THE EFFECT OF AN ELEMENT HAVING A LOGARITHMIC EXPONENTIAL CHARACTERISTIC ON THE PERFORMANCE OF A PROCESS CONTROL SYSTEM

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by

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Abstract

Regulating units in process control are mostly control valves, these usually have either linear or exponential characteristic. The effect of these control valves in comparison with linear valves is investigated.

The loop chosen for investigation consists of a 3-stage RC network representing the process, a 3-term controller and a regulating unit which is considered either linear or exponential.

The characteristic of the exponential regulating unit being a non-linear one results in closed loop equations that are non-linear differential equations. To solve these equations a numerical method which is a modification of that used by Tustin is used.

The closed loop response to step changes in the desired value and load for both regulating units and under various control modes is calculated. The results obtained from the numerical solutions of the linear equations is checked by solutions using the Laplace Transform method. All results obtained from the solution of the linear and non-linear equations are checked experimentally by a simulator built especially for this purpose.

Then the closed loop response to a step change in the desired value is calculated in the presence of a distance-
Velocity lag in the loop for both regulating units using the proportional control mode. Also the closed loop response considering saturation of the regulating units in cases of a large change in the desired value and when derivative control action is present is calculated.

The results proved that an exponential regulating unit is inferior to a linear one in the case of desired value changes and is not markedly superior in case of load changes.

The work is concluded by commenting on the numerical method used in the solution of the differential equations involved. It proved to be a powerful tool in dealing with non-linear control system in particular and with non-linear differential equations in general.
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Summary:

In this introductory chapter process control systems are considered in general. It is shown that the practice of using semi-logarithmic valves as regulating units gives rise to non-linear differential equations for the closed-loop response. This leads to the consideration of known analytical, numerical and graphical methods of solution of non-linear differential equations.

1.1 Process Control Systems in General

A single control loop in process control systems usually comprises the elements shown in the block diagram of figure (1.1). The controlled condition is detected, measured and compared with the desired value. The resulting deviation applied to the control action generator produces a corrective control action which through the regulating unit changes the input to the process, thus affecting the controlled condition.

For the purpose of theoretical analysis the detecting and measuring elements are usually considered as part of the process, thus the block diagram reduces to that of fig. (1.2), which consists of three main parts,
FIG. 1.1 Block Diagram of a Single Loop Control System.
viz. the plant or process, the control action generator and the regulating unit.

The Plant or Process:

While it is difficult to frame a concise definition of the word "process", its meaning is well understood. Usually the words "plant" and "process" are used synonymously, but we will define "plant" as the equipment with which the process is carried out.

It is customary in considering the theoretical analysis of a plant or process to split it up into parts which produce time lags, and to determine the characteristic of these parts. These time lags are either transfer lags or distance-velocity lags. Thus the behaviour of the plant can be described by a mathematical equation; this equation considering only transfer lags can be assumed as linear, provided that the excursions of the variables about the mean working values are not great.

This assumes that the disturbances are sufficiently small to permit consideration of the plant as a linear system, which is reasonably correct to the accuracy required in practice.

The effect of disturbances depends on their point of entry to a control system. In particular, the effect of a disturbance on the supply side differs from that of a
disturbance on the demand side. It is possible to study the effect of a disturbance on the supply side from the response of the controlled condition to desired value changes, and the effect of a disturbance on the demand side from the response to load changes.

The Control Action Generator or Controller.

Usually the word controller is used for the combination of the measuring element, the difference element and the control action generator, but as we have combined the measuring element with the process and the difference element is considered alone, the word "controller" will be used to represent the control action generator for simplicity.

The simplest type of controller has two-step (or "on/off") action, but the inherent difficulties of such discontinuous action led to the development of continuous action controllers. The output signal of these controllers, when it is proportional to the input signal or deviation, is called "proportional action", and when changing at a rate proportional to the deviation, the action is called "integral action", and when proportional to the rate of change of the deviation, the action is called "derivative action".
In practice sufficiently good approximation to the pure actions can be obtained with simple arrangements, either pneumatic, hydraulic or electrical. Also it is possible to generate the actions independently and add together the signals from the proportional, integral and/or derivative action generating units, without altering their individual values appreciably. Therefore pneumatic, electric and hydraulic proportional action and 2- and 3- action compound controllers can be constructed and thus it is possible to get a linear mathematical relationship between output and input of a controller.

The Regulating Unit:

In most process control systems the regulating unit is a control valve operated pneumatically by a diaphragm motor. The control valve has a position of major importance in the control loop, since it is the means by which the controller is applied to the plant.

The characteristic of interest of such valves is the relationship between valve lift and flow for a constant pressure drop across the valve. This characteristic of the valve is as important as that of any other element in the control loop and should be selected with special care, and merits a detailed study.
(1.2) Valve Characteristic:

The selection of valve characteristic is often referred to as if it were possible to obtain any characteristic required, but the valves normally available nearly always have either a "linear" or an "equal percentage" (semilog) characteristic.

Having been able to generate the right corrective action in the controller due to a deviation arising from some disturbance, the control valve should be able to produce a corresponding correction on the process, i.e., a corresponding potential correction.

But while it is a comparatively straightforward matter to select the best characteristic to employ when changes in one set of operating conditions are expected, for example, desired value changes at constant load, this characteristic may happen to be the worst to select for dealing with other changes which may be equally likely to occur in the same plant; for example, load changes at constant desired value.

A small change $\Delta V$ in the signal from the controller will produce a change $\Delta x$ in the valve position, which in turn will produce a change $\Delta q$ in the flow rate, thus producing a change $\Delta \theta q$ in the potential correction.
\[
\frac{\Delta \Phi_H}{\Delta V} = \frac{\Delta \Phi_H}{\Delta Q} \times \frac{\Delta Q}{\Delta x} \times \frac{\Delta x}{\Delta V} \ldots \ldots \ldots \ldots (1.2-1)
\]

\[
\frac{\Delta \Phi_H}{\Delta V}
\]
at each load (i.e. for each value of \( Q \)) is
determined by the process and cannot usually be
changed, i.e. it varies with the load in a manner
determined by the process.

\[
\frac{\Delta Q}{\Delta x}
\]
is determined at each valve position by the valve
design.

\[
\frac{\Delta x}{\Delta V}
\]
is normally a constant determined by the design of
the diaphragm motor.

Putting \( \frac{\Delta \Phi_H}{\Delta V} \) as defined by equation (1.2-1) equal to \( K \),
then assuming a proportional mode of control the open loop
gain can be put as

\[
G = KK_1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1.2-2)
\]

where \( K_1 \) is the proportional action factor.

Thus at the normal operating point corresponding to
a certain load, the open loop gain is adjusted by setting
\( K_1 \) to give optimum performance.

Now assuming a change in load took place, then
\( \frac{\Delta \Phi_H}{\Delta Q} \) will change correspondingly, and thus if a linearly
characterised valve is being used, i.e. \( \frac{\Delta Q}{\Delta x} \) is constant for all valve positions, then from equation (1.2-1) \( K \) will change and correspondingly the loop gain.

Thus to keep the open loop gain constant, either the controller setting must be changed, or a valve must be used having a characteristic such that:

\[
\frac{\Delta Q}{\Delta x} = e^{k'} Q 
\]

so that whenever \( Q \) changes \( \frac{\Delta Q}{\Delta Q} \) and \( \frac{\Delta Q}{\Delta x} \) will change proportionally keeping \( x \) and consequently \( C \) constant.

Equation (1.2-3) could be written as

\[
\frac{\Delta Q}{Q} = k' \Delta x 
\]

where \( k' \) is the constant of proportionality. A valve characterised by equation (1.2-4) is what is known as the "equal percentage" valve because, as seen from the equation, the percentage change in \( Q \) is constant for a given change in \( x \).

This characteristic is sometimes also called "logarithmic" or exponential characteristic, because

\[
\frac{\Delta Q}{\Delta x} \text{ at any valve position corresponding to any value of } Q \text{ is the slope of the valve characteristic. Thus equation (1.2-3) could be written as:-}
\]

\[
\frac{dQ}{dx} = k' Q 
\]

or \( Q = \text{constant} x e^{k'x} \) .................(1.2-5)
(1.3) Effect of Exponential Regulating Unit on Loop Equations.

The previous section has shown that a valve having an exponential characteristic would be more suitable for control systems liable to load changes than a linearly characterized one, but although this may be true as regards the static behaviour of the system, yet it would be of interest to know what is the effect of the valve characteristic, being a non-linear one, on the dynamic behaviour of the control system.

The dynamic behaviour is best studied by transient response analysis. The transient response could be obtained either from the frequency response or from the differential equations of the system.

Now having a nonlinear element in the loop, the frequency response analysis and the techniques of obtaining the transient response from it used in linear systems can no longer be used here.

Describing function methods have been widely used in recent years for the investigation of nonlinear control systems. These are essentially frequency response methods in which it is assumed that the system behaves as a low-pass filter and that at the output all harmonics of the input frequency other than the fundamental are of such small amplitude that they can be neglected. These describing
function methods have been also extended to obtain the transient response of non-linear systems.

Apart from the limited accuracy of the describing function technique, there are several other limitations to its use; for example, the dependence on attenuation of harmonics may be justified in systems having a large power amplification between input and output, but certainly not true of many closed loop control systems used in analogue computing. Thus it is better to study the dynamic behaviour of the control system having exponential regulating unit from the transient response obtained from the differential equations of the system.

The type of transient response analysis that would be of interest here takes one of two forms. The first is the response of the control system having an exponential regulating unit to step changes in the desired value, at constant load, in comparison with the response of the same system having a linear regulating unit. It would be expected, if the slope of the exponential regulating unit at the operating point is the same as the slope of the linear one, that such response will be the same for small changes in desired value, but that could not be true for large changes.

The second form of transient response that would be of interest to know is the response of the control system,
having an exponential regulating unit, to step changes in load in comparison with the response of the same system having a linear regulating unit. The characteristic of an exponential regulating unit enables it to counteract the change produced in the potential correction due to load changes and thus keeps the gain of the loop constant at any load. Load changes not only change the potential correction but also change the coefficients of the differential equation describing the dynamic behaviour of the plant. Thus, although the response of the control system, having an exponential regulating unit, to small disturbances may be expected to be the same at every load, yet the response to step changes of the load cannot be so easily predicted.

(1.4) Loop Equations:

In deriving the differential equations of the control system having an exponential regulating unit, in order to determine the transient response, the equations will be non-linear due to the non-linearity of the valve characteristic. Also they will be of an order higher than the second, because the number of transfer lags of a plant are not usually less than three, that being the minimum number to bring the potential correction vector into the third quadrant, which allows the loop to be unstable when working in the proportional control mode.\textsuperscript{1/1/5}
Now the problem becomes that of the solution of non-linear differential equations of at least the third order.

"Until recently the subject of non-linear differential equations has been a happy hunting-ground populated almost exclusively by the technologist. Research into methods of solving these equations has been neglected by the pure mathematician." That is why nearly all the methods known, especially the analytical methods, for the solution of such equations, are only applicable to equations that are similar in form to those solved by these methods.

In general there exist no methods capable of yielding exact solutions of non-linear differential equations, and the only methods available are those of approximations. The scope of the quantitative or analytical methods of solving such equations available at present is rather limited. It is restricted, in fact, to the class of non-linear differential equations that yield periodic solutions.

1.5 Analytical Methods of Solution of Non-Linear Differential Equations.

These methods could be summarised as follows:

1. A method dealing with equations that are integrable exactly without recourse to elliptic functions, and by using suitable transformations.

2. A method dealing with equations integrable exactly (with some exceptions) in terms of either Jacobian or Weierstrassian elliptic functions.
3. **Approximate periodic solutions by:**  
   (a) successive approximation (iteration),  
   (b) perturbation method used by Lindstedt and Poincare,  
   (c) assuming a Fourier series and determining early coefficients therein.

4. **Approximate periodic solutions, assuming Fourier series**  
   as based upon theory of Mathieu functions.

5. **Approximate periodic and non-periodic solutions by**  
   method of slowly varying amplitude and phase.

6. **Deriving the equivalent linear equations.**

   None of these methods deal with equations of order higher than the second order, and mostly with equations that have periodic solutions and thus they could not be applied to our equations where transient responses to step function inputs are required.

   Thus we could come to the conclusion that analytical solutions to our equations are impossible, and even if possible would be so cumbersome that they become of no real practical use. The type of solutions wanted are those from which the response curves could be obtained, so that they could be compared with curves obtained from the equivalent linear equations of the control system, using a linear regulating unit.
1.6 Numerical Methods:

The numerical solution of a differential equation is a problem which has engaged the attention of mathematicians for many years. Numerous methods have been proposed, and some have actually received practical trial: on the whole, however, there appears to be no authoritative statement as to a best method, or even a set of such recommended methods. Upon examining the literature we find, for example, that Whittaker and Robinson give only one method of solution (that of Bashforth and Adams) and state that this is the best. On the other hand, Hartree makes no mention of this method at all, and a like remark is true of Milne, although in a more recent publication he points out that most methods are in the nature of modifications of the Bashforth-Adams process.

In many of the finite difference methods of solution it is necessary to have one or more values of the solution, and possibly of its derivatives, near to the starting point. These are usually best obtained by a process rather different from that used in the remainder of the solution. An initial approximate solution is usually obtained by means of a Taylor series expansion, which is preferred to other methods, e.g. Piccard and Runge (based on an idea originally due to Euler and has been extended by Kutta, Heun and Piaggio). The Taylor series approach
is more easily applicable and is, furthermore, readily extended to equations of higher order than the first.

The Taylor series is used for increments of the independent variable small enough to make the series converge rapidly, and after thus obtaining a few, generally four, values of the function, these furnish enough data to continue the solution by the Bashforth and Adams method or its modifications based on formulae belonging to the calculus of finite differences.

Numerical solutions are usually tedious, even with the aid of calculating machines, especially when the equations are of order higher than the second, unless digital computers are available. Even then, the time of programming may not be inconsiderable, although this is immaterial if a large number of similar equations are to be solved.

In this thesis a numerical method due to Tustin was found, after slight modification, to be the best for the solution of our equations to a high degree of accuracy, and without the work being so tedious. This so-called "time series" method is fully described in Chapter 2.
1.7 Graphical and Other Methods:

It may be of interest here to mention the phase plane methods. Phase plane analysis provides a method of studying the behaviour of any system, either linear or non-linear, whose motion can be described by an ordinary differential equation of the second order. This restriction to second order systems is a major limitation in the application of the method of analysis to control systems, although recently the study has been extended to third order systems by geometrical representation, in 3-dimensional space instead of 2-dimensional.

Some graphical methods are used in the construction of phase portraits; e.g. the isoline (lines of equal slope) method is a convenient method of solving many second-order non-linear autonomous differential equations of greater complexity than those which can be handled analytically in the phase plane.

Another graphical construction is that of Liénard, regarded as a special case of the more general construction of Jacobsen.

Graphical construction, like numerical solution, is tedious and is restricted to second-order differential equations (first-order equations being treated as a special case).

Also we should not fail to mention electronic
analogue computing techniques or differential analysers, but although systems of order higher than the second and involving different kinds of non-linearities are studied by analogue computers, the accuracy of results has to be checked. A method of substituting a computer solution into the original equation for checking is usually made formidable by the problem of repeated graphical differentiation. Comparison of theoretical and computer solutions of the corresponding linearised equation also provides some verification, but much more is desirable.

* Autonomous equations are those which do not contain time explicitly.
CHAPTER 2

The Time Series Numerical Method

Summary:

In this chapter the time series numerical method briefly mentioned in the previous chapter is fully described. The general rules of manipulating serial numbers and the derivations of differentiating and integrating operator are described. A set of rules for the application of these operators are formulated and their application to different types of functions is explained. The chapter concludes with the solution of linear and non-linear differential equations.

2.1 General Nature of the Method.

In this method time functions are represented by the sequence of numbers giving the heights of successive equally spaced ordinates. Such sets of numbers, represented by a single symbol or regarded as single "multi-place" numbers, may be multiplied, divided, added or subtracted by stated rules, so that the numerical implications of any operational equation may be found by direct calculation.

The rules for combining such multi-place numbers are found to be substantially identical with those of ordinary multiplication, addition, etc., of decimal numbers.
except that "carrying" is barred.

For example, let \( A = (a_1, a_2, a_3 \ldots \) etc.\)

\[ \text{and } B = (b_1, b_2, b_3 \ldots \) etc.\]

\[ \therefore A + B = (a_1 + b_1), (a_2 + b_2), \ldots \) etc.\]

\[ \text{and } A - B = (a_1 - b_1), (a_2 - b_2), \ldots \) etc.\]

In case of multiplication consider the multiplication of the serial number \( \sigma = (\sigma_1, \sigma_2, \sigma_3 \ldots \) etc.\)

by the serial number \( (1, 3, 3, 1) \), the process of multiplication will be the same as that of multiplying two decimal numbers and done as follows:

\[ \sigma_1, \sigma_2, \sigma_3 \ldots \sigma_{n-1}, \sigma_n, \sigma_{n+1} \ldots \]

\[ 3\sigma_1, 3\sigma_2 \ldots \sigma_{n-2}, 3\sigma_{n-1}, 3\sigma_n \ldots \]

\[ 3\sigma_1, 3\sigma_2 \ldots \sigma_{n-3}, 3\sigma_{n-2}, 3\sigma_{n-1} \ldots \]

\[ \sigma_1, \sigma_2 \ldots \sigma_{n-3}, \sigma_{n-2} \]

adding

\[ \sigma_1, (\sigma_2 + 3\sigma_1), (\sigma_3 + 3\sigma_2 + 3\sigma_1) \ldots (\sigma_n + 3\sigma_{n-1} + 3\sigma_{n-2} + 3\sigma_{n-3}) \ldots \]

The \( n \)th ordinate of the resulting serial number will be

\[ \sigma_n + 3\sigma_{n-1} + 3\sigma_{n-2} + \sigma_{n-3} \quad \text{\( n \) being } 1, 2, 3, \ldots \) etc.\]

In the case of division the process is the same as ordinary synthetic division. To illustrate this consider
as a simple example the division of
\((2, 5, 7, 10, 6, 4, 1)\) by \((1, 2, 1, 1)\)

\[
\begin{array}{c|cccccccc}
1, & 2, & 1, & 1 & | & 2, & 5, & 7, & 10, & 6, & 4, & 1 \\
\hline
2, & 4, & 2, & 2 & & 1, & 5, & 8, & 6 \\
1, & 2, & 1, & 1 & & 3, & 7, & 5, & 4 \\
3, & 6, & 3, & 3 & & 1, & 2, & 1, & 1 \\
1, & 2, & 1, & 1 & & \ldots \\
\end{array}
\]

The result being the serial number \((2, 1, 3, 1)\)

Also the differentiating operator \(D\) can be replaced
by a multiplicative operator in the same way as it is
replaced by a complex multiplicative operator
\(S = \phi + jw\) in the Laplace Transform method.

Thus this step-by-step method of calculating the
time function of response consists essentially in deriving
from the differential equation of the system, or whatever
experimental or other data may be available, the so-called
"regression equation" by which each of a sequence of equally
spaced ordinates is related to the immediately preceding
ordinates.
2.2 The Differentiating Operator:

Let \( f(t) \) be a function of \( t \) with continuous derivatives of all orders and let
\[ f_1, f_2, \ldots, f_n, \ldots \text{ etc.} \]
be a sequence of values of this function taken at a sequence of values of \( t \), successive values being separated by a constant interval \( \delta \). Then the sequence of first differences is defined by the relation
\[ \Delta f_n = f_{n+1} - f_n \quad \cdots \cdots \cdots \cdots \quad (2.2-1) \]
The differences of the sequence of first differences just defined are called the second differences of the original sequence.

Thus \[ \Delta^2 f_n = \Delta f_{n+1} - \Delta f_n. \]
This can be extended to differences of any order.

The symbol \( \Delta \) can be regarded as an operator which converts the sequence \( f_n \) into the sequence of differences \( \Delta f_n \) and it is treated as the operator \( D = \frac{d}{dt} \).

Another useful operator is \( E \) defined by the relation:
\[ Ef_n = f_{n+1} \cdots \cdots \cdots \cdots \quad (2.2-2) \]
so that \[ E^2 f_n = Ef_{n+1} = f_{n+2}, \text{ etc.} \]
From (2.2-1) and (2.2-2) we get the operational relation
\[ E = 1 + \Delta \quad \cdots \cdots \cdots \cdots \quad (2.2-3) \]
Now as the values of the function \( f(t) \) are at interval \( \delta \) apart, then (2.2-2) could be written as
\[
E f(t) = f(t + \delta)
\]
Then using Taylor's theorem and assuming \( \delta \) small enough to ensure convergence
\[
E f(t) = f(t + \delta) = f(t) + \delta f'(t) + \frac{\delta^2}{2!} f''(t) + \ldots
\]
\[
= (1 + \delta D + \frac{\delta^2 D^2}{2!} + \ldots) f(t)
\]
Thus
\[
E = 1 + \delta D + \frac{\delta^2 D^2}{2!} + \ldots = e^\delta D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2-4)
\]
The inverse relation giving \( D \) in terms of \( E \) or \( \Delta \) is
\[
D = \frac{1}{\delta} \log E = \frac{1}{\delta} \log (1 + \Delta)
\]
\[
= \frac{1}{\delta} (\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \frac{1}{4} \Delta^4 + \ldots) \ldots(2.2-5)
\]
It is seen that \( D \) can be expanded in powers of \( \Delta \) but not in powers of \( E \) or \( E^{-1} \). Because of this it is impossible to represent \( D \) exactly as a time series. It is, however, possible to obtain approximations in the form of rational fractions in \( E \) whose expansions in powers of \( \Delta \) agree with equations (2.2-5) as far as any arbitrarily chosen term.

The simplest of these has been derived by Tustin without recourse to finite difference operators as follows:

Let \( F = (f_1, f_2, f_3, \ldots, f_{n-1}, f_n, \ldots \text{ etc.}) \)

be the ordinate sequence of a time function and let
\[
F' = (f'_1, f'_2, f'_3, \ldots, f'_{n-1}, f'_n, \ldots \text{ etc.})
\]

be the ordinate sequence of its derivative. Then
over the interval \( \delta \) between the \((n-1)\)th and the \(n\)th ordinate the average value of the derivative is 
\[
(f_n - f_{n-1})/\delta
\]
which follows the ordinate series 
\[
(1, -1)f/\delta.
\]
The average of the values \( f_n \) and \( f_{n-1} \) is \((f_n + f_{n-1})/2\) which forms the ordinate series 
\[
(1, 1)f/2.
\]
Hence for these to correspond 
\[
(1, -1)f/\delta = (1, 1)f/2 \quad \text{and as } D = \frac{f'}{f}
\]
\[
\therefore D = \frac{2}{\delta} \left( \frac{1, -1}{1, 1} \right) \quad \ldots \ldots \ldots \ldots \quad (2.2-6)
\]

E.M. Brown in his paper about finite difference operators states that this is the time series equivalent to the operator 
\[
D = \frac{2}{\delta} \frac{1 - E^{-1}}{1 + E^{-1}} \quad \ldots \ldots \ldots \ldots \quad (2.2-7)
\]
and then writing this as 
\[
\frac{2}{\delta} \frac{E - 1}{E + 1} = \frac{1}{\delta} \frac{\Delta}{1 + \frac{1}{2} \Delta} = \frac{1}{\delta} (\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} \ldots) \quad (2.2-8)
\]
Then added that this agrees with equation (2.2-5) as far as the term in \( \Delta^2 \) and so can be used in general only if the 3rd differences of the functions operated on are negligible.

But when applying (2.2-6) and (2.2-8) to different functions, as will be seen in the next section, they do not give the same answer unless they both give the right answer. Moreover (2.2-8) is formed of forward differences
which take into account the coming ordinates
\((\Delta f_n = f_{n+1} - f_n)\), whereas (2.2-6) which is actually derived from the formula:

\[
f'_n = \frac{2}{\delta} (f_n - f_{n-1}) - f'_{n-1} \quad \ldots \ldots (2.2-9)
\]

where \(f'_n\) is the \(n\)th ordinate of the derivative, is seen to be dependent on the preceding ordinate.

At the first ordinate position, for example,

\[
f'_1 = \frac{2}{\delta} (f_1 - f_0) - f'_0 \quad \ldots \ldots (2.2-10)
\]

i.e. in calculating the 1st ordinate of the derivative the zero ordinate of the function and its derivative are taken into account, while calculating the 1st ordinate of the derivative by (2.2-3) does not take into account the zero ordinate of the function, but the coming ordinates.

Thus (2.2-6) could only be applied to functions that have zero initial slope because, as it is operating on the ordinates of the function, and \(f'_0\) is not zero, then it needs \(f'_{-1}\) and \(f'_{-1}\) to calculate \(f'_{-1}\) and then it needs \(f'_{-2}\) and \(f'_{-2}\) to calculate \(f'_{-1}\) and so on. The initial value of the function \(f_0\) needs also to be known, and if not zero it should be taken into account in calculating the 1st ordinate of the derivative \(f'_1\) (\(f'_0\) is assumed zero.).

Also in deriving (2.2-6) it was assumed that the average of the derivative over the interval \(\delta\) is \((f'_n + f'_{n-1})/2\) which is only true when the derivative is varying linearly, i.e. the 3rd differences are zero or negligible, that is the only common thing between (2.2-6) and (2.2-3).
The 3rd differences could be made negligible by taking \( \delta \) small enough, although that is not possible near the origin and thus errors are inevitable.

Thus it is seen from the above that the serial operator (2,2-6) could only be applied, in general, to functions that have zero initial value and slope and negligible 3rd differences.

The reciprocal of this operator, i.e. \( \frac{\delta}{2} \frac{1}{(1, -1)} \)

can also be used for numerical integration and the formula similar to (2,2-9) is

\[
f_n^{-1} = \delta (f_n + f_{n-1}) + f_{n-1}^{-1} \quad \ldots \ldots \ldots (2,2-11)
\]

where \( f_n \) and \( f_{n-1}^{-1} \) are the integrals of the function up to the \( n^{th} \) and \( (n-1)^{th} \) ordinate positions respectively.

At the 1st ordinate position, for example:

\[
f_1^{-1} = \frac{\delta}{2} (f_1 + f_0) + f_0^{-1} \quad \ldots \ldots \ldots (2,2-12)
\]

As numerical integration is definite, i.e. carried out between 2 limits \( f_0^{-1} \) is considered zero and the initial value of the function is to be taken into account if it is not zero. Also from (2,2-11) the area between the \( n^{th} \) and \( (n-1)^{th} \) ordinate is considered to be equal to

\[(f_n + f_{n-1}) \delta/2 \] which is only true when the function is linear, i.e. 2nd differences are zero or negligible.
The 2\textsuperscript{nd} differences can be made negligible by taking \( \delta \) small enough, but still they will not be negligible near the origin.

Thus the integrating operator could only be applied to functions that have zero initial value and negligible 2\textsuperscript{nd} differences.

2.3 Application of the Operator \[ \begin{pmatrix} 2 \\ \delta \end{pmatrix} \frac{(1, -1)}{(1, 1)} \]
and its reciprocal to various Types of Functions.

1. Functions with zero initial value and slope and negligible 3\textsuperscript{rd} differences:

The simplest of these is the parabolic function \( F = t^2 \)

having an ordinate sequence \( \delta^2 \{1, 4, 9, 16 \ldots \text{etc.}\}. \)

Multiplying it by the operator \[ \begin{pmatrix} 2 \\ \delta \end{pmatrix} \frac{(1, -1)}{(1, 1)} \] results in \( 2 \delta \{1, 2, 3, 4 \ldots \text{etc.}\}, \) but \( \frac{dt^2}{dt} = 2t \), which gives the same ordinate sequence at \( t = \{1, 2, 3 \ldots \text{etc.}\}. \)

Thus the operator gives exact result in this case whatever the value of \( \delta \) used, because the 3\textsuperscript{rd} differences are zero.

Also equation (2.2-3) gives the same result.

Applying the reciprocal of the operator to this function results in \[ \delta^3 \{1.5, 9, 23.5, 66 \ldots \text{etc.}\}, \) but the integral of \( t^2 \) is \( t^{3/2} \), which gives the ordinate sequence \[ \delta^3 \{1, 3, 27, 64 \ldots \text{etc.}\}. \) It is seen that the result
is not exact because the 2nd differences are not zero,
but taking \( \delta \) small makes the percentage error diminish
quickly, because this makes the 2nd differences negligible.

