SOME HIGH FREQUENCY DIFFRACTION PROBLEMS

IN CLASSICAL ELASTODYNAMICS

by

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Summary

The diffraction of high frequency torsion waves by disc-shaped obstacles, situated in solids which are homogeneous, isotropic and of infinite extent, are considered in this thesis. In a high frequency limit these problems are formulated as Fredholm integral equations of the second kind. The thesis is divided into two chapters:

Chapter I: diffraction of high frequency torsion waves by a penny-shaped crack. Explicit asymptotic expressions are obtained for the dynamic stress intensity factors and the scattering coefficients. These results predict an oscillatory behaviour of the stress intensity factors at high frequencies.

Chapter II: diffraction of high frequency torsion waves by a rigid disc. Explicit asymptotic expressions are obtained for the torque resisting the motion of the disc, and for the scattering coefficients.

In both chapters extensive numerical results are given.
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CHAPTER I

DIFFRACTION OF HIGH FREQUENCY TORSION WAVES BY A

PENNY - SHAPED CRACK
Derivation of the dual integral equations

1.1 Introduction to Chapter 1

The problem of the diffraction of time harmonic torsion waves by a penny-shaped crack was first investigated by Sih and Loeber [1]. They adopted an approach which employed the theory of Hankel transforms to obtain a pair of dual integral equations which were eventually cast into a Fredholm integral equation of the second kind and solved numerically. The near field solutions were discussed and in particular the dynamic stress intensity factors were calculated for a range of wavenumbers. It was stated that for high frequencies, i.e. for wavelengths small compared with the radius of the crack, the stress intensity factors rapidly tended to zero. However, the stress intensity factors so obtained are only valid at low and intermediate frequencies.

Mal [2] considered the diffraction of a time harmonic free
 torsion wave by a penny-shaped crack and, using a different method, again obtained a Fredholm integral equation of the second kind which, for intermediate and high frequencies, was solved numerically. Mal showed that the dynamic stress intensity factor is a periodic function of the wavenumber. The physical significance of this result is that for large wavenumbers the waves generated by diametrically opposing points on the edge of the crack reinforce or interfere with each other.

It is proposed to adopt the analysis contained in [1] to obtain a pair of dual integral equations, for the determination, in the plane of the crack, of the Hankel transformed displacement in the scattered field. The diffraction problem is to be solved for large values of the wavenumber \( k = \omega/c_2 \) where \( \omega \) is the circular frequency and \( c_2 \) is the speed of shear waves), and a method due to Noble [3] is used to convert the pair of dual integral equations into a Fredholm integral equation of the second kind which is then solved asymptotically. Explicit
expressions are obtained for the dynamic stress intensity factor $k_3$, valid for large values of $k$ and all physically significant values of the wavenumber $\lambda = \omega / c$ where $c$ is the velocity of propagation of torsion waves in the axial direction), i.e. for $0 < \lambda < k$. It is not possible to obtain a single uniformly valid asymptotic expression for $k$ since there are two wavenumbers in the problem; for this reason it is necessary to obtain three non-uniform asymptotic expressions for the stress intensity factors. The variation of the dynamic stress intensity factors for $2 < k < 10$ and certain values of $\epsilon (\sqrt{1 - \lambda^2} / k^2)$ is displayed in a graphical form. These graphs clearly show that the stress intensity factors, for all values of $\epsilon$ are oscillatory functions of the wavenumber $k$.

Asymptotic expressions for the scattering coefficients are also derived, by using an approach due to Barratt and Collins [5]. These coefficients, for the values of $\epsilon$ considered are oscillatory functions of the wavenumber $k$, which is clearly demonstrated in the accompanying graphs.
1.2 Formulation

Consider an infinite space of homogeneous isotropic elastic material, with shear modulus $\mu_c$ and density $\rho_c$, containing a penny-shaped crack of radius $a$, whose plane faces are stress-free. Introduce cylindrical polar coordinates $(R, \theta, Z)$ with the centre and axis of the crack as origin and $Z$-axis; the crack is then given by $Z=0$, $0 \leq R \leq a$. $R$ and $\theta$ define the position of an element in a plane of a given cross-section at a time $t$. Figure 1 shows the position of the penny-shaped crack in the elastic space with respect to its axes, and also an infinitesimal element of the elastic space.
A torsion wave impinges on the crack so that the particles of the material experience only an angular displacement. The components of the displacement in the radial, tangential and normal directions are denoted by \( u_r^* \), \( u_\theta^* \), and \( u_z^* \) respectively.

Since the deformation is symmetrical about the \( Z \)-axis, the displacement is independent of the angle \( \theta \). Also, there is no displacement in the \( R \) and \( Z \) directions, therefore

\[
\begin{align*}
  u_r^* = u_z^* &= 0, \\
  u_\theta^* &= u_\theta(R, Z, t). 
\end{align*}
\]  

(1.2.1)

Normalise all lengths with respect to the radius of the crack by introducing the affine transformation

\[
R = a\rho, \quad Z = az, \quad u_\theta^*(R, Z, t) = au_\theta^*(\rho, z, t), \quad (1.2.2)
\]

and letting

\[
\sigma_\theta^*(R, Z, t) = \sigma_\theta^*(\rho, z, t) \quad \text{and} \quad \sigma_z^*(R, Z, t) = \sigma_z^*(\rho, z, t). \quad (1.2.3)
\]

From (1.2.1) it follows that all the stress components are zero except two, which after using (1.2.2) and (1.2.3) may be written as:

\[
\sigma_{zz}(\rho, z, t) = \mu c \frac{\partial u_\theta}{\partial z}, \quad (1.2.4)
\]

and
\( \sigma^*(\rho, z, t) = \mu_c \left( \frac{\partial u^*_\theta}{\partial \rho} - \frac{u^*_\theta}{\rho} \right) \) \tag{1.2.5}

Two of the equations of motion are identically satisfied, and the remaining one is
\[
\frac{\partial \sigma^*_\theta}{\partial \rho} + \frac{\partial \sigma^*_\phi}{\partial z} + \frac{2}{\rho} \sigma^*_\theta = \alpha^2 \rho \frac{\partial^2 u^*_\theta}{\partial t^2}. \tag{1.2.6}
\]

Substitute (1.2.4) and (1.2.5) into (1.2.6) in order to obtain an equation wholly in terms of displacements, i.e.
\[
\mu_c \frac{\partial}{\partial \rho} \left( \frac{\partial u^*_\theta - u^*_\phi}{\partial \rho} \right) + \mu_c \frac{\partial}{\partial z} \left( \frac{\partial u^*_\phi}{\partial z} \right) + \frac{2}{\rho} \mu_c \left( \frac{\partial u^*_\theta - u^*_\phi}{\partial \rho} \right) = \alpha^2 \rho \frac{\partial^2 u^*_\theta}{\partial t^2},
\]
and hence
\[
\frac{\partial^2 u^*_\theta}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u^*_\theta}{\partial \rho} - \frac{u^*_\theta}{\rho} + \frac{\partial^2 u^*_\phi}{\partial z^2} = \alpha^2 \frac{\partial^2 u^*_\theta}{\partial t^2}, \tag{1.2.7}
\]
where \( c_s = \left( \frac{\mu_c}{\rho} \right)^{1/2} \), which is the speed of the shear wave.

The time dependence of the stress and displacement components may be taken to vary harmonically as, e.g.
\[
\psi^*(\rho, z, t) = \psi(\rho, z) e^{i\omega t}. \tag{1.2.8}
\]

After substituting (1.2.6) into (1.2.7) we obtain
\[
\frac{\partial^2 u^*_\theta}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u^*_\theta}{\partial \rho} - \frac{u^*_\theta}{\rho} + \frac{\partial^2 u^*_\phi}{\partial z^2} + k^2 \alpha^2 u^*_\theta = 0, \tag{1.2.9}
\]
where the wavenumber \( k = \omega/c_s \).

In the subsequent work we set \( a \), the radius of the crack, equal to unit length.
1.3 The incident field

The stress field caused by the oscillating torques at infinity produces an incident field which will be denoted by the superscript $i$. When the waves reach the crack they produce a scattered field consisting of reflected and refracted elastic waves, and this field will be denoted by the superscript $s$. We can write, by using the principle of superposition

$$u_e(\rho, z) = u_{e}^{(i)}(\rho, z) + u_{e}^{(s)}(\rho, z), \quad (1.3.1)$$

and

$$\sigma_{e}(j, z) = \sigma_{e}^{(i)}(j, z) + \sigma_{e}^{(s)}(j, z), \quad j = \rho, z. \quad (1.3.2)$$

Let the incident torsion wave travel in the positive $z$-direction so that it will be incident on the negative side of the crack, hence take

$$u_{e}^{(i)}(\rho, z) = \sqrt{\rho} e^{i\kappa z}, \quad (1.3.3)$$

where $\kappa = \omega/\alpha$, and $\alpha$ is the velocity of propagation of torsion waves in the axial direction.

In order to determine $v(\rho)$ substitute (1.3.3) into (1.2.9) to give
\[ \frac{d^2 v}{d\rho^2} + \frac{1}{\rho} \frac{dv}{d\rho} + \left( k^2 - \nu^2 \right) v = 0. \]  
(1.3.6)

Let
\[ \lambda = (k^2 - \nu^2)^{1/2}, \]  
(1.3.5)

then the equation for \( v(\rho) \) becomes:
\[ \frac{d^2 v}{d\rho^2} + \frac{1}{\rho} \frac{dv}{d\rho} + \left( \frac{\lambda^2}{\rho^2} - 1 \right) v = 0. \]  
(1.3.6)

For the time being we only consider the case \( \lambda \neq 0 \), so that (1.3.6) is Bessel's equation of order one and argument \( \lambda \rho \) which, has as its solution
\[ v(\rho) = A J_1(\lambda \rho) + B Y_1(\lambda \rho). \]  
(1.3.7)

At \( \rho = 0 \), \( v(0) \) must be finite; hence the only possible value of \( B_1 \) is zero. Set
\[ A_1 = \frac{2 \nu}{\lambda}. \]  
(1.3.8)

The reason for this particular choice of the constant \( A_1 \) will become clear later on.

It is now possible to write \( u_{e}^{(1)} \) and \( \varphi_{e}^{(1)} \) by using (1.2.1),
(1.3.5) and (1.3.7) as
\[ u_{e}^{(1)}(\rho, z) = \frac{2 \nu}{\lambda} J_1(\lambda \rho) e^{i\rho z}, \]  
(1.3.9)

and
\[ \varphi_{e}^{(1)}(\rho, z) = -\frac{2 \nu z}{\lambda} J_1(\lambda \rho) e^{-i\rho z}, \]  
(1.3.10)
where \( \sigma_0 = \mu_c u_0 \alpha \). \hspace{1cm} (1.3.11)

\( \alpha_0 \), the stress on the incident torsion wavefront is assumed to be non-zero as \( \alpha \to 0 \).

Now consider the case \( \lambda = 0 \), i.e. when \( k = \alpha \). It is seen that equation (1.3.6) is equal to

\[
\frac{d^2 v}{d \rho^2} + \frac{1}{\rho} \frac{dv}{d \rho} - \frac{v}{\rho^2} = \sigma,
\]  \hspace{1cm} (1.3.12)

which is not Bessel's equation, but an equation of the Euler type which has solutions of the form

\[
v(\rho) = A'_0(\rho) + B'_0 \frac{1}{\rho},
\]  \hspace{1cm} (1.3.13)

On using reasoning similar to that used to determine the value of \( B_1 \), and after letting

\[
A'_0 = u_0,
\]  \hspace{1cm} (1.5.14)

we have

\[
u^{(i)}_0(\rho, z) = u_0 \rho e^{-i\lambda z},
\]  \hspace{1cm} (1.3.15)

which represents a free torsion wave, and

\[
\sigma^{(i)}_{e0}(\rho, z) = -\sigma_0 \rho e^{-i\lambda z},
\]  \hspace{1cm} (1.3.16)

Let \( \lambda \to 0 \), first in (1.5.9), so that

\[
\lim_{\lambda \to 0} u^{(i)}_0(\rho, z) = 2u_0 \lim_{\lambda \to 0} \frac{1}{\lambda} \frac{\rho e^{-i\lambda z}}{2} = u_0 \rho e^{-i\lambda z},
\]  \hspace{1cm} (1.5.17)
and then in \((1.3.10)\) to give
\[
\lim_{\lambda \to 0} a^{(1)}_{\varepsilon z}(\rho, z) = -\varphi \cdot \rho^{2} e^{i\rho z}. \tag{1.3.18}
\]

It is seen that, because of the particular choice of \(A_{1}\),
\((1.3.15)\) is a member of the same set of solutions as \((1.3.9)\),
therefore it is only necessary to consider an incident wave of
the form \((1.3.9)\), and for the particular case of the free torsion
wave take the limit as \(\lambda\) tends to zero.

For the case \(\lambda > k\), by using \((1.3.5)\) write the incident field
as
\[
u^{(1)}(\rho, z) = \psi(\rho) e^{-i(k^2 \lambda^2)^{1/2} z}. \tag{1.3.19}
\]
Now \(i(k^2 - \lambda^2)^{1/2}\) will be real, and hence \((1.3.19)\) will not represent
a travelling wave since the resultant field will be evanescent;
therefore take \(0 < \lambda \leq k\). When \(\lambda = k\), \((1.3.9)\) represents a
standing wave.
1.4 The scattered field

The scattered part of the solution must satisfy the radiation condition
\[ \rho \left\{ \frac{\partial u_0^{(s)}}{\partial \rho} + i k u_0^{(s)} \right\} \to 0, \text{ as } \rho \to \infty, \]
uniformly with respect to \( \theta \) and \( z \). This solution may be found by the application of a Hankel transform of order one to equation (1.2.9), (c.f. [5])

\[ \int_0^\infty \rho \left( \frac{\partial^2 u_0^{(s)}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u_0^{(s)}}{\partial \rho} \right) J_1(s \rho) d \rho = \int_0^\infty \left( \rho \frac{\partial u_0^{(s)}}{\partial \rho} \right) J_1(s \rho) d \rho. \]  
(1.4.1)

Therefore, after integrating the right-hand side of (1.4.1) once by parts we have

\[ \int_0^\infty \rho \frac{\partial u_0^{(s)}}{\partial \rho} J_1(s \rho) d \rho = \left[ \rho u_0^{(s)} J_1(s \rho) \right]_0^\infty - s \int_0^\infty \frac{\partial u_0^{(s)}}{\partial \rho} J_1(s \rho) d \rho, \]  
(1.4.2)

where

\[ J_1'(X) = \frac{d}{dX} J_1(X). \]  
(1.4.3)

The first term on the right-hand side of (1.4.2) is zero, since there must be no scattered field at infinity, and after a further integration by parts we obtain

\[ \int_0^\infty \rho \frac{\partial u_0^{(s)}}{\partial \rho} J_1(s \rho) d \rho = - s \left[ \rho u_0^{(s)} J_1(s \rho) \right]_0^\infty + \int_0^\infty \rho \frac{\partial}{\partial \rho} J_1(s \rho) d \rho. \]  
(1.4.4)
But

\[
\frac{\partial}{\partial \rho} \rho \tilde{J}_1'(s\rho) = s^2 \rho \tilde{J}_1''(s\rho) + s \tilde{J}_1'(s\rho) \quad (1.4.5)
\]

and from Bessel's equation of order one we have

\[
S^2 \rho \tilde{J}_1''(s\rho) + s \tilde{J}_1'(s\rho) = -\frac{1}{\rho} (s^2 \rho^2 - 1) \tilde{J}_1(s\rho) \quad (1.4.6)
\]

so that (1.4.4) may be re-written as

\[
\int_0^\infty \rho \left( \frac{\partial^2 u_\eta^{(s)}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u_\eta^{(s)}}{\partial \rho} - \frac{u_\eta^{(s)}}{\rho^2} \right) \tilde{J}_1(s\rho) d\rho = -s^2 \int_0^\infty \rho u_\eta^{(s)}(\rho, z) \tilde{J}_1(s\rho) d\rho. \quad (1.4.7)
\]

Define

\[
u_\eta^{(s)}(s, z) = \int_0^\infty \rho u_\eta^{(s)}(\rho, z) \tilde{J}_1(s\rho) d\rho \quad (1.4.8)
\]

and hence the inverse of \( u_\eta^{(s)}(s, z) \) as

\[
u_\eta^{(s)}(\rho, z) = \int_0^\infty s U_\eta^{(s)}(s, z) \tilde{J}_1(s\rho) d\rho. \quad (1.4.9)
\]

From definition (1.4.8), (1.4.7) may be written as

\[
\int_0^\infty \rho \left( \frac{\partial^2 u_\eta^{(s)}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u_\eta^{(s)}}{\partial \rho} - \frac{u_\eta^{(s)}}{\rho^2} \right) \tilde{J}_1(s\rho) d\rho = -s^2 U_\eta^{(s)}(s, z). \quad (1.4.10)
\]

Also

\[
\int_0^\infty \rho \frac{\partial^2 u_\eta^{(s)}}{\partial \rho \partial \rho} \tilde{J}_1(s\rho) d\rho = \frac{2^2}{\rho^2} \int_0^\infty \rho u_\eta^{(s)}(\rho, z) \tilde{J}_1(s\rho) d\rho,
\]

therefore from (1.4.8) we have

\[
\int_0^\infty \rho \frac{\partial^2 u_\eta^{(s)}}{\partial \rho \partial z} \tilde{J}_1(s\rho) d\rho = \frac{2}{\rho^2} \int_0^\infty \rho u_\eta^{(s)}(\rho, z) \tilde{J}_1(s\rho) d\rho. \quad (1.4.11)
\]

Use results (1.4.10) and (1.4.11) to write down the Hankel transform of equation (1.2.9) which is

\[
\frac{d^2}{d z^2} U_\eta^{(s)}(s, z) + (k^2 - s^2) U_\eta^{(s)}(s, z) = 0 \quad (1.4.12)
\]
In (1.4.12) let $\gamma^2 = s^2 - k^2$, then $U^{(s)}(s, z)$ satisfies the following equation

$$\frac{d^2}{dz^2} U^{(s)}(s, z) - \gamma^2 U^{(s)}(s, z) = 0.$$  

(1.4.13)

It is necessary to define the cuts in the s-plane so that $\gamma(s)$ will behave in a certain manner.

![Diagram of the s-plane with cuts](image)

The s-plane is cut along the real axis from $-\infty$ to $-k$, and from $k$ to $+\infty$. The contour for $s$ lies below the cut $-\infty$ to $-k$, and above the cut $k$ to $+\infty$.

Figure 2

Following Noble [6], arrange the cuts in the s-plane as shown in figure 2 so that, as

$$s \to +\infty, \quad \gamma = (s^2 - k^2)^{1/2} \to \infty.$$  

For $s \to \infty$, consider $\gamma = (-s-k)^{1/2}(-s+k)^{1/2}$ as $s \to \infty$, and $\arg(-s-k) = \pi$.
which gives that \(-s-k=(s+k)e^{-\pi i}\). Similarly \(\text{arg}(-s+k)=\pi\) and hence 
\(-s+k=(s-k)e^{\pi i}\). Therefore \(s \to \infty\) as \(s \to -\infty\). It is also necessary to 
know which branch to select when \(s<k\), therefore consider 
\(\gamma=(s-k)^{1/2}(s+k)^{1/2}\). For \(s<k\), \(\text{arg}(s-k)=\pi\) and \(s-k=(s-k)e^{\pi i}\), which 
gives that \(\gamma=(k-s)^{1/2}e^{\pi i/2(k-s)^{1/2}}\). Summarising we have 
\[
\gamma(s) = \begin{cases} 
(s^2-k^2)^{1/2}, & s > \frac{k}{2} \\
i(s^2-k^2)^{1/2}, & s \leq \frac{k}{2}.
\end{cases}
\]
(1.4.14)

With definition (1.4.14) in mind it is obvious that the 
solution of equation (1.4.13), in \(z \to 0\), which ensures outgoing 
waves as \(z \to \infty\), is 
\[
U_{(s, z)} = A(s) e^{\gamma z}.
\]
(1.4.15)

Substitute (1.4.15) into (1.4.9) to obtain 
\[
U_{(s, z)} = \int_{0}^{\infty} A(s) J_1(s \rho) e^{\gamma z} d \rho.
\]
(1.4.16)

Solution (1.4.16) is only valid for \(z \to 0\); the solution for all \(z\) is
\[
U_{(s, z)} = \text{sgn } z \int_{0}^{\infty} A(s) J_1(s \rho) e^{\gamma z} d \rho,
\]
(1.4.17)

where
\[
\text{sgn } z = \begin{cases} 
-1, & z < 0 \\
1, & z \geq 0
\end{cases}
\]
(1.4.18)

(1.4.17) ensures the continuity of \(Q_{(s)}^{(s)}\) across \(z=0\).
Hence, we are justified in only considering the solution of this 
problem for the region \(z \to 0\).
1.5 Boundary conditions

The first boundary condition assumed for the problem under consideration is that there are zero tangential shear stresses on the surfaces of the crack, which implies that there is no friction between opposite faces of the crack, i.e. using the notation given by (1.3.2)

$$q^{(i)} + q^{(e)} = 0, \text{ for } 0 \leq \rho < 1, \ z = \pm 0$$  \hspace{1cm} (1.5.1)

Boundary condition (1.5.1) can be written in terms of displacements by using (1.2.4), and the value of $q^{(i)}$ from (1.5.10) to give

$$u_{\phi} \frac{\partial u_{\phi}}{\partial z} = \frac{2q_{\phi}}{\lambda} \ J_1(\lambda \rho), \ 0 \leq \rho < 1 \text{ and } z = 0.$$  \hspace{1cm} (1.5.2)

The second boundary condition requires the continuity of $u_{\phi}$ across $z = 0$ for $\rho > 1$. From (1.4.17) this is equivalent to

$$u_{\phi}(\rho, z) = 0, \ \rho > 1 \text{ and } z = 0.$$  \hspace{1cm} (1.5.3)

Finally, for large $(\rho^2 + z^2)^{1/2}$, $u(\rho, z)$ must represent outgoing waves.

This last condition has already been used in obtaining (1.4.17).
1.6 The dual integral equations

Before the scattered field can be determined it is necessary to determine the unknown function \( A(s) \), (c.f. (1.4.16)). The approach we adopt to find \( A(s) \) is to obtain a pair of dual integral equations by inserting the integral equation (1.4.16) into the boundary conditions (1.5.2) and (1.5.3). Hence

\[
\mu_c \int_0^\infty \gamma(s) A(s) \tilde{J}_i(s \rho) \, ds = - \frac{2 \kappa \rho \lambda}{\rho^2} \tilde{J}_i(\lambda \rho), \quad 0 < \rho < 1, \quad (1.6.1)
\]

and

\[
\int_0^\infty s A(s) \tilde{J}_i(s \rho) \, ds = 0, \quad \rho > 1. \quad (1.6.2)
\]

Let \( f(\rho) = \begin{cases} \frac{2 \kappa \rho \lambda}{\rho^2} \tilde{J}_i(\lambda \rho), & 0 < \rho < 1 \\ f_1(\rho), & \rho > 1 \end{cases} \) \quad (1.6.3)

where \( f_1(\rho) \) is an unknown function which is to be determined.

Equation (1.6.1) may be inverted by using Hankel's inversion theorem. Inversion gives that

\[
A(s) = \mathcal{F}^{-1} \int_0^\infty r f(\rho) \tilde{J}_i(\rho s) \, d\rho. \quad (1.6.4)
\]

Substitute (1.6.4) into (1.6.2) and interchange the order of integration so that we obtain another integral equation, i.e.

\[
\int_0^\infty r f(\rho) \, d\rho \int_0^\infty \frac{s}{(s^2 - \lambda^2)^{1/2}} \tilde{J}_i(\rho s) \tilde{J}_i(\rho s) \, ds = 0, \quad \rho > 1. \quad (1.6.5)
\]
Let the kernel of integral equation (1.6.5) be represented by

\[ K(\rho, r) = \int_{0}^{\infty} \frac{s}{(s^2 - 4k^2)^{1/2}} J_1(\rho s) J_1(\gamma s) \, ds \]  

(1.6.6)

This kernel is of the form considered by Noble [5].
Section 2

The Fredholm integral equation of the second kind

2.1 Transformation of the kernel

The transformation of the kernel $K(\rho, r)$ is considered in this section. Hence, following Noble [3], we make the change of variable $s = k\xi$ in (1.6.6) to give

$$K(\rho, r) = \frac{k}{\pi} \int_0^\infty \frac{\xi}{(\xi^2 - 1)^{1/2}} J_1(k \rho \xi) J_1(k \nu \xi) d\xi.$$  \hspace{1cm} (2.1.1)

For elastic waves of high frequency, the wavenumber $k$ is large. It would be convenient if we could introduce the asymptotic expansions of the Bessel functions for large $k$, but for the time being this step would not be valid since the integrals of the higher order terms would be divergent near $\xi = 0$. Split the infinite integral for $K(\rho, r)$ into the ranges $(0, 1)$ and $(1, \infty)$, so that (2.1.1) is written as

$$K(\rho, r) = \frac{k}{\pi} \int_0^1 \frac{\xi}{(\xi^2 - 1)^{1/2}} J_1(k \rho \xi) J_1(k \nu \xi) d\xi + \frac{k}{\pi} \int_1^\infty \frac{\xi}{(\xi^2 - 1)^{1/2}} J_1(k \rho \xi) J_1(k \nu \xi) d\xi.$$  \hspace{1cm} (2.1.2)

All the divergence is contained in the range $(0, 1)$, and the next step is to transform this integral to a more suitable...
form. In order to perform this transformation introduce the complex variable \( z = x + iy \). \( J_i(\rho z) \) tends to infinity as \( z \) tends to infinity, in both the upper and lower half-planes, therefore write

\[
J_i(\rho z) J_i(\rho \psi) = \frac{1}{2} \left\{ H_i^{(1)}(\rho z) J_i(\rho \psi) + H_i^{(2)}(\rho z) J_i(\rho \psi) \right\}
\]

(2.1.5)

First consider

\[
\int_{c_1} \frac{z}{(z^2 - 1)^{1/2}} H_i^{(n)}(\rho z) J_i(\rho \psi), \text{ for } \rho > r,
\]

(2.1.4)

where \( c_1 \) is the contour of integration shown in figure 3.

The contour \( c_1 \) is indented around the singularities at \( z = 0 \) and \( z = 1 \).

Figure 3
Let the radius of the circles indented around the singularities at \( f=0 \) and \( f=1 \) tend to zero, and hence by Cauchy's theorem
\[
\oint_{A\beta} df + \oint_{BC} df + \oint_{C\alpha} df = 0,
\]
as \( R \to \infty \).

It is quite easy to show that as \( R \to \infty \) there is no contribution from the arc BC. Let
\[
I_1 = \oint_{A\beta} df,
\]
and
\[
I_2 = \oint_{C\alpha} df.
\]
Hence
\[
I_1 + I_2 = 0.
\]
The integral \( I_1 \) has a branch point at \( f=1 \), and on \( A \) to \( 1 \) it is seen from figure 5 that \( \arg(f-1) = \pi \) and
\[
f-1 = (1-\xi) e^{i\xi}.
\]
Therefore
\[
I_1 = \int_0^\infty \frac{\xi}{\sqrt{1-\frac{\xi^2}{2}}} J_n(k \rho \xi) J_n(k \rho \xi) d\xi + \int_0^\infty \frac{\xi}{(\xi^2-1)^{3/2}} J_n(k \rho \xi) J_n(k \rho \xi) d\xi,
\]
and for \( I_2 \)
\[
I_2 = (i)^n \int_0^\infty \frac{\xi}{\sqrt{1-\frac{\xi^2}{2}}} J_n(i k \rho \xi) J_n(i k \rho \xi) d\xi.
\]
Use the fact that on $C$ to $A$ \( f = (1-e^i)e^i \) so that (2.1.11) may be written as

\[
I_2 = i \int_0^\infty \frac{\gamma}{(1 + \gamma^2)^{3/2}} H_1^{(n)}(i k \rho \gamma) J_0(i k \gamma) \, d \gamma.
\] (2.1.12)

Substitute into (2.1.8) the values of $I_1$ and $I_2$ given by (2.1.10) and (2.1.12) to obtain

\[
-i \int_0^{\sigma} \frac{\gamma}{(1 - \sigma^2)^{1/2}} H_1^{(n)}(k \sigma \xi) J_0(k \xi) \, d \xi + \int_0^{\sigma} \frac{\gamma}{(1 + \sigma^2)^{1/2}} H_1^{(n)}(k \sigma \xi) J_0(k \xi) \, d \xi = 0, \rho > r.
\] (2.1.13)

Next consider

\[
\int \frac{\gamma}{\varepsilon^2 (\varepsilon^2 - 1)^{1/2}} H_1^{(2)}(k \rho \xi) J_0(k \xi) \, d \xi, \text{ for } \rho > r,
\] (2.1.14)

where $C_2$ is the contour of integration shown in figure 4.

The contour $C_2$ is indented around the singularities at $f = 0$ and $f = 1$.

**Figure 4**
By applying reasoning similar to that used in the evaluation of (2.1.4) we obtain

\[
\int_{A_0} df + \int_{D_A} df = 0. \tag{2.1.15}
\]

Similarly it can be shown that on \( A \) to \( 1 \)

\[
\int_1^{\infty} - (1 - \xi) e^{-\xi}, \tag{2.1.16}
\]

and hence

\[
\int_{A_0} df + \int_{D_A} df = e^{\xi/2} \int_0^{\xi/2} H_1^{(2)}(k \rho \eta) J_1(k \sqrt{\eta}) d \xi + \int_0^{\infty} \frac{\xi}{(\xi^2 - 1)^{1/2}} H_1^{(2)}(k \rho \eta) J_1(k \sqrt{\eta}) d \xi
\]

\[
+ \int_{\infty}^0 \frac{-e^{\eta/2}}{(1 + \eta^2)^{1/2}} H_1^{(2)}(-i k \rho \eta) J_1(-i k \sqrt{\eta})(-i) d \eta.
\]

Therefore

\[
i \int_0^{\xi/2} \frac{H_1^{(2)}(k \rho \eta) J_1(k \sqrt{\eta}) d \xi + \int_0^{\infty} \frac{\xi}{(\xi^2 - 1)^{1/2}} H_1^{(2)}(k \rho \eta) J_1(-k \sqrt{\eta}) d \xi}
\]

\[-i \int_{\infty}^0 \frac{\eta}{(1 + \eta^2)^{1/2}} H_1^{(2)}(-i k \rho \eta) J_1(-i k \sqrt{\eta}) d \eta = 0. \tag{2.1.17}
\]

From the relationships between the Bessel functions of different kinds in [7] et al, we have

\[
H_1^{(1)}(-i k \rho \eta) J_1(i k \sqrt{\eta}) = -\frac{2i}{\pi} K_1(k \rho \eta) I_1(k \sqrt{\eta}),
\]

and

\[
H_1^{(2)}(-i k \rho \eta) J_1(-i k \sqrt{\eta}) = \frac{2i}{\pi} K_1(k \rho \eta) I_1(k \sqrt{\eta}). \tag{2.1.18}
\]
Subtract (2.1.17) from (2.1.15) and use (2.1.18), therefore for \( \rho > r \),

\[
-i \int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} \left[ H_{1}^{(1)}(\kappa \rho \xi) + H_{1}^{(2)}(\kappa \rho \xi) \right] J_1(\kappa \nu \xi) \, d \xi
\]

and use (2.1.18), therefore

\[
\int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} \left[ -H_{1}^{(1)}(\kappa \rho \xi) + H_{1}^{(2)}(\kappa \rho \xi) \right] J_1(\kappa \nu \xi) \, d \xi,
\]

or

\[
\int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} J_1(\kappa \rho \xi) J_1(\kappa \nu \xi) \, d \xi = \int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} Y_1(\kappa \rho \xi) Y_1(\kappa \nu \xi) \, d \xi. \tag{2.1.19}
\]

The next step is to substitute result (2.1.19) into (2.1.2), and use (1.4.14) to deal with the square root to give

\[
K(\rho, \nu) = \kappa \int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} \left[ J_1(\kappa \rho \xi) - i Y_1(\kappa \rho \xi) \right] J_1(\kappa \nu \xi) \, d \xi,
\]

or

\[
K(\rho, \nu) = \kappa \int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} H_{1}^{(2)}(\kappa \rho \xi) J_1(\kappa \nu \xi) \, d \xi, \quad \rho > r. \tag{2.1.20}
\]

Obviously

\[
K(\rho, \nu) = \kappa \int_0^{\infty} \frac{\xi}{(1 - \xi^2)^{1/2}} H_{1}^{(2)}(\kappa \nu \xi) J_1(\kappa \rho \xi) \, d \xi, \quad \nu > \rho. \tag{2.1.21}
\]

It is now permissible to introduce the asymptotic expansions of the Bessel functions into (2.1.20), i.e.

\[
H_{1}^{(2)}(\kappa \rho \xi) J_1(\kappa \nu \xi) = \left( \frac{2}{\pi \kappa \rho \xi} \right)^{1/2} e^{-i(\kappa \rho \xi - \frac{3\pi}{4})} \left\{ 1 + O \left( \frac{1}{\kappa} \right) \right\}
\]

\[
\times \left( \frac{1}{2 \kappa \nu \xi} \right)^{1/2} \left[ e^{i(\kappa \nu \xi - \frac{3\pi}{4})} + e^{-i(\kappa \nu \xi - \frac{3\pi}{4})} \right] \left\{ 1 + O \left( \frac{1}{\kappa} \right) \right\},
\]
similar result is obtained for $r > \rho$, and we can combine these two results in the following way:

\[
K(\rho, \nu) = \frac{1}{\pi (\rho \nu)^{\frac{1}{2}} \nu^2} \left\{ K_0(\rho, \nu) - i K_1(\rho, \nu) + O\left(\frac{1}{\rho^2}\right) \right\},
\]

\[\text{(2.1.22)}\]

where

\[
K_0(\rho, \nu) = \int_1^\infty \frac{e^{-i K_1 \rho \nu \xi}}{(\xi^2 - 1)^{\frac{1}{2}}} d\xi,
\]

\[\text{(2.1.23)}\]

and

\[
K_1(\rho, \nu) = \int_1^\infty \frac{e^{-i K_1 (\rho + \nu) \xi}}{(\xi^2 - 1)^{\frac{1}{2}}} d\xi.
\]

\[\text{(2.1.24)}\]

Equations (2.1.23) and (2.1.24) are next transformed by

the substitutions

\[
\xi |_{p-r} = 2\nu - \rho - \nu,
\]

\[\text{(2.1.25)}\]

and

\[
\xi |_{p+v} = 2\nu - \rho + \nu,
\]

\[\text{(2.1.26)}\]

to give

\[
K_0(\rho, \nu) = e^{i K_1 (\rho + \nu)} \int_\max(\rho, \nu)^\infty \frac{e^{2i K_1 \nu}}{(\nu - \rho)^{\frac{1}{2}} (\nu - \nu)^{\frac{1}{2}}} d\nu,
\]

\[\text{(2.1.27)}\]

and

\[
K_1(\rho, \nu) = e^{i K_1 (\rho - \nu)} \int_\rho^\infty \frac{e^{2i K_1 \nu}}{(\nu - \rho)^{\frac{1}{2}} (\nu + \nu)^{\frac{1}{2}}} d\nu.
\]

\[\text{(2.1.28)}\]
If terms of $O\left(\frac{1}{k^2}\right)$ are ignored in (2.1.22), then (1.6.5) may be written as

$$
\int_{0}^{\infty} \psi^2 \{ \zeta, \zeta - i \zeta \} \, d \zeta = 0, \quad (2.1.23)
$$

for $\rho > 1$ and large $k$. 
2.2 Reduction of the integral equation

We shall obtain an alternative form for the integral equation (2.1.29) where \( f(r) \) is known for \( 0 < r < 1 \), and unknown for \( r > 1 \). Insert the expressions for \( K_0(\rho, r) \) and \( K_1(\rho, r) \), from (2.1.27) and (2.1.28) respectively, into (2.1.29) to give

\[
\int_0^{\rho} r^{\frac{1}{2}} e^{i \kappa r} f(v) dv \int_0^{\infty} \frac{\hat{e}^{2 i \kappa v}}{(v-\rho)^{\frac{1}{2}}(v-r)^{\frac{1}{2}}} dv - i \int_0^{\rho} r^{\frac{1}{2}} e^{i \kappa v} f(v) dv \int_{\max(\rho, v)}^{\infty} \frac{\hat{e}^{2 i \kappa v}}{(v-\rho)^{\frac{1}{2}}(v+r)^{\frac{1}{2}}} dv = 0, \quad \rho > 1. \tag{2.2.1}
\]

**Figure 5**

Take the first integral in (2.2.1) and interchange the order of integration, by considering

\[
\lim_{T \to \infty} \int_0^T \int_0^{\max(\rho, v)} dr \, dv,
\]
taken over the shaded area, shown in figure 5, in two
different ways.

Now
\[
\lim_{T \to \infty} \int_0^T \int_0^T dV dW = \lim_{T \to \infty} \int_0^T dV \int_0^T dW + \lim_{T \to \infty} \int_0^T \int_0^T dV dW,
\]
and after interchanging the order of integration we have
\[
\lim_{T \to \infty} \int_0^T dV \int_0^T dW = \lim_{T \to \infty} \int_0^T dV \int_0^T dW + \lim_{T \to \infty} \int_0^T \int_0^T dV dW
\]
(2.2.2)

Add together the two integrals on the right-hand side of
(2.2.2) to obtain
\[
\lim_{T \to \infty} \int_0^T dV \int_0^T dW = \lim_{T \to \infty} \int_0^T dV \int_0^T dW.
\]
(2.2.3)

Interchange the order of integration of the second integral
in (2.2.1), and substitute result (2.2.3) into (2.2.1) to
give
\[
\int_0^\infty \frac{2i \pi v}{(v-r)^{1/2}} \left\{ \int_0^v \frac{e^{ikr}}{(v-r)^{1/2}} f(v) dV - \int_0^\infty \frac{e^{-ikr}}{(v+v+1)^{1/2}} f(v) dV \right\} = 0, \quad r > 1.
\]
(2.2.4)

Equation (2.2.4) is an Abel integral equation which may
written as
\[
\int_0^\infty \frac{\tilde{f}(v)}{r^{(v-r)^{1/2}}} dv = 0, \quad r > 1,
\]
and its solution is
\[ J(\nu) = 0, \quad \rho > 1. \]

Therefore from (2.2.4), and also writing the result so that the resultant integral equation for \( f(r) \) is in a form where the unknown part is on the left-hand side, and the known part is on the right, we obtain

\[
\int_{x}^{y} \frac{e^{ikr}}{(\nu-r)^{\frac{1}{2}}} f(\nu) \, d\nu = -i \int_{x}^{y} \frac{e^{ikr}}{(\nu+\nu)^{\frac{1}{2}}} f(\nu) \, d\nu
\]

\[ = -\left[ \int_{x}^{y} \frac{e^{ikr}}{(\nu-r)^{\frac{1}{2}}} - i \frac{e^{ikr}}{(\nu+\nu)^{\frac{1}{2}}} \right] \frac{1}{y} f(\nu) \, d\nu, \quad \nu > 1. \quad (2.2.5) \]

In (2.2.5) let

\[ P(\nu) = \nu \frac{1}{y} e^{ikr} f(\nu), \quad (2.2.6) \]

and

\[ Q(\nu) = i \int_{y}^{x} \frac{1}{(\nu+\nu)^{\frac{1}{2}}} f(\nu) \, d\nu - \int_{x}^{y} f(\nu) \left\{ \frac{e^{ikr}}{(\nu-r)^{\frac{1}{2}}} - i \frac{e^{ikr}}{(\nu+\nu)^{\frac{1}{2}}} \right\} \, d\nu, \quad \nu > 1. \quad (2.2.7) \]

then (2.2.5) becomes

\[ \int_{x}^{y} \frac{P(\nu)}{(\nu-r)^{\frac{1}{2}}} \, d\nu = Q(\nu), \quad \nu > 1. \quad (2.2.8) \]

Equation (2.2.8) is another Abel integral equation which we now invert, in order to derive an expression for \( P(\nu) \).

The inverse of (2.2.8) is given by

\[ P(\nu) = \frac{1}{\pi} \int_{\nu}^{\infty} \frac{Q(\nu)}{(\nu-r)^{\frac{1}{2}}} \, d\nu, \quad \nu > 1. \quad (2.2.9) \]

Hence from (2.2.6), (2.2.7) and (2.2.9)
\[ \rho^2 e^{i\frac{\pi}{2}\rho} f(\rho) - \frac{i}{\pi} \int d\rho \int_0^\infty v^2 e^{i\frac{\pi}{2}v} f(v) dv \]

Interchange the order of integration in (2.2.10) to give

\[ \rho^2 e^{i\frac{\pi}{2}\rho} f(\rho) - \frac{i}{\pi} \int d\rho \int_0^\infty v^2 f(v) \left\{ e^{i\frac{\pi}{2}v} - i e^{-i\frac{\pi}{2}v} \right\} dv, \rho > 1. \]  

(2.2.10)

Equation (2.2.11) is a Fredholm integral equation of the second kind for \( v^2 f(v) \). Let

\[ C_1(\rho, v) = \frac{d}{d\rho} \int_0^\rho \frac{dv}{(v^2 + \frac{\pi}{2})(\rho^2 + \frac{\pi}{2})}, \]  

and

\[ C_2(\rho, v) = \frac{d}{d\rho} \int_0^\rho \frac{dv}{(v^2 + \frac{\pi}{2})(\rho^2 + \frac{\pi}{2})}. \]  

(2.2.12)

(2.2.13)

then equation (2.2.11) may be re-written as

\[ \rho^2 e^{i\frac{\pi}{2}\rho} f(\rho) - \frac{i}{\pi} \int d\rho \int_0^\infty v^2 e^{i\frac{\pi}{2}v} f(v) C_2(\rho, v) dv \]

\[ = -\frac{1}{\pi} \int d\rho \int_0^\infty v^2 \left\{ e^{i\frac{\pi}{2}v} C_1(\rho, v) - i e^{-i\frac{\pi}{2}v} C_2(\rho, v) \right\} dv, \rho > 1. \]  

(2.2.14)

\( C_1(\rho, v) \) and \( C_2(\rho, v) \) can be evaluated by elementary methods.
to give

\[ C_1(\rho, r) = \frac{(1-r)^{\nu_2}}{(\rho - r)^{\nu_2}(\rho - r)} \]  \hspace{1cm} (2.2.15) 

and

\[ C_2(\rho, r) = \frac{(1+r)^{\nu_2}}{(\rho + r)^{\nu_2}(\rho + r)} \]  \hspace{1cm} (2.2.16) 

Substitute the values of \( C_1(\rho, r) \) and \( C_2(\rho, r) \), given by

(2.2.15), and (2.2.16), into (2.2.14) to obtain

\[ \rho^{\nu_2}(\rho - r)^{\nu_2} e^{ikr} f(r) - \frac{i}{\pi} \int_0^{\infty} r^{\nu_2} \frac{e^{ikr}(1+r)^{\nu_2}}{(\rho + r)} f(r) \, dr 
\]

\[ = - \frac{1}{\pi} \int_0^{\infty} r^{\nu_2} f(r) \left\{ \frac{e^{ikr}(1-r)^{\nu_2}}{(\rho - r)} - i \frac{e^{ikr}(1+r)^{\nu_2}}{(\rho + r)} \right\} \, dr, \rho > 1. \]  \hspace{1cm} (2.2.17) 

Let

\[ F(\rho) = \rho^{\nu_2}(\rho - r)^{\nu_2} e^{ikr} f(r), \rho > 1, \]  \hspace{1cm} (2.2.18) 

and

\[ L(\rho) = -\frac{1}{\pi} \int_0^{\infty} r^{\nu_2} f(r) \left\{ \frac{e^{ikr}(1-r)^{\nu_2}}{(\rho - r)} - i \frac{e^{ikr}(1+r)^{\nu_2}}{(\rho + r)} \right\} \, dr. \]  \hspace{1cm} (2.2.19) 

Insert (2.2.18) and (2.2.19) into (2.2.17), so that

\[ F(\rho) = -\frac{1}{\pi} \int_0^{\infty} \frac{e^{2ikr}(1+r)^{\nu_2} f(r) \, dr}{(\rho - r)^{\nu_2}(\rho + r)} = L(\rho), \rho > 1. \]  \hspace{1cm} (2.2.20) 

Introduce the substitutions

\[ x = \rho - 1, \quad \gamma = r - 1, \quad F(\rho) \equiv L(x), \]  \hspace{1cm} (2.2.21)
into (2.2.20), then
\[ l(x) - \frac{2\pi}{\sqrt{2\pi}} \int_0^\infty e^{-2\pi k^2} (2+\xi^2)^{-\frac{1}{2}} \frac{d\xi}{(2+\xi^2)} = L(x+1), \quad x > 0. \] (2.2.22)

The only place where \( k \) appears explicitly under the integral sign in (2.2.22) is in the term \( \exp(-2\pi k^2) \), and this is convenient for asymptotic expansion, provided that \( l(\xi) \) has a suitable dependence on \( k \). The important part of the integral comes from the region near \( \xi = 0 \), and the asymptotic expansion will depend on the behaviour of \( l(\xi) \), near \( \xi = 0 \) (c.f. [3]).

