SECOND ORDER EFFECTS ON BLADE VIBRATION

A thesis for the degree of
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by

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SUMMARY

This thesis is mainly concerned with the second order effects on cubical oval shaped cross-section blading. A detailed analysis is made on the variations of blade constants with reference to thickness and width of cubical oval cross section blades. Differential equations of motion are derived based on strain, allowing for the effects of second order terms, pretwist and rate of pretwist.

A transformation method of solution of the differential equations which was developed by Dawson (17) is adopted for solving the equations. This method consists of transforming the original equations into a set of simultaneous first order differential equations and solving by a step-by-step finite difference process. The natural frequencies of vibration up to third mode are obtained for two sets of blades of equivalent width to thickness ratios of 15.7:1 and 7.9:1. In either case frequencies are calculated firstly including the second order terms and secondly neglecting the second order terms.

Three programs were developed by the author in ALGOL to calculate

i) the blade parameters

ii) to check the ordinates, and

iii) to compute the natural frequencies.
The theoretical results obtained are compared to measured frequencies and the effect of the second order terms is shown. The cubical oval cross-section blades were manufactured by a modified shaping machine and were excited by means of an electro-magnetic exciter.
ACKNOWLEDGEMENTS

The author wishes to express his thanks to Professor W Carnegie for giving permission to carry out the research work in the Mechanical Engineering Department of the University of Surrey and to both Professor W Carnegie and Dr B Dawson (now at the Polytechnic of Central London) for their valuable guidance and useful discussion on the work and helpful suggestions in preparing this thesis.

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LIST OF SYMBOLS

A : Area of cross section
A₁, A₂, A₃ etc : Constants, notations of equations
B₁, B₂ : Blade Constants, moments of higher order
B₁, B₂, B₃ etc : Constants, notations of equations
B₃, B₄ : Blade coefficients
C : Blade width
C₁, C₂, C₃ etc : Constants, notations of equations
C₀ : Torsional stiffness constant
D₁, D₂, D₃ etc : Constants, notations of equations
E : Modulus of Elasticity
G : Modulus of Rigidity, Earth's gravitational force
I₁, I₂ : Moments of Inertia about principal axes
I₃, I₄, I₅ : Moments of Inertia
ICF : Polar moment of Inertia about centre of flexure
ICg : Polar moment of Inertia about centre of gravity
Iₓₓ, Iᵧᵧ : Moment of Inertia about xx, yy axes
Iₜz : Moment of Inertia about zz axis
Iₓᵧ : Moment of Inertia about xᵧ axis
Iᵧᵧ : Moment of Inertia about yy axis
Iₓz : Moment of Inertia about xz axis
Iₓᵧ : Moment of Inertia about xᵧ axis
Iₓy : Moment of Inertia about xᵧ axis
Iᵧy : Moment of Inertia about yy axis
Iₓz : Moment of Inertia about xz axis
XYZ : Product moment of Inertia about YY and ZZ axes
Iₓ₁, Iᵧ₁ : Moment of Inertia about x₁, y₁ axis
Iz₁z₁ : Moment of Inertia about z₁z₁ axis
J : Torsion, torsional constant, constant
K : Cross section function associated with torsion
K_A : Radius of gyration
L : Length of the beam
M_1, M_2 : Moments
M_x, M_y, M_z : Moments
P : Frequency
Q : Moment, torsion along elastic axis
R : Radius of curvature
T : Torsion, tension
V_x, V_y, V_z : Shears
W : Weight per unit length
X : Co-ordinate X along XX axis
x_1 : Reference axis
x_1 : Reference axis
Y : Co-ordinate Y along YY axis
Y_0 : Reference axis through round nose
Y_1 : Reference axis through centre of gravity
Z : Co-ordinate Z along ZZ axis
Z_1 : Reference axis
Z_1 : Reference axis
a : Co-ordinate of elemental area from centre of flexure axis
a_1, a_2, a_3 etc : Arbitrary constants
a_r : Coefficient
ax, ay, az : Acceleration vectors
\( b \): Co-ordinate of elemental area from centre of flexure axis

\( b/d \): Equivalent width to depth ratio

\( c \): Constant, blade width

\( ds \): Elemental change in length of the fibre

\( e \): Distance between centre of gravity and centre of flexure

\( e_1 \): Strain

\( e_r \): Tensile strain

\( e_y, e_z \): Resolved components of \( e \)

\( f \): Fibre, frequency, stress

\( g \): Gravitational force

\( h \): Distance of centre of gravity from round nose

\( i \): Unit vector

\( j \): Unit vector

\( k \): Unit vector

\( l \): Length

\( m \): Mass per unit length

\( p \): Frequency

\( P_y \): Natural frequency of the fundamental mode

\( P/P_y \): Frequency ratio

\( \vec{P}_x, \vec{P}_y, \vec{P}_z \): Force loadings in \( x, y \) and \( z \) directions respectively

\( \vec{q}_x, \vec{q}_y, \vec{q}_z \): Moment loadings in \( x, y \) and \( z \) directions respectively

\( r, \dot{r}, \ddot{r} \): Acceleration vectors

\( r_x, r_y \): Co-ordinates of centre of flexure in \( x \) and \( y \) directions

\( r_X, r_Y \): Co-ordinates of centre of flexure in \( X \) and \( Y \) directions
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<td>$\beta$</td>
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<td>$\beta'$</td>
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<td>$\phi$</td>
<td>Torsion angle, torsional displacement</td>
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$\phi'$ : First derivative with respect to $x$
$\phi, \ddot{\phi}$ : First and second derivatives with respect to time
$\theta$ : Thickness parameter, angle
$\eta$ : Major axis
$\xi$ : Minor axis
$\eta_{te}$ : Trailing edge
$\eta_{le}$ : Leading edge
$\rho$ : Density
$\sigma$ : Stress, Poisson's ratio
$\omega$ : Frequency
$\Omega$ : Shearing stress function, notation
$\mu$ : Modulus of Rigidity
CHAPTER 1

INTRODUCTION, REVIEW AND SCOPE OF INVESTIGATION
1.0 Introduction, Review and Scope of Investigation

1.1 Introduction

The design engineers of turbo machinery are very concerned about the failure of turbine and compressor blades due to excessive vibration and subsequent fracture due to fatigue. When the frequency of the exciting force coincides with the natural frequency of a blade, resonance occurs causing large vibration amplitude and consequently failure of the blade. The turbo machinery design is such that a failure of a single blade can result in a total failure of the complete system. In general, the inherent damping of the blades is low and hence they respond readily to many sources of periodic and other disturbances inherent in turbo machinery.

In view of the ready response of blades to these excitations, it is essential at the design stage for the designer to determine within a reasonable degree of accuracy, the vibrational characteristics of the blading. A theoretical study of the problem is usually made by assuming the blades are cantilever beams. The problem is to predict the normal mode frequencies of cantilevers having complex shapes including pretwist.
Owing to the complexities each blade has an infinite number of normal mode frequencies each one being of the coupled bending-bending-torsion type. It has been shown by many previous authors that the blade vibration is only serious in the lower modes of vibration and it is therefore only necessary for the designer to attempt to predict the natural frequencies of the lower mode of vibration of blading.

The blading may be considered as cantilever beams and methods of determining natural frequencies of pretwisted cantilever beams has received considerable interest in the recent years due to the direct application to turbine and compressor blades as well as aircraft propellers. Many contributions were made in analysing vibrational characteristics of pretwisted cantilever beams and in particular beams of rectangular cross section. Different methods of deriving the coupled differential equations of motion and of solving those differential equations either analytically or numerically have been considered by most of the previous authors. However, invariably solutions have been given for simplified equations neglecting most of the higher order terms which appear in the differential equations.

The object of the present work is to study the vibrational characteristics of non-rotating straight and pretwisted uniform cantilever beams of aerofoil cross section having an explicitly defined geometrical shape namely a cubical oval cross section and in particular to determine the effect upon the lower
natural frequencies of the terms considered second order and to see how the effect is dependent upon the blade parameters. For cubical oval cross section beams, the theoretical frequency ratio \( P/P_y \) for the first three natural frequencies are obtained for beams of different width to thickness parameter ratio pretwisted within the range of 0-90 degrees. The theoretical results are compared to experimental results, and the effect of the second order terms assessed.

The beams used in the experiment are of drawn mild steel and all the beams were obtained from the same stock.

1.2 Review of previous work, and scope of investigation

Due to failure of turbine and compressor blading because of vibration, considerable interest has been shown in recent years in the study of vibrational characteristics of pretwisted beams. Turbine and compressor blading are of aerofoil cross section and the vibration is of the coupled type. When a uniform asymmetrical aerofoil cross section beam, as shown in figure 1(a), is subjected to vibration the motion consists of displacements in two perpendicular directions at the same time coupled with torsion. This is termed bending-bending-torsion vibration. In the case of beams having one axis of symmetry as shown in figure 1(b), the bending vibration in \( z \) direction is an independent bending vibration. On the other
hand bending vibration in the y direction occurs simultaneously with a torsional vibration and is referred to as bending-torsion vibration.

For the case of rectangular cross-section beams, the bending vibrations in the two perpendicular principal planes occur independently as well as an independent torsional vibration. If a rectangular blade is initially pretwisted then the two bending vibrations couple together resulting in a vibration of the bending-bending type, but the torsional vibration remains uncoupled. The initial pretwist of beams of aerofoil cross-section with one axis of symmetry, causes the bending-torsion vibrations to couple with the previously independent bending vibration resulting in coupled vibration of the bending-bending-torsion type. The initial pretwist of asymmetrical aerofoil cross-section beams will affect the values of the natural frequencies, but the vibration will be of the bending-bending-torsion type.

The factors affecting the vibrational characteristics of aerofoil cross-section beams can be appreciated by studying the differential equations of motion. The equations contain coupling terms dependent upon the co-ordinates of the centre of flexure relative to the centre of gravity of the cross-section of the blade. Complete solution of equations therefore requires a knowledge of the centre of flexure and its co-ordinates. Different terms have been used by different authors such as shear centre, flexural centre, etc, to define the centre of
flexure. Also different definitions have been given by different authors. The definition given by Carnegie (6) is used in the present work and is quoted below.

'When a cantilever blade of elastic material is supported rigidly at the root and loaded at the free end with a concentrated load normal to the longitudinal axis, then in general, lateral displacement and twist of normal cross-sections relative to one another will occur. Application of the load at one particular point in the free end cross-section (considered not to distort) will not cause twist and this point is defined as the centre-of-flexure of that cross-section. When the blade is subjected to distributed transverse loads the centre-of-flexure may depend on the load distribution, but for a long blade the dependence will not be appreciable.'

Another term which is closely related to the centre-of-flexure is known as centre-of-torsion and the definition of the same is quoted from Carnegie (6).

'When a long cantilever blade of elastic material mounted as described above, has a purely torsional couple, that is to say a couple acting in a plane normal to the blade longitudinal axis, applied at its free end then the point of zero displacement of the free end cross-section (considered not to distort) in its own plane will be the centre-of-torsion of that cross-section.'

This phenomenon has been referred to as torsion centre or centre of twist by different authors.
Many authors have considered the relative positions of the
centre-of-flexure and centre-of-torsion. Duncan and others (11)
argued that in an ideal case the centre-of-twist must coincide
with the centre-of-flexure. In the present investigation also
it is assumed that centre-of-flexure coincides with the
centre-of-twist.

A definition similar to that of Carnegie was given by Duncan,
Ellis and Scruton (11) for centre-of-flexure. Osgood (10)
produced a unique definition for centre-of-flexure for any
cross-section. The relevance of Poisson's ratio in determining
the co-ordinates of centre-of-flexure was discussed at length
by Osgood (10), Timoshenko (15) and Duncan (12).

If transverse loads are applied to a beam which has an asymmetric
cross-section then flexure will occur followed by torsion. It is
therefore necessary to understand the response of beams subjected
to torsion. Long ago this torsional problem was assumed to be
covered by the theory of St Venant which predicted the shear
stress distribution and the angle of twist of a beam under
torsion. The assumptions were that the torque along the beam was
constant (ie) equal and opposite couples applied at each end;
that these couples were also the resultants of shear stresses
distributed over every cross-section of the beam, and that the
end sections were free to warp. Analysis based on this theory
is known as St Venant torsion.

A solution of co-ordinates of centre-of-flexure based on
St Venant's theory of torsion was presented by Duncan (12) for
beams having one axis of symmetry. The accuracy of the above theory is verified by Duncan and others (11) by experimentally determining the co-ordinates of centre-of-flexure of aerofoil cross-section having one axis of symmetry.

It is usual to assume that for a uniform cantilever beam the locus of the centres-of-flexure is a straight line parallel to the centroidal axis. Though some authors stated that the locus of centre-of-flexure need not be a straight line but depends on the load conditions and load distribution, the assumption is made for the present work that the locus of centres-of-flexure is a straight line. This locus is also called the shear centre axis or flexural axis. This axis is taken as one of the reference axes in the present work.

In most of the cases the centroid and centre-of-flexure may not coincide and consideration of the dynamics of such a cantilever shows that transverse and torsional vibrations of the beam cannot exist independently. An investigation of this problem was carried out by Garland (25) who applied the Rayleigh-Ritz method for solution. In 1945 Timoshenko (35) presented an excellent unification of theories relating to the bending and torsion of open thin walled members.

A large amount of literature is available concerning in particular the vibrational characteristics of cantilever beams which are considered similar to turbine blades. In general the investigators have been concerned with determination
of the normal mode frequencies and mode shapes of vibration. Different methods of analysis of the vibration problem of cantilever beams have been developed by different authors, namely setting up of the differential equations of motion and their subsequent solution by various means, obtaining solutions to the problem by application of methods based on Rayleigh energy approach and the use of finite element theory etc.

Most turbine blades possess a complex form which may include asymmetric aerofoil section, taper pretwist etc and hence many investigators have attempted to examine the effects of various parameters on the natural frequencies of the blading. By using a variety of approaches the effect of blade pretwist has been examined by a number of investigators \((1, 2, 4, 9, 13, 18, 20, 24, 28, 29, 34)\). The effect of taper has been examined by some authors \((3, 29, 36, 37)\). In many of the analyses higher order solutions have been obtained by considering additional effects of rotary inertia and shear deformation of the blade \((1, 8, 9, 18)\).

Mendalson and Gendler \((34)\) adopted a method for solving the equations of motion by means of station functions. They worked out the first three natural frequencies of a rectangular beam of width to thickness ratio 12:1 for pretwist angle in the range of 0 to 60 degrees. The first two natural frequencies were compared with experimental results.
Rosand (28) determined the natural frequencies experimentally for beams of width to thickness ratios in the range 4:1 to 12:1 and pretwist angle up to 40 degrees. Experimental results were compared to the theoretical results obtained by Myklestad method. Calculated mode shapes were presented for the first three modes of vibration of an 8:1 width to thickness ratio beams with a pretwist angle of 40 degrees. Experimental mode shapes were not determined. The methods of excitation and vibration detection were not that refined and hence the accuracy of the experimental results is open to doubt.

Houbolt and Brooks (1) derived the equations of motion for pretwisted beams whose mass and elastic axes were non collinear. The equations of motion allowed for variable mass per unit length and stiffness and also included many second order terms. A solution by Rayleigh-Ritz method was indicated but no attempt at a solution was made for a practical case.

Applying the Rayleigh-Ritz principle and using fourth order Polynomials as approximations to the dynamic displacement curves, Diprima and Handelman (40) solved the differential equations of motion. Only the fundamental frequencies were obtained for a set of pretwisted beams of various width to thickness ratios with pretwist angles up to 29 degrees. No experimental conformation of the results was presented. Carnegie (5) used an energy method to determine the equations of motion of pretwisted blades and later on extended the method to include the effect of rotation. Dawson (8, 9)
considered in addition the effects of rotary inertia and shear deformation and then used the Rayleigh-Ritz energy method to obtain solutions.

The differential equations of motion for cantilever blading including pretwist were derived by Carnegie (4, 5) and solved by using the calculus of variations. This same theory was extended later to include the effects of rotary inertia, shear deformation, torsion bending and blade deformation.

Carnegie, Dawson and Thomas (3) and Carnegie and Thomas (36, 37) included the use of finite differences to reduce the differential equations to a set of algebraic equations and then used an iterative procedure to obtain the eigenvalues. Much attention has been paid to the finite element technique in recent years. Slyper (20) used this technique in investigating the coupled bending vibration of pretwisted cantilever beams.

Dawson (17) developed a method known as the transformation method to solve the equations of motion of pretwisted aerofoil cross-section beams. The transformation method is of direct interest to this project and is explained in detail in Appendix II. In short the method involves the reduction of the coupled higher order differential equations to a set of simultaneous differential equations of first order. A set of simultaneous solutions of these equations is then obtained on a computer using the Runge-Kutta step-by-step integration
procedure. The process is an iteration procedure which relies on repeated assumptions of the value of the normal mode frequency. The actual normal mode frequency is eventually determined by extrapolation of the results.

Dawson (17) applied the transformation method to solve the differential equations of motion of rectangular and aerofoil cross-section beams and later Ghosh (18) extended the method to allow for shear deformation and rotary inertia. Hemingway (16) used the transformation method to solve the differential equations of coupled vibration of an open tube. He considered only straight tubes and the effects of pretwist were not dealt with by him.

The equations of motion for pretwisted aerofoil cross-section cantilever beams have been derived by Montoya (2) which include second order terms. The derived equations of motion were solved by a similar method to the transformation method presented in this thesis. Natural frequencies were obtained up to the seventh mode of vibration for one actual turbine blade of tapered aerofoil cross-section and 72 cms long. A comprehensive study of the effect of the second order terms was not however made.
The present investigation is concerned with the effect of terms previously considered as second order terms upon the natural frequencies of cantilever blading. In order to study the effects of such terms, cubical oval cross section blades were chosen for the investigation since the position of the centre of flexure of such blading could be calculated theoretically and it has one axis of symmetry, thus reducing the complexity of the coupled equations of motion.

The first part of the work is concerned with an examination of the interrelation of the various parameters of the cubical oval cross section blading for example moments of inertia, centre of flexure position, area, torsional stiffness etc. Analytical formulas for these parameters were derived and a computer program was developed that would yield all parameters and copies of the program are presented for reference in Appendix IIIa. Variations of these parameters were analysed for equivalent b/d ratios ranging from 15.7:1 to 5.2:1 by variation of the thickness parameter and the chord. These results are presented in graphical form in figures 3-12 and the interrelations of the various parameters can be appreciated by reference to these figures.

The differential equations of motion allowing for higher order terms were derived based on the work of Houbolt and Brooks (1) and these equations are compared to the equations derived by Montoya (2) who allowed for higher order terms and Dawson (17)
who did not allow for these terms. The differential equations
of motion were solved by transforming the differential
equations and solving by a Runge-Kutta step-by-step integration
process. This method is referred to throughout this thesis as
the transformation method. The method has the important
advantage on most other methods in that it can be readily
extended to include rotary inertia, shear deformation, linear
taper of a beam and centrifugal tensile effects.

The transformation method has been adopted in this work because
of the adaptability with which it can be extended to allow for
the second order terms. The natural frequencies of vibration
up to third mode have been calculated by the transformation
method for beams of equivalent width to thickness ratios of
15.7:1 and 7.9:1 and pretwist angle from zero to 90 degrees
both with and without allowing for second order terms. The
theoretical results both with and without second order terms
are compared with experimental results obtained on sets of
machined cubical oval cross section pretwisted beams of
corresponding width to thickness ratios and pretwist angles
and the effect of the second order terms assessed.
ASYMMETRIC AEROFOIL CROSS-SECTION BEAM

Figure 1(a)

BEAM WITH ONE AXIS OF SYMMETRY

Figure 1(b)
CHAPTER II

ANALYSIS OF BLADE CONSTANTS
2.0 Analysis of Blade Constants

It can be noticed from the differential equations of motion (Appendix I) that they are complex functions involving the moments of inertia of the cross-section, distance of centre-of-flexure from centre of gravity, torsional stiffness constant, coefficients of coupling terms etc. Variations in any one or all terms would have an effect upon the value of the natural frequencies. It is therefore of interest to establish the inter-relations of the various properties of the section such as area, position of centre-of-flexure, position of centre of gravity, principal moments of inertia, etc. The cross-section shape of the blade was chosen as a cubical oval, since all the blade constants could be derived as explicit mathematical functions and variations in the parameters easily made.

The formula for the cubical oval adopted from Duncan (12) is

\[
\left( \frac{t}{c\theta} \right)^2 = \frac{z}{c} \left( 1 - \frac{z}{c} \right)^2
\]

......... (1)

where \( t \) is the half thickness at a distance of \( z \) measured from round nose, \( c \) is the width or chord of the blade and \( \theta \) is the thickness parameter. A typical shape of cubical oval is shown in figure 1(c). The cubical ordinates worked out from the above formula for two sets of blades are presented in table 1.
The derivations of different blade constants are carried out as follows.

