Some boundary value problems associated
with the Heun equation

by

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Abstract

In this thesis an attempt has been made to complete to a large extent one's knowledge of the solutions of the Heun equation. After the formulation of the equation and a discussion of the convergence of the power series solutions, a new notation is introduced, (Chapter 1). In Chapter 2 integral representations of the solutions of Heun's equation are discussed. Chapter 3 is concerned with certain orthogonality properties, while Chapter 4 deals with new non-linear integral equations satisfied by Heun functions and Heun polynomials. In Chapter 5 Lamé's equation is discussed as a special case of Heun's equation. The objective of this chapter is to correct the statement that Lamé functions of the second kind associated with the Lamé polynomials can be expressed as a finite series of associated Legendre functions of the second kind. In Appendices A and B the work of two previous investigations is summarized.
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Introduction

The Fuchsian equation of the second order designated in this thesis as the Heun equation seems to appear first in mathematical literature in 1889 when Heun (Ref.1) showed that a second order Fuchsian equation with four regular singularities can be reduced to the form

$$\frac{d^2 W}{dz^2} + \left\{ \frac{2}{z} + \frac{\xi}{z-1} + \frac{\epsilon}{z-a} \right\} \frac{dW}{dz} + \frac{\epsilon \zeta (z-a)}{z(z-1)(z-a)} W = 0.$$  

He obtained power series solutions of this equation valid in the neighbourhood of each of the singularities and showed that if \( q \) takes certain "characteristic" values, then solutions can be obtained which are regular at two singularities. Such solutions are now commonly called "Heun functions".

Heun also discovered twenty-four homographic substitutions for this equation which are analogous to the six similar substitutions for the hypergeometric equation. With these substitutions he discovered 43 power series solutions which correspond to Kummer's 24 series for the hypergeometric equation. In fact there are 192 such series although only 96 of them are distinct, (Ref.2a §15.3).

One aspect of Heun's investigations, which seems to have attracted little attention from later workers, is his attempt to obtain relations between contiguous solutions of Heun's equation which correspond to the Gaussian
relations for the hypergeometric functions. The relations obtained by Heun seem to have little application, since the accessory parameter, \( q \), must be restricted to certain values. A brief summary of this part of Heun's work is given in Appendix B.

In 1914 Whittaker (Ref. 3) conjectured that the Heun equation is the simplest equation of Fuchsian type whose solution cannot be represented by a contour integral; instead the nearest approach to such a solution is to find a homogeneous integral equation satisfied by a solution of the differential equation. In the same year, having obtained homogeneous integral equations satisfied by periodic Lamé functions, Whittaker, by restricting the parameters to certain values, obtained homogeneous integral equations satisfied by periodic solutions of Heun's equation. In Chapter 2 of this thesis we reconsider Whittaker's conjecture and show that in fact solutions of Heun's equation can be represented in terms of contour integrals, similar to those of Barnes for the hypergeometric equation. The integrands of these integrals are of a rather complicated nature and cannot be said to involve known or simpler functions, although they do provide expressions for the analytic continuation of Heun functions analogous to those for the hypergeometric functions.

In 1934, Lambe and Ward (Ref. 4) considered
polynomial solutions of Heun's equation and showed that such solutions satisfy certain homogeneous integral equations. By taking particular values of the parameters they were able to deduce integral equations satisfied by polynomials which are solutions of special or limiting cases of Heun's equation. The scope of this work of Lambe and Ward seems to be rather limited in that they only considered one particular solution of the partial differential equation satisfied by the nucleus. Lambe and Ward appear to be the first to consider the ordinary orthogonality of Heun polynomials. In Chapter 3 we extend their work and deduce the ordinary orthogonality of Heun functions and also, by treating the Heun polynomial as a solution of an essentially two-parameter eigenvalue problem, we obtain a certain double orthogonality property. Erdélyi, in 1942, (Ref.5), extended the work of Lambe and Ward and obtained more general nuclei for the integral equation, including the most general nucleus for the equation satisfied by Heun polynomials. With such nuclei, Erdélyi was able to find integral equations satisfied not only by Heun polynomials, but also Heun functions and even by solutions of Heun's equation with arbitrary values of the accessory parameter q. In Chapter 2, Section D, we adapt these integral equations in order to obtain some further integral representations for solutions of Heun's equation.
Up to this time the only series solutions of Heun's equation considered were power series, although in 1939 Svartholm (Ref.6) had considered the solution of Heun's equation in terms of hypergeometric or Jacobi polynomials. In two papers published in 1942 and 1944 respectively, Erdelyi (Refs.7,8) extended the work of Svartholm and solved the Heun equation in terms of series of hypergeometric functions, showing that the results of Svartholm were a particular case of his more general investigations. Because of the importance of Erdelyi's work in this field, especially when applied to special or limiting cases of Heun's equation, it seems appropriate to give a summary of the above papers, and this has been included as Appendix A.

Since 1944, Heun's equation has received little attention and apart from a discussion of the general properties of power series solutions of Heun's equation by Snow (Ref.9) in 1952 and a general survey of the knowledge of Heun's equation existing up to 1953 given in Volume III of the encyclopaedic "Higher transcendental functions," (Ref.2c), there appears to be no fresh contribution.

A primary consideration in this thesis was the introduction of a new notation for solutions of Heun's equation. The reason for this is two-fold; firstly, in all previous literature the symbol used could be confused with that now commonly employed to denote a generalized hypergeometric
function. Thus in order to avoid this possible confusion a new symbol was needed, and secondly, it was found necessary to extend this symbol slightly in order to distinguish clearly between general solutions of Heun's equation, Heun functions and Heun polynomials. This new notation is given at the end of Chapter 1.

Two main sources of ideas ought now to be mentioned. Because of the close analogy of Heun's equation to the hypergeometric equation, it is reasonable to suppose that many of the results established for the hypergeometric equation would find their natural extension in the theory of the Heun equation. This has in fact been the approach of many of the previous writers on this subject, especially Heun. Unfortunately it has been found that this approach, although applicable to a certain extent, soon leads to grave difficulties. The main difficulties are, first, the presence of the arbitrary accessory parameter \( q \) and secondly the fact that the coefficients in a simple power series solution, and indeed also in the case of solutions in series of hypergeometric functions, are related by a three term recurrence relation. Having little knowledge of the exact solution of such a recurrence relation, properties such as analytic continuation, contiguous relations (which one would expect to exist for general values of \( q \)) and integral representations in terms of simpler functions, have, so far, defied all attempts at
discovery. The difficulty of solving such recurrence relations has been overcome to a certain extent by the author in Chapter 2 of this thesis, where analytic solutions are obtained using the Mellin transform method described by Milne-Thomson (Ref.10). The other natural source of ideas has been found in the theory of Lamb's equation, which in its algebraic form is merely a special case of the Heun equation. This approach has been used by the author to obtain non-linear integral equations satisfied by solutions of Heun's equation. The partial differential equation which has to be satisfied by the nucleus turns out to have the remarkable property that if the parameter $\gamma$ takes the value $\frac{1}{2}$, then this equation reduces to Laplace's equation and solutions are obtained in terms of spherical harmonics.

Obviously, in a thesis of this nature it is difficult to distinguish clearly the original parts, although every effort has been made to indicate reliance on previous work. Apart from the main sources of ideas mentioned above, Chapters 2, 3 (with the exception of theorem 2 in Section A), 4 and 5 are, to the author's knowledge, all original, while Chapter 1 is preliminary and contains little that is new.

The plan of this thesis is as follows. Chapter 1 is introductory, showing the formulation of the equation and
discussing the convergence of the power series solution. A new notation for solutions of Heun's equation is also described. Chapter 2 is concerned with integral representations of solutions of Heun's equation. In Chapter 3 we generalize the ordinary orthogonality properties of Heun polynomials to the case of Heun functions and also deduce a certain double orthogonality property. Chapter 4 deals with non-linear integral equations satisfied by solutions of Heun's equation. Chapters 2 and 4 are, in the author's view, the most important fresh contribution to the subject here made. In Chapter 5 Lamé's equation is discussed as a special case of Heun's equation. The objective of this chapter is to correct the statement that Lamé functions of the second kind, corresponding to the Lamé polynomials can be expanded in a finite series of associated Legendre functions of the second kind.

When this thesis was almost complete, it was brought to the author's attention, via (Ref.11), that ideas similar to those exploited in Chapter 2 of this thesis had been used by Ford (Ref.12) when he considered the asymptotic developments of functions defined by Maclaurin Series.

A slightly shortened version of Chapter 5 has been accepted for publication in the Proceedings of the Cambridge Philosophical Society.
A. The formulation of the equation

A generalization of many of the special functions of mathematical physics is Heun's function (Ref. 2c) which satisfies a homogeneous linear differential equation of the second order having four regular singularities.

It is well known that since any Fuchsian equation of the second order with four regular singularities preserves this character under any homographic transformation of the independent variable, three of the singularities can be brought to $z = 0, 1$ and infinity, and the fourth then becomes some finite point $z = a$ (say) which does not coincide with any of the other finite singularities. Without loss of generality we can specify $|a| > 1$. The differential equation must, as shown by Fuchs, be of the form

$$\frac{d^2y_1}{dz^2} + P_1(z) \frac{dy_1}{dz} + Q_1(z)y_1 = 0 \quad (1.1)$$

where

$$P_1(z) = \frac{1-\alpha_0 - \beta_0}{z} + \frac{1-\alpha_1 - \beta_1}{(z-1)} + \frac{1-\alpha_2 - \beta_2}{(z-a)} ,$$

$$Q_1(z) = \frac{1}{\psi(z)} \left[ \alpha_3 \beta_3 + q_1 + \alpha_0 \beta_0 \frac{\psi'(0)}{z} + \alpha_1 \beta_1 \frac{\psi'(1)}{z-1} + \alpha_2 \beta_2 \frac{\psi'(a)}{z-a} \right] ,$$

and

$$\psi(z) = z(z-1)(z-a) ,$$

the constant $q_1$ being arbitrary. The exponents are
\[ \alpha_0, \beta_0 \text{ at } z = 0, \]
\[ \alpha_1, \beta_1 \text{ at } z = 1, \]
\[ \alpha_2, \beta_2 \text{ at } z = a, \]
and \[ \alpha_3, \beta_3 \text{ at } z = \infty. \]

They are not all assignable arbitrarily however, since the Riemann relation

\[ \sum_{r=0}^{3} (\alpha_r + \beta_r) = 2 \]

must be satisfied.

The equation (1.1) takes the standard form when \[ \alpha_0 = \alpha_1 = \alpha_2 = 0. \] If not in that form, it becomes so by the transformation of the dependent variable

\[ y_1 = z^{\alpha_0(z-1)}^{\alpha_1(z-a)}^{\alpha_2} y. \] (1.2)

The differential equation for \( y \) takes the form

\[ \frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha \beta(z-a)y}{z(z-1)(z-a)} = 0. \] (1.3)

where

\[ \alpha = \sum_{r=0}^{3} \alpha_r, \quad \beta - \beta_3 = \alpha - \alpha_3, \]
\[ \gamma = 1 + \alpha_0 - \beta_0, \quad \delta = 1 + \alpha_1 - \beta_1, \]
\[ \epsilon = 1 + \alpha_2 + \beta_2 \]

and

\[ 1 + \alpha + \beta - \gamma - \delta - \epsilon = 0 \] (1.3a)
The constant $q$ is given by
\[
- \alpha \beta q = a_1 + \alpha_0 [\beta_0 (a+1) + (\beta_1-1)a + \beta_2 - 1] + \\
+ \alpha_1 [\beta_0 - 1 + \beta_1 (a-1)] + \\
+ \alpha_2 [\beta_0 - 1 - \beta_2 (a-1)].
\]
The exponents in (1.3) are
\[
0, 1 - \gamma, \text{ at } z = 0 \\
0, 1 - \delta, \text{ at } z = 1 \\
0, 1 - \epsilon, \text{ at } z = a
\]
and $\alpha, \beta$ at $z = \infty$.

Equation (1.3) is called Heun's equation and can be symbolized by the usual Riemannian scheme
\[
P \equiv \begin{pmatrix}
0 & 1 & a & \infty \\
0 & 0 & 0 & \alpha \\
1 - \gamma & 1 - \delta & 1 - \epsilon & \beta
\end{pmatrix}
\]
(1.4)

Heun's equation is of considerable theoretical interest, as it is the simplest equation of Fuchsian type for which the coefficients are not uniquely determined by the singularities and the associated exponents. In fact in Heun's equation there is the accessory parameter $q$ which is quite arbitrary from the point of view of the scheme (1.4). From the practical point of view Heun's equation is of interest because many better-known differential equations are special or limiting cases - for instance the hypergeometric, Lamé and Mathieu differential equations.

It can be shown, (Ref.1), in a manner entirely
similar to the corresponding analysis of the solutions of
the hypergeometric equation, that if \( y(z) \) be that solution
of (1.3) which is analytic at the origin and equal to unity
there, then any other solution of (1.3) can be expressed
simply in terms of functions similar to \( y(z) \). It must be
noted that certain exceptional cases may occur. For
instance the parameters in (1.3) may be such as to give rise
to solutions which involve logarithmic singularities. Such
solutions will not be discussed in this thesis. Clearly
\( y(z) \) can be expanded in the form of a series valid at least
in \( |z| < 1 \).

\[
y(z) = \sum_{n=0}^{\infty} c_n(q)z^n.
\]  

(1.5)

Substituting (1.5) into (1.3) and equating the powers of
\( z \) to zero leads to the following recurrence relations for
the \( G_n(q) \).

\[
a(\gamma \lambda)G_0(q) - \alpha \beta qG_0(q) = 0,
\]

\[
a(n+1)(n+\gamma)G_{n+1}(q) - [n^2(1+\alpha) + n(a \gamma \beta - 1) + \gamma + \beta - 1] + \alpha \beta q]G_n(q) +
\]

\[
+ (n+\alpha - 1)(n+\beta - 1)G_{n-1}(q) = 0, \quad n \geq 1, \quad (1.6)
\]

The condition that \( y(z) \) be unity at the origin requires
that \( G_0(q) = 1 \). Clearly \( G_n(q) \) is a Polynomial of degree
\( n \) in \( q \).
From (1.6) we have

\[ aG_{n+1}(q) - \frac{[n^2(1+a)+n\{a(\gamma+\delta-1)+\gamma+\epsilon-1]+a\beta q]}{(n+1)(n+\gamma)} G_n(q) + \]

\[ + \frac{(n+\alpha-1)(n+\beta-1)}{(n+1)(n+\gamma)} G_{n-1}(q) = 0 , \]

(1.6a)

and as \( n \to \infty \) the coefficients of \( G_n(q), G_{n-1}(q) \to (1+a), -1 \) respectively.

Perron's rule (Ref. 13, vol. 2, §20) shows that in general the radius of convergence of the series (1.5) is \( 1/|t| \), where \( t \) is that root of the quadratic equation

\[ at^2 - (1+a)t + 1 = 0 , \]

(1.7)

which has the larger modulus. Since we have assumed that \( |a| > 1 \), then in general (1.5) only converges within the circle \( |z| = 1 \); (Snow (Ref. 9) has shown that if \( \text{Re}(\gamma+\delta-\alpha-\beta) > 0 \) then (1.5) converges also at \( z = 1 \).

However, as will be seen, if a certain characteristic equation is satisfied, then the series converges within the larger circle \( |z| = |a| \). If a further condition is applied to the equation (1.5) then the series terminates and we obtain polynomial solutions which are valid in the whole of the finite \( z \)-plane.

B  Convergence of the Series Solution

Writing

\[ \lambda_n = -(n+\alpha-1)(n+\beta-1), \quad \mu_n = -[n^2(1+a)+n\{a(\gamma+\delta-1)+\gamma+\epsilon-1)] + \]

\[ + \alpha\beta q] , \]

\[ \nu_{n+1} = a(n+1)(n+\gamma) \]
then the relations (1.6) may be written as
\[ \mu_0 g_0(q) + \nu_1 g_1(q) = 0 \]
\[ -\lambda_n g_{n-1}(q) + \mu_n g_n(q) + \nu_{n+1} g_{n+1}(q) = 0, \quad n > 1. \]  
(1.8)

We now put \( g_n(q) = v_{n-1} g_{n-1}(q) \)
and obtain
\[ v_{n-1} = \frac{\lambda_n}{\mu_n + \nu_{n+1} v_n} \]
and by repeated application we are lead to the infinite continued fraction
\[ v_{n-1} = \frac{\lambda_n}{\mu_n + \frac{\nu_{n+1} \lambda_{n+1}}{\mu_{n+1} + \frac{\nu_{n+2} \lambda_{n+2}}{\mu_{n+2} + \ldots}}} \]
(1.9)

Since \(|a| > 1\) then, (Ref. 13 vol 2 §46) this continued fraction is convergent and
\[ \lim_{n \to \infty} v_{n-1} = 1/a. \]

Meixner and Schöfke (Ref. 14 §1.8 theorem 4) have shown that if the condition
\[ \mu_0 + \nu_1 v_0 = 0 \]
(1.10)
is satisfied then
\[ \lim_{n \to \infty} \frac{g_n(q)}{g_{n-1}(q)} = 1/a \]
and the \( g_n(q) \) are uniquely determined by means of the \( v_n \) given by (1.9). For general values of \( q \), (1.10) is not satisfied and as has already been mentioned the series (1.5) converges only for \(|z| < 1\). When (1.10) is
satisfied however, then (1.5) converges within the
largest circle \(|z| = |a|\).

Equation (1.10) is a transcendental equation
(called the characteristic equation) for the determination
of the \(q\)'s. An exceptional case arises when \(n_{N+1} = 0\) for
some positive integer \(N\); that is when either \(\alpha, \beta\) or both
are negative integers \((-N)\). When \(n < N + 1\), \(V_{n-1}\) is a
finite continued fraction and (1.10) is an algebraic
equation for \(q\). If (1.10) is satisfied in this case then
\(G_{N+1}(q) = 0\) and from (1.8) also \(G_{N+2}(q) = G_{N+3}(q) = \ldots \ldots = 0\).
Thus the series terminates and we obtain \(N + 1\) polynomial
solutions of degree \(N\) of Heun's equation, one for each
value of \(q\).

C. Notation and Definitions

(1) The solution of the Heun equation which is
analytic at the origin and normalized so that it takes the
value one there, will be represented by
\[ H_u(a, q; \alpha, \beta, \gamma, \delta; z). \] (1.11)
Although in all previous works such solutions are
represented by \(F(a, q; \alpha, \beta, \gamma, \delta; z)\), this symbol could be
confused with that now commonly used to represent a
generalized hypergeometric function. Unless absolutely
necessary the notation (1.11) will not be used, but the
more concise symbol
\[ H_u(q; z). \] (1.12)
The usual definition, (Ref. 2c, §15.3) of a Heun function is given as that solution convergent in a region of the $z$-plane which includes at least two singularities of the Heun equation. In this thesis we consider without loss of generality Heun functions convergent in a region containing the singularities $z = 0, 1$. The analysis to follow may be readily adapted to deal with the other types of Heun function. Thus the Heun function discussed in this thesis is defined as follows

(ii) If in addition to the properties given in (1), $H_\mu(q;z)$ is also regular at $z = 1$, i.e. when $q$ is one of the solutions of the characteristic equation (1.10), then $H_\mu(q;z)$ becomes a "Heun function", and will be represented by

$$H_\mu(q_m;z)$$

(1.13)

where $q_m (m = 1, 2, 3, \ldots)$, is a solution of (1.10)

(iii) If either $\alpha$, $\beta$ or both are negative integers, ( $-n$ say), $q$ is a solution of the now reduced equation (1.10) and the properties in (i) apply, then we obtain $n + 1$ "Heun polynomials", i.e. one for each value of $q$. Such functions will be represented by

$$H_\mu(q_m,n;z)$$

(1.14)

where $q_m,n (m = 1, 2, 3, \ldots, n+1)$, is a solution of (1.10). Throughout this thesis, whenever Heun
polynomials are discussed, we shall only consider, without loss of generality, the case where $\alpha = -n$, the remaining parameters being arbitrary.
Chapter 2

Integral Representations

Introduction

All functions which are special or limiting cases of hypergeometric functions, can be represented by integrals of known simpler functions. These integrals normally take the form of those associated with Barnes's contour integrals or Riemann's contour integral (Ref. 15, §§14.5,6). Functions which are special or limiting cases of Heun functions, but still retain the accessory parameter q, as in Lame and Mathieu function theory, do not at first sight appear to be expressible in terms of contour integrals. It has in fact been conjectured that no such representations exist, and that the nearest approach to a relation of this type would be a homogeneous integral equation. The main difficulty which arises is that so far as is known, no form of the coefficients $G_p(q)$ in (1.5) has been found in terms of known functions, except in the very special cases where the recursion system satisfied by the $G_p(q)$ can be reduced to a two-term relation. However, the Mellin-Transform method described by Milne-Thomson (Ref. 10, Ch. 15) leads one to express the Heun functions in terms of contour integrals in which the integrand contains the coefficient $G_p(q)$ and where $r$ is no longer restricted to integral values. These
integrals are similar to those of Barnes for the hypergeometric function and provide expressions for the analytic continuation of Heun functions, although it must be admitted that their practical value is limited because of our lack of knowledge concerning the solution of the recurrence relations satisfied by $G_r(q)$.

In section A, suitable analytic solutions of the recurrence relations $(1.6)$ are found and these are then used in sections B, C to obtain a "Barnes type" integral representation and the analytic continuation of the Heun function. Section D is concerned with adapting the integral equations given by Erdelyi (Ref. 5) in order to obtain solutions of Heun's equation in terms of integrals involving Heun's functions.
A. The analytic solution of the recurrence relation.

In this section we seek a general solution of the difference equation (1.6) where \( n \) is no longer restricted to integral values and is in general complex, so we write \( s \) for \( n \). The method employed is that described by Milne-Thomson (Ref.10, ch.15) and is based on the work of Carmichael (Ref.16) and Birkhoff (Ref.17). In order to keep the analysis within reasonable bounds we shall outline only the main steps of the method.

Write (1.6) in the form

\[
a(s+2)(s+1+\gamma)U(s+2)-[(1+a)s(s+1)+[(1+a)\gamma+a\delta+\epsilon](s+1)+\alpha\beta]U(s+1)+\{s(s+1)+(-1+\alpha+\beta)s+\alpha\beta\}U(s) = 0.
\]  
\[	ag{2.1}
\text{i.e. } P_2(s)U(s+2)+P_1(s)U(s+1)+P_0(s)U(s) = 0 \text{ (say)}.
\]  

We assume (2.1) has a solution of the form

\[
U(s) = \int_{t} t^{s-1}v(t)dt,
\]  
\[
\tag{2.2}
\]  

where \( t \) is a contour in the complex \( t \)-plane, to be chosen appropriately.