If the 3rd differences are not zero as in the function
\( F = t^3 \), for example, then applying the operator to its
ordinate sequence \( \delta^3 (1, 8, 27, 64 \ldots \text{ etc.}) \) results in
\( 3 \delta^2 (0.66, 4, 8.66, 16, \ldots \text{ etc.}) \), but \( \frac{dt^3}{dt} = 3t^2 \)
gives the ordinate sequence \( 3\delta^2 (1, 4, 9, 16 \ldots \text{ etc.}) \).
The result is seen to be not exact, but could be made very
much nearer to the exact answer by taking \( \delta \) small.

Applying equation (2.2-3) gives \( 3 \delta^2 (0.63, 3.33, 8.83 \ldots \text{etc.}) \),
which is different from that obtained by the operator.

2. Functions with zero initial slope and negligible
3rd differences, but not zero initial value:

Consider as an example the function \( F = \cos t \) with
ordinate sequence \( (0.99500, 0.9300, 0.95534, 0.92106 \ldots \text{etc.}) \)
taken at \( \delta = 0.1 \) to make the 3rd differences negligible.

This function has a unity ordinate at \( t = 0 \), and as explained
in section (2.2) it should be taken into account
(equation 2.2-10). This is done in the following way:

On multiplying the ordinate sequence of the function by
the operator \( \frac{2}{\delta} \frac{(1, -1)}{(1, 1)} \), we usually multiply it by \((1, -1)\)
first and then divide by \((1, 1)\). Thus when multiplying
by \((1, -1)\) we should subtract from the resulting first
ordinate the value of the zero ordinate, i.e. unity in this
case.
Applying the operator in this way results in the ordinate sequence \((-0.1, 0.1936, 0.295, ...\) etc.) which is nearly the same as the ordinate sequence \((-0.09833, 0.19367, 0.29552, ...\) etc.) of the true derivative \((-\sin t)\).

Applying (2.2-8) to this function results in the ordinate sequence \((-0.0997, 0.19369, 0.2979 \ldots\) etc.) which is different from that obtained by the operator.

To find the integral of this function by multiplying its ordinate sequence by \(\frac{5}{2}(1, 1)\) the same precaution is taken, i.e. when multiplying by \((1, 1)\) we should add to the first resulting ordinate the value of the function at \(t = 0\), then divide by \((1, -1)\) as seen from equation (2.2-12). This results in the ordinate sequence \((-0.0993, 0.1935, 0.2953 \ldots\) etc.) which is nearly the same as the ordinate sequence of the true integral \((\sin t)\).

3. Functions with zero initial value and negligible 3\textsuperscript{rd} differences but discontinuous at \(t = 0\):-

The simplest of these is the function \(F = t u(t)\) whose ordinate sequence is \((1, 2, 3, 4 \ldots\) etc.). Multiplying this by \(\frac{2}{5}(1, 1)\) results in the ordinate sequence \((2, 0, 2, 0 \ldots\) etc.) which is not the ordinate sequence of a unit step function, the true derivative, while applying (2.2-3) results in \((1, 1, 1, \ldots\) etc.) with a value of unity at \(t = 0\).
And multiplying by the reciprocal of the operator results in \( \delta^2 (\frac{1}{2}) \) which is exactly the ordinate sequence of the integral of the function \( (t^2/2) \) at \( t = \delta(1, 2, 3 \ldots\text{etc.}) \) whatever the value of \( \delta \) used. This is because the function has zero initial value and zero second differences.

For the differentiating operator \( \frac{2}{\delta} (\frac{1}{1, 1}) \) to be applied to such function they should be approximated at \( t = 0 \). It would seem reasonable to replace the broken part of the curve by a continuous one touching both the time axis and the function as shown in fig. (2.1). The simplest form this approximating curve can take is a parabolic form. This is achieved by assuming the function to have an ordinate at \( t = 0 \). Multiplying this by the operator gives \( (0.5, 1, 1, 1, \ldots\text{etc.}) \) with the 0.5 ordinate at \( t = 0 \) which could be taken as the ordinate sequence of the unit step function whose value at \( t = 0 \) is uncertain.

To illustrate the efficacy of this approximation consider its application to the function \( F = \sin t u(t) \) whose ordinate sequence is \( (0.0993, 0.19867, 0.29552, \ldots\text{etc.}) \) with \( \delta = 0.1 \). We assume the function to have an ordinate of \( \frac{\pi}{4} \times 0.0993 = 0.02496 \) at \( t = 0 \) and then multiply the ordinate sequence with the zero ordinate by the operator.
FIG. 2.1 The Approximation to the Slope Function.

FIG. 2.2 The Approximation to the Unit Step Function.
which will result in the ordinate sequence
(0.4992, 0.9932, 0.9786, 0.9534 ...etc.) with the 0.4992
ordinate at \( t = 0 \). Now the derivative of the function
\( \sin t \ u(t) \) is \( \cos t \ u(t) \) which has a step of unity at \( t = 0 \)
and then the ordinate sequence with \( \delta = 0.1 \) as
(0.99500, 0.93007, 0.95534, ...etc.) which is not far from
that obtained by the operator, the main difference being
the zero ordinate which was calculated to be \( \approx 0.5 \) and that
could be taken as representing the unit step at \( t = 0 \).

In case of integration this approximation is not used,
and the reciprocal of the operator can be applied directly
because the conditions for its use are still satisfied.

4. Step Functions:

The simplest of these is the unit step function whose
ordinate sequence is \( (1, 1, 1 \ldots \text{etc.}) \) with the zero ordinate
uncertain. If we multiply this by the operator \( \frac{2}{\delta} \frac{(1, -1)}{(1, 1)} \)
we get the ordinate sequence \( \frac{2}{\delta} (1, -1, 1, -1 \ldots \text{etc.}) \)
whether the zero ordinate is considered as zero or unity.
The true derivative of a unit step function is the unit
impulse function with a unique value of infinity at \( t = 0 \).

The unit step function jumps from value zero to unity
at \( t = 0 \), and as the least interval of time that the operator
can deal with is \( \delta \), then this jump is to be first assumed
to take a time = \( \delta \), and as the function has only a slope
at $t = 0$, then it should be assumed as starting from $-\frac{\delta}{2}$ and attaining the unity value at $t = \frac{\delta}{2}$, thus having a value of 0.5 at $t = 0$ which agrees with the result obtained in calculating the derivative of the slope function above. Also it should have no discontinuities; thus it is approximated parabolically as shown in fig. (2.2) i.e., the ordinate at $t = -\frac{\delta}{2}$ equals 0.125 and that at $t = \frac{\delta}{2} = 0.375$, but the ordinate sequence will be (0.5, 1, 1, 1 ... etc.) with only the 0.5 ordinate at $t = 0$ considered.

If this ordinate sequence is now multiplied by the operator $\frac{2}{5} \left( \frac{1}{6} \right)$, this results in $(\frac{1}{6}, 0, 0, ... etc.)$ with the $\frac{1}{6}$ ordinate at $t = 0$ which taking $\delta$ small approaches the infinity value of the true derivative, and if it is multiplied by the operator $\frac{2}{5} \left( \frac{1}{6} \right)$, this results in $\delta (0.25, 1, 2, 3, 4 ... etc.)$ which is the approximated ordinate sequence for the slope function with 0.25 ordinate at $t = 0$ taken in number 3 above.

It should be noted here that for equation (2.2-3) to give $\frac{1}{5}$ as the derivative of a unit step function, the value of the function is to be considered zero at $t = 0$ although using (2.2-3) to get the derivative of a slope function resulted in a value of unity at $t = 0$ which is a contradiction.

To illustrate the efficacy of the approximation, consider its application to the function $F = \cos t u(t)$ which has a
unit step at \( t = 0 \) and then start with zero slope following a cosine function. Thus considering the ordinate sequence of this function as \((0.5, 0.99500, 0.99997, 0.99993 \ldots \text{etc.})\) with the 0.5 ordinate at \( t = 0 \), then multiplying it by \( \frac{\delta}{5} \) \((1, -1)\)
The ordinate sequence \((10, -0.1, -0.1934, -0.296 \ldots \text{etc.})\) with the 10 ordinate at \( t = 0 \) which is actually \( \frac{1}{5} \) \((\delta \text{ being taken } = 0.1)\) representing an impulse function at \( t = 0 \). The true derivative of this function is \( \cos t u(t) - \sin t u(t) \) giving the ordinate sequence \((\infty, -0.09983, -0.19367, -0.29534 \ldots \text{etc.})\) In the ensuing work the ordinate sequence of a unit step function will be considered as \((0.5, 1, 1, \ldots \text{etc.})\) with 0.5 ordinate at \( t = 0 \).

From the above it is seen that, although the operator \( \frac{\delta}{5} \) \((1, -1)\) could only be applied directly to functions having initial value and slope zero and negligible 3rd differences, yet its application could be extended to other types of functions when these functions are approximated appropriately to make them satisfy these three conditions.

Also the reciprocal of the operator, i.e., \( \frac{\delta}{2} \) \((1, 1)\) although it could only be applied directly to functions having zero initial value and negligible 2nd differences, yet it could be applied to other functions approximated to satisfy these two conditions.
2.4 Numerical Solution of Linear Differential Equations with Constant Coefficients using the Operator \( \frac{2}{\beta} \left( 1, \frac{-1}{1} \right) \).

To illustrate how such equations are solved using the operator \( \frac{2}{\beta} \left( 1, \frac{-1}{1} \right) \) equivalent to \( D \) two examples are considered.

1. Consider a circuit of resistance \( R \) in series with a condenser of capacity \( C \); the differential equation relating the voltage across the condenser \( E_0 \) and the voltage applied to the circuit \( E_1 \) is given by

\[
(T + 1) E_0 = E_1 \quad \text{where} \quad T = \frac{R}{\beta}
\]

putting \( T = 1 \) for simplicity results in

\[
(2 + 1) E_0 = E_1
\]

The analytical solution of this equation for a unit-step input voltage \( E_1 = u(t) \) can be easily found to be

\[
E_0 = 1 - e^{-t} \quad \text{.............(2.4-2)}
\]

The numerical solution is as follows:

Substituting \( \frac{2}{\beta} \left( 1, \frac{-1}{1} \right) \) for \( D \) and for \( E_0 \) and \( E_1 \) by their sets of ordinates, at interval \( \beta \) in equation (2.4-1) results in

\[
\left( \frac{2}{\beta} \left( 1, \frac{-1}{1} \right) + 1 \right) (E_{01}; E_{02}; E_{03} \ldots) = (E_{11}; E_{12} \ldots)
\]

multiplying both sides by \( (1, 1) \).
\[
\begin{align*}
\therefore \left[ \frac{2}{\delta} (1, -1) + (1, 1) \right] (E_0, E_0, \ldots) & = (1, 1)(E_{11}, E_{12}, \ldots) \\
\text{or} \quad \left[ (1 + \frac{2}{\delta}), (1 - \frac{2}{\delta}) \right] (E_0, E_0, \ldots) & = (1, 1)(E_{11}, E_{12}, \ldots)
\end{align*}
\]

The equality sign means that when the multiplications are carried out, the resulting ordinate series are identical element to element. Hence the following equality holds for the \( n \)th elements of the products:

\[
(1 + \frac{2}{\delta}) E_{on} + (1 - \frac{2}{\delta}) E_{on-1} = E_{in} + E_{in-1}
\]

or

\[
E_{on} = \frac{1}{1 + 2/\delta} (E_{in} + E_{in-1}) - \frac{1 - 2/\delta}{1 + 2/\delta} E_{on-1} \quad \ldots \ldots (2.4-3)
\]

\( n = 1, 2, 3, \ldots \)

Thus each ordinate of \( E_o \) is given as a simple linear function of the preceding ordinate and the contemporary and preceding ordinate of \( E_1 \).

This is what is called the "regression equation".

Now to get \( E_o \) for \( E_i = u(t) \), the set of ordinates of \( u(t) \) are taken as \((0.5, 1, 1, 1 \ldots)\) with 0.5 ordinate at \( t = 0 \), this will result in an ordinate of \( E_o \) at \( t = 0 \), which is discarded although used in calculating the ordinate at \( t = \delta \).

Thus the regression equation taking \( \delta = 0.1 \) sec.

becomes:

\[
E_{on} = 0.9043 \ E_{on-1} + 0.0952 \quad \ldots \ldots \ldots (2.4-4)
\]

for \( n = (2, 3, 4 \ldots) \)

The ordinates at \( t = 0 \) and at \( t = \delta \) i.e., at \( n = 0 \) and 1 are still calculated from the regression equation (2.4-4) only with the constant term being equal to 0.0233 for \( n = 0 \) and 0.0714 for \( n = 1 \).
The following table compares the results obtained
by the numerical solution and the analytical solution
equation (2.4-2)

<table>
<thead>
<tr>
<th>t</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0 ) (analy)</td>
<td>0.1813</td>
<td>0.3271</td>
<td>0.4512</td>
<td>0.5507</td>
<td>0.6321</td>
<td>0.6933</td>
<td>0.7554</td>
</tr>
<tr>
<td>( E_0 ) (numerical)</td>
<td>0.1793</td>
<td>0.3281</td>
<td>0.4500</td>
<td>0.5493</td>
<td>0.6315</td>
<td>0.6933</td>
<td>0.7530</td>
</tr>
</tbody>
</table>

If the solution is started by a smaller \( \delta = 0.05 \) say
and increased after getting 10 ordinates to 0.1, closer
results are obtained.

For \( \delta = 0.05 \) the regression equation becomes

\[
E_{on} = 0.9512 \cdot E_{on-1} + 0.0488 \quad \ldots \ldots \ldots (2.4-5)
\]

The constant term being equal to 0.0122 for \( n = 0 \) and
0.0566 for \( n = 1 \). This results in

<table>
<thead>
<tr>
<th>t</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
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<tbody>
<tr>
<td>( E_0 ) (analy)</td>
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<td>0.4512</td>
<td>0.5507</td>
<td>0.6321</td>
<td>0.6933</td>
<td>0.7554</td>
</tr>
<tr>
<td>( E_0 ) (numerical) ( \delta = 0.05 )</td>
<td>0.1303</td>
<td>0.3294</td>
<td>0.4510</td>
<td>0.5506</td>
<td>0.6321</td>
<td>0.6933</td>
<td>0.7534</td>
</tr>
</tbody>
</table>

In this simple example it is possible to calculate the
first ordinate at \( t = \delta \) using the differential equation and
formula (2.3-9) and then equation (2.4-4) is used after this,
thus avoiding the use of the approximation to the unit-step
at \( t = 0 \).
From equation (2.4-1) putting \( E_1 = u(t) \)
\[
\therefore \left( \frac{dE_0}{dt} \right)_{t=0} = 1 \quad E_{oo} \text{ being equal zero.}
\]
Then from equation (2.2-3)
\[
\left( \frac{dE_1}{dt} \right)_{t=0} = \frac{2}{\gamma} ( E_{01} - E_{oo} ) - \left( \frac{dE_0}{dt} \right)_{0}
\]
Thus for \( \gamma = 0.1 \)
\[
\therefore \left( \frac{dE_0}{dt} \right)_{t=0} = 20 E_{01} - 1
\]
substituting this in equation (2.4-1) we get
\[
20 E_{01} - 1 + E_{01} = 1
\]
or \( E_{01} = 0.0952 \)
The regression equation (2.4-4) is then used to get the rest of the ordinates.

The following table compares the results obtained by this method with those from equation (2.4-2).

<table>
<thead>
<tr>
<th>t</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_0 ) (analy)</td>
<td>0.1813</td>
<td>0.3297</td>
<td>0.4512</td>
<td>0.5507</td>
<td>0.6321</td>
<td>0.6933</td>
<td>0.7534</td>
</tr>
<tr>
<td>( E_0 ) (numerical)</td>
<td>0.1813</td>
<td>0.3297</td>
<td>0.4512</td>
<td>0.5507</td>
<td>0.6321</td>
<td>0.6933</td>
<td>0.7535</td>
</tr>
</tbody>
</table>

2. As a second example consider the free oscillation of a mass \( m \) attached to one end of a spring whose other end is fixed and the whole on a smooth plane.
The equation of free vibration of such a system is
\[
m \frac{d^2 x}{dt^2} + F(x) = 0
\]
The mass being pulled a distance $a$ from equilibrium position and then released:

$$ F(x) = Kx \text{ where } K \text{ is the stiffness of the spring} $$

$$ m \frac{d^2x}{dt^2} + Kx = 0 $$

$$ \ddot{x} + w^2x = 0 \quad \text{..........................}(2.4-6) $$

where $w = \left(\frac{K}{m}\right)^{1/2}$

the initial conditions being at $t = 0 \quad x = a \text{ and } \dot{x} = 0$

The analytical solution of this equation is known to be

$$ x = a \cos wt \quad \text{..........................}(2.4-7) $$

To solve this numerically, numerical values must be assigned to $a$ and $w$, for simplicity these were taken as $a = w = 1$

Thus equation (2.4-6) becomes

$$ (D^2 + 1)x = 0 \quad \text{..........................}(2.4-8) $$

and its analytical solution is given by

$$ x = \cos t \quad \text{..........................}(2.4-9) $$

substituting $\frac{2}{\delta} \left(\frac{1}{1+1}\right)$ for $\delta$ in equation (2.4-8)

results in

$$ \left[ \frac{2}{\delta} \left(\frac{1}{1+1}\right)^2 + 1 \right] (x_1, x_2 \ldots \ldots) = 0 $$

from which the regression equation is

$$ x_n = 2^{\left(\frac{1}{\delta} - 1\right)} x_{n-1} - x_{n-2} \quad \text{..........................}(2.4-10) $$

and for $\delta = 0.1$ this becomes

$$ x_n = 1.99 x_{n-1} - x_{n-2} \quad \text{..........................}(2.4-11) $$
In this case where the initial conditions are not zero, the first ordinate should be calculated using these initial conditions, the differential equation and formula (2.2-9) and then equation (2.4-11) is used to get the rest of the solution.

at \( t = 0 \) we have \( x_0 = 1 \) and \( \dot{x}_0 = 0 \)

then from (2.4-3) \( \ddot{x}_0 = -1 \)

and from (2.2-9) \( \dot{x}_1 = \frac{2}{\delta} ( x_1 - x_0 ) - \dot{x}_0 \)

Thus for \( \delta = 0.1 \) \( \dot{x}_1 = 20(x_1 - 1) \)

and using formula (2.2-9) also for second derivatives

\[ \ddot{x}_1 = \frac{2}{\delta} ( \dot{x}_1 - \dot{x}_0 ) - \ddot{x}_0 \]

\[ = 400(x_1 - 1) + 1 \quad \text{for} \ \delta = 0.1 \]

substituting this in equation (2.4-3) we get

\[ 400(x_1 - 1) + 1 + x_1 = 0 \]

\[ \Rightarrow x_1 = 0.9950 \]

Now having the value of \( x_1 \) and \( x_0 \) known the regression equation can be used to complete the solution. The results compared to those obtained from the analytical solution given by equation (2.4-9) are tabulated as follows:
### Linear Differential Equations with Variable Coefficients

Consider the solution of the equation

\[ \ddot{y} + t \dot{y} - y = 0 \]

with initial conditions \( t = 0 \), \( y = 0 \), \( \dot{y} = 1 \).

This equation is solved analytically in Appendix A.

The solution is given by

\[ y = t \]

To solve equation (2.5-1) numerically it is treated in the same way as equations with constant coefficients;

Rewriting equation (2.5-1) as

\[ (D^2 + t D - 1) y = 0 \]

and putting \( D = \frac{2}{\delta} \left( \frac{1}{\beta} - 1 \right) \)

and \( y = (y_1, y_2, \ldots \) etc.) we get
\[
\left[\frac{2}{\delta} \left(\frac{1}{\delta}, \frac{-1}{\delta}\right) + t \frac{2}{\delta} \left(\frac{1}{\delta}, \frac{-1}{\delta}\right) - 1\right] (\dot{y}_1, \dot{y}_2, \ldots \text{etc.}) = 0
\]

\[
\therefore \left[\frac{\delta}{\delta} \left(\frac{1}{\delta}, \frac{-2}{\delta}, \frac{1}{\delta}\right) + \frac{2t}{\delta} \left(\frac{1}{\delta}, \frac{0}{\delta}, \frac{-1}{\delta}\right) - \left(\frac{1}{\delta}, \frac{2}{\delta}, \frac{1}{\delta}\right)\right] (\dot{y}_1, \dot{y}_2, \ldots \text{etc.}) = 0
\]

\[
\therefore \left[\frac{\delta}{\delta} \left(\frac{1}{\delta} + \frac{2t}{\delta} - 1\right), - \left(\frac{\delta}{\delta} + 2\right)\left(\frac{\delta}{\delta} - \frac{2t}{\delta} - 1\right)\right] (\dot{y}_1, \dot{y}_2, \ldots \text{etc.}) = 0
\]

From which the regression equation for \( \delta = 1 \)

is \( \dot{y}_n = \frac{10}{3 + 2t} \dot{y}_{n-1} - \frac{2 - 2t}{3 + 2t} \dot{y}_{n-2} \) \( \ldots \ldots \ldots \ldots \ldots \ldots \ldots \) (2.5-4)

As the initial conditions are not zero, the first
ordinate at \( t = 1 \) is to be calculated using the differential
equation and formula (2.2-9) we have

\( \dot{y}_0 = 0 \) and \( \ddot{y}_0 = 1 \) thus \( \dddot{y}_0 = 0 \)

then from (2.2-9) \( \dot{y}_1 = \frac{2}{\delta} (y_1 - y_0) - \dddot{y}_0 \)

\( = 2y_1 - 1 \) for \( \delta = 1 \)

also \( \ddot{y}_1 = \frac{2}{\delta} (\dot{y}_1 - \dddot{y}_0) - \dddot{y}_0 \)

\( = 4y_1 - 4 \) for \( \delta = 1 \)

substituting these values of \( \dot{y}_1 \) and \( \dddot{y}_1 \) in equation (2.5-1)

we get

\( 4y_1 - 4 + t(2y_1 - 1) - y_1 = 0 \) but \( t = \delta = 1 \)

\( \therefore y_1 = 1 \)

Now having \( y_0 \) and \( y_1 \) known the regression equation

(2.5-4) can be used to get the rest of the solution, but

before doing this we note the following:
In the regression equation (2.5-4)

- $y_n$ is the ordinate at $t = n \delta$
- $y_{n-1}$ is the ordinate at $t = (n-1) \delta$
- $y_{n-2}$ is the ordinate at $t = (n-2) \delta$

Thus the value of $t$ in the regression equation is to be the average of these three, i.e. $(n-1) \delta$.

For example, to calculate $y_2$ which is at $t = 2 \delta = 2$

the value of $t$ in the regression equation is to be $(2-1) \delta = 1$ giving $y_2 = 2$ and if the solution is continued, the ordinates of $y$ are found to be

$y = (1, 2, 3, 4 \ldots \text{etc.})$ which are exactly the set of ordinates of the function $y = t$ at $t = 1, 2, 3 \ldots \text{etc.}$

which is the analytical solution of the differential equation.

2.6. Nonlinear Differential Equations:

Consider the mechanical oscillating system taken in section (2.4). The equation of free vibration was given as

$$\frac{d^2 x}{dt^2} + F(x) = 0$$

now putting $F(x) = kx + bx^3$

$$m \frac{d^2 x}{dt^2} + kx + bx^3 = 0$$

and putting $w = (\frac{k}{m})^{\frac{1}{2}}$ and $\alpha = \frac{b}{m}$

Thus the equation becomes

$$\ddot{x} + w^2 x + \alpha x^3 = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.6-1)$$

with initial conditions $t = 0, x = a, \dot{x} = 0$
Equation (2.6-1) occurs in the theory of non-linear vibrating systems and certain types of non-linear electric systems.

Equation (2.6-1) is solved analytically in Appendix A, the solution taking \( w = a = 1 \) and \( \alpha = 0.2 \) is given by:

\[
x = 0.99453 \cos w_0 t + 0.00544 \cos 3 w_0 t + 0.00003 \cos 5 w_0 t
\]

\[
w_0 = 1.072
\]

Putting the numerical values of \( w, a \) and \( \alpha \) in equation (2.6-1) we get

\[
\ddot{x} + x + 0.2 x^3 = 0 \quad (2.6-3)
\]

with initial conditions \( x_0 = 1 \) and \( \dot{x}_0 = 0 \)

Rewriting equation (2.6-3) as

\[
\left(D^2 + 1\right)x = -0.2 x^3 \quad (2.6-4)
\]

and putting

\[
D = \frac{2}{5} \frac{(1, -1)}{(1, 1)}
\]

\[
x = (x_1, x_2, \ldots \text{etc.})
\]

and \( x^3 = (x_1^3, x_2^3, \ldots \text{etc.}) \) we get

\[
\left(\left(\frac{2}{5} \frac{1, -1}{1, 1}\right)^2 + 1\right) (x_1, x_2, \ldots \text{etc.}) = -0.2 (x_1^3, x_2^3, \ldots \text{etc.})
\]

\[
\therefore \left[\left(\frac{4}{5} \frac{1, -1}{1, 1}\right) + (1, 2, 1)\right] (x_1, x_2, \ldots \text{etc.}) =
-0.2 \left(1, 2, 1\right)(x_1^3, x_2^3, \ldots \text{etc.})
\]

\[
\therefore \left[\left(1 + \frac{4}{5} \frac{1, -1}{1, 1}\right), 2\left(1 - \frac{4}{5} \frac{1, -1}{1, 1}\right), (1 + \frac{4}{5} \frac{1, -1}{1, 1})\right] (x_1, x_2, \ldots \text{etc.}) =
-0.2 \left(1, 2, 1\right)(x_1^3, x_2^3, \ldots \text{etc.})
\]
From which the regression equation for $\delta = 0.1$ is
\[
x_n = -0.000499 (x_n^3 + 2x_{n-1}^3 + x_{n-2}^3) + 1.99 x_{n-1} - x_{n-2}
\]

which differs from the regression equation (2.4-11) by the term
\[-0.000499 (x_n^3 + 2x_{n-1}^3 + x_{n-2}^3)\]
due to the non-linear term in the differential equation.

As before the first ordinate of $x$ at $t = \delta$
is to be calculated using the differential equation and
formula (2.2-9) because the initial conditions are not zero.

We have
\[
x_0 = 1 \quad \dot{x}_0 = 0
\]
thus from
differential equation $\ddot{x}_0 = -1.2$

using formula (2.2-9):
\[
\ddot{x}_1 = \frac{2}{\delta} (x_1 - x_0) - \dot{x}_0
\]
\[
= 20 (x_1 - 1) \quad \delta = 0.1
\]
also
\[
\ddot{x}_1 = \frac{2}{\delta} (\dot{x}_1 - \dot{x}_0) - \ddot{x}_0
\]
\[
= 400 (x_1 - 1) + 1.2 \quad \delta = 0.1
\]
substituting these values of $\dot{x}_1$ and $\ddot{x}_1$ in equation
(2.6-3) results in
\[
4 (x_1 - 1) + 1.2 + x_1 + 0.2 x_1^3 = 0
\]
from which $x_1 = 0.9940$.

Now having $x_0$ and $x_1$ known, the regression equation can be
used to get the rest of the solution. It should be noticed
here that the ordinates of $x$ are now calculated from the
regression equation by trial and error, a value of $x_n$ is
assumed, then substituted in the R.M.S. of equation (2.6-5),
the values of \( x_{n-1} \) and \( x_{n-2} \) being known; if the same value
of \( x_n \) assumed is obtained we proceed to the next ordinate;
if not the procedure is repeated until we get the same value
of \( x_n \) assumed. This may sound laborious, but actually
after getting a few ordinates it is possible to assume the
right value of \( x_n \) and with the aid of a calculating machine
the work is not laborious.

