ABSTRACT

This thesis is concerned with the vibration of a simplified compressor blade system which is stiffened by the introduction of a mid-span shroud, or snubber, constraint. The initial investigation is concerned with the effect of different types of snubber interface constraint, namely:

1) Dry friction rubbing at the snubber interface.
2) Clearance at the snubber interface.
3) Viscoelastic shear joint at the snubber interface.

The results of resonance tests carried out on a single blade for each type of constraint showed the viscoelastic shear joint to stiffen the structure, while giving higher damping than that of the other interface conditions.

A more comprehensive experimental and theoretical investigation is carried out on this type of constraint on a single blade. The complex modulus model is used to describe the viscoelastic material properties which are treated as a function of frequency, and justification is given for its use in free vibration. The results show optimum positions for the snubber for maximum stiffening and damping and that they vary from mode to mode. The results also reveal that heavy damping may be obtained if the parameters are optimised correctly. The experiments verify the theoretical work satisfactorily.

The investigation is extended to the study of a batch of five blades which are connected by a viscoelastic shear joint at the snubber interface. The theoretical analysis however fails to predict one of the five basic modes. The structure exhibits heavy damping with close
natural frequencies. A survey of the methods for separating the response of individual modes from the total transient response of a structure, is carried out. It was found to be impossible to estimate the frequency and damping for one of the heavily damped modes by any of the available techniques. Correlation between experimental and theoretical results is acceptable.
### CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE PAGE</td>
<td>1</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>2</td>
</tr>
<tr>
<td>CONTENTS</td>
<td>4</td>
</tr>
<tr>
<td>LIST OF GRAPHS AND DRAWINGS</td>
<td>10</td>
</tr>
<tr>
<td>LIST OF PLATES</td>
<td>17</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>18</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>24</td>
</tr>
<tr>
<td>CHAPTER 1 INTRODUCTION</td>
<td>25</td>
</tr>
<tr>
<td>CHAPTER 2 SURVEY, APPROACH AND SCOPE OF INVESTIGATION</td>
<td>28</td>
</tr>
<tr>
<td>2.1 Survey</td>
<td>28</td>
</tr>
<tr>
<td>2.2 Approach and Scope of Investigation</td>
<td>28</td>
</tr>
<tr>
<td>SECTION A</td>
<td></td>
</tr>
<tr>
<td>EXPERIMENTAL INVESTIGATION INTO TYPES OF CONSTRAINT OF A SINGLE VIBRATING CANTILEVER BEAM</td>
<td>32</td>
</tr>
<tr>
<td>CHAPTER 3 EXPERIMENTAL WORK</td>
<td>32</td>
</tr>
<tr>
<td>3.1 Nature of Experiments</td>
<td>32</td>
</tr>
<tr>
<td>3.2 Method of Excitation</td>
<td>33</td>
</tr>
<tr>
<td>3.2.1 Calibration of Electromagnet</td>
<td>34</td>
</tr>
<tr>
<td>3.2.2 Comments on Calibration</td>
<td>38</td>
</tr>
<tr>
<td>3.3 Design and Description of Blades and Rig</td>
<td>38</td>
</tr>
<tr>
<td>3.3.1 Description of Blades</td>
<td>38</td>
</tr>
<tr>
<td>3.3.2 Design and Description of Rig</td>
<td>40</td>
</tr>
<tr>
<td>CHAPTER</td>
<td>TITLE</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>7</td>
<td>APPROACH AND SCOPE OF INVESTIGATION</td>
</tr>
<tr>
<td>8</td>
<td>THEORETICAL CONSIDERATIONS AND RESULTS</td>
</tr>
<tr>
<td>8.1</td>
<td>Preliminary Theoretical Considerations</td>
</tr>
<tr>
<td>8.1.1</td>
<td>Application of Complex Modulus Model to Free Vibrations</td>
</tr>
<tr>
<td>8.1.1.1</td>
<td>Justification for Simple Spring-Dashpot Representation of Real Viscoelastic Materials</td>
</tr>
<tr>
<td>8.1.1.2</td>
<td>Vibration Characteristics of Spring-Dashpot Model</td>
</tr>
<tr>
<td>8.1.2</td>
<td>Assumptions Made in Subsequent Analysis</td>
</tr>
<tr>
<td>8.2</td>
<td>Vibration Characteristics of Single Blade with Viscoelastic Shear Restraint</td>
</tr>
<tr>
<td>8.2.1</td>
<td>Frequency and Damping Equations</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Mode Shape Equations</td>
</tr>
<tr>
<td>8.2.3</td>
<td>Upper and Lower Frequency Boundaries for Blade</td>
</tr>
<tr>
<td>8.3</td>
<td>Theoretical Results</td>
</tr>
<tr>
<td>9</td>
<td>EXPERIMENTAL WORK</td>
</tr>
<tr>
<td>9.1</td>
<td>Nature of Experiments</td>
</tr>
<tr>
<td>9.2</td>
<td>Details of Apparatus</td>
</tr>
<tr>
<td>9.2.1</td>
<td>Arrangement of Apparatus</td>
</tr>
<tr>
<td>9.2.2</td>
<td>Linearity of Piezoelectric Strain Gauges</td>
</tr>
<tr>
<td>9.2.3</td>
<td>Response Characteristics of Parallel 'T' Band Rejection Filters to Transient Waveforms</td>
</tr>
<tr>
<td>9.3</td>
<td>Experimental Procedure</td>
</tr>
</tbody>
</table>
SECTION C

THEORETICAL AND EXPERIMENTAL INVESTIGATION OF A
MULTI-BLADE VIBRATING SYSTEM WITH VISCOELASTIC SHEAR CONSTRAINT

CHAPTER 11   SURVEY OF PREVIOUS WORK AND APPROACH TO INVESTIGATION

CHAPTER 12   THEORETICAL CONSIDERATIONS AND RESULTS

12.1 Assumptions and Method of Analysis
12.2 Frequency Equations
12.2.1 Symmetric Modes of Vibration
12.2.2 Asymmetric Modes of Vibration
12.3 Method of Solution of Equations
12.4 Theoretical Results

CHAPTER 13   EXPERIMENTAL WORK

13.1 Experimental Methods for the Determination of the Dynamics of a Structure from Transient Waveforms
13.1.1 Mathematical Analysis of Complex Transient Waveform
13.1.2 Extraction of Required Mode using an Electronic Wave Analyser
### 13.1.2.1. The 'Haaret' Technique

Page No. 198

### 13.1.2.2 The Method of 'Palei and Uspenskii'

Page No. 200

### 13.1.3 Elimination of Unwanted Modes by Use of Band Rejection Filters

Page No. 203

### 13.2 Nature of Experiments

Page No. 204

### 13.3 Description of Rig

Page No. 205

### 13.4 Arrangement of Apparatus

Page No. 207

### 13.5 Shock Pulse Shape and its Frequency Spectrum

Page No. 212

### 13.6 Experimental Procedure

Page No. 215

### 13.6.1 Symmetric Modes

Page No. 217

### 13.6.2 Asymmetric Modes

Page No. 221

### CHAPTER 14 RESULTS AND CONCLUSIONS

Page No. 227

#### 14.1 Experimental Results

Page No. 227

#### 14.2 Conclusions

Page No. 229

### APPENDIX I FREE VIBRATION OF A SINGLE DEGREE OF FREEDOM SYSTEM WITH HYSTERETIC DAMPING.

Page No. 237

### APPENDIX II FREE VIBRATION OF A SINGLE DEGREE OF FREEDOM SYSTEM WITH DAMPING FORCE DIFFERING IN PHASE WITH VELOCITY.

Page No. 239

### APPENDIX III THE EFFECT OF THE DIRECTION OF THE DAMPING FORCE ON THE SINGLE DAMPED BLADE.

Page No. 241

### APPENDIX IV RESPONSE CHARACTERISTICS OF A PARALLEL 'T' FILTER TO TRANSIENT DECAYING INPUT.
| APPENDIX V | COMPARISON OF SHEAR STIFFNESS AND BENDING STIFFNESS OF VISCOELASTIC INTERFACE JOINT. | 245 |
| APPENDIX VI | LIST OF MATHEMATICAL ABBREVIATIONS USED IN SECTION 12.2 | 247 |
| APPENDIX VII | PROOF OF METHOD OF 'PALEI AND USPENSKII' | 252 |
| APPENDIX VIII | FREQUENCY SPECTRUM OF SHOCK PULSE | 255 |
| APPENDIX IX | BIBLIOGRAPHY | 257 |
### List of Graphs and Drawings

#### Chapter 2

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Description</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Simplification of Snubber Blade System</td>
<td>30</td>
</tr>
</tbody>
</table>

#### Chapter 3

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Description</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Electromagnet Calibration Apparatus</td>
<td>35</td>
</tr>
<tr>
<td>3.2</td>
<td>Electromagnet Calibration Curves</td>
<td>39</td>
</tr>
<tr>
<td>3.3</td>
<td>General Arrangement I. Snubber Vibration Rig.</td>
<td>41</td>
</tr>
<tr>
<td>3.4</td>
<td>General Arrangement and Details of Probe Holder</td>
<td>47</td>
</tr>
<tr>
<td>3.5</td>
<td>Arrangement of Apparatus for Resonance Testing.</td>
<td>49</td>
</tr>
</tbody>
</table>

#### Chapter 4

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Description</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Response Curves, Blade Clamped Across Snubber $\alpha=0.2$</td>
<td>54</td>
</tr>
<tr>
<td>4.2</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.4$</td>
<td>55</td>
</tr>
<tr>
<td>4.3</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.6$</td>
<td>56</td>
</tr>
<tr>
<td>4.4</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.8$</td>
<td>57</td>
</tr>
<tr>
<td>4.5</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=1.0$</td>
<td>58</td>
</tr>
<tr>
<td>4.6</td>
<td>Response Curves, Blade with Clearance at Snubbers $\alpha=0.2$</td>
<td>61</td>
</tr>
<tr>
<td>4.7</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.4$</td>
<td>62</td>
</tr>
<tr>
<td>4.8</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.6$</td>
<td>63</td>
</tr>
<tr>
<td>4.9</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.8$</td>
<td>64</td>
</tr>
<tr>
<td>4.10</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=1.0$</td>
<td>65</td>
</tr>
<tr>
<td>4.11</td>
<td>Response Curves Blade, with Viscoelastic Joint at Snubber $\alpha=0.2$</td>
<td>67</td>
</tr>
<tr>
<td>4.12</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.4$</td>
<td>68</td>
</tr>
<tr>
<td>4.13</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.6$</td>
<td>69</td>
</tr>
<tr>
<td>4.14</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=0.8$</td>
<td>70</td>
</tr>
<tr>
<td>4.15</td>
<td>&quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; &quot; $\alpha=1.0$</td>
<td>71</td>
</tr>
</tbody>
</table>
CHAPTER 6

6.1 Complex Shear Modulus Representation 74
6.2 The Effects of Frequency and Temperature Upon the Properties of Real Viscoelastic Materials 74
6.3 Variation of Loss Factor and Shear Moduli with Frequency for 3M No. 466 Viscoelastic Material 77

CHAPTER 8

8.1 Simple Spring-Dashpot Viscoelastic Model, 85
8.2 Force Diagram, Spring-Dashpot Model 85
8.3 Effect of Mass and Rotary Inertia of Snubber upon the Frequency of a Cantilever Blade 92
8.4 Blade Coordinates 95
8.5 Moment Equilibrium Diagram 95
8.6 Upper and Lower Frequency Boundaries for Lateral Vibration of a Blade with Snubber Restraint 108
8.7 Modal Curves Corresponding to Points A and B on Frequency Boundaries in Figure 8.6 110
8.8 Upper and Lower Frequency Boundaries for Vibration of a Pretwisted Blade with Snubber Restraint 112
8.9 Upper and Lower Frequency Boundaries for Lateral and Edgewise Vibration of a Blade with Snubber Restraint 113
8.10 Second Modal Curves for Pretwisted Blade Propped at $\alpha = 0.714$ 114
8.11 Theoretical effect of Snubber Position upon Frequency for the first mode of vibration 117
8.12 Theoretical Effect of Snubber Position upon Damping for the first mode of Vibration 118
8.13 Theoretical Effect of Snubber Position upon Frequency for the second mode of Vibration 119
8.14 Theoretical Effect of Snubber Position upon Damping for the second mode of Vibration.
8.15 Theoretical Effect of Snubber Parameter upon Frequency for the First and Second Modes of Vibration \( \alpha = 0.6 \).
8.16 Theoretical Effect of Snubber Parameter upon Damping for the First and Second Modes of Vibration \( \alpha = 0.6 \).
8.17 Effect on Frequency and Damping of Assuming Damping Force to be in-phase with velocity.
8.18 Non-Dimensional Effects of \( \eta \) and \( \lambda \) (\( 6_{1/2} \)) Upon Frequency and Damping for the First Mode of Vibration, \( \alpha = 0.73 \).
8.19 Variation of Optimised Damping with Viscoelastic Loss Factor.
8.20 Influence of Viscoelastic Material Temp. upon Frequency and Damping.
8.21 Complex Modal Curve for First Mode of Vibration.

**CHAPTER 9**
9.1 Arrangement of Experimental Apparatus
9.2 Linearity of Piezoelectric Strain Gauges for Flexural Vibration.
9.3 Rejection Characteristics of Parallel 'T' Filters to a Decaying Sinusoidal Input.

**CHAPTER 10**
10.1 Experimental Effect of Snubber Position \( \alpha \) upon Frequency for the First Mode of Vibration \( \lambda = 1.92 \times 10^5 \).
10.2 Experimental Effect of Snubber Position \( \alpha \) upon Frequency for the First Mode of Vibration \( \lambda = 7.68 \times 10^5 \).
Experimental Effect of Snubber Position $\alpha$ upon Damping for the First Mode of Vibration $\lambda = 1.92 \times 10^5$.  

Experimental effect of Snubber Position $\alpha$ upon Damping for the First Mode of Vibration $\lambda = 7.68 \times 10^5$.  

Experimental Effect of Snubber Position $\alpha$ upon Frequency for the Second Mode of Vibration $\lambda = 1.92 \times 10^5$.  

Experimental Effect of Snubber Position $\alpha$ upon Frequency for the Second mode of Vibration $\lambda = 7.68 \times 10^5$.  

Experimental Effect of Snubber position $\alpha$ upon Damping for the Second Mode of Vibration $\lambda = 1.92 \times 10^5$.  

Experimental Effect of Snubber Parameter $\lambda$ upon Damping and Frequency for First Mode of Vibration $\alpha = 0.6$.  

Experimental Effect of Snubber Parameter $\lambda$ upon upon Damping and Frequency for Second Mode of Vibration $\alpha = 0.6$.  

CHAPTER 12

Diagram of Bending and Shearing of Viscoelastic Interface Joint.  

Coordinates of Multi-blade System and Force and Moment Equilibrium Diagram.
12.3 First Batch Mode Shapes.

12.4 Theoretical Effect of Snubber position \( \alpha \) upon
Frequency for First Batch Mode \( \lambda = 7.68 \times 10^5 \).

12.5 Theoretical Effect of Snubber Position \( \alpha \) upon
Damping for First Batch Mode \( = 7.68 \times 10^5 \).

12.6 Theoretical Effect of Snubber Position \( \alpha \) upon
Frequency for First Batch Mode \( \lambda = 3.84 \times 10^5 \).

12.7 Theoretical Effect of Snubber Position \( \alpha \) upon
Damping for First Batch Mode \( \lambda = 3.84 \times 10^5 \).

12.8 Theoretical Effect of Snubber Position \( \alpha \) upon
Frequency for First Batch Mode \( \lambda = 1.92 \times 10^5 \).

12.9 Theoretical Effect of Snubber Position \( \alpha \) upon
Damping for First Batch Mode \( \lambda = 1.92 \times 10^5 \).

12.10 Theoretical Effect of Snubber Position \( \alpha \) upon
Frequency for Second Batch Mode \( \lambda = 3.84 \times 10^5 \).

12.11 Theoretical Effect of Snubber Position \( \alpha \) upon
Damping for Second Batch Mode \( \lambda = 3.84 \times 10^5 \).

12.12 Theoretical Effect of Snubber Position \( \alpha \) upon
Frequency for Second Mode \( \lambda = 7.68 \times 10^5 \).

12.13 Theoretical Effect of Snubber Position \( \alpha \) upon
Damping for Second Batch Mode \( \lambda = 7.68 \times 10^5 \).

12.14 Theoretical Effect of Snubber Parameter \( \lambda \)
upon Frequency for First and Second Modes
\( \alpha = 0.6 \).

12.15 Theoretical Effect of Snubber Parameter \( \lambda \)
upon Damping for First and Second Modes
\( \alpha = 0.6 \).
12.16 Non-Dimensional Effects of \( \eta \) and \( \lambda (G_1/E) \) upon Frequency and Damping for First Symmetric Mode. \( \alpha = 0.74 \).

12.17 Non-Dimensional Effect of \( \eta \) and \( \lambda (G_1/E) \) upon Frequency and Damping for Second Asymmetric Mode \( \alpha = 0.73 \).

12.18 Non-Dimensional Effect of \( \eta \) and \( \lambda (G_1/E) \) upon Frequency and Damping for Second Symmetric Mode \( \alpha = 0.725 \).

CHAPTER 13

13.1 Elliott Algol Mk.II Computer Programme for Transient Waveform Analysis.

13.2 Effect of Time Interval upon Accuracy of Waveform Analysis.

13.3 Calibration of Analyser for Transient Decaying Input.

13.4 Effective 'Q' Factor versus Decay Ratio \( r \).

13.5 General Arrangement II of Snubber Blade Vibration Rig.

13.6 Arrangement of Experimental Equipment.

13.7 Frequency Spectrum of Magnetic Shock Pulse.

13.8 Strain Gauge Response for Resonance Test of Five Blade System.

13.9 Effect of Strain Gauge Position upon strain Gauge Response for the Two Blade Symmetric Mode Arrangement.

CHAPTER 14

14.1 Experimental Effect of Snubber Parameter \( \lambda \) upon the Frequency and Damping of the Fundamental and First Asymmetric Modes. \( \alpha = 0.6 \).
| 14.2 | Experimental Effect of Snubber Parameter $\lambda$ upon the Frequency and Damping of the First and Second Symmetric Modes $\alpha = 0.6$. | 232 |
| 14.3 | Experimental Effect of Snubber Position $\alpha$ upon Frequency and Damping for Fundamental Mode. | 233 |
| 14.4 | Experimental Effect of Snubber Position $\alpha$ upon Frequency and Damping of the First Asymmetric Mode. | 234 |
| 14.5 | Experimental Effect of Snubber Position $\alpha$ upon Frequency and Damping of the First Symmetric Mode. | 235 |
| 14.6 | Experimental Effect of Snubber Position $\alpha$ upon Frequency and Damping of the Second Symmetric Mode. | 236 |
**LIST OF PLATES**

<table>
<thead>
<tr>
<th>Plate</th>
<th>Description</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Electromagnet Calibration Apparatus</td>
<td>37</td>
</tr>
<tr>
<td>2.</td>
<td>General View of Rig</td>
<td>43</td>
</tr>
<tr>
<td>3.</td>
<td>Plan view of Rig with Clamping Ring Removed</td>
<td>45</td>
</tr>
<tr>
<td>4.</td>
<td>General View of Experimental Equipment</td>
<td>210</td>
</tr>
<tr>
<td>5.</td>
<td>Shock Pulse Shape</td>
<td>213</td>
</tr>
<tr>
<td>6.</td>
<td>Batch mode shape</td>
<td>213</td>
</tr>
<tr>
<td>7.</td>
<td>View of Rig, Symmetric Mode Arrangement</td>
<td>219</td>
</tr>
<tr>
<td>8.</td>
<td>View of Rig, Asymmetric Mode Arrangement</td>
<td>223</td>
</tr>
<tr>
<td>9.</td>
<td>Transient Response, Fundamental Mode</td>
<td>225</td>
</tr>
<tr>
<td>10.</td>
<td>Transient Response, 2nd Asymmetric Mode</td>
<td>225</td>
</tr>
</tbody>
</table>
NOMENCLATURE

(Excluding Computer Programme Symbols)

- \( a \) Real coefficient of complex eigenvalue \( s \).
- \( a_n \) Polynomial coefficient.
- \( b \) Imaginary coefficient of complex eigenvalue \( s \).
- \( c \) Snubber thickness and classical visions damping coefficient.
- \( d \) Viscoelastic joint thickness or interface clearance.
- \( e \) Exponential constant.
- \( f \) Frequency, cycles per second.
- \( f(t) \) Pulse shape function.
- \( h \) Snubber face width.
- \( i \) Subscript, input wave to analyser.
- \( j \) \( \sqrt{-1} \)
- \( k \) Elastic spring stiffness or arbitrary constant, pulse shape.
- \( k_1 \) Elastic spring stiffness.
- \( l \) Total blade length.
- \( l_1 \) Blade length from root to snubber.
- \( l_2 \) Blade length from snubber to tip.
- \( m \) Mass per unit length of continuous systems, and mass of single degree of freedom system.
- \( m_s \) Mass of Snubbers per blade.
- \( n \) Integer denoting a particular blade in multiblade system and number of cycles.
- \( q \) Constant.
- \( r \) Snubber length per side or decay ratio.
\[ \ddot{\tau} = \frac{k_1}{c} \]

- Complex Eigenvalue.
- \( t \) time.
- \( t_m \) Time duration until signal reaches a maximum.
- \( u \) Real coefficient, particular solution to difference equation.
- \( v \) Imaginary coefficient, particular solution to difference equation.
- \( x \) Coordinate, Appendix V.
- \( x_1 \) Displacement or coordinate from blade root to snubber.
- \( x_2 \) Coordinate from blade tip to snubber.
- \( \dot{x}, \dot{x}_1 \) First derivative with respect to time.
- \( \ddot{x}, \ddot{x}_1 \) Second derivative with respect to time.

\[ \ddot{x}_1 = x_1 \]
\[ \ddot{x}_2 = x_2 \]

- \( y \) Dynamic blade deflection in range \( x \).
- \( y_1 \) Dynamic blade deflection range \( x_1 \).
- \( y_2 \) Dynamic blade deflection in range \( x_2 \).

\[ y_1', y_2' \text{ First derivatives with respect to } x_1 \text{ and } x_2 \text{ respectively.} \]
\[ y_1'', y_2'' \text{ Second derivatives with respect to } x_1 \text{ and } x_2 \text{ respectively.} \]
\[ y_1''', y_2''' \text{ Third derivatives with respect to } x_1 \text{ and } x_2 \text{ respectively.} \]

\[ \alpha = \frac{1}{1} \text{ dimensionless snubber position.} \]
\[ \beta = (1-\alpha) \text{ and coefficient Appendix II.} \]
\( \gamma \) Transverse vibration amplitude at snubber.

\( \delta, \delta_i \) Logarithmic decrement of structure.

\( \delta_{\text{anal}} \) Logarithmic decrement of analyser.

\( \varepsilon \) Strain.

\( \eta \) Viscoelastic loss factor \( \frac{G_2}{G_1} \).

\( \mu \) Damping index, and abbreviation in multiblade system analysis.

\( \nu \) Damping index of blade.

\( \rho \) Decay index of pulse and abbreviation for mathematical function.

\( \sigma \) Abbreviation for mathematical function.

\( \tau \) Time, and duration of shock pulse.

\( \phi \) Loss angle \( \tan^{-1} \eta \), or complex root in difference equation.

\( \phi(t) \) Hereditary function.

\( \phi_n, \phi_p, \phi_v \) Phase angles.

\( \omega \) Frequency rads/sec.

\( \omega_n \) Undamped natural frequency \( \sqrt{k/m} \) and analyser input frequency.

\( \omega_o \) Tune frequency of filter and analyser rads/sec.

\( \omega_1 \) Frequency of fundamental cantilever vibration rads/sec.

\( A, A_1, A_2 \) Arbitrary constants, differential equations.

\( B, B_1, A_2 \) Arbitrary constants, differential equations.

\( C \) Capacitance

\( C_1, C_2 \) Arbitrary constants.
D Differential operator.
D₁, D₂ Arbitrary constants.
E Young's modulus of blade material.
E' Absolute value of complex Young's modulus.
E₁...E₁₇ Abbreviations, mathematical expressions.
F Clamping force across snubber and restoring force.
F Tangential restoring force.
F(ω) Frequency spectrum.
G Absolute value complex shear modulus
G Elastic shear modulus
G₂ Viscous shear modulus.
I Minimum second moment of area of blade section.
M₁ Bending moment at snubber due to blade section 1₁.
M₂ Bending moment at snubber due to blade section 1₂.
M Bending moment due to bending of viscoelastic joint.
M Bonding moment due to shearing of viscoelastic joint.
Mᵥ, Mᵥ, Mᵥ, Mᵥ Bending moment due to viscoelastic restraint.
N Integer, such that number of blades in batch equals (2N+1)
P Arbitrary constant.
P₁...P₈ Abbreviation for mathematical expression.
Q Arbitrary constant, and quality factor.
Q₁...Q₈ Abbreviations for mathematical expression.
Qₑffective Effective quality factor of analyser.
R Electrical resistance.
**Abbreviations for mathematical functions.**

$R_1 \ldots R_5$  

Shear force at snubber due to section $l_1$.

$S_1$  

Shear force at snubber due to section $l_2$.

$T$  

Periodic time $1/f$.

$T_1 \ldots T_5$  

Abbreviations for mathematical functions.

$U$  

Abbreviation for real coefficient.

$U_1 \ldots U_6$  

Abbreviation for mathematical functions, see Appendix VI.

$V$  

Abbreviation for imaginary coefficient.

$V_1 \ldots V_6$  

Abbreviations for mathematical functions, see Appendix VI.

$V_i$  

Input signal to filter or analyser.

$V_o$  

Output signal from filter or analyser.

$V_n$  

Amplitude of input signal $V_i$.

$W$  

Mass ratio of snubber per blade $m_s/ml$, and constant.

Viscoelastic properties equation.

$X$  

Constant Viscoelastic properties equation.

$X_1 \ldots X_4$  

Abbreviations for mathematical functions.

$Y(x)$  

Characteristic equation, heavy beam theory.

$Y_1 \ldots Y_4$  

Abbreviations for mathematical functions.

$Y'$  

Dimensionless distance of exciter from blade root.

$Y''$  

Dimensionless distance of probe from blade root.

$\Theta$  

Angular deflection of snubber.

$\Theta_1$  

Slope of blade at $x_1 = l_1$.

$\Theta_2$  

Slope of blade at $x_2 = l_2$. 
\( \theta_n \) Angular deflection of snubber of blade \( n \).

\( \bar{\phi} \) Abbreviation for complex particular solution to difference equation.

\( \psi \) Imaginary component of \( \phi \).

\( \psi_n \) Phase angle, wave analyser.

\( \Omega \) Real component of \( \phi \) and undamped natural frequency of wave analyser. Rads/sec.
ACKNOWLEDGEMENTS

The author wishes to thank the authorities of The University of Surrey for permission to build and use the apparatus, equipment and workshop facilities.

Thanks also go to Professor W. Carnegie for his supervision of the project and his helpful comments during preparation of the thesis. In addition, thanks go to Rolls-Royce Limited who suggested and supported the investigation.

The author also wishes to express his gratitude to the technicians in the Mechanical Engineering Department for their help and to the following industrial organisations:

Minnesota Mining and Manufacturing Company Limited,
St. Paul,
Minnesota, U.S.A.

Muirhead and Company Limited,
Beckenham,
Kent, England.
CHAPTER 1

INTRODUCTION

The problem of fatigue in the blades of axial flow compressors due to resonant and self-excited vibration is well known to workers in the aircraft gas turbine field (1).

In recent years in an effort to improve the propulsive efficiency of turbo-jet engines the by-pass principle has been evolved. For a given thrust requirement the acceleration of the gas through the engine has been reduced and the mass flow increased. This has necessitated a large increase in annulus area with a consequent increase in the length of the rotor blades, particularly the early stages.

Long slender blades are particularly prone to low engine-order and self-excited vibrations (2, 3). To obviate this and other vibration problems (4) the blade system is stiffened by the introduction of mid-span shrouds or snubbers which are integral with the blade at some distance up the aerofoil. An excellent photograph of such a system is shown in reference 3. Interest has been shown concerning the most effective location for the snubber along the blade length and also the most satisfactory conditions under which the snubbers of adjacent blades meet. These may be broadly classified as follows:

Snubbers locked together thus forming a complete ring.