Putting

\[
\phi_2(t) = at^2 - (1+a)t + 1,
\]
\[
\phi_1(t) = ayt^2 - [(1+a)\gamma + a\delta + \epsilon]t + (-1+\alpha+\beta),
\]
\[
\phi_0(t) = \alpha\delta(1-q),
\]  
\[
\tag{2.3}
\]
the equation \( \phi_2(t) = 0 \) is called the "characteristic equation" and has roots \( t_1 = 1, \ t_2 = 1/a \). On substituting in (2.1) for \( U(s) \) the contour integral given by (2.2) the left hand side of (2.1) becomes

\[
\int t^{s-1} \sum_{r=0}^{2} (-t)^r \phi_r(t) v^{(r)}(t) dt + [I(s,t)]_1,
\]

where

\[
I(s,t) = v(t)\left[t^s \phi_1(t) + \frac{d}{dt}t^{s+1} \phi_2(t)\right].
\]

(2.4)

Milne-Thomson shows that \( I(s,t) = 0 \) when \( t = 0 \) provided \( \text{Re}(s) > \theta \), where \( \theta \) is the largest root of \( P_0(s) = 0 \), i.e., \( \theta = \max(-\alpha, -\beta) \).

It follows that (2.2) provides a solution of the difference equation if \( v(t) \) is a solution of the differential equation

\[
t^2 \phi_2(t) \frac{d^2 v}{dt^2} - t \phi_1(t) \frac{dv}{dt} + \phi_0(t)v = 0,
\]

(2.5)

and if the contour \( l \) is so chosen that \([I(s,t)]_1\) vanishes identically. The singular points of the differential equation (2.5) are \( t = 0, 1, 1/a, \infty \) with exponents \( \alpha, \beta, (0,1-\delta), (0,1-\epsilon), (0,1-\gamma) \) respectively. In the present analysis we are only concerned with the solutions of (2.5) which are valid in the neighbourhood of \( t = 1, 1/a \) and are associated with the exponents \( 1 - \delta, 1 - \epsilon \) respectively. Thus the solution valid near \( t = 1 \), and having the exponent \( 1 - \delta \), these, can be written as
\[ v_1(t) = (t-1)^{1-\delta} f_1(t), \quad (2.6) \]

where
\[ f_1(t) = \sum_{n=0}^{\infty} A_n (t-1)^n \]

is regular at \( t = 1 \). Similarly, the solution valid near \( t = 1/a \), and having the exponent \( 1 - \epsilon \), these, can be written as
\[ v_2(t) = (t-1/a)^{1-\epsilon} f_2(t) \quad (2.7) \]

where
\[ f_2(t) = \sum_{n=0}^{\infty} B_n (t-1/a)^n \]

is regular at \( t = 1/a \).

The functions \( t^{s-1} v_1(t) \), \( t^{s-1} v_2(t) \) are made single valued by introducing a cut in the \( t \)-plane along the real axis from \( t = 0 \) to infinity, and a further cut from \( t = 0 \) through \( 1/a \) to infinity. (See figure 1.)

[Diagram]

fig 1.
Thus if Re(s-\phi) > 0,

\[ U_1(s) = \frac{1}{2\pi i} \int_{l_1} t^{s-1}\dot{v}_1(t)dt, \]

\[ U_2(s) = \frac{1}{2\pi i} \int_{l_2} t^{s-1}\dot{v}_2(t)dt, \]

(2.8)

are solutions of (2.1) provided the loops \( l_1, l_2 \) are drawn to include only one root of the characteristic equation \( \phi_2(t) = 0 \), i.e., the points \( t = 1, t = 1/a \). If \( a \) is real then \( 1/a \) lies on the real axis of the \( t \)-plane and so \( l_2 \) must be deformed so that it does not enclose the point \( t = 1 \).

**Factorial Series Solutions**

We have, from (2.8),

\[ U_1(s) = \frac{1}{2\pi i} \int_{l_1} t^{s-1}(t-1)^{1-\delta}f_1(t)dt. \]

(2.9)

In this, we make the change of variable

\[ t = z^{1/w} \]

(2.10)

where \( w > 1 \). Then

\[ U_1(s) = \frac{1}{2\pi i w} \int_{C} z^{s/w-1}(z^{1/w}-1)^{1-\delta}f_1(z^{1/w})dz, \]

where \( C \) is a loop from the origin round \( z = 1 \). The circle \( |z-1| = 1 \) in the \( z \)-plane transforms into a loop in the \( t \)-plane, round \( t = 1 \) enclosed by two rays inclined at an angle \( \pi/w \). By taking \( w \) sufficiently large we ensure that
t = 0, 1 are the only singular points of \( v_1(t) \) in or on \( C \).

It follows that \( f_1(z^{1/w}) \) is regular within and on the circle \( |z-1| = 1 \) except at \( z = 0 \). We can therefore find an expansion

\[
\frac{1}{2\pi i w} \left( \frac{z^{1/w}}{z-1} \right)^{1-\delta} f_1(z^{1/w}) = \sum_{n=0}^{\infty} c_n (1-z)^n,
\]

which is convergent within the circle \( |z-1| = 1 \), so that

\[
U_1(s) = \int_C z^{s/w-1} (z-1)^{1-\delta} \sum_{n=0}^{\infty} c_n (1-z)^n dz.
\]

Since \( C \) is interior to the circle, we can integrate term by term and since \( \text{arg}(z-1) = -\pi \) at the beginning of the loop

\[
U_1(s) = e^{-i\pi(1-\delta)} \sum_{n=0}^{\infty} c_n \int_C z^{s/w-1} (1-z)^{1-\delta+n} dz
\]

\[
= \frac{\Gamma(s/w) \Omega_1(s, 1-\delta)}{\Gamma(s/w+2-\delta)},
\]

(2.11)

where

\[
\Omega_1(s, 1-\delta) = 1 + \sum_{n=1}^{\infty} \frac{D_n(2-\delta)\ldots(1-\delta+n)}{(s+2w-w\delta)\ldots(s+(1+n)w-w\delta)}
\]

\( U_1(s) \) being determined up to an arbitrary constant.

Similarly, by making the substitution

\[
t = \frac{1}{a} z^{1/w},
\]

and using the above method we find a factorial series
for $U_2(s)$ in the form

$$U_2(s) = \left( \frac{1}{a} \right)^s \frac{\Gamma(s/w)\Omega_2(s,1-\varepsilon)}{\Gamma(s/w+2-\varepsilon)} \quad (2.12)$$

where $\Omega_2(s,1-\varepsilon)$ has a similar form to $\Omega_1(s,1-\delta)$. The coefficients $D_n$ in (2.11) and hence the corresponding coefficients in (2.12) are found by substituting $U_1(s)$, $U_2(s)$ in (2.1) and then employing the usual method of undetermined coefficients.

The series (2.11), (2.12) are single valued analytic functions, convergent in the right half plane defined by $\text{Re}(s-\theta) > 0$. If we write (2.1a) in the form

$$U(s) = \frac{-1}{P_0(s)} [P_1(s)U(s+1) + P_2(s)U(s+2)] \quad (2.13)$$

then by repeated substitution of $U_1(s)$, $U_2(s)$ defined by (2.11), (2.12) into the right hand side of (2.13) we obtain single valued and analytic solutions convergent for $\text{Re}(s-\theta) < 0$, except for the poles

$$s = -\alpha - n, -\beta - n, n = 0, 1, 2, \ldots \ldots \quad (2.14)$$

By considering the asymptotic properties of the gamma function we easily see that

$$U_1(s) = \left( \frac{a}{w} \right)^{\delta-2} [1+\eta_1(s)], \quad \lim_{|s| \to \infty} \eta_1(s) = 0$$

$$U_2(s) = a^{-3} \left( \frac{a}{w} \right)^{\varepsilon-2} [1+\eta_2(s)], \quad \lim_{|s| \to \infty} \eta_2(s) = 0. \quad (2.15)$$
Thus there exists a solution \( G_s(q) \) of (2.1) which is a linear combination of \( U_1(s) \) and \( U_2(s) \), say
\[
G_s(q) = AU_1(s) + BU_2(s).
\]
In general \( A, B \) are arbitrary functions of \( s \) of period one, but for our purposes we take them to be constants. We require \( G_s(q) \) to coincide with \( G_n(q) \) in (1.5) when \( s = n, \ n = 0,1,2, \ldots \), and to vanish identically when \( s = -n \). Thus we have the boundary conditions
\[
1 = AU_1(0) + BU_2(0)
\]
\[
0 = AU_1(-1) + BU_2(-1)
\]
(\( U_1(0), U_2(0) \) found using (2.13), and on using Casorati's determinant (Ref. 10, p. 488) we find that
\[
A = \frac{\Gamma(\gamma)U_2(-1)}{(1-a)\Gamma(\alpha)\Gamma(\beta)}, \quad B = \frac{\Gamma(\gamma)U_1(-1)}{(1-a)\Gamma(\alpha)\Gamma(\beta)}.
\]
Clearly, the condition that \( A \) in (2.18) should vanish can be identified with the condition (1.10); that is, the equation \( U_2(-1) = 0 \) is equivalent to the characteristic equation (1.10). Therefore
\[
G_s(q) = \frac{-\Gamma(\gamma)}{(1-a)\Gamma(\alpha)\Gamma(\beta)} \left\{ U_2(-1)U_1(s) - U_1(-1)U_2(s) \right\},
\]
which when considered as a function of the complex variable \( s \) is single valued and analytic throughout the region \( \text{Re}(s-\alpha) > 0 \), and when extended to the left of this
region by means of (2.13) has the same properties except at the poles (2.14).

By writing

\[ U_1(s) = G_1(s), \quad U_2(s) = a^{-s}G_2(s), \]  

(2.20) may be written as

\[ y = \frac{-\Gamma(\gamma)U_2(-1)}{(1-a)\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} G_1(n)z^n + \frac{\Gamma(\gamma)U_1(-1)}{(1-a)\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} G_2(n)(z/a)^n. \]  

(2.21)

In the case of the Heun function \( H(u; z) \) we have

\[ H(u; z) = \frac{\Gamma(\gamma)U_1(-1)}{(1-a)\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} G_2(n)(z/a)^n. \]  

(2.22)

B. An Integral Representation for the Heun function.

Consider

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G_2(s)(-z/a)^s}{\sin \pi s} \, ds, \]  

(2.23)

where \((-z/a)^s\) is rendered single valued by the restriction \(|\arg(-z/a)| < \pi\), and the path of integration is such that the poles (2.14) of \( G_2(s) \) lie to the left of the path, and the zeros of \( \sin \pi s \) lie to the right of the path. Clearly as \(|s| \to \infty\) on the contour, the integrand is a single valued and analytic function of \( z \) throughout the domain \(|\arg(-z/a)| < \pi - \Delta, \Delta > 0\).
Consider the integral
\[
\frac{1}{2\pi i} \int_{C} \frac{G_2(s)(-z/a)^s}{\sin s\pi} \, ds,
\]
where \( C \) is a semi-circle of radius \( N + \frac{1}{2} \) to the right of the imaginary axis with centre at the origin, and \( N \) is an integer.

Now, using equation (2.15) we see that as \( N \to \infty \),
\[
\frac{G_2(s)(-z/a)^s}{\sin s\pi} = O\left(\frac{(N^{1/2})^{s-2}}{\sin s\pi}\right),
\]
the constant implied in the \( O \) symbol being independent of \( \arg s \) when \( s \) is on the semi-circle; and if \( s = (N+\frac{1}{2})e^{i\theta} \) and \( |z/a| < 1 \), then
\[
(-z/a)^s \csc s\pi = 0\left\{\exp\left[\left(N+\frac{1}{2}\right)\cos \theta \log \left|\frac{z}{a}\right| - (N+\frac{1}{2})\sin \theta \arg\left(-\frac{z}{a}\right) - (N+\frac{1}{2})\pi \sin \theta|\right]\right\}
\]
\[
= 0\left\{\exp\left[\left(N+\frac{1}{2}\right)\cos \theta \log \left|\frac{z}{a}\right| - (N+\frac{1}{2})\Delta|\sin \theta|\right]\right\}.
\]
Hence if \( |z/a| < 1 \), then \( \log |z/a| \) is negative and the integral tends to zero sufficiently rapidly (when \(-\pi/2 < \theta < \pi/2\)) to ensure that
\[
\int_{C} \frac{G_2(s)(-z/a)^s}{\sin s\pi} \, ds \to 0 \text{ as } N \to \infty.
\]
Consider the closed contour $\Gamma$. We have, omitting the integrand

$$\int = \int_{-i\infty}^{i\infty} + \int_{-\infty}^{i\infty} + \int_{C}^{i(N+\frac{1}{2})} + \int_{i\infty}^{i\infty}$$

$$= 2\pi i \sum \text{residues in } \Gamma.$$

Now as $N \to \infty$, the first three integrals on the right hand side of the above expression tend to zero when $|\arg(-z/a)| < \pi$ and $|z/a| < 1$ and so

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{G_2(s)(-z/a)^s}{\sin s \pi} \, ds = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{G_2(s)(-z/a)^s}{\sin s \pi} \, ds$$

$$= \lim_{N\to\infty} \sum \text{residues in } \Gamma. \quad (2.25)$$

The sum of the residues in $\Gamma$ as $N\to\infty$

$$= \sum_{n=0}^{\infty} \lim_{s\to n} \frac{(s-n)G_2(s)(-z/a)^s}{\sin s \pi}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} G_2(n)(z/a)^n,$$
and therefore
\[ \sum_{n=0}^{\infty} G_2(n)(z/a)^n = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{G_2(s)(-z/a)^s}{\sin s \pi} \, ds, \quad (2.26) \]

\[ |\arg(-z/a)| < \pi, |z/a| < 1. \]

Hence we obtain an integral representation of the Heun function in the form
\[ \text{Hu}(q_m; z) = \frac{i\Gamma(y)U_1(-1)}{2(1-a)\Gamma(\alpha)\Gamma(\beta)} \int_{-i\infty}^{i\infty} \frac{G_2(s)(-z/a)^s}{\sin s \pi} \, ds \quad (2.27) \]

where \( |\arg(-z/a)| < \pi \) and \( |z/a| < 1 \).

If, however, \( |\arg(-z)| < \pi \) and \( |z| < 1 \), then by a similar analysis we can show that
\[ \sum_{n=0}^{\infty} G_1(n)z^n = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{G_1(s)(-z)^s}{\sin s \pi} \, ds \quad (2.28) \]

Thus on using (2.19), (2.20), (2.26) and (2.28) we see that \( y \) as defined by (1.5) can be represented as
\[ y = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{G_2(q)(-z)^s}{\sin s \pi} \, ds, \quad (2.29) \]

where \( |\arg(-z)| < \pi, |\arg(-z/a)| < \pi \) and \( |z| < 1 \).

C. The analytic continuation of the Heun function

In order to obtain the analytic continuation of the Heun function for \( |z| > |a| \) we need to find the residues
of $c_2(s)(-z/a)^s/\sin s \pi$ at the poles (2.14) and at the poles $s = -n, n = 1, 2, 3, \ldots, \ldots$. Consider (2.13); in terms of the function $U_2(s)$, we have

$$U_2(s) = -\frac{1}{p_0(s)}[p_1(s)U_2(s+1)+p_2(s)U_2(s+2)].$$

Therefore

$$\frac{U_2(s)(-z)^s}{\sin s \pi} = -\frac{[p_1(s)U_2(s+1)+p_2(s)U_2(s+2)]}{(s+\alpha)(s+\beta)\sin s \pi}(-z)^s.$$

Thus the residue at the pole $s = -\alpha$ is:

$$\lim_{s \to -\alpha} \frac{(s+\alpha)U_2(s)(-z)^s}{\sin s \pi} = e^{i\pi \alpha} \frac{[p_1(-\alpha)U_2(1-\alpha)+p_2(-\alpha)U_2(2-\alpha)](-\alpha)^{-\alpha}}{(\beta-\alpha)\sin \alpha \pi}.$$

In terms of the function $c_2(s)$ defined by (2.20), the residue of $c_2(s)(-z/a)^s/\sin s \pi$ at $s = -\alpha$ is:

$$\frac{e^{i\pi \alpha}p_1(-\alpha)p_2(1-\alpha)}{\pi(\beta-\alpha)a^2} \cdot \left[a\alpha_1(-\alpha)c_2(1-\alpha)+a\alpha_2(-\alpha)c_2(2-\alpha)\right] \left(\frac{a}{z}\right)^\alpha.$$ (2.30)

The residue at the pole $s = -\beta$ is obtained similarly. The residues at the remaining poles (2.14) are found similarly by repeated application of (2.13). Since in the case of flown functions $U_2(-1) = 0$, then clearly $c_2(-n) = 0$, $n = 1, 2, \ldots, \ldots$, and thus the residues at the poles $s = -n$ are all zero.

Consider

$$\int \frac{c_2(s)(-z/a)^s}{\sin s \pi} ds,$$

where $\Gamma$ is the closed contour shown in the figure.
It follows by analysis, similar to that employed in section B, that
\[ \int_{C} \frac{G_{2}(s)(-z/a)^{s}}{\sin s \pi} \, ds \to 0, \]
as \( \rho \to \infty \), provided \( |\arg(-z/a)| < \pi \), \( |z/a| > 1 \) and \( \rho \to \infty \) in such a way that the lower bound of the distance of \( C \) from the poles of the integrand is not zero.

Now symbolically,
\[ \int_{-i\infty}^{i\infty} + \int_{-\infty}^{-i\infty} + \int_{C} - \int_{-i\infty}^{i\infty}. \]
Thus as \( \rho \to \infty \)
\[ \int_{-i\infty}^{i\infty} = 2\pi i \sum \text{residues of the integrand within } \Gamma. \]
On using (2.30) we get
\[ \int_{-i\infty}^{i\infty} \frac{G_{2}(s)(-z/a)^{s}}{\sin s \pi} \, ds \sim \]
\[ \sim 2i e^{i\pi \alpha} r(\alpha) r(1-\alpha) \left[ \frac{aP_{1}(\alpha)G_{2}(1-\alpha) + P_{2}(\alpha)G_{2}(2-\alpha)(a/z)^{\alpha}}{(\rho-\alpha)a^{2}} \times \{1+0(1/z)\} + \right. \]
\[ + 2i e^{i\pi \beta} r(\beta) r(1-\beta) \left[ \frac{aP_{1}(\beta)G_{2}(1-\beta) + P_{2}(\beta)G_{2}(2-\beta)(a/z)^{\beta}}{(\alpha-\beta)a^{2}} \times \{1+0(1/z)\} \right. \]
Thus the Heun function \( H_u(q_m;z) \) is expressible asymptotically as a linear combination of the two solutions of Heun's equation valid in the neighbourhood of \( z = \infty \). Therefore, in the notation of (1.11) we have

\[
H_u(q_m;z) = H_u(a,q_m;\alpha,\beta,\gamma,\delta;z) \sim
\]

\[
- \frac{\Gamma(\gamma)e^{\pi\mu}T(1-\alpha)U_1(-1)}{(1-a)\Gamma(\beta)(\beta-\alpha)a^2} \left[ aP_1(-\alpha)G_2(1-\alpha)+P_2(-\alpha)G_2(2-\alpha) \right] \left( \frac{a}{z} \right)^\alpha \times
\]

\[
\times H_u\left( \frac{1}{a},q_m;\alpha,\alpha-\gamma+1,\alpha-\beta+1,\delta;\frac{1}{z} \right) +
\]

\[
+ \frac{\Gamma(\gamma)e^{\pi\mu}T(1-\beta)U_1(-1)}{(1-a)\Gamma(\alpha)(\beta-\alpha)a^2} \left[ aP_1(-\beta)G_2(1-\beta)+P_2(-\beta)G_2(2-\beta) \right] \left( \frac{a}{z} \right)^\beta \times
\]

\[
\times H_u\left( \frac{1}{a},q_m;\beta,\beta-\gamma+1,\beta-\alpha+1,\delta;\frac{1}{z} \right),
\]

(2.31)

for \(|z| > |a|\), \(|\arg(-z/a)| < \pi\).

The asymptotic sign in (2.31) can be replaced by an equality sign if a cut is made along the line \( \arg z = \arg a \) from \( z = a \) to infinity. Similarly we can obtain the analytic continuation of \( H_u(q;z) \) for \(|z| > 1\). It will be in a similar form to (2.31) except for an additional cut along the positive real axis from \( z = 1 \).

D. Integral representations deduced from a certain homogeneous integral equation.

In this section and in the remainder of this thesis we shall often require the use of the well-known Pochhammer
loop contour \( C \) given in figure 2.

\[ f_{ig.2} \]

\((C_1, C_2)\) are taken to be any pair of the points 0, 1, a while the contour is deformed if necessary in order to exclude the third. The lines A B C D are supposed to coincide with the line \( C_1 C_2 \). If we consider \((C_1, C_2)\) to be the pair \((0, 1)\) then clearly the value of the Heun function \( H \nu \left( q \right| z \right) \) at \( P \)(say) returns to the same value after being continued around \( C \). In the case of Heun polynomials we may take \((C_1, C_2)\) to be any of the pairs of points \((0, 1); (0, a); (1, a)\). For brevity the pachhammer loop contours surrounding the points \((0, 1); (0, a); (1, a)\) will be symbolized by \( C_{0,1} \), \( C_{0,a} \) and \( C_{1,a} \) respectively.

