VIBRATION OF TURBINE BLADE PACKETS
WITH NON-RIGID ROOTS

A thesis submitted to the University of Surrey
for the Degree of Master of Philosophy

By

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SUMMARY

This thesis is concerned with the study of the vibration characteristics of packets of blades having flexible roots and with the determination of forces in the blades and shroud.

The main body of the work is divided into three parts:

First the development of an improved finite element is shown and applied to a fully fixed cantilever blade. Comparisons show the present method to be considerably more accurate, needing a lower-order matrix, than analyses presented by other investigators. Distributions of bending moment, shear force end-load and mode-shapes are included.

In the second part, the finite-element method developed in the first part is extended to apply to packets of blades with fixed roots. Close agreement is shown with experimental results and with results computed by other investigators. The latter comparisons are particularly meaningful when frequencies are presented in parametric form \( \lambda \). Also given, are the development and applications of the frequency-inference diagram and distributions of bending moment, shear force, end-load and mode-shapes for various blade packets.
Part three shows further development of the finite-element method to include blade flexibility at the root. Good agreement is shown with experiment and with results presented by other investigators, though the latter comparison is limited to single blades.

The frequency-inference diagram, to predict the frequencies of blade packets is developed successfully for blades with non-rigid roots.

Mode-shapes, end-loads, shear forces and bending moments are determined.

A comprehensive study of the effect of infinitely variable root-fixation on the dynamic characteristics of blade packets is presented. It is seen that over certain ranges $\lambda$ is very sensitive to changes in stiffness.
DEDICATION

This thesis is dedicated to my wife Elizabeth without whose patience and encouragement it would not have been written.
ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr J Thomas for his encouragement, valuable suggestions and critical comments throughout this investigation.

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CHAPTER 1
INTRODUCTION

1.1 General Introduction
1.2 Short Survey of Vibration Analysis of Turbine Blading
1.3 Present Investigation
1.3.1 Development of an improved finite element
1.3.2 Investigation into the effect of root flexibility on the vibration characteristics of a packet of blades

CHAPTER 2
THE FINITE-ELEMENT METHOD

2.1 Origins of the Finite-Element Method
2.2 Theoretical Considerations
2.2.1 Introduction
2.2.2 Rayleigh's Method
CHAPTER 2 CONTD.

2.2.3 Rayleigh-Ritz Method 50

2.2.4 Derivation of the free vibration equation by the finite-element method 51

2.2.4.1 Strain energy of a finite element 51

2.2.4.2 Kinetic energy of a finite element 54

2.2.4.3 Global stiffness and mass matrices 55

2.2.4.4 Formation of the free vibration equation 55

CHAPTER 3

DEVELOPMENT OF THE FINITE ELEMENT 57

3.1 Introduction 58

3.2 Uniform Beam Element 58

3.2.1 Transverse effects 58

3.2.1.1 Derivation of strain energy 58

3.2.1.2 Determination of kinetic energy 62

3.2.1.3 Derivation of the frequency equation 64

3.2.2 Longitudinal vibration 66

3.2.2.1 Derivation of strain energy 66

3.2.2.2 Determination of kinetic energy 68

3.2.2.3 Derivation of the frequency equation 70

3.2.3 Combined transverse and longitudinal effects 70

3.3 Eigenvalue Solution of Frequency Equation 74

3.4 Single Blade Fixed at the Root 74

3.5 Discussion of Results 76

CHAPTER 4

BLADE PACKETS FIXED AT THE ROOT 94

4.1 Introduction 95

4.2 Finite Element Analysis 95
4.2.1 Conformity and compatibility of nodal co-ordinates
4.2.1.1 Conformity between blade and shroud elements
4.2.1.2 Compatibility between blade and shroud corner elements
4.2.1.3 Compatibility of blade and shroud elements at the 'T' joint
4.2.1.4 Boundary conditions
4.2.2 Assembly of global stiffness and mass matrices
4.2.2.1 Matrix assembly using the code system
4.2.2.1.1 Procedure for generating code numbers
4.2.2.1.2 Procedure for assembly of global matrices
4.3 Multi-Element Blade Shroud Assembly
4.4 The Frequency Inference Diagram
4.4.1 Uncoupled blade and shroud modes
4.4.1.1 Blade Clamped-Free transverse modes (BCF)
4.4.1.2 Blade Clamped-Pinned transverse modes (BCP)
4.4.1.3 Shroud Clamped-Clamped transverse modes (SCC)
4.4.1.4 Blade and shroud longitudinal modes
4.5 Classical Analysis
4.5.1 Determination of the frequency equation
### CHAPTER 4 CONTD.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5.1.1 Range of frequency equations for packet of blades with 'n' bays</td>
<td>115</td>
</tr>
<tr>
<td>4.6 Discussion of Results.</td>
<td>117</td>
</tr>
<tr>
<td>4.6.1 Convergence of Results and Comparisons with Experimental Results and Results Obtained by Other Investigators</td>
<td>117</td>
</tr>
<tr>
<td>4.6.2 Mode Shapes, End-Load, Shear Force and Bending Moment Distributions</td>
<td>122</td>
</tr>
<tr>
<td>4.6.3 Frequency Inference Diagrams</td>
<td>126</td>
</tr>
</tbody>
</table>

### CHAPTER 5

**BLADE PACKETS WITH FLEXIBLE ROOTS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Introduction</td>
<td>167</td>
</tr>
<tr>
<td>5.2 Finite-Element Analysis</td>
<td>167</td>
</tr>
<tr>
<td>5.2.1 Development of the finite element at the blade root</td>
<td>167</td>
</tr>
<tr>
<td>5.2.2 Strain energy due to spring system</td>
<td>169</td>
</tr>
<tr>
<td>5.2.3 Solution of eigenvalue problem</td>
<td>170</td>
</tr>
<tr>
<td>5.2.4 Stiffnesses at the blade root expressed non-dimensionally</td>
<td>171</td>
</tr>
<tr>
<td>5.3 Classical Analysis</td>
<td>172</td>
</tr>
<tr>
<td>5.3.1 The frequency equation</td>
<td>173</td>
</tr>
<tr>
<td>5.3.2 The displacement equation</td>
<td>173</td>
</tr>
<tr>
<td>5.4 Experimental Work</td>
<td>174</td>
</tr>
<tr>
<td>5.4.1 Introduction</td>
<td>174</td>
</tr>
<tr>
<td>5.4.2 Models simulating large springs at the blade roots</td>
<td>174</td>
</tr>
<tr>
<td>5.4.3 Models fitted with medium and small springs at the blade roots</td>
<td>175</td>
</tr>
</tbody>
</table>
CHAPTER 5 CONTD.

5.4.4 Instrumentation and measurements 176
5.5 The Frequency Inference Diagram 178
5.5.1 Blade with spring system at one end 178
and the other end free, transverse
modes (BSF)
5.5.2 Blade with spring system at one 179
end and the other end pinned, transverse
modes (BSP)
5.5.3 Shroud clamped-clamped transverse modes 179
(SCC)
5.6 Discussion of Results 180
5.6.1 Experimental Results and Comparisons 180
with Other Investigators
5.6.2 Relation between Frequency Parameter 183
\( \lambda \) and Root Flexibility of a Packet of
Blades
5.6.3 Modes Shapes, Bending Moment, Shear 188
Force, and End-Load Distributions of a
Blade Packet with Flexible Roots
5.6.4 Frequency Inference Diagrams 188

CHAPTER 6

CONCLUSIONS 251
6.1 Development of an Improved Finite 252
Element
6.2 Effect of Root Flexibility on the 252
Vibration Characteristics of Packets
of Blades
CHAPTER 6 CONTD.

6.3 Frequency Inference Diagrams 253

6.4 Future Developments 254

LIST OF TABLES

| TABLE 3.1 | EIGENVALUES AND EIGENVECTORS FOR A SINGLE-ELEMENT, UNIFORM CANTILEVER BEAM | 82 |
| TABLE 3.2 | CONVERGENCE OF FREQUENCY PARAMETER $\lambda$ FOR A UNIFORM CANTILEVER BEAM | 83 |
| TABLE 3.3 | COMPARISON BETWEEN PRESENT RESULTS AND THOSE OBTAINED BY LECKIE AND LINDBERG (29, 1963) | 84 |
| TABLE 3.4 | COMPARISON BETWEEN PRESENT RESULTS AND THOSE OBTAINED BY CARNEGIE, THOMAS AND DOKUMACI (31, 1969) AND PRASAD, KRISHNA AND RAO (32, 1972) | 84 |
| TABLE 4.1 | TRANSFORMED VALUES OF Y AND U | 96 |
| TABLE 4.2 | CONVERGENCE OF FREQUENCY PARAMETER $\lambda$ FOR SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), (REF. TABLE 5.1), FOR N = 2 | 129 |
| TABLE 4.3 | CONVERGENCE OF FREQUENCY PARAMETER $\lambda$ FOR SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), (REF TABLE 5.1), FOR M = 2, 3 AND 4 | 130 |
4.4 CONVERGENCE OF FREQUENCY PARAMETER \( \lambda \) FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I), FOR \( N = 2 \)

4.5 CONVERGENCE OF FREQUENCY PARAMETER \( \lambda \) FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I), FOR \( M = 2, 3 \) AND 4

1.6 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET (a), REF. (55, 1977)

1.7 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET (b), REF. (55, 1977)

1.8 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKETS FOR \( L_r = 0.5 \), REF. (11, 1965)

1.9 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKETS FOR \( L_r = 1.0 \), REF. (11, 1965)

1.10 COMPUTED FREQUENCY PARAMETER \( \lambda \) FOR DIFFERENT LENGTH RATIOS \( L_r \) FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I) REF. TABLE 5.1

1.11 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1

1.12 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (3) REF. TABLE 5.1

1.13 COMPUTED FREQUENCY PARAMETER \( \lambda \) FOR DIFFERENT LENGTH RATIOS \( L_r \) FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I)
| TABLE 4.13 | COMPUTED FREQUENCY PARAMETER FOR DIFFERENT LENGTH RATIOS $L_n$ FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (II) |
| TABLE 5.1 | DATA RELATING TO EXPERIMENTAL MODELS (1), (2) and (3) (REF. FIGURES 5.4 & 5.5) |
| TABLE 5.2 | DATA RELATING TO EXPERIMENTAL MODELS (4), (5) AND (6) (REF. FIG. 5.6) |
| TABLE 5.3 | RELATIVE DEFLECTIONS OF SINGLE CANTILEVER BLADE, EXPERIMENTAL MODEL (1), REF. TABLE 5.1 |
| TABLE 5.4 | RELATIVE DEFLECTIONS OF TWO-BLADED PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1 |
| TABLE 5.5 | EFFECT OF VARYING ROOT FIXATION OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1 |
| TABLE 5.6 | EFFECT OF VARYING ROOT FIXATION OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (3), REF. TABLE 5.1 |
| TABLE 5.7 | COMPARISON OF COMPUTED VALUES OF FREQUENCY PARAMETER $\lambda$ FOR $G^* = 0.3581$ AND $K^* = 183.3613$ |
| TABLE 5.8 | COMPARISON OF COMPUTED VALUES OF FREQUENCY PARAMETER $\lambda$ FOR $G^* = 3.8964$ AND $K^* = 498.7428$ |
| TABLE 5.9 | COMPUTED AND EXPERIMENTAL FREQUENCIES OF SINGLE BLADE, EXPERIMENTAL MODEL (4), REF. TABLE 5.2 |
TABLE 5.10  COMPUTED AND EXPERIMENTAL  
FREQUENCIES OF SHROUDED BLADE  
PACKET, EXPERIMENTAL MODEL (5),  
REF. TABLE 5.2

TABLE 5.11  COMPUTED AND EXPERIMENTAL  
FREQUENCIES OF SHROUDED BLADE  
PACKET, EXPERIMENTAL MODEL (6),  
REF. TABLE 5.2

LIST OF FIGURES

FIGURE 3.1  NODAL CO-ORDINATES OF BEAM  
ELEMENT WITH EIGHT DEGREES OF  
FREEDOM

FIGURE 3.2  NODAL CO-ORDINATES OF BEAM  
ELEMENT WITH FOUR DEGREES OF  
FREEDOM

FIGURE 3.3  NODAL CO-ORDINATES OF BEAM  
ELEMENT WITH TWELVE DEGREES OF  
FREEDOM

FIGURE 3.4  NODAL CO-ORDINATES OF SINGLE  
ELEMENT CANTILEVER BEAM WITH  
FOUR DEGREES OF FREEDOM

FIGURE 3.5  CONVERGENCE CURVES FOR UNIFORM  
CANTILEVER BLADE (FIRST AND  
SECOND MODES)

FIGURE 3.6  CONVERGENCE CURVES FOR UNIFORM  
CANTILBER BLADE (THIRD AND  
FOURTH MODES)
FIGURE 3.7  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 6)

FIGURE 3.8  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 6)

FIGURE 3.9  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)

FIGURE 3.10  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FOURTH MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)

FIGURE 3.11  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIFTH MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)

FIGURE 4.1  RELATION BETWEEN U-Y AND V-W CO-ORDINATES
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>Nodal Co-ordinates of adjoining blade and shroud corner elements</td>
<td>141</td>
</tr>
<tr>
<td>4.3</td>
<td>Nodal Co-ordinates of blade and shroud elements at a T joint</td>
<td>142</td>
</tr>
<tr>
<td>4.4</td>
<td>Packet of three shrouded blades showing co-ordinate reference numbers and element numbers</td>
<td>142</td>
</tr>
<tr>
<td>4.5</td>
<td>Nodal Co-ordinates at the T joint</td>
<td>143</td>
</tr>
<tr>
<td>4.6</td>
<td>Relation between local and global co-ordinates</td>
<td>143</td>
</tr>
<tr>
<td>4.7</td>
<td>Sign convention for displacements, shear force and bending moment of a uniform beam AB</td>
<td>144</td>
</tr>
<tr>
<td>4.8</td>
<td>End conditions of vertical and horizontal members in a packet of blades with n bays</td>
<td>144</td>
</tr>
<tr>
<td>4.9</td>
<td>Relation between natural frequency, $f$ and $\lambda$ for shrouded blade packet (a)</td>
<td>145</td>
</tr>
<tr>
<td></td>
<td>Ref. (55, 1977)</td>
<td></td>
</tr>
<tr>
<td>4.10</td>
<td>Relation between natural frequency, $f$ and $\lambda$ for shrouded blade packet (b)</td>
<td>146</td>
</tr>
<tr>
<td></td>
<td>Ref. (55, 1977)</td>
<td></td>
</tr>
<tr>
<td>4.11</td>
<td>Relation between natural frequency, $f$ and $\lambda$ for shrouded blade packets Ref. (11, 1965)</td>
<td>147</td>
</tr>
<tr>
<td>4.12</td>
<td>Mode shapes for the first, second &amp; third modes of vibration of experimental packet (2), fully fixed at the root.</td>
<td>148</td>
</tr>
</tbody>
</table>
FIGURE 4.13 BENDING MOMENT SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (2), FULLY FIXED AT THE ROOT

FIGURE 4.14 BENDING MOMENT SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF EXPERIMENTAL PACKET (2), FULLY FIXED AT THE ROOT

FIGURE 4.15 BENDING MOMENT SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF EXPERIMENTAL PACKET (2) FULLY FIXED AT THE ROOT

FIGURE 4.16 MODE SHAPES FOR THE FIRST, SECOND, THIRD AND FOURTH MODES OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT

FIGURE 4.17 BENDING MOMENT DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT

FIGURE 4.18 SHEAR FORCE DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT
LIST OF FIGURES CONTD.

FIGURE 4.19 END-LOAD DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT

FIGURE 4.20 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF THEORETICAL PACKET (I)

FIGURE 4.21 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF THEORETICAL PACKET (I)

FIGURE 4.22 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE SECOND MODE OF VIBRATION OF THEORETICAL PACKET (I)

FIGURE 4.23 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF THEORETICAL PACKET (I)

FIGURE 4.24 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (I)

FIGURE 4.25 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (I)
LIST OF FIGURES CONTD.

FIGURE 4.26 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE FOURTH MODE OF VIBRATION OF THEORETICAL PACKET (I) 162

FIGURE 4.27 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FOURTH MODE OF VIBRATION OF THE THEORETICAL PACKET (I) 163

FIGURE 4.28 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (I) 164

FIGURE 4.29 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (II) 165

FIGURE 5.1 NODAL CO-ORDINATES OF BLADE ROOT ELEMENT 200

FIGURE 5.2 BLADE ROOT ELEMENT WITH TORSIONAL, LONGITUDINAL AND TRANSVERSE SPRING SYSTEM AT THE ROOT 201

FIGURE 5.3 UNIFORM BEAM WITH A TORSIONAL AND TRANSVERSE SPRING SYSTEM AT ONE END AND THE OTHER END FREE 201

FIGURE 5.4 EXPERIMENTAL MODELS (1), (2), AND (3) (SIDE ELEVATION) 202

FIGURE 5.5 EXPERIMENTAL MODELS (1), (2) AND (3) (FRONT ELEVATION) 203

FIGURE 5.6 EXPERIMENTAL BLADE PACKET (6) SHOWING SPRING SYSTEM AT THE ROOT 204
<p>| FIGURE 5.7 | SCHEMATIC REPRESENTATION OF EXPERIMENTAL SET-UP | 205 |
| FIGURE 5.8 | RELATION BETWEEN NATURAL FREQUENCY, ( f ) AND ( \lambda^\frac{1}{2} ) FOR SINGLE CANTILEVER BLADE, EXPERIMENTAL MODEL (1) | 207 |
| FIGURE 5.9 | RATE OF SPRING (a) USED IN EXPERIMENTAL MODELS (4) AND (5) | 208 |
| FIGURE 5.10 | RATE OF SPRING (b) USED IN EXPERIMENTAL MODEL (6) | 209 |
| FIGURE 5.11 | VARIATION OF THE FREQUENCY PARAMETER ( \lambda ) WITH ( K^* ) FOR ( H^* = 1.00 ), FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I) | 210 |
| FIGURE 5.12 | VARIATION OF FREQUENCY PARAMETER ( \lambda ) WITH ( K^* ) FOR ( H^* = 10^{-4} ), FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I) | 211 |
| FIGURE 5.13 | VARIATION OF THE FREQUENCY PARAMETER ( \lambda ) WITH ( K^* ) FOR THE SECOND MODE OF VIBRATION OF THEORETICAL MODEL (I) | 212 |
| FIGURE 5.14 | VARIATION OF THE FREQUENCY PARAMETER ( \lambda ) WITH ( K^* ) FOR THE THIRD MODE OF VIBRATION OF THEORETICAL MODEL (I) | 213 |</p>
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>VARIATION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.15</td>
<td>of the</td>
<td>214</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>fourth mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.16</td>
<td>of the</td>
<td>215</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>fifth mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.17</td>
<td>of the</td>
<td>216</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sixth mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.18</td>
<td>of the</td>
<td>217</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>seventh mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.19</td>
<td>of the</td>
<td>218</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>eighth mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.20</td>
<td>of the</td>
<td>219</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( K^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ninth mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.21</td>
<td>of the</td>
<td>220</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>parameter ( \lambda ) with ( G^* ) for the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>first mode of vibration of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>theoretical model (I)</td>
<td></td>
</tr>
</tbody>
</table>
FIGURE 5.22 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $G^*$ FOR THE SECOND AND THIRD MODES OF VIBRATION OF THEORETICAL MODEL (I)

FIGURE 5.23 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $G^*$ FOR THE FOURTH, FIFTH, SIXTH, SEVENTH, EIGHTH AND NINTH MODES OF THEORETICAL MODEL (I)

FIGURE 5.24 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I)

FIGURE 5.25 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE SECOND MODE OF VIBRATION OF THEORETICAL MODEL (I)

FIGURE 5.26 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE THIRD MODE OF VIBRATION OF THEORETICAL MODEL (I)

FIGURE 5.27 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE FOURTH MODE OF VIBRATION OF THEORETICAL MODEL (I)

FIGURE 5.28 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE FIFTH, SIXTH, SEVENTH AND EIGHTH MODES OF THEORETICAL MODEL (I)
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.29</td>
<td>Variation of the frequency parameter $\lambda$ with $h^*$ for the ninth mode</td>
<td>228</td>
</tr>
<tr>
<td></td>
<td>of vibration of theoretical model (I)</td>
<td></td>
</tr>
<tr>
<td>5.30</td>
<td>Mode shapes for first and second modes of vibration of theoretical packet</td>
<td>229</td>
</tr>
<tr>
<td></td>
<td>(I)</td>
<td></td>
</tr>
<tr>
<td>5.31</td>
<td>Mode shapes for third and fourth modes of vibration of theoretical packet</td>
<td>230</td>
</tr>
<tr>
<td></td>
<td>(I)</td>
<td></td>
</tr>
<tr>
<td>5.32</td>
<td>Mode shape for fifth mode of vibration of theoretical packet (I)</td>
<td>231</td>
</tr>
<tr>
<td>5.33</td>
<td>Bending moment, shear force and end-load distributions for the first mode</td>
<td>232</td>
</tr>
<tr>
<td></td>
<td>of vibration of theoretical packet (I)</td>
<td></td>
</tr>
<tr>
<td>5.34</td>
<td>Bending moment, shear force and end-load distributions for the second mode</td>
<td>233</td>
</tr>
<tr>
<td></td>
<td>of vibration of theoretical packet (I)</td>
<td></td>
</tr>
<tr>
<td>5.35</td>
<td>Bending moment, shear force and end-load distributions for the third mode</td>
<td>234</td>
</tr>
<tr>
<td></td>
<td>of vibration of theoretical packet (I)</td>
<td></td>
</tr>
<tr>
<td>5.36</td>
<td>Bending moment, shear force and end-load distributions for the fourth mode</td>
<td>235</td>
</tr>
<tr>
<td></td>
<td>of vibration of theoretical packet (I)</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF FIGURES CONTD.

FIGURE 5.37 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FIFTH MODE OF VIBRATION OF THEORETICAL PACKET (I) 236

FIGURE 5.38 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (I) FOR G* = 1000, H* = 10 AND K* = 10 237

FIGURE 5.39 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (I) FOR G* = 1000, H* = 10 AND K* = 100 238

FIGURE 5.40 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (I) FOR G* = 1000, H* = 10 AND K* = 1000 239

FIGURE 5.41 VARIATION OF λ WITH ROOT FLEXIBILITY FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (I), SUPERIMPOSED ON THE FREQUENCY INFERENCE DIAGRAM FOR THE FULLY FIXED CASE. 240

FIGURE 5.42 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (II) FOR G* = 1000, H* = 10 AND K* = 10 241

FIGURE 5.43 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (II) FOR G* = 1000, H* = 10 AND K* = 100 242

FIGURE 5.44 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (II) FOR G* = 1000, H* = 10 AND K* = 1000 243
LIST OF FIGURES CONTD.

FIGURE 5.45 VARIATION OF $\chi$ WITH ROOT FLEXIBILITY FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (II), SUPERIMPOSED ON THE FREQUENCY INFERENCE DIAGRAM FOR THE FULLY FIXED CASE

LIST OF PLATES

PLATE 5.1 GENERAL VIEW OF INSTRUMENTS AND EXPERIMENTAL RIG FOR MODELS (1), (2) AND (3) 206

PLATE 5.2 CLOSE-UP VIEW OF EXPERIMENTAL RIG FOR MODELS (1), (2) AND (3) 245

PLATE 5.3 CLOSE-UP VIEW OF PIEZO-ELECTRIC STRAIN GAUGES AND INDUCTIVE TRANSUDER 246

PLATE 5.4 EXPERIMENTAL MODELS (4), (5) AND (6) 247

PLATE 5.5 SPRING ARRANGEMENT OF SINGLE BLADE EXPERIMENTAL MODEL (4) 248

PLATE 5.6 SPRING ARRANGEMENT OF THREE BLADE PACKET EXPERIMENTAL MODEL (5) 249

PLATE 5.7 SPRING ARRANGEMENT OF TWO BLADE PACKET EXPERIMENTAL MODEL (6) 250
LIST OF SYMBOLS

Symbols have been listed in alphabetical order. Every dot over a variable signifies one derivation with respect to time. The symbols not included in the following list are defined in the text.

\[ \alpha = \frac{G^*}{K^*} \]  
Torsional to transverse spring stiffness ratio

\[ \alpha_1 = -\frac{E_s I_s}{E_b A_b} \]  
End load coefficient

\[ \alpha_2 = \frac{E_s I_s}{E_b I_b} \]  
Bending moment coefficient

\[ \alpha_3 = -\frac{E_s A_s}{E_b I_b} \]  
End load coefficient

\[ \alpha_4 = -1 \]  
Coefficient of rotations

\[ \alpha_5 = \alpha_6 = -\frac{E_s I_s}{E_b A_b} \]  
End load coefficient

\[ \alpha_7 = \alpha_8 = \frac{E_s I_s}{E_b I_b} \]  
Bending moment coefficient

\[ \alpha_9 = \alpha_{10} = \frac{E_s A_s}{E_b I_b} \]  
End load coefficient

\[ \alpha_{11} = -1 \]  
Coefficient of rotations

\[ \alpha, \beta \]  
Frequency functions

\[ \beta = \lambda^4 \]  
Frequency parameter

\[ \gamma = \frac{E_s I_s}{E_b I_b} \]  
Flexural rigidity ratio

\[ \Delta = \frac{\rho_s A_s}{\rho_b A_b} \]  
Shroud to blade mass ratio

\[ \delta \]  
Frequency function (Section 5.3.2)
δ 
Side sway (Section 4.5.1)

e 
Direct strain

η 
Beam to end load mass ratio

η = \frac{x}{L} 
Non-dimensional length ratio along the element

θ 
Angular deflection

\lambda = \frac{\rho b^4 \frac{E b}{1 + \nu}}{P^2} 
Frequency parameter for blade

\lambda_s = \frac{\rho_s A_s L_s}{E_s} 
Frequency parameter for shroud

\mu = 4\gamma\rho A\omega^2 
Frequency parameter

ν 
Poisson's Ratio

\{\xi\} 
Nodal vector of local co-ordinates

ρ 
Mass density

G 
Direct stress

τ 
Shear stress

Ø 
Finite difference variable or frequency function

Ω 
Natural circular frequency

A 
Cross-sectional area of beam

A_b 
Cross-sectional area of blade

A_s 
Cross-sectional area of shroud

A, B 
Subscripts indicating ends of beams

A, B, C, D 
Integrating constants

A, B, C, D, E 
Labels (Figures 5.4 and 5.5)