A first approximation to (2.2.22) is
\[ l(x) \approx L(x+1), \]
and since
\[ L(0) \sim L(1) \text{ for } x \to 0, \]
we are interested in the behaviour of
\[ l(\rho) \text{ for } \rho \approx 1. \]

The behaviour of the first part of the integrand of (2.2.19) changes from \( (1-r)^{-\frac{1}{2}} \) to \( (1-r)^{-\frac{1}{2}} \) as \( \rho \to 1 \).

It is necessary to be careful that \( l(\xi) \) does not contain concealed exponential factors depending on \( k \). These difficulties
could be avoided by using an iterative method of solution, but, for a first approximation expand \((2+x)^{1/2}(2+x+\xi)^{-1}\) by the binomial theorem, as a power series about the point \(\xi=0\), and take only the leading term so that

\[
(2+x)^{1/2}(2+x+\xi)^{-1} = \frac{1}{2}(2+x)^{-1} + O(\xi). \tag{2.2.23}
\]

Hence from the above comments, and result (2.2.23) it is seen that the integral in (2.2.22) may be approximated as

\[
\int_0^\infty e^{i k \xi} \frac{(2+x)^{1/2}}{(2+x+\xi)^{1/2}} \frac{d \xi}{(2+x)} \sim \frac{2^{1/2}}{\pi} \int_0^\infty e^{i k \xi} \frac{d \xi}{2-x} \cdot \tag{2.2.24}
\]

Substitute (2.2.24) into (2.2.22) to give

\[
\lambda(x) \approx \lambda(x+1) + \frac{i 2^{1/2} e^{-i k x}}{\pi} \int_0^\infty e^{i k \xi} \frac{d \xi}{(2+x)} \cdot \tag{2.2.25}
\]

In (2.2.25) let

\[
C = \frac{1}{\pi} \int_0^\infty e^{i k \xi} \frac{d \xi}{(2+x)} \cdot \tag{2.2.26}
\]

then, on substituting for \(\lambda(x)\) by using (2.2.18), an alternative form for (2.2.25) is obtained, which is given by

\[
\rho^{1/2} \lambda(\rho+1) = \lambda(\rho) + \frac{C}{\rho+1}, \quad \rho > 1. \tag{2.2.27}
\]

We can determine the value of \(C\) by multiplying equation (2.2.25), by \(x \exp(-2ikx)\), and then integrating with respect
to $x$ from zero to infinity, i.e.

$$
\int_{0}^{\infty} e^{2i\xi x} l(x) \, dx = \int_{0}^{\infty} \frac{e^{2i\xi x} l(x+1)}{x^{1/2}} \, dx - C \int_{0}^{\infty} \frac{e^{2i\xi x}}{x^{1/2} (x+2)} \, dx .
$$

(2.2.28)

In (2.2.28) multiply both sides by $-i2^{1/2} e^{2i\xi /\pi}$ and also let

$$
Q = \int_{0}^{\infty} \frac{e^{2i\xi x}}{x^{1/2} (x+2)} \, dx ,
$$

(2.2.29)

to give

$$
C = -\frac{i}{\pi} 2^{1/2} e^{2i\xi /\pi} \int_{0}^{\infty} \frac{e^{2i\xi x} l(x+1)}{x^{1/2}} \, dx + \frac{i2^{1/2}}{\pi} e^{2i\xi /\pi} C, Q .
$$

(2.2.30)

Redefine $C$ in terms of $f(x)$, by using (2.2.18) to obtain

$$
C = -\frac{i}{\pi} 2^{1/2} \int_{-1}^{\infty} e^{i\xi r} f(r) \, dr ,
$$

and it is seen that before the constant $C$ can be evaluated, $f(x)$ has to be determined for $x > 1$. Clearly an alternative expression will have to be used in order to find $C$. Consider the first integral on the right-hand side of (2.2.30), and from (2.2.19) we have

$$
-\frac{i}{\pi} 2^{1/2} e^{2i\xi /\pi} \int_{0}^{\infty} \frac{e^{2i\xi x} l(x+1)}{x^{1/2}} \, dx
$$

$$
= -\frac{i}{\pi} 2^{1/2} e^{2i\xi /\pi} \int_{0}^{\infty} \frac{e^{2i\xi x}}{x^{1/2}} \left( -\frac{1}{\pi} \int_{0}^{1} \frac{e^{i\xi v}}{(1-x-v)(1+x+v)} f(v) \, dv \right) \, dx
$$

Let

$$
P = -\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{2i\xi x}}{x^{1/2}} \, dx \int_{0}^{1} \frac{e^{i\xi v}}{(1-x-v)(1+x+v)} f(v) \, dv ,
$$

(2.2.31)
and
\[ P_2 = -\frac{1}{\pi} \int_0^\infty \frac{e^{2i\mathbf{k} \mathbf{x}}}{x^{1/2}} \left( \int_\mathbb{R} e^{-i^{1/2}y^2} f(y) \, dy \right) \, dx. \]  
(2.2.32)

With the aid of (2.2.31) and (2.2.32) we may cast (2.2.30)

into
\[ C = -i \frac{2^{1/2}}{\pi} e^{2i\mathbf{k} \mathbf{Q}} (p_1 - i p_2) \left[ 1 - i \frac{2^{1/2}}{\pi} e^{2i\mathbf{k} \mathbf{Q}} \right]^{-1}. \]  
(2.2.33)

The expression given in (2.2.33) for \( C \) is in terms of known quantities, since \( f(x) \) is known for \( 0 < x < 1 \). In order to find \( f(\rho) \) for \( \rho > 1 \), it is only a matter of evaluating integrals asymptotically, for large \( k \). This evaluation is discussed in the next section.
2.3 Discussion of the asymptotic evaluation of certain integrals

Two main difficulties arise in evaluating the inner integrals in (2.2.31) and (2.2.32) asymptotically for large $k$. First of all, in evaluating the former of these two integrals we cannot use the method of steepest descent, or a similar classical technique, since these methods require that the distance of the nearest singularity from the saddle point is reasonably large. In this integral, the distance of the pole from the saddle point is governed by the parameter $x$ (the distance of a point from the edge of the crack along a radius lying in the plane $z=0$), and the range of $x$ to be considered includes the case where this pole can come arbitrarily close to the saddle point. However, if we replace $f(r)$ by $r$, then it is seen that Stallybrass [9] derives an integral of a similar form, and he overcame this difficulty by using the work of Clemmow [10] and Oberhettinger [11], et al.
From (1.6.3) it is obvious that the integrands of these integrals are dependent on $\lambda$, and it is not possible to obtain asymptotic expressions for large $k$, which are uniformly valid in $\lambda$. We can distinguish several different cases when evaluating these integrals asymptotically, e.g. when $\lambda$ is sufficiently small replace the Bessel functions by their power series. If $\lambda$ is large, we can, after suitable contour integrations (c.f. Chapter 3) replace the Bessel functions by their asymptotic expansions, and in the resulting integrals distinguish two cases, which depend on whether the factor $k-\lambda$, which appears in the exponential part of the integrands is small or large. Explicitly, three cases may be defined as follows:

Case I $\frac{k\varepsilon}{\lambda} \gg 1$ and $\frac{k(1-\varepsilon)}{\lambda} \gg 1$, \hspace{1cm} (2.3.1)

Case II $0 \leq \varepsilon \leq \frac{1}{k}$, \hspace{1cm} (2.3.2)

and

Case III $\frac{1}{k^{3/2} \lambda} \leq \varepsilon \leq 1$, \hspace{1cm} (2.3.3)

where $k\varepsilon = \lambda$, \hspace{1cm} (2.3.4)
and hence from (1.3.10), $0 \leq \varepsilon \leq 1$. The upper and lower limits on $\varepsilon$ in (2.3.2) and (2.3.3) respectively, are to a certain extent arbitrary, although in each instance some form of restriction must be imposed.
Section 3

Asymptotic evaluation of \( f \)

3.1 Case I. \((k \gg 1, k(1 - \epsilon) >> 1)\).

In this subsection we evaluate \( f(\rho) \) for the region \( \rho > 1 \) and subject to the conditions given by (2.3.1). By standard methods it is easy to show from (2.2.29) that

\[
Q = \frac{\pi^{\nu_2} e^{-\frac{\nu_2}{\rho}}}{2^{\nu_2} k^{\nu_2}} \left[ 1 + O\left(\frac{1}{k}\right) \right].
\]

(3.1.1)

This result holds for the three cases since it is independent of \( \lambda \).

For (2.2.31) we insert the value of the function \( f \) from (1.6.3) and let

\[
\lambda Q_1 = \int_0^1 \frac{e^{ikr}}{(1-x)} J_1(\lambda r) dr, \quad x > 0.
\]

(3.1.2)

It is not yet permissible to substitute the asymptotic expansion of \( J_1(\lambda r) \) into (3.1.2) since the integrals of the higher order terms would be divergent near \( r = 0 \); therefore an alternative form of \( Q_1 \) is required. We consider

\[
\int_\zeta_3^{\rho} \frac{(1 - f)^{\nu_2}}{(1 + x - f)} e^{ikf} J_1(\lambda f) df, \quad \text{for } x > 0,
\]

for }
where $C_3$ is the contour shown in figure 6.

$$\lambda Q_1 = e^{-(k^2-2\pi)} \int_0^{\infty} e^{-kr\gamma} \frac{(1+i\gamma)\gamma}{(x+i\gamma)} J_1(\lambda r e^{i\gamma/2}) \gamma^2 d\gamma, \quad x > 0.$$  

$$+ e^{2\pi x} \int_0^{\infty} e^{-kr\gamma} \frac{(1-i\gamma)\gamma}{(1+\gamma+i\gamma)} J_1(\lambda r e^{i\gamma/2}) \gamma^2 d\gamma, \quad x > 0.$$  

Similarly we let

$$\lambda Q_2 = \int_0^{\infty} e^{-ikr\gamma} \frac{(1+r)\gamma}{(1+\gamma+r)} J_1(\lambda r) \gamma^2 d\gamma, \quad x > 0,$$  

and consider

$$\int_{C_4} e^{\gamma^2(1+i\gamma)^2} e^{ikf} J_1(\lambda f) df, \quad \text{for} \quad x > 0.$$
where $C_4$ is the contour shown in figure 7.

\[ \text{Figure 7} \]

On letting $R \to \infty$, we obtain

\[
\lambda Q_2 = e^{iKx} \int_0^\infty e^{-kr} \frac{(1-iv)^{\gamma_2}(2-iv)^{\gamma_2}}{(1+x-iv)} \int_1 \{\lambda (1-iv)\} \, dv \\
+ e^{iKx} \int_0^\infty e^{-kr} \frac{(1-iv)^{\gamma_2}}{(2+x-iv)} \int_1 \{\lambda e^{-iv/2}\} \, dv, x > 0. \tag{3.1.5}
\]

From an examination of the forms of $Q_1, Q_2$, and $L(\rho)$ it is seen that it is only necessary to consider $P$, where

\[
P = Q_1 - iQ_2, \tag{3.1.6}
\]

(c.f. equations (3.1.2), (3.1.4) and (2.2.19)). Therefore to (3.1.5) we add minus $i$ times (3.1.5) to give

\[
\lambda P = e^{i(K-\frac{3}{4})} \int_0^\infty e^{-kr} \frac{(1+iv)^{\gamma_2}}{(x-iv)} \int_1 \{\lambda (1+iv)\} \, dv \\
+ e^{iKx} \int_0^\infty e^{-kr} \frac{(1-iv)^{\gamma_2}(2-iv)^{\gamma_2}}{(2+x-iv)} \int_1 \{\lambda (1-iv)\} \, dv, x > 0. \tag{3.1.7}
\]
noting that the second integral of (3.1.3) cancels with the second integral of (3.1.4), since \( J_1(x) = -J_1(-x) \). On substituting the respective asymptotic expansions for \( J_\lambda(\pm ir) \) into (3.1.7) and then using (2.3.4) we obtain

\[
\begin{align*}
\kappa \xi \rho_n e^{i(k - \frac{3\pi}{4})} & \int_0^\infty e^{\frac{k\sqrt{r}}{r - x - i\nu}} \left( \frac{2}{\pi \kappa \xi} \right)^{\nu/2} \cos \left( \kappa \xi (1 + i\nu) - \frac{3\pi}{4} \right) \, dr \\
+ e^{ik} & \int_0^\infty e^{\frac{k\sqrt{r}}{r - x + i\nu}} \left( \frac{2}{\pi \kappa \xi} \right)^{\nu/2} \cos \left( \kappa \xi (1 - i\nu) - \frac{3\pi}{4} \right) \, dr, \quad \kappa > 0.
\end{align*}
\]

We now write the trigonometric functions in terms of exponentials to arrive at

\[
\begin{align*}
2^{\nu/2} \pi^{\nu/2} & \kappa^{\nu/2} \rho_n e^{i(k(1 + \nu) + \nu/2)} \int_0^\infty e^{-\frac{k(1 + \nu)}{r - x - i\nu}} \nu^{\nu/2} \, dr + e^{i(k(1 - \nu) - \nu/2)} \int_0^\infty e^{-\frac{k(1 - \nu)}{r - x - i\nu}} \nu^{\nu/2} \, dr \\
+ e^{i(k(1 - \nu) - \nu/2)} & \int_0^\infty e^{\frac{k(1 - \nu)}{r - x + i\nu}} (2 - i\nu)^{\nu/2} \, dr + e^{i(k(1 + \nu) + \nu/2)} \int_0^\infty e^{-\frac{k(1 + \nu)}{r - x + i\nu}} (2 - i\nu)^{\nu/2} \, dr, \quad \kappa > 0 (3.1.8)
\end{align*}
\]

Consider the first integral on the right-hand side of (3.1.8). It is not a valid step to expand \((r - ir)^{-1}\) in powers of \(r\), since \(x\) may take values arbitrarily close to zero.

This integral exists when \(x\) is zero and may be evaluated by an application of Watson's lemma; but an asymptotic expansion is
desired which is valid for all \( k(1+\varepsilon)x \) and gives the correct result in the limit as \( x \) tends to zero, i.e. the expansion must be uniform with respect to \( x \). The type of expansion which will be obtained is called by Cramow [10] a partial asymptotic expansion. However, it is more convenient to use the work of Oberhettinger [11] to give

\[
\int_0^\infty \frac{\varepsilon^{-k(1+\varepsilon)r} r^{\nu/2}}{(x-i\nu)} dr = x^{\nu/2} e^{\nu/2} \left( x e^{i\nu/2} \right)^{-\nu/4} \left[ e^{x} e^{i\nu/2} \right]^{3/4} \varepsilon^{-k(1+\varepsilon)r/2} W_{\frac{3\nu}{4}, \frac{3\nu}{4}}^{\frac{3\nu}{4}, \frac{3\nu}{4}} \left[ i x^{-k(1+\varepsilon)x} \right].
\]

we can express the Whittaker function in terms of Kummer's function or in terms of the incomplete gamma function, i.e.

from section 7.2.3. of [12]

\[
W_{\frac{3\nu}{4}, \frac{3\nu}{4}}\left( x^{-k(1+\varepsilon)x} \right) = e^{i x^{-k(1+\varepsilon)x} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4}} \left[ i x^{-k(1+\varepsilon)x} \right].
\]

and then, from section 6.7.2 of [12],

\[
W_{\frac{3\nu}{4}, \frac{3\nu}{4}}\left( x^{-k(1+\varepsilon)x} \right) = e^{i x^{-k(1+\varepsilon)x} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4}} \left[ i x^{-k(1+\varepsilon)x} \right].
\]

Hence

\[
\int_0^\infty \frac{e^{-k(1+\varepsilon)r} r^{\nu/2}}{(x-i\nu)} dr = x^{\nu/2} e^{\nu/2} e^{i k(1+\varepsilon)x} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ i x^{-k(1+\varepsilon)x} \right], x > 0. \tag{3.1.9}
\]

Therefore

\[
\frac{e^{-k(1+\varepsilon)+\nu/2} e^{i k(1+\varepsilon)x}}{x} \int_0^\infty \frac{e^{-k(1+\varepsilon)r} r^{\nu/2}}{(x-i\nu)} dr = x^{\nu/2} e^{\nu/2} e^{i k(1+\varepsilon)(x+i) \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ x e^{i\nu/2} \right]^{3/4} \left[ i x^{-k(1+\varepsilon)x} \right]}, x > 0. \tag{3.1.10}
\]
For the second integral on the right-hand side of (3.1.8) it is only necessary to change $\varepsilon$ to $-\varepsilon$ in (3.1.9) to give
\[
\int_{0}^{\infty} \frac{e^{-k(x+\varepsilon)^2} e^{i k (1-\varepsilon)^2} - e^{-k(x-\varepsilon)^2} e^{i k (1+\varepsilon)^2}}{k (x+\varepsilon)} \; dx = 2 \frac{\sqrt{2} \varepsilon}{k} e^{i k (1+\varepsilon)} \left\{ 1 + O \left( \frac{1}{k} \right) \right\},
\]
(3.1.11)

Results (3.1.10) and (3.1.11) are exact and in the limit as $x \to 0$ are finite and agree with the results obtained on applying Watson's lemma when $x=0$. We can evaluate asymptotically the third integral on the right-hand side of (3.1.8) by applying Watson's lemma, i.e.,
\[
\int_{0}^{\infty} \frac{e^{-k(x+\varepsilon)^2} e^{i k (1+\varepsilon)^2}}{k (x+\varepsilon)} \; dx = 2 \frac{\sqrt{2} \varepsilon}{k} e^{i k (1+\varepsilon)} \left\{ 1 + O \left( \frac{1}{k} \right) \right\},
\]
(3.1.12)

Similarly for the last integral we have
\[
\int_{0}^{\infty} \frac{e^{-k(x-\varepsilon)^2} e^{i k (1-\varepsilon)^2}}{k (x-\varepsilon)} \; dx = 2 \frac{\sqrt{2} \varepsilon}{k} e^{i k (1+\varepsilon)} \left\{ 1 + O \left( \frac{1}{k} \right) \right\},
\]
(3.1.13)

From (1.6.3), (2.2.19), (3.1.2), (3.1.5) and (3.1.6) it is seen that

\[
L(\rho) = L(x+1) = \frac{2 \omega \varepsilon}{\mu c} \rho.
\]
(3.1.14)

Hence on using this last result, (3.1.8), (3.1.10), (3.1.11), (3.1.12) and (3.1.13) we obtain
\[
L(\rho) \sim \frac{2 \sqrt{2} \omega \varepsilon}{\mu c} \left[ e^{i k (1+\varepsilon)^2} e^{i k (1-\varepsilon)^2} \left\{ 1 + O \left( \frac{1}{k} \right) \right\} \right]
\]
and
\[
\frac{\sqrt{2} \varepsilon}{2} \left( \rho - 1 \right) \left( \rho + 1 \right) \left[ e^{i k (1+\varepsilon)^2} e^{i k (1-\varepsilon)^2} \left\{ 1 + O \left( \frac{1}{k} \right) \right\} \right].
\]
The error term in the brackets \([\ldots]\) is of \(O(k^{3/2})\) or of \(O(k^3)\) depending on whether \(k(\rho-1)\) is small or large.

Before \(O\) can be evaluated asymptotically we must determine

\[
P_{1} - i P_{2} = \int_{0}^{\infty} \frac{e^{2i-kx}}{x^{1/2}} L'(x+1) \, dx,
\]

(3.1.16)

and hence from (3.1.15)

\[
P_{1} - i P_{2} \sim \frac{2^{1/2} \pi}{\mu \sqrt{k}} \frac{2i}{\kappa^{1/2} \kappa^{1/2} / 2} \left[ \frac{\sqrt{2}}{i \pi} e^{i \pi \kappa(1+i)} \right]
\]

\[
+ \frac{i}{\sqrt{2}} e^{-\pi \kappa(1+i)/4} \int_{0}^{\infty} e^{-i \pi \kappa(1+i)x} \left\{ \frac{\pi}{2} - \frac{\pi}{2} \right\} \frac{e^{2i-kx}}{x^{1/2}} \, dx
\]

\[
+ 2^{1/2} \pi \frac{e^{1/2}}{\kappa} \left\{ \frac{e^{i \pi \kappa(1+i)/4}}{1-i} + \frac{e^{i \pi \kappa(1+i)/4}}{1+i} \right\} \int_{0}^{\infty} e^{2i-kx} \, dx.
\]

(3.1.17)

we deform the path of integration of the first integral into

the negative imaginary axis to give

\[
\int_{0}^{\infty} e^{-i k(1+i)x} \left\{ \frac{\pi}{2} - \frac{\pi}{2} \right\} \frac{e^{2i-kx}}{x^{1/2}} \, dx = -i \int_{0}^{\infty} e^{k(1+i)x} \left\{ \frac{\pi}{2} - \frac{\pi}{2} \right\} \frac{e^{2i-kx}}{x^{1/2}} \, dx.
\]

(3.1.18)

and then use result (31) of section 4.12 of [13] which is

\[
\int_{0}^{\infty} e^{(p-a)t} T(\nu,at) \, dt = \frac{T(\nu)}{(p-a)^{\nu}} \left[ \Gamma(\nu) \left( 1 - \frac{a}{p} \right)^{\nu} \right].
\]

(3.1.19)
for $\Re(\nu)>-1$ and $\Re(p)>0$ to obtain
\[
\int_{0}^{\infty} e^{i(k(1-\nu)x)} \Gamma_{1/2,1/2} i^{k(1+\nu)x} \, dx = \frac{2i^{1/2} \omega}{k(1-\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\}.
\] (3.1.20)

Result (3.1.20) holds if $\nu$ is changed to $-\nu$. The last integral on the right-hand side of (3.1.17) has already been evaluated asymptotically for large $k$ (c.f.Q). Hence from (3.1.20) and (3.1.1)

we have
\[
P_1 - iP_2 = \frac{2^{1/2} \omega}{\mu \epsilon^{3/2} \xi^{3/2} \nu^{1/2}} \left\{ \frac{e^{i(k(1+\nu)x)}}{k(1-\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + \frac{e^{i(k(1+\nu)x)}}{k(1+\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + O\left(\frac{1}{k^{1/2}}\right) \right\}.
\] (3.1.21)

On expanding (2.2.33) we obtain
\[
C = -\frac{i}{\nu} 2^{1/2} \epsilon^{2i k} (P_1 - iP_2) \left[ 1 + O\left(\frac{1}{k^{1/2}}\right) \right]
\] (3.1.22)

and after substituting for $P_1 - iP_2$ from (3.1.21), we have
\[
C = \frac{2^{1/2} \omega}{\mu \epsilon^{3/2} \xi^{3/2} \nu^{1/2}} \left\{ \frac{e^{i(k(1+\nu)x)}}{k(1+\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + \frac{e^{i(k(1+\nu)x)}}{k(1+\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + O\left(\frac{1}{k^{1/2}}\right) \right\}
\] (3.1.23)

Hence from (2.2.27) (3.1.15) and (3.1.23) it is seen that
\[
f(p) \sim \frac{2^{1/2} \omega}{\mu \epsilon^{3/2} \xi^{3/2} \nu^{1/2}} \left\{ \frac{e^{i(k(1+\nu)x)}}{k(1+\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + \frac{e^{i(k(1+\nu)x)}}{k(1+\nu)} \left\{ 1 - \frac{2^{1/2}}{1+\nu} \right\} + O\left(\frac{1}{k^{1/2}}\right) \right\}
\]
\[ + \frac{2e^{ik}}{(\rho+1)k(1-\varepsilon)^{3/2}} \left\{ e^{ik\varepsilon - \frac{\pi}{4}} + e^{ik\varepsilon - \frac{3\pi}{4}} \right\}, \quad \rho > 1, \quad (3.1.24) \]

since the two terms in the brackets \{\} of (3.1.15) cancel with the first terms in each of the brackets \{\} of (3.1.23). For \( \rho \neq 1 \), the error term in the brackets \{\} of (3.1.24) is of \( O(\kappa^2) \) and for large \( k(\rho - 1) \), of \( O(\kappa^{-2}) \).
3.2 Case II. \(0 \leq \varepsilon \leq k^{-1}\)

First of all let us consider the case when \(\lambda = 0\) by taking the limit as \(\lambda \to 0\) in (3.1.7) to give

\[
P = \frac{e^{-\frac{1}{2}(k^2-\varepsilon^2)}}{2} \int_0^\infty e^{kr} r^{\frac{1}{2}} (1+ir)^{\frac{1}{2}} dv + \frac{e^{-i\varepsilon}}{2} \int_0^\infty e^{kr} (1-ir)^{\frac{1}{2}} (2+ir)^{\frac{1}{2}} dr. \tag{3.2.1}
\]

It is required to evaluate (3.2.1) for \(x > 0\) and large \(k\). For the first integral it would be convenient if we could expand \((1+ir)^{\frac{1}{2}}\) as a power series about the origin. In order to justify this step we will adopt the method outlined in section 4 of [10].

We make use of the Maclaurin expansion

\[
(1+ir)^{\frac{1}{2}} = \sum_{s=0}^{n-1} \frac{\Gamma(s+1)}{\Gamma(s+1)} e^{\frac{s}{2}} r^s + r_n \tag{3.2.2}
\]

where Cauchy's form of the remainder \(r_n\), is given by

\[
r_n = \frac{\Gamma(s+1)}{(n-1)!} \Gamma(s+1-n) \int_0^r (r-u)^{n-1} (1+iu)^{\frac{1}{2}} du. \tag{3.2.3}
\]

In (3.2.3) let \(u = r(1+\mu)^{-1}\), so that

\[
r_n = \frac{\Gamma(s+1)}{\Gamma(n-1)!} \int_0^r \frac{(r-u)^{n-1}}{(1+\mu)^{n+1}} \left(1+i\frac{1+\mu}{1+\mu}\right)^{\frac{1}{2} n} d\mu. \tag{3.2.4}
\]

Then from (3.2.2) and (3.2.4)

\[
\int_0^\infty e^{kr} r^{\frac{1}{2}} (1+ir)^{\frac{1}{2}} dv = \sum_{s=0}^{n-1} f_s(k) + R_n, \tag{3.2.5}
\]

where
\[ R_n = \frac{\Gamma(s_2) \cdot e^{\gamma/2}}{\Gamma(n) \cdot \Gamma(s_2-n)} \int_0^\infty \frac{e^{x/r}}{(x-i\tau)^{(n+1)/2}} \left(1+\frac{i\tau}{1+i\tau} \right)^{n/2} d\mu, \quad (3.2.6) \]

and

\[ f_s(-k) = \frac{\Gamma(s_2) \cdot e^{(s+1)/2}}{\Gamma(s_2-s)} \int_0^\infty \frac{e^{x/r}}{(r+i\tau)^{(s+1)/2}} dr \quad (3.2.7) \]

From [11], (3.2.7) is

\[ f_s(-k) = \frac{\Gamma(s_2) \cdot e^{(s+1)/2} \cdot \Gamma(s+\frac{1}{2})}{\Gamma(s_2-s)} \cdot \left(x e^{\frac{x}{2}} \right)^{\frac{1}{2}} \left(\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s_2-s)} \right) e^{i\frac{\pi}{2} \frac{(s+1)}{2}} \mathcal{W}(i-k\tau), \quad \tau = \frac{-\frac{1}{2}(s+\frac{1}{2})}{s+\frac{1}{2}} \]

or alternatively it can be shown, as for (3.1.9), that

\[ f_s(-k) = \frac{\Gamma(s_2) \cdot \Gamma(s+\frac{1}{2})}{\Gamma(s_2-s)} \cdot \left(e^{i\frac{\pi}{2} \frac{(s+1)}{2}} \cdot \left(x e^{\frac{x}{2}} \right)^{\frac{1}{2}} \right) \mathcal{W}(i-k\tau), \quad \tau = \frac{-\frac{1}{2}(s+\frac{1}{2})}{s+\frac{1}{2}} \]

and for \( k \gg 1 \), \( k\tau \ll 1 \) we have

\[ f_s(-k) \approx (-1)^s \frac{\Gamma(s_2) \cdot \Gamma(s+\frac{1}{2})}{\Gamma(s_2-s)} \cdot \frac{e^{(1-s)\pi/2} + i\frac{\pi}{2} \tau}{\frac{1}{2} s+\frac{1}{2}} \]

In the limit as \( x \to 0 \) result (3.2.9) is exact; also Watson's lemma is applicable to the integral and the result obtained agrees with the aforementioned, i.e. the expansion composed of the \( f_s(k) \) is identical with that obtained after an application of Watson's lemma. If only \( f_0(k) \) is taken we have

\[ \frac{\sqrt{\pi}}{2} \int_0^\infty e^{\frac{\pi}{2} \frac{(s+1)}{2}} \left(1+\frac{i\tau}{1+i\tau} \right)^{\frac{3}{2}} d\tau \approx \frac{\sqrt{\pi}}{2} \frac{x^{\frac{1}{2}} e^{i\frac{\pi}{2} \frac{(s+1)}{2}} \Gamma(-\frac{1}{2} + i\tau)}{\frac{1}{2} s+\frac{3}{2}} \]

However, it has not yet been established that (3.2.5) is a valid asymptotic expansion; in order to do this we will have to
obtain an estimate of the remainder \( R_n \). This estimate will be of a form which is due to Jeffreys [14]. Hence, following Clemmow [16], in (3.2.6) we make the substitution \( r = (1 + \mu)z \) to give

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(n) \Gamma(s - \eta - 1/2)} \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + \mu)z + i k} \cdot \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + \mu)z + i k} \cdot dz \cdot dy, \quad (3.2.11)
\]

and after making the further substitution \( \mu = y/z \), we have

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(n) \Gamma(s - \eta - 1/2)} \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + i k)^2} \cdot \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + i k)^2} \cdot dz \cdot dy. \quad (3.2.12)
\]

In \((z + iy)^{1/2} = y + z + i x\), we can put \( y = n/k, \ z = 0\); therefore

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(n) \Gamma(s - \eta - 1/2)} \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + i k)^2} \cdot \int_0^\infty \frac{e^{-r} z^{n-1}}{(1 + i k)^2} \cdot dz \cdot dy. \quad (3.2.13)
\]

Since

\[
\int_0^\infty \frac{e^{-r} z^{n-1}}{y} dy = \frac{\Gamma(n)}{k^n}, \quad (3.2.14)
\]

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(s - \eta - 1/2)} \int_0^\infty \frac{\exp\left[- r + \left(\frac{3}{2} - n\right) \log (1 + i k)\right]}{(1 + i k)^2} dz. \quad (3.2.15)
\]

If we expand the logarithm in (3.2.15) and ignore second and higher order powers of \( z \) in the exponential, we have

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(s - \eta - 1/2)} \int_0^\infty \frac{\exp\left[- \left(\frac{3}{2} - n\right) \log (1 + i k)\right]}{(1 + i k)^2} dz, \quad (3.2.16)
\]

and hence

\[
R_n = \frac{\Gamma(\gamma/2) \Gamma(\eta + 1/2)}{\Gamma(s - \eta - 1/2)} \cdot \frac{\exp\left[- \left(\frac{3}{2} - n\right) \log (1 + i k)\right]}{(1 + i k)^2}. \quad (3.2.17)
\]
when $s$ is large we can apply saddle-point techniques to (3.2.7).

The saddle-point occurs when $r = s/k$, which is not small; therefore let $r = s/k + u$, and hence

$$f_s(-k) \approx \frac{\Gamma(s/2) e^{(s+1)u}}{(s/2-u) s! (s/k-x)} \int e^{s \mu^2} d\mu,$$

i.e.,

$$f_s(-k) \approx \frac{\Gamma(s/2) e^{(s+1)u/2} e^{s s/2} (2\pi)^{s/2} s^{s/2}}{\Gamma(s/2-s) s! (s/k+i-x) k^{s+1}}.$$  \hspace{1cm} (3.2.18)

Now use Stirling's approximation which is

$$s! \approx s^{s+1/2} e^{-s} (2\pi)^{1/2}.$$  

Hence

$$f_s(-k) \approx \frac{\Gamma(s/2) e^{(s+1)u/2} e^{s s/2} (2\pi)^{s/2} s^{s/2}}{\Gamma(s/2-s) s! (s/k+i-x) k^{s+1}},$$

and in particular

$$f_{n-1}(-k) \approx \frac{\Gamma(s/2) e^{u/2} (n/k)^{n/2}}{\Gamma(s/2-n+1) (n/k+i-x) n}.$$  \hspace{1cm} (3.2.19)

If this last expression is substituted into (3.2.17) we obtain

$$R_n \approx \frac{(s/2-n)e^{u/2}}{\gamma k + (n-3/2)i} f_{n-1}(-k),$$  \hspace{1cm} (3.2.20)

and since the smallest term is reached when $n \equiv k$ we may write

$$R_n \approx \frac{i}{(1+i)} f_{n-1}(-k),$$

or, alternatively,

$$R_n + f_{n-1}(-k) \approx \frac{1}{2} (1-i) f_{n-1}(-k).$$  \hspace{1cm} (3.2.21)
This estimate of the remainder shows, according to Jeffreys [14] that the expansion given by expressions (3.2.5) to (3.2.7) is a valid asymptotic expansion. Hence from (3.2.10) and by applying Watson's lemma to the second integral of (3.2.1) we have

\[ P \sim \frac{\Lambda^2 \chi^{1/2}}{4} e^{i k \chi} \left( -\frac{1}{2}, \frac{i k}{2} \chi \right) + \frac{e^{i k}}{2^{1/2} (\chi + 1)^{1/2} k}, \quad \chi > 0. \quad (3.2.22) \]

Now consider the case when \( \lambda \neq 0 \). If the series in powers of \( \lambda^n (1 \pm i r)^n, \quad n=1, 2, \ldots \) are substituted for \( \lambda, \lambda (1 \pm i r) \), respectively, in (3.1.7) and the resultant integrals evaluated asymptotically for large \( k \), then the expansion obtained will be in terms of powers of \( 1/k \), each power of \( 1/k \) being multiplied by an infinite series in powers of \( \lambda \). Clearly, the expansion would only be valid if these series in \( \lambda \) converged, which would be the case if \( \lambda \) was sufficiently small, say of \( O(1/k) \).

It is, however, possible to evaluate (3.1.7) for a much wider range of \( \lambda \) by replacing the Bessel functions by their respective Taylor series about the point \( z=\lambda \), the arguments of the Bessel functions being given by \( z=\lambda (1 \pm i r) \). This will produce an
asymptotic expansion in powers of \((\lambda/k)\) with coefficients involving \(J_1^{(s)}(\lambda)\), where
\[
J_1^{(s)}(\lambda) = \frac{d^s}{d \lambda^s} J_1(\lambda), \quad s = 0, 1, \ldots,
\]
and has the advantage that it should give accurate results when \(\lambda\) is of \(O(1)\).

In (3.1.7) we replace the Bessel functions by the first terms in their respective Taylor series, and taking note of the comments made in evaluating (3.1.8) and (3.2.1), we expand the rest of the integrand in powers of \(r\) to give
\[
\lambda P \sim e^{(k-R)} J_1(\lambda) \int_0^{\infty} e^{k r} x^2 \frac{(x+i)^{1/2}}{(x+2)} \frac{\pi}{\lambda} dr + \frac{2\sqrt{2} e^{i k}}{k(x+2)} J_1(\lambda) \int_0^{\infty} e^{k r} r x^2 \frac{\pi}{\lambda} dr, \quad x > 0.
\]
and hence
\[
P \sim \frac{\sqrt{2} e^{i k(x+1)}}{2/k} J_1(\lambda) x^{1/2} \{1 - i(x, k) - i(\rho-1)\} + \frac{2\sqrt{2} e^{i k} J_1(\lambda)}{k(x+2) \lambda}, \quad x > 0.
\]
In (3.2.25) we can, for small \(\lambda\), replace the Bessel function by the first few terms in its power series and if we let \(\lambda \to 0\), then this result reduces to (3.2.22).

From (3.2.25) and (3.1.14) it is possible to write down the expression for \(L(\rho)\), i.e.,
\[
L(\rho) \sim \frac{2\sqrt{2} e^{i k(\rho-1)}}{\pi} \int_0^{\infty} e^{k r} r x^2 \frac{\pi}{\lambda} dr + \frac{2\sqrt{2} e^{i k}}{k(x+2)} \{1 - i(x, k) - i(\rho-1)\}, \quad \rho > 1.
\]
The error term in the brackets [ ] of (3.2.26) is, for \( \rho \approx 1 \), of \( O(1/k^2) \) and for large \( k(\rho - 1) \) of \( O(1/k^2) \).

As in Case I, from (3.1.16) and (3.2.26) we have

\[
P_1 - iP_2 = \frac{2\alpha_i}{\mu_c \pi i} \frac{J_1(\lambda)}{\lambda} \left[ \int_0^{\pi/2} e^{ikx} \Gamma(-\frac{1}{2}, i k x) dx \right. \\
+ \left. \frac{2\sqrt{2} e^{-ik}}{k} \int_0^{\infty} \frac{e^{x^2}}{x^{1/2}(x+2)} dx + O\left(\frac{1}{k^2}\right) \right]
\]  

(3.2.27)

The first integral on the right-hand side of (3.2.27) can be evaluated if \( \epsilon \) is set to zero in (3.1.20), and the second integral is 0. Hence

\[
P_1 - iP_2 = \frac{2\alpha_i}{\mu_c \pi i} \frac{J_1(\lambda)}{\lambda} \left[ \int_0^{\pi/2} e^{ikx} \Gamma(-\frac{1}{2}, i k x) dx \right. \\
+ \left. \frac{2\sqrt{2} e^{-ik}}{k} \int_0^{\infty} \frac{e^{x^2}}{x^{1/2}(x+2)} dx + O\left(\frac{1}{k^2}\right) \right]
\]  

(3.2.28)

From (3.1.28) and (3.2.28) it is seen that

\[C = \frac{2\alpha_i}{\mu_c \pi i} \frac{J_1(\lambda)}{\lambda} \left[ \frac{2\sqrt{2}}{k} \right] e^{-ik} \Gamma(-\frac{1}{2}, i k) \left[ 1 + O\left(\frac{1}{k^2}\right) \right],
\]  

(3.2.29)

and hence (3.2.27), (3.2.28) and (3.2.29) give

\[
f(\rho) \sim \frac{2\alpha_i}{\mu_c \pi i} \frac{J_1(\lambda)}{\lambda} e^{ik} \rho \left[ \int_0^{\pi/2} (\rho - i)^{\frac{1}{2}} e^{ik} d\rho + O\left(\frac{1}{k^2}\right) \right], \quad \rho > 1,
\]  

(3.2.30)

since the second term in the brackets [ ] of (3.2.26) cancels with the second term in the brackets ( ) of (3.2.29). The error term in the brackets [ ] of (3.2.30) is of \( O(1/k^2) \) for \( \rho \approx 1 \) and of \( O(1/k^2) \) for large \( k(\rho - 1) \).
3.3 Case III. (1-1/k_2^2, ε ≤ 1)

If the respective asymptotic expansions for \( J_{\lambda(1+ir)} \) are substituted into (3.1.7), then in the limit as \( \lambda \to k \) the integrals will be divergent. In order to avoid this trouble we write \( Q_1 \), given by (3.1.2), in the form

\[
\lambda Q_1 = \frac{1}{2} \int_0^\infty e^{ikr} \frac{y_2 r y_2 (1-r y_2 )}{(1+x-r)} \left[ H_1^{(0)}(\lambda r) + H_1^{(1)}(\lambda r) \right] \, dr, \quad x > 0. \tag{3.3.1}
\]

By comparison with (3.1.3) it is seen that

\[
\int_0^\infty e^{ikr} \frac{y_2 r y_2 (1-r y_2 )}{(1+x-r)} \, H_1^{(1)}(\lambda r) \, dr = e^{i(k-\frac{3k}{2})} \int_0^\infty e^{ikr} \frac{y_2 r y_2 (1+ir y_2 )}{(x-i r)} \, H_1^{(1)}(\lambda r e^{i/2}) \, dr, \quad x > 0, \tag{3.3.2}
\]

where \( \frac{\pi}{2} < \arg(r-1) \leq \pi \).

