UNIVERSITY OF SURREY

"COLLAPSE ANALYSIS OF THIN WALLED STRUCTURES"

by

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Summary

A general purpose finite element computer program has been written for the large-deflection, elasto-plastic analysis of plate assemblages up to and including collapse. The approximate Ilyushin's yield criterion has been adopted, where sudden plastic flow in the full section depth of the plate is assumed. The program has been compared with other theoretical and experimental treatments, and gives acceptable results.

In the present work, the computer program has been used to study the interaction between the webs, flanges and diaphragm on the collapse behaviour of single cell rectangular box girders in the support region. A special beam element has been incorporated into the program to represent portions of the structure away from the support that may be assumed to behave in a linear elastic manner. The imperfection sensitivity of the box whose dimensions are shown to be in a critical range regarding interaction between two types of collapse has been established. Suggestions are made for a simplified two dimensional treatment to give safe values of collapse load involving the flange-diaphragm interaction.
TO MY WIFE, SON

AND THE MEMORY OF MY FATHER
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CHAPTER 1
INTRODUCTION

1.1 General Introduction

The objective of this dissertation is to develop a method for the study of the collapse behaviour of spatial plate assemblages. Extensive work has been carried out on the stability and collapse behaviour of individual plates or plate components (26-35, 39-41, 49-52, 65-66, 89-96, 123-126, 142-145) under in plane and lateral loading. It is well established that the behaviour of thin plates in compression is usually characterised by a considerable reserve of post buckling strength, of the order of three or four times the initial buckling strength. Therefore, the buckled form of the plate is generally stable.

This is not the case with cylindrical shell structures where the plate surfaces are curved, and the panel invariably unstable. A cylinder can be represented by an 'n' sided polygon, where n is a very large number. As the number 'n' becomes smaller, it is likely that some post-buckling reserve occurs. In most applications, these thin walled structures are likely to buckle locally at stresses well below yield, and able to carry higher loads by virtue of the post buckling strength of the component panels. As a unique case, it would be worth investigating box sections (n=4), particularly with relevance to steel box girder structures. The behaviour of these structures is very complex (26, 27, 52, 59), and the recent collapses (21, 98, 119, 122) have generated a great deal of interest in their behaviour.
In these box girder structures, high local deformations (particularly in the lateral direction) may be observed when they are loaded to collapse\(^{(39)}\). When some components collapse locally, they may influence overall deformation patterns through sympathetic rotational and membrane action in other parts of the structure, with collapse occurring when critical sections in the redistributed stress fields fail.

In order to obtain a realistic stress and deformation pattern of the structure, upto collapse, account must be taken not only of the large deflection characteristics but also of the plastic flow, which alters the stress-strain relationship and restricts the ultimate load carrying capacity.

Most of the theoretical work so far on the ultimate limit state of box girders has been largely confined to the strength of individual components, albeit under complex loading conditions\(^{(26,27,52,59)}\). Simplifying assumptions have therefore necessarily been made to the boundary conditions in order to obtain solutions. There are two major problems of component interaction which had not been studied till now. The first was the problem relating to component interaction at the support region, where it had yet to be established whether the ultimate strength of the structure was reduced by interaction of the elasto-plastic buckling of the compression flange and of the support diaphragm. This is investigated in some detail in this thesis. Interaction of the web with the other components at the support region is also taken into account. This problem of the interaction between webs and flanges is also currently under investigation at Imperial College, London.
Some early work on the latter problem has been carried out by Graves-Smith\(^{(58)}\) who looked at the large deflection elasto-plastic interaction of the webs and flanges of a box girder subjected to uniform bending moment, where he observed that the beam retained much of its stiffness after local buckling in the flange. Supple and Bunni\(^{(16,135)}\) carried out a finite element investigation into the elastic large deflection and buckling failure of thin walled box girders in cantilever. This work was, however, limited to initial buckling, and the complex interaction between instability and plastic deformation was not considered.

1.2 Non-linear elastic stability

A linearised analysis will yield the critical load of a structure in many important practical problems. However, the behaviour in the post critical range is essentially non-linear due to loss of stability of the original equilibrium configuration. Similarly, the effect of imperfections on the buckling behaviour can only be studied through a non-linear analysis.

Following the early investigations into stability of specific structural systems by, for instance, Bryan\(^{(14)}\), Southwell\(^{(130)}\), Reissner\(^{(112)}\), Cox\(^{(22)}\) and many others, Koiter\(^{(74)}\) presented in his doctoral dissertation (1945), a general theory of elastic stability in which he delineated for the first time the importance of post buckling characteristics. In the non-linear post buckled regions, he examined the characteristic features of three distinct branching points, namely asymmetric, stable symmetric and unstable symmetric, and further established the concept of imperfection sensitivity in relation to these critical points.
Since then, exploitation of the theory using generalised coordinates has been progressing at University College, London (133, 137, 142, 143). Detailed expositions on the general theory are published by Croll and Walker (30), Thompson and Hunt (136), and Supple (134), who discuss further coupled branching modes.

1.3 Method of analysis chosen

The analysis of these plated structures fall into two major classifications - analytical and numerical methods. In general, the former methods seek a closed form solution of the governing differential equation and are impractical for all but the simplest structures. Numerical methods fall into three main categories:

a) Energy methods, such as Rayleigh-Ritz, Timoshenko, Lagrangian multiplier, etc.;

b) Numerical solution of the governing equations, such as finite differences, Dynamic Relaxation, etc.;

c) Discrete element method, including the stiffness (displacement compatibility) and flexibility (force equilibrium) methods.

In the second category, an approximate solution of the governing equations is sought by means of a finite difference or numerical integration technique. The problem must then be such that these equations can be determined, and this clearly limits the generality of the approach. The finite difference
methods use geometric mesh discretisations, and may be summarised as methods for analysing smooth differentiable continua, but the boundary conditions require special treatment.

Since its origin, the discrete element method (finite element method) has been understood as a piecewise Rayleigh-Ritz procedure, where the strain energy of the structure is obtained by piecewise integration over the number of segments or 'elements' into which the structure is divided. These elements are interconnected only at 'nodes'. The variational procedure is carried out on the displacements when the total potential energy is minimised (displacement-compatibility method), and on the stresses when the complementary energy is minimised (force-equilibrium method). The behaviour of a single element having been formulated, the contribution from all the elements are added together and the resulting sets of equations are then solved for all the structural displacements or stresses, with the resulting solution converging to the correct solution as the number of variables is increased.

The finite element formulations for the analysis of continuum structures are extremely powerful, since they enable problems with complex geometry and boundary conditions to be evaluated. The stiffness (displacement compatibility) method, which is commonly preferred for programming on the computer, has been chosen for the present work because of its powerful capability and its generality.
1.4 Linear elastic analyses commonly used for single cell box girders

Until the collapse of steel box girders and the formation of the Merrison Committee of Inquiry in 1971, analytical techniques were largely confined to linear elastic solution. Apart from bending and shear stresses that are associated with simple beam theory, there are stresses due to transversely non uniform loading which cause cross-sectional deformation. This behaviour gives rise to longitudinal stresses due to non uniform warping, and to transverse flexural stress due directly to distortion. Warping stresses due to the torsional component of the load are negligible in comparison with warping associated with distortion. These stresses tend to reduce the advantages anticipated from the high torsional stiffness of the box girder.

Because the longitudinal curvature due to distortion is confined to the loaded region for box girders with large transverse stiffness, the warping stress is significantly reduced if the load is spread over a longer length. In the loaded region, distortional stress increases rapidly with span (or diaphragm spacing) until a limiting value is reached, while the warping stress increase is much slower.

An additional form of warping arises due to symmetric loading (shear lag), which is the effect of shear deformation in redistributing the bending stresses. It leads to a decrease in the longitudinal bending stress away from the webs.
Interior diaphragms are placed to reduce distortional stresses, but are less effective in reducing warping stresses. Kristek\(^{75}\) has shown that as transverse distortional stresses decrease rapidly with an increasing number of rigid diaphragms, the warping stresses decrease more slowly. He formulates an equation for the critical length between diaphragms, above which value they are ineffective in reducing warping stresses.

At the supports, the effect of restraining warping produces warping stresses in an opposite direction to the warping stress at the point of application of the torsional load.

An excellent review and bibliography of these analytical techniques from nearly 300 authors is published by Maisel\(^{81}\), albeit for concrete and steel structures. A few commonly used methods are discussed below.

1.4.1 *Richmond and others\(^{31,113}\)*

Richmond promulgated an approximate method known as the substitute beam theory where only single span rectangular beams of constant cross-section and stiffness could be considered. The webs are analysed as equivalent beams with sinusoidal antisymmetric loads simulating twisting moments. They are assumed to be connected top and bottom by a medium ensuring the correct lateral movement of the equivalent beams. Discrete diaphragms are 'smeared' as an equivalent continuous diaphragm medium.
An exact method of solution is also shown by Richmond, applicable to box girders with a vertical axis of symmetry, where variable wall thickness and second moment of area are allowable. Discrete diaphragms are allowed for. The flexibility and matrix displacement approaches are dealt with, where the former is a hand method and the latter suitable for use on a computer. The flexibility method ignores the effect of torsional shear forces and deflections, and is only applicable to determine external support systems. The matrix displacement method is more versatile.

1.4.2 Wright and others

Here, Wright noted the similarity existing between the differential equation for a continuous diaphragm medium of a box and a beam on elastic foundations. Deflections of the beam on elastic foundations (BEF) represent the distortional displacements, and moments in the BEF represent the warping moments. The foundation modulus of the BEF is analogous to the frame stiffness of the box under torsional load, and the BEF second moment of area to the warping rigidity of the box (i.e. restraint against warping).

This approach only considers the distortional component of loading, and all torsional effects are ignored. Also, the method of calculating the warping stiffness due to bending rotation is arbitrary and not derived from the theory of elasticity as by Richmond.
1.4.3 Folded plate analysis

In 1957, Goldberg and Leve\(^{(57)}\) described a general solution for folded plate structures, using Fourier series as components. This theory was later applied by Scordelis et al\(^{(35,128)}\), and Kristek\(^{(75)}\) for computational purposes. It involved the solution of four simultaneous equations at each longitudinal plate joint for each harmonic of the assumed displacement and force field. The solution is stopped when enough harmonics have been applied to give the desired accuracy.

Scordelis and others have analysed box girder structures with interior rigid diaphragms by modifying them into base structures as above. The redundant forces are the interaction forces between the box girder and the diaphragm, solved by analysing the structure without diaphragms, and equating the boundary conditions to the displacements at the points where the redundants are to act. These box girder diaphragms are assumed to be infinitely rigid in shear and completely flexible normal to their plane, and a force method of analysis is used. This approach becomes very complicated, however, for more than one internal diaphragm.

The disadvantage with the analysis is in the limitation to simply supported spans and determination of transverse normal stresses. This has been overcome by Scordelis and Lo\(^{(80)}\) by means of a 'finite segment analysis', where the ordinary beam theory used above was modified to include continuous supports and isolated supports or loads.
1.4.4 Other computational methods

These have already been mentioned in 1.3, the most commonly used being the finite difference, dynamic relaxation and finite element methods for box girder bridge problems. The latter method will be discussed further in the ensuing chapters, particularly with reference to geometric and material non-linearity. Some of the assumptions and limitations are therefore briefly discussed.

In a continuum structure such as flat plated structures, the continuum is artificially divided into a number of elements interconnected at discrete points (corners, specified points on the boundary, etc.) before matrix methods can be applied. This assumption means that continuity requirements are only satisfied at these 'nodal' points, and the relaxation of this continuity along the element sides would make the structure more flexible than it really is, giving discontinuities along the edges.

In the finite element method, individual elements are constrained to deform to specified patterns of deflection so as to satisfy some, if not all the continuity requirements along the boundaries. The shape and size of the elements depend on the problem being considered and by judgement from past experience on convergence of the solution. The accuracy of the solution increases with the number of elements taken in the structure, but so does the computational cost. By selectively providing more elements at regions of stress concentration, good convergence can be obtained.
1.5 **Interim design rules for steel box girder bridges**

At the time of the formation of the Merrison Committee, BS153 was the only standard design directive for steel bridges in the U.K., whose scope only extended to the superstructure of simply supported bridges of moderate spans. These rules were drafted after the failure of four bridges referred to in the Committee's report, and formed a framework to be included into the British Standard Bridge Code with scope for modifications as new research came to light.

Apart from the new loading clauses, requirements of welding, residual stresses, fatigue, strength and serviceability rules for webs, flanges and diaphragms in the Merrison Rules, the analysis section in Part II contains condensed rules to simplify the treatment of the more complex effects of torsion, distortion and shear lag. Whilst these rules may be adequate to cover the design of most simple structures, alternative sophisticated analytical treatment is not ruled out.

It is possible that with a better understanding of the individual and overall collapse behaviour of panels, some of the more conservative limitations on the stresses could be relaxed. Some of the design clauses could also be simplified once the grey areas of knowledge are better understood.
CHAPTER 2
DEVELOPMENT AND CHOICE OF THE FINITE ELEMENT PROCEDURES

Since the late 1960's, the mathematical literature on the finite element method has grown enormously. Several books and monographs (8, 132, 147) are devoted to the mathematical foundations of the method. A review paper by Oden (102) summarises some of the salient mathematical contributions. A large number of papers, proceedings of conferences and short courses (13, 45, 47, 55, 63, 108, 117, 121, 140, 154) and several books (37, 56, 86, 99, 103, 116, 159) have been published on the subject. A number of recent survey articles also serve as sources for additional references (54, 84, 146, 148, 157).

2.1 Choice of the Variational Principle

Most applications of the finite element method to solid mechanics problems rely on the use of a variational principle to derive the necessary element properties or equations. The three most commonly used variational principles in elasticity problems are the principle of minimum potential energy, the principle of complementary energy, and the Reissner Principle. The first two have already been mentioned in Chapter 1.

To invoke the Reissner Principle, the functional includes an assumed pattern of displacements as well as stresses. Hence this approach is known as a mixed method. Pian and Tong (106) have tabulated these and other variational bases of the finite element method in solid mechanics. Although the displacement method is the most widely used finite element
approach, the equilibrium, hybrid equilibrium and mixed methods are used to some extent. They are therefore briefly summarised below, with reasons given for using the displacement method in favour of the others.

2.1.1 Equilibrium method

The equilibrium method was due to Pian and Tong\(^{106}\) and Fraeijs de Veubeke\(^{46}\). The field variables are the stresses, where the assumed stress distribution satisfies stress equilibrium in the interior, but the corresponding displacements are not fully compatible with adjacent elements. Pian\(^{106}\) derived element stiffness matrices using the minimum complementary energy principle, and observed that by increasing the number of undetermined stress coefficients representing the stress components, the convergence criteria changes from an upper bound to a lower bound solution.

Morley\(^{90}\) formulated a triangular bending element with linearly varying bending moments within the element. Equilibrium conditions are satisfied as above for the normal bending moment, Kirchhoff normal force and twisting moments on the plate boundary. Morley, however, formulates a flexibility matrix.

Pian's complementary energy model for/stiffness matrix of an element has many of the advantages of the displacement model, obtained at the expense of a somewhat more complicated element analysis. The most important properties which are lacking are the lower bound to the direct influence coefficients
and the uniform convergence with decreasing mesh size. However, convergence to the correct result is assured. Elias (43) uses the analogy between plane stress and bending, and the unknown parameters are the nodal values of Southwell's stress functions U and V at the corner nodes. There are difficulties associated with representing loading and boundary conditions, due to lack of a clear interpretation.

2.1.2 Hybrid equilibrium method

The method was expanded by Pian and Tong (106), Pian (105), and Allwood and Cornes (1). Here the field variable model of the equilibrium method is still employed. However, the generalised displacements are chosen differently. The displacements at any point on the element boundary is approximated by an interpolation function in terms of the generalised displacements.

The generalised displacements are chosen independently from the generalised stress coordinates. Hence it is easier to ensure the larger number of generalised stress coordinates than the generalised displacements in order to avoid boundary displacement nodes that cause no stresses in the elements. Also, it is not necessary, as in the equilibrium formulation, to chose interpolation functions satisfying equilibrium since they do not have to satisfy surface tractions.

2.1.3 'Hellinger-Reissner' mixed method

Variations are made here of both the displacement and stress fields. One advantage of this method is the versatility of the associated Hellinger-Reissner functional. By using
various forms of this functional, problem statements requiring interelement continuity of different combination of displacement and stress parameters may be obtained (106). Herrmann (60) has used the interelement continuity of displacement and normal bending moment for the plate bending problem.

Another feature of this method is that neither the equilibrium equations, nor the stress-strain-displacement equations need be satisfied within the element. Fraeijs de Veubeke (46) has shown, however, that this principle does not generally offer any advantage over either the principle of minimum potential energy or the principle of minimum complementary energy.

2.1.4 Comparison of the methods

For most problems in structural analysis, it is principally important to obtain the stresses. Displacement compatibility (the principle of stationary potential energy) does not give smooth stresses across interelement boundaries. Therefore the fact that the other methods mentioned above give a more accurate stress representation would favour the use of these approaches in preference to the displacement method.

In spite of this phenomenon, the displacement method still remains very widely used because the other methods result in a larger number of equations and a larger band-width (46). Also, satisfying equilibrium is not always as easy as formulation of displacement compatible models. Finally, the stiffness matrix formed in the mixed method is not positive definite, and is badly suited for computer analysis in comparison with that obtained from the displacement method.
The equilibrium method gives lower bounds to the element stiffness (and therefore upper bound to displacements), whereas the displacement method gives upper bounds to the stiffness (lower bound to the displacements). The mixed and hybrid methods should therefore give solutions somewhere in between. Desai and Abel\(^{(37)}\) have tabulated the performance of all these methods for a plate bending problem as presented by various authors.

A significant reason for not using mixed or hybrid methods is that very few attempts have been made so far to extend these models to deal with non-linear problems, due to the lack of incremental formulations of the associated variational principles. Horrigmoe and Bergan\(^{(67)}\) have recently described the details of the various incremental variational principles which have been used for large deflection problems and can be extended to include material non-linearity.

### 2.2 Linear Elastic Analysis and the Displacement Function for Three Dimensional Plate Problems

The basic step in the finite element analysis is the unique description of the displacement function (unknown) within each element in terms of parameters associated generally with the values of this function at specified boundary points or internal points of the element.

i.e. \( \{ \delta(x,y) \} = [N] \{ \delta^e \} \)

where \([N]\) is defined as the shape function.
These displacement functions should include

1) All rigid body motions without self straining;

2) Continuity of the function between adjacent elements;

and 3) The ability to represent a constant strain state.

The simplest forms of elements in general use are triangular and rectangular (or generalised quadrilateral). The use of flat plate elements for folded plate and general shell analyses was initiated by Zienkiewicz and Cheung (153), and by Clough and Tocher (20), using rectangular elements. Satisfactory bending stiffness matrices for triangular elements were later developed by Zienkiewicz et al (9) and by Clough and Tocher (19), which enabled triangular folded plate and shell elements to be developed. Tocher (19) used the complete 10 term polynomial displacement expression which he reduced to a 9-degree of freedom system by the Ritz method, obtaining an over flexible element. His earlier element used a 9-degree of freedom system where symmetry was maintained by combining two of the cubic terms in the displacement function. This element, however, gives a singular transformation matrix for certain variations of the element sides with respect to the coordinate axes.

Zienkiewicz et al (9) formed a satisfactory stiffness matrix for the triangular bending element with 9 degrees of freedom by using area coordinates, which are intrinsically related to the element geometry, removing the problem of
invariance. It is a complex function, but fulfils the constant strain condition and has compatibility of displacements but not of slopes. Both authors used the constant strain triangular element with their respective bending elements for the shell analysis.

On the other hand, rectangular bending elements have a simplicity of approach, where the 12-degree of freedom system, although incomplete (15 complete terms for the quartic polynomial), preserves symmetry and possesses geometric isotropy (property of invariance, where the polynomial expansion for the element remains unchanged under linear transformation from one cartesian coordinate system to another). This is achieved by dropping terms that occur in symmetrical pairs. Also, rectangular (or generalised quadrilateral) elements are more amenable to the numerically integrated isoparametric family of elements allied to the natural coordinate system\(^{(158)}\).

