"Sampled-data closed-loop control of linear systems having a variable time delay."

By Mohamed Mahmoud Ali Abou Eid,

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Department of Electrical Engineering Battersea College of Technology

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ABSTRACT

This work deals with sampled-data control of linear systems incorporating variable time delays.

It was found that such systems can be satisfactorily controlled according to two proposed specifications, in which the transient duration can either be dependent upon or independent of the delay incorporated in the system.

General time domain matrix treatment is applied, since to achieve the delay dependent response in particular, it is necessary to provide a controller with adjustable-width output pulses.

The discrete controller used, is designed according to a certain design procedure called the forward loop compensation method.
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Table of contents.

Chapter 1, INTRODUCTION 6

Chapter 2, GENERAL ANALYSIS METHODS. 10

2.1 The Sampling Process 10
2.2 Frequency Domain Analysis 13
2.3 Z-Transform Method 17
2.4 Response Between Sampling Instants 24
2.5 Stability of Sampled-Data Systems in The Z-Domain 33
2.6 Systems with Appreciable Pulse Width 43
2.7 Systems with Pure Time Delay 50
2.8 Time Domain Analysis 55
2.8.1 Conventional sampling time domain analysis 56
2.8.2 General time domain analysis 62

Chapter 3, GENERAL SYNTHESIS PROCEDURES 72

i)Continuous Compensation Methods 73
3.1 Frequency Response in The S-Plane 73
3.2 Frequency Response in The Z-Plane 75
3.3 Root-Locus Method 76
3.4 Time Domain Specifications 80

ii)Discrete Compensation Methods 82
3.5 Conventional Techniques of Continuous Systems 82
3.6 Z-Transform Methods
3.6.1 Characteristic equation approach
3.6.2 Minimal prototype approach
3.6.3 Ripple free response approach
3.7 Time Domain Matrix Method
3.8 Synthesis of Systems Incorporating Variable Time Delay
3.8.1 Controller forward loop compensation
3.8.2 Delay dependent settling time
3.8.3 Delay independent settling time
3.9 Illustrative Example

Chapter 4, ANALOGUE COMPUTER SIMULATION
4.1 The Discrete Controller
4.1.1 Sample-and-hold circuit
4.1.2 Pulse generating devices
4.1.3 Special Auxiliary circuits
4.2 Pure Time Delay Simulation
4.2.1 Speed controlled motor drive
4.2.2 Modulation device
4.2.3 Demodulation device
4.2.4 Effect of speed variations
4.2.5 Suggestions for improvements
4.3 Simulation Set Up and Results
4.3.1 Forward loop synthesis simulation 175
4.3.2 Delay dependent settling time simulation 177
4.3.3 Delay independent settling time simulation 179
4.3.4 Simulation of a system without delay 179

Chapter 5, CONCLUSIONS 183

BIBIOLOGRAPHY 185

APPENDIX 193
CHAPTER ONE

INTRODUCTION

Sampled-data systems may be defined as those in which variables appear at one or more points as pulses or sequence of numbers. In contrast, continuous systems have variables available at all instants of time.

The applications of sampled-data systems can be broadly divided into

(a) Systems in which variables are available only intermittently, e.g. radar tracking systems and digital computers in control systems.

(b) Systems in which sampling is purposely introduced, e.g. the field of process control, where more economical use of measuring devices can be realized, or when several processes are controlled by one digital controller on a time sharing basis.

Recently, the field of digital computers, which are essentially sampled-data systems, has experienced a rapid growth. They provide high accuracy, as well as high computational speeds, flexibility and versatility which analogue computation cannot generally provide.

Nowadays digital computers are not restricted to scientific
laboratories, but play a major role in many large business concerns. There is also a trend of introducing digital controllers to control large processes, e.g. nuclear reactors and other chemical plants, where many variables have to be monitored and utilized for control purposes.

In this work the basic analytic methods of sampled-data systems are surveyed in Chapter 2. These include methods applying conventional sampling, e.g. the Z-transform method and the time domain matrix method. The survey also covers the transform methods used for treating appreciable finite sampling duration systems, e.g. the P-transform method.

A method that can be used for a general treatment, e.g. variable sampling duration and/or variable sampling rate, is developed in section 2.8.2. working directly in the time domain.

Chapter 3 describes the basic techniques for continuous and discrete compensation methods. The importance of the time domain treatment is emphasized, which is particularly useful for systems incorporating variable time delays.

For such systems two design specifications are presented and worked out in section 3.8.

One approach specifies the output transient duration to be dependent upon the delay, where the sampling period is made proportional to the delay, and the output sequence (at sampling instants) independent of it. (Section 3.8.2).
The other approach specifies the transient duration to be independent of the delay, where the sampling period is fixed, and the response has the same shape irrespective of the delay. (Section 3.8.3.)

The two specifications are satisfied by following a given controller design procedure, called the forward loop compensation method, where the discrete controller forward loop shapes the response, and its feedback loop neutralizes the effect of the feedback output response. (Section 3.8.1.)

Some pieces of equipments, designed and constructed to simulate a sampled-data system incorporating variable time delay are considered in Chapter Four.

There are two major equipment items:

(a) The Sampling Equipment, which in conjunction with operational amplifiers and other conventional analogue computer elements simulates a discrete controller.

(b) The Time Delay Simulator, which is a magnetic tape unit applying pulse width modulation. It can simulate continuously variable delays, by varying the speed of the tape across the two fixed magnetic heads.

The agreement between the computed and simulated results was found to be satisfactory within the accuracy limits of the components
used in the simulation. This demonstrates the validity of the synthesis procedures presented here.

It is hoped that this work will provide a useful contribution to the field of sampled-data control systems.
CHAPTER TWO
GENERAL ANALYSIS METHODS

2.1. THE SAMPLING PROCESS

This process converts a continuously varying signal into a train of pulses. It can be expressed mathematically in terms of the sampler characteristics, and the sampled function.

Considering a regular ideal sampler, its time function is a delta function
\[ \delta_T(t) = \sum_{n=\infty}^{\infty} \delta(t - nT) \] (2.1.1)
representing a train of impulses of unit strength.

The output of the sampler \( e^*(t) \) for a given input \( e(t) \) is
\[ e^*(t) = e(t) \delta_T(t) \] (2.1.2)

Assuming \( e(t) \) is Laplace transformable, the L.T. of \( e^*(t) \) can be found by real multiplication or the complex convolution theorem. *

\[ L(e^*(t)) = L(e(t) \delta_T(t)) \]
\[ = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(\lambda) G(s-\lambda) \, d\lambda \] (2.1.3)

* Reference 32, p. 275
where \[ E(s) = L(e(t)) \]

and \[ G(s) = L(\delta_T(t)) \]

The real multiplication can be carried out on the assumption that the samplers start at \( t = 0 \).

C is so chosen that all the poles of \( E(x) \) lie to the left of the imaginary axis displaced by \( C \).

Now \( L(\delta_T(t)) \)
\[ = 1 + e^{-sT} + e^{-2sT} + \ldots \]
\[ = \frac{1}{1 - e^{-sT}} \quad (2.1.4) \]

for \( |e^{-sT}| < 1 \)

Hence (2.1.3) can be written as
\[ L(e^*(t)) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{E(x)}{1 - e^{-(s-x)T}} \, ds \quad (2.1.5) \]

The last equation can be evaluated using contour integration, either in a clockwise or anticlockwise direction, as indicated in fig. (2.1.2) subject to uniform convergence of the integral as \( \text{radius} \rightarrow \infty \).

Cauchy's residue formula gives the line integral as the sum of the residues of the integrand at the poles enclosed within. The sign of the sum depends on the direction of contour integration.

Evaluating the integral in a clockwise direction,
\[ E^*(s) = L(e^*(t)) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(x) \frac{1}{1 - e^{-(s-x)T}} \, ds \quad (2.1.6) \]

(11)
The poles enclosed within this contour are those of

\[ 1 - e^{-(s-\gamma)T} = 0 \]

or \( \gamma = s + j \frac{2\pi}{T} \) \( k = s + jkw_s \) \hspace{1cm} (2.1.7)

\( k \) an integer \( (-\infty < k < \infty) \) and \( \omega_s \), the sampling frequency

\[ E^*(s) = - \sum_{k=-\infty}^{\infty} \frac{E(\gamma)}{d\gamma} \left( 1 - e^{-(s-\gamma)T} \right) \bigg|_{\gamma = s+jkw_s} \] \hspace{1cm} (2.1.8)

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s+jkw_s) \] \hspace{1cm} (2.1.9)

Evaluating the integrand in an anticlockwise direction

\[ E^*(s) = \frac{1}{2\pi j} \oint \frac{E(\gamma)}{1-e^{-(s-\gamma)T}} d\gamma \]

\[ = \sum_{\text{poles of } E(\gamma)} \text{Residues of } \frac{E(\gamma)}{1-e^{-(s-\gamma)T}} \] \hspace{1cm} (2.1.10)

The last two equations are equivalent, but the former is an infinite series, while the latter is expressed in a closed form.

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* Reference 32. P. 277.
2.2. **FREQUENCY DOMAIN ANALYSIS.**

Equation (2.1.9) demonstrates the effect of sampling upon the frequency spectrum of the sampler output:

\[
E^*(s) = \frac{1}{T} \sum_{k = -\infty}^{\infty} E(s + jkw)
\]  

(2.2.1)

\( E^*(s) \) is a periodic function with period \( jws \). This can be further explained by considering the sampler as amplitude modulation of the train of impulses by the input signal *. This is why the sampling frequency should be at least twice the highest input frequency, so that the information can be recovered by linear filtering.

A zero order hold circuit usually follows the sampler, acting as a linear filter. The presence of such a circuit is also necessary for conventional sampling. This is due to the fact that the sampler output pulses are very short and do not carry sufficient energy to actuate the following system components.

The zero order hold circuit, holds the sampler output constant between any two successive sampling instants.

Its L.T. can be described by considering its action as composed of a unit step, followed by another negative one, one sampling period later:

\[
G(s) = \frac{1}{S} - \frac{e^{-Ts}}{S} = \frac{1 - e^{-Ts}}{S}
\]  

(2.2.2)

Its frequency response is a low pass filter with full cut off at \( \frac{n}{T} \) c/s, where \( n \) is an integer. +

---

Transfer function of Sampled-Data System

Consider the system of

\[ X(s) \xrightarrow{G(s)} Y(s) \]

It can readily be seen that \( Y(s) = X^*(s) \cdot G(s) \) (2.2.3)

which follows from the properties of the L.T.

If the output is required at the sampling instants only a fictitious sampler is inserted at the output and from (2.2.1)

\[ Y^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X^*(s+jkw_s) \cdot G(s+jkw_s) \] (2.2.4)

but \( X^*(s+jkw_s) \) is periodic, so that

\[ X^*(s+jkw_s) = X^*(s) \] (2.2.5)

Thus \( Y^*(s) = X^*(s) \cdot \sum_{k=-\infty}^{\infty} G(s+jkw_s) \) (2.2.6)

using equation 2.2.1.

hence \( Y^*(s) = X^*(s) \cdot G^*(s) \) (2.2.7)

or \( G^*(s) = \frac{Y^*(s)}{X^*(s)} \) (2.2.8)

\( G^*(s) \) is known as the starred transfer function, or the transfer function relating the starred input and output.

As an example, consider the closed loop system shown in figure 2.2.2a, which is

redrawn as in figure 2.2.2b.

\[ C(s) = E^*(s) \cdot G(s) \]

\[ C^*(s) = E^*(s) \cdot G^*(s) \]

also

\[ E^*(s) = R^*(s) - E^*(s) \cdot G^*(s) \]

Figure 2.2.2a

Figure 2.2.2b.
Where $G H^*(s)$ is the starred transform of $G(s) H(s)$.

Eliminating $E^*(s)$ from the above equations gives

$$C(s) = \frac{R^*(s)}{1+G H^*(s)} G(s)$$

and

$$C^*(s) = \frac{R^*(s)}{1+G H^*(s)} G^*(s)$$

It can be seen that the exact expression in the frequency domain or S-plane, consists of an infinite series, which means that the inverse time function is also an infinite series in time. However, due to the fact that most controlled systems have low pass filter characteristics a few terms of the series are sufficient to describe the behaviour of the system with reasonable accuracy. Bearing this in mind, the stability analysis can be considered.

**Stability of Sampled-data Systems in the frequency domain.**

As for continuous data systems, the necessary and sufficient condition for stability is that the starred transfer function of the systems does not have any poles in the right half of the S-plane.

Consider the closed loop system above:

$$C(s) = \frac{R^*(s)}{1+G H^*(s)} G(s) \quad (2.2.9)$$

Its characteristic equation is

$$1 + G H^*(s) = 0 \quad (2.2.10)$$

but

$$G H^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G H^*(s+jkw)$$

As already stated it is reasonable to retain only the terms with $k=0$ and $k = \pm 1$.

$$1 + G H^*(s) = 1 + \frac{1}{T} (G H(s) + G H(s+jw_0) + G H(s-jw_0)) = 0 \quad (2.2.11)$$

The Routh criterion can be applied to the last equation, with a slight modification due to the presence of complex coefficients. *

Also Nyquist criterion can be applied by plotting the approximate open loop transfer function

\[ G H^*(s) = \frac{1}{T} (G H(s) + G H(s+jw) + G H(s-jw)) \quad s = jw. \quad (2.2.12) \]

Then the usual test for enclosure of the point (-1,0) on the S-plane can be applied, just as with continuous feedback systems.

The main advantage of frequency or S-plane treatment is that it utilizes the conventional techniques developed for continuous systems. Its main disadvantage is the high degree polynomials involved. Also, the degree of approximation may be in doubt unless checked by other exact methods.

* Reference 109, p 526.
2.3. Z - TRANSFORM METHOD

Another expression for the output of the sampler was derived in section 2.1, that is

\[ E^*(s) = \sum \text{Residues of } \frac{E(\cdot)}{E(\cdot)} \frac{E(s)}{1-e^{-Ts}} \]  \hspace{1cm} (2.3.1)

In the last equation \( E^*(s) \) is a function of \( e^{-TS} \) only while other terms are constants. For notational convenience \( e^{-TS} \) is replaced by \( z \).* This transforms \( E^*(s) \) into a rational function in \( z \), known as the z-transform + \( E(z) \).

Thus the z-transform can be considered as a special case of the Laplace Transform. The last equation can be rewritten as

\[ E(z) = \sum \text{Residues of } \frac{E(s)}{1-e^{-Ts}z^{-1}} \]  \hspace{1cm} (2.3.2)

The evaluation of the residue at the singularities of \( E(s) \) depends upon the form of \( E(s) \). For

\[ E(s) = \frac{A(s)}{B(s)}, \text{ having } Q \text{ first order poles} \]

\[ E(z) = \sum_{q=1}^{Q} \frac{A(sq)}{B'(sq)} \frac{1}{1-e^{-T(s-sq)}} \]  \hspace{1cm} (2.3.3)**+

where \( B'(sq) = \frac{d}{ds} B(s) \bigg|_{s = sq} \text{ a typical pole} \)

For multiple poles of order \( m \), say \( E(s) = \frac{1}{(s+q)^m} \)

\[ E(z) = (-1)^{m-1} \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial a^{m-1}} \frac{1}{1-e^{-aT}} \]  \hspace{1cm} (2.3.4)

It should be recalled that for a system whose L.T. is a finite polynomial in \( s \), the corresponding z-transform is also a finite

polynomial in $z$. Also the degree of the denominator of $E(z)$ in $z^{-1}$ is the same as the degree of the denominator of $E(s)$ in $s$.

$E(z)$ is usually known as the pulse transfer function.\

Writing equation (2.3.2) as

$$E(z) = \frac{P(z)}{Q(z)}$$

(2.3.5.)

where $P(z)$ and $Q(z)$ are finite polynomials in $z$ and carrying out

synthetic division, gives

$$E(z) = E(0) + E(1) z^{-1} + E(2) z^{-2} + \cdots$$

$$= \sum_{n=0}^{\infty} E(n) z^{-n}$$

(2.3.6)

The time inversion of 2.3.6 yields

$$e^*(t) = E(0) \delta(t) + E(1) \delta(t-T) + E(2) \delta(t-2T) + \cdots$$

$$= \sum_{n=0}^{\infty} E(n) \delta(t-nT)$$

(2.3.7)

$E(n)$ represents the sampler output at the $n^{th}$ sampling interval.

(2.3.6) is another way of expressing the z-transform as an infinite

series.

But $E(n)$ for the sampling process,

$$E(n) = e(nT)$$

(2.3.8)

Therefore equations (2.3.6) and (2.3.7) can be rewritten as

$$E(z) = \sum_{n=0}^{\infty} e(nT) z^{-n}$$

(2.3.9)

and $e^*(t) = \sum_{n=0}^{\infty} e(nT) \delta(t-nT)$

(2.3.10)

Pulse Transfer Function

Consider the system shown in

fig. 2.3.1, where $g(t)$ is the

impulsive response of the system.

Reference 2

Fig. 2.3.1
\( e^*(t) \) can be considered as a train of impulses of strength \( e(nT) \).

A particular component of the output is the impulsive response from the \( n^{th} \) sample \( e(nT) \). Its corresponding output at the \( m^{th} \) instant, \( m > n \) is given by,

\[
C_n(mT) = e(nT) \, g(m-n)T
\]  

(2.3.11)

\( g(m-n)T \) is the impulsive response after time \( (m-n)T \).

The total output is found by summation:

\[
C(mT) = \sum_{n=0}^{m} e(nT) \, g(m-n)T
\]  

(2.3.12)

As the impulsive response is zero for \( n > m \), the upper limit of summation can be extended to infinity,

\[
C(mT) = \sum_{n=0}^{\infty} e(nT) \, g(m-n)T
\]  

(2.3.13)

The last summation is known as the convolution summation.*

From equation (2.3.10)

\[
C^*(t) = \sum_{m=0}^{\infty} C(mT) \, \delta(t-mT)
\]  

(2.3.14)

substituting (2.3.13) into (2.3.14) gives

\[
C^*(t) = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} e(nT) \, g(m-n)T \right) \, \delta(t-mT)
\]

and interchanging the summation signs

\[
C^*(t) = \sum_{n=0}^{\infty} e(nT) \, \sum_{m=0}^{\infty} g(m-n)T \, \delta(t-mT)
\]  

(2.3.15)

putting \( k = m-n \)

\[
C^*(t) = \sum_{n=0}^{\infty} e(nT) \, \sum_{k=0}^{\infty} g(kT) \, \delta(t-nT-kT)
\]  

(2.3.16)

Taking the L.T. of both sides of (2.3.16) and considering \( g(kT) = 0 \) for \( k < 0 \) gives

\[
L\left( C^*(t) \right) = \sum_{n=0}^{\infty} e(nT) \sum_{k=0}^{\infty} g(kT) \, e^{-nTs} \, e^{-kTs}
\]

* Reference 86 pp 66 - 69

(19)
\[
E(z) = R(z) - E(z) G(H(z))
\]

\[
C(z) = E(z) G(z)
\]

Eliminating \(E(z)\) gives,

\[
E(z) = 1
\]

*Reference 105, p. 149.*

(20)
\[ C(z) = \frac{R(z)G(z)}{1 + GH(z)} \]

so \[ \frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)} \], the overall pulse transfer function

Whereas for the system shown in fig. 2.3.2, the overall pulse transfer function is found to be

\[ \frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)} \]

Inversion of z-Transformation

This means the evaluation of the system output in the time domain from the z-domain. Two methods are used to effect this

(a) The residue method

\[ C(z) \text{ can be expressed as an infinite series (from (2.3.9))}; \]

\[ C(z) = \sum_{n=0}^{\infty} C(nT) z^{-n} = C(0) + C(T) z^{-1} + C(2T) z^{-2} + \ldots \]

\[ + C(nT) z^{-n} + \ldots \] (2.3.19)

The inversion requires the evaluation of \( C(nT) \)

Multiplying (2.3.19) by \( z^{(n-1)} \)

\[ C(z) z^{n-1} = C(0) z^{n-1} + C(T) z^{n-2} + \ldots \]

\[ + C(nT) z^{-1} + \ldots \] (2.3.20)

The last equation is a Laurent expansion around \( z = 0 \).

The principal part is an infinite series

\[ C(nT) z^{-1} + C(n+1)T z^{-2} + \ldots \]

(21)
The residue of $C(z)z^{m-1}$ at the pole $z = 0$, is the coefficient of $z^{m-1}$ in the expansion. In other words,

$$C(nT) = \frac{1}{2\pi j} \oint C(z) z^{n-1} \, dz$$

where the integration is over a contour enclosing both the origin and all the singularities of $C(z) z^{n-1}$ or the poles of $C(z)$.*

It will be seen later that for a stable system, all the poles of the $z$-transform must lie within a unit circle centered at the origin. Therefore this unit circle is taken as the contour of integration.

This inversion formula, which can be evaluated by the residue method, is useful, if $C(nT)$ is required in a closed form. $C(z)$ is preferably expanded into partial fractions (if possible) and each term is considered. This demonstrates the effect of each pole of $C(z)$ on the output.

(b) Power Series Method

Eq. (2.3.19) can be inverted directly to give

$$C^*(t) = \sum_{n=0}^{\infty} C(nT) \delta(t-nT)$$

(2.3.22)

where $C(nT)$ represents the output at the $n^{th}$ sampling instant. The terms $C(nT)$ are obtained by expanding $C(z)$ into a power series in $z^{n-1}$ by synthetic division.

This method is direct and less involved. It is more frequently used rather than the residue method. It is of advantage if the initial behaviour of the system is of interest.

The output coefficients $C(nT)$ can be obtained from $C(z)$ as follows:

*Reference 86, p. 59.*
The relations between time functions and their corresponding z-transform may also be tabulated on similar lines to the Laplace transform pairs. These tables can be used for convenience and quick results.

It should be pointed out that the above inversion methods yield the output at sampling instants only. This should be adequate for properly designed systems.

However, it may be necessary to know the system response between sampling instants, or to consider the continuous output rather than the discrete form. The methods of finding out the response between sampling instants are considered in the next section.

---

* Reference 49, p. 29.
2.4. RESPONSE BETWEEN SAMPLING INSTANTS

It can be studied using the following approaches:

(a) Multirate Sampling.
(b) z-Transform and Impulsive response.
(c) Real convolution
(d) Modified z-Transform.

(a) Multi-rate Sampling Technique

This technique is based upon introducing a fictitious sampler of \( n \) times the actual sampling frequency, where \( n \) is a positive integer.

By evaluating the response at the new sampling instants more output informations can be obtained.*

Consider the system of figure 2.4.1.

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
\text{T} \\
\downarrow \\
\text{E(z)} \\
\downarrow \\
\text{E(z)} \\
\downarrow \\
\text{G(s)} \\
\downarrow \\
\text{C(z)} \\
\downarrow \\
\text{C(z)} \\
\end{array}
\]

A fictitious sampler is inserted after the actual sampler.

Fig. (2.4.1)

The two samplers close simultaneously every \( n \times \frac{T}{n} \) sec.

Another variable corresponding to the fictitious sampler is introduced.

\[
z_n = e^{\frac{T}{n} s} \quad \text{hence} \quad z = e^{Ts} = (z_n)^n \quad (2.4.1)
\]

The error sequences \( E(z) \) and \( E(z_n) \) must be the same, hence \( E(z_n) \) is found by substituting the above relation between \( z \) and \( z_n \) into the expression for \( E(z) \), or \( E(z_n) = E(z_n^n) \). This

* References (67a); (109) pp. 515 - 516 and (61).
relation can also be proved by writing

\[ E(z) = E_0 + E_1 z^{-1} + E_2 z^{-2} + \]
\[ = E_0 + 0 z_n^{-1} + 0 z_n^{-2} + \cdots + E_n z_n^{-n} + \cdots + E_{2n} z_n^{-2n} + \]
\[ = E(z^n), \text{ which can be designated} \]
\[ E(z^n), \text{ or the z-transform referred to} \]
\[ \text{a period } \frac{T}{n}. \]

Hence \( E(z^n) = E(z^n) \) \hspace{1cm} (2.4.2)

But \( E(z) = \frac{R(z)}{1 + G(z)} \)

Thus \( E(z^n) = \frac{R(z^n)}{1 + G(z^n)} \) \hspace{1cm} (2.4.3)

Hence \( C(z^n) = \frac{R(z^n)}{1 + G(z^n)} \cdot G(z^n) \) \hspace{1cm} (2.4.4)

Again \( G(z^n) \) is the z-transform for a period \( \frac{T}{n} \)

Inverting \( C(z^n) \) by one of the methods mentioned in the preceding section, yields the output at \( (n - 1) \) additional instants between two original sampling instants. By varying \( n \), more information can be obtained, but usually \( n = 2 \) and \( 4 \) are quite adequate.

This method can be used for analysis as well as for checking the final results of design.

(b) **z-Transform and Impulsive Response Method**, or the partial-fraction expansion technique.* It may be outlined as follows:

The component directly preceding the output, and preceded by a sampler, is split into simple partial fractions. The output sequence of the sampler preceding this component is then applied to each partial

*Reference (96a).
fraction, as an impulse, and the sum of the impulsive responses is the continuous output between, say the $n$th and $(n+1)$th sampling instants.

