RC ACTIVE NETWORKS FOR REALIZING HIGH-ORDER

APPROXIMATIONS TO DELAY

by

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Two general methods for realizing high-order delay approximation functions by RC active networks are described. One is to cascade simple low-order sections, each section being terminated by a suitable amplifier for direct cascading; the other is to apply additional feed-forward paths to a cascaded network. The second method has particular advantages over the first for realizing a high-order transfer function having transmission zeros in the right half S-plane. The amplifiers used here are unity-gain and operational amplifiers.

Some basic transposition properties of an active network have been discovered and applied to the synthesis procedures.

A new RC active method of realizing an RL admittance and its application to synthesis, give the minimum sensitivity of the denominator coefficients of a transfer function to variations in the parameters of the active element.

A new two-operational-amplifier method is derived from Millman's theorem, and a realization of an RL admittance using two operational amplifiers is derived as an extension of methods using one finite-gain amplifier.
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LIST OF PRINCIPAL SYMBOLS

A, B, \bar{A}, \bar{B}, C, D, G, H, H': matrices
A(S), B(S), C(S), E(S), Q(S): polynomials each having all zeros simple
A_0, B_0, C_0: constants
a_i, b_i, c_i, e_i, q_i: simple zeros of A(S), B(S), etc.
a_{ij}, b_{ij}, c_{ij}, e_{ij}, g_{ij}, h_{ij}, p_{ij}, q_{ij}: elements of A, B, etc.
C: capacitance
D(S): denominator polynomial of a transfer function
e: column vector of voltages
e_v, e_u: elements of e
F(S): overall transfer function
F_i(S): transfer function of one stage
G: conductance
g, h, k, m: constants
H_r(S) = F_{r+1}(S) F_{r+2}(S) \cdots F_n(S)
I: current
I_i, I_o: input, output currents
i: column vector of currents
i_v, i_u: elements of i
K : gain of an amplifier

M(S), P(S) : polynomials

N(S) : numerator polynomial of a transfer function

P, Q : matrices (operators)

p_r : coefficient of P(S)

\[ p_{2r}(S) = \prod_{i=1}^{\frac{n}{2}} (S^2 + \alpha_i S + \beta_i) \]

R : resistance

S : Laplace transform variable

T : transpose of a matrix (index)

T_j = t_1 t_2 \ldots t_j

T'_j = t_1 t_2 \ldots t_j

T''_j = t_1 t_2 \ldots t_n t_{n-1} \ldots t_{n-j+1}

t_i = C_i R_i

u_m = \sum_{j=0}^{m} G_j v_i

V : voltage

V_i, V_o : input, output voltages

x_i, x_o : input, output quantities

Y, y : two-port admittances

Z : two-port impedance

\[ \alpha, \beta, \gamma, \delta, \eta, \xi : \text{constants} \]

\[ E = (-\xi_{r+1})(-\xi_{r+2}) \ldots (-\xi_n) \]

\[ \varepsilon = 2 \beta / \alpha \]
\[ \Theta : \text{angle} \]

\[ \Lambda : \text{proportional constant} \]

\[ \sigma : \text{real negative root} \]

\[ \tau : \text{time delay} \]

\[ \tau_g, \tau_p : \text{group, phase delays} \]

\[ \phi(\omega) : \text{phase} \]

\[ \omega : \text{angular frequency} \]

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**Open-circuit z-parameters:**

\[ V_1 = z_{11}I_1 + z_{12}I_2 \]

\[ V_2 = z_{21}I_1 + z_{22}I_2 \]

**Short-circuit y-parameters:**

\[ I_1 = y_{11}V_1 + y_{12}V_2 \]

\[ I_2 = y_{21}V_1 + y_{22}V_2 \]
INTRODUCTION

A time-delay unit is frequently required in control-system simulation for transportation-delay. This may also be applied to sound broadcasting to give an artificial delay. The order of such delays is roughly in the range of 1s - 1ms.

Various methods of generating a delay are possible. They include transportation methods, switching methods; and networks, passive or active. The difficulty of the transportation method is the complexity of speed-controlled drive and the modulation and demodulation system; in the switching method the distortion of charging and discharging a chain of capacitors is inevitable. Passive networks involve RLC elements where the inductor is out of favour when working at very low frequencies (e.g. 1c/s). RC active networks consisting of amplifiers and RC elements seem to be preferable from the view point of simplicity and cost.

In this thesis, we use an RC active network to simulate this delay. The first problem involved is approximating to a time-delay. Since the delay function $e^{-\gamma S}$ is a
transcendental function which can only be realized by a
distributed network, and we are using a lumped network, rational-
function approximations are needed. In Chapter 1 a brief
survey of such approximations, and also a new curve-fitting
method suggested by Ream, are described.

The second problem is the realization problem.
To realize a high-order delay approximation function,
two general methods are suggested in this thesis: first,
direct cascading of low-order sections without isolating
amplifiers (Chapters 3 and 5); second, a more elegant way
of applying additional feedforward paths to a directly
cascaded network (Chapter 6). The latter is particularly
favourable for realizing a high-order transfer function
having zeros in the right half S-plane.

Chapter 2 describes some transposition properties
of an active network, which are believed to be original.
These enable us to transform a voltage realization imme-
diately to a current analogue or vice versa. A configuration
using a current-controlled current source is more suitable
for transistor circuitry, because a transistor has a low
input impedance and a high output impedance in the common-
base configuration, and its amplification is then least
sensitive to parameter variations. Such transistor
amplifiers are used in Chapters 3 and 4 to simulate the unity-gain current-controlled current sources.

The RC active network synthesis methods described here are mainly divided into two groups: one using unity-gain amplifiers (Chapters 3 and 4), and the other using operational amplifiers (Chapters 5, 6 and 7).

Chapter 3 consists of methods derived from Bélabanian and Patel's circuit using a finite-gain amplifier. It also establishes the essential property of a unity-gain amplifier and the ease of simulation (Section 3.9). Chapter 4 deals with a new method of realization of a RL driving-point admittance and its application in network synthesis. This gives a possibility of obtaining the minimum sensitivity of the denominator coefficients of a transfer function to variations in the active element parameter.

Methods of using one operational amplifier in Chapter 5 are mainly derived from the finite-gain-amplifier methods of Chapter 3, though when the gain becomes infinite many finite-gain configurations lose their usefulness. Chapter 6 deals with new methods for realizing high-order delay approximations. Operational
amplifiers are used merely for ease of illustration. Two new methods using two operational amplifiers are described in Chapter 7: one is derived from Millman's theorem, and the other is a realization of an RL admittance.

Claim to originality

Chapter 2 is the major original work of the present author. Chapter 4 deals with a new method of realizing an RL admittance using an RC active bridge network, and some applications of such a network in synthesis, but the idea of using an RL admittance is due to Horowitz\textsuperscript{22,23}.

Ream introduced the multiple feedforward method (Section 6.4) to the author who then derived a better method (Section 6.3.2) from a more general system (Section 6.2). The application of Millman's theorem in Section 7.2 was new at the time, although Paul\textsuperscript{46} has since published it independently. Section 7.3 is an extension of the new realization of an RL admittance (Section 4.2) using two operational amplifiers.

Chapter 3 gives the modifications of Balabanian and Patel's circuit to allow unity-gain amplification. Some configurations are new, and in particular Section 3.9 gives a very convenient transfer function and a new simple
synthesis procedure for any fourth-order transfer function. Chapter 5 is just the preliminary work of Chapter 6, but the one-operational-amplifier methods are considered as special cases of finite-gain-amplifier methods.

Experimental details

Simple common-base transistor amplifiers (ACY30, S.T.C.) at audio frequencies were used for the unity-gain current amplifiers (Sections 3.4, 3.8, 3.9, 4.3 and 4.4). The current gain is about 0.98 or better. Input impedance is of the order of $28\Omega$ without external feedback, and the output impedance looking into the collector and base terminals is of the order of $50k\Omega$. The D.C. operating point is kept at $V_{CE} = 12v$ and $I_C = 1mA$. Due to the D.C. bias, blocking capacitances were introduced in the transistor circuits, which limits the frequency response curves of the low-pass filters and the all-pass network. If a network has to be operated down to D.C., a more elaborate direct-coupled unity-gain amplifier has to be used.

The measurements of amplitude and phase, as functions of frequency, for the transistor circuits were obtained by using the Resolved Component Indicator (Model VP 250, 20c/s - 20kc/s, Solartron)
Accuracy of indication: ± 2% of full scale on all ranges

Input impedance: Greater than 50MΩ in parallel with approximately 15pF for both signal and reference input

Noise and harmonic rejection: Better than 40dB down

The operational amplifiers used in Section 6.3.2 and 6.4 were taken from an analogue computer (CD 10, Solartron).

Model: AA 1054 Solartron

Gain: \(10^7\) at D.C.
\(10^4\) at 100c/s

Daily drift: Typically 20μv

Output: 100v at 10mA max.

Noise: Less than 100μv

Frequency response curves were obtained by using the Resolved Component Indicator (Model VP 253, 0.5c/s - 1kc/s, Solartron). The specifications are about the same order as those for the VP 250.
CHAPTER 1

SURVEY OF DELAY APPROXIMATIONS

1.1 General survey of the literature

A pure delay \( \tau \) is represented by the transfer function \( e^{-\tau s} \). This function is transcendental and it can only be realized approximately by a lumped network. Many rational-function approximations have been suggested in the literature.

Although this thesis is not primarily concerned with the approximation problem, a brief study was made of approximations in the frequency domain to \( \arg(e^{-j\omega \tau}) = -\omega \tau \), assuming that the modulus \(|e^{-j\omega \tau}| = 1\) is realized exactly. One such approximation, believed to be new, is described in Section 1.2.

We define a low-pass transfer function as a function whose magnitude decreases with increasing frequency, and an all-pass transfer function as one whose magnitude is constant (normally unity) at all frequencies.

The published methods of approximating a delay can be divided roughly into the following groups according...
to the conditions satisfied.

We first define two kinds of delay. Let a transfer function be $F(S)$ and its argument $\phi(\omega) \equiv \arg F(j\omega)$.

We define phase delay as

$$-\frac{\phi(\omega)}{\omega} = \tau_p \quad \text{say} \quad (1.1.1)$$

and group (envelope) delay as

$$-\frac{d}{d\omega} \phi(\omega) = \tau_g \quad \text{say} \quad (1.1.2)$$

Group delay has been used more frequently because, if $F(S)$ is rational, group delay is also a rational function of $\omega$, whereas phase delay is a transcendental function.

1.1.1 Maximally-flat phase approximation (Bessel Polynomial)

Thomson$^{60,61}$ considered the low-pass case $F(S) = \frac{1}{H_n(\omega S)}$ where $H_n(\omega S)$ is a polynomial of degree $n$. He defined delay by $(1.1.2)^*$, and expanded it in the form

$$\tau_g = \tau(1+a_1\omega^2 + a_2\omega^4 + \ldots) \quad (1.1.3)$$

*The delay here can be taken either as a group delay or a phase delay since if one is maximally flat, so is the other.
where \( \tau \) is the zero frequency delay and the \( a \)'s are constants. To obtain maximal flatness with \( n \) available design parameters, these must be chosen such that \( a_1 = a_2 = \ldots = a_n = 0 \).

This is equivalent to requiring the first \( 2n-1 \) derivatives of the delay with respect to frequency to vanish at the origin. This leads to a polynomial \( H_n(\tau S) \) which is related to Bessel functions\(^47\) and this class of low-pass transfer function is known as a Bessel filter.

Storch\(^54\) and Weinberg\(^69\) also described the properties of Bessel polynomials to approximate delay in a maximally flat manner at the origin.

1.1.2 Pade approximation

The Pade approximation\(^41,66\) minimises the error in \( F(S) \) in the Taylor sense at \( S = 0 \). This leads to a collection of rational functions of different order known as the Pade table. The exponential function may be approximated as

\[
e^{\tau S} = \frac{P_{nm}(\tau S)}{Q_{nm}(\tau S)} \tag{1.1.4}
\]

where

\[
P_{nm}(x) = 1 + \frac{m}{m+n} \frac{x}{1!} + \frac{m(m-1)}{(m+n)(m+n-1)} \frac{x^2}{2!} + \ldots
\]

\[
Q_{nm}(x) = \frac{m(m-1)(m-2)\ldots1}{(m+n)(m+n-1)\ldots(n+1)} \frac{x^m}{m!}
\]
and $Q_{nm}(x) = P_{mn}(-x)$

When $n = 0$, the Pade approximants reduce to the Taylor series. But for $n > 5$, they have zeros in the right-half $S$-plane, and so $\frac{1}{P_{0m}(\tau S)}$ is unstable.

When $n \neq m$ the phase/frequency and gain/frequency characteristics are not themselves maximally flat functions of $\omega$. When $m = n$, however, the gain is unity and the phase/frequency characteristic is, in a sense, maximally flat at $\omega = 0$. The phase/frequency characteristic then has a bandwidth double that of the Bessel filter with the same value of $n$,

e.g., $H_n(\tau S) = P_{nn}(\frac{1}{2}\tau S)$

The Bessel filter has a fairly good step response and poor gain/frequency characteristic; the all-pass Pade approximant has the reverse. We now introduce a compromise approximation whose phase/frequency characteristic is still maximally flat.

1.1.3 Budak's approximation

Budak introduced a parameter $k$ to split $e^{-\tau S}$ into two parts:
and then approximated $e^{k\tau S}$ by $H_n(k\tau S)$ and $e^{(k-1)\tau S}$ by $H_n((k-1)\tau S)$. If $n>m$, (1.1.5) has finite zeros and this gets rid of the initial step (or spike) in the step response to an all-pass $F(S)$. Also the effect of $k$ is to increase the bandwidth of the gain/frequency characteristic over that of the Bessel filter without affecting the maximally-flat nature of the phase.

1.1.4 Scanlan's approximation

Scanlan\textsuperscript{49} introduced an empirical approximation to the Bessel filter. The $n$ poles are located on a circle in the $S$-plane as shown in Fig.1.1. For $n=5$ they are fairly close to those of the Bessel filter and give nearly the same amplitude and phase characteristics. For $n=1$ and $n=2$, they actually coincide.

1.1.5 Equal-ripple approximation

A Chebyshev approximation (to either phase or group delay) minimises the maximum value of error for a given bandwidth, or else provides maximum bandwidth for a given maximum error. The approximation is oscillatory
radius = \((R_1 R_2 \cdots R_n)^{1/n}\)
where \(R_i\) are radii to
Pade poles

Fig.1.1 Pole distribution of Scanlan's empirical
approximation to maximally-flat delay

in the approximation band. This gives a better
approximation over a finite range within given error-
limits whereas the maximally-flat approximation is only
the best approximation to a function at a given point.
The Chebyshev reduces to the maximally-flat when the
error or bandwidth \(\to 0\).

The Chebyshev approximation to the group delay of
a rational transfer function yields a set of nonlinear
simultaneous equations for the coefficients. These
equations have been solved iteratively for the low-pass
case on a digital computer\(^1,19,31,67\). The phase/frequency
characteristic tends to drift away from the straight line $\phi = -\gamma \omega$ (Fig. 1.8).

No equal-ripple phase delay approximation has been produced, but it is not thought that the results will differ a great deal from the equal-ripple group-delay approximation.

Humphreys\(^2\) produced a kind of equal-ripple phase approximation to the straight line $\phi = -\gamma \omega$. The maximum error of phase is varied between two given limits. He has published the results and indicated that the results were obtained by an iterative process.

1.1.6 Approximation derived from a particular pole pattern

This arises from the potential analogue method. Bode\(^9\) first used the "condenser-plate" analogue in the design of phase equalizers. Darlington\(^14\) lumped the distributed charge into point charges in order to realize a given $\phi(\omega)$ by a lumped network. To approximate a delay he arranged the point charges in an infinite array parallel to the imaginary axis in the S-plane. The resulting phase/frequency characteristic is equal-ripple about the straight line,\(^{16,53} \phi = -\gamma \omega$.

A truncation error arises when the array is finite.
1.1.7 Other approximations in the frequency-domain

For a given Nth-order rational function approximation, we may define the bandwidth of the phase/frequency characteristic as

\[ B = \frac{N\alpha}{2\pi} \]  \hspace{1cm} (1.1.6)

where \( N \) is the sum of the degrees of the numerator and the denominator polynomials of \( S \). However good the phase approximation, any signal frequency outside this band will be phase-distorted, and so the constant gain/frequency characteristic is not required for \( \omega > B \). A low-pass filter may therefore be used and so the initial step in the step response of an all-pass transfer function may be avoided.

Bennett\(^8\) described a low-pass filter with linear phase and constant gain approximations in the pass-band. He considered

\[ F(S) = \frac{M(S)}{N^2(S)} \]  \hspace{1cm} (1.1.7)

where \( N^2(S) \) is to approximate a desirable phase characteristic and \( M(S) \) has quadrantal zeros to give constant phase/frequency characteristic for small \( \omega \). \( M(S) \) is chosen so that \( |F(j\omega)| \) is a Chebyshev approximation to unity in the pass-band.
1.1.8 **Mixed frequency- and time-domain approximations**

Kuh\(^{28a}\) set the requirements in the time-domain and divided the filter into two cascaded sections, Fig.1.2. One is a low-pass filter to provide a satisfactory step response, and the other is an all-pass network to give the required time delay. In this way, he improved the rise time of the step response.

![Cascaded low-pass and all-pass sections](image)

**Fig.1.2** Cascade low-pass and all-pass sections

Tomlinson\(^{64}\) adopted Kuh's cascade idea, but optimized the approximation in the frequency-domain. He approximated a constant-gain characteristic by a sharp cut-off low-pass filter and corrected the phase by an all-pass network in least-mean-square-error manner.

1.1.9 **Phase-fitting method**

This is the simplest approach to determine a linear-phase function, although it needs trial and error. Many curve-fitting methods have been described in the
literature$^{13,33}$ and they are all for low-order approximations. Johnson$^{25}$ has given a full account in his book.

Another method, suggested by Ream (private communication) is now described.

1.2 Equal-spacing phase approximation and empirical result

1.2.1 Equal-spacing phase approximation for all-pass $F(S)$

This is a curve-fitting method. The approximation is obtained by making $\phi(\omega)$ pass through equi-distant points on the straight line $\phi = -\omega$, Fig.1.3. It requires only the solution of linear simultaneous equations.