2.1 Maximum thickness of the blade

From the general equation of the boundaries for a cubical oval beam it is possible to derive a formula for maximum thickness

\[ t = c\theta \left( 1 - \frac{Z}{c} \right) \frac{VZ}{c} \]  

................ (2)

Differentiating with respect to \( z \)

\[ \frac{dt}{dz} = c\theta \left[ \frac{1}{2} \frac{c}{VZ} - \frac{1}{2} \frac{c}{Z} \right] \]

\[ = \frac{c\theta}{2c^{1/4}} \left[ \frac{c}{VZ} - 3VZ \right] \]

For maximum value \( \left( \frac{c}{VZ} - 3VZ \right) = 0 \) or \( (c - 3z)z = 0 \)

ie either \( z = 0 \) or \( z = \frac{1}{3} c \)

Hence thickness will be maximum at \( \frac{1}{3} \) of chord or width of the blade measured from round nose. Substituting this value of \( z \) in formula (2)

\[ t_{\text{max}} = c\theta \left( 1 - \frac{1}{3} \right) \frac{1}{3} \frac{1}{3} = \frac{2c\theta}{3\sqrt{3}} \]
Maximum thickness of the blade \( t_{\text{max}} = \frac{2c\theta}{\sqrt{27}} \) ........ (3)

2.2 Area of Cross Section

Consider a small element of \( dz \) as shown in figure 1(c) at a distance of \( z \) measured from round nose. Area of the finite strip is \( 2t\,dz \). Area of the cubical oval would be the integral of such strips between \( z = 0 \) and \( z = c \).

\[
A = \int_{0}^{c} 2t \, dz
\]

Substituting the value of \( t \) from equation (2)

\[
A = \int_{0}^{c} 2c\theta \left( 1 - \frac{z}{c} \right) \sqrt{\frac{z}{c}} \, dz
\]

\[
= 2\theta \sqrt{c} \int_{0}^{c} \left( z - \frac{3}{8} \frac{z^3}{c} \right) \, dz
\]

\[
= 2\theta \sqrt{c} \left[ \frac{z^{3/2}}{3/2} - \frac{3}{8} \frac{z^5}{5c} \right]_{0}^{c}
\]

\[
= 2\theta \sqrt{c} \left[ \frac{2c^{3/2}}{3} - \frac{3}{8} \frac{c^5}{5c} \right]
\]

\[
= \frac{8}{15} \theta \sqrt{c} \cdot \frac{c^3}{3}
\]

Area \( A = \frac{8}{15} \theta c^2 \) .............. (4)
2.3 Position of Centre of Gravity

If the shape of the cross-section was irregular or asymmetrical then both \( y \) and \( z \) co-ordinates have to be determined. Now that the shape is symmetrical about \( z \) axis the \( y \) co-ordinate of centre of gravity namely \( \overline{y} = 0 \). Hence it becomes necessary to find out only \( \overline{z} \). Considering an element \( dz \) as shown in figure 1(c) at a distance of \( z \) from round nose, the position of centre of gravity from round nose is given by

\[
\overline{z} = \frac{\int_0^c z \cdot 2t \, dz}{\int_0^c 2t \, dz}
\]

\[
= \frac{\int_0^c \theta \int_0^c z (1 - \frac{z}{c}) \, dz}{\int_0^c \theta \int_0^c z (1 - \frac{z}{c}) \, dz}
\]

\[
= \frac{\int_0^c z^\frac{3}{2} (1 - \frac{z}{c}) \, dz}{\int_0^c z^\frac{1}{2} (1 - \frac{z}{c}) \, dz}
\]

\[
= \frac{\int_0^c (z^\frac{3}{2} - \frac{z^\frac{5}{2}}{c}) \, dz}{\int_0^c (z^\frac{1}{2} - \frac{z^\frac{3}{2}}{c}) \, dz}
\]

\[
= \frac{\left[ \frac{2}{5} z^\frac{5}{2} - \frac{2}{7} z^\frac{7}{2} \right]_0^c}{\left[ \frac{2}{5} z^\frac{3}{2} - \frac{2}{7} z^\frac{5}{2} \right]_0^c}
\]

\[
= \frac{\left[ \frac{2}{5} c^\frac{5}{2} - \frac{2}{7} c^\frac{7}{2} \right]}{\left[ \frac{2}{5} c^\frac{3}{2} - \frac{2}{7} c^\frac{5}{2} \right]}
\]

\[
= \frac{\left( \frac{2}{5} c^\frac{5}{2} - \frac{2}{7} c^\frac{7}{2} \right)}{\left( \frac{2}{5} c^\frac{3}{2} - \frac{2}{7} c^\frac{5}{2} \right)}
\]

\[
= \frac{3}{7} c
\]

\[\text{(5)}\]
2.4 Co-ordinate System and Moments of Inertia

The general co-ordinate system of a beam of one axis of symmetry is shown in figure 2 with principal axes ZZ and YY. The angle of pretwist is taken as $\beta^{o}$. From the figure it can be noted that

\[ a = Z \cos \beta - Y \sin \beta \quad \text{and} \quad b = Z \sin \beta + Y \cos \beta \]

\[ I_{zz} = \int \int b^2 dA = \int \int b^2 dZ dY \]

where \( Z \) and \( Y \) are co-ordinates with reference to principal axes.

\[ I_{zz} = \int \int (Z^2 \sin^2 \beta + Y^2 \cos^2 \beta + 2ZY \sin \beta \cos \beta) \, dZ dY \]

\[ = I_{yy} \sin^2 \beta + I_{zz} \cos^2 \beta + I_{zy} \sin 2\beta \]

The product moment of inertia \( I_{zy} \) about principal axis is zero.

Hence \( I_{zz} = I_{yy} \sin^2 \beta + I_{zz} \cos^2 \beta \) \hspace{1cm} (6)

Similarly \( I_{yy} = \int \int a^2 dA = \int \int a^2 dZ dY \)

\[ = \int \int (Z^2 \cos^2 \beta + Y^2 \sin^2 \beta - 2ZY \sin \beta \cos \beta) dZ dY \]

\[ = I_{yy} \cos^2 \beta + I_{zz} \sin^2 \beta - I_{zy} \sin 2\beta \]

As \( I_{zy} = 0 \) \( I_{yy} = I_{yy} \cos^2 \beta + I_{zz} \sin^2 \beta \) \hspace{1cm} (7)
Product moment of area about reference axes \( zz \) and \( yy \)
is given by
\[
I_{zy} = \int \int_{A} ab \, dA = \int \int_{A} \, ab \, dZ \, dY \quad \ldots \quad (8)
\]

Substituting the values of \( a \) and \( b \), equation (8) becomes

\[
I_{zy} = \int \int (Z \sin\beta + Y \cos\beta)(Z \cos\beta - Y \sin\beta) \, dZdY
\]
\[
= \int \int (Z^2 \sin\beta \cos\beta - Y^2 \sin\beta \cos\beta + ZY \cos^2\beta - ZY \sin^2\beta) \, dZdY
\]
\[
= I_{YY} \sin\beta \cos\beta - I_{ZZ} \sin\beta \cos\beta + I_{ZY} \cos^2\beta - I_{ZY} \sin^2\beta
\]

As \( I_{ZY} = 0 \);
\[
I_{zy} = I_{YY} \sin\beta \cos\beta - I_{ZZ} \sin\beta \cos\beta
\]

\[
I_{zy} = \frac{I_{YY} - I_{ZZ}}{2} \sin 2\beta \quad \ldots \quad (9)
\]

From figure 2 it can be noted that

\[
\ell_y = \varepsilon \sin\beta \quad \text{and} \quad \ell_z = \varepsilon \cos\beta \quad \ldots \quad (10)
\]

Carnegie's derivation of \( \gamma_x \) and \( \gamma_y \) co-ordinates (4) are

\[
\gamma_x = \gamma_x \cos\alpha - \gamma_y \sin\alpha \quad \text{and} \quad \gamma_y = \gamma_x \sin\alpha + \gamma_y \cos\alpha
\]

owing to one axis of symmetry the term \( \gamma_y = 0 \).

Hence it can be seen that centre of flexure co-ordinates are similar to that of Carnegie.
To find out reference moments of inertia $I_{zz}$, $I_{yy}$ and the product moment of inertia $I_{zy}$ only the moments of inertia about principal axes namely $I_{zz}$ and $I_{yy}$ and the pretwist angle $\beta$ need to be known. This can be seen from equations (6) - (8).

2.4(1) Moment of Inertia about Axes through Centre of Gravity
(but Parallel to axes through centre of flexure)

\[
I_1 = \int_0^C \frac{1}{12} (2t)^3 \, dz
\]

\[
= \frac{8}{12} \int_0^C t^3 \, dz
\]

Substituting the value of $t$ from equation (1)

\[
I_1 = \frac{2}{3} \int_0^C c^3 \theta^2 z (1 - \frac{z}{c})^2 \, dz
\]

\[
= \frac{2}{3} c \frac{1}{12} \theta^3 \int_0^C \left( \frac{z^{11}}{2} - \frac{3z^{23}}{2c} + \frac{3z^{31}}{c^2} - \frac{z^{41}}{c^3} \right) \, dz
\]

\[
= \frac{2}{3} c \frac{1}{12} \theta^3 \left[ \frac{z^{21}}{2^2} - \frac{3z^{31}}{4c} + \frac{3z^{41}}{4c^2} - \frac{z^{51}}{4c^3} \right]
\]

\[
= \frac{2}{3} c \frac{1}{12} \theta^3 \times \left[ \frac{2c^{21}}{5} - \frac{6c^{21}}{7} + \frac{6c^{21}}{9} - \frac{2c^{21}}{11} \right]
\]

\[
= \frac{4}{3} \theta^3 c^4 \left[ \frac{8}{15} - \frac{40}{77} \right]
\]

\[
= \frac{64}{3465} \theta^3 c^4
\]

........................... (11)
I_2, moment of Inertia can be derived as follows.

From figure 2, \( I_{y_1y_1} \) or \( I_2 \) is equivalent to \( I_{y_0y_0} \cdot \text{Area} \times h^2 \)
where \( h \) is the distance of centre of gravity from round nose.

\[
I_{y_0y_0} = \int_{c}^{z^2(2t)} dz
\]

\[
= 2 \Theta \int_{0}^{c} \int_{0}^{c} z^2 (1 - \frac{z}{c}) z^2 dz
\]

\[
= 2 \Theta \int_{0}^{c} \left[ \frac{z^{3\frac{1}{2}}}{3\frac{1}{2}} - \frac{z^{4\frac{1}{2}}}{4\frac{1}{2}c} \right]_0^c
\]

\[
= 2 \Theta \int_{0}^{c} c^{3\frac{1}{2}} \left[ \frac{2}{7} - \frac{2}{9} \right]
\]

\[
= \frac{8}{63} \Theta c^4 \quad \text{.......................... (12)}
\]

\[
I_2 = I_{y_0y_0} - Ah^2 \quad \text{.......................... (13)}
\]

Substituting the values of \( A \) and \( h \) from formulas (4) and (5)

\[
I_2 = \frac{8}{63} \Theta c^4 - \frac{8}{15} \Theta c^2 \left[ \frac{3}{7c} \right]^2
\]

\[
= \frac{8}{63} \Theta c^4 - \frac{8}{15} \Theta c^2 \left[ \frac{9}{49} \right]
\]

\[
= \Theta c^4 \left[ \frac{8}{63} - \frac{72}{15 \times 49} \right]
\]

\[
= \frac{64}{2205} \Theta c^4 \quad \text{.......................... (14)}
\]
Similarly moment of inertia about principal axis YY can be found by substituting the value of distance of centre of flexure from round nose in the formula (13). Now h becomes $(\frac{3}{7}c - \ell)$ where $\ell$ is the distance between centre of gravity and centre of flexure.

\[
I_{YY} = \frac{8}{63} \Theta c^4 - \frac{8}{15} \Theta c^2 \left[ \frac{3}{7}c - \ell \right]^2 \quad \cdots \cdots \quad (15)
\]

2.4(2) Polar Moment of Inertia about Centre of Flexure

By perpendicular axes theorem of inertias, polar moment of inertia about centre of flexure is the sum of principal moments of inertia.

\[
I_{CF} = I_{ZZ} + I_{YY}
\]

\[
= \frac{64}{3465} \Theta^3 c^4 + \frac{8}{63} \Theta c^4 - \frac{8}{15} \Theta c^2 \left( \frac{3}{7}c - \ell \right)^2 \quad \cdots \cdots \quad (16)
\]

2.4(3) Radius of Gyration

Polar radius of gyration can be worked out since the value of polar moment of inertia is known.
\[ I_{CF} = \text{Area} \times (\text{radius of gyration})^2 \]

\[ = A \times K_A^2 \]

\[ \therefore K_A^2 = \frac{I_{CF}}{A} \]

Substituting the values of \( I_{CF} \) and \( A \) from equations (16) and (4)

\[ K_A^2 = \frac{64}{3465} \theta^3 c^4 + \frac{8}{63} \theta c^4 - \frac{8}{15} \left( \frac{3}{7} c - e \right)^2 \theta c^2 \]

\[ = \frac{8}{231} \theta c^2 + \frac{15}{63} c^2 - \left( \frac{3}{7} c - e \right)^2 \]

\[ \therefore K_A = \sqrt{\frac{8}{231} \theta c^2 + \frac{15}{63} c^2 - \left( \frac{3}{7} c - e \right)^2} \]

\[ \ldots \ldots \ldots \ldots \ldots (17) \]

2.5 Centre of Flexure

Many of the previous investigators contributed towards a better understanding of the position of the centre of flexure. However, the definition quoted by Carnegie (6) is adopted throughout this investigation which is quoted here again for reference.

'When a long cantilever blade of elastic material is supported rigidly at the root and loaded at the free end with a
concentrated load normal to the longitudinal axis then, in general, lateral displacement and twist of normal cross-sections relative to one another will occur. Application of the load at one particular point in the free-end cross-section (considered not to distort) will not cause twist and this point is defined as the centre-of-flexure of that cross-section. When the blade is subjected to disturbed transverse loads the centre-of-flexure may depend on the load distribution, but for a long blade the dependence will not be appreciable.*

In short the flexural centre can be defined as the point of application of the load which produces pure flexure without twist. When the stresses are distributed in such a way that the resultant shearing force cuts the shear centre axis, then there occurs bending without twisting. An important solution was obtained by St Venant. This solution has the characteristic that all the shearing stresses are independent of the distance from the base and it must therefore correctly represent the state of stress and strain in a long beam. Hence St Venant's theory may be made a basis of a theoretical determination of the position of centre of flexure of a thin long beam.

A point has to be added here about the selection of co-ordinate axis for the present investigation. From the definition of centre of flexure it is known that when there is bending without twist at a particular load, the resultant shearing force
cuts through the shear centre. If the co-ordinate axis is selected in such a way that x axis coincides with the locus of shear centres or elastic axis, then effects due to the resultant shears can be ignored. This was one of the reasons why co-ordinate axes system is assumed to pass through the locus of centre of flexure.

Early authors quoted in (12) Griffith and Taylor assumed that position of centre of flexure did not depend on the Poissons ratio of the material of the beam whereas later authors like Duncan (12) assumed that the position of centre of flexure depends also on the value of Poisson's ratio for the material of the beam.

Duncan (12) has derived a formula for the position of centre of flexure. When the section of the beam has an axis of symmetry and is free from sharp angles (except on the axis of symmetry at the ends), he showed that the flexural centre lies on the axis at a distance of \( \bar{z} \) from the centroid given by the formula

\[
\bar{z} = \frac{\int z t^3 dz}{\int t^3 dz} - \frac{3 \int \int \sigma_{xy} dz dy}{\int t^3 dz}
\]

Here \( t \) is the half thickness of the section at a distance \( z \) from the centroid and \( \sigma_{xy} \) is a shearing stress function which vanishes on the boundary. In the limiting condition when the thickness is very small \( \bar{z} \) has been proved by Duncan.
and others (11) to be

\[
\bar{z} = (1 + 3\sigma) \int z t^3 dz / (1 + \sigma) \int t^3 dz \quad \ldots \ldots \ldots \quad (19)
\]

If the assumption of Griffith and Taylor (12) namely that Poisson's ratio need not be considered for the calculation of position of centre of flexure then formula (19) reduces to

\[
\bar{z} = \int z t^3 dz / \int t^3 dz
\]

2.5(1) Position of Centre of Flexure based on Griffith's Assumption

Considering the cubical oval section figure 1(c) the position of centre of flexure from round nose becomes

\[
\bar{z} = \int_0^C z t^3 dz / \int_0^C t^3 dz
\]

\[
= \frac{\int_0^C \left[ \frac{c^2 \varphi^2 Z}{C} \left( 1 - \frac{Z}{C} \right)^2 \right] c \varphi \frac{Z}{C} \left( 1 - \frac{Z}{C} \right) dz}{\int_0^C \frac{c^2 \varphi^2 Z}{C} \left( 1 - \frac{Z}{C} \right)^2 \frac{c \varphi \frac{Z}{C}}{C} \left( 1 - \frac{Z}{C} \right) dz}
\]

\[
= \frac{c^2 \varphi^2 c \varphi}{c^2 \varphi^2 c \varphi} \int_0^C \frac{Z}{C} \left( 1 - \frac{Z}{C} \right)^2 \frac{Z}{C} \left( 1 - \frac{Z}{C} \right) dz
\]

\[
= \frac{c^2 \varphi^2 c \varphi}{c^2 \varphi^2 c \varphi} \int_0^C \frac{Z}{C} \left( 1 - \frac{Z}{C} \right)^2 \frac{Z}{C} \left( 1 - \frac{Z}{C} \right) dz
\]
Hence the approximate position of centre of flexure of the cubical oval section of equation (1), is at a distance of \( \frac{5}{13}c \) from round nose.

It has been already shown that the centre of gravity is at a distance of \( \frac{3}{7}c \) from round nose. Hence the distance of centre of flexure from centre of gravity can be written as

\[
\frac{3}{7}c - \frac{5}{13}c = \frac{4}{91}c 
\]

A formula for the position of centre of flexure after the centre of gravity is given by Duncan (12)

\[
z = -\frac{4c}{91(1+\sigma)} \left\{ 1 + 3\sigma - 6\sigma^2 + \left(\frac{2026}{17} - 56\right)\sigma^4 \right\} \quad \ldots \quad (21)
\]

Incidentally it can be noted from the above formula that if one assumes a very thin section and follows Griffiths and Taylor's
assumption then Θ and σ can be regarded as zero. Then the above formula (21) reduces to \( z = -\frac{4c}{9} \) which is the same as (20a), negative sign indicating that centre of flexure is behind centre of gravity. One can safely assume that for very thin beams the position of centre of flexure is directly proportional to the width. On the other hand for non-thin beams the position of centre of flexure becomes a complex function of Θ and c. Variation in value of Θ could considerably affect the centre of flexure co-ordinate. Separate tables and figures in graphical form are presented at the end of this chapter which indicate how the width of the blade and the thickness parameter Θ could affect the position of centre of flexure.

2.6 Torsional Stiffness of Cubical Oval Beam

St Venant's theory of torsion states that the stresses and strains and torsional stiffness of the beam are quite independent of the choice of the axis. According to St Venant torsional moment is given by

\[ M_x = GJ \frac{d\phi}{dx} \]

where G is the modulus of rigidity and J is termed as St Venant's torsional constant. GJ is regarded as torsional
stiffness and in the present investigation it is quoted as
C10. The torsional stiffness C10 of the beam whose cross-
section has the formula (1) is derived by Duncan (12) and
the same is adopted for the present work.

\[
\begin{align*}
C10 &= \frac{256\mu c^4 \phi^3}{3465} \left\{ 1 - \frac{11}{13} \phi^2 + \frac{379}{221} \phi^4 \right\} \\
\end{align*}
\]

(22)

Though Carnegie (5) derived a modified formula for increased
torsional stiffness due to pretwist angle, the formula in
(22) is adopted throughout this investigation.

Roark (33) gives a very simple formula for stiffness in torsion
from the fundamental torsion equation \( \Theta = \frac{TL}{K\phi} \) where \( \Theta \) is the
twist angle, \( T \) is the twisting moment, \( L \) is the length, \( K \) is a
cross-section function and \( G \) is the modulus of rigidity. He
gives a value for \( K \) as

\[
K = \frac{4 I_{zz}}{(1 + \frac{16 I_{zz}}{A c^2})} \\
\]

(23)

for cubical oval cross-section. Computed results of \( K \times G \) for
different sections of cubical ovals compare very well with
the values of C10 calculated from the formula (22).

2.7 Derivation of Coefficients of Coupling B1 and B2

The blade constants B1 and B2 are defined by Houbalt and
and Brook (1) as integral functions of blade dimensions. These integral functions are shown in Appendix ID. They are as follows:

\[ \int \eta \cdot \left( \eta^2 + \frac{t^2}{6} - K^2 \right) \cdot d\eta \quad \text{and} \quad \int \eta \cdot \left( \eta^2 + \frac{t^2}{12} - K^2 \right) \cdot d\eta \]

where \( \eta \) represents principal axis and \( t \) is the thickness of the blade at any section at a distance \( \eta \) from centre of flexure and \( K^2 \) is the square of polar radius of gyration about centre of flexure. The value of \( t \) is taken from equation (1) and the value of \( K^2 \) for a cubical oval is already derived vide equation (17). These blade constants are also termed as moments of higher order or coefficients of coupling. Integration of these terms are carried between the limits 0 to \( c \).