(BCF) 
Blade clamped-free transverse modes

(BCP) 
Blade clamped-pinned transverse modes

(BSF) 
Blade-spring-free transverse modes

(BSP) 
Blade-spring-pinned transverse modes
Finite difference shifting operator

Modulus of elasticity

Modulus of elasticity for blade material

Modulus of elasticity for shroud material

Frequency functions

Torsional stiffness constant at blade-root

Local to global conformity matrix

Blade to shroud transformation matrix

Corner blade to shroud compatibility matrix

T-junction, blade to shroud compatibility matrix

Stiffness transfer matrix

Torsional stiffness at blade-root

Transformation matrix

Longitudinal stiffness constant at blade-root

Longitudinal stiffness at blade-root

Second moment of area

Second moment of area of blade cross-section

Second moment of area of shroud cross-section

Transverse stiffness constant at blade-root

Length ratio

Global stiffness matrix

Local stiffness matrix for transverse vibrations

Local stiffness matrix for longitudinal vibrations

Local stiffness matrix

Element in local stiffness matrix at the blade-root
\[ K_m^4 = \frac{\rho A}{\sigma_s^4} \]  
Mass ratio

\[ K_s^4 = \frac{EI}{E_s I_s} \]  
Stiffness ratio

\[ [KB] \]  
Local blade stiffness matrix

\[ [KG] \]  
Global stiffness matrix

\[ [KG^*] \]  
Modified global stiffness matrix to suit computation

\[ [KL] \]  
Local stiffness matrix for combined transverse and longitudinal vibrations

\[ [KLD] \]  
Local stiffness matrix in dimensional form

\[ [KLE] \]  
Local blade stiffness matrix

\[ [KLB_1] \]  
Transformed local blade stiffness matrix

\[ [KLB_2] \]  
Corner blade transformed stiffness matrix

\[ [KLB_4] \]  
Transformed local stiffness matrix at the blade-root

\[ [KLB_4(R)] \]  
Modified local stiffness matrix at the blade-root

\[ [KS] \]  
Local shroud stiffness matrix

\[ k \]  
Rotational stiffness of foundation (Section 4.5.1)

\[ k \]  
Transverse stiffness at blade-root

\[ L \]  
Length of beam

\[ L_b \]  
Blade length

\[ L_r = \frac{L_s}{L_b} \]  
Shroud to blade length ratio

\[ L_s \]  
Shroud length

\[ \ell \]  
Length of element

\[ \ell_b \]  
Length of blade element

\[ \ell_s \]  
Length of shroud element

\[ M \]  
Bending Moment
M  Number of shroud elements

\[
\begin{bmatrix}
M \\
M_1
\end{bmatrix}
\]
Global mass matrix

\[
\begin{bmatrix}
M_2 \\
M_1
\end{bmatrix}
\]
Local mass matrix for transverse vibrations

\[
\begin{bmatrix}
M_{b1} \\
M_{b2}
\end{bmatrix}
\]
Local mass matrix for longitudinal vibrations

\[
\begin{bmatrix}
M \Phi \\
M \Gamma
\end{bmatrix}
\]
Local, blade mass matrix

\[
\begin{bmatrix}
M G
\end{bmatrix}
\]
Global mass matrix

\[
\begin{bmatrix}
M G^*
\end{bmatrix}
\]
Modified global mass matrix to suit computation

\[
\begin{bmatrix}
M L
\end{bmatrix}
\]
Local mass matrix for combined transverse and longitudinal vibrations

\[
\begin{bmatrix}
M L
\end{bmatrix}
\]
Local mass matrix in dimensional form

\[
\begin{bmatrix}
M L B
\end{bmatrix}
\]
Local blade mass matrix

\[
\begin{bmatrix}
M L B_1 \\
M L B_2
\end{bmatrix}
\]
Transformed local blade mass matrix

\[
\begin{bmatrix}
M S
\end{bmatrix}
\]
Corner blade, transformed mass matrix

\[
\begin{bmatrix}
M S
\end{bmatrix}
\]
Local, shroud mass matrix

N  Number of blade elements

n  Number of bays in a packet of blades

P  Number of blades in a packet or end load

\[
\{P\}
\]
Local force vector

\[
\{P_1\} = \frac{3}{E I} \{P\}
\]
Local force vector, parameter

P, Q  Summation constants

\[
\{Q\}
\]
Global force vector

\[
\{Q_1\}
\]
Local force vector for combined transverse and longitudinal vibrations
\{q\} \quad \text{Nodal vector of global co-ordinates}

\[ R = \frac{kL}{EI} \] \quad \text{Stiffness ratio}

\( r \) \quad \text{Subscript indicating joint number}

\( S \) \quad \text{Shear Force}

(SCC) \quad \text{Shroud clamped-clamped transverse modes}

\( T \) \quad \text{Kinetic energy of bending}

\( \hat{T} \) \quad \text{Maximum kinetic energy}

\( \overline{T} \) \quad \text{Mean kinetic energy}

\( T_I \) \quad \text{Kinetic energy of a finite element I}

\( t \) \quad \text{Time}

\( U \) \quad \text{Potential energy of bending}

\( \hat{U} \) \quad \text{Maximum potential energy}

\( \overline{U} \) \quad \text{Mean potential energy}

\( U_I \) \quad \text{Strain energy of a finite element I}

\( u \) \quad \text{Displacement function}

\( V \) \quad \text{Volume}

\( V_b \) \quad \text{Blade displacement in direction longitudinal to shroud}

\( V_s \) \quad \text{Longitudinal shroud displacement}

\( V_{sL} \) \quad \text{Left-hand shroud longitudinal displacement}

\( V_{sR} \) \quad \text{Right-hand shroud longitudinal displacement}

\( W_b \) \quad \text{Blade displacement in direction transverse to shroud}

\( W_s \) \quad \text{Transverse shroud displacement}

\( W_{sL} \) \quad \text{Left-hand shroud transverse displacement}

\( W_{sR} \) \quad \text{Right-hand shroud transverse displacement}
W-V  Shroud co-ordinate system
x  Variable denoting beam length
Y-U  Blade co-ordinate system
y  Transverse displacement of beam
CHAPTER 1
INTRODUCTION
1.1 General Introduction

The need for further research into the vibration characteristics of turbine blading is a real one. Owing to very high costs of development in producing a prototype turbine, and even greater costs involved in bringing the prototype to production and finally to installation stages, it is essential that a full investigation into the dynamic characteristics of turbine blading be carried out accurately at the design stage. Many recent examples can be quoted where resonant vibration of turbine blading has led to partial or complete failure of turbines in service. A notable example of early in-service failure was that of the Q.E.2, fully reported by Fleeting and Coats (1, 1970) under the title 'Blade Failure in the H.P. Turbines of R.M.S. Queen Elizabeth 2 and Their Rectification'.

Donald (2, 1973) in his paper entitled 'Marine Steam Turbines Some Points of Design and Operation', emphasises the importance and need for more information with regard to vibration of turbine blading in general, and points out that information on blade root flexibility is particularly lacking.

The aim of the present investigation is twofold:
(i) The development of an improved finite element based on displacement and a true state of stress
of the element.

(ii) An investigation into the effect of root flexibility on the vibration characteristics of a packet of blades.

A fuller description of the scope of the present investigation will follow, but first a short survey of vibration analysis of turbine blading is included.

1.2 Short Survey of Vibration Analysis of Turbine Blading

The volume of work, carried out on furthering the understanding and the development of techniques of determining vibration characteristics of turbine blading is considerable.

Stodola (3, 1927) outlined a technique which by present day standards would be regarded as approximate, but at the time, and for a long time to follow, was regarded as a basis for the vibration analysis of turbine blading by many steam and gas turbine designers.

Later Sezawa (4, 1933) and Smith (5, 1948) extended and added to the work of Stodola by investigating in detail the effect of a shroud ring attached to the blade tips, but owing to algebraic complexity were forced to make assumptions allowing the shroud mass to be zero or wholly concentrated at the blade tip.

Timoshenko (6, 1955) and Den Hartog (7, 1956)
have presented texts on the vibration of beams based on the classical Bernoulli-Euler theory and also, where applicable, have allowed for rotary inertia and shear deformation effects.

Prohl (8, 1958) and Weaver and Prohl (9, 1958) have presented two complementary papers dealing with the vibration problems of shrouded turbine blades from a designer's standpoint. Axial and out-of-plane effects were also included and results were obtained with the aid of a digital computer, but the assumption allowing shroud mass to be wholly concentrated at the blade tips still applied.

Ellington and McCallion (10, 1957) investigated the effect of lacing of turbine blades, but it was not until Rieger and McCallion (11, 1965) that there was presented a comprehensive classical solution to packets of shrouded blades. Good results were obtained for the natural frequencies of the packets considered but the mode shapes were determined by experiment only.

Henshell and Warburton (12, 1969) studied and compared results obtained by classical analysis, the lumped-mass method and the finite-element method, for determining the vibration characteristics of structures composed of beam elements. However, in the finite-element method, the co-ordinates were limited to displacements only.

A considerable amount of work has been carried out
by researchers into more-sophisticated methods of determining the dynamic characteristics of uncoupled and coupled cantilever blades, including the effects of pre-twist, taper and rotary inertia.

Myklestad (13, 1944) presented a method of determining uncoupled natural frequencies in bending of aircraft wings. Later, Rosard (14, 1952) adapted this method to pre-twisted cantilever beams. Slyper (15, 1962) conducted a comprehensive study of the vibration characteristics of uniform cantilever beams for coupled bending modes.


At approximately the same time Dawson (20, 1967), Thomas (21, 1968) and Dokumaci (22, 1968) carried out extensive studies on the cantilever blade in the bending-bending and torsional modes. Dawson (20, 1967) investigated the vibration characteristics of rectangular cantilever blades including the effects of shear, rotary inertia and pre-twist. Soon after, Thomas (21, 1968) carried out a very thorough investigation into cantilever blading, including the Timoshenko effect. The effects of pre-twist and taper were also included, which gave a much more realistic solution to
the vibration characteristics of turbine blading as used in
turbo machinery. Dokumaci (22, 1968), a little later,
investigated the vibration characteristics of a Timoshenko
beam in bending and torsion, taking into account the
effects of pre-twist and taper, but used a finite element
method.

A number of papers were published jointly and
individually by the above authors. Carnegie, Dawson and
Thomas (23, 1966) jointly presented a paper covering, in the
first part, with the aid of a finite-difference method the
vibration characteristics of a rectangular pre-twisted
tapering blade. The second part of the paper dealt with a
uniform aerofoil cross-section blade including the effect of
pre-twist. Results showed that the effects of bending -
bending and torsional coupling resulting from the aerofoil
section of the blade were considerable, and especially so in
the higher modes. Carnegie and Thomas (24, 1967), investigated
the effects of width taper only on the flexural-vibration
characteristics of a cantilever beam of uniform thickness.
They concluded that the effect of taper was to increase the
amplitudes of vibration, which would result in increased
stresses. The same two authors published two complementary
papers. The first of these (25, 1971) dealt with the effects
of taper on a pre-twisted blade of square section and the
second (26, 1971) dealt with the effects of shear deformation
and rotary inertia on the lateral frequencies of cantilever
beams in bending. The results were compared and contrasted
with those obtained from the classical Bernoulli-Euler
analysis. It was concluded that in the case of uniform and tapered beams with no pre-twist, the effect of including shear deflection and rotary inertia tend to reduce the frequency parameter. However, as the pre-twist increases results tend to fluctuate with mode order.

Bury (27, 1967) studied and reported the effect of thinning the blade neck on the fatigue life of a turbine blade. It appeared that for a given steady and oscillatory stress, an optimum thickness could be found. Smith, J.E. (28, 1968) presented a paper proposing a design and development procedure based on correlation between the natural frequencies of blading and possible causes of excitation, in order to minimise the possibility of vibrational problems in the blading of axial turbomachinery.

A noteworthy development in the vibration analysis of turbine blading was the introduction and, continuously increasing use, of the finite-element method. Leckie and Lindberg (29, 1963) demonstrated the superiority of the finite-element method by comparing their results for the vibration of a uniform beam with those obtained using a Myklestad model. Dokumaci, Thomas and Carnegie (30, 1967) and Carnegie, Thomas and Dokumaci (31, 1969), published two papers, the first one on bending-bending vibration of pre-twisted blading, and the second showing, by the introduction of internal modes, an improved matrix-displacement analysis for vibrating beams.
Many publications have appeared over the last decade on the finite-element method in general and the vibration analysis of turbine blading in particular. Noteworthy examples are a paper published on the modified Rayleigh-Ritz method by Prasad Krishna and Rao (32, 1972), two papers written by Thomas and Dokumaci (33, 1973) on tapered beams and (34, 1974) on pre-twisted blades, as well as a paper presented by Wilson and Kirkhope (35, 1975) on turbine disc vibration.

It should be noted here that the more traditional methods of vibration analysis are still used successfully by researchers, usually where less complex structures are analysed, or, in the case of more complex structures, where receptance matching techniques are used. Cottney and Ewins (36, 1974) analysed a complete disc blade and shroud assembly by receptance matching. They determined the vibration characteristics of the disc, blade and shroud separately by classical analysis and then by matching forces and displacements at the common joints, determined the dynamic characteristics of the complete system.

Chalk (37, 1976) has studied and presented an analysis, based on the Myklestad method, of determining the effect of shroud extensibility on the vibrational characteristics of turbine blade packets. The experimental techniques used on a wide range of experimental models, including the holographic recording of mode shapes, are of particular interest.
1.3 Present Investigation

1.3.1 Development of an Improved Finite Element

Physical and mathematical models have been used in the past for research or design purposes, and useful information has been obtained. However, it was found that these models lacked flexibility, and in the case of physical models, minor changes in specification of the prototype would involve the scrapping of previous models and hence the need for new ones, so adding considerably to the cost and duration of the investigation. Mathematical models based on classical methods were often found to be lacking in scope and flexibility.

With the advent of the finite-element approach, and the use of the digital computer, it is now possible to obtain a comparatively inexpensive 'tailor-made' mathematical model of turbine blading having considerable in-built flexibility. It is not surprising that a large number of investigations have been, and will continue to be, carried out on how best to relate the physical properties of a particular system to mathematical solutions. The question that has posed itself to many investigators, and is still a real one, is how many degrees of freedom should be associated with a particular element and how many elements should be considered in order to obtain a reasonably accurate solution? In theory, there is no limit to the number of elements which can be used, but in practice it is found that computer capacity and time are the limiting factors.
In the past, there has been a tendency to use either the displacement or the force structural analysis. A full description of these methods is given by Przemieniecki (38, 1968). They give good approximations, but the results obtained are incomplete and convergence based on the numbers of elements is slow.

An attempt is made in this investigation to show that a structural analysis approaching as near as possible the true state of stress of each element, and ensuring stress equilibrium and strain compatibility in conjunction with the appropriate boundary conditions, will not only give an eigenvector solution, from which the complete stress system for each element and mode shapes can be obtained, but will require fewer elements for an accurate solution and provide considerably improved convergence.

From a practical point of view, this investigation also attempts to demonstrate that computer capacity and time are saved by the use of combined force-displacement matrices.

A short introduction and description of the finite-element method is given in Chapter 2. A presentation of the improved finite element, together with results and comparisons of present results with those obtained by other investigators, are given in Chapter 3.
1.3.2 Investigation into the Effect of Root Flexibility on the Vibration Characteristics of a Packet of Blades.

The vibration characteristics of complete blade, disc and shroud assemblies have been investigated by Cottney and Ewins (36, 1974) and by Belek (39, 1977).

However, there still exists a considerable need for more information about the effect of blade root fixation on the vibration characteristics of complete, shrouded blade assemblies.

It is well known that the number of types of root joint used is almost as large as the number of manufacturers of turbo-machinery. Currently in use are the inverted and straddle 'T' root joints. There are fir-tree and inverted fir-tree types and the shank and bulb root joints. Then there are the single and multiple fork types and the lozenge-section root joints. Donald (2, 1973) points out that various assumptions currently made in the determination of natural frequencies of turbine blading, may lead to error, and even more so in the case of short blades where the root part forms a large proportion of the total blade length.

Perkins in 1966 (40, 1966) investigated the effect of support flexibility in the natural frequencies of a uniform cantilever, but considered one degree of freedom only. Chun (41, 1972) investigated the free vibration of a beam with one end spring-hinged and the other free.
Gorman (42, 1975) developed a classical solution for a uniform beam with one end free and the other restricted by a transverse and torsional spring.

In the present investigation no attempt is made to study in detail particular types of blade-root assemblies. The object of the research described in this thesis is to develop a finite-element model of a packet of blades with an infinitely variable spring system at the root. It is hoped that such a system may provide the designer with a blanket solution for a wide variety of blade-root fixations.

The investigation, here, is limited to a two-dimensional model, hence three springs, longitudinal, transverse and torsional, all having stiffnesses that can be varied independently between zero and infinity, are chosen.

The development of the finite-element models, experimental work and comparisons of results obtained in the present investigation with those of other investigators are presented in Chapters 4 and 5.

Conclusions and recommendations for future research related to the present investigation are given in Chapter 6.
CHAPTER 2

THE FINITE-ELEMENT METHOD
2.1 ORIGINS OF THE FINITE ELEMENT METHOD

It is difficult to establish exactly the origins of the finite-element method as now known.

R. Courant (43, 1943) demonstrated the need for modification of the well-established and powerful Rayleigh-Ritz method.

Later Argyris (44, 1955) illustrated his work on matrix structural analysis by developing the stiffness matrix for a plane rectangular panel. At approximately the same time, Turner, Clough, Martin and Topp (45, 1956) presented a paper on the displacement matrix analysis applied to two dimensional rectangular and triangular elements.

Dokumaci, Thomas and Carnegie (30, 1967), have written a paper introducing matrix displacement analysis as a method of determining the vibration characteristics of pre-twisted turbine blading.


2.2 THEORETICAL CONSIDERATIONS

2.2.1 Introduction

The finite-element method is a mathematical
technique for solving a differential equation of the type

\[ [A] \{ x \} = \{ f \} \quad \text{in } \Omega \quad (2.01) \]

where \([A]\) is a matrix of linear operators. \(\{ x \}\) and \(\{ f \}\) are vector functions of \(n\) independent variables \(x_1, x_2, \ldots, x_n\) and \(\Omega\) is the domain boundary of the problem.

Owing to the complexity of equation 2.01 a numerical approach to its solution is usually employed. The most important methods used are those based on the variational principles or finite differences.

In structural analysis, the variational methods are the most important ones, particularly the Rayleigh and Rayleigh-Ritz methods, which are fully described by Oliveira (46, 1968), and Mikhlin (47, 1963). Also, more recently, Zienkiewicz (48, 1971), Desai and Abel (49, 1972), Cook (50, 1974) and Norrie and de Vries (51, 1973) give full accounts of the use of the finite-element method in structural analysis.

For completeness, short descriptions of the Rayleigh and Rayleigh-Ritz methods and a derivation of the free vibration equation by the finite-element method for an isotropic elastic structure are included.

2.2.2 **Rayleigh's Method**

Rayleigh's method is an energy method based on the principles that the potential energy of a system is a minimum at a position of stable equilibrium and that in simple harmonic motion the mean values of the total
potential energy and the total kinetic energy of a vibrating system are equal when averaged over a long period of time.

It follows that in a fundamental mode of an elastic vibration the distributions of potential and kinetic energies are such as to make the frequency a minimum.

The true natural mode of vibration is compared with an assumed mode in which the system is constrained to execute simple harmonic motion by the action of some external forces.

The energies and hence the frequency of vibration are derived from this constrained mode. The frequency cannot be less than the fundamental for the system and will be a much closer approximation to the fundamental frequency than the approximation of the assumed amplitudes to the true amplitudes. Hence, an improvement in the choice of function to represent the fundamental mode of vibration will give rapid convergence of results.

The determination of the natural frequency of longitudinal vibration of a uniform beam, of length L, is given here as a simple illustration of the above method.

It has already been stated that

\[ \overline{T} = \overline{U} \]

where \( \overline{U} \) = the mean potential energy

and \( \overline{T} \) = the mean kinetic energy.
Sometimes it is more convenient to use values of maximum potential and kinetic energies and it can easily be shown that

\[ \hat{T} = \hat{U} \]

where \( \hat{U} \) = the maximum potential energy
and \( \hat{T} \) = the maximum kinetic energy

Quoting the potential and kinetic energies relevant to longitudinal vibration,

\[ U = \frac{EA}{2} \int_0^L \left( \frac{du}{dx} \right)^2 \, dx \quad (2.02) \]

and

\[ T = \frac{\rho A}{2} \int_0^L \hat{u}^2 \, dx \quad (2.03) \]

and assuming a displacement function

\[ u = \hat{u}(x) \sin pt \quad (2.04) \]

the maximum values of potential and kinetic energies are

\[ \hat{U} = \frac{EA}{2} \int_0^L \left( \frac{d\hat{u}}{dx} \right)^2 \, dx \quad (2.05) \]

and

\[ \hat{T} = \frac{\rho A}{2} \int_0^L \hat{u}^2 \, dx \quad (2.06) \]

Hence frequency parameter, \( \lambda_L \), is given by

\[ \lambda_L = \frac{\int_0^L \left( \frac{d\hat{u}}{dx} \right)^2 \, dx}{\int_0^L \hat{u}^2 \, dx} \quad (2.07) \]

where \( \lambda_L = \frac{pD^2}{E} \quad (2.08) \)
2.2.3 Rayleigh-Ritz Method

The limitation of Rayleigh's method is that it can only be used successfully for the determination of the approximate value of the fundamental natural frequency of a vibrating system.

Ritz extended Rayleigh's method and improved it considerably by introducing a deflection curve written as a series containing a set of variable parameters. The magnitudes of the parameters are chosen in such a manner as to give stationary values to the energy functions thus enabling the determination of the frequencies and the modes of vibration. Letting

\[ \hat{u} = \sum_{i=1}^{n} a_i \phi_i(x) \quad (2.08) \]

where \(a_1, a_2, \ldots, a_n\) are constants and \(\phi_i(x)\) are functions of \(x\) carefully chosen so that boundary conditions are satisfied and reasonably good approximations of the mode shapes considered are obtained.

\(\hat{u}\) is then substituted in equation (2.07) giving:

\[ \lambda = \frac{\int_{0}^{L} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \phi_i(x) \phi_j(x) dx}{\int_{0}^{L} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_i(x) \phi_j(x) dx} \quad (2.09) \]

The optimum values of parameters \(a_1, a_2, \ldots, a_n\) are now obtained from a series of equations

\[ \frac{\partial \lambda}{\partial a_i} = 0, \quad \frac{\partial \lambda}{\partial a_{i+1}} = 0, \quad \frac{\partial \lambda}{\partial a_n} = 0 \quad (2.10) \]
The n natural frequencies can now be obtained from equation (2.09).

If mode shapes are required, these are obtained by substituting values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ back into equation (2.08).

2.2.4 Derivation of the free vibration equation by the finite-element method

2.2.4.1 Strain energy of a finite element

The strain energy of a finite element $I$ may be given as

$$ U_I = \frac{1}{2} \int_V \{\varepsilon\}^T \{\sigma\} \, dV \quad (2.11) $$

where

$$ \{\varepsilon\} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad (2.12) $$

is the strain vector, and

$$ \{\sigma\} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \quad (2.13) $$

is the stress vector.

Assuming that the material of the element is isotropic, homogeneous and elastic,

$$ \{\sigma\} = [E] \{\varepsilon\} \quad (2.14) $$
where $E$ the elasticity matrix is given as

$$
\begin{bmatrix}
1-v & v & 0 & 0 & 0 \\
v & 1-v & v & 0 & 0 \\
v & v & 1-v & 0 & 0 \\
o & 0 & 0 & \frac{1-2v}{2} & 0 \\
o & 0 & 0 & 0 & \frac{1-2v}{2}
\end{bmatrix}
$$

(2.15)

substituting for $\{\sigma\}$ from equation (2.14) into equation (2.11),

$$
U_I = \frac{1}{2} \int \{\varepsilon\}^T \left[ E \right] \{\varepsilon\} \, dV .
$$

(2.16)

Expressing strains in terms of displacements,

$$
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
u
\end{bmatrix},
$$

(2.17)

where

$$
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= \begin{bmatrix} u \end{bmatrix}
$$

(2.18)

is the displacement vector,
and letting

$$
\left[ S \right] = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
$$

(2.19)

$$\{ \varepsilon \} = \left[ S \right] \{ u \} .$$

(2.20)

Substituting for $\{ \varepsilon \}$ from equation (2.20) into equation (2.16),

$$U_I = \frac{1}{2} \int_V \left\{ \{ u \}^T \left[ S \right]^T \left[ E \right] \left[ S \right] \{ u \} \right\} dV .$$

(2.21)

Letting

$$\{ u \} = \left[ F \right] \{ b \} ,$$

(2.22)

where

$$\left[ F \right] \{ b \}$$

is a matrix representation of the assumed mode-shape function, and letting

$$\{ \xi \} = \left[ C \right] \{ b \} ,$$

(2.23)

where

$$\{ \xi \}$$

is the assumed nodal vector and

$$\left[ C \right] \{ b \}$$

is a matrix representation of that vector, then

substituting for $\{ b \}$ from equation (2.23) into equation (2.22)

$$\{ u \} = \left[ F \right] \left[ C \right]^{-1} \{ \xi \} .$$

(2.24)
substituting for \( \{ \mathbf{u} \} \) and \( \{ \mathbf{u} \}^T \)

from equation (2.24) into equation (2.21),

\[
U_I = \frac{1}{2} \int_V \{ \xi \}^T \left[ C^{-1} \right]^T \left[ F \right]^T \left[ S \right]^T \left[ E \right] \left[ S \right] \left[ F \right] \left[ C^{-1} \right] \{ \xi \} \, \mathrm{d}V \tag{2.25}
\]

or

\[
U_I = \frac{1}{2} \{ \xi \}^T \left[ K_I \right] \{ \xi \} \tag{2.26}
\]

where

\[
\left[ K_I \right] = \int_V \left[ C^{-1} \right]^T \left[ F \right]^T \left[ S \right]^T \left[ E \right] \left[ S \right] \left[ F \right] \left[ C^{-1} \right] \, \mathrm{d}V \tag{2.27}
\]

is the stiffness matrix of the element.

---

2.2.4.2 Kinetic energy of a finite element

The kinetic energy of a finite element, \( I \), may be given as

\[
T_I = \frac{1}{2} \int_V \rho \{ \mathbf{u} \}^T \{ \dot{\mathbf{u}} \} \, \mathrm{d}V, \tag{2.28}
\]

where \( \rho \) is the mass density of the material of the element.

From equation (2.24),

\[
\{ \dot{\mathbf{u}} \} = \left[ F \right] \left[ C^{-1} \right] \{ \ddot{\xi} \} \tag{2.29}
\]

substituting for

\[
\{ \dot{\mathbf{u}} \} \text{ and } \{ \dot{\mathbf{u}} \}^T
\]

from equation (2.29) into equation (2.28),

\[
T_I = \frac{1}{2} \int_V \{ \xi \}^T \rho \left[ C^{-1} \right]^T \left[ F \right]^T \left[ F \right] \left[ C^{-1} \right] \{ \ddot{\xi} \} \, \mathrm{d}V \tag{2.28}
\]

or

\[
T_I = \frac{1}{2} \{ \ddot{\xi} \}^T \left[ M_I \right] \{ \ddot{\xi} \}, \tag{2.29}
\]

where

\[
\left[ M_I \right] = \int_V \rho \left[ C^{-1} \right]^T \left[ F \right]^T \left[ F \right] \left[ C^{-1} \right] \, \mathrm{d}V \tag{2.30}
\]

is the mass matrix of the element.
2.2.4.3 **Global stiffness and mass matrices**

Before local stiffness and mass matrices can be assembled to form global matrices it is necessary to ensure that all co-ordinates at nodal points conform to a common system of axes, and that strain compatibility, stress equilibrium and boundary conditions are satisfied.

Global conformity is achieved by transforming individual local matrices to a global system of co-ordinates as shown by

\[
\{\xi\} = [G] \{q\} \quad (2.31)
\]

The total strain and kinetic energies of the whole structure are equal to the sum of the energies of all the elements present. Hence

\[
U = \frac{1}{2} \{q\}^T [K] \{q\} \quad (2.32)
\]

and

\[
T = \frac{1}{2} \{\dot{q}\}^T [M] \{\dot{q}\} \quad (2.33)
\]

where \(U\) and \(T\) are the total strain and kinetic energies and

\([K]\) and \([M]\)

are the global stiffness and mass matrices of the structure.

2.2.4.4 **Formation of the free vibration equation**

The equation of motion for a conservative dynamic system is derived from Hamilton's Principle as stated in equation (2.34):

\[
\int_{t_0}^{t_1} (T - U) dt = \text{Minimum}, \quad (2.34)
\]
where \( t_0 \) and \( t_1 \) are two prescribed time intervals.

The equation of Lagrange follows from the minimisation of Hamilton's Principle:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{d}{dt} \left( \frac{\partial U}{\partial q} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = \{Q\} \quad (2.35)
\]

where \( \{Q\} \) is the force vector for the whole structure. Substituting the expressions for the total strain and potential energies given in equations (2.32) and (2.33) into Lagrange's equation (2.35) gives the equation of motion

\[
[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \quad (2.36)
\]

For simple harmonic motion and free vibration, equation (2.36) becomes

\[
\left( [K] - \lambda [M] \right) \{q\} = 0 \quad (2.37)
\]

where \( \lambda \) is a frequency parameter.

Equation (2.37) is solved by an eigenvalue routine. Eigenvalues and eigenvectors of

\[
(K - \lambda M)q = 0
\]

are obtained, where \( K \) and \( M \) are real, symmetric and \( M \) is positive definite.