This is the present design policy (1, 2, 4), the interface being such that as the blades untwist under centrifugal loading, adjacent snubbers lock together forming a complete ring. The dynamics of a snubber blade-disk system are very complicated and have been investi-
gated using rubber models (5). An energy analysis has also been used to investigate the flutter instability of systems of this type (6). One of the major practical disadvantages of this design is the wear caused by relative vibratory motion at the snubber interface. Attempts to overcome this difficulty have been made by designing the snubber such that the relative motion is a minimum (2) and by hard-plating of the snubber interfaces (7). However, reducing this relative motion to prevent wear also reduces the amount of mechanical damping which is helpful in opposing flutter instability.

**Snubbers with small interface clearance.**

Due to variations in the natural frequencies of individual blades and the phasing and harmonic content of the exciting forces, it would be expected that adjacent blades with small interface clearance would impact against each other. This would limit the amplitude of vibration not by damping but by distributing the vibrational energy over a greater frequency range by generating harmonics of the fundamental excitation. It could be expected however that the relative normal motion would also cause deterioration of the snubber faces.

**Snubbers bonded together by a thick viscoelastic joint**

Bonding the snubber interfaces together with an appropriate viscoelastic material would form the snubbers into a complete ring thus stiffening the structure. The relative motion at the snubber interface would then cause a shearing action in the bonding material. In this way it has been suggested (8, 9, 10, 19) that relative motion and damping can take place without surface deterioration.

Improvements in the mechanical integrity of engine components
are obviously important and the purpose of this project is to investigate the dynamic characteristics of the above mentioned types of blade restraint as applied to a much simplified blade system.
CHAPTER 2

2. SURVEY, APPROACH AND SCOPE OF INVESTIGATION

2.1 Survey

No literature has been published on work carried out that is of direct application to the proposed investigation. However, reports on some related investigation by other workers will be referred to from time to time in the text.

2.2 Approach and Scope of Investigation

Axial flow compressors are sophisticated machines. The blades are of complex shape, being twisted, tapered, of asymmetrical aerofoil cross-section and mounted on a flexible rotating drum or disc. The manufacture of these blades is a formidable task and it is even more difficult to predict their dynamic characteristics mathematically to a high degree of accuracy. The addition of snubbers complicates the problem still further. It is necessary therefore as a first approach to simplify the system so as to reduce the number of variables and gain a basic understanding of the dynamics.

A straight cantilever beam is used to represent the actual blade and is of similar general proportions. Simple snubbers are integral with the blade and have their faces parallel to the axis of least second moment of area of the beam section. The root fixing is assumed to be encastre which approximates to the actual condition when the rotor is at speed. Even a rigid rotor fitted with such simplified blades provides a formidable system for an experimental investigation into the stiffening effect of snubbers. Hence it is not unreasonable as a start to the investigation to study the effects of snubbers on the vibration characteristics of a single blade. In an attempt to simulate
the types of restraint envisaged in the actual system the following simplifications are made:

(a) **Single blade clamped rigidly across snubbers.**

The action of centrifugal force on the twisted blades shown inset in Fig. 2.1 is to lock them together at the snubber. This is simulated by a single blade clamped rigidly across the snubbers as shown diagrammatically in Fig. 2.1a. It is proposed to determine experimentally the effect of snubber position, snubber length and clamping force.

(b) **Single blade restrained by rigid stops at snubbers.**

This represents any blade in a rotor where there is a small clearance at the interface of the snubbers of adjacent blades. It is simulated by a single blade which has a small clearance between the snubber faces and a pair of rigid stops as shown in Fig. 2.1b. It is proposed to determine experimentally the effects of snubber position and interface clearance. The effects of snubber length are likely to be small.

(c) **Single blade restrained by a viscoelastic shear joint at snubber interface**

This simulates the viscoelastic bond at the interface of adjacent blades in the real system. It consists of a single blade, the snubbers of which are restrained from rotary motion by bonding to a stationary member as shown in Fig. 2.1c. For a viscoelastic joint of fixed dimensions it is proposed to investigate experimentally the effects of snubber length and snubber position.

While it is appreciated that the motion of the single blade systems described above will not simulate all the possible motions of
FIGURE 2.1 SIMPLIFICATION OF SNUBBER BLADE SYSTEM.

(A) CLAMPING FORCE

(B) CLEARANCE

(C) VISCOELASTIC SHEAR JOINT

PART VIEW OF SNUBBER-BLADE SYSTEM SHOWING ACTION OF CENTRIFUGAL UNTWIST.
the blade in the real system, the experimental work will nevertheless yield a quantitative assessment for comparison purposes.

Engine running experience has shown that fatigue failures have been caused in general by vibration in the lower flexural and first torsional modes. This project is concerned only with flexural vibration and the initial experiments are restricted to the first mode of vibration only. No attempt is made to measure the mode shapes.

After having examined the results of the preliminary experiments for each type of blade restraint, it is proposed to investigate further both theoretically and experimentally the type of restraint which is most suitable for practical application. In addition, it is further proposed to investigate the characteristics of a batch of several blades incorporating the selected type of restraint as this approximates more closely to the actual system.

For convenience therefore, the thesis is divided into three sections, each section dealing with one of the three separate investigations, namely:

SECTION A Experimental investigation into types of constraint for a single vibrating cantilever beam.

SECTION B Theoretical and experimental investigation of a single vibrating blade with a specific type of restraint.

SECTION C Theoretical and experimental investigation of a multi-blade vibrating system with a specific type of restraint.
SECTION A

EXPERIMENTAL INVESTIGATION INTO TYPES OF CONSTRAINT OF A SINGLE VIBRATING CANTILEVER BEAM.

CHAPTER 3

3. EXPERIMENTAL WORK

3.1 Nature of experiments

To compare experimentally the different types of blade restraint, it is necessary to subject each type to the same vibratory environment. The test used was the sweep vibration or resonance test. The blade was subjected to a harmonic excitation force of variable frequency while the amplitude of the force was maintained constant. Because the blade is an elastic system, the amplitude of vibration depends not only on where the amplitude is measured but also on where the system is excited. It is necessary therefore to keep the position of the exciter and amplitude measuring instrument fixed in all tests for a given snubber position. Changing the snubber location necessitates a change in the exciter and amplitude measuring instrument position. It is only possible therefore to compare the response for different snubber positions by shape and position along the frequency axis as the peak amplitudes are arbitrary.

Blades were manufactured of effective length 10.0 inches, width 2.0 inches and thickness 0.125 inches. Two sets were made, one set with snubbers of 1.0 inch length per side and the other with snubbers 0.5 inch length per side. Each set consisted of five blades which had snubbers at different positions. If \( \alpha \) is the ratio of the distance from the root to the snubber, over the effective blade length, then the snubber positions may be specified as \( \alpha = 0.2, 0.4, 0.6, 0.8, \ldots \).
1.0. The amplitude of vibration was measured using a capacitance probe and vibration meter.

3.2 Method of excitation

The requirements of the exciter are as follows:-

1) Facility to maintain forcing amplitude constant while varying the frequency.

2) Non-contacting, to prevent interference with the dynamics of the structure.

3) Compact, because of limited space.

After a survey of the various methods the technique which came closest to fulfilling the above requirements was the electromagnetic method. It is well known that the attractive force of an electromagnet is proportional to the square of the current flowing in the coils. Consequently if a sinusoidal current of frequency \( f \) flows through the coil, the force felt, for example by a steel blade, will be a steady pull modulated by an alternating force of frequency \( 2f \) (12). The static attraction force may be cancelled out by arranging for a permanent magnet to pull in the opposite direction and the frequency doubling effect may be overcome by supplying the coils at half the required excitation frequency. In addition, the forcing amplitude, although unknown, may be maintained constant by keeping the excitation current amplitude constant, in a similar manner to that previously reported (11). However, the attraction forces between two bodies in a magnetic field depend on the distance between them. If one of the objects corresponds to the electromagnet and the other to the blade any change of amplitude of the vibrating blade would in turn produce a change in the force exciting the blade, especially at resonance. An experiment was
carried out therefore to determine how the attractive forces between the electromagnet and a ferrous object vary with the distance between them.

3.2.1 Calibration of electromagnet

It is difficult to measure how the amplitude of the alternating force on a body varies with its distance from the electromagnet producing the alternating field. It is proposed therefore to supply the coils with direct current thus producing a steady magnetic pull which is easier to measure. The validity of this calibration depends solely on the premise that the decay of the attractive force with distance for a steady magnetic field is similar to the case when the field is alternating. It is further assumed that any differences in the magnetic circuit between the calibration arrangement and experimental arrangement do not significantly affect the shape of the curves.

The calibration equipment is arranged as shown in Fig. 3.1 and plate 1. A 12 volt battery provides the necessary current via a reostat and ammeter to the coils of the electromagnet (f). A soft iron anvil (d) is suspended from a spring balance (b) near to the pole face of the electromagnet. The electromagnet is of the type to be used in the experiments and consists basically of 256 turns of 16 S.W.G. copper wire on a 1 inch square ferroxcube core. It is fixed rigidly to the base board. The position of the anvil relative to the pole face is measured using a capacitance pick-up and distance meter.

Because the stiffness characteristics of the spring balance and magnet are very different the system is inherently unstable, the procedure therefore was as follows.

The required current was set using the ammeter and reostat.
ELECTROMAGNET CALIBRATION

APPARATUS.

FIGURE 3.1

ADJUSTING NUT (a)

SPRING BALANCE (b)

STOP NUT (c)

ANVIL (d)

PICK-UP (e)

ELECTROMAGNET (f)

DISTANCE METER
PLATE I. ELECTROMAGNET CALIBRATION APPARATUS
The adjusting nut (a) was slackened off until the anvil was attracted to the electromagnet, the anvil however was prevented from touching the pole face of the magnet by the stop nut (c). The adjusting nut was then tightened until the stop nut was just free to rotate, neglecting friction the two forces were then equal and a reading was taken from the spring balance and distance meter.

3.2.2 Comments on Calibrations

The curves of attraction forces versus distance apart for various current settings are shown in Fig. 3.2. The dotted line indicates the force versus distance characteristics for a permanent magnet. This permanent magnet is to be used to cancel out the steady pull of the electromagnet in the actual blade tests. The slight scatter of experimental points was caused by friction in the anvil guides although the apparatus was mounted vertically to keep this to a minimum.

From the calibration, it may be seen that provided that in the blade tests the electromagnet is placed at a point near the blade where the amplitude of vibration is relatively small, and sufficiently distant from the blade face, the forcing amplitude will be almost constant.

3.3 Design and description of blades and rig

3.3.1 Description of blades

The blades were manufactured from bright mild steel strip. The snubbers were made from the same material and were silver soldered to the blade in a plane at right angles to its longitudinal axis. The end faces of the snubbers were then finish ground to the required length, giving an average surface finish of 190 μ.
FIGURE 3.2 ELECTROMAGNET CALIBRATION

![Graph showing the force of attraction vs. distance for 1 and 2 ampere cases with a permanent magnet.]
3.3.2 Design and description of rig

A general arrangement drawing of the rig is shown at reduced scale in Fig. 3.3 and additional views are provided in plates 2 and 3. In the following description numbers where quoted refer to components shown in the above mentioned plates and drawings. The rig is designed so that the blades are mounted vertically to prevent any deflection of the blade due to its own mass. The base of the rig consists of a root block (R54) and a clamp block (R55) which when bolted together grip the blade firmly providing an encastre root fixing. The root block is bolted rigidly to a cast iron bed plate which in turn is bolted to a concrete block on vibration isolation foundations. The combined weight is of the order of ten tons. A support plate (R56) 1.0 inch thick is bolted to the side of the root block. This carries the support (R57) which provides the restraint for the blade, and the exciter assembly. Mounted opposite to the electromagnet on the exciter assembly is a pot-type permanent magnet, the position of which may be varied by rotating the magnet adjusting screw. The support plate has two vertical slots which allow the support and exciter assembly to be moved to any position along the blade. The support is made from duralumin and has a large central hole which is recessed to house two crescent shaped hardened steel jaws (R63). The shape of the jaws facilitates easy alignment with the snubbers and the faces are ground giving an average surface finish of 15\mu. The jaws are held in position by a clamping ring (R58). The tolerances are such that the jaws may slide in the recess depending on the tightness of the clamping screws. The three types of restraint are produced in the following ways:

(a) Clamping across snubbers

The lever arm is pivoted at one end and has a force applied to it at the other, by the spring balance. The moment produced is
PLATE 2. GENERAL VIEW OF RIG.
PLATE 3. PLAN VIEW OF RIG WITH CLAMPING RING REMOVED.
balanced by the force which the lever arm exerts on a short rod, which
is in contact with one of the two jaws. The dimensions of the rig are
such that the snubber face of the blades when in position just touch
the face of the second jaw. The force provided by the spring balance
and transmitted by the lever arm thus produces a pure clamping action
across the snubber. The magnitude of the clamping force may be adjust-
ed by turning the wing nut on the tension screw and is directly propor-
tional to the load registered on the spring balance.

(b) Clearance at snubber faces

By fitting undersize jaws in the support a clearance may be
set using feeler gauge strip between the snubber and jaw faces on
either side of the blade. The jaws are then held rigidly by tightening
down the clamping ring.

(c) Viscoelastic joint at snubber interfaces

Fitting undersize jaws in the support allows room for an
adhesive joint to be made at the snubber and jaw interfaces. The jaws
are fixed by tightening down the clamping ring.

The capacitance pick-up is held by a probe holder which
together with details of its components is shown in Fig. 3.4 and plate
2. It consists of an "Eclipse Magnetic Base" which supports a pillar
(R86) and an adjustable slide mechanism. The magnetic base allows the
holder to be moved to any position and to be fixed firmly by pressing
the button located at the side. A requirement of the pick-up is that
it should be electrically insulated, therefore the crosshead (R82) is
made from "Fluon" which also has the advantage that it has a low coef-
ficient of friction. Fine position control may be achieved for the
pick-up by rotating the knurled adjusting screw on the crosshead.
The rig was also designed with future experiments in mind such that it has the facility to clamp a maximum of five blades together. The blade roots are held apart by spacers of the appropriate thickness.

3.4 Arrangement of Apparatus

The general arrangement of apparatus is shown diagrammatically in Fig. 3.5. A Decade Oscillator supplies an alternating voltage of variable frequency to a 20 watt Williamson Amplifier which has a variable gain control. The output terminals of the amplifier are connected through an ammeter to the coils of the electromagnet. The capacitance pick-up is connected to a Wayne Kerr Distance and Vibration Meter which displays amplitude of vibration and the mean position of the blade from the pick-up, on two dials. The pick-up is supplied with several interchangeable heads for measuring different amplitude ranges. Full scale deflection on the meter dial corresponds to the range of the head being used and does not require calibration.

To observe the vibration waveform and detect any non-harmonic motion, an output signal is taken from the vibration meter and displayed on one beam of a double beam oscilloscope. The signal is in the form of a 50 kc carrier wave amplitude modulated by the vibration waveform. A reference waveform is fed to the remaining beam of the oscilloscope from a sweep oscillator via a digital frequency meter.

3.5 Viscoelastic shear joint material

It is usual to describe the dynamic properties of viscoelastic materials by their response to a harmonically applied stress. If a purely elastic solid is loaded sinusoidally the resulting strain is in phase with the applied stress. For a Newtonian liquid however stress and strain are always 90 degrees out of phase and all imparted
FIGURE 3.5 ARRANGEMENT OF APPARATUS FOR RESONANCE TESTING.
energy is lost. Viscoelastic materials are partly viscous and partly elastic consequently the response is a combination of the two. All polymers, elastomers and rubbers are viscoelastic in nature and may be either adhesive or non-adhesive. Although these materials are in everyday use comparatively little information is available regarding their dynamic mechanical properties. More recently however great interest has been shown in the high internal damping properties of these materials, particularly in those of an adhesive nature (9, 13, 14, 15). The material selected for these experiments is in tape form and is designated No. 466 by the manufacturers, Messrs. Minnesota Mining and Manufacturing Co., Ltd. It was chosen for the following reasons:

1) The dynamic mechanical properties had been investigated for low strains by the manufacturers and for high strains by Whittier (9) and the data was generally available.

2) The material takes the form of an adhesive layer 0.002 inches thick on a paper backing strip. The layers may be added together to form a material of the required thickness.

Although the mechanical properties viscoelastic materials are in general sensitive to temperature the experiments will be carried out at laboratory ambient temperature and no attempt will be made to maintain it constant from day to day.

3.6 Experimental procedure

The apparatus was arranged as shown in Fig. 3.5. Prior to the experiments the snubber and jaw faces were cleaned with trichloroethane. The blade was inserted between the root block and clamp block and the clamping block bolts were evenly tightened. Before each test the electronic apparatus was switched on and allowed to warm-up
for 20 mins. The experimental procedure for each type of restrain was as follows:-

3.6.1 **Blade clamped across snubbers**

Preliminary tests had revealed that some of the blades were not quite straight and it was necessary to place steel shims between the support and support plate to ensure that the snubber face and the jaw on the support plate side of the blade just touched. The remaining jaw was inserted and the clamping ring lightly fastened down. The nut on the tension screw was then adjusted until the spring balance registered a load corresponding to a clamping force of 27 lb on the jaws. It was possible to spring the jaw away from the snubber face a little against the clamping force. The clamping ring screws were tightened just sufficiently to allow the jaw to return to the snubber face of its own accord. The setting-up procedure was repeated for clamping forces of 9 lb and 3 lb.

3.6.2 **Blade with clearance at snubber faces**

Small helical compression springs were fitted in the holes which lead outward from the recess in the support and undersize jaws were fitted. The jaws were held against the snubber face by the force of the springs. The clamping ring was clamped lightly. The jaws were sprung away from the snubber faces by a small amount and shims of 0.003 inch thickness were inserted. The jaws were then released and the clamping ring screwed down tightly. The shims were then removed leaving a clearance of 0.003 inch at each snubber face. This procedure was repeated for a 0.001 inch clearance.

3.6.3 **Blade with viscoelastic joint at snubber face**

A bonding material of thickness 0.010 inch was built up from
the adhesive and from this strips were cut 0.125 inch wide and a little
more than 2.00 inches long. The backing paper was removed from one
side and the material applied to the snubber faces. The remaining
backing paper was then removed and the jaws inserted and pressed
firmly against the adhesive material. The clamping ring was then
tightened down.

3.6.4 General procedure

By placing steel shims between the support plate and the
electromagnet an air-gap of 0.062 inch was set between the pole face
and the blade using a distance piece of that thickness. The permanent
magnet was moved away from the blade by rotating the adjusting screw.
The reading on the distance meter was set to full scale deflection by
rotation of the fine adjustment on the probe holder. The decade
oscillator was then set to a frequency of 10 c.p.s. and the gain con­
trol on the amplifier adjusted until the ammeter registered 1.0 amps.
(which preliminary tests had shown to be a suitable excitation level).
The steady pull produced by the electromagnet caused the mean position
of the blade vibration to move from the stationary blade position to a
position closer to the electromagnet. This movement was registered on
the distance meter. The adjusting screw for the permanent magnet was
then rotated clockwise until the distance meter again read full scale
deflection. The steady force produced by the electromagnet was then
cancelled out and the mean position of vibration coincided with the
centre line of the stationary blade. A note was made of the oscillator
frequency (half excitation frequency) and the amplitude of vibration
from the meter. The amplifier gain was then reduced to zero and the
oscillator frequency increased by a small amount. The size of the
increment depended upon the expected increase in response of the struc­
ture. The gain control was then reset to give a current of 1.0 amp.
and the procedure repeated until the forcing frequency had passed
through the resonant range of the blade.
4. EXPERIMENTAL RESULTS

The response curves are shown in Figs. 4.1-4.15. The frequency axes on all graphs are to the same scale and the amplitude scales are the same for a given snubber location except for the results of the blade with clearance where the mode shapes are basically dissimilar to the other cases and a comparison of absolute amplitude is meaningless. On each graph is marked the dimensionless distance to the exciter Y' and probe Y" measured from the root.

4.1 Blade clamped across snubber

Figs. 4.1-4.5 show the response curves obtained for blades clamped across the snubber for snubber locations of \( a = 0.2 \) to \( a = 1.0 \). Each graph shows curves for clamping forces of 3 and 27 lbf and snubber lengths of 1 and \( \frac{1}{2} \) inch. The response curves for the 9 lbf clamping force were omitted to avoid crowding of the graphs. In all cases they were intermediate between the 3 lbf and 27 lbf curves and were of similar shape. Several of the curves in Figs. 4.1 and 4.2 are asymmetrical in that the peaks lean to the left. This is characteristic of the non-linear "softening spring" response curve and may be expected for the following reasons:

The moment applied to the snubbers by the blade will increase until it reaches a limiting value of \( 2 \mu_l r F \) where \( \mu_L \) is the coefficient of limiting friction, \( r \) the snubby length and \( F \) the clamping force. When this value is reached interface slip will occur and the torque required to maintain slipping will remain constant or full to \( 2 \mu_D r F \) where \( \mu_D \) is the dynamic coefficient of friction. Thus as the blade amplitude increases past a critical point the restraining moment reduces similar to that of a softening spring. An attempt was made to measure the snubber
**Figure 4.1 Response Curves, Blade Clamped Across Snubbers \( \alpha = 0.2 \)**

- Probe
- Exciter
- \( Y' \)
- \( Y'' \)
- \( r = \text{Snubber Length} \)
- \( F = \text{Clamping Force lbs.} \)

### Key Points
- \( Y' = 0.6 \)
- \( Y'' = 0.97 \)
- \( r = 1.0 \) \( F = 27 \)
- \( r = 0.5 \) \( F = 3 \)
**Figure 4.2 Response Curves, Blade Clamped Across Snubbers α = 0.4**

- $y' = 0.6$
- $y'' = 0.97$

$f = $ Snubber Length'

$F = $ Clamping Force lbs.

- $f = 0.5$
  - $F = 27$
- $f = 1.0$
  - $F = 27$
- $f = 0.5$
  - $F = 3$

Amplitude Thousandths Inch

Excitation Frequency C.P.S.
Figure 4.3 Response Curves, Blade Clamped Across.

- $y' = 0.940$
- $y'' = 0.725$
- $f = \text{clamping force} \text{ lbs.}$
- $f_1 = 0.27$
- $f_2 = 0.3$
Figure 4.4 Response curves, blade clamped across stub.

\[ Y' = 0.48 \]
\[ Y'' = 0.65 \]
\[ F = \text{clamping force} \text{ lbs.} \]

Excitation frequency vs. amplitude in thousands of inches.
Figure 4.5 Response Curves, Blade Clamped Across

\[ Y' = 0.51 \]
\[ Y'' = 0.68 \]

\( r = \text{snubber length} \)
\( F = \text{clamping force lbs.} \)

Excitation Frequency C.P.S.
tip motion to verify the above hypothesis but it was too small to be measured conveniently being of the order of 0.000001 inch.

In order to dissipate dry frictional energy, there must be both relative motion and frictional restraint. If the clamping force is infinitely large there will be no relative slip and if it is zero there will be no frictional restraint, therefore it must have an optimum value. The curves shown in Figs. 4.1 and 4.2 indicate that this optimum value is below 3 lb as maximum damping has not been achieved.

It may be noticed in all the graphs that the higher clamping force produces a response curve which is sharper and more symmetrical with a resonant peak at a higher frequency. This may be expected as an increase clamping force past the optimum value causes less slip and therefore less non-linearity. It also produces a greater frictional restraining force with a consequent rise in frequency. Increasing the snubber length has a similar effect to increasing the clamping force, producing greater restraint with a consequent rise in frequency. The limiting moment to cause interface slip is proportional to the snubber length. If the length is increased the limiting moment is higher and less slipping occurs. The converse is also true. Fig. 4.3 shows that one of the curves is asymmetrical and the peak leans to the right, similar to that of a "hardening spring". It may be shown that at this snubber location the blade is quite sensitive to rotational restraint. The hardening spring effect is caused by the right angled corners of the snubber force rotating against the jaws.

The response curves for $a = 0.8, 1.0$, $r = 1$, $F = 3$, are rather unusual in shape and it is thought that this is perhaps due to combined vibration of the blade and clamping mechanism. A pronounced vibration of the spring balance was noticed on one occasion at 363 c.p.s. and it is at this point that the response curve shows a distinct hump.
The vibration which was continuously monitored showed no particularly marked distortion.

4.2 Blade with clearance at snubber face

The response curves for the blade with a clearance of 0.001, 0.003 inch at each interface are shown in Fig. 4.6-4.10. The response curves for all snubber locations with a clearance of 0.003 are strongly asymmetrical with peaks which lean to the right and exhibit typical "hardening spring" characteristics. This is to be expected as although the blade is free to vibrate linearly within the clearance, when the snubber amplitude is equal to the clearance, impact occurs and the blade feels a sharp increase in stiffness. In all the above cases the peak overhangs the lower part of the curve and two distinct vibration amplitudes are possible over a small frequency range. The motion was noticed to be generally unstable between the upper and lower amplitudes and it was impossible to obtain the intermediate part of the curve. Hence, the break shown in graphs. The two impacts per cycle produced a loud ringing noise and the amplitude jump could easily be recognised by a change in volume. At some snubber locations, particularly those near the blade tip, even off-resonant excitation was sufficient to cause the blade to vibrate at a snubber amplitude equal to that of the interface clearance. These curves therefore are fairly straight and unbroken.

The response curve for $\alpha = 0.6$ with a clearance of 0.003 inch shows two distinct breaks. The vibration waveform showed some harmonic content. With the aid of the reference signal it was possible to ascertain that the additional waveform was at a frequency below the excitation frequency. A photograph of the oscilloscope trace was made and with the aid of an episcopes, ordinates were measured which allowed a fourier analysis of the waveform to be carried out at different points on the response curve. The result is shown in Fig. 4.8 where
Figure 4.6: Response curves, blade with clearance. At snubber $\alpha = 0.2$. $C = 0.003$. $Y = 0.97$. $C = C$ - clearance. 

Excitation frequency (c.p.s.)

Amplitude (thousandths inch)

Graph showing the relationship between excitation frequency and amplitude for different values of clearance.
Figure 4.7 Response curves, blade with clearance at snubbers $\alpha = 0.4$.
Figure 4.8 Response curves, blade with clearance.

\[ \gamma' = 0.04 \]
\[ \gamma'' = 0.72 \]
\[ C = \text{clearance}'' \]

Motion at A is
Motion at B is
Motion at C is
Figure 4.9 Response curves, blade with clearance at snub

\[ Y' = 0.48 \]
\[ Y'' = 0.65 \]
\[ C = \text{CLEARANCE}'' \]
FIGURE 4.10 RESPONSE CURVES, BLADE WITH CLEARANCE

\( Y = 0.51 \)
\( Y'' = 0.68 \)
\( C = \text{CLEARANCE}'' \)

\( C = 0.001 \)
\( C = 0.003 \)
it will be noticed that for the points B and C the predominant motion is at half the excitation frequency. No other sub-harmonics were present. This appears to be a sub-harmonic resonance of order half, which is unusual in practical mechanical systems (12). This is also in complete contrast with the work of Babitskii (16) who showed that theoretically for a single degree of freedom system with limiters, there are a profusion of superharmonics.

A short supplementary test was carried out on five blades clamped together with 0.003" interface clearance. Exciting an end blade revealed that in addition to the complicated antiphase motion of adjacent blades there existed a mode in which the blades all vibrate in-phase at large amplitude.

4.3 Blade with viscoelastic joint at snubber interface

The response curves for different snubber locations are shown in Figs. 4.11-4.15. They are all linear in form and exhibit heavy damping. Particularly heavy damping is shown with the snubber at the \( \alpha = 0.8 \) position. As may be expected the blade with the longer snubber has a higher natural frequency.

4.4 Comparison of types of blade restraint

Both the clamped blade and the viscoelastically jointed blade show a similar stiffening effect in terms of resonant frequency. However, the resonant amplitude for the viscoelastic restraint is consistently lower and the damping generally higher than that of the clamped blade. The snubber blade with clearance produced little stiffening in terms of resonant frequency when the snubber is near the root, that is when \( \alpha = 0.2, 0.4 \). Because of drastic non-linearity the stiffness cannot be expressed in terms of frequency for higher values of \( \alpha \).
Figure 4.11 Response curves, blade with viscoelastic joint at snubber interface

\[ \alpha = 0.2 \]

\[ V' = 0.6 \]
\[ V'' = 0.97 \]
\[ \Gamma = \text{snubber length}'' \]
\[ \text{Shear joint thickness} = 0.010'' \]
Figure 4.12 Response Curves Blade with Viscoelastic Joint at Snubber Interface

\[ \alpha = 0.4 \]

The diagram shows the relationship between excitation frequency (C.P.S.) and amplitude (in thousands of inches) for different values of \( r \). The values are:

- \( Y' = 0.6 \)
- \( Y'' = 0.97 \)
- \( r = \text{Snubber Length}^" \)
- \( \text{Shear Joint Thickness} = 0.010" \)

The curves at different values of \( r \) are marked as follows:

- \( r = 0.5 \)
- \( r = 1.0 \)
**Figure 4.13** Response curves blade with viscoelastic interface $\alpha = 0.6$

$Y' = 0.94$
$Y'' = 0.725$
$r = \text{snubber length}$
Shear joint thickness = 0.010"
FIGURE 4.14 RESPONSE CURVES, BLADE WITH VISCOELASTIC INTERFACE $\alpha = 0.8$.