Consider the system shown in fig. 2.4.2.

The error sequence $E_1(z)$ can be proved to be

$$E_1(z) = \frac{R(z)}{1 + G_1(z)G_2H(z)}$$

and

$$E_2(z) = \frac{R(z)G_1(z)}{1 + G_1(z)G_2H(z)}$$

(2.4.5)

$G_2(s)$ can be expanded into partial fractions,

$$G_2(s) = \frac{A}{s+a} + \frac{B}{s+b} + \ldots + \frac{A'}{s^{n+1}} + \frac{B'}{s^n} + \ldots$$

(2.4.6)

The block diagram can then be redrawn, as in fig. (2.4.3)

The output due to branch A say is found by inversion of
\[ C_A(z) = \frac{R(z) G(z)}{1 + G_1(z) G_2 H(z)} \]  
\[ G_A(z) = Z \left( \frac{A}{s + a} \right) \]

If the inversion is in an infinite series:
\[ C_A(t) = C_0 \delta(t) + C_1 \delta(t-T) + \ldots + C_n \delta(t-nT) + \ldots \] (2.4.8)
then the continuous output between \( nT \) & \( (n+1)T \) for this simple pole \( \left( \frac{A}{s + a} \right) \) is:
\[ C(t) = C_n e^{-a(T-nT)} \] (2.4.9)

For branches containing a simple pole at the origin \( \frac{1}{s} \),
the response between \( nT \) & \( (n+1)T \) is a step:
\[ C(t) = C_n \delta(T-t) \] (2.4.10)

For higher order poles at the origin \( \frac{A}{s^m} \) the time response is:
\[ C_{a,t}(t) = C_{a,n} \frac{(t-nT)^{m+1}}{(m+1)!} \] (2.4.11)

The total output between any two sampling instants can then be found by summing the individual outputs.

This method is straightforward, but sometimes tedious. It has the advantage of demonstrating the effect of each pole location on the output response, and it is also an exact method. It may also be used for checking the design of a certain system.

(c) Real Convolution Method.

For the system shown in fig. 2.4.4.
\[ C(s) = E*(s) G(s) \] (2.4.12)

---

Reference 86, p. 212.

(27)
the corresponding \( C(t) \) may be found from the inverse Laplace transform formula.

\[
C(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} C(s) e^{st} \, ds \quad (2.4.13)
\]

or the convolution integral

\[
C(t) = \int_{0}^{t} e^{-(t-x)} g(t-x) \, dx
\]

as the error is intermittent

\[
C(t) = \sum_{n=m}^{n=m} \left[ e^{(nT)} g(t-nT) \right]_{mT \leq t \leq (m+1)T} \quad (2.4.14)
\]

The real convolution or inverse Laplace transform methods are almost of no practical value, due to the difficulties involved in evaluating the integrals and the amount of labour required.

(d) **Modified z-transform Method.**

The z-transform treatment as considered in section 2.4, does not yield any information about the output except at sampling instants. This is a serious shortcoming of the method. However, by modifying the method, the output at any instant can be determined.

To find the output at instants other than sampling instants, a fictitious sampler is introduced at the output terminal. Unlike the case for the z-transform, this sampler may be staggered by an optional interval of time w.r.t. the input sampler and operate at the same frequency.

The same effect is produced by introducing a pure time delay (or advance) \( \Delta T \) as shown in fig. 2.4.5, and operating the two samplers

*Reference 2 and 46.*
in phase.
Here $\Delta T$ represents the amount of staggering required.

It can be seen from fig. 2.4.6 that the delayed sequence at the sampling instants is the sequence at instants $(nT - \Delta T)$ of the actual output.

By varying $\Delta$ the output at any instant can be obtained.

The pulse transfer function of the system with the pure time delay is

$$Z(G(s) e^{-\Delta Ts}) = G(z, \Delta), \text{ say} \quad (2.4.15)$$

From (2.1.5)

$$G(z, \Delta) = \frac{1}{2 \pi j} \oint G(\xi) e^{-\Delta T \xi} \frac{1}{1-e^{-T(s-\xi)}} d\xi \quad (2.4.16)$$

where the contour of integration encloses all the poles of $G(\xi) e^{-\Delta T \xi}$ in the left half of the $\xi$-plane.

The evaluation of the integrand can be simplified by introducing another variable $m$, such that

$$m = 1 - \Delta \quad \text{or} \quad \Delta = 1 - m \quad (2.4.17)$$
The starred transform of \( G(s) e^{-\Delta T s} \), is found from (2.2.1) to be

\[
G^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s+jkw) e^{mT(s+jkw)} e^{-T(s+jkw)}
\]  

(2.4.18)

But \( e^{-T(s+jkw)} = e^{-Ts} \) for all integer \( k \)

Hence \( G^*(s) = e^{-Ts} \left( \sum_{k=-\infty}^{\infty} G(s+jkw) e^{mT(s+jkw)} \right) \)  

(2.4.19)

The quantity between brackets in the last equation is the starred transform of \( G(s) e^{mT} \), then from (2.1.9), (2.1.10) and (2.3.2).

\[
G(z,m) = z^{-1} \sum \frac{G(\chi)}{1-e^{-\Delta T(\chi)}} d\chi
\]

Residues of \( \frac{G(\chi)}{1-e^{-\Delta T(\chi)}} \)  

(2.4.20)

If in the last equation \( m \rightarrow 1 \), then the z-transform is obtained. However this is only true if the response is continuous, otherwise letting \( m \rightarrow 1 \) would give \( C_n(t) \), and letting \( m \rightarrow 0 \) gives \( C_{n+1}(-) \) for discontinuous response. Thus the z-transform is only valid if the impulsive response \( g(t) \) is continuous.

Another expression is found by applying equation (2.3.9)

\[
G(z,m) = z^{-1} \sum_{n=0}^{\infty} g(nT + mT) z^{-n}
\]  

(2.4.21)*

The evaluation of the modified z-transform follows the same lines as the ordinary z-transform.

For closed loop systems, the introduction of a fictitious delay in the forward loop alters the

*Reference 105, p. 187

(30)
overall pulse transfer function. To restore it another fictitious
advance of equal magnitude is introduced in the feedback loop as shown
in fig. 2.4.7. The output sequence $C(z,m)$ is

$$C(z,m) = E(z) G(z,m)$$

but

$$E(z) = \frac{R(z)}{1 + G H(z)}$$

Hence

$$C(z,m) = \frac{R(z)}{1 + G H(z)} G(z,m) \quad (2.4.22)$$

Bearing in mind that the introduced delays are fictitious they can
be eliminated from the block diagram.

As can be seen, the modified z-transform is considered only for
the component between a sampler and the output terminal required.

Following the same procedure the modified z-transform for any
output can be determined.

**Inversion of the Modified z-Transform.**

There are two methods for inversion, which are exactly the same
as for the ordinary z-transform. This is not surprising as the
parameter $m$ generally affects the coefficients of the equation,
but not its order or form.

(a) **The Residue Method.** Its formula can be derived by the
method used for equation (2.3.21), i.e.

$$C(nT,m) = \frac{1}{2\pi j} \oint C(z,m) z^{n-1} \, dz$$

$$= \sum \text{Residues of } C(z,m) z^{n-1} \quad (2.4.23)$$

*Reference (2)*

(31)
The contour of integration encloses all the singularities of \[ C(z,m) \]. This contour is usually taken as the unit circle with the origin as centre on the \( z \)-plane.

(b) **Power Series Method.** The output sequence \( C(z,m) \) can be expressed as an infinite series, using equation (2.4.21):

\[
C(z,m) = z^{-1} \sum_{n=0}^{\infty} C(nT + mT) z^{-n} = z^{-1} \sum_{n=0}^{\infty} C_n(m) z^{-n}
\]

\[ (2.4.24) \]

\( C_n(m) \) denotes an \( m \)-dependent coefficient.

Direct inversion of (2.4.24) gives

\[
C(nT,m) = C_0(m) \delta(t-T) + C_1(m) \delta(t-2T) + \cdots + C_n(m) \delta(t-nT) + \cdots
\]

\[ (2.4.25) \]

\( C_n(m) \) describes the output between \( t = nT \) and \( t = (n+1)T \) when \( m \) is varied from 1 to \( \sigma \).

The modified \( z \)-transform is very useful in both analysis and synthetis. It can be used for other applications in combination with the \( z \)-transform *, such as the summation of some infinite series. +

---

* Reference 50.
+ Reference 54.
2.5. **STABILITY OF SAMPLED-DATA SYSTEMS IN THE Z-DOMAIN**

Stability is a basic requirement for any feedback control system. If a system exhibits a bounded output for a bounded input, then it is stable. This is the general definition of stability.

For sampled-data systems, the output is only available at discrete instants of time, or expressed as a pulse sequence. Therefore, the stability criterion may be stated as:

A sampled-data system is stable if the pulse sequence at its output, in response to a bounded sequence, is bounded.*

The time domain output can be expressed as

\[ C^*(t) = \sum_{n=0}^{\infty} C(nT) \delta(t-nT) \]  
(2.5.1)

This follows from equation (2.3.10).

The necessary and sufficient condition for \( C^*(t) \) to be bounded, is the absolute convergence of the series \( \sum_{n=0}^{\infty} \left| C(nT) \right| < \infty \)  
(2.5.2)

\( C(nT) \) can be found from (2.3.21):

\[ C(nT) = \frac{1}{2\pi j} \oint_{C} C(z) z^{n-1} \, dz \]

\[ = \sum_{q=1}^{Q} \text{Residues of } C(z) z^{n-1} \quad z = z_q \]  
(2.5.3)

where \( z_q \) is a typical pole of \( C(z) \).

For any pole \( z_q \), \( C(nT) \) is bounded, if \( z_q^{n-1} \) is bounded, or

\[ \left| z_q \right| \leq 1 \]  
(2.5.4)

Therefore \( C(z) \) is stable if all its poles lie within a unit

*Reference 86, p. 94.

+Reference 42 pp. 233-234 and 86 pp 95-96
circle with the origin as its centre on the z-plane.

But \( C(z) = R(z) G'(z) \)

where \( G'(z) \) is an overall pulse transfer function of some system, and \( R(z) \) the input pulse transfer function.

As the input is bounded, the poles of \( R(z) \) must lie within the unit circle. Therefore, if the output is bounded the poles of \( G'(z) \) must lie within the unit circle as well. 

This condition can further be related to the stability condition on the S-plane, by considering the conformal mapping of the left half of the S-plane into the Z-plane, using the transform

\[
z = e^{sT} \quad (2.5.5)
\]

The imaginary axis on the S-plane is \( s = jw \mid w = -\infty \) its corresponding contour is

\[ z = e^{jwt} \]

which is a multivalued function, whose amplitude is unity. Fig. 2.5.1.

As \( w \) increases from \( 0 \) to \( \omega_o \) (The primary component), \( z \) traverses a unit circle with the origin as centre on the z-plane. The negative real axis on the s-plane is transformed into the line along the real axis between \( z = 1 \) (\( s = \infty \)) and \( z = C \) (\( s = -\infty \)) on the z-plane.

Therefore the left half of the s-plane is transformed into the inside of the unit circle on the z-plane.

*Reference 42, p. 243; Reference 2; Reference 49, p. 33; & Ref. 86, p. 97
Hence it can be stated, that if \( G(s) \) is stable on the s-plane, i.e. its poles lie to the left of the imaginary axis, then its corresponding \( G(z) \) is stable on the z-plane, as all the poles will lie within the unit circle.

This result also follows by considering the z-transform as a special case of the Laplace transform.

For a typical closed loop system such as that of figure 2.5.2, the pulse transfer function is,

\[
G'(z) = \frac{G(z)}{1 + G H(z)}
\]

This system is stable if all its poles lie within the unit circle on the z-plane. In other words the zeros of \( 1 + G H(z) \) must lie within the unit circle. The equation

\[
1 + G H(z) = 0
\]

is known as the characteristic equation of the system. The stability and the response can be found by the location of its roots on the z-plane.

The z-transform method yields the output only at sampling instants. This is satisfactory from the stability point of view except in some exceptional situations where the system response is bounded at sampling instants but unbounded in between. This is what may be termed a state of hidden oscillations.\(^*\) The frequency of such

\* References 2 and 47

(35)
oscillations being $n f_s$, where $f_s$ is the sampling frequency and $n$ is an integer.

This case can be detected by using the modified z-transform

$$C (z, m) = \frac{R (z)}{1 + G H (z)} G (z, m)$$

If $G (z) = \frac{A (z)}{B (z)}$ then $G (z, m)$ is usually $\frac{A (z, m)}{B (z)}$, except for the case with hidden oscillations where $G (z, m) = \frac{A_1 (z, m)}{B (z)}$.

$B_1 (z)$ contains more roots than $B (z)$. If any of these roots lie outside the unit circle, the system is unstable. This case arises only if $G (s)$ is unstable.

It may be said that to check for absolute stability the modified z-transform should be used, and the roots of the characteristic equation of $C (z, m)$ should lie within the unit circle. This procedure can take care of any hidden oscillations.

As for continuous systems, the stability may be tested either graphically or analytically as follows.

**Graphical Stability Criterion (Transfer Locus)**

To determine graphically whether any of the roots of the characteristic equation lie outside the unit circle about the origin on the z-plane, Cauchy's mapping theorem is used in a similar manner as for deriving Nyquist's Criterion for continuous-data systems.

Cauchy's mapping theorem states that if a closed contour encloses poles and zeros of a function, then the map of the corresponding closed contour on the function plane encircles the origin.

*References 47 and 89.*
a number of times equal to the difference between the number of zeros and number of poles so enclosed.

The characteristic function (2.5.6) is mapped on a \((1 + G H(z))\)-plane, choosing the closed contour to be the entire region outside the unit circle on the z-plane.

\[
\begin{align*}
\text{Z-plane} & \rightarrow \text{(1+GH(z))-plane} & \text{GH(z)-plane}
\end{align*}
\]

\[\text{Fig. 2.5.3.}\]

It is noticed that as \(z \rightarrow \infty\) \((1 + G H(z)) \rightarrow 1\), and the mapping of the region outside the unit circle, is essentially the mapping of the unit circle on the \((1 + G H(z))\)-plane.

The number of encirclements of the origin by the contour \(1 + G H(z)\), as \(z\) traverses the unit circle, equals the difference between the number of zeros and number of poles of \((1 + G H(z))\) on the z-plane.

The \(G H(z)\) locus is the same as that of \((1 + G H(z))\), except that the origin is to be shifted by unity to the right as shown in fig. 2.5.3.

Hence by plotting the open loop pulse transfer locus $G H(z)$
\[ z = j\omega T, \quad 0 < \omega < \omega_s \], the number of encirclements of the point
\((-1, j0)\) indicates the difference between the number of zeros and poles
of the characteristic equation.

Assuming the system to be open loop stable, the poles of $G H(z)$
lie within the unit circle and these are the same as the poles of
$(1 + G H(z))$. Therefore for no zeros of the characteristic equation
to lie outside the unit circle the transfer locus of the open loop
$G H(z)$ must not pass through or encircle the point $(-1, j0)$ on the
transfer locus plane.

If $G H(z)$ is not stable, then the condition for closed-loop
stability is the encirclement of $(-1, j0)$ by the transfer locus a
number of times equal to the number of poles of $G H(z)$ outside the
unit circle.

In plotting the pulse transfer loci, the argument of the complex
variable $z$ is the only parameter, as the magnitude is constant along
the unit circle. Also conjugate values of $z$ correspond to conjugate
values of $G H(z)$. Therefore it is sufficient to plot only one half
of the transfer locus, say
\[ z = 1 \text{ to } z = -1 \quad (\omega = 0 \text{ to } \omega = \frac{\pi}{T} \text{ rad/sec.}) \]

**Analytical Stability Test.**

There are two methods for analytical stability testing: one is
a modification of the Routh-Hurewitz criterion and the other is known
as Schur-Cohn criterion.
(a) **Modified Routh-Hurwitz criterion**, sometimes known as the bilinear transformation method.

To allow the Routh-Hurwitz criterion to be used, a transformation
\[ z = \frac{w + 1}{w - 1}, \quad w = \frac{z + 1}{z - 1} \tag{2.5.7} \]
is used to map the inside of the unit circle into the entire left half plane of an auxiliary plane, the w-plane.

![Diagram](image)

Fig. 2.5.4

The function of \( z \) is then transformed into another function of \( w \), of the same order.

The Routh-Hurwitz criterion can then be applied in the w-plane. The w-plane may be used for analysis as well applying the techniques of the s-plane.†

However, the stability test takes a good deal of work to correlate between the pole-zero locations on both the z and w-planes.

(b) **Schur-Cohn Criterion** ++

This criterion determines the presence of any roots of the characteristic equation in \( z \) outside the unit circle. It can be stated as follows:

---

References 44; 49 pp 36-38; 86 pp 98-100 & 105 pp 244-247.
†Reference 44
++References 49 pp 34-36 and 105 pp 238-244, also Ref. 51.
If for the polynomial

\[ F(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \quad (2.5.8) \]

all the determinants of the matrices

\[
\begin{vmatrix}
  a_{0} & a & 0 & \cdots & 0 & a_n & a_{n-1} & a_{n-k+1} \\
  a_{1} & a & 0 & \cdots & 0 & a_n & a_{n-k+2} \\
  a_{2} & a_{1} & a & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  a_{k-2} & a_{k-3} & a_n & \cdots & 0 & a_n \\
  a_{k-1} & a_{k-2} & a_{k-3} & a_{0} & 0 & a_{n} \\
  a_{0} & 0 & \cdots & 0 & a_{0} & a_{1} & \bar{a}_{k-1} \\
  a_{n} & 0 & \cdots & 0 & \bar{a}_{0} & a_{n-1} & \bar{a}_{k-2} \\
  \bar{a}_{n-1} & \bar{a}_{n} & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & \bar{a}_{n-k+3} & \bar{a}_{n} & 0 & 0 & a \\
\end{vmatrix}
\]

where \( k = 1, 2, 3, 4, \ldots, n \)

and \( \bar{a}_k = \text{conjugate of } a_k \)

are all different from zero, satisfying \( \Delta_k \leq 0 \) for \( k \) odd,

\( \Delta_k \leq 0 \) for \( k \) even,

then the system is stable.

The number of zeros inside the unit circle equals the number of variations in sign in the determinant sequence \( \Delta_1, \Delta_2, \cdots \).

Thus there are \( n \) variations in sign, i.e. \( n \) zeros inside the unit circle for a characteristic equation of order \( n \).

However, as the coefficients are all real, the conjugate sign is
superfluous, and by suitable transformation the matrix can be put into
a form where the determinant \( \Delta_k \) is reduced to a product of two
k-order determinants.*

\[
\Delta_k = \begin{vmatrix} \bar{X}_k & \bar{Y}_k \\ \bar{X}_k & \bar{Y}_k \end{vmatrix} \quad (2.5.10)
\]

where

\[
\begin{align*}
\bar{X}_k &= \begin{bmatrix}
a_0 & a_1 & a_2 & \ldots & a_{k-1} \\
0 & a_0 & a_1 & \ldots & a_{k-2} \\
0 & 0 & a_0 & \ldots & a_{k-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & a_1 \\
0 & 0 & 0 & \ldots & a_0 
\end{bmatrix} \\
\bar{Y}_k &= \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n 
\end{bmatrix}
\end{align*}
\quad (2.5.11)
\]

and

\[
\begin{align*}
\bar{X}_k &= \begin{bmatrix}
a_{n-k+1} & \ldots & a_{n-1} & a_n \\
0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 \\
\end{bmatrix} \\
\bar{Y}_k &= \begin{bmatrix}
a_{n-k+2} & \ldots & a_n \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\end{bmatrix} \\
\end{align*}
\quad (2.5.12)
\]

\( |\bar{X}_k + \bar{Y}_k| \) & \( |\bar{X}_k - \bar{Y}_k| \) are homogeneous polynomials of dimension
k in the variables \( a_0 \ldots a_n \). The two polynomials are written
as

\[
\begin{align*}
|\bar{X}_k + \bar{Y}_k| &= A_k + B_k \\
|\bar{X}_k - \bar{Y}_k| &= A_k - B_k 
\end{align*}
\quad (2.5.13)
\]

where \( A_k \) (\( B_k \)) is the sum of monomial terms which do not (do) change
sign when \( \bar{Y}_k \) is replaced by \( -\bar{Y}_k \) in \( |\bar{X}_k + \bar{Y}_k| \).

*Reference 50a.
$A_k$ and $B_k$ are designated the stability constants.\footnote{References 51 and 52.}

The condition for stability is then
\[
\begin{align*}
\left| A_k \right| & > \left| B_k \right| \quad \text{for } k \text{ even} \\
\text{and } \left| A_k \right| & < \left| B_k \right| \quad \text{for } k \text{ odd}
\end{align*}
\]
which follows from (2.5.10) and (2.5.13) leading to
\[
\left| \Delta_k \right| = A_k^2 - B_k^2 , \text{ and the condition in the original criterion.}
\]

As an example consider a third order polynomial.
\[
F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \quad a_3 \geq 0
\]

The stability conditions are \footnote{Reference (51)}
\[
\begin{align*}
\left| a_0 \right| & < a_3 \\
a_0^2 - a_3^2 & \leq a_0 a_2 - a_1 a_3 \\
a_0 + a_1 + a_2 + a_3 & > 0 , \quad a_0 - a_1 + a_3 \leq 0
\end{align*}
\]
2.6. SYSTEMS WITH APPRECIABLE PULSE WIDTH.

The preceding sections are based on the assumption that sampling occurs instantaneously. In other words the sampler acts as an impulse modulator. This assumption is valid for digital control systems, where the variables are represented by a sequence of numbers. It is also valid for systems with very short sampling duration compared with the time constants of the system.

For the cases with appreciable pulse width, the sampler function is no longer described by equation (2.1.1)

Several attempts have been made to analyse sampled-data systems with appreciable pulse width. Some methods are approximate while others are exact.

P. Transform Method.

Consider a periodic sampler with constant pulse width $h$, and period $T$. Its output $r_p(t)$ can be viewed as the multiplication of its input $r(t)$ and sampler time function $u_p(t)$ as shown in fig.(2.6.1)

![Diagram](image)

Figure 2.6.1 a.
The sampler time function is

\[ U_p(t) = \begin{cases} 1 & \text{for } nT < t < nT + h \\ 0 & \text{elsewhere} \end{cases} \]  

(2.6.2)

The L.T. of \( r_p(t) \) can be found by the real multiplication theorem.

\[ L(r_p(t)) = \frac{1}{2\pi j} \int \Re(\lambda) U_p(s-\lambda) \, d\lambda \]  

(2.6.3)

where \( R(s) = L(r(t)) \)

and \( U_p(s) = L(U_p(t)) \).

*Reference 27.
The function of the sampler can be considered as a succession of positive and negative steps at \( nT \) and \((nT + h)\) respectively. In a similar way as in establishing the transfer function of the zero order hold device (equation 2.2.2) we have

\[
L (U(t)) = \frac{1 - e^{-hs}}{s} + e^{-Ts} \frac{1 - e^{-hs}}{s} + \ldots
\]

\[
= \sum_{n=0}^{\infty} e^{-nTs} \frac{1 - e^{-hs}}{s}
\]

\[
= \frac{1 - e^{-Ts}}{1 - e^{-hs}} \frac{1 - e^{-hs}}{s}
\]

(2.6.4)

assuming \( |e^{-Ts}| < 1 \)

Substituting in equation (2.6.3) gives

\[
L (r_p(t)) = \frac{1}{2 \pi j} \int R(\chi) \frac{1 - e^{-h(\chi)}}{(\chi - s)(1 - e^{-T(s-\chi)})} d\chi
\]

(2.6.5)

where the contour integration encloses all the poles of \( R(\chi) \) in the left-hand plane of \( s \)-plane.

It can be seen that the sampler operates on the input function in a certain definite manner. This operational transformation is designated the P-transform*, or the \( \mathcal{T} \)-transform.+

In symbolic form

\[
R_p(s) = P (R(s))
\]

and

\[
r_p(t) = P^{-1} (R_p(s))
\]

(2.6.6)

where \( P \), and \( P^{-1} \) denote p-transformation and inverse p-transformation respectively.

---

*Reference 27; and Reference 49, chapter 9.
+References 104; 105 pp. 567 - 569.

(45)
Equation (2.6.5) can be evaluated by the residue method:

\[ \mathcal{R}_p(s) = \sum_{\text{poles}} \text{Residue of } R(\lambda) \frac{1 - e^{-h(s-\lambda)}}{(s-\lambda)(1 - e^{-T(s-\lambda)})} \quad (2.6.7) \]

The output transform of the system of figure (2.6.1) is

\[ C(s) = \mathcal{R}_p(s) \cdot G(s) \quad (2.6.8) \]

The p-transform is actually a special case of the Laplace transform. It is a function of \( e^{-hs}, e^{-Ts} \) and \( s \), expressed in a closed form as seen from (2.6.7)

**Inversion of the P-Transform**

The p-transform may be inverted using the inverse Laplace transform method. But due to the presence of factors of \( e^{-Ts} \) in the denominator, the solution is an infinite series in time.