![Diagram](image)

**Fig.1.3** Phase characteristic of equal-spacing phase approximation function
Use the normalized value \( \zeta = 1 \), then the linear phase characteristic is given by

\[
\phi = -\omega \quad \tag{1.2.1}
\]

Let the approximant be

\[
F(S) = \frac{P(-S)}{P(S)} \quad \tag{1.2.2}
\]

where \( P(S) = \sum_{i=0}^{n} p_i S^i \) with \( p_n = 1 \). Assume \( n \) is even.

Putting \( S = j\omega \) gives

\[
\phi(\omega) = -2\tan^{-1} \frac{\frac{p_1\omega + p_3\omega^3 + \cdots + (-1)^{\frac{1}{2}n} p_{n-1}\omega^{n-1}}{p_0 - p_2\omega^2 + p_4\omega^4 - \cdots (-1)^{\frac{1}{2}n} \omega^{2n}}} \quad \tag{1.2.3}
\]

To find the \( n \) unknowns, \( p_0, \ldots, p_{n-1} \), we have the \( n \) simultaneous equations.

\[
\phi(r\theta) = -r\theta \quad \text{for} \quad r = 1, 2, \ldots, n
\]

where \( \theta \) is constant and \( 0 < \theta < \pi \).

Substituting into (1.2.3), we obtain

\[
\{ p_0 - p_2(r\theta)^2 + p_4(r\theta)^4 - \cdots (-1)^{\frac{1}{2}n} (r\theta)^n \} \sin \frac{1}{2}r\theta
\]

\[
= \{ p_1(r\theta) - p_3(r\theta)^3 + p_5(r\theta)^5 - \cdots
\]

\[
+ (-1)^{\frac{1}{2}n} p_{n-1} (r\theta)^{n-1} \} \cos \frac{1}{2}r\theta
\]

or

\[
\sum_{i=1}^{n-1} p_{i-1}(r\theta)^{i-1} \sin \frac{1}{2}(r\theta - (i-1)\pi) = -(r\theta)^n \sin \frac{1}{2}(r\theta - n\pi)
\]
This can be written in matrix form as

\[ \begin{bmatrix} A & B \\ C \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \]  

(1.2.6)

where \( a_{ri} = (r\theta)^{i-1} \sin \left( \frac{1}{2} r\theta - (i-1)\theta \right) \)

\( b_i = p_{i-1} \)

\( c_r = -(r\theta)^n \sin \left( \frac{1}{2} r\theta - n\theta \right) \)

Therefore

\[ B = A^{-1} C \]  

(1.2.7)

The set of simultaneous equations may be solved on a digital computer by means of a standard matrix-inversion program.

By this means the error in \( \theta \) can be kept reasonably small, up to a frequency close to \( \omega = n\pi/2 \) (Fig.1.4). The only precaution here is that \( \theta \) must not be chosen too small in order to avoid ill-conditioned equations.

1.2.2 Simplified pole-zero pattern

Here we give a set of all-pass transfer functions derived from pole-zero locations, which give phase characteristics near to those of the equal-spacing phase approximations.
Fig. 1.4  Phase errors of tenth-order all-pass equal-spacing phase functions with different $\phi$
Fig. 1.5 Zeros of eighth-order all-pass equal-spacing phase function with different $\theta$ in the first-quadrant of the $S$-plane

Each straight line is divided into eight equal portions.
The zeros in the first-quadrant of the $S$-plane are plotted for various $\theta$ in Fig.1.5 for $n = 8$. They are nearly on the straight lines joining the Pade zeros, which correspond to $\theta = 0^\circ$ and zeros on the $j\omega$-axis $2\pi$ apart, which correspond to $\theta = 180^\circ$. Also, the zeros for a given $\theta$ divide the straight lines in nearly equal ratios. Therefore, Ream (private communication) suggested that for practical purposes, we may take the zeros $z_i$ as in Fig.1.6:

$$z_i = z_{li} + \lambda(\theta) (z_{0i} - z_{li}) \quad (1.2.8)$$

where $z_{0i}$ is the Pade zero, $\lambda$ is independent of $i$, and

$z_{li} = j\pi(2i+1), \quad i = 1, 2, ..., \frac{1}{2}n, \quad n$ even

$z_{li} = j2i\pi, \quad i = 0, 1, 2, ..., \frac{1}{2}(n-1), \quad n$ odd

![Fig.1.6 Zeros in the first-quadrant of the $S$-plane](Section 1.2.2)
The all-pass transfer function which has these zeros gives a phase characteristic which approaches the equal-spacing approximation as $\theta \to \pi$ (Fig. 1.7)

$\lambda = 0.125$ corresponds to $\theta = 170^\circ$ approximately.

The transfer functions with $\lambda = 0.125$ are used in Sections 3.4 and 6.3.2 to represent functions with poles and zeros very near to the $j\omega$-axis.

Fig. 1.8 and 1.9 show the phase errors and zeros in the first-quadrant for tenth-order all-pass Chebyshev group-delay functions with different percentage of ripple, for comparison with the previous results.

Fig. 1.10 shows the step responses of the empirical result with different $\lambda$ for $n = 10$. As $\lambda \to 0$, i.e. as poles and zeros move closer to the $j\omega$-axis, the oscillations in the interval $0 < t < \tau$ become smaller but shift up to a different level. The other oscillations shift their level and become larger. The same phenomenon is shown in Fig. 1.11 for the step responses of tenth-order all-pass Chebyshev group-delay functions with different percentage of ripple.

Fig. 1.12 shows the step response for $\lambda = 0.125$, and $n = 20$ to be a very good approximation to a step.
Fig. 1.7  Phase errors of tenth-order phase approximations with different $\lambda$ (Section 1.2.2)
Fig. 1.8 Phase errors of tenth-order all-pass Chebyshev group-delay functions with different percentage of ripple.
Fig. 1.9 Zeros of tenth-order all-pass Chebyshev group-delay functions with different percentage of ripple
Fig. 1.10 Step responses of tenth-order all-pass phase approximations with different $\lambda$ (Section 1.2.2)
Fig. 1.11  Step responses of tenth-order all-pass Chebyshev group-delay functions with different percentage of ripple.
Fig. 1.12  Step response of twentieth-order all-pass phase approximation with $\lambda = 0.125$ (Section 1.2.2)
1.3 Conclusions

To design a delay network, we first have to know what is the application, whether it is used to delay a step, or a signal within a band of frequency. Then we may decide what kind of rational function will be required.

To delay a step, a low-pass filter will give a good step response and the rise-time may be improved by using Kuh's method$^{28a}$.

To delay a signal within a given bandwidth, the phase/frequency characteristic should be approximately linear and the gain/frequency characteristic approximately constant within this bandwidth. Outside this bandwidth, we are not interested. Therefore, an all-pass network with linear phase/frequency characteristic can meet the requirement. If a part of the bandwidth is to be eliminated a sharp cut-off filter may be followed by an all-pass network to correct the phase to a linear-phase characteristic.
2.1 Introduction; general transposition rules

In the next five chapters we present various synthesis methods, using unity-gain amplifiers, or operational amplifiers as active elements.

It is often useful to have two realizations by the same method, one based on voltages and the other on currents; the former is generally preferred for valve circuitry and the latter for transistors. In examining some simple "duals", Ream and the author noticed a certain reciprocal relationship and thought it might be a special case of a general property. (Stubbs and Single\textsuperscript{56} had noticed it also, but had not investigated further). As a result the author has proved the general results described below, which are believed to be original.
The result concerns ideal voltage- or current-controlled voltage or current sources, whose properties are as shown in Figs. 2.1 to 2.3. A number of such "amplifiers" are assumed to be embedded in a reciprocal network with nodes 1, 2, ..., n in such a way that reasonable operating conditions are satisfied, e.g. no node is fed by more than one voltage source.

Additional currents $i_1, i_2, ..., i_n$ may be supplied from outside the network, although in Figs. 2.1a and 2.3, a current injected at node v will have no effect on the nodal voltages.

We first define a 'modified nodal admittance matrix' for the network as follows. Let $\mathbf{A}$ be the nodal admittance matrix for the reciprocal network without amplifiers:

$$\mathbf{A} \mathbf{e} = \mathbf{i} \quad \text{(2.1.1)}$$

where $\mathbf{A} = (a_{ij})$, $i,j = 1, 2, ..., n$; $\mathbf{e}$ and $\mathbf{i}$ are column vectors of voltages and currents.

Case I. Ideal voltage-controlled voltage source and current-controlled current source, Fig. 2.1

(1) Insert an ideal voltage-controlled voltage source in a reciprocal network as shown in Fig. 2.1a and assume $i_v = 0$. 
Fig. 2.1a An ideal voltage-controlled voltage source embedded in a reciprocal network

Fig. 2.1b An ideal current-controlled current source embedded in a reciprocal network
The amplifier constrains the network such that
\[ e_v = K e_u \quad (2.1.2) \]
So we have
\[ B e = i \quad (2.1.3) \]
where \( B = (b_{ij}) \) is \( A \) with row \( v \) and column \( v \) replaced by zeros, and column \( u \) replaced by column \( u+K \) column \( v \), i.e.
\[ b_{ij} = a_{ij}, \quad i \neq v, \quad j \neq u \]
\[ b_{iu} = a_{iu} + K a_{iv}, \quad i \neq v \]
\[ b_{vj} = 0, \text{ all } j; \quad b_{iv} = 0, \text{ all } i \]
\( B \) may be expressed as
\[ B = Q A P \quad (2.1.4) \]
where \( q_{ij} = 0 \), when \( i \neq j \) or \( i = j = v \), and \( q_{ii} = 1 \) elsewhere \( p_{vu} = K \), otherwise \( p_{ij} = 0 \) when \( i \neq j \) or \( i = j = v \), and \( p_{ii} = 1 \) elsewhere.
\[ i_v = 0 \] is a necessary condition for equations (2.1.4) to have a solution. This is an artificial condition here, consequent on including \( n \) instead of \( n-1 \) equations in (2.1.3), but in the next case the condition is meaningful.
(2) Repeat for an ideal current-controlled current source as shown in Fig. 2.1b and assume $i_v = 0$. The amplifier constrains the network such that

$$e_v = 0, \quad i_s = -K_i$$  \hspace{1cm} (2.1.5)

and $i_u$ is replaced by $i_u + i_s$ where $i_s$ is the additional current injected by the amplifier into node $u$, and

$$I = \sum_{j} a_{vj} e_j,$$

since $i_v = 0$. Then we have

$$B' e = i$$  \hspace{1cm} (2.1.6)

where $B' = (b'_{ij})$ is $A$ with row $v$ and column $v$ replaced by zeros, and row $u$ replaced by row $u + K$ row $v$, i.e.

$$b_{ij} = a_{ij}, \quad i \neq v, j \neq u$$

$$b_{uj} = a_{uj} + Ka_{vj}, \quad j \neq v$$

$$b_{iv} = 0, \text{ all } i; \quad b_{vj} = 0, \text{ all } j.$$

$B'$ may be expressed as

$$B' = P^T A Q$$  \hspace{1cm} (2.1.7)

where $P, Q$ are the matrices already defined.

Since $A^T = A$ (reciprocal network) and $Q^T = Q$ by definition, we have

$$B^T = B'$$  \hspace{1cm} (2.1.8)

Note that this transposition property will only hold if $i_v = 0$ in Fig. 2.1b.
Alternative way of obtaining transposition property

(1) If we write (2.1.2) as
\[ e_u = \frac{e_v}{K} \quad (2.1.9) \]
and substitute into (2.1.1), this results

\[ \mathbf{H} e = i \quad (2.1.10) \]
where \( \mathbf{H} = (h_{ij}) \) is \( \mathbf{A} \) with row \( v \) and column \( u \) replaced by zeros, and column \( v \) replaced by \( \frac{1}{K} \) column \( u \) + column \( v \), i.e.

\[
\begin{align*}
    h_{ij} &= a_{ij}, \quad i \neq v, \; j \neq u \\
    h_{iv} &= \frac{1}{K} a_{iu} + a_{iv}, \quad i \neq v \\
    h_{vj} &= 0, \text{ all } j; \quad h_{iu} = 0, \text{ all } i
\end{align*}
\]
Similarly, \( \mathbf{H} \) may be expressed as

\[ \mathbf{H} = \mathbf{Q} \mathbf{A} \mathbf{P}' \quad (2.1.11) \]
where \( \mathbf{Q} \) is defined in (2.1.4)

\[
\begin{align*}
    p'_{uv} &= \frac{1}{K}, \text{ otherwise } p'_{ij} = 0 \text{ when } i \neq j \text{ or } i = j = v, \\
    p'_{ii} &= 1 \text{ elsewhere. Also } i_v = 0 \text{ is a necessary condition for equations (2.1.10) to have a solution.}
\end{align*}
\]

(2) To obtain the transpose of \( \mathbf{H} \), we replace the voltage-controlled voltage source by a current-controlled current source as shown in Fig. 2.1b and assume \( i_u = 0 \).
As before, \(i_u\) and \(i_v\) are replaced by \(i_u + i_s\) and \(i_v + I\) respectively, where \(i_s\) and \(I\) are the additional currents injected by the amplifier into nodes \(u\) and \(v\), and are related by (2.1.5). Using the assumption \(i_u = 0\) we obtain
\[
I = -\frac{1}{K} i_s = -\frac{1}{K} \sum_j a_{uj} e_j \quad (2.1.12)
\]
Then we have
\[
\mathbf{H'} \mathbf{e} = \mathbf{i} \quad (2.1.13)
\]
where \(\mathbf{H'} = (h'_{ij})\) is \(\mathbf{A}\) with row \(u\) and column \(v\) replaced by zeros and row \(v\) replaced by \(\frac{1}{K}\) row \(u + v\), i.e.
\[
\begin{align*}
h'_{ij} &= a_{ij}, & i \neq v, & j \neq u \\
h'_{vj} &= \frac{1}{K} a_{uj} + a_{vj}, & j \neq v \\
h'_{uj} &= 0, & \text{all } j; & h_{iv} = 0, & \text{all } i
\end{align*}
\]
\(\mathbf{H'}\) may be expressed as
\[
\mathbf{H'} = \mathbf{P}'^T \mathbf{A} \mathbf{Q} \quad (2.1.14)
\]
where \(\mathbf{P}', \mathbf{Q}\) are the matrices defined by (2.1.11).
Therefore it follows that \(\mathbf{H'} = \mathbf{H'}\) (2.1.5) since \(\mathbf{A}'^T = \mathbf{A}\) and \(\mathbf{Q}^T = \mathbf{Q}\) by definition.

Hence this transposition property can hold with either \(i_v = 0\) or \(i_u = 0\).
Case II. Ideal voltage-controlled current source, Fig.2.2

(1) For an ideal voltage-controlled current source inserted in a reciprocal network as shown in Fig.2.2, we have

\[ i_s = g_c u \]  \hspace{1cm} (2.1.15)

and \( i_v \) is replaced by \( i_v + i_s \) where \( i_s \) is the additional current injected by the amplifier into node \( v \).
Hence the modified nodal admittance matrix $\mathbf{B}$ for this network, defined by

$$\mathbf{B} = \mathbf{A} + \mathbf{G}$$

is $\mathbf{A}$ with $b_{vu}$ replaced by $a_{vu} - g$, i.e.

$$\mathbf{B} = \mathbf{A} + G$$

\begin{equation}
\text{(2.1.16)}
\end{equation}

where $g_{vu} = -g$ and $g_{ij} = 0$ elsewhere.

(2) If nodes $u$ and $v$ are interchanged, we have

$$\mathbf{B}' = \mathbf{A} + G^T$$

\begin{equation}
\text{(2.1.17)}
\end{equation}

where $\mathbf{B}'$ is $\mathbf{A}$ with $b_{uv}$ replaced by $a_{uv} - g$, i.e.

$$\mathbf{B}' = \mathbf{A} + G^T$$

Therefore $\mathbf{B}' = \mathbf{B}^T$ since $\mathbf{A}^T = \mathbf{A}$.
Case III. Ideal current-controlled voltage source, Fig. 2.3

Fig. 2.3 An ideal current-controlled voltage source embedded in a reciprocal network

(1) For an ideal current-controlled voltage source inserted in a reciprocal network as shown in Fig. 2.3, we have

\[ e_v = -rI, \quad e_u = 0 \quad (2.1.18) \]

Assume \( i_v = 0 \), and eliminate \( I \) by substituting (2.1.18) into

\[ \sum_{j} a_{uj} e_j = i_u + I, \]
then we have
\[ B e = i \]
where \( B \) is \( A \) with column \( u \) and row \( v \) replaced by zeros, except that \( b_{uv} \) is replaced by \( a_{uv} \frac{1}{r} \), i.e.
\[ B = Q A P + G \]  
(2.1.19)
where \( q_{ij} = 0 \) when \( i \neq j \) or \( i = j = v \), and \( q_{ii} = 1 \) elsewhere
\[ p_{ij} = 0 \] when \( i \neq j \) or \( i = j = u \), and \( p_{ii} = 1 \) elsewhere
\[ g_{uv} = \frac{1}{r}, \text{ and } g_{ij} = 0 \text{ elsewhere}. \]

(2) If nodes \( u \) and \( v \) are interchanged and we assume \( i_u = 0 \), we have, from the above
\[ B' e = i \]
where \( B' \) is \( A \) with column \( v \) and row \( u \) replaced by zeros, except that \( b_{vu} \) is replaced by \( a_{vu} \frac{1}{r} \), i.e.
\[ B' = P A Q + G^T \]  
(2.1.20)
Since \( A = A^T \), \( P = P^T \) and \( Q = Q^T \) by definition, it follows that \( B' = B^T \).
Extension to more than one amplifier

Suppose a reciprocal network with a simple amplifier embedded in it obeys the above transposition rules so that the modified nodal admittance matrices $\mathbf{B}$ and $\mathbf{B}'$ are related by

$$\mathbf{B}^T = \mathbf{B}'$$  \hspace{1cm} (2.1.21)

Now insert a second amplifier and define a further modified matrix $\mathbf{D}$ by the equation

$$\mathbf{D} = \mathbf{Q}_2 \mathbf{B} \mathbf{P}_2$$  \hspace{1cm} (2.1.22)

where $\mathbf{P}_2$ and $\mathbf{Q}_2$ are defined similarly to $\mathbf{P}$ and $\mathbf{Q}$ in (2.1.4). Then

$$\mathbf{D}^T = \mathbf{P}_2^T \mathbf{B}^T \mathbf{Q}_2$$  \hspace{1cm} since $\mathbf{Q}_2^T = \mathbf{Q}_2$

$$= \mathbf{P}_2^T \mathbf{B}' \mathbf{Q}_2$$

$$= \mathbf{D}'$$  \hspace{1cm} say  \hspace{1cm} (2.1.23)

But $\mathbf{D}'$ is the matrix of the network modified in the same way as $\mathbf{B}$ was modified to give $\mathbf{B}'$. Hence the transposition property holds for the network with two amplifiers embedded, and this result can clearly be extended to cover any permissible number of embedded amplifiers.
Summary of results

A number of ideal amplifiers, any one of which may be one of any of the above four types, are embedded in a reciprocal network, the modified nodal admittance matrix being $C$. If the following transformation is made:

1. Amplifier in Fig. 2.1a $\leftrightarrow$ 2.1b with nodes $u$, $v$ unchanged, and either $i_v = 0$ or $i_u = 0$ in Fig. 2.1b.