- $Y' = 0.48$
- $Y'' = 0.65$
- $\tau =$ SNUBBER LENGTH
- SHEAR JOINT THICKNESS = 0.010"
Figure 4.15 Response Curves, Blade with Viscoelastic Interface $\alpha = 1.0$

- $Y' = 0.51$
- $Y'' = 0.68$
- $r = $ Snubber Length$^*$
- Shear Joint Thickness = 0.010

Amplitude (Thousands of Inch)

Excitation Frequency (C.P.S.)
5. CONCLUSIONS

Although the tests were carried out for only a limited range of values for each of the salient variables, it is considered that they were significant. Tests on the blade with interface clearance revealed instability. The supplementary test showed that in-phase motion of a batch of blades could occur, in which case the snubber provided no restraint. In addition, it is thought that the two vicious impacts per cycle would cause extensive snubber damage.

The blade clamped across the snubbers provided good stiffening and high damping in several cases but is sensitive to damping force, which, in a practical system, is a function of speed, and difficult to optimize. Although flame plating (7) may be used on metal snubbers to minimise wear, this cannot be used on plastics which are coming into use as compressor blade materials (17).

The viscoelastic shear joint appears to be the most satisfactory interface condition giving high damping and a low resonant amplitude in comparison with the clamped blade and the blade with interface clearance. Viscoelastic materials are in general temperature sensitive, however materials are being developed to operate over a wider temperature (13, 18, 19). It is proposed therefore to investigate more fully the viscoelastic shear interface type of restraint.
SECTION B

THEORETICAL AND EXPERIMENTAL INVESTIGATION
OF A SINGLE VIBRATING BLADE WITH VISCOELASTIC RESTRAINT

CHAPTER 6

6. SPECIFICATION AND PREVIOUS APPLICATIONS OF REAL VISCOELASTIC MATERIALS

6.1 Specification of real viscoelastic materials

6.1.1 Steady harmonic motion specification

It has been mentioned previously that the dynamic mechanical properties of viscoelastic materials are usually described by their response to harmonic loading. Because they are often used in cyclic shear they are normally specified by the complex modulus of rigidity:

\[ G = G_1 + jG_2 = G_1(1 + j\eta) \]

where the absolute value of \( G \) is given by:

\[ G = \sqrt{G_1^2 + G_2^2} \]

and \( G_1 \) and \( G_2 \) are the elastic (real) and viscous (imaginary) parts of the complex rigidity respectively. The loss factor \( \eta = G_2/G_1 \) is the damping-stiffness ratio for the viscoelastic material and represents the tangent of the phase angle by which the stress vector leads the strain vector during harmonic vibration as shown in Fig. 6.1. Snowdon (20) verified that the complex modulus notation was a realistic mechanical model for rubber-like materials undergoing harmonic loading. In
FIGURE 6.1 COMPLEX SHEAR MODULUS REPRESENTATION

FIGURE 6.2 THE EFFECTS OF FREQUENCY AND TEMPERATURE UPON THE PROPERTIES OF REAL VISCOELASTIC MATERIALS.
general these properties vary with vibration frequency, temperature and shear strain amplitude (9). The variation of the shear moduli \( G, G'_1, G'_2 \) and loss factor \( \eta \) with vibration frequency and temperature are shown in Fig. 6.2. In the rubbery region for low frequency or high temperature both the storage modulus \( G'_1 \) and loss modulus \( G'_2 \) are small and little energy dissipation occurs. In the transitional region for intermediate frequencies and temperatures the loss modulus has its maximum value indicating that maximum energy dissipation is obtained in this region. The maximum value of the loss factor \( \eta \) occurs in the transitional region but at a lower frequency or higher temperature than that at which the loss modulus occurs. In the glassy region for high frequency and low temperature the loss modulus \( G'_2 \) is again small and the elastic modulus \( G'_1 \) approaches its maximum value and the material is almost purely elastic. Loss factors in excess of 2.0 may be obtained under optimum conditions. The elastic modulus is usually in the range 300-3000 lb/ins\(^2\) which is too low for most structural purposes. The high damping capacity of these materials may be exploited by forming a composite structure with the more usual engineering materials. It is becoming possible now by variation of the composition of the viscoelastic material to arrange for the transition region to be in the required temperature-frequency range (15). The properties of the material are found by applying a known harmonic shear strain, the stress-strain hysteresis loop is recorded and the complex moduli are then calculated. The apparatus described by Fitzgerald and Ferry (21) is the most commonly used machine. It applies very low shear strains of the order of 0.1%. The machine used by Whittier (9) is capable of applying strain of 100% and he showed that some adhesive materials can withstand such strains for millions of cycles without failure. Another method suggested by Nicholas and Heller (14) obtains the moduli by calculation, from the measurements of the free decay of a sandwich beam with a viscoelastic core.

A graph of complex shear moduli versus frequency for 3M
No. 466 adhesive material (22) at 22°C is shown in Fig. 6.3, together with a sample of the material. The moduli were determined at very low strain amplitudes using the machine described by Fitzgerald and Ferry (21). It must be pointed out however that the material supplied for this investigation was not from the same batch as that for which the calibration shown in Fig. 6.3 was made.

It is important to note that the complex modulus notation is defined for harmonic vibration only where the velocity vector leads the displacement by π/2 radians. For free damped vibration the motion is not harmonic and the velocity vector leads the displacement vector by \(\pi/2 + \tan^{-1}(6/2\pi)\) where \(\delta\) is the logarithmic decrement of the decay.

### 6.1.2 Transient motion specification

Of the small amount of calibration work that has been conducted on viscoelastic and rubber-like material, almost all of it has been of the steady state variety. Some early work however was carried out by Taylor (23) on the transient impact of polythene specimens. More recently a paper was published by Volterra (24) who showed that for plastics and rubber-like materials the stress-strain relationship could be expressed by the formula:

\[
\sigma = E\varepsilon + \int_0^t \frac{\partial}{\partial \tau} \phi(t-\tau) \frac{d\varepsilon}{d\tau} \, d\tau
\]

where \(\sigma\) is the dynamic stress, \(\varepsilon\) is the strain and \(E\) is the elastic modulus. The function \(\phi(t)\) is known as the hereditary function. Using the experimental results of Taylor (23) he showed that for a hereditary system of the first degree for polythene the function was of the form:

\[
\phi(t) = Ae^{-\alpha t}
\]

where \(A\) and \(\alpha\) are constants.
FIGURE 6.3 VARIATION OF LOSS FACTOR AND SHEAR MODULUS WITH FREQUENCY FOR 3M N0.466 VISCOELASTIC ADHESIVE MATERIAL. TEMP = 22°C.
Thus for polythene the dynamic stress-strain relationship becomes:

\[
\sigma = E \varepsilon + A \int_0^t e^{-\alpha(t-\tau)} \frac{d\varepsilon(\tau)}{d\tau} d\tau
\]

### 6.2 Previous application of complex modulus model

It is well known that many engineering materials have internal damping characteristics in which the energy dissipated per cycle is largely independent of frequency. When representing this mathematically for harmonic motion the classical viscous damping coefficient is made inversely proportional to frequency. The magnitude of the damping force is then only dependent upon the displacement but is in phase with the velocity. For harmonic motion this is identical in form to the complex modulus notation and is sometimes referred to as "complex stiffness" and "structural damping" or "hysteretic damping", Bishop (25).

#### 6.2.1 Application to single degree of freedom systems

The application of structural or hysteresis damping to the harmonic vibration of single degree of freedom systems has received some attention in the past. More recently the application of this type of model to free damped vibration of single degree of freedom systems has caused some considerable controversy. This began with a paper by Soroka (26) who predicted that unlike the case of viscous damping, an increase in the imaginary damping coefficient caused an increase in natural frequency. Since then several papers have appeared on the subject by Bishop (25), Myklestad (27), Pinsker (28), Reid (29), Neumark (30), Lancaster (31), Kawashima (32), Caughey (33), Scanlan and Mendelson (34), each attempting to explain and clarify the mechanics. Bishop (25) suggested a difference between the resonant frequency and the damped natural frequency. Caughey (33) using a spring-dashpot
model attributed to Biot (35) shows that the energy dissipation per cycle is practically constant for this model, and that an increase in the viscous damping coefficient raises the natural frequency. He also points out that some of the authors take great liberties with their mathematics and together with Lancaster (31) mentions that the complex modulus model when applied to a real free vibration problem yields complex solutions. This is shown in Appendix I.

6.2.2 Application to an infinite degree of freedom system

Applications of viscoelastic materials to damp vibrations in continuous elastic systems have been mainly to beams and plates. These may be divided into two types (a) boundary damping, that is at the support only; (b) constrained and unconstrained sandwich layers, forming a laminated structure.

(a) Boundary damping has been investigated theoretically and experimentally by Mentel (36)(37) mainly in terms of optimising the support configuration but only in terms of energy dissipated per cycle. Fu and Mentel (38) and Mead and Wilby (39) and others analysed theoretically the forced vibration of beams with damping due to translation and rotational motion at the support, in terms of energy dissipated per cycle and composite loss factor respectively.

(b) There has been much published work on the constrained and unconstrained layer sandwich beams, by Obserst (40), Kerwin (41), Sobicka (42), Ungar (43), Yi-Yuan Yu (44), Ungar and Kerwin (46), Blasingame and DiTarmo (47), Hertelendy and Goldsmith (48) and others. In these papers the damping is expressed mainly in the form of a composite loss factor for the structure
(41, 43, 44, 45, 46, 47, 48) and in only a few cases do the analyses predict damped natural frequencies (44, 45, 48). Comparatively little experimental work has been done, with only a few papers showing a comparison of theoretical and experimental damping values (40, 41, 46). None of the above authors compare predicted and experimental damped natural frequencies. The theoretical effects and experimental effects of viscoelastic damping on vibration mode shapes have not been shown by any of the above authors. In all cases the non-linear effects of shear strain amplitude are neglected and the complex shear moduli are considered to be independent of frequency.

It is very surprising to find that several authors Nicholas and Helier (14), Yi-Yuan Yu (44), Di Taranto (45), Hertelendy and Goldsmith (48), some going into very detailed analyses, fail to mention the assumption they make in applying the complex modulus model to damped free vibrations. It may be seen in the paper by Di Taranto (45) that his equations 26 and 33 correspond exactly to equations A2 and A3 for the case of free vibration of a single degree of freedom system with hysteretic damping shown in Appendix I. Similarly in the paper by Hertelendy and Goldsmith (48) the frequency is in complex form and their equation 36 corresponds basically to equation 3A of Appendix I. The time function of the solution is therefore complex and identical to that for the single degree of freedom system of Appendix I.

It may be seen from the paper by Di Taranto (45) that an increase in the value of the viscoelastic loss factor produces a rise in the damped natural frequency. Yi-Yuan Yu (44) mentions an apparent increase in frequency due to the increased stiffness caused by the
complex modulus model. Nicholas and Heller (14) remark that theoretically the resonant frequency is not the same as the free vibration frequency but fail to justify the statement.
CHAPTER 7

7. APPROACH AND SCOPE OF INVESTIGATION

It is the purpose of the present work to determine the optimum position of the snubber in terms of stiffening and damping and to investigate how these are dependent upon the geometry of the snubber and the viscoelastic bond thickness. It is desirable that the above characteristics of the structure be specified in a manner that can be readily appreciated by an engineer. The stiffness may best be described by the ratio of the frequency of the blade with snubber restraint to that of the frequency of a free cantilever without restraint. The damping is specified by the logarithmic decrement.

The properties of the viscoelastic material are only specified for harmonic motion and for this reason it would seem appropriate to consider both theoretically and experimentally the blade undergoing forced vibration. It has been suggested by Plunkett (49) however that for the forced vibration of elastic systems with concentrated damping the shape of the response curve and the value of the 'Q' factor depend upon the position of excitation. The 'Q' is an abbreviation for Quality and defines the sharpness of the response curve (50). Moreover he points out that the relationship between the logarithmic decrement $\delta$ and the 'Q' factor $\delta = \pi/Q$, which has been shown to hold for single degree of freedom systems is not necessarily true for continuous elastic systems. In addition, experimental difficulties prevail. Single point excitation by an electrodynamic exciter is the only practical method when it is required to know the amplitude of the forcing function. This entails the mechanical coupling of the armature of the exciter to the structure, with a force transducer between the two. In comparison to the size of the structure the attached mass is large. The inaccuracies in frequency measurements caused by such attachments have been reported by Rissone and Williams (51) and
the effects of the mass on damping measurements by Bishop and Pendered (54). For the above reasons the system is treated as a free vibration problem after making various assumptions and justifying the use of the complex modulus model for free damped vibration. The first two modes of vibration of the system are investigated although no attempt is made to determine the mode shapes experimentally. It was shown in Chapter 4 of Section A that moving the snubber location caused the resonant frequency to vary over a wide range. In view of this and the frequency dependence of the viscoelastic material properties, it is necessary to consider the complex shear moduli $G_1$ and $G_2$ as a function of frequency.
8. THEORETICAL CONSIDERATIONS AND RESULTS

8.1 Preliminary theoretical considerations

8.1.1 Application of complex modulus model to free vibration

8.1.1.1 Justification for simple spring dashpot representation of real viscoelastic materials

The mass \( m \) in Fig. 8.1 is supported by a Zener combination of dashpot and springs. A dashpot in series with a spring is known as the Maxwell model. When this is in parallel with another spring it is called the Zener combination. Considering the impedance of the model, then for the spring \( k_1 \) and dashpot \( c \) in series the motion of the point \( x_1 \) is given by:-

\[
\frac{dx_1}{dt} + \frac{r}{c} x_1 = \frac{dx}{dt}
\]

or

\[
\frac{d}{dt} (x_1 e^{rt}) = \frac{dx}{dt} e^{rt}
\]

where \( r = k_1/c \)

If \( x_1 = 0 \) for \( t < 0 \), then in a similar manner to Caughey (33) for \( t > 0 \):

\[
x_1 = \int_{0}^{t} e^{-r(t-\tau)} \frac{dx}{d\tau} d\tau
\]
FIGURE 8.1 SIMPLE SPRING–DASHPOT VISCOELASTIC MODEL.

FIGURE 8.2 FORCE DIAGRAM, SPRING DASHPOT MODEL.
The damping force \( k_1 x_1 \) is then given by:

\[
k_1 x_1 = k_1 \int_0^t e^{-\dot{r}(t-\tau)} \frac{dx}{d\tau} d\tau
\]

and the total restoring force is:

\[
F = kx + k_1 \int_0^t e^{-\dot{r}(t-\tau)} \frac{dx}{d\tau} d\tau
\]

It has not been previously recognised but it may be seen that the above equation is identical in form to that derived for the dynamic stress in polythene by Volterra (24) and shown in section 6.1.2. This is very significant in justifying the use of this model to simulate real viscoelastic materials as suggested by several authors Ruzicka (53), Alfrey (54), Newton and Mathews (55), Neubert (56). It may also be pointed out that for harmonic motion the variations in the elastic and damping stiffnesses with frequency as shown by Neubert (56) are very similar to those for real viscoelastic materials as shown in Fig. 6.2. By inspection of the model it will be readily appreciated that increasing the damping coefficient \( c \) will raise the natural frequency of the mass \( m \).

8.1.1.2 Vibration characteristics of Zener spring - dashpot model

If it can be assumed that for the real viscoelastic material the values of the shear moduli do not themselves change when undergoing free damped vibration, then it is only the direction of the damping force which is important. The effect of the direction of the damping force on the vibration of a single degree of freedom system is examined in Appendix II. It is shown that a hypothetical damping force which is out of phase with the velocity, causes an increase in natural frequency, regardless of whether it lags or leads the velocity.
Consider the motion of the mass \( m \) suspended by the Zener spring-dashpot model shown in Fig. 8.1. The equations of motion are as follows:

\[ \ddot{x} + c\dot{x} + kx - c\dot{x}_1 = 0 \] ...............................(1)

\[ c\dot{x}_1 + k_1x_1 - c\dot{x}_1 = 0 \] ...............................(2)

From which:

\[ \ddot{x} + \frac{k_1}{c} \dot{x} + \left( \frac{k + k_1}{m} \right) \dot{x} + \frac{k}{mc} x = 0 \] ...............................(3)

Solving equation 3 yields solutions of the form

\[ x = A_1 e^{\mu t} + B_1 e^{(-\nu + j\omega)t} + C_1 e^{(-\nu - j\omega)t} \] ...............................(4)

\[ x_1 = A_2 e^{\mu t} + B_2 e^{(-\nu + j\omega)t} + C_2 e^{(-\nu - j\omega)t} \] ...............................(5)

where \( \mu, \nu \) and \( \omega \) are functions of the constants \( m, c, k \) and \( k_1 \).

Neglecting the aperiodic relaxation motion and considering only one solution for the oscillatory motion such that

\[ x' = B_1 e^{(-\nu + j\omega)t} \] ...............................(6)

and

\[ x'_1 = B_2 e^{(-\nu + j\omega)t} \] ...............................(7)

Substituting the appropriate time derivatives of equations (6) and (7) into equation (2) yields

\[ B_2 = \left[ c(-\nu + j\omega)/ \left[ (k_1 - \nu c) + j\omega \right] \right] B_1 \]
The velocity of the mass $m$ is:

$$\dot{x}' = (-v + j\omega)B_1 e^{(-v + j\omega)t} \tag{10}$$

Considering the motion at the instant when the displacement lies along the real axes of the complex plane, the arrangement of the forces are as shown in Fig. 8.2 where

$$\phi_V = \tan^{-1}\frac{v}{\omega} \quad \phi_F = \tan^{-1}\left[\frac{v}{\omega} - \frac{\nu c - \omega c}{\omega k_1 k_1}\right]$$

Thus the velocity leads the displacement by:

$$\frac{\pi}{2} + \tan^{-1}\frac{v}{\omega} \tag{11}\text{.................................(11)}$$

and the damping force leads the displacement by:

$$\frac{\pi}{2} + \tan^{-1}\left[\frac{v - \nu c - \omega c}{\omega k_1 k_1}\right] \tag{12}\text{.................................(12)}$$

From equations (11) and (12), it may be clearly seen that the damping force lags the velocity, even for harmonic motion, i.e., $\nu = 0$. This was quickly verified by using an analogue computer to simulate the mechanical model. The effect of free vibration therefore is to increase the amount by which the damping force lags the velocity.
Because the damping force for the Zener model lags the velocity in harmonic motion, it does not mean that there is a discrepancy between this model and the complex model. The calibration data for the harmonic shear tests carried out on the real material are only resolved into real and imaginary components for convenience. The complex modulus model does not necessarily represent the true value of the elastic modulus or the true magnitude and direction of the damping modulus, but rather the components in a real and imaginary direction.

For the Zener model the phase lag is a variable, being a function of the characteristics of the composite structure, that is \( \nu \) , and \( \omega \), and a direct function of the properties of the model, \( c \) and \( k_l \). In applying the complex model to free vibrations the phase lag is also a variable but is only dependent on the characteristics of the composite structure \( \nu \), and \( \omega \) since the direction of the damping force remains constant.

In the light of a more comprehensive understanding of the transient material properties, it has been shown that using a simple mechanical model it might be expected that the viscoelastic material damping force would lag the velocity in free vibration. The complex modulus model approximates to this, the phase lag causing a rise in frequency. Practical examples where damping produces a rise in natural frequency are not unknown. Cherry (57) showed experimentally that for electromagnetic induction damping, which is normally assumed to be viscous in nature, a rise in natural frequency occurred.

8.1.2 Assumptions made in subsequent analysis

The following assumptions are made in the analysis of the blade with a viscoelastic restraint:

(1) That the snubber is rigid and does not flex.
That in the light of insufficient information on the transient behaviour of viscoelastic materials and the approximations mentioned in section 8.1.1 that the complex modulus model can be used for free vibration.

The bond material and its thickness are such that transverse deflection of the blade at the snubber can be assumed to be zero.

That the shear strain amplitude of the viscoelastic material at the interface is very small, that is, of the same order as that for which the calibration was made. Although Whittier (9) showed the shear moduli to reduce with increasing shear strain amplitude, he suggested that the material could be considered linear for small amplitudes.

That the relative motion at the snubber interface due to axial shortening of the blade when deflected is small compared to that due to the rotation of the snubber. The axial motion is a non-linear effect and no non-linear effects were exhibited in the response curves for this type of restraint shown in Chapter 4 Section A.

That the internal damping of the blade material is small and can be neglected.

That the effect of rotary inertia of the snubber is small for the lower modes of vibration and can be neglected. A short preliminary analysis was carried out using the classic flexural beam equations for the unrestrained vibration of the blade
taking into account the snubber mass and its rotary inertia. In addition to the snubber position and blade length, the frequency was found to depend upon the following two dimensionless factors:

\[
\left(\frac{S I}{m}\right), \left(\frac{S m}{S}ight)
\]

where \(m_s\) is the mass of the snubber and \(I_c\) is the moment of inertia of the snubber about an axis through the blade centre line and normal to the plane of bending. \(S = \omega^2 m/EI\) where the remaining symbols are as defined in the nomenclature. The problem was programmed for a Ferranti Sirius Computer and the results were given as a percentage reduction in frequency. Fig. 8.3 shows the reduction in frequency versus snubber position for the size of blades considered previously with snubbers of total length 1.0" and 2.0". Neglecting the root position, the point of minimum frequency reduction corresponds to a point where the snubber location and a node of the system are coincident. The reductions in frequency at these points are due to rotary inertia only and are seen to be small, but increasing with the higher modes of vibration. As the snubber is itself restrained from rotating, in the case under investigation, the effects of snubber inertia are neglected.

(8) That pure shear of the viscoelastic band is assumed to take place. It is normally assumed that for viscoelastic materials the Poissons ratio is 0.5 Yin(13)
Figure 8.3 Effect of Mass & Rotary Inertia of Snubber Upon the Frequency of a Cantilever Blade.
Mentel (37). This has been verified recently by Krause, Segretto, Przirembel and Mach (58) although it has been suggested by Gottenburg and Christenson (59) that this also is complex. The shear stiffness of rubber-like materials as affected by the dimensions were investigated by Read (60). He showed that for a material of Poisson's ratio 0.5 and a viscoelastic bond length to thickness ratio \((c/d)\) of less than 3, that the actual stiffness is overestimated by less than \(7 \pm 2\%\) by assuming pure shear to occur. In the light of previous assumptions concerning the material properties and to avoid the introduction of an additional parameter \((c/d)\) this discrepancy was neglected.

![Diagram of a single vibrating blade with viscoelastic restraint]

8.2 Vibration characteristics of a single vibrating blade with viscoelastic restraint

8.2.1 Frequency and damping equations

The transverse vibration of a cantilever beam with a dashpot located at the free end has been considered by McBride (61) using the classic flexural beam theory with complex eigenvalues. Young (62) verified this work making use of normal modes in the solution of the transverse vibration of beams acted upon by elastic, damping, and mass inertia, forces concentrated at any point along the beam length.
Using Dirac Delta functions the effects of point forces and concentrated masses on beams have been investigated in papers by Haddow (63) and Yu Chen (64) respectively. However the application of a rotational restraining force has not been considered previously. For the case under investigation the rotational restraint is coincident with an additional restraint producing zero displacement at the snubber. This causes considerable complication to the Dirac Delta method and introduces another transcendental equation into the trial and error solution of the method proposed by Young (62). The approach utilised here is that of McBride (61) using complex eigenvalues. To account for the discontinuity caused by the restraints, the beam is considered to be made up of two parts which join at the snubber.

The blade, shown in Fig. 8.4, is held encastre at the root and is restrained from transverse motion by the snubber. Let \( m \) be the mass per unit length of the blade and \( EI \) the stiffness.

The characteristic equation of transverse motion of a beam neglecting shear deflection and rotary inertia is given by:

\[
EI \frac{d^4 y}{dx^4} = -m \frac{d^2 y}{dt^2}
\]

or

\[
\frac{d^4 y}{dx^4} = -q \frac{d^2 y}{dt^2}
\]

where

\[
q = \frac{m}{EI}
\]

The solution of equation (13) may be written as;

\[
y = Y(x)e^{j(s/q)^2 t}
\]

where

\[Y(x) = A \sin x + B \cos x + C \sinh x + D \cosh x\]
FIGURE 8.4 BLADE COORDINATES

FIGURE 8.5 MOMENT EQUILIBRIUM DIAGRAM
A, B, C and D are arbitrary constants and s is also a constant to be specified later.

For the portion of the blade between the root and the snubber:

\[ y_1 = (A_1 \sin x_1 + B_1 \cos x_1 + C_1 \sinh x_1 + D_1 \cosh x_1) e^{i(s/q)t} \]  

\[ y_2 = (A_2 \sin x_2 + B_2 \cos x_2 + C_2 \sin x_2 + D_2 \cosh x_2) e^{i(s/q)t} \]

Because the system is damped it may be expected that the motion would be of the form:

\[ y = Y(x) e^{-\nu t} \]

A comparison of equations (15) and (18) yields the identity:

\[ (s/q)^2 = -\nu + j\omega \]

Following the method of McBride (61) such that

\[ s = \frac{1}{l} (a+jb) \]

where l is the total length of the blade and a and b are pure numbers.