Hedéyi (Ref. 5) and Lambe and Ward (Ref. 4) have considered integral equations satisfied by solutions of Heun's equation of the form

\[ y(z) = \lambda \int \frac{t^{\gamma-1}(1-t)^{\delta-1}(1-\frac{t}{a})^{\epsilon-1}}{C} K(z, t)y(t)dt. \]  \( (2.32) \)

The nucleus, \( K(z, t) \) satisfies the partial differential equation

\[ (M_z - M_t)K(z, t) = 0, \]  \( (2.33) \)
where

\[ M_z = z(z-1)(z-a) \frac{d^2}{dz^2} + [\gamma(z-1)+z(z-a)+cz(z-1)] \frac{d}{dz} + \alpha \beta z, \]

and the contour C in (2.32) is chosen so that the "integrated" parts vanish. Erdélyi obtained a wide set of solutions of equation (2.33), by introducing the variables \( \theta \) and \( \phi \), where

\[ \cos \theta = \left( \frac{zt}{a} \right)^{\frac{1}{2}}, \quad \sin \theta \cos \phi = i \left( \frac{(z-a)(t-a)}{a(1-a)} \right)^{\frac{1}{2}} \]

and \( \sin \theta \sin \phi = \left\{ \frac{(z-1)(t-1)}{(1-a)} \right\}^{\frac{1}{2}}. \) \hspace{1cm} (2.35)

Under the substitutions (2.35), equation (2.33) becomes

\[ \sin^2 \theta \left[ \frac{d^2}{d\theta^2} + [(1-2\gamma)\tan \theta + 2(\delta + \epsilon - \frac{1}{2})\cot \theta] \frac{dK}{d\theta} - 4\alpha \beta \Phi \right] + \]

\[ + \left( \frac{d^2}{d\phi^2} + [(1-2\delta)\cot \phi - (1-2\epsilon)\tan \phi] \frac{dK}{d\phi} \right) = 0. \] \hspace{1cm} (2.36)

We shall follow Erdélyi in choosing solutions of (2.36) in the form of a product of a function of \( \theta \) and a function of \( \phi \), a typical solution being:

\[
\begin{bmatrix}
0 & 1 & \infty \\
0 & 1 & \frac{1}{2} - \delta - \epsilon \\
1 - \gamma & \frac{1}{2} - \epsilon + \sigma & \beta \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & \infty \\
0 & 0 & -\frac{1}{2} + \delta + \sigma \\
1 - \epsilon & 1 - \delta & -\frac{1}{2} + \epsilon - \sigma \\
\end{bmatrix}
\]

where \( P\{ \} \) is the Riemann symbol, \( \sigma \) is the arbitrary separation constant and \( \theta, \phi \) are given by (2.35). Erdélyi then discussed the various nuclei suitable for the integral equations satisfied by Heun functions, and Heun polynomials.
For Heun polynomials a possible set of nuclei is provided if
\[ \sigma = \frac{1}{2} - \ell - m, \quad m = 0, 1, \ldots, n, \quad (2.33) \]
\( n \) being the degree of the Heun polynomial considered.

This follows from the bilinear development of the symmetric nucleus of an integral equation, (Ref. 15, §11.7), from which it follows that the nucleus (2.37) must be a Polynomial, giving condition (2.33).

In that which follows we shall denote by
\( \text{Ku}(q_m; z) \) the solution of the equation satisfied by
\( \text{Hu}(q_m; z) \) which is valid in the neighbourhood of infinity and belongs to the exponent \(-\alpha\) or \(-\beta\) there. Such a solution may be obtained by applying the transformation
\( z = 1/\xi \) to Heun's equation and then obtaining a simple power series solution about \( \xi = 0 \). If \( \text{Ku}(q_m; z) \) is normalized so that the coefficient of the highest power of \( z \) is unity then, in the notation of (1.11), we may write
\begin{align*}
\text{Ku}(q_m; z) &= z^{-\alpha} \text{Hu}(1/a, q_m; \alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \delta; 1/z) \\
or &= z^{-\beta} \text{Hu}(1/a, q_m; \beta, \beta - \gamma + 1, \beta - \alpha + 1, \delta; 1/z). \quad (2.39)
\end{align*}

As \( |z| \to \infty \), \( \text{Ku}(q_m; z) \) behaves as \( |z|^{-\alpha} \) or \( |z|^{-\beta} \) according to the branch chosen and therefore it is an obvious step to take as a nucleus, those branches of the \( P \)-functions (2.37) which behave in the same manner for \( |z| \to \infty \). A suitable nucleus is the function
\[(zt-a)^{\frac{1}{2}+\sigma-\delta} \left(\frac{zt}{a}\right)^{-\frac{1}{2}+\sigma-\delta-\alpha} \right. \\
\left. \cdot _2F_1\left(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{-\sigma-\alpha-\gamma}{2},\frac{\alpha+1-\beta}{zt}\right) \right. \\
\times \left. \begin{bmatrix}
0 & 1 & \infty \\
0 & 0 & \frac{(z-a)(t-a)}{(1-a)(zt-a)} \\
1-\epsilon & 1-\delta & \frac{1}{2}+\epsilon-\sigma
\end{bmatrix} \right) \right) \tag{2.40}
\]

or the corresponding nucleus with \(\alpha\) replaced by \(\beta\). With

\[K(z,t)\]

as the function (2.40), the integral

\[\int_C t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1}K(z,t)Hu(q_m;t)dt \tag{2.41}\]

represents a multiple, possibly zero, of \(Ku(q_m;z)\),

provided the contour \(C\) is suitably chosen, and \(z\) is such

that the integral converges inside the contour of integration.

Denoting the integral (2.41) by \(y(z)\) and writing

Heun's equation in terms of \(M_z\) we find that

\[
(M_z - \alpha \beta q_m)y = 0
\]

\[= \int_C t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1}Hu(q_m;t)\left\{M_t - \alpha \beta q_m\right\}K(z,t)dt,
\]

on using (2.33). Now, after rearranging it is easily shown that

\[t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1}M_t = \frac{d}{dt}[at^\gamma(1-t)^\delta(1-t/a)^\epsilon \frac{d}{dt}] \] .

Thus

\[\int_C t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1}Hu(q_m;t) M_t\{K(z,t)\}dt \]

\[= \int_C a Hu(q_m;t) \frac{\partial}{\partial t} [t^\gamma(1-t)^\delta(1-t/a)^\epsilon \frac{\partial K}{\partial t}]dt . \tag{2.42}\]
Integrating by parts, the right hand side of (2.42) becomes
\[
\int \gamma (1-t)^{\delta}(1-t/a)^{\epsilon} \left\{ \frac{\partial}{\partial t} \text{Hu}(q_m; t) - K \frac{\partial \text{Hu}(q_m; t)}{\partial t} \right\} dt +
\int t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1} K(z, t) \{ M_t - \alpha \beta q_m \} \text{Hu}(q_m; t) dt =
\int \left[ \gamma (1-t)^{\delta}(1-t/a)^{\epsilon} \left\{ \frac{\partial}{\partial t} \text{Hu}(q_m; t) - K \frac{\partial \text{Hu}(q_m; t)}{\partial t} \right\} \right] dt.
\]

since \[ M_t - \alpha \beta q_m \] \text{Hu}(q_m; t) = 0.

If C is chosen such that the "integrated" parts vanish, then \( y(z) \) satisfies Heun's equation and is a constant multiple, possibly zero, of \( \text{Ku}(q_m; z) \). Suppose \( \text{Re}(\delta) > 0 \), \( \text{Re}(\epsilon) > 0 \) and C is deformed to the straight line joining 1 to a then we must take \( |z| > |a| \). If \( \text{Re}(\epsilon - \delta - 2\sigma) < 0 \), or \( 0 < \text{Re}(\epsilon - \delta - 2\sigma) < 1 \), so that \( \frac{z}{t} \) in (2.40) is absolutely or conditionally convergent, respectively, at \( |z| = |a| \), then the range of \( z \) can be extended to \( |z| > |a| \).

Hence we have
\[
\text{Ku}(q_m; z) = \lambda \int_{1}^{a} \gamma (1-t)^{\delta-1}(1-t/a)^{\epsilon-1} K(z, t) \text{Hu}(q_m; t) dt,
\]
provided the singularity \( t = a/z \) is left outside the contour and where \( K(z, t) \) is given by (2.40). In this, \( \lambda \) is the characteristic number dependent upon \( q \).

We now turn to integral relations for the solution
\[ \text{Ku}(q_m; z) = \text{Ku}(q_{m,n}; z), \quad (m = 1, 2, \ldots, n+1), \] of the
equation satisfied by the Heun polynomial \( \text{Hu}(q_m, n; z) \).
Here we again use the nucleus (2.40) but where \( \sigma \) takes
the values \( \frac{1}{2} - \delta - m \). The nucleus then takes the form
\[
(zt-a)^m \left( \frac{zt}{a} \right)^{\beta-m} 2F_1(\beta+m, \beta+m+1-\gamma; \beta+1-\alpha; \frac{zt}{a}) \times
\begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & -m \\
1-\epsilon & 1-\delta & \epsilon+\delta-1+m
\end{pmatrix}, \quad \alpha = -n.
\]
(2.44)
The nucleus with \( \beta \) replaced by \( \alpha \) merely gives an
integral equation satisfied by the polynomial \( \text{Hu}(q_m, n; z) \).
If \( \text{Re}(\delta) > 0, \text{Re}(\epsilon) > 0 \) and \( C \) is the straight line joining
1 to \( a \), then provided \( |z| > |a| \) or if \( \text{Re}(\epsilon+\delta-1+2m) < 0 \) or
\( 0 < \text{Re}(\epsilon+\delta-1+2m) < 1 \) so that \( z \) can be extended to the range
\( |z| > |a| \), we find that
\[
\text{Ku}(q_m, n; z) = \lambda \int_1^a t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1} K(z, t) \text{Hu}(q_m, n; t) dt,
\]
(2.45)
where \( K(z, t) \) is of the form (2.44) or is a linear
combination of such nuclei, summed over \( m \). The restrictions
on \( \delta, \epsilon \) are removed if we choose the contour \( C \) to be \( C_{1,a} \).
The simplest possible nucleus in this case is given by
\( m = 0 \), and the solution \( \text{Ku}(q_m, n; z) \) is related to
\( \text{Hu}(q_m, n; z) \) by the integral
\[
\text{Ku}(q_m, n; z) = \lambda \left( \frac{z}{a} \right)^{-\beta} \int_{C_{1,a}} t^{\gamma-1}(1-t)^{\delta-1}(1-t/a)^{\epsilon-1} x
\]
\[ x \frac{2F_1(\beta, \beta+1-\gamma; \beta+1-\alpha; \frac{a}{zt})}{z} \text{Hu}(q_m, n; t)dt. \] (2.46)

By a similar analysis we can obtain integral relations for the solutions of Heun's equation which are valid in the neighbourhood of \( z = 1, a \) in terms of Heun functions or Heun polynomials.

An important special case is the integral representation for the second solution \( E^M_N(z) \) of Lamé's equation when the first solution is a Lamé polynomial \( E^M_N(z) \).

Let
\[ \alpha = -\frac{1}{2}, \beta = \frac{1}{2}(n+1), \gamma = \delta = \epsilon = \frac{1}{2}, a = k^{-2}, \] (2.47)
and choose as nucleus the function
\[ (zt-a)^m \left( \frac{zt}{a} \right)^{-\beta-m} \frac{2F_1(\beta+m, \beta+m+1-\gamma; \beta+1-\alpha; \frac{a}{zt})}{z} \times \]
\[ \frac{(z-1)(t-1)}{(a-1)(zt-a)}. \] (2.48)

For Lamé polynomial solutions of Heun's equation, \( n \) must be an integer, possibly zero, thus \( \alpha \) or \( \beta \) is a negative integer or zero. On using the substitutions (2.47), the nucleus (2.48) becomes
\[ (zt-k^{-2}) \left( zt k^2 \right)^{-\frac{1}{2}n-1-m} \frac{2F_1(\frac{1}{2}(n+1)+m, \frac{1}{2}n+m+1; n+3; \frac{1}{2} k^2/zt)}{zt} \times \]
\[ \frac{k^{-2}(z-1)(t-1)}{(k^2-1)(zt-k^{-2})}. \] (2.49)

In the case of Lamé functions we find it convenient because of the restrictions on \( n \) to replace \( m \) by \( m/2 \).

With this substitution, the nucleus (2.49) can be expressed in terms of associated Legendre functions of the second
kind, and is a constant multiple of the function
\[ q_n^m(\cos \theta) \cos m \phi \]
where
\[ \cos^2 \theta = k^2 z t, \quad \cos^2 \phi = \frac{(z-k^{-2})(t-k^{-2})}{(1-k^{-2})(zt-k^{-2})}. \]

If we take the branch of the P-function in (2.34) which belongs to the exponent 1-\(\delta\) at the singularity 1, then \(q_n^m(\cos \theta)\sin m \phi\) is another suitable nucleus. Thus we have
\[ F_N^M(z) = \lambda \int_C t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} (1-tk^2)^{-\frac{1}{2}} q_n^m(\cos \theta) \left\{ \sin m \phi \frac{E_M^m(t)}{\cos m \phi} \right\} dt. \]
\[ (2.51) \]

We now make a further transformation, writing
\[ z = \sn^2 \alpha, \quad t = \sn^2 \beta; \]
and choose the loop contour C which does not pass through any of the poles of \(\sn \beta\). We further set \(|\sn \alpha| > 1/k\), then (2.51) becomes
\[ F_N^M(\alpha) = \lambda \int_C q_n^m(\kappa \sn \alpha \sn \beta) \left\{ \sin m \phi \frac{E_M^m(\beta)}{\cos m \phi} \right\} d \beta, \]
\[ (2.53) \]
where
\[ \cos^2 \phi = \frac{dn^2 \alpha \ dn^2 \beta}{k^2 (1-k^2 \sn^2 \alpha \sn^2 \beta)}. \]

(2.53) is the integral relation initially obtained for the second solution of Lamé's equation by Arscott (Ref.16).

If we take \(m = 0\), and choose a solution of Lamé's
equation which is a polynomial of the first species

denoting it by \( u_{\frac{M}{n}} \), then \( n \) must be even = 2N say. This

notation is due to Arscott (Ref. 19, §9.3.2). In this

case we choose the nucleus as

\[
Q_{2N}(k\text{sn} \alpha \text{sn} \beta),
\]

and here \( \text{Re}(\delta) > 0, \text{Re}(\epsilon) > 0 \) and \( \text{Re}(\epsilon + \delta - 1) = 0 \), so we

can replace the contour \( C \) by the straight line joining

1 to \( k^{-2} \). Under the transformation (2.52) this becomes

the straight line in the complex \( \alpha \)-plane, running from \( K \)

to \( K + iK' \). The function \( u F_{\frac{M}{2N}}(\alpha) \) can then be expressed

in terms of \( u E_{\frac{M}{2N}}(\alpha) \) by the integral relation

\[
u F_{\frac{M}{2N}} = \lambda \int_{K}^{K+1K'} Q_{2N}(k\text{sn} \alpha \text{sn} \beta) u E_{\frac{M}{2N}}(\beta) d\beta,
\]

(2.54)

for \( |\text{sn} \alpha| > \frac{1}{K} \), which is in full agreement with the

corresponding result obtained by Arscott.
Chapter 3

The Orthogonal Properties of Heun Functions

The ordinary orthogonal properties of Heun polynomials have been studied by Lambe and Ward (Ref. 4), and the corresponding properties for Heun functions have been given by Erdelyi (Ref. 3) who used the expansions in series of hypergeometric functions. In this chapter we generalize the work of Lambe and Ward to the case of Heun functions and deduce the ordinary orthogonality properties directly from Heun's equation. For the sake of completeness we shall incorporate the results of Lambe and Ward within the structure of our own generalizations. Section A deals with certain interesting theorems concerning the orthogonality of the coefficients $G_r(q)$ of the series (1.5) and in Section B we discuss the ordinary orthogonality properties of Heun functions and Heun polynomials. In the case of Heun polynomials we also deduce a certain double orthogonality property.

A. The orthogonality of the coefficients $G_r(q)$

In this section we make use of the functions $U_1(s), U_2(s)$ given, respectively, by equations (2.11), (2.12). As we require to show their dependence on the parameter $q$ we shall write

$$U_1(s) = U_1(q;s), \quad U_2(s) = U_2(q;s).$$
Theorem 1

Let (i) \( q_1, q_2 \) both be roots of the equation
\[ U_2(q_1-1) = 0, \]
which is equivalent to the characteristic equation (1.10), or both be roots of the equation \( U_1(q_1-1) = 0 \),
(ii) \( A_{11}, A_{21} \) be defined by
\[ A_{11} = \frac{\Gamma(1) \Gamma(\beta+1)(1-a)}{\Gamma(\alpha+1) \Gamma(\beta+1)} U_1(q_1;1) U'_2(q_1;1), \]
\[ A_{21} = -\frac{\Gamma(1) \Gamma(\beta+1)(1-a)}{\Gamma(\alpha+1) \Gamma(\beta+1)} U'_1(q_1;1) U_2(q_1;1), \]
and (iii)
\[ P_0 = 1, \quad P_r = \frac{\Gamma(\gamma) \Gamma(\alpha+r) \Gamma(\beta+r)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma+r)(r)! a^r}, \quad r > 1, \]
\[ \alpha, \beta, \gamma \not= 0, -1, -2, \ldots, \]
then
\[ \sum_{r=0}^{\infty} g_r(q_1) g_r(q_2) P_r^{-1} = 0 \quad \text{if} \quad q_1 \neq q_2, \]
\[ = A_{j1} \quad \text{if} \quad q_1 = q_2, \]
where \( j = 1, 2 \).

Proof

Since, for general values of \( q_1 \) and \( q_2 \),
\[ a(r+1)(r+\gamma) g_{r+1}(q_1) - [r^2(1+a)+a(\gamma+\delta-1)+\gamma+\epsilon-1]+\alpha \beta q_1] g_r(q_1) + \]
\[ + (r+\alpha-1)(r+\beta-1) g_{r-1}(q_1) = 0, \]
\[ a(r+1)(r+\gamma) g_{r+1}(q_2) - [r^2(1+a)+a(\gamma+\delta-1)+\gamma+\epsilon-1]+\alpha \beta q_2] g_r(q_2) + \]
\[ + (r+\alpha-1)(r+\beta-1) g_{r-1}(q_2) = 0, \]
it follows, on multiplying the first of these by \( g_r(q_2) P_r^{-1} \),
and the second by \( c_r(q_1) q_r^{-1} \) and subtracting that

\[
(r+\alpha)(r+\beta) [c_{r+1}(q_1) c_r(q_2) - c_{r+1}(q_2) c_r(q_1)] q_r^{-1} -
\]

\[
(\alpha-1)(\beta-1) [c_r(q_1) c_{r-1}(q_2) - c_r(q_2) c_{r-1}(q_1)] q_r^{-1} =
\]

\[
= a\beta(q_1 - q_2) c_{r}(q_1) c_{r}(q_2) q_r^{-1}.
\]

Putting \( r = 0, 1, 2, \ldots, n \) and summing we get

\[
\sum_{r=0}^{n} a_{r}(q_1) c_{r}(q_2) q_r^{-1} =
\]

\[
= a(n+1) (n+\gamma) \frac{[c_{n+1}(q_1) c_{n}(q_2) - c_{n+1}(q_2) c_{n}(q_1)]}{a\beta(q_1 - q_2) q_n^{-1}}
\]

\[
= a(n+1) (n+\gamma) \left\{ \frac{\Gamma(\gamma)}{(1-a) \Gamma(\alpha) \Gamma(\beta)} \right\}^2 \times
\]

\[
\times \left[ \begin{bmatrix} U_2(q_1; n+1) U_1(q_1; n+1) U_2(q_1; n+1) \\ U_1(q_1; n+1) U_2(q_1; n) \end{bmatrix} - \begin{bmatrix} U_2(q_2; n+1) U_1(q_2; n+1) U_2(q_2; n+1) \\ U_1(q_2; n+1) U_2(q_2; n) \end{bmatrix} \right] \times
\]

\[
\times \left[ \begin{bmatrix} U_2(q_1; n+1) U_1(q_1; n+1) U_2(q_1; n+1) \\ U_1(q_1; n+1) U_2(q_1; n) \end{bmatrix} - \begin{bmatrix} U_2(q_2; n+1) U_1(q_2; n+1) U_2(q_2; n+1) \\ U_1(q_2; n+1) U_2(q_2; n) \end{bmatrix} \right], \quad (3.1)
\]

using equation (2.19).

Therefore letting \( n \to \infty \) and employing the asymptotic expressions (2.15) and the asymptotic expansion for the gamma function for large values of \( n \) (Ref. 15, §13.6) we obtain

\[
\sum_{r=0}^{\infty} a_{r}(q_1) c_{r}(q_2) q_r^{-1} = \frac{e^{-\alpha} \Gamma(\gamma)}{\Gamma(\alpha+1) \Gamma(\beta+1)(1-a)} \cdot \frac{U_2(q_2; n+1) U_2(q_1; n) - U_1(q_2; n+1) U_2(q_2; n)}{(q_1 - q_2)} \quad (3.2)
\]

as \( q_2 \to q_1 \).
\[
\sum_{r=0}^{\infty} G_r(q_1)G_r(q_2)g_r^{-1} = \frac{\Gamma(\gamma)\Gamma(\alpha+1)\Gamma(\beta+1)(1-a)}{\Gamma(\alpha+\beta+\gamma+1)} \cdot U_1(q_1;1)U_2(q_1;1) - U_1(q_1;1)U_2(q_1;1)
\]

Hence if \( q_1, q_2 \) are roots of the equation \( U_2(q;1) = 0 \), then

\[
\sum_{r=0}^{\infty} G_r(q_1)G_r(q_2)g_r^{-1} = A_{11} \quad \text{if} \quad q_1 = q_2.
\]

Similarly if \( q_1, q_2 \) are roots of the equation \( U_1(q;1) = 0 \), then we obtain

\[
\sum_{r=0}^{\infty} G_r(q_1)G_r(q_2)g_r^{-1} = A_{21} \quad \text{if} \quad q_1 = q_2.
\]

Therefore we have the results

\[
\sum_{r=0}^{\infty} G_r(q_1)G_r(q_2)g_r^{-1} = A_{j1} \quad \text{if} \quad q_1 = q_2, \quad j = 1,2.
\]

N.B. For the case \( q_1 = q_2 \), the theorem can alternatively be proved by letting \( q_1 \to q_2 \) in (3.1) and then proceeding to the limit as \( n \to \infty \).