With the development of linear strain elements\(^{(5,46)}\) it was soon observed that if the number of degrees of freedom associated with an element is increased while keeping the total number of structural freedoms the same, better results are obtained. Guided largely by physical intuition, elements with more nodes and/or degrees of freedom per node developed, and the parametric representation of strain, stress, or both became more complex. These higher order elements were soon accompanied by criteria for evaluating element performance.
The relative merits of different finite element functions were then debated by various authors by evaluating convergence characteristics. Convergence of energy is assumed to be typified by performance of point displacement but seldom by stresses. The merits of eigenvalue traces (sum of the eigenvalues, where the best strain element has the lowest trace), compatibility or equilibrium across adjacent element boundaries and the need to represent zero strain modes have been discussed \(^{(7, 19, 116)}\). Lately, Irons and Razzaque \(^{(70)}\) introduced the patch test, where a patch of elements is analysed, where the nodes on the extremity of the patch are given displacements that are consistent with a state of constant strain. If the stresses obtained are equal, then it is assumed that any patch of the same elements will behave well in a general structure with a constant stress field.

Zienkiewicz \(^{(159)}\) has shown that triangular plate bending elements with complete continuity criteria have worse convergence characteristics than those with only displacement continuity (non-compatible elements). Also, the Zienkiewicz \(^{(159)}\) rectangular plate bending element has been widely used in spite of its restricted continuity of displacement (and not the derivatives), providing satisfactory solutions.

Desai and Abel \(^{(37)}\) have collected and presented the results of a number of authors who compared the performances of different plate bending elements. This comparison is based upon the solution of two fundamental problems - a centrally loaded square plate with edges either simply supported, or
clamped. Such comparisons reveal the diverse behaviour of element types and individual performance. In the present work, the non conforming element chosen is shown to converge with increasing fineness of mesh. However, although the displacement model (minimisation of the potential energy) should give a lower bound to the correct displacements, the restricted continuity requirements make the element overflexible, so that the net effect on the displacements is an upper bound. The percentage error is shown to be lower than with conforming (bending) elements.

As for the rectangular membrane element, Wilson et al modified the lower order membrane displacement model having a bi-linear displacement, where the convergence was slow. They concluded that the source of slow convergence was the bi-linear shear strain, and by making it constant within the element, the speed of convergence was dramatically accelerated.

Recently, Robinson, in his single element test, tested a single Wilson membrane element against various loading configurations and obtained all but one result to within 94% of their true values. The only discrepancy was found for an element loaded as a cantilever, where for aspect ratios greater than 2 or 3, the vertical displacement was a constant 70% of the true value. The cause of this discrepancy is that the element displacement function produces a constant in plane moment only, whereas the applied loading gives a linearly
varying moment. This deficiency can be overcome by network refinement, although the safest would be that which could satisfy the single element test, where the assumed displacement function would give a bi-linear stress variation within the element.

For the problems at hand, it appears that if the chosen rectangular element fits the given geometry of the structure to be analysed, satisfactory results can be obtained. Also, in order to represent the buckled mode and the spread of plasticity accurately, the structural geometry has to be represented by a fine mesh network of elements, thereby reducing the approximations inherent in the displacement formulation.

2.3 Non-Linearity

Two types of non-linearity were widely recognised. In the first class (the large displacement problem), attempts were made to allow for the effects on both the equilibrium and strain displacement relations of changes in geometry. In the second class (material non-linearity), are the effects of creep and plasticity. Only plasticity is, however, of relevance in this dissertation. All the techniques so far to solve these non-linear problems rely heavily on the linear small displacement formulation, where the basic approach is to discretise the problem into a series of incremental linear steps.
Initially, large displacement problems were solved simply by updating the elastic stiffness with each increment with a 'moving coordinate' approach, where the coordinates were updated to the new position. A second order theory to account for geometric influences was first proposed by Martin et al (141) who referred to it as the 'initial stress stiffness matrix'. Argyris (3) named it the 'tangent stiffness', and others 'geometric stiffness matrix'. Subsequently, this has been extended to include the effects of geometry on loading as well. This large deflection theory provided a basis at first for investigating classical complex instability problems (85). Later, it was extended to other large deformation problems by Argyris (4) and other authors.

Geometric non-linearities may enter the formulation as a result of

1) Deformation on the equilibrium equations;

2) Product terms in the strain displacement relationship;

3) Deformation on the size and shape of the elements.

The first two effects are associated with large deflection problems only (linear elastic material response and small engineering strains).

The geometric stiffness matrix (initial stress matrix) represents the influences on the equilibrium equations of the initial unperturbed stress due to rotations induced by increments in the generalised nodal displacements in the
overall stiffness matrix. These effects are included in classical stability analyses of structures with no appreciable prebuckling deformations and do not account for large deflection behaviour.

Although some authors\(^{16,95}\) have used updated local coordinate systems to represent the large displacement effects, for moderately large displacements that occur in structural plates, this has been found not to be necessary\(^{24,82,109}\) when the initial displacement matrix (coupling between the quadratic and linear terms in the strain displacement expression) is included. The earlier authors retained the quadratic terms in nodal displacements in the strain energy expression, but discarded the higher order terms. The quadratic terms led to the initial stress matrix, whereas Marcal\(^{82}\) presented an alternative development which retained these higher order terms, leading to a hierarchy of stiffness matrices which he called the initial displacement stiffness matrices.

Because the change in geometry is accounted for in the strain displacement relationship, there is no need to update the geometry of the structure after each load increment. This procedure of including the initial stress and initial displacement matrices without updating the coordinates (Lagrangian system) has therefore been adopted in the author's formulation. It may, however, be necessary to update the geometry if very large deformations are to be considered, where the slopes begin to approach 5 degrees.
The general isoparametric concept evolved by Irons (69) and Zienkiewicz (158) in which both geometric shape and internal elastic behaviour is interpolated with identical shape functions associated with numerical integration techniques, led to an efficient and systematic treatment of non-linear effects of plate and shell analysis. The relative and simple ease of application of this procedure on computers as opposed to explicit integration, especially when geometric and material non-linearities (including the higher order initial displacement matrix) are considered, makes it a very attractive proposition, and is used in the formulation.

Non-linear effects due to non-linear constitutive relations have been looked at by numerous authors, amongst whom Pope (107) and Marcal (83) formed the tangent stiffness approach to be followed by both the initial strain and initial stress (156) methods. They all used variations of the Newton-Raphson iterative scheme, and considerable work has gone into improving the rate of convergence. Following this work, a general treatise on the material non-linearity problems was given by Oden (101) where he expounded the possibility of combined material and geometric non-linearities. Zienkiewicz (160) has shown that a properly formulated elasto-viscoplasticity approach is consistent with physical reality and is computationally efficient. The extremes represented by non-linear constitutive relations can be treated in a unified way.

Most authors have used the Prandtl Reuss stress strain relationship for von Mises solids. The Tresca criterion has also been studied by Levy (76). Nayak and Zienkiewicz (97) have
also considered isotropic and kinematic strain hardening. The present treatment is for structural steels which have negligible strain hardening, and so a perfectly plastic (ideally elastic) stress strain relationship is assumed.

The yield function adopted for the present analysis is due to Ilyushin\(^{(68)}\), and was developed for the case of thin shells obeying von Mises yield criterion. Yield is determined by stress resultants rather than stresses (as in von Mises solids), where sudden plastification of the full section depth is assumed. Therefore, surface yielding is ignored in the stress strain laws. The gradual spread of plasticity through the depth of the plate is normally treated using the full von Mises approach, using a layered analysis (with numerical integration stations within the plate depth as well as the surface area). However, since considerably less computer storage is required with the Ilyushin approach, there is a significant advantage in its use. Crisfield\(^{(24)}\) has successfully used this approach, and the treatment is duplicated here due to its success, so that the complex component interaction problem can be solved on the computer.

Although the Ilyushin yield function was derived on the basis of a von Mises solid, and uses the Prandtl Reuss flow rule, the resulting solution is an approximation. However, Robinson\(^{(115)}\) concluded that it is a very good approximation to the exact von Mises solution, and is superior to other
linear approximations. Although Ilyushin's theory is based on deformation theory, it is still applicable when a flow rule is used. Some theoretical objections (129) to non-linear analysis based on deformation theory exist, but Crisfield (24) has shown that the use of such analysis in engineering computations is in agreement with experimental work.

2.4 Non-Linear Solution Procedure

Oden (100), amongst others has reviewed various incremental procedures as well as alternative non-linear formulations such as the Newton-Raphson and modified Newton-Raphson procedures and other gradient methods.

The Newton-Raphson method is one of the most reliable methods of solving systems of non-linear equations, and it is possible to estimate the rate of convergence, existence of solutions, and to find multiple solutions using this method. Brebbia and Connor (12) used the method to study stability and geometrically non-linear behaviour of arbitrary shells. They refer to the incremental formulations used by Argyris (6) and others as 'one step Newton-Raphson procedures with no corrective cycling'. They employ a mixed procedure where incremental loading is used for three steps and Newton-Raphson is introduced to provide successive corrections.

The Newton-Raphson procedure has proved to be very satisfactory for non-linear problems, and can be applied to determine unstable equilibrium configurations. However, although the rate of convergence is very high, it is computationally time consuming, since the total stiffness
matrix has to be solved for every iteration within an increment. The modified Newton-Raphson procedure, where the stiffness matrix is kept constant for certain iterations or load increments, is often preferred, even though convergence is relatively slower. Authors such as Thurston (138) and Brebbia (12) have suggested different schemes for improving convergence of the modified Newton-Raphson procedure.

One of the more popular schemes for assessing convergence, used by several authors (12, 24), is the Euclidean norm,

\[ \| \delta \|^2 = \| \delta \|_2^2 \].

This method is used in the author's formulation.

It may well be that a judicious combination of the Newton-Raphson and incremental schemes will give optimum economy. Therefore, the scheme adopted by the author allows any one of a combination of the schemes to be chosen while incrementing the loads or deflections. The modified Newton-Raphson procedure adopted allows the stiffness matrix to be updated only at the beginning of each increment, subsequent iterations being carried out using this updated matrix. The whole procedure is thus simplified so that changes from one scheme to another can be easily effected from one increment to another.
3.1 Introduction

The analysis procedure utilises the well known displacement formulation and provides a highly accurate mathematical model for general application to three dimensional spatial plate assemblages. The plates are capable of membrane and flexural action, and flat shell rectangular elements are used. Modifications can be performed to the elements to generalise the rectangular element into a quadrilateral element. In the present work, the quadrilateral element was not required and so in order to economise on the computational operations, these modifications were not made.

The plasticity formulation uses an approximate yield criterion given by Ilyushin\(^{(68)}\), which relates to the six generalised stress resultants in the shell element \((N_x', N_y', N_{xy}', M_x', M_y', M_{xy}')\), the assumption being that the full section of the shell undergoes plasticity, similar to that frequently assumed in bending of plates\(^{(2,10)}\). The approximation is considered worthwhile, since appreciable savings in computer time are accomplished and the results obtained for some problems with known solutions were sufficiently accurate.

3.1.2 Basic assumptions

The approximations are based on the assumptions that

1) The shell is thin;
2) Plane sections remain plane - extension of the Bernoulli-Euler-Kirchoff hypothesis of beam theory;

3) The transverse normal stress (shear deformation) is negligible;

4) A Lagrangian (initial coordinate) system is used. This is valid so long as the slopes $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ in the shell are small ($<<1$);

5) The material stress-strain relationship is linear elastic and perfectly plastic (no strain hardening).

The first assumption does not permit transverse shear deformation of the shell, which is included only in thick plate theory. The second assumption allows normals to the mid-plane of the shell to remain normal to it, with no change in length under deformation. Therefore, all strain components normal to the mid-surface vanish. Consequently, no shear deformation occurs.

The fourth assumption permits the referencing of all derivatives and calculations to the original undeformed configuration, thereby allowing only moderately large deflections (small slopes). For the last assumption, since isotropic material is assumed, only the Young's modulus $E$, Poisson's ratio $\nu$ and yield stress $\sigma_0$ are required to specify its behaviour.
3.2 Displacement Functions

The displacement functions are as used by Crisfield(24,25). These are basic functions as used by a number of authors, and were chosen in order to reduce the number of degrees of freedom and the volume of integration that would be necessary with more refined functions. This sacrifice was necessary if complicated problems involving the interaction of plate assemblages necessitated a large number of elements in order to model the buckled shape of the panels accurately. Also, by providing a larger number of smaller elements than normally used in linear elastic analyses, the solution converges to that given by higher order functions.

3.2.1 Membrane behaviour

The simplest (first order) rectangular element(24,25,150,159) is based on taking the corner nodes (see Fig. 1) and working only with displacements as nodal quantities. The dimensionless local cartesian coordinates (or normalised coordinates, $\xi, \eta$) that are used henceforth (with limits of $\pm 1$) are defined as:

$$
\xi = (x - x_c)/a, \quad d\xi = dx/a \quad ......3.2.1.1
$$

$$
\eta = (y - y_c)/a, \quad d\eta = dy/b
$$

The area integral is defined as

$$
\iint_A g(x,y)dA = \int_{-1}^{+1} \int_{-1}^{+1} g(\xi,\eta)d\xi d\eta \quad ......3.2.1.2
$$

Bilinear in plane functions for the element are generated by evaluating

$$
\delta(\xi,\eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi \eta \quad ......3.2.1.3
$$
at the nodal points and solving for the \( \alpha \). We get

\[
\begin{align*}
\mathbf{u}(\xi, \eta) &= \{N\}^T \{\mathbf{u}\}^e \\
\mathbf{v}(\xi, \eta) &= \{N\}^T \{\mathbf{v}\}^e
\end{align*}
\] ....3.2.1.4

where \( \{\mathbf{u}\}^e,\{\mathbf{v}\}^e \) are the vectors of displacements at the element nodes and

\[
N_i = 1/4(1+\xi \xi_i)(1+\eta \eta_i) \\
\] ....3.2.1.5

where \( \xi_i, \eta_i \) are their respective nodal values for \( i = 1,4 \)

i.e.  \( \xi_i = -1,1,1,-1 \)

\( \eta_i = -1,-1,1,1 \)

These displacement functions satisfy the compatibility conditions for a general quadrilateral. However, the constant strain criterion is satisfied only for rectangular elements and not for a general quadrilateral, as can be seen in the strain matrix (strain-nodal displacement relationship) in 3.3.1.

3.2.2 Flexural behaviour

The displacement function is the well known non conforming restricted quartic polynomial for a quadrilateral chosen by Zienkiewicz and Cheung. The displacement function in normalised coordinates is given by

\[
w = a_1 + a_2 \xi + a_3 \eta + a_4 \xi^2 + a_5 \xi \eta + a_6 \eta^2 + a_7 \xi^3 + a_8 \xi^2 \eta
\]

\[
+ a_9 \xi \eta^2 + a_{10} \eta^3 + a_{11} \xi^3 \eta + a_{12} \xi \eta^3
\] ....3.2.2.1

where \( a_1 \) to \( a_{12} \) are arbitrary coefficients.
The relationships between the normalised and cartesian coordinates are given in 3.2.1.

Evaluating the expression 3.2.2.1 at the nodal points and solving for \( a \) gives the shape functions derived by Melosh\(^{(88)}\)

\[
\begin{bmatrix}
\theta_x(\xi, \eta) \\
\theta_y(\xi, \eta)
\end{bmatrix} = \{N\}^T \begin{bmatrix}
w \\
\theta_x \\
\theta_y
\end{bmatrix} \quad \ldots \ldots 3.2.2.2
\]

where \( w \) = lateral plate displacement, \( \theta_x = -\frac{\partial w}{\partial y} \), \( \theta_y = \frac{\partial w}{\partial x} \)

(for sign convention, refer to Fig. 2)

and the shape function for any node \( i \) (with coordinates at centroid) is

\[
\{N_i\} = \frac{1}{8} \begin{bmatrix}
(1+\xi_i \xi)(1+\eta_i \eta)(2+\xi_i \xi+\eta_i \eta-\xi^2-\eta^2), \\
\beta \eta_i (1+\xi_i \xi)(1+\eta_i \eta)(1-\eta^2), \\
\alpha \xi_i (1+\xi_i \xi)(1+\eta_i \eta)(\xi^2-1)
\end{bmatrix}
\]

\( \xi_i \) and \( \eta_i \) are defined in 3.2.1.

The displacement functions satisfy the compatibility conditions for a general quadrilateral. When used as a rectangle or parallelogram, it could model constant curvature, but not when used as a general quadrilateral.

3.3 Strain Matrices

Differentiation of the chosen shape functions (derived from the displacement functions) leads to the strain matrices,
namely \([H]\), the in-plane strain matrix, \([F]\), the bending curvature matrix, and \([G]\) the slope matrix which is due to large deflection effects such as \(\frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2\), etc. These are the strain-nodal displacement relationships, and are derived only for rectangular elements.

3.3.1 Strain matrix (in-plane)

Differentiation of equation 3.2.1.4 leads to:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = \begin{bmatrix}
[H] \\
[u] \\
[v]
\end{bmatrix}
\]

\[\cdots 3.3.1.1\]

where \([H]\) is the strain matrix. This has, however, been modified so that the shear stress is constant over the element and equal to its centroidal value\((25,150)\). This modification is necessary because otherwise the element becomes too stiff in shear. Also, the convergence in the analysis of in plane bending problems is poor. The modified strain matrix is given by

\[
[H] = \begin{bmatrix}
1/a\{1/4\xi_1(1+\eta_1\eta)\}^T & 0 \\
0 & 1/b\{1/4\eta_1(1+\xi_1\xi)\}^T \\
\{\xi_1\}^T & \{\eta_1\}^T
\end{bmatrix}
\]

\[\cdots 3.3.1.2\]

where \(\xi_1, \eta_1\) are their respective nodal values for \(i = 1,4\)

i.e. \(\xi_1 = -1,1,1,-1\)

\(\eta_1 = -1,-1,1,1\)
3.3.2 Curvature matrix

Double differentiation of the restricted cubic shape function 3.2.2.2 gives curvatures $\chi_x$ and $\chi_y$ that vary linearly with $x$ and $y$. However, the twisting curvature variation is quadratic, and inconsistent with the lower order (four point Gauss) numerical integration adopted for the stiffness matrix. The curvature term can be reduced simply by under integrating the twisting strain energy with the desired two by two Gaussian integration scheme, instead of the three by three scheme strictly required. However, Crisfield\(^{(25)}\) has modified the curvature matrix at source using Reissner's variational principle. The effect is the same as that obtained if the quadratic distribution is replaced by a linear distribution that coincides with the former at four two point Gauss stations. This modification has been used in the present formulation, so that

$$\begin{bmatrix}
\chi_x \\
\chi_y \\
\chi_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
\frac{2\partial^2 w}{\partial x \partial y}
\end{bmatrix} = [F] \begin{bmatrix}
w \\
\theta_x \\
\theta_y
\end{bmatrix}$$

\[\ldots 3.3.2.1\]

where $\theta_x = \frac{\partial w}{\partial y}$, $\theta_y = \frac{\partial w}{\partial x}$.

and $[F] = \begin{bmatrix}
\{\frac{3}{4a} \xi_1 \xi (1+\eta_1 \eta)\}^T & 0 & \{\frac{1}{4a} (1+\eta_1 \eta) (3\xi+\xi_1)\}^T \\
\{\frac{3}{4b} \eta_1 \eta (1+\xi_1 \xi)\}^T & \{\frac{1}{4b} (1+\xi_1 \xi) (3\eta+\eta_1)\}^T & 0 \\
\{\frac{1}{2ab} \xi_1 \eta_1\}^T & \{\frac{1}{2a} \xi_1 \eta_1 \eta\}^T & \{\frac{1}{2b} \xi_1 \eta_1 \xi\}^T
\end{bmatrix}$

\[\ldots 3.3.2.2\]
The reduced terms are effectively obtained by replacing the quadratic terms $\xi^2$ and $\eta^2$ in the twisting strain energy:

$$\{x_{xy}\} = \left[\begin{array}{c}
\frac{-1}{4ab}\xi_i\eta_i(3\xi^2+3\eta^2-4) \\
\frac{1}{4a}\xi_i(3\xi^2+2\xi_i\xi-1) \\
\frac{1}{4b}\eta_i(3\xi^2+2\xi_i\xi-1)
\end{array}\right]^{T} \left[\begin{array}{c}
w \\
o_x \\
o_y
\end{array}\right]$$

by the constant 1/3, the values at the four point integration stations.