2. Amplifier in Fig. 2.2 or 2.3 is unchanged but nodes $u$ and $v$ are interchanged, then the 'modified nodal admittance matrix' of the transformed network is $C^T$.

2.2 Applications of the transposition rules

Consider any network of the type assumed above. Delete the zero rows and columns of $B$ to give

$$ C \, e^{(1)}_i = i \quad \text{say} \quad (2.2.1) $$

where $e^{(1)}_i$ and $i$ have corresponding elements deleted. The solution for the $e_i$ is, say,

$$ e_i^{(1)} = \sum_j C_{ij} i_j \quad (2.2.2) $$
The transformed network will have the admittance matrix \( \mathbf{C}_T \) and hence the voltages will be given by

\[
e_i^{(2)} = \sum_j C_{ji} i_j \quad (2.2.3)
\]

(1) If all \( i_j = 0 \), except \( i_a \), then

\[
\begin{align*}
e_a^{(1)} &= C_{aa} i_a \\
e_a^{(2)} &= C_{aa} i_a
\end{align*}
\]  

(2.2.4)

i.e., the driving-point impedances are the same.

(2) If all \( i_j = 0 \) except \( i_a = i \) in the first network and \( i_b = i \) in the transformed network, then

\[
e_b^{(1)} = e_a^{(2)} = C_{ba} i
\]  

(2.2.5)

(3) If in the first network all \( i_j = 0 \), except \( i_a \), then

\[
\frac{e_b^{(1)}}{e_a^{(1)}} = \frac{C_{ba}}{C_{aa}}
\]  

(2.2.6)

If in the second network all \( i_j = 0 \) except \( i_a \), \( i_b \), and \( i_a \) is dependent on the condition \( e_a^{(2)} = 0 \), then

\[
-\frac{i_a^{(2)}}{i_b^{(2)}} = \frac{C_{ba}}{C_{aa}} = \frac{e_b^{(1)}}{e_a^{(1)}}
\]  

(2.2.7)
Example 2.1

Using the property (1), we can replace the voltage-controlled voltage source of the circuit, Fig.4.4 by a current-controlled current source as shown in Fig.4.6. The resulting circuit still has the same driving-point impedance between the input terminals.

Example 2.2

Apply the property (3) to a section which can be cascaded directly without interaction\textsuperscript{46,71}. Balabanian and Patel's circuit, Fig.3.1, has a voltage-controlled voltage source at the output, and its transposed network has the input terminated by the input of a current-controlled current source Fig.3.3. The voltage and current transfer
ratios for the respective circuit are the same [See (3.2.2) and (3.3.1)].

2.3 Particular case for operational amplifier

If there is a feedback impedance connected between the input and the output of an operational amplifier which has infinite input impedance, zero output impedance and infinite negative voltage gain, Fig. 2.5, the whole unit including Z may be considered as an ideal current-controlled voltage source with

\[ V = ZI \quad (2.3.1) \]

\[ \text{Fig. 2.5 Operational amplifier with a feedback impedance} \]

(2.1.3) can be applied when this unit is embedded in a reciprocal network with \( g_{uv} = \frac{1}{Z} \).

As \( Z \) tends to infinity, the unit will simply contain an operational amplifier and (2.1.13) becomes

\[ B = QA + P \quad (2.3.2) \]
(2.3.2) agrees with the equation obtained by Nathan. If the input and output terminals of the operational amplifier are reversed, we have

$$B' = P A Q = B^T$$  \hspace{1cm} (2.3.3)

Hence the transposition rule for an operational amplifier is to interchange the input and output nodes.
CHAPTER 3

ACTIVE RC NETWORK SYNTHESIS BY RC THREE-TERMINAL

NETWORKS AND UNITY-GAIN AMPLIFIERS

3.1 Introduction

In this chapter we introduce several networks to simulate second-order transfer functions. To simulate a high-order delay approximation, one may simply cascade low-order sections. By doing this, the synthesis procedure becomes simpler and it is generally accepted that the transfer function is less sensitive to the parameters of the active elements.

Many methods such as using a negative impedance-converter, gyrator, etc. require isolating amplifiers when similar sections are to be cascaded. To be cascaded directly, sections must be terminated by a suitable amplifier. Several new methods using a unity-gain amplifier in Sections 3.3, 3.4, 3.7 and 3.8 can be cascaded directly.

The advantage of a unity-gain amplifier over other amplifiers is its simple construction and low cost.
A transistor circuit in the common-base configuration has a current gain \( \alpha \) less sensitive to the transistor parameter variations than in any other configuration; \( \alpha = 0.98 \) is representative, but values nearer unity can be obtained with a better transistor or a more elaborate circuit. Hence it is very practical to use a common-base transistor amplifier for simulating a unity-gain current-controlled current source. In valve circuitry, cathode followers have been widely used to simulate ideal unity-gain voltage-controlled voltage sources.

The new methods are mainly derived from Balabanian and Patel's method. The configuration which they used was not original, but they seem to be the first to use three-terminal circuit parameters in the synthesis of active networks using a finite-gain amplifier. Therefore, it is convenient to start with a brief summary of their method (Section 3.2), and then to describe the modifications which allow unity-gain amplification (Sections 3.3, 3.4). The methods are extended to two three-terminal networks (Sections 3.7, 3.8, 3.9). The last section deals with a modified Bach's circuit, the passive components of which are determined successively and directly from the coefficients of a specified transfer function.
Some configurations may result in simple circuits for some particular transfer functions. They cannot be compared by just realizing a particular function. From the point of view of economy in number of passive elements, it is better to use the configurations with one three-terminal network. They also have the advantage of cancelling the unwanted common denominator of the y-parameters, but usually they are more restricted.

3.2 Summary of Balabanian and Patel's method and application to an all-pass network

In this section, we describe briefly Balabanian and Patel's method, and show that an all-pass realization due to McVey is a particular case.

The method realizes a function of the form

\[ F(S) = h \frac{S^2 + \alpha_1 S + \beta_1}{S^2 + \alpha_2 S + \beta_2} \]  

(3.2.1)

with complex poles and zeros, subject to certain restrictions on the constant multiplier \( h \).

\( F(S) \) is realized as the voltage transfer ratio of Fig. 3.1, given by

\[ \frac{V_o}{V_i} = \frac{-Ky_{12}}{(1-K)y_{22} - Ky_{12}} \]  

(3.2.2)
Fig. 3.1 Balabanian and Patel's circuit, consisting of an ideal voltage-controlled voltage source of gain $K$ and a three-terminal network

where $y_{12}$, $y_{22}$ refer to the three-terminal network $B$.

From (3.2.1) and (3.2.2) we have

$$\frac{y_{12}}{y_{22}} = \frac{h(1-K)}{K(1-h)} \frac{S^2 + \alpha_1 S + \beta_1}{S^2 + \frac{d_2 - \alpha_1 h}{1-h} S + \frac{\beta_2 - \beta_1 h}{1-h}}$$

(3.2.3)

If (3.2.3) is to be RC realizable, the denominator must have distinct negative real roots. $h$ must then satisfy the conditions:
either
\[ h < \min \left( 1, \frac{\alpha_2}{\alpha_1}, \frac{\beta_2}{\beta_1} \right) \]  

or
\[ h > \max \left( 1, \frac{\alpha_2}{\alpha_1}, \frac{\beta_2}{\beta_1} \right) \]  

(3.2.4)

In addition
\[ h_1 < h < h_2 \]  

(3.2.5)

where \( h_1 \) and \( h_2 \) are the zeros of
\[
f(h) = (4 \beta_1 - \alpha_1^2)h^2 - 2(2 \beta_1 + 2 \beta_2 - \alpha_1 \alpha_2)h + (4 \beta_2 - \alpha_2^2)\]  

(3.2.6)

It can be shown that \( f(h) \) will never have complex zeros if \( F(S) \) has complex poles and zeros.

To satisfy Fialkow and Gerst's condition for a three-terminal network, \( h \) and \( K \) are further related by
\[ h < K < 1 \quad \text{or} \quad h > K > 1 \]  

(3.2.7)

Choosing \( h = K \) makes the constant multiplier of (3.2.3) unity. To realize (3.2.3), we put
\[
y_{12} = \frac{S^2 + \alpha_1 S + \beta_1}{S + 6} \]  

(3.2.8a)

\[
y_{22} = \frac{(S + \delta_1)(S + \delta_2)}{S + \delta} \]  

(3.2.8b)
where \( \delta_1 \) and \( \delta_2 \) are given by

\[
\delta_1 + \delta_2 = \frac{\alpha_2 - \alpha_1 h}{1 - h} \tag{3.2.9}
\]

\[
\delta_1 \delta_2 = \frac{\beta_2 - \beta_1 h}{1 - h} \tag{3.2.10}
\]

and \( \delta \) is chosen such that \( \delta_1 > \delta > \delta_2 \).

Then the passive three-terminal network \( B \) can be realized by standard synthesis procedure.

Note that \( K \neq 1 \), by (3.2.2), but both \( K > 1 \) and \( K < 1 \) are possible.

3.2.1 Application to second-order all-pass function

McVey\(^3\) has used a symmetrical twin-T RC network (Fig. 3.2) for network \( B \) to realize the all-pass function

\[
F(S) = h \frac{S^2 - \alpha S + \beta}{S^2 + \alpha S + \beta} \tag{3.2.11}
\]

Since the zeros of \( F(S) \) are the transmission zeros of Fig. 3.2 which is a third-order network, the zeros (and therefore poles) of \( F(S) \) are restricted to be within \( 30^\circ \) of the imaginary axis.\(^3\)
Fig. 3.2 Normalized symmetrical RC twin-T network

We have

$$-y_{12} = \frac{(S + 1) \left[ S^2 + \left( \frac{2}{m} - 1 \right)S + 1 \right]}{2 \left[ S^2 + \left( \frac{2}{m} + \frac{m}{2} \right)S + 1 \right]}$$ \hspace{1cm} (3.2.12a)

$$y_{22} = \frac{(S + 1) \left[ S^2 + \left( m + 1 + \frac{2}{m} \right)S + 1 \right]}{2 \left[ S^2 + \left( \frac{2}{m} + \frac{m}{2} \right)S + 1 \right]}$$ \hspace{1cm} (3.2.12b)

Substituting into (3.2.2), we obtain

$$\frac{V_o}{V_i} = \frac{K \left[ S^2 + \left( \frac{2}{m} - 1 \right)S + 1 \right]}{S^2 + \left[ m + 1 + \frac{2}{m} - K(m + 2) \right]S + 1}$$ \hspace{1cm} (3.2.13)

Equating (3.2.11) and (3.2.13) gives

$$m = \frac{2}{1 - \alpha}$$ \hspace{1cm} (3.2.14)
or

\[ K = \frac{4 + m^2}{m(m + 2)} = 1 - \frac{\alpha(1 - \alpha)}{2 - \alpha} \quad (3.2.15) \]

Since \( \alpha < 1, K < 1 \). \( K \) has a minimum value of

\[ 2(\sqrt{2} - 1) = 0.828 \] when \( \alpha = 2 - \sqrt{2}, m = 2 + 2\sqrt{2} \)

3.3 Two modifications of Balabanian and Patel's method allowing unity-gain amplification

We use the current analogue in this chapter in order to use a common-base transistor amplifier to simulate the unity-gain current-controlled current source in an experimental circuit.

Using (2.2.7), we can show that the current analogue of Balabanian and Patel's circuit is Fig.3.3.

![Current analogue of Fig.3.1](image)
The current transfer ratio is
\[
\frac{I_o}{I_i} = \frac{Ky_{12}}{(1 - K)y_{22} - Ky_{12}} \quad (3.3.1)
\]

Section 3.2 shows that the gain $K \neq 1$, but it can be less than 1. Therefore in the next two sections, we use a unity-gain amplifier and attenuate the amplification by shunting admittances, so that the effective gain is still less than 1.

3.3.1 Modification 1 (Fig. 3.4)

The amplifier gain is effectively reduced by shunting the earth a current $I_e$ related to $I_f$ by the equation.

\[
I_e = \frac{1 - k}{k} I_f, \quad 0 < k \leq 1 \quad (3.3.2)
\]

We then have

\[
I_f + I_e = I_o + KI \quad (3.3.3)
\]

\[
I = I_f + I_i \quad (3.3.4)
\]

\[
I_o = K \frac{y_{12}}{y_{22}} I \quad (3.3.5)
\]

By eliminating $I$, $I_e$ and $I_f$ from (3.3.2) to (3.3.5), we obtain
Fig. 3.4 Modification 1 of Fig. 3.3 allowing unity-gain amplification

\[
\frac{I_0}{I_i} = \frac{KY_{12}}{(1 - kK)Y_{22} - kKY_{12}} \quad (3.3.6)
\]

If \( K = 1 \), we have

\[
\frac{I_0}{I_i} = \frac{Y_{12}}{(1 - k)Y_{22} - kY_{12}} \quad (3.3.7)
\]

This is just (3.3.1) with \( K \) replaced by \( k \) and the numerator factor \( K \) omitted. Hence any function realized by Balabanian and Patel's method can be realized with unity D.C. gain.
3.3.2 Modification 2 (Fig.3.5)

In this modification the output of the amplifier is shunted to earth via admittance \( Y \): this has a similar effect to \( I_e \) in Fig.3.4.

We have

\[
\begin{align*}
I_b &= \frac{Y_{22}}{Y + Y_{22}} \times I_1 \tag{3.3.8} \\
I_0 &= \frac{Y_{12}}{Y_{22}} I_b \tag{3.3.9} \\
I &= I_1 + I_f \tag{3.3.10} \\
I_f &= I_b + I_0 \tag{3.3.11}
\end{align*}
\]

\[\text{Fig.3.5 Modification 2 of Fig.3.3 allowing unity-gain amplification}\]
By eliminating \( I_b, I_f \) and \( I \), we obtain

\[
\frac{I_0}{I_1} = \frac{KY_{12}}{(1 - K)y_{22} + Y - Ky_{12}} \tag{3.3.12}
\]

If \( K = 1 \), we obtain

\[
\frac{I_0}{I_1} = \frac{Y_{12}}{Y - Y_{12}} \tag{3.3.13}
\]

This is (3.3.1) with \( Y \) replacing \( (1 - K)y_{22}/K \), and so any transfer function realized by Balabanian and Patel's method can also be realized by this method.

3.4 Realization of all-pass \( F(S) \) by the methods of Section 3.3.1 and 3.3.2

For the first method, it is only necessary to replace \( K \) by \( k \) in Section 3.2.1, then \( F(S) \) is realized as \(-I_0/I_1\) and the multiplier \( h \) is unity.

For the second method, it is convenient to take

\[
Y = \frac{1 - h}{2h} (S + 1) \tag{3.4.1}
\]

(3.2.12a), (3.3.13) and (3.4.1) give

\[
- \frac{I_0}{I_1} = \frac{h[S^2 + (\frac{2}{m} - 1) S + 1]}{S^2 + \left[\frac{2}{m} + \frac{1}{2} - h(1 + \frac{m}{2})\right]S + 1} \tag{3.4.2}
\]
Equating (3.4.2) to (3.2.11) and using (3.2.14) gives

\[
h = \frac{m^2 - 2m + 8}{m(m + 2)} = l - \frac{2\alpha(1 - \alpha)}{2 - \alpha}
\]  

(3.4.3)

\(h < 1\) when \(\alpha < 1\); as before, the minimum occurs when \(\alpha = 2 - \sqrt{2}\), but its value is now \(4\sqrt{2} - 5 = 0.657\).

This realization was obtained independently by McVey\(^3\).

**Example 3.1**

To realize the normalized second-order all-pass function with \(\lambda = 0.125\) (Section 1.2.2):

\[
F(S) = \frac{h S^2 - 0.2509s + 1}{S^2 + 0.2509s + 1}
\]  

(3.4.4)
as \(-I_o/I_i\) of (3.3.7) and (3.3.13) respectively. The zeros and poles lie within \(30^\circ\) of the imaginary axis, therefore we may use the methods of Section 3.4.

(i) Using the first method, by (3.2.14)

\[
m = \frac{2}{1 - 0.2509} = 2.670
\]

By (3.2.15) with \(K\) replaced by \(k\), we have

\[
k = 1 - \frac{0.2509(1 - 0.2509)}{2 - 0.2509} = 0.8926
\]

For this method, \(h = 1\), Fig. 3.6 shows the resulting
\[ R_1 = 3.346 \, \Omega \]
\[ R_2 = 3.749 \, \Omega \]
\[ R_3 = 1.880 \, \Omega \]
\[ C_1 = 0.1 \, \text{F} \]
\[ C_2 = 0.08926 \, \text{F} \]
\[ C_3 = 0.1777 \, \text{F} \]

**Fig. 3.6** Circuit realizing (3.4.5) using the configuration of Fig. 3.4

\[ R_1 = 3.346 \, \Omega \]
\[ R_2 = 1.253 \, \Omega \]
\[ R_3 = 24.42 \, \Omega \]
\[ C_1 = 0.1 \, \text{F} \]
\[ C_2 = 0.2670 \, \text{F} \]
\[ C_3 = 0.01369 \, \text{F} \]

**Fig. 3.7** Circuit realizing (3.4.6) using the configuration of Fig. 3.5
realization of
\[
\frac{I_0}{I_1} = \frac{S^2 - 0.75S + 8.934}{S^2 + 0.75S + 8.934} \quad (3.4.5)
\]

(ii) Using the second method, we have \( m = 2.670 \) as before.