The values obtained by the numerical solution are
compared with those calculated from the analytical solution
given by equation (2.6-2) in the following table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n ) (analy.)</td>
<td>0.9761</td>
<td>0.9060</td>
<td>0.7333</td>
<td>0.6460</td>
<td>0.4703</td>
<td>0.2749</td>
<td>0.0633</td>
</tr>
<tr>
<td>( x_n ) (numerical)</td>
<td>0.9761</td>
<td>0.9061</td>
<td>0.7339</td>
<td>0.6462</td>
<td>0.4704</td>
<td>0.2750</td>
<td>0.0633</td>
</tr>
</tbody>
</table>

| \( \delta = 0.1 \) |

<table>
<thead>
<tr>
<th>( t )</th>
<th>1.6</th>
<th>1.8</th>
<th>2</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n ) (analy)</td>
<td>0.1410</td>
<td>0.3446</td>
<td>0.5341</td>
<td>0.7011</td>
<td>0.8373</td>
<td>0.9355</td>
<td>0.9393</td>
</tr>
<tr>
<td>( \zeta = 0.1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_n ) (numerical)</td>
<td>0.1411</td>
<td>0.3447</td>
<td>0.5343</td>
<td>0.7012</td>
<td>0.8373</td>
<td>0.9354</td>
<td>0.9397</td>
</tr>
</tbody>
</table>
2.7 Conclusion:

In the last 3 sections 3 different types of differential equations were solved numerically using the operator \( \frac{2}{\delta} \frac{(1, -1)}{(1, 1)} \), and it is seen that the work involved in the solution is not so tedious as in the other known numerical methods, with the results still having a high degree of accuracy.

The steps followed in the solution can be summarised as follows:

1. \( \delta \) is substituted by \( \frac{2}{\delta} \frac{(1, -1)}{(1, 1)} \) and the unknown function and the driving function are substituted by their sets of ordinates in symbols.

2. The regression equation is then derived, giving each ordinate of the unknown function as a simple function of the preceding ordinate or ordinates, depending on the order of the differential equation, and the contemporary and preceding ordinate or ordinates of the driving function.

It is preferable to get the regression equation in terms of \( \delta \) so that when different values of \( \delta \) are used, the regression equation corresponding to each value could be obtained easily.

3. If the initial conditions are zero and the driving function satisfies the conditions of applying the integrating operator (section (2.3)), then the regression
equation could be used directly, starting with the ordinate at $t = 0$

4. If the driving function is a step function, then the appropriate set of ordinates of the driving function is to be used, making the necessary approximation at the zero ordinate. The regression equation is then used to get the ordinates of the unknown function starting with the zero ordinate position (at $t = 0$).

The result will thus include an ordinate at $t = 0$ which is discarded if it is of value equal approximately to $\frac{1}{6}$ the ordinate at $t = 0$, and if it is half the 1st ordinate, this means the unknown function has a step at $t = 0$; and if it is of value proportional to $\frac{1}{6}$, then the solution contains an impulse at $t = 0$.

5. In case the initial conditions are not zero, then the 1st ordinate or 1st few ordinates depending on the order of the differential equation are to be calculated using these initial conditions, the differential equation and formula (2.2.9). The regression equation is then used for the rest of the solution.

6. When the driving function is a step function, then for simple equations it is possible to guess the behaviour of the unknown function at $t = 0$, and thus the solution could be obtained as in number 5 above, avoiding the use of the approximation to the step function at $t = 0$. 
The solution should be started with a small value of $\delta$ and this value is to be kept or changed depending on whether the 3rd differences of the solution so obtained is negligible. Also inspection of these differences can be used as a check in detecting arithmetical errors.
CHAPTER 3

Formation of the Loop Equations and their Solution in the Linear Form

Summary:

In this chapter a loop consisting of a 3-stage R.C. network representing the process, a 3-term controller and a constant current generator acting as the regulating unit, is chosen for investigation. The differential equations of the loop both with a linear and an exponential regulating unit are derived. The equations of the loop using the linear regulating unit are solved for step changes in both the desired value and the load, using the Laplace Transform method.

3.1 Transfer Function of the Process in the Form of Differential Equation:

In selecting an example to be used for a general investigation of control performance with basic types of control, it is not possible to cover all the various plant characteristics which may be encountered. However, very many cases are analogous to coupled R.C stages. A zero damped oscillation with proportional control is only possible where the polar curve passes at least into the third quadrant. The minimum number of stages which will produce such a curve is three and it will be convenient to adopt this arrangement for the investigation which will be undertaken in this thesis.
Thus the process will be represented by a 3-stage RC network as shown in figure (3.1) which is seen to be an interacting network. This being preferred to a non-interacting one, which necessitates buffer amplifiers between the stages. Also to simplify calculations the values of resistances and capacities of these 3 stages are taken equal, and \( T = RC \) is taken as unity.

The resistance \( R_L \) across the last condenser represents the load, and a demand change is produced by changing this resistance. The voltage \( V_o \) across it will be the controlled condition. The desired value chosen for this voltage is 100 volts. The process demand at zero deviation is taken as 2 m.a. giving \( R_L = 50,000 \) ohms. \( R \) is taken 10,000 to make the efficiency about 70%. Thus \( R/R_L \) at this operating point = 0.2.

Thus at this load a change of 0.2 m.a. in the input \( I \) to the process produces a change of 10 volts in the controlled condition \( V_o \), while increasing the load by 25%, i.e. decreasing \( R_L \) to 40,000 ohms will make the change of 0.2 m.a. in \( I \) produce a change of 3 volts only in \( V_o \), also decreasing the load by 25%, i.e. increasing \( R_L \) to \( 66,666.6 \) ohms will make the change of 0.2 m.a. in \( I \) produce a change of 13.3 volts in \( V_o \). This shows how the potential correction varies with load.
From fig. (3.1) we can write:
\[ I = (V_1 - V_2)/R + C dV_1/dt , \]
\[ (V_1 - V_2)/R = (V_2 - V_0)/R + C dV_2/dt \]
and \[ (V_2 - V_0)/R = V_0/R_L + C dV_0/dt \]
eliminating \( V_1 \) and \( V_2 \) from these equations considering \( R_L \) as variable and putting \( RC = 1 \) results in:
\[
\left[ D^3 + (4 + R/R_L) D^2 + (3 + 3R/R_L) D + R/R_L \right] V_0 = IR \\
+ 3 V_0 d(R/R_L)/dt + 2 dV_0/dt \cdot d(R/R_L)/dt + V_0 d_2(R/R_L)/dt^2 \\
\cdots \cdots \cdots (3.1-1)
\]
In case of constant load this reduces to
\[
\left[ D^3 + (4 + R/R_L) D^2 + (3 + 3R/R_L) D + R/R_L \right] V_0 = IR \\
\cdots \cdots \cdots (3.1-2)
\]
and at the normal operating point where \( R/R_L = 0.2 \), this becomes
\[ (5 D^3 + 21 D^2 + 13 D + 1) V_0 = 5 IR \cdots \cdots (3.1-3) \]
Equations (3.1-1, 2 and 3) give the relation between the input \( I \) of the process, and the controlled condition \( V_0 \) in the form of differential equations.

3.2. The Transfer Function of the Controller:

As it is desired to find the response using various modes of control, a 3-term controller capable of giving proportional, integral, and derivative actions is considered.

In the absence of deviation a controller is normally
FIG. 3.1 The Process.

FIG. 3.2 The Characteristics of the Regulating Units.
set to give an output which enables the regulating unit to admit just sufficient of the physical quantity effecting control to balance the process at the desired value. This output in normal pneumatic practice is a 9 p.s.i. pressure, if the valve has been suitably sized. In the electrical analogy of the process the 9 p.s.i. will become a fixed value of the voltage, which is applied to a constant current generator, acting as the regulating unit, to supply the process demand at zero deviation. When deviation occurs, the controller is to give a corrective action opposing the deviation, and if this action is proportional to the deviation it is called proportional action and the controller output becomes:

\[ A = K_1 \theta \]  

where \( A \) = controller output in the absence of deviation

\( \theta \) = deviation

\( K_1 \) = constant of proportionality, called the proportional action factor, related to the "proportional band" which is defined as "that range of values of the controlled condition which operates the regulating unit over its full range." This constant \( K_1 \) is determined from considering the stability of the control system and in section (3.6) it will be shown how it is chosen to give optimum response.
Proportional action necessarily produces "offset" defined as "The sustained departure of the control point from the desired value". This can be eliminated by manual adjustment of the controller, but it can be eliminated automatically by adding integral action \( = -K_o \int \Theta \, dt \).

Thus the controller output becomes

\[
\Theta_c = A - K_1 \Theta - K_o \int \Theta \, dt \quad \ldots \ldots \ldots \ldots \ldots (3.2-2)
\]

where \( K_o \) is a constant related to the "integral action time" defined as "The time interval in which the integral action, in a controller having proportional and integral action, increases by an amount equal to the proportional action when the deviation is unchanging".

Thus putting \( \Theta \) equals some constant \( K \) in (3.2-2) results in

\[
\Theta_c = A - K_1 K - K_o \int K \, dt
\]

\[
= A - K_1 K - K_o K t
\]

Then according to the definition, the integral action time \( T_i \) is that time \( t \) such that

\[
K_o K T_i = K_1 K
\]

or

\[
T_i = K_1/K_o
\]

Thus (3.2-2) can be written as

\[
\Theta_c = A - K_1 (\Theta + \frac{1}{T_i} \int \Theta \, dt) \quad \ldots \ldots \ldots \ldots \ldots (3.2-3)
\]
Proportional plus integral control eliminates "offset", but it decreases the damping and the natural frequency, and thus necessitates reduction of $K_1$. This results in reduction of the immediate corrective action and large errors may be developed initially on the occurrence of demand changes.

"The control action which varies directly with the rate at which deviation changes" is called derivative action

$$\Theta_c = A - K_1 \Theta - K_2 \frac{d\Theta}{dt} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.2-4)$$

where $K_2$ is a constant related to the "derivative action time" defined as "The time interval in which the proportional action, in a controller having proportional and derivative action when the deviation is changing at a constant rate". $\frac{d}{dt}$

Thus putting $\Theta = Kt$ in (3.2-4) results in

$$\Theta_c = A - K_1 Kt - K_2 K$$

Then according to the definition, the derivative action time $T_d$ is that time $t$ such that

$$K_1 K T_d = K_2 K$$

or

$$T_d = K_2 / K_1$$

Thus (3.2-4) can be written as

$$\Theta_c = A - K_1 (\Theta + T_d \frac{d\Theta}{dt}) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.2-5)$$

Proportional plus derivative control does not eliminate offset, but it increases the damping and the natural frequency
thus enabling much narrower proportional bands to be used. Also it makes the response to sudden changes more satisfactory.

Adding integral action to this to eliminate offset results in the 3-term controller having an output given by:

$$\Phi_c = A - K_1 (\Theta + \frac{1}{T_1} \int \Theta \, dt + T_3 \frac{d\Theta}{dt}) \ldots (3.2-6)$$

### 3.3 The Regulating Unit:

It has been previously shown in section (1.2) that there are two main types of regulating unit characteristics in general use, the linear and the exponential.

In the present investigation the effect of both desired value and load change disturbances will be considered using identical control loops:

(a) with a linear regulating unit, and

(b) with an exponential regulating unit.

The characteristic curves of these two regulating units are shown in Fig. (3.2).

The characteristics have been so chosen that both regulating units give the same process demand of 2 m.a. at zero deviation, i.e. with the controller output = A. Also the slope of the characteristic of the exponential regulating unit will be made equal to the slope of the characteristic of the linear one, then a small change in the controller output $\Phi_c$ will produce the same potential correction at this operating point.
Thus the open loop gain, at this load, using either of the regulating units and the proportional control mode, will be the same, resulting in nearly the same response for small disturbances.

The linear regulating unit must give a linear relationship between the controller output $Q_c$ and the process input $I$, i.e.

$$I = B \frac{Q_c}{(3.3-1)}$$

where $B$ is a constant which will be determined soon.

In section (1.2) the exponential regulating unit has a characteristic given by equation (1.2-5) which can be written in terms of $I$ and $Q_c$ as

$$I = B' \ e^{Q_c} \quad (3.3-2)$$

where $K'$ is also a constant to be determined.

For these two regulating units to give the same process demand at zero deviation, i.e. at

$$Q_c = A \quad \text{then}$$

$$B \ A = B' \ e^A \quad (3.3-3)$$

Also equating the slope of both characteristics results in

$$B = B' \ e^A \quad (3.3-4)$$

Thus from (3.3-3) and (3.3-4) it is seen that $A$ should be taken as unity and consequently $B' = B/e$ and equation (3.3-2) becomes

$$I = B \ e^{Q_c - 1} \quad (3.3-5)$$
In section (3.1) the process demand at zero deviation
i.e. at \( \Theta_c = A \approx 1 \) was taken 2 m.a.
Then from (3.3.1) or (3.3-5) \( B = 2 \) m.a./volt.
Thus the linear regulating will be defined by
\[
I = 0.002 e^\Theta_c \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.3-6)
\]
and the exponential regulating unit by
\[
I = 0.002 e^{(\Theta_c-1)} \quad \ldots \ldots \ldots (3.3-7)
\]
At the normal operating point a change of 0.1 volt in
\( \Theta_c \) produces a change 0.2 m.a. in \( I \) and consequently a
10 volt change in \( V_o \) for both regulating units. If the
load is increased by 25\%, i.e. \( R_L \) reduced to 40,000 ohms,
then the linear regulating unit will still give a change
of 0.2 m.a. in \( I \) for a change of 0.1 in \( \Theta_c \) but this will
only produce a change of 3 volts in \( V_o \), while the exponential
regulating unit will give a change of 0.25 m.a. in \( I \) for
a change of 0.1 volt in \( \Theta_c \) and thus producing a change of
10 volts in \( V_o \) as before. And if the load is decreased by
25\%, then the linear regulating unit will also give a change of
0.2 m.a. for 0.1 volt change in \( \Theta_c \) and thus producing a
change of 13.3 volts in \( V_o \), but the exponential regulating
unit will give a change of 0.15 m.a. in \( I \) for 0.1 change in
\( \Theta_c \), thus producing a change of 10 volts in \( V_o \).

From this it is seen how the exponential regulating unit
keeps the potential correction constant as the load changes,
thus keeping the open loop gain constant at different loads.
while the linear regulating unit fails to keep the potential correction constant and thus the open loop gain changes as the load changes.

3.4 Open Loop Equations:

There are 2 cases to be considered:

1. **Constant load case:**

   In equations (3.2-1, 3.5 and 6) \( \Phi_c \) can be written as:
   \[ \Phi_c = A - F(\Theta) \]
   the value of A has been assigned to unity in section (3.3)
   then \( \Phi_c = 1 - F(\Theta) \) \hspace{1cm} (3.4-1)
   where \( F(\Theta) = K_1 \Theta + K_2 (\Theta + \frac{1}{T_1} \int \Theta \, dt) \),
   \( K_1 (\Theta + T_d \frac{d\Theta}{dt}) \) or \( K_1 (\Theta + \frac{1}{T_1} \int \Theta \, dt + T_d \frac{d\Theta}{dt}) \)
   according to the control mode used.

   Then in equation (3.1-3) substituting for I from equations (3.3-6 and 7) taking \( \Phi_c \) as given by (3.4-1) and putting \( V_o = 100 + \Theta_o \) \hspace{1cm} (3.4-2)
   where \( \Theta_o \) is the departure of \( V_o \) from the 100 volt desired value.

   This results in:

   **For the linear regulating unit**
   \[ (5D^3 + 21D^2 + 13D + 1) \Theta_o = -100F(\Theta) \] \hspace{1cm} (3.4-3)
   and for the exponential regulating unit
   \[ (5D^3 + 21D^2 + 13D + 1) \Theta_o = 100 (e^{-\frac{\Theta}{1}}) \] \hspace{1cm} (3.4-4)
2. **Variable load case:**

Only step changes in load will be considered and thus $R/R_L$ instead of being a constant of 0.2 becomes

$$R/R_L = 0.2 + C u(t) \quad (3.4-5)$$

where $C$ = the change made in $R/R_L$.

Thus in equation (3.1-1) substituting for $R/R_L$ from equation (3.4-5) and for $V_o$ from equation (3.4-2) results in:

$$\left[ D^3 + (4 + 0.2 \cdot C \cdot u(t)) \right] D^2 + \left[ 3 + 3(0.2 + C \cdot u(t)) \right] D + 0.2 + C u(t) \right] \theta_o$$

$$= \text{IR} - (0.2 + C \cdot u(t)) 100 + 3(100 + \theta_o) \frac{d(R/R_L)}{dt}$$

$$+ 2 \cdot \frac{d\theta_o}{dt} \frac{d(R/R_L)}{dt} + (100 + \theta_o) \frac{d^2 (R/R_L)}{dt^2}$$


\(3.4-6)\)

$u(t)$ appearing in the L.H.S. of this equation can be omitted because $\theta_o$ is zero at and before $t = 0$. Also from (3.4-5) $d(R/R_L)/dt$ and $d^2(R/R_L)/dt^2$ are impulse functions and for $u(t)$ and $d^2u(t)$ and as $\theta_o = 0$ at $t = 0$ therefore equation (3.4-6) reduces to

$$\left( D^3 + (4.2 + C) D^2 + (3.5 + 30) D + 0.2 + C \right) \theta_o$$

$$= \text{IR} - 20 - 100 C (D^2 + 3 D + 1) u(t) \quad (3.4-7)$$

Now substituting for $I$ from equations (3.3-5 & 7) taking $\theta_o$ as given by equation (3.4-1) we get for the linear regulating unit:
( \( D^3 + (4.2 + C) D^2 + (3.6 + 3 C) D + C.2 + C \) ) \( \Theta_0 \)

\[ = -20 \Phi(\Theta) - 100 C (D^2 + 3 D + 1) u(t) \ldots (3.4-3) \]

and for the exponential regulating unit

( \( D^3 + (4.2 + C) D^2 + (3.6 + 3 C) D + C.2 + C \) ) \( \Theta_0 \)

\[ = 20 e^{-\Phi(\Theta)} - 100 C (D^2 + 3 D + 1) u(t) \ldots (3.4-9) \]

3.5 Closed Loop Equations:

In equations (3.4-3, 4, 5 & 9) putting \( \Theta \) as

\[ \Theta = \Theta_o - \Theta_d \ldots (3.5-1) \]

where \( \Theta_d \) is the change in desired value, by this the loop is closed and the closed loop equations are obtained.

The equation obtained from (3.4-3) will be the closed loop equation at constant load using the linear regulating unit. This will then be a linear differential equation that can be solved for a step change in the desired value by the Laplace Transform method.

The equation obtained from (3.4-3) will be the closed loop equation for step load changes at constant desired value using the linear regulating unit. This will also be a linear differential equation and can be solved by the Laplace Transform method.

The equation obtained from (3.4-4) will be the closed loop equation at constant load using the exponential regulating unit. This will be a non-linear differential equation which will be solved for step changes in desired
value by the numerical method of Chapter 2.

And the equation obtained from (3.4-9) will be the closed loop equation for step load changes at constant desired value using the exponential regulating unit. This will also be a non-linear differential equation which will be solved numerically.

3.6. Solution of the Constant Load Linear Equations.

The closed loop equations obtained from equation (3.4-3) by putting \( \theta = \theta_o - \theta_d \) and using various control modes, are solved here for a step change in the desired value by the Laplace Transform method.

1. Proportional Control Mode.

For proportional control \( F(\theta) = K_1 \theta \)

Substituting this in equation (3.4-3) results in

\[
(5 \, D^3 + 21 \, D^2 + 18 \, D + 1) \, \theta_o = -100 \, K_1 \, \theta \quad \ldots \ldots (3.6-1)
\]

Taking the L.T. (Laplace Transform) of this equation results in

\[
(5 \, S^3 + 21 \, S^2 + 18 \, S + 1) \, \theta_o(S) = -100 \, K_1 \, \theta(S) \quad \ldots \ldots (3.6-2)
\]

where \( \theta_o(S) = \text{L.T. of } \theta_o \) and \( \theta(S) = \text{L.T. of } \theta \)

We cannot proceed to the solution of this equation until a value is fixed for \( K_1 \) the proportional action factor.

In the proportional mode this factor will determine the dynamic response of the closed loop. The choice of magnitude for \( K_1 \) therefore depends on what is considered to be a good response. Ideas as to what is a good response
differ widely. Several performance criteria have been formulated in recent years, \(26, 27, 28, 29, 30, 31\).

In the ensuing analysis the Brown and Campbell method of setting the open loop gain in accordance to a specified \(M = \left| \frac{\theta_0}{\theta_d} (jw) \right| \) criterion has been chosen as a convenient criterion. In this method taking \(M_p\) the maximum magnitude of \(\frac{\theta_0}{\theta_d} (jw)\) as 1.3 is considered to be resulting in a good closed loop performance. The open loop gain is determined as follows:

The locus of the \(G^{-1}(jw)\) function is drawn on the complex plane, in our case \(G^{-1}(jw)\) can be deduced from equation (3.6-2) as

\[
G^{-1}(jw) = 5 (jw)^3 + 21(jw)^2 + 13(jw) + 1
\]

and is drawn in fig. (3.3). Then a line is drawn making an angle = \(\sin^{-1} \frac{1}{M_p} = \sin^{-1} \frac{1}{1.3}\), i.e. 50.2 degrees with the negative real axis. Then a circle is drawn that is tangent to both this line and the \(G^{-1}(jw)\) locus with its centre on the negative real axis. Thus the distance from the origin of co-ordinates to the centre of this circle determines the open loop gain which is 100 \(K_1\) in our case. This was found from the graph Fig. (3.3) to be equal to 15.

Thus \(K_1 = 0.15\)

Substituting this value of \(K_1\) in equation (3.6-2) and putting \(\theta(\delta) = \theta_0(\delta) - \theta_d(\delta) \) ............... (3.6-3)

which is the I.T. of \(\theta = \theta_0 - \theta_d\) given in (3.5-1) results in
FIG. 3.3 The Graphical Determination of the Open Loop Gain from the $G^{-1}(j\omega)$ Plot in the Complex Plane.
\[ \Phi_0(s) = \frac{15 \Phi_d(s)}{5 s^2 + 21 s^2 + 18 s + 16} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \text{(3.6-4)} \]

For a unit-step change in desired value \( \Phi_d(s) = \frac{1}{s} \)

substituting this in equation (3.6-4) results in

\[ \Phi_0(s) = \frac{3}{s(s^3 + 4.2 s^2 + 3.6 s + 3.2)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \text{(3.6-5)} \]

By getting the roots of the characteristic equation

\[ s^3 + 4.2 s^2 + 3.6 s + 3.2 = 0 \]

then equation (3.6-5) becomes

\[ \Phi_0(s) = \frac{3}{s(5 + 3.42112)((s + 0.33944)^2 + (0.83525)^2)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \text{(3.6-6)} \]

Then using a table of Laplace Transforms the inverse transform of equation (3.6-6) is found to be

\[ \Phi_0 = 0.9375 - 0.03791 e^{3.4112t} + 1.10948 e^{-0.33944t} \times \]

\[ \sin (0.83525t - 130^\circ 1.5) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \text{(3.6-7)} \]

2. Proportional + Integral Control Mode:

According to Rutherford the integral action time for optimum controller settings is that time equal to the period of oscillation of the response in the proportional control mode.

This period from equation (3.6-7) \[ \frac{2\pi}{0.83525} \approx 7 \text{ secs} \]

But \( T_1 \) will be taken = 10 sec as this would simplify
calculations and is not a serious departure from the conventional value.

Thus \( F(\Theta) = 0.15 (\Theta + 0.1 \int \Theta \, dt) \)

substituting this in equation (3.6-3) results in

\[
(5 b^3 + 21 b^2 + 13 b + 1) \Theta_0 = -15(\Theta + 0.1 \int \Theta \, dt)
\]

\[
(5 s^3 + 21 s^2 + 13 s + 1) \Theta_0(s) = -15(\Theta(s) + 0.1 \Theta(s)/s)
\]

putting \( \Theta(s) = \Theta_0(s) - \Theta_d(s) \) we get

\[
\Theta_0(s) = \frac{(15 s + 1.5) \Theta_d(s)}{5 s^4 + 21 s^3 + 13 s^2 + 16 s + 1.5}
\]

For a unit-step change in desired value this becomes

\[
\Theta_0(s) = \frac{3(s + 0.1)}{2(s^4 + 4.2 s^3 + 3.6 s^2 + 3.2 s + 0.3)}
\]

getting the roots of the characteristic equation; the inverse transform is then found to be

\[
\Theta_0 = 1 - 0.03569 e^{-3.4125 t} + 0.05134 e^{-0.1046 t} + 1.2037 e^{-0.34145 t} \sin (0.3508 t - 127^0 3') \]

3. **Proportional + Derivative Control Mode:**

The derivative action time \( T_d \) is taken arbitrarily as 0.2 secs., partly because it is then easy to get the roots of the characteristic equation and also it does give a
reasonable transient response.

\[ F(\Theta) \text{ now becomes } 0.15 (\Theta + 0.2 \frac{d\Theta}{dt}) \]

substituting this in equation (3.4-3) results in

\[ (5 D^3 + 21 D^2 + 18 D + 1) \dot{\Theta}_0 = -15 (\Theta + 0.2 \frac{d\Theta}{dt}) \]

\[ \theta_0(t) = \frac{15 + 3s}{5 s^3 + 21 s^2 + 21 s + 16} \]

For a unit-step change in the desired value this becomes

\[ \theta_0(s) = \frac{3 (0.2 s + 1)}{3 (s^3 + 4.2 s^2 + 4.2 s + 3.2)} \]

Getting the roots of the characteristic equation, the inverse transform is then found to be:

\[ \Theta_0 = 0.9575 - 0.042 e^{-3.2t} + 1.1196 e^{-0.5t} \sin(3.2t) - 128° 53' \]

4. 3-Term Control Mode:

Now \( F(\Theta) \) becomes 0.15 \( (\Theta + 0.1 \int \Theta dt + 0.2 \frac{d\Theta}{dt}) \)

Substituting this in equation (3.4-3) results in

\[ (5 D^3 + 21 D^2 + 18 D + 1) \dot{\Theta}_0 = -15(\Theta + 0.1 \int \Theta dt + 0.2 \frac{d\Theta}{dt}) \]
L.T. of this equation is

\[(5s^3 + 21s^2 + 13s + 1) \theta_0(s) = 15\theta_d(s) + 0.12\theta_d(s) + 0.229\theta_d(s)\]

\[\text{Putting } \theta(s) = \theta_0(s) - \theta_d(s) \text{ we get}\]

\[\theta_0(s) = \frac{3s^2 + 15s + 1.5}{5s^4 + 21s^3 + 21s^2 + 16s + 1.5} \theta_d(s)\]

For a unit-step change in the desired value this becomes

\[\theta(s) = \frac{3(0.2s^2 + s + 0.1)}{s(s^4 + 4.2s^3 + 4.2s^2 + 3.2s + 0.3)}\]

\[\text{getting the roots of the characteristic equation, the inverse transform is then found to be}\]

\[\theta_0 = 1 - 0.0395 e^{-3.1832t} + 0.0573 e^{-0.1073t} + 1.2594 e^{-0.45225t} \sin(0.82015t - 124^\circ 49')\]

\[\text{3.7. Solution of the Variable Load Linear Equation.}\]

This is the closed loop equation obtained from equation (3.4-3) and will be solved for 25\% step increase and decrease in load using proportional control mode.