### 3.3.3 Slope matrix

This is obtained by differentiating the restricted cubic shape function 3.2.2.2, which gives slopes as high as cubic.

$$\text{Now } \{\theta\} = \left[\begin{array}{c}
\frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial y}
\end{array}\right] = [G] \left[\begin{array}{c}
w \\
o_x \\
o_y
\end{array}\right]$$

In order to make the matrix more consistent with the bilinear variation assumed for the in plane effects, these terms should be reduced (using four Gauss stations), to give linear variation in the $x$ and $y$ directions. The effect is to reduce the cubic terms $\xi^3$ and $\eta^3$ by linear terms $\xi/3$ and $\eta/3$ respectively, and the quadratic terms by the constant 1/3 as before. The full expression for the strain matrix is as given below, but has been modified (3.3.3.3) for application to the computer program.
An explicit expression has been derived (see ref 29) for the strain matrix using the above mentioned smoothing techniques and is given below:

\[
\begin{bmatrix}
{-\frac{1}{8a\eta_1}(1+n_1\eta)(3\xi^2+n^2-\eta_1\eta-3)}^T
& {\frac{b}{8a\eta_1\xi}(1+n_1\eta)(1-\eta^2)}^T
& {\frac{1}{8}(1+n_1\eta)(3\xi^2+2\xi_1\xi-1)}^T
\\
{-\frac{1}{8b\eta_1}(1+\xi_1\eta)(3\eta^2+\xi^2-\eta_1\eta-3)}^T
& {\frac{b}{8b\eta_1}(1+\eta_1\xi)(3\eta^2+2\eta_1\eta-1)}^T
& {\frac{a}{8b\eta_1}(1+\xi_1\eta)(1-\eta^2)}^T
\\
{\frac{1}{12\eta_1}(4\eta_1\xi+3)}^T
& {\frac{b}{12\eta_1\xi}(1+n_1\eta)}^T
& {\frac{\xi_1}{4}(1+n_1\eta)}^T
\end{bmatrix}
\]

......3.3.3.2

3.4 Tangential Elasto-Plastic Modular Matrices

In the absence of yielding, the elastic modular matrix for an isotropic plate is given by:

\[
[D] = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]

......3.4.1

The incremental elastic stress strain laws are given by Hooke's Law as

\[
\{\Delta N\} = t[D]\{(\Delta\varepsilon_t) - (\Delta\varepsilon_p)\}
\]

......3.4.2

\[
\{\Delta M\} = t^3/12[D]\{(\Delta\chi_t) - (\Delta\chi_p)\}
\]

where \{\Delta N\} and \{\Delta M\} are the incremental generalised stress resultants, defined as
\{\Delta N \} = \int_{-t/2}^{t/2} \{\Delta \sigma \} \, dz \quad \text{and} \quad \{\Delta M \} = \int_{-t/2}^{t/2} z \{\Delta \sigma \} \, dz

\{\Delta \varepsilon_\text{t} \}, \{\Delta \chi_\text{t} \} = \text{total strain and curvature increments respectively.}

\{\Delta \varepsilon_\text{p} \}, \{\Delta \chi_\text{p} \} = \text{plastic strain and curvature increments respectively.}

and \( t = \text{plate thickness.} \)

For plasticity, Crisfield\(^{(24)}\) has exploited Ilyushin's\(^{(68)}\) deformation theory which is based upon the quadratic stress intensities \( N_e, M_e \) and \( MN_e \). Robinson\(^{(115)}\) concluded that it should still be applicable when a flow rule is employed. This approach has been used here. The approximate yield criterion so derived is given by:

\[
f = \frac{N_e}{t^2 \sigma_o} + \frac{4s M N_e}{\sqrt{3} t^3 \sigma_o} + \frac{16M_e}{t^4} \leq 1 \quad \ldots \quad 3.4.3
\]

where \( N_e = N_x^2 + N_y^2 - N_x N_y + 3N_{xy}^2 \)

\[
M_e = M_x^2 + M_y^2 - M_x M_y + 3M_{xy}^2 \]

\[
MN_e = M_x N_x + M_y N_y - \frac{1}{2} M_x M_y - \frac{1}{2} M_y M_x + 3M_{xy} N_{xy} \]

\[
s = \frac{MN_e}{|MN_e|} \]

For plastic flow to occur, the generalised stress resultants must remain on the yield surface and \( \delta f = 0 \).

\[
\delta f = \left\{ \frac{1}{2} \frac{3N_e}{2N} \frac{e}{t^2} + \frac{2s}{\sqrt{3} t^3} \left( \frac{e}{3N} \right)^T \{\Delta N \} + \frac{2s}{\sqrt{3} t^3} \left( \frac{e}{3N} \right)^T \{\Delta N \} + \frac{2s}{t^4} \left( \frac{e}{3N} \right)^T \{\Delta N \} + \frac{16}{t^4} \left( \frac{e}{3N} \right)^T \{\Delta N \} \right\} = 0 \quad \ldots \quad 3.4.5
\]
In order to avoid discontinuity in the partial derivative of \( f \) as \( MN_0 = 0 \), \( s \) is made zero when \( MN_0 < 10^{-4} \sqrt{3} \frac{\sigma_{\theta}}{10} \), (ie very small).

Assuming that 3.4.3 is a plastic potential so that the plastic strain rates are proportional to its partial derivatives,

\[
\{\Delta e_p\} = \lambda \{f_v\} \tag{3.4.6}
\]
\[
\{\Delta \chi_p\} = \lambda \{f_m\} \tag{3.4.7}
\]

where \( \{f_v\} = \frac{1}{t^2} \left[ \frac{2s \partial N}{\partial N} + \frac{2s \partial M}{\partial M} \right] \) \tag{3.4.64}

\[
\{f_m\} = \frac{2s \partial N}{\sqrt{3} t^3 \partial N} \left[ \frac{16}{t^4} \frac{\partial M}{\partial M} \right] \tag{3.4.7}
\]

\( \lambda \) maybe defined as a plastic strain rate multiplier, which when assuming a negative value, signifies unloading from the yield surface.

Making use of 3.4.2, 3.4.6 and 3.4.5, we obtain

\[
\lambda = \frac{1}{(m+n)} (t \{f_v\} T [D]\{\Delta e_t\} + \frac{t^3}{12} \{f_m\} T [D]\{\Delta \chi_t\}) \tag{3.4.8}
\]

where \( n = t\{f_v\} T \{f_v\} \) \tag{3.4.9} \]

\[ m = \frac{t^3}{12} \{f_m\} [D]\{f_m\} \]

Therefore, the plastic strain increments maybe related to the total strain increments as

\[
\{\Delta e_p\} = \frac{1}{(m+n)} \left\{ t[G_N][D][\Delta e_t] + \frac{t^3}{12} [G_M][D] \{\Delta \chi_t\} \right\} \tag{3.4.10}
\]

\[
\{\Delta \chi_p\} = \frac{1}{(m+n)} \left[ G_{NM} \right] T [D] \{\Delta e_t\} + \frac{t^3}{12} \left[ G_M[D] \Delta \chi_t \right] \tag{3.4.11}
\]

where

\[
[G_N] = \left\{ f_v \right\} \left\{ f_v \right\} T \tag{3.4.12}
\]

\[
[G_M] = \left\{ f_m \right\} \left\{ f_m \right\} T \tag{3.4.13}
\]

\[
[G_{NM}] = \left\{ f_v \right\} \left\{ f_m \right\} T \tag{3.4.14}
\]
substituting 3.4.10 into 3.4.2 yields the following incremental elasto-plastic stress strain laws.

\[
\{\Delta N\} = [C^*]\{\Delta \varepsilon_t\} + [cd]\{\Delta \chi_t\}
\]

\[
\{\Delta M\} = [cd]^T\{\Delta \varepsilon_t\} + [D^*]\{\Delta \chi_t\}
\]

where \([C^*]\), \([D^*]\), and \([cd]\) are the tangential elasto-plastic modular matrices, given by:

\[
[C^*] = t[D] - \frac{t}{(m+n)} [D][G_N][D]\]

\[
[D^*] = \frac{t^3}{12}[D] - \frac{t^3}{12(m+n)} [D][G_M][D]
\]

\[
[cd] = \frac{-t^4}{12(m+n)} [D][G_{NM}][D]
\]

These tangential elasto-plastic modular (3x3) matrices are each function of the current stress resultants \((N_x, N_y, N_{xy}, M_x, M_y, M_{xy})\).

In the absence of plasticity,

\[
[C^*] = t[D]
\]

\[
[D^*] = \frac{t^3}{12}[D]
\]

and \([cd]\) is non existent.

For consistency, these modular matrices are derived and used at each Gauss point of each element.

3.5 Geometric non linearity (moderately large displacements)

Geometric non linearity is treated by allowing standard linear forms to be used in an iterative way to obtain the solutions.

When displacements are not infinitesimal, but also not excessively large (i.e. \(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \ll 1\)), the lateral displacements...
are responsible for developing membrane strains, and there is a cross coupling of the two actions. The strains are to be defined in terms of mid-surface displacements. Therefore, the displacements at any point in the plate depth is:

\[ u_z = u - z \frac{\partial w}{\partial z} \]  
\[ v_z = v - z \frac{\partial w}{\partial y} \]  

where \( u \) and \( v \) = mid surface displacements  
\( u_z \) and \( v_z \) = displacements at distance \( z \) from mid surface.

The strains may be written as:

\[ \{\varepsilon\} = \{\varepsilon_1\} + \{\varepsilon_\chi\} + 3\{\chi\} \]  

where

\[ \{\varepsilon_1\} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \]  

\[ \{\varepsilon_\chi\} = \begin{bmatrix} \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x}\right) \left(\frac{\partial w}{\partial y}\right) \end{bmatrix} \]  

and \( \{\chi\} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \]

The second term is the non-linear term, and \( u, v \) and \( w \) are the appropriate displacements at the mid-surface. The non-linear strain components (3.5.4) can be written as:
\[
\{\varepsilon_x\} = \begin{bmatrix}
\frac{\partial w}{\partial x} & o \\
0 & \frac{\partial w}{\partial y} \\
\frac{\partial w}{\partial y} & \frac{\partial w}{\partial x}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial y}
\end{bmatrix}
= \frac{1}{2} [A] \{\theta\} \quad \ldots \quad 3.5.6
\]

Taking the variation of 3.5.6 we have

\[
\Delta\{\varepsilon_x\} = \frac{1}{2}[\Delta A]\{\theta\} + \frac{1}{2} [A]\{\Delta\theta\} = [A]\{\Delta\theta\} \quad \ldots \quad 3.5.7
\]

where

\[
\{\Delta\theta\} = \begin{bmatrix}
\frac{\partial \Delta w}{\partial x} \\
\frac{\partial \Delta w}{\partial y}
\end{bmatrix} \quad \ldots \quad 3.5.8
\]

The total incremental strain is therefore given by:

\[
\{\Delta\varepsilon\} = \{\Delta\varepsilon_\parallel\} + [A] \{\Delta\theta\} + \{\Delta\varepsilon_x\} + z \{\Delta\chi\}, \quad \ldots \quad 3.5.9
\]

where \{\Delta\varepsilon_\parallel\}, \{\Delta\varepsilon_x\} and \{\Delta\chi\} are obtained from \{\varepsilon_\parallel\}, \{\varepsilon_x\}, and \{\chi\} by replacing \(u, v\) and \(w\) by the increments \(\Delta u, \Delta v,\) and \(\Delta w\).

This can conveniently be written as

\[
\{\Delta\varepsilon\} = \{\Delta\varepsilon_\perp\} + z \{\Delta\chi\} \quad \ldots \quad 3.5.10
\]

where \{\Delta\varepsilon_\perp\} = \{\Delta\varepsilon_\parallel\} + [A] \{\Delta\theta\} + \{\Delta\varepsilon_x\} \quad \ldots \quad 3.5.11

\{\Delta\varepsilon_\perp\} refers to the total strain increment at the mid-section, i.e. where \(z = 0\).

The total incremental strain and curvature may be stored and used to calculate the stresses from the constitutive stress strain relationships in section 3.4, and are stored at the two by two Gauss points for further analysis.
3.6 Stiffness formulation

The variational principle of minimum potential energy is used in the formulation, where the potential energy is stationary with regard to all kinematically admissible variations in displacements from the state of equilibrium. The potential energy of the plate (ignoring body forces) is given by

\[ \Pi = U - V \quad \ldots \quad 3.6.1 \]

where strain energy \( U = \iint (f^1_\varepsilon \sigma_0 \varepsilon) \, d \text{vol} \quad \ldots \quad 3.6.2 \)

and virtual work of applied loads \( V = \int_s \mathbf{P} \delta \varepsilon \, ds \ldots 3.6.3 \)

An increment of the total potential energy is given by:

\[ \Delta \Pi = \iint \{ \sigma \}^T \{ \Delta \varepsilon \} + \frac{1}{2} \{ \Delta \sigma \}^T \{ \Delta \sigma \} \, d \text{vol} - \int_s (P + AP) \Delta \delta \, ds - \int_s \Delta P (\delta - \delta_0) \, ds \ldots 3.6.4 \]

where the last term may be ignored, since variations with respect to \( \Delta \delta \) makes the term zero.

Substituting 3.5.10 into 3.6.4 gives

\[ \Delta \Pi = \iint \{ N \}^T \{ \Delta \varepsilon_1 \} + A \{ \Delta \theta \} \, dA + \frac{1}{2} \iint \{ M \}^T \{ \Delta X \} \, dA + \frac{1}{2} \int \{ \Delta \theta \}^T [N] \{ \Delta \theta \} \, dA + \frac{1}{2} \iint \{ \Delta M \}^T \{ \Delta X \} \, dA \]

\[ + \int_s (P + AP) \Delta \delta \, ds \ldots 3.6.5 \]

where \([N] = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \]

\[ \{ N \} = \int \frac{t/2}{2} \{ \sigma \} \, dz \quad \text{and} \quad \{ M \} = \int \frac{t/2}{2} z \{ \sigma \} \, dz \]

\[ \ldots \quad 3.6.6 \]
Introducing the stress - strain (elasto-plastic) relationship into 3.6.5 gives:

\[
\Delta \Pi = \frac{1}{2} \iint_A \left[ \{\Delta \varepsilon_i\}^T \{\varepsilon_i\} + \{\Delta \theta\}^T [A]^T [C^*] A \{\Delta \theta\} + \{\Delta \chi\}^T [D^*] \{\Delta \chi\} \\
+ \{\Delta \theta\}^T [A]^T [C^*] A \{\Delta \chi\} + \{\Delta \chi\}^T [C^*] A \{\Delta \theta\} + \{\Delta \varepsilon_i\}^T [C^*] A \{\Delta \varepsilon_i\} \\
+ \{\Delta \varepsilon_i\}^T [C^*] A \{\Delta \varepsilon_i\} + \{\Delta \chi\}^T [C^*] A \{\Delta \varepsilon_i\} \right] \, dA
\]

\[
+ \frac{1}{2} \iint_A \left[ \{\Delta \theta\}^T [N] \{\Delta \theta\} - \phi ((U + \Delta U) \Delta u + (V + \Delta V) \Delta v) \, ds - \iint_A (W + \Delta W) \Delta w \, dA \\
+ \iint_A \left[ \{\Delta \varepsilon_i\} + [A] \{\Delta \theta\} \right] \{\Delta \chi\} \, dA
\]


where all terms involving products with \( \Delta \varepsilon^*_i \) (higher than third order polynomials) have been ignored.

Now for stable equilibrium, the stationary value of the potential energy is an absolute minimum, i.e. \( \delta(\Delta \Pi) = 0 \)

On introducing the strains in terms of the displacements,

\[
ie \{\Delta \varepsilon_i\} = [H] \{\Delta u\} \\
{\Delta \chi} = [F] \{\Delta w\} \]

\[
{\Delta \theta} = [G] \{\Delta w\}
\]

into 3.6.7:

\[
\begin{pmatrix}
U \\
V \\
W
\end{pmatrix}
+ \begin{pmatrix}
\Delta U \\
\Delta V \\
\Delta W
\end{pmatrix}
= \begin{pmatrix}
\iint_A [H]^T [N] dA \\
\{\Delta \varepsilon_i\}^T [N] dA \\
\{\Delta \varepsilon_i\}^T [N] + \{\varepsilon_i\}^T [G] [A] \{\Delta u\} + \{\varepsilon_i\}^T [G] [A] \{\Delta w\} + \{\varepsilon_i\}^T [G] [A] \{\Delta w\} + \{\varepsilon_i\}^T [G] [A] \{\Delta w\}
\end{pmatrix}
\]

\[
= [K] \{\Delta \delta\} 3.6.9
\]

The first term is the vector of total external forces \( \{P_e\} \) prior to applying incremental loads at the nodes, the second term is the vector of nodal values of the incremental applied forces \( \{\Delta P_e\} \), and the third term is the internal load vector \( \{P_i\} \).
\{Δδ\} is the vector of nodal displacements and [K] the tangent stiffness matrix, given by

\[
[K] = [K_{p1}][K_c]
\]

\[
[K_c]^T[K_b]
\]

\[ \ldots \ldots 3.6.10 \]

where \([K_{p1}]\) is the in-plane stiffness = \[ \int \int [H]^T[C^*][H] \, dA \]

\([K_b]\) is the bending stiffness =


and \([K_c]\) is the coupled stiffness, =

\[ \int \int \left( [H]^T[C^*][A][G] + [H]^T[cd][F] \right) \, dA \]

\[ \ldots \ldots 3.6.11 \]

In equation 3.6.9 the term \(\{P_e\} - \{P_i\}\) gives the out of balance loads between the externally applied forces and the internal stress resultants calculated from the displacements. When this term vanishes after a number of iterations, exact equilibrium has been achieved for the current load increment \(\{ΔP_e\}\).
3.7 Rotational in-plane degree of freedom

According to the formulation so far, the element has at each of its nodal points two in-plane degrees of freedom, \(u, v\), and three out-of-plane degrees of freedom, \(w, \theta_x, \theta_y\). For convenience in assembly of the elements, the in-plane rotation of the element, \(\theta_z\) will be introduced. The displacement pattern for in-plane effects have been independent of \(\theta_z\), since this value is negligible on account of the high stiffness (in-plane) of the elements. However, if zero terms are introduced in the appropriate rows and columns of the stiffness matrix, coplanar elements immediately become singular on transformation to global co-ordinates, should the global co-ordinates differ from the local ones.

This problem could be avoided at the outset by accounting for the real rotational stiffness in the in-plane displacement functions, which result in higher order elements. The problem can also be overcome by adding an arbitrary quantity to the diagonal term of the stiffness matrix corresponding to the singularity \(^{(42)}\). Zienkiewicz \(^{(155)}\) et al. suggested a simpler artifice which has been modified in the present approach for rectangular elements.

The nodal rotation \(\theta_z\) at any node is assumed to be responsible only for developing resisting couples \(M_z\) at all the element nodes. The sum of the couples is always zero to ensure equilibrium and made arbitrarily proportional to the young's modulus \(E\), plate thickness \(t\) and element area \(A\).
This is described by:

\[
\begin{bmatrix}
M_{zi} \\
M_{zj} \\
M_{zk} \\
M_{zl}
\end{bmatrix} = \alpha E A t
\begin{bmatrix}
1 & -1/3 & -1/3 & -1/3 \\
-1/3 & 1 & -1/3 & -1/3 \\
-1/3 & -1/3 & 1 & -1/3 \\
-1/3 & -1/3 & -1/3 & 1
\end{bmatrix}
\begin{bmatrix}
\theta_{zi} \\
\theta_{zj} \\
\theta_{zk} \\
\theta_{zl}
\end{bmatrix}
\]

where \( \alpha \) is an undetermined coefficient. This value has been recommended for arch dams as \( 3 \times 10^{-2} \) (ref.155), but is considered unsuitable for box girders due to oscillations about the correct value of the flexural component of longitudinal stress at the web-flange junction\(^{(77)}\). This is due to the cross coupling between the fictitious rotational stiffness of the web and the bending stiffness of the flange.

The value of \( \alpha \) therefore has to be made as small as the computer would allow without making the solution unstable, and the value of \( \alpha \) chosen successfully for the CDC7600 computer used was \( 1.0 \times 10^{-11} \).
3.8 Transformation from local to global coordinates

Each element tangent stiffness matrix, evaluated in local coordinates, has to be transformed into the global coordinate system prior to assembly with the other elements of the structure into an overall structural tangent stiffness matrix. The transformation matrix has to be general purpose to cope with any orientation or combination of plates in three dimensions.