Useful reports on this type of mathematical solution have been written by Martin and Wilkinson (52, 1968) Martin, Reinsch and Wilkinson (53, 1968) and Pestel (54, 1966).
CHAPTER 3

DEVELOPMENT OF THE FINITE ELEMENT
3.1 Introduction

In this chapter a finite element for the vibrational analysis of beams is developed, having twelve degrees of freedom, six at each nodal point. The parameters considered are lateral, longitudinal and rotational displacements coupled with parameters proportional to longitudinal and shear forces and the bending moment at each nodal point. The finite element method is then used in order to obtain an eigenvalue solution to a uniform beam problem.

The effect of the number of elements used on convergence of eigenvalue is studied and results are compared with exact values and references (29, 31, 32).

3.2 Uniform Beam Element

3.2.1 Transverse effects:

The nodal coordinates chosen for a beam element are as shown in figure 3.1.

3.2.1.1 Derivation of strain energy, $U_i^{i+1}$

Assuming that the shear strain energy in a thin beam element is small, the total strain energy is that due to bending. Hence,

$$U_i^{i+1} = \frac{1}{2} EI \int_0^\ell \left( \frac{d^2y}{dx^2} \right)^2 dx$$

Expressing non-dimensionally, by letting,

$$\eta = \frac{x}{\ell}$$

(3.01)
Let \( y \) be of the form,
\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7
\]
Expressing non-dimensionally,
\[
y = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4 + a_5 \eta^5 + a_6 \eta^6 + a_7 \eta^7
\]
or
\[
y = b_0 + b_1 \eta + b_2 \eta^2 + b_3 \eta^3 + b_4 \eta^4 + b_5 \eta^5 + b_6 \eta^6 + b_7 \eta^7
\]
Hence,
\[
y = \begin{bmatrix} 1 & \eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 \end{bmatrix} \{b\}
\]
or
\[
y = \begin{bmatrix} F \end{bmatrix} \{b\}
\]
where,
\[
\begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} 1 & \eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 \end{bmatrix}
\]
The values of the derivatives of \( y \) with respect to \( \eta \) are given below as;
\[
y' = b_1 + 2b_2 \eta + 3b_3 \eta^2 + 4b_4 \eta^3 + 5b_5 \eta^4 + 6b_6 \eta^5 + 7b_7 \eta^6
\]
\[
y'' = 2b_2 + 6b_3 \eta + 12b_4 \eta^2 + 20b_5 \eta^3 + 30b_6 \eta^4 + 42b_7 \eta^5
\]
\[
y''' = 6b_3 + 24b_4 \eta + 60b_5 \eta^2 + 120b_6 \eta^3 + 210b_7 \eta^4
\]
With boundary conditions appropriate to,
\[
y = y_i \text{ and } y = y_{i+1} \text{ at node } i \text{ and } i+1
\]
respectively, the relationship between the nodal co-ordinates and the coefficients is,
\[
\begin{bmatrix}
  y_i \\
  y_i' \\
  y_i'' \\
  y_i'''
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 6 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b_5 \\
  b_6 \\
  b_7
\end{bmatrix}
\]

Let the above equation be represented by,

\[\{\xi\} = \mathbf{C} \{b\}\]

Hence,

\[\{b\} = \mathbf{C}^{-1} \{\xi\}\] \hspace{1cm} (3.03)

\(\mathbf{C}^{-1}\) was obtained analytically in order to ensure an exact result at this stage.

\[
\begin{bmatrix}
  6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \cdots & 3 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  -210 & -120 & -30 & -4 & 210 & -90 & 15 & -1 \\
  504 & 270 & 60 & 6 & -504 & 234 & -42 & 3 \\
  -420 & -216 & -45 & -4 & 420 & -204 & 39 & -3 \\
  -120 & 60 & 12 & 1 & -120 & 60 & -12 & 1
\end{bmatrix}
\]

From equation (3.02)

\[
\frac{d^2 y}{dn^2} = \begin{bmatrix}\mathbf{F}'\end{bmatrix} \{b\}
\]

\[
\left(\frac{d^2 y}{dn^2}\right)^2 = \{b\}^T \begin{bmatrix}\mathbf{F}'\end{bmatrix}^T \begin{bmatrix}\mathbf{F}'\end{bmatrix} \{b\}
\]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 12\eta & 24\eta^2 & 40\eta^3 & 60\eta^4 & 84\eta^5 \\
0 & 0 & 12\eta & 36\eta^2 & 72\eta^3 & 120\eta^4 & 180\eta^5 & 252\eta^6 \\
0 & 0 & 24\eta^2 & 72\eta^3 & 144\eta^4 & 240\eta^5 & 360\eta^6 & 504\eta^7 \\
0 & 0 & 40\eta^3 & 120\eta^4 & 240\eta^5 & 400\eta^6 & 600\eta^7 & 800\eta^8 \\
0 & 0 & 60\eta^4 & 180\eta^5 & 360\eta^6 & 600\eta^7 & 900\eta^8 & 1260\eta^9 \\
0 & 0 & 84\eta^5 & 252\eta^6 & 504\eta^7 & 840\eta^8 & 1260\eta^9 & 1764\eta^{10}
\end{bmatrix}
\]

Integrating with respect to \( \eta \) and taking limits 0 to 1

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 6 & 8 & 10 & 12 & 14 \\
0 & 0 & 6 & 12 & 18 & 24 & 30 & 36 \\
0 & 0 & 8 & 18 & 144/5 & 40 & 360/7 & 63 \\
0 & 0 & 10 & 24 & 40 & 400/7 & 75 & 840/9 \\
0 & 0 & 12 & 30 & 360/7 & 75 & 100 & 126 \\
0 & 0 & 14 & 36 & 63 & 840/9 & 126 & 1764/11
\end{bmatrix}
\]

Letting the above matrix to be represented by,

\[
\begin{bmatrix}
k_1
\end{bmatrix}
\]

the expression for strain energy \( U_{i,i+1} \) of the element becomes

\[
U_{i,i+1} = \frac{EI}{2l^3} \{b^T\} \begin{bmatrix} k_1 \end{bmatrix} \{b\}
\]

From equation (2.3)

\[
U_{i,i+1} = \frac{EI}{2l^3} \left[ \begin{bmatrix} C \end{bmatrix}^{-1} \{\xi\} \right]^T \begin{bmatrix} k_1 \end{bmatrix} \left[ \begin{bmatrix} C \end{bmatrix}^{-1} \right] \{\xi\}
\]

or

\[
= \frac{EI}{2l^3} \{\xi\}^T \left[ \begin{bmatrix} C \end{bmatrix}^{-1}\right]^T \begin{bmatrix} k_1 \end{bmatrix} \left[ \begin{bmatrix} C \end{bmatrix}^{-1}\right] \{\xi\}
\]

or
\[ u^{i,i+1} = \frac{EI}{2a^3} \{\xi\}^T[K_i] \{\xi\} \quad (3.04) \]

where,

\[ [K_i] = [C^{-1}]^T[K_1][C^{-1}] \]

\[ [K_i] \] is given in equation (3.11)

### 3.2.1.2 Determination of kinetic energy \( T^{i,i+1} \)

The kinetic energy of an elemental length of the beam is given by

\[ T^{i,i+1} = \frac{1}{2} \rho A \int_0^L \dot{y}^2 \, dx \]

Expressing non-dimensionally, using equation (3.01)

\[ T^{i,i+1} = \frac{1}{2} \rho A \int_0^1 \dot{y}^2 \, dn \quad (3.05) \]

Substituting for \( y \) from equation (3.02) the kinetic energy becomes

\[ T^{i,i+1} = \frac{1}{2} \rho A \mathcal{L} \{b\}^T \int_0^1 \{F\}^T \{F\} \, dn \{b\} \]

where

\[
\begin{bmatrix}
1 & \eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 \\
\eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\
\eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 \\
\eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} \\
\eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} \\
\eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} \\
\eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} \\
\eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} & \eta^{14}
\end{bmatrix}
\]

and,

\[
\begin{bmatrix}
\eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 \\
\eta^2 & \eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\
\eta^3 & \eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 \\
\eta^4 & \eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} \\
\eta^5 & \eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} \\
\eta^6 & \eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} \\
\eta^7 & \eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} \\
\eta^8 & \eta^9 & \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} & \eta^{14}
\end{bmatrix}
\]
Letting the above matrix be represented by 

\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\
\frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\
\frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\
\frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\
\frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15}
\end{bmatrix}
\]

the kinetic energy of the beam element becomes

\[
T_i, i+1 = \frac{1}{2} \rho A L \{b\}^T \begin{bmatrix} m_i \end{bmatrix} \{b\}
\]

From equation (3.03),

\[
\{b\} = [C^{-1}] \{\xi\}
\]

Hence,

\[
T_i, i+1 = \frac{1}{2} \rho A L \{\xi\}^T \left[ [C^{-1}]^T [m_i] [C^{-1}] \right] \{\xi\}
\]

Let,

\[
[M_i] = \left[ [C^{-1}]^T [m_i] [C^{-1}] \right]
\]

Hence,

\[
T_i, i+1 = \frac{1}{2} \rho A L \{\xi\}^T [M_i] \{\xi\}
\]  \hspace{1cm} (3.06)

\([M_i]\) is given in equation (3.12)
3.2.1.3 Derivation of the frequency equation

Quoting Lagrange's Equation

\[
\left\{ \frac{\partial U}{\partial \{\xi\}} \right\} + \frac{\partial}{\partial t} \left\{ \frac{\partial T}{\partial \{\xi\}} \right\} = \{P\}
\]  \hspace{1cm} (3.07)

From equations (3.04) and (3.06),

\[
\left\{ \frac{\partial U}{\partial \{\xi\}} \right\} = \frac{EI}{\chi^3} [K_1] \{\xi\}
\]

\[
\left\{ \frac{\partial T}{\partial \{\xi\}} \right\} = \rho A \lambda [M_1]\{\ddot{\xi}\}
\]

\[
\frac{\partial}{\partial t} \left\{ \frac{\partial T}{\partial \{\xi\}} \right\} = \rho A \lambda [M_1] \{\dddot{\xi}\}
\]

Substituting into Lagrange's equation

\[
\frac{EI}{\chi^3} [K_1] \{\xi\} + \rho A \lambda [M_1] \{\ddot{\xi}\} = \{P\}
\]

Assuming \( \{\xi\} \) to be of the form,

\[
\{\xi\} = \{\hat{\xi}\} e^{jpt}
\]

and

\[
\{P\} = \{\hat{P}\} e^{jpt}
\]

\[
\frac{EI}{\chi^3} [K_1] - p^2 \rho A \lambda [M_1] = \{P\}
\]  \hspace{1cm} (3.08)

or

\[
[K_1] - \frac{\rho A \lambda^4}{EI} p^2 [M_1] = \frac{\chi^3}{EI} \{P\}
\]

or

\[
[K_1] - \lambda [M_1] = \{P_1\}
\]  \hspace{1cm} (3.09)

where

\[
\lambda = \frac{\rho A \lambda^4}{EI} p^2
\]  \hspace{1cm} (3.10)

\[
\{P_1\} = \frac{\chi^3}{EI} \{P\} \]
\[ [K_1] = \frac{1}{13880} \begin{bmatrix} 352 800 & 176 400 & 16 800 & 630 & -352 800 & 176 400 & -16 800 & 630 \\ 176 400 & 108 000 & 11 370 & 480 & -176 400 & 68 400 & -5 430 & 150 \\ 16 800 & 11 370 & 3 000 & 140 & -16 800 & 5 430 & -30 & -25 \\ 630 & 480 & 140 & 8 & -630 & 150 & 25 & -3 \\ -352 800 & -176 400 & -16 800 & -630 & 352 800 & -176 400 & 16 800 & -630 \\ 176 400 & 68 400 & 5 430 & 150 & -176 400 & 108 000 & -11 370 & 480 \\ -16 800 & -5 430 & -30 & 25 & 16 800 & -11 370 & 3 000 & -140 \\ 630 & 150 & 25 & 3 & -630 & 480 & -140 & 8 \end{bmatrix} \]


\[(3.11)\]
3.2.2 Longitudinal Vibration

The nodal coordinates of a beam element \( i, i+1 \) in longitudinal motion are shown in Fig.3.2

3.2.2.1 Derivation of strain energy, \( U_{L}^{i, i+1} \)

\[
U_{L}^{i, i+1} = \frac{1}{2} EA \int_{0}^{L} \left( \frac{du}{dx} \right)^2 dx
\]

Expressing non-dimensionally, by letting,

\[ \eta = \frac{x}{L} \]

\[
U_{L}^{i, i+1} = \frac{1}{2} \frac{EA}{L^2} \int_{0}^{1} \left( \frac{du}{d\eta} \right) d\eta \tag{3.13}
\]

Let \( u \) be of the form,

\[ u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]

Expressing non-dimensionally,

\[ u = b_0 + b_1 \eta + b_2 \eta^2 + b_3 \eta^3 \]

or

\[ u = [F] \{b\} \tag{3.14} \]

where,

\[
[F] = \begin{bmatrix}
1 & \eta & \eta^2 & \eta^3
\end{bmatrix}
\]
With boundary conditions appropriate to,
\[ u = u_1 \quad \text{and} \quad u = u_{i+1} \]
the relationship between the nodal co-ordinates and the coefficients is,
\[
\begin{align*}
\begin{bmatrix}
  u_i \\
  u_i' \\
  u_{i+1} \\
  u_{i+1}'
\end{bmatrix}
&= 
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 1 & 1 & 1 \\
  0 & 1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\end{align*}
\]

Letting the above expression be represented by,
\[ \{\xi\} = [C]\{b\}, \]
the coefficients are given by
\[ \{b\} = [C]^{-1}\{\xi\} \]  
(3.15)

\[ [C]^{-1} \] was determined analytically as
\[
\begin{bmatrix}
  1 & 0 & -3 & 2 \\
  0 & 1 & -2 & 1 \\
  0 & 0 & 3 & -2 \\
  0 & 0 & -1 & 1
\end{bmatrix}
\]

From equation (3.14),
\[ \frac{d\mathbf{u}}{d\eta} \mathbf{F}' \{b\} \]
\[ (\frac{d\mathbf{u}}{d\eta})^2 = \{b^T\} \mathbf{F}'^T \mathbf{F}' \{b\} \]

\[ \mathbf{F}'^T \mathbf{F}' = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 1 & 2\eta & 3\eta^2 \\
  0 & 2\eta & 4\eta^2 & 6\eta^3 \\
  0 & 3\eta^2 & 6\eta^3 & 9\eta^4
\end{bmatrix} \]

Integrating with respect to \( \eta \) and taking limits 0 to 1
\[
\int_0^1 \left( \frac{dn}{dx} \right)^2 \, dn = \{b^T \} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 30 & 30 & 30 \\ 0 & 30 & 40 & 45 \\ 0 & 30 & 45 & 54 \end{bmatrix} \{b\}
\]

Let the above matrix be represented by,
\[
[k_2]
\]
then, the strain energy is
\[
U_{L, i, i+1} = \frac{EA}{2L} \{b^T \} [k_2] \{b\}
\]

From equation (3.15)
\[
U_{L, i, i+1} = \frac{EA}{2L} \{\xi \}^T [C^{-1}]^T [k_2] [C^{-1}] \{\xi \}
\]

or
\[
U_{L, i, i+1} = \frac{EA}{2L} \{\xi \}^T [k_2] \{\xi \} \quad (3.16)
\]

\([k_2]\) is given by equation (3.19)

3.2.2.2 Determination of Kinetic Energy, \(T_{L, i, i+1}\)

The kinetic energy of an elemental length of a beam is given by
\[
T_{L, i, i+1} = \frac{1}{2} \rho A \int_0^1 \dot{u}^2 \, dx
\]

Expressing non-dimensionally, using equation (1.11), the kinetic energy
\[
T_{L, i, i+1} = \frac{1}{2} \rho A \int_0^1 \dot{u}^2 \, dn \quad (3.17)
\]

\[
= \frac{1}{2} \rho A \{\dot{b}\}^T \int_0^1 [\dot{F}]^T [\dot{F}] \, d\eta \{\dot{b}\}
\]
\[
[F^T] [F] = \begin{bmatrix}
1 & \eta & \eta^2 & \eta^3 \\
\eta & \eta^2 & \eta^3 & \eta^4 \\
\eta^2 & \eta^3 & \eta^4 & \eta^5 \\
\eta^3 & \eta^4 & \eta^5 & \eta^6 
\end{bmatrix}
\]

\[
\int_0^1 [F^T][F] \, dn = \frac{1}{420} \begin{bmatrix}
420 & 210 & 140 & 105 \\
210 & 140 & 105 & 84 \\
140 & 105 & 84 & 70 \\
105 & 84 & 70 & 60
\end{bmatrix}
\]

Let the above matrix be represented by,
\[
[m_2]
\]

then,
\[
T_L^{i, i+1} = \frac{1}{2\rho \lambda} [b^T] [m_2] [b]
\]

From equation (2.13)
\[
b = \begin{bmatrix} C^{-1} \end{bmatrix} \{\xi\}
\]

Hence,
\[
T_L^{i, i+1} = \frac{1}{2\rho \lambda} [\xi]^T [C^{-1}] [m_2] [C^{-1}] \{\xi\}
\]

or
\[
T_L^{i, i+1} = \frac{1}{2\rho \lambda} [\xi]^T [M_2] \{\xi\}
\]

where
\[
[M_2] = [C^{-1}]^T [m_2] [C^{-1}]
\]

and is given by equation (3.20)
\[
[K_2] = \frac{1}{30} \begin{bmatrix}
36 & 3 & -36 & 3 \\
3 & 4 & -3 & -1 \\
-36 & -3 & 36 & -3 \\
3 & -1 & -3 & 4
\end{bmatrix}
\]
3.2.2.3 Derivation of the frequency equation

Using Lagrange's Equation and referring to work, in section 3.2.1.3 it can be easily shown that

\[
\frac{EA}{\lambda} [K_2] - \rho A \lambda^2 [M_2] = \{Q\}
\]  

(3.21)

3.2.3 Combined transverse and longitudinal effects

The nodal co-ordinates of a beam element i, i+1 in transverse and longitudinal motion are shown in Fig.3.3.

By superposition and from (3.11), (3.12), (3.19) and (3.20), the frequency equation for a single element is,

\[
\frac{EI}{\lambda^3 13860} \begin{bmatrix} K_1 & \vdots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & K_2 \end{bmatrix} - \lambda^2 \begin{bmatrix} \frac{\rho A}{12972960} \{M_1^*\} \\ \vdots \\ \frac{\rho A}{420} \{M_2^*\} \end{bmatrix} = \{Q\}
\]

\[
\frac{EI}{\lambda^3 13860} \begin{bmatrix} K_1^* \\ \vdots \\ \frac{A^4L^4}{12972960} K_2^* \end{bmatrix} - \lambda^2 \begin{bmatrix} \frac{\rho A^4}{12972960} \{M_1^*\} \\ \vdots \\ \frac{30888}{12972960} \{M_2^*\} \end{bmatrix} = \{Q\}
\]

Letting \( \lambda = \frac{\rho A^4}{EI} p^2 \)
\[
\begin{bmatrix}
K_1^* & & & \\
& K_2^* & & \\
& & &
\end{bmatrix}
- \begin{bmatrix}
\lambda \\
\frac{462}{936} \\
\end{bmatrix}
\begin{bmatrix}
M_1^* & & & \\
& M_2^* & & \\
& & &
\end{bmatrix}
= \{Q\} \quad (3.22)
\]

or
\[
[KL] - \frac{\lambda}{936} [ML] = \{Q\} \quad (3.23)
\]

\([KL]\) and \([ML]\) are the local non-dimensional stiffness and mass matrices given by equations (3.24) and (3.25) respectively.
\[
[KL] =
\begin{bmatrix}
352800 & 176400 & 16800 & 630 & 0 & 0 & -352800 & 176400 & -16800 & 630 & 0 & 0 \\
176400 & 108000 & 11370 & 480 & 0 & 0 & -176400 & 68400 & -5430 & 150 & 0 & 0 \\
16800 & 11370 & 3000 & 140 & 0 & 0 & -16800 & 5430 & -30 & 25 & 0 & 0 \\
630 & 480 & 140 & 8 & 0 & 0 & -630 & 150 & 25 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha L^2 462}{I} x 36 & \frac{\alpha L^2 462}{I} x 3 & 0 & 0 & 0 & 0 & -\frac{462 \alpha L^2}{I} x 36 & \frac{462 \alpha L^2}{I} x 3 \\
0 & 0 & 0 & 0 & \frac{462 \alpha L^2}{I} x 3 & \frac{462 \alpha L^2}{I} x 3 & 0 & 0 & 0 & 0 & -\frac{462 \alpha L^2}{I} x 3 & -\frac{462 \alpha L^2}{I} x 3 \\
-352800 & -176400 & -16800 & -630 & 0 & 0 & 352800 & -176400 & 16800 & -630 & 0 & 0 \\
176400 & 68400 & 5430 & 150 & 0 & 0 & -176400 & 108000 & -11370 & 480 & 0 & 0 \\
-16800 & -5430 & -30 & 25 & 0 & 0 & 16800 & -11370 & 3000 & -140 & 0 & 0 \\
630 & 150 & -25 & -3 & 0 & 0 & -630 & 480 & -140 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{462 \alpha L^2}{I} x 36 & -\frac{462 \alpha L^2}{I} x 3 & 0 & 0 & 0 & 0 & -\frac{462 \alpha L^2}{I} x 36 & -\frac{462 \alpha L^2}{I} x 3 \\
0 & 0 & 0 & 0 & \frac{462 \alpha L^2}{I} x 3 & -\frac{462 \alpha L^2}{I} x 3 & 0 & 0 & 0 & 0 & -\frac{\alpha L^2 462}{I} x 36 & \frac{462 \alpha L^2}{I} x 4 \\
\end{bmatrix}
\]

(3.24)
\[
\begin{bmatrix}
5251680 & 978480 & 98640 & 4596 & 0 & 0 & 1234800 & -411480 & 55800 & -3126 & 0 & 0 \\
978480 & 237600 & 26460 & 1296 & 0 & 0 & 411480 & -134280 & 17910 & -990 & 0 & 0 \\
98640 & 26460 & 3096 & 156 & 0 & 0 & 55800 & -17910 & 2358 & -129 & 0 & 0 \\
4596 & 1296 & 156 & 8 & 0 & 0 & 3126 & -990 & 129 & -7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 30888x156 & 30888x22 & 0 & 0 & 0 & 0 & 30888x54 & 30888x-13 \\
0 & 0 & 0 & 0 & 0 & 0 & 30888x22 & 30888x4 & 0 & 0 & 0 & 0 & 30888x13 & 30888x-3 \\
1234800 & 411480 & 55800 & 3126 & 0 & 0 & 5251680 & -978480 & 98640 & -4596 & 0 & 0 \\
-411480 & -134280 & -17910 & -990 & 0 & 0 & -978480 & 237600 & -26460 & 1296 & 0 & 0 \\
55800 & 17910 & 2358 & 129 & 0 & 0 & 98640 & -26460 & 3096 & -156 & 0 & 0 \\
-3126 & -990 & -129 & -7 & 0 & 0 & -4596 & 1296 & -156 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 30888x54 & 30888x13 & 0 & 0 & 0 & 0 & 30888x156 & 30888x22 \\
0 & 0 & 0 & 0 & 0 & 0 & 30888x-13 & 30888x-3 & 0 & 0 & 0 & 0 & 30888x-22 & 30888x4 \\
\end{bmatrix}
\]

(3.25)
If more than one or different types of elements are used, stiffness and mass matrices for the whole system, global matrices are assembled from the local matrices. This work is fully described in the next chapter, chapter 4, section 4.3.

The frequency equation is of the same type as equation (3.23) but matrices $[KL]$ and $[ML]$ are now replaced by the global matrices $[KG]$ and $[MG]$.

3.3 **Eigenvalue Solution of Frequency Equation**

The solution to equations of the type shown by (3.23) is obtained using an eigenvalue routine.

Eigenvalues and eigenvectors of

$$(A - \lambda B)x = 0$$

are found where $A$ is real symmetric and $B$ is real, symmetric positive definite, using Cholesky's and Householder's methods, (52, 1968; 53, 1968).

3.4 **Single Blade Fixed at the Root**

The element developed in this chapter is applied to the vibration analysis of a simple cantilever beam to illustrate the accuracy of the element. The nodal co-ordinates are shown in Figure 3.4.

For simplicity transverse effects only are considered and the blade is represented by a single element.

From equations 3.11 and 3.12 and with the appropriate boundary conditions as shown in Figure 3.4, the stiffness and mass matrices are:
\[
[K_1](\xi) = \frac{1}{13860}\begin{bmatrix}
3000 & 140 & -16800 & 8430 \\
140 & 8 & -630 & 150 \\
-1680 & -630 & 352800 & -176400 \\
540 & 150 & -176400 & 108000
\end{bmatrix}
\begin{bmatrix}
y_0'' \\
y_0''' \\
y_1 \\
y_1'
\end{bmatrix}
\]

\[
[M_1](\xi) = \frac{1}{12972960}\begin{bmatrix}
3096 & 156 & 55800 & -17910 \\
156 & 8 & -3126 & -990 \\
55800 & 3126 & 5251680 & -978480 \\
-17910 & -990 & -978480 & 237600
\end{bmatrix}
\begin{bmatrix}
y_0'' \\
y_0''' \\
y_1 \\
y_1'
\end{bmatrix}
\]
3.5 Discussion of Results

The theory developed in this chapter is applied to a simple case, namely that of a uniform cantilever blade. This enables comparison to be made with exact results obtained by the classical theory and results obtained by other investigators.

Table 3.1 shows the results obtained when using one element only. Results for four modes only can be obtained as the stiffness and mass matrices for one element are of order four. Information regarding mode shapes, shear force and bending moment are also limited but nevertheless values of deflection and bending slope at the tip of the blade and values of shear force and bending moment at the root for each of the four modes considered are given.

Table 3.1 also shows a comparison of frequency parameter $\lambda$ and exact values obtained by classical theory. Here, very good agreement is obtained for the first three modes with a maximum error of 0.228%. The error obtained for the fourth mode indicates that more than one element must be used in order to compute results for vibration modes of order greater than three.

Table 3.2 shows the convergence of the frequency parameter, $\lambda$, with increasing $N$, the number of blade elements used. The table shows that up to the sixth mode good agreement is obtained between computed values of $\lambda$ and exact ones, when two blade elements are used. With four elements, satisfactory values of $\lambda$ are obtained up to the ninth or
tenth modes. However small 'rounding-off' errors have occurred in the first and second modes of vibration. A blade with six elements gives good agreement right up to the fifteenth mode where the greatest error from the exact values is 0.88%.

Convergence for the first four modes is shown in Figures 3.5 and 3.6. Again it is illustrated that computed values of $\lambda$ are satisfactory for the first three modes of vibration when the blade consists of one element only, but at least two elements are required in order to compute satisfactorily the value of $\lambda$ for the fourth mode.

Tables 3.3 and 3.4 show comparisons of present results with those obtained by other investigators and exact values of the frequency parameter $\lambda$.

Table 3.3 lists values of $\lambda$ computed by Leckie and Lindberg (29, 1963). Here the authors give a method which is presented as a departure from and improvement of the Myklestad method. In effect the analysis is a finite element one based on the matrix displacement method.

Results in the table show that the present method gives considerably better results than those obtained by Leckie and Lindberg (29) and more markedly in the higher modes of vibration. An error of 2.92% is quoted by Leckie and Lindberg (29) for the value of $\lambda$ in the fourth mode, whereas only 0.06% error is obtained with the use of the present method. In both cases the matrix order was eight.
Table 3.4 gives values of $\lambda$ computed by Carnegie, Thomas and Dokumaci (31, 1961) and Prasad, Krishna and Rao (32, 1972) and comparison is made with values of $\lambda$ computed by the present method and exact values. Both references, mentioned above, give improved presentations of the matrix displacement methods as used by previous investigators.

Carnegie, Thomas and Dokumaci (31) achieve an improvement in convergence by the introduction of internal nodes in an element. However, the present method gives considerably better results. In reference (31) an error of 0.2158% is obtained for the value of $\lambda$ in the fifth mode with matrix order 16. This is compared with an error of 0.0061% obtained using the present method with matrix order 12.