1. \text{25\% Step Increase in Load:}\n
25\% step increase in load is obtained by a step change in \( R_L \) from 50,000 ohms to 40,000 ohms, i.e., \( R/R_L \) is step changed from 0.2 to 0.25. Thus \( C \) in equation (3.4-5) equals 0.05.
Then substituting this value of C in equation (3.4-3) and putting \( P(\theta) = 0.15 \) \( \theta \) for proportional control results in:

\[
(D^3 + 4.25 D^2 + 3.75 D + 0.25) \phi = -39 - 5 (D^2 + 3 D + 1) u(t) \ \cdots \ \cdots (3.7-1)
\]

As the desired value will be kept constant at 100 volts, then \( \phi_d = 0 \) and \( \phi = \phi_o \) putting this in equation (3.7-1) results in:

\[
(D^3 + 4.25 D^2 + 3.75 D + 3.25) \phi_o = 5 (D^2 + 3 D + 1) u(t) \ \cdots \ \cdots (3.7-2)
\]

L.T. of this equation is

\[
(s^3 + 4.25 s^2 + 3.75 s + 3.25) \phi_o(s) = -5 (s^2 + 3 s + 1) \frac{1}{s^3}
\]

From which

\[
\phi_o(s) = \frac{-5(s^2 + 3 s + 1)}{s (s^3 + 4.25 s^2 + 3.75 s + 3.25)} \ \cdots \ \cdots (3.7-3)
\]

Getting the roots of the characteristic equation, then the inverse transform is found to be:

\[
\phi_o = -1.53345 + 0.36485 e^{-3.4335t} -0.40325t -3.83236 e^{-0.3331t} \sin(0.8331t - 1^\circ)
\]

\[
\ \cdots \ \cdots (3.7-4)
\]

2. 25\% Step Decrease in Load:

In this case the load resistance \( R_L \) is step increased from 50,000 to 66,666.6 \( \text{ohms} \), i.e. \( R/R_L \) will be step changed from 0.2 to 0.15. Hence Thus C in equation (3.4-5) becomes equal to -0.05. Then substituting
this value of \( C \) in equation (3.4–3) and putting

\[ F(\theta) = 0.15\theta \] results in

\[ (D^3 + 4.15 D^2 + 3.45 D + 0.15) \theta_0 = -3\theta + 5(D^2 + 3 D + 1) u(t) \] \( (3.7-5) \)

Putting \( \theta = \theta_0 \) and getting the L.T. results in

\[ (s^3 + 4.15 s^2 + 3.45 s + 3.15) \theta_0(s) = 5(s^2 + 35 + 1) \frac{1}{s} \]

Thus

\[ \theta_0(s) = \frac{5(s^2 + 35 + 1)}{s(s^3 + 4.15 s^2 + 3.45 s + 3.15)} \] \( (3.7-6) \)

getting the roots of the characteristic equation; then the inverse transform is found to be

\[ \theta_0 = 1.5873 - 0.3505 e^{-3.409t} + 3.9708 e^{-0.3765t} \sin(0.837t-18^\circ 9') \]

\( (3.7-7) \)
CHAPTER 4

Numerical Solution of Loop Equations

Summary:

In this chapter the loop equations obtained in Chapter (3) with the linear and exponential regulating units are solved numerically, using the time series method described in Chapter 2 for step changes in desired value and load under various control modes. The linear equations solved by the L.T. method in Chapter 3 being solved again, to show the efficacy of the numerical method, results obtained by both methods being compared in tabulated form. Curves obtained from linear and non-linear equations are drawn together for each control mode.

4.1 Constant Load Equations - Proportional Control Mode.

1. Linear Regulating Unit:

In section (3.6) equation (3.4-3) has been solved by the Laplace Transform method for a unit step change in the desired value using various control modes. It will be solved here by the numerical method described in Chapter 2 for comparison, starting with the proportional control mode, i.e. with $F(\Theta)$ put equal to $0.15 \Theta$. Thus equation (3.4-3) becomes:

\[(5D^3 + 21D^2 + 18D + 1) \Theta_0 = -15 \Theta \quad \ldots..(4.1-1)\]
The steps for numerical solution of differential equations given in section (2.7) will be followed here.

Thus replacing $D$ by the serial operator $\frac{2}{\delta} \left( \frac{1}{1} - \frac{1}{1} \right)$ and putting the ordinate sequence of $\phi_0$ and $\phi$ as

$$\phi_0 = (\phi_{00}, \phi_{01}, \phi_{02} \ldots \ldots \ldots \ldots \text{etc.})$$

and

$$\phi = (\phi_0, \phi_1, \phi_2 \ldots \ldots \ldots \ldots \text{etc.})$$

equation (4.1-1) becomes:

$$\left[ \frac{2}{\delta} \left( \frac{1}{1} - \frac{1}{1} \right) \right] \phi = -15 (\phi_0, \phi_1, \phi_2 \ldots \ldots \ldots \ldots \text{etc.})$$

then

$$\left[ \frac{40}{\delta} \left( \frac{1}{1} - \frac{1}{1} \right) + \frac{84}{\delta} \left( \frac{1}{1} - \frac{1}{1} \right) + \frac{36}{\delta} \left( \frac{1}{1} - \frac{1}{1} \right) + 1 \right] (\phi_{00}, \phi_{01}, \phi_{02} \ldots \ldots \text{etc.})$$

$$= -15 (\phi_0, \phi_1, \phi_2 \ldots \ldots \ldots \ldots \text{etc.})$$

Multiplying both sides by $(1, 3, 3, 1)$ we get:

$$\left[ \frac{40}{\delta} (1, -3, 3, -1) + \frac{84}{\delta} (1, -1, -1, 1) + \frac{36}{\delta} (1, 1, -1, -1) + (1, 3, 3, 1) \right] (\phi_{00}, \phi_{01}, \phi_{02} \ldots \ldots \text{etc.})$$

$$= -15 (1, 3, 3, 1) (\phi_0, \phi_1, \phi_2 \ldots \ldots \ldots \ldots \text{etc.})$$

Rearranging then
\[
\begin{align*}
\left[ \left( \frac{40}{\delta} + \frac{84}{\delta^3} + \frac{36}{\delta} + 1 \right), \left( -\frac{120}{\delta^3} - \frac{84}{\delta^2} + \frac{36}{\delta} + 3 \right), \left( \frac{120}{\delta} - \frac{84}{\delta^2} - \frac{36}{\delta} + 3 \right), \right.
\left. \left( -\frac{40}{\delta} + \frac{84}{\delta^2} - \frac{36}{\delta} + 1 \right) \right]
(\theta_{00}, \theta_{01}, \theta_{02} \ldots \text{etc.})
\end{align*}
\]

\[= -15 \left( 1, 3, 3, 1 \right) (\theta_0, \theta_1, \theta_2 \ldots \text{etc.})\]

Putting
\[
\begin{align*}
\left( \frac{40}{\delta} + \frac{84}{\delta^3} + \frac{36}{\delta} + 1 \right) &= a \\
\left( -\frac{120}{\delta^3} - \frac{84}{\delta^2} + \frac{36}{\delta} + 3 \right) &= b \\
\left( \frac{120}{\delta} - \frac{84}{\delta^2} - \frac{36}{\delta} + 3 \right) &= c \\
\left( -\frac{40}{\delta} + \frac{84}{\delta^2} - \frac{36}{\delta} + 1 \right) &= d
\end{align*}
\]

results in
\[
(a, b, c, d) (\theta_{00}, \theta_{01}, \theta_{02} \ldots \text{etc.})
\]

\[= -15 \left( 1, 3, 3, 1 \right) (\theta_0, \theta_1, \theta_2 \ldots \text{etc.}) \quad \text{...(4.1-3)}
\]

Equating the \((n + 1)\)th terms of the product on each side results in the regression equation:

\[
\theta_{on} = -\frac{15}{a} (\theta_n + 3\theta_{n-1} + 3\theta_{n-2} + \theta_{n-3}) - \frac{b}{a} \theta_{on-1} - \frac{c}{a} \theta_{on-2} - \frac{d}{a} \theta_{on-3} \quad \text{...(4.1-4)}
\]

From this regression equation the ordinates of \(\theta_0\) are calculated as follows:
Tables are made each column of which runs down as follows: \( t, \theta_n, \beta \theta_n, \beta \theta_n, \theta_n-3, -\frac{15}{a} \times \text{sum of } \theta \)

\[
- \frac{b}{a} \theta_{n-1} - \frac{c}{a} \theta_{n-2} - \frac{d}{a} \theta_{n-3} = \theta_n
\]

A value of \( \theta_{on} \) is assumed from which \( \theta_n \) is obtained equals \( \theta_{on} = \theta_{dn} \) where \( \theta_{dn} \) is the \( n \)th ordinate of \( \theta_d \) the change in the desired value taken as a 5 \( u(t) \) step function, thus its ordinate sequence is \( 5 (0.5, 1, 1, \ldots, \text{etc.)} \) as explained in part 4, section (2.3). Thus for all ordinates except the zero ordinate

\[
\theta_n = \theta_{on} - 5
\]

the zero ordinate being \( \theta_0 = \theta_{co} - 2.5 \)

Then going down the column, putting the other values, being known from the preceding columns, \( \theta_{on} \) is calculated. This calculated value of \( \theta_{on} \) should agree with the assumed one, otherwise the procedure is repeated.

It should be noted here that this trial and error method of calculating the ordinates of \( \theta_{on} \) could have been avoided by putting \( \theta = \theta_0 - \theta_d \) in equation (4.1-1) and then getting the regression equation in terms of \( \theta_d \) instead of \( \theta \). But this cannot be done in the non-linear case and, as will be seen, doing it this way will save derivation of regression equations for each case and help in comparison during the process of calculation.

To get the coefficients \( a, b, c \) and \( d \) of the regression equation values must be assigned for \( j \). These have been
chosen as $\delta = \%$ from $t = 0$ to $t = 1$ then increased to
$\%$ from $t = 1$ to $t = 7$ and increased again to $\%$ from
$t = 7$ to $t = 11$.

Substituting these values of $\delta$ in equation (4.1-4)
results in:

For $\delta = \%$

$$\Theta_{on} = -0.000574 \left( \Theta_n + 3 \Theta_{n-1} + 3 \Theta_{n-2} + \Theta_{n-3} \right)$$
$$+ 2.5445 \Theta_{on-1} - 2.1335 \Theta_{on-2} + 0.5867 \Theta_{on-3}$$

For $\delta = \%$

$$\Theta_{on} = -0.0037 \left( \Theta_n + 3 \Theta_{n-1} + 3 \Theta_{n-2} + \Theta_{n-3} \right)$$
$$+ 2.1924 \Theta_{on-1} - 1.53 \Theta_{on-2} + 0.3356 \Theta_{on-3}$$

For $\delta = \%$

$$\Theta_{on} = 0.0296 \left( \Theta_n + 3 \Theta_{n-1} + 3 \Theta_{n-2} + \Theta_{n-3} \right)$$
$$+ 0.6749 \Theta_{on-1} - 0.7613 \Theta_{on-2} + 0.0755 \Theta_{on-3}$$
The following table shows how the numerical solution is started using the first of the regression equations (4,1-5) for \( \delta = \frac{1}{8} \).

<table>
<thead>
<tr>
<th>( t ) sec.</th>
<th>0</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.625</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\theta_n )</td>
<td>2.4936</td>
<td>4.9392</td>
<td>4.9593</td>
<td>4.8985</td>
<td>4.7935</td>
<td>4.6580</td>
<td>4.4732</td>
</tr>
<tr>
<td>( -3\theta_{n-1} )</td>
<td>0</td>
<td>7.4958</td>
<td>14.9676</td>
<td>14.8794</td>
<td>14.6955</td>
<td>14.3955</td>
<td>13.9740</td>
</tr>
<tr>
<td>( -5\theta_{n-2} )</td>
<td>0</td>
<td>0</td>
<td>7.4958</td>
<td>14.9676</td>
<td>14.8794</td>
<td>14.6955</td>
<td>14.3955</td>
</tr>
<tr>
<td>( -7\theta_{n-3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.4936</td>
<td>4.9392</td>
<td>4.9593</td>
<td>4.8935</td>
</tr>
</tbody>
</table>

\[0.000574 \times \text{sum} = 0.0014 \times 0.0072 \times 0.0157 \times 0.0214 \times 0.0226 \times 0.0222 \times 0.0217\]

\[2.5445\theta_{on-1} \]

\[\begin{array}{cccccc}
-2.1335\theta_{on-1} & 0 & 0 & -0.0030 & -0.0230 & -0.0853 & -0.2166 & -0.4299 \\
0.5837\theta_{on-2} & 0 & 0 & 0 & 0.0003 & 0.0064 & 0.0237 & 0.0593 \\
\theta_{on} & 0.0014 & 0.0103 & 0.0402 & 0.1015 & 0.2015 & 0.3420 & 0.5213 \\
\end{array}\]

As explained above a value of \( \theta_{on} \) is assumed.

For example at \( t = 0 \) \( \theta_{on} \) is reasonably assumed to be zero, accordingly \( \theta_n = -2.5 \) (in the table above \( \theta_n \) is written instead of \( \theta_n \) and then after suming \( \theta^5 \), the sum is multiplied by \( \frac{15}{a} \) instead of \( \frac{-15}{a} \) ) and as \( \theta_n = 0 \) before \( t = 0 \) then the sum of \( -\theta^5 = 2.5 \) multiplying by 0.000574, which is \( \frac{15}{a} \) for \( \delta = \frac{1}{8} \), results in 0.0014, and as \( \theta_{on} = 0 \) before \( t = 0 \), then \( \theta_{on} = 0.0014 \), the assumed value was zero,
but if the calculations are repeated using $\theta_{on} = 0.0014$
the same result will be obtained. Then proceeding to the
next column $\theta_{on}$ is still reasonably assumed as zero.
Thus $- \theta_n = 5$ and $- 3 \theta_{n-1} = 7.4953$ from the first column.
Then $- 3 \theta_{n-2}$ and $- \theta_{n-3}$ are zero. Thus 0.000574 x sum of $\theta^n$
is equal to 0.0072, then $2.5445 \theta_{on-1} = 0.0036$ and
$\theta_{on-2} + \theta_{on-3}$ being zero results in $\theta_{on} = 0.0072 + 0.0036 = 0.0108$; repeating the calculations with $\theta_{on} = 0.0108$ results
in the same value, then we proceed to the next.

The following table compares the values of the ordinates
of $\theta_0$ as calculated from the L.T. solution of equation (4.1-1)
given by equation (3.6-7), and the numerical solution using
the regression equations (4.1-5).

<table>
<thead>
<tr>
<th>t sec.</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{on}$ by L.T.</td>
<td>0.0333</td>
<td>0.1361</td>
<td>0.4367</td>
<td>0.6335</td>
<td>0.9136</td>
<td>1.0671</td>
<td>1.1466</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>0.0403</td>
<td>0.1969</td>
<td>0.4365</td>
<td>0.6929</td>
<td>0.9133</td>
<td>1.0677</td>
<td>1.1479</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec.</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{on}$ by L.T.</td>
<td>1.1697</td>
<td>1.0696</td>
<td>0.9484</td>
<td>0.8363</td>
<td>0.8365</td>
<td>0.9190</td>
<td>0.9441</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>1.1626</td>
<td>1.0710</td>
<td>0.9479</td>
<td>0.8341</td>
<td>0.8361</td>
<td>0.9179</td>
<td>0.9450</td>
</tr>
</tbody>
</table>
The response in the numerical solution being calculated for \( \Theta_d = 5 \ u(t) \), the ordinates are then divided by 5.

2. Exponential Regulating Unit:

The corresponding equation using the exponential regulating unit is obtained from equation (3.4-4) putting \( F(\Theta) = 0.15 \Theta \) as the same controller settings will be used for linear and exponential regulating units for the reasons explained in section (3.5). Thus we get:

\[
(5 \ D^3 + 21 \ D^2 + 13 \ D + 1) \ \Theta_o = 100 \ (e^{-0.15 \Theta} - 1) \ldots (4.1-6)
\]

putting \( \Theta' = -\frac{100}{15} (e^{-0.15 \Theta} - 1) \ldots (4.1-7) \)

results in

\[
(5 \ D^3 + 21 \ D^2 + 13 \ D + 1) \ \Theta_o = -15 \ \Theta' \ldots (4.1-8)
\]

which is the same as equation (4.1-1) with \( \Theta' \) as given by equation (4.1-7) instead of \( \Theta \).

Thus the regression equations (4.1-5) with \( \Theta' \) instead of \( \Theta \) can be used to get the ordinates of \( \Theta_o \) in this case, the procedure being the same as in the linear equation above with the only difference that after getting

\[
\Theta_n = \Theta_{on} - \Theta_{dn}, \ \Theta'_{n} \text{ is calculated, using a table of}
\]

table of natural logarithms from equation (4.1-7), thus only one step more is added to the steps taken in calculating the ordinates in the linear case.
The response to \( \phi_d = u(t), 5u(t), 10u(t), -5u(t), \) and \(-10u(t)\) has been calculated in this case. The curves obtained are drawn in fig. (4.1) except that for \( \phi_d = u(t) \) as it is close to the curve obtained using the linear regulating unit, which is also drawn in the same figure. The values of the ordinates are tabulated in Appendix B, table (1).

4.2. Constant Load Equations - Proportional + Integral Control Mode.

1. Linear Regulating Unit:

The equation in this case is equation (3.6-3) which has been solved in section (3.6) by the Laplace Transform method and will be solved numerically for comparison. Thus in equation (3.6-3) putting

\[ \phi' = \phi + 0.1 \int \phi \, dt \quad \ldots \ldots \ldots (4.2-1) \]

results in

\[ (5 D^3 + 21 D^2 + 18 D + 1) \phi = -15 \phi' \quad (4.2-2) \]

which is the same as equation (4.1-1) with \( \phi' \) as given by equation (4.2-1) instead of \( \phi \).

Then the regression equations (4.1-5) with \( \phi' \) instead of \( \phi \) can be used and, in calculating \( \phi_n \) from \( \phi_n \), the integral of \( \phi \) at each ordinate position is evaluated using formula (2.2-11) derived in section (2.2) for numerical integration.
FIG. 4.1 Response to Step Changes in the Desired Value - Proportional Control Mode.

(a) Linear Regulating Unit-5% Step Increase.
(b) Exponential Regulating Unit-5% Step Increase.
(c) Exponential Regulating Unit-5% Step Decrease (Reversed)
(d) Exponential Regulating Unit-10% Step Increase.
(e) Exponential Regulating Unit-10% Step Decrease (Reversed)
Thus 3 steps more are added to those taken in calculating the ordinates in the linear case section (4.1), these being $0.1 \frac{\delta}{2} (\theta_n + \theta_{n-1})$,

$$0.1 (\int \sigma)_n = 0.1 \frac{\delta}{2} (\theta_n + \theta_{n-1}) + 0.1 (\int \sigma)_{n-1}$$ using formula (2.2-11) and $\theta'_n = \theta_n + 0.1 (\int \sigma)_n$.

To make this clear the start of the solution is given by the following table using the first of the regression equations (4.1-5) for $\delta = \%$

<table>
<thead>
<tr>
<th>$-\theta_n$</th>
<th>2.4986</th>
<th>4.9392</th>
<th>4.9596</th>
<th>4.9976</th>
<th>4.7959</th>
<th>4.5522</th>
<th>4.4672</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.00525$</td>
<td>0.0156</td>
<td>0.0463</td>
<td>0.0622</td>
<td>0.0616</td>
<td>0.0606</td>
<td>0.0591</td>
<td>0.0570</td>
</tr>
<tr>
<td>$(\theta_n + \theta_{n-1})$</td>
<td>0.0156</td>
<td>0.0624</td>
<td>0.1246</td>
<td>0.1862</td>
<td>0.2463</td>
<td>0.3059</td>
<td>0.3629</td>
</tr>
<tr>
<td>$-0.1(\int \sigma)_n$</td>
<td>2.5142</td>
<td>5.0516</td>
<td>5.0342</td>
<td>5.0333</td>
<td>5.0427</td>
<td>4.9531</td>
<td>4.8301</td>
</tr>
<tr>
<td>$-\theta_n$</td>
<td>0</td>
<td>7.5426</td>
<td>15.1548</td>
<td>15.2526</td>
<td>15.2514</td>
<td>15.1231</td>
<td>14.8743</td>
</tr>
<tr>
<td>$-3 \theta_{n-1}$</td>
<td>0</td>
<td>0</td>
<td>7.5426</td>
<td>15.1548</td>
<td>15.2526</td>
<td>15.2514</td>
<td>15.1231</td>
</tr>
<tr>
<td>$-3 \theta_{n-2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.5142</td>
<td>5.0516</td>
<td>5.0342</td>
<td>5.0333</td>
</tr>
<tr>
<td>$-\theta'_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0.00574 \times \sum \text{sum}$</td>
<td>0.0014</td>
<td>0.0072</td>
<td>0.0159</td>
<td>0.0213</td>
<td>0.0233</td>
<td>0.0232</td>
<td>0.0229</td>
</tr>
</tbody>
</table>

| $2.5445$ | 0 | 0.0036 | 0.0275 | 0.1023 | 0.2606 | 0.5193 | 0.8350 |
| $-2.1335$ | 0 | 0 | -0.0030 | -0.0230 | -0.0362 | -0.2185 | -0.4354 |
| $0.5337$ | 0 | 0 | 0 | 0.0003 | 0.0064 | 0.0233 | 0.0603 |
| $\sigma_{on}$ | 0.0014 | 0.0103 | 0.0404 | 0.1024 | 0.2041 | 0.3473 | 0.5328 |
It is seen from this table how the 3 steps mentioned above are added to those of the previous section. Thus at \( t = 0 \) assuming \( \Theta_{on} = 0 \) then \( \Theta_n = -2.5 \) and as \( S \) at the start is taken as \( \frac{1}{5} \) then \( 0.1 \frac{\Theta}{S} = 0.00625 \), thus \( 0.1 \frac{\Theta}{S} (\Theta_n - \Theta_{n-1}) = -0.0156 \) and as the integral before \( t = 0 \) equals zero, then \( 0.1 (\int \Theta) = -0.0156 \), thus \( \Theta_n = \Theta_{n-1} + 0.1 (\int \Theta) = -2.5142 \) which leads to \( \Theta_{on} = 0.0014 \) and if the calculations are repeated using this value of \( \Theta_0 \) we will get the same result, then we proceed to the next, and so on.

The following table compares the values of the ordinates of \( \Theta_0 \) as calculated from the L.T. solution of equation (3.6-3) given by equation (3.6-12), and the numerical solution. The response in the numerical solution being calculated for \( \Theta_d = 5 \, u(t) \), the ordinates are then divided by 5.

<table>
<thead>
<tr>
<th>t sec.</th>
<th>( \Theta_{on} ) by L.T.</th>
<th>( \Theta_{on} ) numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0337</td>
<td>0.0903</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2022</td>
<td>0.2023</td>
</tr>
<tr>
<td>1</td>
<td>0.4572</td>
<td>0.4574</td>
</tr>
<tr>
<td>1.5</td>
<td>0.7401</td>
<td>0.7391</td>
</tr>
<tr>
<td>2</td>
<td>0.9927</td>
<td>0.9913</td>
</tr>
<tr>
<td>2.5</td>
<td>1.1799</td>
<td>1.1795</td>
</tr>
<tr>
<td>3</td>
<td>1.2875</td>
<td>1.2330</td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec.</th>
<th>( \Theta_{on} ) by L.T.</th>
<th>( \Theta_{on} ) numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.3195</td>
<td>1.2230</td>
</tr>
<tr>
<td>4</td>
<td>1.2285</td>
<td>1.0671</td>
</tr>
<tr>
<td>5</td>
<td>1.0671</td>
<td>0.9526</td>
</tr>
<tr>
<td>6</td>
<td>0.9526</td>
<td>0.9441</td>
</tr>
<tr>
<td>7</td>
<td>0.9441</td>
<td>0.9781</td>
</tr>
<tr>
<td>8</td>
<td>0.9781</td>
<td>1.0182</td>
</tr>
<tr>
<td>9</td>
<td>1.0182</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Exponential Regulating Unit:

The corresponding equation using the exponential regulating unit is obtained from equation (3.4-4) by putting

\[ F(\theta) = 0.15 (\theta + 0.1 \int \theta \, dt) \]

Thus we get

\[ (5 \, d^3 + 21 \, d^2 + 18 \, d + 1) \, \phi = -100 \left( 0.15 (\theta + 0.1 \int \theta \, dt) \right) \]

Putting \( \phi' = - \frac{100}{15} \left( 0.15 (\theta + 0.1 \int \theta \, dt) \right) \)

results in

\[ (5 \, d^3 + 21 \, d^2 + 18 \, d + 1) \, \phi = -15 \, \phi' \]

which is the same as equation (4.1-3) with \( \phi' \) as given by equation (4.2-4); thus the same regression equations (4.1-5) are used and the same procedure of calculating the ordinates in the linear case above is used with \( \phi'_n \) calculated from equation (4.2-4).

The response to \( \phi_0 = u(t), 5 \, u(t) \) and \(-5 \, u(t)\) has been calculated in this case, and the curves obtained for the latter two are drawn in fig. (4.2) together with the curve of linear case. The numerical values of the ordinates are tabulated in Appendix B, table (2). Control Mode.

4.3 Constant Load Equations — Proportional + Derivative Control Mode.

1. Linear Regulating Unit:

The equation in this case is equation (5.6-13). The numerical solution follows the same steps taken in the
FIG. 4.2 Response to 5% Step Change in the Desired Value - Proportional+Integral Control Mode.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) Exponential Regulating Unit - Step Decrease (Reversed...
linear case section (4.2) with
\[ \phi' = \phi + 0.2 \frac{d\phi}{dt} \] .............. (4.3-1)

As we are calculating the response to a step change in the desired value, thus \( \phi \) being equal to \( \phi_o - \phi_d \), then at \( t = 0 \) where \( \phi_o = 0 \), \( \phi = -\phi_d \), i.e. it has a step at \( t = 0 \). This results in \( d\phi/dt \) having an impulse at \( t = 0 \). Thus the solution in this case is to be started with a smaller \( \phi \) than in the last 2 sections. In the period from \( t = 0 \) to \( t = 1 \), is then taken equal to \( 1 \) sec. instead of \( \frac{1}{2} \) sec. and the regression equation for this period becomes:
\[ a_{on} = -0.000001 (\phi_n + 3 \phi_{n-1} + 3 \phi_{n-2} + \phi_{n-3}) + 2.7563 a_{on-1} - 2.525 a_{on-2} + 0.7687 a_{on-3} \] .............. (4.3-2)

Then the regression equations (4.1-5) with the first equation replaced by equation (4.3-2) are used putting \( \phi' \) as given by equation (4.3-1) instead of \( \phi \). The derivative of \( \phi \) at each ordinate position is evaluated using formula (2.2-9). Thus instead of the 3 steps added in section (4.2) to the columns of ordinate calculation in section (4.1), the following 3 steps are added:
\[ 0.2 \frac{2}{5}(\phi_n - \phi_{n-1}), 0.2(d\phi/dt)_n = 0.2 \frac{2}{5}(\phi_n - \phi_{n-1}) - 0.2(d\phi/dt)_{n-1} \] from equation (2.2-9), and \( \phi_n' = \phi_n + 0.2(d\phi/dt)_n \).
The solution is started as given by the following table:

<table>
<thead>
<tr>
<th>t sec</th>
<th>0</th>
<th>1/16</th>
<th>0.125</th>
<th>3/16</th>
<th>0.25</th>
<th>5/16</th>
<th>0.375</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_n$</td>
<td>2.4935</td>
<td>4.9910</td>
<td>4.9723</td>
<td>4.9425</td>
<td>4.8939</td>
<td>4.8454</td>
<td>4.7795</td>
</tr>
<tr>
<td>$-6.4(\phi_n - \phi_{n-1})$</td>
<td>15.9904</td>
<td>15.9520</td>
<td>-0.1153</td>
<td>-0.1946</td>
<td>-0.2726</td>
<td>-0.3483</td>
<td>-0.4218</td>
</tr>
<tr>
<td>$0.2(\phi_n'/dt)_{n-1}$</td>
<td>15.9904</td>
<td>-0.0384</td>
<td>-0.0774</td>
<td>-0.1172</td>
<td>-0.1556</td>
<td>-0.1934</td>
<td>-0.2214</td>
</tr>
<tr>
<td>$-3 \phi_{n-1}$</td>
<td>0</td>
<td>55.4667</td>
<td>14.8573</td>
<td>14.6955</td>
<td>14.4759</td>
<td>14.2335</td>
<td>13.9560</td>
</tr>
<tr>
<td>$-3 \phi_{n-2}$</td>
<td>0</td>
<td>0</td>
<td>55.4667</td>
<td>14.8573</td>
<td>14.6865</td>
<td>14.4759</td>
<td>14.2335</td>
</tr>
<tr>
<td>$-\phi_{n-3}'$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>18.4389</td>
<td>4.9525</td>
<td>4.8955</td>
<td>4.8253</td>
</tr>
<tr>
<td>0.000031 X sum</td>
<td>0.0015</td>
<td>0.0049</td>
<td>0.0061</td>
<td>0.0043</td>
<td>0.0031</td>
<td>0.0031</td>
<td>0.0030</td>
</tr>
<tr>
<td>2.7532 $\phi_{on-1}$</td>
<td>0</td>
<td>0.0041</td>
<td>0.0243</td>
<td>0.0747</td>
<td>0.1535</td>
<td>0.2759</td>
<td>0.4261</td>
</tr>
<tr>
<td>$-2.5254 \phi_{on-2}$</td>
<td>0</td>
<td>0</td>
<td>-0.0033</td>
<td>-0.0227</td>
<td>-0.0594</td>
<td>-0.1452</td>
<td>-0.2523</td>
</tr>
<tr>
<td>0.7537 $\phi_{on-3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0012</td>
<td>0.0069</td>
<td>0.0203</td>
<td>0.0442</td>
</tr>
<tr>
<td>$\phi_{on}$</td>
<td>0.0015</td>
<td>0.0030</td>
<td>0.0271</td>
<td>0.0575</td>
<td>0.1001</td>
<td>0.1546</td>
<td>0.2205</td>
</tr>
</tbody>
</table>

At the zero ordinate position assuming $\phi_{on} = 0$ then $\phi_n = -2.5$ and as $\frac{\phi}{\delta} = \frac{1}{16}$ then $0.2 \frac{\phi}{\delta} = 6.4$, thus $0.2 \frac{\phi}{\delta} (\phi_n - \phi_{n-1}) = -16$ and as the derivative before $t = 0$ is zero then $0.2 (\phi_n'/dt)_{n-1} = -16$. Then $\phi_n' = -2.5 - 16 = -18.5$ leading to $\phi_{on} = 0.0015$ and if the calculations are repeated the values in the column will be changed slightly but the result will still be $\phi_{on} = 0.0015$, then we proceed to the next, and so on.
The following table compares the values of the ordinates calculated from the L.T. solutions given by equation (3.6-17) and the numerical solution where the response to $\theta_d = 5 u(t)$ is calculated; the ordinates are then divided by 5.

