of \( \omega_0 \) and \( \omega_1 \) as the upper and lower 3 dB frequencies.

Choosing \( \delta, e \) and \( \kappa \) in eqn. (26) to satisfy these conditions gives:

If \( n \) is odd,
\[
\left| \frac{G}{G_m} \right| = \left[ (1 + \epsilon) + e T_{2n}(\frac{y}{jkd}) \right]^{-1/2}
\]
where
\[
\epsilon = \frac{1}{2} (10^{8/10} - 1) \simeq 0.115 s, \quad s < 8.7
\]
and
\[
1/\kappa = \cosh \left( \frac{1}{2n} \arccosh \frac{1 - \epsilon}{\epsilon} \right)
\]
If \( n \) is even,
\[
\left| \frac{G}{G_m} \right| = \left[ (1 - \epsilon) + e T_{2n}(\frac{y}{jkd}) \right]^{-1/2}
\]
where
\[
\epsilon = \frac{1}{2} (1 - 10^{-8/10}) \simeq 0.115 s, \quad s < 8.7
\]
and
\[
1/\kappa = \cosh \left( \frac{1}{2n} \arccosh \frac{1 + \epsilon}{\epsilon} \right)
\]
From eqns. (27) and (28), it follows that the appropriate form of eqn. (26) for both odd and even values of \( n \) is
\[
\left| \frac{G}{G_m} \right| = \left\{ [1 + (1) n + 1 \epsilon] + e T_{2n}(\frac{y}{jkd}) \right\}^{-1/2}
\]
where
\[
\epsilon \simeq 0.115 s
\]
and
\[
1/\kappa \simeq \cosh \left( \frac{1}{2n} \arccosh \frac{8.686}{s} \right)
\]
(29)
The singularities \( z_p \) of the minimum phase gain function \( G/G_m \) having a modulus \( |G/G_m| \) defined by eqn. (29) are given by the zeros in the left half \( z \)-plane of the function\(^8\)
\[
H(z) = 1 + (1)^n + 1 \epsilon + e T_{2n}(\frac{z}{jkd})
\]
which reduces to the condition
\[
T_{2n}(\frac{z}{jkd}) = \cos 2n \cos (\frac{z}{jkd})
\]
\[
= \frac{1 + (1)^n + 1 \epsilon}{\epsilon} = -\cosh \theta
\]
(30)
where
\[
\theta = \arccosh \frac{1 + (1)^n + 1 \epsilon}{\epsilon}
\]
(31)
From eqn. (30),
\[
\cos \left[ 2n \arccos \left( \frac{z}{jkd} \right) \right] = -\cosh \theta = \cos \left( j\theta - (2r - 1)\pi \right)
\]
or
\[
z_p = jkd \cos \left( \frac{1}{2n} \left[ j\theta - (2r - 1)\pi \right] \right)
\]
(32)
where \( r = 1, 2, \ldots n \).

Equating the real and imaginary parts of eqn. (32) yields
\[
x_p = -kd \sinh \theta/2n \sin \left( \frac{(2r - 1)\pi}{2n} \right)
\]
\[
y_p = kd \cosh \theta/2n \cos \left( \frac{(2r - 1)\pi}{2n} \right)
\]
(33)
Eqs. (33) are the parametric equations of the ellipse
\[
\frac{x_p^2}{kd \sinh \theta/2n} + \frac{y_p^2}{kd \cosh \theta/2n} = 1
\]
(34)
which shows that the required poles lie on an ellipse having a semi-major axis \( kd \cosh \theta/2n \simeq d \) along the imaginary axis and a semi-minor axis \( kd \sinh \theta/2n \simeq d \sqrt{(1 - \kappa^2)} \) along the real axis.
Fig. 4.—Locating the singularities of a desired response shape.

Assuming for the moment that a value of \( n \) has been established, the required pole positions may be rapidly plotted (see Fig. 4) as follows:

(a) Draw circles of radii \( d \) and \( d\sqrt{1 - \kappa^2} \) for the given specification, with centres at \( x = y = 0 \).

(b) Draw, in the second quadrant of the \( z \)-plane, radial lines making angles \((2r - 1)\pi/2n\) with the \(+jy\) axis.