From principles of analytical geometry, if the equation of a plane is written as:

\[ A_x + B_y + C_z + D = 0 \]  \hspace{1cm} 3.8.1

then the direction cosines of the normal to the plane \((z')\) can be written as:

\[ \lambda_{z'x} = \frac{A_z}{\sqrt{A_x^2 + B_y^2 + C_z^2}} \]  \hspace{1cm} 3.8.2

\[ \lambda_{z'y} = \frac{B_z}{\sqrt{A_x^2 + B_y^2 + C_z^2}} \]  \hspace{1cm} 3.8.2

\[ \lambda_{z'z} = \frac{C_z}{\sqrt{A_x^2 + B_y^2 + C_z^2}} \]  \hspace{1cm} 3.8.2

now the equation of the plane passing through the specified points 1,2,3 (see fig 3.) of the rectangular element 1,2,3,4 can be written as:

\[ \det \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \]  \hspace{1cm} 3.8.3
from which \[ A_z = y_{21}z_{31} - y_{31}z_{21} \]
\[ B_z = -x_{21}z_{31} + x_{31}z_{21} \]
\[ C_z = x_{21}y_{31} - x_{31}y_{21} \] 

..... 3.8.4

where \( x_{21} = x_2 - x_1, y_{21} = y_2 - y_1 \) etc...

A, B & C represent the directional vectors of z' axis.

The equation of a plane passing through line i-z' and point j is given by:

\[
\begin{vmatrix}
  x-x_1 & y-y_1 & z-z_1 \\
  A_z & B_z & C_z \\
  x_2-x_1 & y_2-y_1 & z_2-z_1
\end{vmatrix} = 0
\] 

..... 3.8.5

and the direction cosines given by

\[
\lambda_{y'}^x = \frac{A_y}{\sqrt{A_y^2 + B_y^2 + C_y^2}}
\]

\[
\lambda_{y'}^y = \frac{B_y}{\sqrt{A_y^2 + B_y^2 + C_y^2}}
\] 

..... 3.8.6

\[
\lambda_{y'}^z = \frac{C_y}{\sqrt{A_y^2 + B_y^2 + C_y^2}}
\]

where

\[
A_y = B_z z_{21} - C_y z_{21}
\]

\[
B_y = C_y x_{21} - A_z z_{21}
\] 

..... 3.8.7

\[
C_y = A_z y_{21} - B_z x_{21}
\]
In a similar manner, direction cosines of the $x'$ axis are given by:

$$
\lambda_{x'x} = \frac{A_x}{\sqrt{A_x^2 + B_x^2 + C_x^2}}
$$

$$
\lambda_{x'y} = \frac{B_x}{\sqrt{A_x^2 + B_x^2 + C_x^2}}
$$

$$
\lambda_{x'z} = \frac{C_x}{\sqrt{A_x^2 + B_x^2 + C_x^2}}
$$

where

$$
A_x = B_y C_z - C_y B_z
$$

$$
B_x = C_y A_z - A_y C_z
$$

$$
C_x = A_y B_z - B_y A_z
$$

Therefore, the coordinates of the two systems are related by:

$$
\begin{bmatrix}
  x'_n \\ y'_n \\ z'_n
\end{bmatrix} = \left[ \lambda \right] \begin{bmatrix}
  x_n - x_1 \\ y_n - y_1 \\ z_n - z_1
\end{bmatrix}
$$

where subscript $n$ refers to any of the nodes 1, 2, 3, 4 (fig.3)

and

$$
\left[ \lambda \right] =
\begin{bmatrix}
  \lambda_{x'x} & \lambda_{x'y} & \lambda_{x'z} \\
  \lambda_{y'x} & \lambda_{y'y} & \lambda_{y'z} \\
  \lambda_{z'x} & \lambda_{z'y} & \lambda_{z'z}
\end{bmatrix}
$$
The element stiffness matrix expressed in global coordinates is given by:

\[ [K] = [T]^T [K'] [T] \] \hspace{1cm} \text{(3.8.12)}

where \([K']\) is the element stiffness matrix expressed in local coordinates, \([T]\) is the transformation matrix and \([T]^T\) the transpose of \([T]\).

Each of the above matrices is a 24 by 24 matrix.

where \([T] = \begin{bmatrix}
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda \\
\lambda
\end{bmatrix} \] \hspace{1cm} \text{(3.8.13)}

The element nodal displacements are transformed from the global into local coordinates prior to calculating element stress resultants by the following relationship.

\[ \{\delta'\} = [T] \{\delta\} \] \hspace{1cm} \text{(3.8.14)}
3.9 Special Deep Beam Element

In order to model the stress gradients, buckling deformations and the spread of plasticity to an acceptable degree of accuracy, it is necessary to maintain the fineness of the finite element mesh. At the same time, in order to represent the whole structure with finite elements, the conventional modelling would lead to either unacceptably high aspect ratios of the elements or to excessively large numbers of elements. This would be particularly so in the case of long box girders or plate girders, where it may be necessary to produce realistic moment/shear ratios.

The author decided to model special beam elements that could represent the box girder as a beam in those regions (away from the region of interest) where linear elastic behaviour may be assumed. Shear deformation, which is significant in box beams, is also accounted for in this formulation.

The beam element is modelled to maintain full compatibility of displacements at all nodes on a cross section of the box girder that represents the end of a finite element model and the beginning of a special beam model. Simple Engineering beam theory is used, giving a linear in-plane stress due to bending, and a shear stress distribution as shown in Fig. 4. Fig. 5 shows the special beam element with the arrows representing the positive signs adopted. 'i' represents a typical node in the bottom flange, 'j' a typical node in the web, and 'k' a typical node in the top flange.
Now, the normal stiffness matrix for a beam element (including shear deformation) is given by (see Fig. 6):

\[
\begin{bmatrix}
F_{z1} \\
M_{y1} \\
F_{z2} \\
M_{y2}
\end{bmatrix} =
\begin{bmatrix}
a & b & \text{sym} \\
b & e & -a \\
-a & -b & a \\
b & g & -b & e
\end{bmatrix}
\begin{bmatrix}
w_1 \\
y_1 \\
W_2 \\
y_2
\end{bmatrix}
\]

\[3.9.1\]

where \( a = \frac{EI}{(1+2n)} \frac{12}{L^3} \), \( b = \frac{EI}{(1+2n)} \frac{6}{L^2} \), \( e = \frac{EI}{(1+2n)} \frac{4}{L} \frac{(1+n)}{2} \)

\[ g = \frac{EI}{(1+2n)} \frac{2}{L} (1-n) \]

Dimensional constant \( n = \frac{6EI}{kAGL^2} \)

where \( kA = \) total shear area (area of webs)

\[ E = \text{Young's modulus} \]

\[ I = \text{second moment of area of beam} \]

\[ G, \text{modulus of rigidity} = E \]

\[ \nu = \text{Poisson's ratio} \]

\[ L = \text{length of beam element} \]

Equation 3.9.1 is modified to give the appropriate proportion of shear forces and longitudinal forces at the nodes \( \Sigma i, \Sigma j, \Sigma k \) of the finite element cross section as follows, when the end 2 in figure 6 is replaced by the finite element nodes as in figure 5. Node \( m \approx i, j \) or \( k \). i.e. \( m \) represents end 1.

If \( m \) were a higher node number, a transformation would be required of 3.9.2 which would change the signs of coefficients containing 'b'.
where \( k_1, k_2 \) and \( k_3 \) are the factors to be applied to the matrix in row and column multiplication in order to obtain the nodal stiffness that will be added to the global stiffness matrix. The values of \( k_1, k_2 \) and \( k_3 \) are dependent on the shear and bending stress distribution of figure 4, and vary in sign and magnitude for each node. The derivation of these factors is given in Appendix 1.

The factors are given by:

\[
k_1 = \frac{1}{z} \frac{I_n}{I},
\]

\[
k_2 = \frac{1}{I} t f h \bar{y} \text{ } \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldOTS

\begin{align*}
F_{zm} &= \begin{bmatrix} a \\ M_{ym} \end{bmatrix}, \\
F_{yi} &= \begin{bmatrix} k_1 & b \\ -k_2 & -a \\ F_{yi} \end{bmatrix} \\
F_{xj} &= \begin{bmatrix} k_1 & b \\ -k_2 & -a \\ F_{xj} \end{bmatrix} \\
F_{xk} &= \begin{bmatrix} k_1 & b \\ -k_2 & -a \\ F_{xk} \end{bmatrix} \\
F_{yk} &= \begin{bmatrix} k_1 & b \\ -k_2 & -a \\ F_{yk} \end{bmatrix} \\
\end{align*}

\[
\begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ \end{bmatrix}
\]

$3.9.2$

$3.9.3$
The above factors ensure the shear and bending stress distribution in Fig. 4 in the stiffness formulations. The matrix so formed is added to the global stiffness matrix in the normal manner as will be shown in section 3.10

3.10 Assembly and solution of simultaneous equations

Gaussian elimination and Choleski decomposition are the principal contenders for the direct methods of solution. Gauss requires the equivalent of 10 n (where n = number of equations) less calculations than Choleski's algorithm, because Gauss avoids calculation of square roots. Since Klyuyev and Kokovkin-Shcherbak (73) have proved that no algorithm for equation solution can involve fewer calculations than Gaussian elimination, either of these algorithms is acceptable from the point of view of calculation efficiency.

However, although the Choleski's square root algorithm has certain important data storage advantages, it is more error...
prone than Gaussian elimination. Therefore, a half (semi)
banded Gaussian elimination procedure (159) is used for the
equation solution, where the upper band is operated upon.

For efficiency, the assembly and elimination are carried
out simultaneously, so that the elimination of each node is
done as soon as it is formed. Some of the properties of the
frontal approach are made use of, but only to the extent
that none of the time consuming housekeeping operations are
carried out. The individual element matrices are stored in
ascending nodal order, always using the minimum node number
of each element. The assembly continues so long as the
minimum node number of an element is the same as the pivotal
node, but automatically stops when the next highest (minimum)
node number is encountered.

During the solution, the out of balance forces \( \{P_e\} - \{P_i\} \)
equation 3.6.9) are always ignored, being corrected by
successive iteration. Therefore:

\[
[K] \{\Delta \delta\} = \{\Delta P_e\} \quad ........... \quad 3.10.1
\]

Since the matrix is stored in a banded form, any elimination
of a single node modifies only the terms within the band
width, and so only those elements lying within this band
need be assembled at this stage. This procedure enables usage
of a larger band width, since only the upper triangular matrix
of size \((m+1)(m+2)/2\) (where \(m = \) semi band width) can be
stored at any one time as elimination continues progressively
down the band. If equation 3.10.1 represents a set of \(m\)
equations, where \(m\) is the semi-band width, then it can be
represented in partitioned form as:
\[
\begin{bmatrix}
K_{11} & \{K_{1i}\} \\
\{K_{1i}\}^T & [K_{ii}]^T
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_i \\
\{\Delta \delta\}
\end{bmatrix} =
\begin{bmatrix}
\Delta P_{e1} \\
\{\Delta P_{e}\}
\end{bmatrix}
\]
\ldots \ldots \quad 3.10.2

where \(K_{11}\) is a \(1 \times 1\) matrix
\[i = m - 1\]
\(\{K_{1i}\}\) is a \(1 \times (m-1)\) matrix
\(\{K_{1i}\}^T\) is a \((m-1) \times 1\) matrix
and \([K_{ii}]\) is a \((m-1) \times (m-1)\) matrix

\(\Delta \delta\) is a vector of unknowns and \(\Delta P_{e}\) is a vector of known values.

The Gaussian elimination procedure allows the reduction of matrix \(K\) to be a \((m-1)\) matrix of the form

\[
[K]^*\{\Delta \delta\} = \{\Delta P_{e}\}^* \quad \ldots \ldots \quad 3.10.3
\]

where

\[
[K]^* = [K_{ii}] - \{K_{1i}\}^T K_{11}^{-1} \{K_{1i}\} \quad \ldots \ldots \quad 3.10.4
\]

\[
\{\Delta P_{e}\}^* = \{\Delta P_{e1}\} - \{K_{1i}\}^T K_{11}^{-1} \Delta P_{e1} \quad \ldots \ldots \quad 3.10.5
\]

This procedure is repeated by partitioning \(K^*\) in the same way, the fundamental operation being the triple product

\[
\{K_{1i}\}^T K_{11}^{-1} \{K_{1i}\}. \quad \text{When the}[K]^*\text{matrix is finally reduced to a} \ 1 \times 1 \text{matrix, the whole remaining (reduced) matrix is shifted so that the modified}[K_{ii}]\text{matrix, where} \ i = m - 1, \text{takes up its new position as:}
\]

\[
[K]_{m-1} \{\Delta \delta\}_{m-1} = \{\Delta P_{e}\}_{m-1} \quad \ldots \ldots \quad 3.10.6
\]

in the storage area. The first coefficient of 3.10.6 becomes \(K_{11}\), while the others take up their relevant positions accordingly. The eliminated equations are stored temporarily in a buffer area and then transferred to peripheral storage if and when it is used up, to be brought into use for the
final back substitution phase. The newly formed matrices for the next node number are added to equation 3.10.6, and the whole procedure starting from equation 3.10.1 repeated. This operation is continued until all the equations are eliminated.

When the whole system of equations has been eliminated, back substitution may be applied. Direct solution is possible for the last unknown $\Delta \delta_n$ in a system of a total of $n$ equations.

$$\Delta \delta_n = \frac{1}{K_{nn}} \Delta P_{en} \quad \quad \quad 3.10.7$$

then remaining unknowns are solved by equations of the type

$$\Delta \delta_i = K_{ii} \Delta P_{ei} - K_{ij} \sum_{j=i+1}^{n} (K_{ij} \delta_j) \quad \quad 3.10.8$$

where $\Delta \delta_i$ = unknown for degree of freedom $i$

$K_{ii}$ = pivotal reduced stiffness of degree of freedom $i$.

$\delta_j$ = solved values of unknown from previous operation.

The applied loads are dealt with by the vector $\{ \Delta P_e \}$ However, to apply specified displacements to the boundary, the stiffness matrix and load vector are modified without eliminating appropriate rows and columns, for ease of indexing.

The vectors of incremental loads are modified so that equation 3.10.5 is replaced by $\Delta P_{el} = \Delta \delta_1$

$$\{ \Delta P_{el} \} = \{ \Delta P_{ej} \} - \{ K_{ij} \} \Delta P_j \quad \quad 3.10.9$$

where $\Delta P_j$ = prescribed displacement,

& $j = 2$ to $n$
The corresponding row and column of this matrix are made zero, and the diagonal term unity.

In the back substitution process, the procedure is reversed, so that an unknown load is obtained for a known displacement. The reactions so obtained are produced by replacing equation 3.10.8 by:

$$\Delta P_{ei} = \Delta P_{ei} - \sum_{j=1}^{n} K_{ij} \Delta \delta_{j} \quad \ldots \quad 3.10.10$$

for each degree of freedom i on the n simultaneous equations.

Provided the tangent stiffness matrix $[K]$ remains unchanged during an increment, several load cases, or even iterations for correcting for out of balance forces $\{P_e\} - \{F_i\}$, can be carried out by altering only the right hand sides $\{\Delta P_e\}$ each time and solving the equations to obtain deflections for the new load case (linear-elastic analysis) or the cumulative deflections for the increment at the end of the iterations.

The back substitution (equations 3.10.8 and 3.10.10) is carried out by storing the relevant stiffness terms in a buffer area or backing store, and called into operation when desired.

Externally coupled elastic stiffnesses can be added by inputting the relevant coefficients coupling any two node numbers systematically. Therefore either spring supports or small amounts of substructuring can be provided, where elastic stiffness coefficients of one substructure can be added to the current substructure. The program has, however, not been written with this as an objective, since such operations would require automatic generation of the large stiffness terms involved, with considerable file handling. The present facility allows only manual input of individual stiffness coefficients. Also, substructuring is not suitable for any non-linear geometric
or material effects, since they both imply incremental solutions in which the stiffness coefficients have to be updated at each increment, requiring operations on each substructure.

3.11 **Iterative and Incremental Cycle**

Considering equation 3.6.9, the out of balance forces are restated:

\[
\{P_e\} - \{P_i\} = \left[ \begin{array}{c} U_e \\ v_e \\ \omega_e \end{array} \right] - \left\{ \begin{array}{c} \int [H]^T \{N\} \, dA \\ \int ([G]^T[A]^T\{N\}+[F]^T\{M\}) \, dA \end{array} \right\}
\]

where \([F]\) = curvature matrix.

\{M\} = bending stress resultants

\[H\] = in-plane strain matrix

\{N\} = in-plane stress resultants.

\[G\] = slope matrix

\[A\] = total slope matrix.

The out of balance load vector vanishes if the internal and external forces are in exact equilibrium for any particular load increment. The flow chart for the program solution (Fig. 7) indicates the various paths that can be taken, shown as loops. The out of balance forces can be iterated upon using a straight incremental solution (loop 1, where \(\{P_e\} = \{P_i\}\)), an incremental solution with correction for out of balance forces (loop 2), a modified Newton-Raphson iteration (loop 3) or a Newton-Raphson iteration (loop 4), where the choice is left to the user while incrementing the loads or displacements.
While calculating the internal load vector, although nodal stresses extrapolated from the two by two Gauss points are output for convenience, the vector of stress resultants stored at the Gauss stations are used in equation 3.11.1, as the approach is then more consistent with the derivation of the tangential stiffness matrix.

Convergence is monitored by using the Euclidean norm $||\Delta \delta_1|| = \left[ (\Delta \delta_1)^T \Delta \delta_1 \right]^{1/2}$ of the iterative deflections and compared with the Euclidean norm of the cumulative incremental deflection until the last iteration. This value is expressed as a percentage, and convergence is deemed to have been satisfied if this value is of the order of one percent.

The use of this iterative approach does not always guarantee convergence, and sometimes a combination of two or more methods of solution (incremental, Newton-Raphson etc,) is necessary. Modified Newton-Raphson procedure is the most preferred approach. However, near points of local instability it does not lead to convergence, and a purely incremental approach has to be resorted to. This minimises the possibility of obtaining a negative diagonal term in the tangent stiffness matrix. Also, in order to trace the lowest equilibrium path, moderately small increments of load or deflection are applied to obviate the possibility of convergence on a higher equilibrium path, especially in the latter stages of loading.
It may be borne in mind that the iterations correct for approximations in the linearised treatment of large deflection (geometric non-linearity) and plasticity (material non-linearity). Also, they correct for any residuals in the loads or deflections that could accumulate progressively. Therefore, the increments invariably have to be smaller during the plastic stages of loading when the load deflection plot is highly non-linear, all the above effects having to be accounted for.

3.12 Monitoring of stress resultants

The stress resultants and the stress history are stored at the two by two Gauss points. However, the term:

$$\int \int [G]^T[A]^T[C^*][A][G] \, dA$$

in equation 3.6.11 strictly requires a three by three integration, since it contains the most highly non-linear terms in the tangent stiffness matrix. Also, in the elastic stages, no other terms require more than the two by two integration, since the cross coupling elasto-plastic modular matrix $[cd]$ does not exist. In the plastic stages, however, the existence of this modular matrix term brings in further terms which require a three by three Gaussian integration.

Therefore, as a compromise, a two by two Gaussian integration of the complete stiffness matrix is employed during the elastic increments, and a three by three Gaussian integration during the plastic stages of loading. The values of the terms $[N^*],[C^*],[D^*]$ and $[cd]$ in equation 3.6.11 are then obtained by bilinear extrapolation from the values at the two by two Gauss stations that are already stored or calculated.
The incremental stress resultants are calculated from the constitutive stress strain relationship in equation 3.4.12 and added to the total sum of all incremental stress resultants. The modular matrices are resolved from the average value of the stress resultants at the last increment and the last iteration.

\[ M_{av} = \frac{1}{2}(M_{i-1} + M_{i,t-1}) \quad \ldots \quad 3.12.1 \]
\[ N_{av} = \frac{1}{2}(N_{i-1} + M_{i,t-1}) \]

where \( M_{i-1}, N_{i-1} \) are stress resultants at the previous load or deflection increments.

\( M_{i,t-1}, N_{i,t-1} \) are stress resultants from the last iteration of the present increment.

The total stress resultants (sum of all increments) are given by:

\[ M_{i,t} = M_{i-1} + [C^*] \{ \Delta \varepsilon_t \} + [cd] \{ \Delta x_t \} \quad \ldots \quad 3.12.2 \]
\[ N_{i,t} = N_{i-1} + [cd] \{ \Delta \varepsilon_t \} + [D^*] \{ \Delta x_t \} \]

where \([C^*], [cd], [D^*]\) are functions of \( M_{av} \) & \( N_{av} \).

Points that are elastic at the start of an increment, but become plastic during iterations are dealt with as in reference (24) and (159).