<table>
<thead>
<tr>
<th>t sec.</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{on}$ by L.T.</td>
<td>0.0764</td>
<td>0.2734</td>
<td>0.5177</td>
<td>0.7461</td>
<td>0.9214</td>
<td>1.0387</td>
<td>1.0791</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>0.0769</td>
<td>0.2729</td>
<td>0.5171</td>
<td>0.7461</td>
<td>0.9222</td>
<td>1.0325</td>
<td>1.0813</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec.</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{on}$ by L.T.</td>
<td>1.0312</td>
<td>1.0162</td>
<td>0.9464</td>
<td>0.9156</td>
<td>0.9170</td>
<td>0.9294</td>
<td>0.9397</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>1.0334</td>
<td>1.0174</td>
<td>0.9464</td>
<td>0.9143</td>
<td>0.9158</td>
<td>0.9293</td>
<td>0.9395</td>
</tr>
</tbody>
</table>

2. Exponential Regulating Unit:

The corresponding equation using the exponential regulating unit is obtained from equation (3.4-4) by putting $f(\theta) = 0.15 (\theta + 0.2 \frac{d\theta}{dt})$. Thus we get

$$(5 D^3 + 21 D^2 + 13 D + 1) \theta = 100 (e^{0.15(\theta + 0.2 \frac{d\theta}{dt})})$$

...(4.3-3)

then putting $e^t = -\frac{100}{15} (e^{0.15(\theta + 0.2 \frac{d\theta}{dt})} - 1)$...(4.3-4)

results in

$$(5 D^3 + 21 D^2 + 13 D + 1) \theta = -15 e^t$$

...(4.3-5)

which is the same as equations (4.2-5) and (4.1-3) with $e'$ as given by equation (4.3-4), thus the regression equations (4.1-5) and (4.3-2) with $e'$ as given by (4.3-4) instead of $\theta$ are used, and the procedure of calculating the ordinates is
the same as above in the linear case, i.e. \( \frac{d\phi}{dt} \) is calculated for each ordinate position using formula (2.2-9) and then \( \phi_n^1 \) from equation (4.3-4).

The response to \( \phi_d = u(t), 5u(t) \) and \(-5u(t)\) has been calculated in this case, and the curves obtained from the later two are drawn in fig. (4.5), together with the curve of the linear case. The numerical values of the ordinates are tabulated in Appendix B, table (3).

4.4 Constant Load - 3-term Control Mode

1. Linear Regulating Unit:

The equation in this case is equation (5.6-13). The procedure of numerical solution followed in sections (4.2) and (4.3) is also adopted here, i.e. in equation (5.6-13) we put \( \phi' = \phi + 0.1 \int \phi \, dt + 0.2 \, \frac{d\phi}{dt} \) \hspace{1cm} (4.4-1)

Thus the equation reduces to the same form of equation (4.2-2) with \( \phi' \) as given by equation (4.4-1). Then the regression equations (4.1-5) and (4.3-2) are used with \( \phi' \) as given by (4.4-1) instead of \( \phi \), and instead of the steps added in sections (4.2) and (4.3) to the columns of ordinate calculations, the following 5 steps are added to those of section (4.1):

\[ 0.1 \, \frac{c}{2} (\phi_n + \phi_{n-1}) \]
\[ \text{then } 0.1 \left( \int \phi \right)_n \text{ evaluated using formula (2.2-11), then } 0.2 \, \frac{c}{6} (\phi_n - \phi_{n-1}), \text{ then } 0.2 \left( \frac{d\phi}{dt} \right)_n \]

evaluated using formula (2.2-9), then

\[ \phi_n^1 = \phi_n + 0.1 \left( \int \phi \right)_n + 0.2 \left( \frac{d\phi}{dt} \right)_n \].
FIG. 4.3 Response to 5\% Step Change in the Desired Value - Proportional+Derivative Control Mode.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) Exponential Regulating Unit - Step Decrease (Reversed).
The following table compares the values of the ordinates of \( \Theta_o \) as calculated from L.T. solution of equation (3.6-18) for \( \Theta_d = u(t) \), given by equation (3.6-22), and from the numerical solution where the response to \( \Theta_d = 5 u(t) \) is calculated, and then the ordinates are divided by 5.

<table>
<thead>
<tr>
<th>( t ) sec.</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta_o ) by L.T.</td>
<td>0.0753</td>
<td>0.2792</td>
<td>0.5321</td>
<td>0.7900</td>
<td>0.9937</td>
<td>1.1313</td>
</tr>
<tr>
<td>( \Theta_o ) numerical</td>
<td>0.0775</td>
<td>0.2804</td>
<td>0.5401</td>
<td>0.7924</td>
<td>0.9962</td>
<td>1.1336</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t ) sec.</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta_o ) by L.T.</td>
<td>1.2103</td>
<td>1.1546</td>
<td>1.0616</td>
<td>1.0027</td>
<td>0.9927</td>
<td>1.0032</td>
</tr>
<tr>
<td>( \Theta_o ) numerical</td>
<td>1.2203</td>
<td>1.1545</td>
<td>1.0599</td>
<td>1.0027</td>
<td>0.9906</td>
<td>1.0024</td>
</tr>
</tbody>
</table>

2. **Exponential Regulating Unit**

The corresponding equation using the exponential regulating unit is obtained from equation (3.4-4) by putting:

\[ R(\Theta) = 0.15 (\Theta + 0.1 \int \Theta \, dt + 0.2 \Theta \, \frac{d\Theta}{dt}) \]

thus we get

\[ (5 \, D^3 + 21 \, D^2 + 18 \, D + 1) \, \Theta_o = 100(2^{0.15(\Theta+0.1 \int \Theta \, dt+0.2\Theta \, \frac{d\Theta}{dt})}) \]

\[ \Theta_o = 15 \, \Theta' \]

\[ (5 \, D^3 + 21 \, D^2 + 18 \, D + 1) \, \Theta_o = -15 \, \Theta' \]

results in

\[ (5 \, D^3 + 21 \, D^2 + 18 \, D + 1) \, \Theta_o = -15 \, \Theta' \]

which is the same as equations (4.4-5), (4.2-5) and (4.1-3) with \( \Theta' \) as given by equation (4.4-3), thus the same
regression equations (4.1-5) and 4.3-2) with \( e' \) as given by equation (4.4-3) instead of \( e \) are used, and the procedure of calculating the ordinates is the same as above in the linear case only with \( e' \) as given by (4.4-3).

The response to \( e = u(t), 5u(t) \) and \( -5u(t) \) has been calculated in this case, and the curves obtained for the later two are drawn in fig. (4.4) together with the curve of the linear case. The numerical values of the ordinates are tabulated in Appendix B, table (4).

4.5. Variable Load Equations - Proportional Control Mode.

In section (3.4) equations (3.4-3) and (3.4-5) were derived for step changes in load, then in section (3.7) equation (3.4-8), being a linear equation, was solved by L.T. method for a 25% step increase and decrease in load.

In this section equations (3.4-3) and (3.4-9) will be solved numerically for the same load changes. Equation (3.4-3) will be solved numerically for comparison with the L.T. solution.

1. 25% Step Increase in Load - Linear Regulating Unit:

The equation for this case is equation (3.7-1) obtained from equation (3.4-3) by putting \( F(\theta) = 0.15 \theta \) for proportional control mode.
FIG. 4.4 Response to 5% Step Change in the Desired Value - 3-Term Control Mode.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) Exponential Regulating Unit-Step Decrease(Reversed).
Rewriting this equation for convenience:
\[(x^3 + 4.25 x^2 + 3.75 x + 0.25) e_0 = -3 - 5(x^2 + 3x + 1)u(t)\]
\[\quad \quad \quad \quad \quad (4.5-1)\]
This equation except for the term \(-5(x^2 + 3x + 1)u(t)\)
is of similar form to equation (4.1-1), thus this part isconsidered first.

The ordinate sequence of a unit-step function is givenin section (2.3) as
\[u(t) \equiv (0, 5, 1, 1, \ldots \text{etc.}) \quad (4.5-2)\]
with the 0.5 ordinate at \(t = 0\) and the derivative of aunit-step function was also given in section (2.3) as
\[D u(t) \equiv (\frac{5}{3}, 0, 0, \ldots \text{etc.}) \quad (4.5-3)\]
Now it is not possible to find \(D^2 u(t)\) from this, as ithas only one value at \(t = 0\), and its shape is not defined,i.e. the conditions for applying the operator to it are notsatisfied. But returning to section (2.3), in approximatingthe unit-step function at \(t = 0\) it was made to start from\(t = -\delta/2\) parabolically and attaining the unity value at\(t = \delta\) also parabolically, thus having a value of 0.125 at\(t = -\delta/2\) and a value of 0.875 at \(t = \delta/2\). Thus if thesevalues are considered, the slope of \(D u(t)\) will be definedand then we can proceed to get \(D^2 u(t)\).

To be able to consider these values, the ordinate sequence of a unit step function must be taken as:
\[ u(t) \equiv (0.125, 0.5, 0.875, 1, 1, 1 \ldots \text{etc.}) \ldots (4.5-4) \]

\( \zeta \) should then be taken smaller than if the ordinate sequence was taken as in \((4.5-2)\).

Applying the differentiating operator \( \frac{2}{\zeta} \left( \frac{1}{1}, -1 \right) \) to \((4.5-4)\) results in

\[ D u(t) \equiv \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, 0, \ldots \text{etc.} \right) \ldots (4.5-5). \]

The first ordinate is seen to be half the second one, meaning that the function has a step at \( t = -\delta \) (as the first ordinate is at \( t = -\delta \)) and another step at \( t = \delta \) when the ordinate is half the previous one and the function becomes zero after this.

As the original rule for applying the operator to a step function that at the position where the step occurs the ordinate is made half the value of the step, then the operator can be applied to \((4.5-5)\) resulting in:

\[ D^2 u(t) \equiv \left( \frac{1}{2}, 0, -\frac{1}{2}, 0, 0, 0, \ldots \text{etc.} \right) \ldots (4.5-6) \]

Thus from \((4.5-4, 5 \& 6)\) we get the ordinate sequence of

\[ -5(D^2 + 3D + 1) u(t) \equiv -\left[ (0.625 + \frac{15}{16} + \frac{5}{8}), (2.5 + \frac{15}{16}), (4.375 + \frac{15}{16} - \frac{5}{8}), 5, 5, \ldots \text{etc.} \right] \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots (4.5-7) \]

from which it is seen that this part of equation \((4.5-1)\) will have a constant value after the first few ordinates.

Thus following the same steps taken in section \((4.1)\) in deriving the regression equation \((4.1-4)\) from equation \((4.1-1)\) the regression equation of equation \((4.5-1)\) can be
deduced as:

$$
\phi_{on} = -\frac{3}{a^1} (\phi_n + 3\phi_{n-1} + 3\phi_{n-2} + \phi_{n-3}) - \frac{40}{a^1} - \frac{b^1}{a^1} \phi_{on-1} - \frac{c^1}{a^1} \phi_{on-2} - \frac{d^1}{a^1} \phi_{on-3} \quad \cdots \cdots \cdots \cdots \quad (4.5-3)
$$

with the coefficients $a^1$, $b^1$, $c^1$ and $d^1$ of the same nature as $a$, $b$, $c$ and $d$ in equation $(4.1-4)$ and are given by:

$$
\begin{align*}
\frac{a^1}{\delta^1} &= \left(\frac{8}{\delta^1} + \frac{17}{\delta^1} + 7\frac{5}{\delta^1} + 0.25\right) \\
\frac{b^1}{\delta^1} &= \left(-\frac{24}{\delta^1} - \frac{17}{\delta^1} + 7\frac{5}{\delta^1} + 0.75\right) \\
\frac{c^1}{\delta^1} &= \left(\frac{24}{\delta^1} - \frac{17}{\delta^1} - 7\frac{2}{\delta^1} + 0.75\right) \\
\frac{d^1}{\delta^1} &= \left(-\frac{8}{\delta^1} + \frac{17}{\delta^1} - 7\frac{5}{\delta^1} + 0.25\right)
\end{align*}

\begin{align*}
\frac{a^1}{\delta^1} &= \left(\frac{8}{\delta^1} + \frac{17}{\delta^1} + 7\frac{5}{\delta^1} + 0.25\right) \\
\frac{b^1}{\delta^1} &= \left(-\frac{24}{\delta^1} - \frac{17}{\delta^1} + 7\frac{5}{\delta^1} + 0.75\right) \\
\frac{c^1}{\delta^1} &= \left(\frac{24}{\delta^1} - \frac{17}{\delta^1} - 7\frac{2}{\delta^1} + 0.75\right) \\
\frac{d^1}{\delta^1} &= \left(-\frac{8}{\delta^1} + \frac{17}{\delta^1} - 7\frac{5}{\delta^1} + 0.25\right)
\end{align*}

\end{align*}

and the constant term $-\frac{40}{a^1}$ resulting from multiplying $-5(D^2 + 3D + 1)u(t)$ by $\frac{1}{a^1}(1, 3, 3, 1)$ which from $(4.5-7)$ is seen to be equal to $-\frac{40}{a^1}$ for all ordinates except the first few ones.

To start the solution $\delta$ should be chosen smaller than it has been in section $(4.1)$ where the solution is started by $\delta = \frac{1}{2}$, but reduced to $\frac{1}{16}$ in section $(4.3)$. Thus here the solution is started by $\delta = \frac{1}{32}$ from $t = 0$ to $t = \frac{1}{4}$ then increased to $\frac{1}{16}$ from $t = \frac{1}{4}$ to $t = \frac{1}{2}$, then increased to $\frac{1}{8}$ from $t = \frac{1}{2}$ to $t = 2$ and again to $\frac{1}{8}$ from $t = 2$ to $t = 7$ and finally to $\frac{1}{2}$ from $t = 7$ to $t = 12$ secs.
Thus the regression equations for these periods are

for $\delta = \frac{1}{32}$

$$
\theta_{on} = -0.000011(\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) - 0.0001
+ 2.8721 \theta_{on-1} - 2.7477 \theta_{on-2} + 0.3756 \theta_{on-3}
$$

For the first six ordinates starting at $t = -\delta = -\frac{1}{32}$

the constant term being $= -0.0006, -0.0096, -0.0296, -0.0266,
0.0139, 0.0252, 0.0086$ respectively.

for $\delta = \frac{1}{16}$

$$
\theta_{on} = -0.000051(\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) - 0.0011
+ 2.7534 \theta_{on-1} - 2.5197 \theta_{on-2} + 0.7663 \theta_{on-3}
$$

for $\delta = \frac{1}{8}$

$$
\theta_{on} = -0.000572 (\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) - 0.0076
+ 2.539 \theta_{on-1} - 2.1244 \theta_{on-2} + 0.535 \theta_{on-3}
$$

for $\delta = \frac{1}{4}$

$$
\theta_{on} = -0.00363 (\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) - 0.0491
+ 2.1327 \theta_{on-1} - 1.5164 \theta_{on-2} + 0.3313 \theta_{on-3}
$$

and for $\delta = \frac{1}{2}$

$$
\theta_{on} = -0.02037 (\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) - 0.2716
+ 1.6537 \theta_{on-1} - 0.7453 \theta_{on-2} + 0.073 \theta_{on-3}
$$
The procedure of calculating the ordinates is the same as in section (4,1), tables are made each column of which runs down as follows:

\[ t, \ θ_n, 3 θ_{n-1}, 3 θ_{n-2}, θ_{n-3}, -\frac{2}{a'} \text{ into sum of } θ^S, \]

\[ -\frac{4a}{a'} (\text{the constant term}), -\frac{b'}{a'} θ_{n-1}, -\frac{c'}{a'} θ_{n-2}, \]

\[ -\frac{a'}{a'} θ_{n-3} \]

and finally θ_{on}.

The ordinates are calculated by trial and error as before, although in this linear case it was possible to put \( θ = θ_o - θ_d = θ_o \), \( θ_d \) being equal to zero because the desired value is kept constant, and thus obtaining the ordinates of \( θ_o \) directly, but as this is not possible in case of the exponential regulating unit, it has been done in this way for comparison and to save calculating regression equations for each case.

The following table shows how the solution is started, using the first of the regression equations (4,5-10) for \( \delta = \frac{1}{32} \) starting at \( t = -\delta = -\frac{1}{32} \).
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{t sec} & -1/32 & 0 & 1/32 & 1/16 & 3/32 & 5/32 \\
\hline
- \theta_n & 0.0096 & 0.0572 & 0.1605 & 0.2933 & 0.4406 & 0.5777 & 0.7099 \\
- 3\theta_{n-1} & 0 & 0.0233 & 0.1716 & 0.4815 & 0.9049 & 1.3213 & 1.7331 \\
- 3\theta_{n-2} & 0 & 0 & 0.0233 & 0.1716 & 0.4815 & 0.9049 & 1.3213 \\
- \theta_{n-3} & 0 & 0 & 0 & 0.0096 & 0.0572 & 0.1605 & 0.2933 \\
\hline
0.00001X \text{ sum} & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\hline
\text{Constant term} & -0.0096 & -0.0233 & -0.0233 & -0.0139 & -0.0252 & -0.0086 & -0.0001 \\
\hline
2.87213\theta_{n-1} & 0 & -0.0276 & -0.1643 & -0.4610 & -0.8654 & -1.2654 & -1.5992 \\
2.74773\theta_{n-2} & 0 & 0 & 0.0264 & 0.1572 & 0.4410 & 0.8193 & 1.2105 \\
0.87562\theta_{n-3} & 0 & 0 & 0 & -0.0084 & -0.0501 & -0.1405 & -0.2612 \\
\hline
\theta_{on} & -0.0096 & -0.0572 & -0.1605 & -0.2933 & -0.4406 & -0.5777 & -0.7099 \\
\hline
\end{array}
\]

It is clear from the table that 0.00001X sum of \( \theta \)'s is negligible at the start with \( \lambda = \frac{1}{22} \), thus the ordinates are calculated directly. For example, at \( t = -\frac{1}{22} \), \( \theta_{on} \) equals the constant term in the regression equation, i.e. = -0.0096 at this ordinate position. Then at \( t = 0 \), \( \theta_{on} = \) the constant term + 2.8721 x 0.0096 = -0.0572. Then we proceed to the next ordinate position, and so on.

The following table compares the values of the ordinates of \( \theta \) as calculated from the \( h, T \), solution of equation (4.5-1) given by equation (3.7-4) and from the
### Numerical Solution Using Equations (4.5-10)

<table>
<thead>
<tr>
<th>t sec</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.625</th>
<th>0.75</th>
<th>0.875</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_n$ by L.T.</td>
<td>0.5796</td>
<td>1.0312</td>
<td>1.5174</td>
<td>1.8965</td>
<td>2.2242</td>
<td>2.5041</td>
<td>2.7339</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>0.5777</td>
<td>1.0791</td>
<td>1.5145</td>
<td>1.8929</td>
<td>2.2294</td>
<td>2.5011</td>
<td>2.7374</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_n$ by L.T.</td>
<td>2.9316</td>
<td>3.3269</td>
<td>3.2432</td>
<td>2.8623</td>
<td>2.3572</td>
<td>1.8637</td>
<td>1.4762</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>2.9315</td>
<td>3.3306</td>
<td>3.2473</td>
<td>2.8651</td>
<td>2.3592</td>
<td>1.8663</td>
<td>1.4767</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec</th>
<th>4.5</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_n$ by L.T.</td>
<td>1.2293</td>
<td>1.1240</td>
<td>1.2160</td>
<td>1.4494</td>
<td>1.6053</td>
<td>1.6347</td>
<td>1.5900</td>
</tr>
<tr>
<td>$\theta_{on}$ numerical</td>
<td>1.2283</td>
<td>1.1225</td>
<td>1.2141</td>
<td>1.4493</td>
<td>1.6097</td>
<td>1.6406</td>
<td>1.5940</td>
</tr>
</tbody>
</table>

2. 25% Step Increase in Load - Exponential Regulating Unit.

In equation (3.4-9) putting $G = 0.05$ and

$P(t) = 0.15 4$ to get the corresponding equation to (4.5-11) results in

$$(D^3 + 4.25 D^2 + 3.75 D + 0.25) \theta_0 = 20 \left( e^{-0.15 \theta} - 1 \right) - 5(D^2 + 3 D + 1) u(t) \ldots \ldots \ldots (4.5-11)$$

then putting

$$\theta' = -\frac{20}{2} \left( e^{-0.15 \theta} - 1 \right) \ldots \ldots \ldots \ldots \ldots (4.5-12)$$

results in:

$$(D^3 + 4.25 D^2 + 3.75 D + 0.25) \theta_0 = -32 - 5(D^2 + 3 D + 1) u(t) \ldots \ldots \ldots (4.5-13)$$
which is the same as equation (4.5-1) with \( \theta' \) as given by (4.5-12) instead of \( \theta \), thus the regression equations (4.5-10) also with \( \theta' \) instead of \( \theta \) are used, and in calculating the ordinates only one step more is added to the steps in each column which is the calculation of \( \theta'_n \) from \( \theta_n \) using equation (4.5-12).

The response calculated is drawn in fig. (4.5-3) together with that of the linear case above.

The numerical values of the ordinates are tabulated in Appendix B, table (5).

3. 25\% Step Decrease in Load - Linear Regulating Unit.

The equation for this case is equation (3.7-5), rewriting it for convenience:

\[
(b'^3 + 4.15 b'^2 + 3.45 b' + 0.15) \theta' = -3 \theta + 5 (b'^2 + 3 b' + 1) u(t) \tag{4.5-14}
\]

which is almost the same as equation (4.5-1) with slightly different coefficients; thus the regression equation can be deduced directly as:

\[
\theta_{on} = \frac{3}{a'} \left( \theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3} \right) + \frac{40}{a''}
\]

\[
- \frac{b''}{a''} \theta_{on-1} - \frac{a''}{a''} \theta_{on-2} - \frac{a}{a'} \theta_{on-3} \tag{4.5-15}
\]

for all ordinates except the first six where the constant term is different.
FIG. 4.5 Response to 30% Step Change in Load -
Proportional Control Mode.
The coefficients $a''$, $b''$, $c''$ and $d''$ are given by:

$$
\begin{align*}
  a'' &= \left( \frac{8}{\delta^3} + \frac{16\cdot6}{\delta^6} + \frac{6\cdot2}{\delta} + 0.15 \right) \\
  b'' &= \left( -\frac{24}{\delta^3} - \frac{16\cdot6}{\delta^6} + \frac{6\cdot2}{\delta} + 0.45 \right) \\
  c'' &= \left( \frac{24}{\delta^3} - \frac{16\cdot6}{\delta^6} - \frac{6\cdot2}{\delta} + 0.45 \right) \\
  d'' &= \left( -\frac{8}{\delta^3} + \frac{16\cdot6}{\delta^6} - \frac{6\cdot2}{\delta} + 0.15 \right)
\end{align*}
$$

Taking the same values of $\delta$ as in the first part of this section, the following regression equations are obtained for $\delta = \frac{1}{32}$

$$
\begin{align*}
  \theta_{on} &= -0.000011(\theta_n + 3\theta_{n-1} + 3\theta_{n-2} + \theta_{n-3}) + 0.0001 \\
  &\quad + 2.87514 \theta_{on-1} - 2.75345 \theta_{on-2} + 0.8733 \theta_{on-3} \\
\end{align*}
$$

The constant term for the first six terms starting with the ordinate at $t = -\frac{1}{32}$ are 0.0095, 0.0297, 0.0227, -0.0133, -0.0252, -0.0035.

for $\delta = \frac{1}{6}$

$$
\begin{align*}
  \theta_{on} &= -0.000031(\theta_n + 3\theta_{n-1} + 3\theta_{n-2} + \theta_{n-3}) + 0.0011 \\
  &\quad + 2.7592 \theta_{on-1} - 2.5303 \theta_{on-2} + 0.7711 \theta_{on-3} \\
\end{align*}
$$

for $\delta = \frac{1}{6}$

$$
\begin{align*}
  \theta_{on} &= -0.000075(\theta_n + 3\theta_{n-1} + 3\theta_{n-2} + \theta_{n-3}) + 0.0077 \\
  &\quad + 2.54924 \theta_{on-1} - 2.1426 \theta_{on-2} + 0.5924 \theta_{on-3} \\
\end{align*}
$$

for $\delta = \frac{1}{6}$

$$
\begin{align*}
  \theta_{on} &= 0.000025(\theta_n + 3\theta_{n-1} + 3\theta_{n-2} + \theta_{n-3}) + 0.0497 \\
  &\quad + 2.2022 \theta_{on-1} - 1.5437 \theta_{on-2} + 0.2400 \theta_{on-3} \\
\end{align*}
$$
and for \( j = \frac{1}{2} \)

\[
\theta_{on} = -0.02073 (\theta_n + 3 \theta_{n-1} + 3 \theta_{n-2} + \theta_{n-3}) + 0.2771 \\
+ 1.6914 \theta_{on-1} - 0.7776 \theta_{on-2} + 0.0779 \theta_{on-3}
\]

The procedure of calculating the ordinates is the same as explained in the first part of this section.