(c) Read off the \( x \)-co-ordinate of the points of intersection of the smaller circle and each radial line, and the \( jy \)-co-ordinate of the points of intersection of the larger circle and each radial line. These co-ordinates are

\[
\begin{align*}
\theta_p &= -\kappa d \sinh \frac{(2r - 1)\pi}{2n} \\
\theta_y &= \kappa d \cosh \frac{(2r - 1)\pi}{2n}
\end{align*}
\]

which define the required poles.

It only remains to superimpose curves B and C of Fig. 2 on Fig. 4, to test a particular specification for physical realizability.

(5.3) The Butterworth Response

The Chebyshev response having been defined in terms of a 3 dB bandwidth specification, the Butterworth (maximally flat) response follows by making the passband gain tolerance, \( s \), tend to zero.

As \( s \) tends to zero, \( \kappa \cosh \theta/2n \to \kappa \sinh \theta/2n \to 1 \) so that a Butterworth response requires poles \( z_p \) defined by

\[
\begin{align*}
\theta_p &= -d \sin \frac{(2r - 1)\pi}{2n} \\
\theta_y &= d \cos \frac{(2r - 1)\pi}{2n}
\end{align*}
\]

which is the parametric equation of a circle centred on the origin and of radius \( d \).

(6) CHOOSING THE VALUE OF \( n \)

The only requirement in the original specification yet to be considered is the specified mid-band gain. It is to realize this that the value of \( n \) must be chosen.

Consider first the case of an \( n \)-stage amplifier consisting of either \( n/2 \) inverse-feedback pairs if \( n \) is even, or \((n - 1)/2\) feed-back pairs together with a single stage without feedback if \( n \) is odd. It follows from eqn. (2) and the equation

\[
G = \frac{g_m}{\omega_0 C} \frac{1}{z + g_1/\omega_0 C}
\]

for the gain of a single stage, that

\[
|G| = \left( \frac{\text{Geometric mean } g^e_m}{\omega_0 C} \right)^n \times (y^{2n} + a_{2n-2}y^{2n-2} + \ldots + a_2y^2 + a_0)^{-1/2} \quad (36)
\]

for such amplifiers. In eqn. (36) \( a_{2n-2}, \ldots a_2, a_0 \) are constants, and

\[
\text{(Geometric mean } g^e_m) = g^{n/2}_m \prod_{r=1}^{n/2} (g_m - g_{f_r}) \text{ if } n \text{ is even,}
\]

\[
= g^{(n+1)/2}_m \prod_{r=1}^{(n-1)/2} (g_m - g_{f_{2r}}) \text{ if } n \text{ is odd,}
\]

where \( g_{f_r} \) is the feedback conductance of the \( r \)th pair in the chain.

The gain of an \( n \)-stage stagger-tuned amplifier may be developed from eqn. (17), and is of the form

\[
|G| = \left[ g_m/\omega_0 C \right]^n \times (y^{2n} + a_{2n-2}y^{2n-2} + \ldots + a_2y^2 + a_0)^{-1/2} \quad (37)
\]

The desired response shape is found from eqn. (29) by expressing \( T_2n(y/\kappa d) \) in polynomial form (Section 10.2), when the desired \( |G| \) becomes

\[
|G| = |G_m| \epsilon^{-1/2} \times (y^{2n} + a_{2n-2}y^{2n-2} + \ldots + a_2y^2 + a_0)^{-1/2} \quad (38)
\]

Locating the singularities of the gain functions, eqns. (36) and (37), in the same positions as those of the desired response shape eqn. (38) ensures the identity of the polynomials in these three expressions. Hence

\[
|G_m| = \epsilon^{1/2}(2n-1)/2 \left( \frac{g_m}{2\pi\kappa BC} \right)^n \quad \ldots \quad (39)
\]

for an \( n \)-stage stagger-tuned amplifier with a Chebyshev response, and

\[
|G_m| = \epsilon^{1/2}(2n-1)/2 \left( \frac{g_m}{2\pi\kappa BC} \right)^n \quad \ldots \quad (40)
\]

for an \( n \)-stage amplifier of feedback pairs with a Chebyshev response.