By (3.4.3) \( h = 1 - 2 \times \frac{0.2509(1 - 0.2509)}{2 - 0.2509} = 0.7851 \)

By (3.4.1) \( Y = \frac{1 - 0.7851}{2 \times 0.7851} (S + 1) = 0.1369(S + 1) \)

For this method, \( h = 0.7851 \) and Fig. 3.7 shows the resulting realization of
\[
\frac{I_0}{I_1} = 0.7851 \frac{S^2 - 0.75S + 8.934}{S^2 + 0.75S + 8.934} \quad (3.4.6)
\]

Experimental fourth-order delay

A circuit realizing the fourth-order all-pass function with \( \lambda = 0.125 \) (Section 1.3.2) and 1ms delay was built (Fig. 3.8) by cascading two second-order sections (Fig. 3.5) with common-base transistor amplifiers. The passive elements were of 5% tolerance. The experimental results are shown in Figs. 3.9 and 3.10. Fig. 3.9 shows that the network has poor amplitude/frequency response, the maximum errors of which occur at the frequencies where the (phase characteristic
\[ R_1 = 1.120\, \text{k\Omega} \quad C_1 = 0.1\, \mu\text{F} \\
R_2 = 0.4941\, \text{k\Omega} \quad C_2 = 0.2267\, \mu\text{F} \\
R_3 = 18.04\, \text{k\Omega} \quad C_3 = 0.006209\, \mu\text{F} \\
R_4 = 5.276\, \text{k\Omega} \quad C_4 = 0.1\, \mu\text{F} \\
R_5 = 0.8609\, \text{k\Omega} \quad C_5 = 0.3805\, \mu\text{F} \\
R_6 = 13.49\, \text{k\Omega} \quad C_6 = 0.02428\, \mu\text{F} \\
R = 5\, \text{k\Omega} \quad C = 50\, \mu\text{F} \\
\]

**Fig. 3.8** Circuit realizing the fourth-order all-pass delay with \( \lambda = 0.125 \) and 1ms delay
Fig. 3.9 Frequency response curves of Fig. 3.8
Input: 10μA r.m.s.
Fig.3.10. Step response of Fig.3.8
Input step: 12.5\mu A
crosses the line $\phi = -10^{-3} \omega$ group delay is nearly a maximum. This is because the phase changes rapidly with respect to frequency in the neighbourhood of a pole or zero, so that such frequencies correspond to maxima of group delay. At such a frequency the gain is sensitive to changes in the relative positions of a pole and a zero.

3.5 Direct feedback from output

If a current proportional to the output current is fed back to the input, Fig. 3.11, we have

$$I = I_1 - kI_o$$

$$\frac{(1 + k)I_o}{KI} = \frac{Y_{12}}{Y_{22}}$$

Fig. 3.11 Circuit with direct feedback from output
Substituting (3.5.1) into (3.5.2), we obtain

\[
\frac{I_0}{I_i} = \frac{K y_{12}}{(1 + k)y_{22} + kK y_{12}} \quad (3.5.3)
\]

Here the gain \( K \) can be unity.

3.6 Comparison between the circuits of Figs. 3.4, 3.5 and 3.11

Armstrong and Reza show that if \( F(S) \) is a second-order function, with possibly complex poles, to be realized as

\[
F(S) = \frac{-y_{12}}{y_{22} - g y_{12}} \quad (3.6.1)
\]

where \( g \) is a constant, then the following rules apply:

(i) If

\[
F(S) = \frac{h}{S^2 + \alpha S + \beta} \quad \text{or} \quad \frac{S^2 + \alpha_1 S + \beta_1}{S^2 + \alpha_2 S + \beta_2}
\]

then \( g > 0 \)

(ii) If

\[
F(S) = \frac{h(S + d)}{S^2 + \alpha S + \beta}
\]

then \( g < 0 \)

Hence Fig. 3.4 or 3.5 realize type (i) functions whereas Fig. 3.11 realizes type (ii) functions.
3.7 Extension to two three-terminal RC networks

If we apply an additional feedback through a three-terminal network, Fig. 3.12, we again obtain additional attenuation which allows a unity-gain amplifier to be used.

We have

\[ I_b = \frac{y_{22}^B}{y_{22}^A + y_{22}^B} \text{ KI} \quad (3.7.1) \]

\[ I_a = \frac{y_{22}^A}{y_{22}^A + y_{22}^B} \text{ KI} \quad (3.7.2) \]

Fig. 3.12 Circuit with two feedback three-terminal networks
By eliminating $I_a$, $I_b$, $I_f$ and $I$, we obtain

$$I_o = \frac{y_{12}^B}{y_{22}} I_b \quad (3.7.3)$$

$$I = I_i + I_f - \frac{y_{12}^A}{y_{22}} I_a \quad (3.7.4)$$

$$I_f = I_b + I_o \quad (3.7.5)$$

If $y_{12}^A = 0$, (3.7.6) reduces to (3.3.12) as it should.

When $K = 1$, we have

$$I_o = \frac{y_{12}^B}{y_{22} - Ky_{12}^B + y_{22}^A + Ky_{12}^A} \quad (3.7.6)$$

But $y_{22}^A + y_{12}^A = -y_{20}^A$, the transfer admittance between terminal 2 and the reference terminal 0 (earth), hence

$$I_o = -\frac{y_{12}^B}{y_{12}^B + y_{20}^A} \quad (3.7.8)$$

Since $y_{12}^A$ and $y_{20}^A$ have all coefficients negative, (3.7.8) shows that $I_o/I_i$ can only realize a transfer function whose denominator coefficients are greater than or equal to the corresponding numerator coefficients.

**Example 3.2**

To realize
\[
\frac{I_0}{I_1} = h \frac{S^2 + 2}{S^2 + 0.5S + 1} \tag{3.7.9}
\]

Since the denominator coefficients must be equal to or greater than the corresponding numerator coefficients, we choose \( h = 0.5 \), and write the denominator as \( 2S^2 + S + 2 \).

A standard realization is

\[
-y_{12}^{-B} = \frac{S^2}{S + 1} + \frac{2}{S + 1}
\]

\[
y_{20}^{A} = \frac{S^2}{S + 1} + \frac{S}{S + 1}
\]

A circuit realizing (3.7.9) is shown in Fig. 3.13.

\[\text{Fig. 3.13 Circuit realizing (3.7.9) using the configuration of Fig. 3.12}\]
3.8 Alternative method of using two three-terminal RC networks

If the reference terminal of network B is connected directly to earth, i.e., the feedback is applied only through network A, we have the configuration of Fig. 3.14.

The circuit equations are (3.7.1) to (3.7.4) with $I_f = 0$. By eliminating $I_a$, $I_b$ and $I$, we obtain

\[
\frac{I_o}{I_i} = \frac{K_{y_{12}}^B}{y_{22}^B + y_{22}^A + K_{y_{12}}} \quad (3.8.1)
\]

**Fig. 3.14** Circuit with one feedback three-terminal network
When $K = 1$, (3.8.1) becomes

$$I_0 = \frac{y_{12}^B}{I_1}$$

$$I_i = \frac{y_{22}^B - y_{20}^A}{y_{22}^B}$$ (3.8.2)

Due to the Fialkow and Gerst condition\textsuperscript{15} that the numerator coefficients of $y_{22}^B$ must be greater than or equal to those of $y_{12}^B$, (3.8.2) may give less D.C. gain than (3.7.8). But (3.8.1) is much simpler than (3.7.6) when $K \neq 1$.

This configuration can also be derived from Hakim's method\textsuperscript{17} as shown by the author\textsuperscript{71}.

**Example 3.3**

To realize

$$-\frac{I_0}{I_i} = \frac{h}{S^2 + 1.386S + 1}$$ (3.8.3)

We write

$$-\frac{I_0}{I_i} = \frac{h}{S + 1}$$

$$= \frac{(S + 0.5)(S + 2)}{2(S + 1)} + \frac{S^2 + 0.272S + 1}{2(S + 1)}$$

and take

$$y_{22}^B = \frac{(S + 0.5)(S + 2)}{2(S + 1)}$$, \quad $$y_{12}^B = \frac{0.5}{S + 1}$$
Fig. 3.15 Circuit realizing (3.8.3) using the configuration of Fig. 3.14

Fig. 3.16 Transistor circuit realizing (3.8.3) with $S = 10^3 s^{-1}$
Fig. 3.17 Frequency response curves of Fig. 3.16
Input: 10μA r.m.s.
Fig. 3.18  Step response of Fig. 3.16

Input step: 40μA
\[-y_{20}^A = \frac{S^2 + 0.272S + 1}{2(S + 1)} = \frac{1}{2(S + 1)} + \frac{S(S + 0.272)}{2(S + 1)}\]

Here \(h = 0.5\) by Fialkow and Gerst condition. The resulting network is shown in Fig.3.15. A circuit for this function has been built, Fig.3.16, using a common-base transistor amplifier and with \(S\) equal to \(10^3\) sec\(^{-1}\). Its frequency response curves are shown in Fig.3.17, and the step response in Fig.3.18. The passive elements used were of 1% tolerance, except the blocking capacitances.

The input impedance is of the order of 100\(\Omega\) at \(10^3\) rad/s, and its output impedance has a series component 3.33k\(\Omega\). Hence the loading effect of a similar circuit connected across the output terminals will be negligible.

3.9 Use of one unity-gain amplifier to give a transfer ratio as a ratio of two transfer admittances (Fig.3.19)

3.9.1 Basic network

The voltage at terminal 2 of network B in Fig.3.3 is

\[V_o = \frac{I_o^B}{y_{12}^B}\]

and if the input current to the circuit is obtained from a voltage source \(V_i\) through a three-terminal network A,
we have
\[ I_1 = -V_i y_{12} \]  \hspace{1cm} (3.9.2)

Substituting (3.9.1) and (3.9.2) into (3.3.1), we obtain
\[ \frac{V_o}{V_i} = \frac{-K y_{12}^A}{(1 - K) y_{22}^B - K y_{12}^B} \]  \hspace{1cm} (3.9.3)

If \( K = 1 \), we have
\[ \frac{V_o}{V_i} = \frac{y_{12}^A}{y_{12}^B} \]  \hspace{1cm} (3.9.4)

This is the same transfer function as the one obtained with one operational amplifier and two three-terminal networks (Section 5.3.2)

![Circuit diagram](image)

**Fig.3.19** Circuit giving a voltage transfer ratio as a ratio of two transfer admittances
Sonde used nearly the same configuration, but with a feedforward resistance instead of a three-terminal network \( A \), for a tuned amplifier.

The only disadvantage seems to be that the section cannot be cascaded directly with similar sections.

### 3.9.2 An improved synthesis method using cascaded passive sections

In realizing

\[
F(S) = \frac{N(S)}{D(S)}
\]  

(3.9.5)

as \( V_o/V_1 \) in (3.9.4), the usual approach\(^{44,45}\) is to choose a suitable polynomial \( Q(S) \) and synthesise

\[
-y_{12}^A = \frac{N(S)}{Q(S)} , \quad -y_{12}^B = \frac{D(S)}{Q(S)} ,
\]

(3.9.6)

Although a number of methods are available\(^{7,14,27}\) the calculations may be tedious for polynomials of degree greater than 2, and the choice of \( Q(S) \) is somewhat arbitrary.

We now propose a method based on cascaded sections. If \( a, b \) are two networks connected as in Fig.3.20, the overall transfer admittance is\(^{65}\).
\[-y_{12} = \frac{y^a_{12} y^b_{12}}{y^a_{22} + y^b_{11}} \quad \text{(3.9.7)}\]

If, say, \(D(S)\) is of degree four with two pairs of complex zeros, we may assign one pair to \(y^a_{12}\) and the other to \(y^b_{12}\), using a standard form of network. The resulting \(y^a_{22}\) and \(y^b_{11}\) will determine \(Q(S)\), and the full synthesis procedure will only have to be applied to network A, with a saving of perhaps half the labour of the usual method.

In fact, even network A can be synthesised simply if both \(N(S)\) and \(D(S)\) are of degree four with complex zeros. In this case Fig. 3.20 is replaced by Fig. 3.21 in which

\[\text{Fig. 3.20 Two cascaded passive three-terminal networks}\]
The insertion of the extra admittances enables us to fix the denominator zeros of (3.9.8), by keeping \((Y_1^a + Y_2 + Y_3)\) constant, while varying \(Y_1^a\) and \(Y_2^a\) to move the numerator zeros.

**Example 3.4**

To realize the fourth-order low-pass Bessel filter

\[
F(S) = \frac{105}{S^4 + 10S^3 + 45S^2 + 105S + 105}
\]

\[
= \frac{105}{(S^2 + 4.268S + 11.49)(S^2 + 5.792S + 9.140)}
\]

(3.9.9)

![Fig. 3.21 Two cascaded three-terminal networks with an additional star-connection](image-url)
as $V_o/V_i$ in (3.9.4), using the method just described.

Fig.3.22 shows two four-elements networks, each having a transfer admittance of the form.

$$-y_{12} = \frac{S^2 + \alpha S + \beta}{S + \delta} \quad (3.9.10)$$

![Diagram](image)

(a) $\delta = \alpha$

R = $\alpha/2\beta \Omega$

$C_1 = 1 F$

$C_2 = 4 \beta/\alpha^2 F$

(b) $\delta = \beta/\alpha$

C = 2F

$R_1 = 1/\alpha \Omega$

$R_2 = \alpha/4\beta \Omega$

Fig.3.22 Two four-elements circuits both giving a pair of complex transmission zeros
We choose Fig. 3.22a to realize each pair of the complex poles. The reason for not choosing Fig. 3.22b will be evident later. So, by substituting the values of $\alpha$ and $\beta$, we obtain network B as in Fig. 3.23.

Therefore, for network B, we have

$$Y_1 = C^a_1, \quad Y_2 = C^b_1 \text{ and } Y_3 = 0.\$$

Now, we construct network A whose $Y_{12}^A$ has the same poles as $Y_{12}^B$. Since there are no transmission zeros for this network, we put $Y_1 = Y_2 = 0$, and therefore

$$Y_3 = C^a_1 + C^b_1 \text{ (Fig. 3.24)}$$

![Diagram of network B giving the two pairs of complex poles of the fourth-order low-pass Bessel filter](image)

\[ \begin{align*}
C^a_1 &= C^b_1 = 1F \\
C_2 &= 2.596F \\
C_3 &= 1.090F \\
R_1 &= 0.1831\Omega \\
R_2 &= 0.3169\Omega
\end{align*} \]

Fig. 3.23 Network B giving the two pairs of complex poles of the fourth-order low-pass Bessel filter
Fig. 3.24 Network A giving no transmission zeros for the fourth-order low-pass Bessel filter

The complete circuit is shown in Fig. 3.25. A circuit has been built (Fig. 3.26) with 1ms delay using a common-base transistor amplifier. Figs. 3.27 and 3.28 show the experimental results.

3.10 Extension of Bach's circuit

This circuit selects C's and R's successively to realize the denominator coefficients of a specified function. The current analogue of Bach's circuit is shown in Fig. 3.29 whose current transfer ratio is
$R^1 = 0.1831\,\text{A}$

$R^2 = 0.3169\,\text{A}$

**Fig. 3.25** Circuit realizing the fourth-order Bessel filter (3.9.9) using the configuration of Fig. 3.18

**Fig. 3.26** Transistor circuit realizing the fourth-order Bessel filter (3.9.9) with lms delay

$R_1 = 0.1831\,\text{k}\Omega$

$R_2 = 0.3169\,\text{k}\Omega$

$R = 5\,\text{k}\Omega$

$C_1 = 1\,\text{F}$

$C_2 = 2.596\,\text{F}$

$C_3 = 1.090\,\text{F}$

$C_1 = 0.1\,\mu\text{F}$

$C_2 = 0.2596\,\mu\text{F}$

$C_3 = 0.1090\,\mu\text{F}$

$C = 50\,\mu\text{F}$
Fig. 3.27 Frequency response curves of Fig. 3.26
Input: 4v r.m.s.
Fig. 3.28  Step response of Fig. 3.26
Input step: 2v
Fig. 3.29  Current analogue of Bach's circuit

Fig. 3.30  Circuit of Fig. 3.29 with additional feedforward conductances
\[ I_{\text{out}} = \frac{1}{I_i + \frac{T_1 S + T_2 S^2 + \ldots + T_n S^n}{1}} \]  \hspace{1cm} (3.10.1)

where \( T'_m = t_n t_{n-1} \ldots t_{n-m+1} \), \( t_j = C_j R_j \)

If we interchange each \( C_j \) and \( R_j \), the denominator of (3.10.1) becomes

\[ T_n S^n + T_{n-1} S^{n-1} + \ldots + T_1 S + 1 \]  \hspace{1cm} (3.10.2)

where \( T_m = t_1 t_2 \ldots t_m \),

and the numerator will have \( S \) terms. Since applying feedforward can only modify the numerator of a transfer function, we apply feedforward currents to each summing junction from a voltage source (Fig. 3.30). We also, for convenience, convert the output current to a voltage \( V_o \).