Now \( B_1 = \int_0^c tz^2\left(z^2 + \frac{t^2}{6} - K^2\right) \, dz \)

Substituting the value of \( t \) from equation (1)

\[ B_1 = \int_0^c 2\sqrt{c} \, \theta \sqrt{z} \cdot (1 - \frac{z}{c})z^2 \cdot \left[ z^2 + \frac{2}{3} \cdot \cos^2(z) \cdot (1 - \frac{z}{c})^2 - K^2 \right] \, dz \]

\[ = 2\sqrt{c} \int_0^c \left[ \left(z^\frac{1}{2} - \frac{5z}{c} \right) + \frac{2}{3} \cdot \cos^2 \left(z^\frac{1}{2} - \frac{3z^\frac{1}{2}}{c} + \frac{3z^\frac{5}{2}}{c^2} - \frac{6z^\frac{1}{2}}{c^3} \right) \right] dz \]
\[
B_2 = \int_0^C tz(z^2 + \frac{t^2}{12} - K_A^2) \, dz
\]

Substituting the values of \(t\) from equation (1)

\[
B_2 = \int_0^C 20\sqrt{c} \sqrt{z (1 - \frac{z}{c})} \left[ z^2 + \frac{1}{3} c^2 z (1 - \frac{z}{c})^2 - K_A^2 \right] \, dz
\]

\[
= 20\sqrt{c} \int_0^C \left[ (z^{\frac{3}{2}} - z^{\frac{4}{2}}) + \frac{1}{3} c^2 z (z^{\frac{2}{2}} - \frac{3}{2} z^{\frac{3}{2}} + \frac{3}{2} z^{\frac{4}{2}} - \frac{5}{2} z^{\frac{5}{2}}) - K_A^2 (z^{\frac{1}{2}} - z^{\frac{2}{2}}) \right] \, dz
\]

\[
= 20\sqrt{c} \left[ \left( \frac{z^{\frac{4}{2}}}{4^2} - \frac{z^{\frac{5}{2}}}{5^2 c} \right) + \frac{1}{3} c^2 (z^{\frac{3}{2}} - \frac{3}{4^2} z^{\frac{4}{2}} + \frac{3}{5^2} z^{\frac{5}{2}} - \frac{6}{5^2 c^3}) - K_A^2 \left( z^{\frac{1}{2}} - \frac{z^{\frac{2}{2}}}{2^2} \right) \right]_0^C
\]

\[
= 40c^3 \left[ (\frac{1}{9} - \frac{1}{11})c^2 + \frac{1}{3} c^2 z^2 (\frac{1}{7} - \frac{1}{3} + \frac{3}{11} - \frac{1}{13}) - K_A^2 (\frac{1}{5} - \frac{1}{7}) \right]
\]

\[
= 80c^3 \left[ \frac{1}{99} c^2 + \frac{8}{9009} c^2 z^2 - \frac{1}{35} K_A^2 \right] \quad ............. (25)
\]
2.8 Parameter Evaluation

A computer programme was written in ALGOL to work out the different blade constants discussed in this chapter. A print out of the programme is presented for reference purposes in Appendix IIIA. The computed results are presented in tabular form at the end of this chapter. The results are also shown graphically in figures 3-12. These figures indicate how the blade constants change with reference to thickness parameter $\theta$ and blade width $c$ and are discussed later.
# TABLE 1

ORDINATES WORKED OUT FOR CUBICAL OVAL BLADES

OF WIDTH 1.5 in

<table>
<thead>
<tr>
<th>No</th>
<th>Distance from Round Nose: inches</th>
<th>Ordinate Set 1 Blade - inches $\theta = 0.1$</th>
<th>Ordinate Set 2 Blade - inches $\theta = 0.2$</th>
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# TABLE 2

MATRIX OF AREA OF CUBICAL OVAL BLADES WITH RESPECT TO BLADE WIDTH AND THICKNESS

<table>
<thead>
<tr>
<th>Θ Thickness Parameter</th>
<th>Area x10^-2 sq in c = 1 in</th>
<th>Area x10^-2 sq in c = 1.125 in</th>
<th>Area x10^-2 sq in c = 1.25 in</th>
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<th>Area x10^-2 sq in c = 1.5 in</th>
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<td>θ Thickness Parameter</td>
<td>Radius of gyration $c = 1$ in $x 10^{-1}$ inch$^2$</td>
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<td>Radius of gyration $c = 1.25$ in $x 10^{-1}$ inch$^2$</td>
<td>Radius of gyration $c = 1.375$ in $x 10^{-1}$ inch$^2$</td>
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<td>Centre of flexure $c = 1.25$ in $\times 10^{-2}$ inch</td>
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### TABLE 5

**II MATRIX OF MOMENT OF INERTIA ABOUT MAJOR AXIS WITH RESPECT TO BLADE WIDTH AND THICKNESS**

<table>
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<tr>
<th>Thickness Parameter</th>
<th>Moment of Inertia $c = 1\ in \ x \ 10^{-5}in^4$</th>
<th>Moment of Inertia $c = 1.125\ in \ x \ 10^{-5}in^4$</th>
<th>Moment of Inertia $c = 1.25\ in \ x \ 10^{-5}in^4$</th>
<th>Moment of Inertia $c = 1.375\ in \ x \ 10^{-5}in^4$</th>
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## TABLE 6

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<th>$\Theta$ Thickness Parameter</th>
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<th>Moment of Inertia $c = 1.125 \text{ in} \times 10^{-3} \text{ in}^4$</th>
<th>Moment of Inertia $c = 1.25 \text{ in} \times 10^{-3} \text{ in}^4$</th>
<th>Moment of Inertia $c = 1.375 \text{ in} \times 10^{-3} \text{ in}^4$</th>
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TABLE 7

ICF MATRIX OF POLAR MOMENT OF INERTIA ABOUT CF WITH RESPECT TO BLADE WIDTH AND THICKNESS

<table>
<thead>
<tr>
<th>Θ Thickness Parameter</th>
<th>Moment of Inertia ( c = 1 \text{ in} \times 10^{-3} \text{ in}^4 )</th>
<th>Moment of Inertia ( c = 1.125 \text{ in} \times 10^{-3} \text{ in}^4 )</th>
<th>Moment of Inertia ( c = 1.25 \text{ in} \times 10^{-3} \text{ in}^4 )</th>
<th>Moment of Inertia ( c = 1.375 \text{ in} \times 10^{-3} \text{ in}^4 )</th>
<th>Moment of Inertia ( c = 1.5 \text{ in} \times 10^{-3} \text{ in}^4 )</th>
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TABLE 11

B1 MATRIX OF BLADE CONSTANT WITH RESPECT TO BLADE WIDTH AND THICKNESS
TABLE 12

B2 MATRIX OF BLADE CONSTANT WITH RESPECT TO BLADE WIDTH AND THICKNESS

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<th>Thickness Parameter</th>
<th>B2 c = 1 in x 10^{-3}</th>
<th>B2 c = 1.125 in x 10^{-3}</th>
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CUBICAL OVAL BLADE SHAPE - ENLARGED
GENERAL CO-ORDINATE SYSTEM

Figure 2
VARIATION OF SECTION AREA WITH REFERENCE TO
BLADE WIDTH AND THICKNESS

- Thickness Parameter = 0.3
- Thickness Parameter = 0.25
- Thickness Parameter = 0.2
- Thickness Parameter = 0.15
- Thickness Parameter = 0.1

Figure 3
VARIATION OF CENTRE OF FLEXURE FROM CG WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- Thickness Parameter 0.3
- Thickness Parameter 0.25
- Thickness Parameter 0.20
- Thickness Parameter 0.15
- Thickness Parameter 0.10

Figure 4
VARIATION OF MOMENT OF INERTIA ABOUT MAJOR AXIS
WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- Thickness Parameter = 0.3
- Thickness Parameter = 0.25
- Thickness Parameter = 0.2
- Thickness Parameter = 0.15
- Thickness Parameter = 0.10

Figure 5
VARIATIONS OF MOMENT OF INERTIA ABOUT MINOR AXIS WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- Thickness Parameter = 0.3
- Thickness Parameter = 0.25
- Thickness Parameter = 0.20
- Thickness Parameter = 0.15
- Thickness Parameter = 0.10

Blade Width C

Figure 6
VARIATION OF SQUARE OF RADIUS OF GYRATION WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- Thickness Parameter = 0.1
- Thickness Parameter = 0.15
- Thickness Parameter = 0.2
- Thickness Parameter = 0.25
- Thickness Parameter = 0.3

c = 1in  c = 1.125in  c = 1.25in  c = 1.375in  c = 1.5in

BLADE WIDTH C

Figure 7
VARIATION OF POLAR MOMENT OF INERTIA WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- Thickness Parameter = 0.30
- Thickness Parameter = 0.25
- Thickness Parameter = 0.20
- Thickness Parameter = 0.15
- Thickness Parameter = 0.10

Figure 8
VARIATION OF TORSIONAL STIFFNESS CONSTANT
WITH REFERENCE TO BLADE WIDTH AND THICKNESS

\[ C_{10} \text{ Torsonal Stiffness Constant } \times 10^{-6} \text{ lbs/rad/inch} \]

- ▽ Thickness Parameter = 0.3
- □ Thickness Parameter = 0.25
- △ Thickness Parameter = 0.20
- × Thickness Parameter = 0.15
- ○ Thickness Parameter = 0.10

\[ c = 1 \text{ in} \quad c = 1.125 \text{ in} \quad c = 1.25 \text{ in} \quad c = 1.375 \text{ in} \quad c = 1.5 \text{ in} \]

Figure 9
VARIATION OF BLADE CONSTANT $B_1$ WITH REFERENCE TO BLADE WIDTH AND THICKNESS

Figure 10
VARIATION OF BLADE CONSTANT $B_2$ WITH REFERENCE TO BLADE WIDTH AND THICKNESS

\[ B_2 \text{ BLADE CONSTANT} \times 10^{-3} \]

- ▼ Thickness Parameter = 0.30
- □ Thickness Parameter = 0.250
- △ Thickness Parameter = 0.200
- × Thickness Parameter = 0.150
- ○ Thickness Parameter = 0.10

$C = 1\text{ in}$  $C = 1.125\text{ in}$  $C = 1.25\text{ in}$  $C = 1.375\text{ in}$  $C = 1.5\text{ in}$

BLADE WIDTH $C$

Figure 11
VARIATION OF STIFFNESS FACTOR $K$ WITH REFERENCE TO BLADE WIDTH AND THICKNESS

- ▽ Thickness Parameter = 0.30
- ○ Thickness Parameter = 0.25
- △ Thickness Parameter = 0.20
- × Thickness Parameter = 0.15
- • Thickness Parameter = 0.10

Figure 12
Theoretical Considerations

The vibrational characteristics of uniform straight and pretwisted cantilever beams are best achieved by solving the differential equations of motion of the beams. Solution of these equations give the normal mode frequencies and mode shapes of vibration. A direct analytical solution of the differential equations is not possible owing to the complexity of equations. A general method of solving standard differential equations is presented in Appendix II, which was adopted from Dawson (17).

This method of solving the differential equations of motion requires the transformation of the equations into a set of simultaneous linear first order differential equations. When all the initial boundary conditions are known at the root then integration of these first order differential equations can be carried out by using the Runge Kutta finite difference step by step integration process. When all the initial boundary values are known, the problem is called a marching or initial value problem.

In the present case only half of the initial boundary values are known. Hence arbitrary sets of initial boundary values are assumed and integration carried out by Runge-Kutta finite
difference step by step integration process. The actual solution of the equations is a linear co-ordination of these solved values obtained by assuming arbitrary initial boundary conditions. The condition that the actual solutions must equal the known boundary values at the tip for a cantilever beam, results in a set of homogenous equations. The vanishing of their determinant gives the natural frequencies of the beam.

The differential equations of motion of a pretwisted vibrating beam of cubical oval cross-section with one axis of symmetry as shown in Appendix ID are as follows.

\[- \left\{ C10 + E B_1 (\beta')^2 \right\} \varphi' - E B_2 \beta' (v'' \cos \beta + w'' \sin \beta) \right]'

\[- \frac{w}{g} e p'^2 (v \sin \beta - w \cos \beta) + \frac{ICF}{g} \varphi p^2 = 0 \quad \ldots \ldots \ldots \ldots \quad (26)\]

\[- \left[ (E I_1 \cos^2 \beta + E I_2 \sin^2 \beta) w'' + (E I_2 - E I_1) \sin \beta \cos \beta v'' - E B_2 \beta' \sin \beta \right]''

\[- \frac{w}{g} p^2 (v - e \varphi \sin \beta) = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (27)\]

\[- \left[ (E I_2 - E I_1) \sin \beta \cos \beta w'' + (E I_1 \sin^2 \beta + E I_2 \cos^2 \beta) v'' - E B_2 \beta' \cos \beta \right]''

\[- \frac{w}{g} p^2 (w + e \varphi \cos \beta) = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (28)\]

It has been shown in Appendix ID that moments $M_x, M_y$ and $M_z$ have the following functions:
\[ M_x = \left[ C_{10} + E_{B_1} (\beta^1)^2 \right] \phi^1 - E_{B_2} \beta^1 (v^{11} \cos \beta + w^{11} \sin \beta) \quad \ldots \quad (29) \]

\[ M_y = (E_{I_1} \sin^2 \beta + E_{I_2} \cos^2 \beta)v^{11} + (E_{I_2} - E_{I_1}) \sin \beta \cos \beta w^{11} - E_{B_2} \beta^1 \phi^1 \cos \beta \quad \ldots \ldots \quad (30) \]

\[ M_z = (E_{I_1} \cos^2 \beta + E_{I_2} \sin^2 \beta)w^{11} + (E_{I_2} - E_{I_1}) \sin \beta \cos \beta v^{11} - E_{B_2} \beta^1 \phi^1 \sin \beta \quad \ldots \ldots \quad (31) \]

From equations (26)-(28) and (29)-(31)

\[ -M_x - \frac{w}{g} \epsilon p^2 (v \sin \beta - w \cos \beta) + \frac{ICF}{g} \phi p^2 = 0 \quad \ldots \ldots \quad (32) \]

\[ M_y - \frac{w}{g} p^2 (w + e \phi \cos \beta) = 0 \quad \ldots \ldots \quad (33) \]

\[ M_z - \frac{w}{g} p^2 (v - e \phi \sin \beta) = 0 \quad \ldots \ldots \quad (34) \]

or

\[ M_x = \frac{ICF}{g} \phi p^2 - \frac{w}{g} \epsilon p^2 (v \sin \beta - w \cos \beta) \quad \ldots \ldots \quad (35) \]

\[ M_y = \frac{w}{g} p^2 (w + e \phi \cos \beta) \quad \ldots \ldots \quad (36) \]

\[ M_z = \frac{w}{g} p^2 (v - e \phi \sin \beta) \quad \ldots \ldots \quad (37) \]

Letting \( B_2 \sin \beta = B_3 \) and \( B_2 \cos \beta = B_4 \), equations (29)-(37)
can be written as
\[ \left[ C_1 + \frac{E_B}{B_1} (\beta^1)^2 \right] \phi^1 - E_B^2 \beta^1 v'' - E_B^3 \beta^1 w'' = M_x \quad \ldots \ldots \quad (38) \]

\[ (E_1 \sin^2 \beta + E_2 \cos^2 \beta) v'' + (E_1^2 - E_1) \sin \beta \cos \beta w'' - E_B^4 \beta^1 \phi^1 = M_y \quad \ldots \ldots \quad (39) \]

\[ (E_1 \cos^2 \beta + E_2 \sin^2 \beta) w'' + (E_1^2 - E_1) \sin \beta \cos \beta v'' - E_B^4 \beta^1 \phi^1 = M_z \quad \ldots \ldots \quad (40) \]

Let

\[ I_1 \cos^2 \beta + I_2 \sin^2 \beta = I_3 \]
\[ (I_2 - I_1) \sin \beta \cos \beta = I_5 \]
\[ I_1 \sin^2 \beta + I_2 \cos^2 \beta = I_4 \]

Then equations (38)-(40) take the following shape

\[ \left[ C_1 + \frac{E_B}{B_1} (\beta^1)^2 \right] \phi^1 - E_B^2 \beta^1 v'' - E_B^3 \beta^1 w'' = M_x \quad \ldots \ldots \quad (41) \]
\[ E_1^2 w'' + E_1 v'' - E_B^4 \beta^1 \phi^1 = M_y \quad \ldots \ldots \quad (42) \]
\[ E_3 w'' + E_5 v'' - E_B^4 \beta^1 \phi^1 = M_z \quad \ldots \ldots \quad (43) \]

Considering equations (42) and (43) and multiplying (43) by \( I_5 \) and (42) by \( I_3 \) and dividing throughout by \( E \),

\[ I_3 I_5 w'' + I_5^2 v'' - B_3 \beta^1 \phi^1 I_5 = \frac{M_z}{E} I_5 \quad \ldots \ldots \quad (44) \]
\[ I_3 I_5 w'' + I_4 I_3 v'' - B_4 \beta^1 \phi^1 I_3 = \frac{M_y}{E} I_3 \quad \ldots \ldots \quad (45) \]
Subtracting equation (44) from (45)

\[ v^{ii}(I_4I_3 - I_5^2) = B_4\beta^1\phi^1I_3 + B_3\beta^1\phi^1I_5 = \frac{My}{E}I_3 - \frac{Mz}{E}I_5 \quad \cdots \quad (46) \]

ie

\[ v^{ii}(I_4I_3 - I_5^2) = \frac{My}{E}I_3 - \frac{Mz}{E}I_5 - B_3\beta^1\phi^1I_5 + B_4\beta^1\phi^1I_3 \]

\[ \therefore v^{ii} = \frac{MyI_3}{E(I_4I_3 - I_5^2)} - \frac{MzI_5}{E(I_4I_3 - I_5^2)} - \frac{B_3\beta^1\phi^1I_5}{(I_4I_3 - I_5^2)} \]

\[ + \frac{B_4I_3\beta^1\phi^1}{(I_4I_3 - I_5^2)} \quad \cdots \quad (47) \]

Letting \( \frac{I_3}{I_4I_3 - I_5^2} = c_1 \) and \( \frac{I_5}{I_4I_3 - I_5^2} = c_2 \) then (47) becomes

\[ v^{ii} = \frac{My}{E}c_1 - \frac{Mz}{E}c_2 - \beta^1\phi^1B_3c_2 + B_4\beta^1\phi^1c_1 \quad \cdots \quad (48) \]

\[ = \frac{My}{E}c_1 - \frac{Mz}{E}c_2 + \beta^1\phi^1(B_4c_1 - B_3c_2) \]

Letting \( \beta^1(B_4c_1 - B_3c_2) = c_3 \)

\[ v^{ii} = \frac{My}{E}c_1 - \frac{Mz}{E}c_2 + c_3\phi^1 \quad \cdots \quad (49) \]

Again considering equations (42) and (43) and multiplying (42) by \( I_4 \) and (43) by \( I_5 \) and dividing throughout by \( E \)

\[ I_3I_4v^{ii} + I_5I_4v^{ii} - B_3\beta^1\phi^1I_4 = \frac{Mz}{E}I_4 \quad \cdots \quad (50) \]

\[ I_5^2w^{ii} + I_5I_4v^{ii} - B_4\beta^1\phi^1I_5 = \frac{My}{E}I_5 \quad \cdots \quad (51) \]
Subtracting (51) from (50)

\[(I_3 I_4 - I_5^2) \omega^{11} + B_4 \beta^1 \phi^1 I_5 - B_3 \beta^3 \phi^1 I_4 = \frac{MzI_4}{E} - \frac{MyI_5}{E}\]

\[w^{11} = \frac{Mz}{E} \left(\frac{I_4}{I_3 I_4 - I_5^2}\right) - \frac{My}{E} \left(\frac{I_5}{I_3 I_4 - I_5^2}\right) + \frac{B_3 \beta^3 \phi^1 I_4}{(I_3 I_4 - I_5^2)} - \frac{B_4 \beta^1 \phi^1 I_5}{(I_3 I_4 - I_5^2)}\]

Letting \(I_4/(I_3 I_4 - I_5^2) = C_4\)

\[w^{11} = \frac{Mz}{E} C_4 - \frac{My}{E} C_2 + B_3 \beta^3 \phi^1 C_4 - \beta^1 \phi^1 B_4 C_2\]

\[= \frac{Mz}{E} C_4 - \frac{My}{E} C_2 + \beta^1 \phi^1 (B_3 C_4 - B_4 C_2) \quad \ldots \ldots \quad (52)\]

Letting \(\beta^1 (B_3 C_4 - B_4 C_2) = C_5\)

\[w^{11} = \frac{Mz}{E} C_4 - \frac{My}{E} C_2 + C_5 \phi^1 \quad \ldots \ldots \quad (53)\]

Substituting the values of \(\omega^{11}\) and \(w^{11}\) from (49) and (53) in equation (38)

\[
\left[C_{10} + EB_1 (\beta^1)^2\right] \phi^1 - EB_4 \beta^1 (\frac{My}{E} C_1 - \frac{Mz}{E} C_2 + C_3 \phi^1) - EB_3 \beta^1 (\frac{Mz}{E} C_4 - \frac{My}{E} C_2 + C_5 \phi^1) = Mx \quad \ldots \ldots \quad (54)
\]

Rearranging the equation (54)
\[
\{C_{10} + E_B (\beta')^2 - E_B \beta' C_3 - E_B \beta' C_5\} \varphi' = E_B \beta' \left(\frac{M_C}{E} - \frac{M_C}{E} C_2\right)
\]
\[
- E_B \beta' \left(\frac{M_C}{E} C_4 - \frac{M_C}{E} C_2\right) = M_x \tag{55}
\]
\[
ie \left[\{C_{10} + E_B (\beta')^2 - E_B \beta' (B_4 C_3 + B_3 C_5)\} \varphi'\right] = M_x + E_B \beta' \left(\frac{M_C}{E} C_1 - \frac{M_C}{E} C_2\right)
\]
\[
+ E_B \beta' \left(\frac{M_C}{E} C_4 - \frac{M_C}{E} C_2\right) \tag{56}
\]

Dividing equation (56) throughout by E

\[
\left[\frac{C_{10}}{E} + B_1 (\beta')^2 - \beta' (B_4 C_3 + B_3 C_5)\right] \varphi' = \frac{M_x}{E} + B_4 \beta' \left(\frac{M_C}{E} C_1 - \frac{M_C}{E} C_2\right) + B_3 \beta' \left(\frac{M_C}{E} C_4 - \frac{M_C}{E} C_2\right) \tag{57}
\]

Letting \(B_1 (\beta')^2 - \beta' (B_4 C_3 + B_3 C_5) = C_6\) Equation (57) takes the following shape

\[
\left[\frac{C_{10}}{E} + C_6\right] \varphi' = \frac{M_x}{E} + \frac{M_C}{E} \beta' (B_4 C_1 - B_3 C_2) + \frac{M_C}{E} \beta' (B_3 C_4 - B_4 C_2) \tag{58}
\]

Let \(\beta' (B_4 C_1 - B_3 C_2) = C_7\) and

\(\beta' (B_3 C_4 - B_4 C_2) = C_8\)

then (58) can be written as

\[
\left[\frac{C_{10}}{E} + C_6\right] \varphi' = \frac{M_x}{E} + \frac{M_C}{E} C_7 + \frac{M_C}{E} C_8 \tag{59}
\]
\[ \phi' = \frac{\frac{M_x}{E} + \frac{M_y}{E}C_7 + \frac{M_z}{E}C_8}{C_{10}} \frac{C_1}{E} + C_6 \] ............ (60)

Thus three complex equations are obtained for \( v'' \), \( w'' \) and \( \phi' \) in terms of \( M_x \), \( M_y \) and \( M_z \) which are

\[ v'' = \frac{M_y}{E}C_1 - \frac{M_z}{E}C_2 + C_3\phi' \] ............ (61)

\[ w'' = \frac{M_z}{E}C_4 - \frac{M_y}{E}C_2 + C_5\phi' \] ............ (62)

\[ \phi' = \frac{\left(\frac{M_x}{E} + \frac{M_y}{E}C_7 + \frac{M_z}{E}C_8\right)}{(C_{10}\frac{C_1}{E} + C_6)} \] ............ (63)

In equations (61) to (63) the following are the expansions of the constants used.

\[ E = \text{Modulus of Elasticity} \]
\[ B_1 = \text{Coefficient of coupling} \]
\[ B_2 = \text{Coefficient of coupling} \]
\[ B_3 = B_2 \sin \beta \]
\[ B_4 = B_2 \cos \beta \]
\[ C_1 = I_3/(I_4I_3 - I_5^2) \]
\[ C_2 = I_5/(I_4I_3 - I_5^2) \]
\[ C_3 = C_7 = (B_4C_1 - B_3C_2)\beta' \]
\[ C_4 = I_4/(I_4I_3 - I_5^2) \]
\[ C_5 = C_8 = \beta'(B_3C_4 - B_4C_2) \]
\[ C_6 = B_1(\beta')^2 - \beta'(B_4C_3 + B_3C_5) \]
\[ C_{10} = \text{Torsional stiffness constant} \]
From equations (29)-(31) and (61)-(63) the following six equations are arrived:

\[
M_x' = \frac{ICF}{g} \phi^2 - \frac{w}{g} \epsilon p^2 (v \sin \beta - w \cos \beta) \tag{64}
\]

\[
M_y'' = \frac{w}{g} p^2 (w + e \phi \cos \beta) \tag{65}
\]

\[
M_z'' = \frac{w}{g} p^2 (v - e \phi \sin \beta) \tag{66}
\]

\[
v'' = \frac{My}{E} C_4 - \frac{Mz}{E} C_2 + C_3 \phi' \tag{67}
\]

\[
w'' = \frac{Mz}{E} C_4 - \frac{My}{E} C_2 + C_5 \phi' \tag{68}
\]

\[
\phi' = \frac{\frac{My}{E} C_7 + \frac{Mz}{E} C_9}{\left(\frac{C10}{E} + C_6\right)} \tag{69}
\]