For the second integral comprising \( Q_1 \), we consider a contour integration around a rectangle of the form shown in figure 6, but instead of letting \( R \to \infty \), we set \( R = 1 \). On using an argument similar to that used to obtain (3.1.3) we have

\[
\int_0^\infty e^{ikr} \frac{y_2 r y_2 (1-r y_2 )}{(1+x-r)} \, H_1^{(1)}(\lambda r) \, dr = e^{i(k-\frac{3k}{2})} \int_0^\infty e^{ikr} \frac{y_2 r y_2 (1+ir y_2 )}{(x-i r)} \, H_1^{(1)}(\lambda r e^{i/2}) \, dr + \int_0^\infty e^{ikr} \frac{y_2 r y_2 (1-r y_2 )}{(1-r-x)} \, H_1^{(1)}(\lambda (1+i r)) \, dr.
\]
\( x > 0, \) where \( \frac{\pi}{2} \leq \arg (r-1) \leq \pi. \)  \hspace{1cm} (3.3.3)

Similarly, we can write \( Q_2 \) as

\[
\lambda Q_2 = \frac{1}{2} \int \frac{e^{ikr} r^{1/2} (1+r)^{1/2} \{ H_1^{(1)}(\lambda r) + H_1^{(2)}(\lambda r) \}}{(1+x+r)} \, dr, \quad x > 0,
\]

and hence by consideration of (3.1.15)

\[
\int_0^\infty \frac{e^{ikr} r^{1/2} (1+r)^{1/2} H_1^{(2)}(\lambda r)}{(1+x+r)} \, dr = i \int_0^\infty \frac{e^{ikr} (1-r)^{1/2} (2-r)^{1/2} H_1^{(2)}(\lambda (1-r))}{(2+x-4r)} \, dr
\]

\[
+ i e^{\frac{\gamma i}{2}} \int_0^\infty \frac{e^{ikr} r^{1/2} (1-r)^{1/2} H_1^{(2)}(\lambda r e^{-i\pi/2})}{(1+x-i)} \, dr, \quad x > 0,
\]

where \( -\pi \leq \arg (r-1) \leq -\pi. \) \hspace{1cm} (3.3.5)

For the other integral comprising \( Q_2, \) we once again consider a finite contour, consisting this time of a rectangle of the form shown in figure 7, with \( R=1, \) to give

\[
\int_0^\infty \frac{e^{ikr} r^{1/2} (1+r)^{1/2} H_1^{(1)}(\lambda r)}{(1+x+r)} \, dr = i \int_0^\infty \frac{e^{ikr} (1-r)^{1/2} (2-r)^{1/2} H_1^{(1)}(\lambda (1-r))}{(2+x-4r)} \, dr
\]

\[
+ i e^{\frac{\gamma i}{2}} \int_0^\infty \frac{e^{ikr} r^{1/2} (1-r)^{1/2} H_1^{(1)}(\lambda r e^{-i\pi/2})}{(1+x-i)} \, dr
\]

\[
+ e^{ikx} \int_0^\infty \frac{e^{ikr} (y-r)^{1/2} (y-r+1)^{1/2} H_1^{(1)}(\lambda (y-r))}{(1-y+x)} \, dr, \quad x > 0,
\]

where \( -\pi \leq \arg (r-1) \leq -\pi. \) \hspace{1cm} (3.3.6)
Therefore from (3.3.1) to (3.3.6) and (3.1.6) we obtain

\[ \lambda p = \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 + ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} J_1(\lambda (e^{-iv/y})) dv \]

\[ + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (e^{-iv/y})) dv \]

\[ + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} J_1(\lambda (e^{-iv/y})) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv \]

\[ + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (e^{-iv/y})) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv \]

\[ + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (e^{-iv/y})) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv \]

\[ + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (e^{-iv/y})) dv + \frac{\varepsilon}{2} \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} H_1^{(1)}(\lambda (1 + ivy)) dv \]

\[ x > 0. \quad (3.3.7) \]

The second integral cancels with the seventh and the fourth

adds to the ninth, since

\[ H_1^{(1)}(\lambda e^{-iv/y}) = H_1^{(2)}(\lambda e^{-iv/y}) = - \frac{2}{\pi} k_1(\lambda v). \quad (3.3.8) \]

Hence integrals of the form

\[ \int_0^\infty \frac{e^{-ky}}{y^2} (1 - ivy)^{1/2} K_1(k e^{-iv/y}) dv \]

are produced which may be discarded since the integrands contain an exponential decay factor. The sixth integral is of \( O(1/k) \) and
will be discarded since it is proposed to ignore terms of $O(1/k)$ and above. On replacing the Hankel functions by their asymptotic expansions in the remaining integrals we obtain

\[
2^{\nu/2} \pi^{3/2} \xi^{2\nu} \, \rho \sim e^{i\nu(1+\epsilon)+\frac{3\pi i}{4}} \int_0^\infty e^{-\nu(1-\epsilon)\nu^2} d\nu + e^{i\nu(1-\epsilon)} \int_0^\infty e^{-\nu(1-\epsilon)\nu^2} d\nu
\]

\[
+ e^{i\nu(1-\epsilon)-\frac{3\pi i}{4}} \int_0^\infty e^{-\nu(1-\epsilon)\nu^2} d\nu + e^{i\nu(1-\epsilon)+\frac{3\pi i}{4}} \int_0^\infty e^{-\nu(1-\epsilon)\nu^2} d\nu
\]

\[
+ \int_0^\infty e^{i\nu(1-\epsilon)} \left( 1 + \nu + \right) d\nu \right), \quad x > 0.
\] (3.3.9)

Where it is valid we can expand the integrands in powers of $x$ and also expand the exponential functions in all but the first integral, in powers of $k(1-\epsilon)$ to give

\[
2^{\nu/2} \pi^{3/2} \xi^{2\nu} \, \rho \sim e^{i\nu(1+\epsilon)+\frac{3\pi i}{4}} \int_0^\infty e^{-\nu(1-\epsilon)\nu^2} d\nu + \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu + e^{i\nu(1-\epsilon)+\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu
\]

\[
+ e^{i\nu(1-\epsilon)-\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu + e^{i\nu(1-\epsilon)+\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu
\]

\[
+ e^{i\nu(1-\epsilon)} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu + e^{i\nu(1-\epsilon)+\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu + e^{i\nu(1-\epsilon)-\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu
\]

\[
+ e^{i\nu(1-\epsilon)+\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu + e^{i\nu(1-\epsilon)-\frac{3\pi i}{4}} \int_0^\infty \frac{\nu}{x-\epsilon} \, d\nu
\] (3.3.10)
In (3.3.10) it is possible to ignore terms of order \( k^2 (1-\varepsilon)^2 \)
and above if \( k^2 (1-\varepsilon)^2 \leq 1/k \). This condition is satisfied if
\( \varepsilon \geq 1-1/k^{3/2} \); hence \( \varepsilon \) must satisfy the following condition
\[
1 - \frac{1}{k^{3/2}} \leq \varepsilon \leq 1.
\]
For the time being, the error due to expanding in powers of \( x \) is
of \( O(x) \).

First of all in (3.3.10) let us consider
\[
\int_{0}^{1} \frac{y^{1/2}}{(x-iy)} dy, \quad \text{for } x > 0.
\]
We isolate the dominant part of this integral, so that
\[
\int_{0}^{1} \frac{y^{1/2}}{(x-iy)} dy = -\frac{1}{i} \int_{0}^{1} y^{-1/2} dy + \int_{0}^{1} \frac{y^{1/2}}{(x-iy)} \left( \frac{1}{i} \right) \frac{1}{iy} dy
\]
\[
= 2i + \frac{x}{i} \int_{0}^{1} \frac{dy}{y^{1/2} (x-iy)},
\]
and then let \( x = \lambda y \), to give
\[
\int_{0}^{1} \frac{y^{1/2}}{(x-iy)} dy = 2i - i \frac{x^{1/2}}{x^{1/2}} \int_{0}^{\lambda} \frac{dw}{\lambda^{1/2} (1-\lambda w)}
\]
\[
\sim 2i - i \frac{x^{1/2}}{\lambda^{1/2}} \int_{0}^{\lambda} \frac{dw}{\lambda^{1/2} (1-\lambda w)}, \quad \text{as } x \to 0.
\]
Thus
\[ \int_0^{y^2} \frac{dx}{x-ir} = 2i + O(x^{1/2}), \quad x > 0. \quad (3.3.11) \]

Similarly, we can write
\[ \int_0^{y^2} \frac{dx}{(x-ir)^2} = i \int_0^{y^2} dx - i \int_0^{y^2} \frac{dr}{x-ir} \]
\[ = \frac{2}{3} i + O(x). \quad (3.3.12) \]

For
\[ \int_0^{y^2} \frac{dx}{(2-ir)^{1/2}}, \]
we only need to consider
\[ \int \frac{dz}{z^{1/2}} \]
since the integrand is regular in the whole z-plane, cut along the negative part of the real axis. \( z^{1/2} \) is taken to be real and positive on the positive part of the real axis. Hence
\[ \int_0^{y^2} \frac{dx}{(2-ir)^{1/2}} = 2i \left\{ (2-i)^{1/2} - 2^{1/2} \right\}, \quad (3.3.13) \]
where the square root of \((2-ir)\) is positive as \( r \to 0 \).

Also
\[ \int_0^{y^2} \frac{dx}{(2-ir)^{1/2}} = i \int_0^{y^2} dx - 2i \int_0^{y^2} \frac{dr}{(2-ir)^{1/2}}, \]
so that after consideration of (3.3.13) and \( \int z^{\frac{1}{2}} dz \) we have

\[
\int \frac{\gamma \, dx}{(2 - i \nu)^{\frac{1}{2}}} = -2 \left\{ \frac{1}{3} (2 - i)^{\frac{3}{2}} - \frac{1}{3} 2^{\frac{3}{2}} - 2 (2 - i)^{\frac{1}{2}} + 2^{\frac{1}{2}} \right\}. \tag{3.3.14}
\]

Similarly

\[
\int \frac{\gamma \, dx}{(1 - i + \nu)^{\frac{1}{2}}} = 2 \left\{ (2 - i)^{\frac{1}{2}} - (1 - i)^{\frac{1}{2}} \right\}, \tag{3.3.15}
\]

\[
\int \frac{\gamma \, dx}{(1 - i - \nu)^{\frac{1}{2}}} = \int \frac{dx}{(2 - i)^{\frac{1}{2}}} - (1 - i) \int \frac{dx}{(1 - i + \nu)^{\frac{1}{2}}} = -2 \left\{ \frac{1}{3} (2 - i)^{\frac{3}{2}} - \frac{1}{3} (2 - i)^{\frac{3}{2}} - (1 - i)^{\frac{1}{2}} - 2 (2 - i)^{\frac{1}{2}} + (1 - i)^{\frac{1}{2}} \right\}, \tag{5.3.15}
\]

\[
\int \frac{\gamma \, dx}{(1 - i - \nu)^{\frac{1}{2}}} = 2 \left\{ (1 - i)^{\frac{1}{2}} - e^{-\frac{\pi i}{4}} \right\}, \tag{3.3.17}
\]

and finally

\[
\int \frac{\gamma \, dx}{(1 - i - \nu)^{\frac{1}{2}}} = (1 - i) \int \frac{dx}{(1 - i - \nu)^{\frac{1}{2}}} - \int (1 - i - \nu)^{\frac{1}{2}} d\nu
\]

\[
= 2 \left\{ \frac{1}{3} e^{\frac{2\pi i}{3}} - \frac{1}{3} (1 - i)^{\frac{3}{2}} - e^{\frac{\pi i}{3}} (1 - i) + (1 - i)^{\frac{3}{2}} \right\}. \tag{3.3.18}
\]

Therefore results (3.1.10) and (3.3.11) to (3.3.18), after some cancellation, give the following expression for (3.3.10)

\[
2^{\frac{1}{2}} e^{\frac{\pi^2}{2}} \frac{\Gamma^2 \phi L}{\pi^2} \left\{ 2 \phi \int \frac{dx}{x^{\frac{1}{2}}} e^{\frac{i(x + \nu)}{\nu^{\frac{1}{2}}}} \int \frac{dx}{x} e^{\frac{i(x + \nu)}{\nu^{\frac{1}{2}}}} \int \frac{dx}{x} e^{\frac{i(x + \nu)}{\nu^{\frac{1}{2}}}} + 2^{\frac{1}{2}} e^{\frac{\pi i}{4}} \right\}
\]

\[
= -\frac{2}{3} (1 + i) \hat{k} (1 - \epsilon) + O(x^{\frac{1}{2}}), \quad x > 0, \tag{3.3.19}
\]
and hence from this last result and (3.1.14) we have

\[
L(r) \sim \frac{2^{3/2} \sigma_{e}}{\mu \epsilon^{3/2} k^{3/2} \xi^{3/2}} \left[ \frac{\gamma_{2}^{1/2}}{2} (\rho - i) e^{i(\varepsilon + \xi) \rho - i \gamma_{2}^{1/2}} \xi^{1/2} + 2^{3/2} e^{-3i \gamma_{2}/4} \right]
\]

\[+ \frac{2^{3/2}}{3} (1 - i) k (1 - \varepsilon) + O_{1}((\rho - i)^{1/2})], \rho > 1.
\]

(5.3.20)

From (3.1.16)

\[
P_{i} - P_{2} = \frac{2^{3/2} \sigma_{e}}{\mu \epsilon^{3/2} k^{3/2} \xi^{3/2}} \left[ \frac{2^{3/2} e^{-3i \gamma_{2}/4}}{\pi^{1/2} k^{1/2}} \int_{0}^{\infty} \frac{e^{2i \frac{\gamma_{2}}{k}}}{x^{3/2}} \, dx + O \left( \frac{1}{k} \right) \right]
\]

\[= \frac{2^{3/2} \sigma_{e}}{\mu \epsilon^{3/2} k^{3/2} \xi^{3/2}} \left[ - \frac{2^{3/2}}{k^{1/2}} + O \left( \frac{1}{k^{1/2}} \right) \right], \quad (3.3.21)
\]

and on using (3.1.22) we obtain

\[
C = - \frac{2^{3/2} \sigma_{e}}{\mu \epsilon^{3/2} k^{3/2} \xi^{3/2}} \left[ \frac{2^{3/2} e^{2i \frac{\gamma_{2}}{k} - 3i \gamma_{2}/4}}{\pi^{1/2} k^{1/2}} \xi \right] + O \left( \frac{1}{k^{1/2}} \right).
\]

(3.3.22)

Therefore (2.2.27), (3.3.20) and (3.3.22) give

\[
f(\rho) \sim \frac{2^{3/2} \sigma_{e} e^{-i \frac{\gamma_{2}}{k}}}{\mu \epsilon^{3/2} k^{3/2} \xi^{3/2} \rho^{1/2} (\rho - i)^{1/2}} \left[ \frac{\gamma_{2}^{1/2}}{2} (\rho - i) e^{i(\varepsilon + \xi) \rho - i \gamma_{2}^{1/2}} \xi^{1/2} + 2^{3/2} e^{-3i \gamma_{2}/4} \right]
\]

\[+ \frac{2}{3} (1 - i) k (1 - \varepsilon) + \frac{2^{3/2}}{\pi^{1/2} k^{1/2}} \frac{e^{2i \frac{\gamma_{2}}{k} - \gamma_{2}/2}}{(\rho + i)} + O_{1}((\rho - i)^{1/2})], \rho > 1.
\]

(3.3.23)

The error term in the brackets \( [ \) of (3.3.23), is for large \( k \), of \( O(1/k) \).
Section 4

Dynamic stress intensity factor

4.1 Definition in terms of \( f \)

In predicting how a crack will behave it is necessary to possess a detailed knowledge of the stress distribution in a small region surrounding the crack edge (c.f. [15] and [37]). It has been shown by Sih and Loeber [1] that the stress distribution near the crack edge is given by

\[
\sigma_{\rho\theta}^{(s)} = -\frac{k_3}{(2\rho)^{1/2}} \sin \frac{1}{2} \Theta + O(\rho^{1/2}),
\]

(4.1.1)

and

\[
\sigma_{\varphi z}^{(s)} = \frac{k_3}{(2\rho)^{1/2}} \cos \frac{1}{2} \Theta + O(\rho^{1/2}),
\]

(4.1.2)

where \((\rho, \Theta)\) are polar coordinates, measured from the edge of the crack. We need only consider the scattered field, since there are no singularities in the incident field. \(k_3\), the dynamic stress intensity factor, is given by (c.f. [15])

\[
k_3 = 2^{1/2} \lim_{\rho \to 1+} (\rho-1)^{1/2} \sigma_{\varphi z}^{(s)}(\rho, 0).
\]

(4.1.3)

The next step is to relate \( \sigma_{\varphi z}^{(s)}(\rho, 0) \) to \( f(\rho) \). On differentiating,
(4.1.6) with respect to \( \rho \) we obtain
\[
\alpha_{\theta z}^{(s)} (\rho, \sigma) = - \mu_c \int_0^\infty \gamma'(s) A(s) J_1(s\rho) ds. \tag{4.1.4}
\]

Now substitute for \( A(s) \) from (4.1.4), to give
\[
\alpha_{\theta z}^{(s)} (\rho, \sigma) = - \mu_c \int_0^\infty s J_1(s\rho) ds \int_0^\infty f(r) J_1(rs) dr, \tag{4.1.5}
\]

and hence, from Titchmarsh [16]
\[
\alpha_{\theta z}^{(s)} = - \mu_c f(\rho), \quad \rho \neq 1. \tag{4.1.6}
\]

Therefore (4.1.3) may be written as
\[
\kappa_3 = - 2^{1/2} \mu_c \lim_{\rho \to 1+} (\rho-1)^{1/2} f(\rho). \tag{4.1.7}
\]

It can be deduced from [12] that the corresponding static stress intensity factor, \( \kappa_3 \), is given by
\[
\kappa_3 = - \frac{4 \alpha_{\theta z}^{(s)}}{3\pi}. \tag{4.1.8}
\]

This result will be required later, since in subsection 4.5 the variation of the modulus of \( \kappa_3/N_3 \), with the wavenumber \( \kappa \), for certain values of \( \varepsilon \), is shown.
4.2 Case I. \((k\varepsilon \gg 1, k(1-\varepsilon) \gg 1)\)

From (3.1.24) and (4.1.6) we have

\[
\hat{k}_3 = -\frac{2\sigma_k \varepsilon^{i\hat{\beta}_k}}{k^{3/2} \hat{N}^{3/2} \varepsilon^{3/2}} \left[ \frac{\nu/2}{(1-\varepsilon)^{3/2}} \lim_{\rho \to 1+} \left( \rho^{-1} \right)^{1/2} \sum_{-\nu/2, \pm i\hat{k}(1+\varepsilon)(\rho^{-1})} \right]
\]

\[
+ \frac{\nu/2}{2} e^{i\hat{k}(1-\varepsilon)-3\pi i/4} \lim_{\rho \to 1+} \left( \rho^{-1} \right)^{1/2} \sum_{-\nu/2, \pm i\hat{k}(1-\varepsilon)(\rho^{-1})} \right]
\]

\[
+ \frac{1}{\hat{k}(1-\varepsilon)^{3/2}} \left\{ \frac{e^{i\hat{k}(1-\varepsilon)-\pi i/4} - e^{i\hat{k}(1+\varepsilon)-3\pi i/4}}{(1-\varepsilon)^{3/2}} \right\} + O\left(\frac{1}{\hat{k}^{3/2}}\right) \quad (4.2.1)
\]

It is seen from page 135 of Erdélyi [7] that

\[
\lim_{X \to 0^+} X^{1/2} \ln (-1/2, i b X) = \frac{2\varepsilon^{i/2}}{b^{1/2}} \quad (4.2.2)
\]

Hence

\[
\hat{k}_3 = \frac{2\sigma_k i \varepsilon^{i\hat{\beta}_k}}{k^{3/2} \hat{N}^{3/2} \varepsilon^{3/2}} \left[ \frac{\nu/2}{(1+\varepsilon)^{3/2}} \left\{ \frac{e^{i\hat{k}(1+\varepsilon)}}{(1+\varepsilon)^{3/2}} - \frac{e^{i\hat{k}(1-\varepsilon)}}{(1-\varepsilon)^{3/2}} \right\} \right]
\]

\[
+ \frac{1}{\hat{k}(1-\varepsilon)^{3/2}} \left\{ \frac{e^{i\hat{k}(1-\varepsilon)+\pi i/4} - e^{i\hat{k}(1+\varepsilon)+\pi i/4}}{(1+\varepsilon)^{3/2}} \right\} + O\left(\frac{1}{\hat{k}^{3/2}}\right) \quad (4.2.3)
\]

After some rather laborious algebra we have

\[
\left| \frac{\hat{k}_3}{N_3} \right| = \frac{3}{2^{1/2} \varepsilon^{1/2}} \left\{ 1 - (1-\varepsilon)^{3/2} \sin 2\hat{k}\varepsilon \right\} \left[ \frac{1}{(1-\varepsilon)^{3/2}} \right]
\]

\[
+ \frac{1}{\varepsilon^{1/2} k^{1/2} \left( 1 - (1-\varepsilon)^{3/2} \sin 2\hat{k}\varepsilon \right)^{1/2}} \left( \frac{\cos (2\hat{k}\varepsilon/4)}{ (1-\varepsilon)^{3/2}} + \frac{\sin 2\hat{k}(1-\varepsilon)-\pi/4}{2 (1-\varepsilon)} \right)
\]

\[
- \frac{\sin \frac{\pi}{2} \hat{k}(1+\varepsilon)-\pi/4}{2 (1+\varepsilon)} + O\left(\frac{1}{\hat{k}}\right) \quad (4.2.4)
\]
and this is the expression that will be used when we come to obtain numerical results for this case.
4.3 Case III. \(0 \leq \varepsilon \leq 1/k\)

Similarly from (3.2.30) and (4.1.7) we have

\[
\kappa_3 = - \frac{2^{3/2}}{\pi} \sigma_0 \int_1(\lambda) \left[ \frac{\pi^{1/2}}{k^{1/2}} \hat{e}^{1/2} + \frac{e^{2iK}}{K} + O\left(\frac{1}{K^3}\right) \right],
\]

i.e., after using (4.2.2).

For the case of an incident free torsion wave the dynamic stress intensity factor \(\kappa_3\), is easily found by taking the limit as \(\lambda \to 0\) in (4.3.1). Hence

\[
\kappa_3 = - \frac{2^{3/2}}{\pi} \sigma_0 \left[ \frac{\pi^{1/2}}{k^{1/2}} \hat{e}^{1/2} + \frac{e^{2iK}}{K^{1/2}} + O\left(\frac{1}{K}\right) \right].
\]  

(4.3.2)

On dividing (4.3.1) by the static stress intensity factor \(N_3\), given by (4.1.8) and then taking the modulus we have

\[
\left| \frac{\kappa_3}{N_3} \right| = \frac{3\pi^{1/2}}{2^{3/2} K^{1/2}} \left[ 1 + \frac{1}{K^{1/2}} \cos(2k - \pi/4) + O\left(\frac{1}{K}\right) \right],
\]

and for the case of an incident free torsion wave

\[
\left| \frac{\kappa_3}{N_3} \right| = \frac{3\pi^{1/2}}{2^{3/2} K^{1/2}} \left[ 1 + \frac{1}{K^{1/2}} \cos(2k - \pi/4) + O\left(\frac{1}{K}\right) \right].
\]

(4.3.3)
4.2 Case III. \((1-1/k^{3/2} \leq \varepsilon \leq 1)\)

Finally, from (3.3.23) and (4.1.7)

\[
K_3 = \frac{2\sigma \varepsilon + \varepsilon}{\pi^{1/2} k^{3/2} \varepsilon^{1/2}} \left[ \frac{\varepsilon^{1/2}}{\pi^{1/2} k^{1/2} (1 + \varepsilon)^{1/2}} + \varepsilon^{1/2} \frac{k}{1 + \varepsilon} + \frac{3}{3} \frac{\varepsilon^{1/2} k}{\pi^{1/2} k^{1/2}} + O\left(\frac{1}{k}\right) \right],
\]

(4.4.1)

On dividing (4.4.1) by the static stress intensity factor \(K_3\),
given by (4.1.8), and then taking the modulus we arrive at

\[
\left| \frac{K_3}{K_3} \right| = \frac{3}{\pi^{1/2} k^{3/2} \varepsilon^{1/2}} \left[ \frac{2^{1/2} - \varepsilon^{1/2} \sin \left( \frac{k(1+\varepsilon) - \pi/4} {2^{1/2}} \right) + \cos \left( 2k - \frac{\pi}{4} \right) + O\left(\frac{1}{k}\right) } {2^{1/2} k^{1/2} \varepsilon^{1/2} + \frac{1}{2^{1/2} k^{1/2} \varepsilon^{1/2}}} \right].
\]

(4.4.2)
4.5 Numerical results and discussion

For \( \varepsilon = 0.00(0.10)1.00 \) and \( \varepsilon = 0.98 \), the modulus of the dynamic stress intensity factor, divided by the static stress intensity factor, is plotted as a function of \( k \), in figures 8 to 19. The curves which possess an error of the order given in the appropriate analytical expressions are represented by red lines, whereas curves in which the error term is more significant are represented by green lines. For the purpose of numerical calculation the black lines are to be ignored. This applies to all the graphs in this thesis.

\( \varepsilon = 0.00 \), Figure 6

The dynamic stress intensity factor, for an incident free torsion wave, increases to 32\% above its static value when \( k = 3.5 \). This percentage increase is in close agreement with the result of Sin and Loebor [1] which is 31\%, whereas the result due to Mal [2] of 49\% appears to be far too high. There is a small phase difference between the three graphs, so that the peaks occur at marginally different values of \( k \). The second
peak in figure 8 is in reasonably close agreement with the corresponding peak in [21].

\( \xi = 0.10, \text{figure 9} \)

Obviously, for \( 2 \leq k \leq 10 \) the conditions on Case II are satisfied and this is the case which has been considered.

\( \xi = 0.20, \text{figure 10} \)

The relevant case is Case II. Although Case I is not valid for the smaller values of \( k \), it does however, for \( k = 10 \), give a value of \( |k_3/N_3| \) which is within 6% of the corresponding value given by Case II.

\( \xi = 0.50, \text{figure 11} \)

For \( k = 2 \) we consider Case II and as \( k \) increases we can change over to Case I when \( k = 7.9 \).

\( \xi = 0.40, \text{figure 12} \)

For \( k = 2 \) we consider Case II and as \( k \) increases we can change over to Case I when \( k = 5.3 \).

\( \xi = 0.50, \text{figure 13} \)

For \( k = 2 \) we consider Case II and change over to Case I when
$k \approx 4.0$. There is very close agreement between Cases I and II for a fairly wide range of $k$, e.g. the two cases give results within 1\% of each other for $3.2 < k < 4.2$.

$\varepsilon = 0.60$, figure 14

We have three curves which do not intersect. However, for values of $k$ in excess of 5, Case I should give accurate results.

$\varepsilon = 0.70$, figure 15

We have two curves which do not intersect. Case I is valid for the larger values of $k$ in $2 \leq k \leq 10$ and Case III is only valid for $2.00 \leq k \leq 2.23$. From these observations it appears that there is a "gap" between the ranges of validity of these two cases.

$\varepsilon = 0.80$, figure 16

For the smaller values of $k$, we can consider Case III and for $k \geq 4$, we can consider Case I. There is very close agreement between these two cases for quite a wide range in $k$.

$\varepsilon = 0.90$, figure 17

For $2 \leq k \leq 4.6$ we consider Case III and for $k$ in this range the error is of the order given in (4.4.2) but as $k$ increases the
error term becomes increasingly significant until for \( k \approx 6.3 \)
the error term in the brackets \([ \] \) is of \( O(1/k^2) \). Although
\( k(1-\varepsilon) \leq 1 \), for \( 2 \leq k \leq 10 \), there is surprisingly close agreement
between Cases I and III.

\[ \varepsilon = 0.98, \text{ figure 18} \]

The significance of this value for \( \varepsilon \), is that this is the
lowest one, to two places of decimals, for which Case III is
valid for all \( 2 \leq k \leq 10 \).

\[ \varepsilon = 1.00, \text{ figure 19} \]

The dynamic stress intensity factor is, once again, an oscillatory
function of the wavenumber \( k \). It is seen that for \( k \geq 2 \), \( k_3 \) is
less than the static stress intensity factor, which confirms
an observation made by Sih and Loeger [1]. The oscillatory
nature of \( k_3 \) is not obvious in [1], but there does appear to be
a phase difference of \( 0.009 \) radians between their curve for
\( |k_3/N_3| \) and the curve shown in figure 19. For \( 2.8 < k < 4.3 \), the
gradients of the two curves are in close agreement.

It appears that quite often, for a particular \( \varepsilon \), there is
a "gap" between the two relevant cases. However, it should be realised that unless \( \epsilon = 0 \) or \( 1 \), it only requires \( k \) to be sufficiently large, for the conditions on Case I to be satisfied, i.e. for any \( k = k_0 \), such that \( k_0 (1 - \epsilon) \gg 1 \) and \( k_0 \epsilon \gg 1 \), then Case I is valid for all \( k \gg k_0 \). Hence, in Case I, we have succeeded in covering the largest part of the range, subject to the proviso that \( \epsilon = 0, 1 \).
Section 5

Scattering coefficient

5.1 Definition in terms of $f$

The definition of the scattering cross section is given by Truell and Elbaum [17] as the ratio of the average rate at which energy $E_s$, is scattered away from the obstacle, to the average rate at which energy is incident on unit area normal to the direction of propagation. The above definition is analogous to that used in acoustics and electromagnetism, (e.g., c.f. [16]).

We define the scattering coefficient $Q$ as

$$Q = \frac{E_s}{E_i},$$

(5.1.1)

where $E_i$ is the average rate at which energy is incident upon the obstacle. In order to derive an expression for the scattering coefficient in terms of $f$ we will adopt the approach of Barratt and Collins [4].

First of all we derive an expression for the displacement
at large distances compared to the wavelength. On introducing
spherical polar coordinates \( R, \phi \) defined by

\[
\rho = R \sin \phi \quad \text{and} \quad z = R \cos \phi,
\]

then \((1.5.9)\) can be written as

\[
u^1_\theta (R, \phi) = \frac{2 \mu_0}{\lambda} \mathcal{J}_1(\lambda R \sin \phi) e^{-i R \cos \phi}, \tag{5.1.2}
\]

Since, for large \( R \)

\[
\frac{\partial}{\partial \phi} \sim \mu \frac{\partial \nu_\theta}{\partial R}, \tag{5.1.3}
\]

we have

\[
\frac{\partial}{\partial \phi} \sim \frac{2 \mu_0 i}{\lambda} \cos \phi \mathcal{J}_1(\lambda R \sin \phi) + i \lambda \sin \phi \mathcal{J}_1'(\lambda R \sin \phi) e^{-i R \cos \phi} \tag{5.1.4}
\]

For the scattered far field, after making the substitution

\[ s = k \xi \] in \((1.4.16)\), we obtain

\[
u^{(3)}_\theta (R, \phi) = k^2 \int_0^\infty \xi A(k \xi) \mathcal{J}_1(k R \xi \sin \phi) e^{i k R \xi^2 \cos \phi} d\xi, \quad 0 \leq \phi \leq \pi/2. \tag{5.1.5}
\]

Approximate \( \nu^{(o)}_\theta (R, \phi) \) for large \( R \), i.e.,

\[
u^{(o)}_\theta (R, \phi) \sim \frac{k^{3/2}}{(2\pi R \sin \phi)^{1/2}} \left[ e^{\frac{3i}{8} \int_0^\infty \xi^2 A(k \xi) e^{-i k R \xi^2 \sin \phi + (1-\xi^2)^2 \cos \phi} d\xi} \right], \quad 0 \leq \phi \leq \pi/2. \tag{5.1.6}
\]
From Erdélyi [3] or Copson [19] it is seen that the most
important contributions to the value of \( u_{\phi}(R, \phi) \) occur in the
vicinity of points where \( \left\{ \pm \xi \sin \phi + (1 - \xi^2)^{\frac{1}{2}} \cos \phi \right\} \)
have stationary values. For the moment we impose the restriction \( \phi \neq 0, \pi \).

Let \( \omega_1(\xi) = \xi \sin \phi + (1 - \xi^2)^{\frac{1}{2}} \cos \phi \),
then \( \omega_1'(\xi) = 0 \), when \( \xi = \sin \phi \)
and \( \omega_1''(\sin \phi) = -1/\cos^2 \phi \).

Similarly let \( \omega_2(\xi) = -\xi \sin \phi + (1 - \xi^2)^{\frac{1}{2}} \cos \phi \),
then \( \omega_2'(\xi) = 0 \), when \( \xi = -\sin \phi \).

But if \( 0 < \phi < \frac{\pi}{2} \) and \( \xi \in [0, \infty) \), therefore the integrand of the first
integral in (5.1.3) does not possess a stationary value and its
contribution is of \( O(1/R) \) which may be ignored. Hence from the
principle of stationary phase applied to the second integral
we have

\[
\left(5.1.7\right)
\]

\[
\left(5.1.8\right)
\]

and
Since the leading term is continuous for all real $\phi$, results (5.1.7) and (5.1.8) are valid for $0 \leq \phi \leq \pi$.

The rate at which energy crosses unit area perpendicular to its direction of propagation is given by Truell and Albano [17] as

$$-\Re I_n \hat{e}^\omega \cdot \frac{\partial \Re \hat{u} e^{j\omega t}}{\partial t},$$

where $\hat{e}^\omega$ is the displacement and $I_n e^{j\omega t}$ is the stress vector across an element of the surface at the point $x$, whose normal is in the direction $\hat{n}$. The corresponding average rate $R_E$ is calculated by using the definition given by Barratt and Collins [4], which is

$$R_E = -\frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} R I_n \hat{e}^\omega \cdot \frac{\partial \Re \hat{u} e^{j\omega t}}{\partial t},$$

(5.1.9)

On carrying out the integration in (5.1.9) we have

$$R_E = \frac{1}{2} \omega \Re i (I_n, \overline{u}).$$

(5.1.10)

The total energy $E$ transmitted through a surface $S$ is therefore given by

$$E = \frac{1}{2} \omega \Re i \int_S (I_n, \overline{u}) dS.$$

(5.1.11)

$\overline{u}$ is the complex conjugate of $u$. 
The normal to the surface element $dS$ is taken in the direction of $\hat{n}$. It therefore follows that the incident energy $E_i$ is given by

$$E_i = -\frac{1}{2} \omega a^3 \mathcal{R} i \int_{S_1} \rho u^i_\theta \cdot \sigma^{(i)}_S dS,$$

(5.1.12)

where $\sigma^{(i)}_S$ is the incident tangential shear stress on $S$, and $S_1$ is the "illuminated" side of $S$.

From the symmetry of the stress tensor

$$\sigma_S = \left[ \hat{n} \cdot I_n \right] = \left[ \hat{n} \cdot I_e \right]_S,$$

therefore

$$\sigma_S = \hat{n} \left[ \frac{\partial}{\partial \rho} (\rho, z) \hat{r} + \frac{\partial}{\partial z} (\rho, z) \hat{z} \right]_S,$$

and on using (1.2.4) and (1.2.5) we obtain

$$\sigma_S = \mu_c \hat{n} \left[ \frac{\partial}{\partial \rho} (\rho, z) \hat{r} + \frac{\partial}{\partial z} (\rho, z) \hat{z} \right]_S,$$

(5.1.13)

With the aid of (5.1.13), (5.1.12) may be written as

$$E_i = -\frac{1}{2} \omega a^3 \mu_c \mathcal{R} i \int_{S_1} \rho u^i_\theta \cdot \frac{\partial}{\partial n} \left( \frac{u^i_\theta}{\rho} \right) dS,$$

or alternatively

$$E_i = -\frac{1}{2} \omega a^3 \mu_c \mathcal{R} i \int_{S_1} \rho u^i_\theta \cdot \frac{\partial}{\partial n} \left[ \frac{1}{\rho} \frac{\partial u^i_\theta}{\partial n} - \frac{u^i_\theta}{\rho^2} \frac{\partial \rho}{\partial n} \right] dS.$$

(5.1.14)

The second term in the integrand of (5.1.14) has only a real part therefore
\[ E_i = -\frac{1}{2} \omega a^3 \mu_c R_i \int_{S_1} \left( u_{\theta}^{(i)} \frac{\partial u_{\theta}^{(i)}}{\partial n} \right) dS. \] (5.1.15)

Hence from (1.3.9) and (1.3.10), and after performing the integration with respect to \( \theta \) we obtain

\[ E_i = \frac{4\pi \omega a^3 u_0 \sigma_0}{\lambda^2} \int_0^1 J_1^2(\lambda \rho) d\rho. \]

This last integral may be evaluated by using a result given on page 29 of [26] to give

\[ E_i = \frac{2\pi \omega a^3 u_0 \sigma_0}{\lambda^2} \left[ J_1^2(\lambda) + \left( 1 - \frac{1}{\lambda^2} \right) J_1^2(\lambda) \right]. \] (5.1.16)

The average rate at which energy is scattered away from the crack is equal to the average rate at which energy is transmitted through a closed surface \( S_1 \), which is in the elastic material and surrounds the crack. A similar argument to that applied in obtaining (5.1.11) gives that the scattered energy \( E_s \) is

\[ E_s = \frac{1}{2} \omega a^3 \mu_c R_i \int_{S_1} \sigma_{\kappa \theta}^{(s)} \frac{\partial u_{\theta}^{(s)}}{\partial n} dS. \] (5.1.17)

The average rate at which the energy of the total field is transmitted across the surface \( S_1 \) is zero, since it is assumed that no energy is absorbed by the crack. Therefore

\[ \frac{1}{2} \omega a^3 \mu_c R_i \int_{S_1} \left( \sigma_{\kappa \theta}^{(i)} + \sigma_{\kappa \theta}^{(s)} \right) \left( u_{\theta}^{(i)} + u_{\theta}^{(s)} \right) dS = 0. \] (5.1.18)
Also, the average rate at which the energy of the incident wave is transmitted across $S_1$ is zero, since the incident field has no sources within $S_1$. Hence from (5.1.17) and (5.1.18) we have

$$E_s = -\frac{1}{2} \omega a^3 R \iiint \left[ a_k^{(s)} u_k^{(s)} + a_k^{(h)} u_k^{(h)} \right] dS.$$  \hspace{1cm} (5.1.19)

Let

$$E_s = -\frac{1}{2} \omega a^3 R \iiint (J_1 + J_2),$$  \hspace{1cm} (5.1.20)

where

$$J_1 = \int_{S_1} a_k^{(s)} u_k^{(s)} dS,$$  \hspace{1cm} (5.1.21)

and

$$J_2 = \int_{S_1} a_k^{(h)} u_k^{(s)} dS.$$  \hspace{1cm} (5.1.22)

We can evaluate $J_1$ as $R \to \infty$ by substituting the displacement component of the incident field, given by (5.1.2) and, then using (5.1.8) to obtain the integral

$$J_1 = 2i \mu_c u_0 \frac{k^2}{\lambda} \lim_{R \to \infty} R^2 e^{ikR} \int_0^{2\pi} J_1(\lambda R \sin \phi) A(\lambda R \sin \phi) \sin \phi \cos \phi e^{i\lambda R \cos \phi} d\phi,$$

and on carrying out the integration with respect to $\theta$ we have

$$J_2 = 4\pi i \mu_c u_0 \frac{k^2}{\lambda} \lim_{R \to \infty} R^2 e^{ikR} \int_0^{2\pi} J_1(\lambda R \sin \phi) A(\lambda R \sin \phi) \sin \phi \cos \phi e^{i\lambda R \cos \phi} d\phi,$$

which for large $R$ may be written as

$$J_1 = 2^{3/2} \pi^{1/2} i \mu_c u_0 \frac{k^2}{\lambda^{1/2}} \lim_{R \to \infty} R^{1/2} e^{ikR} \left[ e^{\frac{-i\lambda R}{2}} \int_0^{2\pi} A(\lambda R \sin \phi) \sin \phi \cos \phi e^{i\lambda R \cos \phi} d\phi \right]$$
The most important contributions to the value of $\tilde{J}_1$ occur in the vicinity of points where $\xi \pm \lambda \sin \phi + \alpha \cos \phi$ have stationary values. Let

$$\omega_1(\phi) = \lambda \sin \phi + \alpha \cos \phi,$$

then $\omega_1'(\phi) = 0$, when $\tan \phi = \epsilon/(1-\epsilon^2)^{1/2}$ and $\omega_1''(\phi) = -k$.

Similarly let

$$\omega_2(\phi) = -\lambda \sin \phi + \alpha \cos \phi,$$

then $\omega_2'(\phi) = 0$, when $\tan \phi = -\epsilon/(1-\epsilon^2)^{1/2}$ and $\omega_2''(\phi) = k$.

Hence by the principle of stationary phase

$$\tilde{J}_1 = -\frac{4\pi i \mu_c u_\alpha}{\lambda} \lim_{R \to \infty} \int_0^\pi A(\lambda) \sin \phi \cos \phi e^{iR\sin \phi} d\phi.$$

Similarly for $\tilde{J}_2$, we substitute from (5.1.4) and (5.1.7) into (5.1.22) to give, after performing the integration with respect to $\theta$

$$\tilde{J}_2 = \frac{4\pi i \mu_c u_\alpha}{\lambda} \lim_{R \to \infty} \int_0^\pi \left\{ \alpha \cos \phi \tilde{J}_1(\lambda R \sin \phi) + i \lambda \sin \phi \tilde{J}_1'(\lambda R \sin \phi) \right\}$$

$$\times A(\lambda R \sin \phi) \sin \phi \cos \phi e^{iR \cos \phi} d\phi,$$

which for large $R$ may be written as
\[ J_2 = \frac{2^{\frac{3}{2}} \kappa \frac{1}{\lambda^2}}{\mu_c u_0 \kappa} \lim_{R \to \infty} R^{\frac{1}{2}} e^{ikR} \left[ \int_0^\pi \left( e^{-\frac{3\kappa}{2} \cos \phi + i \lambda \frac{2}{\sin \phi}} \right) A(\kappa \sin \phi) \sin \frac{\phi}{2} d \phi \right] \]

\[ \times \cos \phi e^{-i R \frac{\kappa}{2} \sin \phi + d \cos \phi} \left( \int_0^\pi e^{\frac{3\kappa}{2} \cos \phi + i \lambda \frac{2}{\sin \phi}} A(\kappa \sin \phi) \sin \frac{\phi}{2} d \phi, \right) \]

and, once again, by applying the principle of stationary phase, we obtain

\[ J_2 = -\frac{4\pi i \mu_c u_0 \kappa}{\lambda} \left( 1 + e^{2ik} \right) A(\lambda) \]  \hspace{1cm} (5.1.27)

Hence from (5.1.26) and (5.1.27), (5.1.19) is

\[ E_s = -\frac{2\pi \omega a^3 \mu_c u_0 \kappa}{\lambda} \left[ A(\lambda) + A(\lambda) - e^{2ik} A(\lambda) + e^{2ik} A(\lambda) \right], \]  \hspace{1cm} (5.1.29)

and since the last two terms of \( E_s \) combine to give a quantity which is wholly imaginary we have

\[ E_s = -\frac{4\pi \omega a^3 \mu_c u_0 \kappa \Re A(\lambda)}{\lambda}, \]  \hspace{1cm} (5.1.29)

or alternatively, from (1.6.4)

\[ E_s = -\frac{4\pi \omega a^3 \mu_c u_0}{\lambda} \int_0^\infty f(\eta) J_1(\lambda \eta) d\eta. \]  \hspace{1cm} (5.1.30)

On splitting the range of integration into the ranges \((0,1)\) and
(1,\infty) \text{ in (5.1.30) and then, for } r \in (0,1), \text{ substituting for } f \text{ from (1.6.3) we have}
\begin{equation}
E_s = -\frac{4\pi a^3}{\lambda} \mu_c \omega \left\{ 2\pi i \int_{0}^{\infty} \frac{J_1^2(\lambda r) dr}{\lambda} + \int_{0}^{\infty} \{f(\gamma)J_1(\lambda r) dr \right\}. \tag{5.1.31}
\end{equation}
By comparison with (5.1.16) and on letting \( r = 1 + x \) in the second integral of (5.1.31) it is seen that
\begin{equation}
E_s - 2E_i = -\frac{4\pi a^3}{\lambda} \mu_c \omega \left\{ \int_{0}^{\infty} (1+x) \{f(1+x)J_1(\lambda(1+x)) dr \right\}. \tag{5.1.32}
\end{equation}
The right-hand side of (5.1.32) is in terms of known functions since \( f(1+x) \), for \( x > 0 \), has already been asymptotically evaluated in Section 3. Therefore, we can asymptotically evaluate \( E_s \) and, hence the scattering coefficient. Once again it is necessary to distinguish three separate cases (c.f. 2.3).
Since $\lambda = k \varepsilon >> 1$, we can replace the Bessel function in (5.1.32) by its asymptotic expansion to give, on taking the most significant terms,

$$E_1 - 2E_1 \sim - \frac{2^{3/2} \pi^{3/2} \omega \mu \varepsilon}{k^{3/2} \varepsilon^{1/2}} \int \frac{e^{-i \frac{k \varepsilon}{\varepsilon^2} x^2}}{x} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx$$

$$+ \frac{1}{2} \int_0^\infty \left( \frac{2^{3/2} \pi^{3/2} \omega \mu \varepsilon}{k^{3/2} \varepsilon^{1/2}} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx \right)$$

$$+ \frac{n^{1/2}}{2} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx + \frac{n^{1/2}}{2} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx$$

Hence, on substituting for $f$, from (3.1.84) we have

$$E_1 - 2E_1 = - \frac{4 \omega \mu \varepsilon^3}{k^{3/2} \varepsilon^{1/2}} \int \left\{ e^{i \frac{k \varepsilon x^2}{2}} \int_0^\infty e^{i \frac{k \varepsilon x}{\varepsilon}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx \right\} dx$$

$$+ \frac{1}{2} \int_0^\infty \left( \frac{4 \omega \mu \varepsilon^3}{k^{3/2} \varepsilon^{1/2}} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx \right)$$

$$+ \frac{n^{1/2}}{2} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx + \frac{n^{1/2}}{2} \int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx$$

As for (3.1.20) it can be shown that

$$\int_0^\infty e^{i \frac{k \varepsilon x^2}{2}} f(x) e^{i \frac{k \varepsilon x}{\varepsilon}} dx = \frac{i \varepsilon^{1/2}}{k \varepsilon} \int_0^\infty \left( \frac{1}{(1-\varepsilon)^{1/2}} + O\left( \frac{1}{k \varepsilon} \right) \right).$$

(5.2.3)
Obviously, this last result still holds if $\varepsilon$ is changed to $-\varepsilon$.