At each summing junction, we have

\[ I_j = G_j V_i + \frac{t_{j+1} S}{1 + t_{j+1} S} I_{j+1} + \frac{1}{1 + t_{j-1} S} I_{j-1}, \]

\[ j = 1 \text{ to } n \text{ with } I_{n+1} = 0, \ I_0 = 0. \]  \hspace{1cm} (3.10.3)

Also

\[ I_{\text{out}} = G_0 V_i + \frac{t_1 S}{1 + t_1 S} I_1 = G V_o \]  \hspace{1cm} (3.10.4)

Write

\[ x_j = \frac{1}{1 + t_j S} I_j, \ j = 0 \text{ to } n + 1; \ x_0 = x_n + 1 = 0. \]
Then (3.10.3) becomes

\[(1 + t_j S) x_j = G_j V_i + t_{j+1} S x_{j+1} + x_{j-1}\]

or \[G_j V_i = S(t_j x_j - t_{j+1} x_{j+1}) + (x_j - x_{j-1}), \quad j = 1 \text{ to } n.\]

(3.10.5)

Summing (3.10.5) from \(j = 1\) to \(m\), and writing

\[\sum_{j=0}^{m} G_j V_i = u_m,\]

we get

\[u_m = S(t_1 x_1 - t_{m+1} x_{m+1}) + x_m + G_0 V_i\]

\[= I_{\text{out}} - S t_{m+1} x_{m+1} + x_m, \quad \text{by (3.10.4)}\]

or \[x_m = (u_m - I_{\text{out}}) + S t_{m+1} x_{m+1}, \quad m = 0 \text{ to } n\]

Thus \[0 = x_0 = (u_0 - I_{\text{out}}) + S t_1 x_1\]

\[= (u_0 - I_{\text{out}}) + S t_1 (u_1 - I_{\text{out}}) + S^2 t_1 t_2 x_2\]

\[= \ldots = (u_0 - I_{\text{out}}) + S t_1 (u_1 - I_{\text{out}}) +\]

\[S^2 t_1 t_2 (u_2 - I_{\text{out}}) + \ldots + S^n t_1 t_2 \cdots t_n (u_n - I_{\text{out}})\]

or \[(1 + S t_1 + S^2 t_1 t_2 + \ldots + S^n t_1 t_2 \cdots t_n) I_{\text{out}}\]

\[= u_0 + S t_1 u_1 + S^2 t_1 t_2 u_2 + \ldots + S^n t_1 t_2 \cdots t_n u_n\]

or \[(1 + T_1 S + T_2 S^2 + \ldots + T_n S^n) I_{\text{out}}\]

\[= u_0 + T_1 u_1 + T_2 S^2 u_2 + \ldots + T_n S^n u_n\]
or \[
\frac{V_o}{V_i} = \frac{k_0 + k_1 T_1 S + k_2 T_2 S^2 + \ldots + k_n T_n S^n}{1 + T_1 S + T_2 S^2 + \ldots + T_n S^n}
\] (3.10.6)

where \( k_m = \frac{u_m}{G V_i} = \sum_{j=0}^{m} \frac{G_j}{G} ; \ T_m = t_1 t_2 \ldots t_m \)

3.10.1 All-pass transfer function: sequential adjustment of circuit elements

If negative \( k_m \) are required, a sign-reversing voltage amplifier must be used, A case in point is an all-pass transfer function, for which we may take

\[
G_0 = G
\]
\[
G_{2j-1} = -2G
\]
\[
G_{2j} = 2G
\] (3.10.7)

so that (3.10.6) becomes

\[
\frac{V_o}{V_i} = \frac{\sum_{m=0}^{n} T_m (-S)^m}{\sum_{m=0}^{n} T_m S^m}
\]

say (3.10.8)

If (3.10.7) is replaced by

\[
G_j = (-1)^j G, \ j = m, m+1, m+2, m+3
\]
\[
G_j = 0 \ \text{otherwise}
\]

then the numerator of (3.10.6) becomes
\[ T_m(-S)^m + T_{m+2}(-S)^{m+2} \]

which is zero when
\[ S = j \left( \frac{T_m}{T_{m+2}} \right)^{\frac{1}{2}} = j\omega_m \]

This suggests a possible method (which has not been tried in practice) of setting up the circuit experimentally by adjusting successive \( R_j \) and \( C_j \) to give transmission zeros at frequencies \( \omega_0, \omega_1, \ldots, \omega_{m-2} \). The value of \( t_1 \) must be found by setting \( G_0 = G, G_1 = -2G \), in which case
\[ \frac{V_0}{V_1} = \frac{1 - t_1 S}{1 + t_1 S} \]

### 3.11 Conclusions

Apart from a constant multiplier, any second-order transfer function can be realized by any one of the methods of Sections 3.3, 3.4, 3.5, 3.7 and 3.8. The unity-gain current-controlled current source can be realized particularly simply by a common-base transistor amplifier when there is a D.C. path from the output terminal of the source to earth, otherwise a more elaborate amplifier is required.
The method of Section 3.9 is considered to be the best one using unity-gain amplifiers, since it gives the simpler transfer function; but stages cannot be cascaded without isolating amplifiers.

The current analogue of Bach's circuit can be realized in the same way, and can be extended to synthesise a more general function by means of feedforward conductances.
CHAPTER 4

REALIZATION OF RL DRIVING-POINT ADMITTANCE BY RC ACTIVE NETWORKS, AND APPLICATION IN SYNTHESIS

4.1 Introduction

One application of RC active networks is to realize inductance, so as to avoid using coils at low frequencies. This can be done by using ideal gyrators\(^{34,50}\), but it is uneconomical and inconvenient in some cases\(^{20,35}\). However, a RL driving-point admittance can be realized otherwise with a single active element. This then removes the constraints on the passive RC network which cannot have complex poles.

Horowitz\(^{22,23}\) first introduced the idea of realizing an RL admittance using a RC network and a voltage-controlled current source. His method had the disadvantage that the gain of the active element is very sensitive to the transistor parameters, and he therefore introduced a lengthy optimization procedure to minimize the effect. This is perhaps why little attention has been paid to his method. Then he applied the RL admittance in cascade with an RC
admittance to realize a transfer impedance which seems to be applicable only to fourth-order functions. A similar procedure has been adopted by Balabanian$^6$.

In this chapter, we introduce a new method of realizing an RL admittance by using a RC passive network using a unity-gain voltage or current amplifier, or a high-gain voltage amplifier. This RL admittance is then used in new methods of synthesis.

This type of active network converting an RC admittance to an RL admittance has also been called a non-ideal gyrator$^6,22$.

We summarise Horowitz's RL admittance realization method here, because (4.1.2) is similar to (4.2.5) for the driving-point admittance in a balanced bridge circuit, and so his method of inverting an RC admittance to an RL admittance can be applied.

Horowitz used a voltage-controlled current source in shunt with an RC admittance as shown in Fig. 4.1.

The driving-point admittance is given by

$$Y = G \frac{Y_1 + G}{Y_1 + G} \quad (4.1.1)$$

whence

$$Y_1 = G \frac{Y - G}{G - Y} \quad (4.1.2)$$
Fig. 4.1 Equivalent circuit for realizing an RL admittance

If \( Y \) is a specified RL admittance having the characteristic shown in Fig. 4.2, and \( g \) and \( G \) are chosen so that

\[
g > Y(0); \quad Y(\infty) > G > 0 \quad (4.1.3)
\]

then \( Y_1 \) is an RC admittance with zeros and poles given by the solutions of \( Y(S) = g \) and \( Y(S) = G \) respectively. The poles and zeros are interlaced and the solution nearest to the \( Y \)-axis is a zero.

4.2 Bridge network realizing a RL driving-point admittance

In this section, we introduce a new method of realizing a RL driving-point admittance by means of a balanced bridge network. We first describe three types
$O$ and $x$ are the zero and pole for an RC admittance.

**Fig. 4.2** An RL admittance plotted as a function of real $S$

of balanced bridge using a controlled source, and transform one of these networks using (2.2.4). Then we show that the driving-point admittance of such networks is an RL admittance if the passive elements in the three arms are properly chosen.

Stuart and Lampard\textsuperscript{55} introduced a bridge network balanced at all frequencies by a high-gain voltage-controlled voltage source (Fig. 4.3), and applied it to
Bott-Duffin synthesis of two-terminal impedances and to a constant-resistance bridged-T network.

Parsons\textsuperscript{43} replaced the high-gain voltage-controlled voltage source by a unity-gain voltage-controlled voltage source (Fig. 4.4). The present author noted that balance can be maintained by a unity-gain current-controlled current source (Fig. 4.5). The only disadvantage may be the floating power supply for the amplifier which also occurs in Stuart and Lampard's circuit.

\textbf{Fig. 4.3} Balanced bridge incorporating a high-gain voltage amplifier ($K \to \infty$)
Fig. 4.4 Balanced bridge incorporating a unity-gain voltage amplifier

Fig. 4.5 Balanced bridge incorporating a unity-gain current amplifier
By applying (2.2.4) to the driving-point impedance, we can replace the active element in Fig.4.4 by a unity-gain current-controlled current source (Fig.4.6). Here the amplifier does not need a floating power supply and the circuit is no longer balanced, but the input admittance remains unchanged.

Now consider a circuit equivalent to the balanced bridges as shown in Fig.4.7. The driving-point admittance between the terminals A and B is given by

\[ Y = \frac{Y_1Y_2}{Y_1 + Y_2} + \frac{Y_3Y_4}{Y_3 + Y_4} \quad (4.2.1) \]

![Diagram of Fig.4.6](image)
Fig. 4.7 Equivalent circuit for a balanced bridge incorporating an active source

At balance, the potential across CD is zero and

\[ Y_4 = \frac{Y_2 Y_3}{Y_1} \]  \hspace{1cm} (4.2.2)

Substituting (4.2.2) into (4.2.1), we obtain

\[ Y = Y_2 \frac{Y_1 + Y_3}{Y_1 + Y_2} \]  \hspace{1cm} (4.2.3)

or \[ Y_1 = Y_2 \frac{Y - Y_3}{Y_2 - Y} \]  \hspace{1cm} (4.2.4)

If \( Y_2 \) and \( Y_3 \) are resistive, say \( Y_2 = G_2 \) and
\[ Y_3 = G_3, \text{ we have} \]
\[ Y_1 = G_2 \frac{Y - G_3}{G_2 - Y} \quad (4.2.5) \]

which is (4.1.2) with \( G \) and \( g \) replaced by \( G_2 \) and \( G_3 \) respectively. If \( Y \) is a specified RL admittance, we can therefore choose \( G_2 \) and \( G_3 \) in the ranges

\[ Y(\infty) > G_2 > 0; \quad G_3 > Y(0) \quad (4.2.6) \]

so that \( Y_1 \) is an RC admittance. Hence Figs. 4.3 - 4.6 can realize a RL driving-point admittance.

4.3 Synthesis method with three-terminal RC network terminated by an RL admittance

We now describe a new method of applying a grounded RL admittance in network synthesis. The problem can be formulated mathematically as one of adding or subtracting suitably restricted polynomials so that the resulting polynomial has the complex zeros which are the desired poles of a specified function. In active RC synthesis, this operation is implemented by a controlled source. A method based on addition has the advantage that the coefficients are less sensitive to the variations in the active element than one involving subtraction.
The method to be described uses addition; but it has the disadvantages of (i) a loss in D.C. gain due to the resistive component in the shunt branch (load) and the absence of an active element in the series branch, (ii) surplus elements due to the cancellation of a common denominator. Therefore, apart from the sensitivity, this method is probably no better than most of the existing methods.

If a three-terminal RC network is terminated by an RC admittance, then the poles of the voltage transfer ratio are restricted to the negative real axis of the S-plane. However, if the load is an RL admittance, complex poles can be allowed. The voltage transfer ratio of a three-terminal RC network terminated by a load as shown in Fig.4.8* is given by

$$\frac{V_o}{V_i} = \frac{-Y_{12}}{Y_{22} + Y}$$  \hspace{1cm} (4.3.1)

* This general configuration has also been given by Su\textsuperscript{57}, but Y is not specified.
Let the specified voltage transfer ratio be
\[
\frac{V_o}{V_i} = \frac{N(S)}{D(S)} \quad (4.3.2)
\]
where the degree of \(D(S)\) is not less than the degree of \(N(S)\), \(N(S)\) has no positive real zeros and \(D(S)\) has complex zeros only:
\[
D(S) = \frac{m}{1} (S + S_1)(S + S_2)
\]

The condition for the RC-RL decomposition of \(D(S)\) according to Fig.4.8 is that positive \(A_o, B_o, a_i, b_i\) and \(q_i\) exist such that
\[
D(S) = A_o \frac{2^m}{1}(S + a_1) + B_o \frac{2^m}{1}(S + b_1)
\]
\[ A_0A(S) + B_0B(S) \text{ say} \quad (4.3.3) \]

where \[ \frac{A_0A(S)}{Q(S)} = y_{22}; \quad \text{(RC admittance)} \quad (4.3.4) \]

\[ \frac{B_0B(S)}{Q(S)} = y; \quad \text{(RL admittance)} \quad (4.3.5) \]

\[ Q(S) = \frac{2m}{1}(S + q_1) \quad (4.3.6) \]

Calahan has shown that if

\[ \frac{m}{1} \arg S_1 \leq \frac{\pi}{2} \quad (4.3.7) \]

a RC-RL decomposition is always possible. RC-RL decomposition is therefore always possible when \( m = 1 \).

To complete the synthesis of (4.3.2), we must make

\[ -y_{12} = \frac{N(S)}{Q(S)} \quad (4.3.8) \]

This is always possible to within a constant multiplier since \( y_{12} \) has the same poles as \( y_{22} \).

**Example 4.1**

To realize, as \( \frac{V_0}{V_1} \) in Fig. 4.8, the normalized low-pass Bessel filter

\[ F(S) = \frac{h}{S^2 + \sqrt{3}S + 1} \]

where \( h \) is a constant to be determined.
Fig. 4.9  Circuit for realizing the second-order Bessel filter (4.3.9) with 1s delay using the configuration of Fig. 4.8.
We write the denominator in the form
\[ S^2 + \sqrt{3}S + 1 = (S + 0.2)(S + 1.2) + 0.3321(S + 2.288) \]
and take \( Q(S) = S + 1 \), so that

\[
\frac{V_0}{V_i} = \frac{\frac{h}{S + 1}}{\frac{(S + 0.2)(S + 1.2) + 0.3321(S + 2.288)}{S + 1}}
\]

\[
= \frac{-y_{12}}{y_{22} + Y}
\]

By the Fialkow and Gerst condition\(^{15}\),
\[ h \leq 0.2 \times 1.2 = 0.24 \]
in this realization. We take \( h = 0.24 \).

\( Y \) is an RL admittance which can be realized by Fig.4.6. From (4.2.5), we want to choose \( G_2 \) and \( G_3 \) so that \( Y_1 \) is an RC admittance. We have \( Y(\infty) = 0.3321 \), \( Y(0) = 0.76 \), so (4.2.6) gives \( 0.3321 > G_2 > 0 \) and \( G_3 > 0.76 \).

Choose \( G_2 = 0.15 \), \( G_3 = 4 \)
then \( Y_1 = \frac{3.021(S + 0.8833)}{S + 3.348} = 0.7970 + \frac{2.224S}{S + 3.348} \)

Fig.4.9 shows the resulting realization of
\[
\frac{V_0}{V_i} = \frac{0.72}{S^2 + 3S + 3} \quad \text{(4.3.9)}
\]
\begin{itemize}
\item \textbf{b. Using Fig. 4.5}
\end{itemize}

- $C_1 = 0.1918 \mu F$
- $C_2 = 0.2894 \mu F$
- $C_3 = 0.2894 \mu F$
- $C = 50 \mu F$
- $R_1 = 2.510 k\Omega$
- $R_2 = 13.33 k\Omega$
- $R_3 = 0.5 k\Omega$
- $R_4 = 0.9024 k\Omega$
- $R_5 = 3.333 k\Omega$
- $R_6 = 5 k\Omega$
- $R_7 = 5 k\Omega$

\textbf{Fig. 4.10} Two transistor circuits for realizing the second-order Bessel filter with 1 ms delay using the configuration of Fig. 4.8
Fig. 4.11 Frequency response curves of Fig. 4.10a O and Fig. 4.10b © Input: 3v r.m.s.
For Fig.4.10a
(a) Experimental

For Fig.4.10b
(b) Theoretical

Fig.4.12 Step responses of Fig.4.10a and Fig.4.10b
Input step: 5v
A circuit realizing (4.3.9) with 1 ms delay was built, using a common-base transistor amplifier (Fig.4.10a). An alternative connection (Fig.4.5) for the amplifier is shown in Fig.4.10b. Passive elements of 3\% tolerance were used and the experimental results for these two circuits are shown in Figs.4.11 and 4.12.

4.4 Extension of the method of Section 4.3 to permit 'optimum' decomposition of $D(S)$ with two RC networks of order $m$ (Fig.4.13, 4.18)

Calahan has shown that if (4.3.7) is satisfied, then the decomposition

$$D(S) = A_0 \sum_{i=1}^{m} (S + a_i)^2 + B_0 \sum_{i=1}^{m} (S + b_i)^2$$

$$= A_0 A^2(S) + B_0 B^2(S) \text{ say (4.4.1)}$$

with $a_i < b_i$ for each $i$, is possible. This decomposition, which is not unique, minimizes the sensitivity of the poles, the coefficients of $D(S)$ and the frequency response with respect to the active element parameters. Balabanian also tried to use this 'optimum' decomposition but did not succeed in realizing a general transfer function.

(4.4.1) suggests the possibility of halving the order of the RC network in Fig.4.8 by choosing

$$Q(S) = A(S) B(S) \text{ (4.4.2)}$$
and taking

\[ y_{22}^A = \frac{A_o A^2(S)}{Q(S)} = \frac{A_o A(S)}{B(S)} \]  \hspace{1cm} (4.4.3) \\
\[ Y = \frac{B_o B^2(S)}{Q(S)} = \frac{B_o B(S)}{A(S)} \]  \hspace{1cm} (4.4.4)

\[ y_{22}^A \] will refer to network A in the subsequent discussion. Since \( y_{12}^A \) and \( y_{22}^A \) have the same poles, we can write

\[ -y_{12}^A = \frac{N^A(S)}{B(S)} \], say \hspace{1cm} (4.4.5)

where \( N^A(S) \) is a polynomial of degree m. In Fig.4.8, from (4.3.1) and the above equations, we now have

\[ \frac{V_o^A}{V_i} = \frac{N^A(S) A(S)}{D(S)} \] \hspace{1cm} (4.4.6)

The object of the proposed extension is to remove the unwanted factor \( A(S) \) from the numerator of the voltage transmission. To do this we may:

(i) In Fig.4.4, feed the source V into another three-terminal RC passive network,

or (ii) In Fig.4.5, feed the source I into a second RC network B(Fig.4.13),
or (iii) Reverse the terminal A and B of Fig. 4.5 so that Y may be replaced by a three-terminal network B (Fig. 4.18).

Of these, (iii) has the advantage that the number of elements remains the same since $Y_1$ now serves two functions: first to give the required driving-point admittance between A and C, and second to remove the unwanted factor $A(S)$. But a large decrease in D.C. gain may result due to the resistive component in the shunt branch. (ii) has the advantage of controlling D.C. gain although the number of elements is increased due to the second RC network. (i) has neither increase in D.C. gain nor fewer elements, and so it is not discussed further.