It can be shown (Section 10.3) that

\[
\text{(Geometric mean } g^e_m) = g_m^n \prod_{r=1}^{n/2, or (n-1)/2} \frac{1}{g_m} \left[ 1 + \left( \frac{-4\pi CB^2}{g_m} \cos^2 \frac{(2r - 1)\pi}{2n} \right) \right]^{1/2} \quad (41)
\]

when eqn. (40) becomes

\[
|G_m| = \epsilon^{1/2}(2n-1)/2 \left( \frac{g_m}{2\pi\kappa BC} \right)^n \times \prod_{r=1}^{n/2, or (n-1)/2} \frac{1}{g_m} \left[ 1 + \left( \frac{-4\pi CB^2}{g_m} \cos^2 \frac{(2r - 1)\pi}{2n} \right) \right]^{1/2} \quad (42)
\]

Expressions for the mid-band gain of a Butterworth approximation follow from eqns. (39) and (42) by setting the gain tolerance, \( s \), to zero. This gives

\[
|G_m| = \left( \frac{g_m}{2\pi BC} \right)^n \quad \ldots \quad (43)
\]
for an \( n \)-stage stagger-tuned amplifier with a Butterworth response, and

\[
[G_m] = \left( \frac{\mathcal{E}_m}{2\pi BC} \right)^n
\]

\[
\times \prod_{r=1}^{n/2} \frac{n!}{2} \begin{bmatrix} \left( 1 - \frac{\alpha_n C B}{\mathcal{E}_m} \right) \cos^2 \left( \frac{2\pi \left( 2r - 1 \right)}{2n} \right) \end{bmatrix}^{1/2}
\]

(44)

for an \( n \)-stage amplifier of feedback pairs with a Butterworth response.

Eqns. (39), (42), (43) and (44) contain \( n \) as the only unknown parameter on the right-hand side. The smallest integral value of \( n \) which makes \([G_m] \geq \) (specified mid-band gain) defines the appropriate number of stages.

The value of \( n \) required by eqn. (43) for a given mid-band gain modulus is easily deduced, and suggests values of \( n \) to explore when seeking a solution of any of eqns. (39), (42) and (44).

(7.1) The Design Specification

The above procedures will now be applied to a particular design problem to illustrate the simplicity of the method. It is required to design an amplifier with a passband gain of at least 66 dB and a passband defined by the 3 dB frequencies \( f_1 \) and \( f_m \) where \( f_1 = 7.5 \) Mc/s and \( f_m = 20 \) Mc/s. It is further required that the passband gain shall not vary by more than \( \frac{1}{2} \) dB over as great a part of the passband as possible.

Initial experiments suggest that, for the available components,

\[
\mathcal{E}_m = 7.6 \text{ mV} \quad C = 17 \text{ pF} \\
\mathcal{E}_{min} = 10^{-5} \text{ mho}
\]

for the inverse feedback case, and \( d_{min} = 8 \times 10^{-3} \) for the stagger-tuned amplifier.

(7.2) Choosing \( n \)

The first step in the design is to decide upon a suitable number of stages. Eqn. (43) suggests trying

\[
n = 20 \log_{10} [G_m] \left/ 20 \log_{10} \left( \frac{\mathcal{E}_m}{2\pi BC} \right) \right. = 66\left/ 15.1 \approx 4
\]

With a gain tolerance \( s = \frac{1}{2} \) dB specified, a Chebyshev approximation can be used.

Since \( \epsilon = 0.115s \approx 0.029 \)

\[
B = 20 \times 10^6 \left( \frac{7.5 \times 10^6}{12.5 \times 10^6} \right) = 2\pi \times 10^6 \text{ rad/s}
\]

\[
\omega_m = 2\pi \times 10^6 \sqrt{20 \times 7.5} - 2\pi \times 12.25 \times 10^6 \text{ rad/s}
\]

\[
1/k = \cosh \left( \frac{1}{2n} \arccosh \left( \frac{8.668}{s} \right) \right) = \cosh \left( 2 \cdot 12/n \right)
\]

\[
C = 17 \times 10^{-12} \text{ F}
\]

\[
\mathcal{E}_m = 7.6 \times 10^{-3} \text{ mho}
\]

it follows from eqns. (39) and (42) that, for a Chebyshev approximation, using stagger-tuned amplifier stages [eqn. (39)],

\[
[G_m] \approx 70 \text{ dB for } n = 4
\]

\[
[G_m] \approx 47 \text{ dB for } n = 3
\]

and using a chain of feedback pairs [eqn. (42)]

\[
[G_m] \approx 70 \text{ dB for } n = 4
\]

\[
[G_m] \approx 47 \text{ dB for } n = 3
\]

Hence, whichever circuit configuration is used, four stages are required to achieve the desired mid-band gain.