However, a relation between the inverse Laplace transform and the inverse modified z-transform can be used, resulting in a closed form solution.*

\[ C(t) = \mathcal{L}^{-1}(C(s)) \quad (2.6.8) \]

Also \[ C(t) = \mathcal{Z}_m^{-1}(C(z,m)) \quad (2.6.9) \]

\[ t = (n-1+m)T \]

where \( n = 1, 2, 3, \ldots \)

and \( 0 \leq m < 1 \), \( T = \text{sampling period} \).

The last two equations are identical, thus

\[ \mathcal{L}^{-1}(C(s)) \equiv \mathcal{Z}_m^{-1}(C(z,m)) \quad (2.6.10) \]

\[ t = (n-1+m)T \]

*Reference 27.
The modified z-transform is first found, say by the residue formula:

\[
C(z,m) = z^{-1} \sum \text{poles of } C(\chi) \text{ Residues of } \frac{C(\chi) e^{mT}}{1-e^{-T} \chi z^{-1}} \quad (2.6.11)
\]

\[
\chi = \text{sq}
\]

C(z,m) is a rational function in z.

Then the time function is found from

\[
C(t) = \sum \text{poles of } C(z,m) z^{n-1} \quad (2.6.12)
\]

\[
t = (n-1+m) T
\]

\[
0 < m < 1
\]

The poles of C(z,m) lie within the unit circle about the origin, for a stable system.

The above procedure is only valid for periodic sampling with constant pulse width. For multi-sampling where the pulse width and/or the period are variable, the solution is carried out term by term, transforming to the s-plane, then to the z-plane, and back to t-domain and so on.

Where the loop transfer function has simple definite poles, the exact analysis can be carried out more simply+.

Generally speaking the exact analysis of finite pulse width systems through transformations requires complicated computation, where it may be essential to use computing machines.

When h. is small compared with T and the system time constant, equation (2.6.4) can be approximated by

*References 27a and 28.
+Reference 76a.
which reduces the sampling process into a conventional one. The constant $h$ is incorporated with the gain constants of the following system components. This follows by comparing (2.6.13) with (2.1.4)

Another approximation is to consider the sampler output to be a flat topped pulse, where $h$ is relatively small.*

Referring to fig. 2.6.1, the sampler output can be expressed as

$$R^*(s,h) = R^*(s) \frac{1 - e^{-hs}}{s}$$

(2.6.14)

and $C(s) = (1 - e^{-hs}) R^*(s) G_1(s)$

(2.6.15)

where $G_1(s) = \frac{G(s)}{s}$, taking the modified z-transform

$$C(z,m) = (1 - z^{-h/T}) R(z) G(z,m)$$

(2.6.16)

This approximation utilizes the modified z-transform to convert the case into a conventional one.

Closed loop systems can be analysed using any of the above described methods, whether exact or approximate.

The formation of the closed loop transfer functions follows the general rules for the corresponding transform used. The characteristic equation is found and the stability determined following the same lines as for s-plane or z-plane analysis.

*Reference 105, p. 572.
It may be concluded that the exact analysis of systems with finite pulse width using transform methods is quite laborious. Another simple method is introduced in section 2.8, where the analysis and synthesis are carried out directly in the time domain.
2.7. SYSTEMS WITH PURE TIME DELAY

Now let us consider the presence of pure time delay in sampled-data systems,

\[
\begin{align*}
E(t) &\rightarrow E(z) \\
\rightarrow C_1(t) &\rightarrow C(z)
\end{align*}
\]

Fig. 2.7.1.

A delay of \( T \) follows a conventional sampler as shown in figure 2.7.1. To start with assume \( \lambda = 1 \ T \), where 1 is an integer.

From equation 2.3.10, the sampler output sequence is given by

\[
E^*(t) = \sum_{n=0}^{\infty} e(nT) \delta(t-nT) \quad (2.7.1)
\]

The output of the delay (or after the delay) \( C_1(t) \), is the same sequence but delayed 1 sampling periods.

\[
C_1(t) = \sum_{n=0}^{\infty} e(nT) \delta(t-nT-1T) \quad (2.7.2)
\]

Taking the z-transform gives

\[
G(z) = \sum_{n=0}^{\infty} e(nT) z^{-(n+1)T} = z^{-1} \sum_{n=0}^{\infty} e(nT) z^{-nT} = z^{-1} E(z) \quad (2.7.3)
\]

Therefore the pulse transfer function of the delay is \( z^{-1} \).

The output after the delay is a pulse sequence, whether a fictitious sampler is inserted there or not. This follows from the nature of the delay.
The forward loop pulse transfer function is then \( z^{-1} G(z) \)

If \( \delta = (1-6)T \)

where \( 0 \leq 6 \leq 1 \)

the output after the delay is still a pulse sequence but not at the sampling instants. To simplify the procedure, \( \delta \) is considered as a delay \( 1T \) plus an advance \( 6T \). The pulse transfer function of \( 1T \) is derived as above, while \( 6T \) is incorporated into the following system components.

By following the procedure leading to equation (2.4.20), the pulse transfer function of the forward loop is found to be

\[
\begin{align*}
G(z,m) &= (2.7*4) \\
\end{align*}
\]

The overall closed loop pulse transfer function is

\[
Y(z) = \frac{z^{-1}G(z,m)}{1 + z^{-1}G(z,m)} \bigg|_{m = 5} \quad (2.7.5)
\]

It is immaterial whether the delay is before or after the other system components.

The analysis of such systems can be carried out following the techniques of the preceding sections.

Actually, the presence of delay in a conventional sampled-data system does not create any particular problems, in contrast to its presence in continuous systems where it generally reduces the margin of stability. Also the stability analysis can only be approximate.*

*Reference 109, pp. 546 - 553.
Sampling was even introduced as stabilising factor for some systems incorporating delay.

As for finite pulse width sampled-data systems incorporating delay, the case is rather different. The P-transform method (see section 2.6) is expressed in the $s$-domain. Therefore its application leads to the same difficulties encountered with similar continuous systems treated in the $s$-domain. The difficulty is the presence of a factor $e^{-1TS}$ in the denominator of the transfer function.

Therefore it could be said that the P-transform method is not adequate for satisfactory analysis of sampled-data systems incorporating time delays.

One way of treating such systems is by using the Laplace transform and its inverse in a step by step procedure as illustrated by the example shown in figure 2.7.2.

![Diagram](image)

Fig. 2.7.2.

The system is assumed to have zero initial conditions, and $h \leftarrow T$. Also $r(t) = 0$ for $t < 0$.

---

+Reference 109, p. 546.
For the interval $0 < t < h$, the system can be treated as an open loop one, and the L.T. can be used:

$$C_1(s) = E(s) \times G(s) = R(s) \times G(s) \quad (2.7.6)$$

The time response can be found for this interval as

$$C_1(t) = L^{-1}(C_1(s)) \quad (2.7.7)$$

For $h \leq t < T$, the sampler is open. Again the system is treated as an open loop, and its response can be determined through Laplace transformation taking into consideration the initial conditions at $t = h$.

At the next sampling interval $T \leq t < T+h$, the input to the system $e(t)$ is composed of $r(t)$ and $C(t)$ as

$$e(t) = r(t) - C(t) = r(t) - C_1(t-T+h) \quad (2.7.8)$$

$C_1(t-T+h)$ is completely known, and again the system can be treated as being open loop, whose input is that of equation (2.7.8)

The response transform is the product of the transform of equation (2.7.8) and the transform of the system, taking into consideration the initial conditions at $t = T$.

When the sampler opens again at $t = T+h$, the subsequent response for $T+h \leq t < 2T$ is determined from the initial conditions at $t = T+h$. By repeating this procedure the response can be built up.
Generally speaking step by step procedure is the more appropriate for sampled-data system incorporating time delays.

By iteration the output, say at $t = n T$, can be expressed as a time series. The stability can then be determined by testing the convergence of the series.

The transformation from and to the time-domain is not generally favoured specially for higher order systems. This is because the number of initial conditions to be used in determining the L.T. of the system is the same as the order of the system differential equation.

A more compact method in terms of time domain matrices/therefore introduced in the next section.
2.8. **TIME DOMAIN ANALYSIS**

For a long time control systems were mostly treated by techniques based on transformations, such as the Laplace transform for continuous systems, and the z-transform for sampled-data systems.

Recently time-domain techniques for treating sampled-data systems in particular have been introduced.*

After all, control system variables are usually functions of time. Therefore it is an advantage to carry out the treatment in time-domain. This avoids the trouble of transforming into another domain, and inverse transforming.

In fact the amount of work necessary for proper time domain treatment is considerably less than that needed for corresponding transform treatment. Also the use of digital processing elements favours the time domain treatment.

Another factor for preferring time domain treatment is that the transformation methods proved inadequate for treating, say, finite pulse width sampled-data systems incorporating time delays, and variable sampling rate systems.+ 

Time-domain analysis may further be classified into that for conventional sampling, and the other approach for finite pulse width sampling.

---

*References 30; 57; 33 and 25.

+Reference 30.

This technique is based on the convolution summation, and provides the system response at discrete instants of time, not necessarily the sampling instants.*

From equation 2.3.12, and referring to figure 2.3.1, we have

\[ C(mT) = \sum_{n=0}^{m} g(mT - nT) e(nT) \]  
(2.8.1)

where \( C(mT) \) represents the response at \( t = mT \), and \( g(mT - nT) \), the impulsive response at the \( m \)th instant due to an impulse \( e(nT) \) at the \( n \)th instant.

Equation (2.8.1) can be rewritten in a notational form:

\[ C_m = \sum_{n=0}^{m} g_{mn} e_n \]  
(2.8.2)

from which the output sequence can be found as

\[ C_0 = g_{00} e_0 \]

\[ C_1 = g_{10} e_0 + g_{11} e_1 \]

\[ C_2 = g_{20} e_0 + g_{21} e_1 + g_{22} e_2 \]  
(2.8.3)

\[ \ldots \]

\[ C_m = g_{m0} e_0 + g_{m1} e_1 + \ldots + g_{mm} e_m \]

Equation (2.8.3) can be expressed in matrix notation as

\[ C = G E. \]  
(2.8.4)

where

* Reference 25
If the system is time invariant, then \( g_{ij} \) of equation 2.8.5 is only a function of \( i - j \), which reduces the transfer matrix to

\[
\mathbf{G} = \begin{bmatrix}
    g_0 & 0 & 0 & \cdots & 0 \\
    g_1 & g_0 & 0 & \cdots & 0 \\
    g_2 & g_1 & g_0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    g_m & g_{m-1} & g_{m-2} & \cdots & g_0
\end{bmatrix}
\]  

(2.8.7)

For the last equation the elements are found from the impulsive

---

*Reference 30.
†Reference 25.
response

\[ g_i = g(t) \bigg|_{t = iT} \quad (2.8.8) \]

where \( T \), is the sampling period, which need not necessarily be constant.

The response between sampling instants can be evaluated by introducing a fictitious delay, as for the modified z-transform and formulating another transfer matrix with elements

\[ g_i(\Delta) = g(t) \bigg|_{t = (i - \Delta)T} \quad (2.8.9) \]

where \( \Delta T \) is the delay introduced, \( 0 < \Delta < 1 \).

Taking into consideration that \( g(t) = 0 \) for \( t < 0 \), the new transfer matrix would be

\[ \mathcal{G}(\Delta) = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
g_1(\Delta) & 0 & 0 & \ldots & 0 \\
g_2(\Delta) & g_1(\Delta) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_m(\Delta) & g_{m-1}(\Delta) & g_{m-2}(\Delta) & \ldots & 0
\end{bmatrix} \quad (2.8.10) \]

By varying \( \Delta \), the continuous output can be scanned.

**Systems with Time Delays.**

![Diagram](Fig. (2.8.1))
Consider the forward loop of figure 2.8.1, where a delay follows a system whose transfer matrix is $G$.

If $\lambda$ is a whole number of sampling periods, say $1 T$, then the overall transfer matrix is

$$
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
G_m & G_{m-1} & G_{m-2} & \cdots & G_0
\end{bmatrix}
$$

(2.8.11)

The output sequence $C = G_1 E$, which means that for the first $1$ sampling instants (including $n = 0$) the output sequence is zero. This result should be expected by the nature of the delay.

If $\lambda = 1T + \Delta T$, the transfer matrix would be

$$
G_{\lambda} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
G(\Delta)
\end{bmatrix}
$$

(2.8.12)

where $G(\Delta)$ is defined by equation (2.8.10).

Therefore the presence of time delay adds a number of zero rows at the top of the matrix depending on the amount of the delay in terms of sampling periods.

**Closed Loop Systems.**

Referring to figure 2.8.1, these relations follow

$$
E = R - C
$$

(2.8.13)

$$
C = G_{\lambda} \cdot E
$$

(2.8.14)
The closed loop transfer matrix can be formulated to be

\[ C \mathbf{R}^{-1} = \mathbf{G} \lambda (I + \mathbf{G} \lambda)^{-1} \]  \hspace{1cm} (2.8.15)

where \( I \) is the identity matrix.

The evaluation of the response sequence applying the last equation involves matrix addition, inversion and multiplication.

However, for the present case the output sequence or error sequence can be evaluated step by step, since for the first \( n \) instants the output sequence is zero, and the system is open looped. The procedure can be explained by an example where \( \lambda \) say = \( T \), and 2 or 3 sampling instants are considered.

\[
\begin{bmatrix}
C_0 \\
C_1 \\
C_2 \\
C_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
g_0 & 0 & 0 \\
g_1 & g_0 & 0 \\
g_2 & g_1 & g_0
\end{bmatrix}
\begin{bmatrix}
\mathbf{r}_0 - C_0 \\
\mathbf{r}_1 - C_1 \\
\mathbf{r}_2 - C_2 \\
\mathbf{r}_3 - C_3
\end{bmatrix}
\]

\( C_0 = 0 \)
\( C_1 = g_0 (\mathbf{r}_0 - C_0) = g_0 r_0 \)
\( C_2 = g_1 r_0 - g_0 (\mathbf{r}_1 - C_1) \)

and so on, the output sequence is evaluated term by term in a simple and systematic manner.

If there is no time delay in the closed loop system, the step by step method can still be used, since for all physical systems there cannot be an immediate output, in other words \( C_0 \) can be taken to be zero, or \( g_0 = 0 \). The output sequence is then:

*Reference 25.
The matrices should be of an infinite order to describe the complete behaviour of the system. However for stable systems, the transient error sequence tends to zero after certain number of sampling periods, and the matrices are considered only up to an equivalent order.

The stability can be satisfactorily checked for closed loop systems by testing the convergence of the error sequence. As mentioned above, it should tend to zero.

The above analytic method can be used for non-linear systems, and time variant systems as well. Generally the same procedure is followed, with the proper transfer matrix elements.

Continuous systems can also be analysed using this method but fictitious sampling and hold has to be introduced. The accuracy of analysis depends on the rate of sampling.

\[
\begin{bmatrix}
C_0 \\
C_1 \\
C_2 \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \ddots \\
\vdots & \vdots & \vdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
C_0 \\
C_1 \\
C_2 \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix} = 
\begin{bmatrix}
[r_0 - C_0] \\
[r_1 - C_1] \\
[r_2 - C_2] \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\]

\[C_0 = 0.\]

\[C_1 = g_1 r_0\]

\[C_2 = g_2 r_0 - g_1 (r_1 - C_1)\]

* Reference 110
* Reference 49, Chapter 8.
2.8.2. General Time Domain Analysis

This technique was developed primarily for finite pulse width sampled-data systems incorporating time delays. However, it can be used for general applications.

For the sampling duration, finite pulse width sampled-data systems behave as continuous systems.

Therefore consider a continuous system described by the following set of linear differential equations, assuming zero input,

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
& \quad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{align*}
\]

Equation 2.8.19 can be put in a compact matrix form.

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

Fig. 2.8.2.

Physically \(x_1, x_2, \ldots, x_n\) represents the outputs of the successive integrators on the analogue computer representation of the differential equation, and \(a_{ij}\) represents the coefficients with which the output of the \(j\)th integrator is fed to the \(i\)th integrator*, as shown in figure 2.8.2.

Equation 2.8.19 can be put in a compact matrix form.

*Reference 57.
where $\bar{X}$ is a column matrix or vector of the elements

$$x_1, x_2, \cdots, x_n$$

and $\bar{X}_0$ is also a column matrix of $x_{o1}, x_{o2}, \cdots, x_{on}$, representing the initial conditions.

The variables $x_1, \cdots, x_n$ determine the system behavior at any instant.

The solution of (2.8.20) is given by

$$\bar{X} = e^{\bar{A}t}, \quad \bar{X}_0$$  \hspace{1cm} (2.8.21)*

where

$$e^{\bar{A}t} = \sum_{k=0}^{\infty} \frac{\bar{A}^k t^k}{k!} \hspace{1cm} (2.8.22)*$$

The last equation is known as matrix exponential.

With external input vector $\bar{E}$, equation (2.8.20) is rewritten

$$\frac{d\bar{X}}{dt} = \bar{A} \cdot \bar{X} + \bar{E}, \quad \bar{X}(o) = \bar{X}_0 \hspace{1cm} (2.8.23)$$

Its solution is given by

$$\bar{X} = e^{\bar{A}t} \cdot \bar{X}_0 + \int_0^t e^{\bar{A}(t-\tau)} \bar{E}(\tau) \, d\tau \hspace{1cm} (2.8.24)^+$$

For physical systems $e^{\bar{A}t}$ is a convergent series. However, its evaluation is difficult and it is preferably expressed in a closed form.

An equivalent matrix of the same order as the differential equation can be formulated to replace the exponential matrix. The elements of such a matrix are evaluated from the simultaneous

*Reference 4, p. 165.
^Reference 4, p. 169.
equations governing the system behaviour and its initial conditions.

Two examples are considered to illustrate the formulation of the equivalent matrix, which may be called the transfer matrix or transition matrix of the system, \( \mathcal{G}(t) \).

**Example 1**, a second order differential equation.

\[
\begin{align*}
x &= \frac{Ke}{p^2 + pa} e(t) \\
p &= \frac{d}{dt} 
\end{align*}
\]

Its analogue simulation is as shown in figure 2.8.3. The variables are given by

\[
\begin{align*}
x_2 &= 0 \\
x_2(o) &= x_{o1} \\
x_1 &= -ax_1 + Kx_2 , \quad c_1(o) = c_{o1}
\end{align*}
\]

Applying the Laplace transform

\[
X_2(s) = \frac{x_{o2}}{s}
\]

and

\[
X_1(s) = \frac{K}{s(s+a)} x_{o2} + \frac{1}{s+a} x_{o1}
\]

Inverse Laplace transformation gives

\[
\begin{align*}
x_2(t) &= x_{o2} L^{-1}\left(\frac{-1}{s}\right) \\
x_1(t) &= x_{o2} L^{-1}\left(\frac{K}{s(s+a)}\right) + x_{o1} L^{-1}\left(\frac{1}{s + a}\right)
\end{align*}
\]
In a matrix form

\[
\begin{bmatrix}
  x_2 \\
  x_1
\end{bmatrix} =
\begin{bmatrix}
  L^{-1}\left(\frac{1}{s}\right) & 0 \\
  L^{-1}\left(\frac{k}{s(s+a)}\right) & L^{-1}\left(\frac{1}{s + a}\right)
\end{bmatrix}
\begin{bmatrix}
  x_{o2} \\
  x_{o1}
\end{bmatrix}
\]

\[\bar{x} = G(t) \bar{x}_0\] (2.8.28)

where

\[G(t) =
\begin{bmatrix}
  1 & 0 \\
  K & (1-e^{-st}) e^{-st}
\end{bmatrix}\] (2.8.29)

The last equation can be put into another form.

\[G(t) =
\begin{bmatrix}
  L^{-1}(F_{22}) & 0 \\
  L^{-1}(F_{21}) & L^{-1}(F_{11})
\end{bmatrix}\] (2.8.30)

where \(F_{ij}\) is the transfer function between the input of the \(i^{th}\) integrator and the output of the \(j^{th}\) integrator.

Ex. 2. A third order differential equation.

\[x = \frac{(p + a)(p + \beta)}{p(p + a)(p + b)} e\] (2.8.31)

If the equation is not factorised, it can be put in the above factorised form.

Its analogue simulation is shown in figure 2.8.4.*

---

*Reference 33.
The variables are given by

\[
\begin{align*}
\dot{x}_3 &= a x_3 \\
\dot{x}_2 &= b x_2 + (\alpha - a) x_3 \\
\dot{x}_1 &= (\beta - b) x_2 + (\alpha - a) x_3
\end{align*}
\]

Applying Laplace transform.

\[
\begin{align*}
X_3(s) &= \frac{x_{03}}{s+a} \\
X_2(s) &= \frac{x_{02}}{s+b} + \frac{(\alpha - a) x_{03}}{(s + b)(s + a)} \\
X_1(s) &= \frac{x_{01}}{s} + \frac{\beta - b}{s(s+b)} x_{02} + \frac{(\beta-b)(\alpha-a)}{s(s+b)(s+a)} + \frac{\alpha-a}{s(s+a)} x_{03}
\end{align*}
\]

Inversing the last three equations,

\[
\begin{align*}
x_3(t) &= x_{03} f_{33} \\
x_2(t) &= x_{03} f_{32} + x_{02} f_{22} \\
x_1(t) &= x_{03} f_{31} + x_{02} f_{21} + x_{01} f_{11}
\end{align*}
\]

(2.8.32)

where \(f_{ij}\) is the inverse Laplace transform of the transfer function between the \(i^{th}\) integrator and the output of the \(j^{th}\) integrator.

Therefore the transfer matrix of the system is given by

\[
\bar{G}(t) = \begin{bmatrix} f_{33} & 0 & 0 \\ f_{32} & f_{22} & 0 \\ f_{31} & f_{21} & f_{11} \end{bmatrix}
\]

(2.8.33)

The same procedure can be used for other systems.

From equations (2.8.21) and (2.8.28), it follows that

\[
e^{A t} = \bar{G}(t)
\]

(2.8.34)
\[ X = G(t) X_0 + \int_0^t G(t-\tau) E(\tau) \, d\tau \quad (2.8.35) \]

The integral on the R.H.S. of the last equation may be considered as analogous to a convolution integral and it is evaluated term by term.

As an example let us consider evaluating the response of the system of equation (2.8.28) for a step input, though a finite pulse width sampler, as shown in the forward loop of figure 2.8.5.

\[ \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + \begin{bmatrix} K(1-e^{-at}) \\ \frac{K(1-e^{-at})}{a} + x_0 e^{-at} \end{bmatrix} + \begin{bmatrix} \int_0^t e^{\lambda} \, d\lambda \\ \int_0^t \frac{1-e^{-a(t-\tau)}}{a} e(\tau) \, d\tau \end{bmatrix} \]

as \( E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) for a unit step.

The output of interest is \( x_1 \), and for zero initial conditions the response for \( 0 < t < h \) is found from

\[ C = x_1 = K \left( \int_0^t \frac{1-e^{-a(t-\tau)}}{a} \, d\tau \right) = \frac{K}{a} (t + \frac{e^{-at}}{a} - \frac{1}{a}) \quad (2.8.37) \]

which is exactly the same result obtained by the conventional method of the Laplace transform.

(67)
At \( t = h \), the output vector represents the initial condition vector for the subsequent interval.

\[
\begin{bmatrix}
    x_{o2} \\
    x_{o1}
\end{bmatrix} = \begin{bmatrix}
    h \\
    (h + \frac{e^{-ah}}{a} - \frac{1}{a}) \frac{K}{a}
\end{bmatrix}
\]

(2.8.38)

The output for \( h < t < T \) is found from

\[
\begin{bmatrix}
    x_2 \\
    x_1
\end{bmatrix} = \begin{bmatrix}
    h \\
    \frac{hK}{a} (1-e^{-a\tau}) + e^{-a\tau} \frac{K}{a} \left(h + \frac{e^{-ah}}{a} - \frac{1}{a}\right)
\end{bmatrix}
\]

(2.8.39)

where \( \tau = t - h \), i.e. the time should start from zero each new interval by change of variables.

At \( t = T \) or \( \tau = T - h \), the output vector constitutes the initial conditions vector for the subsequent interval \( T \leq t < T+h \).

Equation (2.8.36) is used to evaluate the response and so on, the response can be built up step by step.

However, the output sequence at any instant say \( t = nT \) can be found by induction, once two or three outputs have been evaluated using the above method.

The output sequence is then expressed as a time series, and the stability is deduced by testing the absolute convergence of the series.

**Presence of the Time Delay**

It can readily be seen that the presence of delay in a closed loop finite pulse width sampled-data system does not add any complications to the procedure. In fact it makes the evaluation of the response \( \mathcal{R}(t) \) much simpler when following the step by step method,
though this can only be done if the delay is greater than the initial pulse duration.

For the closed loop system of figure 2.8.5.

\[ \bar{E}_n = \bar{R}_n - \bar{C}_n \] (2.8.40)

The column vectors of the last equations have all their elements zero except the first row. However, for each sampling duration there are different values for this first row.