If \( f_{i-1} \) = elastic value of yield function at the end of the \( i \)-th increment,

\( f_i \) = plastic value at the current iteration of the current increment,

the strain increments that would cause the stress resultants to remain elastic could be approximated by

\[ A = \frac{(1-f_{i-1})}{(f_i - f_{i-1})} \quad \ldots \quad 3.12.3 \]
The stress resultants required to reach the yield surface are given by:

\[
\{\bar{N}\} = \{N_{i-1}\} + A[\{N_{i-1}\} - \{N_{i-1}\}]
\]

\[
\{\bar{M}\} = \{M_{i-1}\} + A[\{M_{i-1}\} - \{M_{i-1}\}]
\]  

...... 3.12.4

where \{N_{i-1}\} and \{M_{i-1}\} are the first estimate of the 'elastic' stress resultants given by:

\[
\{N_{1,1}\} = \{N_{1-1}\} + [\Delta \varepsilon_t]
\]

\[
\{M_{1,1}\} = \{M_{1-1}\} + [\Delta \chi_t]
\]  

...... 3.12.5

A better estimate of the stress resultants is given by

\[
\{N_{1,2}\} = \{\bar{N}\} + (1-A)[[C^*] \{\Delta \varepsilon_t\} + [cd] \{\Delta \chi_t\}]
\]

\[
\{M_{1,2}\} = \{\bar{M}\} + (1-A)[[cd] \{\Delta \varepsilon_t\} + [D^*] \{\Delta \chi_t\}]
\]  

...... 3.12.6

where \[C^*, cd\] & \[D^*\] are functions of \{\bar{N}\} & \{\bar{M}\}

Subsequent iterations finally stabilise the values of the stress resultants and modular matrices at a fully converged value.

Due to the finite nature of the load or deflection increments adopted, violation of the yield surface occurs, and the tangency condition \(\delta f\), (equation 3.4.5) is no longer zero. The stress resultants are therefore adjusted by a factor \(C\), where:

\[
C = \frac{1.0}{f_i}
\]  

...... 3.12.7

where \(f_i\) = value of the yield function (greater than 1.0) so that the stress resultants are again on the yield surface.
Should any unloading of the stress resultants from the yield surface occur, the plastic strain rate multiplier $\lambda$ (equation 3.4.6 and 3.4.8) becomes negative. In such an event, the stress resultants during the iteration are moved away from the yield surface by making $f < 1.0$ (say 0.995), i.e. reducing the stress resultants by a factor $c$ where:

$$c = \frac{0.995}{f_i}$$

and assuming the Gauss point to be elastic in the next iteration.

A pointer is kept of the Gauss points which are plastic (full section depth) as defined by Ilyushin's approximate theory, where $f = 1.0$. Also, the positions where first fibre yield occur are stored for each increment from the relationship (see equation 3.4.3 for definitions)

$$f_y = N_e + \frac{12sM_N}{t^2\sigma_o} + \frac{36M}{t^3\sigma_o} + \frac{36M}{t^4\sigma_o}$$

where first fibre yield occurs if the yield function $f$ is:

$$f_y \leq f < 1.0$$

The stress resultants at the nodes of each element are output by a bilinear extrapolation from the two by two Gauss stations. Therefore, smoothing of the numerically discontinuous model is not carried out by averaging nodal stress resultants. Should such a stress distribution, better conforming with the 'exact' solution, be required, manual averaging is resorted to.
Such an average does result in serious errors in regions of large stress gradients or where element sizes vary, and therefore interpretation of the stress resultants are left to the user's discretion.

The signs and directions of the stress resultants are given in Fig. 8 where the arrows indicate the positive signs.
Fig. 1. Rectangular Element.

\[ \xi_i = -1, 1, 1, -1 \]
\[ \eta_i = -1, -1, 1, 1 \]

Fig. 2.
Sign convention for displacements and external forces in local and global coordinates (directions shown are positive).

Fig. 3.
Orientation of the local element coordinates with respect to the global coordinates.
Fig. 4. Stress distribution in special beam element

Fig. 5. Basic model of special beam element.

Fig. 6. Beam sign convention.
Read Geometry, material properties and boundary conditions. Check input

Read loads and perform load increments

Order elements with lowest node number in ascending order

Form (global) tangent stiffness matrix and write each element submatrix to a file in the order specified above

Read element data from file created above, forming and eliminating the global matrix using the Gaussian half banded elimination procedure, storing row operations and eliminated equations in a buffer file

Back substitute and output updated total deflections and reactions. Also form incremental total strain at plate mid section. Check for convergence of the Euclidean norms.

Calculate and store stress history, plasticity counters, yield function, etc., for last load increment and last iteration. Output stress resultants, etc., after convergence achieved at last iteration.

Form out of balance load vector \( \{P_e\} - \{P_i\} \)

Carry out operations on the right hand side (loads) for iterations on the current load increment for out of balance forces

1. Carry out operations on the right hand side (loads) for iterations on the current load increment for out of balance forces

2. Form out of balance load vector \( \{P_e\} - \{P_i\} \)

3. Back substitute and output updated total deflections and reactions. Also form incremental total strain at plate mid section. Check for convergence of the Euclidean norms.

4. Yes

Next load increment

Fig. 7. Flow chart for the computer program 'NOLP'
Notes on Fig. 7

The 'loops' of the iterative cycle have been numbered to show the alternative load paths that the solution can take as decided during increments of load.

Loop 1 - This loop will be taken if no iterations are asked for and no correction is carried out for the residual out of balance forces. This procedure is sometimes necessary near bifurcation paths or collapse and is purely incremental solution.

Loop 2. - This loop is incremental as above, but corrects for out of balance forces from the previous load increment (no iterations having been allowed).

Loop 3. - This is the modified Newton-Raphson procedure, where the geometric stiffness matrix is not updated. This is done by carrying out a number of iterations within any load increment until convergence is deemed to have been satisfied. The geometric stiffness matrix is then only updated for each increment of load or deflection.

Loop 4. - This procedure is hopefully to be vary sparingly used, since it involves updating the stiffness matrix (and its time consuming Gaussian elimination) for both the incremental and iterative procedures. This would involve very large amounts of computer time, although convergence is faster. A judicious combination of the above three loops should obviate the necessity of using this "Newton-Raphson" iteration.
All axes refer to local co-ordinates

Fig. 8. Stress Resultants
CHAPTER 4

COMPARISON OF COMPUTER PROGRAM 'NOLP' WITH
ESTABLISHED AND CLOSED FORM NUMERICAL AND
ANALYTICAL TREATMENTS

In order to establish the validity of the computer program to analyse box girder structures in the non-linear elasto-plastic range, it would ideally require an existing theoretical solution in this range. Since no such analytical treatment exists, it will therefore be necessary to carry out two sets of tests.

(1) Use the computer program to analyse existing box girder bridge solutions and compare the results.

(2) Carry out a non-linear elastic and elasto-plastic analysis of plates under various loading and boundary conditions up to collapse. Compare these results with well known and proven analyses conducted by other researchers. It would be necessary to test the program for plates under both in-plane and out of plane loading up to collapse.

Once these tests have confirmed the validity of the computer program, it may be applied with confidence to the more complex problems.

4.1. Linear Elastic Analyses

4.1.1 Single Cantilevering Box

A simple cantilevering square box of constant wall thickness was taken and the displacements and stresses worked out from simple engineering beam theory (including shear deformation) for two loading cases acting at the tip of the
cantilever, namely (see fig.9)

(a) Bending (symmetric load)
(b) Torsional moment (antisymmetric load)

The results were then compared with those obtained using the finite element program 'NOLP' with the mesh indicated in Fig.9.

<table>
<thead>
<tr>
<th></th>
<th>Engineering Beam Theory</th>
<th>Finite Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tip deflection.</td>
<td>5.775 mm</td>
<td>5.5066mm (5% error)</td>
</tr>
<tr>
<td>Web shear stress resultant.</td>
<td>500N/mm</td>
<td>497.6N/mm</td>
</tr>
<tr>
<td>Tip rotation at corner nodes</td>
<td>0.003755rads</td>
<td>0.00376rads</td>
</tr>
<tr>
<td>Average shear stress resultant (at root of cantilever).</td>
<td>250N/mm</td>
<td>246N/mm</td>
</tr>
</tbody>
</table>

4.1.2 Scordelis\(^{(127)}\) four cell box bridge

Of the many different bridge sizes analysed by Scordelis, a four cell right bridge deck of span 60 ft is described here (see Fig.10). A concentrated point load of 1 Kip (1000 lbs) is applied over the outer web at the mid-span, and the results obtained from the computer program compared with results presented by Scordelis. This load position is chosen in order to give the worst stress gradients, and so that we may observe the transverse as well as longitudinal pattern of behaviour. This would therefore be a stringent check on the computer program.

A rigid end diaphragm over each support was assumed in Scordelis's folded plate analysis, and these boundary conditions are duplicated in the computer program. Crisfield\(^{(23)}\) carried out a comparison of the stress resultants and also presented the deflections obtained from his multicell finite element program.
Scordelis did not present the deflections for his folded plate solutions, and so the deflections obtained from program 'NOLP' were compared with Crisfield's solutions (Fig 11(a))

It maybe observed that the bending moments under the point load are lower than those obtained by Scordelis (see Figs 11(b) and 12) using his Fourier series, because of the coarseness of the mesh size of the finite elements under the concentrated point load. Refining the mesh sizes in these areas of steep stress gradient would undoubtedly give more agreeable results. All other stresses in areas other than near the joint load are very satisfactory (Figs 11(b) and 12). The deflections obtained are slightly lower than those obtained by Crisfield's (23) multicell program.

Details of the bridge are given in Fig.10. The finite element mesh was divided longitudinally to give 36 in. size elements. The web depth contained one element, and at the flanges the element widths were 7 ft. wide so that the nodes coincided with the web centre lines. The plate properties assumed were:

Youngs modulus = 2,500,000 lb/in$^2$

Poisson's ratio = 0.15
4.1.3 William & Scordelis (149) right deck two-cell box girder example.

The slab in this case is simply supported at both ends of the bridge and a rigid diaphragm assumed to act only within the outer webs (see Fig.13). The points of support are along the centre line of the webs at the rigid diaphragms, the span between the supports being 60 ft. A concentrated point load of 1 kip (1000lbs) is again applied over the outer web at the mid-span section.

The computer analysis was carried out on one half of the bridge deck, making use of symmetry about the centre line. The analysis involved 130 elements and 132 nodes, the maximum joint difference being 18 (maximum band width = 108 coefficients). Even so, a fairly coarse mesh division was employed, with one element along the depth of the web, four elements representing the width of the bottom flange and six elements along the width of the top flange.

The transverse distribution of vertical deflections is compared with that obtained by William and Scordelis in Fig.14. The transverse distribution of the longitudinal membrane stress resultants compares favourably with that presented by William & Scordelis (Fig.15).

4.2 Non-linear elasto-plastic analysis

4.2.1 Imperfect Plate subjected to uniaxial compression.

Moxham (91,92) carried out both experimental and theoretical work to obtain an elasto-plastic solution for the above plates. His work has therefore been used to test the
accuracy of the program in the elasto-plastic range. Moxham (92) used a Ritz approach with eight Fourier coefficients representing the deflected shape of the plate, while the plate had five divisions in its depth. An iterative minimisation method was utilised in finding the stationary energy configuration. Moxham (92) concluded that long plates would deform in approximately square buckles with aspect ratio approximately 0.875. He therefore analysed plates with these aspect ratios, simply supported on the sides in order to predict the behaviour of longer plates.

Although several breadth to thickness (b/t) ratios were investigated by Moxham, the range chosen for comparison using mild steel plates was a b/t ratio of 55 only. This was because the elastic critical buckling load was close to the squash yield load of the plate exploiting the full large deflection elasto-plastic capabilities of the program.

Bradfield (11) carried out further theoretical work using Moxham's computer programs, and presented results for varying imperfections for the same plates. The imperfections chosen were sinusoidal along the length and breadth of the plate, the imperfection \( \delta_0 \) representing the maximum initial deformation in the centre of the plate. In the present work, an initial imperfection of \( \delta_0 = 0.005b \) was used, where \( b \) = width of the plate. The load shortening curve obtained was satisfactorily sandwiched between the imperfections of \( \delta_0 = 0.003b \) & 0.01b used by Bradfield (see Fig.16)
Crisfield\(^{(27)}\) also produced results for the above case with \(\delta_0 = 0.005b\), using a modified Ilyushin criterion, and his results are also shown in Fig.16. An imperfection sensitivity plot of Bradfield's results gives an estimated maximum average stress value for \(\delta_0/b\) of 0.005 equal to 0.725\(\sigma_0\), which compares more favourably with the value given for the present work (0.74\(\sigma_0\)) than with Crisfield's modified yield criterion (0.68\(\sigma_0\)). \(\sigma_0\) represents the yield stress. Although it is accepted that Ilyushin's\(^{(68)}\) approximate yield criterion gives an upper bound to a more exact von-Mises approach, Crisfield has attempted to correct for this by allowing for intermediate plasticity. The uniaxial moment-plastic curvature relationship is closely approximated by introducing an equivalent plastic curvature.

Fig. 17 shows the variation of the edge reaction with increasing uniformly prescribed displacements. As can be seen, the stress distribution adopts the forms anticipated for the elastic range given by closed form solutions.

4.2.2 Square clamped plate under uniformly distributed load

Hooke and Rawlings\(^{(64)}\) suggest that a permanent set or a limit on the maximum deflection would be a more relevant design criterion in fully clamped plates. This is due to the significantly large reserves of strength possessed by such plates after the onset of yielding. Due to the absence of reliable permanent set data, design procedures have assumed limiting stress conditions instead. They therefore conducted
experiments on a clamped plate, the results of which are compared with the large deflection elasto-plastic solution using the program. The poor comparison obtained is due to the inexact boundary conditions simulated by Hook & Rawlings, giving a more flexible plate than the finite element solution. The clamped edges had 'pulled in' at various stages (note the reversals in the load-deflection plot (Fig.18)) having a significant effect on the plate behaviour. Crisfield's (27) results again show a more flexible plate than the author's on account of the approximate (modified) yield criterion he adopts to account for plastic curvature.

The small deflection elastic and the large deflection elastic finite element results are almost identical with Timoshenko's classical solution. The small deflection elasto-plastic plate solution is again stiffer than Crisfield's (27) as expected, and lies between the largest lower bound and smallest upper bound solutions provided by Hodge and Belytschko (62) and Ranaweera and Leckie (101). The author's solution has not been provided over a satisfactory length of the plateau because of the extremely slow convergence in this region. Also, small deflection elasto-plastic solutions are not being considered in the main body of this dissertation.
All dimensions in mm
Wall thickness = 10 mm
Youngs modulus = 200 000 N/mm²
Poisson's ratio = 0.25

(a) symmetric load  (b) antisymmetric load

Loads applied at the tip of the cantilever

Fig. 9 Square Cantilevering Box Beam
SS = Simply supported

All deflections & stresses plotted refer to midspan cross-section

Fig.10 Scordelis' four cell box girder bridge
Fig. 11(a) Transverse distribution of midspan deflections

Fig. 11(b) Longitudinal moments at midspan cross-section
Fig. 12 Transverse distribution of transverse flange moments at the midspan cross-section
Young's modulus = 4.32 × 10^8 lb/ft^2
Poisson's ratio = 0.15
Deflection (10^{-4})

Fig. 14 Transverse distribution of top flange deflections at the midspan cross-section (William & Scordelis' example)

Flange stress resultants __________ 100 lbs/ft
Web stress resultants __________ 50 lbs/ft

Fig. 15 Transverse distribution of longitudinal membrane stresses at the midspan cross-section of the twin box (William & Scordelis' example)
Fig. 16 Relationship between average load and end shortening for a simply supported plate with $b/t = 55$, under uniaxial compression.
FIG. 17 DISTRIBUTION OF STRESS AT THE LOADED EDGE WITH INCREASE IN UNIFORM PRESCRIBED DISPLACEMENTS. (IN-PLANE)
Fig. 18 Relationship between load and lateral central deflection for a square fully clamped plate under uniformly distributed load.
CHAPTER 5

COMPARISON OF THEORY WITH MODEL BOX GIRDER
TESTS UP-TO COLLAPSE

Not many experiments have been conducted on complete box girders up to collapse where stresses and deformations have been recorded, mainly on account of the expensive fabrication and testing procedures, requiring large testing equipment. In 1972, however, the Department of the Environment commissioned a series of tests at Imperial College, London on four symmetrically loaded unstiffened box girder diaphragms situated at the ends of model box girders. These diaphragms were labelled diaphragms D1, D2, D3 and D4. These have all been investigated in depth by the author in a joint paper by two dimensional idealisation, where it was stated that diaphragm D1 collapsed with pure elastic buckling and diaphragm D4 collapsed under pure squash failure. Diaphragm D2 is in the middle range of breadth to thickness ratios (b/t), producing a combination of instability and plastic deformation, and was therefore chosen for the analysis in three dimensions (3D) using the present program. Diaphragm D4 is identical to D2 but rotated in-plane through 90°.

The experimental diaphragm D2 was located at the end of a model box girder, which has been idealised for analytical purposes as shown in Fig.19. At some distance away from the Diaphragm D2, a substantially rigid intermediate load bearing diaphragm was provided, over which was positioned a high capacity reaction spherical bearing during testing.
This position along the box was represented by the idealised fixed end. Jacks were positioned under a similarly stiff diaphragm further away to provide the balancing moment from the reactions.

Uniformly prescribed displacements were applied to the end diaphragm D2 in the computer analysis at the nodes signifying the bearing. The portion of the box between the end diaphragm D2 and the intermediate diaphragm was given all the measured dimensions and properties of the test specimen, including the square holes positioned on the top flange for access into the box girder.

The computed failure load (2960 kN) was within 7% of the experimental value (2770 kN). The discrepancy was partly due to the fact that the Ilyushin's yield criterion adopted gives an upper bound to the collapse load on account of assuming 'full-section' yielding in the plate depth. Another reason is that, as Dean noted, the ductility in the bearings, lower flange and intervening weld zone caused the peak experimental reaction to move away from the bearing edge towards the centre of the diaphragm. This phenomenon therefore weakens the structure, causing an earlier collapse than anticipated. Curves 1 for the experimental and analytical vertical membrane stress in Fig. 20 indicate the shift in the experimental peak reaction towards the centre of the box during the earlier loading stages. Since the mesh division is too coarse near the edge of the bearing, this behaviour is not observed at higher loads when the stress gradients are very steep (curves 2, Fig. 20).
In spite of the scatter in the measured imperfections of the diaphragm D2, the experimental and predicted out of plane deformations up to collapse agree, as shown in Fig. 21. The idealised out of plane initial imperfection was assumed to be sinusoidal, with an amplitude of 2.03mm. This was assumed to give the best fit to the measured imperfections. The largest variations in the idealised and measured imperfections were just above the diaphragm bearing and in the top half of the diaphragm. This variation is reflected in the out of plane deformations up to collapse (see Fig. 21). The predicted plastic zones are shown in Fig. 22.

The web shear stress adjacent to the Diaphragm D2 for the predicted and experimental cases are shown in Fig. 23, and the shear stress in the diaphragm in Fig. 24. The horizontal membrane stresses in the diaphragm up to collapse for the two cases are shown to be compatible, although the predicted (analytical) neutral axis is slightly above that observed experimentally (see Fig. 25).

It has been commented on elsewhere (29) by the author that the experimental collapse loads of the end diaphragms had been influenced by the rotational restraint provided by the bottom flange, and could be dependent upon such factors as the direction of the geometric imperfections (towards the inside or the outside) of the box. The author decided to analyse the box girder diaphragm with the initial imperfections inside the box (as opposed to the experimentally observed outward imperfection) in order to observe any differences.
While it was observed that the collapse load was identical to that with initial imperfections out of the box, the out of plane deflections, particularly in the lower half of the diaphragm were slightly larger, indicating that this would be the preferred mode of failure. It is likely that a more elastic plate (high breadth to thickness ratio) would present a lower collapse load when the initial imperfections were towards the inside of the box. This is because plasticity would not intervene in the disturbing moment provided by the bottom flange when the diaphragm deforms inside the box.

In conclusion, it would appear that the finite element program developed herein has produced results which compare very favourably with these experimental test results.
Young's modulus (diaphragm) = 209,700 N/mm²
Yield stress (diaphragm) = 389 N/mm²
Yield stress (flanges) = 306 N/mm²
Poisson's ratio = 0.3
Initial central deformation (diaphragm) = 2.03 mm

Fig. 19 Structural idealisation, dimensions and mesh for the analysis of the Imperial College, experimental box with end diaphragm D2.
Fig. 20  Vertical membrane stress in diaphragm, 51 mm from the bottom flange
Fig. 21. Deflections under load at the diaphragm centre line.

Fig. 22. Predicted plastic zones in the diaphragm at collapse (2960kN)
**Fig. 23.** Web membrane shear stress at 51 mm from the mid-thickness of the diaphragm.