Letting

\[
y_1 = v \tag{70}
\]
\[
y_2 = w \tag{71}
\]
\[
y_3 = v' \tag{72}
\]
\[
y_4 = w' \tag{73}
\]
\[
y_5 = My \tag{74}
\]
\[
y_6 = Mz \tag{75}
\]
\[
y_7 = My' \tag{76}
\]
\[
y_8 = Mz' \tag{77}
\]
\[
y_9 = \phi \tag{78}
\]
\[
y_{10} = Mx \tag{79}
\]
From the above assumptions the following equations are readily available.

\begin{align*}
y_1^1 &= v^1 = y_3 \quad \ldots \ldots \ldots \quad (80) \\
y_2^1 &= w^1 = y_4 \quad \ldots \ldots \ldots \quad (81) \\
y_3^1 &= v'' \quad \ldots \ldots \ldots \quad (82) \\
y_4^1 &= w'' \quad \ldots \ldots \ldots \quad (83) \\
y_5^1 &= M_1 = y_7 \quad \ldots \ldots \ldots \quad (84) \\
y_6^1 &= M_2 = y_9 \quad \ldots \ldots \ldots \quad (85) \\
y_7^1 &= M_1'' \quad \ldots \ldots \ldots \quad (86) \\
y_8^1 &= M_2'' \quad \ldots \ldots \ldots \quad (87) \\
y_9^1 &= \phi^1 \quad \ldots \ldots \ldots \quad (88) \\
y_{10}^1 &= M_x^1 \quad \ldots \ldots \ldots \quad (89)
\end{align*}

Values of $v''$, $w''$, $\phi^1$, $M_x^1$, $M_1''$ and $M_2''$ are substituted from equations (64)-(69) in equations (80)-(89) then

\begin{align*}
y_1^1 &= y_3 \quad \ldots \ldots \ldots \quad (90) \\
y_2^1 &= y_4 \quad \ldots \ldots \ldots \quad (91) \\
y_3^1 &= \frac{M_y}{E} C_1 - \frac{M_z}{E} C_2 + C_3 \phi^1 \quad \ldots \ldots \ldots \quad (92) \\
y_4^1 &= \frac{M_z}{E} C_4 - \frac{M_y}{E} C_2 + C_5 \phi^1 \quad \ldots \ldots \ldots \quad (93) \\
y_5^1 &= y_7 \quad \ldots \ldots \ldots \quad (94) \\
y_6^1 &= y_9 \quad \ldots \ldots \ldots \quad (95) \\
y_7^1 &= \frac{w}{g} p^2 (w + e^2 \cos B) \quad \ldots \ldots \ldots \quad (96) \\
y_8^1 &= \frac{w}{g} p^2 (v - e^2 \sin B) \quad \ldots \ldots \ldots \quad (97) \\
y_9^1 &= \frac{M_x}{E} + \frac{M_y}{E} C_7 + \frac{M_z}{E} C_9 / (\frac{C_{10}}{E} + C_6) \quad \ldots \ldots \ldots \quad (98) \\
y_{10}^1 &= \frac{ICF}{g} \phi p^2 - \frac{w}{g} ep^2 (v \sin \beta - w \cos \beta) \quad \ldots \ldots \ldots \quad (99)
\end{align*}
From the assumptions made in equations (70)-(79) (for values of \(v, w, \phi, M_x, M_y\) and \(M_z\)) the equations can be written as:

\[ y_{11} = y_3 \]

\[ y_2 = y_4 \]

\[ y_3 = \frac{v_5}{E} C_1 - \frac{v_6}{E} C_2 + C_3 \phi' \]

\[ y_4 = \frac{v_5}{E} C_4 - \frac{v_6}{E} C_2 + C_5 \phi' \]

\[ y_5 = y_7 \]

\[ y_6 = y_8 \]

\[ y_7 = \frac{w}{g} \ p^2 \ (y_2 + e\phi \cos \beta) \]

\[ y_8 = \frac{w}{g} \ p^2 \ (y_1 - e\phi \sin \beta) \]

\[ y_9 = \frac{v_3}{E} + \frac{v_6}{E} C_7 + \frac{v_7}{E} C_5 / (\frac{C_{10}}{E} + C_6) \]

\[ y_{10} = \frac{I_{CF}}{y} y_5 p^2 - \frac{w}{g} \ e p^2 (y_1 \sin \beta - y_2 \cos \beta) \]

Thus equations (26), (27) and (28) are reduced to ten linear first order simultaneous differential equations, and the ten first order differential equations may be conveniently written in the following form.
y_1' = f_1 (y_3) \quad \ldots \quad (110)

y_2' = f_2 (y_4) \quad \ldots \quad (111)

y_3' = f_3 (x, y_6, y_7, y_{10}) \quad \ldots \quad (112)

y_4' = f_4 (x, y_6, y_7, y_{10}) \quad \ldots \quad (113)

y_5' = f_5 (y_7) \quad \ldots \quad (114)

y_6' = f_6 (y_8) \quad \ldots \quad (115)

y_7' = f_7 (x, y_2, y_9) \quad \ldots \quad (116)

y_8' = f_8 (x, y_1, y_9) \quad \ldots \quad (117)

y_9' = f_9 (x, y_6, y_7, y_{10}) \quad \ldots \quad (118)

y_{10}' = f_{10} (x, y_9, y_1, y_2) \quad \ldots \quad (119)

or

\frac{dy_i}{dx} = f_i (x, y_1, y_2, y_3 \ldots \ldots y_{10})

\text{where } i = 1, 2, 3 \ldots \ldots 10 \quad \ldots \quad (120)

In the above equations only some of the initial boundary values are known and some of the end boundary values are known. They are as follows.

At \( x = 0 \)

\begin{align*}
v &= 0 \text{ or } y_1 = 0 \\
w &= 0 \text{ or } y_2 = 0 \\
v' &= 0 \text{ or } y_3 = 0 \\
w' &= 0 \text{ or } y_4 = 0 \\
\phi &= 0 \text{ or } y_9 = 0
\end{align*}

At \( x = L \)

\begin{align*}
My &= 0 \text{ or } y_5 = 0
\end{align*}
A straightforward integration by computational methods is not possible because not all the initial boundary conditions are known.

Hence the transformation method described in Appendix II is used to solve the equations (100)-(109).

A trial natural frequency \( p \) is assumed. The known initial boundary conditions are set to their respective values.

\[
y_1 = 0, \ y_2 = 0, \ y_3 = 0, \ y_4 = 0, \ y_9 = 0 \quad \text{........... (121)}
\]

The remaining five unknown boundary values are assigned arbitrary values as follows.

\[
y_5 = 1, \ y_6 = 0, \ y_7 = 0, \ y_8 = 0, \ y_{10} = 0 \quad \text{........... (122)}
\]

Starting from the five known boundary values and five assigned boundary conditions integration of equations (100)-(109) is carried out by Runge-Kutta step-by-step integration process. A set of solutions is obtained. This first set of trial solutions at the tip may be written as follows:

\[
y_i^{1}(L) = E_i^{1}(i = 1, 2, 3 \ldots 10) \quad \text{........... (123)}
\]
The second subscript in equation (123) represents first solution of equations with the first set of initial boundary values.

The above procedure is repeated with the five known boundary values kept at their respective values but the five unknown boundary values set in turn to the following values.

\[ y_5 = 0; \quad y_6 = 1; \quad y_7 = 0; \quad y_8 = 0; \quad y_9 = 0 \quad \ldots \ldots \quad (124) \]
\[ y_5 = 0; \quad y_6 = 0; \quad y_7 = 1; \quad y_8 = 0; \quad y_9 = 0 \quad \ldots \ldots \quad (125) \]
\[ y_5 = 0; \quad y_6 = 0; \quad y_7 = 0; \quad y_8 = 1; \quad y_9 = 0 \quad \ldots \ldots \quad (126) \]
\[ y_5 = 0; \quad y_6 = 0; \quad y_7 = 0; \quad y_8 = 0; \quad y_9 = 1 \quad \ldots \ldots \quad (127) \]

Thus five sets of solutions are obtained at the tip which can be represented as follows:

\[ y_{i,r}(L) = E_{i,r}(i = 1, 2 \ldots 10)(r = 1, 2 \ldots 5) \quad \ldots \ldots \quad (128) \]

Equations (100)-(109) are linear and the actual solutions are a linear combination of five sets of solutions. Hence:

\[ y_1(x) = a_1y_{i,1} + a_2y_{2,1} + a_3y_{3,1} + a_4y_{4,1} + a_5y_{5,1} \quad \ldots \ldots \quad (129) \]
\[ y_2(x) = a_1y_{i,2} + a_2y_{2,2} + a_3y_{3,2} + a_4y_{4,2} + a_5y_{5,2} \quad \ldots \ldots \quad (130) \]
\[ y_3(x) = a_1y_{i,3} + a_2y_{2,3} + a_3y_{3,3} + a_4y_{4,3} + a_5y_{5,3} \quad \ldots \ldots \quad (131) \]
\[ y_4(x) = a_1y_{i,4} + a_2y_{2,4} + a_3y_{3,4} + a_4y_{4,4} + a_5y_{5,4} \quad \ldots \ldots \quad (132) \]
\[ y_5(x) = a_1y_{i,5} + a_2y_{2,5} + a_3y_{3,5} + a_4y_{4,5} + a_5y_{5,5} \quad \ldots \ldots \quad (133) \]
\[ y_6(x) = a_1y_{i,6} + a_2y_{2,6} + a_3y_{3,6} + a_4y_{4,6} + a_5y_{5,6} \quad \ldots \ldots \quad (134) \]
\[ y_7(x) = a_1y_{i,7} + a_2y_{2,7} + a_3y_{3,7} + a_4y_{4,7} + a_5y_{5,7} \quad \ldots \ldots \quad (135) \]
\[ y_8(x) = a_1y_{i,8} + a_2y_{2,8} + a_3y_{3,8} + a_4y_{4,8} + a_5y_{5,8} \quad \ldots \ldots \quad (136) \]
\[ y_9(x) = a_1y_{i,9} + a_2y_{2,9} + a_3y_{3,9} + a_4y_{4,9} + a_5y_{5,9} \quad \ldots \ldots \quad (137) \]
\[ y_{10}(x) = a_1 y_{1,10} + a_2 y_{2,10} + a_3 y_{3,10} + a_4 y_{4,10} + a_5 y_{5,10} \ldots \ldots \] (138)

where \(a_1, a_2, \ldots, a_5\) represent the coefficients.

The above equations can be written in a simplified form

\[ y_i(x) = \sum_{r=1}^{5} a_r y_{i,r}(x) \quad (i = 1, 2 \ldots 10) \ldots \ldots \] (139)

The known boundary values for the cantilever beam at \(x = L\) are

\[ y_{i,L} = 0 \quad (i = 5, 6, 7, 8, 10) \ldots \ldots \ldots \ldots \] (140)

Hence

\[
\begin{align*}
    a_1 y_{1,5} + a_2 y_{2,5} + a_3 y_{3,5} + a_4 y_{4,5} + a_5 y_{5,5} &= 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (141) \\
    a_1 y_{1,6} + a_2 y_{2,6} + a_3 y_{3,6} + a_4 y_{4,6} + a_5 y_{5,6} &= 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (142) \\
    a_1 y_{1,7} + a_2 y_{2,7} + a_3 y_{3,7} + a_4 y_{4,7} + a_5 y_{5,7} &= 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (143) \\
    a_1 y_{1,8} + a_2 y_{2,8} + a_3 y_{3,8} + a_4 y_{4,8} + a_5 y_{5,8} &= 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (144) \\
    a_1 y_{1,10} + a_2 y_{2,10} + a_3 y_{3,10} + a_4 y_{4,10} + a_5 y_{5,10} &= 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (145)
\end{align*}
\]

A solution for the equations (100)-(109) is possible when the determinant of the coefficients of the homogenous equations (141)-(145) vanishes ie

\[
\begin{vmatrix} E_{i,r} \end{vmatrix} = 0 \quad (i = 5, 6, 7, 8, 10) \\
\quad (r = 1, 2, 3, 4, 5) \ldots \ldots \ldots \ldots \ldots (146)
\]

The values of determinants given by equation (146) are solved by a standard library program.
The actual value of normal mode frequency is obtained by extrapolation of determinant values against trial frequencies. The frequencies for which the determinant values are zero are the normal mode frequencies of vibration. The method of solution has been programmed in ALGOL and basically consists of four subroutines. The first subroutine sets up the differential equations, the second subroutine sets up the initial boundary conditions, the third performs the Runge-Kutta integration process and the fourth solves the determinant. Differential equations were set up in non-dimensional form and integrations were carried out from 0 to 1. A copy of the program is given in Appendix IIIB for reference purposes. The accuracy of this method is dependent on the size of step length and a step length of 0.05 was found to be satisfactory for all modes of vibration and pretwist angles.

The natural frequencies of vibration up to third mode have been calculated by the transformation method for beams of equivalent width to thickness ratios of 15.7:1 and 7.9:1. In both cases calculations were carried out to obtain the natural frequencies

i) including second order terms and

ii) neglecting the second order terms, for pretwist angle range of zero to 90 degrees.

The theoretical results are compared with the results obtained by experiment on sets of pretwisted beams of corresponding width to thickness ratios and pretwist angles.
CHAPTER IV

EXPERIMENTAL WORK
4.0 Experimental Work

4.1 Nature of Experiments

Standard tests were carried out to obtain the values of Young's Modulus for rectangular cross section beams. The aerofoil beams were machined from the same stock of rectangular beams and the value of Young's Modulus is assumed to be the same as that determined for the rectangular cross section beams.

Two sets of cubical oval aerofoil beams were manufactured, one with thickness coefficient 0.1 and the other with thickness coefficient 0.2. These blades were pretwisted in the rig used by Carnegie (Plate 1). The range of pretwist angle was from 0° to 90°. The blades were mounted in a large block and a series of vibration tests were performed on the sets of beams with a view to obtaining the natural frequencies up to the third mode of vibration.
4.2 Manufacturing Technique of Cubical Oval Blades

4.2(1) Theoretical blade co-ordinates

The cross section of the blade is of cubical oval shape, and the geometrical formula defining the section (12) is

\[
\left( \frac{t}{cG} \right)^2 = \frac{z}{c} \left( 1 - \frac{z}{c} \right)^2
\]

where \( t \) is the half thickness of the blade, \( c \) is the width or chord of the blade, \( \theta \) is the thickness parameter and \( z \) is the co-ordinate along the major principal axis. From the above formula the thickness of the blade at any distance from the round nose can be written as

\[
t = \theta \sqrt{c} \sqrt{z} \left( 1 - \frac{z}{c} \right)
\]

Ordinates of 1\( \frac{1}{2} \) in width blades for thickness parameter \( \theta = 0.1 \) and 0.2 were worked out using the above formula. Results are presented in table 1 (Chapter II). From these results it can be noticed that the maximum thickness of the blade with \( \theta = 0.1 \) is 0.116 in and that of blade with \( \theta = 0.2 \) is 0.232 in.

4.2(2) Material Specification

The material used for blade manufacture was cold rolled bright
steel of free cutting quality to BSS 970 EN1A. Steel of dimensions $1\frac{1}{2}$ in $\times \frac{3}{16}$ in $\times$ 12 in and $1\frac{1}{2}$ in $\times \frac{5}{16}$ in $\times$ 12 in of the above quality was used for production of blades with parameters $\Theta = 0.1$ and $0.2$ respectively, and of length 10 inches. The bar was first normalised by heating up to the upper critical point ($910^\circ$C) and allowing to cool slowly in air. This process relieved stresses and avoided any possible warping. The sample material was descaled before starting the actual machining operation. The Young's Modulus or Modulus of Elasticity was found to be $29 \times 10^6$ lbs per square inch after the appropriate tensile tests.

4.2(3) Machining technique

For machining the blades a standard Invicta shaping machine of 24 in stroke with a magnetic working table and a hydraulically operated profiling attachment was used. As the shape of the blade is of aerofoil cross section, a standard machining process was not possible. The profiling attachment required a copying template and this template was manufactured by using a co-ordinate milling technique to mill the profile accurately. Details of the template are shown in figure 13. Finished templates were checked by dial gauge indicators and were found to be within $1^{\prime}/.$ accuracy. The milling operation was carried out from the above material of initial dimensions 4 in $\times$ 2 in $\times 1\frac{1}{4}$ in.
Normalised blade material was set up on the magnetic work
table of the Invicta shaper. The work piece was aligned by
a dial gauge with reference to the shaper ram and base. The
template was fixed on the same plane as the work piece. The
follower edge was in direct contact with the profile of the
template. Because of the relative attachment, as the job
base moves machining would be carried out in accordance with
the profile of the template. Before the actual machining the
relative positioning of the work piece and the template was
checked by a dial gauge indicator. The cutting tool of the
shaper was ground to the same profile as the follower edge.
A typical set up of template and job arrangement is given in
figure 14 and the general arrangement during actual machining
is shown on plates 2 and 3.

Standard cutting oil was used for efficient machining. A
backlash of 0.005 in was suspected in the cutting system.
Hence care was taken to lift the tool head manually during the
return stroke. When the machining of one side was completed
the work piece was reverted and aligned again. This time the
bottom of the work piece would be uneven and hence a spacer
was kept as shown in figure 14(b) and special care was taken
in aligning. The machining operation was then carried out in
exactly the same way as before. As the trailing edge tended
to chip off at the end, machining was stopped at about 0.050 in
before the trailing edge and finishing was carried out by oil
stones and swiss files. Completed blades were checked for
co-ordinates along the cross section by using a dial gauge and
slip gauges as shown in figure 15, and were also checked in a
two axis measuring machine as described below.
The Ferranti two axial inspection machine basically consists of an inspection table, interchangeable probe tips, a moveable ram to travel along the job and a numerically controlled digital read out display. The machine has a measuring capacity of 25 inches by 15 inches by 6 inches and is accurate to ± 0.0001 inch. No reference standards are required. The travelling table was checked for accuracy and the machined blade was fixed on to the table by means of T slots and clamps. The interchangeable probe tip is brought on to the tip of the blade and the reading on the digital display was recorded. The probe tip was moved along the width of the blade and both X and Y co-ordinates were recorded and these operations were carried out at four different places along the length of the blade. The Ferranti two axis measuring machine is shown together with the cubical oval blade in plate 4. A computer program was written in ALGOL language to compare the theoretical ordinates and the measured ordinates and is shown in Appendix IIIC, and one set of printed results is also presented.
4.2(4) Pretwisting

The pretwist rig designed by Carnegie (24) is shown in plate 1. It consists of two bearings (a) and (b) mounted on base plate (c). The bearings support the housing (d) and (e) which consists of two jaws in cylindrical recesses. The jaws are suitably profiled to grip the cubical oval section material rigidly when the screws on the housings are tightened. The jaw housing is fitted with stops (f) and (g) which limit the motion of the toggle bars (h) and (j). The bearing (a) can slide along the base plate to facilitate the entry and removal of the blade.

The pretwisting operation was carried out in the following manner. A 12 in blade was fixed in the jig so that the distance between two faces was 10 in. The stop (f) was adjusted so that the jaw housing (d) could be rotated in one direction half the pretwist angle required plus an estimated amount to allow for spring back. The stop (g) was also adjusted in the same fashion. The blade was pretwisted by rotating the toggle bars (h) and (j) their full extent by applying uniform torque to avoid any buckling in the middle. The stops were further adjusted if required, until the desired angle was obtained and then the blade was removed from the jig. One of the one inch straight portions was removed leaving one inch straight portion for root fixing.

Two sets of blades were twisted, one set with 1\(\frac{1}{2}\) in chord and \(\theta = 0.1\) and another set with chord 1\(\frac{1}{2}\) in and \(\theta = 0.2\). The
pretwist angles were 30°, 60°, 90° and one blade in each set was included without pretwist.

4.2(5) Root Fixing Method

Originally a rectangular block of dimension 2 in x 2 in x 1 in was chosen and a cavity was milled in the middle of the 2 in x 1 in face, the cavity being slightly wider and longer in dimension of the blade and about 1 in deep. The blade was inserted and checked for perpendicularity. An attempt was made to braze the blade. However, this method was not adopted because at the root the trailing edge of the blade tended to burn away and also the method required heat treatment after the brazing operation. A method of holding the blade in low melting point alloy was finally adopted as the root fixing method.

Fusible cerro alloys are those which melt at low temperatures say below 150°C. The basic metals from which these alloys are made are bismuth, lead, tin, cadmium and where very low melting point is required indium. There is an enormous number of variants that can be produced by using some or all of these metals. These were standardised and first produced some years ago by the cerro corporation of New York. They are also manufactured, under licence, by Mining and Chemical Products
Limited London. Table 13 indicates the range of melting temperatures available of cerro alloys. Almost all cerro alloys contain bismuth, a metal which has the rare property of expanding 3.3% by volume when it solidifies. Mechanical properties of cerro alloys improve with age. Conductivity of electricity and heat is poor and they are diamagnetic. Because they are stable they have the advantage of remelting and using again and again. The low melting range of 47° to 150°C makes the alloy extremely easy to handle. Cerro alloys find some of their many applications on temperature properties alone.

Since it is difficult to grip efficiently turbine blades because of the aerofoil shape casting in cerro alloy proves a very convenient way of gripping the root. The alloy chosen for the present work is termed as cerro bend and its properties are listed in Table 14. As far as the work holding is concerned it can be done by two ways. The work piece can be set up in a defined relationship to a steel member with the intervening space filled by fusible alloys or the work piece is embedded in a block of alloy so shaped as to give a suitable reference shape.

The root fixing operation was carried out as follows. A steel blade of 2 in x 2 in x 1 in was chosen as a mounting block. In the centre of the face 2 in x 1 in a cavity was milled to a depth of about 1 in and an area of about $\frac{1}{2}$ in x $\frac{3}{4}$ in. The
blade is aligned vertically inside the cavity and cerroband alloy was melted at about 85°C and poured in the cavity. On solidification of the alloy the blade is permanently bonded to the rectangular steel block. The block may then be held rigidly on a large mounting block. A set of blades embedded with cerroband alloy into steel rectangular blocks is shown on plate 5. Details of the blade block holding arrangement is shown in figure 16. The mounting block consists of an upper half (a) and lower half (b). The blade (j) is already moulded in block (g) by cerroband process explained earlier. The block (g) is gripped between jaws (f) which are machined to have 45° taper on the sides. Chemically blackened strips are placed on the sides as well as on the top (e and d) and this ensures a grip on all sides of the block. The mounting block is tightened by means of bolts (c). Suitable clearances are provided so that when the nuts on bolts (c) are tightened the blade block is rigidly gripped.