When \( \alpha = -N \) in equation (1.3) and \( q \) is one of the \( N + 1 \) roots of equation (1.10) we are led to the following theorem due to Lambe and Ward (Ref.4).
Theorem 2

If (i) the \((N+1)\) roots of equation (1.10) are all distinct,

\[
A_i^{-1} = \frac{a(N+1)(N+\gamma)g_{N+1}(q_i)g_N(q_i)}{\alpha \beta P_N}
\]

where \(P_0 = 1\), \(P_r = \frac{\alpha(\gamma+1)\ldots(\gamma+r-1)\beta(\gamma+1)\ldots(\gamma+r-1)}{\gamma(\gamma+1)\ldots(\gamma+r-1)} \ r > 1\),

\[
= -N, \quad \beta, \gamma \neq 0, -1, -2, \ldots \ldots,
\]

then

\[
\sum_{r=0}^{N} g_r(q_1)g_r(q_2)P_r^{-1} = 0 \quad \text{if} \quad q_1 \neq q_2,
\]

\[
= A_i^{-1} \quad \text{if} \quad q_1 = q_2 = q_i,
\]

and

\[
\sum_{i=1}^{N+1} A_i g_r(q_1)g_s(q_1) = 0 \quad \text{if} \quad r \neq s,
\]

\[
= P_r \quad \text{if} \quad r = s.
\]

Proof

Proceeding in the same manner as in theorem 1 we easily obtain

\[
\sum_{r=0}^{N} g_r(q_1)g_r(q_2)P_r^{-1} =
\]

\[
= a(N+1)(N+\gamma) \frac{[g_{N+1}(q_1)g_N(q_2)-g_{N+1}(q_2)g_N(q_1)]}{\alpha \beta(q_1-q_2)P_N}
\]

when \(q_2 \to q_1\), the right hand side of this equation approaches a definite limit, and so
\[
\sum_{r=0}^{N} \left[ G_r(q_1) \right]^{2P_r-1} = a(N+1)(N+\gamma)[G_{N+1}(q_1)G_N(q_1) -
\prod_{r=0}^{N} \left[ G_{N+1}(q_1)G_N(q_1) \right]^{2P_r-1}. \]

If \( q_1 \) is a root of (1.10), then \( G_{N+1}(q_1) = 0 \) and we have
the result
\[
\sum_{r=0}^{N} G_r(q_1)G_r(q_2)P_r^{-1} = 0 \quad \text{if} \quad q_1 \neq q_2
\]
\[
= A_i^{-1} \quad \text{if} \quad q_1 = q_2 = q_1. \tag{3.7}
\]
Hence if \( K_{r1} = A_i^{1/2} G_r(q_1)P_r^{-1/2}, \ r = 0, 1, \ldots, N, \)
the matrix \( |K_{r1}| \) is orthogonal and we find that
\[
\sum_{i=1}^{N+1} A_i G_r(q_1)G_s(q_1) = 0 \quad \text{if} \quad r \neq s
\]
\[
= P_r \quad \text{if} \quad r = s. \tag{3.8}
\]

Corollary
\[
\sum_{m=1}^{N+1} A_m H_u(q_{m,N};z) H_u(q_{m,N};t) = \frac{2^F_1(-N, \beta; \gamma; z^2 a)}{a}. \tag{3.9}
\]

Proof
Considering the left hand side of (3.9), we have
\[
\sum_{m=1}^{N+1} A_m H_u(q_{m,N};z) H_u(q_{m,N};t) =
\]
\[
= \sum_{m=1}^{N+1} \sum_{r=0}^{N} \sum_{s=0}^{N} A_m G_r(q_{m,N}) G_s(q_{m,N}) z^r t^s
\]
\[
= \sum_{r=0}^{N} P_r z^r t^r, \ \text{using theorem 2,}
\]
$$= _2F_1(-N, \beta; \gamma; \frac{z^2}{a}).$$

D. Orthogonal Properties of Heun Functions

Theorem 3

Let $H_{\gamma}(q_{m_1}; z)$ and $H_{\gamma}(q_{m_2}; z)$ be distinct Heun functions with the same $\alpha, \beta, \gamma, \delta, \epsilon$ but different accessory parameters $q_{m_1}$ and $q_{m_2}$ satisfying the characteristic equation (1.10). If $C$ is the Pochhammer loop contour $C_{a,1}$, then

$$\int_C z^{\gamma-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\epsilon-1} \frac{H_{\gamma}(q_{m_1}; z)H_{\gamma}(q_{m_2}; z)}{z} dz = 0. \quad (3.10)$$

Proof

Suppose first that $\text{Re}(\gamma) > 0, \text{Re}(\delta) > 0$; let $C$ be the straight line segment $(0,1)$ and let

$$\int_0^1 z^{\gamma-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\epsilon-1} \frac{H_{\gamma}(q_{m_1}; z)H_{\gamma}(q_{m_2}; z)}{z} dz = A. \quad (3.11)$$

On using (2.34), Heun's equation can be written as

$$H_{\gamma}y - \alpha \beta y = 0. \quad (3.12)$$

Multiplying both sides of (3.11) by $\alpha \beta q_{m_1}, \alpha \beta q_{m_2}$ and subtracting we easily find that

$$\alpha \beta (q_{m_1} - q_{m_2})A = \int_0^1 z^{\gamma-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\epsilon-1} x \times$$

$$\times [H_{\gamma}(q_{m_2}; z)H_{\gamma}q_{m_1} - H_{\gamma}(q_{m_1}; z)H_{\gamma}q_{m_2}] dz \quad (3.13)$$
which after rearrangement gives

\[
\alpha\beta(q_{m_1} - q_{m_2})A = \int_0^1 \frac{d}{dz}\{az^\gamma(1-z)^\delta(1-\frac{z}{a})^c [H_{\mu}(q_{m_2};z)H_{\mu}(q_{m_1};z) - \\
- H_{\mu}(q_{m_1};z)H_{\mu}(q_{m_2};z)]\}dz,
\]

\[
= \{az^\gamma(1-z)^\delta(1-\frac{z}{a})^c [H_{\mu}(q_{m_2};z)H_{\mu}(q_{m_1};z) - \\
- H_{\mu}(q_{m_1};z)H_{\mu}(q_{m_2};z)]\}_0^1.
\]

Since \(H_{\mu}(q_m;z)\) is regular at \(z = 0, 1\) and \(\text{Re}(\gamma) > 0, \text{Re}(\delta) > 0\) the right hand side vanishes and

\[
\alpha\beta(q_{m_1} - q_{m_2})A = 0.
\]

But since \(\alpha\beta \neq 0\) and \(q_{m_1} \neq q_{m_2}\) it follows that \(A = 0\). If we now replace \(C\) by \(C_{0,1}\), then by considerations of the analytic continuation of \(H_{\mu}(q_m;z)\) we find, using similar arguments to those of Whittaker and Watson (Ref. 15, §12.4), that the restrictions on \(\gamma\) and \(\delta\) can be removed. When \(m_1 = m_2 = m\) (say) we write, using (3.10),

\[
\int_{C_{0,1}} z^{\gamma-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{c-1} [H_{\mu}(q_m;z)]^2 dz = \theta_{0,1}.
\]

where \(\theta_{C_{0,1}}\) may or may not be zero.

Lambe and Ward (Ref. 4) have shown, using theorem 2, that in the case of Heun Polynomials \(A_m \theta_{C_{0,1}}\) is non-zero,
provided certain restrictions are laid upon the remaining parameters. The method employed can be applied to many special cases of Heun polynomials, and because of its importance we shall demonstrate its use in theorem 4.

It has already been mentioned (Chapter 1) that many of the special functions occurring in mathematical physics are special or limiting cases of Heun functions. Some of these, for example, Mathieu and Lame functions, appear as solutions to many essentially two-parameter eigenvalue problems, and have associated with them certain double orthogonality properties. We now consider a corresponding double orthogonality property for the Heun polynomials, where we choose $\alpha(=\pm)\alpha$ and $q$ as variable parameters. Obviously $\alpha$ cannot be varied at will, since equation (1.3a) must still hold, and therefore if $\alpha$ varies, at least one other parameter must vary also. We choose $\beta$ as this parameter so that if $\alpha = -n$, $\beta = \gamma + \delta + \varepsilon + n - 1$, all the remaining parameters being fixed.

**Theorem 4**

Let $H_{\alpha}(q_{m_1}, n_{1} ; z)$ and $H_{\alpha}(q_{m_2}, n_{2} ; z)$ be distinct Heun polynomials with the same $\gamma$, $\delta$ and $\varepsilon$, but different $q_{m_1}, n_{1}, q_{m_2}, n_{2}$ and different $\alpha, \beta$; let $C_1, C_2$ be any different pairs of the contours $C_{0,1}, C_{0,a}, C_{1,a}$, then
\[
\int_{c_1} \int_{c_2} (t-s) \eta_u(q_{m_1,n_1},t) \eta_u(q_{m_2,n_2},t) \eta_u(q_{m_1,n_1},t) \eta_u(q_{m_2,n_2},t) \times \\
\times t^{\gamma-1}(1-t)^{\delta-1}(1-\frac{\alpha}{a})^{\epsilon-1} s^{\gamma-1}(1-s)^{\delta-1}(1-\frac{\alpha}{a})^{\epsilon-1} \, ds \, dt \\
= 0 \quad \text{if } n_1 \neq n_2 \text{ or } m_1 \neq m_2 \\
= \phi_{c_1} \phi_{c_2} - \phi_{c_2} \phi_{c_1} \quad \text{if } n_1 = n_2 = n, \ m_1 = m_2 = m,
\]

where \( \phi_c \) is defined by (3.14) and

\[
\phi_c = \int_c z^{\gamma}(1-z)^{\delta-1}(1-\frac{z}{a})^{\epsilon-1} \left\{ \eta_u(q_m,n;z) \right\}^2 \, dz. \tag{3.15a}
\]

**Proof**

Under the transformation

\[
y = e^y w \quad 1 - \gamma \quad 1 - \delta \quad 1 - \varepsilon
\]

where

\[
\psi = \log \left\{ z \frac{\varepsilon}{z} (1-z)^{\frac{\delta}{a}} \right\} ,
\]

Heun's equation takes the form

\[
\frac{d}{dz} [z(z-1)(z-a) \frac{d}{dz} w] + \left\{ \Delta(z) + \alpha \beta(z-a) \right\} w = 0 , \tag{3.17}
\]

where

\[
\Delta(z) = z(z-1)(z-a) \left\{ \psi'' + \psi' \right\} .
\]

If we write \( w \) in the form

\[
w = w(q_{m_1,n_1};z),
\]

then if

\[
W_1 = W_1(s,t) = w(q_{m_1,n_1};s)w(q_{m_1,n_1};t) \tag{3.18}
\]

we find, in the usual manner that \( W_1 \) is a solution of
the partial differential equation
\[
\frac{\partial}{\partial t}[t(t-1)(t-a)\frac{\partial W_1}{\partial t}] - \frac{\partial}{\partial s}[s(s-1)(s-a)\frac{\partial W_1}{\partial s}] +
\]
\[
+ \{n_1\beta_1(t-a) + \Delta(t) - \Delta(s)\}W_1 = 0. \tag{3.19}
\]

Similarly if \( W_2 = W_2(s,t) = w(q_{m_2},n_2,s)w(q_{m_2},n_2,t) \) \( (3.20) \)
then \( W_2 \) satisfies a similar equation. Multiplication by \( W_2, W_1 \) respectively and subtracting gives, on integrating,
\[
\int\int_{C_1}^{C_2} \left[ W_2\left\{ \frac{\partial}{\partial t}[t(t-1)(t-a)\frac{\partial W_1}{\partial t}] - \frac{\partial}{\partial s}[s(s-1)(s-a)\frac{\partial W_1}{\partial s}] \right\} -
\]
\[
- W_1\left\{ \frac{\partial}{\partial t}[t(t-1)(t-a)\frac{\partial W_2}{\partial t}] - \frac{\partial}{\partial s}[s(s-1)(s-a)\frac{\partial W_2}{\partial s}] \right\} \right] \, ds \, dt
\]
\[
= (\alpha_2\beta_2 - \alpha_1\beta_1) \int_{C_1}^{C_2} (t-s)W_1W_2 \, ds \, dt. \tag{3.21}
\]

Denoting the left hand side of equation (3.21) by \( I \), we have, on integrating by parts,
\[
I = \int_{C_2}^{C_1} \left[ \left\{ W_2 \frac{\partial W_1}{\partial t} - W_1 \frac{\partial W_2}{\partial t} \right\} t(t-1)(t-a) \right] ds -
\]
\[
- \int_{C_1}^{C_2} \left[ \left\{ W_2 \frac{\partial W_1}{\partial s} - W_1 \frac{\partial W_2}{\partial s} \right\} s(s-1)(s-a) \right] dt.
\]

On rewriting \( W_1, W_2 \) in terms of Heun polynomials and the function \( \psi \) we see that since \( C_1, C_2 \) are Pochhammer loop contours, then when \( n_1 \neq n_2 \), the integrated parts in \( I \)
vanish. Thus the right hand side of (3.21) is identically zero when \( n_1 \neq n_2 \) for all \( m_1, m_2 \).

We now write the integral on the right hand side of (3.21) as
\[
I' = \int_{C_1} \int_{C_2} (t-s)w_1 w_2 \, ds \, dt
\]
\[
= \int_{C_1} \int_{C_2} (t-s)w(q_{m_1}, n_1; t)w(q_{m_1}, n_1; s)w(q_{m_2}, n_2; t) \times w(q_{m_2}, n_2; s) \, ds \, dt.
\]

But \( w(q_{m}, n; t) = e^{-\psi_{\alpha}(q_{m}, n; t)} \) and therefore
\[
I' = \int_{C_1} t \psi_{\alpha}(q_{m_1}, n_1; t)\psi_{\alpha}(q_{m_2}, n_2; t)t^{\gamma-1}(1-t)^{\delta-1}(1-\frac{t}{a})^{\epsilon-1} \, dt \times
\]
\[
\int_{C_2} \psi_{\alpha}(q_{m_1}, n_1; s)\psi_{\alpha}(q_{m_2}, n_2; s)s^{\gamma-1}(1-s)^{\delta-1}(1-\frac{s}{a})^{\epsilon-1} \, ds
\]

- a similar expression with \( s, t \) interchanged.

When \( n_1 = n_2, m_1 \neq m_2 \), then clearly by theorem 3 we see that
\( I' = 0 \). When \( n_1 = n_2 = n, m_1 = m_2 = m \) say, then
\[
I' = \phi_{C_1} \phi_{C_2} - \phi_{C_2} \phi_{C_1}
\]

and the theorem is proved.

It has already been mentioned that Lambe and Ward, by imposing certain restrictions on some of the parameters of Heun's equation, have shown that \( \phi_{C} \) is non-zero. For the
sake of completeness we reproduce the proof here.

We have, using a method analogous to that of theorem 3,

\[
\int \frac{z^{\gamma-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\varepsilon-1}}{c} M_{\varepsilon}(q_{m_1,n};z) M_{\varepsilon}(q_{m_2,n};z) dz = 0 \text{ if } m_1 \neq m_2
\]

\[
= \theta_C \text{ if } m_1 = m_2 = m
\]

\[m_1, m_2 = 1, 2, \ldots, n+1, C = C_0, 1\]

On multiplying by \(A_{m_1}G_r(q_{m_1,n})\), and summing over \(m_1\), it follows from theorem 2 that

\[
A_{m_1} \theta_C G_r(q_{m_1,n}) = \int \frac{M_{\varepsilon}(q_{m_1,n};z) P_r z^{\gamma+r-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\varepsilon-1}}{c} dz
\]

\[= P_r \sum_{s=0}^{n} G_s(q_{m_1,n}) \int \frac{z^{\gamma+r+s-1}(1-z)^{\delta-1}(1-\frac{z}{a})^{\varepsilon-1}}{c} dz
\]

and denoting the integral on the right hand side by \(\psi(r+s)\) we have

\[
A_{m_1} \theta_C G_r(q_{m_1,n}) = P_r \sum_{s=0}^{n} G_s(q_{m_1,n}) \psi(s+r).
\]

If \(A_{m_1} \theta_C = 0\) then \(\sum_{s=0}^{n} G_s(q_{m_1,n}) \psi(s+r)\) vanishes for every \(r\), so that the determinant \(\psi(s+r), (s, r = 0, 1, 2, \ldots, n)\) vanishes. Writing \(m_2\) instead of \(m_1\) in (3.22), it is obvious that \(A_{m_2} \theta_C\) also vanishes, so that
\[ \sum_{s=0}^{n} G_s(q_m,n)\psi(s+r) = 0 \text{ for every } r \text{ and } m. \]

Multiplying this last equation by \( A_m G_t(q_m,n) \) and summing over \( m \), it follows from theorem 2 that \( \psi(s+t) = 0 \) for every \( s, t \) satisfying the inequalities \( 0 < s, t < n \). Hence for every \( s \) satisfying \( 0 < s < 2n \), \( \psi(s) \) vanishes. But (Ref. 2, §2.1.3)

\[
\psi(s) = \int_0^1 x^{\gamma+s-1}(1-x)^{\delta-1}(1-\frac{z}{a})^{c-1} \, dx
\]

\[
= \frac{4\pi^2 e^{i\pi \gamma} \, _2F_1(\gamma+s,1-s;\gamma+\delta+s;1)}{\Gamma(1-\gamma-s)\Gamma(1-\delta)\Gamma(\gamma+\delta+s)},
\]

and provided none of the quantities

\( 1-\gamma-s, 1-\delta, \gamma+\delta+s, \) \( s = 0, 1, \ldots, n \), are negative or zero integers then \( \psi(s) \neq 0 \) and so none of the numbers \( A_m \psi \) vanishes.

In order to obtain an expression for \( \psi \), we write (3.15a) as

\[
\phi_C = \sum_{s=0}^{n-1} \sum_{r=0}^{n} G_s(q_m,n)G_r(q_m,n) \int_0^1 x^{\gamma+r+s}(1-x)^{\delta-1}(1-\frac{z}{a})^{c-1} \, dz,
\]

where \( C = C_{0,1} \); thus

\[
\phi_C = \sum_{r=0}^{n-1} \sum_{s=0}^{n} G_s(q_m,n)G_r(q_m,n)\psi(r+s+1) + \]

\[
+ G_n(q_m,n) \sum_{s=0}^{n} G_s(q_m,n)\psi(s+n+1)
\]

which on using (3.22) gives
\[ \phi_C = A_m \phi_C \sum_{r=0}^{n-1} \frac{g_r(q_m,n)g_{r+1}(q_m,n)}{r+1} + g_r(q_m,n) \sum_{s=0}^{n} g_s(q_m,n) \psi(s+n+1) \]

It is to be noted that if \( \text{Re}(\gamma), \text{Re}(\delta), \text{Re}(\epsilon) > 0 \) then the Pochhammer loop contours may be replaced by straight lines joining any of the pairs of points \( z = (0,1); (1,a); (0,a) \).

An important special case of theorems 3,4 is the case in which \( \alpha = -\frac{1}{2}n, \beta = \frac{1}{2}(n+1), \) \( n \) being a non-negative integer, \( \gamma = \delta = \epsilon = \frac{1}{2} \), and \( a = k^{-2} \), where \( k \) is the modulus of the Jacobian elliptic function \( snz \). Heun's equation now takes the algebraic form of Lame's equation,

\[ \frac{d^2y}{dz^2} + \frac{1}{2}\left[ \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-k^{-2}} \right] \frac{dy}{dz} + \frac{(hk_n^{-2}n(n+1)z)}{4z(z-1)(z-k^{-2})} y = 0, \quad (3.24) \]

where \( hk_n^{-2} = n(n+1)q_\star \).

If we put

\[ z = sn^2 u, \quad (3.25) \]

we obtain the Jacobian form of Lame's equation, namely

\[ \frac{d^2y}{du^2} + [h-n(n+1)k^2 sn^2 u] y = 0 \quad (3.26) \]

If \( h \) is a root of the equation analogous to equation (1.10), we obtain Lame polynomials \( E^m_n(u) \), \( m = 0,1, \ldots, N \), where \( N = [\frac{1}{2}n] \).

Obviously the loop contours of theorems 3,4 can be replaced by straight lines, and under the transformation (3.25) we choose these to be the lines joining the points \((-2K,2K)\) and \((K-2iK',K+2iK')\) in the complex \( U \)-plane. Theorems 3 and 4
then give the following well known results:

(1) \[ \int_{-2K}^{2K} E_n^{m_1}u E_n^{m_2}u du = 0, \quad m_1 \neq m_2 \]

(2) \[ \int_{-2K-2iK_1}^{2K+2iK_1} E_n^{m_1}u E_n^{m_2}u du = 0, \quad m_1 \neq m_2 \]

(3) \[ \int_{-2K}^{2K} \int_{-2K}^{2K} E_n^{m_1}u E_n^{m_2}v E_n^{m_3}E_n^{m_4}v (sn^2u - sn^2v) du dv = 0, \quad \text{unless } n_1 = n_2, \ m_1 = m_2 \]
Chapter 4

Integral Equations for Heun Functions

It has already been mentioned (Chapter 2 section B), that homogeneous integral equations of the type (2.32) have been studied by Erdélyi and by Lambe and Ward. In this chapter we obtain new integral equations for Heun functions which are based on those studied by Schmidt (Ref.20) and Arscott (Ref.21). These integral equations are non-linear and, when the Heun functions are considered as being solutions of essentially two-parameter eigenvalue problems, (the parameters being $\alpha$ and $\sigma$), provide a genuine reduction of the problem to a one-parameter eigenvalue problem.

A The integral equation

Theorem

Let (i) $w(z)$ be a solution of Heun's equation,

(ii) $H(z,s,t)$ be a solution of the partial differential equation

$$(t-z)M_s(H) + (z-s)M_t(H) + (s-t)M_z(H) = 0 \quad (4.1)$$

where $M_z$ is the operator given by (2.34), $w, H$ being analytic in appropriate complex regions.

(iii) $C_1, C_2$ be suitable paths in the complex $s,t$ planes such that both the quantities

$$\left[ s^\gamma(1-s)^\delta(1-s^\alpha)^\epsilon \left\{ w(s) \frac{\partial H}{\partial s} - H \frac{\partial w(s)}{\partial s} \right\} \right] \quad (4.2a)$$

and
(1.2b)
\[ \left[ t^\gamma(1-t)^\delta(1-\frac{t}{a})^\varepsilon \left\{ w(t)\frac{\partial w}{\partial t} - H\frac{w(t)}{\partial t} \right\} \right] \]

vanish,

(iv) the function

\[ W(z) = \int_{C_1}^{C_2} (s-t)(st)^{\gamma-1}\left\{ (1-s)(1-t) \right\}^{\delta-1}\left\{ \left( \frac{s}{a} \right) \left( 1-\frac{t}{a} \right) \right\}^{\varepsilon-1} \times \]

\[ \cdot H w(s)w(t)ds \, dt \]

exist, and if the integral is singular let it converge uniformly with respect to \( z \), when \( z,s,t \) lie in appropriate regions.

Then \( W(z) \) is a solution of Heun's equation.