- Bearing reaction = 2430 kN
- Bearing reaction = 2760 kN
- Expt. (Imperial College)
- Finite element (NOLP)

**Fig. 24.** Diaphragm membrane shear stress at 254 mm from the centre-line of the diaphragm
Fig. 25. Horizontal membrane stress along the diaphragm centre line
CHAPTER 6

INTERACTIVE COLLAPSE OF BOX GIRDER IN THE SUPPORT REGION

6.1 Box Girder Idealisation at the Support

One of the most highly stressed parts of a continuous box girder is the support region, particularly when cantilevered construction methods are employed. The following study is therefore entirely devoted to the nature of the interaction of the diaphragm-flange-web assemblage and its effect on the overall collapse behaviour.

In the present study, the idealisation is simplified and equal spans are assumed on either side of the support. One arm is purported to represent the portion of the box from the support to an assumed point of contraflexure in the continuous portion, which is therefore allowed to carry shear, but no bending. It is assumed that the positions of zero bending moment for all plate panels occur at this hypothetical point of contraflexure of the box section, and the plates are assumed to have simply supported boundary conditions at this location. The other arm is allowed to represent the cantilevering span.

Further, in order to suppress rigid body effects, although in practice the end of the cantilever is likely to deflect more than the point of contraflexure, the increments of displacements are applied equally to both arms. This was however not done for the preliminary studies, where displacements are only incremented at the tip of the cantilever, the point of contraflexure being kept stationary. Rigid body effects will
therefore be observed in the deformation patterns plotted for the preliminary studies. Displacements are incremented rather than loads so that the complete load deflection path can be monitored satisfactorily until collapse.

The finite element mesh division (see Fig. 26) was made as fine as practicably possible so that all the components (including the diaphragm) could be modelled to represent the buckling deformation, and also so that the spread of plasticity could be monitored to a reasonable accuracy.

It may be worth mentioning at this juncture that were it possible to use substructuring facilities satisfactorily at the support so that the 'continuous' arm of the cantilever be analysed first and the relevant eliminated stiffness coefficients fed into the 'cantilever' substructure, then a finer mesh division could have been possible. This is because the band width (at the support) could have been almost halved by the substructuring treatment, allowing more storage area in the CDC 7600 computer than is used for the present treatment. This approach was discarded for two reasons. Firstly, the solution system used is not suitable for substructuring. Although this is not an insurmountable problem, the second reason was sufficient to abandon this approach. The reason is that it would become necessary to treat the continuous arm as linear elastic, so that no large deflection effects, and more important, no plastic flow could be allowed in this region. Even if stocky plates were assumed so that these effects would not be predominant, the treatment would be far from satisfactory.
The more simplified treatment of using special beam elements to represent parts of the box girder away from the support in a linear elastic manner proved more attractive. A refined mesh could therefore be used in the support region for the large deflection elasto-plastic treatment.

Brief sections of the ensuing work have been published in a joint paper by the author (110).

6.2 Plate Parameters

The aspect ratios (breadth to thickness) of most steel plates in bridge design fall within the intermediate range where plasticity intervenes in any elastic buckling failure. Accordingly, the aspect ratios chosen for the idealised structure were in this range. The diaphragm shape was chosen so that it was similar to the Diaphragm D2 discussed in Chapter 5, because its stress range was more likely to be encountered in bridge construction. Fig.27 gives a cross-section of the box girder configuration chosen for the main analysis. The length of the box girder may be varied to suit the particular moment/shear ratio required.

6.3 Imperfections

All the initial deformations are assumed to have a sinusoidal shape and formed in approximately square buckle shapes. The amplitudes of the sinusoidal shape are obtained from the Merrison Part III rules and are based on the Part IV tolerances (36). These are the maximum imperfections allowed by the Rules for each of the panels, and for imperfection sensitivity, all imperfection magnitudes will be proportionately varied. Therefore the amplitudes of the imperfections in the
individual plates will be taken as the same factor 'f' of their maximum values according to the Merrison Rules. The Merrison imperfections (f = 1.0) for the panels in the analyses are given by

- Diaphragm: 2.4mm
- Compression flange: 2.4mm
- Tension flange: 4.8mm
- Web: 3.9mm

The positive directions and form of these imperfections are given in Fig. 26.

6.4 Preliminary Studies

The reasons for this preliminary study are twofold. Firstly, to check whether the plate parameters chosen are acceptable, and in the range where the critical buckling of each component can be easily adjusted to occur close together. Secondly, it is necessary to determine whether the diaphragm or flange exhibit any preferred mode shape at failure, so that the initial deformations can be provided in a sympathetic form.

The initial deformations of the web, which are also represented in square buckle shapes, do not imitate the shear buckles associated with the large deflections in the web. This is partly because such shapes are difficult to form into imperfection modes, and partly that such imperfections are unlikely to occur in web panels. Also, as observed by Dowling et al., the strength of restrained shear panels such as webs is almost independent of the level of geometric imperfection, and therefore the mode shape is unlikely to influence the local buckling or the final overall collapse.
The imperfections in the web are therefore kept in phase with those in the flange so that the rotations of the edges of the web plate are in the same direction as the rotations of the flanges. Fig.26. shows the form of the web imperfections relative to those of the flange and these will be considered henceforth.

As observed before, deflections are only incremented for the preliminary studies at the tip of the cantilever, and therefore rigid body effects will be seen. Also the deformation patterns for this pilot study are only plotted for the 'cantilever' arm.

6.4.1 Check on parameters (Box Girder Al)

Only the web thickness is different for the preliminary studies from that chosen for the main study. All other parameters are identical (See Fig.28). The lever arm either side of the support is provided as 2400mm. The imperfection factor $f$ is unity (i.e., the maximum values of imperfection amplitudes of the sinusoidal shapes allowed by the Merrison Part III rules and based on the Part IV tolerances\(^{36}\)). The positive values of the imperfections are assumed as in Fig. 26 (for $f = 1.0$).

At the load of 2335kN (failure load = 2340 kN), there is noticeable yielding along the shear buckles of the web (see Fig.29). There is appreciable yielding along the shear buckles of the diaphragm. However, most of the yielding in the bottom flange extends outwards from the edge of the bearing and is caused by the local bending effect due to the indentation of the bearing into the box.
Bearing indentation is defined as the in-plane displacement of the diaphragm at the bearing relative to the mean displacement of the diaphragm-web interface. The relationship between this bearing indentation and the load bearing capacity of the box-girder at the bearings is presented in Fig. 30.

This analysis showed that the overall box collapse was almost entirely influenced by diaphragm failure. The bottom flange could only have had a very small effect. There is only extreme fibre yielding in the bottom (compressive) flange on the ridge of the buckle. The results confirm that the parameters chosen for the analysis are close to the range that would influence an interaction between the diaphragm-flange. It also showed that for further studies, it may well be advantageous to chose a slightly higher web thickness, so that the diaphragm-flange interaction could be studied where very little (if any) shear yielding in the web may occur, and not much tension field action is allowed in the webs which might influence the results.

6.4.2. Check on Buckling Loads (Box Girder A2)

The parameters and spans are similar to those of A1, while the imperfection magnitudes are reduced. For compression flanges, if $\delta_0/b$ (imperfection amplitude as a factor of the width) is kept very small, it would behave as a perfectly flat plate, so that any numerical instability would indicate the buckling load of the components. In this way, it would be possible to check whether the parameters chosen gave panel buckling loads within a close range under the true boundary and loading conditions existing in the box girder. Also, any potentially large differences due to the levels of imperfection could be ascertained.
The parameters were initially chosen for the web and flange based on classical buckling theory for idealised simply supported boundary conditions and loading\(^{(15)}\). The elastic critical buckling load of the diaphragm was calculated from numerical integration of the governing differential equation, with the deformation pattern as described by Khan & Walker\(^{(72)}\).

The imperfection amplitudes on all plates were chosen as \(\delta_0 = 0.2\) mm, so that \(\delta_0/b\) for the compression flange was of the order of 0.0002, and considered to be almost flat\(^{(11,92)}\).

It was observed during the incremental procedure that numerical instability appears when the bearing reaction is in the range of 1600 kN to 1900 kN (see fig.30). Extremely small increments of deformation had to be provided so that the solution would not diverge and that negative values in the diagonal term of the overall stiffness matrix were avoided. When the increments (taken on a trial and error basis) are too large, negative terms in the stiffness matrix occur in the nodes of the web, indicating local buckling of the web. After the reaction of 1900 kN was achieved in the bearings, higher increments were taken, until the overall collapse of the structure occurred at a bearing reaction of 2678 kN (see Fig 30).

6.4.3 Preferred mode of failure (Box girder A3)

On account of the symmetry about the support, the direction of the imperfections shown in figure 26. would give similar collapse loads if reversed. However, the region of interest is the arm representing the cantilever span. Since for this study the tip of the cantilever is given increments in displacements, while keeping the end of the other arm (point of contraflexure on the continuous span) stationary, it would be
Interesting to note whether the rigid body effects are likely to affect collapse or stresses.

The imperfections shown in Fig. 26 were reversed in direction, keeping the parameters, spans and imperfection magnitudes the same as for Al. The bearing reaction at collapse is observed to be about the same (2334 kN) as for box girder Al. The rigid body motion only slightly affected the stresses in the diaphragm. However, the spread of plasticity and deformed shape of the bottom (compression) flange within the cantilever span is noticeably less than in Al (see Fig. 31).

This observation proves that in the region of interest, the imperfection pattern shown in Fig. 26 produces more severe stresses in the flanges of the cantilever span on account of the larger deformations produced at collapse. This is because the lever arm between the top and bottom flanges is progressively reduced by the flange plates approaching each other for this assumed imperfection pattern. This consequently results in higher internal forces in the flanges so as to counteract the externally applied bending moment in the box. Inspection of figures 29 and 31 illustrate this point.

Another outcome of this comparison is that the rigid body rotations in the support region affect the stresses and displacements only slightly, and it would be justifiable to do away with this secondary effect altogether, when carrying out the main series of analyses.
6.4.4 Check on the web thickness for lower level of shear yielding (Box Girder Bl)

The web thickness provided in box girders A1 to A3 was increased from 7 mm to 7.5 mm in the hope that shear yielding, which may redistribute stresses, does not occur. The parameters used are shown in Fig. 27 and these values are also used in the main analyses. As for cases A1 to A3, the length of the arm on either side of the support is kept unchanged at 2400 mm. The initial imperfection 'f' = 1.0.

Collapse occurred at a bearing reaction of 2350 kN. The value for box A1 was slightly less at 2340 kN. There are signs of first fibre yield only at the ridges of some of the shear buckles in the web, but not as extensive as in A1. As expected, the out-of-plane deformations in the thicker web are smaller, but there is no appreciable change in the out-of-plane deflection or yielding in the bottom flange (see Fig. 32). Although there is more out-of-plane deformation and extent of yielding in the diaphragm than observed in box girder A1, this was on account of the larger shear stresses that could be taken by the plate prior to collapse. The thicker web allowed less interaction between the flange and diaphragm. This is to be further substantiated later in the chapter.

6.4.5 Representation of a perfectly flat plate (Box girder B2)

It was observed in box girder A2 that the imperfection amplitudes, although very small, were not small enough to show more precisely the bearing reaction where any local instability in one of the components occurs. Zero values cannot be taken as the numerical treatment would terminate at the elastic critical load. The amplitudes of all the panels were given a nominal value of 0.001 mm, with parameters and spans as in B1.
There was numerical instability in this case also, but the treatment was slightly better than for A2. Fig. 33. shows the load-bearing indentation relationship for this case. The small discontinuity shown in the curve is produced by buckling deformations in the web. A few increments in this region had to be taken with a straight incremental procedure that allowed no iterations or carry over of the out of balance forces from one increment to another. After the buckling load, increments were taken normally (as before) using the modified Newton-Raphson procedure with iterations at each increment.

6.5 Variation of the moment/shear ratio

This parametric variation is achieved by lengthening or shortening the spans of the box girder equally on either side of the support. On applying equal displacement increments at the extremities of both spans, the shear stress in the webs and diaphragm remain the same for each magnitude of displacement, while the applied bending moment in the flange (varying linearly away from this support) is changed in proportion to the variation in span. This variation of the moment–shear ratio on the box girder without altering any other parameters, is performed in order to study the interaction between the compression flange and diaphragm on the collapse load of the structure.
The variation in box girder length is achieved by utilising the special beam elements so that the mesh refinement of the finite elements could be maintained. However, in order to allow the panels to assume their desired buckling modes, a large aspect ratio (1:4) was maintained by modelling the appropriate box girder length in finite elements. In this way, the panels are not constrained to follow a particular deflection pattern. The box girder arms on each side of the support (referred to henceforth as lever arm) were varied from 800 mm to 7200 mm in order to study the full effects of interaction.

The box girders analysed are denoted as ABl, B1, C1, D1, E1 & F1, and their 'lever arms' are 800 mm, 2400 mm, 3000 mm, 3600 mm, 4800 mm and 7200 mm respectively. The imperfection factors provided in all these cases is $f = 1.0$ for the purposes of the comparison, while the mode of imperfection is as in Fig. 26.

6.5.1 Simplified treatment of individual components

In order to find some correlation for the overall collapse behaviour, the individual panels that affect collapse (compression flange and diaphragm) are given idealised boundary conditions and simplified loading.

6.5.1.1 Bottom Flange

The bottom flange is treated as a long simply supported plate with aspect ratio 4 and square sinusoidal imperfections are given, with imperfection factor $f = 1.0$ as in the case of the bottom (compression) flange of boxes ABl, B1, C1, D1, E1 and F1. The long plate was subjected to uniaxial shortening, and under load, the aspect ratio of the buckles shortened to a factor of 0.869. It was later observed that
the aspect ratio varied for different imperfection levels. This reinforces Moxham's conclusion that long plates deform in a series of buckles with aspect ratios close to 0.875, so that a single buckle could be treated as a simply supported individual plate.

Fig. 34 gives the total end shortening load \( F \) required to collapse the long plate as 2182.8kN for an imperfection factor \( f = 1.0 \). For the sake of consistency, it will be assumed that the aspect ratio of the deformed buckles at collapse remain standardised at 0.875, so that the peak of the buckle, where the maximum stress precipitates failure is at half that length \( (0.875/2) \). Now, if

\[
\begin{align*}
    b &= \text{width of the flange.} \\
    c &= \text{distance from the support to the peak of the dominant flange buckle, } b \times 0.875/2. \\
    \lambda &= \text{lever arm (length from the support to the tip of the cantilever)} \\
    d &= \text{distance between centre lines of top and bottom flange.} \\
    F &= \text{total end shortening load of the flange at collapse.}
\end{align*}
\]

The bearing reaction required to determine flange collapse at the peak of the dominant flange buckle is calculated from statical principles of equilibrium as:

\[
P = \frac{2Fd}{(\lambda - c)}
\]

This bearing reaction \( P \) was calculated for different lever arms and produced as a flange simply supported (S.S) collapse curve in Fig. 35 as a means of correlating the results of the three dimensional collapse analyses of the box girder.
In a similar manner, the bearing reactions required to achieve the full plastic moment at the support are calculated and plotted in Fig. 35 for comparison.

6.5.1.2 Diaphragm

The treatment of the diaphragm is as described by the author in a joint paper\(^{(29)}\). The two dimensional idealisation is similar to the recommendations in the Merrison Interim Design Rules\(^{(36)}\). The structure is partitioned along the web-diaphragm junction and the web analysed with associated areas of flange in a linear elastic manner. This determines the flexibility coefficients along the web-diaphragm junction that would be used to treat the in-plane boundary conditions for the diaphragm analysis.

The diaphragm is analysed for the effects of large deflection and plasticity, with the associated triangular areas of the flange (see Fig. 36) treated as elasto-plastic line elements. Two idealised boundary conditions were chosen:

1. all sides simply supported.
2. all sides built in.

The imperfections for the simply supported case are exactly similar to those for the diaphragm in the three-dimensional box (i.e. sinusoidal). Therefore, the imperfection factor \( f = 1.0 \) gives the same imperfection amplitudes to both the simply supported diaphragm and the diaphragm in the three-dimensional box. On the other hand, for the diaphragm with built-in edges, the shape of the imperfections is not similar to the above two cases. Although these are sinusoidal the built-in edges allow no slopes in the imperfections at the edges. The shape is therefore as shown below, where the wave

\[\text{imperfection} \]
lengths of the sinusoidal imperfection are half those of the above two cases. The amplitude of each sine wave is also halved, so that the total magnitude of the imperfection at the centre of the panel is the same as for the simply supported case for any imperfection factor (f). The average curvatures of the imperfections in the two cases are therefore the same (for comparison of their behaviour).

It is assumed that the triangular effective areas of flange associated with the diaphragm ($A_{fd}$) are consistent with a line of zero transverse shear flow in the flange, and for the internal diaphragm considered, varies linearly from $bt_f/4$ at the centre to zero at the diaphragm, where

$$b = \text{width of the diaphragm.}$$

$$t_f = \text{flange thickness.}$$

The idealised web thickness ($t_{wi}$) for forming the flexibility matrix is taken as twice the actual web thickness ($t_w$) to represent the structure either side of the diaphragm. The associated effective flange area acting with the idealised web (thickness = $t_{wi}$) is given by

$$A_{fw} = \frac{bt_f}{2}$$

The diaphragm is then 'loaded' in a similar manner to the three dimensional assembly by incrementing constant prescribed displacements. Fig. 37 gives the load-bearing
Indentation relationship for imperfection factor $f = 1.0$ for the two relevant cases. The bearing reaction at collapse is given as

1755 kN for the simply supported case
and 2485 kN for the built-in case.

These two bearing reactions are shown in Fig. 35 as horizontal datum values. Other reference loads are the bearing reactions at shear yield (2910 kN) and elastic critical buckling (2099 kN). The latter value is determined by a finite element eigen-value solution of the geometric stiffness matrix where the general treatment is similar to that described above, assuming the diaphragm sides to be simply supported. The idealised bearing width (450 mm) is assumed to represent a slightly higher actual width of bearing $\approx 650$ mm. The actual width represented however is somewhere between 450 mm & 650 mm, so that squash yield could be anywhere between 2182 kN and 3152 kN, and within range of the shear yield. It is therefore not shown as a reference load in Fig. 35.

6.5.2 Interaction Curve

The collapse loads achieved on the box girder structure are plotted for the varying lever arms (or moment/shear ratios) in Fig. 35. For the short lever arm, where interaction with the flanges during collapse is negligible, the collapse load is the same as with a two dimensional idealisation where the junctions with the webs and flanges are assumed to be built in. The influence of the flange on the behaviour at collapse grows with an increase in the lever arm.
Inspection of Fig. 35 for the larger lever arms suggests that however predominantly the flange influences collapse, there is some reserve of strength past the individual flange collapse load as treated in 6.5.1.1, before it induces collapse in the diaphragm. Except for the most 'interactive' region (where the results for the individual components coincide), the overall box collapse occurs between the individual flange (as in 6.5.1.1) collapse load and the load at which full plastic moment occurs at the root of the cantilever.

The curve appears to be analogous to the column 'tangent modulus' formula (18), where similar behaviour is observed. For the imperfect structure plotted, the curve has a gentle curvature and not a discontinuous kink in the highly interactive region, as would be observed with a perfect structure. On account of the imperfections, this 'interactive' region exhibits an area where both the built in diaphragm and the simply supported flange give unsafe results for the collapse load. The diaphragm may therefore be assumed to be simply supported similar to the flange, and the two independent analyses conducted for determining a safe estimate of the collapse load of the overall structure.

Plotting the Diaphragm S.S (simply supported) and flange S.S. (simply supported) values as in Fig. 35, the curves can now be used as safe lower bounds for determining collapse. Hence the idealisation of the diaphragm as simply supported (29) appears to be a safe assumption. Although at first this may appear to be a statement of fact, it was not obvious until now, because there was the possibility that during interaction, the lower flange could produce a 'disturbing moment' (rather than a restoring moment) at the base of the
As it happens, the short continuous box appears to produce a restoring moment on the diaphragm which restrains the boundary, so that in the case of the short lever arm, a built-in effect is obtained.

The overall collapse is therefore dependent upon the strength of the weaker of the two components (flange and diaphragm), the web appearing to exhibit considerable post-buckling strength.

6.5.3 Detailed Behaviour

For the cases with longer lever arms, (where imperfection factor \( f = 1.0 \)) flange failure appears to trigger off a collapse of the diaphragm by inducing high bending stresses at the intersection of the compression flange and the diaphragm. For the shorter lever arms, the diaphragm exhibits a largely shear failure and can be seen in Figs. 32, 38, 39 and 40 in the form of shear yield lines for Box Girders B1, C1, D1 and E1 respectively. With increasing lever arms, higher bending stresses are induced in the base of the diaphragm, reducing the amount of shear yielding and causing yield to occur closer to the base of the diaphragm.