Prasad, Krishna and Rao (32) assume linear distributions of displacements, shear force and bending moment over an element and in that way achieve more rapid convergence. Here again, the results obtained by the present method are considerably better than those quoted in reference (32). In the fourth mode an error of 0.0051% of the computed value of $\lambda$ with matrix order 20 is compared with an error 0.0003% obtained by the present method with matrix order 12. In the fifth mode the errors are: 0.0461% in reference (32) and 0.0061% using the present method with matrix orders as for the fourth mode, above.

One of the aims of this investigation was to
determine the displacements and forces within the structure considered. The analysis and the development of the finite element, facilitating the achievement of this result is fully described in the preceding sections of this chapter.

A computer program has been developed such that values of the frequency parameter $\lambda$ are printed out together with corresponding values of lateral, longitudinal and rotational displacements, longitudinal and shear forces and bending moments. The longitudinal and shear forces and bending moments are printed out in parametric form, $\frac{P}{AE}$, $\frac{S}{ET}$ and $\frac{M}{EI}$ where $P$, $S$ and $M$ are the true values of the longitudinal force, the shear force and the bending moment respectively.

In the case of a cantilever the longitudinal displacement and force are zero except when the blade is vibrating in one of its independent longitudinal modes. These longitudinal modes, however, occur considerably higher up in the range of natural frequencies. The first longitudinal mode of vibration occurs between the seventh and eight transverse modes and the second longitudinal mode occurs between the 13th and 14th transverse modes for the cantilever considered.

Figures 3.7 to 3.11 show the mode shapes and the shear force and bending moment distributions for a thin uniform cantilever blade. Also, for the first four modes experimental values of transverse displacements are shown.
Figure 3.7 shows that in the first mode a fairly large bending stress is generated at the blade root. In the case of the cantilever, experimental model (1), an amplitude of vibration of 2 mm at the tip would produce an oscillatory bending stress of approximately 50 N/mm² which must be regarded at quite considerable when compared to the tensile strength of mild steel taken as 300 - 400 N/mm².

Figure 3.8 shows an increase in maximum bending moment per unit deflection of approximately 6 when compared to the first mode of vibration. The maximum bending moment still occurs at the blade root but in addition a bending moment of approximately 3/4 of the magnitude of the maximum bending moment occurs at the centre of the blade.

Figures 3.9, 3.10 and 3.11 all show bending moment distributions which are approximately sinusoidal but have maximum values at the blade root. The relative magnitudes of these values based on that of the first mode are approximately 20, 40 and 60 for the third, fourth and fifth modes of vibration respectively, assuming unit maximum deflection of the blade in all the cases considered. The maximum deflections for all five modes of vibration occur at the blade tip.

No attempt is made here at predicting the actual stresses generated in the blade as these would depend on the true amplitudes of the blade in its various modes of vibration, and the amplitudes, in turn, would depend on the magnitudes of the existing forces, which are not known here.
For the above blade shear stresses are small and can be ignored but this is not necessarily the case for other types of structures.

Figures 3.7 to 3.11 show that for all five modes of vibration the maximum shear and bending stresses occur at the blade root. Hence for all five modes of vibration the maximum stress would occur at the root of the blade.

The determination of the stress distribution in a turbine blade corresponding to a particular mode of vibration is essential for fatigue considerations.

The finite-element model described in this chapter represents a new development in vibration analysis.

The use of this element is further extended to the vibration analysis of packets of blades with fixed and flexible roots. The work relating to this further development is given in Chapters 4 and 5 of this thesis.
<table>
<thead>
<tr>
<th>MODE NO</th>
<th>FREQUENCY PARAMETER (λ)</th>
<th>% ERROR</th>
<th>EIGENVALUES AND EIGENVECTORS</th>
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</thead>
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<td>4</td>
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<td>14617.2733</td>
<td>27.059</td>
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**TABLE 3.1** EIGENVALUES AND EIGENVECTORS FOR A SINGLE-ELEMENT, UNIFORM CANTILEVER BEAM
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<th>FREQUENCY PARAMETER ((\lambda))</th>
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<th>N = 3</th>
<th>N = 4</th>
<th>N = 5</th>
<th>N = 6</th>
<th>EXACT VALUES</th>
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<td>1739792.9109</td>
<td>1713016.8887</td>
<td>1706889.0151</td>
<td>1703691.090</td>
<td>1703691.090</td>
<td>1703691.090</td>
</tr>
<tr>
<td>13</td>
<td>NOMINAL</td>
<td>2456017.7090</td>
<td>2404248.6340</td>
<td>2380633.3629</td>
<td>2378151.638</td>
<td>2378151.638</td>
<td>2378151.638</td>
<td>2378151.638</td>
</tr>
<tr>
<td>14</td>
<td>NOMINAL</td>
<td>3596045.2164</td>
<td>3293404.7910</td>
<td>3250413.5733</td>
<td>3235449.050</td>
<td>3235449.050</td>
<td>3235449.050</td>
<td>3235449.050</td>
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<tr>
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<td>NOMINAL</td>
<td>5207398.7528</td>
<td>4400940.6485</td>
<td>4343835.0958</td>
<td>4305974.960</td>
<td>4305974.960</td>
<td>4305974.960</td>
<td>4305974.960</td>
</tr>
</tbody>
</table>

TABLE 3.2 CONVERGENCE OF FREQUENCY PARAMETER \(\lambda\) FOR A UNIFORM CANTILEVER BEAM
TABLE 3.3  COMPARISON BETWEEN PRESENT RESULTS AND
THOSE OBTAINED BY LECKIE & LINDBERG (29, 1963)
FOR A UNIFORM CANTILEVER BEAM

<table>
<thead>
<tr>
<th>MODE NO</th>
<th>FREQUENCY PARAMETER λ</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>REF (29) MATRIX ORDER = 8</td>
<td>PRESENT METHOD MATRIX ORDER = 8</td>
<td>EXACT VALUE</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>λ</td>
<td>% ERROR</td>
<td>λ</td>
<td>% ERROR</td>
<td>λ</td>
</tr>
<tr>
<td>1</td>
<td>12.3634</td>
<td>0.0008</td>
<td>12.3623</td>
<td>0.0000</td>
<td>12.3623</td>
</tr>
<tr>
<td>2</td>
<td>486.651</td>
<td>0.23</td>
<td>485.5188</td>
<td>0.0000</td>
<td>485.5188</td>
</tr>
<tr>
<td>3</td>
<td>3865.72</td>
<td>1.55</td>
<td>3806.5738</td>
<td>0.0007</td>
<td>3806.5462</td>
</tr>
<tr>
<td>4</td>
<td>15044.9</td>
<td>2.92</td>
<td>14626.1004</td>
<td>0.0604</td>
<td>14617.2733</td>
</tr>
</tbody>
</table>

TABLE 3.4  COMPARISON BETWEEN PRESENT RESULTS AND THOSE
OBTAINED BY CARNEGIE, THOMAS AND DOKUMACI (31, 1969)
FIG. 3.1 NODAL CO-ORDINATES OF BEAM ELEMENT WITH EIGHT DEGREES OF FREEDOM

FIG. 3.2 NODAL CO-ORDINATES OF BEAM ELEMENT WITH FOUR DEGREES OF FREEDOM
FIG 3.3 NODAL CO-ORDINATES OF BEAM ELEMENT WITH TWELVE DEGREES OF FREEDOM

FIG 3.4 NODAL CO-ORDINATES OF SINGLE ELEMENT CANTILEVER BEAM WITH FOUR DEGREES OF FREEDOM
FIG. 3.5 CONVERGENCE CURVES FOR UNIFORM CANTILEVER BLADE (FIRST AND SECOND MODES)
FIG. 3.6 CONVERGENCE CURVES FOR UNIFORM CANTILEVER BLADE (THIRD AND FOURTH MODES)
FIG 3.7  MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 6)
FIG. 3.8 MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 6)
FIG. 3.9 MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)
FIG 3.10 MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FOURTH MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)
FIG. 3.11 MODE SHAPE, SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIFTH MODE OF VIBRATION OF A UNIFORM CANTILEVER BEAM (N = 12)
CHAPTER 4

BLADE PACKETS FIXED AT THE ROOT
CHAPTER 4
Blade Packets Fixed at the Root

4.1 Introduction

In this chapter a mathematical finite element model is developed for a shrouded packet of blades fixed at the roots to a disc of infinite radius and infinite rigidity. The finite element used is the one developed in chapter 3, but with appropriate modifications to root, corner and 'T' junction elements in order to satisfy boundary, force equilibrium and displacement compatibility conditions.

Results obtained from the finite element models are compared with experimental ones and also, where possible, with those derived by classical analysis.

Comparison is also made with other investigators, Reiger and McCallion (11, 1965) and Thomas and Belek (55, 1977). Rieger and McCallion (11, 1965) have used the 'classical' method in order to find the natural frequencies of portal frames but have determined the nodal shapes by experiment only. Thomas and Belek (55, 1977) have used a finite element method based on the matrix displacement method only.

The effect of coupling between the independent blade and shroud modes is demonstrated by means of frequency inference diagrams.

4.2 Finite Element Analysis

4.2.1 Conformity and compatibility of nodal co-ordinates

In order to be able to assemble stiffness and mass matrices for complete structures it is necessary to ensure conformity and compatibility of all elements in the structure.
4.2.1.1 Conformity between blade and shroud elements

Conformity is achieved by means of a transformation matrix, $G_1$, derived as shown in Figure 4.1 and table 4.1.

\[
\begin{array}{cccccc}
Y &=& W & \cos \theta - V & \sin \theta \\
Y' &=& W' & \cos \theta - V' & \sin \theta \\
Y'' &=& W'' & \cos \theta - V'' & \sin \theta \\
Y''' &=& W''' & \cos \theta - V''' & \sin \theta \\
U &=& W & \sin \theta + V & \cos \theta \\
U' &=& W' & \sin \theta + V' & \cos \theta \\
\end{array}
\]

<table>
<thead>
<tr>
<th></th>
<th>BLADE, $\theta = 90^\circ$</th>
<th>SHROUD, $\theta = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = W$</td>
<td>$Y = -V$</td>
<td>$Y = W$</td>
</tr>
<tr>
<td>$Y' = W'$</td>
<td>$Y' = -V'$</td>
<td>$Y' = W'$</td>
</tr>
<tr>
<td>$Y'' = W''$</td>
<td>$Y'' = -V''$</td>
<td>$Y'' = W''$</td>
</tr>
<tr>
<td>$Y''' = W'''$</td>
<td>$Y''' = -V'''$</td>
<td>$Y''' = W'''$</td>
</tr>
<tr>
<td>$U = W$</td>
<td>$U = W$</td>
<td>$U = V$</td>
</tr>
<tr>
<td>$U' = W'$</td>
<td>$U' = W'$</td>
<td>$U' = V'$</td>
</tr>
</tbody>
</table>

Table 4.1 Transformed values of $Y$ and $U$

\[
\begin{bmatrix}
y \\
y' \\
y'' \\
y''' \\
u \\
u'
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
W \\
W' \\
V' \\
V \\
V' \\
V'
\end{bmatrix}
\]

Let,

\[
\{Y\}_b = [g_1] \{W\}_b
\]

The complete matrix $|G_1|$ is given below,

\[
\begin{bmatrix}
g_1 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
g_1
\end{bmatrix} = \begin{bmatrix}
G_1 \\
0
\end{bmatrix} \quad (4.01)
\]
Hence,

\[
\{Y\}_b^{[KLB]} \{Y\}_b = \{W\}_b^{[G_1^T \times [KLB] \times [G_1]} \{W\}_b
\]

\[
= \{W\}_b^{[KLB_1]} \{W\}_b
\]

where,

\[
[KLB_1] = [G_1^T \times [KLB] \times [G_1] \quad (4.02)
\]

Similarly

\[
[MLB_1] = [G_1^T \times [MLB] \times [G_1] \quad (4.03)
\]

### 4.2.1.2 Compatibility between blade and shroud corner elements

Compatibility at the corner is achieved with the use of a compatibility matrix \( G_2 \), derived as shown,

\[
\begin{bmatrix}
W \\
W' \\
V'' \\
V'''
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_3 & 0
\end{bmatrix}
\begin{bmatrix}
W \\
W' \\
W'' \\
W'''
\end{bmatrix}
\]

\[
\begin{bmatrix}
V \\
V'
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \alpha_4 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
V \\
V'
\end{bmatrix}
\]

Let,

\[
\{W\}_b = [g_2] \{W\}_S
\]

The complete \( G_2 \) matrix is given below,

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & g_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} = [G_2] \quad (4.03)
\]

where \( I \) is a unit matrix
Hence,
\[
\{W\}_b^T [KLB_1] \{W\}_b = \{W\}_s^T [G_2^T] [KLB_1] [G_2] \{W\}_s
\]
\[
= \{W\}_s^T [KLB_2] \{W\}_b
\]
where,
\[
[KLB_2] = [G_2^T] [KLB_1] [G_2]
\] (4.04)

Similarly,
\[
[MLB_2] = [G_2^T] [MLB_2] [G_2]
\] (4.05)

The nodal co-ordinates of a blade to shroud corner joint are given in Figure 4.2

The coefficients \(a_1\), \(a_2\), \(a_3\), and \(a_4\) are determined as follows:

Equating forces in the \(W\) direction,
\[
E_b A_b W'_b + E_s A_s W'''_s = 0
\]
Hence,
\[
W'_b = - \frac{E_s A_s}{E_b A_b} W'''_s
\]
\[
W'_b = a_1 W'''_s \quad \text{where,} \quad a_1 = \frac{E_s A_s}{E_b A_b}
\] (4.06)

Equating forces in the \(V\) direction,
\[
E_b A_b V'''_b + E_s A_s V'_s = 0 \quad V'''_b = - \frac{E_s A_s}{E_b A_b} V'_s
\]
Hence,
\[
V'''_b = a_3 V'_s \quad \text{where,} \quad a_3 = - \frac{E_s A_s}{E_b A_b}
\] (4.07)

Equating moments,
\[
E_b A_b V'''_b = E_s A_s W''_s \quad V'''_b = \frac{E_s A_s}{E_b A_b} W''_s
\]
Hence,
\[
V'''_b = a_2 W_s \quad \text{where,} \quad a_2 = \frac{E_s A_s}{E_b A_b}
\] (4.08)
Equating rotations

\[
V_b' = -\ W_s' \quad \quad V_b' = -\ W_s'
\]

Hence,

\[
V_b' = \alpha_4 \ W_s' \quad \text{where,} \quad \alpha_4 = -1 \tag{4.09}
\]

4.2.1.3 **Compatibility of blade and shroud elements at the 'T' joint**

The following compatibility matrix, \( G_3 \), was derived as shown,

\[
\begin{bmatrix}
W \\
W' \\
V' \\
V' \\
V' \\
V' \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & \alpha_6 \\
0 & 0 & \alpha_7 & 0 & 0 & 0 & 0 & \alpha_8 & 0 \\
0 & 0 & 0 & 0 & \alpha_9 & 0 & 0 & 0 & \alpha_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
W \\
W' \\
V' \\
V' \\
V' \\
V' \\
\end{bmatrix}
\]

Let,

\[
\{W\}_b = [g_3]\{W\}_{SL,SR}
\]

The complete \( G_3 \) matrix is given below,

\[
\begin{bmatrix}
I & 0 \\
\alpha_3 & \text{6 x 6} \\
\text{6 x 9} & \alpha_3 \\
\end{bmatrix} = [G_3] \tag{4.10}
\]}
Hence,
\[ \{W\}_b^{T}[KLB_1]\{W\}_b = \{W\}_s^{T}[G_3^T][KLB_1][G_3]\{W\}_s \]
\[ = \{W\}_s^{T}[KLB_3]\{W\}_b \]

where,
\[ [KLB_3] = [G_3^T][KLB_1][G_3] \]  \hspace{1cm} (4.11)

Similarly
\[ [MLB_3] = [G_3^T][MLB_1][G_3] \]  \hspace{1cm} (4.12)

The nodal co-ordinates of a blade to shroud 'T' joint are given in Figure 4.3.

The coefficients \( \alpha_5 \), \( \alpha_6 \), \( \alpha_7 \), \( \alpha_8 \), \( \alpha_9 \), \( \alpha_{10} \) and \( \alpha_{11} \) are determined as follows,

Equating forces in the \( \mathbf{W} \) direction,
\[ E_{bA_b}W' + E_{sI_s}W'' + E_{sI_s}W'' = 0 \]
\[ W'_b = \frac{E_{sI_s}}{E_{bA_b}}W'' + \frac{E_{sI_s}}{E_{bA_b}}W'' \]
\[ = \alpha_5W'' + \alpha_6W'' \]

where
\[ \alpha_5 = -\frac{E_{sI_s}}{E_{bA_b}} \quad \text{and} \quad \alpha_6 = -\frac{E_{sI_s}}{E_{bA_b}} \]  \hspace{1cm} (4.13)

When \( I_{SL} = I_{SR} \) and \( \ell_{SL} = \ell_{SR} \), \( \alpha_5 = \alpha_6 \)

Equating moments,
\[ E_{bI_b}V'_b = E_{sI_s}W'' \]
\[ V'_b = \frac{E_{sI_s}}{E_{bI_b}}W'' \]
\[ V'_b' = \alpha_7W'' + \alpha_8W'' \]

where,
\[ \alpha_7 = \frac{E S^I_{SL}}{E b^I_b} \quad \text{and} \quad \alpha_8 = \frac{E S^I_{SR}}{E b^I_b} \]  

When \( I_{SL} = I_{SR} \), \( \alpha_7 = \alpha_8 \)

Equating forces in the \( V \) direction

\[ E b^I_b V'''' + E S^A_S V'_{SL} + E S^A_S V'_{SR} = 0 \]

\[ V'''' = - \frac{E S^A_S}{E b^I_b} V'_{SL} - \frac{E S^A_S}{E b^I_b} V'_{SR} \]

\[ V'''' = \alpha_9 V'_{SL} + \alpha_9 V'_{SR} \]

where,

\[ \alpha_9 = - \frac{E S^A_S}{E b^I_b} \]  

Equating rotations,

\[ V''_b = - W'_b SL \]

\[ V'_b = \alpha_{11} W'_b SL \quad \text{where,} \quad \alpha_{11} = -1 \]

4.2.1.4 Boundary conditions

The boundary conditions follow directly from the assumption that all blades in a packet are fully fixed. In effect the number of co-ordinates at the root is reduced from 6 to 3. The co-ordinates representing longitudinal and lateral deflections and rotation are put equal to zero.

4.2.2 Assembly of global stiffness and mass matrices

Global matrices are assembled from local matrices representing individual elements.

The procedure for the assembly of stiffness and mass matrices is the same.

To a large extent but with the exception of local matrices for root, corner and 'T' junction elements the local matrices
are identical for each group of 'standard' elements. Also, for a particular configuration chosen, it is only necessary to vary the number of standard elements in order to achieve a desired accuracy.

4.2.2.1 **Matrix assembly using the code system**

A typical blade packet configuration, with vertical members representing the blades and horizontal members the shroud, and with nodal points shown, is given in Figure 4.4. Details of the co-ordinates at the 'T' joint are shown in Figure 4.5.

The numbers shown in a circle represent the element number. These numbers are used for reference only but not in the assembly of a global matrix.

The numbers shown at the nodal points represent code numbers. These are global co-ordinate reference numbers which when used together with local co-ordinate reference numbers will determine the position of any local matrix element within the global matrix.

4.2.2.1.1 **Procedure for generating code numbers**

A procedure for generating code numbers is developed for a packet of blades of the type illustrated in Figure 4.4. The procedure allows for variations in P, N and M, the number of blades in a packet and the numbers of blade and shroud elements respectively. The effect of boundary conditions and additional co-ordinates needed at 'T' junctions are also included in the procedure. Figure 4.5 shows the co-ordinates with their corresponding code numbers at the 'T' junction of the typical blade packet illustrated in Figure 4.4.
4.2.2.1.2 Procedure for assembly of global matrices

The method is best illustrated with a simple example as shown in Figure 4.6. This simple example was also used for testing the computer program developed for the assembly of global matrices.

Consider three elements as shown in Figure 4.6. Corresponding boxes give local and global co-ordinates of the same element, e.g., for element (2), \( a_{11} a_{12} a_{13} a_{14} \), the local elements will be given locations \( a_{33} a_{34} a_{35} a_{36} \) in the global matrix.

Similarly,

\[
\begin{array}{cccc}
 a_{22} & a_{23} & a_{24} \\
\end{array}
\]

will correspond to \( a_{44} a_{45} a_{46} \) in the global matrix.

A discussion on the above method is given by S.S. Tezcan (56,1963)

4.3 Multi-element blade shroud assembly

In order to be able to assemble global stiffness and mass matrices for blade and shroud elements it is necessary to express the local matrices dimensionally. It can be easily seen that matrices equations (3.24) and (3.25) assume the dimensional form shown in equations (4.19) and (4.20) by a simple transformation.

Letting \( \{ \ell \} = [ \ell^0 \ell^1 \ell^2 \ell^3 \ell^4 \ell^5 \ell^6 \ell^7 \ell^8 \ell^9 ] \)

\[
[KL_p] = \{ \ell \}^T [KL] \{ \ell \}
\]

and

\[
[ML_p] = \{ \ell \}^T [ML] \{ \ell \}
\]

Matrices \( [KL_p] \) and \( [ML_p] \) are given by equations (4.19) and (4.20)
\[
\begin{bmatrix}
352800 & 176400l & 16800l^2 & 630l^3 & 0 & 0 & -352800 & 176400l & -16800l^2 & 630l^3 & 0 & 0 \\
176400l & 108000l^2 & 11370l^3 & 480l^4 & 0 & 0 & -176400l & 68400l^2 & -5430l^3 & 150l^4 & 0 & 0 \\
16800l^2 & 11370l^3 & 3000l^4 & 140l^5 & 0 & 0 & -16800l^2 & 5430l^3 & -30l^4 & -25l^6 & 0 & 0 \\
630l^3 & 480l^4 & 140l^5 & 8l^6 & 0 & 0 & -630l^3 & 150l^4 & 25l^5 & -3l^6 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{Al^2}{I} x 462 \times 36 & \frac{Al^2}{I} x 3l^2 & 0 & 0 & 0 & 0 & \frac{-Al^2}{I} x 462 \times 36 & \frac{Al^2}{I} x 3l^2 \\
0 & 0 & 0 & 0 & \frac{Al^2}{I} x 3l^2 & \frac{Al^2}{I} x 4l^2 & 0 & 0 & 0 & 0 & \frac{-Al^2}{I} x 3l^2 & \frac{-Al^2}{I} x 3l^2 \\
-352800 & -176400l & -16800l^2 & -630l^3 & 0 & 0 & 352800 & -176400l & 16800l^2 & -630l^3 & 0 & 0 \\
176400l & 68400l^2 & 5430l^3 & 150l^4 & 0 & 0 & -176400l & 108000l^2 & -11370l^3 & 480l^4 & 0 & 0 \\
-16800l^2 & -5430l^3 & -30l^4 & -25l^5 & 0 & 0 & 16800l^2 & -11370l^3 & 3000l^4 & -140l^5 & 0 & 0 \\
630l^3 & 150l^4 & -25l^5 & -3l^6 & 0 & 0 & -630l^3 & 480l^4 & -140l^5 & 8l^6 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-Al^2}{I} x 462 \times 36 & \frac{-Al^2}{I} x 3l^2 & 0 & 0 & 0 & 0 & \frac{Al^2}{I} x 462 \times 36 & \frac{-Al^2}{I} x 3l^2 \\
0 & 0 & 0 & 0 & \frac{-Al^2}{I} x 3l^2 & \frac{-Al^2}{I} x 4l^2 & 0 & 0 & 0 & 0 & \frac{Al^2}{I} x 3l^2 & \frac{-Al^2}{I} x 3l^2 \\
\end{bmatrix}
\]

(4.19)
\[
\left[ M_{L_D} \right] = \\
\begin{bmatrix}
5251680 & 978480 \lambda & 98640 \lambda^2 & 4596 \lambda^3 & 0 & 0 & 1234800 & -411480 \lambda & 55800 \lambda^2 & -3126 \lambda^3 & 0 & 0 \\
978480 \lambda & 237600 \lambda^2 & 26460 \lambda^3 & 1296 \lambda^4 & 0 & 0 & 411480 \lambda & -134280 \lambda^2 & 17910 \lambda^3 & -990 \lambda^4 & 0 & 0 \\
98640 \lambda^2 & 26460 \lambda^3 & 3096 \lambda^4 & 156 \lambda^5 & 0 & 0 & 55800 \lambda^2 & -17910 \lambda^3 & 2358 \lambda^5 & -129 \lambda^6 & 0 & 0 \\
4596 \lambda^3 & 1296 \lambda^4 & 156 \lambda^5 & 8 \lambda^6 & 0 & 0 & 3126 \lambda^3 & -990 \lambda^4 & 129 \lambda^5 & -7 \lambda^6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 156 & 0 & 0 & 0 & 0 & 0 & 0 & 30888 \lambda & 156 & 22 \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 22 \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 30888 \lambda & 13 \lambda & -13 \lambda \\
1234800 & 411480 \lambda & 55800 \lambda^2 & 3126 \lambda^3 & 0 & 0 & 5251680 & -978480 \lambda & 98640 \lambda^2 & -4596 \lambda^3 & 0 & 0 \\
-411480 \lambda & -134280 \lambda^2 & -17910 \lambda^3 & -990 \lambda^4 & 0 & 0 & -978480 \lambda & 237600 \lambda^2 & -26460 \lambda^3 & 1296 \lambda^4 & 0 & 0 \\
55800 \lambda^2 & 17910 \lambda^3 & 2358 \lambda^5 & 129 \lambda^6 & 0 & 0 & 98640 \lambda^2 & -26460 \lambda^3 & 3096 \lambda^4 & -156 \lambda^5 & 0 & 0 \\
-3126 \lambda^3 & -990 \lambda^4 & -129 \lambda^5 & -7 \lambda^6 & 0 & 0 & -4596 \lambda^3 & 1296 \lambda^4 & -156 \lambda^5 & 8 \lambda^6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 156 & 0 & 0 & 0 & 0 & 0 & 30888 \lambda & 54 & 0 & 13 \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & -13 \lambda & 0 & 0 & 0 & 0 & 0 & 30888 \lambda & 13 & \lambda & -13 \lambda & -3 \lambda^2 \\
\end{bmatrix}
\]

(4.20)
Global stiffness matrix,

\[
[K_G] = \frac{E_b I_b}{\ell_b^3 13860} [K_B] + \frac{E_s I_s}{\ell_s^3 13860} [K_S]
\]  \hspace{1cm} (4.21)

where

\[
\ell_b = \text{length of blade element}
\]

\[
\ell_s = \text{length of shroud element}
\]

\[
[K_B] = [K_{LD}](\ell = \ell_b)
\]

\[
[K_S] = [K_{LD}](\ell = \ell_s)
\]

Taking blade elements of equal length and shroud elements of equal length

\[
\ell_b = \frac{L_b}{N}
\]

and

\[
\ell_s = \frac{L_s}{M}
\]

where

\[
L_b = \text{blade length}
\]

\[
L_s = \text{shroud length}
\]

\[
N = \text{number of blade elements}
\]

\[
M = \text{number of shroud elements}
\]

With appropriate substitutions

\[
[K_G] = \frac{E_b I_b}{\ell_b^3 13860} \left[ [K_B] + \left( \frac{E_s}{E_b} \right) \left( \frac{I_s}{I_b} \right) \left( \frac{L_b}{L_s} \times \frac{M}{N} \right)^3 [K_S] \right]
\]

\[
[K_G] = \frac{E_b I_b}{\ell_b^3 13860} \left[ [K_B] + \frac{\gamma}{L_r (\frac{N}{M})^3} [K_S] \right]
\]  \hspace{1cm} (4.22)

where,

stiffness ratio \( \gamma = \frac{E_s I_s}{E_b I_b} \)

and

\[
L_r = \frac{L_s}{L_b}
\]
Global mass matrix,

\[
[M_G] = \frac{\rho_b A_b l_b}{12972960} [MB] + \frac{\rho_s A_s l_s}{12972960} [MS]
\]

\[
[M_G] = \frac{\rho_b A_b l_b}{12972960} \left[ MB + \left( \frac{\rho_s}{\rho_b} \right) \left( \frac{A_s}{A_b} \right) \left( \frac{l_s}{l_b} \right) [MS] \right]
\]

\[
[M_G] = \frac{\rho_b A_b l_b}{12972960} \left[ MB + A_L \left( \frac{N_M}{N} \right) [MS] \right]
\]  \hspace{1cm} (4.23)

where,

mass ratio \( \Delta = \frac{\rho_s A_s}{\rho_b A_b} \)

Letting

\[
[KG^*] = \left[ KB \right] + \frac{\gamma}{L_r \left( \frac{N_M}{N} \right)} [KS]
\]

and

\[
[MG^*] = \left[ MB \right] + \Delta L_r \left( \frac{N_M}{N} \right) [MS]
\]

The frequency equation is

\[
\frac{E_b}{\ell_b} \frac{I_b}{13860} \left[ KG^* \right] - \frac{\rho_b A_b l_b}{12972960} p^2 [MG^*] = \{ P \}
\]

or

\[
[KG^*] - \frac{\rho_b A_b l_b}{E_b \frac{I_b}{936}} p^2 [MG^*] = \{ P_1 \}
\]

From \( \ell_b = \frac{I_b}{N} \)

\[
[KG^*] = \frac{\rho_b A_b l_b}{E_b \frac{I_b}{936}} p^2 [MG^*] = \{ P_1 \}
\]

or

\[
[KG^*] - \frac{\lambda}{936N^4} [MG^*] = \{ P_1 \}
\]  \hspace{1cm} (4.24)

where \( \lambda = \frac{\rho_b A_b l_b}{E_b \frac{I_b}{p^2}} \)  \hspace{1cm} (4.25)

The complete solution to equation (4.24) is obtained by

an eigenvalue routine as outlined in chapter 3.3.
4.4 The Frequency Inference Diagram

The frequency inference diagram is a diagram of uncoupled blade and shroud frequencies superimposed onto the frequency diagram of a complete packet of blades. For given values of $\gamma$ the flexural rigidity ratio, and $\Delta$ the mass ratio, the frequency parameter $\lambda$ is plotted against $L_r$, the shroud to blade length ratio.