4.4.1 Method (ii) (Fig. 4.13)

We may realize the wanted transfer function $N(S)/D(S)$ as

$$\frac{V_o^B}{V_i} = \frac{Y_1}{V_o^A} \cdot \frac{I}{V_o} \cdot \frac{V_o^B}{V_i}$$

Now

$$\frac{I}{V_o^A} = \frac{G_2 G_3}{Y_1 + G_2}, \text{ hence using (4.4.6) we have}$$

$$\frac{V_o^B}{V_i} = \frac{G_2 G_3}{Y_1 + G_2} \cdot \frac{N_o^A(S)}{D(S)} \cdot \frac{A(S)}{A(S)}$$

(4.4.7)
Fig. 4.13 Modification for allowing optimum RL-RC decomposition with controllable D.C. gain (method (ii))

We now show that \( A(S) \) is a factor of \( Y_1 + G_2 \):

by (4.2.3), we have

\[
Y = G_2 \cdot \frac{Y_1 + G_3}{Y_1 + G_2} \quad (4.4.8)
\]

so that the poles of \( Y \) are the zeros of \( Y_1 + G_2 \); hence from (4.4.4) the zeros of \( Y_1 + G_2 \) are the zeros of \( A(S) \), so we can write.
\[
\frac{Y_1 + G_2}{G_2 G_3} = \frac{A(S)}{C_0 C(S)} \quad \text{say} \quad (4.4.9)
\]

where \( C(S) \equiv \prod \frac{m}{1} \) \((S + c_i)\) and all \( c_i > 0 \) since \( Y_1 \) is RC passive. Thus \((4.4.7)\) and \((4.4.9)\) give
\[
\frac{V_o^B}{V_1} = z_{12}^B \cdot \frac{C_0 C(S) N^A(S)}{D(S)} \quad (4.4.10)
\]

and to complete the realization we must have
\[
z_{12}^B = \frac{N(S)}{C_0 C(S) N^A(S)} \quad (4.4.11)
\]

To minimize the order of network \( B \) we should choose \( N^A(S) \) to be a factor of \( N(S) \).

An important feature of this method is that we can vary the impedance level of \( z_{12}^B \) to obtain a specified D.C. gain for \( F(S) \).

**Example 4.2**

To realize the \( F(S) \) of Example 4.1, but with \( h = 1 \), as \( \frac{V_o^B}{V_1} \) in Fig.4.13. Using Calahan's optimum RC-RL decomposition, we take
\[
D(S) = 2(S^2 + \sqrt{3}S + 1)
\]
\[
= (S + 1.366)^2 + (S + 0.366)^2.
\]
and so \( A(S) = S + 0.366 \)
\( B(S) = S + 1.366 \)

By (4.4.4) \( Y = \frac{S + 1.366}{S + 0.366} \)

By (4.4.3) \( y^A_{22} = \frac{S + 0.366}{S + 1.366} \)

Hence \( -y^A_{12} = \frac{0.366}{S + 1.366} \)

so that \( N^A(S) = 0.366 \)

To realize \( Y \), we choose \( G_2 \) and \( G_3 \) according to (4.2.6) with \( Y(\infty) = 1 \) and \( Y(0) = 3.7 \), so we take \( G_2 = 0.5, G_3 = 10. \)

Then by (4.2.5)

\[
Y_1 = \frac{10(S + 0.366) - (S + 1.366)}{2(S + 1.366) - (S + 0.366)} = \frac{9S + 2.294}{S + 2.366}
\]

\[
= 0.9696 + \frac{8.030S}{S + 2.366}
\]

By (4.4.9) \( \frac{Y_1 + G_2}{G_2G_3} = \frac{(9S + 2.294) + 0.5(S + 2.366)}{5(S + 2.366)} \)

\[
= \frac{S + 0.366}{0.5263(S + 2.366)}
\]

and so \( C_0C(S) = 0.5263(S + 2.366) \)

By (4.4.11) \( z^B_{12} = \frac{2}{0.5263(S + 2.366) \times 0.366} \)
Fig. 4.14 Circuit for realizing the second-order Bessel filter with 1s delay using the configuration of Fig. 4.13

Fig. 4.15 Transistor circuit for realizing the second-order Bessel filter with 1ms delay using the configuration of Fig. 4.13
Fig. 4.16  Frequency response curves of Fig. 4.15 O and Fig. 4.20 O
Input: 3v r.m.s.
Fig. 4.17  Step responses of Figs. 4.15 and 4.20
Input step: 2v
which is an RC impedance chosen to make \( h = 1 \). The circuit for the denormalized transfer function is shown in Fig. 4.14.

As before, we use a common-base transistor amplifier and choose the passive elements within 3% tolerance (Fig. 4.15). The experimental results are shown in Figs. 4.16 and 4.17.

4.4.2 Method (iii) (Fig. 4.18)

We may realize the specified transfer function \( \frac{N(S)}{D(S)} \) as

\[
\frac{V_o^B}{V_i} = \frac{V_o^B}{V_i^B} \cdot \frac{V_i^B}{V_o^A} \cdot \frac{V_o^A}{V_i}
\]

Using (4.4.6) we have

\[
\frac{V_o^B}{V_i} = \frac{z_{12}^B}{z_{11}^B} \cdot \frac{z_{11}^B}{R_2 + z_{11}^B} \cdot \frac{N^A(S) A(S)}{D(S)} \quad (4.4.12)
\]

where \( z_{11}^B = \frac{1}{Y_1} \) and \( R_2 = \frac{1}{G_2} \).

From (4.4.9), the zeros of \( Y_1 + G_2 \) are those of \( A(S) \); we have
Fig. 4.18 Modification for allowing optimum RL-RC decomposition with elements (method (iii))

\[ Y_1 + G_2 = \frac{z_{11}^B + R_2}{z_{11}^B R_2} \]  \hspace{1cm} (4.4.13)

so they are also the zeros of \( z_{11}^B + R_2 \). Therefore, we can write

\[ \frac{z_{11}^B}{R^2 + z_{11}^B} = \frac{C(S)/E(S)}{A(S)/E(S)} \quad \text{say} \]  \hspace{1cm} (4.4.14)

where \( C(S) = \frac{m}{1} (S + c_i), \ E(S) = \frac{m}{1} (S + e_i) \) and all \( c_i > e_i > 0 \), since network B is RC passive. (4.4.12) and (4.4.14) give
\[
\frac{V_0^B}{V_1} = \frac{z_{12}^B}{z_{11}^B} \cdot \frac{C_o C(S) N^A(S)}{D(S)}
\]

and to complete the realization, we have

\[
\frac{z_{12}^B}{z_{11}^B} = \frac{N(S)/E(S)}{C_o C(S) N^A(S)/E(S)} \tag{4.4.15}
\]

To minimize the order of network B, \(N^A(S)\) is chosen to be a factor of \(N(S)\), and we let

\[
\frac{N(S)}{C_o N^A(S)} = N^B(S) \tag{4.4.16}
\]

Therefore we have

\[
z_{12}^B = \frac{N^B(S)}{E(S)}, \quad z_{11}^B = \frac{1}{Y_1} = \frac{C(S)}{E(S)} \tag{4.4.17}
\]

To satisfy Fialkow and Gerst's condition\(^{15}\), the coefficients of \(N^B(S)\) cannot be greater than the corresponding coefficients of \(C(S)\), and this limits the allowable D.C. gain of \(F(S)\).

Example 4.3

Following the previous Example 4.2, we obtain

\[
Y_1 = \frac{9S + 2.294}{S + 2.366} \quad \text{i.e.} \quad z_{11}^B = \frac{0.1111S + 0.2629}{S + 0.2549}
\]
Fig. 4.19 Circuit for realizing the second-order Bessel filter using the configuration of Fig. 4.18 with 1s delay.

\[ C_1 = 1.959\mu F \quad R_1 = 1.033\Omega \quad R_4 = 0.1245\Omega \]
\[ C_2 = 0.5787\mu F \quad R_2 = 2\Omega \quad R_5 = 2.717\Omega \]
\[ R_3 = 0.1\Omega \quad R_6 = 1.002\Omega \]

Fig. 4.20 Transistor circuit for realizing the second-order Bessel filter using the configuration of Fig. 4.18 with 1ms delay.

\[ C_1 = 0.4493\mu F \quad R_1 = 4.485k\Omega \quad R_4 = 0.5403k\Omega \]
\[ C_2 = 0.1332\mu F \quad R_2 = 8.685k\Omega \quad R_5 = 11.79k\Omega \]
\[ C = 50\mu F \quad R_3 = 0.4341k\Omega \quad R_6 = 4.338k\Omega \]
Therefore by Fialkow and Gerst condition,

\[ z_{12}^B = \frac{0.2629}{S + 0.2549} \]

whereas (4.4.16)

\[ N^B(S) = \frac{2}{0.4742 \times 0.366} = 11.54 \]

Hence the resultant D.C. gain is

\[ \frac{0.2629}{11.54} = 0.02278 \]

The circuit for the denormalized transfer function is shown in Fig. 4.19 and a circuit realizing this second-order Bessel filter with 1ms delay was built (Fig. 4.20) using a common-base transistor amplifier, and elements of 3% tolerance. The experimental results are shown in Figs. 4.16 and 4.17. Fig. 4.16 shows that Fig. 4.20 gives the better accuracy. Since network B of Fig. 4.20 used exactly the same components as Y_1 of Fig. 4.15 in the experiments, the additional error in the frequency response curves for Fig. 4.15 is thought to be due to non-exact cancellation of the common factor C(S) for Y_1 and Z_{12}^B.

The D.C. gain may be increased by decreasing G_3 and increasing G_2. If we choose G_3 = 5, G_2 = 0.8, then
by (4.2.5)

\[ Y_1 = \frac{5(S + 0.366) - (S + 1.366)}{1.25(S + 1.366) - (S + 0.366)} = \frac{4S + 0.464}{0.25S + 1.342} \]

i.e. \[ z_{11}^B = \frac{0.0625S + 0.3355}{S + 0.116} \]

and so \[ z_{12}^B = \frac{0.3355}{S + 0.116} \]

Hence the resultant D.C. gain in this case is

\[ \frac{0.7634 \times 0.366 \times 0.3355}{2} = 0.04687 \]

The maximum D.C. gain is limited to 0.06778 by the conditions \( G_3 \geq Y(0) = 3.7 \) and \( G_2 \leq Y(\infty) = 1 \).

4.5 Conclusions

Of the three methods of synthesis using the new method of realizing an RL admittance, each has its advantages:

A. The method of Section 4.3 can avoid the use of a floating power supply.

B. The method of Section 4.4.1 has controllable D.C. gain and the transfer function has minimum sensitivity to the variations of the active element.

C. The method of Section 4.4.2 has fewer elements and also the same minimum sensitivity as B, but it does not
control the D.C. gain.

They all suffer from the disadvantage that they cannot be cascaded directly without an isolating amplifier. Therefore, the methods would only be considered when the minimum sensitivity of a specified transfer function to the active element parameters is a major requirement.
CHAPTER 5

ACTIVE RC NETWORK SYNTHESIS USING THREE-TERMINAL NETWORKS AND ONE OPERATIONAL AMPLIFIER

5.1 Introduction

Ideally, an operational amplifier has an infinite input impedance, zero output impedance and infinite negative voltage gain. There must be direct or indirect feedback around the amplifier for stability. The infinite gain causes the input voltage to be at earth potential; this 'virtual earth' may then be used as a summing junction for currents.

Various methods have been described for synthesising transfer functions by operational amplifiers and RC networks. The usual approach is to adopt a particular configuration and then choose element values to realize the given function. In this chapter, a few configurations are studied via their short-circuit admittance functions; this reveals the constraints on the class of realizable transfer function which are imposed by the particular configuration.
Section 5.2 describes two circuits containing one amplifier and one three-terminal RC network; one or other (but not both) is capable of realizing any second-order transfer function. Section 5.3 describes two circuits containing one amplifier and two three-terminal networks.

Most of the circuits which can be derived are limiting cases as \( K \to -\infty \) of finite-gain voltage-amplifier circuits described in Chapter 3. The trans-position rule (Section 2.3) then gives the current analogues, and these are the circuits as described. These current analogues do not result from letting \( K \to -\infty \) in the finite-gain current-amplifier circuits. It does not of course follow that every finite-gain circuit has a useful operational analogue.

5.2 One RC network

5.2.1 Circuit derived from Section 3.3.1

By making \( K \to -\infty \), we obtain Fig.5.1. The current transfer ratio is given by

\[
\frac{I_0}{I_1} = \frac{-y_{12}}{k(y_{22} + y_{12})}
\]

(5.2.1)

where \( k \) is given as before by
Fig. 5.1 Operational analogue of Fig. 3.4

\[ I_e = \frac{1 - k}{k} I_f \]  

(3.3.2)

and \(0 < k < 1\).

Any second-order transfer function realized by (5.2.1) must be of type (ii), Section 3.6.

When \(k = 1\), i.e. \(I_e = C\), we obtain the same configuration by letting \(K \to -\infty\) in (3.3.12). Pande and Shukla have used this to realize second- and third-order transfer functions.

Aggarwal's circuit\(^2\), Fig. 5.2 is also a particular 'voltage' form of this network with \(k = 1\). The limitation on his circuit can be seen immediately from (5.2.1). Since the three-terminal network is a ladder network, \(Y_{12}\) can only have simple negative-real zeros.
Example 5.1

To realize a second-order band-pass function by Fig. 5.1.

\[ F(S) = \frac{-\varepsilon S}{S^2 + \alpha S + \beta} \]  \hspace{1cm} (5.2.2)

where \( \varepsilon = 2 \beta / \alpha \)

We let \(-y_{22} = \frac{\varepsilon}{S + \delta} \), \( k = 1 \)

and so \( y_{22} = \frac{S^2 + \alpha S + \beta}{S + \delta} + \frac{\varepsilon S}{S + \delta} \)

The zeros of \( y_{22} \) are negative real since

\( \alpha + \varepsilon > 0; \quad (\alpha + \varepsilon)^2 - 4 \beta = \alpha^2 + \varepsilon^2 > 0 \)

\( y_{22} \) is therefore RC-realizable if \( \delta \) is chosen to lie between them.
For compactness, we may choose $\sigma = \alpha$, then

$$y_{22} = S + \frac{\alpha S + \beta}{S + \alpha}$$  \hspace{1cm} (5.2.3)

and we have the network of Fig.5.3. This is the current analogue of a well-known circuit.

5.2.2 Application of direct feedback

This is a new approach to realizing second-order transfer functions. If we apply feedback from the output as shown in Fig.5.4, and use the operational amplifier as a summer, we have

$$I_0(1 + k) = VY_{12}$$  \hspace{1cm} (5.2.4)

$$I_1 - kI_0 = VY$$  \hspace{1cm} (5.2.5)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5_3}
\caption{Circuit realizing $\frac{I_0}{I_1} = \frac{\alpha S}{S^2 + \alpha S + \beta}$ using the configuration of Fig.5.1}
\end{figure}
Fig. 5.4 Circuit with direct feedback from output

Therefore \[ \frac{I_0}{I_1} = \frac{Y_{12}}{(1 + k)Y - ky_{12}} \] (5.2.6)

(5.2.6) can realize types (i) and (iii), Section 3.6, although \( Y \neq y_{22} \).

Example 5.2

Use (5.2.6) to realize

\[ F(S) = \frac{-\beta}{S^2 + \alpha S + \beta} \] (5.2.7)

We can put \( k = 1 \), and take

\[ -y_{12} = \frac{2\beta}{S + \alpha}, \quad Y = S \]

Then substituting into (5.2.6), we have
\[ R_1 = (2\beta)^{-\frac{1}{2}} \eta, \quad R_2 = \left(\frac{2}{\beta}\right)^{\frac{1}{2}} \eta, \quad C_1 = 1 \text{F}, \quad C_2 = \frac{2(2\beta)^{\frac{1}{2}}}{\alpha} \text{F} \]

Fig. 5.5 Circuit realizing (5.2.8) using the configuration of Fig. 5.4

\[
\frac{I_0}{I_1} = \frac{2\beta}{2\beta(S + \alpha) + 2\beta} = \frac{\beta}{S^2 + \alpha S + \beta} \quad (5.2.8)
\]

and the circuit is shown in Fig. 5.5. Again it is the current analogue of a well-known circuit.

5.3 Two RC networks

5.3.1 Circuit derived from Section 3.7

Letting \( K \rightarrow -\infty \), and converting to current amplification, gives Fig. 5.6. The current transfer ratio is

\[
\frac{I_0}{I_1} = \frac{-Y_{12}^B}{Y_{22}^B + Y_{12}^B - Y_{12}^A} \quad (5.3.1)
\]
Fig. 5.6. Operational analogue of the circuit, Fig. 3.7

Fig. 5.7. Operational analogue of the circuit, Fig. 3.8
This configuration seems to have no particular advantage over that of the next section.

5.3.2 Circuit derived from Section 3.8

As $K \to -\infty$, we obtain Fig. 5.7, with current transfer ratio

$$\frac{I_o}{I_i} = \frac{y_{12}^B}{y_{12}^A}$$

(5.3.2)

This is the conventional one-operational-amplifier method\textsuperscript{44,45}. (5.3.2) has the same form as (3.9.4), so the procedure in Section 3.9 can be applied here.
CHAPTER 6

SOME NEW METHODS OF REALIZING TRANSFER FUNCTIONS
ANY ORDER USING OPERATIONAL AMPLIFIERS

6.1 Introduction

In realizing high-order transfer functions, single-amplifier methods are unsatisfactory, partly because of
difficulty of synthesis and partly because of sensitivity
to the parameters of the active element. Therefore, the
usual approach is to cascade similar low-order sections.
But this is not always the best approach, particularly
when there are right-half-plane zeros, because it may
need an excessive number of passive elements. In this
chapter we start with a general circuit containing first-
order 'blocks' with feedforward and feedback. Then we
reduce the number of active elements by using second-
order blocks, and the number of passive elements by
omitting the feedback.

Morrill\textsuperscript{33} suggested a method of realizing an all-
pass function using operational amplifiers and RC elements
(Fig. 6.1). The output signal from each stage realizing
$F_i(s)$ is fed into the inputs of the following stages, and also to a summer. The transfer function is

$$\frac{x_o}{x_i} = \frac{n}{1} \{1 + F_i(S)\} \quad (6.1.1)$$

where $F_i(S) = \frac{-2\alpha_i S}{S^2 + \alpha_i S + \beta_i} \quad (6.1.2)$

and $\alpha_i$ and $\beta_i$ are positive constants corresponding to a pair of (generally) complex poles of $F_i(S)$.