When the moment shear ratio is very large the failure load of the box occurs at an intermediate value between the loads calculated for the conventional fully plastic moment and for the elasto-plastic collapse of the flange (see Fig. 35)

Some numerically unstable solutions resulted from the analysis of Box girders B1, C1 and D1, indicating buckling of the web prior to overall structural collapse. The numerical treatment was made to converge onto the correct equilibrium
path by taking smaller increments. If the plate panels were treated as nominally flat \( (f \approx 0) \), small discontinuities would occur in the load-bearing indentation plot at the local buckling load of the web. This treatment was only considered for boxes B1 and C1, and is discussed in section 6.6.3. With imperfections, the small discontinuities are lost. It is however, well known that webs exhibit considerable elastic post-buckling strength, and their initial buckling values would not significantly affect the collapse of the box girder.

The load-bearing indentation curves for all the cases (Boxes AB1, B1, C1, D1, E1 & F1) are presented in Fig. 41.

6.5.3.1. **Box Girder AB1**

This was the smallest moment shear ratio analysed. The collapse load for this lever arm may be observed to be identical to the collapse load of the diaphragm when idealised as built in (as described in 6.5.1.2). This confirms the validity of the approach adopted for the two dimensional treatment applied to analyse the box girder collapse, where the webs are treated in a linear elastic manner. The web thickness was made sufficiently large so that throughout the treatment, very little plastic flow was observed (see Figs 38, 39 & 40). Although buckling deformations occur in the web before collapse producing non-linear geometric effects, they do not seem to have had any noticeable effect on the collapse load. The collapse of the box girder (due to diaphragm failure) with the shortest lever arm occurs at the same load as that of the diaphragm when treated (as in 6.5.1.2) with the sides built-in.
6.5.3.2. **Box Girder B1**

This box has already been discussed in 6.4.4. Fig. 32 shows the out-of-plane deformations and extent of yielding. The two inclined regions of yielding indicate the ridges of the shear buckles that occur in the diaphragm.

6.5.3.3 **Box Girder C1**

Fig. 38 indicates the deformation pattern and extent of yielding at collapse. The effect of shear lag can be observed in the bottom flange, when full section yield first occurred at the web-flange-diaphragm junction and spread rapidly towards the dominant buckle at collapse. Most of the shear yielding in the diaphragm occurs in the dominant shear buckle at the base of the diaphragm. The smaller buckle does not get an opportunity to yield on account of the larger bending stresses induced in the diaphragm by the flange. This forces a premature collapse of the diaphragm by forming a mechanism.

6.5.3.4 **Box Girders D1,E1 & F1**

With increasing lever arm, the smaller influence of shear in the collapse of the diaphragm (and ultimate collapse of the box girder) is evident in Figs. 39 and 40, for boxes D1 and E1. The influence of bending stresses on the collapse can be seen in the concentration of yielding in the vicinity of the bottom flange, where for box E1 (Fig. 40) there is a very narrow bend of yielding which forms the mechanism in the diaphragm.
The bottom flange has yielded fairly extensively in all those cases, with full section yield spreading across the whole flange width at collapse. This indicates that the flange/web girder can carry bending forces at loads in excess of the component flange collapse loads. This happens because the three dimensional box assemblage provides alternative load paths until the eventual collapse mechanism occurs in the overall structure.

6.6 Variation of Imperfection levels

6.6.1 Suitable Parameters for Imperfection studies

The individual plates were designed so that, without interaction, they would each buckle at loads close to yield. Such a structure would be expected to exhibit significant imperfection sensitivity. An example of the possible extreme imperfection sensitivity of plates that bifurcate in the plastic range is given by Onat and Drucker (104). Inspection of Fig.35 shows the way in which the interaction curve (for \( f = 1.0 \)) falls below the idealised curves for flange and diaphragm failure, and particularly so at the 'knee' of the idealised curve. In this region, the collapse load is particularly 'imperfection sensitive'. Of the analyses conducted so far, box girder Cl appears to be the closest to the 'knee' of the idealised interaction curve. Therefore, the study of imperfection sensitivity is carried out only for the parameters considered for box Cl (lever arm = 3000mm).

The imperfection factors \((f)\) chosen were 0, 0.375, 1.0 and 2.0. The perfectly flat plated structure was simulated by assuming all imperfection amplitudes to be very small (0.001mm).
6.6.2 Individual Components (Flange & Diaphragm)

As before, in order to understand the overall behaviour of the box structure, it is also necessary to study the behaviour of the individual components (flange and diaphragm) under idealised boundary conditions.

6.6.2.1 Bottom Flange

The idealisation procedure and treatment of the flange has been described in detail in 6.5.1.1. Fig. 34 gives the compressive load-shortening curves for the different imperfection factors chosen for the investigation, as well as the compressive collapse loads. The bearing reactions in Fig.42 are then calculated (see 6.5.1.1) for the various levels of imperfections.

The imperfection factor \( f = 1.0 \) in the idealised flange corresponds to the initial deformation in the bottom flange of the box girder when the imperfection factor \( f = 1.0 \).

6.6.2.2 Diaphragm

In order to cover the range of the possible collapse loads, the diaphragm was idealised as a plate simply supported as well as built in on all sides. The procedure described in 6.5.1.2 for the two dimensional idealisation was followed in obtaining the bearing reactions at collapse. Fig.37(a) and (b) give the load-bearing indentation plots for the various levels of imperfections chosen for the two cases. The collapse loads are plotted in Fig.42 to provide values for comparison. Section 6.6.3 discusses these results in detail.
The imperfection factor \( f = 1.0 \) in the idealised simply supported diaphragm corresponds to the initial deformation in the box girder diaphragm which has an imperfection factor \( f = 1.0 \). This also applies in the case of the built-in diaphragm, although the shape of the imperfection is different. The wave length (for the built-in case) is half that for the simply supported case, on account of the initially deformed slopes at the built-in edges being maintained zero (see 6.5.1.2).

6.6.3 Imperfection Sensitivity

Fig. 43 describes the load-bearing indentation relationship for the imperfection levels chosen. Analyses for all but the perfect structure were conducted without many computational problems. The modified Newton-Raphson iterative scheme was capable of allowing convergence of the solutions up to collapse. However, iterations had to cease for the perfect box \((f \approx 0)\) just prior to web buckling (note the slight kink in the \( f = 0 \) curve, Fig.43). The dotted line for this curve indicates the purely incremental solution adopted until collapse.

An interesting observation is that the perfectly flat box achieves its full plastic moment capacity at the support. For all other values of imperfections, collapse was achieved by the bottom flange inducing a bending moment at the base of the diaphragm, the influence of the flange increasing with larger imperfections. In this instance, the growth of buckling deformations in the nominally flat plated box \((f \approx 0)\) does not appear to be large enough to produce plastic flow at the buckle crests, and therefore the difference in the structural behaviour between the perfect and imperfect boxes.
is radically different. Failure in the perfect box is brought about by plastic flow due to squash yield in the flanges and webs at the support, largely due to the longitudinal stress induced by the externally applied (linearly varying) bending moment. The realistically imperfect (using Merrison rules) boxes give elasto-plastic buckling deformations, which alter the load paths. Failure of the box can therefore occur anywhere between the individual component failure (flange or diaphragm) and the conventional fully plastic moment capacity of the box, provided the webs are not too thin to affect flange behaviour (see Section 6.10).

The collapse loads for each imperfection level in the box are plotted against the reference values for the individual components in Fig. 42. As surmised, the collapse strength of the three dimensional plate assemblage is more sensitive to imperfections than the strength of the individual panels analysed as in 6.6.2.

In Fig. 44, some of the relevant reference values and curves of Fig. 35 are repeated. In the box (3D) analyses carried out so far, there are only two perfect box examples ($f_o$). These, however, are adequate in showing the formation of the 'knee' of the idealised interaction curve. Lever arms 2400 mm and 3000mm occur close to and on either side of the knee, and the perfect boxes analysed were for these lever arms. The coincidence of these two collapse loads with the idealised curve indicate the analogy with the column tangent modulus formula (18), where the idealised structure forms a knee. With imperfections, the knee disappears, giving rise to flatter curves with increasing imperfections. Maximum imperfection
sensitivity would therefore occur in this region of abrupt change in the interaction curve, synonymous with the observations in the column tangent modulus studies of Calladine\(^{(18)}\).

6.7 **Collapse of the Box C2 in the non-linear elastic range - No plastic flow**

This analysis was carried out in order to verify whether simultaneous collapse of the flange and diaphragm would produce any drastic reduction in strength if, without interaction, the elastic critical buckling loads of the individual panels were much smaller than the yield values. Properties similar to Box C1 (lever arm 3000mm, imperfection factor \(f = 1.0\)) were chosen because for this moment shear ratio, the loads in the diaphragm and flange both approach their individual elastic critical buckling loads almost simultaneously. In each idealised case the sides of the individual plate are assumed simply supported.

A very large yield stress was assumed so that not even first fibre yield was detected at collapse. Any interaction would occur in the elastic range. Consistent with the properties of thin plates subjected to patch loads, \(^{(72)}\) a significant post buckling reserve was demonstrated by the box, collapse occurring at 6716 kN. The load-bearing indentation relationship for Box C2 is compared against that for box C1 (yield stress = 350N/mm\(^2\)) in Fig. 45. Fig. 46 shows the out-of-plane deformations at collapse for C2 drawn to a smaller scale (1/4) than the out-of-plane deformations for C1 (Fig. 38).
At and for a little while before collapse, the deformed slopes of the plate panels were in excess of those within the assumptions of the present thin plate theory, and therefore the collapse load of 6716kN is not reliable. In fact, the tangent stiffness matrix could not be made to form a plateau during the computational process, but diverged wildly for any higher increments, after the bearing reaction of 6716kN was achieved. As can be seen from this section, ignoring plasticity in boxes gives very large post buckling reserve.

6.8. Use of the three-dimensional (3D) results to check the two dimensional (2D) idealisation of the diaphragm

As a by product of the analytical work, it was possible to ascertain the lines of zero transverse shear flow in the flange. The triangular effective widths which in turn determine the triangular effective area of flange assumed to be acting with the diaphragm in the 2D analysis can thus be determined, being consistent with this line of zero shear flow.

These lines have been found by interpolation of shear stress between the relevant nodes for boxes B1, B2 and C1 in the top and bottom flanges (see Figs. 47 and 48).

Observations show that the effective width decreases fractionally from the earlier linear elastic loads until non-linear elasto-plastic collapse (note values for box B2). The effective width however, rapidly diminishes with increasing imperfections in the compression flange. The effect, understandably, is not so marked in the tension flange. As expected, the imperfect flange becomes weaker transversely under compression, allowing a smaller area of flange to act
with the diaphragm.

We see therefore, that the effective flange width used in the two dimensional idealisation of the diaphragm (b/4) (as shown in Figs. 47 and 48) is a conservative value even for imperfect (f = 1.0) boxes in the bottom (compression) flange. For the chosen parameters and overhang of the diaphragm from the bearing edge, an effective flange width of twice the values assumed in the analysis give a more representative figure when compared with Figs. 47 and 48. Analyses were therefore conducted on the two dimensional diaphragm (see 6.5.1.2) using effective widths of b/2. The collapse loads attained with the increased effective flange area were only about 2% higher.

Since the collapse load in the 2D idealisation is insensitive to the area of flange assumed to act with the diaphragm, these values are identical to the collapse loads achieved for the three dimensional box when the diaphragm failure does not occur due to any interaction with the other components. (as observed in fig.35 box AB1, and in Fig 44, box B2) Therefore, collapse of the diaphragm can reliably be predicted by the two dimensional idealisation.

As observed elsewhere (6.5.3.1) the collapse load of box AB1 (lever arm 800 mm, f = 1.0) is identical to that observed when the diaphragm (with f = 1.0) is assumed built in on all sides and treated as in 6.5.1.2. The out-of-plane displacement and extent of yield is very similar for the two cases (Fig.49).
There is a tendency, however, for the areas of yield to be nearer to the bottom flange for Box AB1. This would appear to be on account of the finite length of lever arm which does not give a complete built in restraint at the bottom flange. This is borne out by the comparison of the vertical bending moment (Fig.50) and the displacements (Fig.49) in this region.

The shear stress distributions for the two cases are identical, leaving little doubt as to the validity of the two dimensional method of analyses conducted in 6.5.1 and elsewhere (29). As the influence of effective areas on the collapse load (see.6.8) appear not to be significant, this method of analysis could form a basis for a design method for the diaphragm. The individual treatment of the diaphragm and the flange could be used to form an interaction curve for the safe design of the complete structure.

6.9. Variation of Web Thickness

So far, the contribution of the flange and diaphragm to the overall collapse has been studied. It is now necessary to ascertain how much the web contributes to the collapse in the structure.

Analyses were therefore conducted on the box keeping all the parameters (including lever arm of 3000 mm) the same, but varying the web thickness. For convenience, the imperfection factor was kept constant at $f = 1.0$. This meant that in spite of a varying web thickness (which would give different allowable imperfections to Merrison rules) the web imperfections were kept constant.
The web thicknesses for which the box was analysed were 10mm, 7.5mm and 4mm (referred to as boxes CO, Cl & C3 respectively). The thickest web had an elastic critical buckling load well above the collapse load. Box C3 allowed the web to buckle well below the collapse load.

Figure 51 shows the load-bearing indentation plots for the various web thicknesses. Collapse loads were 2270 kN, 2184 kN and 1938 kN for boxes CO, Cl & C3 respectively.

The analysis of the weakest box (C3) showed numerical instability at 1266 kN, beyond which it was not possible to carry out any iterations and a purely incremental approach was adopted until collapse. Also, the weak web produced highly deformed slopes in the web panel (see Fig.52), thereby going beyond the range of the present thin plate theory. All deformations (except of the web) are shown to the same out-of-plane scale as hitherto for comparison. Note however, that the web out-of-plane deformations are shown to a smaller (1/4) scale than the other panels for clarity in presentation.

At collapse, virtually all of the web had undergone plastic flow (full section yield), although deformations and yield in the flanges and the diaphragm were of the same order as those observed for box Cl (7.5mm thick web), as in Fig.38. The weaker web allowed larger deformations in the dominant flange buckle, and consequently a far greater region of yield. The deformations in the diaphragm in Fig.38 (7.5mm web) and Fig.52 (4mm web) are close to each other.
In spite of the fact that the observations on the thin webbed box are pure extrapolation of the moderately large deformation theory, it is worth commenting on some possible interaction behaviour. On comparing Fig.38 with Fig.52 at the junction of the flange and web near the dominant buckle, this edge shows a distinctly pronounced curvature. In other words, the flange cannot any longer carry loads above that of the simply supported flange (idealised as in 6.5.1.1.) which assumes simple supports along the junctions with the web. This is aptly demonstrated by the collapse load of the overall structure (1938 kN) being below that calculated for the simply supported flange as in 6.5.1.1 (2073 kN). This is to be expected on account of the large tension field action and associated large deformation that occurs in the thin web. As demonstrated by Fig.53, for the same load, the shear boot in the diaphragm for the 7.5mm and 4 mm webs are identical.

This behaviour for thin webbed boxes is reminiscent of observations on plate girders, where the flange is considerably stockier than the web, and failure is brought about by the creation of a plastic hinge on the flange. Indeed the webs on their own appear to demonstrate post-buckling strength, up to a point where practically the whole web has undergone plastic flow (Fig.52).

Some of the above observations are reinforced by very recent experimental work carried out at Liege, where it was commented that the behaviour of thin unstiffened webs (as in plate girders) comes close to that of a tie-rod girder, while webs with stiffeners tend to behave like trusses. It was concluded that it is not adequate to use the shear theory for
stiffened girders in the case of girders without stiffeners
The unstiffened webs present a high ultimate load due to post
critical resistance, and linear buckling theory does not
apply to such structures.
Fig. 26. Exploded view of a typical box, showing magnitudes and directions of the sinusoidal imperfections for imperfection factor $f = 1.0$.
Yield stress = 350 N/sq.mm.
Modulus of Elasticity = 210000 N/sq.mm.
Diaphragm thickness = 12 mm.

(all dimensions in mm.)

Fig. 27 Box-girder cross section for main analyses

Yield stress = 350 N/sq.mm.
Modulus of Elasticity = 210000 N/sq.mm.
Diaphragm thickness = 12 mm.

(all dimensions in mm.)

Fig. 28. Box girder cross section for preliminary analyses
Fig. 29. Exploded view of the cantilevering portion of box A1, with out-of-plane deformations and plastic zones at collapse (2340kN)
Fig. 30. Load - 'Bearing indentation' relationship for boxes A1, A2, and A3.
Fig. 31. Exploded view of the cantilevering portion of Box A3, with out-of-plane deformations and plastic zones at collapse (2334 kN)
Fig. 32. Exploded view of the cantilevering portion of box Bl, with out-of-plane deformations and plastic zones at collapse (2351 kN)
Fig. 33. Load - 'Bearing Indentation' relationship for boxes B1 and B2.
Fig. 34. Flange under uniaxial shortening (aspect ratio of buckles $\approx 0.875$)
Fig. 35. Interaction curve for imperfection factor $f = 1.0$
Fig. 36. Two dimensional idealisation for diaphragm analysis
Fig. 37(a). Load/"bearing indentation" relationship for the diaphragm two dimensional idealisation (simply-supported sides).
Figure 37(b). Load/"bearing indentation" relationship for the diaphragm two dimensional idealisation (built in sides)
Fig. 38. Exploded view of box Cl, with out-of-plane deformations and plastic zones at collapse (2184 kN) for imperfection factor $f=1.0$. 

- Fibre yield
- Full plastic zone (Ilyushin)
- Diaphragm
- Web
- Top flange
- Bottom flange

Box dimensions: 600 mm
Deformations (exaggerated scale): 30 mm
Fig. 39. Exploded view of box D1, with out-of-plane deformations and plastic zones at collapse (1879 kN) for imperfection factor $f = 1.0$. 

Box dimensions

Deformations (exaggerated scale)
Fig. 40. Exploded view of box El, with out-of-plane deformations and plastic zones at collapse (1402 kN) for imperfection factor $f = 1.0$. 

- Fibre yield
- Full plastic zone (Il'yushin)

Top flange

Bottom flange

Diaphragm

Web

In plane deformations not shown

Box dimensions

Deformations (exaggerated scale)

<table>
<thead>
<tr>
<th>600 mm</th>
</tr>
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<tbody>
<tr>
<td>30 mm</td>
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</table>
Fig. 41. Load/bearing indentation relationship for boxes with varying moment/shear ratios (imperfection factor $f = 1.0$)
Fig. 42. Comparison of Imperfection sensitivity of the collapse loads of the box with that of the individual panels.
Fig. 43. Load - 'bearing indentation' relationship for boxes with different imperfections 'f'.
Figure 44. Box results for the perfect case ($f=0$)

- B1 = Built in
- SS = Simply supported
- Θ = Box collapse

Diaphragm (B1) collapse
Diaphragm (SS) collapse
Plastic hinge at support

Lever arm (mm)
Load (kN)
Diaphragm shear yield

3000
2000
1000
0
Fig. 45. Load—'bearing indentation' relationship comparing elastic and elasto-plastic behaviour of the box (imperfection factor $f = 1.0$)
Fig. 46. Exploded view of Box C2, (no plasticity) with out-of-plane deformations at collapse (6716 kN) for imperfection factor $f = 1.0$. 

Box dimensions: 600 mm
Deformations (exaggerated scale): 120 mm

In plane deformations not shown.
Fig. 47. Lines of zero shear on the top flange at collapse
Fig. 48. Lines of zero shear on the bottom flange at collapse
Diaphragm deformations at collapse of Box AB1 (3D)  

Diaphragm deformations of Diaphragm idealised with built in sides (2D)

Box dimensions  --- 100 mm  
Deformations  --- 10 mm

Fig. 49. Comparison of out-of-plane deformations and plastic zones of Box AB1 with the two dimensional idealisation of the diaphragm (Built in on the sides)
Fig. 50(a) Comparison of diaphragm stresses between the Box AB1 and the two dimensional idealisation (built in sides) at collapse.
Fig. 50 (b) Comparison of diaphragm stresses between the Box AB1 and the two-dimensional idealisation (built-in sides) at collapse.
Load - bearing indentation relationship for boxes with variable web thicknesses (f = 1.0).