Substituting equations (14) and (20) into equation (19) and separating real and imaginary parts yields;

\[ \omega = (a^2 - b^2) \sqrt{\frac{EI}{lm^4}} \]

\[ \nu = 2ab \sqrt{\frac{EI}{lm^4}} \]
Now the fundamental circular frequency $\omega_1$ of the cantilever beam without snubbers is given by:

$$\omega_1 = 3.516 \sqrt{\frac{EI}{mL^4}}$$

hence the frequency ratio is given by:

$$\frac{\omega}{\omega_1} = \frac{(a^2-b^2)^2}{3.516} \cdots \cdots (22)$$

During the time interval between consecutive cycles the amplitude of vibration diminishes from:

$$Y(x)e^{-\nu t} \text{ to } Y(x)e^{-\frac{(t+2\pi)}{\omega}}$$

thus the ratio between adjacent peaks is a constant and equal to

$$e^{-\frac{2\pi \nu}{\omega}}$$

From which the logarithmic decrement is:

$$\delta = \log \frac{y_n}{y_{n+1}} = \frac{4\pi ab}{(a^2-b^2)} \cdots \cdots (23)$$

From Fig. 8.5 it may be seen that for snubber rotation $\Theta$, snubber length $r$ and viscoelastic material thickness $d$, the shear strain in the viscoelastic material is given by $r\Theta/d$ and the shear stress $r\Theta/d$ ($G_1 + jG_2$). If the width of the snubber face is $h$, then the sheared area is $ch$, and the tangential restoring force at the snubber interface is:

$$\bar{F} = \frac{chr}{d} (G_1 + jG_2)\Theta$$
Thus the total restoring moment on the blade is given by:

\[ M_v = \frac{2ch}{d}(G_1 + jG_2) \theta \]  \hspace{1cm} (24)

where \( \theta = y_1'(1_1) = -y_2'(1_2) \)

Applying the curve-fitting technique of Kravitz (65) the shear moduli of the viscoelastic material may be expressed over the frequency range \(30-1000 \text{ c.p.s.}\) by

\[ G_1, G_2 = Z-W \left[ e^{x(1000-f)} \right] - 1 \]  \hspace{1cm} (25)

where \( Z, W \) and \( X \) are constants and \( f \) is the frequency in \( \text{c.p.s.} \) given by:

\[ f = \left( \frac{a^2-b^2}{2\pi} \right) \sqrt{\frac{EI}{ml^4}} \]

Considering the geometric boundary conditions of the blade section 1\(_1\),

\[ y_1(0) = y_1'(0) = y_2'(1_2) = 0 \]

Thus;

\[ y_1 = A_1 \left[ \frac{(\sin x_1 - \sinh x_1) + (\sin x_1 - \sinh x_1)(\cos x_1 - \cosh x_1)}{\cosh x_1 - \cos x_1} \right] e^{(v+j\omega)t} \]  \hspace{1cm} (26)

Considering the geometric boundary conditions of the blade section 1\(_2\),

\[ y_2(1_2) = 0 \]
And the dynamic boundary conditions of the section $1_2$ are given by

$$y''''(0) = y'''(0) = 0$$

Hence:

$$y_2 = A_2 \left[ \frac{(\sin s_{1_2} + \sinh s_{1_2}) - (\sin s_{1_2} + \sinh s_{1_2})(\cosh s_{1_2} + \cosh s_{1_2})}{(\cosh s_{1_2} + \cosh s_{1_2})} \right] e^{-\nu + j\omega} t$$

\[ \text{...............(27)} \]

It may be seen from Fig. 8.5 that at the junction of the two blade sections $1_1$ and $1_2$ the slopes are equal, such that:

$$y'_1(1_1) = -y'_2(1_2) \text{ .................(28)}$$

Taking moments at this point reveals that:

$$M_1 - M_2 - M_v = 0 \text{ .................(29)}$$

where $M_1 = EIy''(1_1)$ and $M_2 = EIy''(1_2)$

Substituting the appropriate derivatives of equations (26) and (27) and also equation (24) into equations (28) and (29), and simplifying yields:

$$\frac{(\cosh s_{1_1} \sin s_{1_1} - \cosh s_{1_1} \sin h s_{1_1})}{(\cosh s_{1_1} \cosh s_{1_1} - 1)} - \frac{(\sinh s_{1_2} \cosh s_{1_2} - \sin s_{1_2} \cosh s_{1_2})}{(\cosh s_{1_2} \cosh s_{1_2} + 1)}$$

$$\frac{2 \text{ch}(\delta')}{\text{sd}l} \left( \frac{G_1}{E} \frac{1 + j\eta}{1 - j\eta} \right) = 0 \text{ .................(30)}$$
Putting \( l_1 = \alpha l \) and \( l_2 = (1-\alpha)l = \beta l \) into equation (20) such that:

\[
\begin{align*}
sl_1 &= \alpha(a+jb) \\
sl_2 &= \beta(a+jb)
\end{align*}
\]

Expanding the necessary hyperbolic and trigonometric terms into real and imaginary parts yields:

\[
\begin{align*}
\cosh s l_1 \sin s l_1 &= P_1 + jQ_1 \\
\cosh s l_2 \sin s l_2 &= P_4 + jQ_4
\end{align*}
\]

where \( P_1 = \cosh aa \cos qa \cosh ab \cos ab + \sinhab \sinab \sinhaa \sinaa \)

and \( Q_1 = \sinhab \sinaa \cosh ab - \sinaa \sinhab \cosh aa \sinaa \coshab \cosab \)

\[
\begin{align*}
\cosh s l_1 \sin s l_1 &= P_2 + jQ_2 \\
\cosh s l_2 \sin s l_2 &= P_4 + jQ_4
\end{align*}
\]

where \( P_2 = \cosh aa \sinaa \coshab \cosab - \sinhab \cosaa \sinab \)

and \( Q_2 = \sinhab \sinaa \coshab \sinab + \coshaa \cosab \cosaa \sinhab \)

\[
\begin{align*}
\sin s l_1 \cosh s l_1 &= P_3 + jQ_3 \\
\sin s l_2 \cosh s l_2 &= P_4 + jQ_4
\end{align*}
\]

where \( P_3 = \sinhab \cosaa \coshab \cosab + \coshaa \sinaa \sinhab \sinab \)

and \( Q_3 = \coshaa \cosqa \coshab \sinab - \sinhab \sinaa \sinhab \cosab \)

\[
\begin{align*}
\sin s l_2 \cosh s l_2 &= P_4 + jQ_4
\end{align*}
\]

where \( P_4 = \coshba \sinba \coshb \cosb - \sinhab \cosba \sinhb \sinb \)
and \( Q_4 = \sinh a \sin b \cosh b \sin b + \cosh a \cos b \cos a \sinh b \)

\[
\sinh s_1 \cos s_1 = P_5 + jQ_5
\]

where \( P_5 = \sinh a \cos b \cosh b \cos b + \cosh a \sin b \sin b \)

and \( Q_5 = \cosh a \cos b \cosh b \sin b - \sinh a \sin b \sin b \cos b \)

\[
\cosh s_1 \cos s_1 = P_6 + jQ_6
\]

where \( P_6 = \cosh a \cosh b \cos b \cos a + \sinh a \sin b \sinh b \sin b \)

and \( Q_6 = \sinh a \sin b \cosh b \cos a - \sin b \sin b \cos b \cosh b \cos b \)

Substituting the above abbreviations into equation(30) and separating real and imaginary parts yields;

\[
\frac{(P_1 + P_3)(P_1 - 1) + Q_1 (Q_2 - Q_3)}{(P_1 - 1)^2 + Q_1^2} - \frac{(P_5 + P_4)(P_5 + 1) + Q_5 (Q_6 - Q_4)}{(P_5 + 1)^2 + Q_6^2} = \frac{2\lambda}{{a^2 + b^2}^2} \left| \frac{g_1}{E} \right|
\]

.................(32)

\[
\frac{(Q_2 - Q_3)(P_1 - 1) + Q_1 (P_2 - P_3)}{(P_1 - 1)^2 + Q_1^2} - \frac{(Q_5 - Q_4)(P_6 + 1) + Q_6 (P_5 - P_4)}{(P_6 + 1)^2 + Q_6^2} = \frac{2\lambda}{{a^2 + b^2}^2} \left| \frac{g_1}{E} \right|
\]

.................(33)
where \( \lambda = \frac{\text{ch}^2 r_1}{\text{Id}} \) and is called the "snubber parameter".

Equation (32) is derived by equating the real part of the complex equation (30) and it is this equation which in general specifies the frequency. Equation (33) vanishes when \( \eta = 0 \). The right hand term of equation (32) represents the stiffness of the restraint and as its value increases the natural frequency rises. It may be readily seen that this term contains the material loss factor \( \eta \), therefore as expected, the natural frequency increases, with increased damping. The effect of assuming that the damping force is in phase with the velocity is treated in Appendix III.

The above equations are of the form \( f_1(a, b) = f_2(a, b) = 0 \). A computer programme was written for a Ferranti Sirius Computer to automatically determine the roots \( a \) and \( b \). The procedure was to substitute values of \( a \) and \( b \) into the equations, keeping one constant while the value of the other was varied. The results of both functions \( f_1(a, b), f_2(a, b) \) were then tested for being approximately zero by checking for a change in sign. If both functions changed sign between the new substituted values and the old values, then the machine automatically concentrated the search for the roots in that area, until the root was located to the required accuracy.

It soon became apparent however that the functions \( f_1(a, b) \) and \( f_2(a, b) \) were very erratic and that the values of \( a \) and \( b \) could only be varied by very small increments to avoid missing the roots in the search. The problem was further complicated by the difficulty in estimating the search area for the values \( a \) and \( b \). To avoid wasting machine time, it was arranged for the computer to print the values of the functions for each pair of substituted values. By monitoring the output it soon became apparent as to whether the estimated values of \( a \) and \( b \) should be reduced or increased. The machine printed out the
values of the functions in the region of the roots. The values of a and b to satisfy equations (32) and (33) were then found by graphical interpolation.

8.2.2 Mode shape Equations

a) Section 1\textsubscript{2}

For unity tip deflection that is \( y_2(0) = 1 \) equation (27) becomes:

\[
y_2 = -\frac{1}{2} \left[ \frac{(\coss_2 + \cosh s_2)}{(\sins_2 + \sinh s_2)} - (\coss_2 + \cosh s_2) \right] e^{(-v + j\omega)t}
\]

Putting \( \hat{x}_2 = x_2/l_2 \) such that \( sx_2 = \hat{x}_2\alpha(a+jb) \)

Expanding the necessary hyperbolic and trigonometric functions into real and imaginary parts yields;

\[
\coss_2 + \cosh s_2 = X_1 + jY_1
\]

where \( X_1 = \cos\beta_a \cosh \beta_b + \cosh \beta_a \cos \beta_b \)

\[ Y_1 = \sinh \beta_a \sin \beta_b - \sin \beta_a \sinh \beta_b \]

\[
\sins_2 + \sinh s_2 = X_2 + jY_2
\]

where \( X_2 = \sin \beta_a \cosh \beta_b + \sinh \beta_a \cos \beta_b \)

\[ Y_2 = \cosh \beta_a \sin \beta_b + \cos \beta_a \sinh \beta_b \]
\[ \sin x_2 + \sinh x_2 = X_3 + jY_3 \]

where \( X_3 = \sin x_2^a \cosh x_2^b + \sinh x_2^a \cos x_2^b \)

\[ Y_3 = \cos x_2^a \sinh x_2^b + \cosh x_2^a \sin x_2^b \]

\[ \cos x_2 + \cosh x_2 = X_4 + jY_4 \]

where \( X_4 = \cos x_2^a \cosh x_2^b + \cosh x_2^a \cos x_2^b \)

\[ Y_4 = \sinh x_2^a \sin x_2^b - \sin x_2^a \sinh x_2^b \]

Using the above abbreviations the deflection curve for the section \( L_1 \) is given by:

\[ y_2 = -\frac{1}{2} \left[ \frac{X_2 X_3 X_4 + X_1 Y_2 X_3 + Y_2 Y_1 X_3 - Y_1 Y_3 X_2}{X_2^2 + Y_2^2} + X_4 \right] + \]

\[ j \left[ \frac{Y_2 X_2 X_4 + X_1 Y_2 X_3 - X_2 X_1 Y_3 + Y_1 Y_2 X_3}{X_2^2 + Y_2^2} - Y_4 \right] e^{-\nu + j\omega t} \ldots \ldots (35) \]

b) Section \( L_2 \)

Using equation (35) in relation (28) the deflection curve over the section \( L_1 \) is;
\[
y_1 = \frac{1}{2} \frac{\left( \cosh \omega \cos x_1 + 1 \right)}{\left( \sinh \omega \sin x_1 \right)} \left( \cosh \omega \cos x_1 \right) \left( \sinh \omega \sin x_1 \right) + \left( \sinh \omega \sin x_1 \right) \left( \cosh \omega \cos x_1 \right) e^{-\gamma + j\omega t}
\]

Putting \( x_1 = \frac{x_1}{\sqrt{\omega}} \), where \( x_1 = x_1/l_1 \) and expanding the necessary hyperbolic and trigonometric functions into real and imaginary parts yields:

\[
(Cosh_{x_1} - Cos_{x_1}) = R_{x_1} + jT_{x_1}
\]

where \( R_{x_1} = \cosh_x \cos_y - \cos_x \cosh_y \)

\( T_{x_1} = \sinh_x \sin_y + \sin_x \sinh_y \)

\[
(Sinh_{x_1} - Sin_{x_1}) = R_{x_1} + jT_{x_1}
\]

where \( R_{x_1} = \sinh_x \cosh_y - \sin_y \cosh_x \)

\( T_{x_1} = \cos_x \sin_y - \cos_y \sin_x \)

\[
(Sinx_{x_1} - Sinh_{x_1}) = R_{x_1} + jT_{x_1}
\]

where \( R_{x_1} = \sin_x \cosh_y - \sinh_x \cosh_y \)

\( T_{x_1} = \cos_x \sinh_y - \cosh_x \sin_y \)

\[
(Cos_{x_1} - Cosh_{x_1}) = R_{x_1} + jT_{x_1}
\]

where \( R_{x_1} = \cos_x \cosh_y - \cosh_x \cos_y \)

\( T_{x_1} = \sin_x \sinh_y - \sinh_x \sin_y \)
Substituting the above abbreviations and others defined previously into equation (36) and separating real and imaginary parts yields:

\[
y_1 = \frac{1/2(R_5 + jT_5) e^{(-\nu + j\omega)t}}{(X_2(P_1 - 1) - Q_1 Y_2)^2 + (P_1 - 1)Y_2 + Q_1 X_2)^2} \tag{37}
\]

where:

\[
R_5 = \left[ (P_6 + 1)(X_2(P_1 - 1) - Q_1 Y_2) + Q_6((P_1 - 1)Y_2 + Q_1 X_2) \right] \left[ R_1 R_3 + R_2 R_4 - T_1 T_3 + T_2 T_4 \right] - \\
\left[ Q_6(X_2(P_1 - 1) - Q_1 Y_2) - (P_6 + 1)((P_1 - 1)Y_2 + Q_1 X_2) \right] \left[ T_1 R_3 + T_3 R_1 + T_2 R_4 - T_4 R_2 \right]
\]

\[
T_5 = \left[ Q_6(X_2(P_1 - 1) - Q_1 Y_2) - (P_6 + 1)((P_1 - 1)Y_2 + Q_1 X_2) \right] \left[ R_1 R_3 + R_2 R_4 - T_1 T_3 + T_2 T_4 \right] + \\
\left[ (P_6 + 1)(X_2(P_1 - 1) - Q_1 Y_2) + Q_6((P_1 - 1)Y_2 + Q_1 X_2) \right] \left[ T_1 R_3 + T_3 R_1 + T_2 R_4 - T_4 R_2 \right]
\]

8.2.3 Upper and lower frequency boundaries

As a check for errors in the theoretical analyses and computer programme, the frequency results were examined to ensure that they lay within, or coincided with the upper and lower frequency boundaries. The calculated values of damping were also checked to ensure that for snubber positions close to the root the predicted damping approached zero, which for physical reasons must be the case.
a) **Lower frequency boundary**

The lower boundary corresponds to the frequency of a cantilever beam which is propped at an intermediate position along its length. Values of frequency ratio for such a system have been calculated by Peek (66) for prop locations in the range $0.5 < \alpha < 1.0$. These values however are displayed on a graph which has a very small scale and it was not possible to verify the results to a high degree of accuracy. However for $\alpha = 1.0$, comparison with the results for a clamped-pinned beam showed good agreement.

b) **Upper frequency boundary**

The upper boundary corresponds to the case when the restraint stiffness is infinite, that is, the blade is encastré at the snubber. Thus two separate vibrating systems are formed, one being a clamped-beam and the other a clamped-free beam. The upper boundary therefore is defined by two frequencies. This may be verified by inspection of equation (30). For an encastré condition at the snubber the term on the right hand side must be infinite. For finite frequency this suggests that the denominators of the remaining two terms are zero. It may be seen that if this is true the two denominators represent the frequency equations for the clamped-clamped and clamped-free beams.

Graphs of upper and lower frequency boundaries versus $\alpha$ for the first two modes of vibration are shown in Fig. 8.6. It may be noticed that the upper boundaries for the lower modes continue to become the upper boundaries for successively higher modes.

The points where the upper boundaries cross each other, are coincident with the lower boundary for the subsequently higher mode, for example, points A, B and C. These points of which there are an infinite number correspond to three possible systems namely, a clamped-clamped beam, a clamped-free beam and the higher mode of a clamped-propped beam.
FIGURE 8.6  UPPER & LOWER FREQUENCY BOUNDARIES FOR A BLADE WITH SNUBBER RESTRAINT.
The mode shapes at the points A and B are shown in Fig. 8.7. They were obtained by substituting the value of 'a' from equation (22) corresponding to the appropriate value of \( \omega/\omega_1 \) into the mode shape equations (35) and (37). It can be seen that the mode shapes do represent each of the three separate systems. The physical significance of these points is that they are positions where a node of the system and the prop location coincide.

The vertical distance between the upper and lower boundaries for a given snubber position \( \alpha \), gives an indication of the sensitivity of the blade dynamics to any type of rotational restraint. It may be seen that the value of \( \alpha \) for maximum sensitivity in a given mode always corresponds to minimum sensitivity in the mode immediately above. It is possible therefore to estimate the optimum position for any tuning, stiffening or damping device which achieves its effect by rotational means.

It was mentioned by Armstrong (1) that for the purposes of fatigue testing snubber blades, the outboard portion of the blade is treated as a cantilever. It has been shown that for a simple straight cantilever beam propped along its length that for certain prop positions in a given mode the outboard portion of the blade does behave exactly as a cantilever. The effect of propping a pre-twisted cantilever beam has not been investigated. However a Computer Programme to do these calculations was available. It had been written by Dokumaci using the method he described in a paper by Dokumaci, Thomas and Carnegie (67). It was necessary to alter the programme slightly in order to set the deflection of the beam to zero in both lateral and edgewise directions at points corresponding to the prop location. With this method the boundary conditions may only be imposed at the end of each blade element. In an effort to achieve a large number of prop locations it was necessary to vary the number of blade elements
FIGURE 8.7 MODAL CURVES CORRESPONDING TO POINTS A & B ON FREQUENCY BOUNDARIES SHOWN IN FIGURE 8.6
from five to eight. It may be expected therefore that the accuracy will vary slightly. Calculations for the lower frequency boundaries were carried out for a beam of total pre-twist 90° and of rectangular cross section such that \( I_{xx}/I_{yy} = 1/16 \). These dimensions give particularly strong coupling effects. The upper frequency boundaries were calculated using the above programme in its original form, where the effect of change of effective twist with snubber position \( \alpha \), for each portion of the blade, were taken into account. The upper and lower boundaries for the pre-twisted blade are shown in Fig. 8.8. The upper and lower boundaries for a straight beam of the same proportions are shown in Fig. 8.9 for comparison purposes. It may be seen from Fig. 8.8 that for the pretwisted blade the intersection D of the upper boundaries for the fundamental mode is not coincident with the lower boundary for the second mode. The deflection curve corresponding to the prop location \( \alpha = 0.714 \) is shown in Fig. 8.10. It may be seen that the deflection curve for motion in the lateral direction exhibits similar "zero slope at the prop" characteristics to that of the straight blade. The calculated points on the graph nearest to the points E and F were at \( \alpha = 0.6 \) and \( \alpha = 0.8 \) and were obtained using five blade elements. The method gives the deflection at the end of each blade element. Excluding the deflection at the prop which is zero, the curves are therefore specified by only four points. This is insufficient to specify the curves accurately, but it appears that the deflections curves in one of the directions exhibit "encastre at the prop" characteristics. However this particular study was not pursued further.

### 8.3 Theoretical results

The graphs of frequency and damping versus snubber position \( \alpha \) for the first mode of vibration are shown in Fig. 8.11 and Fig. 8.12 respectively, for three values of snubber parameter \( \lambda \). It must be pointed out however that these are not truly non-dimensional as the
FIGURE 8.8 UPPER & LOWER FREQUENCY BOUNDARIES FOR VIBRATING BLADE WITH SNUBBER RESTRAINT. TOTAL PRETWIST = 90°. $I_{xx}/I_{yy}$

KEY

- MODE 1
- MODE 2
- MODE 3

FREQUENCY RATIO $\omega_1/\omega_0$ vs. SNUBBER POSITION $\alpha$

- $\alpha l$

MODE 3

MODE 2

MODE 1
FIGURE 8.9 UPPER & LOWER FREQUENCY BOUNDARIES FOR LATI
VIBRATION OF BLADE WITH SNUBBER RESTRAINT.
FIGURE 8.10 SECOND MODAL CURVES FOR PRETWISTED BLADE PROPPED AT $\alpha = 0.714$

$I_{xx}/I_{yy} = 1/16$

TOTAL PRETWIST = 90°
Viscoelastic properties depend upon the absolute frequency (c.p.s.). The frequency ratio increases with snubber parameter $\lambda$ as may be expected. The damping is a pronounced maximum with the snubber position $\alpha = 0.73$. This is consistent with the position of greatest sensitivity to rotational restraint mentioned in the previous section 8.2.3. The graphs of frequency and damping versus snubber position $\alpha$ for the second mode of vibration are shown in Fig. 8.13 and Fig. 8.14 respectively. The damping curves show two distinct optimum snubber positions $\alpha = 0.5$ and $\alpha = 0.8$, they each corresponding to the region of greatest sensitivity as specified by the upper and lower frequency boundaries. The damping falls almost to zero at $\alpha = 0.71$ corresponding to the point of zero sensitivity of the system to rotational restraint. This may also be explained by the mode shape B for a cantilever propped at $\alpha = 0.71$ shown in Fig. 8.7.

The slope at the snubber is zero and as no rotation of the snubber occurs no energy is dissipated, consequently the damping is zero. Fig. 8.15 and Fig. 8.16 show frequency and damping versus snubber parameter $\lambda$ respectively for the first and second modes of vibration when $\alpha = 0.6$. There is an optimum value of $\lambda$ for maximum damping and it is interesting to note that the optimum value of $\lambda$ is the same for both modes of vibration. The logarithmic decrement is proportional to the energy dissipated per cycle over the total strain energy of the system. As the snubber parameter increases the rotational restraint increases until maximum damping is reached. At this point a further increase in the restraint causes the strain energy to increase at a faster rate than the energy dissipated and therefore the damping reduces. The frequency increases with snubber parameter $\lambda$ and gradually becomes asymptotic to the upper frequency boundary. The broken line depicts the characteristics if it is assumed that the damping force is in phase with the velocity. In Fig. 8.17a and b the broken line on the graph indicates the frequency and damping versus snubber position $\alpha$, assuming the damping force to be in phase with the velocity. The
effect in both cases is to raise the damping and lower the natural frequency as suggested in Appendix III.

Fig. 8.18 shows the truly non-dimensional effect on damping and frequency of varying the material loss factor $\eta$ and the dimensionless factor $\lambda \left( G_{1/E} \right)$ for the optimum snubber position $a = 0.73$. As was expected the frequency may be seen to increase with rising values of material loss factor $\eta$. The optimum combination of loss factor $\eta$ and $\lambda \left( G_{1/E} \right)$ is shown by the broken line A-B.

Fig. 8.19 shows the optimum damping $\delta_{\text{opt}}$ versus loss factor $\eta$. The curve is derived from the damping and loss factor values corresponding to the line A-B in Fig. 8.18. Assuming that the relationship between the logarithmic decrement and composite loss factor ($\delta = \pi \eta$) quoted by Lazan (68) holds, then the broken line represents the relationship obtained by Di Taranto and Blasingame (47) for a viscoelastic sandwich beam. It may be seen that the curves are very similar in shape, although slightly higher damping is predicted for the system under investigation. The predicted effects of temperature on the natural frequency and damping are shown in Fig. 8.20. The variations in frequency and damping are large as this particular material has properties which are very sensitive to temperature.

It may be seen from section 8.2.2 that the mode shape equations are complex. The deflection is therefore represented by two standing waves which are 90 degrees out-of-phase with each other and they are shown in Fig. 8.21. Thus no two points on the blade are exactly in-phase or 180 degrees out-of-phase with each other but have a fixed phase difference.
FIGURE 8.11 THEORETICAL EFFECT OF SNUBBER POSITION $\alpha$ UPON THE FIRST MODE OF VIBRATION.

The graph shows the frequency ratio $\frac{\omega}{\omega_1}$ as a function of snubber position $\alpha$. Three curves are plotted for different values of $\lambda$:

- $\lambda = 7.68 \times 10^5$
- $\lambda = 3.84 \times 10^5$
- $\lambda = 1.92 \times 10^5$

The frequency ratio increases as the snubber position $\alpha$ increases.
FIGURE 8.12 THEORETICAL EFFECT OF SNUBBER POSITION \( \alpha \) UPON FIRST MODE OF VIBRATION.

\[ \lambda = 7.68 \times 10^5 \]
\[ \lambda = 3.89 \times 10^5 \]
\[ \lambda = 1.92 \times 10^5 \]
FIGURE 8.13 THEORETICAL EFFECT OF SNUBBER POSITION $\alpha$ ON SECOND MODE OF VIBRATION.

\[
\begin{align*}
&\lambda = 7.68 \times 10^5 \\
&\lambda = 3.84 \times 10^5 \\
&\lambda = 1.92 \times 10^5
\end{align*}
\]
FIGURE 8.14 THEORETICAL EFFECT OF SNUBBER POSITION $\alpha$ UN SECOND MODE OF VIBRATION.

$\lambda = 7.68 \times 10^5$
$\lambda = 3.84 \times 10^5$
$\lambda = 1.92 \times 10^5$

LOG. DEC. $\delta$

SNUBBER POSITION $\alpha$
FIGURE 8.15 THEORETICAL EFFECT OF SNUBBER PARAMETER $\lambda$ UPON FREQUENCY $\omega = 0.6$

SECOND MODE

FIRST MODE

WITH DAMPING IN PHASE WITH VELOCITY
FIGURE 8.16  THEORETICAL EFFECT OF SNUBBER PARAMETER \( \lambda \) UPON

\( \rho = 0.6 \)

WITH DAMPING IN PHASE WITH VELOCITY

LOG. DEC.  \( \xi \)

FIRST MODE

SECOND MODE

SNUBBER PARAMETER \( \lambda \times 10^5 \)
Figure 8.17: Theoretical effect on damping and frequency of assuming the damping force to be in-phase with velocity.

$\lambda = 1.92 \times 10^5$
FIGURE 8.18 NON-DIMENSIONAL EFFECTS OF $\eta$ AND DAMPING FOR THE FIRST MODE OF VIBRATION.
FIGURE 8.19 VARIATION OF OPTIMISED DAMPING WITH VISCOELASTIC LOSS FACTOR.

From Dittranto & Blasingame (47) for infinitely long or simply supported beam.
FIGURE 8.20 INFLUENCE OF VISCOELASTIC MATERIAL TEMP.
UPON DAMPING & FREQUENCY. $\alpha=0.7, \ \lambda=1.92 \times 10^5$.
Figure 8.21 Complex Modal Curve for First Mode of Vibration, $\alpha = 0.6$

\[ \lambda = 1.92 \times 10^5 \]
CHAPTER 9

9. EXPERIMENTAL WORK

9.1 Nature of Experiments

A preliminary tensile test was carried out on two specimens designed in accordance with B.S.I. 18, 1962 and cut from the blade material to determine the Young's Modulus for calculation purposes.

The natural frequencies were determined by a resonance test using the magnetic exciter described in section 3.2.1. It was assumed that the natural frequency and the resonant frequency were the same, as reported by Nicholas and Heller (14), for a viscoelastically damped sandwich beam. The amplitude of vibration was kept to a minimum so that the viscoelastic material behaved in a linear manner. The damping was determined from the free decaying vibration of the blade. A piezoelectric strain gauge mounted on the blade near the root gave a signal proportional to the amplitude of vibration. This type of gauge was used principally because piezoelectric gauges are extremely sensitive to dynamic strains and do not require complex electronic apparatus. The decaying signal was fed to an oscilloscope via a rejection filter and recorded on a polaroid camera. The filter was used to reject unwanted signals corresponding to harmonics of the blade. Large variations in damping and frequency with temperature were predicted and are shown in Fig. 8.20. It was necessary therefore to maintain the temperature of the system constant, at a value of 22°C, corresponding to that for which the theoretical calculations were made.

Tests were carried out for a viscoelastic joint thickness of \(d = 0.010\), for two values of snubber length \(r = 1, \frac{1}{2}\) and nine values of snubber position from \(\alpha = 0.2\) to \(\alpha = 1.0\).
The variations of frequency and damping with snubber parameter $\lambda$ were also investigated experimentally by varying the thickness of the joint, for a snubber position corresponding to $\alpha = 0.6$.

9.2 Details of apparatus

9.2.1 Arrangement of apparatus

The apparatus is as shown diagrammatically in Fig. 9.1. A signal generator supplies a sinusoidal voltage to a Williamson 15 watt amplifier which has a variable gain control. The amplifier is connected to the electromagnetic exciter via an ammeter and a double pole double throw toggle switch. With the switch in position 1, the electromagnet is connected to the amplifier and the trigger circuit to the oscilloscope is completed. Altering the switch to position 2 cuts off the current to the exciter and opens the trigger circuit, thus triggering the sweep on the oscilloscope. The trigger circuit comprises a 9 volt battery with a 22 KΩ resistor in series to reduce the current drain on the battery. A load resistance of 15Ω is provided for the amplifier when the switch is in position 2, to prevent damage to the output transformer. The arrangement for clamping the blade and making the adhesive joint is as described previously in section 3.6. The temperature of the system is sensed by an iron-constantan thermocouple, the tip of which is located in a small pocket in the jaw adjacent to the viscoelastic material. The cold junction is maintained at 0°C in melting ice and the voltage produced is measured on a potentiometer. A tungsten filament lamp, mounted close to the blade is used to raise the temperature to 22°C when required.

The piezoelectric strain gauge is fixed to the blade near the root using Durofix Cement. Coaxial wire connects the strain gauge to the cathode follower. The wires are attached to the strain gauge using
FIGURE 9.1 ARRANGEMENT OF EXPERIMENTAL APPARATUS
low temperature solder, to prevent overheating of the gauge. The cathode follower has a minimum input impedance of 100 MΩ and provides a suitable load for the strain gauge. The signal is amplified in a small variable gain transistor amplifier which has an output impedance of 2.5 kΩ. The output of the amplifier is connected to a tunable band rejection filter. The filter is designated D 925 A by the manufacturers, Messrs. Muirhead Limited, and consists of a parallel T resistance-capacitance network. The filter has a control which when set to "flat" allows the signal to by-pass the rejection network. A band pass filter was not used here because the transient oscillatory response of this type of instrument interferes with the required waveform. The output from the filter is connected to a double beam oscilloscope which has an input impedance of 1 MΩ and is fitted with a Polaroid oscilloscope camera. The photograph takes 10 seconds to develop and provides a permanent record of the trace.