\( \bar{E}_n \), \( \bar{R}_n \) and \( \bar{C}_n \) represent time vectors, \( n \) being the number of sampling periods.

By the nature of the delay, the output \( \bar{C}_1 \), referred to the same time origin as \( \bar{R}_n \), is \( \bar{C}_n (t - \lambda) \), where \( \lambda \) is the delay.

Therefore \( \bar{E}_n(t) = \bar{R}_n(t) - \bar{C}_n(t - \lambda) \) (2.8.41)

Since \( \bar{C}_n (t - \lambda) = 0 \) for \( t < \lambda \), \( \bar{C}' \) can be determined by the step by step procedure. \( \bar{C}' \) is the same as \( \bar{C}_n \) but delayed by \( \lambda \).

\[ \bar{C}_o' = \left[ \bar{G}(t) \right] \bar{E}_o = \left[ \bar{G}(t) \right] \bar{R}_o \] (2.8.42)

where the \[ \left[ \right] \] sign means performing all the mathematical operations of equations (2.8.36) for a whole period. \( \bar{E}_o \) and \( \bar{R}_o \) represent the output of the sampler whether it is zero or not, \( \bar{C}_o' \) the continuous output during the whole period.

After \( \lambda \) the output \( \bar{C}' \) reaches the input terminal but it does not affect the system until the following sampling interval. Assuming \( \lambda = (1 - \Delta) T_1 \), then

\[ \bar{C}_{1} = \left[ \bar{G}(t) \right] (R_1 - \bar{C}_o \Delta) \] (2.8.43)
where \( C_0 A \) is the output at the first sampling period for 
\( \Delta < t < \Delta + h \), as illustrated by figure 2.8.6.

The response is determined by repetitious step by step. It is noticed that the time output is composed of exponential functions which do not introduce any difficulties in evaluating the convolution integrals of equation (2.8.33) or (2.8.43).

The output sequence usually follows some regular pattern, and by induction the sequence can be put into a time series which indicates a stable system if the series converges.

This method of testing stability in the time domain seems to be the natural trend, otherwise complex matrix operations have to be performed to check the stability from the transfer matrix elements and their behaviour.*

In any case, the absolute stability is determined by testing the convergence of the output sequence at regular intervals. The marginal stability has not been investigated as far as I know, by time-domain treatment.

*Reference 57.
The general time domain method discussed above can be used to analyse and synthesise systems with variable sampling rate and/or variable pulse width, since those two parameters do not affect the procedure followed. The method is also of interest in designing systems for certain specification as will be seen later.

It can also be used for conventional sampling.
CHAPTER THREE
GENERAL SYNTHESIS PROCEDURES

The basic analytic techniques for linear-sampled data systems were considered in the preceding chapter. These techniques are used to evaluate system performance and determine stability.

Generally speaking analysis of uncompensated sampled-data systems reveals unsatisfactory overall response. Therefore compensation, or addition of certain networks, is needed to reshape the response in accordance with the design requirements.

The main objective of synthesis is the design of some system complying with certain specifications, e.g. of the overall response; or the improvement of inadequate performance by proper compensation.

The type of synthesis procedure depends upon the controlled system characteristics and the design criteria applied. These criteria vary from system to another, and they also depend on what disturbances, the overall system is subject to. Such disturbances may be periodic, aperiodic or random.

Compensating devices are usually cascaded to the control system following two distinct configurations:

1) Continuous compensation, where the compensator is connected directly to the controlled system, as shown in figure 3.0.1.
ii) DISCRETE compensation, where the compensator, or controller is separated from the controlled system by a sampler, as shown in figure 3.0.2.

The basic techniques and advantages of each configuration are discussed in the following sections.

i) CONTINUOUS COMPENSATION METHODS

3.1. FREQUENCY RESPONSE IN THE S-PLANE.*

This method is based on the assumptions considered in section 2.2., where the high frequency components generated by the sampler are neglected.

Consider the system of figure 3.1.1 where \( G(s) \) is the controlled system transfer function, \( N(s) \) the compensating network transfer function.

In the absence of \( N(s) \), the overall transfer function is found to be

\[
Y(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G^*(s)}
\]  

Its frequency function is

\[
Y(j\omega) = \frac{G(j\omega)}{1+G^*(j\omega)}
\]  

Now the locus of \( G^*(j\omega) \) on its plane can be plotted by considering the primary component only \( G(j\omega) \). Then by applying any associated design criterion, e.g. M-circle, a compensating network \( N(j\omega) \) is formulated. The overall transfer function is then

\[
Y(j\omega) = \frac{N G(j\omega)}{1+N G^*(j\omega)}
\]  

To check whether the choice of \( N(j\omega) \) was appropriate,

\[
N G^*(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} N G(j\omega + jkw)
\]  

is plotted for \( k = 0, \pm 1 \). The resultant plot is then inspected to find whether it is satisfactory or not. One or more trials may be needed to get satisfactory results.

Although, it is approximate, the method can be applied if the frequency responses of the components are known.
However, due to the approximations involved, the effect of the pole-zero pattern of the compensating network cannot be easily evaluated.

3.2. FREQUENCY RESPONSE IN THE Z-PLANE

Referring to the system of figure 3.1.1.

\[ C(s) = \frac{N G^*(s)}{1 + N G^*(s)} R(s) \]  \hspace{1cm} (3.2.1)

If the input \( R(s) \) is a sinusoidal function, the output \( C(s) \) is also a sinusoidal function of the same frequency. The relation between the output and input sinusoids is the overall transfer function

\[ Y^*(s) = \frac{N G^*(s)}{1 + N G^*(s)} \bigg|_{s = jw} \]  \hspace{1cm} (3.2.2)

Using the z-transform notation, gives

\[ Y(z) = \frac{N G(z)}{1 + N G(z)} \bigg|_{z = e^{j\omega T}} \]  \hspace{1cm} (3.2.3)

The transfer locus \( Y(e^{j\omega T}) \) on the \( N G(z) \)-plane \((\omega = 0 \text{ to } \omega = \frac{w}{2})\) represents the steady state behaviour of the system. (As discussed in section 2.2., the highest input frequency should be half the sampling frequency).

The transfer loci can be of value for design provided that,

i) The relation between the transient response and the corresponding transfer locus can be established. This has been done for some systems.\(^*\) The transfer locus is

\(^*\)References 46 and 49, pp. 227-240.
then shaped satisfying special requirements in the transient response.

ii) The effect of the pole zero pattern of \( N(s) G(s) \) on \( N G(z) \) is also known, * to choose the appropriate compensating transfer function \( N(s) \), leading to the required shape of the transfer locus as formulated in (i) above.

This approach is systematic and exact. It is of advantage if general purpose design charts and diagrams are available. But it is based on the response at sampling instants only. The intersample response is generally a function of \( N G(s) \). Trial and error procedure may be required to obtain satisfactory continuous response.

### 3.3. ROOT LOCUS METHOD

Continuous control systems are usually represented by rational functions in \( s \), while sampled-data systems are represented by rational functions in \( z \). Therefore the construction of the root loci for sampled-data systems on the \( Z\)-plane + follows the same rules applied for continuous systems on the \( s\)-plane.

For a typical unit feedback sampled-data system the overall pulse transfer function is given by,

\[
Y(z) = \frac{G(z)}{1+G(z)}
\]  

---

*References 44a and 49, pp. 219-226.
+References 109, pp.534-538; 72; 49 Ch.3 & 105, pp.479-482, also 89.
As considered earlier (section 2.5) for a stable system all the roots of its characteristic equation must lie within the unit circle. Therefore if \( G(z) \) contains an adjustable gain, its maximum allowable value is where the locus intersects the unit circle. If \( G(z) = K \frac{(z+a)}{(z-l)(z-b)} \) (3.3.2)*

The root locus of \( 1 + G(z) = 0 \) would be as shown in figure (3.3.1)

The root locations are related to the transient response of sampled data systems†, in a similar manner as for continuous systems. However, the above relations are only valid at sampling instants. For intersample behaviour the modified z-transform is used. The overall pulse transfer function is written as

\[
Y(z,m) = \frac{G(z,m)}{1+G(z)} = L(z,m) \frac{G(z)}{1+G(z)}
\]

where \( L(z,m) = \frac{G(z,m)}{G(z)} \) (3.3.3)**

* Reference 72
† References 45 and 89.
** Reference 89.
The poles and zeros of $\frac{G(z)}{1+G(z)}$ are treated in the usual way, whereas poles of $L(z,m)$ are independent of $K$ and $m$, and its zeros determine the intersample response. As $m$ moves to unity a zero of $L(z,m)$ moves to the origin, and as $m$ moves to zero, a zero of $L(z,m)$ moves to infinity. Thus at every sampling instant ($m \rightarrow 1$), the zero moves to the origin. If there are discontinuities in the response at sampling instants, a zero will arrive from infinity into the finite plane at such instants, and a zero will jump from some point to the origin. *

Synthesis using the root locus method can be carried out either with a continuous cascaded compensator, or a discrete cascaded compensator. Either way the relation between the root locations and the transient response specifications must be available, e.g. constant overshoot loci. +

The reshaping of the root locus can be done by adding poles and zeros, or cancellation of the existing ones.

The procedure can be outlined by referring to figure 3.1.1 and assuming that the appropriate open loop pulse transfer function is $NG(z)$. By using transform pairs or any other method, the corresponding $NG(s)$ can be found. Knowing $G(s)$, then $N(s)$ is found from

$$N(s) = \frac{NG(s)}{G(s)}$$  \hspace{1cm} (3.3.4)

*Reference 89
+Reference 46.
As for discrete compensation, consider the system of figure (3.3.2)

The pulse transfer function of the open loop is the product of $D(z)$ and $G(z)$, where $D(z)$ is the discrete controller pulse transfer function.

$D(z)$ is formulated by adding poles and zeros to those of $G(z)$, according to the locus shaping required.

The important factor to be noticed in this respect is that both $N(s)$ and $D(z)$ must be physically realizable. Actually this condition applies to any compensating device. Also $N(s)$ and $D(z)$ should be practically feasible, which may involve some approximation in their implementation. The continuous response should be checked. In any case it is usually satisfactory for continuous cascaded compensation since the input to the controlled system is a continuous one.
This approach looks logical for synthesis of sampled-data systems, where a specified time output is assumed, \( C(t) \). The corresponding Laplace transform \( C(s) \) is derived, also its z-transform \( C(z) \).

\[ R(s) \overset{N(s)}{\rightarrow} Y(z) \overset{G(z)}{\rightarrow} C(z) \]

Knowing \( C(z) \) for a particular \( R(z) \), the overall pulse transfer function \( Y(z) \) is found to be

\[ Y(z) = \frac{C(z)}{R(z)} \quad (3.4.1) \]

But

\[ Y(z) = \frac{N G(z)}{1 + N G(z)} \quad (3.4.2) \]

Hence

\[ N G(z) = \frac{Y(z)}{1 - Y(z)} \quad (3.4.3) \]

From (3.4.3) \( N G(s) \) can be evaluated, and the compensating network \( N(s) \) found from the relation

\[ N(s) = \frac{N G(s)}{G(s)} \quad (3.4.4) \]

This procedure applies an important approach of forcing the system to behave in a specified manner, in other words defining the pole-zero pattern of the required pulse transfer function.

However, it is claimed that this procedure involves some difficulties. One is in finding \( Y(z) \) such that the resultant

N G (s) is physically realizable. Nevertheless, this difficulty may be superfluous since N G (s) can be realizable if Y (z) is a finite polynomial in z⁻¹ as will be seen later in discussing finite settling time response.

The other difficulty is that the response is only specified at sampling instants. The intersample response may be checked by any method of section 2.4, and another trial can be done if necessary.

Besides the above discussed approaches, there are some methods based on the approximation of the sample-and-hold by some equivalent continuous transfer functions.

The accuracies of such approximations depend upon the complexity of the approximating network used.

The Continuous compensating elements are usually RC networks whether passive or active. The cost of such components is relatively low. But continuous compensation is usually inflexible and is of limited practical value for a wide range of applications.

*References 49 p 264 and 86 pp 133-136
†References 49 pp 251-261 and 86 pp 123-124
ii) DISCRETE COMPENSATION METHODS

The preceding sections outlined the continuous compensation principles. The main drawback for such approaches is that the open loop pulse transfer function is not the product of the controller and system pulse transfer functions, leading to difficulties in correlating the controller characteristics and the overall response.

This difficulty is removed by discrete compensation, where the controller is connected to the system through a sampler, as shown in figure 3.3.1. Its overall pulse transfer function is given by

\[ Y(z) = \frac{C(z)}{R(z)} = \frac{D(z)G(z)}{1+D(z)G(z)} \]

Earlier discrete compensation methods applied techniques developed for continuous systems, e.g. root locus and frequency response.

The important and promising approach is the time domain procedure, which may apply z-transform and modified z-transform or time domain matrices.

3.5. CONVENTIONAL TECHNIQUES OF CONTINUOUS SYSTEMS

The approaches commonly used are, the frequency response Bode diagrams and root locus methods.

The root locus method was discussed in section 3.3., where the discrete compensation case was also considered.

The frequency response in the z-plane was discussed in section 3.2.

*Reference 86, Ch. 6.
The same procedure for discrete compensation is followed, which is even more simpler as the open loop locus is then the product of the controller and system loci. Therefore, choosing a certain locus, the controller locus and hence its pulse transfer function can be determined.

The design can also be carried out in the w-plane, from the transform of equation 2.5.7, transforming the unit circle of the z-plane into the left-hand side of the w-plane. The rational function in z is then transformed into a rational function in w. The synthesis can be carried out using asymptotic plots or Nichols charts, following conventional continuous system rules, by putting \( w = jv \) where \( v \) is a fictitious frequency.

However, all these methods are based on the response at sampling instants only. The intersample response should be checked. More than one trial may be needed for satisfactory continuous output response.

Nevertheless, the last method (w-plane) can take into consideration the intersample response, by transforming the modified z-transform instead of the z-transform into the w-plane. The resultant function (a rational function in w) will have \( m \) as parameter, where \( m \) is as assumed in section 2.4.d.

As mentioned earlier one common condition must be satisfied, that

\[ \text{Reference 67} \]
\[ + \text{ Reference 44} \]
\[ ++ \text{ Reference 105, pp. 465-479.} \]
\[ +++ \text{Reference 49, p. 172.} \]
is the controller must be physically realizable, i.e. process only present and past informations.

3.6. Z-TRANSFORM METHODS

Many synthesis methods are based on the z-transform and the modified z-transform. Here some of the basic methods of approach are discussed.

3.6.1. Characteristic Equation Approach

The characteristic equation, as its name implies, determines the system performance and its stability. So it can be specified for a required overall response. However, the relations between the characteristic equation root location and time response should be evaluated.

For the closed loop system of figure 3.6.1, the overall pulse transfer function is given by

$$Y(z) = \frac{D(z)G(z)}{1+D(z)G(z)} = \frac{P(z)}{Q(z)}$$

(3.6.1)

Figure 3.6.1

Q (z), the characteristic equation is specified. To simplify the evaluation of D (z), an equivalent open loop system of the same performance is used, as that of figure 3.6.2.

\[
\frac{D_o(z) \cdot G(z)}{Q(z)} = \frac{P(z)}{Q(z)} = \frac{Q(z) - 1}{Q(z)}
\]

Hence

\[
D_o(z) = \frac{Q(z) - 1}{G(z) \cdot Q(z)}
\]

and

\[
D(z) = \frac{D_o(z)}{1 - D_o(z) \cdot G(z)}
\]

The main disadvantage of this method, is that it is based on the response at sampling instants only, which may not be adequate. The intersample response should be checked using any of the appropriate methods, e.g. the modified z-transform.

3.6.2. Minimal Prototype Approach

This is based on the error pulse sequence being modified by the discrete controller, such that the resultant response satisfies certain requirements, which are:

1. The steady state error must be zero after a finite number of sampling periods.
2. The transient response should be as fast as possible.

The procedure can be explained by referring to figure 3.6.1, from which

\[ E(z) = R(z) - C(z) \]
\[ = R(z) - R(z)Y(z) \]
\[ = R(z) (1 - Y(z)) \quad (3.6.5) \]

The condition that the steady state error equals zero after a finite number of sampling periods, is found by applying the final value theorem, giving

\[ e(\infty) = \lim_{z \to 1} (1 - z^{-1}) E(z) \]
\[ = \lim_{z \to 1} (1 - z^{-1}) R(z) (1 - Y(z)) \quad (3.6.6) \]

But \( R(z) = \frac{A(z)}{(1 - z^{-1})^l} \)

Equation 3.6.7 is for certain aperiodic test inputs, e.g. step.

From equations 3.6.6 and 3.6.7, it can be seen that the steady state error = 0 if

\[ (1 - Y(z)) = (1 - z^{-1})^l F(z) \quad (3.6.8) \]

where \( F(z) \) is a function of \( z \), to be determined.

The minimal prototype response is defined such that \( F(z) \) is unity, resulting in a minimum order of \( Y(z) \) in \( z^{-1} \). The number of sampling periods after which the error is zero (at sampling instants) is determined by the order 1.

Knowing the type of the input, the minimal \( Y(z) \) is determined,

*References 86, p.63 and 105, p.163.
†Reference 86, p. 150. (86)
from which the controller pulse transfer function is derived as

\[
D(z) = \frac{Y(z)}{G(z)(1-Y(z))} = \frac{1 - (1-z^{-1})^{-1} F(z)}{G(z)(1-z^{-1})^{-1} F(z)}
\]

(3.6.9)

It should be pointed out that the minimal \( Y(z) \) is a finite polynomial in \( z^{-1} \), which in fact is the condition for finite settling time response. This is due to the fact that the impulsive response of a system containing only a numerator polynomial in \( Y(z) \) is of finite duration.*

\( Y(z) \) is evaluated from equation 3.6.8, by putting \( F(z) = 1 \) for minimal response, from which \( D(z) \) can be determined by equation 3.6.9.

This method is based at the response at sampling instants only. Generally speaking the minimal response contains lot of intersample ripples, which can be detected.

Another limitation of this method is that it can only be applied for open loop stable systems. This can be demonstrated by considering the overall pulse transfer function.

\[
Y(z) = \frac{E(z)}{R(z)} \cdot D(z) \cdot G(z)
\]

= \( (1 - Y(z)) \cdot D(z) \cdot G(z) \)

= \( (1 - z^{-1})^{-1} F(z) \cdot D(z) \cdot G(z) \).

(3.6.10)

If \( G(z) \) contains any poles outside the unit circle, then these poles are contained in the overall pulse transfer function, leading to instability. These poles may be cancelled either by \( D(z) \) or \( F(z) \).

*Reference 86, p. 152.
However, this is not advisable, as incomplete cancellation, due to some drift of the parameters, will still result in instability.

Anyway, unstable systems rarely exist. But the system may contain a multiple pole at the origin in the s-plane, which transforms into a multiple pole at $z = 1 + j0$ on the z-plane. For a finite settling time response, at least one of these poles requires cancellation, which may be accomplished by continuous feedback around the original system.*

Because of these limitations, this method can only be used for preliminary studies and where the ripple content is permissible.

3.6.3. Ripple Free Response Approach

The ripple content can be reduced by multirate sampling, where the controller output is sampled at a multiple rate of its input. This generally reduces the ripples but does not eliminate it.

Ripple free responses can be obtained by extending the finite settling time method considered in the previous section.

The condition for ripple free response can be established from the fact that the controlled system must be capable of generating a continuous output equalling that of the input between sampling instants. This, of course, is after the settling interval, where the output must equal the input at any instant.

*Reference 49, pp. 207-208.
Considering the intersample response to be the impulsive response of the system, then for the output to follow the input, after the settling interval, the controlled system transfer function $G(s)$ must contain $R(s)$, the input transfer function as a factor.

Usually $R(s) = \frac{1}{s^n}$ for the test aperiodic inputs used. Hence the conditions for ripple free response are:

i) The steady state error must be zero after the shortest possible time.

ii) The overall pulse transfer function must be a finite polynomial in $z^{-1}$.

iii) The controlled system transfer function $G(s)$ must contain the poles of $R(s)$.

Based on these assumptions, the ripple free or deadbeat approach is developed. $G(s)$ is assumed to satisfy condition (iii) above.

![Figure 3.6.3](image)

Referring to figure 3.6.3, and assuming no controller, then the corresponding overall pulse transfer function is found to be

$$Y'(z) = \frac{G(z)}{1+G(z)} = \frac{P(z)}{P(z) + Q(z)} \quad (3.6.11)$$

where $G(z) = \frac{P(z)}{Q(z)}$
The action of the controller is to modify equation 3.6.11, into one satisfying condition (ii) above. This means that the denominator of $Y'(z)$, must be made independent of $z^{-1}$. Hence the required overall pulse transfer function would be

$$Y(z) = P(z) H(z) \quad (3.6.12)$$

where $H(z)$ is a polynomial in $z^{-1}$.

Applying condition (i) above, leads to

$$(1 - Y(z)) = (1 - z^{-1})^1 F(z) \quad (3.6.13)$$

Which is the same as equation 3.6.8.

If $H(z)$ of equation (3.6.12) is made constant, the response is then deadbeat (no ripples) for the class of input determined by 1 of equation (3.6.13).

By equating the coefficients of equal powers of $z^{-1}$ in equations 3.6.12 and 3.6.13, the coefficients of $F(z)$, or $H(z)$ are determined.

Generally speaking, the response of the system to other inputs, different from that designed for, is not satisfactory. The response can then be improved by adding more terms to $H(z)$, depending on the new requirements of the response. Naturally this would lead to a longer settling time for the original input.

The controller action is actually cancelling the unwanted poles of the overall pulse transfer function and replacing it by others at the origin of the $z$-plane.

The controller pulse transfer function is found from

$$D(z) = \frac{Y(z)}{1-Y(z)} \frac{1}{G(z)} \quad (3.6.14)$$
The restrictions imposed on the controller, are physical realizability, and non cancellation of any pole outside the unit circle on the z-plane.

So far the synthesis is based on specifications at sampling instants only. The intersample response has been found to be quite adequate for ripple free or deadbeat response, which is a satisfactory one.

The response can also be specified in between sampling instants, e.g. given maximum overshoot at say 2.5 T, where T is the sampling period. These and other general response specifications can be dealt with using the modified z-transform, or even better using time-domain matrices as will be considered in the following section.

3.7. **TIME DOMAIN MATRIX METHOD**

This approach is based on the time domain analysis of section 2.8.1.

Consider the system shown in figure 3.7.1.

![Figure 3.7.1](image_url)

*Reference 49
+Reference 25.
The relation between the output and input sequence is found to be:

\[ \mathcal{C} = \mathcal{G} \mathcal{D} (\mathbb{I} + \mathcal{G} \mathcal{D})^{-1} \mathcal{R} \]  

(3.7.1)

The output \( \mathcal{C} \) is specified for a certain \( \mathcal{R} \), and the controller matrix can be evaluated.

However, this involves complicated matrix processes. A much easier method is by evaluating the controller determinant term by term as follows:

\[ \mathcal{C} = \mathcal{G} \mathcal{\tilde{e}} \]  

(3.7.2)

or

\[ \mathcal{C} = \mathcal{G} \Delta \mathcal{e} \]  

(3.7.3)

where \( \Delta \) is as specified in section 2.8.1.

\[ \mathcal{\tilde{e}} = \mathcal{\Delta} \mathcal{E} \]  

(3.7.4)

\[ \mathcal{E} = \mathcal{R} - \mathcal{C} \]  

(3.7.5)

As an example assume \( C_1, C_2, C_{2.5} \) to be specified, where the suffix indicates the time in terms of sampling periods.

From equation 2.8.7 and 3.7.2, taking into consideration that \( \mathcal{E}_0 \) of equation 2.8.7 equals zero, we have

\[ C_0 = 0 \]

\[ C_1 = \mathcal{G}_1 \mathcal{e}_0 \quad \text{or} \quad \mathcal{e}_0 = \frac{C_1}{\mathcal{E}_1} \]  

(3.7.6)

The controller matrix \( \mathcal{D} \) is expressed as

\[
\begin{bmatrix}
d_0 & o & o & . \\
d_1 & d_0 & o & . \\
d_2 & d_1 & d_0 & . \\
. & . & . & .
\end{bmatrix}
\]  

(3.7.7) a
Its equivalent \( D(z) = d_0 + d_1 z^{-1} + d_2 z^{-2} + \ldots \) \( (3.7.7.b) \)

As \( C_0 = 0 \) hence \( E_0 = R_0 \), giving, from equation \( 3.7.4 \).

and \( 3.7.7.a \).

\[
e_0 = d_0 R_0 \quad \text{or} \quad d_0 = \frac{e_0}{R_0} = \frac{C_1}{\varepsilon_1 R_0} \quad (3.7.8)
\]

At the second sampling instant, the same procedure is followed leading to

\[
C_2 = e_2 e_0 + e_1 e_1
\]

or

\[
e_1 = \frac{C_2 - e_2 e_0}{e_1} \quad (3.7.9)
\]

\[
E_1 = R_1 - C_1 = R_1 - e_1 e_0 \quad (3.7.10)
\]

\[
e_1 = d_1 R_0 + d_0 E_1
\]

\[
= d_1 R_0 + d_0 (R_1 - e_1 e_0) \quad (3.7.11)
\]

From \( 2.7.9 \) and \( 2.7.11 \). \( d_1 \) is determined.