Morrill adopted a simple 'differential-analyser' approach and needed three operational amplifiers to realize (6.1.2) in each stage. Later, Stubbs and Single improved Morrill's method by reducing the number of operational amplifiers to one, using a circuit similar to Fig. 5.3, for each stage.

The main disadvantage of their method is the large number of feedforward admittances. To reduce the number of passive elements, the first step is clearly to remove some of the feedforward paths. Some such systems are described in the following sections.

In section 6.2, we described a general feedforward and feedback system. First, we use a first-order transfer function in each stage, and an all-pass network
turns out to be particularly simple. Then we use a second-order transfer function in each stage to reduce the number of active elements.

The system is then simplified further to have feedforward only (section 6.3) the passive elements are reduced by using a second-order band-pass transfer function in each section, and also the number of feedforward admittances is halved when the transfer function to be realized has poles and zeros on a circle. An alternative method using feedforward suggested by Ream (private communication) is described in section 6.4.

![Diagram of Morrill's method with the output signal from each stage fed into the input of every subsequent and a summer](image-url)
Apart from the general feedforward and feedback system using first-order functions in each stage, the circuits in this chapter are believed to be original.

6.2. General feedforward and feedback system (Fig. 6.2)

Let $F(S)$ be the function to be realized and $F_i(S)$ the transfer function of the $i$th stage.

The numerator of $F(S)$ is controlled by feedforward and the denominator by feedback. We consider low-pass $F_i(S)$, (a) first-order and (b) second-order. The feedforward and feedback coefficients are obtained by equating the overall transfer function to $F(S)$.

In Fig. 6.2 the transfer function is

$$\frac{x_0}{x_i} = \frac{\sum_{r=0}^{n} k_r H_r(S)}{1 - \sum_{r=0}^{n} k_r H_r(S)} = F(S) \quad (6.2.1)$$

where $H_r(S) = F_{r+1}(S)F_{r+2}(S)\ldots F_n(S)$, $r = 0$ to $n-1$

$H_n(S) = 1 \quad (6.2.2)$

where the $k_r$ and $k'_r$ are respectively feedforward and feedback coefficients.

6.2.1 First-order transfer function (integration) in each stage

The simplest method is to put
Fig. 6.2 General feedforward and feedback system

\[ F_1(S) = \frac{1}{t_i S} \]

For general (positive and negative) values of \( k \) and \( k' \), two sign-reversing amplifiers may be required, one connected in the feedforward path and the other in the feedback path.

(a) **Realization of an all-pass function (Fig.6.3)**

If we put
\[ t_i = CR_i \]  
(6.2.3)
and make all \( k = G \), \( k' = (-1)^{n-r+1} G \), and \( k_n = 0 \),
the voltage transfer ratio is the all-pass function.
Fig. 6.3 An all-pass network with adjustable \( R_i \) for realizing the coefficients of an all-pass transfer function

\[
\frac{V_o}{V_i} = (-1)^{n+1} \frac{1 - T_1 S + T_2 S^2 - \cdots + T_n (-S)^n}{1 + T_1 S + T_2 S^2 + \cdots + T_n S^n}
\]

(6.2.4)

where \( T_r = t_1 t_2 \cdots t_r \)

Tomlinson\(^{64}\) has used the current analogue of Fig. 6.3, which requires one extra operational amplifier to convert output current back to voltage.
(b) **Realization of a low-pass function**

To realize a low-pass transfer function, all feed-forward coefficients are made zero except $k_0$.

### 6.2.2 Second-order transfer function in each stage

One way of reducing the number of amplifiers is to increase the order of $F_1(s)$, e.g.

$$F_1(s) = -\frac{\beta_1}{s^2 + \alpha_1 s + \beta_1} \quad (6.2.5)$$

where $\alpha_1$ and $\beta_1$ are arbitrary positive constants, and make

$$k_r = G_r + C_r S, \quad k'_r = G'_r + C'_r S, \quad r = 0 \text{ to } n - 1 \quad (6.2.6)$$

If the poles of $F_1(s)$ are real, $(6.2.5)$ can be realized by a passive three-terminal RC network; if complex, Fig. 5.5 may be used.

If $F(s)$ has $2n$ poles and $2n$ zeros, the values of the $G$'s and $C$'s are obtained as follows. Let

$$F(S) = \frac{N(S)}{D(S)} \quad (6.2.7)$$

and write

$$p_{2r}(S) = \frac{r}{1} (S^2 + \alpha_1 s + \beta_1) \quad r = 1 \text{ to } n \quad (6.2.8)$$

$$p_0(S) = 1$$
From (6.2.2), (6.2.5) and (6.2.8), we have
\[ H_r(S) = \frac{B_r p_{2r}(S)}{p_{2n}(S)}, \quad r = 0 \text{ to } n - 1 \quad (6.2.9) \]
where \( B_r = (-\beta_{r+1})(-\beta_{r+2})\ldots(-\beta_n) \)

Substituting (6.2.9) and (6.2.6) into (6.2.1), we have
\[
\begin{align*}
x_0 &= \frac{\sum_{r=0}^{n-1} (\gamma_r S + \delta_r) p_{2r}(S) + k_n p_{2n}(S)}{x_1 (1 - k'_n) p_{2n}(S) - \sum_{r=0}^{n-1} (\gamma'_r S + \delta'_r) p_{2r}(S)} \\
x_1 &= (1 - k'_n) p_{2n}(S) - \sum_{r=0}^{n-1} (\gamma'_r S + \delta'_r) p_{2r}(S) 
\end{align*}
\]
(6.2.10)
where \( \gamma_r = B_r C_r, \quad \delta_r = B_r G_r, \quad \gamma'_r = B_r C'_r, \) and \( \delta'_r = B_r G'_r \)
(6.2.11)

Since numerator and denominator of (6.2.10) are of degree 2n, we set
\[ k_n = G_n, \quad k'_n = G'_n \quad (6.2.12) \]

Equating the numerators of (6.2.7) and (6.2.10) gives
\[ N(S) = \sum_{r=0}^{n-1} (\gamma_r S + \delta_r) p_{2r}(S) + k_n p_{2n}(S) \quad (6.2.13) \]

and by equating coefficients of powers of S, we obtain a set of \( n + 1 \) linear equations. This results in a triangular matrix and the values of \( \gamma_r \) and \( \delta_r \) can be
obtained successively (see Example 6.1). A similar procedure is applied to find \( \gamma'_r \) and \( \delta'_r \).

The advantage of this method over the later ones is that there is no need to factorize high-order polynomials. However, it requires more elements.

Example 6.1

To realize the fourth-order all-pass Padé delay

\[
F(S) = \frac{S^4 - 20S^3 + 180S^2 - 840S + 1680}{S^4 + 20S^3 + 180S^2 + 840S + 1680}
\]

We choose \( \alpha_1 = \alpha_2 = 2 \)
\( \beta_1 = \beta_2 = 1 \)

therefore \( p_2(S) = S^2 + 2S + 1 \)
\( p_4(S) = S^4 + 4S^3 + 6S^2 + 4S + 1 \)

Using (6.2.13), we have

\[
S^4 - 20S^3 + 180S^2 - 840S + 1680 = \gamma_0S + \delta_0 + (\gamma_1S + \delta_1)(S^2 + 2S + 1) + k_2(S^4 + 4S^3 + 6S^2 + 4S + 1)
\]

i.e. \( k_2 = 1 \)

\[
4k_2 + \gamma_1 = -20
\]
\[6k_2 + 2\gamma_1 + \delta_1 = 180
\]
\[4k_2 + \gamma_1 + 2\delta_1 + \gamma_0 = -840
\]
\[k_2 + \delta_1 + \delta_0 = 1680
\]
Fig. 6.4 Network realizing the fourth-order Pade delay
(Example 6.1), by the method of Section 6.2.2

\[ G_0 = 1457 \mu \text{F} \quad C_0 = 1264 \mu \text{F} \quad R_1 = 0.7071 \Omega \]
\[ G_1 = 222 \mu \text{F} \quad C_1 = 24 \mu \text{F} \quad R_2 = 1.414 \Omega \]
\[ G_2 = 1 \mu \text{F} \quad C_2 = 1 \mu \text{F} \]
\[ G'_0 = 1537 \mu \text{F} \quad C_3 = 1.414 \mu \text{F} \]
\[ G'_1 = 142 \mu \text{F} \quad C'_0 = 1144 \mu \text{F} \]
\[ G = 1 \mu \text{F} \quad C'_1 = 16 \mu \text{F} \]
These give

\[ k_2 = 1 \quad \quad G_2 = 1 \]
\[ \gamma_1' = -24 \quad \delta_1 = 222 \quad C_1 = 24 \quad G_1 = -222 \]
\[ \gamma_0' = -1264 \quad \delta_0 = 1457 \quad C_0 = -1264 \quad G_0 = 1457 \]

Similarly, the feedback coefficients are

\[ k_2' = 0 \quad \quad G_2' = 0 \]
\[ \gamma_1' = -16 \quad \delta_1' = -142 \quad C_1' = 16 \quad G_1' = 142 \]
\[ \gamma_0' = 1144 \quad \delta_0' = -1537 \quad C_0' = 1144 \quad G_0' = -1537 \]

Negative coefficients of \( C \) and \( G \) mean that a sign-reversing amplifier is required. The circuit realizing this transfer function as \( -V_o/V_i \) is shown in Fig. 6.4.

This \( F(S) \) does not show this method to advantage, because Fig. 6.3 requires only one more operational amplifier and has fewer passive elements. However, the example has shown the general idea and the method will be more advantageous with a high-order transfer function.

6.3 Systems with feedforward only

6.3.1 With second-order low-pass transfer function in each stage (Fig. 6.5)

Since the \( \alpha_i \) and \( \beta_i \) in (6.2.5) are arbitrary, we can choose them so that \( S^2 + \alpha_i S + \beta_i \) is a factor of
the denominator of the specified transfer function, i.e. 
p_{2n}(S) = D(S), so the feedback paths are not required. 
Then (6.2.1) becomes

\[
x_0 = \sum_{k=0}^{n} k_{r} H_{r}(S) = F(S) \quad (6.3.1)
\]

\( F_1(S) \) can be realized by Fig. 5.5. In this case 
(6.2.10) - (6.2.12) are used, with all dashed coefficients 
zero.

Example 6.2

To realize the sixth-order all-pass Padé delay

\[
F(S) = \frac{S^6 - 42S^5 + 840S^4 - 10080S^3 + 75600S^2 - 332640S + 665280}{S^6 + 42S^5 + 840S^4 + 10080S^3 + 75600S^2 + 332640S + 665280}
\]

Fig. 6.5 System with feedforward only
The denominator factorized as

\[ D(S) = (S^2 + 16.99S + 75.20)(S^2 + 14.94S + 83.41)(S^2 + 10.06S + 106.05) \]

We choose: \( \beta_3 = 106.05, \beta_2 = 83.41, \beta_1 = 75.20 \)

Then \( p_6(S) = S^6 + 42S^5 + 840S^4 + 10080S^3 + 75600S^2 + 332640S + 665280 \)

\[ p_4(S) = S^4 + 31.9S^3 + 411.8S^2 + 2555S + 6270 \]

\[ p_2(S) = S^2 + 16.99S + 75.20 \]

Substituting \( p_{2r}(S) \) into (6.2.10) and equating the numerator coefficients, we have

\[ k_3 = 1 \]
\[ 42k_3 + \gamma_2 = -42 \]
\[ 840k_3 + 31.9 \gamma_2 + \delta_2 = 840 \]
\[ 10080k_3 + 411.8 \gamma_2 + 31.9 \delta_2 + \gamma_1 = -10080 \]
\[ 75600k_3 + 2555 \gamma_2 + 411.8 \delta_2 + 16.99 \gamma_1 + \delta_1 = 75600 \]
\[ 332640k_3 + 6270 \gamma_2 + 2555 \delta_2 + 75.20 \gamma_1 + 16.99 \delta_1 + \gamma_0 = -332640 \]
\[ 665280k_3 + 6270 \delta_2 + 75.20 \delta_1 + \delta_0 = 665280 \]

These give

\[ k_3 = 1 \quad G_3 = 1 \]
\[ \gamma_2 = -84 \quad \delta_2 = 2,680 \quad G_2 = 0.7920 \quad G_2 = -25.27 \]
\[ \gamma_1 = -70,981 \quad \delta_1 = 317,700 \quad G_1 = -8.023 \quad G_1 = 33.91 \]
\[ \gamma_0 = 7,049,000 \quad \delta_0 = -40,690,000 \quad c_0 = 10.60 \quad G_0 = 61.17 \]
The circuit realizing this transfer function is shown in Fig. 6.6.

6.3.2 With second-order band-pass transfer function in each stage

To reduce the number of the passive elements further, we can use a simpler circuit to realize $F_i(S)$. Fig. 5.3 has one element fewer than Fig. 5.5 and gives

$$F_i(S) = \frac{\xi_i S}{S^2 + \alpha_i S + \beta_i} \quad (6.3.2)$$

where the poles of $F_i(S)$ can be complex.

Due to the $S$ term in the numerator, the set of $n+1$ linear equations will be in a slightly different form, but the procedure of obtaining the feedforward coefficients is basically the same.

Using (6.2.8) and (6.3.2), we have

$$H_r(S) = \frac{p_{2r}(S)}{p_{2n}(S)} E_r S^{n-r} \quad (6.3.3)$$

where $E_r = (-\xi_{r+1})(-\xi_{r+2}) \ldots (-\xi_n)$

(6.2.10) becomes

$$\frac{x_o}{x_i} = \frac{1}{p_{2n}(S)} \left\{ \sum_{0}^{n-1} \left( \gamma_r S + \delta_r \right) S^{n-r} p_{2r}(S) + k_n p_{2n}(S) \right\} \quad (6.3.4)$$
Fig. 6.6 Network realizing the sixth-order Padé delay
(Example 6.2), by the method of Section 6.3.1
where \( \gamma_r S + d_r = E_r (c_r S + g_r), \ r = 1 \) to \( n-1 \), \( k_n = g_n \)

\[ (6.3.5) \]

**Example 6.3**

To realize the sixth-order all-pass Padé delay as before. The equations are now

\[
\begin{align*}
k_j + \gamma_2 & = 1 \\
42k_j + 31.9 \gamma_2 + d_2 + \gamma_1 & = -42 \\
840k_j + 411.8 \gamma_2 + 31.9 d_2 + 16.99 \gamma_1 + d_1 + \gamma_0 & = 840 \\
10080k_j + 2555 \gamma_2 + 411.8 d_2 + 75.20 \gamma_1 + 16.99 d_1 + d_0 & = -10080 \\
75600k_j + 6270 \gamma_2 + 2555 d_2 & = 75600 \\
332640k_j & = 665280 \end{align*}
\]

These give

\[
\begin{align*}k_3 & = 1 \\
\gamma_2 & = 0 \\
\gamma_1 & = 22.05 \\
\gamma_0 & = -571.5 \\
d_2 & = -106.1 \\
d_1 & = 3584 \\
d_0 & = -38970 \\
G_3 & = 1 \\
G_2 & = 5.031 \\
G_1 & = 15.23 \\
G_0 & = 18.72
\end{align*}
\]
Fig. 6.7 Network realizing the sixth-order Padé delay (Example 6.3), by the method of Section 6.3.2

\[ G_0 = 18.72 \mu \text{V} \]
\[ G_1 = 15.23 \mu \text{V} \]
\[ G_2 = 5.031 \mu \text{V} \]
\[ G_3 = 1 \mu \text{V} \]
\[ G = 1 \mu \text{V} \]

\[ C_0 = 0.2744 \text{F} \]
\[ C_1 = 0.09373 \text{F} \]
\[ C_2 = 0.1356 \text{F} \]
\[ C_3 = 0.08957 \text{F} \]
\[ C_4 = 0.04745 \text{F} \]
\[ C_5 = 0.1177 \text{F} \]
\[ C_6 = 0.1338 \text{F} \]
\[ C_7 = 0.1987 \text{F} \]

\[ R_1 = 1 \Omega \]
\[
R_0 = 50,12\, \text{M} \\
R_1 = 5.639\, \text{M} \\
R_2 = 1\, \text{M} \\
R = 1\, \text{M} \\
C_0 = 0.01121\, \mu\text{F} \\
C_1 = 0.07769\, \mu\text{F} \\
C_2 = 0.006600\, \mu\text{F} \\
C_3 = 1.381\, \mu\text{F} \\
C_4 = 1.901\, \mu\text{F}
\]

Fig. 6.8 Network realizing the fourth-order phase approximation function, with \( \lambda = 0.125 \) and 0.1s delay, by the method of Section 6.3.2
Fig. 6.9 Frequency response curves of Fig. 6.8
Input: 5v r.m.s.
Fig. 6.10  Step response of Fig. 6.8
Input step: 20v
In this case all feedforward coefficients turn out to have positive signs, so no extra sign-reversing amplifier is required in the feedforward paths (Fig. 6.7). The results for all-pass Padé delays of even orders up to 20 are given in Table 6.1. The zeros of the transfer functions must be taken in the given order, otherwise sign-reversing amplifiers are required.

Table 6.2 shows the results when the method is applied to a class of all-pass transfer functions whose poles and zeros are very near to the \( j\omega \)-axis (\( \omega = 0.125 \), Section 1.2.2). Here, sign-reversing amplifiers are needed for \( n \geq 3 \), in wherever order the zeros are taken.

A circuit realizing such a function with \( n = 2 \) and 0.1s delay (Fig. 6.8) was built with 3% tolerance elements. The frequency response curves are shown in Fig. 6.9 and the step response in Fig. 6.10. The gain response shows maximum deviations nearly at the maxima of the group delay (compare Fig. 3.9).