We next deform the path of integration of the second

integral in (5.2.2) into the negative imaginary axis to obtain

$$\int_{-\infty}^{\infty} \Gamma(-\varepsilon^2, i(1+i)x) \, dx = -i \int_{0}^{\infty} \Gamma(-\varepsilon^2, i(1+i)x) \, dx.$$  \hspace{1cm} (5.2.4)

We now use result (6) of section 9.3 of 7 which is

$$\int_{0}^{\infty} e^{it} e^{-\epsilon t} \frac{\Gamma(a,t)}{\beta(1+\varepsilon)^{a+\epsilon}} \, dt = \frac{\Gamma(a+\beta)}{\beta(1+\varepsilon)^{a+\epsilon}} \, F_1 \left( 1, a+\beta ; \beta+1 ; \frac{s}{1+\varepsilon} \right),$$  \hspace{1cm} (5.2.5)

for $R(\beta) > 0$, $R(a+\beta) > 0$ and $R(s) > -1$, to obtain

$$\int_{0}^{\infty} \Gamma(-\varepsilon^2, i(1+i)x) \, dx = -i \frac{\pi^{\varepsilon^2}}{\kappa(x+\varepsilon)}.$$  \hspace{1cm} (5.2.6)

By standard methods it can be shown that

$$\int_{0}^{\infty} e^{i\kappa(1-\varepsilon)x} \frac{\pi^{\varepsilon^2}}{\kappa(x+\varepsilon)} \left( 1 + O \left( \frac{1}{\kappa} \right) \right).$$  \hspace{1cm} (5.2.7)

(5.2.6) and (5.2.7) are valid if $\varepsilon$ is changed to $-\varepsilon$.

hence by using (5.2.3), (5.2.6) and (5.2.7), (5.2.2) can be

written as

$$E_5 - 2E_6 = -\frac{4 \omega a^3 \mu c}{\kappa^2 \varepsilon^3} \int \frac{\pi^{\varepsilon^2}}{2\kappa \varepsilon} \left[ 1 - \left( \frac{1-\varepsilon}{1+\varepsilon} \right)^{\varepsilon^2} \right] + \frac{\pi}{2\kappa(1+\varepsilon)} + \frac{\pi}{2\kappa(1-\varepsilon)}$$

$$+ \frac{\pi^{\varepsilon^2}}{2\kappa \varepsilon} \left[ 1 - \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{\varepsilon^2} \right]$$

$$+ \frac{\pi^{\varepsilon^2}}{2\kappa(1-\varepsilon)^{\varepsilon^2}} \left[ \frac{e^{i\kappa(1-\varepsilon)\mu c}}{\kappa(1-\varepsilon)^{\varepsilon^2}} + \frac{e^{i\kappa(1+\varepsilon)\mu c}}{\kappa(1+\varepsilon)^{\varepsilon^2}} \right] \left[ \frac{e^{-i\kappa(1-\varepsilon)\mu c}}{(1-\varepsilon)^{\varepsilon^2}} + \frac{e^{-i\kappa(1+\varepsilon)\mu c}}{(1+\varepsilon)^{\varepsilon^2}} \right] + O \left( \frac{1}{\kappa^2} \right),$$  \hspace{1cm} (5.2.8)
and for large $k\varepsilon$ (5.1.16) can be written as

$$E_i = \frac{4 \omega a^3 u_0 \sigma}{k^3 \varepsilon^3} \left\{ 1 + O\left(\frac{1}{k^2}\right) \right\}.$$  (5.2.9)

Therefore we have

$$Q = 2 - \left\{ 1 - \frac{1}{(1 - \varepsilon^2)^{1/2}} \right\} \frac{\cot 2k\varepsilon}{k\varepsilon}$$

$$+ \frac{2}{\pi^{1/2} k (1 - \varepsilon^2)^{1/2}} \left\{ \frac{\sin (2k(1 + \varepsilon)) - \sin (2k(1 - \varepsilon))}{2(1 + \varepsilon)} - \frac{\cos (2k(1 - \varepsilon)) - \cos (2k(1 + \varepsilon))}{2(1 + \varepsilon)} \right\} + O\left(\frac{1}{k^3}\right).$$  (5.2.10)
5.3 Case II. \((0 < \varepsilon < \frac{1}{4})\)

On substituting for \(f\) from (5.2.30) into (5.1.32) we obtain

\[
E_5 - 2E_1 = - \frac{\lambda \omega \epsilon_w \sigma_0}{\lambda^2} \int J_1(\lambda) \int_0^\infty x^{\nu} e^{ix} \int_0^\infty (1+x)^{\nu} \Gamma(-\nu, i k x) J_1(\lambda (1+x)) \, dx \, d\alpha
\]

\[
+ \frac{2 - 2i k + i}{k} \int_0^\infty \frac{e^{i k x}}{x^{\nu/2} (x+2)} J_1(\lambda (1+x)) \, dx \, d\alpha + O \left( \frac{1}{k^2} \right).
\]  

(5.3.1)

For the first of these integrals we can deform the path of integration into the negative imaginary axis to arrive at

\[
\int_0^\infty (1+x)^{\nu/2} \Gamma(-\nu/2, i k x) J_1(\lambda (1+x)) \, dx = -i \int_0^\infty (1-ix)^{\nu/2} \Gamma(-\nu/2, k x) J_1(\lambda (1-ix)) \, dx
\]

\[
\sim J_1(\lambda) \int_0^\infty \Gamma(-\nu/2, k x) \, dx,
\]  

(5.3.2)

since

\[
\int_0^\infty x^n \Gamma(-\nu/2, k x) \, dx = O \left( \frac{1}{k^{n+1}} \right), \quad n > -\frac{1}{2}.
\]

Hence from (5.2.5)

\[
\int_0^\infty (1+x)^{\nu/2} \Gamma(-\nu/2, i k x) J_1(\lambda (1+x)) \, dx = -i \frac{k^{\nu/2}}{k} J_1(\lambda).
\]  

(5.3.3)

For the second integral comprising (5.3.1) we have, on deforming the path of integration into the positive real axis,

\[
\int_0^\infty \frac{e^{ix}}{(1+x)^{\nu/2} (x+2)} J_1(\lambda (1+x)) \, dx = \frac{e^{i k}}{2} \int_0^\infty \frac{e^{ix}}{x^{\nu/2} (x+2i)} J_1(\lambda (1-ix)) \, dx
\]

\[
= \frac{e^{i k}}{2} \int_0^\infty J_1(\lambda) \frac{e^{k x}}{x^{\nu/2}} \, dx + O \left( \frac{1}{k^{3/2}} \right)
\]
we substitute (5.3.4) and (5.5.3) into (5.3.1) to give
\[E_s - 2E_i = -\frac{4\pi \omega^3 u_0 \phi}{\lambda^2} \left\{ 1 + 2 \frac{e^2 \epsilon}{\pi^2 \frac{\omega}{k^2}} + O \left( \frac{1}{k^2} \right) \right\} \int_{\lambda \to 0} \frac{J_1^2(\lambda)}{\frac{\lambda}{2} J_0(\lambda) J_2(\lambda) + J_1^2(\lambda)} \, d\lambda \right]. \tag{5.5.5}

From (5.1.1), (5.1.16) and (5.3.5) we obtain
\[Q = 2 \left[ 1 + \frac{2 J_1^2(\lambda)}{\pi \frac{1}{2} k^2 \lambda} \int_{\lambda \to 0} \frac{J_1^2(\lambda)}{J_0(\lambda) J_2(\lambda) + J_1^2(\lambda)} \, d\lambda \right] \sin \left( 2k - \pi/4 \right) + O \left( \frac{1}{k^2} \right). \tag{5.5.6}

since
\[J_1'(\lambda) = J_0(\lambda) - \frac{1}{\lambda} J_1(\lambda),\]
and for the case of an incident free torsion wave we let \( \lambda \to 0 \)
in (5.5.5) to give
\[Q = 2 \left[ 1 + \frac{2}{\pi k^2 \lambda} \sin \left( 2k - \pi/4 \right) + O \left( \frac{1}{k^2} \right) \right]. \tag{5.5.7}
In this subsection the asymptotic evaluation of the scattering coefficient, for \( 1-1/k^{3/2} \leq \varepsilon \leq 1 \) is discussed.

Definition (5.1.32) relies on \( f(1+x) \) being known for \( x > 0 \). This function was evaluated in section 3.3 subject to the requirements of Case III and for \( x \approx 0 \). However, before an approximation for small \( x \) was made, the Hankel functions in the integrals of (3.3.5) were replaced by their respective asymptotic expansions and as \( k \) increases the accuracy of these expansions improves. Unfortunately, as \( k \) increases, the range of \( \varepsilon \) decreases, e.g. for \( k = 2 \), \( 0.65 \leq \varepsilon \leq 1 \) whereas for \( k = 10 \), \( 0.98 \leq \varepsilon \leq 1 \).

The great advantage of the method used in evaluating (3.3.5) is that we can obtain asymptotic expressions for quantities of physical interest, which possess a high degree of accuracy, when \( k \) is large and \( \varepsilon = 1 \). However, when \( \varepsilon = 1 \), the average rate at which energy is incident upon the crack is zero, and the scattering coefficient does not have any physical interpretation. Also, by having already asymptotically evaluated the scattering
coefficients for Cases I and II, we have succeeded in covering the larger part of the range of \( \varepsilon \).

It is for the reasons set out in the above paragraph and because of the difficult analytical techniques required that it has been decided not to attempt the evaluation of the scattering coefficient for Case III.
For $\varepsilon = 0.0(0.1)0.9$, the scattering coefficient $\sigma$ is plotted as a function of $k$ in figures 20 to 29.

$\varepsilon = 0.0$ and $\varepsilon = 0.1$, figures 20 and 21

Both of these are examples of Case II.

$\varepsilon = 0.2$, figure 22

The relevant case is Case II. Although Case I is not valid for the smaller values of $k$, it does however, for the larger values of $k$, give results within a few percent of Case II.

$\varepsilon = 0.3$, figure 23

For $k < 6.7$ we consider Case II, but for $6.7 \leq k \leq 10$ the curves corresponding to Cases I and II are nearly identical and it appears to be irrelevant which case is considered.

$\varepsilon = 0.4$, figure 24

For $k > 2$ we consider Case II until $k = 6.7$, then we change over to Case I. Between $k = 6.7$ and $k = 9$ the two cases give results which are within 4% of each other.
For $2 \leq k \leq 4$ we consider Case II and in the vicinity of $k=4$ we change over to Case I. Between $k=4.0$ and $k=10.0$ the two cases give results which are within approximately $3\%$ of each other.

For $k \geq 2$ we consider Case II until $k \approx 3.4$, then we change over to Case I. The two curves do not cross or touch but when $k \approx 3.4$ they are within $0.75\%$ of each other. There appears to be a "gap" between Cases I and II since it is doubtful that Case I will give accurate results unless $k \geq 5.0$.

These are examples of Case I and give accurate results for sufficiently large $k$. 

$\varepsilon = 0.5$, figure 25

$\varepsilon = 0.6$, figure 26

$\varepsilon = 0.7$, $0.8$, and $0.9$, figures 27, 28, and 29 (respectively).
CHAPTER II

DIFFRACTION OF HIGH FREQUENCY TORSION WAVES BY A RIGID DISC
Section 6

Derivation of the Fredholm integral equation of the first kind

6.1 Introduction to Chapter II

The problem of the diffraction of time harmonic torsion waves, by an immovable rigid disc situated in an isotropic and homogeneous body of infinite extent, is considered in this Chapter. The material experiences only an angular displacement. A special case of this problem is similar to the Reissner-Sagoci problem ([21], [22]), which for high frequencies has been solved asymptotically by Thomas [23]. The corresponding impulsive problem, for small time ranges, has been considered by Shail [39]. The low frequency and exact solutions of the problem are well documented by Thomas [23].

It is proposed to solve the problem, for high frequencies, i.e. for large \(k\) and all physically significant \(\alpha\), where \(k\) and \(\alpha\) are defined in Chapter I, by adopting the approach used by Thomas [23], who obtains a Fredholm integral equation of the first kind, which is then converted into a Fredholm integral
equation of the second kind. This integral equation is solved asymptotically, by using a method that Jones ([24], [25],[26]) has devised to solve problems of high frequency diffraction by a circular disc.
6.2 Formulation

The equations of subsections 2 and 3 of Section 1 are applicable. The coordinate system which was adopted in Chapter I, is again used. The time dependence of the various field quantities, is again taken to be proportional to \( \exp(i\omega t) \). The rigid disc is taken to occupy that portion of the elastic space \( z=0, 0 \leq \rho \leq 1 \).

The boundary conditions assumed, are

\[ u_\theta (\rho, z) = 0, \ 0 \leq \rho \leq 1, z = \pm 0, \quad (6.2.1) \]

and hence from (1.3.1) and (1.3.9) we have

\[ u_\theta^{(s)} (\rho, z) = -\frac{2u_0}{\lambda}\frac{J_1 (\lambda \rho)}{\lambda}, \ 0 \leq \rho \leq 1, z = \pm 0 \quad (6.2.2) \]

Also, continuity of shear stress across \( z=0 \) for \( \rho>1 \) is equivalent to

\[ \phi_\theta^{(s)} (\rho, z) = 0, \ \rho>1, z = 0, \quad \text{i.e. } \frac{\partial u_\theta^{(s)}}{\partial z} = 0, \ \rho>1, z=0, \quad (6.2.3) \]

and, as in Chapter I, \( u_\theta^{(s)} \) must satisfy a Sommerfeld type radiation condition. \( u_\theta^{(s)} \) must also satisfy an edge condition [38].
6.3 Derivation of the integral equation

Equation (1.2.9) may be written in a different form, i.e.
as

$$\left( V^2 + k^2 \right) u_0 (\rho, z) \cos \theta = 0.$$  \hspace{1cm} (6.3.1)

and (6.2.3)

From Green's theorem (c.f. Morse and Feshbach [29]), we obtain for the scattered field at a general point \( \mathbf{z} = (\rho, \theta, z) \) in the elastic space

$$u_s (\rho, z) \cos \theta = -\frac{1}{4\pi} \int_{\partial S} \frac{\partial u_0 (\rho', z')}{\partial \eta'} G(\mathbf{z}; \mathbf{z}') \cos \theta' dS'$$  \hspace{1cm} (6.3.2)

where the surface \( S' \) is taken to be both faces of the disc, and \( \partial / \partial \eta' \) denotes differentiation with respect to the source coordinates, in the direction of the normal to \( S' \), pointing into the elastic space. The appropriate Green's function, \( G(\mathbf{z}; \mathbf{z}') \), is given by

$$G(\mathbf{z}; \mathbf{z}') = \frac{e^{-ik|z-z'|}}{|z-z'|}.$$  \hspace{1cm} (6.3.3)

In (6.3.2), we set \( \theta = 0 \), and let \( z \) tend to a point on the surface of the disc, hence, after using (6.2.2) we have

$$\frac{2u_0}{\lambda} J_1 (\lambda \rho) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \left[ \frac{\partial u_0 (\rho', \theta') - \partial u_0 (\rho', 0)}{\partial z'} \right] \cos \theta' \rho d\theta' d\rho'.$$  \hspace{1cm} (6.3.4)
Hence, the integral equation which the problem gives rise to may be written as

\[ \int_0^1 \sigma(t) \psi(t) \, dt = \frac{8\pi u_0 J_1(\lambda r)}{\lambda}, \quad 0 \leq r \leq 1, \]  \hspace{1cm} (6.3.5)

where

\[ \sigma(\rho) = \left[ \frac{\partial u_\theta(\rho, z)}{\partial z} \right]_{z=0^+} - \left[ \frac{\partial u_\theta(\rho, z)}{\partial z} \right]_{z=0^-}. \]  \hspace{1cm} (6.3.6)

and

\[ \psi_n(\rho; \theta) = \frac{1}{2\pi} \int_0^{2\pi} \exp \left\{ -i \frac{\rho^2 + \rho'^2 - 2\rho \rho' \cos \theta'}{2} \right\} \cos n \theta' \, d\theta'. \]  \hspace{1cm} (6.3.7)

Equation (6.3.5) is a Fredholm integral equation of the first kind.
6.4 The transformed integral equation

In this subsection equation (6.3.5) is converted into

an integral equation of a more convenient type and in order
to do this we use the result

\[ \psi_i (\gamma, t) = -2\pi \left( \frac{1}{r} \frac{d}{dr} \right) \left[ \int_0^{\min(\gamma, t)} J_{\lambda/2} \left( \frac{t^2 - x^2}{y_2} \right) \frac{(t^2 - x^2)^{y_2}}{(r^2 - x^2)^{y_4}} \, dx \right. \]

\[ \left. - i \int_0^r J_{\lambda/2} \left( \frac{t^2 - x^2}{y_2} \right) \frac{(t^2 - x^2)^{y_2}}{(r^2 - x^2)^{y_4}} \, dx \right] \]  \quad (6.4.1)

which is proved in [36]. With the aid of (6.4.1), (6.5.5)

becomes

\[ r \left( \frac{1}{r} \frac{d}{dr} \right) \left[ \int_0^1 \alpha(t) \left( \int_0^{\min(\gamma, t)} J_{\lambda/2} \left( \frac{t^2 - x^2}{y_2} \right) \frac{(t^2 - x^2)^{y_2}}{(r^2 - x^2)^{y_4}} \, dx \right. \right. \]

\[ \left. \left. - i \int_0^r \alpha(t) \left( \int_0^{\min(\gamma, t)} J_{\lambda/2} \left( \frac{t^2 - x^2}{y_2} \right) \frac{(t^2 - x^2)^{y_2}}{(r^2 - x^2)^{y_4}} \, dx \right) \right] \right] - \frac{4\mu_0}{\lambda} J_1 (\lambda r), \]  \quad (6.4.2)

\[ 0 \leq r \leq 1. \]
We take the first integral in (6.4.2) and interchange the order of integration by considering
\[ \int_0^1 dt \int_0^{\min(y,t)} dx, \]
taken over the black shaded area, shown in figure 50, in two different ways. We split the ranges of integration of this repeated integral in the following way
\[ \int_0^1 dt \int_0^{\min(y,t)} dx = \int_0^1 dt \int_0^y dx + \int_0^1 dt \int_y^1 dx, \]  
(6.4.3)
and after interchanging the order of integration we have
\[ \int_0^1 dt \int_0^{\min(y,t)} dx = \int_0^y dx \int_0^1 dt + \int_0^1 dx \int_0^y dt. \]  
(6.4.4)
We add the two repeated integrals to obtain
\[ \int_0^1 dt \int_0^{\min(y,t)} dx = \int_0^y dx \int_0^1 dt. \]  
(6.4.5)

For the second repeated integral in (6.4.2), the order of integration may be inverted, if in the previous result we set
\[ r = 1, \text{ i.e.} \]
\[ \int_0^1 dt \int_0^t dx = \int_0^t dx \int_0^1 dt. \]  
(6.4.6)

Therefore, on inverting the order of integration, (6.4.2) becomes
\[ r \left( \frac{1}{r} \frac{d}{dr} \right) \left[ \int_0^r J_{1/2} \left( \frac{\sqrt{1 - x^2} h(x)}{\sqrt{x^2}} \right) dx \right] - \int_0^r J_{1/2} \left( \frac{\sqrt{1 - x^2} h(x)}{\sqrt{x^2}} \right) dx \right] = \frac{\mu_0 J_1(\lambda)}{\lambda}, \]
\[ 0 \leq r \leq 1, \]  
(6.4.7)
where
\[ h(x) = \int_x^1 \frac{1}{y^2} \left\{ \frac{k(t^2-x^2)^{v/2}}{(t^2-x^2)^{v/4}} \right\} dt. \] (6.4.8)

We define a new function \( \mathcal{K} \) by the relationship
\[ \mathcal{K}(t) = \int_t^1 \varphi(u) du, \quad 0 \leq t \leq 1; \] (6.4.9)

hence
\[ \frac{1}{t} \frac{d}{dt} \mathcal{K}(t) = \frac{1}{t} \frac{d}{dt} \int_t^1 \varphi(u) du. \] (6.4.10)

The edge condition [38] requires \( \varphi(\rho) \) to be no more singular than \((1-\rho^2)^{v/2}\) as \( \rho \to 1 \) and hence \( \mathcal{K}(\rho) \) must be proportional to \((1-\rho^2)^{v/2}\) as \( \rho \to 1 \). Further, since \( \varphi(\rho) \) is bounded at \( \rho = 0 \), \( \mathcal{K}(\rho) \) is bounded at \( \rho = 0 \). Therefore, on carrying out the integration we have
\[ \frac{1}{t} \frac{d}{dt} \mathcal{K}(t) = -\frac{1}{t} \varphi(t). \] (6.4.11)

We use this last result to re-write (6.4.8) as
\[ h(x) = -\int_x^1 \frac{1}{y^2} \{ k(t^2-x^2)^{v/2} \} (t^2-x^2)^{v/4} t \left( \frac{1}{t} \frac{d}{dt} \right) \mathcal{K}(t) dt. \] (6.4.12)

We modify this formula by integrating by parts, in such a way as to remove the derivative on \( \mathcal{K}(t) \). Since
\[ \frac{1}{z} \left( \frac{d}{dz} \right) \{ z^v J_v(z) \} = z^{v-1} J_{v-1}(z), \] (6.4.13)
we have
\[ h(x) = - \left[ k \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ K(t) \right] \]

\[ + \frac{k}{x} \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt. \]  \tag{6.4.14}

The first term of (6.4.14) vanishes because of the behaviour of the Bessel function near \( t = x \), and of \( K(t) \) near \( t = 1 \). Hence the last line becomes

\[ h(x) = \frac{k}{x} \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt. \]  \tag{6.4.15}

with the aid of this last result (6.4.7) becomes

\[ \frac{k}{x} \frac{d}{dy} \left[ \int_{0}^{r} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ dx \right] \left[ \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt \right] \]

\[ - i \int_{0}^{r} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ dx \left[ \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt \right] = - \frac{4 \pi a}{\lambda} J_1(\lambda r), \]  \tag{6.4.16}

We interchange the order of integration in the second term on the left-hand side and this gives

\[ \frac{k}{x} \frac{d}{dy} \left[ \int_{0}^{r} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ dx \right] \left[ \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt \right] \]

\[ - i \int_{0}^{r} \left[ \int_{-1}^{1} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ dx \right] \left[ \int_{0}^{r} \frac{k(t^2 - x^2)^{1/2}}{(t^2 - x^2) \sqrt{4}} \ t \ K(t) \ dt \right] = - \frac{4 \pi a}{\lambda} J_1(\lambda r), \]  \tag{6.4.17}

\[ 0 \leq r \leq 1. \]
In the second term on the left-hand side of (6.4.17) we let 

\[ x = t \cos \theta ; \text{ then} \]

\[
\int_0^t \frac{J_{\nu_2} \left\{ \frac{K(t^2 - r^2)^{\nu_2}}{(t^2 - r^2)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, dx
\]

\[
= \left( \frac{t}{\nu_2} \right)^{\nu_2} \int_0^{\nu_2} \frac{r^{\nu_2} \left\{ \frac{K(r^2 - t^2 \cos^2 \theta)^{\nu_2}}{(r^2 - t^2 \cos^2 \theta)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, d\theta
\]

(6.4.18)

Now from Jones [28] we have

\[
\int_0^{\nu_2} \frac{r^{\nu_2} \left\{ \frac{K(r^2 - t^2 \cos^2 \theta)^{\nu_2}}{(r^2 - t^2 \cos^2 \theta)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, d\theta
\]

\[
= \left( \frac{\nu \lambda}{\nu_2} \right)^{\lambda + \nu_2} \int_0^{\lambda + \nu_2} \frac{r^{\lambda + \nu_2} \left\{ \frac{K(z^2 - r^2 \cos^2 \theta)^{\lambda + \nu_2}}{(z^2 - r^2 \cos^2 \theta)^{\lambda + \nu_2}} \right\}}{(z^2 - r^2 \cos^2 \theta)^{\lambda + \nu_2}} \, d\theta,
\]

when \( \Re \lambda \geq -\frac{1}{2}, \ Re \mu > Re (\lambda - 1), \ Re \nu > -\frac{1}{2} \). Therefore we let \( \lambda = 0 \)

\( \mu = -\frac{1}{2}, \nu = \frac{1}{2}, r = kt \) and \( z = kr \) in the above result so that

\[
\int_0^t \frac{J_{\nu_2} \left\{ \frac{K(t^2 - r^2)^{\nu_2}}{(t^2 - r^2)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, dx = \int_0^{\nu_2} \frac{r^{\nu_2} \left\{ \frac{K(r^2 - t^2 \cos^2 \theta)^{\nu_2}}{(r^2 - t^2 \cos^2 \theta)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, d\theta.
\]

(6.4.19)

From this result it is obvious that we can interchange the roles \( r \) and \( t \). Hence

\[
\int_0^t \frac{J_{\nu_2} \left\{ \frac{K(t^2 - r^2)^{\nu_2}}{(t^2 - r^2)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, dx = \int_0^r \frac{J_{\nu_2} \left\{ \frac{K(r^2 - x^2)^{\nu_2}}{(r^2 - x^2)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, dx.
\]

(6.4.20)

Therefore (6.4.17) can be written as

\[
K \frac{d}{dr} \left[ \int_0^r \frac{J_{\nu_2} \left\{ \frac{K(r^2 - x^2)^{\nu_2}}{(r^2 - x^2)^{\nu_4}} \right\}}{(r^2 - x^2)^{\nu_4}} \, dx \right] = K(0) dt.
\]
\[ -i \int_0^r k(t) \, dt \int_0^r \frac{J_{\nu_2} \left( \frac{(r^2 - x^2)^{\nu_2}}{x^4} \right)}{(r^2 - x^2)^{\nu_4}} \, dx \frac{J_{\nu_2} \left( \frac{(r^2 - x^2)^{\nu_2}}{x^4} \right)}{(r^2 - x^2)^{\nu_4}} \, dx = -\frac{4u_0}{\lambda} J_1(\lambda r), \quad 0 \leq r \leq 1, \]  

(6.4.21)

and on inverting the order of integration in the second term on the left-hand side we have

\[ -i \int_0^r \frac{J_{\nu_2} \left( \frac{(r^2 - x^2)^{\nu_2}}{x^4} \right)}{(r^2 - x^2)^{\nu_4}} \, dx \frac{J_{\nu_2} \left( \frac{(r^2 - x^2)^{\nu_2}}{x^4} \right)}{(r^2 - x^2)^{\nu_4}} \, dx \frac{k(t)}{x} \, dt = -\frac{4u_0}{\lambda} J_1(\lambda r), \quad 0 \leq r \leq 1. \]  

(6.4.22)

We can re-write equation (6.4.22) as

\[ -\frac{d}{d\rho} \int_0^\rho \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \, dx \lambda(x) \, dx = \frac{4u_0}{\lambda} J_1(\lambda \rho), \quad 0 \leq \rho \leq 1, \]  

(6.4.23)

where

\[ \lambda(x) = \frac{1}{\rho} \int_0^\rho \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \, dx \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \, dx - i \int_0^\rho \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \, dx \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \, dx \]  

(6.4.24)

Since

\[ \frac{d}{d\rho} J_1(\lambda \rho) = \frac{d}{d\rho} \int_0^\rho J_1(\lambda u) \, du, \]  

(6.4.25)

we have, on integrating equation (6.4.23) with respect to \( \rho \)

\[ \int_0^\rho \frac{J_{\nu_2} \left( \frac{(\rho^2 - x^2)^{\nu_2}}{x^4} \right)}{(\rho^2 - x^2)^{\nu_4}} \lambda(x) \, dx = C_0 - \frac{4u_0}{\lambda k} \int_0^\rho J_1(\lambda u) \, du, \]  

(6.4.26)

where \( C_0 \) is an arbitrary constant. We replace the Bessel function
by its trigonometric equivalent to give
\[
\int_0^\phi \frac{\cos \frac{1}{2} (\rho^2 - x^2)^{1/2}}{(\rho^2 - x^2)^{1/2}} \lambda(x) \, dx = C_0 - \frac{2\sqrt{2}}{\sqrt{\pi}} u_0 \int_0^\phi J_1(\lambda u) \, du. \tag{6.4.27}
\]

Now from Jones [28], if
\[
\int_0^y \frac{f(x) \cos \frac{1}{2} (y^2 - x^2)^{1/2}}{(y^2 - x^2)^{1/2}} \, dx = f(y);
\]
then its solution is given by
\[
g(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x f(y^{1/2}) \cos \frac{1}{2} \frac{(x^2 - y^{1/2})}{(x^2 - y^{1/2})} \, dy.
\]

Hence the solution of (6.4.27) is
\[
\lambda(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^{x^2} \frac{\cos \frac{1}{2} \frac{(x^2 - y^{1/2})}{(x^2 - y^{1/2})}}{(x^2 - y^{1/2})} \left[ C_0 - \frac{2\sqrt{2}}{\sqrt{\pi}} u_0 \int_0^y J_1(\lambda u) \, du \right] \, dy. \tag{6.4.28}
\]

From (6.4.24) we have
\[
\int_0^x \frac{\cos \frac{1}{2} (\rho^2 - x^2)^{1/2}}{(\rho^2 - x^2)^{1/2}} \rho' K(\rho') \, d\rho' - i \int_0^x \frac{\sin \frac{1}{2} (\rho^2 - x^2)^{1/2}}{(\rho^2 - x^2)^{1/2}} K(\rho') \, d\rho' = \left( \frac{\sqrt{2}}{2} \right) \lambda(x), \tag{6.4.29}
\]
and since \(\lambda(x)\) is known it is seen that it is now possible

to determine \(K(\rho')\). We let \(z = x^2\), \(t = \rho^2\) \; \text{then equation}

(6.4.29) becomes
\[
\int_0^t \frac{\cos \frac{1}{2} (t^2 - z)^{1/2}}{(t^2 - z)^{1/2}} K(t^{1/2}) \, dt - i \int_0^t \frac{\sin \frac{1}{2} (t^2 - z)^{1/2}}{(t^2 - z)^{1/2}} K(t^{1/2}) \, dt = \left( 2\pi \right)^{1/2} \lambda(z^{1/2}),
\]

for \(0 < z < 1\). \tag{6.4.30}

Equation (6.4.30) is of a similar form to equation (6) in
Jones [25] and, with this in mind we re-write the last equation as

\[ \int_{0}^{x} \mathcal{K}(t^{1/2}) \frac{e^{iK(t-z)^{1/2}}}{(t-z)^{1/2}} dt - \int_{0}^{x} \mathcal{K}(t^{1/2}) \cos \frac{K}{(t-z)^{1/2}} dt = \left( 2\pi K \right)^{1/2} \lambda(z^{1/2}), \quad (6.4.31) \]

so that it will be possible to obtain an integral equation which is suitable for high frequencies. \((t-z)^{1/2}\) is taken to be equal to \(i(z-t)^{1/2}\) when \(z > t\). We multiply both sides of (6.4.31) by

\[ \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}}, \]

and integrate with respect to \(z\) from 0 to \(x\), i.e.

\[ \int_{0}^{x} \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} d \int \mathcal{K}(t^{1/2}) \frac{e^{iK(t-z)^{1/2}}}{(t-z)^{1/2}} dt - \int_{0}^{x} \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} \int \mathcal{K}(t^{1/2}) \cos \frac{K}{(t-z)^{1/2}} dt \]

\[ = \left( 2\pi K \right)^{1/2} \int_{0}^{x} \lambda(z^{1/2}) \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} d z. \quad (6.4.32) \]

We interchange the order of integration to give

\[ \int_{0}^{x} \mathcal{K}(t^{1/2}) \frac{e^{iK(t-z)^{1/2}}}{(x-z)^{1/2}} dt \int_{0}^{x} \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} d z - \int_{0}^{x} \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} \int_{0}^{x} \mathcal{K}(t^{1/2}) \frac{e^{iK(t-z)^{1/2}}}{(t-z)^{1/2}} dt \]

\[ = \left( 2\pi K \right)^{1/2} \int_{0}^{x} \lambda(z^{1/2}) \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} d z, \quad (6.4.33) \]

Now, from [25] we use the results

\[ \int \frac{\cos \frac{K}{(x-z)^{1/2}}}{(x-z)^{1/2}} e^{iK(t-z)^{1/2}} dz = \int_{(t-x)^{1/2}}^{(t+x)^{1/2}} e^{iKy^{1/2}} dy - i H(x-t). \]
and
\[ \int_{(x-z)^{1/2}}^{(x-z)^{1/2}} \frac{e^{i k \sqrt{t} (t-z)}}{(t-z)^{1/2}} \, dz = \pi \text{sgn}(x-t), \]
so that (6.4.33) can be written as
\[ \int_{(t^{1/2}+x^{1/2})}^{(t^{1/2}+x^{1/2})} \frac{e^{i k \sqrt{t} y}}{(t^{1/2}+x^{1/2})} \, dy - \pi H(x-t) \int_{0}^{\infty} \chi(t^{1/2}) \text{sgn}(x-t) \, dt \]
\[ = (2\pi k)^{1/2} \int_{0}^{\infty} \frac{\chi(x)}{(x-z)^{1/2}} \, dz. \quad (6.4.34) \]

Now
\[ \int_{0}^{\infty} \chi(t^{1/2}) H(x-t) \, dt = \int_{0}^{\infty} \chi(t^{1/2}) \, dt, \quad (6.4.35) \]
and
\[ \int_{0}^{\infty} \chi(t^{1/2}) \, \text{sgn}(x-t) \, dt = \int_{0}^{\infty} \chi(t^{1/2}) \, dt. \quad (6.4.36) \]

Hence (6.4.34) becomes
\[ \int_{(t^{1/2}+x^{1/2})}^{(t^{1/2}+x^{1/2})} \frac{e^{i k \sqrt{t} y}}{(t^{1/2}+x^{1/2})} \, dy = (2\pi k)^{1/2} \int_{0}^{\infty} \frac{\chi(x)}{(x-z)^{1/2}} \, dz. \quad (6.4.37) \]

On differentiating both sides of (6.4.37) with respect to \( t \), and since
\[ \frac{d}{d t} \int_{(t^{1/2}+x^{1/2})}^{(t^{1/2}+x^{1/2})} \frac{e^{i k \sqrt{t} y}}{(t^{1/2}+x^{1/2})} \, dy = \frac{e^{i k \sqrt{t} (t^{1/2}+x^{1/2})}}{t^{1/2}+x^{1/2}} \cdot \frac{d}{dx} \left( \frac{e^{i k \sqrt{t} y}}{t^{1/2}-x^{1/2}} \right) + \frac{e^{i k \sqrt{t} (t^{1/2}-x^{1/2})}}{(t^{1/2}-x^{1/2})} \cdot \frac{d}{dx} \left( \frac{e^{i k \sqrt{t} y}}{t^{1/2}+x^{1/2}} \right), \quad (6.4.38) \]
we have
\[ \int_{0}^{\infty} \chi(t^{1/2}) \left[ \frac{e^{i k \sqrt{t} (t^{1/2}+x^{1/2})}}{t^{1/2}+x^{1/2}} + \frac{e^{i k \sqrt{t} (t^{1/2}-x^{1/2})}}{t^{1/2}-x^{1/2}} \right] \, dt = (2\pi k)^{1/2} \int_{0}^{\infty} \frac{\chi(x)}{(x-z)^{1/2}} \, dz. \quad (6.4.39) \]
The change of variables, \( x = v^2 \) and \( t = \omega^2 \) leads to

\[
\int_{0}^{1} \omega K(\omega) \left\{ \epsilon^{-iK(\omega+v)} + \epsilon^{-iK(\omega-v)} \right\} d\omega = K^{(0)}(\nu), \quad 0 \leq \nu \leq 1, \quad (6.4.40)
\]

where

\[
K^{(0)}(\nu) = \left( \frac{kr}{2} \right)^{\nu^2} \frac{d}{d\nu} \int \frac{\lambda(z^{\nu^2}) \cos \frac{\nu^2 (\nu^2 - z^{\nu^2})}{\nu^2 - z} d\nu}{(\nu^2 - z)^{\nu^2}}. \quad (6.4.41)
\]

The principal value of the integral in (6.4.40) is to be taken here and elsewhere, where necessary. For large \( k \) the dominant contribution will come from the neighbourhood of the principal value, corresponding to the physical situation that to a first approximation, the source distribution on the disc is determined locally by the incident field. The most significant values will tend to occur near the edge, i.e. near \( \nu \approx 1 \). There, the neighbourhood of the principal value is about to move out of the interval of integration (the \( \omega \) in the numerator prevents \( \nu = 0 \) being similarly important). The first term represents the effect of secondary radiation. Note that as \( v \) approaches 0, both terms in (6.4.40) are approximately of the same order of significance.

We let \( z = t^2 \) in (6.4.41) to obtain
and hence from (6.4.28), with \( y = x^2 \), we have

\[
K^{(0)}(\nu) = (2\pi - k)^{\nu_2} \frac{d}{\omega_0} \int_0^\infty \frac{\lambda(t) \cos \frac{\lambda}{\nu_2} (\nu^2 - \nu_2)^{\nu_2}}{(\nu^2 - \nu_2)^{\nu_2}} \, dt,
\]

(6.4.42)

where \( C'' \) is an arbitrary constant.
Section 7

The Fredholm integral equation of the second kind

7.1 Conversion to a Fredholm integral equation of the second kind

In view of the relative significance of the two terms on
the left-hand side of (6.4.40), when \( k \) is large, we let

\[
\int_0^1 \omega k(\omega) \frac{e^{ik(\omega-v)}}{(\omega-v)} \, d\omega = K(v),
\]

(7.1.1)

which may be expressed as

\[
\int_0^1 \omega k(\omega) \frac{e^{ik\omega}}{(\omega-v)} \, d\omega = K(v) e^{ikv}.
\]

(7.1.2)

This is a solvable singular integral equation; from Mikhlin

[50], it is seen that (7.1.1) has a general solution of the form

\[
\omega k(\omega) e^{ik\omega} = -\frac{1}{\pi^2 \omega^2 (1-\omega)^2} \int_0^1 \frac{\sqrt{\omega^2 (1-\omega)^2} K(v) e^{iv\omega}}{(\omega-v)} \, d\omega + \frac{C}{\omega (1-\omega)^2},
\]

(7.1.3)

where \( C' \) is a constant. \( C' \) is determined by applying a condition
at either \( \omega=0 \) or \( \omega=1 \). Now \( K(\omega) \) must be bounded at \( \omega=0 \);
therefore

\[
C' = \frac{1}{\pi^2} \int_0^1 \sqrt{1-\omega^2} \frac{K(v) e^{iv\omega}}{\sqrt{\omega^2 (1-\omega)^2}} \, d\omega,
\]

(7.1.4)

and hence

\[
K(\omega) = -\frac{e^{ik\omega}}{\pi^2 \omega^{3/2} (1-\omega)^{1/2}} \left[ \int_0^1 \frac{\sqrt{\omega^2 (1-\omega)^2} K(v) e^{iv\omega}}{(\omega-v)} \, d\omega - \int_0^1 \frac{K(v) e^{iv\omega}}{(\omega-v)^{1/2}} \, d\omega \right].
\]

(7.1.5)
\[ K(\omega) = -\frac{e^{ik\omega}}{\kappa^2 \omega^2 (1-\omega)^2} \int_0^1 \frac{(1-\omega)^{\nu/2}}{\nu^2} K(\nu) e^{i\nu \omega} d\omega. \quad (7.1.6) \]

By substituting for $K(\omega)$ from (7.1.6) we find that
\[ \int_0^1 \omega K(\omega) \frac{e^{-iK(\omega+v)}}{(\omega+v)} d\omega = -\frac{e^{ik\nu}}{\kappa^2} \int_0^1 \frac{d\omega}{(1-\omega)^{\nu/2} (1-\omega)^2} \int_0^1 \frac{(1-\nu)^{\nu/2}}{(1-\omega)^{\nu/2}} e^{i\nu t} K(t) dt. \quad (7.1.7) \]

and on inverting the order of integration we have
\[ \int_0^1 \omega K(\omega) \frac{e^{-iK(\omega+v)}}{(\omega+v)} d\omega = -\frac{e^{ik\nu}}{\kappa^2} \int_0^1 \frac{d\omega}{(1-\omega)^{\nu/2} (1-\omega)^2} \int_0^1 \frac{(1-\nu)^{\nu/2}}{(1-\omega)^{\nu/2}} e^{i\nu t} K(t) dt \quad (7.1.8) \]

The inner integral may be evaluated by letting $\omega = x^2/(1+x^2)$

to give
\[ \int_0^1 \frac{d\omega}{(1-\omega)^{\nu/2} (1-\omega)^2} = -\frac{\nu!}{(\nu+t)(\nu+t)^2} \nu > 0, \quad 0 < t < 1. \quad (7.1.9) \]

Hence
\[ \int_0^1 \omega K(\omega) \frac{e^{-iK(\omega+v)}}{(\omega+v)} d\omega = -\frac{e^{ik\nu}}{\kappa^2} \int_0^1 \frac{d\omega}{(1-\omega)^{\nu/2} (1-\omega)^2} \int_0^1 \frac{(1-\nu)^{\nu/2}}{(1-\omega)^{\nu/2}} e^{i\nu t} K(t) dt. \quad (7.1.10) \]

and with the aid of (7.1.1) and (7.1.10), (6.4.40) may be written as
\[ K(\nu) + \frac{1}{\kappa} \left( \frac{\nu}{\nu+t} \right)^{\nu/2} \int_0^1 \frac{(1-t)^{\nu/2}}{(1+t)^{\nu/2}} e^{i\nu (t+v)} dt = K'(\nu), \quad 0 < \nu \leq 1. \quad (7.1.11) \]

When (7.1.11) has been solved for $K(\nu)$, we can then determine $K(\omega)$ from (7.1.6) and hence the whole field.