4.3 General Arrangement of Vibration Apparatus

The general arrangement of the apparatus for measuring the natural frequencies is shown in plate 6 and figure 17 is a diagrammatic layout. In the following paragraphs the description letters refer to items shown in plate 6 unless otherwise stated.
The vibrator is a magnetic exciter (e) held by a magnetic stand (d) close to the tip of the blade (c). The magnetic exciter has two magnet faces of size $\frac{1}{2}\text{in} \times \frac{1}{2}\text{in}$ and wound by 22 SWG copper wire with 110 turns. The supply to the magnetic exciter is drawn from the mains via a signal generator (f) and a power amplifier (g) of 25 watts capacity. A lead zincornate strain gauge (k) is fixed very near the root of the blade and is fed directly to the oscilloscope (j). The purpose of the strain gauge is to detect the resonance frequencies. A digital counter (i) is also connected across the lead zincronate strain gauge to give the natural frequencies of the vibrating beam.

### 4.4 Experimental Procedure

The general arrangement of the apparatus is shown in figure 17 and plate 6. The signal generator was set to a frequency of 15 Hz. The voltage output level control was adjusted to give a sufficient input voltage to the magnetic exciter. The frequency was increased until a position of maximum amplitude was shown on the cathode ray oscilloscope. The frequency as shown on the digital counter was recorded. To obtain higher modes the output level to the gauges was increased and a scan mode over a large frequency range until the required number of modes were obtained.
The first three natural frequencies were determined by this method for the two sets of cubical oval aerofoil cross section beams of length 10 in and pretwists 0°, 30°, 60° and 90°. The experimental natural frequencies, together with the theoretically predicted values of natural frequency, are plotted against pretwist angle in figures (18) to (21) (end of Chapter V) and are tabulated in tables 15 and 16.
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<thead>
<tr>
<th>ALLOY</th>
<th>MELTING POINT °C</th>
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</thead>
<tbody>
<tr>
<td>Tin gallium</td>
<td>20</td>
</tr>
<tr>
<td>Bismuth-Tin-lead, cadmium, indium</td>
<td>47</td>
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<tr>
<td>Bismuth-tin-lead, indium</td>
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<tr>
<td>Bismuth-tin-lead, cadmium</td>
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<tr>
<td>Bismuth-tin, indium</td>
<td>78</td>
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<tr>
<td>Bismuth-tin, cadmium</td>
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<tr>
<td>Bismuth, tin, lead</td>
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</tr>
<tr>
<td>Bismuth, tin, cadmium</td>
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<tr>
<td>Tin-indium</td>
<td>117</td>
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<tr>
<td>Bismuth, lead</td>
<td>124</td>
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<tr>
<td>Bismuth, tin, zinc</td>
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<tr>
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<tr>
<td>Bismuth-cadmium</td>
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<tr>
<td>Tin, lead, cadmium</td>
<td>145</td>
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<tr>
<td>Tin-cadmium</td>
<td>176</td>
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<tr>
<td>Property</td>
<td>Value</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>---------------</td>
</tr>
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<tr>
<td>Specific gravity at 20°C</td>
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<tr>
<td>Specific heat - solid</td>
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<tr>
<td>Latent heat of fusion BTQ/lb</td>
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<tr>
<td>Coefficient of thermal expansion °C x 10⁻⁶</td>
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<tr>
<td>Thermal conductivity solid</td>
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<td>cal/cm²/cm/°C/sec (copper 0.94)</td>
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<td>Electrical conductivity compared with copper</td>
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<td>Resistivity, ohms based on volume standard metre x mm²</td>
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<td>Brinell Hardness</td>
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<td>Tensile strength lbs/sq inch</td>
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<td>Elongation in 2 in slow loading %</td>
<td>200%</td>
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<tr>
<td>Max load 30 secs lbs/in²</td>
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</tr>
<tr>
<td>Max load 5 minutes lbs/in²</td>
<td>4,000</td>
</tr>
<tr>
<td>Safe load sustained lbs/in²</td>
<td>300</td>
</tr>
</tbody>
</table>
x = 0.125in

y (a) for template with
\[ \Theta = 0.1 \]
\[ = x + 0.058 \]
\[ = 0.183 \]

y (b) for template with
\[ \Theta = 0.2 \]
\[ = x + 0.116 = 0.241 \]

**TEMPLATE DETAILS**

Figure 13
Follower Guide  
Template  
Magnetic Table

Cutting Tool

I SETTING

Figure 14(a)

II SETTING

Magnetic Table

Work Piece

Spacer

JOB ARRANGEMENT FOR MACHINING

Figure 14(b)
INSPECTION OF CO-ORDINATE MEASUREMENT

Figure 15
GENERAL ARRANGEMENT OF VIBRATION APPARATUS WITH ELECTROMAGNETIC EXCITER

Figure 17
PLATE 5

A SET OF BLADES
DISCUSSION OF RESULTS
CHAPTER V

Discussion of Results

The differential equations of motion of twisted blading under arbitrary loading have been derived based on the analysis due to Houbolt (1). A special feature of the analysis is the consideration of second order terms. The axis of centre of flexure is chosen as one of the reference axes which allowed for coupling terms in a more general form. This theory can therefore be applied to problems where second order terms are significant and to evaluate the less complete theories which are more easily applied.

A comprehensive comparison of the author's equations of motion based on Houbolt's analysis and Montoya's theory and Dawson's analysis is made in Appendix IE. Looking into the three sets of equations in Appendix IE, it can be seen that there is very close agreement between the author's terms and Montoya's terms. The second order terms included by the author $EB_2 \beta \sin \phi'$ compares with $EvJx\phi'$ as used by Montoya. Similarly $EB_2 \beta' \phi' \cos \beta$ of the author's term compares with Montoya's $EvJy\phi'$. Another second order term included by the author, namely $EB_1 (\beta')^2 \phi'$ compares with Montoya's term namely $Ev^2J\phi'$. In Dawson's equations none of these second order terms are included.
It can also be noticed from Appendix IE and equations (24) and (25) in Chapter II that the moments of higher order $B_1$ and $B_2$ include the effect of the radius of gyration.

It should be noted from the strain equation (A20) with respect to $v_1$ and $w_1$ that the strain is not only proportional to the derivatives $v_1''$ and $w_1''$ but also to the additional terms which involve $\beta'$, the rate of change of pretwist. In Dawson's equations this effect is not apparent.

The derivations for strain is based on the assumption that $u$, $v$, $w$ and $\phi$, the displacements in three perpendicular directions and the angle of twisting deformation are small. For longer beams when the angle of twisting deformation is large then non-linear effects could be present and cause errors. A closer look at the equations for the second derivatives of $v''$, $w''$ and derivative $\phi'$ reveals that they are complex functions involving such parameters as the distance between the centre of gravity and centre of flexure, moments of inertia and the moments of higher $B_1$ and $B_2$ etc.

The derivations of these blade constants for a range of thickness parameters have been fully described in Chapter II and computed values based on the derived formulas are presented in tables 2-12. They are also presented graphically in figures 3-12. These tables and figures cover a set of blades having
widths ranging from 1 in to 1½ in with increments of 0.125 in.
The thickness parameter θ is varied between 0.1 and 0.3 for
the above blades. This gives a range of analysis for
equivalent breadth to depth b/d ratio for rectangular blading
of 15.7 to 1 to 5.2 to 1.

From figures 3-12 and tables 2-12 it can be noticed that for
a particular blade width the increase in blade thickness for
a constant width reduces the distance between centre of
gravity and centre of flexure. Thus the thicker the beam the
closer is the distance between centre of gravity and centre
of flexure and assuming that the thickness affected no other
parameter the smaller the amount of coupling between the
bending and torsion motion. It is also seen that for a
particular breadth to thickness ratio (b/d) beam or for a
particular value of θ the increase in blade thickness increases
the distance between centre of gravity and centre of flexure.

The value of the radius of gyration increases steeply with the
increase of blade width as can be seen in figure 7. Other
blade constants such as I₁, I₂, I Cf, area, torsional coefficient,
torsional stiffness constant, coefficients of coupling B₁ and
B₂ can be seen to increase steadily with the increase of width
or thickness parameter. It is apparent from these results
that the coupling between bending and torsion is a complex
function of the various parameters and the individual effect
of the various parameters and the individual effect of a single
parameter on the amount of coupling cannot be estimated.
An interesting fact to be noted about the derivation of the position of centre of flexure is that the value $4/91C$, the distance between centre of gravity and centre of flexure, is based on Griffith's assumption namely Poisson's ratio is negligible. This value $4/91C$ is the same as derived by Duncan (12) if Poisson's ratio is omitted from his equation. Hence it can be concluded that for very thin beams the position of centre of flexure relative to the centre of gravity is directly proportional to the width of the beams (provided the shape of the cross section is cubical oval). On the other hand for non thin beams the position of centre of flexure becomes a complex function of $\theta$ and $C$. Variation of centre of flexure with reference to width and thickness parameter can be seen in table 4 and figure 4.

The object of the investigation has been to study the importance of allowing for second order terms when calculating the natural frequencies of vibration of straight and pretwisted cubical oval cross section beams and to compare the results with appropriate practical tests. A convenient method presenting the natural frequencies of pretwisted beams is in terms of frequency ratio $P/Py$ where $P$ is the natural frequency of vibration and $Py$ is the fundamental natural frequency of an identical beam without pretwist. The frequency ratio $P/Py$ is a function of pretwist angle, second moments of area and second order terms and is independent of material properties.

Theoretical frequencies were obtained for two sets of beams with pretwist angles 0, 15, 30, 45, 60, 75 and 90 degrees and
experimental natural frequencies were determined up to the third mode. The graphs of P/Py ratios against pretwist angle for the two sets of beams are presented in figures 18 and 19 and the calculated results are shown in tables 15 and 16.

Referring to figures 18-21 it can be seen that no apparent constant trend is discernible regarding the effect of allowing for second order terms in the differential equations of motion of the blading. The agreement between theoretical and experimental results is not that satisfactory and one possible explanation for this is that the method is numerically ill-conditioned and that inaccurate results are being obtained at particular pretwist angles. It was considered that the determinant routine was one possible source of numerical ill-conditioning and in order to test this another determinant routine was substituted into the main programme and the comparative test results obtained. It was found however that there was no difference in natural frequency values and it was concluded that if ill-conditioning was present it was in the integration part of the procedure rather than the determinant evaluation procedure. It was found that numerical instability occurred at the 4th and 5th modes as evidenced by an oscillation of the determinant against frequency graph within the region of a natural frequency. Neither Dawson (17), Ghosh (18) or Hemingway (16) experienced this problem for modes lower than the 5th and it is considered that for the particular parameters associated with this problem the method is numerically
sensitive and the results for the lower modes may not be accurate at particular pretwist angles. Further work is required to develop an alternative method of solution for the equations of motion in order to verify the results obtained in this work.

Considering the theoretical and experimental results presented in figures 18-21 it can be seen that the measured fundamental frequencies of vibration show a slight increase of frequency with the increase of pretwist angles. This is applicable to both sets of cubical oval cross section beams. On the other hand when the second order terms are included in the calculations the variation of the theoretical frequencies from the practical values for the I mode, increases with the increase of pretwist angle.

The percentage variation from the practical values is from 12.1% to 31.25% on the thinner blade. This variation rate is still more relevant in the thicker blading the rate steadily increasing. Variation goes from 18.64% to 54.85%. When the second order terms are dropped from the calculations of the 1st natural frequencies the results obtained are closer to the practical values. The variation rate is as low as 3.22% at 60 degrees pretwist on thicker blade. Average variation is less than 20% on thin blades whereas the average variation is less than 10% for thicker blade. However, one point to be noted here is that on 90 degree pretwist the variation rate is 43.7% for the thinner blade. The variation rate drops steadily with the increase of pretwist angle for the thicker
blading however closer agreement in both cases are obtained without second order terms. The variation between thick and thin blades can be attributed to the increase or decrease in values of the blade constants with the increase of thickness. These can be seen from the figures 3-12 and tables 2-12 but because of the interrelation of the various parameters it is not possible to determine the effect of any one single parameter on the natural frequencies. The variations of the theoretical natural frequencies with reference to the measured values can be seen from tables 15-16 and figures 18-21. Also percentage variations are presented in tables 11-18 and figures 22-27.

When the second natural frequencies are considered the results for the thinner blade including the second order terms, show a decreasing trend. However at 90 degrees pretwist percentage variation is 21.55%. The lowest variation is 2.89% at 60 degrees pretwist. The percentage variation of frequencies of thin blades when second order terms are dropped from the calculations show a zig-zag trend with the highest variation being 18.24% at 60 degrees pretwist angle. In the case of the thicker blading the variation rates are low and results are closer to the practical values when the second order terms are included in the calculations. When the second order terms are dropped the difference between theoretical and practical values do not seem to follow any particular pattern. However, from the tables 15-18 and figures 18-27 it can be seen that closer agreement is reached with practical values when the second order terms are included in the computations for the second natural frequencies.
Considering the third mode frequencies for the thinner blading close agreement is reached when second order terms are excluded from calculations. The highest percentage variation being 9.84% at 60 degrees pretwist. However, the lowest percentage variation is recorded (0.24%) when the second order terms are included in the calculations. For the thicker blading the agreement is closer to the practical results when the second order terms are included in the calculations, the variations being 1.26% and 1.57% at 30 degrees and 60 degrees pretwist respectively.

An analysis is also presented to compare the variations on a percentage basis between the calculated frequencies including second order terms and the frequencies calculated excluding second order terms. The results in this form are presented in table 19 and figures 28 and 29. The variation of the natural frequencies both of the thick and thin blading when second order terms are included can be compared with that of frequencies when second order terms are excluded.

One factor to be noted is that for straight beams or beams of 0 degree pretwist, the calculated frequencies are the same whether the second order terms are included or excluded. In these cases it can thus be concluded that the fundamental frequencies of straight beams are not affected by second order terms and that the second order terms are present as a result of the pretwist. This is also apparent by examination of the frequency equations.
It can be stated in general that the theoretical natural frequency ratios of the fundamental mode can be considered as independent of the centre of flexure co-ordinates but is dependent on second order terms. The theoretical frequency ratios for II and III mode frequencies are dependent on both the centre of flexure co-ordinates and the second order terms, and that the effect of the second order terms is a complex function of the various parameters of the blading and the pretwist angle.
### Table 15

<table>
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<tr>
<th>Pretwist Angle</th>
<th>Freq Hz</th>
<th>P/Py ratio</th>
<th>Freq Hz</th>
<th>P/Py ratio</th>
<th>Freq Hz</th>
<th>P/Py ratio</th>
<th>Freq Hz</th>
<th>P/Py ratio</th>
<th>Freq Hz</th>
<th>P/Py ratio</th>
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<td></td>
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<td>75°</td>
<td>90°</td>
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<td><strong>Practical Results</strong></td>
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<td>I Mode</td>
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<td>1.03</td>
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<td>1.09</td>
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<td>25</td>
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<td>I Mode</td>
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<td>35</td>
<td>1.12</td>
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<td>1.12</td>
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<td>13.07</td>
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<td>13.10</td>
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<td>13.42</td>
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NATURAL FREQUENCIES OF CUBICAL OVAL BLADES EQUIVALENT b/d RATIO 15.7:1; θ = 0.1
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<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>62°</th>
<th>61°</th>
<th>60°</th>
<th>59°</th>
<th>1.01</th>
<th>1.03</th>
<th>1.04</th>
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<td>62</td>
<td>61</td>
<td>60</td>
<td>60</td>
<td>1.18</td>
<td>1.19</td>
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<td></td>
<td>283</td>
<td>231</td>
<td>195</td>
<td>159</td>
<td>1.04</td>
<td>1.03</td>
<td>1.01</td>
</tr>
<tr>
<td>III Mode</td>
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<td>P/Py ratio</td>
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<td>6.1</td>
<td>4.78</td>
<td>3.9</td>
<td>3.42</td>
<td>1.83</td>
<td>1.01</td>
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</table>

**TABLE 16**

NATURAL FREQUENCIES OF CUBICAL OVAL BLADES OF EQUIVALENT B/D RATIO 7.9:1; θ = 0.2

DEPLETED TERMS ARE SECOND ORDER INCLUDING RESULTS

PRACTICAL CALCULATED TERMS ARE SECOND ORDER INCLUDING RESULTS WHEN

CALCULATED RESULTS
### TABLE 17

PERCENTAGE VARIATIONS OF CALCULATED NATURAL FREQUENCIES FROM MEASURED FREQUENCIES

**EQUIVALENT b/d RATIO 15.7:1; θ = 0.1**

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<th>Pretwist Angle</th>
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<th>30</th>
<th>60</th>
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<td><strong>% VARIATIONS OF FREQUENCIES FROM PRACTICAL RESULTS WHEN SECOND ORDER TERMS ARE INCLUDED</strong></td>
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### Table 18

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<td><strong>% Variations of Practical Results when Second Order Terms are Included</strong></td>
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<td>3.27</td>
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<td>13.21</td>
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TABLE 19

PERCENTAGE VARIATIONS BETWEEN CALCULATED NATURAL FREQUENCIES SECOND ORDER EFFECTS INCLUDED AND OMITTED

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<tr>
<th>Pretwist Angle</th>
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<th>30</th>
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<td>11.37</td>
<td>11.64</td>
<td>12.32</td>
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</tbody>
</table>
FREQUENCY RATIO $P/P_y$ of pretwisted cubical oval blade of equivalent $b/d$ ratio 15.7:1, $\theta = 0.1$

- ○ Results with second order terms
- △ Results without second order terms
- × Practical Results

Figure 18
P/Py RATIO OF PRETWISTED CUBICAL-OVAL BLADE OF EQUIVALENT
b/d RATIO 7.9:1 Θ = 0.2

○ Results with Second Order Terms
△ Results without Second Order Terms
× Practical Results

Figure 19
Frequency of pretwisted cubical oval blade of equivalent b/d ratio 15.7:1 θ = 0.1

○ Results with second order terms
△ Results without second order terms
× Practical Results

Figure 20.
FREQUENCY OF PRETWISTED CUBICAL OVAL BLADE OF EQUIVALENT

b/d RATIO 7.9:1  Θ = 0.2

○ Results with second order terms
△ Results without second order terms
× Practical Results

Figure 21
EQUIVALENT b/d RATIO 15.7:1; $\Theta = 0.1$ FIRST MODE

PERCENTAGE VARIATION OF FREQUENCIES FROM PRACTICAL RESULTS

- Second Order terms included
- Second Order terms not included

Figure 22
Percentage variation of frequencies from practical results

○ Second Order terms included
× Second Order terms not included

Figure 23
EQUIVALENT b/d RATIO 15.7:1; \( \theta = 0.1 \) SECOND MODE
PERCENTAGE VARIATION OF FREQUENCIES FROM PRACTICAL RESULTS

○ Second Order terms included
× Second Order terms not included

Figure 24
EQUIVALENT b/d RATIO 7.9:1; θ = 0.2 SECOND MODE

PERCENTAGE VARIATION OF FREQUENCIES FROM PRACTICAL RESULTS

○ Second Order terms included
× Second Order terms not included

Figure 25
EQUIVALENT b/d RATIO 15.7:1; $\theta = 0.1$ THIRD MODE
PERCENTAGE VARIATION OF FREQUENCIES FROM PRACTICAL RESULTS

- O Second Order terms included
- x Second Order terms not included

Figure 26
EQUIVALENT b/d RATIO 7.9:1; θ = 0.2 THIRD MODE
PERCENTAGE VARIATION OF FREQUENCIES FROM PRACTICAL RESULTS

○ Second Order terms included
× Second Order terms not included

Figure 27
EQUIVALENT b/d RATIO 15.7:1; Θ = 0.1
PERCENTAGE VARIATION BETWEEN CALCULATED FREQUENCIES SECOND ORDER TERMS INCLUDED AND DROPPED.

I Mode
II Mode
III Mode

PERCENTAGE VARIATION BETWEEN CALCULATED FREQUENCIES SECOND ORDER TERMS INCLUDED AND DROPPED.
EQUIVALENT b/d RATIO 7.9:1; Ω = 0.2
PERCENTAGE VARIATION BETWEEN CALCULATED FREQUENCIES SECOND ORDER TERMS INCLUDED AND DROPPED

- I Mode
- II Mode
- III Mode

PERCENTAGE VARIATION

PRETWIST ANGLE
CHAPTER VI

CONCLUSIONS
CHAPTER VI

Conclusions

1) The interrelation of the various parameters of cubical oval cross section blading such as moments of inertia, moments of higher order terms, position of centre of flexure with respect to the centre of gravity, area etc in relation to the thickness parameter has been determined. It can be stated that the thicker the beam the closer is the distance between centre of gravity and centre of flexure and assuming that the thickness affected no other parameter, the smaller the amount of coupling between the bending and torsion motion. Also for very thin blades the position of centre of flexure relative to the centre of gravity is directly proportional to the width of the beams.

2) The differential equations of motion allowing for second order terms of twisted cubical oval cross section blading have been developed based on the work of Houbolt and compared to the equations developed by Montoya and Dawson.

3) The transformation method initially presented by Dawson for the solution of the differential equations of motion neglecting higher order terms has been extended to the solution of the differential equations of motion allowing for the second order terms.
4) A comparison has been made of the theoretical natural frequencies derived with and without allowance for the second order terms against experimental natural frequencies for cubical oval cross section beams of equivalent breadth to depth ratios (b/d) of 15.7:1 and 7.9:1 and pretwist angles in the range 0-90 degrees. It may be concluded in general that the theoretical natural frequency ratios of fundamental mode are independent of centre of flexure co-ordinates but dependent on the second order terms and the theoretical natural frequency ratios for the II and III mode frequencies are dependent on both the centre of flexure co-ordinates and the second order terms and that the effect of the second order terms is a complex function of the various parameters of the blading and the pretwist angles.