The following table compares the values of the ordinates of \( \theta_0 \) as calculated from the L.T. solution of equation (4.5-14) as given by equation (3.7-7) and from the numerical method using equations (4.5-17)

<table>
<thead>
<tr>
<th>t sec</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.625</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{on} \text{ by L.T.} )</td>
<td>0.5537</td>
<td>1.0943</td>
<td>1.5445</td>
<td>1.9406</td>
<td>2.2872</td>
<td>2.5572</td>
</tr>
<tr>
<td>( \theta_{on} \text{ numerical} )</td>
<td>0.5522</td>
<td>1.0950</td>
<td>1.5451</td>
<td>1.9406</td>
<td>2.2865</td>
<td>2.5565</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_n \text{ by L.T.} )</td>
<td>3.0557</td>
<td>3.5187</td>
<td>3.4675</td>
<td>3.0749</td>
<td>2.5221</td>
<td>1.9623</td>
</tr>
<tr>
<td>( \theta_n \text{ numerical} )</td>
<td>3.0556</td>
<td>3.5203</td>
<td>3.4696</td>
<td>3.0763</td>
<td>2.5225</td>
<td>1.9622</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec</th>
<th>4.5</th>
<th>5.0</th>
<th>6.0</th>
<th>7.0</th>
<th>8.0</th>
<th>9.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{on} \text{ by L.T.} )</td>
<td>1.2062</td>
<td>1.0711</td>
<td>1.1757</td>
<td>1.4743</td>
<td>1.6349</td>
<td>1.7263</td>
</tr>
<tr>
<td>( \theta_{on} \text{ numerical} )</td>
<td>1.2016</td>
<td>1.0653</td>
<td>1.1725</td>
<td>1.4728</td>
<td>1.6399</td>
<td>1.7353</td>
</tr>
</tbody>
</table>

4. 25\% Step Decrease in Load - Exponential Regulating Unit

The corresponding equation using the exponential regulating unit is obtained from equation (3.4-9) by putting
\[ C = -0.05 \] and \[ F(\theta) = 0.15 + \theta \] resulting in
\[
(D^3 + 4.15D^2 + 3.45D + 0.15) \phi_0 = 20 \left( e^{-0.15\theta} - 1 \right) + 5 \left( D^2 + 3D + 1 \right) u(t)
\]
and putting \[ \phi_1 = -\frac{20}{2} \left( e^{-0.15\theta} - 1 \right) \]
results in
\[
(D^3 + 4.15D^2 + 3.45D + 0.15) \phi_0 = -3\phi_1 + 5(D^2 + 3D + 1) u(t)
\]
which is the same as equation (4.5-14) with \[ \phi_1 \] as given by (4.5-19) instead of \[ \phi \], thus the regression equations (4.5-17) also with \[ \phi_1 \] instead of \[ \phi \] can be used.

The response curve calculated is drawn in fig. (4.5-b) together with that obtained from the corresponding linear case above. The numerical values of the ordinates are tabulated in Appendix B, table (5).

4.6 Variable Load Equations - Proportional Plus Integral Control Mode.

In this section equations (3.4-3) and (3.4-9) will be again solved for 25% step increase and decrease in load using proportional + integral control mode.

1. 25% Step Increase in Load - Linear Regulating Unit

The equation for this case is obtained from equation (3.4-3) by putting \[ C = 0.05 \] and \[ F(\theta) = 0.15(\theta + 0.1 \int \theta dt) \]
resulting in

\((D^3 + 4.25D^2 + 3.75D + 0.25) \Theta_o \)

\[ = -3(\Theta + 0.1 \int \Theta \, dt) - 5(D^2 + 3D + 1) u(t) \]...

(4.6-1)

and putting \(\Theta_1 = (\Theta + 0.1 \int \Theta \, dt)\)...

(4.6-2)

results in

\((D^3 + 4.25D^2 + 3.75D + 0.25) \Theta_o = -3\Theta_1 - 5(D^2 + 3D + 1) u(t)\)...

(4.6-3)

which is the same as equation (4.5-1) with \(\Theta_1\) as given by (4.6-2) instead of \(\Theta\), \(0.1 \int \Theta \, dt\) being evaluated at each ordinate position using formula (2.2-11), thus 3 more steps are added to the steps in the columns of ordinate calculation explained in the first part of section (4.5). These are

\[ 0.1 \frac{\delta}{2} (\Theta_n + \Theta_{n-1}), 0.1 (\int \Theta)_n \] using formula (2.2-11) and

\[ \Theta_1 = \Theta_n + 0.1 (\int \Theta)_n. \] The regression equations (4.5-10) with \(\Theta_1\) instead of \(\Theta\) are used.

2. Step Increase of Load - Exponential Regulating Unit:

The corresponding equation in this case is obtained from equation (3.4-2) by putting \(C = 0.05\) and \(f(\Theta) = 0.15(\Theta + 0.1 \int \Theta \, dt)\) resulting in

\((D^3 + 4.25D^2 + 3.75D + 0.25) \Theta_o \)

\[ = 20 \left( e^{-0.15(\Theta + 0.1 \int \Theta \, dt)} - 1 \right) - 5(D^2 + 3D + 1) u(t) \]...

(4.6-4)
and putting \( \Theta' = -\frac{20}{3} (e^{-0.15(\Theta + 0.1) \int \Theta dt}) \) ........(4.6-5)

\[(D^3 + 4.25 D^2 + 3.75 D + 0.25) \Theta_0 = -3\Theta' - 5(D^2 + 3D + 1) u(t) \]

which is the same as equation (4.5-13) only with \( \Theta' \) as given by (4.6-5), thus the regression equations (4.5-10) are used with \( \Theta' \) calculated from (4.6-5) instead of \( \Theta \) and the steps of calculating the ordinates are the same as in the first part of this section above.

The response curve is drawn in fig. (4.6-a) together with that of the corresponding linear case.

3. 25% Step Decrease in Load - Linear Regulating Unit:

The equation for this case is obtained from equation (3.4-8) by putting \( C = -0.05 \) and \( R(\Theta) = 0.15 (\Theta + 0.1) \int \Theta dt \) resulting in

\[(D^3 + 4.15 D^2 + 3.45 D + 0.15) \Theta_0 = -3(\Theta + 0.1) \int \Theta dt + 5(D^2 + 3D + 1) u(t) \]

which is the same as equation (4.5-14) with \( \Theta' \) as given by (4.6-3) instead of \( \Theta \), thus the regression equations (4.5-17) are used with \( \Theta' \) instead of \( \Theta \), the procedure of calculating the ordinates being the same as explained in the first part of this section above.
FIG. 4.6 Response to 25% Step Change in Load - Proportional+Integral Control Mode.
4. Step Decrease in Load - Exponential Regulating Unit:

The corresponding equation in this case is obtained from equation (3.4-9) by putting $\theta = -0.05$ and $F(\theta) = \theta + 0.1\int_0^t \theta dt$ resulting in

$$\begin{align*}
(d^3 + 4.15 d^2 + 3.45 d + 0.15) \theta_0 &= 20 \left( e^{-0.15(\theta + 0.1\int_0^t \theta dt)} - 1 \right) + 5(d^2 + 3d + 1) u(t) \\
\theta^1 &= -\frac{20}{3} \left( e^{-0.15(\theta + 0.1\int_0^t \theta dt)} - 1 \right) \\
(d^3 + 4.15 d^2 + 3.45 d + 0.15) \theta_0 &= -3\theta^1 + 5(d^2 + 3d + 1) u(t)
\end{align*}$$

which is the same as equation (4.6-20) with $\theta^1$ as given by (4.6-11). Thus the regression equations (4.5-17) are used with $\theta^1$ calculated from (4.6-11) instead of $\theta$, and the produce of calculating the ordinates is the same as explained in the first part of this section.

The response curve calculated is drawn in fig. (4.6-b) together with that obtained from the corresponding linear case above.

The numerical values of the ordinates of all the curves calculated in this section is tabulated in Appendix B, Table (6).

Also the response curves for the exponential regulating unit to both step increase and decrease in load are drawn together for each control mode in fig. (4.7) for comparison.
FIG. 4.7 Response to 25% Step Change in Load for the Exponential Regulating Unit.
CHAPTER 5
SIMULATOR DESIGN AND EXPERIMENTAL VERIFICATION
OF THEORETICAL RESULTS

Summary

The design of the simulated process and regulating units is described. The controller was constructed using operational amplifiers. The performance of the parts of the loop were checked experimentally and the response of the closed loop to desired value and load changes under various control modes was determined experimentally, and the results compared with those obtained theoretically in chapter (4).

5.1 Introduction

In chapter (3) the response to step changes in desired value and load has been calculated for the linear regulating unit, by the L.T. method using various control modes. Then in Chapter(4) the same linear equations were solved numerically, the results obtained have been very close, thus showing that this numerical method of solution of differential equations can be depended upon.

Also in chapter (4) the non-linear loop equations, resulting from the use of the exponential regulating unit, have been solved numerically, and as it was not possible to solve them analytically, the results are then to be checked.
practically. However, a simple non-linear differential equation which was solved numerically in section (2.6) and which it was possible to solve it analytically proved that the results are very close, as in the linear equations.

An electrical simulator of the control system used for the investigation has been built and the response to step changes in the desired value and load were recorded for linear and exponential regulating units using various control modes, the records obtained show exact agreement with the calculated curves.

5.2 Process Simulation

The process which was investigated mathematically is equivalent to a three-stage RC network with a load resistance across the condenser of the last stage as shown in fig.(3.1). In the practical simulation of this process it is convenient to change the time scale and RC which when treated mathematically was taken as unity, has been simulated by R=10 K-Ohms and C=10 μF giving a time constant τ of 0.1 sec.

With such a process the response curves obtained using a proportional control mode, should be the same as obtained theoretically, only with the time scale of the theoretical curves changed in the ratio 10 : 1. In case of other control modes, the integral action time $T_i$ and the derivative action time $T_d$ should be taken one tenth of their theoretical
values, and then the response curves will be the same as the theoretical ones with the time scale changed in the ratio 10:1.

It is shown in the circuit diagram fig. (5.1) that the load resistance is made from 3 parts: 40, 10 & 16.5 k-ohms so that at the normal operating point, switch S₁ is closed shorting the 16.6 k-ohms resistance and switch S₂ is open, thus the load resistance is then 50 k-ohms. To make the 25% step increase in load, switch S₂ is closed quickly shorting the 10 k-ohms resistance, thus R/Rₜ is step increased from 0.2 to 0.25 and the current from 2 m.a. to 2.5 m.a.

And to make the 25% step decrease in load from the normal operating point, i.e. with S₁ closed and S₂ opened, S₁ is opened quickly, thus R/Rₜ is step decreased from 0.2 to 0.15 and the current decreased from 2 m.a. to 1.5 m.a.

The output voltage of the process is compared with the desired voltage of 100 volts which is obtained from a stabilized voltage supply of 105 volts. Two resistances 100 k-ohms and 5 k-ohms are connected in series across this voltage and a double switch S₃ is connected across the 5 k-ohms resistance as shown in the circuit diagram fig. (5.1) which is connected from the other side to the controller. By this switch the desired value is step changed from 100 to 105 volts giving \( \theta_d = 5 \ u(t) \) and from 105 to 100 volts giving \( \theta_d = -5u(t) \).
FIG. 5.1  Schematic Diagram of the Simulator.
5.3 The Controller

A three term electronic controller is built from 4 operational amplifiers to give the various control actions of proportional, integral and derivative actions and a summing amplifier. These operational amplifiers are of the parallel feed-back type one of which is shown in fig.(5.2-a) mainly composed of a high gain d-c amplifier. The theory of operation is briefly as follows:

In fig.(5.2-a) $Z_1$ is the input impedance connected to the amplifier, $Z_o$ the feedback impedance, $X_1$ the input voltage to earth, $X_o$ the output voltage to earth and $A$ is the forward gain of the amplifier. The transfer function of this operational amplifier is:

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)Z_1(s) + Z_o(s)}$$

and if $A \gg 1$ then this reduces to $\frac{X_o}{X_1} = \frac{Z_o(s)}{Z_1(s)}$

The negative sign means that the output is of opposite sign to the input, i.e. the amplifier acts as a phase inverter.

Now if $X_1$ is to be multiplied by a constant, then $Z_o$ and $Z_1$ would be two resistances whose ratio equals to this constant, thus to get a proportional action in the controller two resistances are used whose ratio equals to the proportional action factor which is 0.15 in our case.
FIG. 5.2-a  General Operational Amplifier.

FIG. 5.2-b  Function Generation.
It is seen from fig.(5.1) that the input to the controller equals the difference between the desired voltage and the output voltage of the process i.e. \(-\Theta\) (the deviation) thus taking \(Z_1\) as a one meg-ohm resistance and \(Z_0\) as 150 k-ohms resistance, the proportional action factor obtained will be 0.15\(\Theta\).

And if \(X_0\) is to be proportional to the time integral of \(X_1\) i.e. \(Z_0(s)/Z_1(s)\) is proportional to \(1/s\), then \(Z_0\) would be a condenser of capacity \(C\) and \(Z_1\) a resistance \(R\) such that \(1/RC\) equals to the constant of proportionality, and by this arrangement integral action is obtained in the controller with \(1/RC\) equals to integral action factor \(K_0 = K_1 T_1\) (section (3.2)).

The integral action time \(T_1\) was taken theoretically in section 3.6 as 10 secs, but as explained in section 5.2 only one tenth of this is to be taken in the simulator and \(K_1\) being 0.15 then \(K_0\) equals 0.15. Thus taking \(Z_1\) as an 0.6 meg-ohm resistance and \(Z_0\) as a condenser of capacity 10 \(\mu\)F The integral action obtained will be 0.15 integral \(\Theta\).

And if \(X_0\) is to be proportional to the time derivative of \(X_1\) i.e. \(Z_0(s)/Z_1(s)\) is proportional to \(s\), then \(Z_0\) will be a resistance \(R\) and \(Z_1\) a condenser of capacity \(C\) such that \(RC\) equals the constant of proportionality, and by this arrangement derivative action is obtained in the controller with \(RC\) equals to the derivative action factor \(K_0 = K_1 T_1\).
$$K_2 = K_1 T_d \text{ (section 3.2)}$$

The derivative action time $T_d$ was taken theoretically in section(3.6) as 0.2 sec, and thus in the simulator it is taken as 0.02 sec, and $K_1$ being 0.15 then $K_o = 0.003$. Thus taking $Z_1$ as a condenser of capacity 0.02 mF and $Z_o$ as a resistance of 150 K-ohms, the derivative action obtained will be 0.003 d$\theta$/dt.

To be able to sum these control actions a summing amplifier is used, its theory being the same as that of fig.(5.2-a). Assuming various input voltages $X_1, X_2 \ldots$ etc. connected to the amplifier through impedances $Z_1, Z_2 \ldots$ etc. then the output of the summing amplifier will be the sum of the outputs resulting from applying the inputs separately.

i.e. $X_o = -(\frac{X_1}{Z_1(s)} + \frac{X_2}{Z_2(s)} + \ldots \text{ etc } ) Z_o(s)$

and if only the algebraic sum is wanted then $Z_o, Z_1, Z_2 \ldots$ etc would be equal resistances thus giving

$$X_o = -(X_1 + X_2 + \ldots \text{ etc } )$$

Thus in the controller taking the summing resistances and feedback resistance as 100 K-ohms each, the output of the controller will be $-0.15(\theta + \int \theta dt + 0.02 \frac{d\theta}{dt})$ but as explained in sections (5.2 & 3), the output of the controller is to include a constant term of unity, this can be obtained by adjusting the summing amplifier to give one volt output for zero input.
The high gain d-c amplifiers used in these operational amplifiers were designed and built by F.C. PonnaPPa, a research worker in the same department. The circuit diagram of one of these amplifiers is shown in fig.(5.3) from which it is seen that it is a single stage amplifier with positive feedback, giving a gain of about 2000, and having an input and output cathode followers, making it of high input and low output impedences.

5.4 The Regulating Units

1. The linear regulating unit

The linear regulating unit, as explained in section 5.3 must supply the process demand of 2 m.a. at zero deviation and the slope of its characteristic is to be 2 m.a. per volt, thus a pentode tube having a mutual conductance of 2 m.a./volt and which can be biased to give 2 m.a. at zero deviation and having a linear anode current - grid voltage characteristic for ±1 volt change in grid voltage from the bias voltage, would be required.

A (6SN7) Valve is chosen, operated at a 100 volt stabilized screen voltage, and with a cathode resistance adjusted to give the 2 m.a./volt mutual conductance required. The bias voltage giving 2 m.a. at zero deviation is about -1.3 volt which is obtained by adjusting the summing amplifier of the controller to give this bias voltage with zero input.
FIG. 5.3 Circuit Diagram of Operational Amplifier.
The actual characteristic obtained from this valve is shown in fig. (5.4) together with the theoretical one.

2. The Exponential Regulating Unit

For the exponential regulating unit the same pentode tube is used and a function generator is used to shape the grid voltage in accordance with the exponential characteristic required. The output of the exponential regulating unit is given by equation (3.3-7) as

\[ I = 2 e^{(\theta_c - 1)} \text{ m.a.} \]

and as the pentode tube has a mutual conductance of 2 m.a./volt thus the function generator would be required to have an output voltage \( \theta_N = e^{(\theta_c - 1)} \)

Tabulating the values of \( \theta_N \) corresponding to those of \( \theta_c \) taken in steps of 0.2 volt, then 0.1 volt as the characteristic becomes steeper we get

<table>
<thead>
<tr>
<th>( \theta_c ) volts</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_N ) Volts</td>
<td>0.363</td>
<td>0.449</td>
<td>0.549</td>
<td>0.67</td>
<td>0.819</td>
<td>1</td>
<td>1.221</td>
</tr>
<tr>
<td>( \theta_N' = \theta_N - 0.358 )</td>
<td>0</td>
<td>0.081</td>
<td>0.181</td>
<td>0.282</td>
<td>0.451</td>
<td>0.632</td>
<td>0.853</td>
</tr>
<tr>
<td>( 4 \theta_N' )</td>
<td>0</td>
<td>0.324</td>
<td>0.724</td>
<td>1.203</td>
<td>1.804</td>
<td>2.528</td>
<td>3.412</td>
</tr>
</tbody>
</table>
5.4 - The Practical and Theoretical Characteristics of the Regulating Units.

---

**FIG. 5.4** The Practical and Theoretical Characteristics of the Regulating Units.
It is then required to simulate the curve $\theta^i_N$ versus $\theta_c$ as given by the tabulated values above, $\theta_N$ can be easily obtained from $\theta^i_N$ by adding a constant voltage of 0.363 volt to the output of the function generator.

It is usual practise to use thermionic diodes for function generation, but as we are dealing with very low voltages, germanium diodes are used, the generator is designed as follows:

In fig. (5.2-b) the diode $d_1$ will only conduct when the input voltage $\theta_c$ exceeds the back voltage $V_{b1}$, thus the voltage $V_{d1}$ appearing across $R_1$, neglecting the forward resistance of the diode with respect to $R_1$ is

$$V_{d1} = \theta_c - V_{b1}$$

and similarly the voltage $V_{d2}$ appearing across $R_2$ is

$$V_{d2} = \theta_c - V_{b2}$$

$$V_{d3} = \theta_c - V_{b3} \quad \text{... etc.}$$

The sum of these voltages $V_{d1}, V_{d2}, \text{... etc.}$ can be made to follow any prescribed values by suitably choosing $V_{b1}, V_{b2}, \text{... etc.}$, thus it can be made to follow the values

<table>
<thead>
<tr>
<th>$\theta_c$ volts</th>
<th>1.4</th>
<th>1.6</th>
<th>1.7</th>
<th>1.8</th>
<th>1.9</th>
<th>2</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_N$ volts</td>
<td>1.492</td>
<td>1.822</td>
<td>2.014</td>
<td>2.226</td>
<td>2.460</td>
<td>2.718</td>
<td>3.004</td>
</tr>
<tr>
<td>$\theta^i_N = \theta_N - 0.363$</td>
<td>1.124</td>
<td>1.454</td>
<td>1.646</td>
<td>1.858</td>
<td>2.092</td>
<td>2.350</td>
<td>2.636</td>
</tr>
<tr>
<td>$4\theta^i_N$</td>
<td>4.496</td>
<td>5.816</td>
<td>6.534</td>
<td>7.432</td>
<td>8.368</td>
<td>9.400</td>
<td>10.544</td>
</tr>
</tbody>
</table>
of \( \theta'\n \) corresponding to those of \( \theta_c \) in the above table.

Assuming \( V_{b1}, V_{b2}, V_{b3}, \ldots \) etc., then as \( \theta_c \) increases from zero there will appear no voltage across the resistances until \( \theta_c \) exceeds \( V_{b1} \), and then there will only be a voltage

\[
V_{d1} = \theta_c - V_{b1},
\]

until \( \theta_c \) exceeds \( V_{b2} \), and then there will be \( V_{d1} \) and \( V_{d2} \) and so on. But as \( \theta'\n \) is to have a value as \( \theta_c \) increases from zero, then \( V_{b1} \) must be made zero, thus the sum of the voltages \( V_{d1}, V_{d2}, \ldots \) etc., when \( \theta_c \) has a value between zero and \( V_{b2} \), equals \( \theta_c \), and as seen from the table \( \theta'\n \) for \( \theta_c = 0.2 \) is only 0.031, thus the sum of the voltages can never be made equal to \( \theta'\n \), but can be made proportional to \( \theta'\n \), the constant of proportionality being at least 2.5. This constant is taken as 4, as it was not possible to continue the curve to the end by taking it 2.5, thus the sum of the voltage \( V_{d1}, V_{d2}, \ldots \) etc., is to be made equal to 4 \( \theta'\n \).

Now the curve will start with \( 4\theta'\n = V_{d1} = \theta_c \)

as \( V_{b1} \) is made zero, but at \( \theta_c = 0.2 \), \( 4\theta'\n = 0.32 \) from the table thus the second diode must conduct adding a voltage \( V_{d2} \), then \( 4\theta'\n = V_{d1} + V_{d2} = 2\theta_c - V_{b2} \)

i.e., \( 0.32 = 2 \times 0.2 - V_{b2} \) giving \( V_{b2} = 0.03 \) volt then at \( \theta_c = 0.4 \), \( 4\theta'\n = 2 \times 0.4 - 0.03 = 0.72 \) giving \( \theta'\n = 0.18 \)

which is almost the same as given in the table.
At \( \theta_c = 0.6 \), \( 4 \theta_N^i = 1.2 - 0.03 = 1.12 \) giving \( \theta_N^i = 0.28 \), which is less than 0.302 the value given in the table, therefore the third diode must conduct, then

\[
4 \theta_N^i = V_{d1} + V_{d2} + V_{d3} = \theta_c + \theta_c - V_{b2} + \theta_c - V_{b3}
\]

\[
= 3 \theta_c - 0.03 - V_{b3}
\]

i.e., \( 4 \times 0.302 = 3 \times 0.6 - 0.03 - V_{b3} \) giving

\( V_{b3} = 0.5 \) volt, then using this value of \( V_{b3} \) to calculate \( \theta_N^i \) at \( \theta_c = 0.6 \) results in

\[
4 \theta_N^i = 3 \times 0.6 - 0.03 - 0.51 = 1.81
\]

\( \theta_N^i = 0.452 \) which is almost the same as given in the table. But at \( \theta_c = 1 \), \( 4 \theta_N^i = 3 - 0.03 - 0.51 = 2.41 \) giving \( \theta_N^i = 0.602 \) while in the table it is 0.632, therefore the fourth diode must conduct, then

\[
4 \theta_N^i = V_{d1} + V_{d2} + V_{d3} + V_{d4} = 4 \theta_c - 0.03 - 0.51 - V_{b4}
\]

i.e., \( 4 \times 0.632 = 4 - 0.03 - 0.51 - V_{b4} \) giving

\( V_{b4} = 0.33 \) volt, then using this value of \( V_{b4} \) to calculate \( \theta_N^i \) at \( \theta_c = 1.2 \) results in

\[
4 \theta_N^i = 4 \times 1.2 - 0.03 - 0.51 - 0.33 = 3.33
\]

\( \theta_N^i = 0.842 \) which is less than the corresponding value in the table 0.8353, thus the fifth diode must conduct adding a voltage \( V_{d5} \) resulting in \( 4 \theta_N^i = 5 \theta_c - 1.47 - V_{b5} \) where

\( 1.47 = V_{b1} + V_{b2} + V_{b3} \) giving \( V_{b5} = 1.12 \), then if this is used to calculate \( \theta_N^i \) at \( \theta_c = 1.4 \), it will not give the tabulated value and thus the sixth diode must conduct whose
back voltage \( V_{b0} \) is calculated in the same way to be 1.31 volt

The procedure is repeated for \( \theta_c = 1.6 \) resulting in the addition of \( d_7 \) with \( V_{b7} = 1.48 \)

Now if the next step of \( \theta_c \) is taken 1.8, then with 7 diodes: \( 4 \theta^\text{th}_N = 7 \times 1.8 - 5.33 = 7.22 \) (where 5.33 = \( V_{b1} \) + . . . \( V_{b7} \)), from the table it is 7.43, thus the 8th diode must conduct adding a voltage \( V_{d8} \), but calculating the back voltage of this diode \( V_{b8} \) from

\[
7.43 = 8 \times 1.8 - 5.33 - V_{b8} \quad \text{giving}
\]

\( V_{b8} = 1.59 \) which means that \( d_8 \) is added before the step of \( \theta_c = 1.6 \), but this has already been fixed up correctly without \( d_8 \) in a table.

Thus after \( \theta_c = 1.6 \) values of \( \theta_c \) must be taken in steps of 0.1 volt, as the curve is becoming more steep and more than one diode is required between the 0.2 steps of \( \theta_c \) after \( \theta_c = 1.6 \), \( \theta^\text{th}_N \) corresponding to \( \theta_c = 1.7 \) is 1.646, then \( 4 \times 1.646 = 8 \times 1.7 - 5.33 - V_{b8} \) giving

\( V_{b8} = 1.64 \); the procedure is then repeated for the rest of the values of \( \theta_c \) in the table giving \( V_{b9} = 1.75 \)

\( V_{b10} = 1.86 \), \( V_{b11} = 1.97 \) and \( V_{b12} = 2.06 \)

The broken line curve obtained is shown in fig. (5.5) from which it is seen that if the points of the theoretical curves are joined together, the resulting curve will almost coincide with the broken line one.

The actual circuit diagram of the function generator
FIG. 5.5  The Designed Characteristic of the Function Generator.
is shown in fig. (5.6), from which it is seen that the back voltage are taken from one supply using divider resistances. The resistances in series with the diodes are all taken equal each of 150 k-ohms, this value being chosen high to keep the current small so that the back voltages are not affected. The voltages across these resistances \( V_{d1}, V_{d2}, \ldots \)

etc., are then summed by a summing amplifier, the summing resistances being chosen high, 3 meg-ohm each, so that they do not have a shunting effect on the 150 k-ohm resistances. The feedback resistance of the summing amplifier is taken 750 k-ohms so that the output is \( e'_{N} \) instead of \( e_{N} \).

It should be noted here that the voltage summed by this arrangement are not only \( V_{d1} + V_{d2} + V_{d3} \ldots \) etc., but \( (V_{d1} + V_{b1}) + (V_{d2} + V_{b2}) + (V_{d3} + V_{b3}) \ldots \) etc., because these are the voltages between the points where the summing resistances are connected and earth.

Thus the output of the summing amplifier will be

\[-(e'_{N} + 1/4 (V_{b1} + V_{b2} + V_{b3} \ldots \text{ etc.}))\]

and as the function generator is required to give an output equals to \( e'_{N} \) i.e. equals to \((e'_{N} + 0.363)\), thus the quantity

\[1/4(V_{b1} + V_{b2} \ldots \text{ etc.})\]

which is a constant equals to 3.15 volts (only 11 diodes were used and \( V_{b1} \) is zero), is to be reduced to 0.363; this is done by adjusting the
FIG. 5.6 Circuit Diagram of the Function Generator.
the amplifier to give this voltage at zero input. The negative sign is then corrected for by using another operational amplifier as a phase inverter as shown in fig. (5.6).

The actual characteristic of the exponential regulating unit is shown in fig. (5.4) together with the theoretical characteristic, their numerical values being compared in the following table.

<table>
<thead>
<tr>
<th>$\theta_c$ volts</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Im. a. theoretical</td>
<td>0.74</td>
<td>0.9</td>
<td>1.1</td>
<td>1.34</td>
<td>1.64</td>
<td>2</td>
<td>2.44</td>
</tr>
<tr>
<td>I m. a practical</td>
<td>0.9</td>
<td>1</td>
<td>1.13</td>
<td>1.75</td>
<td>1.65</td>
<td>2</td>
<td>2.45</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta_c$ volts</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I m. a Theor.</td>
<td>2.93</td>
<td>3.64</td>
<td>4.45</td>
<td>5.44</td>
</tr>
<tr>
<td>I m. a Pract.</td>
<td>3.9</td>
<td>3.65</td>
<td>4.42</td>
<td>5.35</td>
</tr>
</tbody>
</table>

5.5 The Recorder

The response of the simulator, whose various parts have been described in the last 3 sections, to a 5 volt step change in the desired value and to a 25% step increase and decrease in load, using the linear and the exponential regulating units and various control modes, was recorded by an "Evershed Duplex Quick Response Recorder" type QU/RD9 with amplifier unit PA 10L.
The recorder is connected in series with a 100 volt stabilised voltage across the output \( V_o \) of the process, as seen in fig. (5.1), so that only \( \theta_o = V_o - 100 \) is recorded.