9.2.2 Linearity of Piezoelectric strain gauges

The strain gauges are of dimensions 1.00" x 0.125" x 0.020" and they are made from lead zirconate. They are manufactured by British Clevite Company Limited and are claimed to be approximately twice as sensitive as barium titanate gauges. The sensitivity is quoted as $3.7 \times 10^5$ volts per unit strain and the maximum dynamic strain for linear operation is quoted as $2.5 \times 10^{-4}$. This corresponds to a maximum output voltage of approximately 100 volts for linear operation. No work has been published on the dynamic calibration of lead zirconate gauges although a certain amount has been carried out on the calibration of barium titanate gauges, in which the mechanism is identical. Ripperger (69) calibrates the gauge by mounting it on a small bar which is subjected to a longitudinal impact. He reports that the sensitivity varies from gauge to gauge. He also mentions the need for the gauge to be connected to a very high impedance load in order that the response to harmonic strain be independent of frequency.
Mark and Goldsmith (70) also consider the calibration of the gauge mounted on a bar but in this case the bar is subjected to a longitudinal harmonic force of variable frequency. They report the characteristics to be independent of frequency in the range considered, that is 20 - 600 c.p.s., although they virtually assume a linear relationship between output voltage and strain. Both the above papers are concerned with longitudinal motion where the stress is uniform across the gauge. For gauges mounted on beams in flexure this is not the case. Not only does the stress amplitude vary across the section of the gauge it also varies along the length especially if the ratio of gauge length to beam length is large. Dawson (71) shows, by plotting the theoretical strain amplitude at the centre of the gauge for a given tip amplitude, against the strain gauge output for the same tip amplitude, that a linear relationship does not exist above 400 c.p.s. Petrovsky (72) assumes a linear relationship between output voltage and strain, in measuring the dynamic stresses in steam turbine blades.

A short preliminary test was carried out on a cantilever beam of approximately 6 inches in length with two lead zirconate strain gauges mounted side by side close to the root. The strain gauges were connected to a valve voltmeter of impedance 1MΩ. The beam was excited at a natural frequency by an electromagnet and the tip amplitude was measured using a capacitance probe. It was apparent from the voltmeter readings that there was a variation in sensitivity between the gauges. Fig. 9.2a shows the output voltage for one of the gauges versus tip amplitude for the first and second modes of vibration. The curves are not drastically non-linear for this range of tip amplitude. However, the curve for the first mode shows a slight increase in sensitivity with increasing amplitude while that for the second mode shows a reduction in sensitivity similar to that reported by Dawson (71). The results of a test carried out on a thin beam at much lower tip amplitudes are shown in Fig.9.2b. This graph shows a definite linear relationship.
FIGURE 9.2 LINEARITY OF PIEZOELECTRIC STRAIN GAUGES FOR FLEXURAL VIBRATION

(a) 2\textsuperscript{nd} MODE (970 C.P.S.)

(b) 1\textsuperscript{st} MODE (80 C.P.S.)
As the gauges are intended for use on longer beams and where the amplitudes of vibration (and therefore strains) are very small a linear relationship was assumed between amplitude of vibration and output voltage.

9.2.3 Response characteristics of a parallel T filter to a transient waveform

There is no published analysis on the response of parallel T rejection filters to transient waveforms. The circuit is as shown inset in Fig. 9.3, where $V_i$ and $V_o$ are the input voltages and output voltages respectively. For zero source impedance and infinite load impedance the transfer function governing the input-output relationship is as specified in Appendix IV. The solution is determined for a transient input of the form $V_i = V_0 e^{-vt} \sin \omega t$. It reveals that the output consists of a signal which is proportional to the input, plus the non-oscillatory transient response of the filter. The variations of the coefficient of the particular solution with input frequency $\omega$ and damping $\delta$ are shown in Fig. 9.3. As may be expected the rejection characteristics of the filter are slightly reduced when the input is non-harmonic. The broken line indicates the response of two filters in cascade to a harmonic input. Preliminary tests revealed that the transient response of the filter did not interfere in any way with the forced response.

9.3 Experimental Procedure

The blade was mounted in the clamp block and the viscoelastic joint made at the snubber face as described in section 3.6. The electronic apparatus was switched on and allowed to warm-up for approximately 20 minutes. The tungsten filament lamp was used if necessary to warm the assembly. The temperature was maintained as close to $22^\circ$C as
FIGURE 9.3 REJECTION CHARACTERISTICS OF PARALLEL 'T' FILTERS TO A DECAYING SINUSOIDAL INPUT.
possible throughout the experiment.

The rejection filter was set to "flat" and the toggle switch moved to position 1. The current was maintained constant by using the gain control on the amplifier while the excitation frequency was varied in the vicinity of the expected fundamental resonance. As resonance was approached the waveform on the oscilloscope increased in amplitude until it reached a maximum. The natural frequency was then taken to be twice the signal generator frequency at which maximum amplitude occurred.

The oscilloscope was adjusted for external sweep triggering, the shutter of the camera was opened, and the toggle switch moved to position 2. The decaying wave was displayed instantaneously on the oscilloscope screen, and was recorded by the camera. The shutter was closed. The toggle switch was returned to position 1 and the oscilloscope returned to automatic sweep. The rejection filter was set to tune and adjusted until the trace on the screen had minimum amplitude. The filter was now set to reject the fundamental frequency of the blade. The second resonant frequency was found and the decay photographed in an identical manner to that described above.

In several instances usually when the damping was heavy, the decaying waveform for the fundamental mode was obscured by the superimposed decay of the second mode. When this occurred the procedure was to locate the second mode frequency and reject it with the filter in the manner described previously.

The photographs were placed in an episcope and projected at approximately ten times full size on to squared paper. Measurements were made over several cycles of the peak amplitudes from which the logarithmic decrement was calculated.
CHAPTER 10

10. RESULTS AND CONCLUSIONS

10.1 Results

In the foregoing comparison of experimental and theoretical results the theoretical values are represented by a broken line. In general a curve representing the experimental results is not shown as there are insufficient points to specify it precisely.

Experimental results of frequency and damping for two values of $\lambda$, for the first mode of vibration are shown in Fig. 10.1-10.4. The frequencies show fairly good agreement with predicted values although there is a tendency for them to be higher for low values of $\alpha$ and lower for higher values of $\alpha$. The discrepancy is small, however more uniform correlation would probably occur with the curve for damping in-phase with the velocity as shown in Fig. 8.17. The effect of heavy damping is to reduce the resonant response considerably which makes it more difficult to locate the resonant frequency exactly. The experimental damping results verify the predicted snubber location for optimum damping although the points are somewhat scattered. The experimental results for $\lambda = 1.92 \times 10^5$ verify the shape of the predicted curves in general to within an average of 10%, however for $\lambda = 7.68 \times 10^5$ the average discrepancy is slightly higher being of the order of 20%.

The experimental results of frequency and damping versus snubber position $\alpha$ for the second mode of vibration are shown in Fig. 10.5-10.8. The frequencies show good correlation but are in general slightly higher than the predicted values. The reason for this is not apparent and could be due to several of the approximations made in the assumptions. There are insufficient points to specify the variation
of damping with snubber position precisely, however the results for 
\( \lambda = 1.92 \times 10^5 \) do show a trend corresponding to the predicted curve.
The experimental damping results for \( \lambda = 7.68 \times 10^5 \) shown in Fig. 10.8 give poor correlation with the predicted curves.

The experimental results of damping and frequency as affected by snubber parameter \( \lambda \) for the first mode of vibration are shown in Fig. 10.9. The experimental frequency results are higher than predicted for small values of \( \lambda \) and lower for larger values of \( \lambda \). The reason for the larger frequency values for small values of \( \lambda \) may be due to several causes. However for large values of \( \lambda \) the frequencies are lower and almost constant. This is certainly caused by flexibility of the snubber. Regardless of how stiff the shear joint is made, the snubber flexibility is the limiting stiffness factor. This accounts for the constant experimental frequency values for large values of \( \lambda \), and the reason for the frequencies being below the predicted curve, which was calculated assuming the snubber to be rigid. The experimental damping curve is of similar shape and magnitude to that of the predicted curve. However, the experimental values do not reduce with increasing values of \( \lambda \) at the rate given by the theory. This is because the actual restraint felt by the blade is less, due to snubber flexibility. Therefore it does not cause the strain energy of the blade to increase and the damping to reduce at the predicted rate.

Fig. 10.10 shows the results of damping and frequency as affected by snubber parameter \( \lambda \) for the second mode of vibration. The experimental and predicted damping curves show reasonable agreement although the experimental points are more widely scattered than those of the fundamental mode. The experimental and theoretical frequency curves cross over with increasing values of snubber parameter \( \lambda \) in a similar manner to the fundamental mode. However, the effects of snubber flexibility are not as pronounced. This is because the sensitivity to rotational restraint at this snubber position for the second
mode is much less than for the fundamental mode.

Considering the theoretical approximations and that the complex shear modulus data were not for the same batch of material as used in the tests, the agreement between theoretical and experimental results is regarded as satisfactory.

10.2 Conclusions

Snubber positions for maximum stiffening and maximum damping are coincident. The optimum snubber position for maximum stiffening and damping vary from mode to mode. Peak damping is particularly sensitive to snubber position especially in modes above the fundamental. The number of snubber locations for peak stiffness and damping correspond to the mode number, so that in the second mode of vibration there are two peaks.

The optimum value of the snubber parameter λ is the same for each mode.

Heavy damping may be obtained especially if the parameters are optimised correctly.

Snubber flexibility must be taken into account for accurate results particularly for large values of snubber parameter λ.

The application of the complex modulus model to free damped vibration gives acceptable results.
Figure 10.1 Experimental effect of snubber position $\alpha$ upon the first mode of vibration. $\lambda = 1.92 \times 10^5$
FIGURE 10.2 EXPERIMENTAL EFFECT OF SNUBBER POSITION $\alpha$ UPON THE FIRST MODE OF VIBRATION. $\lambda = 7.68 \times 10^5$
FIGURE 10.3 EXPERIMENTAL EFFECT OF SNUBBER POSITION \( \alpha \) UPON THE FIRST MODE OF VIBRATION. \( \lambda = 1.92 \times 10^5 \)
FIGURE 10.4 EXPERIMENTAL EFFECT OF SNUBBER POSITION $\alpha$ UPON FIRST MODE OF VIBRATION. $\lambda = 7.68 \times 10^5$
FIGURE 10.5 EXPERIMENTAL EFFECT OF SNUBBER POSITION $\alpha$ UPON SECOND MODE OF VIBRATION. $\lambda = 1.92 \times 10^5$
FIGURE 10.6 EXPERIMENTAL EFFECT OF SNUBBER POSITION \( \alpha \) THE SECOND MODE OF VIBRATION. \( \lambda = 7.68 \times 10^5 \)
FIGURE 10.7 EXPERIMENTAL EFFECT OF SNUBBER POSITION upon SECOND MODE OF VIBRATION. $\lambda = 192 \times 10^5$
FIGURE 10.8 EXPERIMENTAL EFFECT OF SNUBBER POSITION ON THE SECOND MODE OF VIBRATION. $\lambda = 7.68 \times 10^5$
FIGURE 10.9 EXPERIMENTAL EFFECT OF SNUBBER PARAMETER $\lambda$ DAMPING & FREQUENCY FOR FIRST MODE OF VIBRATION. $\alpha = 0.6$
FIGURE 10.10 EXPERIMENTAL EFFECT OF SNUBBER PARAMETER DAMPING & FREQUENCY FOR SECOND MODE OF VIBRATION.
As an approximation to the vibration of a set of blades with snubbers mounted around the periphery of a disc, it is proposed to investigate the dynamics of a relatively small batch, not exceeding five blades. This considerably simplifies the problems by reducing the complexity and the amount of experimental equipment required without basically affecting the dynamics of the system. Smith (73) showed for a batch of shrouded blades that the number of modes of vibration was directly proportioned to the number of blades. It could be expected therefore that investigating the motion of a much reduced number of blades would further simplify the problem in terms of the number of modes of vibration present and keep computational effort to a minimum.

The dynamics of a system where the relative motion of adjacent blades causes damping of the structure has not been investigated previously. Many workers have considered the vibration of a batch of several blades which are coupled by an elastic shroud at the tip. These are akin to multi-bay portal frames. A paper by Rieger and McCallion (74) contains an historical review of work on structures of this type. In the vast majority of this work the blades are considered to be of uniform cross section with the shroud located at the tip. Considering the shroud or lacing wire to be at any other position other than at the tip causes additional complication and has not been
investigated using the classical type of solution described by Smith (73), Rieger and McCallion (74), and Ellington and McCallion (75). Smith (73) and Rieger and McCallion (74) give numerical results and show the existence of vibrational modes with close natural frequencies for multi-blade or multi-bay systems. Smith (73) showed that except for the fundamental batch mode as the interblade stiffness approached zero, the frequency of the remaining modes became coincident. Comparitively little experimental work has been done on such systems. Rieger and McCallion (74) verified their frequency results satisfactorily. However, their method of recognizing and determining the batch mode shapes must require some prior information on the phase relationships between different parts of the structure.

From the results of the work mentioned above and from the work of the previous Section B, it could be expected that the system at present under investigation would exhibit close natural frequencies and heavy damping. The experimental difficulties involved in resonance testing such systems has recently received some attention from Bishop and Gladwell (76) and Pendered (77, 78). Pendered (77) remarks that "it is difficult if not impossible to detect the modes, frequencies and damping by means of resonance testing." Bishop and Gladwell (76) mention that due to the large contribution to the response from off-resonant modes, great discrepancies may exist between the natural frequency and the frequency at which peak amplitude occurs. In addition the contribution from off-resonant modes may obscure a heavily damped mode completely, and cause large errors in the estimates of damping. Several methods have been suggested for separating the individual response from the total response and these have been investigated in the above reports. Pendered (77, 78) although assuming the damping to be light shows in a comparison of resonance criteria that the maximum frequency spacing method suggested by Kennedy and Pancu (79) is the superior technique. Bishop and Gladwell (76) also conclude
that this method would give reliable results even with heavy damping and close natural frequencies. However this technique necessitates measuring the phase difference between the harmonic excitation and the structure response. The only practical way of achieving this is by use of an electrodynamic exciter with a force transducer intermediate between the armature and the structure. The inaccuracies in frequency and damping measurements caused by coupling the relatively large mass of the armature and transducer to the structure have been reported by Rissone and Williams (51) and Bishop and Pendered (52) respectively. As discussed in section 7, Plunkett (49) suggested that for the forced vibration of elastic systems with concentrated damping that the shape of the response curve and hence the measure of damping Q depend upon the position of excitation. Because of this he suggests that the relationship \( \delta = \frac{\pi}{Q} \) although shown to hold for single degree of freedom systems is not true for elastic systems with concentrated damping. Bishop and Pendered (52) show that the damping \( \mu_1 \) is obtained in the Kennedy and Pancu (79) method by an analogous procedure to the Q factor and is equal to its reciprocal. Therefore the Kennedy and Pancu (79) method is unsuitable for elastic systems with concentrated damping for the same reasons as that of the amplitude-frequency method.

Much effort has been spent in attempts to separate the individual modes from the total response by steady harmonic excitation methods. However little effort has been concentrated in attempting to obtain the individual damping and frequencies by transient testing. As the above steady state techniques are unsatisfactory for the reasons outlined above, it is proposed to use transient testing methods to determine the dynamic properties of the structure.

It is proposed to investigate the dynamics of a batch of five blades. An odd number of blades is chosen as it could be expected that such a system would have some modes symmetrical about the centre blade.
Bishop and Gladwell (76) remark that often a mode can be eliminated by making use of any symmetrical or anti-symmetrical properties that the structure possesses. Therefore an odd number of blades may be helpful in separation of the modes. No attempts will be made to measure the mode shapes of individual blades. The visco-elastic material properties will be treated as being a function of frequency as in Section B.
12. THEORETICAL CONSIDERATIONS AND RESULTS

12.1 Assumptions and method of analysis

The assumptions made for the single blade in Section 8.1, with the exception of number 3, are applicable to the present analysis and in addition two further assumptions are postulated.

1) That the snubbers and shear joint are inextensible such that the blade pitch at the snubbers is constant.

2) That the contribution to blade restraint from bending of the viscoelastic joint is small and can be neglected. This has been examined in Appendix V, where the bending restraint was found to be small in comparison to the restraint produced by relative shear.

The method of analysis used is based on that described by Ellington and McCallion (75) for purely elastic interblade coupling. The technique is extended here to allow for interblade viscoelastic damping, at any position along the blade length.

12.2 Frequency Equations

The blade system is shown diagrammatically in Fig. 12.2a. Each blade is considered to be made up of two sections which join at the snubber.

It is shown in Section 8.2.1 that the equations governing each portion of the blade are as follows:
FIGURE 12.1 DIAGRAM OF BENDING & SHEARING OF VISCOELASTIC INTERFACE JOINT.

(a) BENDING OF INTERFACE JOINT.

(b) SHEARINoG OF INTERFACE JOINT.
FIGURE 12.2 COORDINATES OF MULTIBLADE SYSTEM & FORCE AND MOMENT EQUILIBRIUM DIAGRAM.

(a) COORDINATE SYSTEM

(b) EQUILIBRIUM DIAGRAM
For the section between the root and the snubber;

\[ y_1 = \left( A_1 \sin x_1 + B_1 \cos x_1 + C_1 \sinh x_1 + D_1 \cosh x_1 \right) e^{(-\nu + j\omega)t} \]  

\text{(1)}

For the portion between the snubber and the tip;

\[ y_2 = \left( A_2 \sin x_2 + B_2 \cos x_2 + C_2 \sinh x_2 + D_2 \cosh x_2 \right) e^{(-\nu + j\omega)t} \]  

\text{(2)}

where \( A_1, A_2, B_1, B_2 \) etc., are arbitrary constants and

\[ s = \frac{1}{1} \]  

\text{(3)}

The frequency \( \omega \) is given by;

\[ \omega = (a^2 - b^2) \sqrt{\frac{EI}{ml^4}} \]

and the damping index by;

\[ \nu = 2ab \sqrt{\frac{EI}{ml^4}} \]

From which the frequency ratio is;

\[ \frac{\omega}{\omega_1} = \frac{(a^2 - b^2)}{3.516} \]

and the logarithmic decrement is given by;

\[ \delta = \frac{4(ab)}{(a^2 - b^2)} \]
The boundary conditions at the root are;

\[ y_1(0) = y_1'(0) = 0 \]

and the boundary conditions at the tip are;

\[ y_2''(0) = y_2'''(0) = 0 \]

from which equations (1) and (2) reduce to:

\[ y_1 = \left[ A_1 \left( \sin x_1 - \sinh x_1 \right) + B_1 \left( \cos x_1 - \cosh x_1 \right) \right] e^{(-\nu + j\omega)t} \]

and

\[ y_2 = \left[ A_2 \left( \sin x_2 + \sinh x_2 \right) + B_2 \left( \cos x_2 + \cosh x_2 \right) \right] e^{(-\nu + j\omega)t} \]

The bending moments at the snubber due to the inboard and outboard portions of the blade are given by;

\[ M_1 = EIy_1''' \quad (1_1) \]

\[ M_2 = EIy_2''' \quad (1_2) \]

Similarly the shear forces at the snubbers due to each section of the blade are given by;

\[ S_1 = EIy_1''' \quad (1_1) \]

\[ S_2 = EIy_2''' \quad (1_2) \]

Setting the transverse deflection at the snubber to \( \gamma \) such that;

\[ \gamma = y_1(1_1) = y_2(1_2) \]
and putting

\[ \varphi_1 = y'_1(l_1) \] and \[ \varphi_2 = y'_2(l_2) \]

then the bending moment and shear force amplitudes acting at the snubber due to each portion of the blade may be written as:

\[
M_1 = E I s^2 \left[ \gamma (\sin s_1 \sinh s_1) + \varphi_1 / (\sinh s_1 \cosh s_1 - \sin s_1 \cosh s_1) \right] \tag{11}
\]

\[
M_2 = E I s^2 \left[ \frac{\varphi_2 / s (\cosh s_1 \sinh s_1 - \sin s_1 \cosh s_1)}{(\cosh s_1 \cosh s_1 + 1)} \right] \tag{12}
\]

\[
S_1 = E I s^3 \left[ \gamma (\sin s_1 \cosh s_1) + \cosh s_1 \sinh s_1 - \varphi_1 / (\sinh s_1 \cosh s_1 - 1) \right] \tag{13}
\]

\[
S_2 = E I s^3 \left[ \gamma (\sin s_1 \cosh s_2 + \cosh s_1 \sinh s_2) - \varphi_2 / (\cosh s_1 \cosh s_1 + 1) \right] \tag{14}
\]

Substitution of \( l_1 = \alpha l \) and \( l_2 = (1-\alpha)l = \beta l \) into equation (3) gives:

\[
s_1^1 = \alpha(a+jb) \] \[
s_1^2 = \beta(a+jb) \tag{15}
\]

Substituting equations (15) into the above expressions for bending moments and shear forces it is then necessary to expand the trigonometric and hyperbolic factors into real and imaginary parts. These are as defined in Section 8.2.1 with the exception of the following:
\[ S_{\sin 12} \sin h_{\sin 12} = P_8 + jQ_8 \]

where \[ P_8 = (\sin \alpha \cosh \beta \sinh \alpha \cos \beta - \cos \alpha \sinh \beta \cos \alpha \sin \beta) \]

and \[ Q_8 = (\cos \alpha \sinh \beta \sin \alpha \cosh \beta + \sin \alpha \cosh \beta \cos \alpha \sin \beta) \]

The expressions for bending moment and shear force amplitudes now become:

\[ M_1 = EI \left[ \frac{\gamma (E_1 + jE_3)}{1^2} + \frac{\theta_1 (E_2 + jE_4)}{1} \right] \]

\[ M_2 = EI \left[ \frac{\gamma (E_5 + jE_7)}{1^2} + \frac{\theta_2 (E_6 + jE_8)}{1} \right] \]

\[ S_1 = EI \left[ \frac{\gamma (E_{10} + jE_{11})}{1^3} - \frac{\theta_1 (E_{12} + jE_{13})}{1^2} \right] \]

\[ S_2 = EI \left[ \frac{\gamma (E_{14} + jE_{17})}{1^3} - \frac{\theta_2 (E_{15} + jE_{16})}{1^2} \right] \]

where the abbreviations \( E_1, E_2, \ldots, E_{17} \) are functions of \( a, b \) and \( \alpha \) only and are listed in Appendix VI.

From Fig. 12.2b it may be seen that the shear strain in the
viscoelastic shear joint to the right of the \( n \)th blade is given by 

\[
\frac{d}{dx} \left[ r(\theta_n + \theta_{n+1}) \right]
\]

and the shear stress is given by

\[
\frac{G_1 r}{d} (\theta_n + \theta_{n+1})(1 + j\eta)
\]

where \( \eta = \frac{G_2}{G_1} \) is the material loss factor. Hence the restoring moment on the \( n \)th blade due to the right hand shear joint of width \( h \) is:

\[
M_{VR} = \frac{\text{chr}^2 G_1 (\theta_n + \theta_{n+1})(1 + j\eta)}{d}
\]

Similarly the restoring moment on the \( n \)th blade due to the shear joint to the left is:

\[
M_{VL} = \frac{\text{chr}^2 G_1 (\theta_n + \theta_{n-1})(1 + j\eta)}{d}
\]

Using the curve fitting method of Kravitz (65) the elastic and loss moduli of the viscoelastic material may be expressed over the frequency range 30 - 1000 c.p.s. in the form:

\[
G_1, G_2 = Z - W \left[ e^{\frac{x(1000-f)}{970}} - 1 \right]
\]

where \( Z, W \) and \( X \) are constants and \( f \) is the frequency in cycles per second given by:

\[
f = \frac{(a^2 - b^2)}{2\pi} \sqrt{\frac{EI}{m l h}}
\]

Taking moments about the snubber in Fig. 12.2b then
\[ M_1 + M_{VL} + M_{VR} - M_2 = 0 \]  \hspace{1cm} \text{(22)}

Setting \( \theta_n = \theta_1 = -\theta_2 \) and substituting equations (16), (17), (20) and (21) into equation (22) yields after simplifying

\[ \theta_{n+1} + \left[ \frac{\rho(E_e)}{\lambda G_1} + 2 + j \frac{\mu}{\lambda} \right] \theta_n + \theta_{n-1} = \frac{\gamma}{\lambda} \left( u + jv \right) \]  \hspace{1cm} \text{(23)}

This is a second order difference equation with complex coefficients.

In the above equation the abbreviations are as follows:-

\[ \rho = \frac{(E_2 + E_6) + \eta(E_4 + E_8)}{1 + \eta^2} \]

\[ \mu = \frac{(E_4 + E_8) - \eta(E_2 + E_6)}{1 + \eta^2} \]

\[ u = \frac{(E_5 - E_1) + \eta(E_7 - E_3)}{1 + \eta^2} \]

\[ v = \frac{(E_7 - E_3) - \eta(E_5 - E_1)}{1 + \eta^2} \]

\( \lambda \) is the snubber parameter previously defined in Section 8.2.1 for the single blade as;

\[ \lambda = \frac{\chi r^2}{\text{Id}} \]

Assuming a trial solution of the form;

\[ \theta_n = (-1)^n p e^{\phi n} \]
for the complementary function of equation (23) (where $P$ is a constant) yields:

$$\cosh\theta = \left[ \frac{P}{2\lambda} \left( \frac{E}{G_1} \right) + 1 + j \frac{\mu}{2\pi} \left( \frac{E}{G_1} \right) \right] \quad \text{(24)}$$

From equation (24) $\theta$ must be complex. Putting $\theta = \varpi + j\psi$ in equation (24) and separating real and imaginary parts gives:

$$\frac{P}{2\lambda} \left( \frac{E}{G_1} \right) + 1 - \cosh\varpi \cos\psi = 0 \quad \text{(25)}$$

$$\frac{\mu}{2\pi} \left( \frac{E}{G_1} \right) - \sinh\varpi \sin\psi = 0 \quad \text{(26)}$$

which are true for all values of $\mu$ and $\varpi$ provided that $\varpi$ and $\psi$ are finite. However if $\varpi$ and $\psi$ are zero simultaneously then a breakdown case occurs and this solution is not valid.

The complementary solution is given by:

$$\Theta_n = (-1)^n \left[ P(\cosh\varpi \cos\psi_n + j \sinh\varpi \sin\psi_n) + Q(\sinh\varpi \cos\psi_n + j \cosh\varpi \sin\psi_n) \right] \quad \text{(27)}$$

The particular solution $\bar{\theta}$ is complex and independent of $n$ and can be shown to be:

$$\bar{\theta} = \gamma \left[ \frac{E}{2\lambda} \left( \frac{E}{G_1} + 1 \right) + \frac{\mu}{2\pi} \left( \frac{E}{G_1} \right) \right] \frac{u(\cosh\varpi \cos\psi + 1) + \nu(\sinh\varpi \sin\psi) + j \left( \nu(\cosh\varpi \cos\psi + 1) - u\sinh\varpi \sin\psi \right)}{(\cosh\varpi \cos\psi + 1)^2 + (\sinh\varpi \sin\psi)^2}$$

$$\quad \text{(28)}$$
Thus the complete solution to equation (23) is the sum of the separate solutions given by equations (27) and (28).

Smith (73) showed that for a blade system connected at the tip by a shroud two distinct classes of vibration modes exist.

(a) Those modes where there is no transverse motion at the tip.

(b) Modes where there is transverse motion of the blade tip.

For a batch of \((2N+1)\) blades which is an odd number, these modes of vibration may be considered separately and can be defined as follows:

a) Symmetric modes where there is symmetry of motion about the centre blade.

b) Asymmetric modes where there is no symmetry of motion about the centre blade.

For physical reasons this must be true for the present case.