Proof:

Consider the integral

\[ W(z) = \int_{C_1}^{C_2} (s-t)(st)^{\gamma-1}\left\{ (1-s)(1-t) \right\}^{\delta-1}\left\{ \left( \frac{s}{a} \right) \left( 1-\frac{t}{a} \right) \right\}^{\varepsilon-1} \times \]

\[ \cdot H w(s)w(t)ds \, dt \]

Then \( M_z(w) = \)

\[ = \int_{C_1}^{C_2} (st)^{\gamma-1}\left\{ (1-s)(1-t) \right\}^{\delta-1}\left\{ \left( \frac{s}{a} \right) \left( 1-\frac{t}{a} \right) \right\}^{\varepsilon-1} w(s)w(t) \times \]

\[ \cdot \left\{ (z-t)M_s(H)+(s-z)M_t(H) \right\} ds \, dt, \]

using (4.1)

Consider the integral
\[
\int \int (st)^{\gamma-1}((1-s)(1-t))^{\delta-1}((1-\frac{s}{a})(1-\frac{t}{a}))^{\epsilon-1}w(s)w(t)M(s)(H)ds
dt, \quad C_1 \quad C_2
\]

which on using the fact that
\[
s^{\gamma-1}(1-s)^{\delta-1}(1-\frac{s}{a})^{\epsilon-1}M(s) = \frac{\partial}{\partial s}\{as^{\gamma}(1-s)^{\delta}(1-\frac{s}{a})^{\epsilon}\partial\}
\]
becomes
\[
a \int \int t^{\gamma-1}(1-t)^{\delta-1}(1-\frac{t}{a})^{\epsilon-1}w(s)w(t)(z-t)^{\delta}M(s)M(t)ds
dt, \quad C_1 \quad C_2
\]

Integrating (4.5) with respect to s by parts gives
\[
\int aw(s)^{\partial}\{as^{\gamma}(1-s)^{\delta}(1-\frac{s}{a})^{\epsilon}\partial\}ds = C_1
\]

\[
= \left[as^{\gamma}(1-s)^{\delta}(1-\frac{s}{a})^{\epsilon}\left(w(s)^{\partial H\partial s} - H\partial w(s)^{\partial s}\right)\right] + \int s^{\gamma-1}(1-s)^{\delta-1}(1-\frac{s}{a})^{\epsilon-1}HM(s)w(s)ds,
\]

If C_1 is chosen so that the integrated part vanishes then
(4.5) becomes
\[
\int \int (st)^{\gamma-1}((1-s)(1-t))^{\delta-1}((1-\frac{s}{a})(1-\frac{t}{a}))^{\epsilon-1}(z-t)M(w(t)M(s)w(s))ds
dt, \quad C_1 \quad C_2
\]

Similarly if C_2 is such that
\[
\left[t^{\gamma}(1-t)^{\delta}(1-\frac{t}{a})^{\epsilon}\left(w(t)^{\partial H\partial t} - H\partial w(t)^{\partial t}\right)\right] \text{ vanishes, then}
\]
(4.4) can be rewritten as

\[ M_z(w) = \int \int (st)^{\gamma-1} \{ (1-s)(1-t) \}^{\delta-1} \{ (1-\frac{s}{a})(1-\frac{t}{a}) \}^{\epsilon-1} \]

\[ C_1 \quad C_2 \]

\[ \times \{ (z-t)w(t)M_z(w(s)) + (s-z)w(s)M_z(w(t)) \} \] ds dt.

If we write Heun's equation as

\[ M_z - \alpha \beta \gamma y = 0 \]

then on substituting for \( W(z) \) we have

\[ M_z - \alpha \beta \gamma y = \]

\[ = \int \int (st)^{\gamma-1} \{ (1-s)(1-t) \}^{\delta-1} \{ (1-\frac{s}{a})(1-\frac{t}{a}) \}^{\epsilon-1} \]

\[ C_1 \quad C_2 \]

\[ \times \{ (z-t)w(t)\{M_z(w(s)) - \alpha \beta \gamma w(s)\} + \]

\[ + (s-z)w(s) \{ M_z(w(t)) - \alpha \beta \gamma w(t) \} \] ds dt = 0,

since \( w(s), w(t) \) are solutions of Heun's equation. Consequently \( W(z) \) is a solution of Heun's equation.

B. The solution of the Partial differential equation satisfied by the Nucleus

We introduce new variables \( u, v, w \) by the relations

\[ u = \frac{k}{a} (stz)^{\frac{1}{2}} \quad v = l \left\{ \frac{(t-1)(z-1)}{a(1-a)} \right\}^{\frac{1}{2}} \]

\[ w = m \left\{ \frac{(z-a)(t-a)(z-a)}{(1-a)} \right\}^{\frac{1}{2}} \quad (4.7) \]

In terms of these new variables equation (4.1) becomes,

after tedious but straightforward algebra

\[ \nabla^2 H + (2\gamma-1) \left( \frac{1}{u} \frac{\partial H}{\partial u} + \frac{1}{v} \frac{\partial H}{\partial v} + \frac{1}{w} \frac{\partial H}{\partial w} \right) = 0 \quad (4.8) \]
where
\[ \nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2}. \]

In the special case \( \gamma = \frac{1}{2} \), (4.8) reduces to Laplace's equation so that it is easy to write down suitable nuclei.

We now make the further transformation to the variables \( r, \theta, \phi \) related to \( u, v, w \) by
\[ v = r \cos \theta, \quad v = r \sin \theta \sin \phi, \quad w = r \sin \theta \cos \phi. \]

Under this transformation (4.8) becomes
\[ \frac{\partial^2 H}{\partial r^2} + \frac{6 \gamma - 1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \left[ \frac{\partial^2 H}{\partial \theta^2} + \{ (4 \gamma - 1) \cot \theta - (2 \gamma - 1) \tan \theta \} \frac{\partial H}{\partial \theta} \right] + \]
\[ + \frac{1}{r^2 \sin^2 \theta} \left[ \frac{\partial^2 H}{\partial \phi^2} + 2(2 \gamma - 1) \cot 2 \phi \frac{\partial H}{\partial \phi} \right] = 0. \]

Consequently, in the case \( \gamma = \frac{1}{2} \) we obtain suitable nuclei for (4.3) in terms of spherical harmonies in \( r, \theta, \phi \).

A separated solution of (4.10)
\[ H = R(r)G(\theta)K(\phi), \]

is obtained if \( R, G \) and \( K \) satisfied by the differential equations
\[ r^2 \frac{d^2 R}{dr^2} + (6 \gamma - 1) r \frac{dR}{dr} - \lambda R = 0. \]
\[ \sin^2 \theta \left[ \frac{d^2 G}{d\theta^2} + \{ (4 \gamma - 1) \cot \theta - (2 \gamma - 1) \tan \theta \} \frac{dG}{d\theta} + \lambda G \right] - \mu G = 0, \]
and
\[ \frac{d^2 K}{d\phi^2} + 2(2 \gamma - 1) \cot 2 \phi \frac{dK}{d\phi} + \mu K = 0. \]
\( \lambda, \mu \) being separation constants.

A fundamental pair of solutions of (4.12) are

\[
m_1 \quad m^2, \quad r^m_1, \quad r^m_2, \tag{4.15}
\]

where \( m_1, m_2 \) are the roots, assumed distinct, of the equation

\[
m^2 + 2(3\gamma - 1)m - \lambda = 0. \tag{4.16}
\]

If we put \( \xi = \cos^2 \theta \) then (4.15) takes the form

\[
\xi(1-\xi) \frac{d^2 G}{d\xi^2} + (\gamma - 3\gamma \xi) \frac{dG}{d\xi} + \left( \frac{\lambda}{4} - \frac{\mu}{1-\xi} \right) G = 0. \tag{4.17}
\]

If \( p \) satisfies the equation

\[
p^2 + (2\gamma - 1)p - \frac{\mu}{4} = 0, \tag{4.18}
\]

then

\[
W(\xi) = (1-\xi)^{-p} g(\xi) \tag{4.19}
\]

satisfies the hypergeometric equation

\[
\xi(1-\xi) \frac{d^2 W}{d\xi^2} + \left[ (\gamma - (3\gamma + 2p)\xi) \right] \frac{dW}{d\xi} + \left( \frac{\lambda}{4} + p - 3\gamma p - p^2 \right) W = 0. \tag{4.20}
\]

Thus (4.13) has the solutions

\[
\begin{bmatrix}
0 & 1 & \infty \\
\sin^2 \theta & P & 0 & 0 & a & \cos^2 \theta \\
1-\gamma & 1-2p\gamma - 2\gamma & b
\end{bmatrix}, \tag{4.21}
\]

where

\[
a, b = \frac{(3\gamma + 2p - 1) \pm [(3\gamma - 1)^2 + \lambda]^{1/2}}{2} \tag{4.22}
\]

and \( p \) is a root of (4.18). Similarly (4.14) has the solutions
The variables \( r, \theta, \phi \) are given by
\[
r^2 = \frac{k^2}{a} (s+t+z-1-a),
\]
\[
\cos^2 \theta = \frac{stz}{a(s+t+z-1-a)},
\]
\[
\cos^2 \phi = \frac{(s-a)(t-a)(z-a)}{(1-a)(stz-a(s+t+z-1-a))}.
\]

Thus typical solutions of the equation for the nucleus are
\[
r^m \sin^2 \theta \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \cos^2 \theta \\ 1-\gamma & 1-2\rho-2\gamma & b \end{pmatrix} P \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a^* \cos^2 \phi \\ 1-\gamma & 1+\gamma & b^* \end{pmatrix},
\]

The expression \((4.26)\) gives a very wide range of nuclei suitable for integral equations for solutions of the Heun equation. The nuclei can be chosen as combinations of the various branches of the \(P\)-functions involved and may even be linear combinations summed over the separation constants \(\lambda\) and \(\mu\). We shall demonstrate the use of \((4.26)\) in constructing suitable nuclei for integral equations satisfied by Heun functions and Heun polynomials. We shall take \(C_1\) to be a contour enclosing the points \(s = 0,1\) and \(C_2\) is a similar contour enclosing the points \(t = 0,1\).
Heun function regular at the singularities $z = 0, 1$ we choose that nucleus which, when considered as a function of $z$, is also regular at $z = 0, 1$. This can be achieved by taking that branch of the first P-function in (4.26) which, as a function of $z$, is regular at the origin and belongs to the exponent 0 there, and that branch of the second P-function, which as a function of $z$ is regular at $z = 1$ and belongs to the exponent 0 there. Such a nucleus is of the form

$$r^n \sin^2 \theta \, _2F_1(a, b; \gamma; \cos^2 \theta) \, _2F_1(a', b'; \gamma; \sin^2 \phi).$$  (4.27)

The contours $C_1, C_2$ must be chosen so that the singularities where $\cos^2 \theta = 1$ or $\infty$ are excluded, i.e. where

$$a(a+t+z-1-a) = stz \text{ or zero}. \quad (4.28)$$

Then with the nucleus (4.27) the "integrated" parts, (4.2a,b), vanish if $C_1, C_2$ are each taken to be the contour $C_{0,1}$. The integral (4.3) thus represents a constant multiple (possibly zero) of the Heun function. If, however, $\Re(\gamma) > 0$ and/or $\Re(\delta) > 0$, and the separation constants $\lambda$ and $\mu$ are chosen so that $\Re(\gamma-a-b) > 0$ and $\Re(\gamma-a'-b') > 0$, so that the hypergeometric functions in (4.27) are regular at $\cos^2 \theta = \sin^2 \phi = 1$, then the contours of integration may be deformed into simple loop contours or straight lines.

In the case of Heun polynomials the nucleus, when considered as a function of $z$, must be finite for finite values of $z$ and also regular at the origin. Further, if $\text{He}(q, z)$ is
a polynomial of degree \( n \) in \( z \), then the nucleus must be a polynomial of degree \( n \) in \( z \). If we again consider the nucleus (4.27) then at least one of \( a, b \) and at least one of \( a', b' \) must be negative integers.

If

\[
\lambda = 2(3\gamma-1)n + n^2, \quad n = 0,1,2,\ldots, \tag{4.29}
\]

then

\[
a = \frac{2(3\gamma-1)+2n+n}{2} \quad \text{and} \quad b = \frac{2n-n}{2}. \tag{4.30}
\]

If

\[
\mu = 2(2\gamma-1)s + s^2, \quad s = 0,1,2,\ldots, \tag{4.31}
\]

then

\[
a' = \frac{2(2\gamma-1)+s}{2} \quad \text{and} \quad b' = -\frac{s}{2}, \tag{4.32}
\]

and \( m = n \) or \( -2(3\gamma-1) - n \), \( (4.33) \)

and \( p = \frac{s}{2} \) or \( -\frac{(2(2\gamma-1)+s)}{2} \), \( (4.34) \)

Hence

\[
a = \frac{2(3\gamma-1)+s+n}{2}, \quad b = \frac{s-n}{2}, \tag{4.35}
\]

or

\[
a = \frac{2s+n}{2}, \quad b = -\frac{2(2\gamma-1)-s-n}{2}. \tag{4.36}
\]

By taking

\[
m = n, \quad p = \frac{s}{2}, \quad a = \frac{2(3\gamma-1)+s+n}{2}, \quad b = \frac{s-n}{2}
\]

and

\[
a' = \frac{2(2\gamma-1)+s}{2}, \quad b' = -\frac{s}{2},
\]
then if $\frac{s-n}{2}$ and $\frac{n}{2}$ are negative integers, the nucleus, considered as a function of $z$ is finite for finite values of $z$, and is regular at $z = 0$. It is therefore a suitable nucleus for the Heun polynomial of degree $n$ in $z$ if $s$ takes one of the values $s = 0, 1, \ldots, n$. In fact we could choose a nucleus which is the sum (over $s$) of nuclei of the form (4.27). Thus we see that for one type of Heun polynomial of degree $n$, there are $(n+1)$ possible nuclei of the form (4.27). If $\Re(\gamma) > 0$, $\Re(\delta) > 0$ then the contours $C_1, C_2$ may be replaced by the straight line intervals $s = (0, 1)$ and $t = (0, 1)$ respectively. Obviously, since the hypergeometric functions in (4.27) are now polynomials, the further restrictions $\Re(\gamma-a-b) > 0$ and $\Re(\gamma-a'-b') > 0$ do not apply. In the above analysis we have taken $C_1, C_2$ to be contours surrounding the points $s = 0, 1; t = 0, 1$, or with certain restrictions on the parameters, straight lines joining them. If, in the case of Heun functions, the restrictions $\Re(\gamma) > 0, \Re(\delta) > 0, \Re(\varepsilon) > 0$, are applied then $C_1, C_2$ may be taken to be any pair of the straight lines joining the points $(0, 1), (0, a), (1, a)$ in the complex $s,t$-planes. When $C_1, C_2$ are the same then only two of the above restrictions are necessary. For Heun polynomials, the same comments apply, except that here the restrictions may be removed if $C_1, C_2$ are Pochhammer loop contours surrounding any of the above pairs of points.
Once more we obtain the corresponding integral equations, which are satisfied by Lame polynomials. In the case of Lame's equation $\gamma = \frac{1}{2}$ and thus for the nucleus (4.27) we take

$$m = n, \quad p = \frac{s}{2}, \quad a = \frac{s+n+1}{2}, \quad b = \frac{s-n}{2},$$

$$a' = \frac{s}{2}, \quad b' = \frac{s}{2},$$

and thus (4.27) becomes a constant multiple, dependent on $n$ and $s$, of the solid spherical harmonic

$$r^2 \frac{s}{n}(\cos \theta) \left\{ \begin{array}{c} \sin s \phi \\ \cos s \phi \end{array} \right\}.$$  (4.37)

the choice of $\sin s \phi$ depending on the parity of $s$ and $n$.

If in (4.25) we put

$$z = \sin^2 \alpha, \quad s = \sin^2 \beta, \quad t = \sin^2 \gamma, \quad a = k^{-2},$$

we find that

$$r^2 = k^2 \left( k^2 \sin^2 \alpha - k^2 \sin^2 \beta - k^2 \sin^2 \gamma \right),$$

$$\cos^2 \theta = \left( \frac{k^2 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma}{(k^2 \sin^2 \alpha - k^2 \sin^2 \beta - k^2 \sin^2 \gamma)} \right),$$

and

$$\cos^2 \phi = \left( \frac{\sin^2 \alpha \sin^2 \beta \sin^2 \gamma}{k^2 \left( k^4 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma - (k^2 \sin^2 \alpha - k^2 \sin^2 \beta - k^2 \sin^2 \gamma) \right)} \right).$$  (4.41)

Obviously we may take $C_1, C_2$ to be the straight lines joining $s = (0,1), t = (0,a)$, and in the transformed plane these are taken to be the segments.
\((-2K, 2K)\) for \(\beta\)

and

\((K-2iK', K+2iK')\) for \(\gamma\).

Thus for Lamé polynomials we obtain the integral equation

\[
E_n^m(a) = \int_{-2K}^{2K} \int_{K-2iK'}^{K+2iK'} r^n P_n^{(s)}(\cos \theta) \sin \phi (\text{sn}^2 \beta - \text{sn}^2 \gamma) E_n^m(\beta) E_n^m(\gamma) d\beta d\gamma.
\]

for any \(s < n\).

It is important to point out that in the case of Lamé's equation, the equation satisfied by the nucleus is Laplace's equation, and consequently it is a simple matter to write down suitable nuclei without appealing to the form (4.26).

Integral equations of the type (4.42) have been discussed by Arscott (Ref.13).
Chapter 5
The Expansions of Lamé Functions into Series of Associated Legendre Functions of the second kind.

A Introduction

In this chapter a study is made of the solutions of Lamé's differential equation as series of associated Legendre functions. The particular feature studied is the representation of a second solution corresponding to the case when the first is a Lamé polynomial; i.e. the representation of a Lamé function of the second kind.

It has been stated that a representation of \( F_n^s(u) \), a second solution of Lamé's equation associated with the corresponding Lamé polynomial \( F_n^s(u) \), can be obtained by taking the known representation of \( F_n^s(u) \) as a finite series of associated Legendre polynomials \( P_n^m(z) \), \( z = \text{snu, cmu, dmu, k; ksmu, dmu/k', ikemu/k'} \) and replacing each \( P_n^m \) by \( Q_n^m \). This however is incorrect, as is illustrated by the case \( n = 0 \).

For if the above statement were true, \( F_0^0(u) \) would be represented by a constant multiple of

\[
Q_0(\text{snu}) = \frac{1}{2} \log \frac{1+\text{snu}}{1-\text{snu}},
\]

But we easily verify that this does not satisfy Lamé's equation. We shall show that in fact the second solution is represented by an infinite series of \( Q_n^m \).

When considering Lamé functions of the first kind, Erdelyi (Ref.22) attacked the problem using the integral
equations for Lamé polynomials given by Whittaker and Watson (Ref. 15), Lambe and Ward (Ref. 4), Ince (Ref. 23) and others. In the case of Lamé functions of the second kind, we approach the problem from the point of view of the integral relations for Lamé functions of the second kind given by Arscott (Ref. 18) and the subsequent unpublished work of Mr. R. S. Taylor.

In §B we give some preliminary formulae relating to the associated Legendre functions. §§ C, D deal with the various types of solutions of Lamé's equation in series of associated Legendre functions, and the convergence properties of these series. In §E we discuss the case of the second solution of Lamé's equation when the first is a Lamé polynomial, and also deduce some relations which exist between the various series representing the same Lamé function.

**Notation**

Lamé's equation in its Jacobian form is

\[ L(y) = \frac{d^2 y}{du^2} + [h - \nu(n+1)k^2 sn^2 u]y = 0, \quad (5.1) \]

where \( sn = sn(u,k) \). We adopt the convention that when \( \nu \) is an integer, we write \( n \) for \( \nu \) and moreover specify \( n > -\frac{1}{2} \) so that \( n > 0 \).

A Lamé polynomial of degree \( n \) is a solution of (5.1) which is a polynomial in \( sn^2 u \), possibly multiplied by one or more of the functions \( sn, cn, dn \). There are
thus eight types of such polynomials; usually denoted by the symbol $E_n^{S}(u)$, but here we use an extended notation due to Arscott (Ref.19) in which we prefix the letter $E$ by the letters $u,s,c,d,sc,sd,cd,scd$. According as the functions consist of a polynomial in $sn^2 u$ multiplied by unity, $snu, cnu, \ldots, snu cnu dnu$ respectively. In the symbol $E_n^{S}(u)$ the suffix $n$ indicates the degree of the polynomial and the upper index $s$ is specified as the number of zeros in the interval $0 < u < k$. The second solution of (5.1) is defined as that solution which is of the form

\[ sn^2 u cnu dnu V(snu), \]

where

\[ V(snu) = \sum_{r=0}^{\infty} a_r(snu)^{-n-l-\rho-\sigma-\tau-2r} \]

and $\rho, \sigma, \tau = 0$ or $1$. The second solution, i.e. a Lame' function of the second kind, corresponding to a particular Lame' polynomial $E_n^{S}(u)$ is denoted by $F_n^{S}(u)$.

§ B Preliminaries

For brevity we adopt the notation $s = smu,$ $c = cnu, d = dnu,$ for the Jacobian elliptic functions, which unless otherwise stated have modulus $k$. We now turn to some results required regarding associated Legendre functions. We adopt Hobson's definition of the associated Legendre functions (Erdelyi uses Ferrers' definition) and
\[ P^m_\nu(z) = (z^2-1)^{-1/2} \frac{d^m P_\nu(z)}{dz^m}, \quad Q^m_\nu = (z^2-1)^{-1/2} \frac{d^m Q_\nu(z)}{dz^m} \quad (5.2) \]

where
\[ P_\nu = P(1+\nu,-\nu;1;\frac{1}{2}-i\bar{z}), \]
\[ Q_\nu = 2^{-\nu} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{5}{2})} z^{-1-\nu} P\left(\frac{1}{2}+i\nu,1+i\nu; \nu+\frac{5}{2}; z^{-2}\right), \]

the quantity \( m \) being a non-negative integer. Further, we introduce a modified form \( \tilde{Q}^m_\nu \) given by Schäfke (Ref. 24 §3.64, 65) of the associated Legendre functions. This is defined by
\[ (-1)^m \Gamma(\nu+m+1) \tilde{Q}^m_\nu = Q^m_\nu, \quad (5.3) \]

a further property being
\[ \left[ \frac{\tilde{Q}^m_\nu - \tilde{Q}^m_{\nu-1}}{\Gamma(\nu+m+1)} - \frac{\tilde{Q}^m_{\nu+1} - \tilde{Q}^m_\nu}{\Gamma(-m-\nu)} \right] \sec \nu \pi = P^m_\nu. \quad (5.4) \]

(1) Recurrence formulae.