Fig. 51.
Fig. 52. Exploded view of box C3 (with 4mm webs) showing out-of-plane deformations and plastic zones at collapse (1933 kN) for imperfection factor $f = 1.0$. In-plane deformations not shown. Web deformations, being very large, are shown to a reduced scale.
Shear stress (N/mm²) at 325 mm from \( \xi \)

Fig. 53. Comparison of diaphragm shear stress of box C1 with box C3 at the lower (C3) collapse load
CHAPTER 7

CONCLUSIONS AND FUTURE WORK

7.1 The formulation of a theoretical method for analysing the large deflection elasto-plastic behaviour of plate assemblages typical of box girders has been presented. The method gives the behaviour up to and including collapse of the structure. The finite element displacement method is utilised, together with an approximate full section yield function and associated flow rules.

7.2 The resulting non-linear finite element program has been compared with established closed form and analytical solutions as well as available experimental data. The predicted strength of an unstiffened box girder with an end diaphragm is within 7% of the strength of the experimental model. These comparisons have confirmed the soundness of the analytical procedure.

7.3 The theory has been applied to an unstiffened box girder structure with a load bearing diaphragm, in order to ascertain the interactive behaviour of the main components in the support region. By providing special beam elements to represent the remote parts of the structure, a refined mesh could be chosen to analyse the support region. The special beam elements were modelled to represent only linear elastic behaviour.
7.4 Buckling of the compression flange into (rather than out of) the box adjacent to the support diaphragm produces more severe stresses and is the preferred mode at collapse. Therefore failure of a box girder near an interior support is likely to be associated with an upward buckle in the lower (compression) flange at the weakest section.

7.5 Failure of the box results from an interactive collapse involving the compression flange and the base of the diaphragm. Collapse of the box under this interaction occurs when the flange induces a bending moment at the base of the diaphragm. This effect gets larger with increasing moment shear ratio at the support.

7.6 Diaphragm-flange interaction curves for collapse loads of the box demonstrate the dependence of the overall collapse on the individual collapse behaviour of the component diaphragm and compression flange in the support region. It therefore appears that in some cases the diaphragm and flange can be treated individually by a simply supported two dimensional idealisation, and the lowest bearing reaction necessary to cause collapse of either component plate taken as a conservative estimate of the strength of the box girder. The 'knee' of the diaphragm-flange interaction curve indicates the moment shear ratio at which the flange begins to affect collapse of the box.
7.7 The two dimensional treatment of diaphragms agrees very favourably with the three dimensional analysis of the box girder assemblage, provided the effects of diaphragm-flange interaction are small (low moment-shear ratio). The three dimensional large-deflection elasto-plastic analysis therefore confirms the validity of the idealised two dimensional approach in these cases.

7.8 For the cases considered, the collapse load of the box was always greater than that predicted from the strength of the individually treated simply supported panels (see Figs. 35 & 42).

7.9 The dimensions of the panels had been chosen in an attempt to obtain simultaneous yielding and buckling of the box components, so as to give high imperfection sensitivity. Also, imperfection sensitivity is highest at the 'knee' of the diaphragm-flange collapse interaction curve. Consequently, the strength of such a structure possessing this 'local' and 'global' imperfection sensitivity was significantly reduced by imperfections. At this chosen moment-shear ratio, the collapse strength of the overall assemblage is more imperfection sensitive than that of the panels analysed by two dimensional idealisation.
For the parameters chosen in section 7.9, in the vicinity of the 'knee' of the interaction curve, it was observed that with a perfect box (with nominally flat plates) yielding occurs in the compression and tension flanges at the support due to longitudinal membrane forces in the flanges, causing collapse with a conventional fully plastic moment. Introduction of imperfections causes the buckles to grow, with yield occurring at the crests of the buckles in the compression flange, causing a decrease in the load carrying capacity (see Figs. 38,39,42).

A simplified design method could be adopted for the box where the flange collapse strength (aspect ratio 0.875, with simply supported boundaries and uniaxial shortening) could be obtained from charts or curves (11,49,50,92) giving stress-strain relationships for various levels of single sinusoidal imperfections. The diaphragm would at present require to be treated by a two dimensional elasto-plastic (large-deflection) idealisation. However, the author is currently working at TRRL towards establishing simple (approximate) design rules for determining elastic critical buckling loads of diaphragms as well as their collapse loads. The approximate interaction curves would, by this simplified treatment be obtained for design purposes.
7.12 The analysis of the box as an elastic plate assemblage (assuming a very high yield stress) taking into account the effects of large deflection, indicates considerable post-buckling strength. Therefore the collapse of the box for the elasto-plastic cases is brought about primarily by the formation of yield mechanisms rather than overall or local buckling failure.

7.13 It has been shown that webs exhibit considerable post-buckling strength. The variation of web thickness has been investigated and the results plotted in Fig 51, where it can be seen that by halving the web thickness the collapse load is reduced by approximately 12%.

7.14 Large tension field action in a very thin web can deform the corners of the box sufficiently to weaken the flange. The flange can therefore no longer be treated as simply supported in determining individual behaviour. Collapse will occur as in plate girders, with a plastic hinge in the flange.

7.15 **Future Work**

The present theory embraces moderately large plate deformation theory. This theory is sufficiently accurate to encompass the behaviour up to collapse of most plated assemblages occurring in Civil Engineering applications. It would, however, sometimes be necessary to model very large deformations, as might occur when local elastic buckling appears very early in the loading sequence.
In such instances, the panels would show significant deformations before collapse. A possible extension of the existing work would be to account for these extra large deformation effects.

Another extension could be the addition of stiffener elements in order to predict the behaviour of stiffened plate structures. The way of effecting this extension would be to include beam elements which would take into account in-plane buckling (strut-action) deformations. A more sophisticated (but much more complicated) approach would be to model plate elements that could, in addition to the present out-of-plane geometric stiffness matrix (accounting for lateral buckling,) account for the in-plane geometric stiffness matrix (accounting for in-plane buckling). The behaviour of stiffened plates can then be fully observed by modelling the panels and their stiffeners as plate elements, and the full in-plane and torsional (tripping) buckling behaviour of stiffeners and stiffened panels assessed.
APPENDIX A

Factors for matrix operations on special beam element

For a beam

\[ M = K\theta \] .................................. Al.1

In a cross-section of a beam, if there were several contributions from many nodes \( n \) in that cross section, each contributing moment \( m_i \) so that

\[ M = \sum_{i=1}^{n} m_i \] .................................. Al.2

we get:

\[ \Delta m_i = \sigma_i t_i \bar{z} \Delta z \] .................................. Al.3

where \( \sigma_i = \frac{M\bar{z}}{I} \) .................................. Al.4

\( \sigma_i \) = stress on node in question (node \( i \))

\( \bar{z} \) = distance from neutral axis to node in question (node \( i \))

\( t_i \) = wall thickness at node \( i \)

\( M \) = total bending moment on the beam.

\( I \) = total second moment of area of beam cross-section.

\[ m_i = \frac{M}{I} \int z^2 t_i dz \] ..... Al.5

For node \( i \), representing a node in the web of the beam,

\( t_i = t_w \) and:

\[ m_i = \frac{M}{I} \left[ \frac{z^3 t_w}{3} \right] \frac{\bar{z} + \bar{\ell}/2}{z - \bar{\ell}/2} \] .................. Al.6
where \( \overline{L} \) = length of wall representing node \( i \)

or \[ \overline{M}_i = \frac{M_{tw}}{3I} \left[ (\overline{z} + \overline{L}/2)^3 - (\overline{z}-\overline{L}/2)^3 \right] \]

from which, \[ \overline{M}_i = \frac{M_{tw}}{I} \left( \overline{L}/12 + \overline{L} \overline{z}^2 \right) \] ....... Al.7

Similarly, for node 'i', representing a node in the flange of the beam, where \( t_i = t_f \) we get:

\[ \overline{m}_i = \frac{M_{tw}}{I} \left( t_f^3/12 + t_f^2 \overline{z} \right) \] ....... Al.8

or, generalising,

\[ \overline{m}_i = \frac{M_{tw}}{I} I_n \] ............ Al.9

where \( I_n \) = second moment of area of the section of the box represented by node \( i \) about the neutral axis of the beam.

Now, work done due to each node is \( m_i \theta_i \)

\[ \therefore \sum_{i=1}^{n} (m_i \theta_i) = M\theta \]

or \[ \sum_{i=1}^{n} \left( \frac{I_n}{I} M \cdot \theta_i \right) = M\theta \]

or \[ \sum_{i=1}^{n} \left( \frac{I_n}{I} \theta_i \right) = \theta \] ............ Al.10

substituting Al.9 and Al.10 into Al.1,

\[ \sum_{i=1}^{n} \frac{m_i}{(I_n/I)} = K.E \left( \frac{I_n}{I} \theta_i \right) \] ....... Al.11
where \( F_i \) and \( u_i \) are force and displacement of the fibres of node \( i \) due to bending and rotation of the beam.

\[
\sum \frac{F_i}{I_{n/z.I}} = k \frac{I_{n/z.I}}{u_i} \quad \text{.... Al.12}
\]

The above equation can be rewritten in matrix form as

\[
\begin{bmatrix}
F_i \\
I_{n/z.I}
\end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} \frac{I_{n/z.I}}{u_i} \end{bmatrix} \quad \text{.... Al.13}
\]

now \( K \) (bending stiffness matrix) is a diagonal matrix

\[
\therefore F_i = \frac{I_{n}}{z.I} \cdot K_i \cdot \frac{I_{n}}{z.I} \cdot u_i \quad \text{.... Al.14}
\]

where \( \frac{I_{n}}{z.I} \) represents the row and column multiplications to the appropriate terms in the bending stiffness matrix, = factor \( k_1 \).

Now shear flow in any section \( xx \) is given by \( F \cdot Q \)

\[
Q = \int \frac{y_2 - y_1}{y} dA \quad \text{.............. Al.15}
\]

\( F \) = shearing force

\( I \) = total second moment of area of the beam.

Therefore, if

\( B \) = effective width of flange acting in shear with the web

\( h \) = height from beam N.A to flange mid-thickness, flange being on the same side of the N.A. as the node in question.

\( t_w \) = web thickness.
then \[ Q = t_f B h + t_w \left[ \frac{(h - t_f)^2 - \left( \frac{z}{2} \right)^2}{z} \right] \]

\[ = B h t_f + t_w \left[ \frac{(h - t_f)^2 - \left( \frac{z}{2} \right)^2}{z} \right] \] \quad \quad \text{Al.16}

Shear flow in webs = \( \frac{FQ}{I} \) = \( B h t_f \cdot F + t_w \cdot \frac{F}{I} \left[ \frac{(h - t_f)^2 - \left( \frac{z}{2} \right)^2}{z} \right] \)

\quad \quad \quad \quad \quad \quad \text{Al.17}

If \( S = \) shear force for node in question

\[ \Delta S = \frac{FQ}{I} \Delta z \] \quad \quad \quad \text{Al.18}

or \( S = \int_{\frac{z}{2} - \frac{h}{2}}^{\frac{z}{2} + \frac{h}{2}} \frac{FQ}{I} \, dz = B h t_f \bar{y} \cdot F + (h - t_f)^2 \cdot t_w \bar{y} F \)

\[ \frac{I}{I} \left[ \frac{2}{2} \cdot \frac{I}{I} \right] \cdot \frac{I}{I} \left[ \frac{t_w}{I} \left( \frac{\bar{y}^3}{12} - \frac{\bar{y} z^2}{2} \right) \right] \]

Shear force in a node \( i \) in the web = \( \frac{B h t_f \bar{y}}{I} + \left[ (h - t_f)^2 - \left( \frac{z}{2} \right)^2 \right] \frac{t_w \bar{y}}{I} - \frac{t_w \bar{y}^3}{12 I} \) \quad \quad \text{Al.19}

where the term within {} represents the factor \( k_3 \).

Shear force in flange = \( \frac{FQ}{I} \) where \( Q = B h t_f \cdot F \)

Shear force in a node \( i \) in the flange = \( \frac{B h t_f \bar{y} F}{I} \cdot \frac{\bar{y}}{B} = \frac{\bar{y} t_f h \cdot \bar{y} F}{I} \) \quad \quad \text{Al.20}

where the term within brackets {} represents the factor \( k_2 \).
APPENDIX B

The Gaussian Quadrature Numerical Integration

This highly efficient method integrates a polynomial of degree 2n-1 exactly as a weighted mean of its n particular values at specified 'Gauss' points (or stations), and has been found most useful in the finite element formulation.

The shape functions for the rectangular elements are given in terms of coordinates \( \xi \) and \( \eta \), which take up values of \( \pm 1 \) at the sides of the element.

The definite integral

\[
\int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) \, d\xi \, d\eta
\]

is replaced by the summation

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{H}_i \mathcal{H}_j f(a_i b_i)
\]

where \( \mathcal{H}_i \) and \( \mathcal{H}_j \) are the weighting coefficients, \( f(a_i b_i) \) the value of the function at specified coordinates \( (a_i, b_i) \) and \( n \) the number of Gauss points used in each coordinate direction.

For a three term Gauss rule, the full expansion of equation Bl.2 can be written as

\[
\int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) \, d\xi \, d\eta = \mathcal{H}_1^2 f(a_1, b_1) + \mathcal{H}_1 \mathcal{H}_2 f(a_1, b_2) + \mathcal{H}_1^2 f(a_1, b_3) + \mathcal{H}_2^2 f(a_2, b_2) + \mathcal{H}_2 \mathcal{H}_3 f(a_2, b_3) + \mathcal{H}_3^2 f(a_3, b_3)
\]
and this will integrate \( f(\xi, \eta) \) up to the fifth power of \( \xi \) and \( \eta \) correctly.

Similarly, the two term Gauss rule will integrate the third power of \( \xi \) and \( \eta \) correctly.

Abscissae and Weight coefficients of the Gaussian Quadrature formula
\[
\int_{-1}^{1} f(x) \, dx = \sum_{i=1}^{n} H_i f(a_i)
\]

are given as:

<table>
<thead>
<tr>
<th></th>
<th>( a )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two point Gauss rule (n=2)</td>
<td>0.57735 02691 89626</td>
<td>1.0000000000000000</td>
</tr>
<tr>
<td>Three point Gauss rule (n=3)</td>
<td>0.77459 66692 41483</td>
<td>0.5555555555555556</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000</td>
<td>0.8888888888888889</td>
</tr>
</tbody>
</table>
Geometric Imperfections

The maximum values of each panel imperfection are calculated from the Merrison Part III rules and based upon Part IV tolerances. The imperfection magnitudes for the main analysis are calculated as shown below, where the figures, tables and clauses refer to the Merrison Rules (36).

**Bottom flange**

Gauge length $G = 2b$ for long panels where $a > 3b$ (Fig.A23.1 and Table 23.1(e))

Tolerance $[\Delta_x] = \frac{G}{30t} \left( \frac{1+b}{5000} \right)$ millimetres

or 1 mm, whichever is greater ... Table 23.1(a)

$\therefore [\Delta_x] = \frac{1800}{30 \times 18} \left( 1 + \frac{900}{5000} \right) = 3.933$ mm

or $[\Delta_x] = 4$ mm, rounded up to the nearest 0.5 mm ... Table 23.1(e)

Initial imperfection $\delta_1 = \frac{1.2b[\Delta_x]}{G} \left[ \frac{3}{N+1} \right]$ mm ... cl.18.1.2(a)(i)

$= \frac{1.2 \times 900 [\Delta_x]}{1800} = 2.4$ mm

**Top flange**

Increase the tolerance for bottom flange by 100% cl.23.2.5 tolerance $[\Delta_x] = 8$ mm.

Initial imperfection $\delta_o = \frac{1.2 \times 900 [\Delta_x]}{1800} = 4.8$ mm

**Web**

Tolerance $[\Delta_x] = \frac{1200}{30 \times 7.5} \left( 1 + \frac{600}{5000} \right) = 6.4$ mm

$= 6.5$ mm, rounded up to the nearest 0.5 mm

... Table 23.1(e)
: Initial imperfection \( \delta_o = \frac{1.2b[A_x]}{G} = \frac{1.2 \times 600 \times 6.5}{1200} = 3.9 \text{mm} \)

Diaphragm

Tolerance \( [A_x] = G \frac{1+b}{30t} \) mm or 1 mm, whichever is greater \( \ldots \) Table 33.1(a)

\( G = \) longer side for diaphragms \( \ldots \) Cl.23.2.2b(i)

\( b = \) shorter side for plate panel \( \ldots \) Table 23.1(e)

\( [A_x] = \frac{900}{30 \times 12} \left(1 + \frac{600}{5000}\right) = 2.8 \text{mm} \)

\( = 3.0 \text{mm}, \) rounded up to the nearest 0.5mm...Table 23.1(e)

Initial imperfection \( \delta_o = \frac{1.2b[A_x]}{G} \) \( \ldots \) Cl.18.1.2.1(a)(i)

\( = 1.2 \times 600 \times \frac{3.0}{900} = 2.4 \text{mm} \)

The above maximum imperfections are made to form usually in the centre of the panels and the plate imperfections are represented as sinusoidal functions. A small computer program has been written to form the initial imperfections so that not only the initial deflections, but also the initial slopes are formed at each node for the doubly sinusoidal function representing the deformed plate. The program is so written as to cope with the full three dimensional assemblage of plates.
A small computer program has been written to generate sinusoidal initial imperfections (deflections and slopes) in the plate panels of the structure. These are stored on file and recalled by the main program when required.

The non-linear treatment, being accomplished by piecewise linearised analyses, requires storage of values after each load increment, to be re-used in calculating the tangent stiffness matrix. On account of the large amount of Central Processor (C.P) time involved for each increment of load (or deformation), not more than a few increments are accomplished in one computer run. Intermediate values between job runs are stored temporarily on file to restart the incremental procedure, until the non-linear analysis is completed on collapse of the structure. For the main analytical treatment of the box girder in the support region, collapse for the imperfect structures was achieved by between 50 and 60 increments (10 or 12 jobs involving 5 increments each).

Computer Storage

The program has been written in standard NCC FORTRAN as far as practicable, allowing for the idiosyncrasies of the C.D.C.6600 and 7600 computers. It has been used and tested on both computers. A unique artifice has been used in addressing the storage, so that a certain amount of 'dynamic storage' (as in ALGOL) is possible. All integer variables are stored in a single dimensional array and the real variables on another single dimensional array, in the master segment of the
program. These arrays are stored in the BLANK COMMON areas of storage, which occupies the end of the user storage area on the computer. Provided the user can calculate the storage requirements of a particular problem that requires analysis and as long as he requests this amount of storage while running the job, the program will not abort. This storage is also calculated by the program and output for the user's convenience. The job can use all the area between this beginning of the BLANK COMMON to the end of the user allocation requested. The program has been written so that storage is compact. The variables that determine the size of each array in the SUBROUTINES are input by the user, so that they are set at the beginning of the job. They are then addressed into an appropriate slot into the large single dimensional (integer or real, as the case may be) array in the BLANK COMMON area by the master program.

The semi-dynamic storage is possible on the CDC computers because it does not abort the job if the BLANK COMMON arrays overflow, as long as the job does not need more storage than requested by the user on the job cards. This may not be possible on other machines. However, the storage can be altered very easily by replacing one card.

The computer program requires a core storage of 28K words on the CDC6600 and 18K words on the CDC7600 preceding and in addition to the BLANK COMMON storage area.
Further storage is dependent upon the job. As a guide, the analytical treatment of the support region of the box girder which had 326 nodes, 295 elements, and a maximum joint difference of 27 (half band width of 168 coefficients) required 52K words in the BLANK COMMON. The CDC 6600 can take a maximum core storage of 98K words, while the CDC 7600, a total core storage of 124K words. However, should the full core storage of the computer be used, the jobs would become very slow and very time consuming. Also, special requests would be necessary for operations by the computer staff.

Error checking

On account of the expense in running the jobs, rigorous data checking subroutines are incorporated in the program, and the job made to abort if any errors are traced before commencing the computer analysis.

Computational Problems

In the majority of cases, the modified Newton-Raphson iterative procedure was employed up to collapse, with smaller increments being necessary to achieve convergence while approaching collapse. In certain cases, however, where nominally flat plated assemblages are analysed, it becomes difficult to achieve convergence, especially near bifurcation points (local buckling loads of individual panels). Convergence in one case was achieved using modified Newton-Raphson iterations, but only at the expense of excessive computing time, on account of the infinitesimally small increments that were taken.
It was therefore decided in such cases to use a purely incremental solution where no iterations were performed. This seems acceptable because load-deflection curves for nominally flat plates remain almost linear up to bifurcation points, and therefore the tangent stiffness matrix is unaffected, except by plasticity, which is accounted for by the small increments.


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