Further work is required to develop an alternative method of solution for the equations of motion in order to verify the results obtained in this work.
A Derivation of Longitudinal Strains

When a twisted beam undergoes both lateral and twisting deformations longitudinal strain occurs. An imaginary plane is considered to cut through the beam perpendicular to the elastic axis. The location of a fibre $f$ of the cross section both before and after deformation may be given as per sketches A and B. The $x$ axis or longitudinal axis is normal to the plane of the paper and is assumed to coincide with the undeformed position of the elastic axis. The initial position of fibre $f$ is assumed to be at a distance of $z$ and $y$ along $z$ and $y$ axes. The position of fibre $f$ after deformation is assumed to be at a distance of $z_1$ and $y_1$ along $z$ and $y$ axes. An equation for longitudinal strain is developed as follows:

\[
\begin{align*}
  z &= \eta \cos \beta - \{ \sin \beta \} \\
  y &= \eta \sin \beta + \{ \cos \beta \}
\end{align*}
\]

The rate of change of these positions with respect to $x$ are

\[
\begin{align*}
  z' &= -\eta \beta' \sin \beta - \{ \beta' \cos \beta \\
  &= -\beta'y \\
  y' &= \eta \beta' \cos \beta - \{ \beta' \sin \beta \\
  &= \beta'z
\end{align*}
\]
Considering the beam displacements to occur such that the point of intersection of the elastic axis and the cutting plane moves the distances, $u$, $v$ and $w$ in the directions of $x$, $z$ and $y$ respectively. It is assumed that the cutting plane remains perpendicular to the elastic axis and moved around it by an angle $\phi$. Assuming the angle $\phi$ is small, the following trigonometric equations are arrived at

$$
\cos(\beta + \phi) = \cos \beta - \phi \sin \beta \quad \text{and} \quad \sin(\beta + \phi) = \sin \beta + \phi \cos \beta
$$

The new position of the fibre is defined by the following equations:

$$
x_1 = x + u - v^1 (z^1 - v) - w^1 (y^1 - w) = x + u - v^1 z - w^1 y \tag{A5}
$$

$$
y_1 = w + y^1 (\sin \beta + \phi \cos \beta) + \{ (\cos \beta - \phi \sin \beta) = w + y + z \phi \tag{A6}
$$

$$
z_1 = v + z^1 (\cos \beta - \phi \sin \beta) - \{ (\sin \beta + \phi \cos \beta) = v + z - y \phi \tag{A7}
$$

The derivatives of $x_1$, $y_1$ and $z_1$ with respect to $x$ given as follows:

$$
X^1_1 = 1 + u^1 - v^{1^1} z - z^1 v^1 - w^{1^1} y - y^1 w^1
$$
\[ 1 + u^1 - v^1 y - w^1 \beta^1 y - w^1 \beta^1 z \]
\[ = 1 + u^1 - v^1 z + v^1 \beta^1 y - w^1 y - w^1 \beta^1 z \] \text{ .......... A8}

Similarly

\[ y_1^1 = w^1 + y^1 + z^1 \phi + \phi^1 z \]
\[ = w^1 + \beta^1 z + \phi^1 z + \phi^1 \]
\[ = w^1 + z (\beta^1 + \phi^1) - \phi^1 y \] \text{ .......... A9}

\[ z_1^1 = v^1 + z^1 - y^1 \phi - \phi^1 y \]
\[ = v^1 - \beta^1 y - \beta^1 z^1 - \phi^1 y \]
\[ = v^1 - y (\beta^1 + \phi^1) - z \phi \beta^1 \] \text{ .......... A10}

The longitudinal strain developed in a fibre may be found from these equations by considering the amount an elastic fibre of length \( ds \) changes in length because of deformation. The final length \( ds_1 \) of a fibre can be expressed in terms of differential components of length in \( x, y \) and \( z \) directions.

\[ ds_1^2 = dx_1^2 + dy_1^2 + dz_1^2 \] \text{ .......... A11}

\[ (\frac{ds_1}{dx_1})^2 = (x_1^1)^2 + (y_1^1)^2 + (z_1^1)^2 \] \text{ .......... A12}

Substituting the values of \( x_1^1, y_1^1 \) and \( z_1^1 \) from equations A8-A10

\[
\frac{(ds_1)}{(dx)} = 1 + 2 \left\{ u^1 - z (v^1 + \beta^1 w^1) - y (w^1 - \beta^1 v^1) \right\} + \\
y^2 (\beta^1)^2 - 2 y \beta^1 (v^1 - z \beta^1 \phi - y \phi^1) + \\
z^2 (\beta^1)^2 + 2 z \beta^1 (w^1 + z \phi^1 - y \beta^1 \phi)
\]
\[ = 1 + (y^2 + z^2) (\beta^1)^2 + 2 \left[ u^1 - zv^{11} - z\beta^1 w^1 - yw^{11} + y\beta^1 v^1 \right. \\
- y\beta^1 v + y (\beta^1)^2 z\phi + y^2 \beta^1 \phi^1 + z\beta^1 w^1 + z^2 \beta^1 \phi^1 - z (\beta^1)^2 y\phi \left. \right] \]

\[ = 1 + (y^2 + z^2) (\beta^1)^2 + 2 \left[ u^1 - zv^{11} - yw^{11} + \beta^1 \phi^1 (y^2 + z^2) \right] \]

\[ \frac{ds_1}{dx} = \sqrt{1 + (y^2 + z^2) (\beta^1)^2 + 2 \left[ u^1 - zv^{11} - yw^{11} + \beta^1 \phi^1 (y^2 + z^2) \right]} \]

To find out the analogous equation of the original length, let \( u = v = w = \phi = 0 \)

then \( \frac{ds}{dx} = \sqrt{1 + (y^2 + z^2) (\beta^1)^2} \)

Tensile strain in the fibre considered may be written as

\[ e_1 = \frac{ds_1}{ds} - \frac{ds_1}{ds} - 1 \]

\[ = \frac{ds_1}{ds} - 1 \]

\[ = \left\{ 1 + (y^2 + z^2) (\beta^1)^2 + 2 (u^1 - zv^{11} - yw^{11} + \beta^1 \phi^1 [y^2 + z^2]) \right\}^{\frac{1}{2}} - 1 \]

\[ = \left\{ 1 + \frac{2}{1 + (y^2 + z^2) (\beta^1)^2} (u^1 - zv^{11} - yw^{11} + \beta^1 \phi^1 [y^2 + z^2]) \right\}^{\frac{1}{2}} - 1 \]

\[ = + \frac{1}{1 + (y^2 + z^2) (\beta^1)^2} \left[ u^1 - zv^{11} - yw^{11} + \beta^1 \phi^1 (y^2 + z^2) \right] \]

Generally the term \( \frac{1}{1 + (y^2 + z^2) (\beta^1)^2} \) can be regarded as unity because the value of \( (y^2 + z^2) (\beta^1)^2 \) will have values less than 0.03.
Hence \( e_1 = u^i - zv^{ii} - yw^{ii} + (y^2 + z^2)\beta'^1\phi'^1 \)  

Making use of the equations A1 and A2 the strain can be expressed in terms of cross sectional co-ordinates \( \eta \) and \( \xi \) as follows:

\[
e_1 = u^i - v^{ii}(\eta \cos\beta - \xi \sin\beta) - w^{ii}(\eta \sin\beta + \xi \cos\beta) \\
+ \left\{ \eta^2 \cos\beta + \xi^2 \sin^2\beta - 2\eta\xi \sin\beta \cos\beta + \eta^2 \sin^2\beta \\
+ \xi^2 \cos^2\beta - 2\eta\xi \sin\beta \cos\beta \right\} \beta'^1 \phi'^1
\]

\[
e_1 = u^i - v^{ii}\eta \cos\beta + v^{ii}\xi \sin\beta - w^{ii}\eta \sin\beta - w^{ii}\xi \cos\beta + \beta'^1 \phi'^1 (\eta^2 + \xi^2)
\]

The cross section is symmetrical about major axis. The integral of the longitudinal stress over the cross section will be equal to total tension. Considering the equilibrium condition \( u^i \) can be eliminated as follows:

\[
T = E \iint_{\eta\in \mathbb{R}} \iint_{\xi\in \mathbb{R}} e_1 d\xi d\eta
= E \iint \left[ u^i - \eta (v^{ii}\cos\beta + w^{ii}\sin\beta) - \xi (w^{ii}\cos\beta - v^{ii}\sin\beta) \\
+ \beta'^1 \phi'^1 (\eta^2 + \xi^2) \right] d\xi d\eta
\]

\[
\frac{T}{E} = \iint u^i d\xi d\eta - \iint \eta (v^{ii}\cos\beta + w^{ii}\sin\beta) d\xi d\eta \\
- \iint \xi (w^{ii}\cos\beta - v^{ii}\sin\beta) d\xi d\eta \\
+ \iint (\xi^2 + \eta^2) \beta'^1 \phi'^1 d\xi d\eta
\]
\[ u^1 = -Ae(v^1 \cos \beta + w^1 \sin \beta) + \beta^1 \varphi^1 AK^2 \] \hspace{1cm} \text{A16}

\[ \frac{T}{EA} = u^1 - e(v^1 \cos \beta + w^1 \sin \beta) + K_A^2 \beta^1 \varphi^1 \] \hspace{1cm} \text{A17}

\[ u^1 = \frac{T}{EA} + e(v^1 \cos \beta + w^1 \sin \beta) - K_A^2 \beta^1 \varphi^1 \] \hspace{1cm} \text{A18}

Let \( \frac{T}{EA} = e_T \)

then \[ u^1 = e_T + e(v^1 \cos \beta + w^1 \sin \beta) - K_A^2 \beta^1 \varphi^1 \] \hspace{1cm} \text{A19}

From equations A15 and A19

\[ e_1 = e_T + e(v^1 \cos \beta + w^1 \sin \beta) - K_A^2 \beta^1 \varphi^1 - \eta (v^1 \cos \beta + w^1 \sin \beta) \]
\[ - \xi (-v^1 \sin \beta + w^1 \cos \beta) + (\eta^2 + \xi^2) \beta^1 \varphi^1 \]
\[ = e_T + (e - \eta)(v^1 \cos \beta + w^1 \sin \beta) + \xi (v^1 \sin \beta - w^1 \cos \beta) \]
\[ + (\eta^2 + \xi^2 - K_A^2) \beta^1 \varphi^1 \]

Hence the complete expression for the strain of any fibre in the cross section can be expressed as

\[ e_1 = e_T + (e - \eta)(v^1 \cos \beta + w^1 \sin \beta) + \xi (v^1 \sin \beta - w^1 \cos \beta) \]
\[ + (\eta^2 + \xi^2 - K_A^2) \beta^1 \varphi^1 \] \hspace{1cm} \text{A20}

and the corresponding stress can be expressed as

\[ f = E \left\{ e_T + (e - \eta)(v^1 \cos \beta + w^1 \sin \beta) + \xi (v^1 \sin \beta - w^1 \cos \beta) \right\} \]
\[ + (\eta^2 + \xi^2 - K_A^2) \beta^1 \varphi^1 \] \hspace{1cm} \text{A21}
It is of some interest to show the development of strain in terms of displacements of the chord and perpendicular to that of chord. These are shown in an enlarged form in figure C. From general trigonometric relationships

\[ v = v_1 \cos \beta - w_1 \sin \beta \] \hspace{1cm} B1

\[ w = v_1 \sin \beta + w_1 \cos \beta \] \hspace{1cm} B2

\[ v' = v_1' \cos \beta - v_1 \sin \beta \beta' - w_1' \sin \beta - w_1 \cos \beta \beta' \] \hspace{1cm} B3

\[ v'' = v_1'' \cos \beta - v_1' \beta' \sin \beta - v_1 \beta'' \sin \beta - v_1 (\beta')^2 \cos \beta \] 
\[ - w_1'' \sin \beta - w_1' \beta' \cos \beta - w_1 \beta' \cos \beta - w_1 \beta'' \cos \beta + w_1 (\beta')^2 \sin \beta \] \hspace{1cm} B4

\[ w' = v_1' \sin \beta + v_1 \cos \beta \beta' + w_1' \cos \beta - w_1 \sin \beta \beta' \] \hspace{1cm} B5

\[ w'' = v_1'' \sin \beta + v_1' \beta' \cos \beta + v_1 \beta' \cos \beta - v_1 (\beta')^2 \sin \beta + v_1 \beta'' \cos \beta \] 
\[ + w_1'' \cos \beta - w_1' \beta' \sin \beta - w_1 \beta' \sin \beta - w_1 (\beta')^2 \cos \beta + w_1 \beta'' \sin \beta \] \hspace{1cm} B6

\[ v'' \cos \beta + w'' \sin \beta = v_1'' \cos \beta - v_1' \beta' \sin \beta \cos \beta - v_1 \beta'' \sin \beta \cos \beta \] 
\[ - v_1' \beta' \sin \beta \cos \beta - v_1 (\beta')^2 \cos \beta - w_1'' \sin \beta \cos \beta \] 
\[ - w_1' \beta' \cos \beta - w_1 \beta' \cos \beta - w_1 \beta'' \cos \beta + w_1 (\beta')^2 \sin \beta \] 
\[ + w_1 (\beta')^2 \sin \beta \cos \beta \] 
\[ + v_1' \sin \beta' + v_1' \beta' \sin \beta \cos \beta + v_1 \beta' \sin \beta \cos \beta \] 
\[ - v_1 (\beta')^2 \sin \beta + w_1' \beta' \sin \beta \cos \beta + w_1 \beta' \sin \beta \cos \beta \] 
\[ - w_1 (\beta')^2 \sin \beta \cos \beta + w_1 \beta'' \sin \beta \] 
\[ = v_1'' (\sin \beta + \cos \beta) - v_1 (\beta')^2 \cos \beta \]
\[ w'' \cos \beta - v'' \sin \beta = -v'' \sin \beta \cos \beta + v_1 \beta' \sin \gamma \beta + v_1 \beta'' \sin \gamma \beta + v_1 \beta' \sin \beta + w_1 \beta' \sin \beta \cos \beta + w_1 \beta'' \sin \beta \cos \beta \]
\[ = w_1\beta'' (\sin \beta + \cos \gamma \beta) + 2v_1 \beta' (\sin \gamma \beta + \cos \gamma \beta) + v_1 \beta'' - w_1 (\beta')^2 (\sin \gamma \beta + \cos \gamma \beta) \]
\[ = w_1\beta'' + 2v_1 \beta' + v_1 \beta'' - w_1 (\beta')^2 \quad \ldots \ldots \quad B8 \]

These results obtained by B7 and B8 are substituted in equation A20. The value of strain \( e_t \) is then given by

\[ e_t = e_1 + (e - \eta) \left[ v_1\beta'' - 2\beta' w_1 - \beta'' w_1 - (\beta')^2 v_1 \right] \]
\[ - \left[ w_1\beta'' + 2\beta' v_1 + \beta'' v_1 - (\beta')^2 w_1 \right] + (\eta^2 + \xi^2 - K_\alpha^2) \beta \phi \]
\[ \ldots \ldots \quad B9 \]

The interesting fact to be noted here is that the strain is not only proportional to the derivatives \( v_1'' \) and \( w_1'' \) but also to the additional terms which involve the rate of change of pretwist.
The figure d shows the x, y, z axes system and deformed positions \( x_1, y_1 \) and \( z_1 \) of the mass particle considered in Appendix IA.

Let \( i, j \) and \( k \) be the unit vectors and let \( ax, ay \) and \( az \) be the acceleration vectors in x, y and z directions respectively.

\[
\mathbf{r} = x_1 \mathbf{i} + z_2 \mathbf{j} + y_1 \mathbf{k}
\]

Differentiating with respect to time,

\[
\dot{\mathbf{r}} = \dot{x}_1 \mathbf{i} + \dot{z}_2 \mathbf{j} + \dot{y}_1 \mathbf{k}
\]

Differentiating again with respect to time,

\[
\ddot{\mathbf{r}} = \ddot{x}_1 \mathbf{i} + \ddot{z}_2 \mathbf{j} + \ddot{y}_1 \mathbf{k} \quad \text{or} \quad \ddot{\mathbf{r}} = a_x \mathbf{i} + a_z \mathbf{j} + a_y \mathbf{k}
\]

From Appendix IA

\[
x_1 = x + u - v^t z - w^t y
\]
\[
z_2 = v + z - y\phi + e \quad (\therefore z_2 = z_1 + e)
\]
\[
y_1 = w + y + z\phi
\]

Differentiating the above three equations with respect to time.
\[ x_1 = \ddot{u} - \dot{v}'z - \dot{w}'y \] ........................................... C2

\[ \dot{z}_2 = \dot{v} - y\ddot{\theta} \] ........................................... C3

\[ \dot{y}_1 = \dot{w} + z\ddot{\theta} \] ........................................... C4

Differentiating again with respect to time.

\[ \ddot{x}_1 = \dddot{u} - \dot{v}'z - \dot{w}'y \] ........................................... C5

\[ \ddot{z}_2 = \ddot{v} - y\dddot{\theta} \] ........................................... C6

\[ \ddot{y}_1 = \dddot{w} + z\dddot{\theta} \] ........................................... C7

The desired components \( a_x, a_y, a_z \) of the acceleration vector in \( x, y \) and \( z \) directions can be written in the following form:

\[ a_x = \dddot{u} - \dot{v}'z - \dot{w}'y \] ........................................... C8

\[ a_y = \dddot{w} + z\dddot{\theta} \] ........................................... C9

\[ a_z = \dddot{v} - y\dddot{\theta} \] ........................................... C10

It is already known by trigonometrical relationship that

\[ y = \eta \sin\beta + \xi \cos\beta \] and \( z = \eta \cos\beta - \xi \sin\beta \)

Substituting the values of \( y \) and \( z \) in equations C8-C10

\[ a_x = \dddot{u} - \eta (\dot{v}'\cos\beta + \dot{w}'\sin\beta) + \xi (\dot{v}'\sin\beta - \dot{w}'\cos\beta) \] ...... C11

\[ a_y = \dddot{w} + \eta \cos\beta\ddot{\theta} - \xi \sin\beta\ddot{\theta} \] ........................................... C12

\[ a_z = \dddot{v} - \eta \sin\beta\ddot{\theta} - \xi \cos\beta\ddot{\theta} \] ........................................... C13

The force and moment loadings in \( x, y \) and \( z \) directions, namely \( \bar{P}_x, \bar{P}_y, \bar{P}_z \) and \( \bar{q}_x, \bar{q}_y \) and \( \bar{q}_z \) are then given by the
following integrals.

\[
\bar{P}_x = - \iiint ax f \, d\xi d\eta \\
\bar{P}_z = - \iiint ay f \, d\xi d\eta \\
\bar{P}_y = - \iiint az f \, d\xi d\eta
\]

\[
\bar{q}_x = - \iiint [\text{[- ax(y1 - w) + ay(z1 - v)]}] f \, d\xi d\eta \\
\bar{q}_z = - \iiint ax(y1 - w)f \, d\xi d\eta \\
\bar{q}_y = - \iiint ax(z1 - v)f \, d\xi d\eta
\]

For practical purposes ax is considered to be negligible and hence \(\bar{P}_x, \bar{q}_z\) and \(\bar{q}_y\) becomes = 0 and the other three values are given by the following equations.

\[
\begin{align*}
\bar{P}_z &= - m(\ddot{v} - e\dot{\theta} \sin\beta) \\
\bar{P}_y &= - m(\ddot{w} + e\dot{\theta} \cos\beta) \\
\bar{q}_x &= m(e(\dot{v} \sin\beta - \dot{w} \cos\beta) - mK_{an}^2 \dot{\theta})
\end{align*}
\]
D Derivation of Differential Equations

It is assumed that the x axis of the x, y, z co-ordinate axes system lies outward along the length of the blade and is coincident with the undeformed position of the elastic axis. Tension in the beam is denoted by T. The t and t axes, with the origin at the elastic axis (t being the major axis) move with the cross section. The blade deformations are denoted by a displacement v along z axis, a displacement w along y axis and a rotation $\phi$ about the elastic axis. The built in twist $\beta$ and also $\phi$ are positive when the leading edge is up. The figures E, F and G represent co-ordinates, displacements and moments. The stress distributions over the cross section may be resolved into effective internal resisting moments at the elastic axis positions. In order to determine these moments, the inclination relative to the elastic axis of the general beam fibre due to initial twist and twisting deformation must be considered. The stress along this fibre is resolved into two components one parallel to elastic axis and one in a plane perpendicular to the elastic axis. These components are represented by figure H.

From the longitudinal component, the flapwise bending moment $M_1$, and chordwise bending moment $M_2$ can be given as follows:

\[
M_1 = - \int_{\eta_{te}}^{\eta_{te} + \beta} \int_{\xi_{te} + \beta}^{\xi_{te} - \beta} f_1 \xi d\xi d\eta \\
M_2 = - \int_{\eta_{te}}^{\eta_{te} + \beta} \int_{\xi_{te} + \beta}^{\xi_{te} - \beta} f_1 \eta d\eta d\eta 
\]

\[\text{D1, D2}\]
The component in the plane normal to elastic axis leads to an effective torsional resisting moment. This includes St Venant's type torsional term C10. And the resisting moment Q can be given by

\[ Q = C10 \phi' + \int_{\eta_{te}}^{t/2} \int_{-t/2}^{t/2} f(\beta + \phi)' (\eta^2 + \xi^2) d\xi d\eta \] ...... D3

The choice of the elastic axis as a reference axis is significant. Elastic axis does not necessarily coincide with centroidal axis. If some other axis was chosen as reference axis then the shearing stresses associated with longitudinal stresses would contribute to the total resisting torque. Such terms should have to be included in equation D3 and would have led to complications in analysis. No such term appears, because elastic axis is defined here as the axis about which the resultant torque of the shearing stresses due to longitudinal stress is nil.

It was shown in Appendix IA that stress

\[ f = E \left[ e_t + (e - \eta) (v^{11} \cos \beta + w^{11} \sin \beta) + \xi (v^{11} \sin \beta - w^{11} \cos \beta) \\
+ (\xi^2 + \eta^2 - K_A^2) \beta' \phi' \right] \]

In the above expression, the longitudinal stresses connected with expression \( E[e_t + e(v^{11} \cos \beta + w^{11} \sin \beta) - K_A^2 \beta' \phi'] \) are uniform across the cross section and will not produce any shearing stresses.
Considering the expression \( E \left[ - \eta (v'' \cos \beta + w'' \sin \beta) + (\xi^2 + \eta^2) \beta^1 \phi^1 \right] \)
the differential stresses would be symmetrical about the major axis (\( \eta \) axis) and since the cross section is symmetrical the resultant shear would lead to shear along the major axis. The only remaining term \( E \xi (v'' \sin \beta - w'' \cos \beta) \) can lead to shearing stress which can produce torque.