(7.3) The Desired Response

With \( n = 4 \), the pole positions of the desired response shape may be determined by the method already described (see Fig. 4). The radii appropriate to this problem are \( r = \frac{B f_m = 12.5/12.25 - 1.02} {1.02} \), and \( d_{1} (1 - x^2) = 1.02 \sqrt{\left(1 - 0.53 \right)^2} - 0.5 \), while the radial lines make angles \( \pi/8 \) and \( 3\pi/8 \) with the \(+y\) axis.

Carrying out this construction and reading off the appropriate co-ordinates gives

\[
\tau_{p1}, \tau_{p1}^* = -0.188 \pm j0.950 \quad . . . (45)
\]

\[
\tau_{p2}, \tau_{p2}^* = -0.455 \pm j0.395 \quad . . . (46)
\]

Plotting the boundaries of the regions of realizable singularities in Fig. 4 immediately shows that the specification can be realized with both stagger-tuned and feedback amplifier circuits.

(7.4) The Stagger-Tuned Realization

The first pair are designed to realize \( \tau_{p1} \) and \( \tau_{p1}^* \), and the second pair subsequently designed to realize \( \tau_{p2} \) and \( \tau_{p2}^* \).

Substituting \( x_p = -0.188, y_p = \pm 0.950 \) in eqn. (23) yields the appropriate value of \( \mu_1 \) for the first pair. Thus

\[
\mu_1 = \left[ \frac{1}{2} \left( \frac{0.188^2 + 0.95^2 + 4}{\sqrt{\left(0.188^2 + 0.95^2 + 4\right)^2 - 16 \cdot 0.188^2}} \right) \right]^{1/2}
\]

\[
= 2.23 = \mu_1 = \omega_m / \omega_1
\]

Therefore \( (\omega_1/\omega_m)^2 - 2.23(\omega_1/\omega_m) + 1 = 0 \) . . . . (47)

and \( \omega_1 = 1.62 \omega_m \) or \( 0.62 \omega_m \) and hence \( \omega_2 = \omega_m / \omega_1 = 0.62 \omega_m \) or \( 1.62 \omega_m \).

We next determine \( d_1 \). From eqn. (20), \( d_1 = 2xp/\mu_1 = 0.376/2.23 = 0.168 \). Hence the load circuit of one stage of the first pair is tuned to \( 0.62 \omega_m \) (= 7.6 Mc/s), and has \( Q = 1/d_1 = 5.9 \), while the other stage is tuned to \( 1.62 \omega_m \) (=19.75 Mc/s) and has the same Q-factor.

The calculations of this Section can now be repeated for the next pair of singularities, eqn. (46), giving \( \omega_1 = 0.65 \omega_m \) \( \omega_2 = 1.575 \omega_m \) and \( d_1 = 0.41 \) as the appropriate values for the second pair of stages of the 4-stage amplifier. One stage of the second pair is thus tuned to \( 7.7 \) Mc/s and has a Q-factor of \( 2.45 \), while the second stage is tuned to \( 19.5 \) Mc/s, also with a Q-factor of \( 2.45 \).

Although it is convenient in the design process to consider the stages in pairs in this way, thereby realizing each pair of complex conjugate singularities by a separate circuit, the stages, once designed, may be cascaded in any order.