Equation (7.1.11) differs from (6.4.40) in being a Fredholm
integral equation of the second kind. Unfortunately, $K^0(v)$ is of such a form that it is impossible to solve (7.1.11) by iteration for large $k$. It is necessary to transform the integral equation into a more suitable form. Jones ([24],[25] and[26]) has succeeded in transforming the integral equation when the incident field is a plane wave. Thomas [23], by using a generalised function approach, has transformed the integral equation when the incident field is a free torsion wave, and Newby [31] has considered the problem of a plane wave at oblique incidence. However, all these methods rely, very heavily on the simple form of the incident field and do not seem to be applicable in this instance. However, Williams [32] has devised a general method of approach so that it becomes relatively simple to transform equation (7.1.11) to a form which is more amenable to solution. Therefore we let

$$K(v) = K_2(v) + \int_0^\infty \Psi_1(\omega) \frac{e^{ik(\omega-v)}}{(\omega-v)} d\omega,$$  \hspace{1cm} (7.1.12)$$

and substitute into (7.1.11) to obtain
\[ K_2(v) + \int_0^\infty \frac{\Psi_1(w) e^{i \frac{v}{(w-v)}}}{(w-v)} \, dw + \frac{1}{\pi} \left( \frac{\nu}{1+\nu} \right)^{\frac{1}{2}} \int_0^1 \frac{K_2(t) e^{i \frac{v}{(t+v)}}}{(t+v)} \, dt \]

\[ \times e^{i \frac{\nu}{(t+v)}} \, dt = K_0^{(0)}(v), \quad 0 \leq v \leq 1, \quad (7.1.13) \]

\[ K_2(v) + \int_0^\infty \frac{\Psi_1(w) e^{i \frac{v}{(w-v)}}}{(w-v)} \, dw + \frac{1}{\pi} \left( \frac{\nu}{1+\nu} \right)^{\frac{1}{2}} \int_0^1 \frac{K_2(t) e^{i \frac{v}{(t+v)}}}{(t+v)} \, dt \]

\[ + \frac{1}{\pi} \left( \frac{\nu}{1+\nu} \right)^{\frac{1}{2}} \int_0^\infty \frac{\Psi_1(w) e^{i \frac{v}{(w-v)}}}{(w-v)} \, dw \int_0^1 \frac{K_2(t) e^{i \frac{v}{(t+v)}}}{(t+v)} \, dt = K_0^{(0)}, \quad 0 \leq v \leq 1. \quad (7.1.14) \]

Now from Appendix B of [25] we have

\[ \int_0^1 \frac{(1-t)^{\frac{1}{2}}}{(1+t)(w-t)} \, dt = - \frac{\pi}{(w+\nu)} \left\{ \left( \frac{w-1}{w} \right)^{\frac{1}{2}} \left( \frac{1+\nu}{\nu} \right)^{\frac{1}{2}} \right\}, \quad \nu > 0 \text{ and } t > 0. \quad (7.1.15) \]

Hence

\[ K_2(v) = K_0^{(0)} - \int_0^\infty \frac{\Psi_1(w) e^{i \frac{v}{(w-v)}}}{(w-v)} \, dw \]

\[ + \frac{1}{\pi} \left( \frac{\nu}{1+\nu} \right)^{\frac{1}{2}} \int_0^1 \frac{K_2(t) e^{i \frac{v}{(t+v)}}}{(t+v)} \, dt, \quad 0 \leq v \leq 1. \quad (7.1.16) \]

Obviously, if \( \Psi_1(w) \) satisfies

\[ K_0^{(0)}(v) = \int_0^\infty \frac{\Psi_1(w) e^{i \frac{v}{(w-v)}}}{(w-v)} \, dw, \quad (7.1.17) \]
then $K_2(v)$ satisfies

$$K_2(v) = \left( \frac{\nu}{2} \right)^{\nu} \int_0^\infty \frac{\psi(v+\omega) \psi(-\omega) d\omega}{(\omega + v)^{1/2}} \int_0^1 \frac{t^{1/2}}{t + v} K_2(t) e^{iK(t+v)} dt,$$

where $0 \leq \nu \leq 1$. \hfill (7.1.18)

Integral equation (7.1.18) is the kind of integral equation obtained by Jones, which can be solved approximately when $k$ is large. The problem of determining the correct transformation reduces to solving equation (7.1.17). This can be done as follows, by considering the problem of finding a solution of

$$(V^2 + k^2)\nu(\rho, z) \cos \theta = 0,$$ \hfill (7.1.19)

in $z > 0$, satisfying the radiation condition and such that

$$\nu(\rho, 0) = \frac{2\mu_0 \int_0^\omega (\lambda \rho)}{\lambda}. \hfill (7.1.20)$$

We take as the Green's function

$$\exp \left[ -i k \left( \rho^2 + \rho'^2 - 2 \rho \rho' \cos (\theta - \theta') + (z - z')^2 \right)^{1/2} \right]$$

$$\left( \rho^2 + \rho'^2 - 2 \rho \rho' \cos (\theta - \theta') + (z - z')^2 \right)^{1/2}$$

$$+ \exp \left[ -i k \left( \rho^2 + \rho'^2 - 2 \rho \rho' \cos (\theta - \theta') + (z + z')^2 \right)^{1/2} \right]$$

$$\left( \rho^2 + \rho'^2 - 2 \rho \rho' \cos (\theta - \theta') + (z + z')^2 \right)^{1/2}$$

whose derivative with respect to $z'$ vanishes on $z' = 0$. 
We apply Green's theorem in a way analogous to that in §6.3 to obtain

\[ \frac{2i}{\lambda} \int_0^{2\pi} \rho \, \sigma_i(\rho') \exp \left( -i \frac{k}{2} \rho^2 + i \rho \rho' \cos \theta' \right) \left( \frac{1}{\rho^2 + \rho'^2 - 2 \rho \rho' \cos \theta'} \right)^{1/2} \cos \theta' \, d\rho' \, d\phi' \]

(7.1.21)

where

\[ \sigma_i(\rho) = \left( \frac{\partial V}{\partial z} \right)_{z=0} \]  

(7.1.22)

Alternatively, the integral equation may be written as

\[ \int_0^\infty t \sigma_i(t) \Psi_i(r; t) \, dt = -\frac{k}{\lambda} u_0 J_i(\lambda r), \]  

(7.1.22a)

where \( \Psi_i(r; t) \) is defined by (6.3.7). Equation (7.1.22a) is of similar form to equation (6.3.5); hence by the same process we have

\[ \int_0^\infty \omega K_i(\omega) \left\{ \frac{e^{iK(\omega+\nu)}}{(\omega+\nu)} + \frac{e^{iK(\omega-\nu)}}{(\omega-\nu)} \right\} \, d\omega = -\frac{1}{2} K_i(\nu), \]

(7.1.23)

where

\[ K_i(\omega) = \int_0^1 \sigma_i(\omega) \, d\omega. \]

(7.1.24)

Therefore, after comparison with (7.1.17) it is seen that

\[ \Psi_i(\omega) = -2\omega K_i(\omega). \]

(7.1.25)

Now

\[ \sigma_i(\rho) = -\frac{2i \, \alpha u_0}{\lambda} J_{1}(\lambda \rho), \]

(7.1.26)

and from (7.1.24) we have

\[ \Psi_i(\omega) = \frac{4}{\lambda} i \, \alpha u_0 \int_0^1 J_{1}(\lambda u) \, du, \]

(7.1.27)
\[ i.e. \quad \frac{\psi_1(\omega)}{\lambda^2} = \frac{4\pi u_0}{\lambda^2} \left[ J_0(\lambda \omega) - J_0(\lambda) + C \right], \quad (7.1.28) \]

where \( C \) is an arbitrary constant.

We substitute (7.1.28) into (7.1.18) to obtain

\[ K_2(v) = \frac{4\pi u_0}{\lambda^2} \left[ (\frac{\nu}{1 + \nu}) \int_0^{\infty} e^{-i \nu \omega} \frac{\nu}{\nu + \omega} J_0(\omega) \, d\omega \right] \quad (7.1.29) \]

On deforming the path of integration into the negative imaginary axis we have

\[ \int_0^{\infty} \frac{\nu}{\nu + \omega} \frac{\nu}{\nu + \omega} e^{-i \nu \omega} \frac{\nu}{\nu + \omega} J_0(\omega) \, d\omega = \frac{e^{-i \nu \omega}}{(\nu + \omega)^2} \int_0^{\infty} \frac{\nu}{\nu + \omega} \frac{\nu}{\nu + \omega} e^{-i \nu \omega} J_0(\omega) \, d\omega, \quad 0 \leq \nu \leq 1. \quad (7.1.30) \]

where \(-\nu_2 \leq \arg \omega \leq 0\) and \(-\nu_2 \leq \arg(\omega - 1) \leq 0\). This result is still valid if we replace \( J_0 \) by \( 1 \). Hence

\[ K_2(v) = \frac{4\pi u_0}{\lambda^2} \left[ (\frac{\nu}{1 + \nu}) \int_0^{\infty} e^{-i \nu \omega} \frac{\nu}{\nu + \omega} J_0(\omega) \, d\omega \right] \quad (7.1.31) \]

We let

\[ K_2(v) = \left( \frac{\nu}{1 + \nu} \right)^{\nu} e^{-i \nu} \Im(v), \quad (7.1.32) \]

then
\[ I(\nu) = \frac{4\pi\lambda_0}{\lambda^2} \oint e^{i(k - \frac{\nu}{2}) - \frac{\nu}{2} - \frac{i\nu}{2} - k \omega} \left[ C - J_0(\lambda) + J_0(\lambda(1 - i\omega)) \right] d\omega \]

\[ - \frac{1}{\pi} \int_0^1 \frac{I(\omega)}{(1 + \omega)(\nu + \omega)} d\omega, \quad 0 \leq \nu \leq 1. \]  

The approximate solution of (7.1.33) for large \( k \) is considered in the next subsection.
7.2 Approximate solution for large $k$

We write (7.1.35) in the form

\[
I(v) = h_1(v) - \frac{1}{2} \int_0^1 \frac{(1 - \omega)^{1/2} e^{-2i\omega} I(\omega) d\omega}{(v + \omega)}
\]  

(7.2.1)

where

\[
h_1(v) = \frac{4d\mu_0}{\lambda^2} e^{-i\frac{\pi}{4}} \int_0^\infty \frac{\omega^{1/2} e^{-i\omega} [C - J_0(\lambda) + J_1(\lambda)] d\omega}{1 + v - i\omega}.
\]

(7.2.2)

We examine the iterative process

\[
I = \sum_{n=1}^\infty \hat{I}_n
\]

(7.2.3)

where

\[
\hat{I}_1(v) = h_1(v),
\]

(7.2.4)

and

\[
\hat{I}_n(v) = -\frac{1}{2} \int_0^1 \frac{(1 - \omega)^{1/2} e^{-2i\omega} \hat{I}_{n-1}(\omega) d\omega}{v + \omega}
\]

(7.2.5)

The presence of the factor $e^{2i\omega}$ suggests that the iterated terms will be of diminishing order of magnitude, for large $k$. However, a check shows that this is not so. In order to show this it is necessary to define three cases, as in subsection 2.3.

First of all, for Case I ($\lambda = k \gg 1$ and $k(1 - \varepsilon) > 1$), and
since we are only interested in the leading term in the asymptotic expansion for \( h_1 \), we replace the Bessel functions in (7.2.8) by the first terms in their respective asymptotic expansions and

\[
\frac{\omega^{\frac{1}{2}} (1-i\omega)^{\frac{1}{2}}}{(1+\nu-i\omega)} \quad \text{by} \quad \frac{\omega^{\frac{1}{2}}}{(1+\nu)}, \quad \text{to give}
\]

\[
h_1(\nu) \sim \frac{4\pi u}{k^2 \varepsilon_x} \frac{e^{-i(k_x-x_0)}}{(1+\nu)^{\frac{1}{2}}} \left[ \frac{2}{\pi k_c} \right]^2 \int_0^\infty e^{-k_c \omega} \omega^{\frac{1}{2}} d\omega \]

\[
+ \int_0^\infty \omega^{\frac{1}{2}} e^{-k_c \omega} \cos \{ k_c (1-i\omega) - \pi/4 \} d\omega
\]

(7.2.6)

where the constant \( C \) has been suitably modified. The trigonometric function can be replaced by exponential functions to give

\[
h_1(\nu) \sim \frac{4\pi u}{k^2 \varepsilon_x} \frac{e^{-i(k-x_0)}}{(1+\nu)^{\frac{1}{2}}} \left[ \frac{2}{\pi k_c} \right]^2 \int_0^\infty e^{-k_c \omega} \omega^{\frac{1}{2}} d\omega \]

\[
+ \frac{1}{2} e^{-i(k_x-x_0)} \int_0^\infty \omega^{\frac{1}{2}} e^{-k_c \omega} d\omega + \frac{1}{2} e^{i(k_x-x_0)} \int_0^\infty \omega^{\frac{1}{2}} e^{-k_c \omega} d\omega
\]

(7.2.7)

Hence

\[
h_1(\nu) \sim \frac{\omega^{\frac{1}{2}}}{k^4 \varepsilon_x^{\frac{1}{2}}} \frac{e^{-i(k-x_0)}}{(1+\nu)} \left[ \frac{2}{\pi k_c} \right]^2 \left[ \frac{\left( C - \cos(k_c - \pi/4) \right)}{2} + \frac{1}{2} \left( \frac{e^{-i(k_x-x_0)/2} + e^{i(k_x-x_0)/2}}{(1-c)^{\frac{1}{2}} (1+c)^{\frac{1}{2}}} \right) \right], \nu \in [0,1].
\]

(7.2.8)
For Case II \((0 \leq \varepsilon \leq 1/k)\) we replace \(J_0(1-i\omega)\) by the first term in its Taylor series, to give

\[
\tilde{h}_1(v) \sim \frac{k\lambda u_0}{\lambda^2} \frac{e^{i\frac{\pi}{4}}}{(1 + v)} \left[ \varepsilon \mathcal{C} - J_\varepsilon(\lambda) \right] \int_0^\infty e^{i(k\omega - \frac{\pi}{4})} w^2 \, dw + J_\varepsilon(\lambda) \int_0^\infty e^{i(k\omega + \frac{\pi}{4})} w^2 \, dw. \tag{7.2.9}
\]

Hence

\[
\tilde{h}_1(v) \sim \frac{k\lambda u_0}{\lambda^2} \frac{e^{i\frac{\pi}{4}}}{(1 + v)} \frac{v^2}{2}, \quad v \in [0, 1]. \tag{7.2.10}
\]

A different approach will be adopted for Case III \((1 - 1/k^2) \leq \varepsilon \leq 1\) and since this is considered in subsection 8.4 we only concern ourselves with Cases I and II, for the time being.

From results (7.2.8) and (7.2.10) we observe that \(\hat{h}_2, \hat{h}_3, \ldots\) are of increasing order of magnitude near \(v = 0\). Therefore a less direct form of iteration will have to be adopted, so that the trouble at the origin can be avoided. We let

\[
\hat{I}(v) = \left(1 + \frac{v}{\varepsilon}\right)^{\frac{v^2}{2}} \left( \int_0^\infty \frac{H_i(t)}{t - v} \, dt + j(v) \right), \tag{7.2.11}
\]

where \(H_i(t)\) will be specified shortly and \(j\) is a new unknown, and substitute expression (7.2.11) into (7.2.1) to obtain

\[
\hat{J}(v) = \hat{h}_1(v) - \left(1 + \frac{v}{\varepsilon}\right)^{\frac{v^2}{2}} \left( \int_0^\infty \frac{H_i(t)}{t - v} \, dt \right)
\]

\[
- \frac{1}{\pi} \left[ \int_0^1 \frac{(1 - \omega)}{1 + \omega} \left( \frac{(1 - \omega)}{\omega + \omega} \right) \left( \frac{(1 + \omega)}{\omega + \omega} \right) \left( \frac{(1 + \omega)}{\omega + \omega} \right) \right] \int_0^\infty \frac{H_i(t)}{t - \omega} \, dt + j(\omega) \right] \, dw. \tag{7.2.12}
\]
Now
\[
\int_0^\infty \frac{e^{-i\omega t}}{\omega} \frac{1}{(\omega+w)} dt = \int_0^\infty e^{2i\omega t} \frac{1}{\omega} dt = \int_0^\infty \frac{V^2}{\omega} d\omega \tag{7.2.13}
\]
and from Appendix B of [25] we have
\[
\int_0^\infty \frac{e^{-i\omega t}}{\omega} \frac{1}{(\omega+w)} dt = -\int_0^\infty e^{2i\omega t} \left\{ \frac{1}{\omega} \right\} dt.
\tag{7.2.14}
\]
On using this result (7.2.12) becomes
\[
\int \left( h_1(v) - \frac{1}{v} \right) \frac{1}{(v+w)} dt = \int_0^\infty e^{2i\omega t} \left\{ \frac{1}{\omega} \right\} dt
\]
\[
+ \int_0^\infty \frac{1}{t} \frac{h_1(v) e^{2i\omega t}}{(v+w)} dt = \int_0^\infty \frac{1}{(v+w)} \frac{V^2}{\omega} \int \left( h_1(v) e^{2i\omega t} \right) dt.
\tag{7.2.15}
\]
We choose \( h_1(t) \) so that
\[
\int_0^\infty h_1(t) e^{i\omega t} \left\{ \frac{1}{(t-v)} + \frac{1}{(t+v)} \right\} dt = \left( \frac{v}{v^2} \right) h_1(v) e^{i\omega v}.
\tag{7.2.16}
\]
Integral equation (7.2.16) has been considered by Jones ([34] and [35]) and its solution is
\[
h_1(y) = \frac{i}{4} e^{-iky} \int_0^\infty \left( \frac{x}{1+x} \right) h_1(x) e^{-ikx} M^{(2)}(x, y) dx,
\tag{7.2.17}
\]
where \( M^{(2)}(x, y) \) is defined as
\[
M^{(2)}(x, y) = \frac{4xy}{(x^2 - y^2)} h_1^{(2)}(-kx) J_0(-ky) - 4y^2 \frac{1}{k} h_1^{(2)}(-kx) J_0(-ky).
\tag{7.2.18}
\]
Hence, with this choice of \( H_1 \), (7.2.15) reduces to

\[
\mathcal{I}(v) = h_2(v) - \frac{1}{\pi} \int_0^1 \frac{(1-w)^{\nu} J(w)}{(w+v) e^{-2i\pi w}} d\omega, \quad (7.2.19)
\]

where

\[
h_2(v) = \int_0^\infty \frac{(t-1)^{\nu} H_1(t)}{(t+v)} e^{2i\pi t} dt. \quad (7.2.20)
\]

The value of \( v \) in the integral for \( h_2(v) \) causes no trouble since \( 0 \leq v \leq 1 \) and \( t \gg 1 \). The factor \( (t-1)^{\nu} \) suggests that \( h_2 \) will be smaller than \( H_1 \) by a factor of \( \frac{1}{2}\sqrt{k} \) and therefore smaller than \( h_1 \) provided that the difference between the orders of magnitude of \( h_1 \) and \( H_1 \) is not too large. This indicates that the above process reduces the size of the known term. We formulate the iteration scheme

\[
\mathcal{I} = \sum_{n=1}^{\infty} \mathcal{I}_n, \quad \text{where}
\]

\[
\mathcal{I}_n(v) = \left( \frac{1+v}{v} \right)^{\nu} \int_0^\infty \frac{H_n(t) e^{-2i\pi t}}{(t-v)} dt, \quad n \geq 1, \quad (7.2.21)
\]

\[
H_n(y) = \left\{ \begin{array}{ll}
\frac{4i\pi u_0}{\lambda^2} e^{i\pi y} [\mathcal{L}_n(\lambda) + \mathcal{J}_n(\lambda y)], & n = 0 \\
\frac{i}{4\pi} e^{i\pi y} \int_0^\infty \left( \frac{x}{1+x} \right)^{\nu} h_n(x) e^{-i\pi x} M^a(x,y) dx, & n \geq 1
\end{array} \right. \quad (7.2.22)
\]

and
\[ h_n(v) = \int_1^\infty \left( \frac{t-1}{t} \right)^2 \frac{H_{n-1}(t) e^{2i\lambda t}}{(t+v)} dt, \quad n \geq 1. \] (7.2.24)

A discussion of the error caused in stopping the iteration scheme will be found in the next subsection.

It is necessary to check that \( h_n \) is smaller, by a factor of \( O(k^{-1}) \) than \( h_{n-1} \) and similarly for \( H_n \). We assume that \( h_n(f) \) is regular for \( \Re f > -1 \) of the complex \( f \)-plane and that \( h_n(f) \leq M \), as \( |f| \to \infty \), in that region, where \( M \) is a positive constant.

We consider

\[ i \frac{e^{iky}}{k} y \int_{C_5} \left( \frac{f}{1+f} \right)^\frac{1}{2} h_n(f) e^{ikf} \left\{ \frac{\int H_1^{(1)}(kf) J_0(k\gamma) - y H_0^{(2)}(k\gamma) J_1(k\gamma)}{(f^2 - \gamma^2)} \right\} df, \]

where the contour \( C_5 \) is shown in figure 31.

\[ f = Re^{i\theta} \]

The contour is indented at the origin and below the simple pole at \( f = y \).

Figure 31
Hence

\[ H_n(y) + \frac{iK e^{k y}}{\pi} \lim_{R \to \infty} \int_0^{\pi/2} \left( \frac{R e^{i \theta}}{1 + R e^{i \theta}} \right)^{y/2} h_n(Re^{i \theta}) e^{-k R e^{i \theta}} \]

\[ \times \left\lbrace \frac{R e^{i \theta} H_1^{(1)}(k R e^{i \theta}) J_0(k_y) - y H_0^{(2)}(k R e^{i \theta}) J_1(k_y)}{(R^2 e^{i \theta} - y^2)} \right\rbrace - i \frac{R e^{i \theta}}{1 - i \chi} e^{R x} \left\lbrace \frac{-i \chi H_1^{(1)}(k x e^{\frac{i \pi}{2}}) J_0(k_y) - y H_0^{(2)}(k x e^{\frac{i \pi}{2}}) J_1(k_y)}{-(x^2 + y^2)} \right\rbrace \]

\[ = -\frac{\pi i}{\pi} \frac{iK e^{k y}}{\pi} \left( \frac{y}{1 + y} \right)^{y/2} h_n(y) e^{-k y} \left\lbrace H_1^{(1)}(-k y) J_0(k y) - H_0^{(2)}(-k y) J_1(k y) \right\rbrace. \]

The first integral on the left-hand side tends to zero as

\( R \) tends to infinity. In the second integral we replace the Hankel functions by modified Bessel functions, by using the relationship

\[ \pi i H_1^{(2)}(z e^{i \pi/2}) = -2 e^{\pi i/2} K_0(z). \]

The quantity on the right-hand side can be substantially modified by using the Wronskian relationship

\[ H_1^{(2)}(-k y) J_0(k y) - H_0^{(2)}(-k y) J_1(k y) = \frac{2i}{\pi k y}. \]

Therefore
where \(-\pi/2 \leq \text{arg}(1-ix) \leq 0\).

In order to evaluate (7.2.25) asymptotically we use the formula

\[
\int_0^\infty e^{-kx} \left\{ x^2 K_0(kx) J_0(kx) - y^2 K_0(kx) J_1(kx) \right\} dx,
\]

and we note that the main contribution comes from a neighbourhood of the origin, provided that \(h_n\) does not depend exponentially on \(k\). Thus, so long as \(ky \gg 1\), the integral in (7.2.25) is asymptotically

\[
\int_0^\infty \left\{ x^{1/2} \left[ h_n(o) + i x \left\{ 1/2 h_n(o) - h_n'(o) \right\} \right] \left\{ x y K_0(kx) J_0(kx) - y^2 K_0(kx) J_1(kx) \right\} e^{-kx} dx,
\]

when only the dominant terms are retained. Hence after using (7.2.26) we obtain

\[
H_n(y) = \frac{i}{\pi} \left( \frac{y^2}{1+y^2} \right)^{1/2} h_n(y) - \frac{15 h_n(o)}{2^{1/2} \pi^{1/2} \kappa^{3/2} y} \frac{e^{k y + \frac{y}{2}}}{2^{3/2} \pi^{1/2} \kappa^{3/2}} \left( 1 + O \left( \frac{1}{\kappa} \right) \right)
\]

\[
+ \frac{J_1(\kappa y)}{2^{1/2} \pi^{1/2} \kappa^{3/2}} \left[ h_n(o) + \frac{15}{16} \frac{h_n'(o) - h_n(o)}{\kappa} + O \left( \frac{1}{\kappa^2} \right) \right].
\]
when $k y \gg 1$.

For $k y \gg 1$, the Bessel functions in (7.2.7) can be replaced by their asymptotic expansions to give

\[ H_n(y) = \frac{i}{\pi} \left( \frac{y}{1+y} \right)^{\frac{1}{2}} h_n(y) - \frac{15 h_n(o) e^{iky + \frac{k^2}{y}}}{2^6 \pi k^3 y^{\frac{3}{2}}} \cos(\frac{k y - \pi}{4}) \]

\[ + \frac{e^{iky + \frac{k^2}{y}}}{2^{\frac{7}{2}} \pi k y^{\frac{3}{2}}} h_n(o) \left[ \cos(\frac{k y - 3\pi}{4}) - \frac{3}{8k y} \sin(\frac{k y - 3\pi}{4}) \right] \]

\[ + \frac{e^{iky + \frac{k^2}{y}}}{2^6 \pi k y^{\frac{3}{2}}} \frac{q y}{16k} \left[ \frac{1}{2} h_n(o) - h_n(o)^2 \right] \cos(\frac{k y - 3\pi}{4}) + O\left( \frac{h_n}{k^3} \right). \quad (7.2.28) \]

We let

\[ h_{n,0} = \frac{i}{8} \left[ h_n(o) + \frac{q y}{16k} \left( \frac{1}{2} h_n(o) - h_n(o)^2 \right) \right], \quad (7.2.29) \]

then

\[ H_n(y) = \frac{i}{\pi} \left( \frac{y}{1+y} \right)^{\frac{1}{2}} h_n(y) - \frac{1}{\pi k y^{\frac{1}{2}}} \left\{ \left( e^{2iky/y} - i \right) h_{n,0} + \frac{q h_n(o)(e^{2iky/y} + i)}{128 ky} \right\} + O\left( \frac{h_n}{k^3} \right). \quad (7.2.30) \]

It can be seen from (7.2.24) that $h_{n+1}(v)$ has those properties of $h_n(v)$ which were used in deducing (7.2.30). Hence there is a similar formula for $H_{n+1}$, since $h_1(v)$ has those properties it follows, by induction, that (7.2.30) is valid.
for all n. Also, the dominant contribution of (7.2.30) to $h_{n+1}(v)$, for large k is of $O(h_n, o(k^{-1}))$. Thus in the relevant range, $h_{n+1}$ is smaller than $h_n(v)$ by a factor of $O(k^{-1})$ and hence, by (7.2.30) $H_{n+1}(v)$ is smaller than $H_n(v)$ by a similar factor. Therefore the iteration scheme is suitable for large k.
7.3 Uniqueness

In this subsection the problem of the uniqueness of solutions of the integral equation for the sound-soft disc, (6.4.40), is discussed. The integral equation is

\[ \int_0^1 \omega K(\omega) \left( \frac{e^{iK(\omega)} + e^{iK(\omega-\nu)}}{(\omega+\nu)(\omega-\nu)} \right) d\omega = K^{(\omega)}(\nu), \]  

(7.3.1)

where \( K^{(\omega)}(\nu) \) is determined by the incident field. We have

\[ K(\omega) = -\frac{e^{iK\omega}}{\pi} \int_0^1 \left( \frac{1}{\nu^2} \right) K(\nu) \frac{e^{iK\nu}}{(\nu-\omega)} d\omega, \]  

(7.3.2)

and

\[ K(\nu) = K^{(\omega)}(\nu) - \frac{\nu^2 e^{iK\nu}}{\pi} \int_0^1 \left( \frac{1}{t^2} \right) K(t) \frac{e^{iKt}}{t+\nu} dt, \]  

(7.3.3)

from (7.1.6) and (7.1.11) respectively.

It is easy to show that if \( K \) is any bounded solution of (7.3.3), then \( K \) defined by (7.3.2) satisfies (7.3.1). However, from (7.1.1) any appropriate field gives a \( K \), i.e.

\[ K(\nu) = \int_0^1 \omega K(\omega) \frac{e^{iK(\omega-\nu)}}{(\omega-\nu)} d\omega, \]  

(7.3.4)

which is bounded and satisfies (7.3.3). Thus there is equivalence between (7.3.1) and (7.3.3) for the types of solution under consideration. Therefore it is sufficient to
show that (7.3.3) possesses a unique solution.

\[ K(\omega) \text{ is continuous in } 0 \leq \omega < 1, \text{ (7.3.4) shows that } K(v) \]

is bounded and continuous in \( 0 < v < 1 \). Near \( v = 0 \)

\[ K(v) = K(0) \int_0^\eta \frac{\omega d\omega}{\omega - v} + o(1), \quad (7.3.5) \]

where \( v << \eta << 1 \). Hence

\[ K(v) = K(0) v \log \left( \frac{\eta}{v} \right) + o(1), \quad (7.3.6) \]

and therefore \( K \) is bounded and continuous at \( v = 0 \). Near \( v = 1 \)

\[ K(\omega) = A (1 - \omega)^{\gamma} + o(1), \quad (7.3.7) \]

and so

\[ K(v) = A \int_0^1 \frac{(1 - \omega)^{\gamma}}{(1 - \omega - v)^{\gamma}} d\omega + o(1) = o(1), \quad (7.3.8) \]

since

\[ \int_0^1 (1 - \omega)^{\gamma} d\omega = \pi, \quad (7.3.9) \]

which differs from the given integral by \( o(1) \).

A solution of (7.3.3) is required in which \( K(v) \) is

continuous in \( 0 \leq v < 1 \). Suppose there are two such solutions of

(7.3.3). Then the difference \( K_0(v) \) is of the same type and

satisfies
\[ K_0(v) = -\frac{\nu_2^2 - i k v}{\nu (1 + \nu)^{\frac{1}{2}}} \int_0^1 \frac{(1-w)^{\frac{1}{2}}}{w^{\frac{1}{2}} (w+\nu)} K_0(w) e^{-ikw} dw, \quad \text{for } v \in (0, 1). \] (7.3.10)

Hence

\[ |K_0(v)| \leq \frac{\nu_2}{\nu (1 + \nu)^{\frac{1}{2}}} \max |K_0| \int_0^1 \frac{(1-w)^{\frac{1}{2}}}{w^{\frac{1}{2}} (w+\nu)} \]

\[ \leq \left\{ 1 - \left( \frac{\nu_0}{1 + \nu_0} \right)^{\frac{1}{2}} \right\} \max |K_0|, \] (7.3.11)

i.e. after using result (B8) from [25]. If \( \max |K_0(v)| \) occurs at \( v = v_0 \) we have

\[ \max |K_0| \leq \left\{ 1 - \left( \frac{v_0}{1 + v_0} \right)^{\frac{1}{2}} \right\} \max |K_0|. \] (7.3.12)

If \( \max |K_0| = 0 \), there is nothing to prove and if \( \max |K_0| \neq 0 \) we must have \( v_0 = 0 \). Now as \( v \to 0 \)

\[ K_0(v) = -\frac{\nu_2^2}{\nu} \int_0^{\frac{v_0}{\nu}} K_0(w) dw + o(1) \]

\[ = -\frac{\nu_2^2}{\nu} K_0(\xi) \int_0^{\frac{\nu_0}{\nu}} \frac{dw}{w^{\frac{1}{2}} (w+\nu)} + o(1), \] (7.3.13)

where \( 0 \leq \xi \leq v_0 \), by the mean value theorem, hence

\[ K_0(v) = -\frac{2}{\nu} K_0(\xi) \tan^{-1} \left( \frac{\nu}{\nu_0} \right)^{\frac{1}{2}} + o(1), \]

and

\[ K_0(v) + K_0(\xi) = o(1). \] (7.3.14)
It follows from (7.3.14) that \( K_0(0) = 0 \). Therefore \( \max |K_0| = 0 \), i.e. there is at most one solution of (7.3.3) continuous in \( 0 \leq v \leq 1 \) and uniqueness has been proved. The corresponding problem for the sound hard disc is dealt with in [33].

A similar method may be used to determine the error caused by stopping the iteration at a certain stage. We denote the result of the \( m \)th iteration by \( K_{m+1} \). Then, after the \( (n-1) \)th iteration the error \( K_n \) satisfies

\[
K_n(v) = \frac{\sqrt{2} \frac{1}{h_n(v)}}{(1+v)^{1/2}} e^{ikv} - \frac{\sqrt{2} \frac{1}{h_n(v)}}{\kappa (1+v)^{1/2}} \int_0^1 \frac{(1-\omega)^{1/2} K_0(\omega) e^{ik\omega} d\omega}{\omega^{1/2} (\omega+v)}
\]

(7.3.15)

where \( h_n \) is defined by (7.2.24). \( h_n \) is bounded and of \( O(h_n k^{1-n}) \) in \( (0,1) \). Therefore

\[
|K_n(v)| \leq 2^{1/2} 2^{1/2} |h_n(v)| + 2^{1/2} |h_n(v)| |\max |K_n| |
\]

(7.3.16)

If \( |K_n| \) has its maximum at \( v = v_n \neq 0 \),

\[
\max |K_n| \leq 2^{1/2} (1+v_n)^{1/2} |h_n(v_n)| = O \left( \frac{h_n}{k^{n-1}} \right)
\]

(7.3.17)

However, if \( |K_n| \) has its maximum at \( v = 0 \) then, as \( v \to 0 \),

\[
K_n(v) + K_n(v) = o(1)
\]

by an argument similar to that leading to
(7.3.14) and $K_n = 0$, i.e. the iteration has come to an exact termination. In both cases we can say that the error caused by stopping after the $(n-1)$th iteration is no worse than $O(h_{1,k}^{1-n})$. Since from (7.3.17) $\max|K_n| \leq \max|h_n|$ a more precise bound for the error can be found by using the specific form of $h_n$. 
From (7.1.25) and (2.3.18) we have
\[ \Psi_1(\omega) = H_0(\omega) e^{i\kappa \omega}. \] (7.4.1)

Therefore we can define a new function \( I_0(\nu) \) given by
\[ I_0(\nu) = (\frac{1 + \nu}{\nu})^{\nu} \int_0^\infty \frac{H_0(t) e^{i \kappa (t - \nu)}}{t - \nu} dt, \] (7.4.2)
and hence the condition on \( n \) in (7.2.21) can be replaced by
the condition \( n \geq 0 \), since definitions (7.2.21) and (7.4.2)
are of a similar form.

By using the definition of \( I_0(\nu) \), we can find an
expression for \( K(\nu) \) in terms of an infinite series of \( I_n(\nu) \)'s.
Hence, after reference to (7.1.12) and (7.1.32) it is seen
that
\[ K(\nu) = (\frac{\nu}{1 + \nu})^{\nu} \sum_{n=0}^\infty I_n(\nu), \] (7.4.3)
or alternatively from (7.2.21) and (7.4.2)
\[ K(\nu) = e^{i\kappa \nu} \sum_{n=0}^\infty \int_0^\infty \frac{H_n(t) e^{i \kappa t}}{(t - \nu)} dt. \] (7.4.4)

To determine \( K \), we insert (7.4.4) into (7.1.6) to obtain
\[ \omega K(\omega) = -\frac{1}{\pi} \left( \frac{\omega}{1 - \omega} \right)^{\nu} e^{i\kappa \omega} \int_0^\infty \frac{(1 - \nu)}{\nu} \frac{\nu}{(\nu - \omega)} \sum_{n=0}^\infty \int_0^\infty \frac{H_n(t) e^{i \kappa t}}{(t - \nu)} dt. \] (7.4.5)
We interchange the order of the summation and the integrations to give

\[ \omega \kappa(\omega) = \frac{1}{\pi^2} \left( \frac{\omega}{1-\omega} \right)^{1/2} e^{ik\omega} \sum_{n=0}^{\infty} \int_0^\infty H_n(t)e^{2ikt} dt \left( \frac{\omega}{1-\omega} \right)^{1/2} \frac{d\omega}{\sqrt{\omega^2 - \omega}} \right], \quad (7.4.6) \]

and on using a result from Appendix B of [25] we have

\[ \omega \kappa(\omega) = \frac{1}{\pi^2} \left( \frac{\omega}{1-\omega} \right)^{1/2} e^{ik\omega} \sum_{n=0}^{\infty} \int_0^\infty H_n(t)e^{2ikt} \left[ \frac{2}{\omega} \delta(t-\omega) + \frac{1}{\sqrt{\omega-\omega^2}} \frac{dH(t-\omega)}{dt} \right] dt, \]

i.e.

\[ \omega \kappa(\omega) = \sum_{n=0}^{\infty} \left\{ H_n(\omega)e^{ik\omega} + \frac{1}{\pi} \left( \frac{\omega}{1-\omega} \right)^{1/2} e^{ik\omega} \int_0^\infty \left( \frac{t-\omega}{1-\omega} \right) \frac{dH_n(t)e^{2ikt} dt}{(t-\omega)} \right\}. \quad (7.4.7) \]

The significance of (7.4.2) is that the terms corresponding to \( n = 0 \) in equations (7.4.4) and (7.4.7) have been obtained in the same way as the terms corresponding to \( n > 1 \). In the original derivation by Jones ([24] and [25]) they were obtained in a special way. The above modification is due to Thomas [23] and it is seen that this shortens the derivation of the iterative solution \( \kappa(\omega) \) given by (7.4.7).
Section 8

The angle of rotation, stress couple and applied torque

6.1 Definitions

We let $M^*$ be the stress couple holding the rigid disc fixed.

Then

$$M^* = -2\pi a^3 \varepsilon \omega t \int_0^a \rho^2 \alpha_{xz} \, d\rho,$$  \hspace{1cm} (8.1.1)

and hence from (6.3.6) we have

$$M^* = -2\pi \mu_c a^3 \varepsilon \omega t \int_0^a \rho^2 \sigma(\rho) \, d\rho.$$  \hspace{1cm} (8.1.2)

We integrate by parts once and we use (6.4.9) which leads to

$$M^* = -4\pi \mu_c a^3 \varepsilon \omega t \int_0^a K(\rho) \, d\rho.$$  \hspace{1cm} (8.1.3)

We let

$$M^* = a M \varepsilon \omega t,$$  \hspace{1cm} (8.1.4)

then

$$M = -4\pi \mu_c \int_0^a K(\rho) \, d\rho.$$  \hspace{1cm} (8.1.5)

However, we can replace (8.1.5) by

$$M = -8\pi \mu_c \lim_{\kappa \to 0} \frac{1}{\sin \phi} \int_0^a K(\rho) J_1(\kappa \rho \sin \phi) \, d\rho.$$  \hspace{1cm} (8.1.6)

At this stage it appears that by defining $M$ in this way we
Making the problem longer and more difficult. However, later on an expression for the scattering coefficient will be derived, and by evaluating \[ \int_0^\infty k(\rho) J_i(\kappa \rho \sin \phi) d\rho \] now, for \( 0 \leq \sin \phi \leq k^{-1} \) and \( 0 \leq \epsilon \leq k^{-1} \), we can obtain the scattering coefficient for Case II, with very little effort.

Expressions (8.1.5) and (8.1.6) contain an arbitrary constant since, from (7.1.17) and (7.1.28), we have

\[ k^{(0)}(\nu) = \frac{4i\kappa}{\lambda^2} \int_0^\infty \left[ J_0(\lambda \omega) - J_0(\lambda) \right] \left[ \frac{-i^k(\omega - \nu)}{(\omega - \nu)} + \frac{\epsilon - i^k(\omega + \nu)}{(\omega + \nu)} \right] d\omega. \]  

(8.1.7)

We let

\[ k(\omega) = \frac{4i\kappa}{\lambda^2} \left[ \frac{1}{\rho} - J_0(\rho) \right] f(\omega) + p(\omega), \]  

(8.1.8)

then

\[ \int_0^1 \omega f(\omega) \left[ \frac{-i^k(\omega - \nu)}{(\omega - \nu)} + \frac{\epsilon - i^k(\omega + \nu)}{(\omega + \nu)} \right] d\omega = F^{(0)}(\nu), \quad 0 \leq \nu \leq 1, \]  

(8.1.9)

and

\[ \int_0^1 \omega p(\omega) \left[ \frac{-i^k(\omega - \nu)}{(\omega - \nu)} + \frac{\epsilon - i^k(\omega + \nu)}{(\omega + \nu)} \right] d\omega = P^{(0)}(\nu), \quad 0 \leq \nu \leq 1, \]  

(8.1.10)

where

\[ F^{(0)}(\nu) = \int_0^\infty \omega \left[ \frac{-i^k(\omega - \nu)}{(\omega - \nu)} + \frac{\epsilon - i^k(\omega + \nu)}{(\omega + \nu)} \right] d\omega, \]  

(8.1.11)
and
\[
p^{(v)}(\omega) = \int_0^\infty J_0(\lambda\omega) \left[ e^{iK_0(\omega-v)} + e^{iK_0(\omega+v)} \right] d\omega. 
\] (8.1.12)

Hence it is obvious that \( f(\omega) \) and \( p(\omega) \) satisfy equations of similar form to (7.1.1) and (7.1.11). Therefore, we can set up an iteration scheme which consists of equations of similar form to (7.2.21) to (7.2.24), i.e.
\[
F_n(v) = (1 + v)^{\nu^2} \int_0^\infty F_n(t) e^{2iK(t-v)} dt, \quad n > 0, 
\] (8.1.13)

\[
P_n(v) = (1 + v)^{\nu^2} \int_0^\infty P_n(t) e^{2iK(t-v)} dt, \quad n > 0, 
\] (8.1.14)

where
\[
E_n(y) = \begin{cases} 
\gamma e^{iKy}, & n = 0, \\
\frac{iK}{4\pi} \int_0^\infty \left( \frac{x}{1+x} \right)^{\nu^2} e_n(x) e^{iKx} M^{(2)}(x,y) dx, & n \geq 1,
\end{cases}
\] (8.1.15)

\[
S_n(y) = \begin{cases} 
\gamma J_0(\lambda y) e^{iKy}, & n = 0, \\
\frac{iK}{4\pi} \int_0^\infty \left( \frac{x}{1+x} \right)^{\nu^2} S_n(x) e^{iKx} M^{(2)}(x,y) dx, & n \geq 1,
\end{cases}
\] (8.1.17)
and
\[ e_n(v) = \sum_{n=0}^{\infty} \left\{ E_n(t) e^{i\omega t} + \frac{1}{\pi} \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{E_n(t)}{t-\omega} e^{i\omega t} \, dt \right\}, \quad (8.1.19) \]

\[ s_n(v) = \sum_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{E_{n-1}(t) e^{i\omega t}}{(t+\omega)^{\frac{1}{2}}} \, dt \right\}, \quad (8.1.20) \]

Therefore we deduce from (7.4.7), (8.1.8) and the last few lines that
\[ \omega f(\omega) = \sum_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{E_n(t)}{t-\omega} e^{i\omega t} \, dt \right\}, \quad (8.1.21) \]

and
\[ \omega \rho(\omega) = \sum_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{s_n(t) e^{i\omega t}}{(t-\omega)^{\frac{1}{2}}} \, dt \right\}. \quad (8.1.22) \]

It is now convenient to evaluate the constant \( C \), appearing in (8.1.8). The edge condition assumed is that \( \lambda(\omega) \) is proportional to \((1-\omega)^{\frac{1}{2}}\), as \( \omega \to 1 \). Appendix B of [26] suggests that we are justified in assuming
\[ f(\omega) = \mu (1-\omega)^{\frac{1}{2}} + O \left\{ (1-\omega)^{\frac{3}{2}} \right\}, \quad (8.1.23) \]

and
\[ \rho(\omega) = \nu (1-\omega)^{\frac{1}{2}} + O \left\{ (1-\omega)^{\frac{3}{2}} \right\}, \quad (8.1.24) \]

where
\[ \mu = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} \frac{E_n(t) e^{i\omega t}}{(t-\omega)^{\frac{1}{2}}} \, dt, \quad (8.1.25) \]

and
\[ o = \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{2}} \]
and
\[
\nu = \sum_{n=0}^{\infty} \frac{e^{i k}}{n!} \int_{\frac{L}{2}}^{\infty} \frac{J_n(t) e^{-it}}{(t-1)^{\frac{3}{2}}} \, dt. \tag{6.1.26}
\]

In order to satisfy the edge condition we must have
\[
\{(-J_0(\lambda))\mu + \nu = 0. \tag{6.1.27}
\]

For convenience we let
\[
\Omega(\phi) = \int_{0}^{\phi} \omega J_1(\lambda \omega \sin \phi) \, d\omega, \tag{8.1.28}
\]

and
\[
\Phi(\phi) = \int_{0}^{\phi} \omega J_1(\lambda \omega \sin \phi) \, d\omega, \tag{8.1.29}
\]
so that (6.1.6), with the aid of (6.1.8) and (6.1.27) may be written as
\[
M = \frac{2i e^{i \phi}}{k} \lim_{\phi \to 0} \sin \phi \left\{ \left( \frac{\nu}{\mu} \right) \Omega(\phi) - \Phi(\phi) \right\}. \tag{8.1.30}
\]

We let the angle of rotation due to the incident field of an infinitesimal element of the solid at the origin be \( \Psi(t) \), which is given by
\[
\Psi(t) = u_0 e^{i \omega t}. \tag{8.1.31}
\]

Also, we let
\[
M e^{i \omega t} = \mu_c \Psi(t) (h_1 + i h_2); \tag{8.1.32}
\]
then
\[
\Psi(t) = \frac{M e^{i \omega t}}{\mu_c} (q_1 + i q_2), \tag{8.1.33}
\]
where
\[
g_1 = \frac{h_1}{(h_1^2 + h_2^2)} \tag{6.1.34}
\]
and
\[
g_2 = \frac{-h_2}{(h_1^2 + h_2^2)} \tag{6.1.35}
\]

If the stress couple is written as \( M \cos \omega t \), the real part of the angle of rotation is given by

\[
\Re \{ \Psi(t) \} = \frac{M}{\mu_c} \left( g_1 \cos \omega t - g_2 \sin \omega t \right)
\]

\[
= \frac{M}{\mu_c} \left( \frac{g_1^2 + g_2^2}{\sqrt{g_1^2 + g_2^2}} \cos (\omega t + \Delta) \right) \tag{6.1.36}
\]

where

\[
tan \; \Delta = -\frac{h_2}{h_1} = \frac{g_2}{g_1} \tag{6.1.37}
\]

\( \Delta \) is the phase angle between \( \Psi \) and \( M \).
8.3 Case I. \((\lambda = k\epsilon \gg 1, k(1-\epsilon) \gg 1)\)

In this subsection we consider the asymptotic evaluation of \(\Omega\) and \(\Phi\), having due regard to the restrictions of Case I.