This instrument is a recording voltmeter with a very rapid response and has a very small pen to paper friction error. It faithfully records sine wave voltages varying from 0 to 10 cycles/sec, with a maximum amplitude of 1.5" peak to peak.

Investigating the theoretical curves obtained in chapter (4) shows that the frequency of oscillation is only about 0.15 cycles/sec and the rise time of the curves at the start does not exceed 1 sec. In the simulator these are 1.5 cycle/sec and 0.1 sec, respectively because of the reduction of the time scale in the ratio 10 : 1. Also the sensitivity of the recorder was found to be 2.8 chart divisions per volt. Thus the recorder is seen to be suitable for our simulator and proved to be quite satisfactory.

In the photograph of the simulator fig. (5.7) the recorder with its amplifier unit is seen to the left, to its right is seen the function generator, then the operational amplifiers, the constant current generator and the process, then finally to the right are the power supplies.
FIG. 5.7 Photograph of the Simulator.
5.6 Open Loop Response to Step input

In order to check the accuracy of the simulation the response of the simulated linear regulating unit and process was determined for a step input. The transfer function of these 2 parts in the form of differential equation is given by equation (3.4-3) with \(-F(\theta)\) representing the input. Applying a step voltage of 0.1 volt then means putting 
\(-F(\theta) = 0.1 u(t)\) in equation (5.4-3) resulting in

\[ (5\theta^3 + 21\theta^2 + 13\theta + 1) \theta_o = 10 u(t) \]  \(\cdots\) (5.6-1)

The L.T. of this equation gives

\[ \theta_o(s) = \frac{2}{s(s^3 + 4.2s^2 + 3.6s + 0.2)} \]

Then getting the roots of the characteristic equation the inverse transforms found to be

\[ \theta_o = 10 - 0.1146 e^{-3.0539t} + 0.8974 e^{-1.1045t} \]

\[ 10.7824 e^{-0.05985t} \cdots (5.6-2) \]

Equation (5.6-2) also can be solved easily by the numerical method, as this equation is similar to equation (4.1-1) with 10 \(u(t)\) instead of \(-15\theta\), thus the regression equation can be deduced, from equation (4.1-4) to be

\[ \theta_{on} = \frac{80}{a} - \frac{b}{a} \theta_{on} - 1 - \frac{c}{a} \theta_{on} - 2 - \frac{d}{a} \theta_{on} - 3 \cdots (5.6-3) \]

with the constant term \(80/a\) instead of the \(e^s\) in equation (4.1-4) and that results from taking the \((n+1)^{th}\) term of the product \(10(1,3,3,1)(0.5,1,1, \ldots \text{etc.})\) and then
dividing by a, (a, b, c & d) being the same as given by equation (4.1-2) ... δ was taken as 1/4 for the first 2 seconds then increased to 1/2 for the next 18 secs, and then to one second for the next 20 secs. The regression equation for these 3 values of δ are for δ = \*\*
\[
\theta_{on} = 0.0198 + 2.1924 \theta_{on-1} - 1.527 \theta_{on-2} + 0.3356 \theta_{on-3}
\]
the constant term for the first four ordinates from t=0 is 0.0012, 0.0062, 0.0135 & 0.0185 and for δ = \*\*\*
\[
\theta_{on} = 0.1097 + 1.6794 \theta_{on-1} - 0.7613 \theta_{on-2} + 0.0755 \theta_{on-3}
\]
and for δ = 1
\[
\theta_{on} = 0.4969 + 1.0248 \theta_{on-1} - 0.0136 \theta_{on-2} - 0.0593 \theta_{on-3}
\]

The following table compares the numerical values of some ordinates obtained from the L.T. solution given by equation (5.6-2) and the numerical solution using the regression equation (5.6-4).

<table>
<thead>
<tr>
<th>T secs</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ_{on} by L.T.</td>
<td>2.002</td>
<td>4.062</td>
<td>5.593</td>
<td>6.730</td>
<td>7.573</td>
<td>8.199</td>
<td>8.664</td>
</tr>
<tr>
<td>θ_{on} numerical</td>
<td>1.997</td>
<td>4.063</td>
<td>5.603</td>
<td>6.747</td>
<td>7.587</td>
<td>8.210</td>
<td>8.672</td>
</tr>
</tbody>
</table>

A photo copy of the record taken from the simulator response to this step input is shown in fig.(5.3) the chart speed was \*\* per second, the calculated response is drawn on a piece of the chart paper to the same scale with the time
scale reduced in the ratio 10 : 1 i.e. the ordinate at 
t = 10 in the theoretical curve corresponding to that at 
t = 1 in the record. Comparison shows almost exact agreement.

5.7 The Records

The response of the open loop having been checked 
and found in agreement with the theoretical results, the 
loop was then closed and the closed loop response was 
recorded. Photo copies of the records taken are shown in 
fig. (5.9 to 15). The first four figures show the response 
to 5 volt step change in the desired value, the records 
obtained for each control mode using the linear and exponen-
tial regulating units are put together. The other four figures 
show the response to 25% step increase and decrease in load.
The chart speed used for all records is 3"/sec.

To be able to compare the practical curves with 
the theoretical ones, the theoretical curves are drawn on 
the same recording paper to the same scale as the records 
the paper being sufficiently transparent for them to be 
superposed and thus compared. These are put in a pocket at the end.

Comparison between the theoretically obtained curves 
and those obtained from the simulator show almost exact 
agreement except in 2 curves, these are those of fig. (5.11-b) 
and fig. (5.12-b) which are the response to 5 volt step 
increase in the desired value for the exponential regulating 
unit under proportional + derivative control mode in case
of fig. (5.11) and proportional + integral + derivative
control mode in case of fig. (5.12).

This is due to the presence of derivative action which
is proportional to $\Omega^2/dt$, and as $\Omega$ has a step at $t=0$
($\Omega=\Omega_0 - \Omega_d = \Omega_0$ at $t=0$) then the derivative action includes
an impulse at $t=0$. Thus theoretically the derivative action
attains an infinite value at $t=0$, but as the step function
practically takes a finite time, then the resulting impulse
will also have a finite value, though this value will still
be large. The regulating unit were designed to give the
required characteristic only over a limited range between
$\Theta_c = 0$ and 2 volts, thus if the controller output is greater
than 2, as in the case of derivative action, then the
practical response cannot be in exact agreement with the
theoretical one.

With the linear regulating unit, however, the pentode
tube used gives a linear characteristic up to $\Theta_c = 2.8$ volts
thereby the response obtained is close to the theoretical
one, but with the exponential regulating unit, the function
generator was designed to simulate the exponential
characteristic up to $\Theta_c = 2$ and after this the curve becomes
linear and that is why the practical response in these two
cases failed to rise as quickly as the theoretical response.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Exponential Regulating Unit - Step Decrease

FIG. 5.9 Response to 5 volt Step Change in the Desired Value - Proportional Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Exponential Regulating Unit - Step Decrease

FIG. 5.10 Response to 5 volt Step Change in the Desired Value - Proportional+Integral Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Exponential Regulating Unit - Step Decrease

FIG. 5.11 Response to 5 volt Step Change in the Desired Value - Proportional+Derivative Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Exponential Regulating Unit - Decrease

FIG. 5.12 Response to 5 volt Step Change in the Desired Value - 3-Term Control Mode.
FIG. 5.13 Response to 25% Step Change in Load—Proportional Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Linear Regulating Unit - Step Decrease

d) Exponential Regulating Unit - Step Decrease

FIG. 5.14 Response to ±5% Step Change in Load - Proportional+Integral Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Linear Regulating Unit - Step Decrease

d) Exponential Regulating Unit - Step Decrease

FIG. 5.15 Response to a step change in load - Proportional+Derivative Control Mode.
a) Linear Regulating Unit - Step Increase

b) Exponential Regulating Unit - Step Increase

c) Linear Regulating Unit - Step Decrease

d) Exponential Regulating Unit - Step Decrease

FIG. 5.16 Response to 25% Step Change in Load - 3-Term Control Mode.
CHAPTER 6
The Effect of a Distance-Velocity Lag
and of over-ranging the Regulating Units—
General Comments and Conclusion

Summary

As many process control loops have distance-velocity
lags as well as transfer lags, a distance-velocity lag is
introduced into the loop investigated in the previous chapters.
Also, as practical regulating units have limited ranges,
the effect of over-ranging the regulating units is considered.

6.1. The Presence of a Distance-Velocity Lag in the Loop

In the last 3 chapters a control loop with 3 transfer
lags was investigated showing the effect of an exponential
regulating unit in comparison with a linear one under
various control modes. Many process control loops have
distance-velocity lags as well as transfer lags, arising
either from within the process itself or introduced by the
measurement system.

Having been able by the aid of the numerical method
to solve the various kinds of differential equations, linear
and non-linear, involved in the previous investigations, it would now be interesting to investigate a loop including a distance-velocity lag.

As an example of the treatment involved a distance-velocity lag of 0.5 sec. has been added to the 3 stage FC process used in the previous investigations, the lag occurring between the output \( V_o \) of the process and the difference element, see fig. (1.2). Thus it can be considered either due to the process or to the detecting element. Calling the output after the distance-velocity lag \( V'_o \), this now becomes the controlled condition, and the variation in \( V'_o \) as \( \varepsilon'_o \), then the relation between \( \varepsilon'_o \) and \( \varepsilon_o \) (the variation in \( V_o \)) is such that if \( \varepsilon_o = f(t) \) and D-V lag = 0.5 sec,

\[
\varepsilon'_o = f(t - 0.5)u(t-0.5)
\]

the multiplication by \( u(t-0.5) \) means that \( \varepsilon'_o \) is zero before \( t = 0.5 \) sec.

or if \( \varepsilon'_o(s) \) is the L.T. of \( \varepsilon'_o \)

and \( \varepsilon_o(s) \) is the L.T. of \( \varepsilon_o \)

then \( \varepsilon'_o(s) = e^{-0.5s} \varepsilon_o(s) \) \hspace{1cm} (6.1-1)

As the rest of the loop is kept unchanged, then the open loop equations (3.4-3), 4, 8, & 9) will be unchanged and it is only when closing the loop that instead of \( \theta \) being equal to \( \varepsilon_o - \varepsilon_d \) it will be equal to \( \varepsilon'_o - \varepsilon_d \).
Only the constant load equations with proportional control mode will be considered and the proportional action factor will be kept unchanged at 0.15.

Thus in equation (3.4-3) for the linear regulating unit \( F(\vartheta) = 0.15 \vartheta \) and \( \vartheta = (\vartheta_0 - \vartheta_d) \) we get

\[
(5D^3 + 21D^2 + 13D + 1) \vartheta_0 = -15(\vartheta_0 - \vartheta_d) \quad \ldots \quad (6.1-2)
\]

The I.T. of this equation is

\[
(5s^3 + 21s^2 + 13s + 1) \vartheta_0(s) = -15(\vartheta_0(s) - \vartheta_d(s)) \quad \ldots \quad (6.1-3)
\]

substituting the value of \( \vartheta_0(s) \) given by equation (6.1-1) in equation (6.1-3) and rearranging results in

\[
(5s^3 + 21s^2 + 13s + 1 + 15e^{-0.5s}) \vartheta_0(s) = 15\vartheta_d(s) \quad \ldots \quad (6.1-4)
\]

now expanding \( e^{-0.5s} \) results in

\[
e^{-0.5s} = 1 - 0.5s + \frac{(0.5s)^2}{2!} - \frac{(0.5s)^3}{3!} + \ldots \quad (6.1-5)
\]

substituting this in equation (6.1-4) taking only the terms up to the third power of \( s \) results in

\[
\vartheta_0(s) = \frac{15\vartheta_d(s)}{4.6975s^3 + 22.375s^2 + 10.5s + 16}
\]

then for a unit step change in the desired value i.e.,

\( \vartheta_d(s) = 1/s \) this becomes

\[
\vartheta_0(s) = \frac{3s^2}{s(s^3 + 4.83s^2 + 2.24s + 3.413)} \quad \ldots \quad (6.1-6)
\]
getting the roots of the characteristic equation, the
inverse L.T. of (6.1-6) is found to be

\[ \theta_0 = 0.9375 - 0.03517 e^{-4.5527 t} + \\
0.97227 e^{-0.16367 t} \sin(0.85027 t - 111.52°) \] .... (6.1-7)

Then as explained above in the relation between \( \theta_0 \) and
\( \theta_0 \), \( \theta_0 \) can be directly deduced from (6.1-7) as,

\[ \theta_0 = 0.9375 - 0.03517 e^{-4.5527(t-0.5)} + \\
0.97227 e^{-0.16367(t-0.5)} \sin(0.85027(t-0.5)-111.52°) \] .... (6.1-8)

for \( t \) equals or more than 0.5 sec., as \( \theta_0 = 0 \) before this.

This equation gives the response of the loop containing
0.5 sec. D-V lag to a step change in the desired value using
the linear regulating unit and proportional control mode.

Although it was possible to solve equation (6.1-2)
by the Laplace Transform method it is not always possible to
do so with the presence of a D-V lag, as in our case the
expansion of \( e^{-0.5s} \) converged quickly and terms in power
of \( s \) greater than the 3rd were negligible, but if they were
not negligible, then the L.T. solution fails.
6.2 Numerical Solution

1. Linear regulating unit

Equation (6.1-2), solved by the L.T. above, will be solved numerically,
putting \( \theta' = (\theta'_0 - \theta_d) \) \hspace{1cm} (6.2-1)
in equation (6.1-2) results in
\[
(5 D^3 + 21 D^2 + 13 D + 1)\theta'_0 = -155' \hspace{1cm} (6.2-2)
\]
This equation is the same as equation (4.1-1) with \( \theta' \)
instead of \( \theta \), thus the regression equations (4.1-5) can
be used with \( \theta' \) as given by (6.2-1) instead of \( \theta \). In this
case the calculation of the ordinates of \( \theta'_0 \) will not be by
trial and error as before, because knowing that \( \theta'_0 \) equals
to zero up to \( t=0.5 \) sec., then in this period \( \theta' = -\theta_d \)
and thus the ordinates of \( \theta'_0 \) in this period will be
calculated directly. For example taking
\[
\theta_d = 5 u(t) = 5(0.5, 1, 1, 1 \ldots \text{ etc.}) \text{, at } t=0 \theta' = -2.5,
\]
thus in the first column of the table shown in section
(4.1) for ordinate calculation \( \theta'_n = -2.5 \) then \( \theta'_n-1, \theta'_n-2, \theta'_n-3 \) are all zero.
also, \( \theta_{on-1}, \theta_{on-2} \) and \( \theta_{on-3} \) then
\[
\theta_{on} = -0.000574 \times -2.5 = 0.0014; \text{ then in the second}
column \( \theta'_n = -5 \) (which is then ordinate of \( \theta_d \) at \( t=\delta =\% \))
and \( 3\theta'_{n-1} = -7.5 \), then \( \theta'_n-2 \) is \( \theta'_n-3 \) are zero, and \( 2.5445\theta'_{on-1} = \)
$0.0036$ then $\theta_{on-2}$ and $\theta_{on-3}$ are zero, thus we get

$$\theta_{on} = -0.000574(-5) + 0.0036 = 0.0103$$

and so on up to $t = 0.5$ where $\theta_o$ starts to have a value and then

$$\theta' = \theta_o - \theta_d$$

but the ordinates of $\theta'$ are those of $\theta_o$ shifted by $0.5$ sec., then the ordinate of $\theta'$ at $t=0.5$ equals that of $\theta_o$ at $t=0$, thus the ordinates of $\theta'$ in the period from $t=0.5$ to $t=1$ are already known as they are the ordinates of $\theta_o$ calculated in the period from $t=0$ to $t=0.5$, thus the ordinates of $\theta_o$ in the second half second can be calculated directly, those will then be the ordinates of $\theta'$ in the third half second and so on.

It is seen from the above explanation how easy the numerical solution is in this case where distance-velocity lag is present. Actually the introduction of the distance-velocity lag made the numerical solution more easy than without it. In the following table it is shown how the numerical solution is started, $\delta$ being taken as $\frac{1}{2}$, from which it is seen that the first four values of $\theta'_{on}$ are zero as the time delay is $0.5$ sec. and $\delta = \frac{1}{2}$, thus in this period $\theta'_n = \theta'_{on} - \theta_d = -\theta_d$, then at $t = 0.5$, $\theta'_{on}$ starts to have the values of $\theta_{on}$ from $t=0$. 

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The following table compares the values of the ordinates of \( \theta_0 \) calculated from the L.T. solution given by equation (6.1-3) and from the numerical solution where the response to \( \theta_d = 5u(t) \) is calculated, the ordinates are then divided by 5.

<table>
<thead>
<tr>
<th>t sec.</th>
<th>0</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.625</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\theta_n )</td>
<td>2.5</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td>4.9985</td>
<td>4.9992</td>
<td>4.9997</td>
</tr>
<tr>
<td>( -3\theta_{n-1} )</td>
<td>0</td>
<td>7.5</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0000</td>
<td>14.9998</td>
<td>14.9996</td>
</tr>
<tr>
<td>( -3\theta_{n-2} )</td>
<td>0</td>
<td>0</td>
<td>7.5</td>
<td>15.0</td>
<td>15.0000</td>
<td>15.0000</td>
<td>14.9998</td>
</tr>
<tr>
<td>( -\theta_{n-3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.5</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>0.000574 x sum</td>
<td>0.0014</td>
<td>0.0072</td>
<td>0.0153</td>
<td>0.0215</td>
<td>0.0230</td>
<td>0.0230</td>
<td>0.0229</td>
</tr>
<tr>
<td>( 2.5445\theta_{on-1} )</td>
<td>0</td>
<td>0.0036</td>
<td>0.0275</td>
<td>0.1025</td>
<td>0.2590</td>
<td>0.5150</td>
<td>0.8766</td>
</tr>
<tr>
<td>( -2.1335\theta_{on-2} )</td>
<td>0</td>
<td>0</td>
<td>-0.0030</td>
<td>-0.0230</td>
<td>-0.0360</td>
<td>-0.2172</td>
<td>-0.4318</td>
</tr>
<tr>
<td>( 0.5335\theta_{on-3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0003</td>
<td>0.0064</td>
<td>0.0237</td>
<td>0.0599</td>
</tr>
<tr>
<td>( \theta_{on} )</td>
<td>0.0014</td>
<td>0.0103</td>
<td>0.0403</td>
<td>0.1013</td>
<td>0.2024</td>
<td>0.3445</td>
<td>0.5276</td>
</tr>
<tr>
<td>( \theta_{on} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0014</td>
<td>0.0100</td>
<td>0.0403</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t sec.</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{on} ) by L.T.</td>
<td>0</td>
<td>0.0337</td>
<td>0.2013</td>
<td>0.4610</td>
<td>0.7624</td>
<td>1.0439</td>
<td>1.2722</td>
</tr>
<tr>
<td>( \theta_{on} ) numerical</td>
<td>0</td>
<td>0.0405</td>
<td>0.2014</td>
<td>0.4611</td>
<td>0.7623</td>
<td>1.0478</td>
<td>1.2716</td>
</tr>
</tbody>
</table>
It is noticed here that the numerical solution agrees almost exactly with the L.T. solution up to t=7 sec. and after that when the solution approaches the steady state, the ordinates of the numerical solution tend to be always less than the ordinates of the L.T. solution by about 0.3% while the error in the numerical solution is usually less than 0.2%. This is due to the approximation made in the L.T. solution when $e^{-0.5s}$ is replaced by its expansion in powers of s up to the third only.

2. Exponential regulating unit

The equation for this case is obtained from equation (5.4-4) by putting $F(s) = 0.15\theta$ and $e=\theta' - \theta_d$ resulting in

\[
(5 D^3 + 21 D^2 + 18 D + 1) \theta^o = 100(e^{-0.15(\theta' - \theta_d)} - 1) \quad \text{... (6.2-3)}
\]

Then putting $\theta' = -100/15(e^{-0.15(\theta' - \theta_d)} - 1) \quad \text{... (6.2-4)}$

results in

\[
(5 D^3 + 21 D^2 + 18 D + 1) \theta^o = -15\theta' \quad \text{... (6.2-5)}
\]

which is the same as equation (6.2-2) above, with $\theta'$ as given by equation (6.2-4), thus the procedure of calculating the
ordinates of $\theta_o$ is the same as explained above in the case of the linear regulating unit, only after getting $\theta'_o = \theta'_d$ a table of Napierian logarithms is used to get $\theta'_d$ from equation (6.2-4), thus calculating the ordinates of $\theta'_o$; those of $\theta'_o$ are the same shifted half a second.

The response to $\theta_d = 5u(t)$ and $-5u(t)$ has been calculated, the curves obtained are drawn in Fig. (6.1) together with the curve of the linear regulating unit. The numerical values of the ordinates are tabulated in Appendix B table (7).

6.3 Over-Range (Saturation) of the regulating units

In practice control valves have a fixed range over which the output is a function of the control input and at a certain minimum input the valve is fully closed, thus if the input becomes less than this minimum, the output will remain at that of the fully closed position. Also at a certain maximum input the valve is fully open and if the input exceeds this maximum the output will remain at that of the fully open position. Thus the valve saturates when the input is beyond the range of the valve.

It is always possible that the valve is over-ranged as it is usually impossible to choose a valve for a control
FIG. 6.1 Response to 5% Step Change in the Desired Value in the Presence of Distance-Velocity Lag - Proportional Control Mode.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) " " " -Step Decrease(Reversed).
system to cope with all the likely disturbances. Thus it would be interesting to know the effect of over-ranging the valve on the response of the control system under investigation.

The controller output is given by equation (3.4-1) as

\[ \vartheta_c = 1 - F(\vartheta) \]

considering the normal operating position at zero deviation midway between the fully open and fully closed positions, and the fully closed position at \( \vartheta_c = \vartheta \), then the valve range in our case is between \( \vartheta_c = 0 \) and \( \vartheta_c = 2 \) volts, thus the regulating unit is considered to control only for \( F(\vartheta) = \pm 1 \). Thus in the proportional control mode, the proportional action factor being \( 0.15 \), the proportional band is then equal to \( 2/0.15 = 13.3 \) volts, i.e. \( 13.3\% \) as the desired value = 100 volts. Thus the deviation from the operating point under proportional control mode should not exceed \( \pm 6.6\% \).

Thus in calculating the response to step change in the desired value more than \( 6.6\% \) the saturation of the regulating units should be taken into account. In section (4.1) the response to \( 10\% \) step increase and decrease in the desired value was calculated for the exponential regulating unit \( \vartheta_1 \) where it was considered that the range of the regulating unit is not limited, now the saturation effect will be considered for both linear and exponential regulating units.
I. Linear regulating unit

In section (4.1) equation (4.1-1) for the linear reg.
unit was solved numerically for \( \theta_d = 5u(t) \), now taking
\( \theta_d = 10u(t) \), then to take the saturation effect into
account the same regression equations (4.1-5) are used only
that \( \theta_n \) in those equations should not exceed \( \pm 6.5 \), to
make this clear, the following table shows how the Solution
saturation is started

<table>
<thead>
<tr>
<th>( t ) sec.</th>
<th>0</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.625</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -3\theta_n-1 )</td>
<td>0</td>
<td>14.9913</td>
<td>20.0000</td>
<td>20.0000</td>
<td>20.0000</td>
<td>20.0000</td>
<td></td>
</tr>
<tr>
<td>( -3\theta_n-2 )</td>
<td>0</td>
<td>0</td>
<td>14.9913</td>
<td>20.0000</td>
<td>20.0000</td>
<td>20.0000</td>
<td></td>
</tr>
<tr>
<td>( -\theta_n-3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4.9971</td>
<td>6.6667</td>
<td>6.6667</td>
<td></td>
</tr>
<tr>
<td>( 0.0000574 \times \text{sum} )</td>
<td>0.0029</td>
<td>0.0124</td>
<td>0.0239</td>
<td>0.0297</td>
<td>0.0366</td>
<td>0.0366</td>
<td></td>
</tr>
<tr>
<td>( 2.5445\theta_{on-1} )</td>
<td>0</td>
<td>0.0074</td>
<td>0.0504</td>
<td>0.1733</td>
<td>0.4135</td>
<td>0.7901</td>
<td>1.5031</td>
</tr>
<tr>
<td>( -2.1335\theta_{on-2} )</td>
<td>0</td>
<td>0</td>
<td>-0.0062</td>
<td>-0.0422</td>
<td>-0.1453</td>
<td>-0.3467</td>
<td>-0.6625</td>
</tr>
<tr>
<td>( 0.5835\theta_{on-3} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0017</td>
<td>0.0117</td>
<td>0.0401</td>
<td>0.0957</td>
</tr>
<tr>
<td>( \theta_{on} )</td>
<td>0.0029</td>
<td>0.0193</td>
<td>0.0581</td>
<td>0.1625</td>
<td>0.3105</td>
<td>0.5141</td>
<td>0.7713</td>
</tr>
</tbody>
</table>

The ordinate sequence of \( \theta_d = 10u(t) \) is (5, 10, 10, 10 .... etc)
with the first ordinate at \( t=0 \). Thus \( \theta_n = \theta_{on} - \theta_{on} = -5 \)
at \( t=0 \) assuming \( \theta_{on} = 0 \) this being less than the limit of
6.5 is then considered as it is, but at the second ordinate
position \( t=0.125 \) \( \theta_n = \theta_{on} -10 \) and as \( \theta_{on} \) at \( t=0.125 \) is
still very small then $\theta_n$ equals $-10$ approximately and as this is beyond the limit then it is considered to be equal only to $-6.6$. Thus the solution proceeds in this manner and whenever $\theta_n$ exceeds $-6.6$ it is then taken equal to this value.

2. **Exponential regulating unit**

The equation in this case is equation (4.1-3) which has been solved in section (4.1) for $\theta_d = \mp -10 u(t)$ without considering the valve limit. Saturation is considered in this case also by making $\theta$ not to exceed $+6.6$, thus from equation (4.1-7) $\theta'$ is not to exceed $-\frac{100}{15} \left( e^{-0.15(\pm 6.6)} - 1 \right)$ i.e. to say not to exceed $-11.4553$ when $\theta_d = 10 u(t)$ and 4.214 when $\theta_d = -10 u(t)$, thus the procedure of solution is the same as in the linear case above with $\theta'$ as given by equation (4.1-7) instead of $\theta$, but whenever $\theta_n'$ exceeds the values equivalent to $\theta = \pm 6.6$ it should then be made equal to these values. 

The curves obtained for $\theta_d = \mp -10 u(t)$ are drawn in fig. (6.2) together with that using the linear regulating unit. The numerical values of the ordinates are tabulated in appendix B, table 7.
FIG. 6.2 Response to 10% Step Change in the Desired Value Considering Saturation - Proportional Control Mode.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) " " " -Step Decrease(Reversed).
6.4 Consideration of Saturation in the presence of Derivative Control Action.

In section (6.3) the range of the regulating units, the linear and the exponential, was fixed so that $F(0)$ falls between +1 and -1 and thus under the proportional control mode where $F(0) = 0.15\theta$ the deviation $\theta$ was to be between +6.6 and -6.6. In the proportional + derivative control mode, however, $F(\theta) = 0.15(\theta + 0.2\dot{\theta}/\dot{t})$ and in case of a step change in the desired value $F(\theta)$ will be having an impulse at $t=0$ as explained in sections (4.3) & (5.7), thus theoretically it will be of infinite value at $t=0$ i.e. in cases of step changes in $\theta$ the desired value and in the presence of derivative control action, the regulating units will always be over-ranged, although instantaneously, whatever the value of the step change is. The same occurs in the proportional + integral + derivative control mode.

Thus in cases under these control modes, the saturation of the regulating units should be taken into account. In the numerical solution of the equations of such cases in sections (4.3) and (4.4), the solutions were started using $\delta = 1/16$ giving the derivative control action $0.15 \times 0.2 (\dot{\theta}/\dot{t})$ at $t=0$ and $\theta_d = 5 \ u(t)$ as $0.15 \times 16 = 2.4$.
which is alone greater than unity, thus \( F(\theta) \) in the presence of derivative control action is greater than unity at \( t=0 \) and to consider saturation, it should be made unity.