The output \( C_{2.5} \) will not alter the controller element \( d_0 \) and \( d_1 \), but is determined by \( d_2 \). For its evaluation equation \( 3.7.3 \) is used, where \( \bar{G} \) is the same as that of equation \( 2.8.10 \).

Following the procedure gives

\[
C_{2.5} = e_{2.5} e_0 + e_{1.5} e_1 + e_{0.5} e_2
\]

from which \( E_2 \) is determined to be

\[
e_2 = \frac{C_{2.5} - e_{2.5} e_0 - e_{1.5} e_1}{e_{0.5}} \quad (3.7.12)
\]

\[
E_2 = R_2 - C_2 \quad (3.7.13)
\]

\[
e_2 = d_2 E_0 + d_1 E_1 + d_0 E_2 \quad (3.7.14)
\]

From the last three equations, \( d_2 \) is determined.
It should be noticed that \( d_0 \) influences the response for \( 0 < t < T \), and generally \( d_n \) controls the response for \( nT < t < (n+1)T \). Therefore it is only possible to specify one constraint at each sampling instant or between any two sampling instants.

The controller elements are evaluated as above term by term, and put in the form of equation 3.7.7.a.

The real difficulty here lies in expressing \( D \) in a closed form. This may be overcome by choosing a certain controller pulse transfer function \( D(z) \), which is the ratio of two polynomials in \( z^{-1} \), and by synthetic division is expressed in an infinite series in \( z^{-1} \).

By inversion the coefficient of \( z^{-n} \) stands for \( d_n \) in the controller matrix. By relating \( d_n \) to the coefficients of \( D(z) \), the latter can be formulated. In practice only a few columns of the controller matrix are adequate for satisfactory response.

The choice of the proper \( D(z) \) depends on experience with similar situations in the \( z \)-plane.

Another method for determining the controller polynomials is introduced in section 3.8.2, following systematic and straightforward approaches.

The time domain synthesis method is usually more practical and logical than working in other domains. Moreover it can be used for a wider range of configurations as indicated towards the end of section 2.8.1.

*Reference 25.*

(94)
3.8. SYNTHESIS OF SYSTEMS INCORPORATING VARIABLE TIME DELAYS

3.8.1. Controller Forward loop Compensation

Consider the system of figure 3.8.1, assuming for the moment that the delay is constant = $lT$, where $l$ is an integer.

\[ R(z) + \frac{D(z)}{G(z)} \rightarrow C(z) \]

\[ y \]

Figure 3.8.1

A deadbeat response is taken as the required response, and the system $G(z)$ is assumed to satisfy the conditions of section 3.6.3.

If $G(z) = \frac{P(z)}{Q(z)}$, then from equation 3.6.12, the overall pulse transfer function is given by

\[ Y(z) = P(z) z^{-l} H(z) \]

and

\[ D(z) = \frac{H(z) Q(z)}{1 - H(z) P(z)} z^{-l} \]

\[ = \frac{A(z)}{1 - B(z)} \]

(3.8.2)

As discussed earlier, for a step response $H(z)$ is taken as a constant, determined from equations 3.6.12 and 3.6.13.

The corresponding transient behaviour lasts for $(p+1)$ periods, where $p$ is the order of the finite polynomial $P(z)$. Consequently the output can be expressed as

\[ C(z) = \sum_{n=0}^{p+1} C_n z^{-n} + \sum_{n=p+1+1}^{\infty} r_n z^{-n} \]

(3.8.3)

where the suffix $n$ denotes the amplitude at the $n$th sampling instant.

(95)
The last equation follows from the specifications of the deadbeat response.

The first part of the R.H.S. of equation (3.8.3) represents the transient behaviour of the response and the latter the steady state behaviour.

Inversion of equation 3.8.3. gives,

\[ C(nT) = \sum_{n=0}^{p+1} c_n \delta(t-nT) + \sum_{n=p+1+1}^{\infty} r_n \delta(t-nT) \] (3.8.4)

The block diagram of figure 3.8.1. can be redrawn as that of figure 3.8.2.

![Figure 3.8.2](image)

The controller pulse transfer function is represented as shown in figure 3.8.2. This follows from,

\[ \frac{e(z)}{E(z)} = \frac{A(z)}{1-B(z)} \]

or \[ e(z) = A(z)E(z) - B(z)e(z). \]

Assume for the time being that the delay \( 1T \) is larger than the finite settling time of \( C'(z) \) (which is exactly the same as \( C(z) \)).

Therefore \( C'(z) \) is given by \( C'(z) = G(z)A(z)R(z) \) (3.8.5.)

This follows since \( B(z) \) contains \( z^{-1} \) as a factor. In other words the system is practically open looped and its output response is governed by the controller forward loop \( A(z) \).

(96)
C'(z) will eventually attain its steady state value, which as specified for deadbeat response equals the input.

The first detected feedback signal around the closed loop, is at \( t = (1+1)T \), \( (C'_o = 0) \). Also at the same instant, the first detected feedback signal around the controller loop is contributing to \( e (1+1 \ T) \), the plant input, say, at \((1+1)th \) instant.

To maintain the steady state response as it is, the above two signals must cancel each other at that instant and at the subsequent instants.

From the above argument, it can be deduced that the controller forward loop \( A(z) \) shapes the response of the system, while the controller feedback loop \( B(z) \) neutralizes the feedback sequence such that the output of the controller is always \( A(z)R(z) \).

This method of synthesing the controller, on an open loop basis, can be used for sampled data systems in general, not necessarily incorporating time delays. Also, time domain synthesis can be carried out systematically using this method, which may be called the forward loop compensation procedure.

3.8.2. Delay Dependent Settling Time

Now let us consider the effect of delay variation upon the system response, particularly the transient component.

As seen from equation 3.8.4, the transient component is completely
determined by the order and amplitude of its sequence, i.e. the strength of the impulses and their positions along the time axis.

Delay variation directly affects the order of the sequence, as $l$ varies correspondingly, and indirectly affects the sequence amplitude.

A specification is set such that the order and amplitude of the transient sequence is made independent of the delay variation. To start with, the order of the sequence can be made the same, by making the sampling period proportional to the delay, thus $l$ is the same. But the transient sequence amplitude is a function of the sampling period $T$. Thus to maintain the sequence amplitude, it must be made independent of $T$.

This specification may be sound for some processes where the response is related to the delay. A quick response may be required for short delays, and relatively slower response for longer delays, e.g. flow processes where the delay is inversely proportional to the flow.

The synthesis problem may be summarised as; given a controlled system incorporating a variable time delay and assuming the sampling period $T$ to be proportional to the delay; how can a predetermined output sequence be made independent of $T$?

It may be added that the rate of change of the delay is so slow, with no appreciable variation over a settling time interval.

In this case the sampling period $T$ acts only as a time scaling
factor. This specification may be termed delay dependent settling time.

It can be seen that the transient output amplitude have to satisfy two simultaneous constraints at any one sampling instant. Therefore two independent parameters are needed to control the output amplitude at any sampling instant.

For this purpose, finite pulse width sampling is applied, where the two independent parameters are the sampler output amplitude and the sampling duration.

It is clear that $C'(nT)$ of figure 3.8.3 depends upon the amplitude of the input to $G$, $e(m-1)T$ and the corresponding pulse width.

\[ E(nT + h_n) \]

\[ C(t) \]

Figure 3.8.3

The general time domain treatment of section 2.8.2 is used.

The discrete controller used here is implemented by using unit delays as explained in section 4.1.

For the finite pulse width case, the delayed signal is $E(nT + h_n)$, where $h_n$ is the sampling duration at the $n^{th}$ sampling instant.
Controller Forward Loop Evaluation

Following the argument of section 3.8.1, the controller forward loop parameters are evaluated, so that the output $C'(t)$ satisfies the imposed constraints, i.e. a specified sequence $C_1, C_2, \ldots$ independent of $T$.

The controller forward loop may be as shown in figure 3.8.4.

![Diagram](image)

Figure 3.8.4

The controller output $e$ assumes two states, one during the hold interval, where it is a constant held signal. The other is that during the sampling interval containing a component representing the corresponding continuous input.

$E(nT + h_n)$ is denoted by $E_n$, and $E_0, E_1, E_2, \ldots$ are represented by a vector, or a column matrix $\bar{E}$. The corresponding output column matrix is $\bar{e}$.

For the sampling interval $E(t)_{nT < t < nT + h_n}$ is denoted by $E_{cn}$ and $E_{c0}, E_{c1}, \ldots$ are represented by $\bar{E}_c$. The corresponding continuous output is $\bar{e}_c$.
The relations between the controller output and input for the two intervals are,

\[ e = \bar{A} \cdot \bar{E} \quad (3.8.6) \]

and \[ \bar{e}_c = \bar{A}_o \cdot \bar{E}_c + (\bar{A} - \bar{A}_o) \cdot \bar{E} \quad (3.8.7) \]

where

\[ \bar{A} = \begin{bmatrix} A_0 & 0 & 0 & 0 & 0 \\ A_1 & A_0 & 0 & 0 & 0 \\ A_2 & A_1 & A_0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_k & A_{k-1} & A_{k-2} & \cdots & A_0 \end{bmatrix} \quad (3.8.8) \]

\[ \bar{A}_o = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \]

diagonal matrix of order k \( (3.8.9) \)

The parameters are evaluated by a step by step procedure, assuming zero initial conditions and continuous output response.

For \( 0 < t < h_o \), we have

\[ e_{co} = A_o \cdot E_{co} = A_o \cdot R_{co} \quad (3.8.10) \]

Thus from equation 2.8.35, \( \bar{X} \) is given by

\[ \bar{X}(t) = A_o \int_0^t \bar{G}(t-\chi) \cdot R_{co} \cdot (\chi) \ d\chi \quad (3.8.11) \]

From the last equation \( \bar{X}(h_o) = \bar{X} \bigg|_{t=h_o} \quad (3.8.12) \)

which constitutes the initial condition for the subsequent interval.

For \( h_o < t < T \), the input to \( \bar{G} \) is zero, then from equations 2.8.21 and 2.8.34, and by change of variables
\[ \bar{x}(\tau) = \bar{g}(\tau) \bar{x}(h_o) \quad (3.8.13) \]

where \( \tau = t - h_o \) and \( 0 \leq \tau \leq (T-h_o) \).

The required output \( C'(T) \) at \( t = T \) is selected from the last equation by putting \( \tau = T - h_o \). \( C'(T) \) is a function of \( A_o \) and \( h_o \), which are determined from the two equations

\[
C'(T) = C_1 \quad (3.8.14)
\]

and \[ \frac{d C'(T)}{d T} = 0 \quad (3.8.15) \]

For \( T < t < T+h_1 \) we have

\[
e_{c_1} = A_o E_{c_1} + A_1 E_o = A_o R_{c_1} + A_1 R_o \quad (3.8.16)
\]

Knowing the initial conditions at \( t = T \), then

\[ \bar{x}(\tau) = \bar{g}(\tau) \bar{x}(T) + \int_0^{\tau} \bar{g}(\tau - \chi) e_{c_1}(\chi) \, d\chi \quad (3.8.17) \]

where \( \tau = t - T \) and \( 0 \leq \tau \leq h \).

From equation 2.8.17 \( \bar{x}(T + h_1) = \bar{x} \bigg|_{t = h_1} \quad (3.8.18) \)

For \( T+h_1 < t < 2T \), again the input to \( \bar{g} \) is zero, from which

\[ \bar{x}(\tau) = \bar{g}(\tau) \bar{x}(T + h_1) \quad (3.8.19) \]

where \( \tau = t - T - h_1 \).

Again the required output \( C'(2T) \) at \( t = 2T \) is selected from the last equation by putting \( \tau = T - h_1 \).

\( C'(2T) \) is a function of \( A_o, h_o, A_1 \) and \( h_1 \). But the first two are already determined, therefore \( C'(2T) \) is only a function of \( A_1 \) and \( h_1 \). These are determined from,

(102)
\[ C'(2T) = C_2 \]

and

\[ \frac{d C'(2T)}{d T} = 0 \]

By repeating the procedure all the parameters can be determined one by one, for the specified response.

**Controller Feedback Loop Evaluation**

Referring to figure 3.8.3. and taking the controller feedback loop into consideration, its output during the sampling interval is given by

\[ \tilde{e}_C = A_0 \cdot \tilde{E}_C + (A - A_0) \cdot \tilde{E} + B \cdot \tilde{e} \]

(3.8.20)

where \( B \) is a matrix to be determined.

As explained in section 3.8.1, the last term of the R.H.S. of equation (3.8.20) is to neutralize the effect of the overall feedback, such that \( \tilde{e}_C \) is the same as that of equation 3.8.7, with \( E = R \).

\( B \cdot \tilde{e} \) is a staircase function, by the nature of the sample-and-hold elements simulating unit delays as explained in section 4.1. Hence the matching output feedback signal should also be of the same nature, so that the two can cancel each other. In other words a conventional sample-and-hold element is used in the feedback loop as shown in figure 3.8.5, to generate a staircase function.
By virtue of the superposition theorem, the block diagram of figure 3.8.5 may be redrawn as in figure 3.8.6. This is to simplify the evaluation of $\bar{B}$.

The delay is generally taken as

$$\lambda = (1 - \delta) T$$

(3.8.21)

where $\lambda$ is an integer and $0 < \delta < 1$

From figure 3.8.6, $\bar{e}_C$ is given by

$$\bar{e}_C = \bar{A}_o \bar{R}_e + (\bar{A} - \lambda \bar{A}_o) \bar{R} + \bar{B} \bar{e} - \bar{A} \bar{C}(\delta)$$

(3.8.22)

where $\bar{C}(\delta)$ is a vector representing $C(t) \big|_{t = (nT + \delta T)}$

Therefore $\bar{e}_C$ can be made the same as that of equation 3.8.7.
(replacing E by \( R \)), if,
\[
\hat{B} \cdot \hat{e} - \hat{A} \cdot \hat{c}(\delta) = 0
\]
or
\[
\hat{B} \cdot \hat{e} = \hat{A} \cdot \hat{c}(\delta)
\]  \hspace{1cm} (3.8.23)

The only unknown in the last equation is \( \hat{B} \), which is evaluated term by term, taking into consideration that \( C_n(\delta) = 0 \) for \( n \ll 1 \).

The above discussed approach for the controller design, is actually independent of the delay present (in terms of sampling periods), and can even be used for systems incorporating no delay.

The technique can also be used for synthesis of finite pulse width discrete systems in general.

This method can be used successfully for discrete controller design in the time domain applying conventional sampling. The controller is first expressed as
\[
\frac{\hat{A}}{1 + \hat{B}}
\]  \hspace{1cm} (3.8.24)

where \( \hat{A} \) is determined by the response specifications and \( \hat{B} \) from equation 3.8.23, using the appropriate \( \hat{c}(\delta) \).

This approach removes the difficulty encountered in formulating the discrete controller in a closed form mentioned towards the end of section 3.7.

3.8.3. Delay Independent Settling Time.

In the preceding section the controller forward loop parameters, as well as the sampling durations were made such that the output sequence is independent of \( T \). This procedure was followed because
the sampling period was made proportional to the delay.

However, a closer look at equations 3.8.21, 3.8.22 and 3.8.23 reveals that $\tilde{e}_0$ and consequently $C'(t)$ can be made the same for a constant sampling period, irrespective of the delay variation by controlling $\tilde{B}$ only. This is because as the delay varies $\tilde{C}(s)$ varies consequently. Equation 3.8.23 can therefore be satisfied by adjusting or controlling the corresponding parameters of $\tilde{B}$ only.

This approach simplifies the evaluation of $\tilde{A}$ and the successive sampling durations, since the output sequence complies with one constraint only. Even conventional sampling can be used, applying the time domain matrices of section 2.8.1.

In applying either specifications, any response can be selected. The term settling time may then be replaced by the transient duration.

However, there is a condition that must be satisfied, i.e. the steady state error equals zero. This is the same as condition (iii) of section 3.6.3, where $G(s)$ must contain the poles of $R(s)$.

The two approaches introduced for variable delay systems are best illustrated by considering an example given in the following section.
The example considered here is worked out for delay dependent settling time.

The controlled system block diagram is that of figure 3.9.1.

\[ \begin{array}{c}
\hat{G} \\
\text{Figure 3.9.1}
\end{array} \]

The delay variation is assumed to be limited to some range, e.g. a 1:2 variation. Therefore the design procedure is limited to such a range.

The forward loop parameters as well as the sampling durations are generally functions of \( T \).

Mathematically, \( A_n \) and \( h_n \) can be formulated in terms of \( T \) such that the corresponding output at \((n+1)T\) is independent of \( T \), within its range of variation.

However, these mathematical relations may not be of practical value, if their implementation is not feasible. Therefore a numerical procedure is followed, where the controller forward loop parameters are formulated as functions of \( T \), over the working range. The procedure is simplified by making the ratio \( \frac{h}{T} \) constant.

\( \hat{G} \) is taken as that of a second order system which is the same as example 1, of section 2.8.2., where
The test function is a step, and zero initial conditions are assumed.

For the interval $0 \leq t < h$, the output is found from equations 3.8.6 and 3.8.11, and figure 3.8.4.

\[ e^{C_0} = A_0 \quad E^{C_0} = A_0 \]  

since $E^{C_0} = R^{C_0} = 1$ for a step.

\[ X(t) = \int_0^t \left[ \frac{t}{k} \left( 1 - e^{-a(t-\tau)} \right) e^{-a(t-0)} \right] d\tau \]

Therefore

\[ X(t) = \begin{bmatrix} A_0 t \\ \frac{A_0 k}{a} (a t + e^{-at} - 1) \end{bmatrix} \]

(3.9.3)

$X(h)$ is found by putting $t = h$ in the last equation.

Thus

\[ X(h) = \begin{bmatrix} A_0 h \\ \frac{A_0 k}{a} \left( aT \left( \frac{h}{T} \right) + e^{-ah} - 1 \right) \end{bmatrix} \]

(3.9.4)

The quantity $(aT(h/T) + e^{-ah} - 1)$ is termed $U_o$, therefore

\[ C'(h) = \frac{KA_0}{a^2} U_o \]

(3.9.5)
For the interval $h < t < T$, the input to the system is zero and from equation 3.8.13 $X(T)$ is given by

$$X(T) = \begin{bmatrix} 1 & 0 \\ \frac{K}{a} \left(1 - e^{-at}\right) & e^{-at} \end{bmatrix} \begin{bmatrix} A_0 h \\ \frac{K A_0}{a^2} \end{bmatrix}$$

(3.9.6)

where $\tau = t - h$

By putting $\tau = T - h$ $X(T)$ is obtained from the last equation to be

$$X(T) = \begin{bmatrix} A_0 h \\ \frac{K A_0}{a^2} \left( a T \left(\frac{h}{T}\right) + e^{-at} - e^{-at\left(1 - \frac{h}{T}\right)} \right) \end{bmatrix}$$

(3.9.7)

The output $C'(T) = \frac{K A_0}{a^2} \left( a T \left(\frac{h}{T}\right) + e^{-at} - e^{-at\left(1 - \frac{h}{T}\right)} \right)$

(3.9.8)

$h/T$ was assumed constant, it can be anything less than unity. It was taken here as 0.25.

The quantity $\left(\frac{h}{T} a T + e^{-at} - e^{-at\left(1 - \frac{h}{T}\right)}\right)$ is termed $V_1$, and is computed and tabulated against $aT$, for $aT = 1.4 \ (0.2) \ 3$ (See table 3.9.1)

Taking $C'(T) = 0.7$, the required value for $\frac{A_0 K}{a^2}$ was found for each $aT$, and its function formulated in terms of $aT$.

It was noticed that $\frac{A_0 K}{a^2}$ is inversely proportional to $aT$, therefore its reciprocal was approximated by a straight line. In fact $V_1$ is proportional to the reciprocal of $\frac{A_0 K}{a^2}$. The straight line approximation is to simplify the relation between $\frac{A_0 K}{a^2}$ and $aT$.

However, this results in some loss of accuracy, whereby a maximum deviation of about 1.5% occurs in the response over the working range.
\[
\frac{A_{o}K}{a^{2}} \text{ was found to be } \frac{2.55}{aT-0.49}
\] (3.9.9)

For the interval \( T \leq t < T+h \)

\( e_{C1} = A_{o} + A_{1} \) and \( X(T) \) is given by

\[
X(T) = \begin{bmatrix}
1 & 0 \\
\frac{K}{a} (1-e^{-aT}) & e^{-aT}
\end{bmatrix}
\begin{bmatrix}
\frac{A_{o}h}{a} \\
A_{K} V_{1}
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
\int_{T}^{T+h} (A_{o} + A_{1}) dX \\
\int_{T}^{T+h} \frac{A_{o} + A_{1}}{a} K (1-e^{-a(T-x)}) dX
\end{bmatrix}
\]
\text{(3.9.10)}

where \( \tau = t - T \)

By putting \( \tau = h \), \( X(T+h) \) can be obtained as

\[
X(T+h) = \begin{bmatrix}
(2A_{o} + A_{1}) h \\
\frac{A_{o} K}{a^{2}} \left( aT \left( \frac{h}{T} \right) + e^{-aT(1+\frac{h}{T})} - e^{-aT} + U_{o} \right) + \frac{A_{1} K}{a^{2}} U_{o}
\end{bmatrix}
\]
\text{(3.9.11)}

where \( U_{1} = aT \left( \frac{h}{T} \right) + e^{-aT(1+\frac{h}{T})} - e^{-aT} + U_{o} \) \text{(3.9.12)}

From equation 3.9.11, \( C'(T+h) \) is given by

\[
C'(T+h) = \frac{K}{a^{2}} \left( A_{o} U_{1} + A_{1} U_{o} \right)
\] (3.9.13)

For the interval \( T+h \leq t < 2T \), \( X(T) \) is given by,
\[
X(\tau) = \begin{bmatrix}
1 & 0 \\
\frac{K}{a} (1-e^{-a\tau}) & e^{-a\tau}
\end{bmatrix}
\begin{bmatrix}
(2A_o + A_1) h \\
\frac{K}{a^2} (A_o U_1 + A_1 U_0)
\end{bmatrix}
\] (3.9.14)

where \( \tau = t - h - T \), since the input is zero. \( X(2T) \) is found by putting \( \tau = T - h \) in the last equation, and is given by

\[
X(2T) = \begin{bmatrix}
\frac{(2A_o + A_1) h}{a^2} \\
\frac{A_o K}{a^2} (aT \left(\frac{h}{T}\right) + e^{-2aT} - e^{-aT(2-\frac{h}{T})} + V_1) + \frac{A_1 K}{a^2} V_1
\end{bmatrix}
\] (3.9.15)

where \( V_2 = aT \left(\frac{h}{T}\right) + e^{-2aT} - e^{-aT(2-\frac{h}{T})} + V_1 \) (3.9.16)

From equation 3.9.15 the output at the second sampling instant is given by

\[
C'(2T) = \frac{K}{a^2} (A_o V_2 + A_1 V_1)
\] (3.9.17)

Taking \( C'(2T) = 1 \), \( \frac{A_1 K}{a^2} \) is then computed by a similar procedure as that used for \( \frac{A_o K}{a^2} \). The maximum deviation from the set value over the range is about 1.5%.

\[
\frac{A_1 K}{a^2} = \frac{1.5216}{aT-0.767}
\] (3.9.18)

From equations 3.9.5 and 3.9.13 it can been seen that there is a correlation between the two, and by induction

(111)
\[ C'(2T+h) = \frac{K}{a^2} \left( A_0 U_2 + A_1 U_1 + A_2 U_0 \right) \quad (3.9.19) \]

where \[ U_2 = aT \left( \frac{h}{T} \right) + e^{-aT(2+\frac{h}{T})} - e^{-2aT} + U_1 \quad (3.9.20) \]

Also from 3.9.8 and 3.9.15
\[ C'(nT) = \frac{K}{a^2} \left( A_0 V_n + A_1 V_{n-1} + A_2 V_{n-2} \right) \quad (3.9.21) \]

and \[ V_n = aT \left( \frac{h}{T} \right) + e^{-anT} - e^{-aT(n-\frac{h}{T})} + V_{n-1} \quad (3.9.22) \]

as \( n \) tends to infinity
\[ V_n = aT \left( \frac{h}{T} \right) + V_{n-1} \quad (3.9.23) \]

also \[ C'(nT) = C'(n-1T) \quad (3.9.24) \]

The last equation is valid for step response only.

From 3.9.21 to 3.9.24 it is deduced that for zero steady state error,
\[ A_0 + A_1 + A_2 = 0 \quad (3.9.25) \]

from which \( A_2 \) can be found.

However, applying \( A_0, A_1 \) and \( A_2 \) in equation 3.9.19 for \( n = 3 \) results in unacceptable deviation on one side of the \( aT \) range.