First of all, let us evaluate \(e_1(\nu)\). Now from (8.1.15) and (8.1.19) we have

\[
e_1(\nu) = \int_{1}^{\infty} \frac{\nu^{1/2} e^{-it\nu}}{(t+\nu)} dt, \tag{8.2.1}
\]

which, on deforming the path of integration into the straight line joining 1 to 1-i\(\infty\), becomes

\[
e_1(\nu) = \int_{0}^{\infty} e^{iit\nu} \frac{\nu^{1/2} e^{-it\nu}}{(1-i \nu t + \nu)} dt. \tag{8.2.2}
\]

We expand the integrand as a power series in \(t\), to give

\[
e_1(\nu) \sim \int_{0}^{\infty} e^{iit\nu} \frac{\nu^{1/2} e^{-it\nu}}{1+i \nu t} \left( \frac{1}{(1+i \nu t)} - \frac{1}{2} \right) dt,
\]

\[
+ \frac{t^2}{8} \left( \frac{1}{2(1+i \nu t)} - \frac{1}{(1+i \nu t)^2} \right) dt,
\]

and hence

\[
e_1(\nu) = \int_{0}^{\infty} e^{iit\nu} \frac{\nu^{1/2} e^{-it\nu}}{2(1+i \nu t)^{3/2}} \left[ 1 + \frac{3i}{2i} \left\{ \frac{1}{(1+i \nu t)} - \frac{1}{2} \right\} \right.
\]

\[
+ \frac{15}{4i \nu^2} \left\{ \frac{1}{8} \left[ \frac{1}{2(1+i \nu t)} - \frac{1}{(1+i \nu t)^2} \right] + O\left( \frac{1}{\nu^3} \right) \right\} dt. \tag{8.2.4}
\]
It can be deduced from (3.2.4) that

$$e_{1}(t) = \frac{V_{2}}{\sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 1 + O\left( \frac{1}{P} \right) \right\}$$ \hspace{1cm} (3.2.5)

$$e'_{1}(t) = -\frac{V_{2}}{8 \sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 1 + O\left( \frac{1}{P} \right) \right\}$$ \hspace{1cm} (3.2.6)

$$e_{1}(0) = \frac{V_{2}}{2 \sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 1 + \frac{3}{4} \frac{1}{\sqrt{P}} \left( 45 - \frac{45}{4 \sqrt{P}} \right) \right\}$$ \hspace{1cm} (3.2.7)

$$e'_{1}(0) = -\frac{V_{2}}{2 \sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 1 + q_{1} \frac{1}{4 \sqrt{P}} + O\left( \frac{1}{P} \right) \right\}$$ \hspace{1cm} (3.2.8)

$$e''_{1}(0) = \frac{V_{2}}{2 \sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 2 + O\left( \frac{1}{P} \right) \right\}$$ \hspace{1cm} (3.2.9)

and in a manner analogous to $h_{n,0}$ (c.f. (7.2.20)) we define a function $e_{n,0}$ which, when $n=1$, is given by

$$e_{1,0} = \frac{V_{2}}{2 \sqrt{P}} e^{i \frac{3 \pi}{4} t} \left\{ 1 + \frac{5}{3 \sqrt{P}} \frac{1}{2 \sqrt{P}} \right\}$$ \hspace{1cm} (3.2.10)

In order to express $e_{n}$ in terms of $e_{n-1}$ for $n \geq 2$, we consider (6.1.19) and, after replacing $E_{n-1}(t)$ by an expression of similar form to (7.2.30), (i.e. we replace $H_{n}$ by $E_{n}$ and $h_{n}$ by $e_{n}$), we arrive at

$$e_{n}(u) = \int_{0}^{\infty} \int_{1}^{(t-u)} \left\{ e^{i \frac{3 \pi}{4} t} \left\{ 1 + \frac{1}{\sqrt{P}} \frac{1}{2 \sqrt{P}} \right\} \right\} dt$$

$$+ O\left( \frac{e_{n-1}}{P^{2}} \right)$$
or

\[ \varepsilon_n(v) = - \frac{1}{\pi} \int \frac{V}{(t+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt - Q \int \frac{V}{(t+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt - \frac{k}{128 \pi^2 \nu^2} \int \frac{V}{(t+\nu)^2} \left( \frac{1}{2} - \frac{V}{(t+\nu)^{1/2}} \right) e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt + O\left( \frac{\varepsilon_{n-1}}{\nu^{5/2}} \right). \]  

(8.2.11)

Those terms in the square brackets, containing no exponential factor, give rise to quantities of \( O\left( \varepsilon_{n-1}/\nu^{3/2} \right) \) and above, and have therefore been discarded.

After letting \( t = 1 + x^2 \) it can be shown that

\[ \int \frac{V}{(1+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt = - \frac{\pi}{\nu} \left\{ \frac{V}{(1+\nu)^{1/2}} \right\}^4, \]  

(8.2.12)

and

\[ \int \frac{V}{(1+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt = \frac{\pi}{\nu} \left\{ \frac{1}{2} + \frac{V}{(1+\nu)^{1/2}} \right\} . \]  

(8.2.13)

Also, by deforming the real axis from 1 to \( \infty \) into the straight line joining 1 to \( 1 - i \infty \), we have

\[ \int \frac{V}{(1+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt = e^{-2i\pi k - \frac{3i}{4}\nu} \int \frac{V}{(1+\nu)^{1/2}} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(1+i t) dt, \]  

(8.2.14)

and hence

\[ \int \frac{V}{(1+\nu)} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(t) dt = e^{-2i\pi k - \frac{3i}{4}\nu} \int \frac{V}{(1+\nu)^{1/2}} e^{-\frac{2i\pi k}{\nu} t} \varepsilon_{n-1}(1+i t) dt \]  

(8.2.15)

Therefore (8.2.4) becomes

\[ \varepsilon_n(v) = - \frac{2i\pi k}{8\pi^2 \nu^{3/2}} \varepsilon_{n-1}(1+i t) + \frac{\varepsilon_{n-1}}{\pi} \left\{ \frac{V}{(1+\nu)^{1/2}} \right\} - \frac{Q}{128 \pi^2 \nu^2} \left\{ \frac{1}{2} \right\} + O\left( \frac{\varepsilon_{n-1}}{\nu^{5/2}} \right). \]  

(8.2.16)
From (8.2.16) it can be deduced that

\[ e_n(t) = e_{n-1,0}(1 - 2^{1/2}) + \frac{e^{2i\beta - \frac{\pi}{4}}}{16\pi^2} e_n(t) - \frac{q e_{n-1}(0)}{128 \pi^2} \left( \frac{3}{2} - 2^{1/2} \right) + O(\frac{e_{n-1}}{\pi^{3/2}}), \]  

(8.2.17)

\[ e_n'(t) = -\frac{e_{n-1,0}}{2^{1/2}} \left( \frac{3}{2^{3/2}} - 1 \right) - \frac{e^{2i\beta - \frac{\pi}{4}}}{2^{3/2}} e_n(t) - \frac{q e_{n-1}(0)}{256 \pi^2} \left( \frac{7}{2} - 5 \right) + O(\frac{e_{n-1}}{\pi^{3/2}}), \]  

(8.2.18)

\[ e_n(0) = -\frac{e_{n-1,0}}{2^{1/2}} + \frac{e^{2i\beta - \frac{\pi}{4}}}{8\pi^2} e_n(t) - \frac{q e_{n-1}(0)}{1024 \pi^2} + O(\frac{e_{n-1}}{\pi^{3/2}}), \]  

(8.2.19)

\[ e_n'(0) = -\frac{e_{n-1,0}}{8^{1/2}} - \frac{e^{2i\beta - \frac{\pi}{4}}}{8^{3/2}} e_n(t) + \frac{q e_{n-1}(0)}{2048 \pi^2} + O(\frac{e_{n-1}}{\pi^{3/2}}), \]  

(8.2.20)

and

\[ e_{n,0} = -\frac{i e_{n-1,0}}{16 \pi^2} + \frac{e^{2i\beta + \frac{\pi}{4}}}{64 \pi^2} e_n(0) + O(\frac{e_{n-1}}{\pi^{3/2}}). \]  

(8.2.21)

Clearly we can replace \( e_n \) by \( e_n \) in expressions (8.2.17) to (8.2.21). The sign of the third term in (8.2.19) disagrees with the sign of the corresponding term in expression (61) of Jones [25].

The next step is to evaluate \( \mu \) for large \( k \). We let

\[ \mu = \sum_{n=0}^{\infty} \mu_n, \]  

(8.2.22)

and

\[ \mu_n = \frac{e^{i\beta}}{\pi} \int_{1}^{\infty} \frac{E_n(t) e^{2i\beta t}}{t^{3/2} (t-1)^{3/2}} dt. \]  

(8.2.23)
From (8.2.23) and (8.1.15) we have, for \( n = 0 \),

\[
\mu_0 = \frac{e^{ik\lambda}}{\pi} \int_0^{2\pi} e^{-i\lambda t} t^{n/2} dt,
\]

(8.2.24)

and on deforming the path of integration in the usual way, we obtain

\[
\mu_0 = \frac{e^{ik\lambda}}{\pi} \int_0^\infty e^{-k t} \frac{(1- i t)^{n/2}}{t^{n/2}} dt.
\]

(8.2.25)

For large \( k \) we write

\[
\mu_0 \sim \frac{e^{ik\lambda}}{\pi} \int_0^\infty e^{-k t} \frac{(1- i t)^{n/2}}{t^{n/2}} \left( \frac{1}{2} + \frac{t^2}{8} \right) dt,
\]

(8.2.26)

and hence

\[
\mu_0 = \frac{e^{ik\lambda}}{\pi^{n/2} k^{n/2}} \left[ 1 - \frac{i}{4k} + \frac{3}{32k^2} + \frac{15}{128k^3} + O\left(\frac{1}{k^4}\right) \right].
\]

(8.2.27)

Now consider (8.2.23) for \( n = 1 \). After replacing \( E_n(t) \) by an expression of similar form to (7.2.30), we have

\[
\mu_n = \frac{e^{ik\lambda}}{\pi} \int_0^\infty e^{-k t} \frac{(1- i t)^{n/2}}{t^{n/2}} \left[ \frac{1}{\pi} \left( \frac{t}{1+t} \right)^{n/2} \frac{\varepsilon_n(t) - \frac{1}{\pi} \frac{1}{(1+t)^{n/2}} \left\{ e^{2ikt} \varepsilon_n(0) + \frac{q}{128k^2} \varepsilon_n(0) / (e^{ik} + i) \right\} \right] dt,
\]

and after deforming the path of integration into the straight line joining 1 to 1+i\infty for those integrals whose integrands contain a factor of \( e^{-2ik\lambda} \), we arrive at

\[
\mu_n = \frac{e^{ik\lambda}}{\pi^2} \int_0^\infty e^{-2ik t} \frac{\varepsilon_n(1- i t) dt}{t^{n/2} (1- i t)^{n/2}} - \frac{1}{2} \frac{e^{ik\lambda}}{\pi^2} \int_0^\infty \frac{dt}{t} \frac{1}{k} \frac{\varepsilon_n(0) e^{ik\lambda}}{t^{n/2} (1- i t)^{n/2}} + O\left(\frac{1}{k^2}\right).
\]

(8.2.28)
On expanding $e_n(1-it)$ as a Taylor series the first integral can be written as

\[
\frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \int_0^\infty e^{2\pi t} dt = \frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \left[ e^{2\pi t} + \frac{1}{4} e^{2\pi t} \right] e^{-iK+\frac{\alpha^2}{4k}} \int_0^\infty e^{2\pi t} dt
\]

\[
= \frac{e^{-iK+\frac{\alpha^2}{4k}}}{2\pi^{1/2} k^{1/2}} \left[ e^{2\pi t} + \frac{1}{4} e^{2\pi t} \right] + \mathcal{O}\left(\frac{e_n}{k^{3/2}}\right)
\]

\[
= \frac{3}{32\pi^2} \left[ e^{2\pi t} + \frac{1}{4} e^{2\pi t} \right] + \mathcal{O}\left(\frac{e_n}{k^{3/2}}\right)
\]

(8.2.29)

By putting $v = -1$ in (8.2.12) and (8.2.13) we obtain

\[
\int_1^\infty \frac{dt}{t (t-1)^{1/2}} = \pi,
\]

and

\[
\int_1^\infty \frac{dt}{t^2 (t-1)^{1/2}} = \frac{\pi}{2}.
\]

For large $k$, we approximate the third integral on the right-hand side of (8.2.29) as

\[
\frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \int_0^\infty e^{-2\pi t} dt = \frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \left[ \frac{1}{2\pi^{1/2} k^{1/2}} \left( 1 + \frac{i}{4k} \right) + \mathcal{O}\left(\frac{e_n}{k^{3/2}}\right) \right].
\]

Therefore, we have

\[
\frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \int_0^\infty e^{-2\pi t} dt = \frac{e^{-iK+\frac{\alpha^2}{4k}}}{n_0} \left[ \frac{1}{2\pi^{1/2} k^{1/2}} \left( 1 + \frac{i}{4k} \right) + \mathcal{O}\left(\frac{e_n}{k^{3/2}}\right) \right].
\]

(8.2.30)
Similarly for the last integral it can be shown that

\[
- \frac{q}{128 \pi^2} \int_0^\infty \left( -\frac{2e^{i\pi^2}}{1 - 1t^4} + e^{i\pi^2} \right) \left( -\frac{q}{128 \pi^2} \varepsilon_\infty + O \left( \frac{1}{\pi^3} \right) \right).
\]  

With the aid of the last few results, (8.2.26) becomes

\[
\mu = e^{i\pi^2} \left[ e_{\infty} + \frac{i}{4} \left( \frac{1}{4} e_\infty \right) - e'(1) \right] \quad + \frac{q^2}{128 \pi^2} \varepsilon_\infty + O \left( \frac{1}{\pi^3} \right).
\]

Hence from (8.2.22) and (8.2.32) we have

\[
\mu = \mu_1 + e^{i\pi^2} \left[ e_{\infty} + \frac{i}{4} \left( \frac{1}{4} e_\infty \right) - e'(1) \right] \quad + \frac{q^2}{128 \pi^2} \varepsilon_\infty + O \left( \frac{1}{\pi^3} \right).
\]

From (8.2.17), (8.2.10) and (8.2.5)

\[
e_2(1) = \frac{i}{8 \pi^2} \left( 1 - 2^{1/2} \right) e^{i\pi^2} \left( \frac{1}{4} - \frac{1}{4} e^{i\pi^2} \right) \frac{2\pi}{16 \pi^2} \left( \frac{1}{4} - \frac{1}{4} e^{i\pi^2} \right) + O \left( \frac{1}{\pi^3} \right),
\]

and from (8.2.21) and (8.2.10)

\[
e_2(0) = \frac{1}{128 \pi^2} \left( \frac{1}{4} - \frac{1}{4} e^{i\pi^2} \right) + O \left( \frac{1}{\pi^3} \right).
\]

Hence from (8.2.33) to (8.2.35), (8.2.27) and (8.2.5) to
\[
\mu = \frac{e^{ik\cdot\frac{i}{2}}}{\sqrt[4]{2}} \left\{ 1 - \frac{i}{4} + \frac{3}{32} \frac{k^2}{k^2} + \frac{15i}{128} \frac{k^3}{k^3} \right\} + \frac{e^{ik+\frac{i}{2}}}{4} \cdot \frac{\sqrt[4]{2} e^{i\frac{k^2-2i\frac{k^2}{4}}{4}}}{4} \left[ 1 + \frac{i}{k} \left( 1 + \frac{i}{4} + \frac{3}{32} \frac{k^2}{k^2} \right) + \frac{1}{4} \cdot \frac{2i \cdot i}{4} \left( 1 - 2 \frac{i}{2} \right) \frac{i}{4} + \frac{e^{2i\frac{k^2}{4}}}{16} \right]
\]

which reduces to

\[
\mu = \frac{e^{ik\cdot\frac{i}{2}}}{\sqrt[4]{2}} \left\{ 1 - \frac{i}{4} - \frac{e^{2i\frac{k^2}{4}}}{8} \frac{k^2}{k^2} - \frac{1}{32} \frac{k^2}{k^2} + \frac{7e^{2i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} + \frac{e^{2i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} - \frac{e^{4i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} \right\} + O \left( \frac{k^4}{k^4} \right)
\]

\[(8.2.36)\]

It can be shown from this last result that

\[
\mu' = \frac{1}{2} e^{ik\cdot\frac{i}{2}} \left\{ 1 + \frac{i}{4} + \frac{e^{2i\frac{k^2}{4}}}{8} \frac{k^2}{k^2} - \frac{3}{32} \frac{k^2}{k^2} - \frac{15e^{2i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} - \frac{e^{2i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} - \frac{e^{4i\frac{k^2}{4}}}{128} \frac{k^3}{k^3} \right\} + O \left( \frac{k^4}{k^4} \right)
\]

\[(8.2.37)\]

The next quantity to be evaluated is \( \nu \). We let

\[
\nu = \sum_{n=0}^{\infty} \nu_n
\]

\[(8.2.38)\]

where

\[
\nu_n = \frac{e^{ik\cdot\frac{i}{2}}}{\sqrt[4]{2}} \left\{ \int_{t_{\frac{2}{4}}}^{t_{\frac{4}{4}}} S_n(t) \left( \frac{2i}{4} \right) e^{2i\frac{k^2}{4}} \, dt \right\}
\]

\[(8.2.39)\]

First of all let us consider \( s_1(v) \). From \((8.1.17)\) and
(8.1.20), we have

$$S_1(v) = \int_{-\infty}^{\infty} e^{i(k^2 t + t^2)} \int_0^{\infty} e^{-i\phi} \left( e^{-i\phi t} \frac{j_0(k_1 t)}{(t+\nu)} \right) dt, \quad (8.2.40)$$

and on deforming the path of integration as for $c_1(v)$, we arrive at

$$S_1(v) = \int_{-\infty}^{\infty} e^{i(k^2 t + t^2)} \int_0^{\infty} e^{-i\phi} \left( e^{-i\phi t} \frac{j_0(k_1 t)}{(t+\nu)} \right) dt. \quad (8.2.41)$$

Now for $k\Phi x >> 1$, (c.f. Watson [27])

$$I_0(k\Phi x) = \left( \frac{2}{\pi k\Phi x} \right)^{1/2} \left( 1 - \frac{9}{128 k^2 \Phi^2 x^2} \right) e^{k\Phi x - \pi/4}$$

$$+ \frac{1}{8 k^2 \Phi^2 x^2} \left( 1 - \frac{75}{128 k^2 \Phi^2 x^2} \right) \sin(k\Phi x - \pi/4) + O(1/k^4). \quad (8.2.42)$$

Hence, after replacing the Bessel function in (8.2.41) by its asymptotic representation, and then replacing the trigonometric functions by their exponential equivalents, we finally arrive at

$$S_1(v) = \int_{-\infty}^{\infty} e^{i(k^2 t + t^2)} \int_0^{\infty} e^{-i\phi} \left( e^{-i\phi t} \frac{j_0(k_1 t)}{(t+\nu)} \right) dt. \quad (8.2.43)$$

where only the dominant terms are retained. Now
\[ \int_{0}^{\infty} \frac{e^{-i(\pm \varepsilon)t}}{1 - i t + \nu} \, dt = \frac{1}{(1 + \nu)} \left[ \int_{0}^{\infty} \frac{e^{-i(\pm \varepsilon)t}}{(1 - i t + \nu)^{3/2}} \, dt \right] \]

\[ = \frac{\nu^{1/2}}{2(1 + \nu) \kappa^{3/2} (1 + \varepsilon)^{3/2}} \left\{ 1 + \frac{3i}{2(1 + \nu) \kappa (1 + \varepsilon)} + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right) \right\}, \quad (8.2.44) \]

and similarly

\[ \int_{0}^{\infty} \frac{e^{-i(\pm \varepsilon)t}}{1 - i t + \nu} \, dt = \frac{\nu^{1/2}}{2(1 + \nu) \kappa^{3/2} (1 + \varepsilon)^{3/2}} \left\{ 1 + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right) \right\}. \quad (8.2.45) \]

Hence

\[ S_{1}(\nu) = \frac{e^{-i(\pm \varepsilon)t} - 3i}{2^{3/2} (1 + \varepsilon)^{3/2}} \left[ \frac{e^{-i(\pm \varepsilon)t}}{\kappa^{3/2} (1 - \varepsilon)^{3/2}} \right] \left\{ 1 + \frac{3i}{2(1 + \nu) \kappa (1 + \varepsilon)} - \frac{i}{8\kappa \varepsilon} \right\} \]

\[ + \frac{e^{-i(\pm \varepsilon)t + \frac{\pi}{4}}}{\kappa^{3/2} (1 + \varepsilon)^{3/2}} \left\{ 1 + \frac{3i}{2(1 + \nu) \kappa (1 + \varepsilon)} + \frac{i}{8\kappa \varepsilon} \right\} + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right). \quad (8.2.46) \]

It can be deduced from (8.2.46) that

\[ S_{1}(1) = \frac{e^{-i(\pm \varepsilon)t} - 3i}{2^{3/2} (1 + \varepsilon)^{3/2}} \left[ \frac{e^{-i(\pm \varepsilon)t}}{\kappa^{3/2} (1 - \varepsilon)^{3/2}} \right] \left\{ 1 + \frac{3i}{4 \kappa (1 + \varepsilon)} - \frac{i}{8\kappa \varepsilon} \right\} \]

\[ + \frac{e^{-i(\pm \varepsilon)t + \frac{\pi}{4}}}{\kappa^{3/2} (1 + \varepsilon)^{3/2}} \left\{ 1 + \frac{3i}{4 \kappa (1 + \varepsilon)} + \frac{i}{8\kappa \varepsilon} \right\} + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right), \quad (8.2.47) \]

\[ S_{1}'(1) = -\frac{e^{-i(\pm \varepsilon)t} - 3i}{2^{3/2} (1 + \varepsilon)^{3/2}} \left[ \frac{e^{-i(\pm \varepsilon)t}}{\kappa^{3/2} (1 - \varepsilon)^{3/2}} + \frac{e^{-i(\pm \varepsilon)t + \frac{\pi}{4}}}{\kappa^{3/2} (1 + \varepsilon)^{3/2}} + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right) \right], \quad (8.2.48) \]

\[ S_{1}(0) = \frac{e^{-i(\pm \varepsilon)t} - 3i}{2^{3/2} (1 + \varepsilon)^{3/2}} \left[ \frac{e^{-i(\pm \varepsilon)t}}{\kappa^{3/2} (1 - \varepsilon)^{3/2}} \right] \left\{ 1 + \frac{3i}{2 \kappa (1 + \varepsilon)} - \frac{i}{8\kappa \varepsilon} \right\} \]

\[ + \frac{e^{-i(\pm \varepsilon)t + \frac{\pi}{4}}}{\kappa^{3/2} (1 + \varepsilon)^{3/2}} \left\{ 1 + \frac{3i}{2 \kappa (1 + \varepsilon)} + \frac{i}{8\kappa \varepsilon} \right\} + \mathcal{O}\left( \frac{1}{\kappa^{5/2}} \right), \quad (8.2.49) \]
\[ S'_{1}(0) = -\frac{e^{-\frac{\kappa}{2}(1 - i)^{1/2}}}{2^{1/2}(\kappa \xi)^{1/2}} \left[ \frac{e^{\frac{\kappa}{2}(1 - i)^{1/2}}}{\kappa^{1/2}(1 - i)^{1/2}} + \frac{e^{\frac{\kappa}{2}(1 + i)^{1/2}}}{\kappa^{1/2}(1 + i)^{1/2}} + O\left(\frac{1}{\kappa^{3/2}}\right) \right], \quad (8.2.50) \]

and

\[ S'_{1,0} = \frac{e^{-\frac{\kappa}{2}(1 + i)^{1/2}}}{2^{1/2}(\kappa \xi)^{1/2}} \left[ \frac{e^{\frac{\kappa}{2}(1 - i)^{1/2}}}{\kappa^{1/2}(1 - i)^{1/2}} \left\{ 1 + \frac{3i}{2\kappa(1 - i)} - \frac{i}{8\kappa i} + \frac{27i\xi}{32\kappa} \right\} \right] + O\left(\frac{1}{\kappa^{3/2}}\right). \quad (8.2.51) \]

From (8.2.39) and (8.1.17) we have, for \( n = 0 \),

\[ \nu_0 = \frac{\epsilon^{-\frac{\kappa}{2}}}{1} \int_0 \frac{e^{-\frac{\kappa}{2}t(1 + i)^{1/2}}}{(1 - t)^{1/2}} J_0(\kappa \xi t) \, dt, \quad (8.2.52) \]

and on deforming the path of integration in the usual way,

we obtain

\[ \nu_0 = \frac{\epsilon^{-\frac{\kappa}{2}}}{1} \int_0^\infty \frac{e^{-\frac{\kappa}{2}t(1 - i)^{1/2}}}{t^{1/2}} J_0(\kappa \xi(1 - it)) \, dt. \quad (8.2.53) \]

We replace the Bessel function by (8.2.42), to give

\[ \nu_0 = \frac{e^{-\frac{\kappa}{2}}}{1} \int_0^\infty \frac{e^{-\frac{\kappa}{2}t(1 - i)^{1/2}}}{t^{1/2}} \left[ \left(1 - \frac{9}{128\kappa^2 t^2(1 - it)^2}\right) \cos \left(\frac{\kappa \xi(1 - it)}{t^{1/4}}\right) \right] \, dt + O\left(\frac{1}{\kappa^{3/2}}\right). \quad (8.2.54) \]

Next, we express the trigonometric functions in terms of exponential functions, and after further approximation we arrive at
\[ \nu_0 = \frac{2^{1/2} e^{-3i/4}}{\pi^{1/2} (k \varepsilon)^{1/4}} \left[ e^{i k \varepsilon - \frac{3i}{4}} \right] \int_0^\infty e^{-\frac{3i}{8} k (1-\varepsilon)^2} \left( 1 - \frac{q_i t}{128 k e^2} - \frac{q_i k}{64 k e^2} - \frac{i t}{8 k e} \left( 1 - \frac{75}{128 k e^2} + i t - t^2 \right) \right) dt \]

\[ + e^{i k \varepsilon + \frac{3i}{4}} \int_0^\infty e^{-\frac{3i}{8} k (1+\varepsilon)^2} \left( 1 - \frac{q_i t}{128 k e^2} + \frac{q_i k}{64 k e^2} - \frac{i t}{8 k e} \left( 1 - \frac{75}{128 k e^2} + i t - t^2 \right) \right) dt + O \left( \frac{1}{k^{1/2}} \right) \]

(8.2.55)

and hence it can be shown that

\[ \nu_0 = \frac{2^{1/2} e^{-3i/4}}{2^{1/2} (k \varepsilon)^{1/4}} \left[ \frac{e^{ik\varepsilon - \frac{3i}{4}}}{\pi^{1/2} (1-\varepsilon)^{1/2}} \left\{ -i \frac{q_i}{128 k e^2} - \frac{q_i k}{64 k e^2} + \frac{1}{8 k e} \left( 1 - \frac{75}{128 k e^2} + i t - t^2 \right) \right\} + O \left( \frac{1}{k^{1/2}} \right) \right] \]

(8.2.56)

In (8.2.35) we can replace \( \mu, \mu_0 \) and \( e_n \) by \( \nu, \nu_0 \) and \( e_n \) respectively. From (8.2.47), (8.2.51) and (8.2.57)

\[ S_2(1) = \frac{e^{i k \varepsilon - \frac{3i}{4}}}{2^{3/2} (k \varepsilon)^{3/4}} \left[ e^{i k \varepsilon - \frac{3i}{4}} \left\{ \frac{-i}{8 k e} (1 - \varepsilon)^{1/2} + \frac{e^{2i k \varepsilon - \frac{3i}{4}}}{32 \varepsilon^{1/2} (k \varepsilon)^{3/2}} \right\} \right. \]

\[ + \left. e^{i k \varepsilon + \frac{3i}{4}} \left\{ i \frac{q_i}{8 k e} (1 - \varepsilon)^{1/2} + \frac{e^{2i k \varepsilon - \frac{3i}{4}}}{32 \varepsilon^{1/2} (k \varepsilon)^{3/2}} \right\} + O \left( \frac{1}{k^{1/2}} \right) \right] \]

(8.2.57)

and from (8.2.21) and (8.2.51)

\[ S_{2,0} = \frac{e^{i k \varepsilon - \frac{3i}{4}}}{2^{3/2} (k \varepsilon)^{3/4}} \frac{1}{128 k e} \left[ e^{i k \varepsilon - \frac{3i}{4}} + \frac{e^{2i k \varepsilon - \frac{3i}{4}}}{32 \varepsilon^{1/2} (k \varepsilon)^{3/2}} \right] + O \left( \frac{1}{k^{1/2}} \right) \]

(8.2.58)
We can now write down $V$. After reference to (8.2.55), we have from (8.2.47), (8.2.48), (8.2.51), (8.2.57), (8.2.49) and (8.2.53)

$$
V = \frac{-e^{\frac{\pi i}{4}}}{2^{\frac{1}{2}} \alpha (\alpha)^{\frac{1}{2}}} \left[ e^{\frac{i k \pi}{2}} \left( \frac{1}{(1-\epsilon)^{\frac{1}{2}}} \right)^{\frac{3}{2}} \left( 1 - i \right) - \frac{1}{2} \left( 1 - \frac{7}{4} \right) + \frac{1}{32 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}} \left[ \frac{7}{4} \frac{1}{2} + \frac{1}{32 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}} \right] \right] + \frac{7 \frac{1}{2}}{4 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}}
$$

which reduces to

$$
V = \frac{-e^{\frac{\pi i}{4}}}{2^{\frac{1}{2}} \alpha (\alpha)^{\frac{1}{2}}} \left[ e^{\frac{i k \pi}{2}} \left( \frac{1}{(1-\epsilon)^{\frac{1}{2}}} \right)^{\frac{3}{2}} \left( 1 - i \right) - \frac{1}{2 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}} \left[ \frac{7}{4} \frac{1}{2} + \frac{1}{32 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}} \right] \right] + \frac{7 \frac{1}{2}}{4 \alpha \alpha^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}}
$$
\[
\frac{32^{-1/2} \zeta^{1/2} \zeta (1 + \xi)}{4 \zeta^2 \zeta (1 + \xi)^2} \left( \frac{3}{2} + \frac{1}{2} \zeta^2 \zeta + \frac{7}{4} \zeta^3 \zeta \right) + \frac{2 i}{32} \left( \frac{75}{32 \zeta^2 \zeta^3} - \frac{9}{4 \zeta^2 \zeta^2 \zeta (1 + \xi)} \right)
\]

\[
- \frac{13}{4 \zeta \zeta (1 + \xi)} - \frac{1}{4 \zeta^2 \zeta (1 + \xi)^2} \left\{ \frac{4 i}{128 \zeta^2 \zeta^2 \zeta (1 + \xi)} \right\} + O \left( \frac{1}{\zeta^4} \right).
\]  

(8.2.59)

We let

\[
\Omega_\zeta (\phi) = \sum_{n=0}^{\infty} \Omega_\zeta (\phi);
\]  

(8.2.60)

then from (8.1.23) and (7.4.7) we deduce that

\[
\Omega_\zeta (\phi) = \int_0^{\infty} \zeta (\omega) e^{-i \omega \zeta \sin \phi} d\omega + \int_0^{\infty} e^{-i \omega \zeta \sin \phi} d\omega \int_1^{(t-1)^2} \frac{E_n(t) e^{-i \omega \zeta \sin \phi}}{(t-w)^2} dt.
\]  

(8.2.61)

It is necessary to consider two cases, when \( n = 0 \) and when \( n > 0 \).

First of all for \( n = 0 \), we have from (8.1.15)

\[
\Omega_0 (\phi) = \int_0^{\infty} J_0 (\omega \zeta \sin \phi) d\omega + \int_0^{\infty} e^{-i \omega \zeta \sin \phi} d\omega \int_1^{(t-1)^2} \frac{E_n(t) e^{-i \omega \zeta \sin \phi}}{(t-w)^2} dt.
\]  

(8.2.62)

The first integral is equal to

\[
\frac{1}{\zeta \sin \phi} \left\{ 1 - J_0 (\zeta \sin \phi) \right\}.
\]

For the inner integral in (8.2.62), we deform the path of integration into the straight line joining 1 to 1-\( i \infty \), so that

\[
\int_1^{(t-1)^2} \frac{E_n(t) e^{-i \omega \zeta \sin \phi}}{(t-w)^2} dt = \frac{e^{-i \omega \zeta \sin \phi}}{1 (t-w-i t)} \int_0^{\infty} \int_0^{\infty} \frac{e^{-i \omega \zeta \sin \phi}}{(1-w-i t)} dt
\]

\[
\sim e^{-i \omega \zeta \sin \phi} \int_0^{\infty} \left( 1 - \frac{1}{2} + \frac{2}{i t} \right) dt.
\]  

(8.2.63)
We cannot replace \( t \) by zero in the denominator, since \( 1 - \omega \)
vanishes at the edge of the disc \( \omega = 1 \). However, let us consider
\[
\int_0^\infty t^\beta e^{-xt} \, dt, \quad \text{for } \beta > -1 \text{ and } v > 0.
\]

Now
\[
\frac{1}{(v - it)} = i \int_0^\infty e^{iu(t+iv)} \, du,
\]
and on interchanging the order of integration we arrive at
\[
\int_0^\infty t^\beta e^{-xt} \, dt = i \int_0^\infty e^{iuv} \, du \int_0^\infty e^{(u+k)v} \, dv \int_0^\infty t^\beta \, dt
\]
\[
= i \Gamma(1+\beta) \int_0^\infty \frac{e^{iuv}}{(u+k)^{\beta+1}} \, du,
\]
which gives, after setting \( u + k = \alpha \),
\[
\int_0^\infty t^\beta e^{-xt} \, dt = i \Gamma(1+\beta) e^{k\alpha} \int_0^\infty \frac{e^{i\alpha v}}{\alpha^{\beta+1}} \, d\alpha.
\]

Therefore we let \( v = 1 - \omega \), and we let \( \beta \) take the values \( 1/2, 3/2 \) and \( 5/2 \), so that
\[
\int_0^\infty \frac{1}{2} e^{-x \sqrt{1 - \omega}} \, dt \sim \frac{1}{2} \int_0^\infty \frac{e^{-i(1-\omega)t}}{t^{3/2}} \left\{ 1 - \frac{3i}{4t} + \frac{15\gamma}{32t^2} \right\} \, dt.
\]
(8.2.65)
(8.2.62) becomes
\[
\omega_0(\phi) = \frac{1 - J_0(\sin \phi)}{2\pi \sqrt{1 - \omega^2}} + \frac{i}{\sqrt{1 - \omega}} \int_0^\infty \frac{I_0(k \omega \sin \phi) \sin \phi}{\omega^2 (1 - \omega)^2} \, d\omega \int_0^\infty \frac{e^{-i(1-\omega)t}}{t^{3/2}} \left\{ 1 - \frac{3i}{4t} + \frac{15\gamma}{32t^2} \right\} \, dt.
\]
(8.2.66)
and on inverting the order of integration we have

$$W_0(\phi) = \frac{1}{k \sin \phi} \int_{-\infty}^{\infty} e^{it} \left( \frac{1}{2\pi} \right)^{1/2} \left( 1 - \frac{3t}{4t - 32t^2} \right) \frac{e^{i\omega t}}{\omega} \frac{1}{\sqrt{1 - \omega^2}} d\omega.$$  \hfill (8.2.67)

We use the integral representation

$$J_n(z) = \frac{1}{\pi i} \int \frac{e^{izs}}{\cos s} \cos s ds,$$  \hfill (8.2.68)

so that we write the inner integral as

$$\int_{-\infty}^{\infty} e^{it} J_n(k \sin \phi \cos \omega) d\omega = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\cos \omega} \cos \omega d\omega.$$  \hfill (8.2.69)

on inverting the order of integration. Since $t > k$ and $0 \leq \sin \phi \leq k^{-1}$ we deform the path of integration into the straight lines joining 1 to $1 + i\infty$ and the origin to $i\infty$, to give

$$\int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 - \omega)^{1/2}} d\omega = e^{i(t + k \sin \phi \cos \omega)} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 + i\omega)^{1/2}} d\omega + e^{i(t + k \sin \phi \cos \omega)} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 - i\omega)^{1/2}} d\omega$$

$$= e^{i(t + k \sin \phi \cos \omega)} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 - \omega)^{1/2}} d\omega + e^{i(t + k \sin \phi \cos \omega)} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 + i\omega)^{1/2}} d\omega$$

$$- \frac{i}{2} \left( 1 - \frac{3\omega^2}{8} \right) \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 - \omega)^{1/2}} d\omega + e^{i(t + k \sin \phi \cos \omega)} \int_{-\infty}^{\infty} \frac{e^{i(t + k \sin \phi \cos \omega)}}{\omega^{1/2} (1 + i\omega)^{1/2}} d\omega$$

$$= e^{i(t + k \sin \phi \cos \omega)} \frac{\lambda^{1/2}}{(t + k \sin \phi \cos \omega)^{1/2}} \left[ 1 - \frac{i}{4(t + k \sin \phi \cos \omega)} - \frac{q}{32(t + k \sin \phi \cos \omega)^2} \right]$$

$$+ \frac{\lambda^{1/4}}{(t + k \sin \phi \cos \omega)^{1/2}}.$$  \hfill (8.2.70)
Hence, we write (6.2.67) as

\[\Omega_0(\phi) \sim \frac{1}{k \sin \phi} \left\{ 1 - \frac{e^{i \phi}}{2\pi} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} \right\} \]

\[\frac{e^{i t}}{k \sin \phi \cos \phi} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} \]

\[- \frac{3i}{4} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} + \frac{15}{32} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} \]

\[- \frac{9}{32} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} - \frac{3}{16} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} + \frac{3}{16} \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} \]

\[= \int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} \]

(8.2.71)

after discarding terms giving a contribution which is of higher order, or of the same order, as the error term.