The uncoupled modes for blade and shroud are given below

4.4.1 Uncoupled blade and shroud modes

4.4.1.1 Blade clamped-free transverse modes (BCF)

The blade is clamped at one end with an apportioned part of the shroud assumed to act as a concentrated mass at the free end.

The frequency equation given in (55, 1977) is

$$\eta = \frac{\beta \left( \sin \beta \cosh \beta - \sinh \beta \cos \beta \right)}{1 + \cos \beta \cosh \beta}, \ldots$$

where $\eta$ is the beam to end load mass ratio and $\beta$ is the eigenvalue parameter.

Assuming the shroud mass and blade mass to be uniformly distributed over shroud and blade respectively

$$\eta = \frac{A_b L_b \rho_b}{A_s L_s \rho_s} \frac{P}{P-1}$$

where $P$ is the number of blades in a packet

or

$$\eta = \frac{1}{\Delta L_r} \frac{P}{P-1}$$

where

$$\Delta = \frac{A_s \rho_s}{A_b \rho_b}$$
Hence

\[ \Delta L_r \cdot \frac{P-1}{P} = 1 + \cos \beta \cosh \beta \frac{e^{(\sin \beta \cosh \beta - \sinh \beta \cos \beta)}}{E} \]  \hspace{1cm} (4.29)

The frequency parameter \( \lambda \) is determined from

\[ \lambda = \beta^4 \]  \hspace{1cm} (4.30)

### 4.4.1.2 Blade clamped-pinned transverse modes (BCP)

These modes are independent of \( \Delta \gamma \) and \( L_r \)

\[ \lambda_1 = 237.7210 \]
\[ \lambda_2 = 2496.4874 \]

### 4.4.1.3 Shroud clamped-clamped transverse modes (SCC)

The frequency parameter for a shroud is

\[ \lambda_s = \frac{\frac{\rho_s A_s L_s^4}{E_s I_s} \omega^2}{\lambda} \]  \hspace{1cm} (4.31)

\[ \lambda_s = \lambda \cdot \frac{\lambda_s}{\lambda} = \lambda \cdot \left( \frac{E_s I_s}{E_b I_b} \right) \left( \frac{\rho_s A_s}{\rho_b A_b} \right) \left( \frac{L_s}{L_b} \right)^4 \]

\[ \lambda_s = \lambda \left( \frac{\Delta}{\gamma} \right)^4 \]  \hspace{1cm} (4.32)

where

\[ \gamma = \frac{E_s I_s}{E_b I_b} \]
\[ L_r = \frac{L_s}{L_b} \]

Hence

\[ \lambda = \lambda_s \left( \frac{\gamma}{\Delta} \right)^4 \frac{1}{L_r^4} \]  \hspace{1cm} (4.33)

### 4.4.1.4 Blade and shroud longitudinal modes

The frequency parameters for longitudinal modes are considerably higher than corresponding transverse modes and therefore are not included in the inference diagrams, Figures 4.28 and 4.29.
4.5 'Classical' Analysis

4.5.1 Determination of the frequency equation

The positive directions of displacements, shear force and bending moment for a uniform beam AB are shown in Figure 4.7.

The equation of motion for free transverse vibrations of a uniform beam in the absence of axial forces and assuming that both shear and rotary inertia effects are small enough to be ignored is

\[
\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI} \frac{\partial^2 y}{\partial t^2} = 0
\]  

(4.34)

For transverse modes the solution to equation (4.34) is of the form

\[
y = u(x) \cos \omega t
\]  

(4.35)

where

\[
u(x) = A \cos \mu x + B \sin \mu x + C \cosh \mu x + D \sinh \mu x
\]  

(4.36)

and

\[
\mu^4 = \frac{\rho A \omega^2}{EI}
\]  

(4.37)

\[
\theta = \frac{\partial y}{\partial x}
\]  

(4.38)

\[
\frac{M}{EI} = \frac{\partial^2 y}{\partial x^2}
\]  

(4.39)

\[
\frac{S}{EI} = \frac{\partial^3 y}{\partial x^3}
\]  

(4.40)

From equations (4.36) (4.38) (4.39) and (4.40) and with the appropriate boundary conditions A, B, C and D are eliminated and the values of shear force and bending moment at stations A and B are found
\[ M_A = \frac{\theta_A}{\mu} F_5 + \frac{\theta_B}{\mu} F_8 + u_A \frac{F_1}{F_3} + u_B \frac{F_{10}}{F_3} \cos \omega t \quad (4.41) \]

\[ M_B = \frac{\theta_A}{\mu} F_5 + \frac{\theta_B}{\mu} F_8 + u_A \frac{F_1}{F_3} + u_B \frac{F_{10}}{F_3} \cos \omega t \quad (4.42) \]

\[ S_A = \frac{\theta_A}{\mu} F_1 + \frac{\theta_B}{\mu} F_6 + u_A \frac{F_1}{F_3} + u_B \frac{F_7}{F_3} \cos \omega t \quad (4.43) \]

\[ S_B = \frac{\theta_A}{\mu} F_5 + \frac{\theta_B}{\mu} F_8 + u_A \frac{F_1}{F_3} + u_B \frac{F_{10}}{F_3} \cos \omega t \quad (4.44) \]

where from (57, 1956)

\[ F_1 = \sin \mu L \sinh \mu L \]

\[ F_2 = \cos \mu L \cosh \mu L \]

\[ F_3 = \cos \mu L \cosh \mu L - 1 \]

\[ F_4 = \cos \mu L \cosh \mu L + 1 \]

\[ F_5 = \cos \mu L \sinh \mu L - \sin \mu L \cosh \mu L \]

\[ F_6 = \cos \mu L \sinh \mu L + \sin \mu L \cosh \mu L \]

\[ F_7 = \sin \mu L + \sinh \mu L \]

\[ F_8 = \sin \mu L - \sinh \mu L \]

\[ F_9 = \cos \mu L + \cosh \mu L \]

\[ F_{10} = \cos \mu L - \cosh \mu L \]

Figure 4.8 gives end conditions of vertical and horizontal members in a packet of blades with \( n \) bays.

For vertical members conditions at the blade root are

\[ M_A = k \theta_A \cos \omega t \quad (4.45) \]

\[ u_A = 0 \quad (4.46) \]

where \( k \) is the rotational stiffness at the blade root.

From equation (4.41) and (4.45)

\[ k \theta_A = EI \mu^2 \left\{ - \frac{\theta_A}{\mu} F_5 - \frac{\theta_B}{\mu} F_8 + u_B \frac{F_{10}}{F_3} \right\} \]

This gives
\[
\theta_A = \frac{EI\mu F_8}{(k F_3 + EI\mu F_5)} \theta_B + \frac{EI\mu^2 F_{10}}{k F_3 + EI\mu F_5} u_B
\]

Substituting for \(\theta_A\) in equations (4.42) and (4.44)

\[
M_B = EI\mu^2 \left\{ \frac{F_8 \left[ -\frac{EI F_8}{k F_3 + EI\mu F_5} + \frac{F_5}{\mu F_3} \right]}{F_3} \theta_B + \frac{F_8 \left[ EI\mu F_{10} \right]}{k F_3 + EI\mu F_5} + \frac{F_3}{F_3} \right\} u_B \]

or

\[
M_B = EI\mu^2 \left\{ \phi_1 \frac{\theta_B}{\mu} + \phi_2 u_B \right\} \cos \omega t \quad (4.47)
\]

Similarly

\[
S_B = EI\mu^3 \left\{ \phi_2 \frac{\theta_B}{\mu} + \phi_3 u_B \right\} \cos \omega t \quad (4.48)
\]

where

\[
\phi_1 = \frac{-2\mu LF_1 + RF_5}{\mu LF_5 + RF_3}
\]

\[
\phi_2 = \frac{\mu LF_2 + RF_1}{\mu LF_5 + RF_3}
\]

\[
\phi_3 = \frac{2\mu LF_2 + RF_6}{\mu LF_5 + RF_3}
\]

and

\[
R = \frac{kL}{EI}
\]

For horizontal members end conditions are

\[
u_A = 0
\]

\[
u_B = 0
\]

Hence from equations (4.41) and (4.42)

\[
M_A = -E_s I_s \frac{F_8}{F_3} \left\{ \phi_4 \theta_A + \phi_5 \theta_B \right\} \cos \omega t \quad (4.49)
\]

\[
M_B = E_s I_s \frac{F_8}{F_3} \left\{ \phi_5 \theta_A + \phi_4 \theta_B \right\} \cos \omega t \quad (4.50)
\]

where

\[
\phi_4 = \frac{F_5(s)}{F_3(s)} \quad \text{and} \quad \phi_5 = \frac{F_8(s)}{F_3(s)}
\]
At the \( r^{th} \) joint
\[ M_r + M_r L - M_r R = 0 \quad (4.51) \]

Substituting from equations (4.47) (4.49) and (4.50) into equation (4.51) and letting
\[ u_B = \delta \]
in equation 4.47

\[
EI\mu^2\{\phi_1 \frac{\theta_r}{\mu} + \phi_2 \delta\} + E_s I_s \mu_s \{\theta_{r-1} \phi_5 + \theta_r \phi_4\} + E_s I_s \mu_s \{\theta_r \phi_4 + \theta_{r+1} \phi_5\} = 0
\]

Letting
\[
K_s^4 = \frac{E I}{E_s I_s}, \quad K_m^4 = \frac{\rho A}{\rho_s A_s}, \quad \frac{L}{L_s} = K_1, \quad \mu^4 = \frac{\rho A \omega^2}{E I}
\]

\[
\theta_r + \{2 \frac{\phi_4}{\phi_5} + K_s^3 K_m \phi_1 \} \theta_r + \theta_{r-1} = -K_m K_s^3 \mu_L \frac{\phi_2}{\phi_5} \delta \quad (4.52)
\]

Using the finite difference "shifting" operator \( E \), equation (4.52) may be expressed as

\[
(E^2 + 2\beta E + 1) \theta_r = -\alpha \quad (4.53)
\]

where
\[
\alpha = K_m K_s^3 \mu_L \frac{\phi_2}{\phi_5} \delta \quad \frac{L}{L}
\]

and
\[
\beta = \left\{ \frac{\phi_4}{\phi_5} + \frac{K_m K_s^3}{2} \cdot \frac{\phi_1}{\phi_5} \right\}
\]

The complete solution to equation (4.53) is

For \( \beta > 1 \)
\[
\theta_r = -\frac{\alpha}{2(1+cosh\phi)} + (-1)^r\{P \cosh \phi r + Q \sinh \phi r\} \quad (4.54)
\]

where \( \cosh \phi = \beta \).
For $\beta = 1$

$$\theta_r = \frac{\alpha}{4} + ( -1 )^r ( P + Qr ) \quad (4.55)$$

For $1 > \beta > -1$

$$\theta_r = - \frac{\alpha}{2(1 + \cos \phi)} + ( -1 )^r \{ P \cos \phi r + Q \sin \phi \} \quad (4.56)$$

where $\cos \phi = \beta$

For $\beta = -1$

$$\theta_r = - \frac{\alpha}{2} r^2 + ( P + Qr ) \quad (4.57)$$

For $\beta < -1$

$$\theta_r = - \frac{\alpha}{2(1 \cosh \phi)} + \{ P \cosh \phi r + Q \sinh \phi r \} \quad (4.58)$$

where $\cosh \phi = -\beta$

For a blade packet with $n$ bays and $n+1$ blades shown in figure 4.8 the boundary conditions are

$$M_o - M_{OR} = 0 \quad (4.59)$$
$$M_n + M_{nL} = 0 \quad (4.60)$$

Assuming the amplitude at the blade tips, $\phi$, to be the same for all blades and from Newton's second law

$$\sum_{r=0}^{n} S_r = - n \rho_s A_s L_s \omega^2 \phi \quad (4.61)$$

From equations (4.47) (4.48) (4.49) and (4.50), equations (4.59) (4.60) and (4.61) become

$$(K_s^3 K_m \phi_1 + \phi_4) \theta_0 + \phi_5 \theta_1 + K_s^3 K_m \phi_2 \mu \delta = 0 \quad (4.62)$$

$$(K_s^3 K_m \phi_1 + \phi_4) \theta_n + \phi_5 \theta_{n-1} + K_s^3 K_m \phi_2 \mu \delta = 0 \quad (4.63)$$

$$\sum_{r=0}^{n} \theta_r = \frac{\mu}{\phi_2} \left\{ (n+1) \phi_3 + \frac{n}{K_m^4 K_1} \mu L \right\} \delta \quad (4.64)$$

The frequency equations are obtained by eliminating $P$, $Q$, and $\delta$ from equations (4.54) to (4.58) and equations (4.62) to (4.64)
4.5.1.1 Range of frequency equations for packets of blades

with 'n' bays

The full range of frequency equations differentiating for
cases with \( n \) even or \( n \) odd is

for \( \beta > 1 \), \( n \) even

\[
(\phi_4 \sinh \frac{n \phi}{2} - \phi_5 \sinh \frac{n+2 \phi}{2})((\phi_4+\phi_5)(\cosh \frac{n+1 \phi}{2}/\cosh \frac{1 \phi}{2})
- (n+1)(\phi_4 \cosh \frac{n \phi}{2} - \phi_5 \cosh \frac{n+2 \phi}{2})[1+2\phi_6(1+cosh\phi)] = 0
\] (4.65)

for \( \beta > 1 \), \( n \) odd

\[
(\phi_4 \cosh \frac{n \phi}{2} - \phi_5 \cosh \frac{n+2 \phi}{2})((\phi_4+\phi_5)(\sinh \frac{n+1 \phi}{2}/\cosh \frac{1 \phi}{2})
- (n+1)(\phi_4 \sinh \frac{n \phi}{2} - \phi_5 \sinh \frac{n+2 \phi}{2})[1+2\phi_6(1+cosh\phi)] = 0
\] (4.66)

for \( \beta = 1 \), \( n \) even

\[
\{n\phi_4 - (n+2)\phi_5\}((\phi_4+\phi_5) - (n+1)(\phi_4-\phi_5)(1+4\phi_6)) = 0
\] (4.67)

for \( \beta = 1 \), \( n \) odd

\[
(\phi_4-\phi_5)((\phi_4+\phi_5) - (n+2)\phi_5(1+4\phi_6)) = 0
\] (4.68)

for \( 1 > \beta > -1 \), \( n \) even

\[
(\phi_4 \sin \frac{n \phi}{2} - \phi_5 \sin \frac{n+2 \phi}{2})((\phi_4+\phi_5)(\cos \frac{n+1 \phi}{2}/\cos \frac{1 \phi}{2})
- (n+1)(\phi_4 \cos \frac{n \phi}{2} - \phi_5 \cos \frac{n+2 \phi}{2})[1+2\phi_6(1+\cos\phi)] = 0
\] (4.69)

for \( 1 > \beta > -1 \), \( n \) odd

\[
(\phi_4 \cos \frac{n \phi}{2} - \phi_5 \cos \frac{n+2 \phi}{2})((\phi_4+\phi_5)(\sin \frac{n+1 \phi}{2}/\cos \frac{1 \phi}{2})
- (n+1)(\phi_4 \sin \frac{n \phi}{2} - \phi_5 \sin \frac{n+2 \phi}{2})[1+2\phi_6(1+\cos\phi)] = 0
\] (4.70)

For \( \beta = -1 \) and all values of \( n \)

\[
\{n\phi_4+(n+2)\phi_5\}\left[\frac{n}{3}[(5n+1)\phi_4+(5n+7)\phi_5]+4\phi_6(\phi_4+\phi_5)\right] = 0
\] (4.71)

For \( \beta < -1 \) and all values of \( n \)
For $\beta < -1$ all all values of $n$

\[
(\phi_4 \sinh \frac{n \phi}{2} + \phi_5 \sinh \frac{n+2 \phi}{2})((\phi_4 + \phi_5)(\sinh \frac{n+1 \phi}{2} / \sinh \frac{\phi}{2})
\]

\[- (n+1)(\phi_4 \cosh \frac{n \phi}{2} + \phi_5 \cosh \frac{n+2 \phi}{2})[1+2\phi_5(1-\cosh \phi)] = 0 \]

(4.72)
4.6.1 Convergence of Results and Comparisons with Experimental Results and Results Obtained by Other Investigators

Table 4.2 shows the convergence of frequency parameter $\lambda$ for experimental packet (2) described in Figures 5.4 and 5.5 and Table 5.1.

It is seen from Table 4.2 that for a constant value of $N$, the number of elements on the blade ($N = 2$) and $M$, the number of elements on the shroud varying between $M = 2$ and $M = 5$, convergence is very rapid, and a very good value of $\lambda$ is obtained for the 15th mode, even when only two blade and two shroud elements are used. The percentage variation in $\lambda$ between $M = 2$ and $M = 5$ is 0.04%.

Results in Table 4.3 show that as $N$ is varied for constant values of $M$, very rapid convergence is also achieved. The variation in $\lambda$ for the 15th mode when $N = 2$ and $M = 2$ and $N = 4$ and $M = 4$ is 0.254%. Table 4.3 also shows that slightly better results are obtained when $N = 4$ and $M = 4$ rather than with $N = 5$, $M = 3$ and also the choice of $N = 4$, $M = 4$ results in a smaller matrix size.

Tables 4.4 and 4.5 show convergence of $\lambda$ for theoretical packet (I). This packet has a shroud which is considerably less stiff than that of experimental model (2). In the case of experimental mode (2) the values of the shroud
to blade flexural rigidity, \( \gamma \), and the shroud to blade mass ratio, \( \Delta \), are both unity, whereas for theoretical model (I), \( \Delta \) is 0.3 and \( \gamma \) is only 0.005. It is seen that the value of \( \lambda \) is more affected by variations in \( N \) than in \( M \).

Results show that a combination of \( N = 4 \) and \( M = 4 \) gives the best value of \( \lambda \) for a particular matrix order.

The variation of \( \lambda \) between \( N = 2, M = 2 \) and \( N = 4, M = 4 \) for the 15th mode of vibration is 1.58% which is small.

Table 4.6 shows a comparison of values of natural frequencies obtained by a finite-element method and experiment by Thomas and Belek (55, 1977) and by the present method. Considering that comparison is made with experimental values of frequency involving possible manufacturing errors in the construction of the model, agreement is good. Also the value of \( E/\rho \) was not available in Reference (55, 1977). For the purpose of determination of natural frequencies by the present method \( E/\rho \) was obtained as shown in Figure 4.9. The greatest variation of the analytical value obtained by the present method from the corresponding experimental value is -2.70% and the least -0.20%. The analytical method of Reference (55, 1977) gives greatest and least variations from experimental values of 2.52% and 0.66% respectively. However, three out of five results obtained by the present method give closer results than those obtained by analysis in Reference (55, 1977).

Comparisons presented in Table 4.7 show very good
agreement except for the first mode of vibration where in
the case of the present method the variation of the analytical
value from the experimental one is 6.51% and in the case
of Reference (55, 1977) the corresponding variation is
6.32%. However, agreement between the analytical results
is good. Hence it is suspected that there is an error in
the experimental value. The value of E/\rho was not available
in the above Reference and for the purpose of analysis by
the present method E/\rho was derived as shown in Figure 4.10.

Tables 4.8 and 4.9 show comparisons of natural
frequencies presented by Rieger and McCallion (11, 1965),
Thomas and Belek (55, 1977) and the present method. Here
again for the purpose of computation by the present method
it was necessary to derive a value of E/\rho as shown in Figure
4.11.

All analytical results compare well with experimental
results. Direct comparison between the three analytical
methods used, here, is difficult since basic data such as
the value of E/\rho and packet dimensions may differ slightly
for the various analyses used.

Tables 4.10 and 4.11 give experimental and analytical
results using the present finite element and classical
methods for determining the natural frequencies of two and
three bladed packets of blades, experimental models (2) and
respectively. These two models are fully described in Chapter 5, where it is seen that the stiffness at the root can be varied. Here the compressive stress at the root is 85.39N/mm² which corresponds to the fully fixed condition.

Good agreement between theoretical and experimental results is obtained for both packets tested. For the two-bladed packet the main minimum variation from the experimental result is 0.65% when using the finite element method and 0.30% for classical analysis. The greatest variation occurs in the fifth mode with 3.76% for the finite element method and 3.21% for classical analysis. Both analytical methods gave almost identical results with slightly more accurate results in the case of classical analysis.

Even better results are obtained for the three bladed packet. For the finite-element method a greatest variation of 1.9% occurs in the 12th mode and by classical analysis a greatest variation of 2.15% occurs in the 10th mode. Again the two analytical methods agree very closely, but the classical analysis, here, appears to be slightly less accurate than when applied to the two bladed packet.

The comparisons of results in Tables 4.10 and 4.11 are far more meaningful than those based on results in Tables 4.6, 4.7, 4.8 and 4.9, as here the true value of \( E/\sigma \) is determined experimentally and is then used in the
two analytical methods listed.

\[ E/\rho \] is determined experimentally by vibrating a cantilever blade, experimental model (1) which is made of the same material as experimental models (2) and (3), (See Chapter 5.4 and also Figures 5.4, 5.5, 5.8 and Table 5.1).

Table 4.12 gives the frequency parameter \( \lambda \) for length ratios \( L_r \) varying from 0.2 to 1.0, for theoretical two-bladed packet, model (I). Direct comparison is made between results obtained by the present method and those obtained by Thomas and Belek (55, 1977). Here the comparison is particularly meaningful as it is made between values of \( \lambda \) which are independent of \( E/\rho \). Results obtained by the present method show very close agreement and an improvement over those presented by Thomas and Belek (55, 1977). It is seen that using the present method with \( N = 2 \) \( M = 2 \) more accurate results are obtained than with the method developed by Thomas and Belek (55, 1977) with \( N = 5 \) \( M = 4 \).

Table 4.13 lists values of \( \lambda \) for \( L_r = 0.1 \) to 1.0 for theoretical packet (II). Here direct comparison is not possible since a table of values of \( \lambda \) for the above packet is not given in Reference (55, 1977). However, it will be shown later in this discussion that frequency inference diagrams based on theoretical packet (II) can be compared (see paragraph 4.6.3).
4.6.2. Mode Shapes, End-Load, Shear Force and Bending Moment Distributions

Figure 4.12 shows the mode shapes of experimental packet (2) for the first three modes of vibration. Figures 4.13, 4.14 and 4.15 also show the bending moment shear force and end-load distributions for the same three modes of the above packet.

It has already been stated in the 'Discussion of Results' of Chapter 3 that a computer program had been developed such that values of $\lambda$, lateral, longitudinal and rotational displacements and values of $\frac{P}{AE}$, $\frac{S}{ET}$, $\frac{M}{EI}$ are printed out for a number of modes considered.

The analysis and the program have now been extended to provide the same facility for multi-blade shrouded packets, fully fixed at the root. In order to give more meaning to the various plots it was decided to use the same scale for the blade and shroud and also to plot all forces to the same scale. The scales chosen were such as to require the least modification to the data printed out. In the case of bending moment and shear force distributions, the scales chosen were $\frac{M}{E_b I_b}$ and $\frac{S}{E_b I_b}$ respectively. The chosen scale for the end-load was $\frac{F}{E_b T_b}$.

The corresponding value of maximum linear displacement
is also quoted so that the actual values of bending moment, shear force and end-load can be determined for a particular value of maximum displacement which in turn would depend on a particular excitational force present.

The mode shapes for experimental packet (2) show that the first and third modes of vibration represent the first and second main modes and the second mode represents a batch mode. Some experimental values of deflections were also plotted for the above packet and show good agreement with the theoretical plots, (see table 5.4). Experimental packet (2) has values of $\gamma$ and $\Delta$ equal to unity and $L_r = 0.168446$. This means that the shroud is fairly stiff and hence will have a bending moment distribution of a similar order to that of the blade. Also large shear forces and end-loads may be expected in the shroud. Figures 4.13, 4.14, and 4.15 show that this is the case. In spite of the relatively high shear forces and end-loads the bending stresses are still considerably larger than the shear or direct stresses. For the first mode of vibration and for a maximum deflection of 1mm, the maximum bending stress occurs at the blade-root and is approximately 56 N/mm$^2$, whereas maximum values of shear and direct stresses occur in the shroud and are approximately 7 N/mm$^2$ and 4 N/mm$^2$ respectively.

In the second and third modes of vibration (see figures 4.14, and 4.15) relative orders of magnitudes of the bending, direct and shear stresses remain approximately
the same, however, the magnitudes of stress when related to a maximum deflection of 1mm are approximately five and seven times larger than those of the first mode, for the second and third modes respectively.

Figure 4.16 shows the first four modes of vibration for experimental packet (3). This packet consists of three shrouded blades but the values of \( \gamma, \Delta \) and \( L_r \) are the same as for experimental packet (2). Here, the first and fourth modes of vibration represent the main first and second modes and the second and third modes are batch modes.

Figures 4.17, 4.18 and 4.19 give bending moment, shear force and end-load distributions for the first mode of vibration. It is seen that the values of maximum stress, here, are very similar to those obtained for the two-bladed packet and therefore plots of bending moment shear force and end-load for the other modes of vibration are not included.

For completeness figures 4.20, 4.21, 4.22, 4.23, 4.24, 4.25, 4.26 and 4.27 give the mode shapes, end-load, shear force and bending moment distributions for the first four modes of theoretical packet (I). This packet is considerably less stiff than experimental packets (2) and (3) with parameters, \( \gamma = 0.005, \Delta = 0.3 \) and \( L_r = 0.6 \). It is seen from the mode-shapes that the first and fourth modes are the first and second main modes of vibration and the second and third modes are batch modes. Packet (I) is two-bladed but because of the combination of the values of parameters
γ, ∆ and L, two batch modes are present here. This point is fully discussed in section 4.6.3.

Figures 4.20 and 4.21 show that for the first mode of vibration the end loads are small but the shear force and bending moment distributions markedly resemble corresponding distributions of a single cantilever blade. The shear force and bending moment distributions in the shroud are almost negligible.

The second mode is a batch mode. It can be seen from Figure 4.22 that most of the vibration occurs in the shroud causing comparatively large end loads in the blades and large bending moment and shear force distributions in the shroud. The maximum bending stress in the shroud for 1mm maximum deflection was found to be approximately 240 N/mm² which would be classed as a large oscillatory stress.