12.2.1 Symmetric Modes

There is no transverse motion at the snubber and for symmetry;

\[ \gamma = \theta_0 = 0 \]

The complete solution to equation (23) reduces to:

\[ \theta_n = (-1)^n Q \left[ \sinh n \cos \psi_n + j \cosh n \sin \psi_n \right] \quad \cdots \cdots (29) \]
For equilibrium of the bending moment at the end blade $n = N$

$$M_1 + M_{VL} - M_2 = 0$$

which reduces to:

$$\phi_N \left[ \frac{\phi}{\lambda} \left( \frac{E}{G_1} \right) + 1 + j \frac{E}{\lambda} \left( \frac{E}{G_1} \right) \right] + \phi_{N-1} = 0 \quad \text{(30)}$$

Substituting the appropriate $\phi_n$ from equation (29) into equation (30) and separating real and imaginary parts yields:

$$(2 \cosh \lambda \cos \psi - 1) \sinh \lambda \cos \psi \cosh \lambda \sinh \lambda - 2 \sinh \lambda \sin \psi \cosh \lambda \sinh \lambda = 0 \quad \text{(31)}$$

$$(2 \cosh \lambda \cos \psi - 1) \cosh \lambda \sin \psi \sinh \lambda + 2 \sinh \lambda \sin \psi \sinh \lambda \cosh \lambda = 0 \quad \text{(32)}$$

In considering the non-trivial roots of equations (31) and (32) there are three possibilities:

(1) $\Omega$ and $\psi$ are both finite

It may be shown that there are no two finite values of $\Omega$ and $\psi$ that satisfy the simultaneous equations (31) and (32).

(2) $\Omega$ is finite and $\psi$ is zero

For $\psi = 0$ equation (32) vanishes and equation (31) reduces to:

$$\sinh(N+1)\Omega - \sinh N \Omega = 0$$

which has no non-trivial root.
(3) \( \Omega = 0 \) and \( U \) is finite

For \( \Omega = 0 \) equation (31) vanishes and equation (32) reduces to:

\[
\sin(N+1)\Psi - \sin N\Psi = 0 \quad \text{..................(33)}
\]

Equation (33) has \( N \) roots which together with \( \Omega = 0 \) when substituted in equations (25) and (26) yield \( N \) frequencies and damping values. The snubber slopes are given by equation (29), from which it can be seen that there are no phase differences caused by damping.

The results of case 2 and 3 give equations which are similar in form to that shown by Ellington and McCallion (75). However the \( \sinh N\Psi \) in their result is multiplied by 2. This discrepancy is due to the different type of blade coupling.

12.2.2 Asymmetric Modes

The particular part of the finite difference solution equation (28) can be abbreviated to:

\[
\bar{\delta} = \frac{\gamma}{2\lambda I} \left( \frac{E}{G_1} \right) (U + JV) \quad \text{..................(34)}
\]

where

\[
U = \frac{u(\cosh N\cos\Psi + 1) + v\sinh N\sin\Psi}{(\cosh N\cos\Psi + 1)^2 + (\sinh N\sin\Psi)^2}
\]

and

\[
V = \frac{v(\cosh N\cos\Psi + 1) - u\sinh N\sin\Psi}{(\cosh N\cos\Psi + 1)^2 + (\sinh N\sin\Psi)^2}
\]

Hence the complete solution can be written:

\[
\theta_n = (-1)^n \left[ P(\cosh \Omega n\cos\Psi_n + j\sinh \Omega n\sin\Psi_n) + Q(\sinh \Omega n\cos\Psi_n + j\cosh \Omega n\sin\Psi_n) \right] + \frac{\gamma}{2\lambda I} \left( \frac{E}{G_1} \right) (U + JV) \quad \text{..................(35)}
\]
For physical reasons and because the snubber rotation is specified by a complex quantity it could be expected that there would be a phase difference between the blades due to damping. Therefore it is not the slopes of the blades on either side of the centre blade that are equal but the moduli of the slopes that are equal, such that

\[ |\theta_1| = |\theta_{-1}| \quad \text{.................(36)} \]

From equation (35) it may be shown that for \( n = 1 \) and \( n = -1 \)

\[ \theta_1^2 = \left[ \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) U - (PCosh + QSinh)Cos\psi \right]^2 + \left[ \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) V - (PSinh - QCosh)Sin\psi \right]^2 \]

\[ \theta_{-1}^2 = \left[ \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) U - (PCosh - QSinh)Cos\psi \right]^2 + \left[ \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) V - (PSinh + QCosh)Sin\psi \right]^2 \]

Substituting the above into equation (36) yields after simplifying;

\[ Q \left[ \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) (USinhCos\psi + VCoshSin\psi) - PCoshSinh\psi \right] = 0 \]

From which either;

1) \( Q = 0 \) \( P \neq 0 \)

2) \( Q \neq 0 \) \( P = \frac{\gamma}{2\lambda i} \left( \frac{E}{G_1} \right) \left( \frac{CosU}{Cosh\psi} + \frac{V Sin\psi}{Sinh\psi} \right) \)
Case 1, \( Q = 0 \) \( P \neq 0 \)

The complete solution reduces to

\[
\Theta_n = (-1)^n P \left[ \cosh n \cos \Psi_n + j \sinh n \sin \Psi_n \right] + \bar{\Theta} \quad \quad \quad (37)
\]

For equilibrium of the bending moment at the end blade (\( n=N \)) then;

\[
M_1 + M_N - M_2 = 0 \quad \quad \quad \quad \quad (38)
\]

Substituting the appropriate difference solution from equation (37) into equation (36) in a similar manner to that for symmetric modes, for the case where \( N \) is even;

\[
\left[ P(-1)^N \left( \cosh N \cos \Psi_N + j \sinh N \sin \Psi_N \right) + \bar{\Theta} \right]\left[ (2 \cosh \angle \cos \Psi - 1) + 2 j \sinh \angle \sin \Psi \right] - 
\]

\[
P(-1)^N \left( \cosh(N-1) \cos(N-1) \Psi + j \sinh(N-1) \sin(N-1) \Psi \right) + \bar{\Theta}
\]

\[
= \gamma \lambda \left( \frac{E}{G_1} \right) (U+jV) \quad \quad \quad \quad \quad (39)
\]

The case for \( N \) is odd yields a similar equation but with different signs.

If \( m_s \) is the mass of each snubber pair which all have a transverse vibration amplitude \( \gamma \), then the sum of the differences in shear force across the snubber is equal to the inertia force of the snubbers such that;

\[
\sum_{-N}^{N} (S_1 + S_2) = -(2N+1)m_s (s/q)^{\frac{4}{k}} \quad \quad \quad \quad \quad (40)
\]

From Section 8.2.1, \( (s/q)^{\frac{4}{k}} = \frac{EI}{m_1 k} \left[ (a^2-b^2)^2 - 4a^2 b^2 + 4jab(a^2-b^2) \right] \)
Substitution of the above equation and equations (18) and (19) into equation (40) gives:

\[
\sum_{n=-N}^{N} \Theta_n = \gamma^{(2N+1)} \frac{\left[ W \left\{ (a^2-b^2)^2 - 4a^2b^2 \right\} + \left( E_1 + E_1^+ \right) + j \left( E_1^+ + E_1 \right) + 4abW(a^2-b^2) \right]}{(E_{12} - E_{14}) + j(E_{13} - E_{15})}
\]

\[ \text{.................(41)} \]

where \( W \) is the ratio of snubber mass to blade mass \( m_s/m_l \).

The summation of equation (37) may be expressed as:

\[
\sum_{n=-N}^{N} \Theta_n = P(-1)^N \left[ 2 \sum_{n=0}^{N} (-1)^n (e^{(\alpha + j\psi)n} + e^{-(\alpha + j\psi)n}) - 1 \right] + (2N+1) \Theta \]

Summing the two geometric progressions:

\[
\sum_{n=-N}^{N} \Theta_n = -P(-1)^N \left[ \frac{(\cosh N \alpha \cos \psi + \cosh (N-1) \alpha \cos (N-1) \psi)}{(1 + \cosh \alpha \cos \psi) + j \sinh \alpha \sin \psi} \right.
\]

\[
\left. \frac{j(\sinh N \alpha \sin \psi + \sinh (N-1) \alpha \sin (N-1) \psi)}{(1 + \cosh \alpha \cos \psi) + j \sinh \alpha \sin \psi} \right] + (2N+1) \Theta \quad \text{.................(42)}
\]

Substitution of equation (42) into equation (41) gives a complex equation in terms of transverse deflection at the snubber \( \gamma \) and \( P(-1)^N \) which may be abbreviated to:

\[
P(-1)^N(U_1 + jV_1) = \frac{\gamma}{\lambda \mu} \left( U_2 + jV_2 \right) \quad \text{.................(43)}
\]
where the abbreviations \( U_1, U_2, V_1, V_2 \) are listed in Appendix VI.

Similarly separating real and imaginary parts of equation (39) and abbreviating, it may also be written in the form:

\[
P(-1)^N(U_3 + jV_3) = \gamma \left( \frac{E}{\lambda_1 G_1} \right) (U_4 + jV_4) \quad \text{..............(44)}
\]

where \( U_3, U_4, V_3, V_4 \) are listed in Appendix VI.

Eliminating \( P(-1)^N \) from equations (43) and (44) and separating real and imaginary parts;

\[
V_1 V_4 - U_1 U_4 - V_2 V_3 + U_2 U_3 = 0 \quad \text{..............(45)}
\]

\[
V_4 U_1 + U_4 V_1 - U_3 V_2 - V_3 U_2 = 0 \quad \text{..............(46)}
\]

Although \( \gamma \) is itself complex it may be eliminated without reducing to real and imaginary parts.

**Case 2.** \( Q \neq 0 \quad P = (U \cos \psi + V \sin \psi) \left( \begin{array}{c} E \\ \frac{Cosh}{Sinh} \end{array} \right) \frac{\gamma}{2\lambda L} \)

Putting \( \delta = \frac{U \cos \psi}{Cosh \eta} + \frac{V \sin \psi}{Sinh \eta} \)

Then the solution to the difference equation can be written as;

\[
\Theta_n = (-1)^n \left[ \frac{\gamma \delta}{2\lambda L G_1} \right] (\text{Cosh} \eta \cos \psi_n + j \text{Sinh} \eta \sin \psi_n) + Q(\text{Sinh} \eta \cos \psi_n + j \text{Cosh} \eta \sin \psi_n) + \delta \quad \text{..............(47)}
\]
For the case where \( N \) is even, substitution of the appropriate value of the above solution into the bending moment equilibrium equation (38)
in a similar manner to the previous case gives after simplifying;

\[
\left[ \frac{\gamma}{2\lambda_1} \left( \frac{E}{G_1} \right) \right] (U_3 + jV_3) + Q(U_5 + jV_5) = \frac{\gamma}{\lambda_1} \left( \frac{E}{G_1} \right) (U_4 + jV_4) \quad \ldots (48)
\]

This is similar to equation (44) for case 1, but with an additional term. The abbreviations for \( U^*, U^t \) and \( V^*, V^t \) are as listed in Appendix VI.

Considering the transverse shearing and inertia forces acting at the snubbers then equations (40) and (41) are still valid for the present solution.

The summation of equation (47) may be written as follows:

\[
\sum_{-N}^{N} \varrho_n = \frac{\gamma}{2\lambda_1} \left( \frac{E}{G_1} \right) \left[ \sum_{n=0}^{N} (-1)^n \left( e^{j(\omega + j\psi)n} + e^{-(\omega + j\psi)n} \right) - 1 \right] +
\]

\[
Q \left[ \sum_{n=0}^{N} (-1)^n \left( e^{j(\omega + j\psi)n} - e^{-(\omega + j\psi)n} \right) - 1 \right] + (2N+1)\vartheta
\]

Carrying out the summation gives:

\[
\sum_{-N}^{N} \varrho_n = -\frac{\gamma d}{2\lambda_1} \left( \frac{E}{G_1} \right) (-1)^N \left[ \frac{(\cosh\omega \cos\psi + \cosh(N-1)\omega \cos(N-1)\psi)}{(1 + \cosh\omega \cos\psi) + j\sinh\omega \sin\psi} \right.
\]

\[
+ \frac{j(\sinh\omega \cos\psi + \sinh(N-1)\omega \sin(N-1)\psi)}{(1 + \cosh\omega \cos\psi) + j\sinh\omega \sin\psi}\]

\[
Q(-1)^N \left[ \frac{(\sinh\omega \cos\psi + \sinh(N-1)\omega \cos(N-1)\psi)}{(1 + \cosh\omega \cos\psi) + j\sinh\omega \sin\psi} \right]
\]

\[
\frac{j(\cosh(N-1)\omega \sin(N-1)\psi + \cosh\omega \sin\psi)}{(1 + \cosh\omega \cos\psi) + j\sinh\omega \sin\psi} \right] \ldots (49)
\]
Substituting equation (49) into the left hand side of equation (41) and abbreviating it can be written as:

\[
\left[ \frac{\gamma \xi}{2\alpha_1} \left( \frac{E}{E_1} \right) \left( U_1 + jV_1 \right) + Q\left( U_6 + jV_6 \right) \right] = \frac{\gamma}{\lambda_1} \left( \frac{E}{E_1} \right) \left( U_2 + jV_2 \right) \quad \text{Equation (50)}
\]

where \( U_1, U_2, U_6 \) and \( V_1, V_2, V_6 \) are abbreviations listed in Appendix VI.

Eliminating the arbitrary constant \( Q \) from equations (48) and (50) and separating real and imaginary parts:

\[
\left[ U_6 \left( U_4 + \frac{\Delta U_3}{2} \right) - V_6 (V_4 + \frac{\Delta V_3}{2}) + U_5 (U_2 + \frac{\Delta U_1}{2}) - V_5 (V_2 + \frac{\Delta V_1}{2}) \right] = 0 \quad \text{Equation (51)}
\]

\[
\left[ U_6 (V_4 + \frac{\Delta V_3}{2}) + V_6 (U_4 + \frac{\Delta U_3}{2}) + U_5 (V_2 + \frac{\Delta V_1}{2}) + V_5 (U_2 + \frac{\Delta U_1}{2}) \right] = 0 \quad \text{Equation (52)}
\]

12.3 Method of Solution

The problem was solved using a Ferranti Sirius computer which is a small low speed machine. Because of the erratic nature of the functions the method of solution was basically a trial and error technique as described in Section 8.2.1. Estimated values of the roots are substituted into the machine as data. The values of the variables and the values of the functions are then printed by the computer for values of the functions close to zero. The exact values of the variables to satisfy the functions are then obtained by graphical interpolation.
Solving equation (33) gives N values of \( \Psi \). Substituting in the frequency equations (25) and (26) and solving, yields N set of values of a and b (see equation 3) from which the frequencies and damping of each mode are calculated. Substitution in equation (29) gives N batch mode shapes, the deflection curves for individual blades corresponding to those of a propped cantilever.

### Symmetric Modes

Substituting estimated values of a and b into equations (45) and (46) and solving gives N roots of \( \lambda \) and \( \Psi \). Substituting each pair of values of \( \lambda \) and \( \Psi \) into equation (25) and (26) produces N closer approximations to a and b. Substituting the closer approximation to a and b into equations (45) and (46) and solving again for the roots, gives a better value of \( \lambda \) and \( \Psi \). However it soon became apparent that the roots \( \lambda \) and \( \Psi \) of equations (44) and (45) were very insensitive to changes in the values of the salient variables, namely a, b, \( \alpha \), \( \lambda \), and \( \omega \). This is in agreement with the work of Smith (73). He reported that for the corresponding modes of vibration for an elastic shroud, that the frequency results were very insensitive to changes of shroud mass. In all calculations the roots \( \lambda \) and \( \Psi \) were found for average values of the above variables and were thereafter regarded as constant.

The batch mode shapes may be determined approximately by substitution of the roots \( \lambda \) and \( \Psi \) into equation (37). The contribution from the particular solution \( \bar{\theta} \) due to transverse motion is small and can be neglected.

### Asymmetric Modes Case 1

Substituting estimated values of a and b into equations (45) and (46) and solving gives N roots of \( \lambda \) and \( \Psi \). Substituting each pair of values of \( \lambda \) and \( \Psi \) into equation (25) and (26) produces N closer approximations to a and b. Substituting the closer approximation to a and b into equations (45) and (46) and solving again for the roots, gives a better value of \( \lambda \) and \( \Psi \). However it soon became apparent that the roots \( \lambda \) and \( \Psi \) of equations (44) and (45) were very insensitive to changes in the values of the salient variables, namely a, b, \( \alpha \), \( \lambda \), and \( \omega \). This is in agreement with the work of Smith (73). He reported that for the corresponding modes of vibration for an elastic shroud, that the frequency results were very insensitive to changes of shroud mass. In all calculations the roots \( \lambda \) and \( \Psi \) were found for average values of the above variables and were thereafter regarded as constant.

The batch mode shapes may be determined approximately by substitution of the roots \( \lambda \) and \( \Psi \) into equation (37). The contribution from the particular solution \( \bar{\theta} \) due to transverse motion is small and can be neglected.

### Asymmetric Modes Case 2

Substituting estimated values of a and b, (see equation 3), into equations (51) and (52) and solving as in the previous case pro-
duced no roots. The search for roots was extended for all possible combinations of positive and negative values of $\omega$ and $\Psi$ for a wide range of magnitudes, but no roots were located.

Several independent checks on the analysis were made although no error was found. The computer programme was rewritten using different locations in the store. However the machine gave exactly the same results. Simplifying the system by setting the damping to zero did not make any improvement. This analysis therefore did not predict a mode of vibration where all the blades vibrate in-phase, with a deflection curve similar to that of the fundamental cantilever mode.

The reason for this is not apparent.

12.4 Theoretical Results

Although the batch mode shape equations are not given in the previous section because of their length and complexity they may be readily derived, and were in fact calculated. The individual blade deflection curve equations are complex as may be expected and represent two standing waves 90 degrees out of phase with each other. They are very similar in shape to those for the single blade shown in Fig. 8.21. For a batch of five blades the batch mode shapes are shown in Fig. 12.3. The individual curves represent the absolute values of the dynamic deflections. The roots of the equations are such that the phase difference between blades due to damping is either zero or very small. The motion between any two blades can be regarded as in-phase or 180 degrees out-of-phase. The fundamental mode which was not predicted by this analysis is indicated by the broken lines. It can be seen that the batch mode shapes correspond basically to those determined experimentally by Rieger and McCallion (74) for a four bay portal frame, with the exception of their mode IIc. It is considered that this is a mistake on their part as this mode shape does not agree with the work of
FIGURE 12.3 FIRST BATCH MODE SHAPES

ASYMMETRIC

2\textsuperscript{nd} ASYM.

2\textsuperscript{nd} SYM.

SYMMETRIC

1\textsuperscript{st} SYM.

1\textsuperscript{st} ASYM.

FUNDAMENTAL
Ellington and McCallion (75) such that $\theta_{-1} = \theta_1$ and could easily have occurred by their method of determining the batch mode shapes.

Graphs of frequency and damping versus snubber position $\alpha$ for the first modes of vibration are shown in Figs. 12.4-12.9. It should be pointed out that although the curves are plotted in non-dimensional terms the curves are not strictly non-dimensional as the viscoelastic material properties depend upon the absolute frequency in cycles per second. As may be expected the frequency and damping curves are very similar to those for the single blade shown previously. The higher frequencies correspond to the higher batch modes. This is understandable from Fig. 12.3 which shows that for the higher batch modes there is more interface shearing and therefore the system is stiffer for motion in these modes. For the first asymmetric mode it may be observed that there is zero damping and the frequency is unaffected by variations in snubber parameter $\lambda$. This is reasonable because the bending restraint of the viscoelastic joint is assumed to be zero, and in this mode no interface shearing occurs because all the blades vibrate with the same amplitude.

Curves of frequency and damping versus snubber position $\alpha$ are shown for the second batch mode vibration in Figs. 12.10-12.13. These are also very similar to those for the single blade shown previously. The maximum and minimum damping values occur at the same snubber positions $\alpha$ as for the single blade.

Figs. 12.14 and 12.15 show curves of frequency and damping respectively versus snubber parameter $\lambda$ at $\alpha = 0.6$ for the first two batch modes of vibration. Fig. 12.15 shows that the optimum value of $\lambda$ for a given mode is the same for the first and second mode. All the curves for the first and second batch modes reach peak damping values of $\delta = 0.78$ and $\delta = 0.30$ respectively. These peak values are approximately the same as the peak values for the single blade.
Figs. 12.16-12.18 show the effects of viscoelastic loss factor \( \eta \) and the dimensionless factor \( \frac{\lambda G_i}{E} \) upon the damping and natural frequency. These curves are truly non-dimensional. The results are for the first batch mode only with the snubber at its optimum position for each mode. The graph shows an increase in natural frequency with rising values of viscoelastic loss factor \( \eta \). This is to be expected as mentioned earlier in the thesis.
FIGURE 12.4 THEORETICAL EFFECT OF SNUBBER POSITION

FOR FIRST BATCH MODE. \( \lambda = 7.68 \times 10^5 \)

- \( \omega_1 / \omega \)
- \( = 2^{\text{ND}} \text{SYM.} \)
- \( = 2^{\text{ND}} \text{ASYM.} \)
- \( = 1^{\text{ST}} \text{SYM.} \)
- \( = 1^{\text{ST}} \text{ASYM.} \)

SNUBBER POSITION \( \alpha \)
FIGURE 12.5 THEORETICAL EFFECT OF SNUBBER POSITION

FOR FIRST BATCH MODE, $\lambda = 7.68 \times 10^5$

- $= 2^{\text{ND}}$ SYM.
- $= 2^{\text{ND}}$ ASYM.
- $= 1^{\text{ST}}$ SYM.
- $= 1^{\text{ST}}$ ASYM.

LOG. DEC. $\delta$

SNUBBER POSITION $\alpha$
FIGURE 12.6 THEORETICAL EFFECT OF SNUBBER POSITION
FOR FIRST BATCH MODE \( \lambda = 3.84 \times 10^5 \)

\[ \text{FREQUENCY RATIO } \frac{\omega}{\omega_0} \]

- \( = 2.\text{ND SYM.} \)
- \( = 2.\text{ND ASYM.} \)
- \( = 1.\text{ST SYM.} \)
- \( = 1.\text{ST ASYM.} \)

SNUBBER POSITION \( \alpha \)
FIGURE 12.7 THEORETICAL EFFECT OF SNUBBER POSITION
FOR FIRST BATCH MODE. $\lambda = 3.84 \times 10^5$

- $= 2^{\text{ND}} \text{SYM.}$
- $= 2^{\text{ND}} \text{ASYM.}$
- $= 1^{\text{ST}} \text{SYM.}$
- $= 1^{\text{ST}} \text{ASYM.}$

LOG. DEC. $\delta$

SNUBBER POSITION $\alpha$
FIGURE 12.8  THEORETICAL EFFECT OF SNUBBER POSITION

FREQUENCY FOR FIRST BATCH MODE. \( \lambda = 1.92 \times 10^5 \)

- \( = 2^{\text{ND}} \text{SYM.} \)
- \( = 2^{\text{ND}} \text{ASYM.} \)
- \( = 1^{\text{ST}} \text{SYM.} \)
- \( = 1^{\text{ST}} \text{ASYM.} \)

FREQUENCY RATIO \( \omega / \omega_0 \)

SNUBBER POSITION \( \alpha \)
FIGURE 12.9 THEORETICAL EFFECT OF SNUBBER POSITION FOR FIRST BATCH MODE. \( \lambda = 1.92 \times 10^5 \)

- \( = 2^{\text{nd}} \text{ SYM.} \)
- \( = 2^{\text{nd}} \text{ ASYM.} \)
- \( = 1^{\text{st}} \text{ SYM.} \)
- \( = 1^{\text{st}} \text{ ASYM.} \)

LOG DEC \( \delta \)

SNUBBER POSITION \( \alpha \)
FIGURE 12.10 THEORETICAL EFFECT OF SNUBBER POSITION FOR SECOND BATCH MODE. \( \lambda = 3.84 \times 10^5 \)

- Solid line = 2\(^{nd}\) SYM.
- Dotted line = 2\(^{nd}\) ASYM.
- Dash line = 1\(^{st}\) SYM.
- Dashed line = 1\(^{st}\) ASYM.

FREQUENCY RATIO \( \frac{\Omega}{\Omega_0} \)

SNUBBER POSITION \( \alpha \)
Figure 12.11 Theoretical Effect of Snubber Position $\alpha$

For second batch mode, $\lambda = 3.84 \times 10^5$

- $\alpha = 2^{nd}$ Sym.
- $\alpha = 2^{nd}$ Asym.
- $\alpha = 1^{st}$ Sym.
- $\alpha = 1^{st}$ Asym.
FIGURE 12.12 THEORETICAL EFFECT OF SNUBBER POSITION FOR SECOND BATCH MODE. $\lambda = 7.68 \times 10^5$

\begin{align*}
\text{Frequency Ratio} & \quad \text{Snubber Position} \\
\text{2$^{\text{nd}}$ Sym.} & \quad \alpha \\
\text{2$^{\text{nd}}$ Asym.} & \quad \alpha \\
\text{1$^{\text{st}}$ Sym.} & \quad \alpha \\
\text{1$^{\text{st}}$ Asym.} & \quad \alpha
\end{align*}
FIGURE 12.13 THEORETICAL EFFECT OF SNUBBER POSITION

FOR SECOND BATCH MODE. $\lambda = 7.68 \times 10^5$

The graph shows the theoretical effect of snubber position for second batch mode with $\lambda = 7.68 \times 10^5$. The graph plots the log decrement ($\delta$) against snubber position ($\alpha$) with four curves representing different modes:
- Solid line: 2nd SYM.
- Dashed line: 2nd ASYM.
- Dotted line: 1st SYM.
- Dashed-dotted line: 1st ASYM.
FIGURE 12.14 THEORETICAL EFFECT OF SNUBBER PARAMETER $\lambda$
FREQUENCY RATIO $\omega/\Omega$. FREQUENCY RATIO $C/\theta$.

SECOND BATCH MODE

FIRST BATCH MODE

SNUBBER PARAMETER $\lambda \times 10^5$
Figure 12.15 Theoretical Effect of Snubber Parameter $\lambda$ on Damping for First & Second Batch Modes. $\alpha = 0.6$
FIGURE 12.16 NON-DIMENSIONAL EFFECT OF \( \sqrt{\alpha} \) FIRST SYMMETRIC MODE. \( \alpha = 0.74 \)
FIGURE 12.17 NON-DIMENSIONAL EFFECT OF THE SECOND ASYMMETRIC MODE, $\alpha = 0$

FREQUENCY RATIO $\frac{\omega}{\omega_0}$

LOSS FACTOR $\eta$

$\lambda (G_i/E)$
FIGURE 12.18 NON-DIMENSIONAL EFFECT OF THE SECOND SYMMETRIC MODE. $\alpha = 0.7$

FREQUENCY RATIO $\frac{\omega}{\omega_1}$

LOSS FACTOR $\eta$

$\lambda \left( \frac{G_1}{E} \right)$
CHAPTER 13

13. EXPERIMENTAL WORK

13.1 Experimental methods for the determination of the dynamic characteristics of structures from transient waveforms.

Unlike resonance testing methods there are no published surveys or comparisons of the techniques for the determination of structural dynamic properties from transient tests. Much of the work that has been done has been directed to aircraft applications particularly to in-flight flutter testing.

Methods of extracting the amounts of damping and frequencies of the individual modes of vibration from the total transient response may be classified as follows:

(1) Mathematical analysis of the complex transient response.

(2) Selection of the required transient wave from the complex signal by use of electronic wave analyser.

(3) Rejection of unwanted components of the complex signal by electrical rejection filters leaving the required waveform unaffected.

A description of each method is given in the following sections.

13.1.1 Mathematical analysis of complex transient waveform

Ellington and McCallion (80) suggested a method of analysis
for extracting the properties of individual modes from the total response where beating occurs due to two or more close natural frequencies. For a transient decay comprised of \( \frac{n}{2} \) modes of vibration the technique requires the measurement of \( 2n \) coordinates of the recorded signal, at equal time intervals. The frequency and logarithmic decrement are then calculated from the complex roots of a polynomial of order \( n \).

To investigate the accuracy of the method a computer programme was written to analyse a waveform comprised of any number of individual decaying waves. The programme was written for an Elliott 503 machine in Elliott Algol Mk II language. This is shown in Fig. 13.1. The method was to calculate the polynomial coefficients by evaluation of their determinants. This was done by using one of two library procedures that were available (81), (82). Both of these procedures were checked for accuracy by setting the elements of the determinant equal to the elements of Pascal's Triangle. These elements have the property of having a determinantal value of unity. The library procedures were found to have an acceptable accuracy provided that the size of the determinant was not greater than 6 by 6. The complex zero's of the polynomial were determined using another library procedure (83). This procedure is based on the Bats-Decay-Hitchcock iterative method.