Therefore \( \frac{A_2K}{a^2} \) is determined by equating \( C'(3T) \) to unity.

\[ C'(3T) = \frac{K}{a^2} \left( A_0 V_3 + A_1 V_2 + A_2 V_1 \right) \quad (3.9.26) \]

By similar procedure as evaluating \( \frac{A_0K}{a^2} \), the last parameter is given by
\[ \frac{A_2K}{a^2} = - \frac{1}{aT - 0.18} \quad (3.9.27) \]

Applying this formula gives a maximum deviation from the required value of about 2.5\% at \( t = 3T \).
To satisfy the steady state requirement another parameter $A_z$ is added such that

$$A_0 + A_1 + A_2 + A_3 = 0 \quad (3.9.28)$$

or

$$-\frac{KA_3}{a^2} = (A_0 + A_1 + A_2) \frac{K}{a^2} \quad (3.9.29)$$

Therefore equation 3.9.19 is rewritten as

$$C'(nT) = \frac{K}{a^2} (A_0 V_n + A_1 V_{n-1} + A_2 V_{n-2} + A_3 V_{n-3}) \quad (3.9.30)$$

Now let us consider the controller feedback loop. The sampling period is adjusted to bear a certain relation to the pure time delay. In this example the delay

$$\delta = 0.6 T$$

Therefore $\delta$ of equation 3.8.21 equals 0.4 and 1 of the same equation equals zero.

The sequence $C(nT + \delta T)$ is required to be used in equation 3.8.23.

$C_0(\delta)$ is obtained from equation 3.9.6. by putting

$$\tau = (\delta T - h), \text{ resulting in }$$

$$C_0(\delta) = \frac{KA_0}{a^2} \left( aT \left( \frac{R}{T} \right) + e^{-\delta aT} - e^{-aT\left( \frac{\delta - h}{T} \right)} \right)$$

$$= \frac{KA_0}{a^2} W_0 \quad (3.9.31)$$

where

$$W_0 = aT \left( \frac{h}{T} \right) + e^{-\delta aT} - e^{-aT\left( \frac{\delta - h}{T} \right)} \quad (3.9.32)$$

Similarly $C_1(\delta)$ is obtained from 3.9.14 by putting

$$\tau = (\delta T - h), \text{ resulting in }$$

(113)
\[ C_1(\delta) = \frac{K A_0}{a^2} W_1 + \frac{K A_1}{a^2} W_0 \]

where
\[ W_1 = aT \left( \frac{h}{T} \right) + e^{-(1+\delta) \frac{h}{T}} aT - e^{-(1+\delta - \frac{h}{T})} aT + W_0 \]

Following the same procedure
\[ C_2(\delta) = \frac{K A_0}{a^2} W_2 + \frac{K A_1}{a^2} W_1 + \frac{K A_2}{a^2} W_0 \]

where
\[ W_0 = aT \left( \frac{h}{T} \right) + e^{-(2+\delta) \frac{h}{T}} aT - e^{-(2+\delta - \frac{h}{T})} aT + W_1 \]

Now applying equation 3.8.23.

\[
\begin{bmatrix}
B_0 & 0 & 0 & 0 \\
B_1 & B_0 & 0 & 0 \\
B_2 & B_1 & B_0 & 0 \\
B_3 & B_2 & B_1 & B_0
\end{bmatrix}
\begin{bmatrix}
e_0 \\
e_1 \\
e_2 \\
e_3
\end{bmatrix} =
\begin{bmatrix}
A_0 & 0 & 0 & 0 \\
A_1 & A_0 & 0 & 0 \\
A_2 & A_1 & A_0 & 0 \\
A_3 & A_2 & A_1 & A_0
\end{bmatrix}
\begin{bmatrix}
C_0(\delta) \\
C_1(\delta) \\
C_2(\delta)
\end{bmatrix}
\]

\( e_0, e_1, \ldots \) are as defined in connection with equation 3.8.6.

Expanding and equating similar components

\[ B_0 = 0 \]
\[ B_1 e_0 = A_0 C_0(\delta) \]
\[ B_2 e_0 + B_1 C_1 = A_1 C_0(\delta) + A_0 C_1(\delta) \]
\[ B_3 e_0 + B_2 e_1 + B_1 e_2 = A_2 C_0(\delta) + A_1 C_1(\delta) + A_0 C_2(\delta) \]

From the above set of equations \( B_1, B_2 \) and \( B_3 \), are evaluated.

From equations 3.8.6 and 3.8.8 for a step function

\[ e_0 = A_0 \]
\[ e_1 = A_1 + A_0 \]
\[ e_2 = A_2 + A_1 + A_0 \]

(114)
Substituting in equation 3.9.37 gives

\[ B_1 = C_0(\delta) \]
\[ B_2 = C_1(\delta) - C_0(\delta) \]
\[ B_3 = C_2(\delta) - C_1(\delta) \] (3.9.39)

The last set of equations demonstrates the fact that the parameters \( B_1 \), \( B_2 \) and \( B_3 \) are directly related to the delay. This relation enables delay independent response to be carried out as discussed in the preceding section.

From the values computed for \( C_0(0.4) \) (see table 3.9.5), it is seen that it varies with \( aT \), in spite of the fact that \( C'(T) \) is practically constant. Therefore \( B_1 \) has to be slightly adjusted with the variation of \( aT \).

\( C_1(0.4) \) and \( C_2(0.4) \) are practically constant. Therefore \( B_3 \) is almost independent of \( aT \), and \( B_2 \) depends on \( aT \) to some extent.

Ideally if the responses are the same at any \( t \), then the controller feedback loop would be independent of \( aT \), and consequently the delay dependent response requires adjustable controller forward loop parameters. On the other hand delay independent response only requires adjustable controller feedback loop parameters.

So either way the amount of parameters controlling is roughly the same.
Numerical tables are attached at the end of this section. Also three representative responses, one on either side of the range and the third in between, are shown on figure 3.9.2.

As indicated earlier, the sampling period acts as a time scaling factor, therefore the three responses are drawn on the same time bases.

The simulation of the above worked example is considered in the following chapter.
### TABLE 3.9.1

<table>
<thead>
<tr>
<th>aT</th>
<th>V₁</th>
<th>V²</th>
<th>V³</th>
<th>U₀</th>
<th>U¹</th>
<th>U²</th>
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<td>0.40376</td>
<td>0.79032</td>
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<td>0.08763</td>
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<td>0.14881</td>
<td>0.70788</td>
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<td>2.1948</td>
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</table>

### TABLE 3.9.2

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<th>aT</th>
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<th>$\frac{A_{1,K}}{a^2}$</th>
<th>$\frac{A_{2,K}}{a^2}$</th>
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(116a)
### TABLE 3.9.3

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<th>C₁</th>
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</tr>
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### TABLE 3.9.4

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<th>W₁(0.4)</th>
<th>W₂(0.4)</th>
<th>W₀(0.6)</th>
<th>W₁(0.6)</th>
<th>W₂(0.6)</th>
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(116b)
### Table 3.9.5

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<th>C₁.₄</th>
<th>C₂.₄</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
</tr>
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<th>C₁.₆</th>
<th>C₂.₆</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
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(116c)
Figure 3.9.2.
CHAPTER FOUR
ANALOGUE COMPUTER SIMULATION

The numerically worked example was then simulated on an analogue computer to check the agreement between the numerical and simulated results.

In the example the pulse width over the sampling period was made constant, while the sampling period was made proportional to the pure time delay. This means that for any delay, the sampling period has to be adjusted proportionally, and the pulse width to be adjusted correspondingly.

However, the simulation can be greatly simplified if the delay, sampling period, and pulse width are kept constant, while varying the time constant $a$ of the controlled system. This follows, because as we have seen, the controller parameters are functions of $aT$ and $\frac{h}{T}$ only, for a constant ratio between the delay and the sampling period. But $\frac{h}{T}$ is taken as constant, so the parameters are only functions of $aT$, which can be varied by either varying $a$ or $T$.

In the present application, it is much easier to vary $a$ and adjust the controller parameters accordingly.

It should be pointed out that the controller parameters can be controlled automatically with respect to $a$ using proper analogue techniques. However, this is not done here as it is irrelevant to the immediate object of this work.
The block diagram of the analogue computer simulation is shown in figure 4.0.1:

\[ 
\text{DISCRETE CONTROLLER} \quad \text{CONTINUOUS TRANSFER FUNCTION} \quad \text{DELAY} 
\]

Controlled System

Figure 4.0.1

The implementation of each of the block diagram components is considered in the following sections.

4.1 THE DISCRETE CONTROLLER

The implementation of the controller requires some means of,

(a) Accepting and storing information

(b) Generating an output which is a suitably weighted function of present and past information.

\( \tilde{A}(t) \), as described earlier, may be represented by the block diagram of figure 4.1.1

(118)
The important feature of the implementation is the unit delay, capable of delaying the signal supplied to it by one sampling period.

The realisation of this unit delay may be achieved by using sample-and-hold circuits operating in a particular manner.*

Consider the chain of sample-and-hold circuits of figure 4.1.2, operated by $P_1$ and $P_2$, which are staggered by half the sampling period as shown in figure 4.1.3.

The input to switch 1, say, is sampled and held as indicated by trace 1 of figure 4.1.3.c.

For a finite pulse width sampler, the value held is that of the input at the instant the switch opens.

Half a sampling period after 1 was closed, switch 2 is closed for a while, allowing the signal held on $H_1$ to pass on to $H_2$, where it is again held. This is indicated by trace 2 on figure 4.1.3.c.

After a further half sampling period, or a complete sampling

*Reference 98.
Figure 4.1.2

Figure 4.1.3

(120)
period referred to switch 1, switches 1 and 3 are closed. This allows the present signal to appear at $H_1$ and the signal held on $H_2$ to pass on to $H_3$. Thus the present and immediate past signals are available for processing. By using more and more sample-and-hold circuits more past information can be made available.

The staggering between the two pulse sequences $P_1$ and $P_2$ is not critical, but it must ensure that the two sets do not coincide.

4.1.1. Sample-and-Hold Circuit

This is composed of the sampler and the hold device, which is usually a zero order hold. The term sample-and-hold is short hand for a sampler followed by a zero-order-hold circuit and is used in this sense throughout.

The sampler is in fact a switch or a gate with controlled closure duration. Many types of gates can be found in the literature ranging from the relay type switch to the electronic diodes.\textsuperscript{+}

Generally speaking the electronic switches are superior to the mechanical ones (e.g. relays) because of the much higher speed of operation. Also the power needed to drive a relay is much higher than that required for an electronic switch, whether it uses valves or semiconductors.

In the present application, electronic diode switches were used.

As the sampler is seldom used alone, the sample-and-hold circuit are discussed together as one combination.

\textsuperscript{Reference 15 and 59.}
\textsuperscript{Reference 34 and 98.}
The choice of any particular configuration of the sample-and-hold is decided upon by considering the special requirements for a given application.

![Circuit Diagram]

Figure 4.1.4

For the present case the requirements call for:

i) Unit gain or very nearly, during the sampling interval.
ii) Bidirectional operation, accepting positive and negative signals.
iii) Maintenance of the d.c level of the sampled signal.
iv) No appreciable change of the signal held during the hold interval.

The symbolic circuit shown in figure 4.1.4 satisfies the above requirements.

In this indirect sample-and-hold circuit, the switch is connected between the virtual earth of the operational amplifier, and the junction of the input and feedback resistors.

During the sampling interval, the output follows the input with a time constant $RC_h$, which should be as small as possible. During the hold interval, the output is held at its initial value because the current drawn by the virtual earth of the amplifier is very small indeed.

*Reference 15, 34 and 98.
The actual circuit used is that of figure 4.1.5.*

![Diagram of the circuit](image)

*Figure 4.1.5*

When positive and negative voltages $E_c$ are applied to the free terminals of $D_3$ and $D_4$ respectively, $D_3$ and $D_4$ are back biased. $D_1$ and $D_2$ are then conducting by virtue of the bleeding resistors $R_b$, and the sampler is closed.

The current to the virtual earth is contributed by the input $e_i$, output $e_o$ and plus and minus H.T supplies. With the input terminal earthed, the potentiometer $r$ can be adjusted, so to make the output potential zero. This balances inequality in the bleed resistors and/or the H.T. supplies. Then the gain can be adjusted to be very close to unity by the potentiometer $r'$, balancing any inequality between the input and feedback resistors and/or the diode characteristics. In any case the change in diode characteristics

---

*Reference 98.*
(forward resistance) is very small as the two diodes are in the same envelope and carry the same current.

When negative and positive voltages $E_b$ are applied to the free terminals of $D_3$ and $D_4$ respectively, $D_3$ and $D_4$ conduct. The anode of $D_1$ and the cathode of $D_2$ are then held at negative and positive potentials respectively. Consequently $D_1$ and $D_2$ are nonconducting and the sampler is open.

For satisfactory operation, the currents or potential differences due to the input signal must not alter the state of any of the diodes.

When the switch is closed, the current in $D_1$ or $D_2$ is given by
\[ i_c = \frac{V}{R_b} + \frac{e_i}{R} \]  
\[ (4.1.1) \]

where $R = R_1 = R_2$.

The limiting case is when $i_c = 0$, in which case
\[ e_i_{\text{max}} = \frac{R}{R_b} V \]  
\[ (4.1.2) \]

When the switch is open, the current in $D_3$ or $D_4$ is given by
\[ i_o = \frac{V + Eb}{R_b} + \frac{Eb + e_i}{R} \]  
\[ (4.1.3) \]

assuming the pulse source output impedance to be small. Again the limiting case is when $i_o = 0$, resulting in
\[ e_i_{\text{max}} = \frac{V R}{R_b} + Eb \left( \frac{R}{R_b} + 1 \right) \]  
\[ (4.1.4) \]

Hence the maximum input signal is determined by equation 4.1.2,
\[ e_i_{\text{max}} = \frac{R}{R_b} V \]
\[ \frac{R}{R_b} \] is usually greater than unity, so the maximum permissible input signal can be greater than the H.T. supply voltage (300 volt). However, in practice the input signal does not exceed 100V, which is the maximum output from the operational amplifiers used.

The control voltage levels \( E_c \) and \( E_b \) need be only a few volts sufficient to maintain the state of the diodes. This is because the potential difference across say, \( D_1 \) or \( D_2 \), when conducting is in practice a fraction of a volt.

The Choice of the Hold Capacitor

This is governed by two factors.

a) The speed of response, calling for a small capacitor

b) The accuracy, calling for a large capacitor.

As usual a compromise is worked out.

a) **Speed of Response.** The equivalent circuit when the sampler is closed is shown in figure 4.1.6.

![Figure 4.1.6](image)

If \( e_i \) is a step at \( t = 0 \) and \( e_o = 0 \) for \( t < 0 \), then

\[ e_o = -e_i \left(1 - e^{-\frac{t}{RC_h}}\right) \quad (4.1.5) \]

If the sampling duration is \( \gamma \), say, then \( e_o = e_i \) if \( \gamma \) is several times \( RC_h \). For \( \gamma = 7 RC_h \), the error is less than 0.1%.
Alternatively, where the switch closes for a considerable duration, the choice of the capacitance may be considered in terms of the highest frequency \( w \) contained in \( e_i \).

We have

\[
E_o(jw) = \frac{1}{1 + jwR_Ch} E_i(jw)
\]  

where \( E_o, E_i \) are Laplace transforms of \( e_o, e_i \) respectively.

For an error of less than 0.1%, \( w \leq \frac{0.05}{R_Ch} \)

b) **Accuracy.** During the hold interval, there should not be any connection between the input and output, also the held signal must not change appreciably. These and other similar effects are best illustrated by considering the equivalent circuit for the hold interval, as shown in figure 4.1.7.

![Figure 4.1.7](image)

The input signal is usually of low frequency and as the stray capacitance \( C_d \) is of the order of 5 pf, the input signal cannot reach the output terminal. The input signal is also greatly attenuated at the junction between \( R \) and \( R_b \) due to the relatively small output impedance of the pulse source.
A far more important effect due to these stray capacitances is that due to imbalance in diode capacitors and/or imbalance in the control voltage pulses.

This effect is examined by shortening all the supplies except the control voltage sources and finding the change in the amplifier output due to the imbalances mentioned above.

The currents to the virtual earth of figure 4.1.7 (with the supplies earthed) are through $C_h$, $C_{d1}$ and $C_{d2}$.

The current through $C_{d2}$ is contributed by $V_{2c}$ and by $e_o$ acting through $R_f$. However, the contribution due to $e_o$ is negligible due to the relatively low impedance of the pulse source. Hence the equivalent circuit can be reduced to that of figure 4.1.8, where

![Figure 4.1.8](image)

$R'$ is the pulse source impedance, the input resistance and the bleeding resistance, all in parallel and

$$V_o' = V_o \frac{RfR}{Rs + \frac{RbR}{Rb+R}} \quad \frac{1}{2} V_c \quad (4.1.7)$$

$Rs$ being the source resistance.

From the virtual earth equation

$$-i_o = i_1 + i_2$$
and taking the L.T. assuming \( V_c \) to be a step applied at the start of the hold interval gives

\[
-I_o = I_1 + I_2 = \frac{V_{1c} C_d}{1 + SR' C_d} - \frac{V_{2c} C_d}{1 + SR' C_d}
\]

\[
\lambda = \frac{V_{1c} C_d}{1 + SR' C_d} - \frac{V_{2c} C_d}{1 + SR' C_d}
\]

as \( R'C_d \ll 1 \).

\( I_o \) results in a change of potential across \( C_h \) equal to

\[
\delta E_o = \frac{I_o}{\delta C_h}
\]

(4.1.8)

\( \delta E \) is found by taking the inverse transform

\[
\delta E = \frac{V_{1c} C_d}{C_h} - \frac{V_{2c} C_d}{C_h}
\]

(4.1.9)

Putting \( V_{1c} = V_{2c} \pm \delta V \) and \( C_d = C_d \pm \delta C \), and neglecting infinitesimal of second order, gives

\[
|\delta E_o| = \frac{V_c \delta C_d}{C_h} + \frac{C_d \delta V_c}{C_h}
\]

so the maximum error is

\[
\delta E_o_{\text{max.}} = \frac{V_c \delta C_d}{C_h} + \frac{V_c \delta C_d}{C_h}
\]

(4.1.10)

This error is readily reduced by increasing \( C_h \).

Another important source of error, is that the controlling pulses may not be simultaneously applied. This means that only one diode is conducting for a very short period. In this case a maximum

*Reference 98.
current of \( \frac{V}{R_b} + \frac{e_i \text{ (max)}}{R} \) may continue to charge or discharge the hold capacitor for an interval \( \delta t \) resulting in a change of potential

of \( \delta e_{\text{max.}} = \frac{I_{\text{max}}}{C_h} \delta t \) \hspace{1cm} (4.1.12)

But \( \frac{V}{R_b} = \frac{300 \text{ Volts}}{300 \text{ K.ohm}} = 1 \text{ m A.} \)

and \( \frac{e_i \text{ max}}{R} = \frac{100 \text{ vclts}}{1 \text{ M.ohm}} = 0.1 \text{ m A.} \)

so \( \delta e_{\text{max.}} = \frac{1.1 \text{ m A}}{C_h} \delta t \)

which is almost independent of the input signal.

Limiting \( \delta e_{\text{max.}} \) to about 20 mV and taking \( C_h = 1000 \text{ pf} \) then \( \delta t \ll 0.02 \text{ U.S.} \)

Again increasing \( C_h \) would reduce this effect.

Effect of the Operational Amplifier Input Current and Back Biased Diode Currents.*

The combined effect of these currents results in some change of the potential difference across \( C_h \) during the hold interval. Quantitatively this effect can be found by equation (4.1.12) with the current and time interval replaced by their corresponding values.

However, the input current of the amplifier is very small, of the order of \( 1 \times 10^{-10} \text{ A.} \) and the back biased current of the thermionic diodes used can be neglected.

Applying equation (4.1.12) the maximum hold interval for a change of 20 mV with a 1000 pf capacitor is found to be

*Reference 98.
This figure of $1 \times 10^{-10}$ A may be difficult to achieve for semiconductor devices. However, it may be argued that for semiconductor circuits the component values can be reduced, particularly the resistors which would allow an increase in the capacitor value to overcome the relatively large back biased current. The increase in the capacitance is accompanied by reduction in the working voltage which in general reduces the size of the components for semiconductor devices.

The input and feedback resistors were taken as 1.0 M, most suitable for the operational amplifiers used. This results in a somewhat slow response, the time constant being $10^{-3}$ S. Therefore the minimum pulse width for conventional sampling is about 7 m.s. and the maximum frequency for appreciable pulse width is about 8 c/s for 0.1% error.

This performance can be improved, without affecting the accuracy, by reducing the input and feedback resistors. The resistors can be reduced to, say, 0.5 M or even smaller without appreciable loss of amplifier performance.

4.1.2. Pulse Generating Devices

As indicated earlier, two staggered sets of pulses are required to implement the discrete controller.

A special equipment was designed and built for this purpose. This equipment incorporates the pulse generating devices, and the switch
Waveforms

Reference
Square-wave

Differentiator

A.C. Phase
Inverter

Clipper

Phantas-tron

Amplifier

D.C. Phase
Inverter

TO THE SWITCHES

Figure 4.1.89
components except for the operational amplifiers, which are part of a conventional analogue computer.

The operation of this equipment is best explained with the aid of the block diagram of figure 4.1.9.

A square wave of the sampling frequency (not necessarily of exactly unity mark/space ratio) is differentiated. The output, which is a succession of positive and negative spikes, is fed into an a.c. phase inverter, producing two outputs in phase opposition (Third trace on figure 4.1.9 a and c). The positive spikes of each output are then retained by clipping.

Each sequence is then used to trigger a phantastron circuit generating pulses whose width is linearly proportional to a d : c level, which can be controlled.

As seen from the fifth trace on figure 4.1.9 a and c, the pulse sequences of the two phantastrons are staggered by half the sampling period. These pulses are then applied to the switches after being properly amplified and levelled, using a d.c amplifier and phase inverter.

The phantastron used in the present application is a cathode coupled version*, whose circuit is shown in figure 4.1.10.

---

*Reference 13, pp. 195-199, 70, Ch.7 and 85, Ch.5.
It is essentially a self-gated Miller integrator, with one stable state, where the plate is cut off by holding the suppressor at a negative potential relative to the cathode. The screen, which is at positive potential relative to the cathode, draws all the space current, hence its potential is relatively low.

On applying a sufficiently positive trigger (short pulse) to the suppressor, the plate is pulled out of cutoff, upon which its potential drops. This drop is coupled to the grid through the capacitor coupling $C$, thus decreasing the cathode current, which in turn increases the suppressor potential resulting in more plate current. This action is regenerative and the plate quickly starts the Miller sweep. Due to the decrease in cathode current, the screen current decreases, and its potential rises.

The Miller-sweep run-down continues, until the plate bottoms, upon which the grid side of the coupling capacitor rises exponentially towards H.T.+. This rise increases the cathode current, reducing the
suppressor potential relative to the cathode and resulting in further reduction of the plate current. Again the action is regenerative. The plate cuts off quickly and the space current is transformed to the screen, dropping its potential.

The plate potential then rises exponentially towards H.T.+ with a time constant RC, but it is clamped to the control potential e_c, applied to the plate through a clamping diode.

A pentode with a sharp cut off suppressor characteristic is more suitable for this application, such as 6AS6 type, which is used here.

To reduce the recovery time, for higher frequency operation, a cathode follower is inserted between the plate and the plate side of the coupling capacitor.*

The stability of the sweep duration, and consequently the screen positive pulse duration may be improved by stabilising the initial grid potential using a catching diode.+

The linearity of the sweep, depends on the amplifier gain which should be fairly high. It can be proved that the maximum deviation from linearity is less than \( \frac{1}{8A} \), where A is the gain of the amplifier (between g, and plate).

For a cathode coupled phantastron, the gain is reduced due to the presence of R_k. However, even a gain of about 65 is sufficiently high, since it results in a maximum deviation from linearity of about 0.2%, which is very satisfactory. (This is only true for the linear

* Reference 85, p. 186.
+ Reference 65, p. 127.
++ Reference 70, p. 225.
part of the sweep not for the part where bottoming takes place, which
is of almost constant duration, anyway).

Two pulses of opposite polarities are obtained at the screen and
cathode, with duration practically proportional to the control voltage.
The operating pulse is usually that of the screen, as it is a free
electrode and can be loaded without affecting the function of the
circuit.

The amplifier following the phantastron is directly coupled so
are the phase inverters. No problem arises due to the fact that the
tubes are operated between bottoming and cut off states, since the
signals are all pulses.

There is an amplifying stage following each phantastron. This
stage supplies more than one phase inverter, which is a cathode
coupled version. Each phase inverter controls three samplers in
parallel.

The pulse widths of both sequences can be controlled through
the phantastron plate level, as indicated earlier.

Moreover, the pulse width may be made proportional to some
external signal; also the sampling period can be made proportional
to another function. This arrangement can be used for the
multiplication of up to 3 analogue functions. (The third being the
pulse height or amplitude). This is additional to the original
purpose of the equipment, which is the simulation of a discrete
controller.
The pulse generating devices considered above are for general use, but they are not sufficient for the simulation of the numerical example solved in the preceding chapter, as it calls for special requirements.