Following the above discussions the moments \( M_1 \), \( M_2 \) and \( \Omega \) can be expressed as follows.

\[
M_1 = -E \int \int (v'' \sin \beta - w'' \cos \beta) \xi^2 \xi \eta d\eta
= -E(v'' \sin \beta - w'' \cos \beta) \int \xi^2 \xi d\eta
\]

As \( \int \int \xi^2 \xi \eta d\eta = I_1 \)
\[
M_1 = -EI_1 (v'' \sin \beta - w'' \cos \beta) \\
\]

\[
M_2 = -E \left\{ \int (e_t \xi \eta d\eta)_{\xi=\xi_1} + e \int \left[ (v'' \cos \beta + w'' \sin \beta) \eta \xi d\eta \right]_{\eta=\eta_1}^{\eta_2}
+ \int \left[ -\eta^2 (v'' \cos \beta + w'' \sin \beta) \xi d\eta \right]_{\eta=\eta_1}^{\eta_2}
+ \int \left[ \frac{\xi^2}{2} \eta (v'' \sin \beta - w'' \cos \beta) d\eta \right]_{\eta=\eta_1}^{\eta_2}
+ \int \left[ \frac{\xi^2}{2} \eta d\eta \right]_{\eta=\eta_1}^{\eta_2}
+ \int \left[ \frac{\eta^2}{2} \xi d\eta \beta^1 \phi^1 \right]_{\eta=\eta_1}^{\eta_2}
- \int \left[ K_A \eta^2 \xi d\eta \beta^1 \phi^1 \right]_{\eta=\eta_1}^{\eta_2} \right\}
\]

\[
M_2 = -E \left\{ -\int \eta^2 (v'' \cos \beta + w'' \sin \beta) d\eta + \frac{1}{3} \int \eta^2 t^2 \beta^1 \phi^1 d\eta
+ \int \eta^3 t \beta^1 \phi^1 d\eta - K_A^2 \int \eta d\eta \beta^1 \phi^1 \right\}
\]

As \( \int \eta^2 \xi d\eta \) is moment of inertia \( I_2 \)

\[
M_2 = -E \left[ -I_2 (v'' \cos \beta + w'' \sin \beta)
+ \int \eta \frac{1}{12} t^3 \beta^1 \phi^1 d\eta
+ \int \eta^3 t \beta^1 \phi^1 d\eta - K_A^2 \eta \right]
\]

\[
M_2 = -E \left[ -I_2 (v'' \cos \beta + w'' \sin \beta) + \beta^1 \phi^1 \int \eta \frac{t^2}{12} + \eta^2 - K_A^2 d\eta \right]
\]
The term
\[
\int_{\eta e} \eta \left( \frac{t^2}{12} + \eta^2 - K_A^2 \right) d\eta
\]
is an additional coupling term. Hereafter this term is represented by symbol $B_2$.

\[M_2 = EI_2 (v'' \cos \beta + w'' \sin \beta) - EB_2 \beta' \phi \]

\[\Omega = C_1 \phi' + \int_{\eta e} \left[ \int_{\eta e} f (\beta + \phi) \left( \eta^2 + \xi^2 \right) d\xi d\eta \right]
\]

\[= C_1 \phi' + E (\beta + \phi) \left[ - \int \eta \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta 
+ \int \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta 
+ \int \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta \right]
\]

\[= C_1 \phi' + E (\beta + \phi) \left[ \int \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta 
+ \int \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta 
+ \int \left( \frac{v'' \cos \beta + w'' \sin \beta}{\xi} \right) d\eta \right]
\]

The additional coupling term \[\int \eta^2 t \left( \frac{t^2}{12} + \eta^2 - K_A^2 \right) d\eta\]
will be hereafter named as $B_1$ and hence the equation reduces to

$$Q = C_1\phi + E(\beta + \phi)\beta [-(v''\cos \beta + w''\sin \beta)B_2 + \beta' \phi' B_1]$$

Dropping small terms

$$Q = C_1\phi + E(\beta^2)\beta' B_1 - EB_2\beta' (v''\cos \beta + w''\sin \beta)$$

$$= [C_1\phi + EB_1(\beta^2)\beta] - EB_2\beta' (v''\cos \beta + w''\sin \beta) \ldots \ldots \text{D6}$$

In considering the equilibrium between moments, shears and tension it is more convenient to express the moments parallel to $x$, $y$ and $z$ axes ie $M_x$, $M_y$ and $M_z$. Hence transformation of $M_1$, $M_2$ and $Q$ are necessary into $M_x$, $M_y$ and $M_z$.

When $\phi$ is small $\sin(\beta + \phi) = \sin \beta + \phi \cos \beta$ and

$\cos(\beta + \phi) = \cos \beta - \phi \sin \beta$

Resolving the moments:

$$M_x = Q = [C_1 + EB_1(\beta^2)]\beta - EB_2\beta' (v''\cos \beta + w''\sin \beta)$$

$$\ldots \ldots \ldots \text{D7}$$

$$M_y = -M_1(\sin \beta + \phi \cos \beta) + M_2(\cos \beta - \phi \sin \beta)$$

$$= EI_1(v''\sin \beta - w''\cos \beta)(\sin \beta + \phi \cos \beta)$$

$$+ [EI_2 (v''\cos \beta + w''\sin \beta) - EB_2\beta' \phi'] (\cos \beta - \phi \sin \beta)$$

$$= (EI_2 - EI_1)w''\sin \beta \cos \beta + (EI_1 \sin \beta + EI_2 \cos \beta) v''$$

$$- EB_2\beta' \phi' \cos \beta$$

$$\ldots \ldots \ldots \text{D8}$$
Mz = M₁(\cosβ - \phi\sinβ) + M₂(\sinβ + \phi\cosβ) \\
= -EI₁(\ddot{v}\sinβ - \ddot{w}\cosβ)(\cosβ - \phi\sinβ) \\
+ [EI₂(\ddot{v}\cosβ + \ddot{w}\cosβ) - EB₂ξ' \phi'](\sinβ + \phi\cosβ) \\
= +EI₂(\dot{v}\sinβ\cosβ + \dot{w}\sin²β - EB₂ξ' \phi' \sinβ \\
- EI₁(\ddot{v}\sinβ\cosβ - \ddot{w}\cos²β) \\
= \dot{v}\sinβ\cosβ(EI₂ - EI₁) + (EI₁\cos²β + EI₂\sin²β)\ddot{w} \\
- EB₂ξ' \phi' \sinβ

Thus Mx, My and Mz are expressed in terms of displacements.

Mx = [C₁₀ + EB₁(ξ')²]ξ' - EB₂ξ' (\ddot{v}\cosβ + \ddot{w}\sinβ)....... D10

My = (EI₂ - EI₁)\sinβ\cosβ\ddot{w} + (EI₁\sin²β + EI₂\cos²β)v' \\
- EB₂ξ' \phi' \cosβ

Mz = (EI₂ - EI₁)\sinβ\cosβ\ddot{w} + (EI₁\cos²β + EI₂\sin²β)\ddot{w} \\
- EB₂ξ' \phi' \sinβ

Considering the equilibrium of forces and moments that act on 
a differential beam element of dx is considered.

Forces and moments acting on the differential beam element 
are shown on figure j. Summation of the forces in the x, 
y, and z directions and summation of the moments about x, 
y, and z axes lead to the following equilibrium conditions 
for shear and moment.

T¹ + \ddot{P}_x = 0

V¹z + \ddot{P}_z = 0

\ddot{V}_y + \ddot{P}_y = 0
Eliminating $V_y$ and $V_z$ from the above equations the following basic equilibrium conditions are obtained.

$$M'x - M'z v' + M'y w' - qz v' + qy w' + q_x = 0 \quad \ldots \quad D19$$

$$M'z + q'_z - P_y = 0 \quad \ldots \quad D20$$

$$M'y + q'_y - P_z = 0 \quad \ldots \quad D21$$

Substituting the values of $M_x$, $M_y$ and $M_z$ from D10-D12

$$- \left\{ \left[ C_{10} + E\beta_1 (\beta')^2 \right] \phi' - E\beta_2 (v'' \cos \beta + w'' \sin \beta) \right\}$$

$$- qz v' - qy w' + q_x = 0 \quad \ldots \quad D22$$

$$\left[ (E\beta_1 \cos^2 \beta + E\beta_2 \sin^2 \beta) w'' + (E\beta_2 - E\beta_1) \sin \beta \cos \beta v'' \right]$$

$$- E\beta_2 \phi' \sin \beta \right]'' + q'_y - P_z = 0 \quad \ldots \quad D23$$

$$\left[ (E\beta_2 - E\beta_1) \sin \beta \cos \beta w'' + (E\beta_1 \sin^2 \beta + E\beta_2 \cos^2 \beta) v'' \right]$$

$$- E\beta_2 \phi' \cos \beta \right]'' + q'_z - P_y = 0 \quad \ldots \quad D24$$

From Appendix IC

$$P_x = q_z = q_y = 0 \text{ and}$$

$$P_z = -m(\dot{v} - e\beta \sin \beta)$$

$$P_y = -m(\dot{\omega} + e\beta \cos \beta) \text{ and}$$

$$\bar{q}_x = m \left( \dot{v} \sin \beta - \dot{\omega} \cos \beta \right) - mK_m \ddot{\beta}$$
Substituting the above conditions in equations D22-D24.

\[-\left\{\left[C_{10} + E_{B_1}(\beta^1)^2\right]\ddot{\theta}' - E_{B_2}(v''\cos\beta + w''\sin\beta)\right\} \right\}
+ m\left(\ddot{v}\sin\beta - \ddot{w}\cos\beta\right) - mK^2_m\ddot{\theta} = 0 \quad \ldots \ldots \quad D25

\[\left((E_{I_1}\cos^2\beta + E_{I_2}\sin^2\beta)w'' + (E_{I_2} - E_{I_1})\sin\beta\cos\beta v''\right) - E_{B_2}\beta'^1\sin\beta]'' + m(\ddot{v} - e\ddot{\theta}\sin\beta) = 0 \quad \ldots \ldots \quad D26
\]

\[\left[(E_{I_2} - E_{I_1})\sin\beta\cos\beta w'' + (E_{I_1}\sin^2\beta + E_{I_2}\cos^2\beta) v''\right) - E_{B_2}\beta'^1\cos\beta]'' + m(\ddot{w} + e\ddot{\theta}\cos\beta) = 0 \quad \ldots \ldots \quad D27
\]

Assuming simple harmonic motion of the form

\[x = \hat{x}\sin pt \text{ for } v, w \text{ and } \theta \text{ displacements}\]
\[v = \hat{v}\sin pt, v = \hat{v}\cos pt\]
\[\ddot{v} = -\hat{v}\sin pt p^2 = -p^2v\]

Similarly \[\ddot{w} = -p^2w \text{ and } \ddot{\theta} = -p^2\theta\]

Substituting these values in equations D25-D27

\[-\left\{\left[C_{10} + E_{B_1}(\beta^1)^2\right]\ddot{\theta}' - E_{B_2}(v''\cos\beta + w''\sin\beta)\right\} \right]\]
+ m\left(\ddot{v}\sin\beta - \ddot{w}\cos\beta\right) - mK^2_m\ddot{\theta} = 0 \quad \ldots \ldots \quad D28

\[\left((E_{I_1}\cos^2\beta + E_{I_2}\sin^2\beta)w'' + (E_{I_2} - E_{I_1})\sin\beta\cos\beta v''\right) - E_{B_2}\beta'^1\sin\beta]'' + m(\ddot{v} - e\ddot{\theta}\sin\beta) = 0 \quad \ldots \ldots \quad D29
\]

\[\left[(E_{I_2} - E_{I_1})\sin\beta\cos\beta w'' + (E_{I_1}\sin^2\beta + E_{I_2}\cos^2\beta) v''\right) - E_{B_2}\beta'^1\cos\beta]'' + m(\ddot{w} + e\ddot{\theta}\cos\beta) = 0 \quad \ldots \ldots \quad D30\]
As \( m = \frac{w}{g} \) and \( mK^2 \dot{m} = \frac{ICF}{g} \)

the final differential equations can be written as

\[
-\left[ (C_{10} + EB_1 \beta' \beta'') \dot{\varphi}'' - EB_2 \beta' \left( v'' \cos \beta + w'' \sin \beta \right) \right]'
- \frac{w}{g} p^{2} (v \sin \beta - w \cos \beta) + \frac{ICF}{g} \dot{p}^{2} = 0 \quad \cdots \quad D31
\]

\[
\frac{w}{g} p^{2} (v \sin \beta - w \cos \beta) + \frac{ICF}{g} \dot{p}^{2} = 0 \quad \cdots \quad D32
\]

\[
\left( (E_1 \cos ^2 \beta + E_2 \sin ^2 \beta) w'' + (E_2 - E_1) \sin \beta \cos \beta v''\right)
- EB_2 \beta' \dot{\varphi}'' \sin \beta'' - \frac{w}{g} p^{2} (v - e \dot{\varphi} \sin \beta) = 0 \quad \cdots \quad D32
\]

\[
\frac{w}{g} p^{2} (v - e \dot{\varphi} \sin \beta) = 0 \quad \cdots \quad D33
\]

Similar equations allowing for second order terms have been derived by Montoya (2), and a comparison of the above equation to those of Montoya and those of Dawson (17) (neglecting second order terms) may be made by reference to Appendix IE.
Differential Equations:

I. \( m \ddot{y} + (E_2 - E_1) \phi = 0 \)

II. \( \frac{d}{dt} \left( \frac{d}{dt} (E_2 - E_1) \phi \right) = 0 \)

III. \( -E_2 \ddot{\psi} \sin \phi + \int_{h}^{y} (E_1 - E_2) \sin \phi \, dx = J_x \theta \)

IV. \( -E_2 \ddot{\psi} \sin \phi + \int_{h}^{y} (E_1 - E_2) \sin \phi \, dx = J_y \theta \)

Moments:

\( M_x = (E_1 - E_2) \phi \sin \phi \cos \phi \)

\( M_y = (E_1 - E_2) \phi \sin \phi \cos \phi \)

Reference axes:

\( x \) and \( y \)
\[ (EI_2 - EI_1) \sin \phi \cos \beta \psi \]
\[ (EI_1 \sin^2 \beta + EI_2 \cos^2 \beta) \psi \]
\[ m (\dot{\psi} - \varepsilon \phi \sin \beta) \]

\[ -EB_2 \phi \cos \beta \]
\[ B_2 \cos \beta \]
\[ B_2 = \int R \left( \frac{\dot{\eta}^2}{2} - \frac{\eta^2}{2} - k_2 \right) d\eta \]

\[ - \left[ c_{10} + EB_1 (p')^2 \right] \phi' \]
\[ EB_2 (\psi'' \cos \beta + \omega \sin \phi \sin \beta) \]
\[ m \dot{k}_m \phi \]
\[ - m_c (\dot{\psi} \sin \phi - \ddot{\omega} \cos \beta) \]
\[ EB_1 \left( p' \right)^2 \phi' \]
\[ C \left( p' \right)^2 \phi' \]
\[ B_1 = \int R \left( \frac{\dot{\eta}^2}{2} - \frac{\eta^2}{2} - k_2 \right) d\eta \]

\[ -EI_{a,x} \psi'' = -E \left[ (I_{a,x} \ddot{\omega} + \dot{I}_{a,x} \dot{\omega}) \right] \]
\[ -EI_{a,y} \psi'' = -E (I_{a,y} \ddot{\psi} + \dot{I}_{a,y} \dot{\psi}) \]
\[ m \omega^2 x = m \omega^2 (u + y_T) \]
\[ m \omega^2 y = m \omega^2 x_T \]

\[ -E \varphi J_y \phi' \]
\[ E \varphi \phi' \]

\[ J_T = -J_g \sin \phi + J_T \]
\[ = - \int_{A}^{A} \dot{\eta} \left( \frac{\dot{\eta}^2}{2} - \frac{\eta^2}{2} - k_2 \right) d\eta \]

\[ = \int_{A}^{A} \left( \ddot{\eta} \sin \phi + \omega \cos \phi \right) \phi' \]
\[ = \int_{A}^{A} \left( \ddot{\eta} \sin \phi + \omega \cos \phi \right) \phi' \]

\[ (EI_T + EU^2 J) \phi' \]
\[ \varphi \left( J_T u'' + J_T \psi'' \right) \]

\[ m \omega^2 \frac{J_T}{J_T} = m \omega^2 \left( u_{y_T} - \dot{x}_{x_T} \right) \]
\[ = \frac{E \varphi^2 J_T \phi'}{J_T} \]
\[ = \frac{E \varphi^2 J_T \phi'}{J_T} \]

\[ J_T = J_T - \frac{1}{2} \int_{A}^{A} \frac{2}{2} dA \]
\[ = \int \left( \dot{\xi}^2 + \dot{\eta}^2 \right) dA - \int \xi^2 \dot{\xi}^2 dA \]
\[ = \int \xi^2 (\dot{\xi}^2 + \dot{\eta}^2) \]
POSITION OF FIBRE $f$ BEFORE DEFORMATION

Figure A

POSITION OF FIBRE $f$ AFTER DEFORMATION

Figure B
DISPLACEMENTS IN AND NORMAL TO CHORD

Figure C

CO-ORDINATE AXES SYSTEM USED IN DERIVATION OF EQUATIONS

Figure D
INTERNAL ELASTIC MOMENTS

\[ \int \sqrt{\eta^2 + \xi^2} \, d(\beta + \phi) \frac{d(A)}{dx} \]

\[ \sqrt{\eta^2 + \xi^2} \frac{d(\beta + \phi)}{dx} \]

Figure H
EQUILIBRIUM OF FORCES AND MOMENTS

FORCES

\[ V_y + dV_y \quad V_z \quad T \]
\[ T + dT \quad dW \quad dV \]
\[ V_y + dV_y \quad V_z + dV_z \]

MOMENTS

\[ M_y + dM_y \quad M_z \quad M_x + dM_x \]
\[ q_x \quad q_y \]

Figure J
APPENDIX II

Solution of Higher Order Simultaneous Differential Equations
by the Transformation Method

This method of solving the coupled differential equations of motion of a turbine blade was introduced by Carnegie, Dawson and Thomas (3) and the approach is outlined below in the solution of a set of equations which is of similar form to those for a vibrating cantilever blade.

Such a set of equations may be represented by:

\[ A_1 y'''' + B_1 w^2 y = C_1 w^2 \theta \]  \hspace{1cm} E1

\[ A_2 x'''' + B_2 w^2 x = C_2 w^2 \theta \]  \hspace{1cm} E2

\[ A_3 \theta'' + B_3 w^2 \theta = C_3 w^2 x + D_3 w^2 y \]  \hspace{1cm} E3

where primes denote derivatives with respect to \( z \), the distance from the root measured along the blade and \( A_1, B_1, C_1, A_2, B_2, C_2 \) etc are constants.

The normal frequency of vibration being represented by \( w \) and \( x, y, \) and \( \theta \) represents the translations and rotations of the blade. Equations (E1-E3) represent an eigenvalue problem for which a set of solutions is possible, each of which is associated with a normal mode of vibration of the blade at frequency \( w \).
These equations may be reduced to a set of ten ordinary
differential equations of first order in $z$ by suitable
substitutions. These equations can be expressed in the
following convenient form:

$$y'_i = f_i(z, y_1, y_2, \ldots, y_{10}) \ (i = 1, 2, \ldots, 10) \quad \text{(E4)}$$

For this particular problem it would be necessary to know ten
boundary conditions of the blade and these may be at the root
$(z = 0)$.

$$y_i(0) = 0 \ (i = 1, 2, 3, 4, 5) \quad \text{(E5)}$$

at the tip $(z = L)$

$$y_i(L) = 0 \ (i = 6, 7, 8, 9, 10) \quad \text{(E6)}$$

The method of solution of equation (E4) is then as follows:

a) A value for $w$, the normal mode frequency, is assumed.

b) The five known boundary values at the root are set to
their respective values namely
$$y_1 = y_2 = y_3 = y_4 = y_5 = 0 \quad \text{(E7)}$$

c) The remaining five unknown boundary values at the root
are arbitrarily assigned values namely
$$y_6 = 1, y_7 = y_8 = y_9 = y_{10} = 0 \quad \text{(E8)}$$

d) Starting from these ten initial boundary conditions, a
set of solutions for the ten first order simultaneous
differential equations (E4) are obtained using the Runge
Kutta step by step numerical integration process. This
yields a set of solutions at the tip or free end of the blade which may be expressed as follows:

\[ y_i, 1(L) = E_i, 1(i = 1, 2, 3 \ldots 10) \] \hspace{1cm} E9

The second subscript indicates the solution of the equations with the first set of initial boundary values.

e) This procedure is then repeated, keeping the five known initial boundary values at the root unaltered, but the unknown initial boundary values at the root are set in turn to the following values:

\[ y_6 = 0, y_7 = 1, y_8 = 0, y_9 = 0, y_{10} = 0 \] \hspace{1cm} E10
\[ y_6 = 0, y_7 = 0, y_8 = 1, y_9 = 0, y_{10} = 0 \] \hspace{1cm} E11
\[ y_6 = 0, y_7 = 0, y_8 = 0, y_9 = 1, y_{10} = 0 \] \hspace{1cm} E12
\[ y_6 = 0, y_7 = 0, y_8 = 0, y_9 = 0, y_{10} = 1 \] \hspace{1cm} E13

Thus five sets of independent solutions are obtained at the tip or free end \((z = L)\) of the beam namely:

\[ y_{i,r}(L) = E_{i,r} \quad (i = 1, 2, 3 \ldots 10)(r = 1, 2 \ldots 5) \] \hspace{1cm} E14

f) Since the equations (E4) are linear, the complete solution of the equations is a linear combination of five sets of independent solutions, thus:

\[ y_i(z) = \sum_{r=1}^{5} a_r y_{i,r}(z) \quad (i = 1, 2, 3 \ldots 10) \] \hspace{1cm} E15

(where \(a_r\) are coefficients).

But the known boundary values at the tip \((z = L)\) are from equations E6:

\[ y_i(L) = 0 \quad (i = 6, 7, 8, 9, 10) \] therefore

\[ \sum_{r=1}^{5} a_r E_{i,r} = 0 \quad (i = 6, 7, 8, 9, 10) \] \hspace{1cm} E16

The condition that the equations E4 have a solution satisfying the boundary values of the problem is that,
the determinant of the coefficient of the homogenous
equation E16 vanishes, thus:
\[ \left| E_{i,r} \right| = 0 (i = 6, 7, 8, 9, 10) (r = 1, 2, 3, 4, 5) \ldots \text{ E17} \]
Different values of the assumed normal mode frequency are
then utilised until the conditions in equations E17 are
satisfied.