The third mode of vibration is also a batch mode but here most of the vibration occurs in the blades. It is seen from Figure 4.24 that the end loads in the blades and shroud are almost negligible. The shear force and bending moment distributions are symmetrical and fairly large in the blades but very small in the shroud as seen from Figure 4.25.

Figures 4.26 and 4.27 give the mode-shape end-load,
shear force and bending moment distributions for the fourth mode, which is also the second main mode of vibration. All these distributions, except for end-loads show a great deal of similarity with corresponding distributions for the single cantilver blade, (Ref. Figure 3.8).

Results for the various packets considered show that the flexural stiffness, mass and length of a shroud have a considerable effect on the overall vibration characteristics of a packet of blades. In order to present a clear picture of the effect of the above parameters frequency inference diagrams can be drawn for whole blade packets.

4.6.3. Frequency Inference Diagrams

Figures 4.28 and 4.29 show frequency inference diagrams for theoretical packets (I) and (II) respectively, (Ref. Tables 4.12 and 4.13). The derivation of these diagrams is fully explained in section 4.4.

It is seen that the broken lines representing the frequencies of vibration of the packet follow closely the continuous lines which are plots of the independent modes of blade and shroud. Hence it is possible to predict the dynamic characteristics of a packet of blades by simply plotting the frequency parameters of uncoupled blade and shroud modes.
It is seen from equation 4.28 that as $\Delta$, $L_r$ and $P$ (the number of blades in a packet) decrease the value of $\lambda$, for the BCF (Blade-Clamped-Free) modes, will decrease. It is also evident from the equation that when $P$ is large changes in $P$ will not affect greatly the value of $\lambda$.

Equation 4.33 shows that for SCC (Shroud-Clamped-Clamped) modes $\lambda$ is proportional to $\gamma/\Delta$ and inversely proportional to $L_r^4$ for a particular mode of vibration. This means that for a particular value of $\gamma/\Delta$ and for a particular mode of vibration the value of $\lambda$ will fall very rapidly for a small increase in $L_r$. Also it follows that if a smaller value of $\gamma/\Delta$ is used more SCC modes of vibration will be present within a particular range of values of $\lambda$.

The BCP (Blade-Clamped-Pinned) modes are completely independent of parameters associated with the shroud.

All the above characteristics are very well illustrated by Figures 4.28 and 4.29, where the values of $(\gamma/\Delta)$ are 0.01667 and 0.005 respectively. In the case of Figure 4.28 three CSS modes are present but in Figure 4.29 four CSS modes are shown for the same range of values of $L_r$.

Also it is seen from the frequency inference diagrams that the rule that the number of batch modes, that is the number of modes between BCF and $BCF_2$ is equal to $(P-1)$ is
only true in the region where the frequency of SCC modes is greater than the range of frequencies considered. Otherwise the number of batch mode frequencies will depend on the number of blades in a packet but also on \((\gamma/\Delta)\) and \(L_r\). A reduction in \((\gamma/\Delta)\) and an increase of \(L_r\) will increase the number of batch mode frequencies present.

Referring to the frequency inference diagrams presented here it is seen that for \(L_r = 0.6\), two batch modes are present in the case of theoretical model (I) with \((\gamma/\Delta) = 0.01667\) but three batch modes are present for model (II) with \((\gamma/\Delta) = 0.005\). When \(L_r = 1.0\) the corresponding numbers of batch modes for models (I) and (II) are four and five respectively.

Good agreement is observed between inference diagrams shown here (Figures 4.28 and 4.29) and those presented by Thomas and Belek (55, 1977). Also in the present final element analysis only two blade and two shroud elements were needed in order to compute accurately all the shroud modes for high \(L_r\) and low \((\gamma/\Delta)\) ratios, whereas in Reference (55, 1977) five blade and four shroud elements were required.

A simple rule for predicting the number of batch modes \(n\), for a packet with \(P\) blades would be \(n = Z(P-1)\) where \(Z\) is the number of batch modes for a two bladed packet.
<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>M = 2</th>
<th>M = 3</th>
<th>M = 4</th>
<th>M = 5</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>24.9139</td>
<td>24.8585</td>
<td>24.7943</td>
</tr>
<tr>
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<td>402.9844</td>
<td>401.8444</td>
<td>400.9109</td>
</tr>
<tr>
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<td>761.3774</td>
<td>758.5723</td>
<td>756.4684</td>
</tr>
<tr>
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<td>3173.0006</td>
<td>3165.7142</td>
<td>3159.6330</td>
<td>3155.0878</td>
</tr>
<tr>
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<td>4686.5347</td>
<td>4668.6971</td>
<td>4655.7792</td>
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<tr>
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<td>12403.5005</td>
<td>12392.2391</td>
</tr>
<tr>
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<td>16356.7983</td>
<td>16298.8217</td>
<td>16257.0968</td>
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<td>34291.0509</td>
<td>34266.9506</td>
<td>34249.2020</td>
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<tr>
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<td>42312.4893</td>
<td>42218.3763</td>
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<tr>
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<td>77169.8345</td>
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<td>208667.2100</td>
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<td>255708.8309</td>
<td>255659.0005</td>
<td>255623.7505</td>
<td>255599.5799</td>
</tr>
</tbody>
</table>

NB = 2

ROOT CONDITION : FULLY FIXED

γ = 1.0

Δ = 1.0

L_r = 0.168446

TABLE 4.2 CONVERGENCE OF FREQUENCY PARAMETER λ FOR
SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2),
(REF. TABLE 5.1), FOR N = 2.
<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>N = 2 M = 2</th>
<th>N = 3 M = 2</th>
<th>N = 4 M = 2</th>
<th>N = 5 M = 2</th>
<th>N = 5 M = 3</th>
<th>N = 4 M = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>404.3627</td>
<td>402.8120</td>
<td>401.8263</td>
<td>401.1757</td>
<td>400.7545</td>
<td>400.6389</td>
</tr>
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<td>3</td>
<td>764.8655</td>
<td>761.0055</td>
<td>758.6075</td>
<td>757.0490</td>
<td>756.0204</td>
<td>755.7342</td>
</tr>
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<td>4659.5239</td>
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<td>4651.2714</td>
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<td>12389.0091</td>
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<td>16274.9111</td>
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<td>34255.8004</td>
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<td>42220.6515</td>
<td>42405.0546</td>
</tr>
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<td>77238.0178</td>
<td>76199.2829</td>
<td>76169.3908</td>
<td>76162.7932</td>
<td>76153.4575</td>
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<td>91326.5563</td>
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<td>142592.9690</td>
<td>142544.1392</td>
<td>142497.1765</td>
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<tr>
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<td>255073.5917</td>
<td>255060.0973</td>
<td>255059.2384</td>
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</tbody>
</table>

**NB = 2**

**γ = 1.0**

**Δ = 1.0**

**Lr = 0.168446**

**TABLE 4.3** CONVERGENCE OF FREQUENCY PARAMETER λ FOR SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), (REF. TABLE 5.1), FOR M = 2, 3 AND 4.

**ROOT CONDITION**: FULLY FIXED
<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>FREQUENCY PARAMETER $\lambda$ (N = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M = 2</td>
</tr>
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</tr>
<tr>
<td>2</td>
<td>238.9601</td>
</tr>
<tr>
<td>3</td>
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<td>2480.7549</td>
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<tr>
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<td>3488.7951</td>
</tr>
<tr>
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<td>5177.1928</td>
</tr>
<tr>
<td>7</td>
<td>10932.3143</td>
</tr>
<tr>
<td>8</td>
<td>13404.5649</td>
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<td>31778.7796</td>
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<td>74243.0186</td>
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<td>14</td>
<td>74824.3523</td>
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<tr>
<td>15</td>
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</table>

$\gamma = 0.005$  $\omega_b = 0.01m$

$\Delta = 0.3$  $t_b = 0.005m$

$L_r = 0.2$

**TABLE 4.4** CONVERGENCE OF FREQUENCY PARAMETER $\lambda$ FOR SHROUDED BALDE PACKET, THEORETICAL MODEL (I), FOR N = 2.
<table>
<thead>
<tr>
<th>MODE</th>
<th>FREQUENCY PARAMETER $\lambda$</th>
</tr>
</thead>
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<tr>
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<td>12.0234</td>
</tr>
<tr>
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<td>12.0222</td>
</tr>
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<td>12.0227</td>
</tr>
<tr>
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<td>12.0230</td>
</tr>
<tr>
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<td>12.0231</td>
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<td>8</td>
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<td>15</td>
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</table>

**Table 4.5** CONVERGENCE OF FREQUENCY PARAMETER $\lambda$ FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I), ROOT CONDITION: FULLY FIXED.

**NB:**
- $\gamma$ = 0.005
- $\Delta = 0.2$ m
- $L_b = 0.25$ m
- $w_b = 0.01$ m
- $t_b = 0.005$ m
<table>
<thead>
<tr>
<th>MODE</th>
<th>FINITE-ELEMENT PRESENT METHOD (N = 4 M = 4)</th>
<th>% VARIATION FROM EXPERIMENTAL VALUES</th>
<th>FINITE-ELEMENT REF. (55) (N = 5 M = 4)</th>
<th>% VARIATION FROM EXPERIMENTAL VALUES</th>
<th>EXPERIMENTAL RESULTS REF. (55)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.66</td>
<td>212</td>
</tr>
<tr>
<td>2</td>
<td>826.3459</td>
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<td>833.6</td>
<td>0.68</td>
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</tr>
<tr>
<td>3</td>
<td>837.8324</td>
<td>-</td>
<td>834.4</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>1199.1784</td>
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<td>0.67</td>
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</tr>
<tr>
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<tr>
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<td>2654.3980</td>
<td>1.97</td>
<td>2667.</td>
<td>2.44</td>
<td>2602</td>
</tr>
</tbody>
</table>

ROOT CONDITION: FULLY FIXED

\[E = 2.4394 \times 10^7 \text{Nm/kg} \quad \text{(REF. FIG. 4.9)}\]
\[\gamma = 0.0171\]
\[L_b = 0.142875m\]
\[\Delta = 0.06944\]
\[w_b = 0.0254m\]
\[L_r = 0.177778\]
\[t_b = 0.0047625m\]

TABLE 4.6 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET (a), REF. (55, 1977)
<table>
<thead>
<tr>
<th>MODE</th>
<th>FINITE-ELEMENT PRESENT METHOD (N = 4 M = 4)</th>
<th>% VARIATION FROM EXPERIMENTAL VALUES</th>
<th>FINITE-ELEMENT REF. (55) (N = 5 M = 4)</th>
<th>% VARIATION FROM EXPERIMENTAL VALUES</th>
<th>EXPERIMENTAL RESULTS REF. (55)</th>
</tr>
</thead>
<tbody>
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<td>72.3</td>
<td>6.32</td>
<td>68</td>
</tr>
<tr>
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<td>251.8576</td>
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<td>252</td>
<td>0.00</td>
<td>252</td>
</tr>
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<td>0.25</td>
<td>257</td>
<td>0.00</td>
<td>257</td>
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<td>387.6</td>
<td>1.73</td>
<td>381</td>
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<td>-1.00</td>
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</tr>
<tr>
<td>6</td>
<td>754.7966</td>
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<td>756</td>
<td>-0.79</td>
<td>762</td>
</tr>
</tbody>
</table>

ROOT CONDITION: FULLY FIXED

\[
\begin{align*}
\text{NB} &= 3 \\
\frac{E}{\rho} &= 2.6024 \times 10^7 \text{Nm/kg} \quad \text{REF. FIG. 4.10} \\
\gamma &= 0.111 \\
L_b &= 0.2286 \text{m} \\
\Delta &= 0.111 \\
w_b &= 0.0254 \text{m} \\
L_r &= 0.11111111 \\
t_b &= 0.003175 \text{m}
\end{align*}
\]

**TABLE 4.7** COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKED (b), REF. (55, 1977).
<table>
<thead>
<tr>
<th>No. of Blades</th>
<th>Mode</th>
<th>Natural Frequencies (Hz)</th>
</tr>
</thead>
<tbody>
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<td>Finite-element Present Method (N = 4) (M = 4)</td>
<td>Finite-element Ref. (55) (N = 5) (M = 4)</td>
</tr>
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<tr>
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</tr>
</tbody>
</table>

Root condition: Fully Fixed

\[
\begin{align*}
    \gamma &= 1.0 \\
    \Delta &= 1.0 \\
    L_r &= 0.5 \\
    E &= 2.5786 \times 10^7 \text{Nm/kg} \quad \text{(Ref. Fig. 4.11)} \\
    \rho &= 0.15478125 \text{m} \\
    w_b &= 0.0079375 \text{m} \\
    t_b &= 0.0047625 \text{m}
\end{align*}
\]

Table 4.8 Computed and Experimental Frequencies of Shrouded Blade Packets for \(L_r = 0.5\), Ref. (11, 1965)
## EXPERIMENTAL RESULTS

**Ref. (11)**

<table>
<thead>
<tr>
<th>NO. OF BLADES</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODE</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>150.0167</td>
<td>152.7</td>
<td>-</td>
<td>152.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>587.3601</td>
<td>604.2</td>
<td>-</td>
<td>602.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>961.8427</td>
<td>980.0</td>
<td>-</td>
<td>986.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1057.0359</td>
<td>1069.0</td>
<td>-</td>
<td>1069.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Finite-Element Present Method**

(N = 4 M = 4)

**Finite-Element Ref. (55)**

(N = 5 M = 4)

**Classical Solution Ref. (11)**

**Experimental Results Ref. (11)**

**TABLE 4.9 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKETS FOR L_r = 1.0, Ref. (11, 1965)**

**Root Condition: Fully Fixed**

\[
\gamma = 1.0, \quad E = 2.5786 \times 10^7 \text{N/m}^2, \quad \rho = \text{(Ref. Fig. 4.11)}
\]

\[
\Delta = 1.0, \quad I_p = 0.15478125 \text{m}^4, \quad L_r = 1.0
\]

\[
\omega_p = 0.0079375 \text{m}, \quad t_p = 0.0047625 \text{m}
\]
<table>
<thead>
<tr>
<th>MODE</th>
<th>EXPERIMENTAL RESULTS</th>
<th>FINITE-ELEMENT METHOD (N = 4 M = 4)</th>
<th>% VARIATION FROM EXPERIMENTAL RESULTS</th>
<th>CLASSICAL ANALYSIS (BLADE FULLY FIXED AT THE ROOT)</th>
<th>% VARIATION FROM EXPERIMENTAL RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80.3</td>
<td>80.8219</td>
<td>0.65</td>
<td>80.5387</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>320.4</td>
<td>324.8977</td>
<td>1.40</td>
<td>323.0957</td>
<td>0.84</td>
</tr>
<tr>
<td>3</td>
<td>431.5</td>
<td>446.2250</td>
<td>3.41</td>
<td>443.8707</td>
<td>2.87</td>
</tr>
<tr>
<td>4</td>
<td>898.7</td>
<td>911.5214</td>
<td>1.43</td>
<td>908.2299</td>
<td>1.06</td>
</tr>
<tr>
<td>5</td>
<td>1066.9</td>
<td>1107.0193</td>
<td>3.76</td>
<td>1101.1710</td>
<td>3.21</td>
</tr>
<tr>
<td>6</td>
<td>1789.2</td>
<td>1806.7067</td>
<td>0.98</td>
<td>1802.6461</td>
<td>0.75</td>
</tr>
<tr>
<td>7</td>
<td>2004.5</td>
<td>2068.9521</td>
<td>3.22</td>
<td>2058.4789</td>
<td>2.69</td>
</tr>
<tr>
<td>8</td>
<td>2991.6</td>
<td>3003.0917</td>
<td>0.38</td>
<td>3000.3099</td>
<td>0.29</td>
</tr>
<tr>
<td>9</td>
<td>3253.3</td>
<td>3334.6602</td>
<td>2.50</td>
<td>3319.4005</td>
<td>2.03</td>
</tr>
</tbody>
</table>

NB = 2
\[ \gamma = 1.0 \quad \frac{E}{\rho} = 2.5850 \times 10^7 \text{Nm/kg} \] (Ref. Fig. 5.8)
\[ \Delta = 1.0 \]
\[ L_r = 0.168446 \]

**TABLE 4.10** COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1
<table>
<thead>
<tr>
<th>MODE</th>
<th>EXPERIMENTAL RESULTS</th>
<th>FINITE-ELEMENT METHOD (N = 4, M = 4)</th>
<th>CLASSICAL ANALYSIS (BLADES FULLY FIXED)</th>
<th>% VARIATION FROM EXPERIMENTAL RESULTS</th>
<th>% VARIATION FROM EXPERIMENTAL (BLADES FULLY FIXED AT THE ROOT)</th>
<th>CLASSICAL ANAYLISIS</th>
<th>% VARIATION FROM EXPERIMENTAL RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>78.2</td>
<td>78.2627</td>
<td>78.2627</td>
<td>0%</td>
<td>0%</td>
<td>85.39 N/mm²</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>327.03</td>
<td>328.2314</td>
<td>328.2314</td>
<td>0.26%</td>
<td>0.26%</td>
<td>435.6711</td>
<td>0.56%</td>
</tr>
<tr>
<td>3</td>
<td>340.891</td>
<td>340.891</td>
<td>340.891</td>
<td>0.56%</td>
<td>0.56%</td>
<td>338.9126</td>
<td>0.42%</td>
</tr>
<tr>
<td>4</td>
<td>433.8</td>
<td>433.8</td>
<td>433.8</td>
<td>-0.6%</td>
<td>-0.6%</td>
<td>1096.7363</td>
<td>-0.81%</td>
</tr>
<tr>
<td>5</td>
<td>924.7</td>
<td>924.7</td>
<td>924.7</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>927.0735</td>
<td>0.26%</td>
</tr>
<tr>
<td>6</td>
<td>945.0</td>
<td>945.0</td>
<td>945.0</td>
<td>0.1%</td>
<td>0.1%</td>
<td>940.9872</td>
<td>-0.3%</td>
</tr>
<tr>
<td>7</td>
<td>1058.3</td>
<td>1058.3</td>
<td>1058.3</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>1829.4642</td>
<td>2.11%</td>
</tr>
<tr>
<td>8</td>
<td>1840.3</td>
<td>1840.3</td>
<td>1840.3</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>1855.0233</td>
<td>0.73%</td>
</tr>
<tr>
<td>9</td>
<td>2037.2</td>
<td>2037.2</td>
<td>2037.2</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>2046.7968</td>
<td>2.15%</td>
</tr>
<tr>
<td>10</td>
<td>2003.8</td>
<td>2003.8</td>
<td>2003.8</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>3031.7730</td>
<td>1.65%</td>
</tr>
<tr>
<td>11</td>
<td>2982.7</td>
<td>2982.7</td>
<td>2982.7</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>3080.7734</td>
<td>1.61%</td>
</tr>
<tr>
<td>12</td>
<td>3031.9</td>
<td>3031.9</td>
<td>3031.9</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>3309.3341</td>
<td>1.52%</td>
</tr>
<tr>
<td>13</td>
<td>3259.8</td>
<td>3259.8</td>
<td>3259.8</td>
<td>-0.2%</td>
<td>-0.2%</td>
<td>3293.6048</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

**NB:** = 3, \( \frac{E}{\rho} = 2.585 \times 10^7 \text{N/m}^2 \) (REF. FIG. 5.8)
<table>
<thead>
<tr>
<th>$L_r$</th>
<th>FREQUENCY PARAMETER $\lambda$</th>
<th>FINITE-ELEMENT PRESENT METHOD</th>
<th>FINITE-ELEMENT REF. (55)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N=2$ $M=2$</td>
<td>$N=4$ $M=4$</td>
<td>$N=5$ $M=4$</td>
</tr>
<tr>
<td>0.2</td>
<td>12.0235</td>
<td>12.0222</td>
<td>12.02</td>
</tr>
<tr>
<td></td>
<td>238.9601</td>
<td>238.9507</td>
<td>239.1</td>
</tr>
<tr>
<td></td>
<td>449.8920</td>
<td>449.8696</td>
<td>450.2</td>
</tr>
<tr>
<td>0.4</td>
<td>10.4234</td>
<td>10.4229</td>
<td>10.42</td>
</tr>
<tr>
<td></td>
<td>230.3291</td>
<td>230.2952</td>
<td>230.4</td>
</tr>
<tr>
<td></td>
<td>334.1954</td>
<td>333.8475</td>
<td>334.0</td>
</tr>
<tr>
<td></td>
<td>409.0552</td>
<td>409.0487</td>
<td>409.3</td>
</tr>
<tr>
<td>0.6</td>
<td>9.3534</td>
<td>9.3531</td>
<td>9.35</td>
</tr>
<tr>
<td></td>
<td>63.2560</td>
<td>63.2138</td>
<td>63.3</td>
</tr>
<tr>
<td></td>
<td>740.6274</td>
<td>240.5945</td>
<td>240.7</td>
</tr>
<tr>
<td></td>
<td>369.2536</td>
<td>369.2294</td>
<td>307.1</td>
</tr>
<tr>
<td>0.8</td>
<td>8.5125</td>
<td>8.5123</td>
<td>8.51</td>
</tr>
<tr>
<td></td>
<td>20.1420</td>
<td>20.1322</td>
<td>20.16</td>
</tr>
<tr>
<td></td>
<td>151.7605</td>
<td>151.6892</td>
<td>154.2</td>
</tr>
<tr>
<td></td>
<td>238.2987</td>
<td>238.3583</td>
<td>238.5</td>
</tr>
<tr>
<td></td>
<td>370.1357</td>
<td>370.1109</td>
<td>370.6</td>
</tr>
<tr>
<td>1.0</td>
<td>7.8143</td>
<td>7.8143</td>
<td>7.81</td>
</tr>
<tr>
<td></td>
<td>8.2719</td>
<td>8.2688</td>
<td>8.28</td>
</tr>
<tr>
<td></td>
<td>62.7935</td>
<td>62.7695</td>
<td>64.84</td>
</tr>
<tr>
<td></td>
<td>223.7349</td>
<td>223.6570</td>
<td>227.5</td>
</tr>
<tr>
<td></td>
<td>259.1806</td>
<td>259.0452</td>
<td>265.5</td>
</tr>
<tr>
<td></td>
<td>350.7287</td>
<td>350.7023</td>
<td>351.8</td>
</tr>
</tbody>
</table>

ROOT CONDITION: FULLY FIXED

$NB = 2$  \hspace{1cm} $L_b = 0.25$ m
$\gamma = 0.005$  \hspace{1cm} $w_b = 0.01$ m
$\Delta = 0.3$  \hspace{1cm} $t_b = 0.005$ m

TABLE 4.12 COMPUTED FREQUENCY PARAMETER $\lambda$ FOR DIFFERENT LENGTH RATIOS $L_r$ FOR SHROUDED BLADE PACKET, THEORETICAL MODEL (I), (SEE DETAILS ABOVE)
<table>
<thead>
<tr>
<th>$L_r$</th>
<th>$\lambda$ (N=4 M=4)</th>
<th>$L_r$</th>
<th>$\lambda$ (N=4 M=4)</th>
</tr>
</thead>
</table>
| 0.1  | 12.0683  
240.8069  
437.2886 | 0.7  | 5.2323  
10.2846  
77.8924  
233.4796  
305.1932  
310.5565 |
| 0.2  | 9.6310  
238.1155  
384.4861 | 0.8  | 4.8016  
6.0398  
45.9008  
173.3212  
244.8754  
295.3557  
490.7930 |
| 0.3  | 8.1886  
226.3203  
321.0985  
353.8084 | 0.9  | 3.7758  
4.4343  
28.7718  
109.7018  
241.0271  
273.4981  
324.4343 |
| 0.4  | 7.1609  
94.6706  
243.4586  
330.8355 | 1.0  | 2.4800  
4.1164  
18.9499  
72.2501  
195.0553  
239.2027  
293.3672  
446.4961 |
| 0.5  | 6.3740  
39.2259  
240.6082  
275.2327  
351.9232 | 0.6  | 5.7467  
13.0025  
142.8310  
238.7239  
316.8589 |
| 0.6  | 5.7467  
13.0025  
142.8310  
238.7239  
316.8589 | 0.7  | 5.2323  
10.2846  
77.8924  
233.4796  
305.1932  
310.5565 |

$\text{NB} = 2 \quad L_b = 0.25 \text{ m}$

$\text{w}_b = 0.005 \text{ m} \quad t_b = 0.005 \text{ m}$

$\text{ROOT CONDITION : FULLY FIXED}$

**Table 4.13** Computed Frequency Parameter $\lambda$

For different length ratios $L_r$ for shrouded blade packet, theoretical model (II) (see details above)
FIG. 4.1 RELATION BETWEEN U-Y AND V-W CO-ORDINATES

FIG. 4.2 NODAL CO-ORDINATES OF ADJOINING BLADE AND SHROUD CORNER ELEMENTS
FIG. 4.3 NODAL CO-ORDINATES OF BLADE AND SHROUD ELEMENTS AT A T JOINT

| 22 | 28 | 58 | 64 | 70 |
| 23 | 29 | 59 | 65 | 71 |
| 24 | 30 'T' Joint | 60 | 66 | 72 |
| 25 | 31 | 61 | 67 | 73 |
| 26 | 32 | 62 | 68 | 74 |
| 27 | 33 | 63 | 69 | 75 |

FIG. 4.4 PACKET OF THREE SHROUDED BLADES SHOWING CO-ORDINATE REFERENCE NUMBERS AND ELEMENT NUMBERS
FIG. 4.5 NODAL CO-ORDINATES AT THE T JOINT

FIG. 4.6 RELATION BETWEEN LOCAL AND GLOBAL CO-ORDINATES
FIG. 4.7 SIGN CONVENTION FOR DISPLACEMENTS, SHEAR FORCE AND BENDING MOMENT OF A UNIFORM BEAM AB

FIG. 4.8 END CONDITIONS OF VERTICAL AND HORIZONTAL MEMBERS IN A PACKET OF BLADES WITH n BAYS
\[ \frac{F}{\rho} = \frac{f^2}{\lambda} \cdot \frac{4\pi^2 L_B^4}{t_b^2} \]

\[ = 2.4394 \times 10^7 \text{Nm/kg} \]

**FIG. 4.9** RELATION BETWEEN NATURAL FREQUENCY \( f \) AND \((\lambda)^{1/2}\) FOR SHROUDED BLADE PACKET (a) REF. (55, 1977)
\[
\frac{E}{\rho} = \frac{f^2}{\lambda} \cdot \frac{48 \pi^2 L_D^4}{t_b^2}
\]

\[
= 2.6024 \times 10^7 \text{Nm/kg}
\]

**Figure 4.10** Relation between natural frequency \( f \) and \( (\lambda)^{\frac{1}{2}} \) for shrouded blade packet (b) Ref. (55, 1977)
FIG. 4.11 RELATION BETWEEN NATURAL FREQUENCY $f$ AND $(\lambda)^{\frac{1}{2}}$ FOR SHROUDED BLADE PACKETS

REF. (11, 1965)

\[ \frac{E}{\rho} = \frac{f^2}{\lambda} \cdot \frac{48\pi^2Lb^4}{t_b^2} \]

\[ = 2.5786 \times 10^7 \text{Nm/kg} \]
FIG. 4.12 MODE SHAPES FOR THE FIRST SECOND AND THIRD 
MODES OF VIBRATION OF EXPERIMENTAL PACKET (2), 
FULLY FIXED AT THE ROOT, REF. TABLE 5.1

Experimental results
FIGURE 4.13  BENDING MOMENT SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (2), FULLY FIXED AT THE ROOT, REF. TABLE 5.1
FIGURE 4.14 BENDING MOMENT, SHEAR FORCE, AND END-LOAD DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF EXPERIMENTAL FACET (2), FULLY FIXED AT THE ROOT, REF. TABLE 5.1
Figure 4.15  Bending moment shear force and end-load distributions for the third mode of vibration of experimental packet (2) fully fixed at the root, ref. Table 5.1
FIGURE 4.16 MODE SHAPES FOR THE FIRST, SECOND, THIRD AND FOURTH MODES OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT, REF. TABLE 5.1
FIGURE 4.17  BENDING MOMENT DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT, REF. TABLE 5.1

$V_{\text{max}} = 1.00$
FIGURE 4.18 SHEAR FORCE DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF EXPERIMENTAL PACKET (3), FULLY FIXED AT THE ROOT, REF. TABLE 5.1
$V_{\text{max}} = 1.00$

**Figure 4.19** End-Load Distribution for the First Mode of Vibration of Experimental Packet (3), Fully Fixed at the Root, Ref. Table 5.1
MODE SHAPE

\[ NB = 2 \]
\[ L_b = 0.25m \]
\[ w_b = 0.01m \]
\[ t_b = 0.005m \]
\[ \gamma = 0.005 \]
\[ \Delta = 0.3 \]
\[ L_r = 0.6 \]

ROOT CONDITION:
FULLY FIXED

\[ V_{\text{max}} = 1.3786 \times 10^{-4} \]

FIGURE 4.20 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE FIRST MODE OF VIBRATION OF THEORETICAL PACKET (1), (SEE DETAILS ABOVE)
\[ V_{\text{max}} = 1.3786 \times 10^{-4} \]

**SHEAR FORCE**

**BENDING MOMENT**

*FIGURE 4.21 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)*
FIGURE 4.22 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE SECOND MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)
FIGURE 4.23 SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)
FIGURE 4.24 MODE SHAPE AND END-LOAD DISTRIBUTION FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)
\[ V_{\text{max}} = 1.1557 \times 10^{-4} \]

**FIGURE 4.25** SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)
FIGURE 4.26  MODE SHAPE AND END-LOAD DISTRIBUTION
FOR THE FOURTH MODE OF VIBRATION OF
THEORETICAL PACKET (1)  (SEE FIGURE 4.20)
V = 1.3829 \times 10^{-4}

\[ V_{\text{max}} = 1.3829 \times 10^{-4} \]

**SHEAR FORCE**

**BENDING MOMENT**

\[ \frac{S}{E_{b_1}} \times 10 \]

\[ \frac{M}{E_{b_1} b} \times 10^2 \]

**FIGURE 4.27** SHEAR FORCE AND BENDING MOMENT DISTRIBUTIONS FOR THE FOURTH MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIGURE 4.20)
FIGURE 4.28 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (1) (SEE TABLE 4.12)
FIGURE 4.29 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (11) (SEE TABLE 4.13)
CHAPTER 5
Blade Packets with Flexible Roots

5.1 Introduction

In the present chapter a finite element method is developed to include the effects of flexibility at the roots of a packet of blades.