Using the recurrence formulae for the associated Legendre functions (Ref. 2a, §3.8(1), 9), we have
\[ \tilde{Q}^m_\nu = \frac{mz}{(z^2-1)} \tilde{Q}^m_\nu - (z^2-1)^{-\frac{1}{2}}(\nu+m+1) \tilde{Q}^{m+1}_\nu, \quad (5.5) \]

(\text{where } ^t \equiv \frac{d}{dz}),
\[ 2mz(z^2-1)^{-\frac{1}{2}} \tilde{Q}^m_\nu = (\nu+m+1) \tilde{Q}^{m+1}_\nu - (\nu-m+1) \tilde{Q}^{m-1}_\nu, \quad (5.6) \]

giving
\[ 3mz \tilde{Q}^m_\nu = -\frac{1}{4}(m+2)(\nu+m+1)(\nu+m+2) \tilde{Q}^{m+2}_\nu + \]
\[ + \frac{1}{2}(\nu(\nu+1) + \frac{z^2+1}{2}(m^2+2)) \tilde{Q}^m_\nu. \]
\[-\frac{1}{2}(m-2)(\nu-m+1)(\nu-m+2) Q_{\nu}^{m-2}. \tag{5.7}\]

From Ref. 2a, §3.3,1(2)
\[Q_{\nu}^{m}(z) = \tilde{Q}_{\nu}^{m}(z). \tag{5.8}\]

(ii) **Differential equation**

\[Q_{\nu}^{m}(z) \text{ satisfies the differential equation} \]
\[\tilde{Q}_{\nu}^{m} = \frac{2z}{1-z^2} \tilde{Q}_{\nu}^{m} - \left\{ \frac{\nu(\nu+1)}{1-z^2} - \frac{m^2}{(1-z^2)^2} \right\} \tilde{Q}_{\nu}^{m}. \tag{5.9}\]

(iii) **Addition Formula**

From (Ref. 15, §15.71) we have
\[\Gamma(\nu+1) \tilde{Q}_{\nu}(zt-(z^2-1)^{\frac{1}{2}}(t^2-1)^{\frac{1}{2}}) \cos w = \]
\[= \Gamma(\nu+1) \tilde{Q}_{\nu}(z)P_{\nu}(t) + 2 \sum_{m=1}^{\infty} \Gamma(\nu+m+1) \tilde{Q}_{\nu}(z)P_{m-\nu}(t) \cos mw, \tag{5.10}\]

where
\[\left| \frac{z+1}{z-1} \right| < \left| \frac{t+1}{t-1} \right|, \nu \neq -1, -2, \ldots, w \text{ real.}\]

(iv) **Asymptotic formula**

The asymptotic representation of \(Q_{\nu}^{m}\) for large positive integer values of \(m\), (Ref. 25, §26), is
\[Q_{\nu}^{m}(z) \sim \frac{\Gamma(m-\nu)}{\Gamma(m+1)} \left[ \left( \frac{z+1}{z-1} \right)^{\frac{1}{2}m} - e^{\pm \pi \nu i} \left( \frac{z-1}{z+1} \right)^{\frac{1}{2}m} \right], \tag{5.11}\]

where the upper or lower sign has to be taken according as \(\text{Im} z \gtrless 0\).

We shall now give a list of the integral relations we shall employ. Lamé polynomials satisfy integral
equations of which the one given is typical:

\[ u_{2n}^S(\alpha) = \lambda_s \int_{-2K}^{2K} P_{2n}(ksn\alpha sn\beta)u_{2n}^S(\beta)d\beta. \]  \hspace{1cm} (5.12)

A complete list of such integral equations has been given by Arscott (Ref.18). Lame functions of the second kind corresponding to the Lame polynomials can be represented by integrals such as

\[ u_{n}^S(\alpha) = \lambda_s \int_{K}^{K+iK'} q_{2n}(ksn\alpha sn\beta)u_{2n}^S(\beta)d\beta, \]  \hspace{1cm} (5.13)

valid for all \( \alpha \) except where Re(\( \alpha \)) = (2p+1)K, \( p \) integral.

These integrals have been initially worked out by Arscott (Ref.18), and a complete list of nuclei for functions of different species were calculated by Taylor in some unpublished work. (The author wishes to express his sincere thanks to Mr. R.S. Taylor for permission to use these results). We have, writing

\[ S = ksn\alpha sn\beta, \quad C = \frac{4\pi}{k^2} cna cn\beta, \quad D = \frac{1}{k} d\alpha d\beta, \]

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<th>Nucleus ( q_{2n}(z) )</th>
<th>Limits of Integration</th>
</tr>
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<td>( q_{2n}(s) )</td>
<td>( K, K + iK' )</td>
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<tr>
<td>2</td>
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<td>( q_{2n}(c) )</td>
</tr>
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<td>( q_{2n}(d) )</td>
<td>( 0, K )</td>
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<td>4</td>
<td>( q_{2n+1}(s) )</td>
<td>( K, K + iK' )</td>
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<tr>
<td>5</td>
<td>( s_{2n+1}^S(\alpha) )</td>
<td>( s q_{2n+1}^S(c) )</td>
</tr>
<tr>
<td>Type</td>
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<td>Limits of integration</td>
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<tr>
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<td>$0, , K$, $K, , K + i , K'$,</td>
</tr>
<tr>
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<td>$0, , K$,</td>
</tr>
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</table>

C Solutions in Series and Convergence Properties

In the following analysis it will be shown that
Lame's equation has solutions of the form

\[ y = \sum_{m=0}^{\infty} Y_m Q_m^m(z) \]  \hspace{1cm} (5.22)

and

\[ y = \left( \frac{1-A^2}{1-z^2} \right)^{1/2} \sum_{m=0}^{\infty} mY_m Q_m^m(z), \]  \hspace{1cm} (5.23)

\[ (z = s, c, d, k, d/k', 1k c/k'). \]  \hspace{1cm} (5.23a)

Here, \( A \) is a constant dependent on \( k \) only, (if \( z = smu \)
then \( A = k^2 \), and \( Y_m, X_m \) depend on \( h, k \) and \( \nu \).

If we write

\[ \left( \begin{array}{c} X_m \\ X_m' \end{array} \right) \Gamma(\nu-m+1) = \left( \begin{array}{c} Y_m \\ Y_m' \end{array} \right) \frac{1}{\Gamma(\nu+m+1)}, \]  \hspace{1cm} (5.24)

then we obtain, apart from a few minor sign changes due
to the different definition of \( \Gamma\nu \), the expansions given
by Erdelyi (Ref. 22). In the case of integral \( \nu, (\nu=n) \), we
shall understand by \( X_m \Gamma(n-m+1) \) the limiting value of
\( X_m \Gamma(\nu-m+1) \) as \( \nu \to n \), and similarly for \( X_m' \Gamma(n-m+1) \).

If we substitute (5.22) into (5.1) the relation

\[ \sum_{m=0}^{\infty} Y_m \{(\nu+m+2)(\nu+m+1)Q_{\nu+2}^m(z) - (a-bm^2)Q_m^m(z) + \\
+ (\nu-m+1)(\nu-m+2)Q_{\nu-2}^m(z)\} = 0 \]  \hspace{1cm} (5.25)

is obtained, which has to be satisfied identically in \( z \).

Here \( a, b \) are constants depending on \( h, k \) and \( \nu \) but
independent of \( m \) and \( z \). In particular if \( z = smu \), then
Equation (5.25) may be written as

\[ a = 2v(v+1) - 4\{v(v+1) - h\}/k^2, \quad b = -2 \frac{h^2}{1-k^2}. \]

Equation (5.25) may be written as

\[
\sum_{m=2}^{\infty} Y_{m-2}(v+m)(v+m-1) Q^m_v(z) - \sum_{m=0}^{\infty} Y_m(a-bm^2)Q^m_v(z) +
\sum_{m=-2}^{\infty} Y_{m+2}(v-m)(v-m-1)Q^m_v(z) = 0.
\]

In the first two terms \((m = -2, -1)\) of the last summation we use (5.28) and write the last equation as

\[
\sum_{m=0}^{\infty} (v+m)(v+m-1)Q^m_v(z)\{\epsilon_{m-2} Y_{m-2} - \delta_{m-2} Y_m - \gamma_m Y_{m+2}\} = 0, \quad (5.26)
\]

where

\[
\gamma_m = -\frac{(v-m)(v-m-1)}{(v+m)(v+m-1)}; \quad m \neq 1,
\]

\[
\delta_m = \frac{a - bm^2}{(v+m)(v+m-1)}; \quad m \neq 1,
\]

\[
\epsilon_{-2} = \epsilon_{-1} = 0, \quad \epsilon_0 = 2, \quad \epsilon_m = 1, \quad m > 1. \quad (5.26a)
\]

In order that (5.26) be identically satisfied in \(z\), the \(Y_m\) must satisfy the recurrence relations

\[
\epsilon_{m-2} Y_{m-2} - \delta_{m-2} Y_m - \gamma_m Y_{m+2} = 0, \quad m = 0, 1, 2, \ldots. \quad (5.27)
\]

Now the relations (5.27) with even and odd \(m\) form independent sets. Thus for even subscripts we get

\[
\epsilon_{2m-2} Y_{2m-2} = \delta_{2m-2} Y_{2m} + \gamma_{2m} Y_{2m+2} \quad m = 0, 1, 2, \ldots. \quad (5.28)
\]

while for odd subscripts we have
\[ \varepsilon_{2m-1} Y_{2m-1} = \delta_{2m-1} Y_{2m+1} + \gamma_{2m+1} Y_{2m+3}, \quad m = 0, 1, 2, \ldots \quad (5.28a) \]

From (5.27) we formally write down the continued fraction

\[ \frac{Y_m}{Y_{m-2}} = \frac{1}{\delta_{m-2}^+} \frac{\gamma_m}{\delta_m^+} \frac{\gamma_{m+2}}{\delta_{m+2}^+} \cdots \quad (5.29) \]

This continued fraction is convergent unless \( b \) is real and

\[ -2 < b < 2, \quad (\text{Ref. 13, \S 20}), \quad \text{and further} \]

\[ \lim_{m \to \infty} \frac{Y_m}{Y_{m-2}} = \frac{1}{2} \left\{ b \pm (b^2 - 4)^{1/2} \right\} \quad (5.30) \]

In (5.30) we must take, in general, the branch of \( (b^2 - 4)^{1/2} \) which gives the smaller modulus, \( b_0^{-1} \) say, in \( \frac{1}{2} \left\{ b \pm (b^2 - 4)^{1/2} \right\} \).

In particular, if \( b \) is real \( (b^2 > 4) \), then the sign opposite to that of \( b \) must be taken.

Starting with \( Y_0 \) the relations (5.28) define the \( Y_{2m} (m > 0) \) uniquely and \( Y_{2m} \to 0 \) as \( m \to \infty \) if the expression derived from (5.29) for \( Y_2 / Y_0 \) is equal to the value of this quotient as given by the first equation of (5.23). Hence we have the condition

\[ \delta_{-2} + \frac{\gamma_0}{\delta_0^+} \frac{\gamma_2}{\delta_2^+} \frac{\gamma_4}{\delta_4^+} \cdots = 0, \quad (5.31) \]

which is a transcendental equation for \( h \). Similarly, for the system (5.28a) the condition is

\[ \delta_{-1} + \frac{\gamma_1}{\delta_1^+} \frac{\gamma_3}{\delta_3^+} \frac{\gamma_5}{\delta_5^+} \cdots = 0. \quad (5.32) \]

By substituting (5.23) into (5.1) we find that the \( Y_m' \) satisfy recurrence relations similar to those for the \( Y_m \) except that \( \delta_{-1} \) is replaced by \( \delta_{-1}' \), where
\[ \delta_{-1} = \frac{a-b}{\nu(\nu+1)} + 1. \] (5.33)

Expressing (5.22) in terms of hypergeometric functions we get (Ref. 2a, §3.3.2 (15))

\[
2y(z) = \sum_{m=0}^{\infty} Y_m \frac{\Gamma(m-\nu)}{\Gamma(m+1)} \left( \frac{z+1}{z-1} \right)^{\frac{1}{2}m} F(-\nu, \nu+1; m+1; \frac{1+z}{2}) - \sum_{m=0}^{\infty} Y_m \frac{\Gamma(m-\nu)}{\Gamma(m+1)} e^{\frac{i\pi}{2}} \left( \frac{z-1}{z+1} \right)^{\frac{1}{2}m} F(-\nu, \nu+1; m+1; \frac{1-z}{2}),
\]

where the upper or lower sign has to be taken according as the series A is convergent if

\[
I_m z > 0.
\]

Clearly the series A is convergent if

\[
\lim_{m \to \infty} \left| \frac{Y_m (m-\nu-1)(m-\nu-2)(z+1)F(-\nu, \nu+1; m+1; \frac{1+z}{2})}{Y_{m-2}(m)(m-1)(z-1)F(-\nu, \nu+1; m+1; \frac{1-z}{2})} \right| < 1,
\]

which, on using (5.11) is equivalent to

\[
\lim_{m \to \infty} \left| \frac{Y_m}{Y_{m-2}} \cdot \frac{z+1}{z-1} \right| < 1,
\]

i.e.,

\[
\left| \frac{z+1}{z-1} \right| < b_0.
\]

Similarly B is convergent for

\[
\left| \frac{z-1}{z+1} \right| < b_0.
\]

Thus the series (5.22) is convergent in that part of the complex z-plane which lies outside both the circles $C_1$ and $C_2$ where
where
\[ z_1 = \frac{1 + 2b_2^0}{1 - 2b_2^0}, \quad r = \frac{4b_0}{2b_0^2 - 1}. \]

Similar results are obtained for (5.23).

If in (5.22) we write \( \hat{Q}^m_\nu \) in terms of \( \hat{\alpha}^m_\nu \), we obtain the expansion
\[
y(u) = \sum_{m=0}^{\infty} \frac{Y_m(-1)^m \hat{Q}^m_\nu(z)}{\Gamma(\nu+m+1)),
\]
where the recurrence formulae for the \( \hat{Q}^m_\nu \) are easily obtained via (5.3) and (5.5) – (5.8). These new recurrence formulae remain true if \( Q \) is replaced by \( P \), (cf. Ref. 2a, §3.8) and so another solution will be
\[
y(u) = \sum_{m=0}^{\infty} \frac{Y_m(-1)^m P^m_\nu(z)}{\Gamma(\nu+m+1)),
\]
or, using (5.4)
\[
y(u) = \sum_{m=0}^{\infty} \frac{Y_m(-1)^m \sec \nu \tau \left\{ \hat{Q}^m_\nu - \hat{\alpha}^m_\nu \frac{\Gamma(-m-\nu)}{\Gamma(-m-\nu)} \right\}}{\Gamma(\nu+m+1)).}
\]
Further, (5.24) shows that (5.34) and (5.35) may be written as
\[
y(u) = \sum_{m=0}^{\infty} X_m(-1)^m \Gamma(\nu-m+1) Q^m_\nu(z),
\]
\[ y(u) = \sum_{m=0}^{\infty} x_m (-1)^m \Gamma (\nu-m+1) P_{\nu}^m (z) , \]

which apart from the factor \((-1)^m\), due to our different definition of \(P_{\nu}^m\) and \(Q_{\nu}^m\), are the series considered by Erdélyi. Similar statements can be made about (5.23).

All the above analysis is valid if \(\nu\) is not an integer; in the case of integral \(\nu\), certain difficulties arise in connection with (5.22) and (5.23). This case will be dealt with in detail later.

D. Solutions in Series of \(Q_{\nu}^m(s)\) and \(Q_{\nu}^m(ks)\)

In this section we formally show how certain Lamé functions of the second kind may be expanded in series of \(Q_{\nu}^m(s)\), \(Q_{\nu}^m(ks)\), \((\nu = 2n \text{ or } 2n+1)\). From the results obtained we suggest that there exist similar series valid for non-integral \(\nu\). Series in terms of \(Q_{\nu}^m(z)\) where \(z\) takes any one of the remaining arguments given by (5.23a) are deduced in a similar manner.

From the first integral relation in (5.14) we have

\[ u_{2n}^{S}(u) = \lambda_s \int_{K+ik'}^{K} \Gamma (2n+1) \mathfrak{Q}_{2n}^{m}(k s n u s n v) u_{2n}^{E}(v) dv. \quad (5.37) \]

Substitution of (5.10) with \(w = \pi/2\), \(z = k s m u\), \(t = s n v\), \(\nu = 2n\) in (5.37) leads, formally, to the expression

\[ u_{2n}^{S}(u) = \lambda_s \int_{K}^{K+ik'} \{ \Gamma (2n+1) \mathfrak{Q}_{2n}^{m}(k s m u) P_{2n}(s n v) + \]
\[ + 2 \sum_{m=1}^{\infty} \Gamma(2n+m+1)(-1)^{m/2} \mathcal{O}_{2n}^m(ksmu)P_{2n}^{-m}(smv) \] \[ \times u_{2n}^S(v)dv. \]

On putting
\[ \epsilon_m A_m = 2\lambda \Gamma(2n+m+1)(-1)^{m/2} \int_{K+1K'} P^{-m}(smv) u_{2n}^S(v)dv, \]
where \( \epsilon_0 = 2, \epsilon_m = 1, \quad m > 1 \), we see that \( u_{2n}^S(u) \) can be formally expanded in a series of the form
\[ u_{2n}^S(u) = \sum_{m=0}^{\infty} A_m \mathcal{O}_{2n}^m(ksmu). \quad (5.38) \]

By differentiating (5.10) with respect to \( w \) and putting \( w = \pi/2, \quad z = ksmu, \quad t = smv, \quad \nu = 2n+1 \), we see on using the last integral relation in (5.15) that \( s_{2n+1}^F(u) \) can be formally expanded in a series of the form
\[ s_{2n+1}^F(u) = \frac{c_{mu}}{d_{mu}} \sum_{m=1}^{\infty} B_m m \mathcal{O}_{2n+1}^m(ksmu). \quad (5.39) \]

By the same arguments we find that if in (5.10) we put
\[ w = \pi/2, \quad z = smu, \quad t = ksmv, \]
then the expansions
\[ y(u) = \sum_{m=0}^{\infty} A'_m \mathcal{O}_{2n}^m(smu), \quad (5.40) \]
\[ y(u) = \frac{d_{mu}}{c_{mu}} \sum_{m=1}^{\infty} B'_m m \mathcal{O}_{2n+1}^m(smu), \quad (5.41) \]
also represent formally certain Lamé functions of the second kind.
Having obtained the expansions of certain Lamé functions of the second kind in series of associated Legendre functions of the second kind, we now generalize (5.33) – (5.41) to the case of non-integral values of $\nu$ and suggest that (5.1) has solutions of the form

\[ y(u) = \sum_{m=0}^{\infty} A_m \mathcal{Q}_\nu^m(ksu), \quad (5.42) \]

\[ y(u) = \sum_{m=0}^{\infty} A_m^t \mathcal{Q}_\nu^m(snu), \quad (5.43) \]

\[ y(u) = \frac{c_{nu}}{c_{nu}} \sum_{m=1}^{\infty} B_m \mathcal{Q}_\nu^m(ksu), \quad (5.44) \]

\[ y(u) = \frac{d_{nu}}{c_{nu}} \sum_{m=1}^{\infty} B_m^t \mathcal{Q}_\nu^m(snu). \quad (5.45) \]

By considering (5.42) and using the abbreviations adopted in § B we have, on differentiating with respect to $u$,

\[ \frac{dy}{du} = \sum_{m=0}^{\infty} A_m \mathcal{Q}_\nu^m k_\nu d \]

and

\[ \frac{d^2y}{du^2} = k^2 c^2 d^2 \sum_{m=0}^{\infty} A_m \mathcal{Q}_\nu^m \mathcal{Q}_\nu^m - ks(d^2 + k^2c^2) \sum_{m=0}^{\infty} A_m \mathcal{Q}_\nu^m \quad (5.46) \]

Substituting (5.46) in (5.1) and using (5.7) and (5.9) we
obtain after some simplification

\[
L(y) = \sum_{m=2}^{\infty} A_{m-2} \left\{ \frac{k^2}{4} (v+m)(v+m-1) \right\} q_m^m - \\
- \sum_{m=0}^{\infty} A_m \frac{1}{2} \left[ v(v+1)(2-k^2) - 2h - m^2(2-k^2) \right] q_m^m + \\
+ \sum_{m=-2}^{\infty} A_{m+2} \left\{ \frac{k^2}{4} (v-m-1)(v-m) \right\} q_m^m = 0.
\]

In the first two terms of the last summation we use (5.3) and write the last equation in the form

\[
\sum_{m=0}^{\infty} \left\{ \epsilon_{m-2} A_{m-2} - \delta_{m-2} A_m - \gamma_m A_{m+2} \right\} (v+m)(v+m-1) q_m^m = 0,
\]

where \( \epsilon_{m-2}, \delta_{m-2}, \gamma_m \) are of the form (5.26a) and \( a \) and \( b \) take the forms

\[-2v(v+1) + 4\left\{ v(v+1) - h \right\}/k^2, 2\left( \frac{1+k^2}{1-k^2} \right),\]

respectively. (5.47) is a relation of the form (5.26) and hence for the coefficients we obtain the recurrence relations (5.27) with \( A_m = Y_m \).

From the results of § C it follows that a solution of the form

\[
\sum_{r=0}^{\infty} A_{2r} q_{2r}^{2r}(ks) \text{ or } \sum_{r=0}^{\infty} A_{2r+1} q_{2r+1}^{2r+1}(ks)
\]

exists if \( h \) satisfies the transcendental equation (5.31).
or (5.32) respectively. The coefficients of these expansions are given by the recurrence formulae (5.28) and (5.28a) respectively. Proceeding in the same manner with (5.43), (5.44), (5.45) we find these series satisfy (5.1) with

\[ A_m = Y_m, \quad B_m = B_m' = Y_m' \]

and together with

\[ a = 2\nu(n+1) - 4[n(n+1) - h]/k^2, \]
\[ b = -2\left(\frac{1+k^2}{1-k^2}\right). \]

The values of \( h \) in each of these expansions satisfy the appropriate transcendental equation.

Other series which suggest themselves from the integral relations in § B are

\[ y(u) = \sum_{m=0}^{\infty} C_m \xi_{\nu}^m (a), \quad (5.48) \]
\[ y(u) = \frac{c}{ks} \sum_{m=1}^{\infty} D_m m \xi_{\nu}^m (d), \quad (5.49) \]
\[ y(u) = \sum_{m=0}^{\infty} C_m \xi_{\nu}^m \left(\frac{d}{k^2}\right), \quad (5.50) \]
\[ y(u) = \frac{n}{ik\sigma} \sum_{m=1}^{\infty} D_m m \xi_{\nu}^m \left(\frac{d}{k^2}\right), \quad (5.51) \]
\[ y(u) = \sum_{m=0}^{\infty} C_m \xi_{\nu}^m (c), \quad (5.52) \]
where

\[ y(\nu) = \frac{ds}{d\nu} \sum_{m=1}^{\infty} H_m \, \bar{q}_m^m(\nu), \quad (5.53) \]

\[ y(\nu) = \sum_{m=0}^{\infty} G_m^m \, \bar{q}_m^m(\frac{ik_0}{k}), \quad (5.54) \]

\[ y(\nu) = \frac{s}{d} \sum_{m=1}^{\infty} H_m^m \, \bar{q}_m^{m\nu}(\frac{ik_0}{k}), \quad (5.55) \]

The convergence properties of the above series can be easily deduced from the results of § 3, and since they can be cast into the form of (5.34), which, for general non-integral
values of \( \nu \) has already been discussed by Erdélyi (Ref. 22), we shall not consider them further, but rather concentrate on the important special case where \( \nu \) is a non-negative integer.