Some simple dividing circuits are used in addition as explained in the next section.

4.1.3. Special Auxiliary Circuits

In the example considered the pulse width was taken as one-quarter the sampling period. Also, the step input was assumed to coincide with the commencement of a sampling interval. These two conditions must, therefore, be satisfied for accurate simulation of the example.

The basic circuit used for generating pulses as required is a multivibrator frequency divider with adjustable repetition rate.

By injecting synchronising pulses to the multivibrator, its frequency can be made a multiple or submultiple of the synchronising frequency.

If the multivibrator is made un asymmetrical, i.e. with unsymmetrical time constants, it is possible by injecting synchronising pulses to both grids to have a mark/space ratio of \( \frac{1}{n} \) (or \( n \)) where \( n \) is an integer. The synchronising frequency would then be \( (n+1)m \), where \( m \) is also an integer.

The frequency division principle can also be used to generate the square waveform, whose period is several times the sampling period.

Waveforms

Reference Square waveform

Differentiator and Clipper

Unsymmetrical Multivibrator

Differentiator and Clipper

Multivibrator I

Multivibrator II

Waveforms

Figure 4.111

(137)
This waveform is used as the step input to the system.

The operation of these circuits are explained in more detail with the aid of the block diagram and associated waveforms of figure 4.1.11.

The reference pulses are from a square wave source, differentiated and clipped, retaining only the positive spikes. These are then injected to the grids of both valves forming the unsymmetrical multivibrator. The mark/space ratio can be locked to one-third by suitable amplitude adjustment of the synchronising pulses. The waveforms indicate a synchronising frequency of 4 times the sampling frequency.

The stability of the unsymmetrical multivibrator frequency and its mark/space ratio depends to some extent on the amplitude of the synchronising pulses (assuming the reference frequency to be fairly stable). In general the synchronising frequency should be slightly larger than 4 m*, where m* is the would be free multivibrator frequency.

The sampling frequency is again divided using a similar procedure, producing a square wave of much lower frequency indicated by the last but one waveform of figure 4.1.11. The usual levelling and clipping operations are applied to shape it to the required level and polarity.

Another symmetrical multivibrator is synchronised to the unsymmetrical one, generating a square wave of approximately unity mark/space ratio. This square wave is used as the reference frequency waveform fed to the pulse generating device, whose block diagram is shown in figure 4.1.9. However, it serves only to
Reference Square-wave \( f \text{ c.p.s.} \)

- Differentiator and Clipper
- Unsymmetrical Multivibrator \( f/4 \text{ c.p.s.} \)
  - Amplifier (2 Stage)
  - D.C. Phase Inverter
  - Group A Switches

- Differentiator and Clipper
  - Multivibrator II \( f/4 \text{ c.p.s.} \)

- Multivibrator I

Test Signal (Square-wave, \( f/24 \) say.)

Differentiator and Inverter
- Clipper
- Phantastron
- Amplifier
- Differentiator and Clipper
- Phase-Inverter
- Group B Switches
- Group C Switches

Figure 4.1.12
generate the staggered pulse sequence, as part of the samplers are directly controlled by the output of the unsymmetrical multivibrator.

The block diagram of all the pulse generating circuitry is shown in figure 4.1.12.

Group C switches are conventional ones with short uncontrolled durations. They operate in phase with Group A.

The complete circuit diagrams of the pulse generating circuits and switches are in the Appendix.
4.2. PURE TIME DELAY SIMULATION

Simulation of appreciable pure time delay, or transportation lag, can be done by various methods, which are considered briefly in the following paragraphs.

1. Network Simulation. Applying either passive or active networks. The transfer functions of such simulations approximate that of exp.(-SL), where L is the pure-time-delay.

   The accuracy and degree of approximation, depends on the components used, and the frequency band width required. The approaches may be low pass, all pass filter, * or Pade' approximations.+ 

2. Switching Simulation. This can be done by moving or passing the signal along a line of capacitors separated by switches and buffers.

   The sample-and-hold chain of figure 4.3. is a very good example. It can readily be seen that trace (3) of figure 4.4c is a staircase, whose envelope is the input signal delayed by one sampling period. By smoothing and filtering, the original signal can be recovered.

   Similar arrangements have been used for delay line synthesizer.++

   The accuracy and frequency-bandwidth depends to some extent on the number of sections and the amount of delay.

   Variable pure-time-delay can also be generated by varying the

*References 16, 73 and 95.
+References 16, 68, 73, 91, 95 and 103.
++References 37 and 39.
3. Transportation Methods. Where the signal is carried or stored on some medium and picked off after the appropriate time delay. One method is by charging a condenser bank connected as shown in figure 4.2.1. and collecting the signal after the required delay.

Another method is by using magnetic tape or drum, recording the signal and picking it off later using two heads. The delay can be controlled through the speed of the magnetic medium and/or the distance between the two magnetic heads.

The magnetic tape technique is the one applied for the pure-time delay simulation of this work.

The block diagram of the delay generator is shown in figure 4.2.2.

References 23 and 99

References 17, 23 and 90.
The information or input signal is continuously recorded on the tape by the recording head, and reproduced after the required delay. The amount of delay is directly proportional to the distance between the two heads and inversely proportional to the speed of the tape. The delay can be controlled through either of the two parameters. Usually the distance between the two heads is fixed, and the delay controlled through the tape speed. Another method of controlling the amount of the delay is suggested towards the end of this Chapter.

Each block of figure 4.2.2 is now considered in more detail.

4.2.1. Speed Controlled Motor Drive

A constant-armature-current, field controlled d.c. motor is used for speed control of the tape. The machine incorporates a d.c. generator as well, the armatures of both being on the same shaft. (Type 74 motor-generator Velodyne).

The motor field windings are connected as the lead of a push pull
power amplifier which is preceded by an operational amplifier acting as summator and voltage amplifier. The operational amplifier is particularly useful in reducing the drift.

Feedback is used for speed control, where a voltage proportional to the speed (from the generator at constant exitation) is compared with the input voltage, and the resultant difference fed to the amplifier thus producing some field flux which develops the required torque in conjunction with the armature flux.

The block diagram of the speed controlled drive is shown in figure 4.2.3.

The feedback network H is essential for system optimisation as the uncompensated system is usually unacceptable.

Assuming linearity, the transfer function of the system can be found as follows:

The motor field current $i_f$ and the error $e$ are related by

$$i_f \frac{R_f}{R_f + L_f} + L_f \frac{\text{diff}}{\text{dt}} = e k_1 k_2 = e k$$  \hspace{1cm} (4.2.1)

where $R_f$ and $L_f$ are the resistance and inductance of the field windings respectively, and,

$k_1$ and $k_2$, the voltages gains of the voltage amplifier and power amplifier respectively.

Taking the L.T., gives

$$I_f = \frac{k}{R_f + SL_f} E$$  \hspace{1cm} (4.2.2)

As the armature flux is constant, the torque developed is directly proportional to field flux, which in turn is directly proportional to
Figure 4.2.3.

Figure 4.2.4.
field current.

\[
\text{Torque } T = N I_f = \frac{N k}{R_f + SL_f} \quad (4.2.3)
\]

where \( N \) is the torque constant in Nm/A.

This torque is developed to overcome the friction and momentum of the rotating parts.

\[
\text{Torque} = J \frac{dw}{dt} + F w \quad (4.2.4)
\]

where \( J \) and \( F \) are the moment of inertia in kg m\(^2\) of the rotating parts, and the friction in Nm/(rad./S).

From equations (4.2.3) and (4.2.4) and taking L.T. of the latter

\[
w = \frac{N k}{(R_f + SL_f) (F + SJ)} \quad (4.2.5a)
\]

or \[
w = \frac{K}{(1 + ST_f) (1 + ST_m)} \quad (4.2.5b)
\]

K constant in (rad/sec)/volt.

\[
T_f = \frac{L_f}{R_f} \quad \text{field windings Time Constant in seconds.}
\]

\[
T_m = \frac{J}{F} \quad \text{motor (and other rotating parts) time constant in seconds.}
\]

The transfer function block diagram is shown in figure 4.2.4.

where \( k_g \) denotes the output of the generator in volts/rad per sec.

for constant excitation.

\[
T_f \text{ is small } = \frac{6}{2500} = 2.4 \text{ m.s.}, \text{ and may be neglected. The equation 4.2.5 can be simplified to}
\]

\[
w = \frac{K}{1 + ST_m} \quad (4.2.6)
\]

(146)
The overall transfer function of figure 4.2.4 is found to be
\[ w = \frac{K}{1+STm + KH kg}. \]  
(4.2.7)

H is a transfer function to be determined according to the requirements of the system.

Generally, for such an application, the specifications would be a short rise time and slightly underdamped response.

These specifications can be satisfied by adjusting the feedback network components, and/or the overall loop gain K kg.

After some trials, H was chosen to be the network shown in figure 4.2.5.

![Figure 4.2.5](image)

which is a second order network as the relation between \( i_g \) and \( v_g \), the generator output is given by
\[ I_G = \frac{1}{R} \frac{1 + S T_2}{1 + S \alpha + S^2 \beta} \frac{V}{g} \]  
(4.2.8)

where
\[ R = R_1 + R_2 + R_3 \]
\[ \alpha = \frac{R_2 + R_3}{R} T_1 + \frac{R_1 + R_2}{R} T_2 \]
\[ \beta = \frac{R_3}{R} T_1 T_2 \]
\[ T_1 = R_1 C_1, \quad T_2 = R_2 C_2 \]
The voltage gain of the operational amplifier is adjusted by varying the amount of feedback through \( P \), as shown in figure 4.2.6.

![Figure 4.2.6](image)

The gain between the generator terminal and the output of the operational amplifier, is then given by

\[
\frac{V_o}{V_g} = -\frac{1}{pR}
\]

\( R \) in megohm

\( p \) the setting of potentiometer \( P \).

Keeping \( R_1, R_2 \) and \( R_3 \), i.e. \( R \) the same, the loop gain can then be adjusted by \( P \). Also the feedback network \( H \) singularities are adjusted by \( C_1 \) and \( C_2 \).

The settings for optimum step response were found to be

- \( R_1 = 50 \, \text{K} \)
- \( R_2 = 0.7 \, \text{M} \)
- \( R_3 = 0.1 \, \text{M} \) & \( R = 0.85 \, \text{M} \) and \( p = 0.25 \)
- \( C_1 = 0.05 \, \text{uf} \) & \( C_2 = 0.035 \, \text{uf} \)

Incorporating the term \( \frac{1}{R} \) of equation 4.2.8, with the forward loop gain, then \( K \) of figure 4.2.4. is

\[
K \propto \frac{4}{0.85}
\]

(148)
Optimum Response (a)

Uncompensated Response (b)

Figure 4.2.7.
and $H$ of the same figure, is
\[ H = \frac{1 + 0.0245S}{1 + 0.025S + 0.72 \times 10^{-5}S^2} \]

The step response corresponding to these settings is shown in figure 4.2.7.a as compared with the step response for the uncompensated system, with $K \propto \frac{6}{0.85}$ and $H = 1$, shown in figure 4.2.7.b.

The optimum response has a rise time of about 0.08 secs, an overshoot of about 6%, and a settling time of about 0.11 sec.

These responses represent the generator output e.m.f. passed through a low pass filter having a flat response up to about 15 c/s. This filter is necessary to eliminate or reduce the ripple content in the generator output, thus facilitating the recording of the system response.

The accuracy and stability of the speed of the motor is a function of the system characteristics, e.g. amplifier gain, armature current and generator excitation.

The d.c. amplifier basically suffers from drift and variations in valve characteristics. However, drift is greatly reduced, by using a drift corrected operational amplifier as the voltage amplification stage. It is followed by the power amplification stage (a push-pull), which is of low gain and any drift at its input is practically of little significance.

*Reference 109, p. 79.*

(150)
The motor armature current is taken from a large d.c. rectified supply through a large dropping resistance. The voltage variations of the supply are ±10% over several hours, and much less for shorter durations. The back e.m.f. of the motor armature is rather small compared to the supply voltage (about 10 v at 2000 r.p.m.) resulting in small variations of the armature current. However, any such variations are disturbances neutralized by the closed loop effect.

The generator excitation current is of importance in stabilising the speed, whose monitoring is wholly dependent upon it. This current should be more stable than the degree of accuracy required for the speed. In this application it is supplied from a stabilised power pack through a regulating resistance. The stabilisation factor of such power packs is usually about 500 : 1.

The tape is driven directly by the motor shaft, developing high torque as the shaft is about 0.25" diameter. However, the driving shaft must be turned carefully to avoid speed fluctuations due to nonuniformity of the driving shaft. The resulting amount of wow depends on the eccentricity in the shaft and its speed, as will be considered later.

Wow* which is the variations in the demodulated tape output due to speed fluctuations of the tape is unwanted noise, and should be reduced. It has several causes, such as nonuniformity of the driving shaft, and slipping between the shaft and tape. The wow due to the

*Reference 10, p. 142.
first cause can be reduced by proper turning and machining of the shaft, while that due to the second can be reduced by ensuring good contact between the tape and the shaft all the time.

Two spring loaded rollers are used to ensure good contact between the tape and the shaft.

![Diagram](image)

The two rollers can press the tape to the shaft at any two different points, but for more effective contact the tape is preferred to be around most of the periphery of the shaft as shown in figure 4.28.

Another source of noise is the vibrations of the tape in motion which leads to spontaneous separation between the tape and either of the magnetic heads. The equivalent noise in the demodulated output is known as the flutter. This can be reduced by not stretching the tape, but running it with little slackening. The flutter is specially noticed at high speeds, where the tape has more tendency to vibrate. Guard pins and pressure pads are used to keep the tape in contact with the heads.
4.2.2. Modulation Device

The signal to be delayed is usually of low frequency not suitable for direct recording on the tape. Therefore, the signal is applied to the recording head in a modulated or coded form.

The types of modulation conventionally used for magnetic recording are

i) Amplitude modulation

ii) Frequency modulation

iii) Pulse width modulation

iv) Pulse coding modulation.

In deciding which type to be used the main features of each are considered in view of the required accuracy, linearity and band width required.

Briefly, the amplitude modulation has poor accuracy and is sensitive to speed variations.

Pulse coding modulation, e.g. using binary code, presents on the other hand the best accuracy, and is independent of speed variations. But it requires complicated circuitry, and is therefore, not recommended for the present application.

The other two types, frequency and pulse width modulation are generally similar, so long as the recording and reproduction tape speeds are the same. But the pulse width modulation system is superior to the frequency modulation system, if the recording and reproduction speeds are different, which would be the case for a continuously variable delay.
Therefore, for the present application pulse-width-modulation is adopted.

The basic function of a pulse-width-modulator is to generate pulses whose width is proportional to the modulated signal. These pulses are at certain rate, usually fixed. Pulse-width-modulation is sometimes known as time modulation.

A reliable and accurate modulator, would generate pulses whose width is accurately and linearly proportional to the modulated signal.

The circuit preferred for such application is a screen coupled phantastron, which is not complicated and of satisfactory linearity.

The diagram of such a circuit is shown in figure 4.2.9.

![Diagram of a screen coupled phantastron](image)

Figure 4.2.9

Usually the suppressor is sufficiently negative, and the plate is cut-off. Upon the application of a positive trigger of sufficient amplitude to the screen or the suppressor, the plate conducts and the Miller action starts. Its mechanism is almost the same as that of the

References 13; 65 Ch.6; 70 Ch.7; and 100.
cathode coupled phantastron considered in section 4.1.2, except that the self gating pulse is now provided by the screen and not by the cathode.

The important advantage of this circuit compared with the cathode coupled version is the much higher gain, as there is no cathode resistance. This results in greater linearity and much better performance. A linearity of better than 0.1% is easily obtained, specially if using proper tubes such as 6AS6.

The pulse is taken from the screen and applied to the grid of the output tube through direct coupling. The recording head windings are connected as the cathode impedance of the output value. This ensures the required high current for saturating the magnetic material in either polarity.

The triggering pulses are generated by a free running phantastron (screen coupled as well), where its suppressor quiescent bias is set within the cathode potential of figure 4.2.9. As soon as the plate bottoms the screen and suppressor voltages drop, thus cutting the tube off momentarily. But when the grid recovers, the suppressor potential rises causing the tube to start conducting again and so on. The output of the screen is then in the form of short negative pulses, whose period is proportional to the control voltage \( e_c \). These pulses are then amplified (practically for phase reversal) and applied as positive triggering pulses to the modulating device.

The pulse rate used is about 500 c.p.s.

*Reference 100.
The block diagram of the modulating device is shown in figure 4.2.10.

![Block Diagram of Modulating Device](image)

Figure 4.2.10

The recording head used should have a small inductance and consequently fewer turns, which requires higher currents. The recording head used was actually home made, from Mullard fenoxocube material, which has low core loss. The inductance is about 1 mH and the current required to produce sufficient flux in the gap was about 26 mA. The output stage supplies current of about 30 mA into the recording head winding.

4.2.3. Demodulation Device

The input to the recording head is a succession of pulses at fixed rate, with modulated pulse durations. The time intervals corresponding to these pulse durations and corresponding to the rest of the modulating period (marks and spaces) are transformed into segments of opposite magnetic orientation on the tape. The tape is assumed to be moving across the recording head.

After a time delay, the part of the tape with these alternate segments on, reaches the reproducing head. The change of magnetic...
From Reproducing Head.

- Single-Stage Amplifier
  - Cathode-Coupled Bistable Multivibrator
    - Single-Stage Amplifier
      - Limiter
        - Filter
          - Demodulated Output

Figure 4.2.11.
orientation produces spikes of related polarity at the reproducing head.

These alternate spikes are used to trigger a cathode-coupled bistable multivibrator (Schmitt's Trigger Circuit),* generating rectangular pulses whose duration is the same as the interval between a positive and the following negative spike.

The output of the cathode coupled bistable multivibrator is then amplified, limited and filtered (being average detected), thus recovering the original function.

The block diagram and associated waveforms of the demodulating device are shown in figure 4.2.11.

A diagram of the bistable circuit used is shown in figure 4.2.12.

Assuming $V_g$, to be zero then $T_2$ is on, as $T_1$ is off because of the voltage drop across $R_k$. $V_{g2}$ is practically clamped to its cathode.

Applying a positive spike of sufficient amplitude to overcome the potential across $R_k$ and slightly drive $V_{g1}$ above the common cathode potential, $T_1$ is turned on, dropping $V_{g2}$. This drop reduces the

*Reference 11, p. 81.
common cathode current and potential $V_{g1}$ is increased further. The action is regenerative and $T_2$ is quickly turned off.

Upon application of a negative spike to $g_1$, again sufficient to overcome the voltage drop across $R_k$, $T_1$ current is reduced and its plate potential rises, thus allowing $T_2$ to start conduction. The current due to $T_2$ increases the common cathode potential, and $V_{g1}$ is even more negative. Again the action is regenerative and $T_1$ is quickly off, while $T_2$ is quickly turned on.

The triggering level, or the difference between the levels required for change over is seen to be a function of $R_k$. This level, and consequently $R_k$ should be sufficiently large not to allow the circuit to be affected by the noise content from the tape.

The triggering level should also be low enough to allow proper triggering at low speeds, within the speed range used. This is because the spikes amplitude is dependent upon the speed, being proportional to the rate of change of flux across the reproducing head. Usually a compromise is made, and generally speaking higher speeds are preferred.

The performance of the circuit is improved by making $T_2$ a heavy current tube and returning its cathode to a point only part way up $R_k$.

The other important component in the demodulating device is the filter. It should have a fairly flat response up to some appropriate frequency, and considerable attenuation at the modulating frequency.

* Reference 100.
An active filter (Fig. 4.2.13) having the transfer function

\[
\frac{V_o}{V_1} = \frac{-1}{1 + sC_2(R_1+2R_2)+s^2C_1C_2R_1R_2} = \frac{-1}{1 + at + bs^2 + ts^2}
\]

was used for this purpose.

For a flat characteristic up to about 30 c/s, the parameters \( a, b \) and \( T \) were evaluated to be as follows:

\[
\begin{align*}
  a &= 0.436 \\
  b &= 0.1 \\
  T &= 0.0053
\end{align*}
\]

From which \( R_1 = 1.0 \) Megohm and \( R_2 = 0.425 \) Megohm.

\[
\begin{align*}
  C_1 &= 5000 \text{ and } C_2 = 1800 \text{ microfarad.}
\end{align*}
\]

The actual filter was composed of two cascaded units to produce sufficient attenuation at the modulating frequency of about 500 c/s.

*Reference 84.*

(160)
4.2.4: Effect of Speed Variations

It may be required to simulate continuously variable delays using the magnetic delay device described in this section, through varying the tape speed.

Therefore the effect of varying the tape speed on the recovered signal has to be considered before deciding whether the equipment is useful in this respect.

As considered earlier, the time intervals of the pulsed input to the recording head are transformed into segments of magnetic orientation on the tape. The length of any segment depends on the speed of the tape during the corresponding interval.

Consider the input to the recording head at any arbitrary instant to be a pulse of duration $\gamma$, whereas the period is of duration $T$.

The modulated signal amplitude at this instant is proportional to $\frac{\gamma}{T}$.

![Figure 4.2.14](image)

Figure 4.2.14

![Figure 4.2.15](image)

Figure 4.2.15

(161)
Assuming the speed of the tape across the head to be $V(t)$, then $l$ and $L$, the lengths of the segments corresponding to $\gamma$ and $T$ respectively are

$$l = \int_{0}^{\gamma} V(t) \, dt \quad \text{(4.2.10)}$$

$$L = \int_{0}^{T} V(t) \, dt$$

After a delay $\Delta$, the foregoing portion of the tape comes in contact with the reproduction head, where the tape segments are transformed back into time intervals, say $\gamma'$ and $T'$ as shown on figure 4.2.15.

Through the demodulator the amplitude corresponding to $\gamma'$ and $T'$ is proportional to $\frac{\gamma'}{T'}$.

Now $\gamma'$ and $T'$ are found from,

$$l = \int_{\Delta + \gamma'}^{\Delta + T'} V(t) \, dt$$

$$L = \int_{\Delta}^{\Delta + T'} V(t) \, dt \quad \text{(4.2.11)}$$

where $l$ and $L$ are those of equation 4.2.10. The last equations are implicit in $\gamma'$ and $T'$.

Now let us consider the general case of $V(t)$, where it is expressed as a power series in time.

$$V(t) = V + at + bt^2 + \ldots \quad \text{(4.2.12)}$$

From equations 4.2.10.

$$l = V \gamma + \frac{a}{2} \gamma^2 + \frac{b}{3} \gamma^3 + \ldots$$

$$L = VT + \frac{a}{2} T^2 + \frac{b}{3} T^3 + \ldots \quad \text{(4.2.13)}$$

(162)
and from equations 4.2.11.

\[ l = v y' + \frac{a}{2} (y'^2 + 2\Delta y') + \frac{b}{3} (y'^3 + \Delta y^2 + 2\Delta^2 y) + \cdots \]
\[ L = v T' + \frac{a}{2} (T'^2 + 2\Delta T') + \frac{b}{3} (T'^3 + \Delta T'^2 + 2\Delta^2 T') + \cdots \quad (4.2.14) \]

From the last two equations and by division,

\[ \frac{\gamma}{T} \left( 1 + \frac{a}{2} \gamma + \frac{b}{3} \gamma^2 + \cdots \right) = \frac{\gamma'}{T'} \left( 1 + \frac{a}{2} \gamma' \left( 1 + \frac{2\Delta}{\gamma} \right) + \frac{b}{3} \gamma'^2 \left( 1 + \frac{\Delta}{\gamma'} + \frac{2\Delta^2}{\gamma'^2} \right) + \cdots \right) \]

where \( \alpha = \frac{a}{v} \) and \( \beta = \frac{b}{v} \)

which can be put as

\[ \frac{\gamma'}{T'} \times \frac{T}{\gamma} = \frac{1 + \frac{a}{2} \gamma + \frac{b}{3} \gamma^2 + \cdots}{1 + \frac{a}{2} \gamma \left( 1 + \frac{2\Delta}{\gamma} \right) + \frac{b}{3} \gamma^2 \left( 1 + \frac{\Delta}{\gamma} + \frac{2\Delta^2}{\gamma^2} \right) + \cdots} \]
\[ \times \frac{1 + \frac{a}{2} T' \left( 1 + \frac{2\Delta}{T'} \right) + \frac{b}{3} T'^2 \left( 1 + \frac{\Delta}{T'} + \frac{2\Delta^2}{T'^2} \right) + \cdots}{1 + \frac{a}{2} T + \frac{b}{3} T^2 + \cdots} \quad (4.2.15) \]

For satisfactory reproduction, the R.H.S. of the last equation should be as close as possible to unity.