In all other respects the packet is of the same type as described in chapter 4 and the assumptions of infinite disc radius and infinite rigidity of the disc itself still apply. The blade root flexibility is represented by transverse, longitudinal and torsional springs at the root of every blade in a packet.

A comprehensive study of the effects of root flexibility on the dynamic characteristics of packets of blades is made. Most of the plots and tabulated results are in non-dimensional form in order to facilitate comparisons between the various results.

Where possible comparison is made with experimental results and also with results obtained by classical analysis, though the latter comparison has only been possible for the simpler cases considered.

5.2 Finite Element Analysis

5.2.1 Development of the finite element at the blade root

The nodal co-ordinates of a blade element transformed to conform with a shroud element are as shown in figure 5.1.

Representing the co-ordinates at the blade root in vector form as shown in equation 5.1 gives the following column matrix,
Letting $G$, $H$ and $K$ be constants proportional to the torsional, longitudinal and transverse stiffnesses as shown in figure 5.2, it follows that,

\[
\begin{align*}
G V_0' &= V_0'' \\
H W_0 &= W_0' \\
K V_0 &= V_0'''
\end{align*}
\]  

(5.2)

On substitution expression (5.1) becomes

\[
\begin{bmatrix}
W_0 \\
W_0' \\
V_0'' \\
V_0'''
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
H & 0 & 0 \\
G & 0 & 0 \\
K & 0 & 0
\end{bmatrix}
\begin{bmatrix}
W_0 \\
V_0 \\
V_0'
\end{bmatrix}
\]

(5.3)

Conformity with the standard blade element is achieved by the following transformation

\[
\begin{bmatrix}
W_0 \\
H W_0 \\
G V_0' \\
K V_0 \\
V_0 \\
V_0'
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
H & 0 & 0 \\
G & 0 & 0 \\
K & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
W_0 \\
V_0 \\
V_0'
\end{bmatrix}
\]

(5.4)

Letting,
\[ \{\mathbf{W}\}_b(G,H,X) = [g_h]\{\mathbf{W}\}_b \]

The complete \( G_4 \) matrix is given below,

\[
\begin{bmatrix}
g & \ 0 \\
6 \times 3 & \\
\vdots & \vdots \\
\end{bmatrix}
= \begin{bmatrix} G_4 \end{bmatrix}
\]

Hence,

\[
\{\mathbf{W}\}_b^T (G,H,K) \{\mathbf{W}\}_b(G,H,K) = \{\mathbf{W}\}_b^T [G_4^T] [K \ L \ B_1] [G_4] \{\mathbf{W}\}_b \\
= \{\mathbf{W}\}_b^T [K \ L \ B_4] \{\mathbf{W}\}_b
\]

where,

\[
[K \ L \ B_4] = [G_4^T] [K \ L \ B_1] [G_4]
\]

similarly

\[
[M \ L \ B_4] = [G_4^T] [M \ L \ B_1] [G_4]
\]

5.2.2 Strain energy due to spring system

The strain energy generated at the root of each blade is

\[
\frac{1}{2} g \ V_o'^2, \ \frac{1}{2} h \ W_o^2 \ \text{and} \ \frac{1}{2} k \ V_o^2
\]

where \( g, h \) and \( k \) are the torsional, longitudinal and transverse stiffnesses at a blade root.

These energy terms are added to the stiffness matrix

\([k \ L \ B_4]\) as shown,
Expressing $g$, $h$ and $k$ in terms of $G$, $H$ and $K$

$$g = G E_b I_b$$
$$h = H A_b E_b$$
$$k = K E_b I_b$$

From equation (4.22)

$$K_{11} = \frac{E_b I_b}{\ell_b^3 13860} \left[ KB_{11} \right]$$

For blade root element

$$K_{11}(R) = \frac{E_b I_b}{\ell_b^3 13860} \left[ KB_{11} + \frac{H A_b E_b \ell_b^3 13860}{E_b I_b} \right]$$

$$\therefore K_{11}(R) = \frac{E_b I_b}{\ell_b^3 13860} \left[ KB_{11} + \frac{H A_b \ell_b 13860}{I_b} \right]$$

Similarly

$$K_{22}(R) = \frac{E_b I_b}{\ell_b^3 13860} \left[ KB_{22} + K \ell_b^3 13860 \right]$$

and,

$$K_{33}(R) = \frac{E_b I_b}{\ell_b^3 13860} \left[ KB_{33} + G \ell_b^3 13860 \right]$$

5.2.3 Solution of eigenvalue problem

Thus for each blade root a new local stiffness matrix
and a new local mass matrix
\[
[MLB_4]
\]
is formed.

It is assumed that all the other blade and shroud elements are of the type as described in chapter 4.

Global stiffness and mass matrices are assembled as described in chapter 4.2.2 giving an equation similar to (4.24) which is solved using an eigenvalue routine as described in chapter 3.3.

5.2.4 **Stiffnesses at the blade root expressed non-dimensionally**

The non-dimensional stiffnesses \( G^* \), \( H^* \), and \( K^* \) are defined,

\[
\begin{align*}
g &= \frac{G^* E_b I_b}{\lambda_b} \\
h &= \frac{H^* E_b A_b}{\lambda_b} \\
k &= \frac{K^* E_b I_b}{\lambda_b^3}
\end{align*}
\]

From equations (5.5) and (5.9)

\[
\begin{align*}
G &= \frac{G^*}{\lambda_b} \\
H &= \frac{H^*}{\lambda_b} \\
K &= \frac{K^*}{\lambda_b^3}
\end{align*}
\]

Substitutions are made for \( G \), \( H \), and \( K \) in equation (5.04) giving
Similar substitutions are made in equations (5.06) (5.07) and (5.08) giving,

\[
\begin{bmatrix}
1 & 0 & 0 \\
\frac{H^*}{\xi_b} & 0 & 0 \\
0 & 0 & \frac{G^*}{\xi_b} \\
0 & K^* & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]  

(5.11)

Global stiffness and mass matrices are assembled and the eigenvalue problem is solved as in section 5.2.3.

5.3 Classical Analysis

Figure 5.3 shows a uniform beam, free at one end and having a lateral and torsional spring at the other end. K* and G* represent the non-dimensional transverse and torsional stiffnesses respectively and are defined

\[
K^* = \frac{kL^3}{EI}
\]

\[
G^* = \frac{gL}{EI}
\]

where 'k' and 'g' are the stiffnesses of the two springs expressed as force per unit linear displacement and torque per unit angular displacement respectively.
5.3.1 The frequency equation

The frequency equation for free transverse vibrations of a uniform beam with end conditions as given in Figure 5.3, ignoring shear and rotary inertia effects is given in reference (42, 1975) as

\[
K^* + \frac{K^*}{\alpha} \cdot \left\{ \frac{(-\beta/\alpha)(\sinh(\beta\cos\theta) - \sin(\beta\cosh\theta)) + \beta^3 (\cos(\beta\sinh\theta) + \cosh(\beta\sin\theta))}{1 - \cos\beta \cosh\beta} \right\} = 0
\]

where \( \beta = \lambda^\frac{1}{2} \)

and

\[ \alpha = \frac{G^*}{K^*} \]

5.3.2 The displacement equation

The displacement equation is given in reference (42, 1975) as

\[ r(\eta) = \sin(\beta\eta) + \sinh(\beta\eta) + \delta(\cos(\beta\eta) + \cosh(\beta\eta)) \]

where

\[
\delta = \frac{\cos\beta - \cosh\beta + (K^*/\beta^3)(\sin\beta + \sinh\beta)}{\sin\beta + \sinh\beta - (K^*/\beta^3)(\cos\beta + \cosh\beta)}
\]
5.4 EXPERIMENTAL WORK

5.4.1 Introduction

Two types of model suitable for experimental measurement were fabricated. Models of the first type were designed to simulate springs with large stiffnesses, and models of the second type were fitted with medium and small springs at the blade-roots.

5.4.2 Models Simulating Large Springs at the Blade-Roots

Figures 5.4, 5.5 and Table 5.1 show the arrangements and details of experimental models (1), (2) and (3) with a heavy machined mild steel block, A. The blades and shrouds were also made of mild steel. The shrouds were brazed to the blades rather than welded. It was found that a brazed joint was satisfactory from strength considerations but less heat was needed for making the joint which helped to avoid distortion. Also it was easier to remove excess material after fabrication. At the root the blades were fitted into the slot machined in the heavy block, A. Spacing blocks, C, machined to give correct separation between the blades at the root were inserted and the whole assembly, consisting of a lower block A and upper block B was bolted together by means of four large bolts, D. The uppermost spacing block, E, was machined to form an I-section beam and acted as a load-cell for the system, (see Figure 5.5).
At the two extreme ends of the web of the I-section strain gauges were fixed. Thus it was possible to relate spanner torque in the bolts to measured values of strain in the load cell and hence determine the compressive stress at the blade-roots. The magnitude of the compressive stress present, gave an indication of the degree of fixation at the blade-roots. The practical limits of the compressive stress that could be applied, were, almost zero to a maximum value not exceeding the yield stress in the web of the load-cell. Plates 5.1, 5.2 and 5.3 give a general layout and details of the blades and instrumentation used.

5.4.3 Models Fitted with Medium and Small Springs at the Blade-Roots

Table 5.2 gives details of the three experimental models used. Model (4) is a single blade, whereas (5) and (6) are three and two-bladed packets respectively. Figure 5.6 is a diagrammatic representation of experimental packet (6), but also illustrates the spring arrangement and shows how leading dimensions were determined for all the three models considered here. It was discovered that the transverse springs which also acted differentially as rotational springs were extremely stiff in the longitudinal direction of the blade. By comparison the existing longitudinal springs, seen in Figure 5.6 were very soft and their effect could be ignored. Hence in the case of
packets (5) and (6) an assumption was made, in each case, that a longitudinal spring of infinite stiffness was present. In the case of the cantilever blade the magnitude of the longitudinal stiffness was of no consequence as transverse vibrations depended only on transverse and torsional springs. During vibration measurements the heavy machined blocks containing the springs and blade-roots were clamped securely to a heavy steel bed.

Plates 5.4, 5.5, 5.6 and 5.7 show a general view and details of experimental models (4), (5) and (6).

5.4.4 Instrumentation and Measurements

Overlay, Figure 5.7 gives a schematic representation of the instrumentation used in conjunction with Plate 5.1 which helps to identify the individual instruments. It is seen that two types of transducer are used: the piezoelectric dynamic strain gauge and an inductive displacement transducer.

The piezoelectric gauge was found to be particularly versatile as it could be used for both excitation and frequency measurement. However for low frequency excitation (below 50Hz) it was not suitable but still satisfactory as a pick-up. Low frequency excitation was provided by an electromagnet. Very low frequencies (below 15Hz) were measured by means of a stroboscope.
The inductive displacement transducer was used for the determination of mode shapes for the various modes of vibration. A special rig was constructed to enable the blades and shroud to be traversed by displacement transducers mounted vertically and horizontally on perspex slides. Fine adjustment was attained by means of micrometers as shown in plates 5.2 and 5.3.

Mode-shape measurement was limited to experimental models (1), (2) and (3).
5.5 THE FREQUENCY INFERENCE DIAGRAM

The principle and construction of the frequency inference diagram is fully discussed in Chapter 4.4. Here the uncoupled transverse modes for blades with flexible roots are included.

5.5.1 Blade With Spring System at One End and the Other End Free, Transverse Modes (BSF)

The blade has transverse and rotational springs at one end with an approportioned part of the shroud assumed to act as a concentrated mass at the free end.

The kinetic energy due to the concentrated mass is given by:

\[ K.E. = \frac{1}{2} \left( \rho_S A_s L_s \cdot \frac{P-1}{P} \right) \dot{y}^2 \quad (5.17) \]

with \[ \frac{A_s}{A_b} = \Delta, \] the shroud to blade mass ratio

and \[ \frac{L_s}{L_b} = L_r, \] the shroud to blade length ratio

\[ K.E. = \frac{1}{2} \left( \rho_b A_b \Delta L_b L_r \cdot \frac{P-1}{P} \right) \dot{y}^2 \quad (5.18) \]

From equation (4.23)

\[ M_{(\text{SIZE} - 1, \text{SIZE} - 1)} = \frac{\rho b^2 A_b b}{12972960} \left[ M_{B(SIZE - 1, SIZE - 1)} \right] \]

where 'SIZE' represents the total size of the global mass matrix.
Including the kinetic energy term of the concentrated mass at the blade tip,

\[ M(SIZE - 1, SIZE - 1) = \frac{\rho b^2 b^2}{12972960} \left[ M_B(SIZE - 1, SIZE - 1) + \frac{MB(SIZE - 1, SIZE - 1)}{12972960, \Delta I N, \frac{P-1}{P}} \right] \]

(5.19)

With the modification to the mass matrix as shown in equation (5.19) an equation of the type shown in (4.24) is formed and the frequency parameter, \( \lambda \) is determined by an eigenvalue routine as outlined in Chapter 3.3.

5.5.2 Blade With Spring System at One End and the Other End Pinned, Transverse Modes (BSP)

These modes are determined by analysing a blade with transverse and torsional springs at one end and with boundary conditions at the other end compatible with a fixed pinned joint.

5.5.3 Shroud Clamped–Clamped Transverse Modes (SCC)

Uncoupled shroud modes are independent of blade fixation and are determined as outlined in Chapter 4.4.
5.6 DISCUSSION OF RESULTS

5.6.1 Experimental Results and Comparisons with Other Investigators

Tables 5.1 and 5.2 give full details of the six experimental models used in this investigation.

Table 5.3 gives the measured relative deflections of cantilever blade, experimental model (1), for the first four modes of vibration. These deflections are plotted in Figures 3.7, 3.8, 3.9 and 3.10, showing good agreement with computed mode shapes.

Table 5.4 gives experimental relative deflections of two-bladed packet, experimental model (2), for the first three modes of vibration. Plots of these deflections are shown in Figure 4.12 and they agree well with computed mode-shapes.

Tables 5.5 and 5.6 show the effect of root fixation on the natural frequencies of two and three-bladed packets, experimental models (2) and (3) respectively. The stress of 85.39 N/mm$^2$ at the blade-root corresponded to considerably higher stresses in the vibration rig bolts and the load cell shown in Figures 5.4 and 5.5. The stress in the bolts was approximately four times and that in the load cell eight times the stress in the blade-root. Hence a stress of order 85.39 N/mm$^2$ was chosen as the higher
stress at the blade-root. The lower stress of 18.88 N/mm\(^2\) at the blade-root was chosen arbitrarily as the lowest value at which a stable load-cell reading could be taken.

It is seen that the effect of changing root fixation from a compressive stress of 85.39 N/mm\(^2\) to 18.88 N/mm\(^2\) on the natural frequency of the two packets tested is small. However a consistent reduction of natural frequency of approximately 1 to 2% is observed. Later in the discussion it will be shown that when the order of stiffness is high at the blade-root, changes in stiffness have little or no effect on the natural frequencies of blade packets.

Tables 5.7 and 5.8 give comparisons between values of \(\lambda\) computed by the present finite-element method and classical analysis as given by Gorman (42, 1975). The model considered was a single blade, free at one end and having a torsional and transverse spring at the other. A longitudinal spring was not included as the presence of one would have had no effect on transverse vibrations.

Excellent agreement is seen between the two computational methods for the two spring systems considered, based on experimental models (1) and (6). Results in Table 5.7 show a greatest variation of 0.091% and those in Table 5.8 a greatest variation 0.158%.
Tables 5.9, 5.10 and 5.11 give measured and computed natural frequencies of experimental models (4), (5) and (6) respectively.

It is seen from Table 5.9 that very good agreement is obtained between computed and experimental values for the first four modes of vibration. In the higher modes of vibration the results are less satisfactory. At the time of the measurements it was found that due to coupling between torsional and transverse modes in the blade and also due to instability at the blade-root there was some difficulty in promoting pure transverse vibration modes.

Computed values of frequencies for the fully-fixed case are included for all three experimental models in order to facilitate the appreciation of the effects of the different spring systems used.

Table 5.10 gives measured and computed values of natural frequencies of experimental packet (5). All results show good agreement except for the 14th and 15th modes. Here again some difficulty existed in the higher ranges of frequency of inducing pure transverse vibrations.

It is seen here, that the longitudinal stiffness, $H$, is given as $H = \infty$. Originally a longitudinal spring of known stiffness was included. However it was discovered
that the transverse springs due to bending and possibly shear deformation acted as very stiff springs in the longitudinal direction and the effect of the existing, comparatively soft longitudinal springs could be ignored. It was also discovered by experiment (see Tables 5.5 and 5.6) and computation (see Figures 5.11 to 5.29) that for large values of $H$ changes in $H$ had little or no effect on the natural frequency of a packet of blades. Here, it was assumed that $H$ is infinitely large.

Table 5.11 gives computed and measured values of natural frequencies for experimental packet (6). Good agreement is seen for all the modes quoted. Possibly due to stiffer springs the packet tended to be more stable at higher frequencies and there was less difficulty in locating the higher transverse modes. The assumption of an infinitely stiff longitudinal spring also applies here.

Figure 5.8 shows how the value of $E/\rho$ was determined for experimental models (2) and (3).

Figures 5.9 and 5.10 show the determination of the spring stiffnesses of the springs used in experimental models (4), (5) and (6).

5.6.2 Relation between Frequency Parameter $\lambda$ and Root Flexibility of a Packet of Blades
One of the objects of this investigation was a comprehensive study of the effect of root flexibility on the dynamic characteristics of blade packets. In this investigation the root flexibility is completely defined by torsional, longitudinal and transverse springs with non-dimensional stiffnesses $G^*$, $H^*$ and $K^*$ respectively for greater generality of results. The packet investigated here is theoretical packet (I), thus enabling comparisons to be made with relevant results of the previous Chapter.

Figures 5.11 to 5.20 give the relation between $\lambda$ and $K^*$ for a family of curves with various values of $G^*$ and $H^*$, for the first nine modes of vibration.

Figure 5.11 shows that for the first mode of vibration $\lambda$ is very sensitive to changes in $K^*$ in the range $K^* = 1.0$ to $K^* = 100$, when $H^* = 1.0$.

Figure 5.12 shows that when $H^*$ has a very low value, also in the first mode of vibration and $G^*$ and $K^*$ are greater than 10, $\lambda$ is independent of $G^*$ and $K^*$ and remains at a constant value of approximately 3.

Figure 5.13 shows that in the second mode of vibration for low values of $H^*$ $\lambda$ is very small and for large values of $K^*$ independent of $K^*$. For high values of $H^*$ $\lambda$ is very sensitive to changes in $K^*$ in the range $K^* = 1.0$ to $K^* = 1000$ and when $K^*$ is greater than 10,000, $\lambda$ is independent of $K^*$.
Figure 5.14 shows that in the third mode of vibration $\lambda$ is very sensitive to changes in $K^*$ for values of $K^*$ between 10 and 1000, when $H^*$ is large. Also for combinations of low $G^*$ and high $H^*$ to the left of the sensitive region that is for values of $K^*$ less than 10, and for high $G^*$ and low $H^*$ to the right of the sensitive region that is for values of $K^*$ greater than 1000 it is seen that $\lambda$ has a constant value and therefore is independent of $K^*$.

It is seen from Figures 5.15 to 5.20 that this particular result also applies to the remaining five modes considered, though the actual values of $K^*$ and $\lambda$ differ from mode to mode. The characteristics of $G^*$, however vary considerably between the modes and therefore will be discussed separately for each mode considered.

Figure 5.15 shows that in the fourth mode when $H^* = 1.0$ and for one particular value of $K^*$, approximately 100, $\lambda$ is independent of $G^*$.

Figure 5.16 shows that in the fifth mode when $H^* = 1.0$ again for $K^* = 100$ $\lambda$ is independent of $G^*$. It is also seen that when $H^* = 10^{-4}$ for all values of $K^*$ less than 100 $\lambda$ is independent of $G^*$.

Figure 5.17 shows that in the sixth mode when
$H^* = 10^{-4}$ is independent of $G^*$ when $K^* = 100$. However when $H^* = 1.0$ is independent of $G$ for all values of $K^*$ greater than 100.

Figure 5.18 shows that in the seventh mode when $H^* = 10^{-4}$ is independent of $G^*$ when $K^* = 100$, but when $H^* = 1.0 \lambda$ is independent of $G^*$ when $K^* = 1000$.

Figure 5.19 shows that in the eighth mode when $H^* = 10^{-4}$, $\lambda$ is independent of $G^*$ for values of $K^*$ greater than 1000, whereas when $H^* = 1.0 \lambda$ is independent of $G^*$ for all values of $K^*$ less than 1000.

Finally in Figure 5.20 for the ninth mode of vibration it is seen that $\lambda$ is independent of $G^*$ when $K^* = 1000$ for all values of $H^*$. It is also seen here that for $G^* = 10,000$, $K^* = 100,000$, it was necessary to raise the value of $H^*$ to 10 in order to reach the fully-fixed value of $\lambda$.

Figures 5.21, 5.22 and 5.23 show the variation of $\lambda$ with $G^*$ when $H^* = 1.00$ and $K^* = 100,000$.

Figure 5.21 shows a large variation in $\lambda$ with $G^*$ in the first mode of vibration.
Figure 5.22 shows that the variation of $\lambda$ with $G^*$ in the second and third modes is considerably less than that observed for the first mode.

Figure 5.23 shows the effect of $G^*$ on $\lambda$ for the fourth, fifth, sixth, seventh, eighth and ninth modes of vibration. Here similarities are noticed in the fourth, fifth, seventh and eighth modes where the increase in $\lambda$ between low and high values of $G^*$ is of the order of 30 to 40 per cent. However in the sixth and ninth modes $\lambda$ is independent of $G^*$. It is seen that in the sixth mode $\lambda$ coincides with $\lambda^{(F)}$ but in the ninth mode $\lambda$ is considerably less than $\lambda^{(F)}$. The latter two results could have been predicted from Figures 5.17 and 5.20 respectively.

Finally Figures 5.24, 5.25, 5.26, 5.27, 5.28 and 5.29 show the behaviour of $\lambda$ when $G^* = 10000$, $K^* = 100,000$ and $H^*$ is varied between $10^{-6}$ and 1000. Here the leading characteristics are very similar for all nine modes considered. With the exception of a very narrow band of values of $H^*$, $\lambda$ is independent of $H^*$. Also when $H^*$ has very low values, $\lambda$ is zero in the first mode or has low values in the higher modes.

It is also seen from Figures 5.24, 5.25, 5.26, 5.27 and 5.29 that when $G^* = 10$ and $K^* = 100$ and for large values of $H^*$, $\lambda$ is independent of $H^*$.
5.6.3 Modes Shapes, Bending Moment, Shear Force and End-Load Distributions of a Blade Packet with Flexible Roots

Figures 5.30, 5.31 and 5.32 show the relative deflections of two-bladed packet, theoretical model (II), with $L_r = 0.2$ and spring stiffnesses, $G^* = 250$, $H^* = 250$ and $K^* = 15.625$. It is seen that the first, third and fifth modes are main modes and the third and fourth modes are batch modes.

Figures 5.33, 5.34, 5.35, 5.36 and 5.37 show the bending moment, shear force and end-load distributions for the first five modes of the above packet.

It is seen that bending moment and shear force distributions in the blade are considerably larger than the corresponding distributions in the shroud, probably because the flexural rigidity ratio between shroud and blade is only 0.005. The end-loads in the blades are larger in the main modes of vibration but very low in the batch modes. The end-loads in the shroud are very low in all five vibration modes.

5.6.4 Frequency Inference Diagrams

The construction and significance of the frequency inference diagram is fully discussed in Chapters 4.4.
In this Chapter the use of the frequency inference diagram is extended to blade packets with flexible roots as shown in section 5.5. Transverse and rotational springs are fitted at the blade root for the blade-spring-free (BSF) and the blade-spring-pinned (BSP) cases, which are independent of the shroud modes. The independent shroud modes are not affected by blade flexibility and are determined as outlined in Chapter 4.4. The effect of varying the spring stiffness at the roots of whole blade packets is studied and comparison is made between coupled and uncoupled modes of vibration.

Figure 5.38 shows that very good agreement is obtained between the coupled and uncoupled blade and shroud modes of theoretical packet (I) with spring system, $G^* = 1000$, $H^* = 10$ and $K^* = 10$.

In the case of the uncoupled blade $H^*$ has no meaning but for the complete packet a large stiffness $H^* = 10$ is chosen in order to satisfy the condition of zero longitudinal deflection at the blade root.

In order to be able to appreciate the effect of the spring system an inference diagram for the same packet of blades with roots fully fixed is included. It can be seen that root-flexibility has no effect on the uncoupled shroud modes.
Figures 5.39 and 5.40 show frequency inference diagrams for the same packet of blades with spring systems $G^* = 1000$, $H^* = 10$, $K^* = 100$ and $G^* = 1000$, $H^* = 10$, $K^* = 1000$ respectively.

In both cases very good agreement is seen between uncoupled modes and modes for the complete packet.

Figure 5.41 shows the variation of $\lambda$ with changes in root flexibility for the third mode of vibration.

Figures 5.42, 5.43 and 5.44 are of the same type as Figures 5.38, 5.39 and 5.40 but for theoretical blade packet (II). Very good agreement is seen between the results for the uncoupled blade and shroud modes and the results for the complete packet for all the cases considered.

Figure 5.45 shows the effect on $\lambda$ of varying root flexibility in the third mode of vibration.

The frequency-inference diagram is a very powerful tool in the hands of the designer, especially at early design stages. It enables the prediction of the dynamic behaviour of a multibladed packet from the uncoupled modes of a single blade and shroud. It is seen from Tables 4.10 and 4.11 that the two batch-modes of the three-bladed packet are very close to the single batch-mode of the two-bladed packet. It is also seen that the values of the main
vibration modes of the two packets are very close with lower values for packets with larger numbers of blades. However, here, it is possible to improve the prediction of natural frequencies by appropriating the concentrated mass carried by each uncoupled blade, according to the number of blades present in a packet, see equation (5.17).