E. Lame functions of integral order

In this section we consider even values of \( m \), writing \( m = 2r \), and when \( \nu \) is an integer we take it to be even also. The cases where \( m \) or \( \nu \) or both are odd can be treated in exactly the same way. We have from (5.24)

\[
X_{2r} \Gamma(\nu-2r+1) = \frac{Y_{2r}}{\Gamma(\nu+2r+1)},
\]

(5.56)

where the coefficients \( X_{2r} \) satisfy the recurrence relations

\[
-ax_0 + (\nu+1)(\nu+2)x_2 = 0,
\]

\[
2\nu(\nu-1)x_0 - (a-4b)x_2 + (\nu+3)(\nu+4)x_4 = 0,
\]

\[
(\nu-2r+1)(\nu-2r+2)x_{2r-2} - (a-4r^2b)x_{2r} + (\nu+2r+1)(\nu+2r+2)x_{2r+2} = 0
\]

\[r = 2, 3, \ldots (5.57)\]

The difficulty which arises is that when \( \nu \) is integral, \( (= 2n \text{ say}) \), \( \Gamma(\nu-2r+1) \) is undefined for \( r > n+1 \) and so the transformation (5.56) is invalid. However, it is precisely for these values of \( r \) that the coefficients \( X_{2r} \) and the continued fraction terminates giving a polynomial equation in \( h \) with \( (4n+1) \) distinct roots. For these values of \( h \), \( X_{2r}(r>n+1) \) when considered as a function of \( \nu \), vanishes at \( \nu = 2n \). Moreover there exists a set of these \( X_{2r} \) which have only a simple zero at \( \nu = 2n \). For suppose it possible that
$X_{2n+2}$ has a zero of multiplicity $P(>1)$ and $X_{2n+4}$ has a zero of multiplicity $q(>1)$ where $q \neq P$; then if we substitute in (5.57), divide through by $\nu - 2n$ and let $\nu \to 2n$, we obtain the result $X_{2n} = 0$ which is false. Hence either $P$ or $q$ or both must be unity. Suppose $P = 1$ then substitution in (5.57) shows that $X_{2n+6}$ has a simple zero at $\nu = 2n$. Hence by repeated argument, the result is proved. If both $P$ and $q$ are unity we prove that all the $X_{2r}$ ($r \geq n+1$) have only simple zeros at $\nu = 2n$. Thus for $r \geq n+1$, $X_{2r}$ may be written as

$$X_{2r} = (\nu-2n)X^*_2 \nu;$$

and since $\nu - 2n$ is independent of $r$, the $X^*_2 \nu$ satisfy the same recurrence relations as do the $X_{2r}$, except when $r = n+1$, which provides the "link" between them, namely

$$(\nu-2n-1)X_{2n} - (a-4(n+1)^2b)X^*_2 + (\nu+2n+3)(\nu+2n+4)X^*_2 = 0;$$

as $\nu \to 2n$ we get

$$X_{2n} + (a-4(n+1)^2b)X^*_2 = 0.$$  (5.58)

From (5.58) it might appear that either $X^*_2$ or $X^*_2$ can be chosen arbitrarily, the other one and subsequent $X^*_2$ being determined uniquely. This however is not true and we now show how the $X^*_2$ may be best evaluated in terms of the $Y^*_2$. Consider (5.56); this may be written as, (Ref. 2a, §1.2,3),

$$\frac{-X^*_2}{\Gamma(2r-\nu)\sin \nu \pi} = \frac{Y^*_2}{\Gamma(\nu+2r+1)};$$

and for $r \geq n+1$
\[ Y_{2r} = -\pi \Gamma (2n+2r+1) \lim_{\nu \to 2n} \frac{(\nu-2n)X^2_{2r}}{\Gamma (2r-\nu) \sin \nu \pi} \]

\[ = \frac{\Gamma (2n+2r+1)}{\Gamma (2r-2n)} X^2_{2r} . \]

Thus

\[ y(u) = \sum_{r=0}^{\infty} Y_{2r} \psi^2_{2n}(a) \]  

is a solution of Lame's equation. However, the domains of convergence given in § C have to be supplemented by the further condition \(|z| > 1\). (5.36) shows that

\[ y(u) = \sum_{r=0}^{\infty} \frac{Y_{2r} \phi^2_{2n-1}}{\Gamma (2n+2r+1) \Gamma (2n-2r+1)} \]

is also a solution of Lame's equation and represents a certain Lame polynomial. In the case of transcendental Lame functions of order 2n, (i.e., non-terminating series solutions of Lame's equation for integral \(\nu\)),

\[ X_{2r} = 0 \quad r < n \]

\[ X_{2r} \neq 0 \quad r > n . \]

Here (5.56) is invalid for \(\nu = 2n\), but if we multiply both sides by \(P^2_r\) and let \(\nu \to 2n\), then

\[ \lim_{\nu \to 2n} \frac{Y_{2r} P^2_r}{\Gamma (\nu+2r+1)} = X_{2r} P^{-2r} \]

\[ = \frac{-X_{2r} \phi^2_{2n}}{\Gamma (2r-2n)} , \]

(\text{using (5.4) and hence (cf. 5.34)})
y(u) = - \sum_{r=n+1}^{\infty} \frac{x_{2r} \phi_{2r}^{(2r)}}{1(2r-2n)}\]

is a transcendental Lamé function of order 2n. The $x_{2r}$ satisfy the relations (5.57) although the consistency conditions must be modified slightly.

We have seen that each of the twelve types of solutions, (5.42) - (5.45), (5.48) - (5.55), give rise to two distinct sets of solutions according as to whether $n$ is even or odd. For example (5.42) provides the pair of solutions

$$y(u) = \sum_{r=0}^{\infty} A_{2r} \phi_{2r}^{(2r)}(ks), \quad y(u) = \sum_{r=0}^{\infty} A_{2r+1} \phi_{2r+1}^{(2r+1)}(ks),$$

which we shall denote by $(5.42)_e$ and $(5.42)_o$ respectively.

Let $V_n$ be a series of descending powers of $\eta$, $(\eta = s, c$ or $d), \text{ of the form}$

$$V_n = \sum_{s=0}^{\infty} a_s \eta^{-n-1-2s}.$$

Corresponding to a given integral value of $\nu$ there are in general $2\nu + 1$ Lamé's functions of the second kind corresponding to as many different values of $n$; also there are eight different types of these functions, each one corresponding to one of the eight types of Lamé polynomial. When $\nu$ is even we have Lamé functions of the second kind and of the first species of the form $V_n$ and of the third species of the
forms cd \( V_{n-2} \), sd \( V_{n-2} \) or sc \( V_{n-2} \). For odd values of \( \nu \) we have Lame functions of the second kind of the second species of the forms s \( V_{n-1} \), c \( V_{n-1} \), d \( V_{n-1} \), and of the fourth species of the form scd \( V_{n-3} \). As expected from the integral relations, each type of Lame function of the second kind may be represented in six different ways by an infinite series of \( \Phi^\nu \). Consider for example (5.42)\(_e\); it is of the form \( V_n \) if \( n \) is even and is of the form \( sd \ V_{n-2} \) if \( n \) is odd. Correspondingly (5.42)\(_o\) is of the form \( sd \ V_{n-2} \) if \( n \) is even and is of the form \( d \ V_{n-1} \) if \( n \) is odd. The same reasoning can be applied to the remaining eleven types, and thus we obtain the following table of representations of Lame functions of the second kind.

<table>
<thead>
<tr>
<th>Type uF</th>
<th>Type sF</th>
<th>Type CF</th>
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<tbody>
<tr>
<td>(5.42)(_e)</td>
<td>(5.42)(_o)</td>
<td>(5.44)(_e)</td>
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<tr>
<td>(5.48)(_e)</td>
<td>(5.48)(_o)</td>
<td>(5.49)(_o)</td>
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<tr>
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<th>Type dF</th>
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<tr>
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<td>(5.44)(_o)</td>
<td>(5.42)(_o)</td>
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<tr>
<td>(5.48)(_e)</td>
<td>(5.49)(_e)</td>
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<tr>
<td>(5.53)(_o)</td>
<td>(5.52)(_o)</td>
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<tr>
<th>Type scF</th>
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<tbody>
<tr>
<td>(5.44)(_e)</td>
<td>(5.44)(_e)</td>
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<tr>
<td>(5.49)(_o)</td>
<td>(5.49)(_e)</td>
</tr>
<tr>
<td>(5.53)(_o)</td>
<td>(5.53)(_e)</td>
</tr>
</tbody>
</table>
The corresponding Lame polynomials are obtained from (5.36) and (5.61). Thus each Lame function of the second kind can be represented by six infinite series of \( \xi_n^m(z) \),
\[
(z = s, c, d, ks, \frac{d}{k'}, \frac{1}{k'})
\]
respectively, in each of the six cases.

Consider (5.42) and (5.43); they are
\[
\sum_{r=0}^{\infty} A_{2r} \xi_n^{2r}(ks) \quad \text{and} \quad \sum_{r=0}^{\infty} A_{2r}^* \xi_n^{2r}(s)
\]
respectively. In their common domain of convergence they both represent solutions of the form \( V_n \) (n even) and of the form \( V_{n-1} \) (n odd). The corresponding values of \( a \) and \( b \) differ only in sign and hence \( \gamma_m \) are the same in both cases while the \( \delta_{m-2} \) differ only in sign if \( m \neq 1 \), \( \delta_{-1} \) being essentially different in both cases. Hence under the transformation (5.56) we obtain the same consistency condition (a terminating continued fraction) in both cases, giving the same values of \( h \). Since (Ref. 15, § 25.47) there is no value of \( h \) for which (5.1) is satisfied by two Lame polynomials of different species, then clearly the same applies to the corresponding Lame functions of the second kind. Thus with the same characteristic value of \( h \) in (5.42) and (5.43) we obtain, apart from a constant multiplier, the same Lame function of the second kind. Hence
\[ \sum_{r=0}^{\infty} A_{2r} \tilde{a}_{n}^{2r} (ks) = \lambda \sum_{r=0}^{\infty} A_{2r}' \tilde{a}_{n}^{2r} (s), \]  \hspace{1cm} (5.62)

where \( \lambda \) is a constant.

From (5.27) it is obvious that, (the recurrence relations for \( A_{2r} \) and \( A_{2r}' \) differing only in the sign of \( \delta_{m-2} \)), we may put \( A_{2r}' = (-1)^{r} A_{2r} \). Thus (5.62) becomes

\[ \sum_{r=0}^{\infty} A_{2r} \tilde{a}_{n}^{2r} (ks) = \lambda \sum_{r=0}^{\infty} (-1)^{r} A_{2r} \tilde{a}_{n}^{2r} (s). \]  \hspace{1cm} (5.63)

There is no corresponding relation between (5.42) and (5.43), since from (5.61) they clearly represent functions of different types. By similar arguments to the above we obtain the following relations

\[ \sum_{r=0}^{\infty} g_{2r} \tilde{a}_{n}^{2r} (c) = \lambda \sum_{r=0}^{\infty} (-1)^{r} g_{2r} \tilde{a}_{n}^{2r} \left( \frac{ik_{c}}{k} \right), \]  \hspace{1cm} (5.64)

for \( u^{E}_{n} \) (n even) or \( c^{E}_{n} \) (n odd).

\[ \sum_{r=0}^{\infty} c_{2r} \tilde{a}_{n}^{2r} (d) = \lambda \sum_{r=0}^{\infty} (-1)^{r} c_{2r} \tilde{a}_{n}^{2r} \left( \frac{d}{k} \right), \]  \hspace{1cm} (5.65)

for \( u^{E}_{n} \) (n even) or \( c^{E}_{n} \) (n odd).

\[ \frac{c}{d} \sum_{r=1}^{\infty} B_{2r} r \tilde{a}_{n}^{2r} (ks) = \lambda \frac{d}{c} \sum_{r=1}^{\infty} (-1)^{r} B_{2r} r \tilde{a}_{n}^{2r} (s), \]  \hspace{1cm} (5.66)

for \( c^{E}_{n} \) (n even) or \( s^{E}_{n} \) (n odd).
\[
\sum_{r=1}^{\infty} \frac{H_{2r}}{r} \varphi_n^{2r}(c) = \lambda \frac{\mu}{d} \sum_{r=1}^{\infty} (-1)^r \frac{\varphi_n^{2r}(\frac{4kr}{\mu})}{r}, \quad (5.67)
\]

for \(scF_n\) (n even) or \(scdF_n\) (n odd).

\[
\sum_{r=1}^{\infty} \frac{D_{2r}}{k^2} \varphi_n^{2r}(d) = \lambda \frac{kd}{c} \sum_{r=1}^{\infty} (-1)^r \frac{\varphi_n^{2r}(\frac{d}{k^2})}{r}, \quad (5.68)
\]

for \(scF_n\) (n even) or \(scdF_n\) (n odd).

Erdélyi (Ref. 22) has shown that the same relations hold for corresponding Lame polynomials, and are obtained, in our notation, by simply replacing \(\varphi_n^{2r}\) in equations (5.63) - (5.68) by

\[
(-1)^n \frac{\varphi_{-n-1}^{2r}}{\Gamma(n+2r+1)\Gamma(n-2r+1)},
\]

and letting \(r\) range from 0 or 1 to \([\frac{1}{2}]\).
Appendices

Introduction

In the following appendices we give a brief review of two major contributions to the theory of Heun's equation which have not been discussed in the main body of this thesis.

Appendix A deals with the work of Erdélyi (Refs. 7, 8) and Svartholm (Ref. 6) on the expansions of the solutions of Heun's equation in series of hypergeometric functions. Svartholm's paper is concerned with solutions in terms of hypergeometric or Jacobi polynomials, whilst Erdélyi (Ref. 7) solves the Heun equation in terms of series of certain types of hypergeometric functions. In reference (8) Erdélyi generalizes the results of both the aforementioned papers and shows that all solutions in terms of hypergeometric functions must, apart from transformations of the Heun equation or of the hypergeometric functions in question, be series either of the type given by Svartholm or of the type dealt with in his earlier paper. Since reference (8) incorporates all the results of references (6, 7) we shall give a brief review of this work and only mention the other papers when required.

In appendix B we review the important work of Heun (Ref. 1, §4) on the relations between solutions of Heun's equation which correspond to the Gaussian "relations between
contiguous functions" associated with the hypergeometric functions.

Truesdell (Ref. 26) has shown that all the special functions of mathematical physics which are special or limiting cases of the hypergeometric function can be studied in terms of the functional equation

\[
\frac{\partial \Phi(a,x)}{\partial x} = F(a+1,x),
\]  

the properties of the solutions of which, provide a unified theory for all the above special functions. Since equation (1) is a kind of contiguous relation, perhaps a fruitful line of future research towards a corresponding unification of the properties of functions which are special or limiting cases of Heun's equation may be via the contiguous relations obtained in Heun's paper.

For ease of reference, when a formula is quoted from any of the above mentioned papers, we shall place the reference number of the paper concerned to the left of the page in double brackets.
Appendix A

A Review of the Work of Erdélyi (Ref.6)

Erdélyi's paper contains ten paragraphs, of which the first two give a brief introduction to the results established in the main part of the paper, and also the definition of a Heun function.

In §3, Erdélyi supposes a solution of Heun's equation, in terms of hypergeometric functions whose exponents at the singularities $z = 0, 1$ are made to coincide with those of Heun's equation, to be of the form

$$P = \sum_{m=0}^{\infty} C_m P_m, \quad (2)$$

where

$$P_m = P \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \lambda + m \\ 1 - \gamma & 1 - \delta & \mu - m \end{pmatrix}, \quad (3)$$

and where

$$\lambda + \mu = \gamma + \delta - 1 = \alpha + \beta - \epsilon \quad (4)$$

The main problem is thus to determine suitable values of $\lambda, \mu$ for which (2) satisfies Heun's equation. In reference (7), Erdélyi considers $P_m$ to be of the form

$$P_m = P \begin{pmatrix} 0 & 1 & \infty \\ m & 0 & \alpha \\ \delta - \alpha - \beta - m & 1 - \delta & \beta \end{pmatrix}, \quad (5)$$

which is merely a linear transformation of (3) for certain
values of $\lambda$, $\mu$, whilst Svartholm, using our notation, takes $P_m$ to be the hypergeometric function

$$2F_1(-m, m-1+y+\delta; y; z).$$

(Erdélyi, § 4, then takes the six significant branches of (3) to be

$$
\begin{align*}
P^1_m &= \frac{(-1)^m \Gamma(\lambda+m)}{\Gamma(1-\mu+m)} \cdot 2F_1(\lambda+m, \mu-m; y; z), \\
P^2_m &= \frac{(-1)^m \Gamma(1-\gamma+\lambda+m)}{\Gamma(\gamma-\mu+m)} z^{1-\gamma} \cdot 2F_1(1-\gamma+\lambda+m, 1-\gamma+\mu-m; 2-\gamma; z), \\
P^3_m &= \frac{\Gamma(\lambda+m) \Gamma(1-\gamma+\lambda+m)}{\Gamma(1-\mu+m) \Gamma(\gamma-\mu+m)} z^{-\gamma} \cdot 2F_1(\lambda+m, \mu-m; \delta; 1-z), \\
P^4_m &= (1-z)^{1-\delta} \cdot 2F_1(1-\delta+\lambda+m, 1-\delta+\mu-m; 2-\delta; 1-z), \\
P^5_m &= \frac{\Gamma(\lambda+m) \Gamma(1-\gamma+\lambda+m)}{\Gamma(1+\lambda-\mu+2m)} z^{-\lambda-m} \cdot 2F_1(\lambda+m, 1-\gamma-\lambda+m; 1+\lambda-\mu+2m; \frac{1}{z}), \\
P^6_m &= \frac{\Gamma(\lambda-m+2m)}{\Gamma(1-\mu+m) \Gamma(\gamma-\mu+m)} z^{-\mu+m} \cdot 2F_1(\mu-m, 1-\gamma+\mu-m; 1-\lambda+\mu-2m; \frac{1}{z}),
\end{align*}
$$

and assumes

$$
P_m = \sum_{\nu=1}^{6} \Pi_{\nu} P_{m}^{\nu},
$$

where the $[\Pi_{\nu}]$ are independent of $m$, but may depend on $\alpha, \beta, \ldots, \epsilon, \lambda, \mu$. He then shows that (8) satisfies the differential equation

$$
z(z-1) \left\{ \frac{d^2 P_{m}}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} \right) \frac{d P_{m}}{dz} \right\} + (\lambda+m)(\mu-m) P_{m} = 0,
$$

(9)
and the functional equation
\[ \epsilon z(z-1) \frac{dP_m}{dz} + \{ \alpha \beta(z-\alpha)-(\lambda+m)(\mu-m)(z-\alpha) \} P_m = \]
\[ ((4.4)) \quad = K_m P_{m+1} + L_m P_m + M_{m-1} P_{m-1}, \quad (10) \]

where
\[ K_m = \frac{(m+\alpha-\mu-1)(m+\beta-\mu-1)(m+\gamma-\mu-1)(m-\mu)}{(2m+\lambda-\mu-1)(2m+\lambda-\mu-2)} , \]
\[ L_m = a(\lambda+m)(\mu-m) - qa \beta + \]
\[ + \frac{(m+\alpha-\mu)(m+\beta-\mu)(m+\gamma-\mu)(m+\lambda)}{(2m+\lambda-\mu)(2m+\lambda-\mu+1)} + \]
\[ + \frac{(m-\alpha+\lambda)(m-\beta+\lambda)(m-\gamma+\lambda)(m-\mu)}{(2m+\lambda-\mu)(2m+\lambda-\mu-1)} , \]
\[ ((4.5)) \quad M_m = \frac{(m-\alpha+\lambda+1)(m-\beta+\lambda+1)(m-\gamma+\lambda+1)(m+\lambda)}{(2m+\lambda-\mu+1)(2m+\lambda-\mu+2)} . \quad (11) \]

In the last part of §4, Erdélyi deduces from the results of Watson (Ref.27) that for large integer values of \( m \),
\[ ((4.6)) \quad \lim_{m \to \infty} \frac{P_{m+1}}{P_m} = \frac{1 - (1 - \frac{1}{Z})^{\frac{1}{2}}}{1 + (1 - \frac{1}{Z})^{\frac{1}{2}}} , \quad \text{Re}(1 - \frac{1}{Z})^{\frac{1}{2}} > 0 , \quad (12) \]

where if \( P_m \) is not a multiple of \( P_m \),
\[ ((4.7)) \quad \lim_{m \to \infty} \frac{P_{m+1}}{P_m} = \frac{1 + (1 - \frac{1}{Z})^{\frac{1}{2}}}{1 - (1 - \frac{1}{Z})^{\frac{1}{2}}} , \quad \text{Re}(1 - \frac{1}{Z})^{\frac{1}{2}} > 0 . \quad (13) \]

In §5, Erdélyi substitutes (2) into Heun's equation, and by making use of the equations (9), (10) shows that the condition
\[ ((5.2)) \quad M_{-1} P_{-1} = 0 , \quad (14) \]
must hold and that the \( \{ C_m \} \) satisfy the system of recurrence
relations

\[ L_0 C_0 + M_0 C_1 = 0 , \]  

(15)

\[ K_m C_m - L_m C_m + M_m C_{m+1} = 0 , \quad (m = 1, 2, \ldots ) . \]  

(16)

Here (15) together with (4) determines the admissible values of \( \lambda, \mu \) and also the admissible branches of \( P_m \). § 6 deals with the convergence of (2) and Erdélyi shows that if \( q \) is arbitrary \( \sum C_m P_m \) diverges unless \( P_m \) is a multiple of \( P_m^5 \), in which case it is convergent outside the ellipse which has foci at \( z = 0, 1 \) and passes through the point \( z = a \). Outside this ellipse \( \sum C_m P_m^5 \) is convergent and represents a solution of the Heun equation that belongs to the exponent \( \lambda \) at infinity. If \( q \) satisfies the characteristic equation

\[ L_0 C_0 - K_1/1 = K_2/2 - \ldots = 0 , \]  

(17)

(2) is convergent inside the above ellipse, with the possible exception of the line joining the foci: \( \sum C_m P_m \) represents in this domain the general solution of the Heun equation. In this case \( \sum C_m P_m^5 \) is convergent in the whole \( z \)-plane cut along the line joining \( z = 0, 1 \), with the possible exception of the cut itself. In this cut \( \sum C_m P_m^5 \) represents the Heun function which arises from the coincidence of the branch regular at \( z = a \) and the branch belonging to the exponent \( \lambda \)
at infinity.