Each factor affecting this quantity is considered as follows:

i) The modulating period \( T \). It is quite obvious that the smaller the period \( T \), the smaller will be the terms associated with it. (Also \( \gamma \), since \( \gamma \) is definitely smaller than \( T \).) In this application \( T \) is about 0.002 s, therefore quadratic and higher orders of \( T \) and \( \gamma \) can be neglected.

ii) The speed function \( V(t) \). The slower the speed variation, the smaller will be \( a \) and \( b \), thus reducing the magnitude of the terms associated with them.
\( T' \) and \( \gamma' \) are not known yet, but roughly,

\[ \frac{T'}{T} = \frac{V(t)}{V(t+\Delta)} = \frac{\gamma'}{\gamma} \quad (4.2.16) \]

Hence \( T' \) and \( \gamma' \) are smaller than \( T \) and \( \gamma \) for higher speeds and vice versa (with respect to \( V(t) \)).

Smaller \( T' \) and \( \gamma' \) results in diminishing the terms associated with them, and following the same argument as in i), quadratic and higher order terms in \( T' \) can be neglected.

Larger \( T' \) and \( \gamma' \) makes the terms associated with them more significant, but still \( T' \) would be small compared to unity for a reasonable reduction in speed, say 2 : 1, where \( T' \) would be 0.004 S.

iii) The amount of the delay. As can be seen, smaller delays diminish the terms associated with it.

Generally speaking, the increase of the modulating frequency and slower speed variations reduce the distortion as the R.H.S. of equation 4.2.15 is then very close to unity. Higher average speeds are preferred, since it increases the range of delay variations, such that the maximum \( T' \) is, say about twice the original \( T \). Another effect for higher speeds is the reduction of \( \alpha \) and \( \beta \) for the same \( a \) and \( b \).

The exact solution of equation 4.2.15 can be carried out for various speed variations with the help of computing machines, but the above analysis indicates that the described magnetic tape delay simulator can be used for satisfactory simulation of variable time delays.
Speed Calibration.
Nominal 25mm/sec.

Input

Output

a) Step Function

b) Sinewave Function, 1 c.p.s.

Figure 4.2.16.
Some records of the delay device as a whole are shown in figure 4.2.16 a and b. On (a) the input was a step function, the delayed output is very close to the input. On (b) a sinusoidal function is shown, at a frequency of 1 c/s. This low frequency was chosen only for being easily recorded on the pen recorder. Much higher frequencies up to about 30 c/s can be delayed without any attenuation, due to the filter.

On the whole the performance of the device may be considered satisfactory.

The only serious drawback is that, as things stand at the moment, there is no accurate means of controlling the amount of the delay.

However, the delay can be adjusted by a rather simple method with reasonable accuracy, as follows:

Let it be assumed that the delay required is say, 0.1 sec. The tape length between the two heads and its speed are then set such that the delay is about 0.1 s, e.g. at a speed of about 40 inches/sec., a separation of about 4 inches would be all right.

The accurate adjustment of the delay is done by delaying a periodic function whose period = 0.1 S.

The output of the delay device would then be as shown in figure 4.2.17.
By observing the two traces on a double beam C.R.T., the tape speed can be manipulated, such that the phase difference between the two traces is reduced to zero. A further correction is by connecting the input and output to the $Y$ and $X$-axis of a single beam, where the trace would be an inclined straight line, if the two traces are in phase.

Once the delay is adjusted it is unlikely to vary, say in the next few minutes, as the speed stability is fairly satisfactory. The input to the device can then be switched to the function to be delayed.

4.2.5. **Suggestions for Improvement**

The above method of delay adjustment can be pushed a little further, using a phase comparater to detect the phase difference between the input and output wave forms. This phase difference is then applied to the motor amplifier, automatically setting the speed for zero phase difference.

The block diagram of such a system would be as shown in figure 4.2.18.
Figure 4.2.18.
At least two modulating and demodulating devices are required, one set for delay control, the other for the function to be delayed. More than one function can be delayed, by dividing the tape into more tracks and using wider tape if necessary.

The recording and reproducing heads, should also be of multiple track construction.

Moreover, if the periodic function (whose period = delay) is taken from a controlled frequency source, then the amount of delay can be continuously controlled, through controlling the periodic function period.

By this method, the magnetic tape delay device would have a much higher quality performance compared with other methods used for delay simulation.

The main features would be:

i) Wide range of delay

ii) Wide band width, can even be increased by increasing the modulating frequency and improving filtering.

iii) Easily adjusted and continuously controlled delay, with negligible effect on signal amplitude.

The modulating and demodulating devices, as well as the motor field power amplifier circuit diagrams are shown in the Appendix.
4.3. SIMULATION SET UP AND RESULTS

In the example considered at the end of the last Chapter, the pure time delay was taken as 0.6 T, where T is the sampling period.

The step response was satisfactory, with $\mathbf{A}$ and $\mathbf{B}$ as following

$$
\mathbf{A} = \begin{bmatrix}
A_0 & 0 & 0 & 0 \\
A_1 & A_0 & 0 & 0 \\
A_2 & A_1 & A_0 & 0 \\
A_3 & A_2 & A_1 & A_0
\end{bmatrix}
$$

and

$$
\mathbf{B} = \begin{bmatrix}
B_1 & 0 & 0 & 0 \\
B_2 & B_1 & 0 & 0 \\
B_3 & B_2 & B_1 & 0
\end{bmatrix}
$$

The parameter values computed above for $\mathbf{A}$, were

$$ f_n = A_n \frac{K}{a^n}, \text{ i.e. } A_n = f_n \frac{a^2}{K} $$

where $n = 0, 1, 2, \text{ and } 3$.

The transfer function $\frac{K}{s \left( s + a \right)}$ can be put as

$$
\frac{K/a^2}{s \left( 1 + \frac{s}{a} \right)}
$$

Therefore $\frac{K}{a^2}$ is eliminated from the overall simulation as shown in figure 4.3.1., where the controller assumes a more compact configuration. The function $\frac{1}{s \left( 1 + \frac{s}{a} \right)}$ is then simulated as shown in figure 4.3.2., where the two potentiometers are ganged ($j_2$ and $j_1$). The transfer function is then,

$$
\frac{E_o}{E_i} = \frac{1}{s \alpha RC \left( 1 + s \alpha RC \right)}
$$

$\alpha$ being the potentiometer setting

*Reference 105, p. 454.
CONTROLLED SYSTEM

\[ \frac{1}{s^2 + \frac{1}{T} s + \frac{1}{T^2}} \]

\[ T \times H \]

\[ S \times H \]

\[ \text{INPUT} \]

\[ \text{OUTPUT} \]

FIGURE 4.3.1
hence \( a = \frac{1}{\alpha RC} \)

Knowing the range of a required, the time constant RC can be determined.

\[ T \text{ is taken as } 0.25 \text{ and } 1.4 \ll aT \ll 3.0 \]

therefore \( \frac{1.4}{0.2} \ll a \ll \frac{3.0}{0.2} \)

or \( \frac{1}{7} \gg aRC \gg \frac{1}{15} \), \( a \ll 1 \).

A suitable value for RC is 0.2 sec.

\[ R = 0.2 \, \text{M} \, , \, \, \, C = 1.0 \, \text{uf}. \]

The analogue computer sheet for the complete simulation is shown in figure 4.3.2. The symbols associated with each operational amplifier indicates its position on the general purpose computer used, e.g. \( 4\), which is connected for sign reversal.

The samplers are represented symbolically, where the associated number indicates its position on the sampling equipment, e.g. 41 denotes the sampler number one of the fourth phase inverter.

Samplers 23, 21, 13, 12 and 11 are all in phase and operate with a sampling duration of one quarter the sampling period. Sampler 62 is also in phase with the above mentioned group, but is a conventional one, with short sampling duration. Samplers 51, 43 and 41 are staggered by about half the sampling period from the above mentioned switches. The sampling duration was made about one quarter the sampling period as well, which is not so critical anyway.

All resistors and capacitors shown in figure 4.3.2. are in M \( \Omega \) and pf respectively, unless otherwise indicated. P and H indicates potentiometers and precision helical potentiometers respectively.
The computer was operated at the auto 1 position, where it switches from problem set to compute by the positive step of the output of multivibrator I of figure 4.1.11.

**Initial Adjustments.** Before taking any results, each sampler must be properly balanced by potentiometers r and r' of figure 4.1.5. The d.c. balancing by potentiometer r was found to be so critical that any slight unbalance of potentiometer r results in an initial sample-and-hold output of several volts. As the potentiometers are not highly stable, some drift was noticed during the simulation procedure. This drift amounts to a considerable value at \( \Sigma_2 \) and \( \Sigma_3 \), which is not surprising due to the chain of sample-and-hold elements.

Another important factor affecting the accuracy of simulation is the interaction between various sample-and-hold elements. This interaction can be reduced by proper screening and also by increasing the hold capacitor (in the same time reducing the input and feedback resistors of the sample-and-hold elements).

The combined effect of the above mentioned factors results in slight disagreement between the computed and actual settings of the controller parameters.

Also it results in some reduction of the initial slope of the responses, except for the responses recorded at the very beginning. (figure 4.3.3).
4.3.1. *Forward loop Synthesis Simulation.*

The open loop response controlled by $A$ only was first checked. Figure 4.3.3. shows the outputs of $f_1$ (for a step input) for different $aT$ within the range specified above.

A comparison between the computed settings and the actual ones reveals close agreement between the two as indicated in table 4.3.1.

<table>
<thead>
<tr>
<th>$aT$</th>
<th>$a$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.714</td>
<td>2.802</td>
<td>2.404</td>
<td>0.820</td>
<td>0.421</td>
</tr>
<tr>
<td></td>
<td>0.714</td>
<td>0.56</td>
<td>0.24</td>
<td>0.82</td>
<td>0.84</td>
</tr>
<tr>
<td></td>
<td>0.56</td>
<td>0.23</td>
<td>0.82</td>
<td>0.81</td>
<td>Actual settings</td>
</tr>
<tr>
<td>2.2</td>
<td>0.454</td>
<td>1.491</td>
<td>1.062</td>
<td>0.495</td>
<td>0.066</td>
</tr>
<tr>
<td></td>
<td>0.454</td>
<td>0.298</td>
<td>0.106</td>
<td>0.495</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>0.298</td>
<td>0.092</td>
<td>0.495</td>
<td>0.13</td>
<td>Actual settings</td>
</tr>
<tr>
<td>3.0</td>
<td>0.333</td>
<td>1.016</td>
<td>0.681</td>
<td>0.355</td>
<td>0.020</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.203</td>
<td>0.068</td>
<td>0.355</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>0.203</td>
<td>0.063</td>
<td>0.355</td>
<td>0.04</td>
<td>Actual settings</td>
</tr>
</tbody>
</table>

where $a$ is the potentiometers settings regarding $aT$.

The responses of figure 4.3.3. may be compared with those of figure 3.9.2., where close agreement is noticed.

N.B. The central response of figure 3.9.2. is for $aT = 2$, not 2.2 as is its counterpart in figure 4.3.3.

(175)
\[ aT = 1.4 \]

\[ aT = 2.2 \]

\[ aT = 3.0 \]

**FIGURE 4.3.3**

(176)
4.3.2. Delay Dependent Settling Time Simulation

Two cases were considered to demonstrate this approach for $aT = 1.4$ and 2.2.

The delay was adjusted by the phase comparison method discussed in connection with figure 4.2.18, to be $0.6T$. Then the controller forward loop parameters were adjusted for satisfactory open loop response, as the output of $\sqrt{5}$. (The connection between $\sqrt{4}$ and $E_9$ being broken.)

A practical method for evaluating the controller feedback parameters is used. This is by comparing the output of FB3, which is a staircase function with the output of $E_3$, (The connection between FB3 and $E_3$ being disconnected), on a double beam cathode ray oscillograph. The helical potentiometers $H5$, $H7$ and $H6$ are then adjusted (from zero) in succession until the two traces are the same. Thus when FB3 is connected to $E_3$, the output of the latter becomes zero. Now the loop can be closed by connecting $\sqrt{4}$ to $E_9$, producing no effect on the system response as shown in figure 4.3.4.

The controller parameters are tabulated against the computed values in table 4.3.2.

<table>
<thead>
<tr>
<th>$aT$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.56</td>
<td>0.24</td>
<td>0.82</td>
<td>0.84</td>
<td>0.31</td>
<td>0.55</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>0.56</td>
<td>0.224</td>
<td>0.835</td>
<td>0.85</td>
<td>0.25</td>
<td>0.55</td>
<td>0.2</td>
</tr>
<tr>
<td>2.2</td>
<td>0.298</td>
<td>0.106</td>
<td>0.495</td>
<td>0.132</td>
<td>0.37</td>
<td>0.5</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>0.298</td>
<td>0.106</td>
<td>0.385</td>
<td>0.132</td>
<td>0.27</td>
<td>0.53</td>
<td>0.2</td>
</tr>
</tbody>
</table>

(177)
Figure 4.3.4

\[ aT = 1.4, \quad \text{Open Loop} \]

\[ aT = 1.4, \quad \text{Closed Loop} \]

\[ aT = 2.2, \quad \text{Open Loop} \]

\[ aT = 2.2, \quad \text{Closed Loop} \]
The agreement between the computed and actual settings is acceptable.

4.3.3. Delay Independent Settling Time Simulation

This simulation can be carried out using conventional sampling. But as the samplers of the controller forward loop were operating at a certain sampling duration, there was no harm in using the same controller forward loop settings, and adjusting the feedback loop settings to suit the new conditions.

The simulation was for \( aT = 1.4 \), and two delays were considered, one is \( 0.6T \) as above, the other is \( 0.4T \).

The responses for the two cases are shown in figure 4.3.5, and the corresponding controller settings are shown in table 4.3.3.

| TABLE 4.3.3. \( aT = 1.4 \) |
|-----------------|--|--|--|--|--|--|
| Delay          | \( f_0 \) | \( f_1 \) | \( f_2 \) | \( f_3 \) | \( B_1 \) | \( B_2 \) | \( B_3 \) |
| 0.6T           | 0.56    | 0.224 | 0.835 | 0.85    | 0.25    | 0.55    | 0.20    |
| 0.4T           | 0.56    | 0.224 | 0.835 | 0.85    | 0.36    | 0.54    | 0.10    |

The close similarity between the responses concerned in figure 4.3.5, is clear.

4.3.4. Simulation of a System without Delay

It was pointed out earlier that systems without delay can be designed following the same procedure as that of section 3.8, assuming that the output response is zero for \( t = 0^+ \).
Delay = 0.6T, Open Loop

Delay = 0.6T, Closed Loop

Delay = 0.4T, Open Loop

Delay = 0.4T, Closed Loop

FIGURE 4.3.5
(180)
An example is considered, demonstrating the step by step design of the controller. First the controller forward loop parameters were adjusted for satisfactory open loop response. Then the feedback parameters were evaluated term by term as discussed in section 4.3.3.

For this simulation the delay components were by-passed, i.e. \( f_1 \) is connected directly to the sampler associated with FB3, also \( f_4 \) is by-passed.

As the open loop response settles in two sampling periods, only two controller feedback parameters are needed i.e. \( B_1 \) and \( B_2 \). The settings are shown in table 4.3.4.

<table>
<thead>
<tr>
<th>( aT )</th>
<th>( f_0 )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.56</td>
<td>0.23</td>
<td>0.84</td>
<td>0.835</td>
<td>0.7</td>
<td>0.3</td>
</tr>
</tbody>
</table>

The resultant responses are shown in figure 4.3.6.

It is noticed from the responses of figures 4.3.4. - 4.3.6. that there is no change in the response due to the closure of the loop, which indicates the validity of the synthesis procedure applied.
Open Loop

$\alpha T = 1.4$

Closed Loop

FIGURE 4.3.6

(182)
4.3.5. **Uncompensated Step Response**

It is of interest to compare the uncompensated system response with those of the compensated or controlled cases.

The block diagram of the uncompensated system is that of figure 2.8.5 (page 67). This is simulated on the analogue computer by by-passing the controlled and the feedback conventional sampler.

Several step responses were taken for various delays and/or exponential lags, also varying the gain $K$. The sampling period was taken as 0.2 S and the pulse width $T/4$.

The responses were found to be of much longer settling time, or transient duration as compared with any of the controlled responses (5 periods against 2 for the controlled response).

Note, the term settling time used in the preceding pages means the significant settling time or the transient duration. It is defined here as the time which elapses before the response remains within 5% of its steady state value, excluding the pure time delay of the system.

Two typical uncompensated step responses are shown on figure 4.3.7, showing the transient duration to be much longer than that of any of figure 4.3.4.

Therefore, the compensating controller produces a significant decrease in the transient duration.
UNCOMPENSATED STEP RESPONSES

\( aT = 1.4, \)  \( \text{Delay} = 0.6T \)

**Figure 4.3.7.**

\((182b)\)
4.3.6. Ramp Response of a Step-compensated System

Now let us consider, the ramp response of one case whose step response is shown on figure 4.3.4., say for aT = 1.4.

A unit ramp was applied to the system with the controller parameters as those of table 4.3.2. for aT = 1.4. The output response and the corresponding error response are shown on figure 4.3.8. Also, an uncontrolled unit ramp response (about the critical damping) is shown on figure 4.3.9.

The output ripples are to be expected, since condition iii) for ripple-free response is not satisfied, i.e. the controlled system transfer function \( \frac{K}{s(s+a)} \) does not contain a double pole at the origin. (see page 89)

The steady-state error at sampling instants was found to be 1.8 T and the transient duration 3 sampling periods for the controlled response, as compared with an offset of 2.15 T and transient duration of 4 sampling periods for the uncontrolled response of figure 4.3.9.

Here, the transient duration is the time \( t_0 \) such that
\[
|T - c(t) + c(t-T)| < 0.05T \quad \text{for all} \quad t \geq t_0.
\]
This is for a unit ramp.

The unit ramp response can be computed following the step by step procedure outlined on pages 68-71. The output at any sampling instant is found to be
\[
C_{nT} = \frac{A K}{a^3} W_n + \frac{K}{a^{2m+1}} \sum_{s=m}^{n-1} S_{n-m} N_m
\]
(18.2c)
UNIT RAMP RESPONSE

\( aT = 1.4 \quad \text{Delay} = 0.6T \)

Controller parameters as Table 4.3.2.

Figure 4.3.8.
UNCONTROLLED UNIT RAMP RESPONSE

\[ aT = 1.4 \quad \text{Delay} = 0.6T \quad \text{Gain} \, K/a = 9 \]

Figure 4.3.9
Where \( W_n = \frac{n}{2} a^2 h^2 - (e^{-a\delta T} - (1-ah)e^{-a(\delta T-h)}) \times \)
\( (e^{-a(n-1)T} + e^{-a(n-2)T} + \cdots + 1) \)
\( N_n = ah - e^{-a\delta T}(e^{ah} - 1) \)
\( S_n = A_3(n^3 - C_{n-3}) + A_2(n^2 - C_{n-2}) + A_1(R_{n-1} - C_{n-1}) \)
\( + A_0(n T - C_n) + B_3 e_{n-3} + B_2 e_{n-2} + B_1 e_{n-1} \)
and \( e_n = S_n + A_0(R_n-nT) \)

As and Bs are the controller parameters, and \( R_n = r(t) \) for \( t=nT+h \)
(See figure 3.8.5)

The response can be evaluated step by step, bearing in mind that \( C_{nT} \) is the same as \( C'_{(n-1+\delta)T} \), where the delay = \((1-\delta)T\).

The delay is taken as 0.6T, at as 1.4, and taking the controller parameters as those of table 4.3.2, the steady state error is found to be 1.857T for a unit step. The transient duration is 3 sampling periods.

The agreement between the numerical and analogue solutions is acceptable.

It may be of interest to compare the above controlled ramp response with that corresponding to a conventionally sampled-data system compensated by applying the ripple-free response criterion of section 3.6.3. (This criterion is applied for a step response)

As the system incorporates a delay \((1-\delta)T\), the modified Z-transform is applied. We have

\[ G(s) = K \frac{1-e^{-sT}}{s^2(s+a)} \]

The system preceded by a zero order hold device has the transform
\[ G(z, \delta) = \frac{K}{a^2} \frac{\alpha z^{-1} + \beta z^{-2} + \gamma z^{-3}}{(1-z^{-1})(1-e^{-aTz^{-1}})} \]

where \( \alpha = a\delta T + e^{-a\delta T} - 1 \)
\[ \beta = (1-a\delta T)(1+e^{-aT}) + aT - 2 e^{-a\delta T} \]
and \( \gamma = e^{-a\delta T} - e^{-aT}(1+aT-a\delta T) \)

From equations 3.6.12 and 3.6.13, by putting 1 equal to unity the overall system transfer function is found to be
\[ Y(z) = \frac{\alpha z^{-1} + \beta z^{-2} + \gamma z^{-3}}{\alpha + \beta + \gamma} \]

Applying a unit ramp, the error sequence is given by
\[ R(z) - C(z) = \frac{T z^{-1}}{(1-z^{-1})^2} (1 - Y(z)) \]
from which the steady state error is found to be 1.984T for 0.6T delay and aT=1.4. The transient duration is 3 sampling periods.

So, the finite pulse width system exhibits less steady state error and a similar transient duration compared with a conventionally sampled system. On the other hand the conventionally sampled system ramp response would be ripple-free as the transfer function with the zero order hold, has a double pole at the origin.

Generally speaking, the ramp responses of the step-compensated systems considered in section 4.3.2 are acceptable (except for the offset) since the ripple content is relatively small, as seen from figure 4.3.8.
In this work, a synthesis approach for finite pulse width sampled-data systems is presented. It has proved to be of value for sampled-data systems incorporating variable delays in particular (section 3.8), and sampled-data systems in general. This forward loop compensation approach takes into consideration the controlled system characteristics and is quite systematic and straightforward. It does not require any prior knowledge of the controller functions, as each parameter is evaluated in turn according to the response specifications.

In the example considered in section 3.9, \( \frac{h}{T} \) was assumed constant for the delay dependent response. Probably the \( \frac{h}{T} \) value taken may not be the best for a certain range of delay variation. Also whether \( \frac{h}{T} \) is to be constant or variable needs further investigation following the lines of section 3.8.2.

The principle of making the response satisfy two simultaneous constraints at any sampling instant can be extended to other applications, e.g. self adjustment or optimisation procedures.

It may be of interest to see whether the forward loop compensation procedure of section 3.8.2 can be extended to continuous systems, by introducing fictitious sampling and letting \( \frac{h}{T} \) tend to unity.
Two approaches were considered for the design of systems incorporating variable time delays (sections 3.8.2 and 3.8.3). The delay independent response may be preferable, specially if conventional sampling is used. This would enable a properly programmed digital processing unit to be applied to the system.

As for the analysis, the time domain treatments of section 2.8. need some investigation concerning the marginal stability in particular.

Proposals for improving the performance of the magnetic tape delay simulator were discussed in section 4.2.5. The discrete controller simulation can also be improved by reducing the size of the sampling equipment to reduce pulse interaction. Semiconductor components may be of advantage in this respect.

Because there is no approximation involved, the time domain technique presented here proved to be of advantage in treating finite-pulse-width systems in general and those with time delays in particular. Also, compared with conventional sampling control, the finite-pulse-width control has some advantages in some respect, (last paragraph but one, p. 182g).

The technique provides the response at any instant of time and can be used successfully for various configurations, e.g. variable pulse-width and/or variable sampling rate. Another important application is feasible, that is the treatment of multivariable systems.
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46--------


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APPENDIX

This contains the circuit diagrams of the devices built for this work as well as some photographs.

<table>
<thead>
<tr>
<th>Circuit Diagrams</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circuit No.1   Pulse Generating Device (section 4.1.2)</td>
<td>194</td>
</tr>
<tr>
<td>Circuit No.2   Pulse Amplifier and Phase Inverter</td>
<td>195</td>
</tr>
<tr>
<td>showing a switch circuit</td>
<td></td>
</tr>
<tr>
<td>Circuit No.3   Auxiliary Circuit (section 4.1.3)</td>
<td>196</td>
</tr>
<tr>
<td>Circuit No.4   Motor-field Power Amplifier (section 4.2.1)</td>
<td>197</td>
</tr>
<tr>
<td>Circuit No.5   Modulation Circuit (section 4.2.2.)</td>
<td>198</td>
</tr>
<tr>
<td>Circuit No.6   Demodulation Circuit (section 4.2.3)</td>
<td>199</td>
</tr>
</tbody>
</table>

Photographs

<table>
<thead>
<tr>
<th>Photographs</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate No.1 Sampling Device</td>
<td>200</td>
</tr>
<tr>
<td>Plate No.2 Delay Simulator</td>
<td>201</td>
</tr>
<tr>
<td>Plate No.3 General View of the Apparatus</td>
<td>202</td>
</tr>
</tbody>
</table>
PULSE GENERATING DEVICE.

To a similar circuit generating the staggered pulses.

CIRCUIT NO. 1
CIRCUIT NO. 2

PULSE AMPLIFIER AND PHASE INVERTER

SWITCH CIRCUIT

For Groups A and B

For Group C

To a similar circuit as above

(195)
AUXILIARY CIRCUIT.

1. Trace 3 of Figure 4.1.11
2. After levelling, trace 6 of Figure 4.1.11.
3. Connected to auto connection of the computer (trace 5 of Figure 4.1.11).
DEMODULATION CIRCUIT.

CIRCUIT NO. 6