The frequency-inference diagram will prove to be useful in Industry, where speed of operation and cost are of paramount importance.
### Table 5.1 Data Relating to Experimental Models (1), (2), and (3) (Ref. Figures 5.4 and 5.5)

<table>
<thead>
<tr>
<th>MODEL NO</th>
<th>NUMBER OF BLADES NB</th>
<th>BLADE LENGTH $L_b$ (m)</th>
<th>SHROUD LENGTH $L_s$ (m)</th>
<th>BLADE THICKNESS $t_b$ (m)</th>
<th>BLADE THICKNESS $t_s$ (m)</th>
<th>BLADE WIDTH $w_b$ (m)</th>
<th>SHROUD WIDTH $w_s$ (m)</th>
<th>FLEXURAL STIFFNESS RATIO $\gamma$</th>
<th>MASS RATIO $\Delta$</th>
<th>LENGTH RATIO $L_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.15079</td>
<td>-</td>
<td>0.00158</td>
<td>0.00158</td>
<td>0.02515</td>
<td>0.02515</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.15079</td>
<td>0.0254</td>
<td>0.00158</td>
<td>0.00158</td>
<td>0.02515</td>
<td>0.02515</td>
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<td>1.0</td>
<td>0.168446</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.15079</td>
<td>0.0254</td>
<td>0.00158</td>
<td>0.00158</td>
<td>0.02515</td>
<td>0.02515</td>
<td>1.0</td>
<td>1.0</td>
<td>0.168446</td>
</tr>
</tbody>
</table>

### Table 5.2 Data Relating to Experimental Models (4), (5) and (6) (Ref. Figure 5.6)

<p>| MODEL NO | NO OF BLADES NB | $L_1$ (m) | $L_2$ (m) | $L_3$ (m) | $L_s$ (m) | $t_b$ (m) | $t_s$ (m) | $w_b$ (m) | $w_s$ (m) | TORSIONAL STIFFNESS $G = \frac{q}{EI}$ (Nm/rad) PER EI | LATERAL STIFFNESS $H = \frac{h}{EA}$ (N/m) PER EA | TRANSVERSE STIFFNESS $K = \frac{k}{EI}$ (N/m) PER EI | FLEXURAL STIFFNESS RATIO $\gamma$ | MASS RATIO $\Delta$ | LENGTH RATIO $(L_s/L_1)$ | $L_r$ |
|----------|-----------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|---------------------------------|---------------------------------------------|--------------------------------------------|-----------------------------|------------------|------------------|
| 4        | 1               | 0.25589   | 0.24765   | 0.24130   | -         | 0.003175  | -         | 0.0254   | -         | 1.4099                          | -                                           | 11189.39                                  | 11189.39                                | 1.0              | 1.00             | 0.149068         |
| 5        | 3               | 0.25589   | 0.24765   | 0.24130   | 0.0381    | 0.003175  | 0.003175  | 0.0254   | 0.0254   | 1.4099                          | LARGE                                      | 11189.39                                  | LARGE                                    | 1.00             | 1.00             | 0.149068         |
| 6        | 2               | 0.25400   | 0.23813   | 0.22587   | 0.0762    | 0.003175  | 0.0015875 | 0.0254   | 0.0254   | 15.3403                         | LARGE                                      | 30435.15                                  | 30435.15                                | 1.25             | 1.50             | 0.3000           |</p>
<table>
<thead>
<tr>
<th>DISTANCE FROM BLADE-ROOT (mm)</th>
<th>RELATIVE DISPLACEMENT FROM NEUTRAL POSITION MODE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0.000</td>
</tr>
<tr>
<td>25</td>
<td>0.046</td>
</tr>
<tr>
<td>50</td>
<td>0.176</td>
</tr>
<tr>
<td>75</td>
<td>0.306</td>
</tr>
<tr>
<td>100</td>
<td>0.472</td>
</tr>
<tr>
<td>125</td>
<td>0.648</td>
</tr>
<tr>
<td>150</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**TABLE 5.3**

RELATIVE DEFLECTIONS OF SINGLE CANTILEVER BLADE, EXPERIMENTAL MODEL (1). REF. TABLE 5.1

<table>
<thead>
<tr>
<th>DISTANCE FROM BLADE-ROOT (mm)</th>
<th>RELATIVE DISPLACEMENT FROM NEUTRAL POSITION MODE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>UPPER BLADE</td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>37.5</td>
<td>-</td>
</tr>
<tr>
<td>75</td>
<td>0.58</td>
</tr>
<tr>
<td>112.5</td>
<td>-</td>
</tr>
<tr>
<td>150</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**TABLE 5.4**

RELATIVE DEFLECTIONS OF TWO-BLADED PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1
<table>
<thead>
<tr>
<th>MODE</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>EXPERIMENTAL RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Finite-Element Method</td>
<td>Comp. Stress at the Root</td>
</tr>
<tr>
<td></td>
<td>Blade Roots</td>
<td>$N=4 ; M=4$</td>
</tr>
<tr>
<td></td>
<td>Fully Fixed</td>
<td>$85.39 ; \text{N/mm}^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$18.88 ; \text{N/mm}^2$</td>
</tr>
<tr>
<td>1</td>
<td>80.8219</td>
<td>80.3</td>
</tr>
<tr>
<td>2</td>
<td>324.8977</td>
<td>320.4</td>
</tr>
<tr>
<td>3</td>
<td>446.2250</td>
<td>431.5</td>
</tr>
<tr>
<td>4</td>
<td>911.5214</td>
<td>898.7</td>
</tr>
<tr>
<td>5</td>
<td>1107.0193</td>
<td>1066.9</td>
</tr>
<tr>
<td>6</td>
<td>1806.7067</td>
<td>1789.2</td>
</tr>
<tr>
<td>7</td>
<td>2068.9521</td>
<td>2004.5</td>
</tr>
<tr>
<td>8</td>
<td>3003.0917</td>
<td>2991.6</td>
</tr>
<tr>
<td>9</td>
<td>3334.6602</td>
<td>3253.3</td>
</tr>
</tbody>
</table>

$\text{NB} = 2 \quad \quad \frac{E}{S} = 2.5850 \times 10^7 \; \text{Nm/kg} \; \text{(REF. FIG. 5.8)}$

$\gamma = 1.0$

$\Delta = 1.0$

$Le = 0.168446$

**TABLE 5.5** EFFECT OF VARYING ROOT FIXATION OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (2), REF. TABLE 5.1
<table>
<thead>
<tr>
<th>MODE</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>EXPERIMENTAL RESULTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Finite-Element Method</td>
<td>Comp. Stress</td>
</tr>
<tr>
<td></td>
<td>Blade Roots</td>
<td>at the Root</td>
</tr>
<tr>
<td></td>
<td>Fully Fixed (N=4, M=4)</td>
<td>85.39(^{\text{N}})/\text{mm}</td>
</tr>
<tr>
<td>1</td>
<td>78.0005</td>
<td>78.2</td>
</tr>
<tr>
<td>2</td>
<td>328.2314</td>
<td>327.0</td>
</tr>
<tr>
<td>3</td>
<td>340.8891</td>
<td>339.0</td>
</tr>
<tr>
<td>4</td>
<td>435.6211</td>
<td>433.8</td>
</tr>
<tr>
<td>5</td>
<td>917.2604</td>
<td>924.7</td>
</tr>
<tr>
<td>6</td>
<td>945.3950</td>
<td>945.0</td>
</tr>
<tr>
<td>7</td>
<td>1085.7363</td>
<td>1068.3</td>
</tr>
<tr>
<td>8</td>
<td>1811.1757</td>
<td>1816.3</td>
</tr>
<tr>
<td>9</td>
<td>1862.1729</td>
<td>1840.3</td>
</tr>
<tr>
<td>10</td>
<td>2037.2140</td>
<td>2003.8</td>
</tr>
<tr>
<td>11</td>
<td>2992.8017</td>
<td>2982.7</td>
</tr>
<tr>
<td>12</td>
<td>3089.4505</td>
<td>3031.9</td>
</tr>
<tr>
<td>13</td>
<td>3293.6048</td>
<td>3259.8</td>
</tr>
</tbody>
</table>

\[
\text{NB} = 3 \quad \frac{E}{G} = 2.5850 \times 10^{7}\text{Nm/kg (REF, FIG.5.8)}
\]

\[
\gamma = 1.0
\]

\[
\Delta = 1.0
\]

\[
L_r = 0.168446
\]

**TABLE 5.6 EFFECT OF VARYING ROOT FIXATION OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (3), REF, TABLE 5.1**
<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>FREQUENCY PARAMETER $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Finite Element Method (N=5)</td>
</tr>
<tr>
<td>1</td>
<td>.9863</td>
</tr>
<tr>
<td>2</td>
<td>202.6695</td>
</tr>
<tr>
<td>3</td>
<td>1361.5394</td>
</tr>
<tr>
<td>4</td>
<td>4800.7761</td>
</tr>
<tr>
<td>5</td>
<td>15601.8929</td>
</tr>
</tbody>
</table>

$L_b = 0.254 \text{ m}$
$w_b = 0.0254 \text{ m}$
$t_b = 0.003175 \text{ m}$

**TABLE 5.7 COMPARISON OF COMPUTED VALUES OF FREQUENCY PARAMETER $\lambda$ FOR $G^* = 0.3581$ AND $K^* = 183.3613$**

<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>FREQUENCY PARAMETER $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Finite Element Method (N=5)</td>
</tr>
<tr>
<td>1</td>
<td>6.0018</td>
</tr>
<tr>
<td>2</td>
<td>283.2329</td>
</tr>
<tr>
<td>3</td>
<td>2036.7990</td>
</tr>
<tr>
<td>4</td>
<td>6659.0364</td>
</tr>
<tr>
<td>5</td>
<td>18231.1611</td>
</tr>
</tbody>
</table>

$L_b = 0.254 \text{ m}$
$w_b = 0.0254 \text{ m}$
$t_b = 0.003175 \text{ m}$

**TABLE 5.8 COMPARISON OF COMPUTED VALUES OF FREQUENCY PARAMETER $\lambda$ FOR $G^* = 3.8964$ AND $K^* = 498.7428$**
<table>
<thead>
<tr>
<th>MODE NO.</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>Finite Element Method</th>
<th>Experimental Results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>N = 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fully Fixed</td>
<td>G = 1.4099</td>
<td>K = 11189.39</td>
</tr>
<tr>
<td>1</td>
<td>40.0188</td>
<td>11.3845</td>
<td>13.6</td>
</tr>
<tr>
<td>2</td>
<td>251.8650</td>
<td>163.0324</td>
<td>164</td>
</tr>
<tr>
<td>3</td>
<td>705.2300</td>
<td>423.5767</td>
<td>429</td>
</tr>
<tr>
<td>4</td>
<td>1381.9693</td>
<td>793.5856</td>
<td>795</td>
</tr>
<tr>
<td>5</td>
<td>2284.4944</td>
<td>1428.5451</td>
<td>1648</td>
</tr>
<tr>
<td>6</td>
<td>3412.6391</td>
<td>2314.9152</td>
<td>2411</td>
</tr>
<tr>
<td>7</td>
<td>4766.4158</td>
<td>3435.4031</td>
<td>3673</td>
</tr>
<tr>
<td>8</td>
<td>6345.8315</td>
<td>4784.9286</td>
<td>4857</td>
</tr>
<tr>
<td>9</td>
<td>8150.9432</td>
<td>6361.7356</td>
<td>7864</td>
</tr>
</tbody>
</table>

\[
\frac{E}{\rho} = 2.62025 \times 10^7 \text{ Nm/kg}
\]

**TABLE 5.9** COMPUTED AND EXPERIMENTAL FREQUENCIES OF SINGLE BLADE, EXPERIMENTAL MODEL (4), REF. TABLE 5.2
TABLE 5.10  COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (5), REF. TABLE 5.2
<table>
<thead>
<tr>
<th>MODE NO</th>
<th>NATURAL FREQUENCIES (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FINITE ELEMENT METHOD</td>
</tr>
<tr>
<td></td>
<td>N = 4  M = 4</td>
</tr>
<tr>
<td>FULLY FIXED</td>
<td>G = 15.3403, H = Ω, K = 30435.15</td>
</tr>
<tr>
<td>1</td>
<td>48.7017</td>
</tr>
<tr>
<td>2</td>
<td>186.9880</td>
</tr>
<tr>
<td>3</td>
<td>265.5651</td>
</tr>
<tr>
<td>4</td>
<td>580.0251</td>
</tr>
<tr>
<td>5</td>
<td>691.4146</td>
</tr>
<tr>
<td>6</td>
<td>1137.5956</td>
</tr>
<tr>
<td>7</td>
<td>1329.8387</td>
</tr>
<tr>
<td>8</td>
<td>1435.5527</td>
</tr>
<tr>
<td>9</td>
<td>2125.5966</td>
</tr>
<tr>
<td>10</td>
<td>2178.2451</td>
</tr>
<tr>
<td>11</td>
<td>3172.9911</td>
</tr>
<tr>
<td>12</td>
<td>3189.9082</td>
</tr>
<tr>
<td>13</td>
<td>3794.8729</td>
</tr>
<tr>
<td>14</td>
<td>4483.7356</td>
</tr>
<tr>
<td>15</td>
<td>4700.2256</td>
</tr>
</tbody>
</table>

NB = 2

\[ \frac{E}{\rho} = 2.62025 \times 10^7 \text{ Nm/kg} \]

\[ \Delta = 0.50 \]

L = 0.300

TABLE 5.11 COMPUTED AND EXPERIMENTAL FREQUENCIES OF SHROUDED BLADE PACKET, EXPERIMENTAL MODEL (6), REF. TABLE 5.2
FIG. 5.1 NODAL CO-ORDINATES OF BLADE ROOT ELEMENT
FIG. 5.2  BLADE ROOT ELEMENT WITH TORSIONAL, LONGITUDINAL AND TRANSVERSE SPRING SYSTEM AT THE ROOT

FIG. 5.3  UNIFORM BEAM WITH A TORSIONAL AND TRANSVERSE SPRING SYSTEM AT ONE END AND THE OTHER END FREE
FIG. 5.4 EXPERIMENTAL MODELS (1), (2) AND (3) (SIDE ELEVATION)
FIG. 5.5  EXPERIMENTAL MODELS (1), (2) AND (3)  
(FRONT ELEVATION)
FIG. 5.6 EXPERIMENTAL BLADE PACKET (6) 
SHOWING SPRING SYSTEM AT THE ROOT
\[ \frac{E}{J} = \frac{x^2}{\beta} \cdot \frac{48 \pi^2 L^4}{t^2 \rho} \]

\[ = 2.5850 \times 10^7 \text{ Nm/kg} \]

**FIG. 5.8** RELATION BETWEEN NATURAL FREQUENCY, \( f \) AND \(( \lambda)^{\frac{1}{3}}\) FOR SINGLE CANTILEVER BLADE, EXPERIMENTAL MODEL (1), REF. TABLE 5.1
RATE OF SPRING = 0.1 TON/in OR 39,228.397 N/m GIVES,

\[ k = 4 \times 39,228.397 = 156913.588 \text{ N/m} \]

\[ g = 2 \times 39,228.397 \left( L_1 - L_2 \right) \times 2 \]

FOR EXPERIMENTAL MODELS (4) & (5)

\[ K = 11189.39 \text{ (N/m) PER (EI)} \]

\[ G = 1.0499 \text{ (Nm/rad) PER (EI)} \]

\[ g = 19.772367 \text{ Nm/rad} \]

FIG. 5.9 RATE OF SPRING (a) USED IN EXPERIMENTAL MODELS (4) AND (5), REF TABLE 5.2 AND FIG 5.6
RATE OF SPRING = 0.272 TON/in
OR 106701.2394 N/m
GIVES
k = 4 \times 106701.2394
= 426804.9576 N/m
\[ g = 2 \times 106701.2394 \left( L_1 - L_2 \right) \times 2 \]^2 \]
FOR EXPERIMENTAL MODEL (6)
g = 215.1230363 Nm/rad
K = 30435.15039 (N/m) PER(EI)
G = 15.3402669 (Nm/rad) PER(EI)

FIG 5.10 RATE OF SPRING (b) USED IN EXPERIMENTAL MODEL (6),
REF. TABLE 5.2 AND FIG 5.6
FIG. 5.11 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR $H^* = 1.00$, FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I) (SEE DETAILS ABOVE)
FIG. 5.12 VARIATION OF FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR $H^* = 10^{-4}$, FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.13 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE SECOND MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.14 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE THIRD MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.15 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE FOURTH-MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)

$\lambda = 2480.3941$ (BLADE-ROOTS FULLY FIXED)
FIG. 5.16 VARIATION OF THE FREQUENCY PARAMETER $\tilde{\lambda}$ WITH $K^*$ FOR THE FIFTH MODE OF VIBRATION OF THEORETICAL MODEL (I), (REF. FIG. 5.11)
FIG. 5.17 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE SIXTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11).
FIG. 5.18 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE SEVENTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.19 VARIATION OF THE FREQUENCY PARAMETER \( \lambda \) WITH \( K^* \) FOR THE EIGHTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.20 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $K^*$ FOR THE NINTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
\( \lambda = 12.0222 \) (BLADE - ROOTS FULLY FIXED)

\[ H^* = 1.00 \]
\[ K^* = 100000 \]

FIG. 5.21 VARIATION OF THE FREQUENCY PARAMETER \( \lambda \) WITH \( G^* \) FOR THE FIRST MODE OF VIBRATION OF THEORETICAL MODEL (I)
(REF. FIG. 5.11)
\( \lambda = 449.8696 \) (BLADE-ROOTS FULLY FIXED)

\( \lambda = 238.9507 \) (BLADE-ROOTS FULLY FIXED)

**FIG. 5.22 VARIATION OF FREQUENCY PARAMETER \( \lambda \) WITH \( G^* \) FOR THE SECOND AND THIRD MODES OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)**

\[ H^* = 1.00 \]
\[ K^* = 100000 \]
FIG. 5.23 VARIATION OF FREQUENCY PARAMETER $\lambda$ WITH $G^*$ FOR THE FOURTH, FIFTH, SIXTH, SEVENTH, EIGHTH AND NINTH MODES OF THEORETICAL MODEL (I) (REF. FIG. 5.11)

$H^* = 1.00$

$K^* = 100000$

$\lambda(F) =$ VALUE OF $\lambda$ FOR BLADE-ROOTS FULLY FIXED

<table>
<thead>
<tr>
<th>$G^*$</th>
<th>$\lambda(F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2480.3941</td>
</tr>
<tr>
<td>0.1</td>
<td>3488.7191</td>
</tr>
<tr>
<td>1.0</td>
<td>5163.9211</td>
</tr>
<tr>
<td>10</td>
<td>10928.5548</td>
</tr>
<tr>
<td>100</td>
<td>13401.3058</td>
</tr>
<tr>
<td>1000</td>
<td>31743.5957</td>
</tr>
</tbody>
</table>

FREQUENCY PARAMETER $\lambda$
FIG. 5.24 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$
FOR THE FIRST MODE OF VIBRATION OF THEORETICAL
MODEL (I) (REF. FIG. 5.11)

$\lambda = 12.0222$ (BLADE-ROOTS FULLY FIXED)

$G^* = 10000$
$K^* = 100000$
Fig. 5.25 Variation of the frequency parameter $\lambda$ with $H^*$ for the second mode of vibration of theoretical model (I) (Ref. Fig. 5.11)

$\lambda = 238.9507$ (blade-roots fully fixed)

$G^* = 10000$

$K^* = 100000$
FIG. 5.26 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE THIRD MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)

$\lambda = 449.8696$ (BLADE-ROOTS FULLY FIXED)

$G^* = 100000$

$K^* = 1000000$
\[ \lambda = 2480.3941 \text{ (BLADE-ROOTS FULLY FIXED)} \]

\[ G^* = 10000 \]
\[ K^* = 100000 \]

FIG. 5.27 VARIATION OF THE FREQUENCY PARAMETER \( \lambda \) WITH \( H^* \) FOR THE FOURTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. FIG. 5.11)
FIG. 5.28 VARIATION OF THE FREQUENCY PARAMETER $\lambda$ WITH $H^*$ FOR THE FIFTH, SIXTH, SEVENTH AND EIGHTH MODES OF VIBRATION OF THEORETICAL MODEL (I) (REF. 5.11)
FREQUENCY PARAMETER

\( \lambda = 31743.5957 \) (BLADE-ROOTS FULLY FIXED)

\( G^* = 100000 \)
\( K^* = 1000000 \)

**FIG. 5.29 VARIATION OF THE FREQUENCY PARAMETER \( \lambda \) WITH \( H^* \)
FOR THE NINTH MODE OF VIBRATION OF THEORETICAL MODEL (I) (REF. 5.11)**
\[ \varphi = 0.005 \quad L_R = 0.25m \quad G^* = 250 \]
\[ \Delta = 0.3 \quad W_r = 0.01m \quad H^* = 250 \]
\[ L_T = 0.2 \quad t_b = 0.005m \quad K^* = 15.625 \]

**FIG. 5.30** MODE SHAPES FOR FIRST AND SECOND MODES OF VIBRATION OF THEORETICAL PACKET (I) (SEE DETAILS ABOVE)
FIG. 5.31 MODE SHAPES FOR THIRD AND FOURTH MODES OF VIBRATION OF THEORETICAL PACKET (I) (SEE FIG. 5.30)
FIG. 5.32 MODE SHAPE FOR FIFTH MODE OF VIBRATION OF THEORETICAL PACKET (I) (SEE FIG. 5.30)
FIG. 5.33 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FIRST MODE OF VIBRATION OF THEORETICAL PACKET (I) (SEE FIG.5.30)

\[ \frac{M}{E_b I_b} \times 10^{-3} \]

\[ \frac{S}{E_b I_b} \times 10^{-2} \]

\[ \frac{P}{E_b I_b} \times 10^{-2} \]

\[ v_{\text{MAX}} = 1.5398 \times 10^{-4} \]
FIG. 5-34 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE SECOND MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIG. 5-30)
FIG. 5.35 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (1) (SEE FIG. 5.30)

BENDING MOMENT $\cdot \quad V_{\text{MAX.}} = 1.3944 \times 10^{-4}$

SHEAR FORCE

END-LOAD

$\frac{M}{E_b I_b} \times 10^{-3}$

$\frac{S}{E_b I_b} \times 10^{-2}$

$\frac{P}{E_b Y_b} \times 10^{-2}$
FIG. 5.36 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FOURTH MODE OF VIBRATION OF THE RETICAL PACKET (1) (SEE FIG. 5.30)
FIG. 5.37 BENDING MOMENT, SHEAR FORCE AND END-LOAD DISTRIBUTIONS FOR THE FIFTH MODE OF VIBRATION OF THEORETICAL PACKET (I) (SEE FIG. 5.30)
FREQUENCY PARAMETER

$\begin{align*}
NB &= 2 & L_b &= 0.25 \text{ m} \\
\gamma &= 0.005 & w_b &= 0.01 \text{ m} \\
\Delta &= 0.3 & t_b &= 0.005 \text{ m}
\end{align*}$

---

FIG. 5.38 FREQUENCY INFEERENCE DIAGRAM FOR THEORETICAL PACKET (1) FOR $G^* = 1000, H^* = 10$ AND $K^* = 10$
(SEE DETAILS ABOVE AND FIGURE 4.28)
FIGURE 5.39 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (1) FOR $G^* = 1000$, $H^* = 10$ AND $K^* = 100$ (REF. FIGURE 5.38)
FIGURE 5.40 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (1) FOR $G^* = 1000$, $H^* = 10$ AND $K^* = 1000$
(REF. FIGURE 5.38)
FIG. 5.41 VARIATION OF $\lambda$ WITH ROOT FLEXIBILITY FOR THE THIRD MODE OF VIBRATION OF THEORETICAL PACKET (I) (REF. 5.38), SUPERIMPOSED ON THE FREQUENCY INFERENCE DIAGRAM FOR THE FULLY FIXED CASE
FIG. 5.42 FREQUENCY INFEERENCE DIAGRAM FOR THEORETICAL PACKET (II) FOR $G^* = 1000$, $H^* = 10$ and $K^* = 10$ (SEE DETAILS ABOVE AND FIG. 4.29)
FIG. 5.43 FREQUENCY INFERENC DIAGRAM FOR THEORETICAL PACKET (II) FOR $G^* = 1000$, $H^* = 10$ AND $K^* = 100$ (REF. FIG. 5.42)
FIG. 5.44 FREQUENCY INFERENCE DIAGRAM FOR THEORETICAL PACKET (II) 
FOR $G^* = 1000$, $H^* = 10$ AND $K^* = 1000$ (REF. FIG. 5.42)
FIG. 5.45 VARIATION OF $\lambda$ WITH ROOT FLEXIBILITY FOR THE THIRD MODE OF THEORETICAL PACKET (II) (REF. FIG. 5.42), SUPERIMPOSED ON THE FREQUENCY INFERENCE DIAGRAM FOR THE FULLY FIXED CASE.
PLATE 5.2 CLOSE-UP VIEW OF EXPERIMENTAL RIG FOR MODELS (1), (2) AND (3)
PLATE 5.4 EXPERIMENTAL MODELS (4), (5), AND (6)
PLATE 5.5  SPRING ARRANGEMENT OF SINGLE BLADE
EXPERIMENTAL MODEL (4)
PLATE 5.6  SPRING ARRANGEMENT OF THREE
BLADE PACKET
EXPERIMENTAL MODEL (5)
PLATE 5.7  SPRING ARRANGEMENT OF TWO BLADE PACKET
EXPERIMENTAL MODEL (6)
Results of the present investigation are presented and fully discussed in Chapters 3, 4 and 5 of this thesis. Conclusions and recommendations for possible future work on turbine-blade packets are as follows:

6.1 DEVELOPMENT OF AN IMPROVED FINITE ELEMENT

The finite element developed in Chapters 3, 4 and 5 is seen to be a considerable improvement over analyses presented by other investigators. The eigenvalue solutions computed by the present method are more accurate and require matrices of lower order than those currently used. Also convergence of results with the number of blade and shroud elements is fast. In addition the eigenvectors provide a complete two-dimensional solution of displacements shear and axial forces and bending moments from which the state of stress in the structure can be found, for all vibration modes considered. The latter facility must be regarded as a considerable improvement over the currently adopted, rather tedious method, of determining forces from mode shapes.

6.2 EFFECT OF ROOT FLEXIBILITY ON THE VIBRATION CHARACTERISTICS OF PACKETS OF BLADES.

Experimental and computed results, included in Chapter 5, show good agreement for a single blade and also for two and three bladed packets with flexible roots. Comparison with other investigators was limited to the single
blade only. Here, very close agreement is shown between results obtained by the present method and those computed by classical analysis, suggested by Gorman (42, 1975) for a cantilever blade with rotational and transverse springs at the root.

In the case of shrouded blade packets the elements at blade-roots were identical to the one used in the single blade. All other elements had already been used and tested in packets with fully fixed blade roots. Hence it is concluded that the present method is satisfactory in determining vibration characteristics of shrouded blade packets with flexible roots. Mode-shapes, shear force, axial force and bending moment distributions were determined for the various blade packets considered. It is seen that the natural frequencies and also the values of bending moment shear and axial forces have been considerably reduced as a result of greater root flexibility.

A comprehensive study of the effect of blade-root flexibility in a packet of blades is presented. It is hoped that this study will help the designer to choose the type of blade-root joint to suit particular vibration requirements.

6.3 FREQUENCY INFERENCE DIAGRAMS

These diagrams help to predict the vibration characteristics of packets of blades from plots of uncoupled blade and shroud modes.
Diagrams for packets of blades with fully-fixed and flexible roots have been developed successfully.

In the case of a packet fully fixed at the root an uncoupled fully-fixed blade is analysed. In the case of a packet with flexible roots an uncoupled blade with identical root conditions to that of the packet is analysed. The analyses of uncoupled shrouds is independent of blade-root conditions.

The frequency-inference diagram is a powerful aid to the designer or investigator. The vibration characteristics of packets consisting of small or large numbers of blades can be predicted with a single inference-diagram.

6.4 FUTURE DEVELOPMENTS

There is considerable scope of extending further the investigation presented in this thesis.

The present analysis can be extended to include vibration problems of non-uniform, pre-twisted asymmetric beams and to plates and shells.

Also the effect of shroud-to-blade flexibility on the vibration characteristics of blade packets is a possible area of future research.
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