By using elementary methods we show that

\[\int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} = \frac{2}{k^2 \sin \phi \cos \phi} \left\{ (1 + \sin \phi \cos \phi)^{1/2} - 1 \right\} \]

(8.2.72)

\[\int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} = \frac{2}{k^2 \sin \phi \cos \phi} \left\{ (1 + \sin \phi \cos \phi)^{1/2} + \frac{1}{(1 + \sin \phi \cos \phi)^{1/2}} - 2 \right\} \]

(8.2.73)

\[\int_0^\infty \frac{dt}{t^{3/2} (t + k \sin \phi \cos \phi)^{1/2}} = \frac{2}{3k^2 \sin \phi \cos \phi} \left\{ (1 + \sin \phi \cos \phi)^{1/2} - 3(1 + \sin \phi \cos \phi)^{1/2} + 2 \right\} \]

(8.2.74)
\[
\int_{k}^{\infty} \frac{dt}{t^{3/2}(t+K\sin \phi \cos u)^{2}} = \frac{2}{15K^{3} \sin \phi \cos u} \left\{ 3(1+\sin \phi \cos u)^{3/2} - 10(1+\sin \phi \cos u)^{1/2} + 15(1+\sin \phi \cos u)^{1/2} - 8 \right\}, \\
(8.2.75)
\]
\[
\int_{k}^{\infty} \frac{dt}{t^{3/2}(t+K\sin \phi \cos u)^{2}} = \frac{2}{3K^{3} \sin \phi \cos u} \left\{ (1+\sin \phi \cos u)^{3/2} - 6(1+\sin \phi \cos u)^{1/2} - \frac{3}{(1+\sin \phi \cos u)^{1/2}} + 8 \right\}, \\
(8.2.76)
\]
\[
\int_{k}^{\infty} \frac{dt}{t^{5/2}(t+K\sin \phi \cos u)^{2}} = \frac{2}{3K^{3} \sin \phi \cos u} \left\{ (1+\sin \phi \cos u)^{3/2} - 6(1+\sin \phi \cos u)^{1/2} - \frac{3}{(1+\sin \phi \cos u)^{1/2}} + 8 \right\}, \\
(8.2.77)
\]

and for the final integral in (8.2.71) we deform the path of integration into the straight line joining \( k \) to \( k - ik\infty \) to give
\[
\int_{k}^{\infty} \frac{\dot{e}^{it} dt}{t^{3/2}(t+K\sin \phi \cos u)^{2}} = -\frac{i}{k} \int_{k}^{\infty} \frac{\dot{e}^{kt} dt}{(1-it)^{3/2}(1+\sin \phi \cos u - it)^{1/2}} \\
\sim -\frac{i}{k^2} \frac{\dot{e}^{ik}}{(1+\sin \phi \cos u)^{1/2}}. \\
(8.2.78)
\]

On using these last few results (8.2.71) becomes
\[
\Omega_{\phi}(\phi) \propto \left\{ \frac{1 - j_{0}(K\sin \phi)}{K \sin \phi} \right\} + \frac{3j_{-1}(K\sin \phi)}{4K} + \frac{1}{K} \int_{0}^{\infty} \frac{\dot{e}^{iK\sin \phi \cos u}}{\sin \phi \cos u} \left\{ (1+\sin \phi \cos u)^{1/2} - j_{1}(K\sin \phi \cos u) \right\} du \\
- \frac{i}{4K^{2}} \int_{0}^{\infty} \frac{\dot{e}^{iK\sin \phi \cos u}}{\sin \phi \cos u} \left\{ (1+\sin \phi \cos u)^{3/2} + \frac{1}{(1+\sin \phi \cos u)^{1/2}} - 2 \right\} du \\
- \frac{i}{4K^{2}} \int_{0}^{\infty} \frac{\dot{e}^{iK\sin \phi \cos u}}{\sin \phi \cos u} \left\{ (1+\sin \phi \cos u)^{3/2} - 3(1+\sin \phi \cos u)^{1/2} + 2 \right\} du \\
+ \frac{1}{32K^{3}} \int_{0}^{\infty} \frac{\dot{e}^{iK\sin \phi \cos u}}{\sin \phi \cos u} \left\{ 3(1+\sin \phi \cos u)^{3/2} - 10(1+\sin \phi \cos u)^{1/2} + 15(1+\sin \phi \cos u)^{1/2} - 8 \right\} du.
\]
The next step is to evaluate asymptotically the integrals in (8.2.79) whose integrands contain an exponential factor.

The integrand is expanded as a power series in $\sin \phi$, which is a valid step since $0 \leq \sin \phi \leq k^{-1}$. For the first integral we have

\[
- \frac{3}{32 k^2} \int \frac{e^{ik \sin \phi \cot \theta}}{\sin^3 \phi \cot^3 \theta} \left\{ 3(1 + \sin \phi \cot \theta)^{\frac{1}{2}} + \frac{6}{(1 + \sin \phi \cot \theta)^{\frac{1}{2}}} - \frac{1}{(1 + \sin \phi \cot \theta)^{\frac{3}{2}}} - 8 \right\} d\theta
\]

\[
- \frac{1}{16 k^2} \int \frac{e^{ik \sin \phi \cot \theta}}{\sin^3 \phi \cot^3 \theta} \left\{ 3(1 + \sin \phi \cot \theta)^{\frac{1}{2}} - 6(1 + \sin \phi \cot \theta)^{\frac{3}{2}} + 3 \right\} d\theta
\]

\[
+ \frac{e^{-ik \phi \cot \theta}}{2k^2} \int \frac{\cot \theta d\theta}{(1 + \sin \phi \cot \theta)^{\frac{1}{2}}}
\]

(8.2.79)

and hence from the integral representation for $J_n(z)$ given by

(8.2.68), we have

\[
\int \frac{e^{ik \sin \phi \cot \theta}}{\sin \phi \cot \theta} \left\{ (1 + \sin \phi \cot \theta)^{\frac{1}{2}} - 1 \right\} d\theta = \frac{\pi}{2} \int_0^\infty \frac{J_1(k \sin \phi) - \pi \sin \phi \int J_0(k \sin \phi) - \frac{1}{16} J_2(k \sin \phi)}{2}
\]

\[+ \frac{\pi \sin^2 \phi \int \frac{3}{8} J_1(k \sin \phi) - \frac{1}{16} J_2(k \sin \phi)}{64} + O(\sin^3 \phi).
\]

(8.2.30)

Similarly
\[
\int_{e^{\pi i \sin \phi \cos \theta}} \left\{ (1+\sin \phi \cos \theta)^{\frac{1}{2}} - \frac{1}{1+\sin \phi \cos \theta} \right\} d\phi 
\approx \int_{e^{\pi i \sin \phi \cos \theta}} \left\{ \frac{1}{4} \right\} d\phi 
\]

\[
= \frac{\pi}{4} \left( J_1(k \sin \phi) - \frac{1}{8} \sin \phi \left( J_0(k \sin \phi) - J_2(k \sin \phi) \right) \right),
\]

\[\text{(8.2.81)}\]

\[
\int_{e^{\pi i \sin \phi \cos \theta}} \left\{ (1+\sin \phi \cos \theta)^{\frac{1}{2}} - 3(1+\sin \phi \cos \theta) + 2 \right\} d\phi 
\approx \int_{e^{\pi i \sin \phi \cos \theta}} \left\{ \frac{3}{4} - \frac{1}{4} \sin \phi \right\} d\phi 
\]

\[
= \frac{3\pi}{4} \left( J_1(k \sin \phi) - \frac{1}{8} \sin \phi \left( J_0(k \sin \phi) - J_2(k \sin \phi) \right) \right),
\]

\[\text{(8.2.82)}\]

\[
\int_{e^{\pi i \sin \phi \cos \theta}} \left\{ 3(1+\sin \phi \cos \theta) - 10(1+\sin \phi \cos \theta)^{\frac{3}{2}} + 15(1+\sin \phi \cos \theta)^{\frac{5}{2}} - 5 \right\} d\phi 
\approx \frac{5\pi}{4} J_1(k \sin \phi),
\]

\[\text{(8.2.83)}\]

\[
\int_{e^{\pi i \sin \phi \cos \theta}} \left\{ 3(1+\sin \phi \cos \theta)^{\frac{3}{2}} + \frac{6}{(1+\sin \phi \cos \theta)^{\frac{3}{2}}} - \frac{1}{(1+\sin \phi \cos \theta)^{\frac{5}{2}}} - \frac{8}{(1+\sin \phi \cos \theta)^{\frac{7}{2}}} \right\} d\phi 
\approx \frac{\pi}{2} J_1(k \sin \phi),
\]

\[\text{(8.2.84)}\]

and finally

\[
\int_{e^{\pi i \sin \phi \cos \theta}} \left\{ (1+\sin \phi \cos \theta)^{\frac{3}{2}} - 6(1+\sin \phi \cos \theta)^{\frac{5}{2}} - \frac{3}{(1+\sin \phi \cos \theta)^{\frac{3}{2}}} + 8 \right\} d\phi 
\approx \frac{\pi}{2} J_1(k \sin \phi),
\]

\[\text{(8.2.85)}\]
For the integral whose integrand does not contain an exponential factor, we use 3.7.14 from Erdélyi [36], i.e.

\[ P_{\nu}^{m}(z) = \frac{\Gamma(\nu+m+1)}{\pi \Gamma(\nu+1)} \int_0^\infty \left( z^{2} + (z^2-1)^{2} \right)^{\nu-\delta} \sin \left( \frac{\pi}{2} z \right) \cos \left( \frac{\pi}{2} t \right) dt, \quad \Re z > 0, \quad (8.2.86) \]

so that

\[ \int_0^\infty \frac{\cos \nu du}{(1+\sin \phi \cos u)^{\nu+1}} = \frac{2\pi}{\cos^{\nu/2} \phi} P_{\nu/2}^{1} (\sec \phi). \quad (8.2.87) \]

For \( \phi \approx 0 \) result (4) of section 3.9.2 of [16] shows that

\[ P_{\nu/2}^{1} (\sec \phi) \approx \sin \phi. \quad (8.2.88) \]

Hence on using these results (8.2.79) becomes

\[ \Omega_{\nu}(\phi) = \frac{1 - J_0(k \sin \phi)}{k \sin \phi} + \frac{\pi}{2k} \int \left[ i J_2(k \sin \phi) + \frac{1}{k} \left( k J_2(k \sin \phi) + k^2 J_0(k \sin \phi) \right) \right] \]

\[ + \frac{i}{32 k^2} \left( k^2 J_0(k \sin \phi) + k J_2(k \sin \phi) \right) \]

\[ \quad + \frac{\pi}{k^2 \cos^{\nu/2} \phi} P_{\nu/2}^{1} (\sec \phi) + O \left( \frac{1}{k^4} \right). \quad (8.2.89) \]

We next consider the first part of (8.2.61), for \( n \geq 1 \). Now

\[ \int_0^\infty e^{i k \omega} J_0(k \sin \phi) d\omega = \int_0^{\infty} E_n(\omega) e^{i k \omega} J_0(k \sin \phi) d\omega \]

\[ - \int_0^{\infty} E_n(\omega) e^{i k \omega} J_1(k \sin \phi) d\omega. \quad (8.2.90) \]
It is obvious from (7.2.16) and subsection 8.1 that we can write

\[ \int_{0}^{\infty} E(\omega) e^{i k \omega (-i k (\omega - \chi) + e^{-i k (\omega + \chi) \frac{\chi}{\omega + \chi}}) \omega = \left( \frac{\chi}{1+\chi} \right)^{\frac{1}{2}} E_n(\chi) e^{i k \chi}. \]  

(8.2.91)

We multiply both sides by \( J_1(kx \sin \phi) / x \), and integrate with respect to \( x \) from 0 to \( \infty \), so that after interchanging the order of integration on the left-hand side we have

\[ \int_{0}^{\infty} E(\omega) e^{i k \omega} \left\{ \int_{0}^{\infty} \frac{e^{-i k \chi}}{(\omega - \chi) x} J_1(kx \sin \phi) \, d\chi + \int_{0}^{\infty} \frac{e^{-i k \chi}}{(\omega + \chi) x} J_1(kx \sin \phi) \, d\chi \right\} \, d\omega \]

\[ = \int_{0}^{\infty} \frac{E_n(\chi) e^{i k \chi} J_1(kx \sin \phi) \, d\chi}{x^{1/2}(1+\chi)^{1/2}}, \quad |\sin \phi| < 1. \]  

(8.2.92)

For the second integral in the brackets \( \{ \} \) we put \( x = -x \), to give

\[ \int_{0}^{\infty} \frac{e^{-i k \chi}}{(\omega - \chi) x} J_1(kx \sin \phi) \, d\chi + \int_{-\infty}^{0} \frac{e^{-i k \chi}}{(\omega + \chi) x} J_1(kx \sin \phi) \, d\chi \]

\[ = \int_{-\infty}^{0} \frac{e^{-i k \chi}}{(\omega + \chi) x} J_1(kx \sin \phi) \, d\chi. \]  

(8.2.93)

We close the contour of integration by using a semi-circle which occupies that part of the complex \( x \)-plane for which \( 0 \leq \text{arg}x < \pi \). We indent the contour of integration above the simple pole at \( x = \omega \). The only contribution to the value of the
integral comes from the simple pole at $x = \omega$. Hence for $\sin \phi < 1$ we have

$$\int_{-\infty}^{\infty} e^{-ikx} J_1(kx \sin \phi) dx = -\frac{ie^{-ik\omega}}{\omega} J_1(k\omega \sin \phi), \quad (8.2.91)$$

and on using this result (8.2.92) becomes

$$\int_0^\infty E_n(\omega) e^{-ik\omega} J_1(k\omega \sin \phi) d\omega = \frac{i}{\pi} \int_0^1 \frac{E_n(x) e^{-ikx} J_1(kx \sin \phi)}{x^{1/2} (1+x)^{1/2}} dx. \quad (8.2.95)$$

This result has been obtained by Jones [25], but the alternative derivation given here, due to W. E. Williams, is simpler and shorter.

In

$$\int_0^\infty E_n(\omega) e^{-ik\omega} J_1(k\omega \sin \phi) d\omega,$$

we replace $E_n$ by an expression of similar form to (7.2.30) to give

$$\int_0^\infty E_n(\omega) e^{-ik\omega} J_1(k\omega \sin \phi) d\omega = \int_{-\infty}^{\infty} J_1(kx \sin \phi) \left( \frac{i}{\pi} \frac{1}{x} \right) \left( \frac{V \sin \phi}{x^{1/2} (1+x)^{1/2}} \right) e^{-ikx} e^{-ik\omega} \frac{dx}{x^{1/2}}. \quad (8.2.96)$$

On combining (8.2.95) and (8.2.95) we obtain

$$\int_0^\infty E_n(\omega) e^{-ik\omega} J_1(k\omega \sin \phi) d\omega = \frac{i}{\pi} \int_0^1 \frac{e^{-ikx} J_1(kx \sin \phi) dx}{x^{1/2} (1+x)^{1/2}} + \int_{-\infty}^{\infty} e^{-ikx} e^{-ik\omega} \frac{dx}{x^{1/2}} \int_0^\infty J_1(kx \sin \phi) dx \quad (8.2.97)$$
In the right-hand side of (8.2.97) we replace the Bessel function by the integral representation (8.2.68), so that after inverting the order of integration we have

\[
\int_0^\infty \frac{F_n(x) \cdot e^{-ikw} \cdot \int f(k \omega \sin \phi) \, d\omega}{\omega} = \frac{1}{\pi} \int_0^\infty \left[ i \int_0^\infty \frac{F_n(x)}{x^{1/2} (1 + x)^{1/2}} \, dx \right] \, du
\]

\[
+ \frac{e^{-ik \omega} - i e^{-ik \omega} \int_0^\infty \frac{d\omega}{x^{1/2} (1 + x)^{1/2}}}{1 + x}
\]

We deform the path of integration of the first of the inner integrals into the straight line joining 1 to 1-i\infty, and the negative imaginary axis to give

\[
\int_0^\infty \frac{F_n(x) \cdot e^{-ik(1-\sin \phi \cos \omega)} \cdot \int f(1-\sin \phi \cos \omega) \, d\omega}{x^{1/2} (1 + x)^{1/2}} \, dx
\]

\[
= \frac{\pi}{2} \int_0^\infty \frac{F_n(x)}{x^{1/2} (1 + x)^{1/2}} \, dx
\]

\[
+ \frac{e^{-ik \omega} - i e^{-ik \omega} \int_0^\infty \frac{d\omega}{x^{1/2} (1 + x)^{1/2}}}{1 + x}
\]

(8.2.98)
Also
\[
\int e^{-i k (1 + \sin \phi \cos \theta)} \frac{d\alpha}{\sqrt{x}} = -i e^{-i k (1 + \sin \phi \cos \theta)} \int e^{-i k (1 + \sin \phi \cos \theta)} \frac{d\alpha}{(1+i \alpha)^{3/2}}
\]
\[
= -i e^{-i k (1 + \sin \phi \cos \theta)} \frac{e^{-i k (1 + \sin \phi \cos \theta)}}{k (1 + \sin \phi \cos \theta)}.
\]  \hspace{1cm} (8.2.100)

and
\[
\int e^{-i k (1 - \sin \phi \cos \theta)} \frac{d\alpha}{\sqrt{x}} \sim -i e^{-i k (1 - \sin \phi \cos \theta)} \frac{e^{-i k (1 - \sin \phi \cos \theta)}}{k (1 - \sin \phi \cos \theta)}.
\]  \hspace{1cm} (8.2.101)

Hence on using results (8.2.99) to (8.2.101), (8.2.98) becomes
\[
\int \mathcal{K}(\omega) e^{-i k \omega} \frac{d\omega}{\omega} = \frac{\mathcal{K}(0)}{k} \left[ e_n(0) \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1+\sin \phi \cos \theta)^{3/2}}
\]
\[
+ \frac{i}{2k} \left\{ \frac{1}{2} e_n(0) - e_n'(0) \right\} \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1-\sin \phi \cos \theta)^{3/2}} \right] + \frac{i}{2k} \left\{ \frac{1}{2} e_n(0) - e_n'(0) \right\} \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1-\sin \phi \cos \theta)^{3/2}}
\]
\[
+ \frac{i}{k} \left\{ \frac{3}{4} e_n(0) - e_n'(0) \right\} \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1+\sin \phi \cos \theta)^{3/2}} + \frac{i}{k} \left\{ \frac{1}{4} e_n(0) - e_n'(0) \right\} \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1+\sin \phi \cos \theta)^{3/2}}
\]
\[
+ i e^{-i k} \int e \frac{\cos \phi \cos \theta \, d\phi \, d\theta}{(1+\sin \phi \cos \theta)^{3/2}} + O \left( \frac{e_n}{k^2} \right).
\]  \hspace{1cm} (8.2.102)

since \( e_n \) can be replaced by \( i e_n(0)/8 \) to the order, to which (8.2.102) is accurate.

Now, we consider the second integral of (8.2.61) for \( n \approx 1 \).
We can use an expression for $E_n(t)$ of similar form to (7.2.30), for the asymptotic evaluation of this integral, since only the range $t \geq 1$ is involved. Hence the asymptotic evaluation of

$$
\frac{1}{\pi \sqrt{1 - \omega^2}} \int_{0}^{\infty} \frac{e^{ikw} J_1(kw \sin \phi) dw}{(t - i\omega)^{\frac{3}{2}}} \left[ \frac{4}{\pi} \left( \frac{\frac{1}{2}}{1 + t} \right) e^{-\frac{1}{2}} e(t) \right]
$$

$$
- \frac{1}{\pi k t^{\frac{3}{2}}} \left[ (e^{i/k t} - i) e^{-\omega t} + \frac{q e(t)}{128 k t} (e^{i/k t} + i) \right] dt,
$$

is required. As before we replace the Bessel function to obtain

$$
\int_{0}^{\infty} \frac{e^{ikw} J_1(kw \sin \phi) dw}{(t - i\omega)^{\frac{3}{2}}} = \frac{1}{\pi t} \int_{0}^{\infty} \frac{e^{k(1 + \sin \phi \cos \omega)} dw}{(t - i\omega)^{\frac{3}{2}}} + \frac{1}{\pi k t^{\frac{3}{2}}} \left[ (e^{i/k t} - i) e^{-\omega t} + \frac{q e(t)}{128 k t} (e^{i/k t} + i) \right] dt,
$$

(8.2.103)

On deforming the path of integration in the usual way we arrive at

$$
\int_{0}^{\infty} \frac{e^{k(1 + \sin \phi \cos \omega)} dw}{(t - i\omega)^{\frac{3}{2}}} = e^{i/k t} \int_{0}^{\infty} \frac{e^{-k(1 + \sin \phi \cos \omega)} dw}{(t - i\omega)^{\frac{3}{2}}}
$$

$$
+ e^{i/k t} \int_{0}^{\infty} \frac{e^{-k(1 + \sin \phi \cos \omega)} dw}{(t - i\omega)^{\frac{3}{2}}} (1 - \frac{i}{2}) dw.
$$

(8.2.104)

Now from (8.2.84)

$$
\int_{0}^{\infty} \frac{e^{k(1 + \sin \phi \cos \omega)} dw}{(t - i\omega)^{\frac{3}{2}}} = i \sqrt{2 \pi} e^{i/k t} \int_{x^{\frac{1}{2}}}^{\infty} \frac{e^{ikx}}{x^{\frac{1}{2}}} dx, \quad \phi(1 + \sin \phi \cos \omega),
$$

(8.2.105)
and
\[ \int_0^\infty \frac{e^{-k(1+\sin \phi \cos \omega)w}}{\omega^{1/2} (t-1-i\omega)} (1-\frac{1}{2}i\omega) d\omega = i \kappa e^{-\frac{k}{2} (1+\sin \phi \cos \omega) (t-1)} \int_0^\infty e^{\frac{i(t-1)x}{x^{1/2}}} \left(1 - \frac{ix}{x}ight) dx. \]

(8.2.107)

Consequently on using (8.2.105) to (8.2.107), the expression under consideration is asymptotic to

\[ \frac{i}{\kappa} \int_0^\infty \cos u \ du \int_0^x \frac{e^{ix}}{x^{1/2}} \left[ i \frac{e^{-k(1+\sin \phi \cos \omega)u + i\phi}}{2\sqrt{\pi} e^{1/2} u} \right] dt, \]

when only the terms in the square brackets of (8.2.103), which do not involve \( e^{2ikt} \), are taken. We deform the path of integration of the inner integral into the straight line joining 1 to 1-i\omega,

so that it becomes equal to

\[ -i \frac{k}{2} \int_0^\infty e^{-i\pi + k(1-\sin \phi \cos \omega)u} \left[ i \frac{e^{-k(1+\sin \phi \cos \omega)u + i\phi}}{2\sqrt{\pi} e^{1/2} u} \right] dt, \]

\[ + \frac{i}{\pi k (1-ikt)} \left\{ \epsilon_n - \left( \frac{q \epsilon_n (0)}{128 k (1-ikt)} \right) \right\} dt, \]

\[ \sim -i \frac{k}{2} \int_0^\infty e^{-i\pi + k(1-\sin \phi \cos \omega)u} \left[ i \frac{e^{-k(1+\sin \phi \cos \omega)u + i\phi}}{2\sqrt{\pi} e^{1/2} u} \right] \]

\[ + \frac{i}{2\sqrt{2}} \left\{ \frac{1}{4} \epsilon_n (1) - \epsilon_n' (1) \right\} + \frac{1}{i k} \epsilon_n, dt, \]
Hence that part of (8.2.103) under consideration is asymptotic to

\[
\frac{\xi^{-\frac{3}{2}}}{2^{3/2}} \left[ e^{\xi^{1/2}} \left( \frac{2\xi^{1/2}}{\xi^{1/2} + \xi^{1/2}} \right) + \frac{\xi^{1/2}}{2} \right] - \frac{\xi^{1/2}}{2} \left( \frac{2\xi^{1/2}}{\xi^{1/2} + \xi^{1/2}} \right) + \frac{\xi^{1/2}}{2} \left( \frac{2\xi^{1/2}}{\xi^{1/2} + \xi^{1/2}} \right)
\]

after discarding terms not required.

The integrals in the square brackets of (8.2.109) are of similar form to the integrals in the square brackets of (8.2.71), which have already been evaluated. Therefore by similar reasoning

\[
\int_0^\infty \frac{d\alpha}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} = \frac{2}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} \left[ 1 - \frac{1}{2} \left( \xi + \xi^{1/2} \right)^2 \right],
\]

\[
\int_0^\infty \frac{d\alpha}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} = \frac{2}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} \left[ 1 - \frac{1}{2} \left( \xi + \xi^{1/2} \right)^2 \right],
\]

\[
\int_0^\infty \frac{d\alpha}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} = \frac{2}{\xi^{1/2} \left( \xi + \xi^{1/2} \right)^2} \left[ 1 - \frac{1}{2} \left( \xi + \xi^{1/2} \right)^2 \right],
\]

Hence (8.2.109) is asymptotic to
\[
\begin{align*}
&\frac{e^{-ikx}}{2^{1/2}} \frac{\sqrt{\pi}}{\kappa} \int_{0}^{\infty} \frac{e^{i\sqrt{\kappa} u \cos \omega}}{1 - \sin \phi \cos \omega} \left[ e_{\kappa}(1) \left( 1 - \frac{1}{2^{1/2}} (1 + \sin \phi \cos \omega)^{1/2} \right)^{2} - \frac{i}{\kappa} e_{\kappa}(1) \frac{1}{\sqrt{\kappa}} (1 + \sin \phi \cos \omega)^{1/2} \right. \\
&+ \frac{2^{1/2}}{(1 + \sin \phi \cos \omega)^{1/2}} \left[ 2 - \frac{3}{2} (1 + \sin \phi \cos \omega)^{1/2} \right] \\
&\left. \left( 1 - \frac{1}{2^{1/2}} (1 + \sin \phi \cos \omega)^{1/2} \right)^{2} \right] \, du, \\
\end{align*}
\]

(8.2.113)

after replacing \( e_{\kappa,0} \) by \( i e_{\kappa,0}/\kappa \).

The remaining part of (8.2.103) is
\[
\begin{align*}
&\frac{i}{\pi^{3/2}} \frac{\sqrt{\pi}}{\kappa} \int_{0}^{\infty} \frac{e^{i\sqrt{\kappa} u \cos \omega}}{w^{1/2} (1 - w)^{1/2}} \, dw \int_{t}^{\infty} \left( \frac{t - w}{t} \right)^{1/2} \left[ e_{\kappa,0} + \frac{q e_{\kappa}(e)}{128 \kappa t} \right] \, dt. \\
\end{align*}
\] (8.2.114)

For \( w \leq 1 \) we have
\[
\int_{t}^{\infty} \left( \frac{t - w}{t} \right)^{1/2} \, dt = \frac{\pi}{\sqrt{t}} \left( 1 - (1 - w)^{1/2} \right),
\] (8.2.115)

and
\[
\int_{t}^{\infty} \left( \frac{t - w}{t} \right)^{1/2} \, dt = \frac{\pi}{\sqrt{t}} \left\{ 1 - \frac{1}{2} w - (1 - w)^{1/2} \right\}.
\] (8.2.116)

Hence (8.2.114) becomes
\[
\begin{align*}
&\frac{i}{\pi^{3/2}} \frac{\sqrt{\pi}}{\kappa} \int_{0}^{\infty} \frac{e^{i\sqrt{\kappa} u \cos \omega}}{w^{1/2} \left( 1 - w \right)^{1/2}} \left\{ 1 - \left( \frac{1}{2} w - (1 - w)^{1/2} \right) \right\} \, dw \\
&\quad + \frac{q e_{\kappa}(e)}{128 \kappa} \int_{0}^{\infty} \frac{e^{i\sqrt{\kappa} u \cos \omega} \omega^{1/2} (1 - \omega)^{1/2}}{w^{1/2} (1 - w)^{1/2}} \left\{ 1 - \frac{1}{2} \omega - (1 - \omega)^{1/2} \right\} \, dw \, du. \\
\end{align*}
\] (8.2.117)

By deforming the path of integration in the usual way, we have
for the first of the inner integrals

\[ \int_0^\infty \frac{e^{iK(1+\sin \phi \cos \mu)\omega}}{\omega^{3/2}} \left\{ \frac{1}{(1-\omega)^{1/2}} - 1 \right\} d\omega = \int_0^\infty \frac{e^{iK(1+\sin \phi \cos \mu)\omega}}{\omega^{3/2}} \left\{ \frac{1}{(1+i\omega)^{1/2}} - 1 \right\} d\omega \]

\[ + \frac{-\pi i}{e^{iK(1+\sin \phi \cos \mu)\omega}} \int_0^\infty \frac{1}{(1+i\omega)^{1/2}} d\omega \]

\[ \sim e^{iK(1+\sin \phi \cos \mu)\omega} \int_0^\infty \frac{1}{(1+i\omega)^{1/2}} d\omega \]

\[ = \frac{\sqrt{\pi}}{e^{iK(1+\sin \phi \cos \mu)\omega}} \left[ \frac{\pi}{4K(1+\sin \phi \cos \mu)^{1/2}} \right] + \frac{1}{2} \]

(8.2.118)

Similarly

\[ \int_0^\infty \frac{e^{iK(1+\sin \phi \cos \mu)\omega}}{\omega^{3/2}} \left\{ \frac{1-1}{2} \right\} d\omega = \int_0^\infty \frac{e^{iK(1+\sin \phi \cos \mu)\omega}}{\omega^{3/2}} \left\{ \frac{1-1}{2} \right\} d\omega \]

\[ - \frac{1}{2} (1+i\omega) \frac{1-1}{2} d\omega \]

\[ \sim \frac{\sqrt{\pi}}{e^{iK(1+\sin \phi \cos \mu)\omega}} \frac{1}{2K(1+\sin \phi \cos \mu)^{1/2}} \]

(8.2.119)

On using (8.2.118) and (8.2.119), (8.2.117) is equal to

\[ \frac{\sqrt{\pi}}{e^{iK(1+\sin \phi \cos \mu)\omega}} \left[ e^{-\frac{i\pi}{2}} \right] \int_0^\infty \frac{\cos \mu}{\mu^{3/2}} \left( \frac{e^{iK(1+\sin \phi \cos \mu)\omega}}{e^{iK(1+\sin \phi \cos \mu)\omega}} \right) \left\{ 1 - \frac{3i}{4\pi} \frac{\mu^{1/2}}{\mu^{1/2}} \right\} d\omega \]

\[ - \frac{1}{\pi^{1/2} K^{1/2}} \left( \frac{1}{\mu^{1/2}} \right) ^2 + \frac{i}{256 \pi} \left( \frac{e^{-iK(1+\sin \phi \cos \mu)\omega}}{e^{iK(1+\sin \phi \cos \mu)\omega}} \right) d\omega \]

(8.2.120)
Therefore from (8.2.102), (8.2.113) and (8.2.120) it is seen that, after some cancellation

\[
\psi_n(y) = -\frac{\xi_n}{\pi^{3/2}} \left[ e^{(o)} \int_0^{\pi} \frac{\cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} + \frac{1}{2} e^{(o)} \int_0^{\pi} \frac{\cos u \, du}{(1 - \sin \phi \cos u)^{1/2}} \right] e^{-i(\phi - \phi_n y)}
\]

\[
- \frac{i}{16} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(o)} \int_0^{\pi} \frac{\cos u \, du}{(1 + \sin \phi \cos u)^{3/2}} \right] + \frac{1}{2} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{\cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} \right]
\]

\[
+ \frac{i}{16} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} \right] + \frac{1}{2} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 + \sin \phi \cos u)^{3/2}} \right]
\]

\[
+ \frac{1}{16} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} \right] + \frac{1}{2} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 + \sin \phi \cos u)^{3/2}} \right]
\]

\[
+ \frac{3}{32} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} \right] + \frac{9}{256} \frac{\xi_n}{\pi^{3/2}} \left[ e^{(1)} \int_0^{\pi} \frac{i \sin \phi \cos u \, du}{(1 + \sin \phi \cos u)^{3/2}} \right] + O(\frac{\xi_n}{\pi^{3/2}})
\]

(8.2.121)

On letting \( u = \pi - t \), it can be shown that

\[
\int_{-\pi}^{\pi} \frac{\cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} = -\int_{-\pi}^{\pi} \frac{\cos u \, du}{(1 + \sin \phi \cos u)^{3/2}}
\]

(8.2.122)

Therefore (c.f. 3.7.14 of [36])

\[
\int_{-\pi}^{\pi} \frac{\cos u \, du}{(1 - \sin \phi \cos u)^{3/2}} = -2\pi \frac{p' (\sec \phi)}{\cos^{3/2} \phi} \left( \sec \phi \right)
\]

(8.2.123)
and  
\[
\int_0^{\infty} \frac{\cos u}{\left(1 - \sin \phi \cos u\right)^{3/2}} du = \frac{2\pi}{\cos \phi} \psi'\left(\sec \phi\right).
\]  
(8.2.124)

Also  
\[
\int_0^{\infty} e^{-i \sin \phi \cos u} \left(1 + \sin \phi \cos u\right)^{3/2} du \sim \int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{3}{2} \sin \phi \cos u\right] du
\]
\[
= \pi \int_0^{\infty} J_1(-k \sin \phi) + \frac{3}{4} \sin \phi \left[J_2(k \sin \phi) - J_2(k \sin \phi)\right],
\]  
(8.2.125)

\[
\int_0^{\infty} e^{-i \sin \phi \cos u} \left(1 + \sin \phi \cos u\right)^{3/2} du \sim \int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{2}{(1 + \sin \phi \cos u)^{1/2}}\right] du
\]
\[
= \frac{3}{2} \int_0^{\infty} e^{-i \sin \phi \cos u} du = \frac{3\pi}{2} J_1(k \sin \phi),
\]  
(8.2.126)

\[
\int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{3}{2} \sin \phi \cos u\right] du \sim \int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{3}{2} \frac{1}{2}\left(1 + \sin \phi \cos u\right)^{1/2}\right] du
\]
\[
= \frac{5}{2} \int_0^{\infty} e^{-i \sin \phi \cos u} du = \frac{5\pi}{2} J_1(k \sin \phi),
\]  
(8.2.127)

\[
\int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{3}{2} \sin \phi \cos u\right] du \sim \int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 - \frac{1}{2} \sin \phi \cos u\right] du = \pi J_1(k \sin \phi) + \frac{3}{4} \sin \phi \left[J_2(k \sin \phi) - J_2(k \sin \phi)\right],
\]  
(8.2.128)

and  
\[
\int_0^{\infty} e^{-i \sin \phi \cos u} \left[1 + \frac{3}{2} \sin \phi \cos u\right] du \sim \int_0^{\infty} e^{-i \sin \phi \cos u} du = \pi J_1(k \sin \phi).
\]  
(8.2.129)

Hence on using results (8.2.122) to (8.2.129) and on replacing  
\[e_n^0\]  
by  
\[i e_n^0(c)/\delta\]  
in (8.2.121), we obtain
\[
\begin{align*}
\Omega_n(\phi) &= -\frac{2e^{\frac{1}{2}\phi}}{\pi^2 2^n \eta^2 \cos^2 \phi} \left[ e_n(0) \left[ 1 + \frac{i}{16k} \frac{\text{sec} \phi}{2k \cos \phi} \right] - \frac{i}{16k} \left( 5 e_n(0) - e_n(0) \right) \frac{\text{sec} \phi}{2k \cos \phi} \right] \\
&\quad - \frac{e^{\frac{1}{2}\phi}}{2\pi^2 \phi^2} \left[ e_n(1) J_1(\phi \sin \phi) + \frac{i}{4k} \left( 5 e_n(1) - 5 e_n(1) \right) \frac{\phi \sin \phi}{2k \cos \phi} \right] \\
&\quad - 3 e_n(1) \frac{\phi \sin \phi}{8k^2 \phi^2} \left[ J_0(\phi \sin \phi) - J_2(\phi \sin \phi) \right] \\
&\quad - \frac{i}{4k} \left( 3 e_n(0) J_0(\phi \sin \phi) - e_n(0) J_0(\phi \sin \phi) \right) J_0(\phi \sin \phi) \left[ J_2(\phi \sin \phi) \right] + O\left( \frac{e_n}{k^2} \right).
\end{align*}
\]

Therefore from (8.2.130) and (8.2.89) we have

\[
\Omega(\phi) = \Omega_\phi(\phi) - \frac{2e^{\frac{1}{2}\phi}}{\pi^2 2^n \eta^2 \cos^2 \phi} e_n(0) \frac{\text{sec} \phi}{2k \cos \phi} \\
- \frac{e^{\frac{1}{2}\phi}}{2\pi^2 \phi^2} e_n(1) J_1(\phi \sin \phi) - \frac{i}{4k} e_n(0) \frac{\phi \sin \phi}{2k \cos \phi} J_0(\phi \sin \phi) + O\left( \frac{1}{k^2} \right).
\]

After substituting for \( e_1(0) \) and \( e_1(1) \) from (8.2.7) and (8.2.5) respectively, we obtain

\[
\begin{align*}
\Omega(\phi) &= \left[ 1 - \frac{J_0(\phi \sin \phi)}{\phi \sin \phi} + \frac{e^{\phi}}{2k} \right] J_0(\phi \sin \phi) - \frac{i}{8k} \left( 2 J_1(\phi \sin \phi) - \phi \sin \phi \right) J_1(\phi \sin \phi) + O\left( \frac{1}{k^2} \right) \\
&\quad + \frac{i e^{\phi}}{4k^2} \left( 2 J_0(\phi \sin \phi) - \phi \sin \phi \right) J_0(\phi \sin \phi) - \frac{i}{32k^2} \left( 4 J_1(\phi \sin \phi) - \phi \sin \phi \right) J_1(\phi \sin \phi) \\
&\quad - k^2 \left( 3 J_1(\phi \sin \phi) - J_1(\phi \sin \phi) \right) + O\left( \frac{1}{k^2} \right).
\end{align*}
\]
since the coefficient of $\frac{1}{\varrho^2} (\sec \phi)$ is identically zero.

It now remains to evaluate $\Phi_n(\phi)$. When $k\varphi/\lambda > 1$ there is no point in evaluating $\Phi_n(\phi)$, for $0 < \sin \phi < k^{-1}$ (c.f. 9.1).

Therefore from (8.1.29), (7.4.7) and (6.1.17) we consider,

for $n = 0$

$$2 \lim_{k \phi \to 0} \Phi_n(\phi) = \int_0^1 \omega J_0(\lambda \omega) d\omega + \int \{ \frac{\lambda}{\omega} \} e^{\frac{k}{\lambda} \omega} \int_0^\infty \frac{e^{i k t} J_0(k \omega t)}{1 - e^{i \omega}} d\omega,$$

(8.2.133)

where

$$\Phi(\phi) = \sum_{n=0}^{\infty} \Phi_n(\phi).$$

(8.2.134)

The first integral is equal to

$$\frac{1}{k} \frac{\omega}{\omega} \int \frac{\omega}{J_1(k \omega)} d\omega,$$

whereas, for the second integral we deform the path of integration of the inner integral in the usual way, so that

$$\int \frac{\omega}{J_1(k \omega)} d\omega = \sum_{n=0}^{\infty} \frac{e^{i n \omega}}{n} J_0(k \omega),$$

(8.2.135)

On replacing the Bessel function by the first few terms in its asymptotic expansion we have

$$\int \frac{\omega}{J_1(k \omega)} d\omega \approx \frac{\omega}{k} \int \frac{\omega}{J_1(k \omega)} d\omega - \frac{\omega}{k} \int \frac{\omega}{J_1(k \omega)} d\omega \int_0^\infty \left[ 1 - \frac{\omega}{12 \pi \omega (1-\omega)} \right] J_0(k \omega) d\omega.$$

(8.2.136)
which, after further approximation and also replacing the trigonometric functions by exponential functions, can be written as

\[
\left\{ \frac{\sqrt{2\pi}}{2^{\frac{3}{2}}} \int_{0}^{\infty} e^{-\frac{1}{2}(1-\omega)^2} d\omega \right\}^{\frac{3}{2}} \int_{0}^{\infty} e^{-\frac{1}{2}(1+\omega)^2} d\omega
\]

and also replacing the trigonometric functions by exponential functions, can be written as

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1-\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1+\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

We use (8.2.64) to obtain

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1-\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1+\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

Therefore, with the aid of this last result and after interchanging the order of integration (8.2.135) becomes

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1-\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1+\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

We deform the path of integration of the inner integrals, so that

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1-\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]

\[
\int_{0}^{\infty} e^{-\frac{1}{2}(1+\omega)^2} d\omega = \frac{\sqrt{\pi}}{2}
\]
\[
\frac{y_2}{r_2} \left( \frac{i(t + \varepsilon)}{(t + \varepsilon)^2} \right)^{-\frac{3}{4}} \left[ 1 + \frac{i}{4(t + k\varepsilon)} + \frac{3}{2(t + k\varepsilon)^2} \right] + \frac{e^{\frac{3\pi i}{4}}}{2(t + k\varepsilon)^{\frac{3}{2}}} = (8.2.140)
\]

Hence
\[
\lim_{k \to 0} \frac{\Phi_k(\phi)}{\sin \phi} \approx \frac{1}{-k\varepsilon} \left[ \frac{e^{-\frac{3\pi i}{4}}}{2\sqrt{2}(t - k\varepsilon)^{\frac{3}{2}}} \right] \left[ \left( 1 - \frac{i}{8k\varepsilon} - \frac{q}{128k^2\varepsilon^2} \right) \right]^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}}
\]

\[
+ \frac{i}{4} \left( 1 - \frac{i}{8k\varepsilon} \right) \int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}} + \frac{3}{32} \int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}} + \frac{3}{16k\varepsilon} \int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}}
\]

\[
+ \frac{1}{2} e^{-\frac{3\pi i}{4}} \int_{k(1+\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t - k\varepsilon)^{\frac{3}{2}}} + \frac{1}{2} e^{-\frac{3\pi i}{4}} \int_{k(1+\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t - k\varepsilon)^{\frac{3}{2}}}
\]

after discarding terms which do not make a significant contribution.