(7.5) Realization with Inverse-Feedback Pairs

Eqns. (11) are appropriate for the solution of the realization problem with this circuit configuration. Substituting \( x_p = -0.188, y_p = \pm 0.950 \) in eqn. (11) yields

\[
B_f = \frac{1}{2} \left[ \frac{7 \times 10^{-3} - \sqrt{\left(7 \times 10^{-3} - (4\pi \times 12.25 \times 10^6 + 17 \times 10^{-12} \times 0.950^2) \right)^2}}{10^6} \right]^{1/2}
\]

\[
= 0.22 \times 10^{-3} \text{ mho}
\]

and

\[
g = 2\pi \times 12.25 \times 10^6 \times 17 \times 10^{-12} \times 0.188 - 0.22 \times 10^{-3}
\]

\[
= 0.024 \times 10^{-3} \text{ mho}
\]

Hence both load circuits of the first feedback pair have a Q-factor \( \left(= \omega_m C g_f \right) \) of 53, and are tuned to 12.25 Mc/s. The feedback resistance \( (= 1/g_f) \) is 4.5 kilohms.

\*\* It is a general property of eqn. (47), that the product of the roots of this quadratic is unity, so that the two values of \( \omega_1 \) have \( \omega_m C \) or their geometric mean. Thus either root may be selected as the appropriate \( \omega_1 \), when the other root automatically becomes the corresponding \( \omega_2 \).
Repeating this procedure for the pair of singularities \( z_2, z_2^* \) of eqn. (46) gives \( g_f = 0.035 \times 10^{-3} \) mho and \( g = 0.556 \times 10^{-3} \) mho. The second pair of the 4-stage feedback chain require identical load circuits, also tuned to 12.25 Mc/s, having Q-factors of 2.3. The required feedback resistance for this pair \((1/g_f)\) is 28.6 kilohms. Connecting these two feedback pairs in cascade provides the desired overall response.

(8) CONCLUSIONS

The above examples demonstrate the simplicity of this design procedure. This simplicity results from designing stages in pairs, each pair realizing a pair of conjugate singularities. In this way the complete network is broken down into a number of basic elements from which the whole circuit is developed. These basic elements, though designed separately, have only to be connected in cascade to produce the required overall response.

No matter how large the value of \( n \) required to satisfy a given design specification, no extra complication is introduced, because the basic element remains unchanged throughout. It is this fact which makes this method of design particularly useful when \( n \) is large.

Determination of the regions of the \( z \)-plane within which conjugate singularities are realizable enables the designer to test the realizability of any Chebyshev or Butterworth response which he may require. The specification of regions of realizable singularities also serves to compare different circuit configurations. The circuit configuration which permits the greatest freedom of choice of positions of singularities is to be preferred. Fig. 2 clearly shows that any response shape which can be realized by the use of cascaded feedback pairs can also be realized by the use of stagger-tuned stages, while the converse is not true.

(9) REFERENCES

(5) CHEBYSHEV, S.: 'Sur les fonctions qui s’écartent peu de zero', Oeuvres, 1899, 1.

(10) APPENDICES

(10.1) The Chebyshev Polynomial

The Chebyshev polynomial \( T_n(x) \) of the first kind and of order \( n \) is defined by

\[
T_n(x) = \cos(n \arccos x) \quad \ldots \quad (48)
\]

It follows that

\[
T_n(-x) = \cos(n \pi \arccos x) \quad \ldots \quad (49)
\]

\[
T_n(x) = \cos(n \arccos x) \quad \ldots \quad (49)
\]

Hence \( T_n(x) \) is an odd function of \( x \) if \( n \) is odd, and is an even function of \( x \) if \( n \) is even.

Since the desired amplitude response function is an even function, only even-order Chebyshev polynomials are of interest here, and hence it is only necessary to study

\[
T_{2n}(x) = \cos(2n \arccos x) \quad \ldots \quad (49)
\]

where \( n \) is a positive integer. From eqn. (49),

\[
\frac{d}{dx} T_{2n}(x) = 2n \sin(2n \arccos x) / \sqrt{(1 - x^2)} = 0
\]

if

\[
x = \cos(r \pi/2n), r = 1, 2, \ldots (2n - 1) \quad (50)
\]

Hence, \( T_{2n}(x) \) has \( 2n - 1 \) stationary values in the range \(|x| < 1\).