§ 7 returns to the discussion of (14) and Erdélyi shows that if

\[(7.1) \quad \lambda = \alpha, \quad \mu = \beta - \epsilon, \quad (18)\]

or if

\[(7.2) \quad \lambda = \beta, \quad \mu = \alpha - \epsilon, \quad (19)\]

then (14) vanishes for all branches of \( P_m \), identically in \( z \). Erdélyi calls the series arising from any of the above two admissible values of \( \lambda, \mu \), series of type I. If \( q \) is not a root of (17) then with \( \lambda = \alpha, \quad \mu = \beta - \epsilon \), the branch belonging to the exponent \( \alpha \) at infinity will be represented by a series of Type I, and since a suitable linear transformation brings any singularity to infinity, Erdélyi concludes that every solution of the Heun equation can be represented by a series of type I. In fact, the series suggested by (5) are of type I. He then goes on to show that if \( q \) satisfies (17) then every Heun function can be represented by series of type I convergent in the whole \( z \)-plane cut along a straight line or circular arc joining the two singularities \( z = 0, 1 \) of the Heun equation.

In § 8 Erdélyi shows that if

\[(8.1) \quad \lambda = \gamma + \delta - 1, \quad \mu = 0; \]
\[(8.2) \quad \lambda = \gamma, \quad \mu = \delta - 1; \]
\[(8.3) \quad \lambda = \delta, \quad \mu = \gamma - 1; \]
\[(8.4) \quad \lambda = 1, \quad \mu = \gamma + \delta - 2, \quad (20)\]
then three branches of $P_m$, one of which is always $P_m^6$, coincide and for these branches (14) vanishes. The other three branches, of which one is always $P_m^5$, also coincide but here (14) does not vanish. The series arising from these values of $\lambda, \mu$ are called series of type II. Since $P_m^6$ is not a multiple of $P_m^5$, there can be no series of type II in the general case. If $q$ satisfies (17) and we choose $\lambda = \gamma + \delta - 1, \mu = 0$ then clearly $P_m^6$ is a polynomial of degree $m$ in $z$ and the series $\sum C_m P_m^6$ is convergent inside the above mentioned ellipse and represents the Heun function which is regular at $z = 0, 1$. Similarly, in the other three cases (20) there is only one series of type II in each case representing a Heun function belonging respectively to the pairs of exponents $0, 1-\delta; 1-\gamma, 0$; and $1-\gamma, 1-\delta$ at $z = 0, 1$. By a linear transformation it follows that every Heun function, but no other solution of the Heun equation, can be represented by a series of type II. In each of the four cases (20), $P_m^6$ is either a Jacobi polynomial or can be expressed in terms of a Jacobi polynomial. In fact the series used by Svartholm are of type II, and in particular it is easy to see that (6) is a constant multiple of $P_m^6$ with $\lambda = \gamma + \delta - 1, \mu = 0$.

§9 is devoted to a short discussion on the advantages and disadvantages of series of type I and II. In the case of Heun functions, Erdélyi points out that a series
of type I is superior to the corresponding series of type II in that its domain of convergence is more extensive, and in that unlike type II it is capable of representing the general solution of Heun's equation. The chief advantage of series of type II is that they are series of orthogonal functions and this property is useful when we wish to normalize Heun functions. Erdélyi considers, as an example, the Heun function regular at $z = 0, 1$; its expansion in a series of type II is of the form

$$y = \sum A_m \, _2F_1(-m, \gamma + \delta + m - 1; \gamma; z), \quad (21)$$

or alternatively, using the Riemannian scheme of the Heun equation

$$y = (a-z)^{1-\epsilon} \sum B_m \, _2F_1(-m, \gamma + \delta + m - 1; \gamma; z). \quad (22)$$

On using the orthogonal property of Jacobi polynomials, Erdélyi shows that

$$\int_0^1 z^{\gamma-1}(1-z)^{\delta-1}(a-z)^{\epsilon-1} y^2 \, dz =$$

$$((9.4)) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\gamma) \Gamma(\delta + m)}{(y + \delta + 2m - 1) \Gamma(y + \delta + m - 1) \Gamma(y + m)} A_m B_m. \quad (23)$$

The recurrence relations (15), (16) determine the $A_m$, $B_m$, while $\sum A_m = \sum B_m$ in conjunction with (23) determines the numerical values for the normalized Heun function. The transcendental equation for $q$ connected with the expansions (21), (22) are the same, so that the ratios of the $A_m$ and
the ratios of the $B_m$ are found from the same continued fraction.

In the final paragraph, §10, Erdélyi points out that there is a relation between series of type I and type II representing the same Heun function. As an example, he considers the Heun function regular at $z = 0, 1$, and for simplicity takes the case where $\Re(y) > 0$, $\Re(\delta) > 0$. If the Heun function is $F(z)$, $(Hu(q, z) \text{ in our notation})$ then (Ref.5)

$$((10.1)) \quad F(z) = \frac{\lambda(1 - \frac{z}{a})^\delta - 1}{\int_0^1 t^{y-1}(1-t)^{\delta-1}F(\alpha-\epsilon+1, \beta-\epsilon+1; y; \frac{2t}{a}) \times F(t) dt.}$$

Using (2.1) and performing the integration he gets

$$F(z) = \lambda'(1 - \frac{z}{a})^{\epsilon-1} \sum (-1)^m \frac{\Gamma(\alpha-\epsilon+m+1)\Gamma(\beta-\epsilon+m+1)\Gamma(\delta+m)}{\Gamma(y+m)\Gamma(y+\delta+2m)} A_m \times$$

$$\times (\frac{z}{a})^m F(\alpha-\epsilon+m+1, \beta-\epsilon+m+1; y+\delta+2m; \frac{z}{a}),$$

which is in fact an expansion of type I, and further the coefficients of (25) are multiples of the coefficients of (21).
Appendix B

The relations between Heun functions which correspond to the Gaussian "Relations between continuous hypergeometric functions".

Ref. (1, § 4)

Instead of the differential equation satisfied by the Heun functions, Heun considers the more general equation

\[ z^2(z-1)^2(z-a)^2 F_2(h) \frac{d^2y}{dz^2} + z(z-1)(z-a) F_1(h+2) \frac{dy}{dz} + F_0(h+4)y = 0, \]

where \( F_0, F_1 \) and \( F_2 \) are rational functions of \( z \), the degree of which is indicated by the expressions in the brackets. The characteristic exponents at each of the four branch points are denoted by \( \lambda_1', \lambda_2' \), and the solutions of (1) are represented in the Riemannian scheme

\[
\begin{pmatrix}
0 & 1 & a & \infty \\
\lambda_1' & \lambda_2' & \lambda_3 & \lambda_4' \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4
\end{pmatrix}
\]

so that

\[ h = 2 - \sum_{i=1}^{4} (\lambda_1' + \lambda_2') \]

In this general case Heun represents the branch points by \( \xi_1', \xi_2', \xi_3', \) and \( \xi_4 \) instead of \( 0, 1, a, \) and \( \infty \). Heun writes

\[ \psi(z) = z(z-1)(z-a), \]
and so when \( h = 0 \), Heun's equation becomes

\[
((6)) \quad (\psi(z))^2 \frac{d^2 \psi}{dz^2} + \psi(z)P_1(2) \frac{d\psi}{dz} + P_0(4)\psi = 0 .
\]  

(5)

The characteristic exponents at each of the branch points of (5) are denoted by \( \lambda_{11} \), \( \lambda_{21} \) and so the solutions of (5) can be represented by the scheme

\[
P = \begin{pmatrix}
0 & 1 & a & \infty \\
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24}
\end{pmatrix},
\]

(6)

so that

\[
0 = 2 - \sum_{1}^{4} (\lambda_{11} + \lambda_{21}^1).
\]

(7)

Heun then writes the scheme of the differences of the exponents as

\[
\begin{pmatrix}
\lambda_{11}' - \lambda_{11} \\
\lambda_{12}' - \lambda_{12} \\
\lambda_{13}' - \lambda_{13} \\
\lambda_{14}' - \lambda_{14} \\
\lambda_{21}' - \lambda_{21} \\
\lambda_{22}' - \lambda_{22} \\
\lambda_{23}' - \lambda_{23} \\
\lambda_{24}' - \lambda_{24}
\end{pmatrix},
\]

(8)

where the differences are whole numbers. Denoting the difference in the \( i \)th vertical column of (8) which is less than the other by a non-negative integer by \( \delta_i \), Heun shows ( ACTA MATH XI p. 105 ) that

\[
y = z (z-1) (z-a) \left\{ p_0(h-D)y_0 + \psi P_1 \left( h-D-2 \right) \frac{d\psi}{dz} \right\},
\]

(9)

where \( y, y_0 \) represent solutions of (1) and (5) respectively and where

\[
D = \delta_1 + \delta_2 + \delta_3 + \delta_4 , \quad \psi = z(z-1)(z-a).
\]

(10)
The degrees of the rational functions \( \rho_0 \) and \( \rho_1 \) are denoted by the brackets provided. It is important to note that since \( \rho_0 \) and \( \rho_1 \) are rational functions of \( z \)

\[
\begin{align*}
\frac{h - D}{h - D - 2} & > 0
\end{align*}
\]

(11)

Heun then makes the substitution

\[
y = z (z - 1) (z - a) w
\]

and by making use of the identity

\[
P \begin{bmatrix}
0 & 1 & a & \infty \\
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24}
\end{bmatrix} z = z^{\lambda_{11}} (z - 1)^{\lambda_{12}} (z - a)^{\lambda_{13}} x
\]

and

\[
((7)) \times P \begin{bmatrix}
0 & 1 & a & \infty \\
0 & 0 & 0 & \lambda_{14} + \sum \lambda_{11} z \\
\lambda_{21} - \lambda_{11} & \lambda_{22} - \lambda_{12} & \lambda_{23} - \lambda_{13} & \lambda_{24} + \sum \lambda_{11}
\end{bmatrix}
\]

(13)

does that \( w \) must satisfy an equation of the form

\[
(5a) \psi^2 \frac{d^2 \psi}{dz^2} + \psi \phi_1 (h+2) \frac{d \psi}{dz} + \phi_0 (h+4) \psi = 0.
\]

(14)

By using (9) it is easy to see that \( w \) may be written in the form

\[
w = \rho_0 (h-D) y_0 + \rho_1 (h-D-2) \frac{dy_0}{dz}
\]

(15)

Heun then briefly shows that there are 2(2h-D)+3 conditions for determining as many parameters. We shall, however, give the reasoning behind this conclusion in more detail.
Differentiating (15) with respect to $z$ we get

$$w' = p'_0y_0 + (p_0 + \psi'p_1 + \psi p'_1)y_0' + \psi p_1y_0'' ,$$

and therefore

$$\psi w' = \psi p'_0y_0 + \psi(p_0 + \psi'p_1 + \psi p'_1)y_0' + \psi^2 p_1y_0'' .$$

Eliminating $y_0''$ between this equation and (5) then

$$\psi w' = (\psi p'_0 - p_1 F_0)y_0 + \psi(p_0 + \psi'p_1 + \psi p'_0 - p_1 F_1)y_0' .$$

Writing

$$\psi p'_0 - p_1 F_0 = f_1 ,$$

and

$$p_0 + \psi'p_1 + \psi p'_1 - p_1 F_1 = f_2 .$$

(17)

it is not difficult to see that both $f_1$ and $f_2$ are rational functions of degree $h - D + 2$. Using the simplification of (17) in (16) we have

$$\psi w' = f_1 y_0 + \psi f_2 y_0' .$$

(18)

A further differentiation gives

$$\psi w'' = f'_1 y_0 + (f_1 + \psi'f_2 + \psi f'_2)y_0' + \psi f_2 y_0'' - \psi' w' ,$$

which on using (5) and (18) gives

$$\psi^2 w'' = (\psi f'_1 - \psi' f_1 - f_2 F_0)y_0 + \psi(f_1 + \psi f'_2 - f_2 F_1)y_0' .$$

Writing

$$\psi f'_1 - \psi' f_1 - f_2 F_0 = g_1$$

$$f_1 + \psi f'_2 - f_2 F_1 = g_2$$

(19)

it is easy to see that $g_1, g_2$ are rational functions of
Thus

\[ \psi^2 w'' = \psi g_1 y_0 + \psi g_2 y'_0. \]  

(20)

Substituting (15), (18) and (20) into (14) we find that

\[ (G_2 g_1 + G_1 f_1 + G_0 p_0) y_0 + \psi (G_2 g_2 + G_1 f_2 + G_0 p_1) y'_0 = 0 \]

and writing

\[
\begin{align*}
G_2 g_1 + G_1 f_1 + G_0 p_0 &= H_0, \\
G_2 g_2 + G_1 f_2 + G_0 p_1 &= H_1,
\end{align*}
\]

(21)

where \( H_0 \) and \( H_1 \) are rational functions of \( z \) and of degree \( 2h - D + 4 \) and \( 2h - D + 2 \) respectively. Thus

\[ H_0 (2h - D + 4) y_0 + \psi H_1 (2h - D + 2) y'_0 = 0. \]  

(22)

If equation (5) is considered as being irreducible, then we must have

\[ ((A)) \quad H_0 = H_1 = 0. \]  

(23)

Heun then argues that since \( H_0, H_1 \) must vanish identically in \( z \), there are \( 2(2h - D) + 8 \) conditions for determining the various parameters. If these \( 2(2h - D) + 8 \) equations are independent of each other, then they are sufficient for the determination of as many unknown quantities. This is in fact the case when not wholly particular conditions are imposed between the branch exponents of the function \( y_0 \).

Heun then discusses the following cases. Firstly he regards as unknowns

z and of degree \( h - D + 4 \) and \( h - D + 2 \) respectively.
(i) $2(h-d)-1$ coefficients of the $F_0$ and $F_1$,
(ii) $2h + 8$ coefficients in the functions $G_0$ and $G_1$,
(iii) the $h$ roots of the equation $G_2(h) = 0$,
which can be known as a consequence of the individual parameters of the equation (14).

In order that this number of unknowns is equal to the number of equations in question, the conditions must become

$$5h - 2D + 7 = 4h - 2D + 8,$$

that is

$$h = 1.$$  

Secondly Heun considers

(i) $2(h - d) - 1$ coefficients in the functions $F_0$, $F_1$;
(ii) the $8$ coefficients in the functions $F_0$, $F_1$;
(iii) $2h + 1$ coefficients in the functions $G_0$, $G_1$,

which are not determined by the exponents $\lambda'_{p1}$ ($p = 1, 2$; $i = 1, 2, 3, 4$).

The number of unknowns is thus equal to the number of equations which come from (23) and whose value depends on $h$. These equations are generally independent from each other.

Lastly Heun considers the case where $h = 0$, then

$$\sum_{1=1}^{4} (\lambda'_{11} + \lambda'_{21}) = \sum_{1=1}^{4} (\lambda_{11} + \lambda_{21}) = 2.$$  \hspace{1cm} (24)
(i) \(-2D - 1\) coefficients in the functions \(P_0\) and \(p_1\);

(ii) 8 coefficients in the functions \(G_0\) and \(G_1\);

(iii) the accessory parameter in the function \(F_0\).

- \(2D + 8\) unknowns can be determined by as many equations which are generally independent from one another.

As an example, Heun illustrates the method in the special case of reduction from \(Hu(a, q'; \alpha, \beta, \gamma + 1, \delta + 1; z)\) to \(Hu(a, q; \alpha, \beta, \gamma, \delta; z)\) and its derivative. In this case \(\delta_1 = -1, \delta_2 = -1, \delta_3 = \delta_4 = 0\) and so \(D = -2\). Also since \(h = 0\), \(p_0\) is of degree two while \(p_1\) is merely a constant.

Heun therefore puts

\[
W = \phi \, Hu(q, z) + \psi \, Hu'(q, z),
\]

where

\[
\phi = P_0 + p_1 z + p_2 z^2, \quad P_0, \, p_1, \, p_2 \quad \text{to be determined.}
\]

He then supposes \(w\) to satisfy an equation of the form

\[
z(z-1) \psi \frac{d^2w}{dz^2} + z(z-1) \theta \frac{dw}{dz} + \sigma w = 0
\]

and from the equations

\[
(8) \quad P_1(z) = (1-\lambda_{11}-\lambda_{21})(z-1)(z-a)+(1-\lambda_{12}-\lambda_{22})z(z-a) +
\]

\[
+ (1-\lambda_{13}-\lambda_{23})z(z-1),
\]
\[ F_0(z) = -[\lambda_{11}(z-1)(z-a)+\lambda_{12}z(z-a)+\lambda_{13}z(z-1)]^2 + \]
\[ + \lambda_{11}(z-1)^2(z-a)^2+\lambda_{12}^2z^2(z-a)^2+\lambda_{13}^2z^2(z-1)^2 - \]
\[ - [\lambda_{11}(z-1)(z-a)+\lambda_{12}z(z-a)+\lambda_{13}z(z-1)]F_1(z) + \]
\[ + [\lambda_{14}+\sum_{i=1}^{3}\lambda_{1i}][\lambda_{2i}+\sum_{i=1}^{3}\lambda_{1i}]\psi(z-q) \]

he finds that

\[ \theta = (y-1)(z-1)(z-a)+(\delta-1)z(z-a)+(\alpha+\beta+\gamma+\delta-1)z(z-1) \]
\[ \sigma = -2\psi - (2z-1) \theta + \alpha\beta z(z-1)(z-q^*) \]  

The function \( \psi_0 = H_{\alpha}(a, q; \alpha, \beta, y, \delta; z) \) satisfies the equation

\[ ((12)) z(z-1)(z-a) \frac{d^2\psi_0}{dz^2} + [(\alpha+\beta+1)z^2 - (\alpha+\gamma+\delta+1)z + \]
\[ + ay] \frac{d\psi_0}{dz} + \alpha\beta(z-q)\psi_0 = 0, \]

which Heun writes as

\[ \psi \frac{d^2\psi_0}{dz^2} + \chi \frac{d\psi_0}{dz} + \tilde{\omega}\psi_0 = 0, \]

where

\[ \chi = ay-[\alpha+\beta+\gamma+1(a-1)\delta+1]z+(1+\alpha+\beta+1)z^2, \]
\[ \tilde{\omega} = \alpha\beta(z-q). \]

Equations (23) then become

\[ [\sigma-z(z-1)\tilde{\omega}]\phi+z(z-1)[\theta\phi'+\psi\psi'] - z(z-1)[(\psi\tilde{\omega})'+(\theta-\chi)\tilde{\omega}] = 0, \]
\[ (\theta-\chi)\phi+2\psi\phi'+(\theta-\chi)(\psi'+\psi)+(\psi''-\chi'-\tilde{\omega})+(z-a)\sigma = 0. \]

These equations must be identically zero for all \( z \), and thus by equating the coefficients to zero, Heun, after much simplification and heavy algebra, finds that
\[ p_0 = 0, \quad p_1 = -p_2 = y + \delta - \alpha - \beta \]

\[ q = a + \frac{(y + \delta - \alpha - \beta)(a-1)(y+\alpha\delta)}{\alpha\beta} \]

\[ q' = a + \frac{(y + \delta - \alpha - \beta + 1)(a-1)(y+a+\delta)}{\alpha\beta}. \]

If we write \( H_u(a, q; \alpha, \beta, y, \delta; z) = H_u(q; z) \) then the reduction equation takes the form

\[ m H_u(a, q'; \alpha, \beta, y+1, \delta+1; z) = (a+\beta-y-\delta)H_u(q; z) + (z-a)H_u(q; z), \]

where \( m \) is a constant. If \( z = 0 \), then the above equation shows that

\[ m = (a + \beta - y - \delta) - \frac{\alpha\beta a}{y}, \]

and thus finally

\[ [y(a+\beta-y-\delta)-\alpha\beta a]H_u(a, q'; \alpha, \beta, y+1, \delta+1; z) = \]

\[ = y(a+\beta-y-\delta)H_u(q; z) + y(z-a)H_u'(q; z). \]

The values of \( q \) and \( q' \) from ((c)), ((d)) are put in this relation. Moreover, the condition \( a > 1 \) is to be considered a necessary one. Only in the special case \( q = 1, \delta = 0 \), does \( a = 1 \). The relation above then gives the well known Gaussian relation between hypergeometric rows, namely

\[ (y-\alpha)(y-\beta) \quad _2F_1 \quad (\alpha, \beta; y+1; z) = \]

\[ = y(y-\alpha-\beta) \quad _2F_1 \quad (\alpha, \beta; y; z-1) - \alpha\beta(z-1) \quad _2F_1 \quad (\alpha+1, \beta+1; y+1; z). \]

Heun then mentions that the above example is sufficient to justify the assumption that the equations (23) are generally independent of each other. From the results
obtained from the conditions (23), Heun concludes with the following propositions

If the function \( H_u(a, q'; \alpha', \beta', \gamma', \delta'; z) \) can be expressed by the function \( H_u(a, q; \alpha, \beta, \gamma, \delta; z) \) and its first derivative, it is not sufficient that the differences \((\alpha' - \alpha, \beta' - \beta, \gamma' - \gamma, \delta' - \delta)\) be whole numbers, but the characteristic parameters \( q \) and \( q' \), must be calculated functions of \( a, \alpha, \beta, \gamma, \) and \( \delta \). On the other hand, if the Riemannian function whose differential equation is of the form

\[
(z-\rho)(z-1)(z-a)z \frac{d^2y}{dz^2} + G_1(3) \frac{dy}{dz} + G_2(5)y = 0
\]

is sufficient, then it can be generally expressed rationally by the function \( H_u(a, q; \alpha, \beta, \gamma, \delta; z) \) and its first derivative, provided the individual parameter \( \rho \) which must not coincide with a branch point, is suitably determined. The characteristic parameter \( q \) can thus be arbitrarily adopted.

A second order Riemannian function with four branch points and any arbitrary number of individual parameters, can be rationally expressed in terms of the function \( H_u(a, q; \alpha, \beta, \gamma, \delta; z) \) and its first derivative; in which case \( q \) is determined.

If equations are proposed for the functions investigated here, and which correspond to the Gaussian relations between contiguous functions then one individual
parameter must be given to at least two of the same, and the value of the parameter cannot be arbitrarily adopted. Otherwise the exponents of the branch points must suffice to determine the conditions. Thus these functions can be determined intrinsically from the hypergeometry. This discrepancy can be expressed as:

All second order Riemannian functions with three branch points can be expressed by the hypergeometric function $2F_1(\alpha, \beta; \gamma; z)$. Functions of the same order with four branch points can be expressed by the function $Hu(a, q; \alpha, \beta, \gamma, \delta; z)$ and its first derivative.
References