Thus to calculate the response considering saturation for the linear regulating unit under proportional + derivative control mode, the procedure is the same as in section (4.3) except that at \( t=0 \), in the table showing the start of the solution there, \( \epsilon_n \) is made equal to 6.6 so that \( F(\theta) = 0.156 \) not to be more than unity. Similarly under the 3-term control mode

\[
\epsilon_n = \epsilon_n + 0.1(\theta)_n + 0.2(\theta/dt)_n \text{ at } t=0 \text{ is made equal to 6.6}.
\]

It should be noticed here that when \( \theta_d \) is less than 6.6 \( u(t) \), \( \epsilon_n \) will only be greater than 6.6 at \( t=0 \) where the impulse occurs, but when \( \theta_d \) is greater than 6.6 then \( \epsilon_n \) will be greater than 6.6 at other ordinate positions not only at \( t=0 \), and thus should be made equal to 6.6 whenever it exceeds this value if saturation is considered.

In case of the exponential regulating unit

\[
\epsilon_n = -\frac{100(e^{-F(\theta)} - 1)}{15} \text{ the limits of } F(\theta) \text{ being } 1 \text{ then}
\]

\( \epsilon_n \) at \( t=0 \) and \( \theta_d = 5u(t) \) should be made -11.4553 as it will exceed this value and when \( \theta_d = -5u(t) \) \( \epsilon_n \) at \( t=0 \) is made 4.214, and in case \( \theta_d \) exceeds \( \pm 6.6 u(t) \), \( \epsilon_n \) is made equal to -11.4553 and 4.214 respectively whenever it exceeds.
these values not only at $t=0$,

The response to $\theta_d=5u(t)$, considering saturation, for
the linear regulating unit under proportional + derivative
control mode and 3-term control mode has been calculated,
also the response to $\theta_d = 5u(t)$ and $-5u(t)$ for the exponential
regulating unit under the same control mode. The curves
obtained for each control mode are drawn together as shown
in fig. (6.3) and (6.4). The numerical values of the
ordinates are tabulated in appendix B tables (3) & (4).

6.5 Comments on the Response Curves Obtained to Step
Changes in the desired value.

Investigating the response curves drawn in figures
(4, 1, 2, 3, & 4) and (6.1, 2, 3, & 4) and tables (1,2,3,4&7)
appendix B we notice the following.

1. It is clear from fig. (4.1) that the shape of the
response for the exponential regulating unit depends on
the magnitude of the step change in the desired value.
while the response to $\theta_d = u(t)$ is close to the response
for the linear regulating unit as can be seen from table 1
appendix B, the larger $\theta_d$ is the larger is the departure
from this response, as can be seen from the response to
$5u(t)$ and $10u(t)$ in fig. (4.1).

It is seen that as $\theta_d$ becomes larger the overshoot
increases and the response becomes less damped. But as
FIG. 6.3 Response to 5% Step Change in the Desired Value - Proportional + Derivative Control Mode, Considering Saturation.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) " " " - Step Decrease (Reversed).
FIG. 6.4 Response to 5% Step Change in the Desired Value - 3-Term Control Mode, Considering Saturation.

(a) Linear Regulating Unit - Step Increase.
(b) Exponential Regulating Unit - Step Increase.
(c) " " " -Step Decrease(Reversed).
in practice the regulating valves have a limited range
the response will not continue to become worse as \( \theta_d \)
increases, because when \( \theta_d \) exceeds a certain value
depending on the proportional band, saturation of the valve
will come into effect and as can be seen from comparing
the response to \( \text{lo}u(t) \) for the exponential regulating
unit in fig.(4.1) where saturation is not considered and
the corresponding response in fig.(6.2) where saturation
is considered, the response seems to be improved \& by
saturation from the point of view of reduced overshoot
and increased damping.

2. It is also noticed from all the response curves to
step changes in the desired values that the response for
the exponential regulating unit to step increase in the
desired value is different from that to step decrease.
While the response to step increase in the desired value
has a larger overshoot and less damped compared with the
response for the linear regulating unit, the response to
step decrease in the desired value has a smaller overshoot
and more damped.

Also in the absence of integral control action, the
"offset" in case of step increase in the desired value is
less than in case of step decrease, for example under the
proportional control mode, the "Offset" in case of 5% 
step increase in the desired value is 0.3055 and in
case: of 5% step decrease is 0.3195, for the linear regulating unit the equivalent "offset" is 0.3125. The percentage "offset" also varies with the magnitude of the change in the desired value for the exponential regulating unit while it is constant for the linear regulating unit.

3. It is seen from figures (6.3) and (6.4) that the response for the exponential regulating unit to 5% step increase in the desired value departs very much at the start from the corresponding response for the linear regulating unit and if the derivative action was larger the departure would have been still greater. It may seem advantageous that the response rises quickly, but considering the saturation of the regulating unit the response as seen from figures (6.3) and (6.4) does no longer rise quickly.

4. In fig. (6.1) where the response in the presence of a distance-velocity lag is shown, it can be reasoned from this figure what is the effect of increasing the open loop gain, as the introduction of a time delay in the loop is similar to an increase in the open loop gain. This effect is seen to make the response for the exponential regulating unit worse.

5. The determination of which regulating unit characteristic gives the best performance in the control loop depends on the criterion adopted for the comparison. Two criteria have been used that of the integral of error-squared $\int \theta^2 \, dt$. 

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and also the integral of the first moment in time of error-squared \( \int_0^t \theta^2 dt \). These calculations have been carried out for the response curves of Fig. (4.2), but only from \( t=0 \) to \( t=10 \) secs, as the curves are almost completely damped after \( t=10 \), the results were found to be as follows:

For the linear regulating unit

\[
\int_0^t \theta^2 dt = 1.403 \quad \text{and} \quad \int_0^t t \theta^2 dt = 1.624
\]

and for the exponential regulating unit in case of step increase in \( \theta_d \)

\[
\int_0^t \theta^2 dt = 1.374 \quad \text{and} \quad \int_0^t t \theta^2 dt = 1.996
\]

and in case of step decrease in \( \theta_d \)

\[
\int_0^t \theta^2 dt = 1.502 \quad \text{and} \quad \int_0^t t \theta^2 dt = 1.946
\]

From which it is seen that the response for the linear regulating unit, having smaller values of \( \int_0^t \theta^2 dt \) and \( \int_0^t t \theta^2 dt \) is then considered better than the response for the exponential regulating unit. Thus we can come to the conclusion that the exponential regulating unit is inferior to the linear one in case of desired value changes.

6.6 Comments on the response Curves obtained to step changes in load.

Investigating figures (4.5, 6, & 7) and tables (5) & (6) Appendix B we notice the following:

1. The response curves for the exponential regulating unit are seen to be better than those for the linear one in case of step increase in load. This is proved by calculating \( \int_0^t \theta^2 dt \) and \( \int_0^t t \theta^2 dt \) for the response curves which for the
case of proportional + integral control mode resulted in:

For the linear regulating unit:
\[ \int_0^\infty \theta^2 \, dt = 25.13 \quad \text{and} \quad \int_0^\infty t \theta^2 \, dt = 55.8 \]

and for the exponential regulating unit:
\[ \int_0^\infty \theta^2 \, dt = 20.23 \quad \text{and} \quad \int_0^\infty t \theta^2 \, dt = 45.56 \]

2. In case of step decrease in load the contrary is noticed (i.e.,) the response curves for the linear regulating unit are better than those of the exponential one. The reason for this is clear when we remember that increasing the load has the effect of decreasing the open loop gain when the linear regulating unit is being used, while this gain is maintained constant with the exponential regulating unit, as explained in section (3.3). And decreasing the load results in increasing the open loop gain when the linear regulating unit is used, and it is kept constant with the exponential one.

3. As the exponential regulating unit maintains the open loop gain constant, it would have been expected that it gives the same response for the same step increase and decrease in load, but fig. (4.7) where these response curves are drawn together, shows that this is not true. The reason is also clear when we remember that although the exponential regulating unit is capable of countering the change in potential correction due to the
change in load, it cannot counteract the change in the coefficients of the transfer function of the process produced by the step change in load as seen from equations (3.4-3.9).

With the linear regulating unit, the response curves to step increase in load are also different from those to step decrease, because of the change in the coefficients of the transfer function of the process, but the difference is very small compared to that in case of the exponential regulating unit as can be seen from tables (5) and (6) Appendix B. Thus the change in the loop gain in case of the linear regulating unit resulting from the change in load has counteracted to a certain extent the change in the coefficients of the transfer function of the process.

Thus it is seen that the nonlinearity of the characteristic of the exponential regulating unit has the same effect on the response to step changes in both load and desired value (viz.,) different response to step increase and to step decrease and this becomes more clear the larger the step is.

Also we cannot come to the conclusion here that the exponential regulating unit is superior to the linear one, because although the response to step increase in load is found to be better in case of the exponential regulating unit, it is not so to step decrease. But we can say that a regulating unit having a linear characteristic, the slope
of which varies according to the load to keep the open loop gain constant would be the best.

6.7 General Conclusion

It will be useful to make some comments on the generality of the numerical method used in this thesis for the solution of the linear and nonlinear systems.

The loop chosen for investigating the effect of the exponential regulating unit comprised a process whose output-input relation was expressed by a linear differential equation with constant coefficients, a 3-term controller whose output-input relation was also expressed by an integro-differential equation with constant coefficients and the regulating unit which was taken either of linear or exponential characteristic.

Thus the non-linearity in the loop equation, arose from the time-independent non-linearity of the regulating unit characteristic. The solutions obtained for the non-linear differential equations using the numerical method was checked experimentally and the numerical solution of the linear equations was checked by solution with the Laplace Transform method and by experiment. The checking showed almost exact agreement.

Then, in this chapter other types of non-linearities were considered, (viz) the presence of a distance-velocity lag in the loop, and the saturation of the regulating units. The addition of these non-linearities to the non-linearity of the exponential regulating unit characteristic did
not result in more work in the numerical solution. To the contrary the addition of the distance-velocity lag made the solution by the numerical method more easy and direct. As explained in section (6,2) its inclusion wipes out the trial and error form of the normal calculations.

Actually any number of time-independent non-linearities can be introduced in the loop without greatly increasing the labour of calculation. For example, a square law measuring element (as in flow control) and saturated exponential regulating unit can be considered together with a distance-velocity lag.

In the numerical solutions the distance and integral control actions were calculated separately at each ordinate position using formula (2,2-9 all) and then added to the proportional control action. Thus any control action can be non-linear. For example, the derivative control action can be dependent on \( \theta \) as well as \( \frac{d\theta}{dt} \), i.e. equals \( K_2 \frac{d\theta}{dt} \), in which case \( \frac{d\theta}{dt} \) can be calculated as usual using formula (2,2-9) then multiplied by \( \theta \) at each ordinate position, thus this type of non-linearity can be dealt with easily as it adds only one step more to the steps of ordinate calculations.

Also the differential equation giving the relation between output and input of the process can have non-linear terms, but this will greatly increase the labour of calculations as each of such terms has to be calculated separately and if
these terms contain derivatives of the variable they have
to be evaluated at each ordinate position using formula
(2.2-9) once, twice , etc. according to the order of the
derivative. This arises because it is not possible in
such terms as \( \frac{d^2 \theta}{dt^2} \) or \( (\frac{d^2 \theta}{dt^2})^2 \) to replace \( \frac{d}{dt} \) by it equivalent serial operator, this can only be done for
the linear terms. A simple example is the non-linear
differential equation solved in section (2.6).

Thus such non-linear terms in the differential equation
of the process can be dealt with in the normal way if it is
simple as in the example of section (2.6), but if it involves
higher derivatives, the method can still be used though
the work will be laborious, as at each ordinate position
the first derivative will be evaluated using formula(2.2-9)
then the second derivative is evaluated from the first and
so on. But if a digital computer ia available it can be
programmed easily to deal with such cases.

Thus this numerical method can be used in the analysis
of non-linear control systems, the work involved is not
laborious especially when the process can be represented by
RC network and a distance-velocity lag as considered in this
thesis. The accuracy of results is high, less than 0.2%.

In general this numerical method, explained in chapter
(2) , can be used in the solution of nonlinear differential
equations of the form
\[(AD^3 + BD^2 + CD + E) \theta_n = f(\theta) \quad \ldots \quad (6.7-1)\]

where \( \theta \) is a function of \( t \) and \( \theta_0 \) and \( f(\theta) \) can take any form that can be evaluated at each ordinate position, for example it can be of the form \( e^{k_1 \theta} \), \( e^{(k_1 \theta + k_2 \frac{d\theta}{dt})} \), etc.

or \( k_1 \theta^2 \), \( k_1 \theta^3 \), \( k_1 \theta^2 + k_2 \frac{d\theta}{dt} \), \( k_1 \theta + k_2 \frac{d\theta}{dt} \),

\( k_1 \theta + k_2 \frac{d\theta}{dt} \) ... etc.

The L.H.S. of equation (6.7-1) can be of any order.

The coefficients \( A, B, C, \& E \) can be either constants or functions of \( t \).

Such equations can be solved numerically by substituting for \( D \) with its equivalent serial operator \( \frac{2}{\delta} \frac{(1 - \frac{1}{\delta})}{(1, \frac{1}{\delta})} \)

and then deriving the regression equation which for equation (6.7-1) will be of the form,

\[ \theta_{on} = k \left( f(\theta)_n + 2f(\theta)_{n-1} + 3f(\theta)_{n-2} + f(\theta)_{n-3} \right) - a\theta_{on-1} \]

\[ -b\theta_{on-2} - c\theta_{on-3} \quad \ldots \quad (6.7-2) \]

which gives the \( n \)th ordinate of \( \theta_\circ \) in terms of the \( n \)th ordinate of \( f(\theta) \) and the few preceding ordinates of \( f(\theta) \) and \( \theta_\circ \) depending on the order of the differential equation.

\( k, a, b, \& c \) are coefficients depending on \( A, B, C, \& E \) and \( \delta \) the chosen interval of time between the ordinates.

The ordinates of \( \theta_\circ \) are calculated from the regression equation (6.7-2) by trial and error because \( \theta \) and thus \( f(\theta) \) is a function of \( \theta_\circ \) i.e., we assume a value for \( \theta_{on} \) from which we can evaluate \( f(\theta) \) and if \( f(\theta) \) contains \( \frac{d\theta}{dt} \) or \( \int \theta dt \) then these can be evaluated at each ordinate position.
using the following formulae

\[
\frac{d\theta}{dt}_n = \frac{2}{5} (\theta_n - \theta_{n-1}) - \frac{1}{5}(\theta/\theta)_n \quad (6.7-3)
\]

\[
(\phi)_n = \frac{1}{2} (\theta_n + \theta_{n-1}) + (\int\theta)_{n-1} \quad (6.7-4)
\]

then proceed to calculate \(\theta_{on}\) which should agree with the assumed value otherwise the procedure is repeated.

If now the L.H.S. of equation (6.7-1) contains non-linear terms in \(\theta\) or its derivatives like \(C\theta \cdot \theta/\theta \) or \(B \cdot \theta/\theta^2 \) \(d\theta/\theta^3 \) etc., then these terms should be evaluated separately. For example \(C\theta \cdot \theta/\theta\) is evaluated by calculating \(d\theta/\theta\) using formula (6.7-3) and then multiplied by \(C\theta\) at each ordinate position and \(B \cdot \theta/\theta^2 \) is evaluated by calculating \(d\theta/\theta\) using formula (6.7-3) and then \(d^2\theta/\theta^2\) from \(d\theta/\theta\) using the same formula and then multiplied at each ordinate position.

This is necessary as it is only in the linear terms that we can substitute \(\theta\) for \(D\) by its equivalent serial operator. Thus even if \(f(\theta)\) in this case is a simple function of \(t\) only, the calculations will increase greatly as the number of these non-linear terms or the number of differentiations done using formula (6.7-3) is increased. However, a simple digital computer programmed on the basis of this method should be able to deal with such equations where the calculations are too great to be done in the usual way.
1. The Analytical Solution of Equation (2.5-1) Section (2.5).

The equation is:

\[ \ddot{Y} + t \dot{Y} - Y = 0 \]

subject to the initial conditions

at \( t = 0 \) \( Y = 0 \) and \( \dot{Y} = 1 \)

The L.T. (Laplace Transform) of equation (1) is:

\[ s^2Y - 1 - \frac{d}{ds}(sy) - y = 0 \]

where \( Y = \text{L.T. of } Y \), rearranging we get

\[ \frac{dy}{ds} + \frac{2-s^2}{s} = -\frac{1}{s} \]

The Integrating Factor from equation (2) is;

\[ \text{I.F.} = e^{\int \frac{s}{2-s}ds} = s^2 e^{-\frac{s^2}{2}} \]

then from equations (2) & (3)

\[ \frac{d}{ds}(ys^2 e^{-\frac{s^2}{2}}) = -s e^{-\frac{s^2}{2}} \]

then on integrating we get

\[ ys^2 e^{-\frac{s^2}{2}} = e^{-\frac{s^2}{2}} + C \]

i.e.

\[ y = \frac{1}{s^2} + \frac{C}{s^2} e^{\frac{s^2}{2}} \]

and as at \( t = 0 \) \( Y = 0 \) then at \( s = \infty \) \( y = 0 \)

thus \( C = 0 \) and accordingly

\[ y = 1/s^2 \]

The inverse L.T. of this gives the solution of equation (1) as;

\[ Y = t \]
2. The Analytical Solution of equation (2.6-1) Section (2.6)

The equation is:
\[ \ddot{x} + w^2 x + \alpha x^3 = 0 \] ...........................(4)

subject to the initial conditions
at \( t = 0 \) \( x = a \) and \( \dot{x} = 0 \)

This equation is solved by Pipes and the solution is given
here briefly and in a slightly different way.

L.T. of equation (4) is:
\[ s^2 x = s a + w^2 x + \alpha lx^3 = 0 \]

or \( (s^2 + w^2) x = a s - \alpha lx^3 \) ...........................(5)

where \( x = L.T. \) of \( x \) and \( lx^3 = L.T. \) of \( x^3 \)

Let
\[ x = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 \] ...........................(6)

\[ w^2 = w_0^2 + c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 \] ...........................(7)

\[ x = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 \] ...........................(8)

In equation (6) \( x_\tau \) are functions of \( t \) to be determined.
In equation (7) \( w_0 \) is the frequency to be determined later
and the \( C_\alpha \) quantities are constants which are chosen to
eliminate resonance conditions in a manner that will become
clear as we proceed. In equation (8) the \( x_\tau \) quantities are
the L.T. of the \( x_\tau \) quantities. \( \alpha \) being small, calculations
are then going to be limited by omitting all terms
containing \( \alpha \) to a power higher than the third.
Substituting (6), (7) & (8) in equation (5) results in
\[ s^2(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3) + \\
(w_0^2 + c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3)(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3) \\
= a s - \alpha x_4 x_0 + \alpha^2 x_2 x_1 + \alpha^3 x_3 x_2 \]

neglecting all terms containing \( \alpha \) to powers higher than

the third results in
\begin{align*}
(s^2 x_0 + w_0^2 x_0) + \alpha(s^2 x_1 + w_0^2 x_1 + c_1 x_0 + L x_2^3) + \\
\alpha^2(s^2 x_2 + w_0^2 x_2 + c_2 x_0 + c_1 x_1 + L x_0^2 x_1) + \\
\alpha^3(s^2 x_3 + w_0^2 x_3 + c_3 x_0 + c_2 x_1 + c_1 x_2 + L(3x_0^2 x_2 + 3x_0 x_1^2)) \\
= a s
\end{align*}

This equation must hold for any value of the quantity \( \alpha \).

This means that each factor for each of the three powers

of \( \alpha \) must be zero.

Hence equation (9) splits into:
\begin{align*}
(s^2 + w_0^2) x_0 &= a s \\
(s^2 + w_0^2) x_1 &= -c_1 x_0 - L x_0^3 \\
(s^2 + w_0^2) x_2 &= -c_2 x_0 - c_1 x_1 - L x_0^2 x_1 \\
(s^2 + w_0^2) x_3 &= -c_3 x_0 - c_2 x_1 - c_1 x_2 - L(3x_0^2 x_2 + 3x_0 x_1^2)
\end{align*}

From equation (10)
\[ x_0 = \frac{a s}{s^2 + w^2} \]

the inverse L.T. of which is:
\[ x_0 = a \cos w_0 t \]
This represents the first approximation to the solution of equation (4). Then from (15)
\[ \mathbf{L} \mathbf{x} = \mathbf{L} ( a^3 \cos^3 w_t ) \]
\[ = \mathbf{L} \left( \frac{3}{4} ( \cos w_t + \cos 3w_t ) \right) \]
\[ = \frac{3}{4} \frac{s^3}{s^2 + w^2} + \frac{3w^2}{s^2 + 3w^2} \]
\[ \text{.................(16)} \]
Substituting now for \( x_0 \) from (14) and \( \mathbf{L} \mathbf{x}_0 \) from (16) in equation (11) results in
\[ x_1 = - \frac{s}{(s^2 + w_0^2)^2} (C_1 a + \frac{3}{4} a^3) - \frac{3w^2}{(s^2 + w_0^2)(s^2 + 3w_0^2)} \]
\[ \text{.................(17)} \]
From a table of Laplace Transforms \[ L^{-1} \frac{s}{(s^2 + w_0^2)^2} = \frac{t}{2w_0} \sin w_0 t \]
Hence the first term of the R.H.S. of equation (17) corresponds to a condition of resonance, to eliminate this \( (C_1 a + \frac{3}{4} a^3) \) should be equal to zero
i.e. \[ C_1 = - \frac{3}{4} a^2 \]
\[ \text{...............(18)} \]
Thus equation (17) reduces to
\[ x_1 = - \frac{\frac{3}{4} a^3}{(s^2 + w_0^2)(s^2 + 3w_0^2)} \]
\[ \text{...............(19)} \]
the inverse L.T. of which gives the second approximation
\[ x_1 = \frac{s^3}{2w_0^2} (\cos 3w_o t - \cos w_o t) \]
\[ \text{...............(20)} \]
Then from (15) & (20)
\[ L(3x_0^2x_1) = L \left( \frac{3a^5}{32w_0^2} \cos^2 w_0 t \left( \cos 3w_0 t - \cos w_0 t \right) \right) \]

\[ = L \left( \frac{3a^5}{128w_0^2} \left( \cos 5w_0 t + \cos 3w_0 t - 2 \cos w_0 t \right) \right) \]

\[ = \frac{3a^5}{128w_0^2} \left( \frac{s}{s^2 + (5w_0)^2} + \frac{s}{s^2 + (3w_0)^2} - \frac{2s}{s^2 + w_0^2} \right) \]

Substituting this and also the values of \( x_0 \) and \( x_1 \) from (14) & (19) in equation (12) results in

\[ x_2 = -\frac{s}{(s^2 + w_0^2)^2} \left( C_2 a - C_1 \frac{a^3}{32w_0^2} - \frac{6a^5}{128w_0^2} \right) \]

\[ = \frac{a^3}{32w_0^2} \left( \frac{C_1 + \frac{4a^2}{3}}{s^2 + w_0^2} \right) \]

\[ = \frac{3a^5}{128w_0^2} \left( \frac{s}{s^2 + w_0^2 (s^2 + 9w_0^2)} \right) \]

\[ \text{..........................(21)} \]

To eliminate the resonance term the coefficient of \( \frac{s}{(s^2 + w_0^2)^2} \) is equated to zero i.e.

\[ C_2 a - C_1 \frac{a^3}{32w_0^2} - \frac{6a^5}{128w_0^2} = 0 \]

then substituting the value of \( C_1 \) from (18) results in

\[ C_2 = \frac{3a^4}{128w_0^2} \]

\[ \text{..........................(22)} \]

Thus the first term of the R.H.S. of equation (21) is now zero, and if the values of \( C_1 \) and \( C_2 \) from (18) & (22), equation (21) then reduces to
\( x_2 = - \frac{3a^5}{128w_0^2} \frac{\alpha}{(s^2 + \frac{\alpha}{w_0})(s^2 + 25w_0^2)} \) \hspace{1cm} (23)

The inverse transform of which gives the third approximation as

\( x_2 = \frac{a^5}{1634w_0^4} (\cos 5w_0 t - \cos w_0 t) \) \hspace{1cm} (24)

Thus from equations (6), (15), (20) and (24), the solution of equation (4) up to the third approximation is:

\[ X = a \cos w_0 t + \frac{\alpha^3 a^2}{32 w_0^2} (\cos 3w_0 t - \cos w_0 t) \]

\[ + \frac{\alpha^2 a^4}{1024 w_0^4} (\cos 5w_0 t - \cos w_0 t) \] \hspace{1cm} (25)

and from equations (7), (18) and (22)

\[ w_0^2 = w^2 + \frac{3\alpha^2 a^2}{128 w_0^2} - \frac{3\alpha^2 a^4}{128 w_0^2} \] \hspace{1cm} (26)

Now putting the numerical values for \( w, \alpha \), and \( \alpha \) taken in section (2.6) results in

\[ X = 0.99453 \cos w_0 t + 0.00844 \cos 3w_0 t \]

\[ + 0.00003 \cos 5w_0 t \] \hspace{1cm} (27)

with \( w_0 = 1.072 \)
Appendix D

Table 1 - Response to Step Changes in the Desired Value - Proportional Control Mode.

<table>
<thead>
<tr>
<th>t (sec.)</th>
<th>Linear</th>
<th>Exponential Regulating Unit</th>
</tr>
</thead>
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<td>1%</td>
</tr>
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<td>0.5</td>
<td>0.0403</td>
<td>0.0427</td>
</tr>
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N.B. The ordinates are divided by $C_d$ in each case, and the sign of the step decrease ordinates is reversed.
Table 2 - Response to Step Changes in the Desired Value - Proportional-Integral Control Mode.

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<th>t (sec)</th>
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<th>Exponential</th>
<th>Regulating Unit</th>
</tr>
</thead>
<tbody>
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<td>5%</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0403</td>
<td>0.0435</td>
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N.B. The ordinates are divided by \( Q_d \) in each case, and the sign of the step decrease ordinates is reversed.
Table 3 - Response to Step Changes in the Desired Value - Proportional + Derivative Control Mode.

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N.B. The ordinates are divided by $S_1$ in each case, and the sign of the step decrease ordinates is reversed.
Table 4 — Response to Step Changes in the Desired Value — 3-Term Control Mode.

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<tr>
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<td>1.0165</td>
<td>1.0143</td>
</tr>
</tbody>
</table>

N.B. The ordinates are divided by $\omega_n$ in each case, and the sign of the step decrease ordinates is reversed.
Table 5 - Response to Step Changes in Load - Proportional Control Mode.

<table>
<thead>
<tr>
<th>sec.</th>
<th>25% Step Increase</th>
<th>25% Step Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>Exp.</td>
</tr>
<tr>
<td>0.25</td>
<td>1.0791</td>
<td>1.0735</td>
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<td>1.3929</td>
<td>1.3913</td>
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<td>2.4963</td>
</tr>
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<td>3.1242</td>
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<td>2.8651</td>
<td>2.5509</td>
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<td>2.3592</td>
<td>1.6936</td>
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<td>1.8563</td>
<td>1.1691</td>
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<tr>
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<td>1.2233</td>
<td>0.7523</td>
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<td>1.6222</td>
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</tr>
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<td>1.5940</td>
<td>1.3651</td>
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N.B. The sign of the step increase ordinates is reversed.
Table 5 - Response to Step Changes in Load -
Proportional+Integral Control Mode.

<table>
<thead>
<tr>
<th>t (sec)</th>
<th>25% Step Increase</th>
<th>25% Step Decrease</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>Exp.</td>
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<td>1.0735</td>
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<td>0.4357</td>
<td>-0.0479</td>
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Note: The sign of the step increase ordinates is reversed.
Table 2 - Response to Step Changes in the Desired Value in the presence of Distance-Velocity Lag, and Response to Large Step Changes in the Desired Value Considering Saturation - Prop, Control Mode.

<table>
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<td>Exponential</td>
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<tr>
<td></td>
<td>5%</td>
<td>5%</td>
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<td>0</td>
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<tr>
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<td>0.8963</td>
<td>0.9566</td>
</tr>
</tbody>
</table>

N.B. The ordinates are divided by \( \theta_d \) in each case, and the sign of the step decrease ordinates is reversed.
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