Test data were generated to four significant figures from the sum of two decaying waves. The waveforms were of frequency 200 and 210 c.p.s. with logarithmic decrements of 0.7 and 0.2 respectively. Arbitrary time intervals between coordinates were taken since Ellington and McCallion (80) do not suggest values to be used. However it soon became apparent that for heavy damping and close natural frequencies, the accuracy of the results were markedly dependent upon the time interval between coordinates. This is shown graphically in Fig. 13.2. For time intervals outside the range shown, the results were wildly inaccurate and almost meaningless particularly for values of damping.
FIGURE 13.1 ELLIOT ALGOL MK II COMPUTER PROGRAMME FOR TRANSIENT
WAVEFORM ANALYSIS.

HALL
WAVEFORM ANALYSIS
12 005 001 004;

begin integer n;
    switch s:=again;
    read n;
again: begin integer array ex,nat[1;n div 2+1];
      array E[0:n;1:n],A[1:n;1:n],W[0:2*n-1],a[0:n],x,y[1:n div 2+1];
      integer b,c,d,e,f,j,k,u,v,m,t,v,K;
      real hold,omega,delta,T,eps0,eps1,eps2,eps3,q,theta;
      switch ss:=C1,C2,C3;
      comment insert F3/1 or F3/2 and C2/1;
      read eps0,eps3,K,T;
      for p:=0 step 1 until (2*n-1) do
        read W[p];
      for f:=1 step 1 until n do
        for e:=0 step 1 until n do
          B[e,f]:=checkr(W[e+f-1]);
          u:=0;
          for j:=0 step 1 until n do
            begin for k:=1 step 1 until n do
              begin hold:=B[u,k];
                B[u,k]:=B[j,k];
                B[j,k]:=hold;
              end;
              for d:=1 step 1 until n do
                for b:=1 step 1 until n do
                  A[d,b]:=checkr(B[d,b]);
                  a[n-j]:=checkr(determinant(A,n));
                end;
              for c:=1 step 2 until n do
                a[c]:=-a[c];
              eps1:=0;
              for v:=0 step 1 until n do
                eps1:=eps1+abs(a[v]);
                eps1:=-7*eps1/(1+n);
                eps2:=-7*(1/abs(a[0]))/(1/n);
                BAIRSTW(n,a,eps0,eps1,eps2,eps3,K,m,x,y,nat,ex);
                print $$\{ t \}$$
                for t:=1 step 1 until m do
                  begin if t mod 7=0 then
                    a[n-j]:=-a[n-j];
                  end;
FIGURE 13.2 EFFECT OF TIME INTERVAL UPON ACCURACY OF WAVEFORM ANALYSIS.

PERCENTAGE ERROR.

TIME INTERVAL BETWEEN COORDINATES SECS.

\( f = 210 \text{ c.p.s.} \)
\( f = 200 \text{ c.p.s.} \)

\( \delta = 0.2 \)
\( \delta = 0.7 \)
For the analysis of a single waveform the results were extremely accurate regardless of time interval. However for the analysis of a waveform comprised of three transient decays the results were worse than for two transient decays.

The method of finding the roots of the polynomials was suspected as a possible cause of error. However another programme was written for a Ferranti Sirius Computer using a trial and error solution by substitution of complex roots. The results were exactly the same as for the Algol programme. Further investigation revealed that it was the value of the polynomial coefficients that vary with time interval between coordinates. For the case of the two waveforms considered above, if the polynomial is written in the form:

\[ x^n a_n + \ldots + x^2 a_2 + x a_1 + 1 = 0 \]

then the coefficients \( a_n \) vary as shown below:

<table>
<thead>
<tr>
<th>POLY Coef.</th>
<th>TIME INTERVAL SECS.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.002</td>
</tr>
<tr>
<td>ELLIOTT 503 PROGRAMME</td>
<td></td>
</tr>
<tr>
<td>( a_1 )</td>
<td>4.171139</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>7.862668</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>7.307011</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>2.898799</td>
</tr>
<tr>
<td>FERRANTI SIRIUS PROGRAMME</td>
<td></td>
</tr>
<tr>
<td>( a_1 )</td>
<td>4.171662</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>7.863345</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>7.307573</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>2.899003</td>
</tr>
</tbody>
</table>
It can be seen that there is considerable variation in the coefficients with time interval although in general good agreement is shown between the coefficients obtained from the two different programmes. Because of the inherent sensitivity of this method to optimum time interval, which is unknown for a given complex waveform this technique was not investigated further.

13.1.2 Extraction of the required mode from the total transient response using an electronic wave analyser.

A wave analyser cannot be used to separate damped transient signals in the same way that it is used to extract waves from a steady input. This is because as mentioned in Section 9.2.1 the transient response of the instrument masks the required component of the forcing signal. This difficulty has been overcome however by the following techniques which are discussed in detail in the subsequent sections.

1) The "Mazet" Technique.

2) The method of "Palei and Uspenskii".

13.1.2.1 The "Mazet" Technique

This method, devised by Mazet (84) for the purpose of flight flutter testing of aircraft was first reported in 1956. The transient signal is recorded on magnetic tape. The signal is then re-recorded backwards; that is beginning at the end of the first recording, on to an endless magnetic tape. The loop of tape is such that the signal occupies only a small portion of its length. The loop is then placed on a rotating drum which can rotate at variable speed, this is shown diagrammatically by Piazzola (85). The signal is then played back into a wave analyser which is tuned to a fixed frequency and is connected to
an oscilloscope. By varying the playback speed the signal may be demodulated. The diverging signal on the screen is the required component of the structural response and the subsequent decaying portion is the transient response of the analyser. By measuring the maximum amplitude of the envelope at the time corresponding to the start of the transient (t=0) for different playback speeds, it has been shown by Mazet (86) that the frequency response of the signal can be determined. The curve of frequency response resembles the resonance curve of the structure. However the shape of the curve must depend upon where the transducer is mounted on the structure and on the position and frequency spectrum of the shock excitation. In this respect and for heavy damping with close natural frequencies the method has similar shortcomings to the normal amplitude-frequency response of the structure, in the sense that individual mode properties cannot easily be estimated. It is from this curve that the natural frequency is obtained and the damping is determined from the diverging portion of the analyser output. Mazet (86) reports frequencies obtained to an accuracy of 0.2% and damping to within 2.0%. However this is for light damping with well separated natural frequencies.

The effects of heavier damping and interference from a close harmonic wave, on the method were investigated experimentally by Hunt (87). He showed that the errors could be much larger than those suggested by Mazet (86) when the frequencies were within 10% of each other. The addition of heavy damping caused errors in measurements of the damping of the order of 40%. The further investigation that he proposed has not been carried out (88). It has been reported only recently by Skingle (89) that the selectivity Q of a wave analyser to a transient signal from a damped structure cannot be greater than the Q of the structure. This may be seen immediately from the expression for response of the analyser shown by Mazet (84, 86), Hunt (87) and others. The damping index of the decaying input adds to the internal damping of the analyser to reduce the effective selectivity. However
the selectivity may be increased by using several analysers in cascade as shown by Piazzoli (85) and Beals and Hurley (90). Beals and Hurley (90) also report errors in damping measurements greater than 10\% for decay rates of greater than $\delta = 0.35$.

13.1.2.2 The method of "Palei and Uspenskii"

This method recently reported by "Palei and Uspenskii" (91) does not involve the specialized equipment of the "Mazet" technique. However it does require the shock excitation of the structure. A proof of the method which the authors do not include in their report is given in Appendix VII which allows some interesting observations to be made. A calibration graph of the expression $A_{26}$ is shown in Fig. 13.3. Inset is shown the response of the analyser to a decaying sinusoidal input. The damping of the analyser circuit $\delta_{\text{anal}}$ may be determined from the tail of the analyser response. By counting the number of cycles $n$ to the maximum amplitude of the envelope the decay ratio $r$ of the system may be determined from Fig. 13.3.

It may be observed from the denominator of equation $A_{24}$ that, unlike the "Mazet" technique, the damping index of the decaying input and the internal damping index of the analyser subtract. For systems where the damping of the structure is equal to the damping of the analyser it would at first appear that the effective $Q$ factor is infinite. However, the numerator of expression $A_{24}$ shows that as the damping indices become the same the response reduces to zero. Fig. 13.4 shows the effect of decay ratio $r$ on the effective $Q$ factor $Q_{\text{eff}}$ for both techniques. The Palei and Uspenskii method has better selectivity than the Mazet technique particularly in the vicinity of $r=1$. However the difficulty arises as to the optimum value of analyser damping, because the damping of the structure is unknown and is the object of the test. The selectivity of the method cannot be improved by connecting several analyses in cascade as in the "Mazet" technique. Palei and Uspenskii
FIGURE 13.3 CALIBRATION OF ANALYSER FOR TRANSIENT DECAYING INPUT

\[ r = \frac{\delta}{\delta_{\text{anl.}}} \]

TRANSIENT RESPONSE OF ANALYSER

DECAY RATIO \( r = \frac{\delta}{\delta_{\text{anl.}}} \)
FIGURE 13.4 EFFECTIVE 'Q' FACTOR VERSUS DECAY RATIO $r$

METHOD OF
PALEI & USPENSKII.

'MAZET' TECHNIQUE

$Q_{eff}$ vs. $r = \delta_1 / \delta_{anal}$.
have not considered the application of their method to systems with close natural frequencies and heavy damping. They assume the resonant frequency of the structure under test to be known. The main problem is that of frequency tuning the analyser when there are close natural frequencies which are of unknown value. If the signal was to be recorded on an endless loop of magnetic tape and played back into the analyser, then the frequency response of the transient signal could be obtained in a similar manner to that of the Mazet technique (86). However, as mentioned in the previous section, this frequency response curve is similar in shape to that obtained by resonance testing the structure and has the same failings. It can be shown that the effect of a close off-tune frequency component is to destroy the smooth shape of the envelope shown inset in Fig. 13.3. This gives an indication of whether a pure mode is being extracted or not.

13.1.3. Elimination on unwanted modes by use of band rejection filters.

The method of rejecting unwanted frequency components from transient signals by use of parallel T band rejection filters has been treated previously in Section 9.2.3. It was shown in Appendix IV that the transient response of this type of filter is aperiodic and does not interfere with the decaying waveform. The effect of a rapidly decaying input was shown to reduce the rejection characteristic as compared to a harmonic input. The basic shape of rejection curve was unaffected as shown in Fig. 9.3. It may also be seen from Fig. 9.3 that the rejection characteristics are not very sharp. Unlike selective filters, connecting rejection filters in cascade only reduces the sharpness of the rejection characteristics. It is necessary to have one filter for each frequency component that is to be rejected. In addition, these filters cause attenuation of off-tune frequencies. While this method may be satisfactory where there are only one or two interfering waves, for waveforms with many unwanted frequency components the technique is impracticable.
13.2 **Nature of experiments**

Of the techniques for separating the transient response of the structure outlined above, it was found that the mathematical analysis technique was unsatisfactory. The "Mazet" technique requires specialised precision equipment. In addition the natural frequencies are obtained from the frequency response of the transient signal and this method has the same disadvantages as the resonance test technique. The "Mazet" technique is not as frequency selective as the method of Palei and Uspenskii, unless two or more analyses are used. For the above reasons an attempt is made to separate the total transient response of the blade system by using either the 'Palei and Uspenskii' method or rejection filter technique whichever proves most satisfactory.

The apparatus is arranged such that the structure may be excited harmonically by two electromagnets connected in series. Each electromagnet is identical to that used in previous experiments. Alternatively the structure may be shock excited which is a requirement of the 'Palei and Uspenskii' method. This is achieved electrically by charging a capacitor and then discharging it through the coils of the electromagnets.

Several sets of blades were manufactured, each with snubbers at different positions along the blade. Each set consisted of five blades with snubbers of length 1.00 inch per side. The blades are held encastrate by clamping together in the root block of the rig. Lead zirconate strain gauges, usually mounted on each blade near the root give a signal proportional to the blade amplitude. Lead zirconate strain gauges are used because of their sensitivity to very small dynamic strains. The signals from the structure are displayed on an eight beam oscilloscope. The batch vibration mode shapes are easily recognizable from the phase relationship of the signals when the structure is undergoing steady harmonic excitation. Measurement of strain
gauge output voltage while varying the frequency of harmonic excitation yields a response curve for each blade. During transient tests the strain gauge signals are fed to the oscilloscope and recorded by a Polaroid camera after passing through either rejection filters or a wave analyser.

The experimental investigation was restricted to the first batch mode of the structure only.

Tests are carried out for two values of viscoelastic shear joint thickness. These correspond to values of snubber parameter $\lambda$ of $7.68 \times 10^5$ and $3.84 \times 10^5$. The snubber position $\alpha$ is varied from $\alpha = 0.2$ to $\alpha = 1.0$ in increments of $0.1$. Tests are carried out to determine the variations in frequency and damping with snubber parameter $\lambda$ for $\alpha = 0.6$.

Because of the temperature sensitivity of the viscoelastic material properties, the structure is maintained at $22^\circ$C throughout the experiments.

13.3 Description of Rig

The rig is basically as described in Section 3.3.2; however, for the present tests the support R57 and its attachments are removed. A general arrangement drawing of the rig as assembled for present tests is shown in Fig. 13.5. The blades are clamped together at the root by tightening the clamping bolts. The blades are held apart at the root by spacers of fixed thickness. Slight adjustments can be made to the blade spacing by placing steel shims next to the spacers as shown. These shims allow the interface distance to be set to the appropriate value for different joint thicknesses. They also account for irregular spacing due to slight variations in the straightness of individual
blades. One electromagnet is fixed to the support plate and may be moved up or down in the slots provided. The second electromagnet is attached to an adjustable pedestal support which allows the magnet to be fixed at any point along the blade length. A test was carried out to determine the amount of coupling through the root block between individual blades. Five identical straight blades were clamped in the root block and the end blade was excited at its natural frequency. The response of the remaining four blades was extremely small and the coupling through the root can be considered negligible.

13.4 Arrangement of Apparatus

A diagrammatic arrangement of the apparatus and the actual arrangement are shown in Fig. 13.6 and plate 4 respectively. The apparatus is connected to provide harmonic or shock excitation, either of which may be selected by operating switch SW3. For harmonic excitation a signal generator provides a variable frequency signal which is amplified in a Williamson 20 watt amplifier. The amplifier has an infinitely variable gain control, and is connected to the electromagnets via an ammeter and a double pole double throw toggle switch SW1. With the switch in position 1 the electromagnet and the amplifier are connected and the trigger circuit is completed. The trigger circuit consists of a 9 volt battery with a 22KΩ resistor in series, to reduce the current drain. Altering the switch SW1 to position 2 cuts-off the excitation current to the electromagnet and opens the trigger circuit, thus triggering the oscilloscope sweep. A load resistance of 15Ω is provided for the amplifier with the switch in position 2 to prevent damage to the output transformer.

Shock excitation is obtained by charging up the 16.4 μF capacitor from the 330 volt D.C. power supply, with the switch SW2 in position 1. Altering the switch to position 2 discharges the capacitor through the electromagnets and the variable damping resistor, at the same time
FIGURE 13.6 ARRANGEMENT OF EXPERIMENTAL EQUIPMENT

- SIGNAL GENERATOR
- 330 VOLT POWER SUPPLY
- DIGITAL FREQUENCY METER
- POWER AMPLIFIER
- EIGHT BEAM OSCILLOSCOPE AND CAMERA
- WAVE ANALYSER OR REJECTION FILTERS
- SW1
- SW2
- SW3
- 22kΩ
- 9V
- 0-10Ω
- 164AF

This diagram illustrates the arrangement of experimental equipment for a specific experiment, showing the connections and components involved.
PLATE 4  GENERAL VIEW OF EXPERIMENTAL EQUIPMENT.
triggering the oscilloscope sweep. A removable link short circuits the interconnection of the two electromagnets when only one is required. The blades are clamped rigidly at the root by the clamp block providing an encastré root fixing. The temperature of the structure is sensed by an iron-constantan thermocouple. The cold junction is maintained at 0°C in melting ice and the voltage is measured on a potentiometer. A tubular lamp with a tungsten filament is mounted close to the structure to raise the temperature when necessary. A lead zirconate strain gauge is bonded to each blade near the root. It is in the same position on each blade. The wires from the strain gauges are screened for most of their length to prevent pick-up of extraneous signals. The strain gauges are assumed to have a negligible effect upon the dynamics of the structure.

The strain gauges are connected directly to the input terminal of an eight beam oscilloscope and camera. Alternatively, the signal from any blade may be selected and fed to either a voltmeter, or to the oscilloscope via a rejection filter or wave analyser. The cathode follower has a minimum input impedance of 100MΩ and has the purpose of keeping the frequency sensitivity to a minimum (69). However when using the voltmeter it is necessary to disconnect the strain gauge inputs to the oscilloscope to prevent the cathode follower being ineffective as the input impedance of the oscilloscope is only 1MΩ. If the method of Palei and Uspenskii is adopted then the wave analyser is used. This instrument is manufactured by Messrs. Muirhead Limited and is designated D 669 B, for which the transfer function quoted in Appendix VII holds (92). This analyser has a variable frequency selectivity adjustment such that variations may be made in the range 5<K<500. Alternatively, if unwanted frequency components are to be eliminated using rejection filters then the analyser is replaced by one or perhaps two parallel T filters of the type described in Sections 9.2.3. and 13.1.3. If two filters are used an additional amplifier is usually necessary to boost the signal after it has passed
through the first filter. A reference signal from the signal generator is fed to the oscilloscope. A digital frequency meter provides an accurate measurement of the excitation and reference frequency.

13.5 Shock pulse shape and its frequency spectrum

A shock pulse may be considered to be made up of a sum of sinusoidal wave shapes of various amplitudes and covering the entire frequency band. The response of a linear system to such an input can be regarded as its response to the transient sine waves contained in the shock. It is important therefore to ascertain that the shock pulse used has strong frequency components in the range of the expected natural frequencies of the structure. To check the shape of the magnetic shock pulse, the transient voltage drop across part of the damping resistor shown in Fig. 13.6 was displayed on an oscilloscope and recorded. The magnetic force is proportional to the square of the current and the current is proportional to the voltage drop displayed on the oscilloscope. The circuit has little inductance and the oscillation of the circuit can be easily eliminated by adjustment of the damping resistance. The pulse shape is shown in plate 5. The peak current is approximately 25 amps. The frequency spectrum of a magnetic pulse produced by such a transient current is examined in Appendix VIII. The frequency spectrum is shown graphically in Fig. 13.7 and gives a curve approaching that of the ideal pure impulse shown by Rubin (93). This pulse shape gives a better spectrum than those for the pulse shapes cited by Smith and Triplett (94). From plate 5 the duration of the pulse is approximately 10 milliseconds. For this pulse duration it may be seen from the frequency scale of Fig. 13.7 that the pulse spectrum covers the expected range of natural frequencies of the structure.
PLATE 5. SHOCK PULSE SHAPE

TIME BASE = 5 MILLISECONDS/CM.

PLATE 6. BATCH MODE SHAPES (1st ASYM.)

BLADE + 2
BLADE + 1
BLADE 0
BLADE - 1
BLADE - 2
FIGURE 13.7 FREQUENCY SPECTRUM OF MAGNETIC SHOCK PULSE
13.6 Experimental Procedure

The blades were mounted in the root block. The interblade spacing was adjusted using shims to set the appropriate interface distance for the viscoelastic material. The viscoelastic adhesive material was applied to each interface joint after cleaning the surfaces with trichloroethylene. The blades were then reclamped tightly. A simple resonance test was carried out to determine the dynamic characteristics of the structure, and investigate the effects of strain gauge and exciter positions upon the strain gauge response.

The structure was excited harmonically. The signal generator frequency was varied while maintaining the excitation current amplitude constant. The output from each strain gauge was measured using the voltmeter and noted, as an indication of the response of individual blades. The response curves produced were found to be strongly dependent upon the following:

a) Position of strain gauge on the blade.

b) Which blade the signal came from.

c) Position and number of electromagnetic exciters.

The response of the structure to harmonic excitation from one electromagnet is shown in Fig. 13.8. The strain gauges were located near the root. Mounting the gauge close to the root gives a better indication of the blade response. The amplitude of the signal from the blade +2 is seen to be smaller than from the remaining blades. This is due to it being farthest away from the exciter and because the gauge was less sensitive than the gauges on other blades. The fundamental mode is very prominent at 76 c.p.s. The next group of four modes are contained in the response in the region of 150-220 c.p.s. The batch modes are easily recognizable from the oscilloscope traces of the signal from each blade. Plate 6 shows an example of the first asymmetric batch mode shape.
It was found that the symmetric modes particularly the second symmetric could not be excited exactly as predicted. The centre blade always had some residual motion which was due to asymmetry of a) The excitation forces, b) The structure.

It may be noticed from Fig. 12.3 that for symmetric modes if the left hand three blades are replaced by a rigid support, then the motion of the remaining two blades will be unaffected. Doing this would simplify the system experimentally and allow the symmetric modes to be isolated from the asymmetric modes. For this reason the two types of vibration mode are examined separately.

13.6.1 Experimental procedure, symmetric modes.

The complete structure was reduced to two blades, the remaining portion of the structure being replaced by a rigid support as shown in plate 7. A simple resonance test for the reduced structure was carried out for three different strain gauge positions. Only one electromagnet was used. The response curves are shown in Fig. 13.9. It may be seen that for the first symmetric mode the most suitable gauge position for maximum relative response is given by positions a or b. However for the second symmetric mode the most suitable position for maximum relative response is position c. The signals were taken from the blade +2 only. This blade was chosen because in changing phase from the first to second symmetric mode, there is an intermediate frequency where the strain due to interblade coupling is almost zero. This gives better discrimination between modes in the response curves. The general experimental procedure for symmetric modes was as follows:

The apparatus was arranged as shown in Fig. 13.6 using the rejection filter in place of the wave analyser. The procedure was similar to that described in Section 9.3 for the single blade. The electronic apparatus was allowed to warm-up. The excitation frequency
FIGURE 13.9 THE EFFECT OF STRAIN GAUGE POSITION UPON STRAIN GAUGE RESPONSE FOR THE TWO BLADE SYMMETRIC MODE ARRANGEMENT.
was increased keeping the excitation current amplitude constant, to the expected resonant range of the first symmetric mode. Either one or two electromagnets were used depending upon the damping of the structure. A signal from a strain gauge, mounted in position either a or b shown in Fig. 13.9, was fed to the filter which was set at 'flat'. With the filter adjusted to the flat position the signal by-passes the rejection circuit. When an apparent resonance was reached as indicated by maximum amplitude of the oscilloscope trace or voltmeter reading, the oscilloscope was set to external triggering. The camera shutter was opened and the switch SW1 moved to position 2. The camera recorded the transient decay and reference signal. Hence the damping, and the natural frequency were calculated by simple proportion. The filter was then tuned to reject this frequency and the signal from gauge c was selected. The approximate resonance peak for the second symmetric mode was located and the transient decay recorded as described above. For cases of particularly heavy damping it was sometimes necessary to introduce an additional filter and amplifier to reject transient components from the second batch mode. It was also necessary on occasions to reject the second symmetric batch mode when attempting to determine the transient decay of the first symmetric batch mode.

13.6.2 Experimental procedure, asymmetric modes.

It is not possible to eliminate the symmetric modes while examining the asymmetric modes. However by selecting signals from strain gauges mounted on the centre blade, it was possible to reduce the interference from the symmetric modes to a minimum as could be expected from Fig. 12.3. The apparatus was arranged as shown in Fig. 13.6 and plates 4 and 8. The wave analyser was incorporated in the circuit and the analytical technique of 'Palei and Uspenskii' was adopted. Since the mode density was high, their elimination by rejection filters would be extremely difficult. One electromagnet was used to excite the fundamental mode and two electromagnets were used to
excite the first and second asymmetric modes and were located as shown in plate 8. The system was excited in the vicinity of the required asymmetric mode while the excitation current was maintained constant. The modes of vibration were recognized by the oscilloscope traces as shown in plate 6. Vibration amplitude was monitored on either the oscilloscope or the value voltmeter. The fundamental mode of vibration is sufficiently far away from the remaining modes for the peak amplitude to be taken as the resonant frequency. The switch SW2 was set to position 1 in order to charge the capacitor, and the switch SW3 was set to position 2 for pulse excitation. The oscilloscope was set for external triggering and the camera shutter was opened. Altering the switch SW2 to position 2 shock excited the structure and simultaneously triggered the oscilloscope sweep. Plate 9 shows the typical shock response of the blade system and analyser for the fundamental mode. The logarithmic decrement was determined by the method described in Section 13.1.2.2. The response of the first asymmetric mode to harmonic excitation was large in comparison to the response from adjacent modes and the peak amplitude was assumed to correspond to the resonant frequency. The analyser was tuned to this frequency and the transient recorded as in the previous case. Considerable difficulty was encountered in locating the second asymmetric mode as there was no pronounced resonant peak due to the heavy damping. Even with the signal taken from the gauge located on the snubber, which was shown in the previous section to give better response for highly damped modes, there was little improvement in response. The analyser was tuned to an estimated value of the natural frequency of this mode and the system shock excited. The analyser response was examined to see if the amplitude envelope was smooth indicating that a pure mode had been extracted. The analyser was retuned and the procedure repeated without any positive result. Plate 10 shows a typical transient waveshape for the second symmetric mode. The blade response shows considerable beating with neighbouring modes. The analyser is insufficiently selective to separate the signals and consequently has a much distorted envelope in comparison to that
PLATE 9. TRANSIENT RESPONSE, FUND. NTL.
MODE.

PLATE 10. TRANSIENT RESPONSE, 2. ASYM.
MODE.
shown in plate 9. Although many different strain gauge positions were tried in conjunction with several settings of the analyser selectivity Q, the transient response of the second asymmetric mode could not be extracted from the total response. Hence no experimental results were obtained for this mode.
CHAPTER 14

14. RESULTS AND CONCLUSIONS

14.1 Experimental results

Fig. 14.1 shows the effect of snubber parameter $\lambda$ upon the damping and frequency of vibration for the fundamental and first asymmetric batch modes. For the fundamental mode there are no theoretical curves. The damping results for the fundamental mode reach a pronounced maximum and then reduce in a similar manner to that for the single blade. The frequency results show some scattering of experimental points, particularly in the region of maximum damping. This is to be expected as the resonance peak is less distinct for heavy damping and the estimation of the resonant frequency is more susceptible to errors. The results show that considerable stiffening and damping may be achieved in this particular mode due to the large interface shear amplitudes.

The experimental results for the first asymmetric mode show the frequency to be slightly higher than predicted and increasing slightly with snubber parameter $\lambda$. This is due to bending of the viscoelastic interface joint which was neglected in the theoretical analysis. The frequency could be expected to rise with reductions in the viscoelastic joint thickness, corresponding to increasing values of snubber parameter $\lambda$. The experimental damping values are very much smaller than for the fundamental mode as would be expected as the interface shearing is zero in this mode. The experimental points are particularly close to the predicted value of zero damping at $\lambda = 3.04 \times 10^5$. The increases in damping for lower values of $\lambda$ are due to thickness compressibility of the viscoelastic joint since it is for these values of $\lambda$ that the joint is particularly thick. The increase in experimental damping for higher values of $\lambda$ is due to the damping provided by the bending restraint of the viscoelastic joint which increases with $\lambda$ as mentioned above.
Fig. 14.2 shows the effect of snubber parameter $\lambda$ upon the damping and frequency of vibration for the symmetric modes. The frequencies show reasonable correlation with predicted curves, however the experimental and theoretical curves cross over in a similar manner to those for the single blade. As was explained in Section 10.1, this is caused by flexibility of the snubby. The experimental damping results show good correlation with the predicted curve for the second symmetric mode. The correlation between theoretical and experimental damping results for the first symmetric mode is reasonable. The reason for the shape of the experimental curve is not easily explained. Consider the increase in damping due to shear to be much less than but of the same general form as that predicted. This is reasonable as it can be seen from the damping curve of the second symmetric mode that the experimental maximum does not occur until a higher value of $\lambda$. This suggests that the analysis overestimates the interface shear restraint. Then the addition of damping due to bending of the joints as shown for the first asymmetric mode would give a very similar shaped curve to that obtained.

Fig. 14.3 shows the effect of snubby position $\alpha$ upon the frequency and damping of the fundamental mode of the structure. Maximum stiffening and damping are achieved in this mode with the snubby at $\alpha = 0.5$. This agrees with the position of maximum sensitivity to rotational restraint for this type of mode. The position of maximum sensitivity may be shown in an analogous manner to that of Section 8.2.3. The results show particularly heavy damping for $\lambda = 7.68 \times 10^5$. For snubby locations close to the root no experimental results were obtained as all the frequencies converge and become very close, making separation of individual modes impossible.