6.3.3 Simplification of the method of Section 6.3.2 when \( F(S) \) has 2n poles and 2n zeros on a circle

The radius of the circle can be normalised to unity. In this case \( F(S) \) is reciprocal, i.e.
\[ F(S) = F\left(\frac{1}{S}\right) \]  \hspace{1cm} (6.3.6)

Also \[ D(S) = S^{2n} D\left(\frac{1}{S}\right), \quad N(S) = S^{2n} N\left(\frac{1}{S}\right) \]  \hspace{1cm} (6.3.7)

Each quadratic factor of \( F(S) \) is a factor of \( F\left(\frac{1}{S}\right) \), hence

\[ p_{2r}(S) = S^{2r} p_{2r}\left(\frac{1}{S}\right), \quad r = 1 \text{ to } n \]  \hspace{1cm} (6.3.8)

If \( F(S) \) is given by \( x_0/x_1 \) in (6.3.4) then replacing \( S \) by \( \frac{1}{S} \) and using (6.3.8) gives

\[ F\left(\frac{1}{S}\right) = \frac{n-1}{S} \left( \gamma_r S + \gamma_r^* \right) S^{n-r} p_{2r}(S) + k_n p_{2n}(S) \]

\hspace{1cm} p_{2n}(S) \hspace{1cm} (6.3.9)

From (6.3.4), (6.3.6) and (6.3.9) we have immediately

\[ \gamma_r = 0, \quad \text{all } r \]  \hspace{1cm} (6.3.10)

Therefore for realizing such a \( F(S) \) the method of Section 6.3.2 results in feedforward admittances consisting of conductance only.

**Note:** \( F(S) \) need not be all-pass, though in practice it is unlikely to be chosen otherwise.

**Example 6.4**

To realize a sixth-order all-pass function having poles and zeros all on the unit circle:
Fig. 6.11  Network realizing (6.3.11) (Example 6.4)
\[ F(S) = \frac{(S^2 - 0.5S + 1)(S^2 - S + 1)(S^2 - 1.5S + 1)}{(S^2 + 0.5S + 1)(S^2 + S + 1)(S^2 + 1.5S + 1)} \]

(6.3.11)

Since there is no feedback, we have
\[ \alpha_1 = 0.5, \quad \alpha_2 = 1, \quad \alpha_3 = 1.5; \quad \beta_1 = \beta_2 = \beta_3 = 1 \]

\[ p_6(S) = S^6 + 3S^5 + 5.75S^4 + 6.75S^3 + 5.75S^2 + 3S + 1 \]

\[ p_4(S) = S^4 + 1.5S^3 + 2.5S^2 + 1.5S + 1 \]

\[ p_2(S) = S^2 + 0.5S + 1 \]

Substituting \( p_2(S) \) into (6.3.4) and equating the numerators and using (6.3.10), we obtain

\[ k_3 = 1 \]

\[ 3k_3 + \delta_2 = -3 \]

\[ 5.75k_3 + 1.5 \delta_2 + \delta_1 = 5.75 \]

\[ 6.75k_3 + 2.5 \delta_2 + 0.5 \delta_1 + \delta_0 = -6.75 \]

These give

\[ k_3 = 1; \quad \delta_2 = -6, \quad \delta_1 = 9, \quad \delta_0 = -2.5 \]

and so

\[ G_3 = 1, \quad G_2 = 6, \quad G_1 = 9, \quad G_0 = 2.5 \]

The complete circuit is shown in Fig. 6.11.

6.4 Alternative method using feedforward

We now introduce a slightly different type of circuit which contains only conductances in the feedforward paths but uses a more complicated \( F_1(S) \) given by (6.4.2). This
method involves finding a real root of a possibly high-order polynomial in each cycle.

The method was suggested by Ream (private communication) and used the configuration of Fig. 6.5, but (6.3.1) is written as

\[
F(S) = \frac{N(S)}{D(S)} = k_n + F_n(S) \left\{ k_{n-1} + F_{n-1}(S) \left[ k_{n-2} + \ldots \right. \right. \\
\left. \left. + F_2(S)(k_1 + k_0 F_1(S)) \right] \right\} \quad (6.4.1)
\]

We take

\[
F_i(S) = \frac{-\xi_i(S + \eta_i)}{S^2 + \alpha_i S + \beta_i} \quad (6.4.2)
\]

where the denominator is a factor of D(S). The decomposition of F(S) in the form of (6.4.1) is shown in Example 6.5. In each cycle, a root \( -\eta_i \) of an odd-degree polynomial has to be found. If \( \xi_i > 0 \) and \( \eta_i > 0 \), \( F_i(S) \) can be realized by several configurations suggested in Chapter 5, such as Figs. 5.1, 5.6 and 5.7. The requirement of positive \( \xi_i \) and \( \eta_i \) limits the application of this method, though all-pass Padé delays up to order twenty can be realized (Table 6.3).

F(S) has poles and zeros on a circle

We have shown that when F(S) has poles and zeros
on a circle, the circuit of Section 6.3.2 has only conductances in the feedforward paths, as in Fig. 6.11, so it is the same type as the circuit just discussed. Hence the two types of $F_1(S)$ must be of the same form i.e. $\eta_1 = 0$. Stubbs and Single's network will also reduce to the same type of circuit.

This confirms a result derived otherwise by Ream (private communication) who has also shown that the $\zeta_1$ are positive if such an $F(S)$ has all its zeros in the right half plane.

**Example 6.5**

To realize the sixth-order all-pass Padé delay as before (Example 6.2). We choose $\alpha_1$ to $\beta_3$ as they appear in Table 6.1, i.e. $\alpha_1 = 10.0637$ etc.

Therefore

$$p_6(S) = S^6 + 42S^5 + 840S^4 + 10080S^3 + 7560S^2 + 33264S + 66528$$

$$p_4(S) = S^4 + 25S^3 + 339.85S^2 + 2424S + 8846.3$$

$$p_2(S) = S^2 + 10.064S + 106.06$$

**First cycle:**

Take $k_3 = 1$

$$p_6(S) - p_6(-S) = -2(42S^5 + 10080S^3 + 33264S)$$

$$= -84S(S^4 + 240S^2 + 7920)$$

$$= -84S q_4(S) \text{ say}.$$
\[ \therefore \gamma_3 = 84, \quad \delta_3 = 0 \]

Second cycle:

Take \( k_2 = 1 \)

\[
p_4(S) - q_4(S) = -(25S^3 + 99.85S^2 + 2424S + 926.3) \\
= -25(S^3 + 3.993S^2 + 96.94 + 37.04) \\
= -25(S + 0.3877)(S^2 + 3.605S^2 + 95.54) \\
= -25(S + 0.3877)q_2(S) \quad \text{say} \]

\[ \therefore \gamma_2 = 25, \quad \delta_2 = 0.3877 \]

Third cycle:

Take \( k_1 = 1 \)

\[
p_2(S) - q_2(S) = -6.459(S + 1.628) \\
\text{Take } k_0 = 0.1, \quad \gamma_1 = 64.59 \text{ and } \delta_1 = 1.628 \]

(\( k_0 \) is chosen to be 0.1 in order that \( F_1(S) \) can have a larger constant multiplier and be realized by a simple network.)

\[ F_1(S), \ F_2(S) \text{ and } F_3(S) \text{ are realized by Fig.5.1.} \]

The complete circuit is shown in Fig.6.12. The results for all-pass Padé delays of order 4(2)20 are given in Table 6.3. Zeros are chosen in such order that no sign-reversing amplifiers are required. When the poles and zeros are near the \( j\omega \)-axis, e.g. for the function of
Fig. 6.12 Network realizing the sixth-order Padé delay (Example 6.5), by the method of Section 6.4.
Fig. 6.13  Network realizing the fourth-order one-second fade delay by the method of Section 6.4

\begin{align*}
R_0 &= R_1 = R_2 = 1\, \text{Ma} & \quad & C_1 = 0.0295\, \text{mF} \\
R_3 &= 1.831\, \text{Ma} & \quad & C_2 = 0.02668\, \text{mF} \\
R_4 &= 10.94\, \text{Ma} & \quad & C_3 = 0.1\, \text{mF} \\
R_5 &= 4.024\, \text{Ma} \\
R_6 &= 0.9371\, \text{Ma} \\
R_7 &= 21.30\, \text{Ma} \\
R &= 1\, \text{Ma}
\end{align*}
Fig. 6.14 Frequency response curves of Fig. 6.13
Input: 5v r.m.s.
Fig. 6.15  Step response of Fig. 6.13
Input step: 20v
Fig. 6.16 Network realizing the tenth-order one-second Padé delay by the method of Section 6.4
Fig. 6.17 Frequency response curves of Fig. 6.16
Input: 5V r.m.s.
Fig. 6.18  Step response of Fig. 6.16
Input step: 20v
Table 6.2, negative $\gamma_i$ appear. Therefore no data for this function are given.

Circuits realizing the fourth-order and tenth-order all-pass Padé delays with 1s delay were built with 3% tolerance elements (6.13 and 6.16). The frequency response curves are shown in Figs. 6.14 and 6.17 respectively, and the step response in Figs. 6.15 and 6.18.

6.5 Conclusions

Although the method of Section 6.2.2 has the advantage of avoiding factorization, it results in too many elements and also a wide spread of values. The latter disadvantage also applies to the method of Section 6.3.1. The method using a band-pass $F_1(S)$, Section 6.3.2, seems to be the best method with regard to number of elements and of values, mathematical procedure, and also there is no limitation on realizability. The method of Section 6.4 involves root-finding and is unrealizable when a positive root is encountered.

Operational amplifiers are used here merely for convenience of illustration; each method can also be realized using other kinds of amplifiers, e.g. $F_1(S)$ can be realized by Sections 3.3, 3.5, 3.7 and 3.8, which can be cascaded directly.
Table 6.1 and 6.2 Parameters for all-pass delays using the method of Section 6.3.2

The 2nth-order all-pass transfer function is

\[ F(S) = \frac{\prod_{i=0}^{n}(s^2 - \alpha_r s + \beta_r)}{\prod_{i=0}^{n}(s^2 + \alpha_r s + \beta_r)} \]

The transfer function of the rth stage is

\[ F_r(S) = \frac{-\varepsilon_r S}{s^2 + \alpha_r S + \beta_r} \quad r = 1 \text{ to } n \]

where \( \varepsilon_r = 2\beta_r / \alpha_r \), and the rth pair of feedforward coefficients are given by \( \gamma_r S + \delta_r \) with \( \gamma_0 = 0, \delta_0 = 1 \).

Entries are in floating point, i.e. \( a \cdot 10^b \)

Table 6.1 Padé delays of order 4(2)20

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Table 6.3 Parameters for all-pass Padé delays of order 4(2)20 using the method of Section 6.4

The 2nth-order all-pass transfer function is

\[ F(S) = \frac{1}{\prod_{i=1}^{n} (S^2 - \alpha_i S + \beta_i)} \]

\[ \prod_{i=1}^{n} (S^2 + \alpha_i S + \beta_i) \]

The transfer function of the ith stage is

\[ F_i(S) = \frac{-\xi_i(S + \eta_i)}{S^2 + \alpha_i S + \beta_i}, \quad i = 1 \text{ to } n \]

and all the feedforward coefficients are taken to be unity.

Entries are in floating point,

i.e. \[ a \cdot 10^b \]

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CHAPTER 7
SYNTHESIS USING TWO OPERATIONAL AMPLIFIERS

7.1 Introduction

The main disadvantage of the methods using one operational amplifier (Chapter 5) is the restriction to negative feedback, and hence transmission zeros should preferably be in the left half S-plane to avoid excessive numbers of elements. If there is positive feedback or negative feedforward, this difficulty may be removed. An additional operational amplifier may be introduced in such networks to produce the extra negative sign.

Lovering\textsuperscript{29} introduced an elegant two-operational amplifier method. His method can derived by simplifying Matthews and Seifert's\textsuperscript{31a} three-operational-amplifier method, which used two amplifiers for sign-reversing.

In this chapter, we introduce two new methods using two operational amplifiers. The first\textsuperscript{*} is an application

\textsuperscript{*} This method was discovered independently by Paul\textsuperscript{46}.
of Millman's theorem\textsuperscript{8a} (nodal voltage equation for a number of parallel branches). The second is an RL admittance realization derived from Section 4.2.

7.2 Application of Millman's theorem

Millman's theorem\textsuperscript{8a} states that the output voltage $V_0$ in Fig. 7.1 is given by

$$V_0 \sum_{i}^{n} y_r = \sum_{i}^{n} y_r e_r$$  \hspace{1cm} (7.2.1)

where the $e_r$ are voltage sources and the $y_r$ are admittances.

![Fig. 7.1: A number of parallel branches, each consisting of an admittance and a voltage source](image)

If $n = 3$, and

$$e_1 = V_i, \quad e_2 = -\frac{Y_2}{G}V_i + K \frac{Y_3}{G}V_o, \quad e_3 = 0$$  \hspace{1cm} (7.2.2)

$$y_1 = Y_1, \quad y_2 = y_3 = G,$$
then we have

\[ V_o = \frac{Y_1 V_1 - Y_2 V_1 + KY_3 V_o}{Y_1 + 2G} \]

i.e. \[ \frac{V_o}{V_1} = \frac{Y_1 - Y_2}{Y_1 + 2G + (1-K)Y_3} \]  \hspace{1cm} (7.2.3)

If we take \( K = 2 \), and write

\[ Y_3 = 2G + Y_4 \]  \hspace{1cm} (7.2.4)

(7.2.3) becomes the useful form

\[ \frac{V_o}{V_1} = \frac{Y_1 - Y_2}{Y_1 - Y_4} \]  \hspace{1cm} (7.2.5)

\[ \begin{array}{c}
\text{Fig. 7.2 Three-amplifier circuit realizing (7.2.2)}
\end{array} \]

Fig. 7.2 shows a direct realization of (7.2.2) using operational amplifiers.
Since \( Y_2 \) and \( Y_3 \) feed currents to the inputs of the two amplifiers having the same output terminal, one of these amplifiers may be removed by connecting \( Y_3 \) to point \( A \). The resulting circuit with \( K = 2 \) and \( Y_3 \) given by (7.2.4), is shown in Fig. 7.3.

![Two-amplifier circuit realizing (7.2.5)](image)

**Fig. 7.3 Two-amplifier circuit realizing (7.2.5)**

If the output voltage is taken from \( C \) in Fig. 7.3, this section can be directly cascaded with a similar section, and (7.2.5) is replaced by

\[
\frac{V_0}{V_1} = -2 \frac{Y_1 - Y_2}{Y_1 - Y_4} \quad (7.2.6)
\]

**Example 7.1**

To realize the second-order all-pass Padé delay,

\[
F(S) = \frac{S^2 - 6S + 12}{S^2 + 6S + 12} \quad (7.2.7)
\]
as $V_o/V_i$ using Fig. 7.3

We may write \( F(S) \) as

\[
\frac{V_o}{V_i} = \frac{S + 3.4641 - \frac{12.93S}{S+3.4641}}{S + 3.4641 - \frac{0.9282S}{S+3.4641}}
\]

using Horowitz decomposition\(^{21a}\), and take

\[
Y_1 = S + 3.4641, \quad Y_2 = \frac{12.93S}{S+3.4641}, \quad Y_4 = \frac{0.9282S}{S+3.4641}
\]

The complete circuit is shown in Fig. 7.4.

---

**Fig. 7.4 Circuit realizing (7.2.7) in the form of (7.3.1) of Fig. 7.3**

### 7.3 Realization of a RL driving-point admittance using two operational amplifiers

In this section, we introduce two new methods of realizing a RL driving-point admittance using two operational
amplifiers. This is an extension of Section 4.2. Having realized the RL admittance, the synthesis procedure given in Section 4.3 can be applied immediately.

The first method (Section 7.3.1) uses the principle of a balanced bridge and has the same disadvantages as given in Section 4.3: it cannot be directly cascaded and there is no control in D.C. gain. The second method (Section 7.3.2) without these disadvantages, is a transformation of Fig. 4.6 by means of a Thevenin voltage generator.

7.3.1 By means of a balanced bridge with a common earth for the amplifiers (Fig. 7.5)

The high-gain amplifier of Fig. 4.3 suggested by Stuart and Lampard\textsuperscript{55} does not have a common earth for input and output. We now introduce an alternative way of amplifying the potential across CD, using two operational amplifiers having the same common earth.

To obtain $V_C - V_D$, we first obtain $-V_D$, by means of a sign-reversing amplifier, and since E is at virtual earth potential, we have
\[ -V_D G_1' + V_C G_2' = 0 \]  
(7.3.1)

If \( G_1' = G_2' \), then \( V_D = V_C \)  
(7.3.2)

Therefore the balance condition is again obtained.

![Diagram of balanced bridge using two operational amplifiers](image)

**Fig. 7.5** Balanced bridge using two operational amplifiers

### 7.3.2 Method derived from Fig. 4.6 (Fig. 7.6)

If we replace the current source of Fig. 4.6 by an equivalent Thevenin voltage generator, the source is now a current-controlled voltage source with a common earth. This may be realised by two operational amplifiers with appropriate feedback (Fig. 7.7).
Fig. 7.6 Fig. 4.6 with the current source replaced by a Thevenin voltage generator

Fig. 7.7 Fig. 7.6 with the current-controlled voltage source realized by two operational amplifiers

The network may be cascaded with a RC three-terminal network as described in Section 4.3. If we take the output voltage from F instead of A, the section can be cascaded directly with similar sections, and the D.C. gain is controlled by $G_3/G$ where $G$ can have any desired value.
CONCLUSIONS

In general, to realize a high-order delay transfer function, direct cascading of simple low-order sections (Chapters 3 and 5) should be used. However, when there are transmission zeros in the right half S-plane, the system with the additional feedforward paths (Chapter 6) is preferable in order to avoid excessive numbers of passive elements. For all-pass networks, the amplitude/frequency responses are sensitive at nearly the frequencies where maximum group-delays occur, specially where the poles and zeros are near to $j\omega$-axis.

The new method of using an RL admittance (Chapter 4) can give the minimum sensitivity of the denominator coefficients of a transfer function to variations in the active element parameters. The realization of an RL admittance can also be achieved by using two operational amplifiers (Chapter 7).

The transposition properties described in Chapter 2 give an important facility in active network synthesis, and they have been proved and applied in the synthesis procedures.
Suggestions for further investigations

Chapter 2 shows that adding a controlled source to a reciprocal network is effectively applying operators to a nodal admittance matrix. Therefore, to realize a transfer function we can first write the transfer function as a ratio of two determinants and then transform them to a matrix satisfying passive network properties by introducing operators. This is a possible method of RC active network synthesis which is worth looking into.

Sensitivity of a transfer function to active element parameter variations is an important problem in RC active network synthesis. The author has taken for granted the performance of stable common-base transistor amplifiers. In practice, it is important to investigate what tolerances are allowable for active and passive elements of a network to give responses within specified limits, and as yet there is no satisfactory method available in the literature.
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