Substituting eqn. (50) into eqn. (49) shows that

\[
T_{2n}(x) = \cos(2n \arccos x) / \sqrt{(1 - x^2)} = \cos r \pi
\]

\[
= \pm 1 \quad \text{at the stationary points.}
\]

Again, from eqn. (49) \( T_{2n}(x) = 0 \) when \( 2n \arccos x = (2r - 1) \pi/2 \), i.e. when

\[
x = \cos((2r - 1) \pi/4n) ; r = 1, 2, \ldots (2n - 1) \quad (51)
\]

Finally, from eqn. (49),

\[
T_{2n}(0) = \cos(2\pi \arccos(0)) = \cos(2\pi (2r - 1) \pi/2)
\]

\[
= -1 \quad \text{for } n \text{ odd, and } +1 \quad \text{for } n \text{ even} \quad (52)
\]

Using the results of eqns. (50)–(52), the shape of the functions \( T_{2n}(x) \) may be sketched by the method illustrated in Fig. 5 for \( n = 3 \) and \( n = 4 \) respectively.

Fig. 5.—Chebyshev responses \( T_{2n}(x) \).

(a) \( n \) odd \((-3)\),

(b) \( n \) even \((-4)\)
(10.2) The Expansion of Chebyshev Polynomials

From the trigonometric identity
\[ \cos 2n\Phi = 2 \cos (2n - 2)\Phi \cos 2\Phi - \cos (2n - 4)\Phi \]
by setting \( \Phi = \arccos x \) it follows that
\[ T_{2n}(x) = 2T_2(x)T_{2n-2}(x) - T_{2n-4}(x) \] ........................................ (53)

From eqn. (49),
\[ T_2(x) = \cos (2 \arccos x) = \cos^2(\arccos x) - \sin^2(\arccos x) = x^2 - (1 - x^2) \]
and
\[ T_0(x) = \cos (0 \arccos x) = 1 \] ........................................ (54)

From eqns. (53) and (54),
\[ T_{2n}(x) = 2(2x^2 - 1)T_{2n-2}(x) - T_{2n-4}(x) \] ........................................ (56)

Putting \( n = 2 \) in eqn. (56)
\[ T_4(x) = 2(2x^2 - 1)T_2(x) - T_0(x) = 2(2x^2 - 1)^2 - 1 = 2x^4 - 8x^2 + 1 \] ........................................ (57)

Putting \( n = 3, 4, \ldots \) in succession in eqn. (56) gives the higher-order Chebyshev polynomial expansions of \( T_{2n}(x) \).

The following observations follow from inspection of eqn. (56) (or may be proved by mathematical induction if required):

(a) \( T_{2n}(x) \) is an even polynomial of order \( 2n \) in \( x \).
(b) The coefficient of \( x^{2n} \) in \( T_{2n}(x) \) is \( 2^{2n-1} \).

Thus,
\[ T_{2n}(x) = 2^{2n-1}x^{2n} + \alpha_{2n-2}x^{2n-2} + \ldots + \alpha_0x^0 \] ........................................ (58)
The coefficients \( \alpha_{2n-2} \) for \( r = 1, 2, \ldots n \) may be determined, if required, through the recurrence relation (56).
Eqn. (38) follows immediately from eqns. (29) and (58).

(10.3) The Value of [Geometric Mean \( g_m' \)]

From the definition,
\[ g_m' = \sqrt[n]{\prod_{r=1}^{n} (g_m - g_{f,2r})} \] if \( n \) is even
\[ g_m' = \sqrt[n/2]{\prod_{r=1}^{(n+1)/2} (g_m - g_{f,2r})} \] if \( n \) is odd

From eqn. (11),
\[ g_{f,2r} = g_m \left( 1 - \left( \frac{2\omega_mC}{g_m} \right)^2 \right)^{1/2} \]
\[ = g_m \left( 1 - \frac{2\omega_mC}{g_m} \right)^{1/2} \] ........................................ (59)

from eqn. (33).
But \( \kappa \cosh \frac{\theta}{2n} \approx 1 \) and hence
\[ g_{f,2r} = g_m \left( 1 - \left( \frac{4\pi BC}{g_m} \right)^2 \right)^{1/2} \] ........................................ (60)

since \( d = Bf_m \).

Eqn. (41) follows immediately from eqns. (59) and (60).