Fig. 14.4 shows the effect of snubby position $\alpha$ upon the damping and frequency of the first asymmetric mode. The experimental damping is comparatively small but is a maximum at approximately the snubby location for optimum damping shown previously for the single blade.
The experimental frequency results are slightly higher than predicted due to the interface joint bending stiffness which was neglected in the theory.

The effects of snubber position upon the frequency and damping of the first and second symmetric modes are shown in Fig. 14.5 and Fig. 14.6 respectively. In general the experimental frequency results are lower than the predicted curves. This is due to approximations made in the assumptions regarding the rigidity of the snubber and direction of the damping force. The experimental damping results show considerable scatter, and poor correlation with the predicted curves, although a similar trend to the predicted curves may be recognized. The scatter suggests that the deviations from the mean are due to the methods of measuring the individual mode damping. This is quite likely. The scatter is much worse than that for the single blade. This is to be expected as it has been shown that close natural frequencies and heavy damping considerably complicate the determination of the properties of individual modes. Much better correlation is obtained between theoretical and experimental frequencies than between theoretical and experimental damping. This is understandable and has also been reported by Beals and Hurley (90) who made damping measurements using the 'Mazet' technique.

14.2 Conclusions

Snubber stiffening and damping vary considerably from one batch mode to another.

The snubber positions for maximum stiffening and maximum damping of the structure are coincident, in all modes of vibration.

Optimum snubber position varies with the mode of vibration such that the optimum position for the first set of modes is not the
same as for the first harmonic of the first set of modes.

For the first set of modes the optimum snubber position is different for each basic blade deflection curve, being \( \alpha = 0.5 \) for the fundamental batch mode and \( \alpha = 0.73 \) for the subsequent group of batch modes.

Snubber flexibility must be taken into account for accurate results particularly for high values of \( \lambda \frac{G}{E} \).

The effects of compression and bending stiffness of the viscoelastic joint although small in comparison to the shear stiffness, can have a significant effect being most apparent in modes where there is little interface shear.

For continuous systems with heavy damping and close natural frequencies the determination of the dynamic properties of individual modes by the transient methods at present available is difficult if not impossible.
FIGURE 14.1 EXPERIMENTAL EFFECT OF SNUBBER PARAMETER $\lambda$ UPON THE FREQUENCY & DAMPING OF THE FUNDAMENTAL AND 1st ASYM. MODE. $\alpha = 0.6$. 

Log dec. $f$
Figure 14.2: Experimental effect of snubber parameter $\lambda$ upon the frequency & damping of the first and second symmetric modes.
FIGURE 14.3  EXPERIMENTAL EFFECT OF SNUBBER POSITION UPON FREQUENCY & DAMPING FOR FUNDAMENTAL MODE.
FIGURE 14.4 EXPERIMENTAL EFFECT OF SNUBBER POSITION UPON THE FREQUENCY & DAMPING OF THE FIRST ASYMMETRIC MODE.

LOG. DEC. $\delta$

0.3
0.2
0.1

FREQUENCY & DAMPING

FREQUENCY C.P.S.

$\lambda = 7.68 \times 10^5$

$\lambda = 3.84 \times 10^5$

THEORY

EXP.
Figure 14.5: Theoretical and Experimental Effect of Snubber Position upon the Frequency & Damping of the First Symmetric Mode.
FIGURE 14.6 EXPERIMENTAL EFFECT OF SNUBBER POSITION \( \alpha \) UPON THE FREQUENCY & DAMPING FOR THE SECOND SYMMETRIC MODE.

\[ \lambda = 7.68 \times 10^5 \]

\[ \lambda = 3.84 \times 10^5 \]
APPENDIX I

1. FREE VIBRATION OF A SINGLE DEGREE OF FREEDOM SYSTEM WITH HYSTERETIC DAMPING.

Consider the following equation of motion for a system with hysteretic damping:

\[ m \ddot{x} + k(1+j\eta)x = 0 \]

or \[ \ddot{x} + \omega_n^2(1+j\eta)x = 0 \] \[ \text{(A1)} \]

where \( \omega_n = \frac{k}{m} \)

Making a trial solution of the form

\[ x = A_1 e^{j\omega t} \] \[ \text{(A2)} \]

where \( \omega \) may be real, imaginary, or complex gives the following equation:

\[ \omega^2 = \omega_n^2(1+j\eta) \] \[ \text{(A3)} \]

where \( \omega \) must be complex if \( \eta \neq 0 \).

In polar form equation A3 becomes:

\[ \omega^2 = \omega_n^2(1+\eta^2)^{1/2} e^{2j\alpha} \]

where \( 2\alpha = \tan^{-1} \eta \)

hence \[ \omega = \omega_n(1+\eta^2)^{1/4} e^{j\alpha} \] \[ \text{(A4)} \]
Letting $a = (1 + \eta^2)^{1/4} \cos \alpha$ and $b = (1 + \eta^2)^{1/4} \sin \alpha$ then from equations A2 and A4

$$x = A_1 e^{-(b+ja)\omega_n t}$$

or

$$x = A_2 e^{bt} \left[ \cos a \omega_n t + j \sin a \omega_n t \right] + B e^{bt} \left[ \cos a \omega_n t - j \sin a \omega_n t \right]$$

...............(A5)

Both parts of the above solution are complex.
APPENDIX II
FREE VIBRATION OF A SINGLE DEGREE OF FREEDOM SYSTEM
WITH DAMPING FORCE DIFFERING IN PHASE WITH VELOCITY

Consider the following equation of motion which contains a hypothetical damping term in addition to the classical viscous damping force:

\[ m\ddot{x} + c\dot{x} + (j\omega\beta + k)x = 0 \quad \text{(A6)} \]

where \( \omega \) is a pure number and corresponds to the damped natural frequency and \( \beta \) is a constant which may be positive or negative.

Assume a solution of the form

\[ x = Ae^{(-\nu + j\omega)t} \quad \text{(A7)} \]

Substituting equation (A7) and its time derivatives into equation (A6) and solving for \( \omega \) and \( \nu \) yields;

\[ \nu = \frac{2}{cm} (1+\beta) \quad \text{(A8)} \]

and

\[ \omega = \pm \sqrt{\frac{k - \frac{c^2}{m} (1-\beta^2)}{\frac{4m^2}{}} \quad \text{(A9)} \]

The velocity \((-\nu + j\omega)x\) leads the displacement by;

\[ \phi_v = \pi + \tan^{-1} \frac{\nu}{\omega} \quad \text{(A10)} \]

The damping force is given by:

\[ F = c(-\nu + j\omega(1+\beta))x \]
and leads the displacement by:

\[ Q_F = \pi + \tan^{-1} \frac{\nu}{\omega (1+\beta)} \] ..............................(A1l)

Considering equations (A10) and (A11) it may be observed that:

If; \(-1 < \beta < 0\);  The damping force leads the velocity vector.

If; \(\beta = 0\);  The damping force is in phase with the velocity vector.

If; \(\beta > 0\);  The damping force lags the velocity vector.

It is evident therefore that any phase difference between the damping force and the velocity produces an increase in the damped natural frequency \(\omega\).
APPENDIX III

THE EFFECT OF THE DIRECTION OF THE DAMPING FORCE ON
THE SINGLE DAMPED BLADE

In an effort to determine how the direction of the damping force affects the theoretical results, the following case is considered where it is assumed that the viscoelastic damping force is in phase with the velocity. A similar approach was taken by Bishop (25) which he attributed to Collar, for a single degree of freedom system.

From section 8.2.1 equation (18), the direction of the velocity may be shown to be:

\((-v + j\omega)\) or \(\left(\frac{-2ab}{a^2 - b^2} + j\right)\)

Replace \(j\) in the right hand term of equation (30) by:

\(\left(\frac{-2ab}{(a^2 - b^2)} + j\right)\).

Although this results in a slight increase in the damping force, equation (30) becomes:

\[
\frac{2\text{chr}^2}{\pi \text{Id}} \left(\frac{G_1}{E}\right) \left[1 + \eta \left(\frac{-2ab}{(a^2 - b^2)} + j\right)\right] \quad \text{which, if } b^2 \ll a^2,
\]

gives

\[
\frac{2\text{chr}^2}{\pi \text{Id}} \left(\frac{G_1}{E}\right) \left[1 + \eta \left(-\frac{2b}{a} + j\right)\right]
\]

When real and imaginary parts are separated, the right hand terms of
equations (32) and (33) become:

$$\frac{-2\lambda}{a^2 + b^2} \left( G \right) (a-\eta b) \quad \text{and} \quad \frac{-2\lambda}{a^2 + b^2} \left( G \right) \left[ \eta (a+2b^2) - b \right]$$

respectively. By comparison with the present terms in equations (32) and (33), it will be seen that the change in sign of the right hand term of equation (32) causes the frequency to decrease with increasing damping. An extra term, namely $2\eta b^2$ in equation (33) causes the damping force to increase slightly. This is analogous to the result obtained by Bishop (25) for a single degree of freedom system. The calculated results are shown by the broken line in Figs. 8.15, 8.16 and 8.17.
APPENDIX IV

RESPONSE CHARACTERISTICS OF A PARALLEL T FILTER TO A TRANSIENT DECAYING WAVEFORM

For zero source impedance and infinite load impedance the transfer function for the circuit shown inset in Fig. 9.3 is given by:

\[
\frac{V_o}{V_i} = \frac{D^2 + \omega_o^2}{D^2 + 4\omega_o D + \omega_o^2} \tag{A12}
\]

where \( D \) is the differential operator and \( \omega_o = 1/RC \).

If the input voltage is of the form:

\[ V_i = \bar{V}_e^{-vt} \sin \omega t \]

where \( \bar{v} = \frac{6\omega}{2\pi} \) then the transfer function may be written in the form:

\[ (D^2 + 4\omega_o D + \omega_o^2)V_o = \bar{V}_e^{-vt} \left[ (\omega^2 + 2\omega_o \omega^2) \sin \omega t - 2\bar{v} \omega \cos \omega t \right] \tag{A13} \]

Putting the R.H.S. equal to zero and solving for the complementary function yields:

\[ V_o = A e^{-\left(2+\sqrt{3}\right)\omega_o t} + B e^{-\left(2-\sqrt{3}\right)\omega_o t} \tag{A14} \]

where \( A \) and \( B \) are arbitrary constants.

Assuming a trial solution for the particular integral of the form:

\[ V_o = A_1 e^{-vt} \sin \omega t + B_1 e^{-vt} \cos \omega t \]
and solving produces the following particular solution:

\[ V_0 = \left( \frac{\omega_0^2 - \omega^2 + v^2}{\omega_0^2 - \omega^2 + v^2 - 4\omega \nu} \right)^{\frac{1}{2}} v e^{-\nu t} \sin(\omega t + \varphi) \]

where \( \varphi = \tan^{-1} \left[ \frac{\frac{1}{4} v \omega_0^2 - 4(\omega_0^2 - \omega^2) \nu}{(\omega_0^2 - \omega^2)^2 + 2 \omega_0^2 (\nu^2 + \omega^2)} - \frac{1}{4}(\omega_0^3 \nu + \nu^3 \omega_0^2) - 4\nu \omega_0^2 \nu + \frac{1}{4} \right] \]

For the case of a steady harmonic input this reduces to the well known form:

\[ V_0 = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 16\omega_0^2 \nu^2}} \sin(\omega t + \varphi) \]

where \( \varphi = \tan^{-1} \left[ -4\omega \nu \right] \)

The complete solution to equation (A13) is:

\[ V = A e^{-\left(2+\sqrt{3}\right)\omega_0 t} + B e^{-\left(2+\sqrt{3}\right)\omega_0 t} + \left( \frac{\omega_0^2 - \omega^2 + v^2}{\omega_0^2 - \omega^2 + v^2 - 4\omega \nu} \right)^{\frac{1}{2}} \times \]

\[ \bar{V} e^{-\nu t} \sin(\omega t + \varphi) \]
APPENDIX V

COMPARISON OF SHEAR STIFFNESS AND BENDING STIFFNESS OF VISCOELASTIC JOINT

To compare the magnitudes of the bending stiffness and the shear stiffness of the viscoelastic joint, it is necessary to consider separately the conditions of pure shear and pure bending. This is shown in Fig. 12.1 where the angular deflections of the snubbers to the left and right of the interface joint are equal in magnitude but may be opposite in direction.

a) Restraint due to bending of the joint

Assuming the snubber material to be rigid in comparison to the viscoelastic material. Then for small deflections the strain produced by the application of a moment, to the element dx of width h is 2θx/h. The stress is given by dP/hdx where dP = M_b/x.

Taking the value of the Poisson's ratio for the viscoelastic material to be 0.5 as is usual, Mentel (37), Yin (13), Krause, Segretto, Przinemel and Mach (58), and substituting this in the classical relation between Young's modulus and the shear modulus reveals that E' = 3G. Where G is the absolute value of the complex shear modulus and E' is the absolute value of the complex Youngs modulus.

Therefore,  \[ E' = 3G = \text{stress/strain} = \frac{dM_b}{dx \cdot 2hx^2} \]

From which \[ M_b = \frac{6Gh\theta}{d} \int_{-c/2}^{c/2} \frac{x^2 dx}{2d} = \frac{Ghc^3\theta}{2d} \]  \[ \text{...............(A16)} \]
In the simple analysis above the end effects although probably large have been neglected. However it is considered to give a quantitative assessment of the stiffness due to bending of the joint.

b) Restraint due to shearing of the joint

By a similar procedure to that of Section 8.2.1 it may easily be shown that the restraining moment felt by the blades due to pure shear is:

\[ M_s = \frac{2\pi r^2 G\theta}{d} \] \hspace{1cm} (A17)

c) Comparison of restraint due to bending and restraint due to shearing

From equations (A16) and (A17), it may be shown that the ratio of shearing restraint to bending restraint is given by:

\[ \frac{M_s}{M_b} = 4 \left( \frac{r}{c} \right)^2 \] \hspace{1cm} (A18)

Due to aerodynamic considerations the snubbers of real blades are thin in comparison to their length.

The snubbers of the blades manufactured for use in this investigation have a length to thickness ratio of 8:1. Even allowing for a considerable underestimation of the bending stiffness due to end effects, for snubbers of the above proportions the bending stiffness is very small when compared with the shear stiffness.
\[ E_7 = \left\{ \frac{(p_6+1)[Q_8(a^2-b^3) + 2abP_8] - Q_6[Q_8(a^2-b^3) - 2abQ_8]}{(p_6+1)^2 + Q_6^2} \right\} \]

\[ E_8 = \left\{ \frac{(p_6+1)[a(Q_8 - Q_4) + b(p_5-p_3)] - Q_6[a(p_5-p_3) - b(Q_8-Q_4)]}{(p_6+1)^2 + Q_6^2} \right\} \]

\[ E_{10} = \left\{ \frac{(p_1-1)[(a^2-3ab)(p_2+p_3) - (3a^2-b^3)(Q_2+Q_3)] + Q_1[(3a^2-b^3)(p_2+p_3) + (a^2-3ab)(Q_2+Q_3)]}{(p_1-1)^2 + Q_1^2} \right\} \]

\[ E_{11} = \left\{ \frac{(p_1-1)[(3a^2-b^3)(p_2+p_3) + (a^2-3ab)(Q_2+Q_3)] - Q_1[(a^2-3ab)(p_2+p_3) - (3a^2-b^3)(Q_2+Q_3)]}{(p_1-1)^2 + Q_1^2} \right\} \]

\[ E_{12} = \left\{ \frac{(p_1-1)[Q_7(a^2-b^3) - 2abQ_7] + Q_1[Q_7(a^2-b^3) + 2abP_7]}{(p_1-1)^2 + Q_1^2} \right\} \]

\[ E_{13} = \left\{ \frac{(p_1-1)[Q_7(a^2-b^3) + 2abP_7] - Q_1[Q_7(a^2-b^3) - 2abQ_7]}{(p_1-1)^2 + Q_1^2} \right\} \]
\[ E_{14} = \frac{(P_6+1)[P_8(a^2-b^2) - 2abQ_8] + Q_6[Q_6(a^2-b^2) + 2abP_8]}{(P_6+1)^2 + Q_6^2} \]

\[ E_{15} = \frac{(P_6+1)[Q_6(a^2-b^2) + 2abP_8] - Q_6[P_8(a^2-b^2) - 2abQ_8]}{(P_6+1)^2 + Q_6^2} \]

\[ E_{16} = \frac{(P_6+1)[(a^3 - 3ab^2)(P_4 + P_5) - (3a^2b - b^3)(Q_4 + Q_5)] + Q_6[(3a^2b - b^3)(P_4 + P_5) + (a^3 - 3ab^2)(Q_4 + Q_5)]}{(P_6+1)^2 + Q_6^2} \]

\[ E_{17} = \frac{(P_6+1)[(3a^2b - b^3)(P_4 + P_5) + (a^3 - 3ab^2)(Q_4 + Q_5)] - Q_6[(a^3 - 3ab^2)(P_4 + P_5) - (3a^2b - b^3)(Q_4 + Q_5)]}{(P_6+1)^2 + Q_6^2} \]
\[ U_1 = \frac{(\cosh(N \alpha \cos \psi) + \cosh(N-1 \alpha \cos(N-1) \psi)(1 + \cosh \alpha \cos \psi) + (\sinh(N \alpha \sin \psi) + \sinh(N-1 \alpha \sin(N-1) \psi) \sinh \sin \psi)}{(1 + \cosh \alpha \cos \psi)^2 + (\sinh \sin \psi)^2} \]

\[ U_2 = \frac{1}{2} \left\{ \frac{u (\cosh \alpha \cos \psi + 1 \sinh \sin \psi)}{(\cosh \alpha \cos \psi + 1)^2 + (\sinh \sin \psi)^2} \right\} - \frac{1}{\lambda (G)} \left\{ \frac{(E_{12} - E_{14}) [(a_1^2 - b_1^2)(-4a_1^2 b_1^2) + (E_{10} + E_{14})] + (E_{13} - E_{15}) [(E_{11} + E_{13}) + 4W_a b (a_1^2 - b_1^2)]}{(E_{12} - E_{14})^2 + (E_{13} - E_{15})^2} \right\} \]

\[ U_3 = \cosh(N \alpha \cos \psi) (2 \cosh(N \alpha \cos \psi - 1) - 2 \sinh \sin \psi \sinh(N \alpha \sin \psi) - \cosh(N-1 \alpha \cos(N-1) \psi) \]

\[ U_4 = u - \left\{ \frac{u (\cosh \alpha \cos \psi + 1 \sinh \sin \psi)}{(\cosh \alpha \cos \psi + 1)^2 + (\sinh \sin \psi)^2} \right\} \cosh \alpha \cos \psi + \left\{ \frac{v (\cosh \alpha \cos \psi + 1 \sinh \sin \psi) - u \sinh \sin \psi}{(\cosh \alpha \cos \psi + 1)^2 + (\sinh \sin \psi)^2} \right\} \sinh \sin \psi \]

\[ U_5 = \cosh(N \alpha \sin \psi) (2 \cosh(N \alpha \sin \psi - 1) + 2 \sinh \sin \psi \cosh(N \alpha \cos \psi) - \cosh(N-1 \alpha \sin(N-1) \psi) \]

\[ U_6 = \frac{(\sinh(N \alpha \cos \psi) + \sinh(N-1 \alpha \cos(N-1) \psi)(1 + \cosh(N \alpha \cos \psi) - (\sinh(N \alpha \sin \psi) + \sinh(N-1 \alpha \sin(N-1) \psi) \sinh \sin \psi)}{(1 + \cosh \alpha \cos \psi)^2 + (\sinh \sin \psi)^2} \]
\( V_1 = \frac{(\sinh n\alpha \sin n\psi + \sin (n-1)\alpha \sin (n-1)\psi)(1 + \cosh n\alpha \cos n\psi) + (\cosh n\alpha \cos n\psi + \cosh (n-1)\alpha \cos (n-1)\psi) \sinh n\alpha \sin n\psi}{(1 + \cosh n\alpha \cos n\psi)^2 + (\sinh n\alpha \sin n\psi)^2} \)

\( V_2 = V_2 \left\{ \frac{V(\cosh n\alpha \cos n\psi + 1) - u \sinh n\alpha \sin n\psi}{(\cosh n\alpha \cos n\psi + 1)^2 + (\sinh n\alpha \sin n\psi)^2} \right\} - \frac{1}{\lambda (\theta)} \left[ \left( E_{12} - E_{14} \right) + 4abW(a^2 + b^2) \right] (E_{12} - E_{14}) - \frac{W((a^2 - b^2)^2 - 4a^2 b^2) + (E_{10} + E_{16}) (E_{13} - E_{15})}{(E_{12} - E_{14}) + (E_{13} - E_{15})^2} \)

\( V_3 = \sinh n\alpha \sin n\psi (2 \cosh n\alpha \cos n\psi + 1) + 2 \sinh n\alpha \sin n\psi \cosh n\alpha \cos n\psi - \sinh (n-1)\alpha \sin (n-1)\psi \)

\( V_4 = v - \left\{ \frac{u(\cosh n\alpha \cos n\psi + 1) + v \sinh n\alpha \sin n\psi}{(\cosh n\alpha \cos n\psi + 1)^2 + (\sinh n\alpha \sin n\psi)^2} \right\} \sinh n\alpha \sin n\psi + \left\{ \frac{V(\cosh n\alpha \cos n\psi + 1) - u \sinh n\alpha \sin n\psi}{(\cosh n\alpha \cos n\psi + 1)^2 + (\sinh n\alpha \sin n\psi)^2} \right\} \cosh n\alpha \cos n\psi \)

\( V_5 = \cosh n\alpha \sin n\psi (2 \cosh n\alpha \cos n\psi + 1) + 2 \sinh n\alpha \sin n\psi \cosh n\alpha \cos n\psi - \cosh (n-1)\alpha \sin (n-1)\psi \)

\( V_6 = \frac{(\cosh n\alpha \sin n\psi + \cosh (n-1)\alpha \sin (n-1)\psi)(1 + \cosh n\alpha \cos n\psi) - (\sinh n\alpha \cos n\psi + \sin (n-1)\alpha \cos (n-1)\psi) \sinh n\alpha \sin n\psi}{(1 + \cosh n\alpha \cos n\psi)^2 + (\sinh n\alpha \sin n\psi)^2} \)
APPENDIX VII

PROOF OF THE METHOD OF "PALEV AND USPENSKII"

In general the transfer function for a wave analyser may be written as;

\[ \frac{V_o}{V_i} = \frac{1}{D^2 + 2\mu D + \Omega^2} \] ........................(A19)

where \( D \) is the differential operator. If the input \( V_i \) from the structure is of the form;

\[ V_i = \sum \tilde{V}_n e^{-\nu n t} \cos(\omega_n t + \phi_n) \]

the equation (A19) may be written;

\[ (D^2 + 2\mu D + \Omega^2)V_o = \sum \tilde{V}_n e^{-\nu n t} \cos(\omega_n t + \phi_n) \] ........................(A20)

Solving for the complementary solution yields;

\[ V_o = Ae^{-\mu t} \sin \omega_o t + Be^{-\mu t} \cos \omega_o t \] .................................(A21)

where \( \omega_o^2 = \Omega^2 - \mu^2 \) and is the damped natural frequency of the circuit.

By making a trial solution for the particular integral of the form;

\[ V_o = \tilde{A}_1 e^{-\nu n t} \sin(\omega_n t + \phi_n) + \tilde{B}_1 e^{-\nu n t} \cos(\omega_n t + \phi_n) \]

the complete solution to equation (A20) may be written as;

\[ V_o = e^{-\mu t} \text{ASin} \omega_o t + \text{BCos} \omega_o t + \]

\[ \sum \frac{\tilde{V}_n e^{-\nu n t} \sin(\omega_n t + \phi_n + \psi_n)}{\sqrt{\left( \nu n - \omega_n^2 + \Omega^2 - 2\mu \nu_n \right)^2 + 4\omega_n^2 (\mu - \nu_n)^2}}} \] ..........................(A22)
where \( \psi_n = \tan^{-1}\left[ \frac{v_n^2 - \omega_n^2 + \omega_n^2}{2\omega_n(\mu - v_n)} \right] \) \hspace{1cm} \text{(A23)}

Now considering the \( i \)th component of the forced response to be at tune, such that \( n = i \). The \( \omega_0^2 = \omega_i^2 = \omega_n^2 - \mu^2 \) and equations (A22) and (A23) reduce to:

\[
V_o = e^{-\mu t}[A \sin \omega_i t + B \cos \omega_i t] + \frac{V_i e^{-\mu t} \sin(\omega_i t + \phi_i + \omega_i)}{\sqrt{(\mu - \nu_i)^2 + 4\omega_i^2(\mu - \nu_i)^2}}
\]

where \( \psi_i = \tan^{-1}\left[ \frac{\mu - \nu_i}{2\omega_i} \right] = \tan^{-1}\left[ \frac{\delta_{\text{anal}} - \delta_i}{2\pi} \right] \)

If \( (\delta_{\text{anal}} - \delta_i)/2\pi \ll 1 \) then \( \psi_i \approx 0 \) where \( \delta_{\text{anal}} \) is the logarithmic decrement of the analyser decay. Consider the structure to be shock excited at \( t = 0 \). If the transient amplitude of each component of the structural response is assumed to be a maximum at \( t = 0 \) then \( \phi_i = 0 \).

In addition, at \( t = 0 \) \( \frac{dV}{dt} = 0 \) from which it can be shown that

\[
V_o = \frac{V_i(e^{-\nu_i t} - e^{-\mu t}) \sin \omega_i t}{\sqrt{(\mu - \nu_i)^2 + 4\omega_i^2(\mu - \nu_i)^2}} \]

The curve represented by equation (A24) is shown inset in Fig. 13.3 and is a maximum at \( t = t_m \). Differentiating the expression for the envelope and equating to zero for \( t = t_m \) then

\[
e^{(\nu_i - \mu)t_m} = \frac{\nu_i}{\mu} \quad \text{or} \quad \mu(r-1)t_m = \log r \quad \text{..................(A25)}
\]
where \( r = \frac{\nu_i}{\mu} = \frac{\delta_i}{\delta_{\text{anal}}} \).

Now \( t = nT \) where \( n \) is the number of cycles between \( t = 0 \) and \( t = t_m \) and \( T \) is the periodic time. Equation (A25) may then be written

\[
\log_e r = n \delta_{\text{anal}}(r-1)
\]  
\[\text{(A26)}\]
APPENDIX VIII

FREQUENCY SPECTRUM OF A SHOCK PULSE

The fourier transform of a shock pulse is a continuous function of frequency and is an indication of the excitation that is applied to the structure at any frequency. It has been shown by Rubin (93) that the frequency spectrum of a shock of time history $f(t)$ which is initiated at $t = 0$ and is of duration $\tau$ is given by:

$$ F(\omega) = \int_{0}^{\tau} f(t) e^{-j\omega t} dt $$ \hspace{1cm} (A27)

If from plate 5 the current decay is taken to be approximately exponential with zero rise time, then the magnetic pulse will also be of the same form such that

$$ f(t) = k e^{-\frac{t}{\tau}} \quad 0 \leq t < \tau $$

where $k$ and $\rho$ are constants.

For $f(t) = 0$ at $t = \tau$ then $\rho > 10$.

The zero frequency component of the frequency spectrum of a shock is equal to the area under the shock time history curve $f(t)$. Setting the area of the shock pulse to unity gives

$$ f(t) = \frac{k}{\tau} e^{-\frac{t}{\tau}} \quad 0 \leq t < \tau $$ \hspace{1cm} (A28)

From (A27) the frequency spectrum becomes;

$$ F(\omega) = \frac{k}{\tau} \int_{0}^{\tau} e^{(j\omega - \rho) \frac{t}{\tau}} dt $$
Integrating and taking the absolute value of the complex result yields:

\[ F(\omega) = \frac{\rho}{\sqrt{\omega^2 + \tau^2}} \]  \hspace{1cm} \text{(A29)}

Setting \( \rho = 10 \) for the lower boundary of the frequency spectrum then

\[ F(\omega) = \frac{10}{\sqrt{100 + \omega^2 \tau^2}} \]  \hspace{1cm} \text{(A30)}

Equation (A30) is shown graphically in Fig. 13.7.
APPENDIX IX

BIBLIOGRAPHY


51. R.F. Rissoné, J.J. Williams. Vibration of non-
   uniform Cantilever Beams. The Engineer.
   September 24th 1965.

   to Some Industrial Resonance Testing Techniques.

53. J.E. Ruzicka. Part I of ref. 42.

54. T. Alfrey. Mechanical Behaviour of High Plymers


56. H.K.P. Neubert. A simple Modal Representing Internal
   Damping of Solid Materials. Aeronautical Quart.

57. L.B. Cherry. Electro-magnetic Induction Damping of
   1960.

   Poissons Ratio for viscoelastic Materials. Mats.