\textit{g)} When a particular normal mode frequency has been evaluated
the normal mode can be obtained by setting say \( a_1 \) equal
to unity in equation E16 and solving for four of the
simultaneous equations to obtain \( a_2, a_3, a_4 \) and \( a_5 \).
The normal mode deflection curves can then be obtained
from the relevant equations contained in the set:
\[ y_i(z) = \sum_{r=1}^{5} a_r y_{i,r}; \ (i = 1, 2, 3 \ldots 10) \ (r = 1, 2 \ldots 5) \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \text{ E18} \]
APPENDIX IIIA

Print Out Program - Blade Constants

100 0 BEGIN
200 1 INTEGER N, K;
300 1 REAL C, THETA;
400 2 BEGIN
500 3 REAL ARRAY C1, THETA1[1:6], AREA, I1, I2, X, KSQ, ICF, C10, B1, B2, K, GJ, J1;
600 ,
700 3 J2, J3, JX, I3, JY, J[1:6, 1:6];
800 3 PROCEDURE MATRIXWRITE(X);
900 5 ARRAY X;
1000 6 BEGIN
1100 6 INTEGER K, L;
1200 6 NEWLINE(1);
1300 8 WRI TE TEXT(( WITH RESPECT TO BLADE WIDTH AND THICKNESS));
1400 9 NEWLINE(3);
1500 10 SPACE(??));
FOR L=1 STEP 1 UNTIL 5 DO BEGIN
  PRINTC1[I],1,6); SPACE(12); END;

1600 11
1700 12
1800 15
1900 16
2000 17
2100 19
2200 20
2300 21
2400 23
2500 24
2600 25
2700 26
2800 28
2900 29
3000 30
3100 31
3200 32 WRITETEXT((-));
3300 33 NEWLINE(1);
3400 34 END;
3500 35 SPACE(22);
3600 36 FOR K:=1 STEP 1 UNTIL 94 DO WRITETEXT((-));
3700 38 PAPERTHROW;
3800 39 NEWLINE(5);
3900 40 END WRITEMATRIX;
4000 40 N:=0;
4100 42 FOR THETA:=0.10 STEP 0.05 UNTIL 0.301 DO BEGIN
4200 43 N:=N+1;
4300 45 THETA1[N]:=THETA;
4400 46 N:=0;
4500 47 FOR C:=1.0 STEP 0.125 UNTIL 1.501 DO
4600 48 BEGIN
4700 49 M:=M+1;
4800 50 C1[M]:=C;
4900 51 AREA[N,M]:=R/15*THETA*C*2;
5000 52  \[ X[N,M] := \frac{4}{118.3} \times (1.9 - 1.8 \times \theta^2 + 4.6/17 \times \theta^4) \times C; \]
5100 53  \[ I1[N,M] := \frac{64}{3465} \times \theta^3 \times C^4; \]
5200 54  \[ I2[N,M] := \left( \frac{8}{63} \times \theta \times C^4 \right) - \frac{3}{15} \times \theta^2 \times C^2 \times (3/7 \times C \times X[N,M]) \times 2; \]
5300 55  \[ I3[N,M] := \frac{64}{2205} \times \theta^2 \times C^4; \]
5400 56  \[ RSQ[N,M] := \frac{8}{231} \times \theta \times \theta \times C \times X[N,M] + 15/63 \times C^2 \times (3/7 \times C \times X[N,M]) \times 2; \]
5500 57  \[ ICF[N,M] := I1[N,M] \times I2[N,M]; \]
5600 58  \[ C10[N,M] := \frac{256}{3465} \times 10.8125 \times 6 \times C^4 \times \theta^3 \times (1 - 11/13 \times \theta^2); \]
5700 58  \[ + \theta^4 - \frac{379}{221} \times \theta^6); \]
5800 58  \[ B1[N,M] := 4 \times \theta^3 \times C^4 \times C^2 \times (1/11 - 1/13) + 2/3 \times \theta^2 \times C^2 \times (1/9 - 3/11; \]
5900 59  \[ + \frac{3}{13 - 1/15} \times \theta \times K[SQ[N,M]] \times (1 - 1/9); \]
6000 60  \[ B2[N,M] := 4 \times \theta^3 \times C^2 \times (1 - 1/11) + 1/3 \times \theta \times C^2 \times (1 - 7/11); \]
6100 60  \[ + \frac{3}{11 - 1/13} \times \theta \times K[SQ[N,M]] \times (1/5 - 1/7); \]
6200 61  \[ K[N,M] := 4 \times I1[N,M] / (1 + 16 \times I1[N,M] / (AREA[N,M] \times C^2)); \]
6300 62  \[ B[N,M] := 10.8125 \times 6 \times K[N,M]; \]
6400 63  \[ J1[N,M] := 4 \times \theta \times \theta \times C^5 \times (4 \times \theta^2 \times (1 - 7/13 - 3/11 - 1/13) + 1/9 - 1/11); \]
6500 64  \[ J2[N,M] := 4 \times \theta \times \theta \times \theta \times C^5 \times (4 \times \theta^2 \times (1/105 + 1/60)); \]
8400 76  MATRIXWRITE( I1  );
8500 77  WRITETEXT('  I2  MOMENT OF INERTIA (MINOR) ABOUT C.F.  ');
8600  });
8700 78  MATRIXWRITE( I2  );
8800 79  WRITETEXT('  KSQ RADIUS OF GYRATION (SQUARED)  ');
8900  });
9000 80  MATRIXWRITE( KSQ  );
9100 81  WRITETEXT('  ICF POLAR MOMENT OF INERTIA ABOUT C.F.  ');
9200  });
9300 82  MATRIXWRITE( ICF  );
9400 83  WRITETEXT('  C10 TORSIONAL STIFFNESS CONSTANT 1  ');
9500  });
9600 84  MATRIXWRITE( C10  );
9700 85  WRITETEXT('  B1 BLADE CONSTANT 1  ');
9800  });
9900 86  MATRIXWRITE( B1  );
10000 87   WRITETEXT(( K STIFFNESS CONSTANT
10100 ));
10200 88   MATRIXWRITE( K );
10300 89   WRITETEXT(( I3 MOMENT OF INERTIA (MINOR) ABOUT C OF G
10400 ));
10500 90   MATRIXWRITE(I3 );
10600 91   WRITETEXT(( B2 BLADE CONSTANT 2
10700 ));
10800 92   MATRIXWRITE( B2 );
10900 93   WRITETEXT(( GJ BLADE CONSTANT 3
11000 ));
11100 94   MATRIXWRITE(GJ );
11200 95   WRITETEXT(( J1 BLADE CONSTANT 4
11300 ));
11400 96   MATRIXWRITE( J1 );
11500 97   WRITETEXT(( J2 BLADE CONSTANT 5
11600 ));
11700 98   MATRIXWRITE( J2 );
11800 99  WRITETEXT(( J3. BLADE CONSTANT 6)

11900  ;;

12000 100  MATRIXWRITE( J3);

12100 101  WRITETEXT(( JX BLADE CONSTANT 7)

12200  ;;

12300 102  MATRIXWRITE( JX);

12400 103  WRITETEXT(( JY BLADE CONSTANT 8)

12500  ;;

12600 104  MATRIXWRITE( JY);

12700 105  WRITETEXT(( J BLADE CONSTANT 9)

12800  ;;

12900 106  MATRIXWRITE( J);

13000 107  END;

13100 108  END;
100 BEGIN REAL X,L,SL,I0,I1,I2,I3,I4,I5,I6,I7,I8,ICF,C1,C2;
200 C3,C4,C5,C6,C7,C8,C10,EPSILON,E,G,W,A,P,D,J1,J2,J3,JJ,ETA;
250 B1,B2,IKP,SLIP,F1N;
300 BETA,BETADASH,AC,AS,AT,H,B;
400 INTEGER I,J,KK,M,N,R;
450 FILE TTY REMOTE(2,9);
500 L:=10; E:=30000000; G:=386; A:=.24; W:=.06792;
510 I1:=.000748051948; I2:=.023877551; ICF:=.032199722;
520 EPSILON:=.0927353985; C10:=3130.6428;
525 B1:=-.124920035; B2:=-.022104286; J3:=.205527731;
530 WRITE(TTY,""TYPE IN VALUE OF BETA ");
540 READ(TTY,""BETA");
545 WRITE(TTY,""HIGHER TERMS INCLUDED ");
550 SL:=.5;
900 BEGIN ARRAY Y,F[1:10],YY[1:10,1:60];
1000 PROCEDURE FUNCTION(X,Y,F);
1100 REAL X;ARRAY Y,F[1];
1200 BEGIN
1300 ETA:=BETADASH*X;
1400 AC:=COS(ETA); AS:=SIN(ETA);
1500 I3:=I1*AC*AC+I2*AS*AS;
1600 I4:=I1*AS*AS+I2*AC*AC;
1700 I5:=(I2-I1)*SIN(ETA*2)/2;
1800 I6:=I0*I3/(I3*I4-I5*2);
1900 I7:=I0*I4/(I3*I4-I5*2);
2000 I8:=I0*I5/(I3*I4-I5*2);
2100 C1:=ICF/(L*2*A); C2:=C10/(E*10);
2200 C3:=-EPSILON*X*COS(ETA)/L; C4:=-EPSILON*X*SIN(ETA)/L;
2300 C5:=L*4*N/(E*10*G);
2400 C6:=BETADASH*(I8*J1-I6*J2)/I0;
2500 C7 := BETADASH*(I8*J2-I7*J1)/I0;
2600 C8 := BETADASH*(BETADASH*J3+C6*J2+C7*J1)/I0;
2700 F[1] := Y[3];
2800 F[2] := Y[4];
3100 F[5] := Y[8];
3200 F[6] := Y[7];
3400 F[8] := -C5*P*2*(Y[2]-Y[9]*C3);
3600 F[10] := C5*P*2*(Y[3]*C3-Y[1]*C4-Y[9]*C1);
3700 END;
3800 PROCEDURE RKFOUR(X,Y,F,H,N);
3900 VALUE H,N; INTEGER N; REAL X,H;
4000 ARRAY Y,F[1];
4100 BEGIN REAL HBY2,HBY6;
4200 INTEGER I;
4300 ARRAY YBAR,P[1:N];
4400 PROCEDURE STIP(A,B,K1,K2);
4500 VALUE K1,K2;
4600 REAL K1,K2;
4700 ARRAY A,B[1];
4800 FOR I:=1 STEP 1 UNTIL N DO
4900 BEGIN P[I]:=P[I]+K1*F[I];
5000 A[I]:=Y[I]+K2*B[I];
5100 END STIP;
5200 HBY2:=H/2.0;HBY6:=H/6.0;
5300 FOR I:=1 STEP 1 UNTIL N DO P[I]:=0.0;

5400 FUNCTION(X,Y,F);STIP(YBAR,F,1.0,HBY2);
5500 X:=X+HBY2;
5600 FUNCTION(X,YBAR,F);STIP(YBAR,F,2.0,HBY2);
5700 FUNCTION(X,YBAR,F);STIP(YBAR,F,2.0,H);
5800 X:=X+HBY2;
5900 FUNCTION(X,YBAR,F);STIP(Y,P1.0,HBY6);
6000 END RKFOUR;
6100 REAL PROCEDURE DETERMINANT(A,N);
6200 VALUE N; ARRAY A[1,1]; INTEGER N;
6300 BEGIN REAL PRODUCT,FACTOR,TEMP,DAV,PIV,ABPIV,MAXPIV;
6400 INTEGER SSIGN,I,J,R,IMAX;
6450 LABEL RETURN,RESUME;
6500 SSIGN:=1;
6600 PRODUCT:=1.0;
6700 FOR R:=1 STEP 1 UNTIL N-1 DO
   6800 BEGIN MAXPIV:=0.0;
   6900 FOR I:=R STEP 1 UNTIL N DO
      7000 BEGIN PIV:=A[I,R];
      7100 ABPIV:=ABS(PIV);
      7200 IF ABPIV>MAXPIV THEN
         7300 BEGIN MAXPIV:=ABPIV;
7400 DAV:=PIV;
7500 IMAX:=1;
7600 END;
7700 IF MAXPIV NEQ 0 THEN
7800 BEGIN IF IMAX=R THEN GO RESUME ELSE
7900 BEGIN FOR J:=R STEP 1 UNTIL N DO
8000 BEGIN TEMP:=A[IMAX,J];
8100 A[IMAX,J]:=A[R,J];
8200 A[R,J]:=TEMP;
8300 END;
8400 END;
8500 SSIGN:=SIGN;
8600 GO RESUME;
8700 END;
8800 END;
8900 DETERMINANT:=0.0;
9000 GO RETURN;
9100 RESUME; FOR I:=R+1 STEP 1 UNTIL N DO
9200 BEGIN FACTOR:=A[I,R]/DAV;
9300 FOR J:=R+1 STEP 1 UNTIL N DO
9500 END
9600 END;
9700 FOR I:=1 STEP 1 UNTIL N DO
9800 PRODUCT:=PRODUCT*A[I,I];
9900 DETERMINANT:=SIGN*PRODUCT;
10000 RETURN:
10100 END OF DETERMINANT;
10200 PROCEDURE FREQ(P,YY,D); VALUE P; REAL P,D;
10300 ARRAY YY[1,1];
10400 BEGIN INTEGER PP,JJ,NN,I; ARRAY A[1:5,1:5];
10500 P:=P*2*3.1416;
10600 FOR M:=5,6,7,8,10 DO
10700 BEGIN FOR NN:=1 STEP 1 UNTIL 10 DO
10800 Y[NN]:=0; Y[M]:=1; X:=0;
10900 FOR I:=1 STEP 1 UNTIL 10 DO
11000 BEGIN FOR J:=1 STEP 1 UNTIL \((1/SL+0.0001)\) DO
11100 RKFOUR(X,Y,F,SL,10);
11200 JJ:=I;
11300 FOR KK:=1 STEP 1 UNTIL 10 DO
11400 YY[JJ,(M-5)×10+KK]:=Y[KK];
11500 END
11600 END;
11700 FOR PP:=1 STEP 1 UNTIL 4 DO
11800 BEGIN
11900 A[1,PP]:=YY[10,5+10×(PP-1)];
12000 A[2,PP]:=YY[10,6+10×(PP-1)];
12100 A[3,PP]:=Y[10,7+10*(PP-1)];
12200 A[4,PP]:=Y[10,8+10*(PP-1)];
12300 A[5,PP]:=Y[10,9+10*(PP-1)];
12400 END ;
12500 A[1,5]:=Y[10,5];
12600 A[2,5]:=Y[10,56];
12700 A[3,5]:=Y[10,57];
12800 A[4,5]:=Y[10,58];
12900 A[5,5]:=Y[10,59];
13000 D:=DETERMINANT(A);  
13100 WRITE(ITY<"D"="E12,A","P,D");
13400 END FREQ;
13500 BEGIN
13600 I0:=0,1;
13700 BETADASH:=BETA*CR2/1260/L;
13800 WRITE(ITY<"<"="E12,A","P,D");
13720 READ(ITY<"<"="E12,A","P,D");
13730 END;
13800 FOR P:=INP STEP SLIP UNTIL FIN DO FREQ(P,YY,D);
14100 END ;
14200 END ;
14400 END.
BEGIN
INTEGER N,N1,N2,N3,N4,N5;
BOOLEAN SAM;
BLOCK 1
REAL A,THER,E,YS1,YS2,YS3,D,DELX,YBAR,YS4,YS5,YS6,YM1,YM2,X;
R,
ALPHA,YS7,YS8,F,SUMF,PI;

PROCEDURE SORT(A,B,I,J,Y);

BLOCK 2

VALUE I,J;
INTEGER I,J;
ARRAY A,B;
REAL Y;
BEGIN
INTEGER CT1,CT2;
REAL TEMP;
Y:=0;
FOR CT1:=I STEP 1 UNTIL J-1 DO
BEGIN FOR CT2:=CT1+1 STEP 1 UNTIL J DO

ACUBE2/VSSK 11/24/72 02:36 PM
1700 18 BEGIN :
1800 18 TEMP:=A[CT2]; A[CT2]:=A[CT1]; A[CT1]:=TEMP;
1900 22 TEMP:=B[CT2]; B[CT2]:=B[CT1]; B[CT1]:=TEMP;
2000 25 END;
2100 26 IF Y<B[CT1] THEN Y:=B[CT1];
2200 27 END;
2300 28 IF Y<B[J] THEN Y:=B[J];
2400 29 END;
2500 29
2600 29 PI:=4*ARCTAN(1);
2700 31 N:=READ;
2800 32 FOR N1:=1 STEP 1 UNTIL N DO :
2900 33 BEGIN :
3000 33 A:=READ*25.4; THETA:=READ;
3100 36 N2:=READ; N3:=READ; N4:=N2+N3; D:=READ;
3200 40 \textbf{BEGIN}\
3300 40 \textbf{ARRAY} \textbf{X} \textbf{Y[1:N4];}\
3400 \textbf{BLOCK} 3\
3500 40\
3600 40 \textbf{PROCEDURE} \textbf{HEAD;}\
3700 \textbf{BLOCK} 4\
3800 41 \textbf{BEGIN}\
3900 41 \textbf{WRITETEXT}((\textbf{ALPHA=})) \textbf{PRINT(}\textbf{ALPHA} \times 180/\textbf{PI},3,3);\
4000 44 \textbf{WRITETEXT}((\textbf{THEORETICAL ORIGIN IS}));\
4100 45 \textbf{PRINT(DELX,3,3)};\
4200 46 \textbf{WRITETEXT}((\textbf{AFT OF MODELS LEADING EDGE(C)DISTANCE FROM NOSE});\
4300 55)\
4400 46 \textbf{MEASURED Y(5S)CALCULATED Y(15S)ERROR(C5S)MM(20S)MM}\
4500 46 \textbf{14S)MM(24S)MM(C)));\
4600 47 \textbf{SUMF:=0;}\
4700 48 \textbf{FOR N5:=1 STEP 1 UNTIL N4 DO}\
4800 49 \textbf{BEGIN}\
4900 49 \textbf{PRINT(X[N5],3);SPACE(13));}
5000 52   XR:=(X[N5]-DELX)/A;
5100 53   F:=Y[N5]-YBAR-ALPHA*X[N5];
5200 54   PRINT(F,3,3);  SPACE(6);
5300 56   XR:=THETA*SQRT(XR)*(1-XR);
5400 57   PRINT(XR*SIGN(F),3,3);  SPACE(15);
5500 59   XR:=XR*SIGN(F)-F;
5600 60   PRINT(XR 3,3);
5700 61   NEWLINE(1);
5800 62   SUMF:=SUMF+XR;
5900 63   END;
6000 64   WRITETEXT(((51S)ERROR SUM IS ));  PRINT(SUMF,3,3);
       
6100 66   WRITETEXT(((5S)PREVIOUS ERROR SUM));  PRINT(YS1,3,3);
6200 ;
6300 68   WRITETEXT(((51S)SUM SOS));
6400 69   PRINT(YS4,3,3);  WRITETEXT((PREVIOUS VALUES));
6500 71   PRINT(YS7,3,3);  PRINT(YS6,3,3);
6600 73    NEWLINE(2);
6700 74    END;
6800 74    FOR N5:=1 STEP 1 UNTIL N2 DO
6900 76    BEGIN
7000 76    X[N5]:=READ -6,3475;
7100 78    Y[N5]:=1000-READ;
7200 79    END;
7300 80    SORT(X,Y,1,N2,YM1);
7400 81    FOR N5:=N2+1 STEP 1 UNTIL N4 DO
7500 82    BEGIN
7600 82    X[N5]:=READ -6 3475;
7700 84    Y[N5]:=1000-READ;
7800 85    END;
7900 86    SORT(X,Y,N2+1,N4,YM2);
8000 87    D:=YM1+YM2-D;
8100 88    FOR N5:=N2+1 STEP 1 UNTIL N4 DO
8200 89    Y[N5]:=D-Y[N5];
8300 90    IF N1>1 THEN PAPERTHROW;
8400 91   WRITETEXT((SECTION));   PRINT(N1,2,0);   NEWLINE(1);   
8500   ;   
8600 94   WRITETEXT((CHRD));   PRINT(A,3,3);   
8700 96   WRITETEXT((55)THETA));   PRINT(THETA,3,3);   
8800 98   WRITETEXT((55)MAX, THICKNESS));   
8900 99   THETA:=A*THETA;   PRINT(A*THETA/SQR(27),3,3);   
9000 101   NEWLINE(1);   
9100 102   YS7:=YS8:=YS4:=0;   
   
9200 103   FOR ALPHA:=1,ALPHA+,005 WHILE NOT (SIGN(YS6-YS7):   
9300 103   =SIGN(YS7-YS4)) OR YS8=0 DO   
9400 104   BEGIN   
9500 104   YS8:=YS7;   YS7:=YS4;   
9600 107   YS6:=YS5:=YS4:=YS2:=YS1:=0;   
9700 108   FOR DEX:=0,DEX-,1,DEX+,001 WHILE   
9800 108   (NOT (SIGN(YS6-YS5)=SIGN   
9900 108   (YS5-YS4)) OR YS6=0) DEELX-,001 DO
10000 109 BEGIN
10100 109 IF DELX=-1 THEN YS5:=0;
10200 111 YS6:=YS5; YS5:=YS4;
10300 113 SAM:=TRUE;
10400 114 FOR YEAR:=6,YBAR+2 WHILE SAM,YBAR-.1,
10500 114 YBAR+.1 WHILE SAM,YBAR-.21,
10600 114 YBAR+.01 WHILE SAM,YBAR-.021,
10700 114 YBAR+.001 WHILE SAM DO
10800 115 BEGIN
10900 115 YS3:=YS2; YS2:=YS1; YS1:=YS4:=0;
11000 119 FOR N5:=1 STEP 1 UNTIL N4 DO
11100 120 BEGIN
11200 120 XR:=(X[N5]-DELX)/A;
11300 122 E:=YBAR+ALPHA*X[N5];
11400 123 IF Y[N5]-E GE 0 THEN
11500 123 F:=E + THETA*SORT(XR)*(1-XR)-Y[N5];
11600 ELSE
11700 123 F:=E - THETA*SORT(XR)*(1-XR)-Y[N5];
11800 124  \[YS1:=YS1+E;\]
11900 125  \[YS4:=YS4+E*X;\]
12000 126  END;
12100 127  IF SIGN(YS1)=SIGN(YS2) OR YBAR=-68-70 THEN;
12200 127  OR (SIGN(YS1)=
12300 127  -SIGN(YS2) AND SIGN(YS1)=SIGN(YS3)) THEN:
12400 127  SAV:=TRUE ELSE SAV:= FALSE;
12500 128  IF YS1=0 THEN GOTO HALT;
12600 129  END;
12700 130  YBAR:=YBAR-.001;
12800 131  HALT:IF DELX<-70 AND DELX>-70 AND(SIGN(Y58-Y57) =
12900 131  -SIGN(Y57-Y54) AND Y58#0) THEN HEAD;
13000 132  END;
13100 133  END;
13200 134  HEAD;
13300 135  END;
13400 136  END;
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<td>1)</td>
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