By using elementary methods we show that
\[
\int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}} = -\frac{2}{k\varepsilon} \left[ \frac{1}{(1-\varepsilon)^{\frac{3}{2}}} \right], \quad (8.2.142)
\]

\[
\int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t - k\varepsilon)^{\frac{3}{2}}} = -\frac{2}{k\varepsilon} \left[ 2 - \frac{1}{(1-\varepsilon)^{\frac{3}{2}}} \right], \quad (8.2.143)
\]

\[
\int_{k(1-\varepsilon)}^{\infty} \frac{dt}{\sqrt{2}(t + k\varepsilon)^{\frac{3}{2}}} = -\frac{2}{3k\varepsilon} \left[ 8 + \frac{3}{2} - \frac{1}{(1-\varepsilon)^{\frac{3}{2}}} \right]. \quad (8.2.144)
\]
Results (8.2.142) to (8.2.145) are still valid if we replace $\varepsilon$ by $-\varepsilon$. Also, by deforming the path of integration into the straight line joining $k(1-\varepsilon)$ to $k(1-\varepsilon)(1-i\infty)$ we have

$$
\int_{t^\gamma/(1+t)\gamma}^\infty e^{i \varepsilon} e^{i (1-\varepsilon) t} t^{1/2} \left(1 - i (1-\varepsilon) t\right)^{1/2} dt,
$$

and in a similar manner we show that

$$
\int_{-2\varepsilon/\varepsilon} \int_{1/2}^\infty e^{i \varepsilon} e^{i (1+\varepsilon) t} t^{1/2} \left(1 + i (1+\varepsilon) t\right)^{1/2} dt.
$$

Hence after a little algebra we have, for (8.2.141)

$$
\frac{2}{k} \lim_{\phi \to 0} \frac{\Phi(\phi)}{\sin \psi} = \frac{1}{k^\varepsilon} J_\varepsilon \left(k_\varepsilon\right) + \frac{e^{i/2}}{2^{1/2} k^\varepsilon} \left[ -2 \frac{k_\varepsilon - \varepsilon i}{k^\varepsilon} \left\{ \frac{1}{1 - i\nu} + \frac{i}{8k^\varepsilon} \left[ 17 - 2(1-\varepsilon) \right]\right\} + \frac{i}{2^{1/2} k^\varepsilon} \left( 15 + 4(1-\varepsilon) - 2S(1-\varepsilon) \right) \right]
$$

$$
+ \frac{2e^{-i\varepsilon\varepsilon}}{k^\varepsilon} \left\{ \frac{1}{1 + i\nu} - \frac{i}{8k^\varepsilon} \left[ 3 - 2(1+i) \right] \right\} + \frac{1}{128k^2} \left( 15 + 4(1+i) - 2S(1+i) \right)
$$

$$
+ \frac{17}{(1+i)^2} - \frac{8}{(1+i)^2} \right\} + \frac{i}{2^{1/2} k^\varepsilon} \left( 15 + 4(1+i) - 2S(1+i) \right) + \frac{1}{8k^\varepsilon} \left( \frac{1}{1+i} \right) \right].
$$

(8.2.148)
Finally, in order to evaluate \( x_n \) asymptotically, for \( n \geq 1 \) we can replace \( \Omega_n \) and \( e_n \) in (8.2.130) by \( \Phi_n \) and \( s_n \) respectively, since this equation was determined independently of the particular form of the incident field. Now

\[
\frac{2}{k} \lim_{\phi \to 0} \frac{P_{1/2}(\sec \phi)}{\sin \phi} = - \frac{1}{4k^2}, 
\]

and

\[
\frac{2}{k} \lim_{\phi \to 0} \frac{P_{1/2}(\sec \phi)}{\sin \phi} = \frac{3}{4k^2}. 
\]

These last two results are obtained after reference to section 3.9.2 of [36]. Therefore

\[
\frac{2}{k} \lim_{\phi \to 0} \frac{\Phi_n(\phi)}{\sin \phi} = \frac{e^{i\pi k}}{2\pi k^{3/2}} \left\{ s_n(0) + \frac{i}{4k} \left( \frac{125}{16} s_n'(0) - \frac{3}{2} s_n''(0) \right) \right\} 
\]

\[
- \frac{e^{-i\pi k}}{2\pi k^{3/2}} \left\{ s_n(1) - \frac{7}{16} s_n'(1) + \frac{5}{8} s_n''(1) + \frac{5}{4} s_n'''(1) \right\} 
\]

\[
- \frac{e^{i\pi k}}{8\pi k^{3/2}} \left\{ s_n(0) - \frac{i}{4k} \left( s_n(0) + \frac{3}{4} s_n'(0) \right) \right\} + O\left( \frac{s_n}{k^3} \right). 
\]

From (8.2.148) and (8.2.151) correct to order \( k^{-7/2} \), we have

\[
2 \lim_{k \to \infty} \frac{\Phi_n(\phi)}{\sin \phi} = 2 \lim_{k \to \infty} \frac{\Phi_n(\phi)}{\sin \phi} + e^{i\pi k} s_n(0) - e^{-i\pi k} s_n'(0) - e^{i\pi k} s_n''(0) + O\left( \frac{1}{k^4} \right). 
\]

We substitute for \( s_1(0) \) and \( s_1(1) \) from (8.2.49) and (8.2.47)
respectively, and we replace $J_1(k \varepsilon)$ by the first few terms
in its asymptotic expansion to give, after some cancellation

$$
\frac{2 \lim_{\phi \to 0} \Theta(k \varepsilon)}{\pi} = \frac{e^{i k (1-\varepsilon)^{1/2}}}{2^{1/2} \pi k (k \varepsilon)^{1/2}} \left[ 1 + \frac{1}{8k^2 \varepsilon} \left( 2(1-\varepsilon)^{1/2} + \frac{1}{(1-\varepsilon)^{1/2}} \right) + \frac{1}{8k^2 \varepsilon^2} \left( 2(1+\varepsilon)^{1/2} + \frac{1}{(1+\varepsilon)^{1/2}} \right) + O \left( \frac{1}{(k \varepsilon)^{3/2}} \right) \right] \cdot (8.2.153)
$$

It is seen from (8.1.30), (8.2.37), (8.2.59), (8.2.132) and
(8.2.153) that we are now able to write down the value of $M$.

However the algebra involved is rather laborious, so only a
few intermediate results will be given. On multiplying (8.2.37)
by (8.2.132), we obtain

$$
\frac{1}{\pi} \sum_{n=1}^{\infty} \left( J_n(k \varepsilon) \right)^2 = \frac{e^{i k (1-\varepsilon)^{1/2}}}{2^{1/2} \pi k (k \varepsilon)^{1/2}} \left[ 1 + \frac{1}{8k^2 \varepsilon} \left( 2(1-\varepsilon)^{1/2} + \frac{1}{(1-\varepsilon)^{1/2}} \right) + \frac{1}{8k^2 \varepsilon^2} \left( 2(1+\varepsilon)^{1/2} + \frac{1}{(1+\varepsilon)^{1/2}} \right) + O \left( \frac{1}{(k \varepsilon)^{3/2}} \right) \right] \cdot (8.2.153)
$$

It is seen from (8.1.30), (8.2.37), (8.2.59), (8.2.132) and
(8.2.153) that we are now able to write down the value of $M$.

However the algebra involved is rather laborious, so only a
few intermediate results will be given. On multiplying (8.2.37)
by (8.2.132), we obtain

$$
\mu \sum_{n=1}^{\infty} \left( J_n(k \varepsilon) \right)^2 = \frac{e^{i k (1-\varepsilon)^{1/2}}}{2^{1/2} \pi k (k \varepsilon)^{1/2}} \left[ 1 + \frac{1}{8k^2 \varepsilon} \left( 2(1-\varepsilon)^{1/2} + \frac{1}{(1-\varepsilon)^{1/2}} \right) + \frac{1}{8k^2 \varepsilon^2} \left( 2(1+\varepsilon)^{1/2} + \frac{1}{(1+\varepsilon)^{1/2}} \right) + O \left( \frac{1}{(k \varepsilon)^{3/2}} \right) \right] \cdot (8.2.153)
$$

It is seen from (8.1.30), (8.2.37), (8.2.59), (8.2.132) and
(8.2.153) that we are now able to write down the value of $M$.
From this last result it is quite easily shown that

\[
\mu^2 \lim_{\phi \to 0} \frac{\frac{\mu^2}{2}}{\sin \phi} \left[ 1 - i \frac{2i \epsilon + \phi}{2\epsilon \sin \phi} \right] _{1 - i \phi} + \frac{3i}{4 \epsilon^2} + \frac{3i}{32 \epsilon^2} + \frac{3i}{128 \epsilon^2} + O \left( \frac{1}{\epsilon^{3/2}} \right) 
\]

\[
= \frac{2i}{128 \epsilon^3} - \frac{2i \epsilon + \phi}{128 \epsilon^3} + O \left( \frac{1}{\epsilon^{3/2}} \right) 
\]

During the remainder of this subsection result (8.2.155) will be used; however (8.2.154) will be used when we consider Case II.

In order to obtain the stress couple \( \mathbf{M} \), holding the rigid disc fixed it only remains to substitute (8.2.59), (8.2.155) and (8.2.148) into (8.1.30) to give, after a certain amount of manipulation,

\[
\mathbf{M} = i \left( \frac{2^{5/2}}{\epsilon \epsilon} \right) \mathbf{r}_0 \left[ e^{i \epsilon \tau - \frac{\pi}{4}} \epsilon \left( \frac{1}{1 - \epsilon} \right) \left( \frac{2(1 - \epsilon)^{3/2}}{(1 - \epsilon)^{3/2}} + \frac{3}{128 \epsilon^2} + \frac{14}{(1 - \epsilon)^{3/2}} \right) \right. 
\]

\[
- \frac{2i \epsilon + \phi}{128 \epsilon^2} \left( \frac{4(1 - \epsilon)^{3/2}}{(1 - \epsilon)^{3/2}} + \frac{13}{128 \epsilon^2} + \frac{18}{(1 - \epsilon)^{3/2}} \right) 
\]

\[
+ \frac{3i}{128 \epsilon^2} \left( \frac{4(1 - \epsilon)^{3/2}}{(1 - \epsilon)^{3/2}} + \frac{13}{128 \epsilon^2} + \frac{18}{(1 - \epsilon)^{3/2}} \right) 
\]

\[
- \frac{198}{(1 - \epsilon)^{3/2}} + 14 \epsilon (1 - \epsilon)^{-1} + 56 (1 - \epsilon)^{-1} \left( \frac{1}{(1 - \epsilon)^{3/2}} \right) \right]
\]

\[
\left. + \frac{2i \epsilon + \phi}{128 \epsilon^2} \left( \frac{4(1 - \epsilon)^{3/2}}{(1 - \epsilon)^{3/2}} + \frac{13}{128 \epsilon^2} + \frac{18}{(1 - \epsilon)^{3/2}} \right) \right]
\]

\[
+ \frac{3i}{8 \epsilon^2} \left( \frac{2(1 + \epsilon)^{3/2}}{(1 + \epsilon)^{3/2}} + \frac{3}{8 \epsilon^2} + \frac{1}{\epsilon^{3/2}} \right) 
\]

\[
+ \frac{2i \epsilon + \phi}{8 \epsilon^2} \left( \frac{4(1 + \epsilon)^{3/2}}{(1 + \epsilon)^{3/2}} + \frac{13}{8 \epsilon^2} + \frac{18}{(1 + \epsilon)^{3/2}} \right) \right] 
\]
\[
\begin{align*}
- \frac{3}{128 \pi \varepsilon^2} & \left(4 \left(1 + \varepsilon\right)^{3/2} + 12 \left(1 + \varepsilon\right) + \frac{19}{(1 + \varepsilon)^{3/2}} + \frac{3 \varepsilon^2 e^{2i \phi + \frac{3}{2}}}{128 \pi \varepsilon^{3/2} \left(1 + \varepsilon\right)^{3/2}} \left( \frac{4}{(1 + \varepsilon)^{3/2}} - \frac{13}{(1 + \varepsilon)^{3/2}} \right) \right) \\
+ \frac{11}{(1 + \varepsilon)^{3/2}} & + \frac{3 i}{1024 \pi \varepsilon^3} \left( \frac{1}{(1 + \varepsilon)^{3/2}} - 148 \left(1 + \varepsilon\right)^{3/2} + 148 \left(1 + \varepsilon\right)^{3/2} - 56 \left(1 + \varepsilon\right)^{3/2} \right)
\end{align*}
\]

\[
\frac{-4 i e^{i \phi + \frac{3}{2}} \cdot \frac{e}{128 \pi \varepsilon^3} (1 + \varepsilon)^{3/2} + O \left( \frac{1}{\varepsilon^{3/2}} \right)}{(8.2.156)}
\]

From this last equation and (8.1.32) we deduce that

\[
h_i = \left( \frac{2}{\varepsilon} \right)^{5/2} \sqrt{\frac{\varepsilon}{x}} \alpha \left[ \sin \left( \frac{\varepsilon}{R \varepsilon - \pi/4} \right) \sin \left( \frac{\varepsilon}{R \varepsilon - \pi/4} \right) - \frac{3}{8 \varepsilon} \right] \left( 2(1 - \varepsilon)^{3/2} - 3 \right)
\]

\[
- \frac{e}{8 \varepsilon^{3/2}} \left[ (1 - \varepsilon)^{3/2} \sin \left( R (2 + \varepsilon) \right) - (1 + \varepsilon)^{3/2} \cos \left( R (2 - \varepsilon) \right) \right]
\]

\[
+ \frac{3}{128 \pi \varepsilon^2} \frac{\sin \left( \varepsilon R - \pi/4 \right)}{\left( 1 - \varepsilon^2 \right)^{3/2}} \left[ 4 \left( 1 - \varepsilon \right)^{3/2} - (1 + \varepsilon)^{3/2} + 12 \left( 1 - \varepsilon \right)^{3/2} - (1 + \varepsilon)^{3/2} \right]
\]

\[
- \frac{19}{(1 - \varepsilon)^{3/2}} \left[ (1 - \varepsilon)^{3/2} - (1 + \varepsilon)^{3/2} \right] - \frac{3}{128 \pi \varepsilon^2 \left( 1 - \varepsilon^2 \right)^{3/2}} \left[ 14 \left( 1 - \varepsilon \right)^{3/2} - 13 \left( 1 - \varepsilon^2 \right)(1 + \varepsilon)^{3/2} \right]
\]

\[
+ 11 \left( 1 - \varepsilon^2 \right)(1 - \varepsilon)^{3/2} \cos \left( R (2 + \varepsilon) \right) - \frac{3}{8 \varepsilon} \left( R (2 - \varepsilon) \right) \sin \left( 1 - \varepsilon \right)^{3/2} - 13 \left( 1 - \varepsilon^2 \right)(1 + \varepsilon)^{3/2}
\]

\[
+ 11 \left( 1 - \varepsilon^2 \right)(1 + \varepsilon)^{3/2} \sin \left( R (2 - \varepsilon) \right) - \frac{3}{8 \varepsilon} \left( R (2 + \varepsilon) \right) \sin \left( 1 + \varepsilon \right)^{3/2} - 56 \left( 1 - \varepsilon \right)^{3/2} - (1 + \varepsilon)^{3/2} \right] \left( 1 \right)
\]

\[
+ 198 \left( 1 - \varepsilon^2 \right)^{3/2} - 148 \left( 1 - \varepsilon \right)^{3/2} - (1 + \varepsilon)^{3/2} + 56 \left( 1 - \varepsilon \right)^{3/2} - (1 + \varepsilon)^{3/2} \right] \left( 1 \right)
\]
+ \frac{\varepsilon}{128 \pi R^3 (1-\varepsilon)^{3/2}} \left\{ (1-\varepsilon)^{3/2} \cos \{ \frac{\pi}{4} (4+\varepsilon) - \frac{\pi}{4} \} + \left( 1 + \varepsilon \right)^{3/2} \cos \{ \frac{\pi}{4} (4-\varepsilon) - \frac{\pi}{4} \} \right\} + O \left( \frac{1}{R^{3/2}} \right),

(8.2.157)

\begin{align*}
h_2 &= \left( \frac{2}{R \varepsilon} \right)^{3/2} \left\{ (1-\varepsilon)^{3/2} \left\{ \cos \left( \frac{\pi}{4} \right) - \frac{3}{8} \varepsilon \sin \left( \frac{\pi}{4} \right) \right\} \left( 2 (1-\varepsilon)^{3/2} + 3 \right) + \frac{3}{8} \varepsilon (1-\varepsilon)^{3/2} \right\} \\
&\quad - \frac{3}{128 \pi R^2 \varepsilon^2} \left\{ (1-\varepsilon)^{3/2} \left\{ \cos \frac{\pi}{4} (2+\varepsilon) \right\} + \left( 1 + \varepsilon \right)^{3/2} \varepsilon \sin \frac{\pi}{4} (2+\varepsilon) \right\} \\
&\quad + \frac{3}{128 \pi R^2 \varepsilon^2} \left\{ \left( \frac{4}{1-\varepsilon} \right)^{3/2} - 13 (1-\varepsilon)^{3/2} (1+\varepsilon)^{3/2} + 11 (1-\varepsilon)^{3/2} (1+\varepsilon)^{3/2} \right\} \left\{ \left( 1 + \varepsilon \right)^{3/2} - \frac{3}{2} \varepsilon \sin \frac{\pi}{4} \right\} \left\{ \left( 1 + \varepsilon \right)^{3/2} + (1+\varepsilon)^{3/2} \right\} \\
&\quad - \frac{19 \varepsilon (1-\varepsilon)^{3/2} + 14 \left( \frac{1}{1-\varepsilon} \right)^{3/2} \left( (1-\varepsilon)^{3/2} + (1+\varepsilon)^{3/2} \right) - \frac{56}{128 \pi R^2 \varepsilon^2} \frac{\varepsilon \sin \frac{\pi}{4} (4+\varepsilon) - \frac{\pi}{4} \right]\right\} + O \left( \frac{1}{R^{3/2}} \right),

(8.2.158)

and from these two expressions we can calculate \( \xi_1, \xi_2 \) and \( \Delta \)
for Case I.
8.3 Case II, \((\lambda = k^\xi, \ 0 \leq \xi \leq k^{-1})\)

Since the asymptotic expressions obtained for \(\mu\) and \(\Omega (\sin \phi)\) in subsection 8.2 are independent of \(\xi\), they are also valid for the current case.

First of all let us evaluate \(s_1(v)\). In (8.2.41) we approximate \(J_0[\lambda(1-it)]\) by using the formula

\[
\int_0^{\infty} J_0[\lambda(1-it)] = J_0(\lambda) \int_0^{\infty} J_0(\lambda \xi) + \sum_{s=1}^{\infty} J_s(\lambda) J_s(\lambda \xi).
\] (8.3.1)

Hence for small \(t\)

\[
\int_0^{\infty} J_0[\lambda(1-it)] \approx J_0(\lambda) + \lambda it J_1(\lambda) + \frac{\lambda^2 t^2}{4} \{J_2(\lambda) - J_2(\lambda \xi)\} + \frac{\lambda^3 t^3}{12} \{3J_3(\lambda) - J_3(\lambda \xi)\},
\] (8.3.2)

where \(J_s(\lambda \xi)\) has been replaced by the first few terms of its power series expansion. After further approximation we have

\[
s_1(v) \sim \frac{e^{-\frac{\lambda}{1+v}}}{(1+v)^{-\frac{1}{2}}} \left[ J_0(\lambda) + i t \left\{ \frac{J_0(\lambda)}{(1+v)^{\frac{1}{2}}} + \frac{\lambda J_1(\lambda)}{2} \right\} \right]
\]

\[
+ \frac{t^2}{4} \left[ J_0(\lambda) \left\{ \frac{1}{2} \frac{1}{(1+v)^{\frac{1}{2}}} - \frac{1}{2} \right\} - \lambda J_1(\lambda) \left\{ \frac{1}{2} \frac{1}{(1+v)^{\frac{1}{2}}} - \frac{1}{2} \right\} + \frac{\lambda^2}{4} \{J_2(\lambda) - J_2(\lambda \xi)\} \right] e^{-\frac{\lambda t}{1+v}},
\] (8.3.3)

and hence

\[
s_1(v) = \frac{\gamma \xi^\frac{1}{2} e^{-\frac{\lambda}{1+v}}}{(1+v)^{\frac{1}{2}}} \left[ J_0(\lambda) + \frac{3}{2} \left( \frac{J_0(\lambda)}{1+v} + \frac{\lambda J_1(\lambda)}{2} \right) + \frac{15}{4} \frac{\lambda^2 J_2(\lambda)}{1+v} \right]
\]
\[- \frac{1}{(1 + \nu^2)^2} - \lambda J_1(\lambda) \left( \frac{1}{(1 + \nu^2)^2} - \frac{1}{2} \right) + \frac{2}{4} \left( J_1(\lambda) - J_2(\lambda) \right) + O\left( \frac{1}{k^2} \right) \] \quad (8.3.4)

We deduce from (8.3.4) that

\[ s_1(t) = \frac{\sqrt{2} e^{-i k \cdot \xi}}{4 \sqrt{\nu^2 - k^2}} \left\{ J_0(\lambda) + \frac{3 i}{2} \lambda J_1(\lambda) + O\left( \frac{1}{k^2} \right) \right\} \quad (8.3.5) \]

\[ s_1'(t) = -\frac{\sqrt{2} e^{-i k \cdot \xi}}{8 \sqrt{\nu^2 - k^2}} \left\{ J_0(\lambda) + O\left( \frac{1}{k^2} \right) \right\}, \quad (8.3.6) \]

\[ s_2(t) = \frac{\sqrt{2} e^{-i k \cdot \xi}}{2 \sqrt{\nu^2 - k^2}} \left\{ J_0(\lambda) + \frac{3 i}{4} \lambda J_1(\lambda) + 2 \lambda J_2(\lambda) + O\left( \frac{1}{k} \right) \right\} \quad (8.3.7) \]

\[ s_2'(t) = -\frac{\sqrt{2} e^{-i k \cdot \xi}}{2 \sqrt{\nu^2 - k^2}} \left\{ J_0(\lambda) + O\left( \frac{1}{k} \right) \right\}, \quad (8.3.8) \]

and

\[ s_{1,2}(t) = \frac{i \sqrt{2}}{8} e^{\frac{-i k \cdot \xi}{4 \sqrt{\nu^2 - k^2}}} \left\{ J_0(\lambda) + \frac{3 i}{4} \lambda J_1(\lambda) + \frac{9 i}{4} \lambda J_2(\lambda) + O\left( \frac{1}{k^2} \right) \right\}. \quad (8.3.9) \]

For \( \nu, \) in (8.3.8) we approximate \( J_0(\lambda - i t) \) as above, so that

\[ \nu = \frac{\pi}{\nu^2 - k^2} \left\{ J_0(\lambda) + \frac{i k}{2} \left( 2 \lambda J_1(\lambda) - J_0(\lambda) \right) + \frac{k^2}{8} \left( J_0(\lambda) + 4 \lambda J_1(\lambda) + 2 \lambda^2 J_2(\lambda) - J_2(\lambda) \right) \right\} dt + O\left( \frac{1}{k^2} \right) \]

\[ + \frac{i k^3}{4 \pi} \left( \frac{3}{8} \left( J_0(\lambda) + 6 \lambda J_1(\lambda) - 6 \lambda^2 J_2(\lambda) \right) + 2 \lambda^3 J_3(\lambda) - J_3(\lambda) \right) \right\} dt + O\left( \frac{1}{k^2} \right) \]

\[ = \frac{\sqrt{2} e^{-i k \cdot \xi}}{2 \sqrt{\nu^2 - k^2}} \left\{ J_0(\lambda) + \frac{i k}{4 \lambda} \left( 2 \lambda J_1(\lambda) - J_0(\lambda) \right) \right\} + \frac{3}{4 \lambda} \left( J_0(\lambda) + 4 \lambda J_1(\lambda) + 2 \lambda^2 J_2(\lambda) - J_2(\lambda) \right) \]
\[
\phi_{\lambda}(\lambda) + 6\lambda \mathcal{J}_1(\lambda) - 6\lambda^2 \mathcal{J}_0(\lambda) + 2\lambda^3 \mathcal{J}_2(\lambda) \right) + O\left(\frac{1}{\kappa^2}\right) \right].
\]

From (8.3.10), (8.3.9) and (8.3.5)
\[
\phi_{\lambda}(\lambda) = \frac{\gamma^2}{2} e^{i\frac{3}{\kappa}} \phi_{\lambda}(\lambda) + \frac{\gamma^2}{8\kappa} (1 - \frac{1}{\gamma^2}) \mathcal{J}_0(\lambda) \right) + O\left(\frac{1}{\kappa^2}\right),
\]
and from (8.2.21) and (8.3.9)
\[
\phi_{\lambda}(\lambda) = \frac{1}{128\kappa^3} \frac{\gamma^2}{2} e^{i\frac{3}{\kappa}} \mathcal{J}_0(\lambda) + O\left(\frac{1}{\kappa^2}\right)
\]

We can now write down \( \nu \), since in (8.3.3) we can replace \( \mu_n \), \( \mu_n \) and \( \nu_n \) by \( \nu \), \( \nu_n \) and \( \nu_n \) respectively. Therefore, we have

from (8.3.9), (8.3.5), (8.3.6), (8.3.10), (8.3.11), (8.3.12) and (8.3.12)
\[
\nu = \frac{\gamma^2}{\kappa^2} \left( \mathcal{J}_0(\lambda) + \frac{i}{4\kappa} \left( 2\lambda \mathcal{J}_1(\lambda) - \mathcal{J}_0(\lambda) \right) - \frac{i}{8\kappa^4} \frac{\gamma^2}{\kappa^2} \mathcal{J}_0(\lambda) \right)
\]
\[
+ \frac{1}{32\kappa^2} \left( \mathcal{J}_0(\lambda) + 12\lambda \mathcal{J}_1(\lambda) + 6\lambda^2 \mathcal{J}_0(\lambda) - \mathcal{J}_2(\lambda) \right) + \frac{i}{128\kappa^4} \left( 7\mathcal{J}_0(\lambda) + 24\lambda \mathcal{J}_1(\lambda) \right)
\]
\[
+ \frac{i}{128\kappa^6} \left( 5\mathcal{J}_0(\lambda) + 18\lambda \mathcal{J}_1(\lambda) - 30\lambda^2 \mathcal{J}_0(\lambda) - \mathcal{J}_2(\lambda) \right) + 10\lambda^3 \mathcal{J}_0(\lambda) - \mathcal{J}_3(\lambda) \right)
\]
\[
- \frac{i}{128\kappa^4} \frac{\gamma^2}{\kappa^2} \mathcal{J}_0(\lambda) + O\left(\frac{1}{\kappa^2}\right) \right].
\]
In a similar manner to (8.2.133) we have

\[
\Phi(\phi) = \int_0^\infty J_0(\lambda) J_1(k \omega \sin \phi) d\omega + \frac{i}{\pi} \int_{\omega^2}^\infty \frac{e^i k \omega}{(1 - \omega)^{1/2}} \int_0^\infty \frac{t \gamma (t - 1)}{t - \omega} e^{-i k t} J_0(\lambda t) dt \, dt.
\]

(8.3.14)

We let

\[
\beta(\phi) = \int_0^\infty J_0(\lambda) J_1(k \omega \sin \phi) d\omega.
\]

(8.3.15)

For the inner integral in (8.3.14) we use (8.2.135) and

we then replace \( J_0(\lambda(1 - it))^2 \) by the first three terms of (8.3.2) to give

\[
\int_0^\infty \frac{t \gamma (t - 1)}{t - \omega} e^{-i k t} J_0(\lambda t) dt \sim e^{i k \omega \sin \phi} \int_0^\infty \frac{e^{i k t}}{t^{1/2}} \left[ J_0(\lambda t) - \frac{i}{2} \{ J_0(\lambda t) - 2\lambda J_2(\lambda t) \} \right] dt,
\]

(8.3.16)

on using (8.2.64). Hence on inverting the order of integration in (8.3.14) we have

\[
\Phi(\phi) \sim \beta(\phi) + \frac{e^{i k \omega \sin \phi}}{2 \pi^{1/2}} \int_0^\infty \frac{e^{i k t}}{t^{1/2}} \left[ J_0(\lambda t) - \frac{3i}{4} \{ J_0(\lambda t) - 2\lambda J_2(\lambda t) \} \right] dt + \frac{15}{32 \lambda^2} \left( J_0(\lambda) + 4 \lambda J_2(\lambda) \right)
\]
\[+2\lambda^2\{J_0(\lambda)-J_2(\lambda)\}\int_0^{\pi} \frac{e^{iut}}{\omega\sqrt{1-\omega^2}} \frac{J_1(\sqrt{\omega^2-1})}{\sqrt{\omega^2(1-\omega^2)}} \, d\omega. \tag{8.3.17}\]

The inner integral in (8.3.17) has already been evaluated (c.f. (8.2.69) and (8.2.70)), and therefore the last line can be written as

\[
\bar{\Phi}_0(\phi) = \frac{e^{-\frac{2\phi}{\pi}}}{\pi} \int_0^{\pi} e^{i\sqrt{\frac{\lambda}{\omega^2-1}} \sin \phi} \frac{d\phi}{\pi} \left\{ J_0(\sqrt{\omega^2-1}) \int_0^{\pi} e^{i\sqrt{\frac{\lambda}{\omega^2-1}} \sin \phi} \sin \phi \, d\phi \right\}.
\]

Fortunately the integrals in this last expression have already been evaluated. Therefore on using results (8.2.72) to (8.2.75) we obtain

\[
\bar{\Phi}_0(\phi) = \frac{e^{-\frac{2\phi}{\pi}}}{\pi} \int_0^{\pi} e^{i\sqrt{\frac{\lambda}{\omega^2-1}} \sin \phi} \frac{d\phi}{\pi} \left\{ J_0(\sqrt{\omega^2-1}) \int_0^{\pi} e^{i\sqrt{\frac{\lambda}{\omega^2-1}} \sin \phi} \sin \phi \, d\phi \right\}.
\]
\[
- \frac{i}{4 K^2} \int_0^\pi e^{i k \sin \phi \sin \psi} \left\{ \frac{1}{\sin^2 \phi \cos^2 \psi} \left[ (1+\sin \phi \cos \psi)^{1/2} + \frac{1}{(1+\sin \phi \cos \psi)^{1/2}} \right] - 2 \right\} d\psi \\
- \frac{i}{4 K^2} \left\{ J_0(\lambda) - 2 \lambda J_1(\lambda) \right\} \int_0^\pi e^{i k \sin \phi \sin \psi} \left\{ (1+\sin \phi \cos \psi)^{3/2} - 3(1+\sin \phi \cos \psi)^{1/2} + 2 \right\} d\psi \\
- \frac{3}{32 K^3} \int_0^\pi \frac{e^{i k \sin \phi \sin \psi}}{\sin \phi \cos^3 \psi} \left\{ 3(1+\sin \phi \cos \psi)^{1/2} + \frac{6}{(1+\sin \phi \cos \psi)^{1/2}} - \frac{1}{(1+\sin \phi \cos \psi)^{1/2}} - 8 \right\} d\psi \\
- \frac{1}{16 K^3} \left\{ J_0(\lambda) - 2 \lambda J_1(\lambda) \right\} \int_0^\pi \frac{e^{i k \sin \phi \sin \psi}}{\sin^3 \phi \cos^3 \psi} \left\{ (1+\sin \phi \cos \psi)^{3/2} - 6(1+\sin \phi \cos \psi)^{1/2} - \frac{3}{(1+\sin \phi \cos \psi)^{1/2}} + 8 \right\} d\psi \\
+ \frac{1}{32 K^3} \left( J_0(\lambda) + 4 \lambda J_1(\lambda) + 2 \lambda^2 \right) \int_0^\pi \frac{e^{i k \sin \phi \sin \psi}}{\sin^3 \phi \cos \phi} \left\{ 3(1+\sin \phi \cos \psi)^{1/2} ight\} d\psi \\
- 10(1+\sin \phi \cos \psi)^{1/2} + 15(1+\sin \phi \cos \psi)^{1/2} - 8 \right\} d\psi \left\{ e^{i K \sin \phi} \int_0^\pi \frac{\cos \phi d\phi}{\sin \phi \cos \phi} - \frac{1}{2\pi K^2} \int_0^\pi \frac{J_0(\lambda) \int_0^\pi e^{i k \sin \phi \sin \psi} \frac{\cos \phi d\phi}{\sin \phi \cos \phi} }{\sin \phi \cos \phi} \right\}
\]

(E.3.19)

and on using results (3.2.30) to (3.2.37) the last line becomes

\[
\bar{F}_o(\phi) = F(\phi) + \frac{e^{-i \phi}}{2 K} \left[ i \int J_0(\lambda) J_1(ksin\phi) + \frac{1}{8 K} \left( \frac{1}{2} J_0(\lambda) J_1(ksin\phi) - 6 \lambda J_1(\lambda) J_1(ksin\phi) \right) \right] + \frac{1}{2 K^2} \left( \frac{1}{2} J_1(\lambda) J_1(ksin\phi) \right) + \frac{1}{2 K^2} \left( \frac{1}{2} J_1(\lambda) J_1(ksin\phi) \right)
\]

+ 10 \lambda^2 \left( J_0(\lambda) - J_1(\lambda) \right)^2 + 4k \sin \phi \left( J_0(\lambda) - 2 \lambda J_1(\lambda) \right) \left( J_0(k \sin \phi) - J_1(k \sin \phi) \right)
In order to obtain an expression for $\Phi_n(\beta)$ it is only necessary to replace $\Omega_n(\beta)$, and $e_n$ by $\Phi_n(\beta)$, and $s_n$ respectively, in (8.2.130). Therefore from (8.3.20), (8.3.7) and (8.3.5)

\[
\Phi(\phi) = \frac{-i\beta}{2} \left[ 2i\beta(\phi) + \frac{1}{k} J_0(\lambda) J_1(k \sin \phi) - i \frac{1}{8k^2} \left( 4J_0(\lambda) J_1(k \sin \phi) \right) \right]
\]

\[
+ 6J_1(\lambda) J_1(k \sin \phi) - k \sin \phi J_0(\lambda) \left[ \frac{1}{k} J_0(k \sin \phi) - J_2(k \sin \phi) \right] + \frac{i e^{-2i(k+\phi)}}{4k^2} \left( J_0(\lambda) J_1(k \sin \phi) \right)
\]

\[
+ \frac{1}{32k^3} \left[ \left( J_1(k \sin \phi) \right) - 4J_0(\lambda) + 2i\beta J_1(\lambda) + 10J_0(\lambda) - J_2(\lambda) - 2J_1(k \sin \phi) \right] + \frac{1}{4k^2} \left( J_0(\lambda) - 2J_1(\lambda) \right)
\]

\[
\times \left\{ J_0(k \sin \phi) - J_2(k \sin \phi) + k^2 \sin \phi J_0(\lambda) \left[ 3J_0(k \sin \phi) - J_1(k \sin \phi) \right] \right\} + O \left( \frac{1}{k^{1/2}} \right)
\]

(8.3.20)

We use (8.2.154), (8.3.13) and (8.3.21) in (8.1.30) to obtain

\[
M = \frac{16\pi i}{k \lambda^2} \lim_{\phi \to 0} \sin \phi \left[ 2J_0(\lambda) \left( 1 - \frac{1}{k \sin \phi} \right) - 2\beta(\phi) + \frac{i e^{-2i(k+\phi)}}{k \sin \phi} \left( 4J_0(\lambda) J_1(k \sin \phi) \right) \right]
\]

\[
+ \frac{1}{8k^2} \left\{ \left( 4J_1(\lambda) + 3k^2 \sin \phi J_0(\lambda) - J_2(\lambda) \right) \left[ 1 - \frac{1}{k \sin \phi} \right] - 2\beta(\phi) J_1(k \sin \phi) \right\}
\]

(8.3.20)
\[ + \frac{e^{2i\lambda + \frac{\pi}{2}}}{4 \pi^{\frac{1}{2}} \pi^{\frac{3}{2}}} \lambda J_1(\lambda) \left[ 1 - J_0(k\sin \psi) + \frac{i}{64\pi^3} \right] \]  

\[ + \frac{e^{2i\lambda + \frac{\pi}{2}}}{4 \pi^{\frac{1}{2}} \pi^{\frac{3}{2}}} \lambda J_1(\lambda) \left[ 1 - J_0(k\sin \psi) + \frac{i}{64\pi^3} \right] \frac{1}{k \sin \psi} \left\{ 24 \lambda J_1(\lambda) - 24 \lambda^2 \left[ J_0(\lambda) - J_2(\lambda) \right] + 102 \lambda^2 \left[ 3 J_1(\lambda) - J_3(\lambda) \right] \right\} \]  

\[- 4 \cdot \frac{k \sin \psi \lambda J_1(\lambda) \left[ J_0(k\sin \psi) - J_2(k\sin \psi) \right]^2}{(k^2)^{\frac{1}{2}}} + O \left( \frac{1}{k^2} \right) \].

Result (8.3.22) is found to be invaluable when, in subsection 9.3, we evaluate the scattering coefficient for Case II. Now from (8.3.15)

\[ \frac{2}{\pi} \lim_{\psi \to 0} \left. \frac{\beta(\psi)}{\sin \psi} \right|_{\psi=0} = \int_0^1 \omega J_0(\lambda \omega) d\omega \]

\[ = \frac{1}{\lambda} J_1(\lambda), \quad (8.3.23) \]

and on proceeding to the limit in (8.3.22) we have

\[ M = - \frac{2\pi i}{\lambda^2} \left[ \frac{2}{\lambda} J_1(\lambda) - 4 J_0(\lambda) + \frac{2i\lambda}{k} J_1(\lambda) - \frac{3\lambda^2}{4k^2} \left[ J_0(\lambda) - J_2(\lambda) \right] \right] \]

\[- \frac{2i\lambda + \pi h}{2\pi \sqrt{\pi} \pi^{\frac{3}{2}}} \lambda J_1(\lambda) - \frac{i}{16k^3} \left( 20 \lambda J_1(\lambda) - 4\lambda^2 \left[ J_0(\lambda) - J_2(\lambda) \right] + 5\lambda^3 \left[ 3 J_1(\lambda) - J_3(\lambda) \right] \right) + O \left( \frac{1}{k^2} \right) \]

\[ (8.3.24) \]

For the case of an incident free torsion wave the stress couple \( M \), holding the rigid disc fixed is given by
\[ M = -2\pi i \varepsilon \left\{ \frac{1}{2} - \frac{i}{k} - \frac{3}{4k^2} \cdot \frac{e^{i\frac{\pi}{4}}}{k^{1/2} \sqrt{2}} - \frac{2i}{8k^3} + O \left( \frac{1}{k^{3/2}} \right) \right\}. \]  

From (8.3.24), and (8.1.32)

\[ h_1 = -\frac{2\pi k}{\lambda^2} \left[ \frac{2\lambda \mathcal{J}_1(\lambda)}{k} - \frac{\lambda \mathcal{J}_2(\lambda)}{2\pi^{1/2} k^{3/2}} \sin (2k - \pi/4) + \frac{1}{16k^3} \right] \left( 20\lambda \mathcal{J}_1(\lambda) \right) \]

\[ - \lambda^2 \left\{ \mathcal{J}_0(\lambda) - \mathcal{J}_2(\lambda) \right\} + 5 \lambda^3 \left\{ 3 \mathcal{J}_1(\lambda) - \mathcal{J}_3(\lambda) \right\} + O \left( \frac{1}{k^{3/2}} \right) \],

\[ h_2 = -\frac{2\pi k}{\lambda^2} \left[ \frac{\lambda \mathcal{J}_1(\lambda)}{2\pi^{1/2} k^{3/2}} \cos (2k - \pi/4) + O \left( \frac{1}{k^{3/2}} \right) \right], \]

and when the incident field is a free torsion wave, we have,

letting \( \lambda \to 0 \) in the last two expressions,

\[ h_1 = -2\pi k \left\{ \frac{1}{k} - \frac{1}{4k^2} \cdot \frac{\sin (2k - \pi/4) + \frac{3}{8k^3} + O \left( \frac{1}{k^{3/2}} \right)}{} \right\}, \]

and

\[ h_2 = -2\pi k \left\{ \frac{1}{2} - \frac{3}{4k^2} - \frac{1}{4\pi^{1/2} k^{3/2}} \cos (2k - \pi/4) + O \left( \frac{1}{k^{3/2}} \right) \right\}. \]

Hence from these results, (8.1.34), (8.1.35) and (8.1.37),

\[ -2\pi q_1 = \frac{2\pi k}{\lambda} \left( \frac{8 \mathcal{J}_1(\lambda) - 4 \mathcal{J}_0(\lambda)}{\lambda} \right)^{-2} \left[ 2\lambda \mathcal{J}_1(\lambda) - \frac{\lambda \mathcal{J}_2(\lambda)}{2\pi^{1/2} k^{3/2}} \sin (2k - \pi/4) \right] \]

\[ + \frac{1}{16k^3} \left( 20\lambda \mathcal{J}_1(\lambda) - 4\lambda^2 \left\{ \mathcal{J}_0(\lambda) - \mathcal{J}_2(\lambda) \right\} + 5\lambda^3 \left\{ 3 \mathcal{J}_1(\lambda) - \mathcal{J}_3(\lambda) \right\} \right) \]
\[ - \frac{2\lambda^2 J_1(\lambda)}{K^2} \left( \frac{4 J_1^2(\lambda)}{\lambda} - \frac{3}{2} \left[ J_0(\lambda) - J_2(\lambda) \right] + \mathcal{O}\left( \frac{1}{K^{5/2}} \right) \right], \]  

\[ (8.3.30) \]

\[ 2\pi q_2 = \frac{\lambda^2}{\alpha} \left( \frac{8}{\lambda} J_1(\lambda) - 4 J_0(\lambda) \right)^2 \left[ \frac{8}{\lambda} J_1(\lambda) - 4 J_0(\lambda) - \frac{4\lambda^2 J_2(\lambda)}{K^2} \left( \frac{8}{\lambda} J_1(\lambda) - 4 J_0(\lambda) \right) \right] 
+ \frac{3\lambda^2}{2K} \left( J_0(\lambda) - J_2(\lambda) \right) + \frac{\lambda J_1(\lambda)}{2\pi^{3/2} K^{3/2}} \cos(2k-\pi/4) + \mathcal{O}\left( \frac{1}{K^{1/2}} \right), \]  

\[ (8.3.31) \]

and

\[ -2\pi q_1 = \frac{\lambda^2}{K^2} \left[ 1 + \frac{\sin(2k-\pi/4)}{4\pi^{3/2} K^{3/2}} - \frac{5}{8K^2} + \mathcal{O}\left( \frac{1}{K^{5/2}} \right) \right], \]  

\[ (8.3.33) \]

\[ -2\pi q_2 = \frac{2}{K} \left[ 1 - \frac{5}{2K^2} + \frac{\cos(2k-\pi/4)}{2\pi^{1/2} K^{3/2}} + \mathcal{O}\left( \frac{1}{K^{1/2}} \right) \right], \]  

\[ (8.3.34) \]

and

\[ -\tan \Delta = \frac{K}{2} \left[ 1 + \frac{\sin(2k-\pi/4)}{4\pi^{1/2} K^{3/2}} - \frac{15}{8K^2} + \mathcal{O}\left( \frac{1}{K^{5/2}} \right) \right]. \]  

\[ (8.3.35) \]

When the incident field takes the form of a free torsion wave we have, taking the limit \( \lambda \to 0 \) in \((8.3.30)\) to \((8.3.32)\),

\[ -2\pi q_1 = \frac{4}{K^2} \left[ 1 - \frac{\sin(2k-\pi/4)}{4\pi^{3/2} K^{3/2}} - \frac{5}{8K^2} + \mathcal{O}\left( \frac{1}{K^{5/2}} \right) \right], \]  

\[ (8.3.33) \]

\[ -2\pi q_2 = \frac{2}{K} \left[ 1 - \frac{5}{2K^2} + \frac{\cos(2k-\pi/4)}{2\pi^{1/2} K^{3/2}} + \mathcal{O}\left( \frac{1}{K^{1/2}} \right) \right], \]  

\[ (8.3.34) \]

and

\[ -\tan \Delta = \frac{K}{2} \left[ 1 + \frac{\sin(2k-\pi/4)}{4\pi^{1/2} K^{3/2}} - \frac{15}{8K^2} + \mathcal{O}\left( \frac{1}{K^{5/2}} \right) \right]. \]  

\[ (8.3.35) \]
Results (8.3.25), and (8.3.33) to (8.3.35), are of a similar form to the results obtained by Thomas [23].
We cannot use the method devised by Williams [38] to transform equation (7.1.11), to a form which is more amenable to solution. If this method is used for $\varepsilon = 0$, then it is seen from (7.1.28) that a null solution is obtained. Therefore, an alternative approach will have to be adopted. In (6.4.43) we carry out the integration with respect to $u$ to obtain

$$K^{(0)}(v) = \frac{d}{d\omega_0} \int_{0}^{v} \left[ \frac{k'v^2 + k(t^2 - x^2)v^2}{(v^2 - t^2)^2} \right] d\omega_0 \int_{0}^{t} \left[ \frac{k\omega_0 - 2u_0}{k'} + 8u_0 J(k,\omega) \right] d\omega dt. \tag{8.4.1}$$

We now refer to equation (27) of Jones [26]. Obviously, if we set $\varepsilon = 1$ in (8.4.1); then $K^{(0)}(v)$ is of similar form to the $B^{(0)}(v)$ of [26]. Therefore (6.4.40) can be solved by using the method set-out in the pages immediately following equation (27) of [26]. However, it is desired to obtain an expression for the couple resisting the motion of the disc which, besides being valid for $\varepsilon = 1$, is also valid for, say $1 - k^{-3/2} \leq \varepsilon \leq 1$.

First of all we consider that part of (8.4.1) which is independent of $\varepsilon$, i.e., we consider
\[
\frac{d}{dt} \left( \int_0^t \frac{\cos \sqrt{k} \left( t^2 - l^2 \right)^{1/2}}{(t^2 - l^2)^{1/2}} \, dt \right) = \frac{d}{dt} \left( \int_0^t \frac{x \cosh \sqrt{k} \left( t^2 - l^2 \right)^{1/2}}{(t^2 - l^2)^{1/2}} \, dx \right).
\]

Now
\[
\int_0^t \frac{x \cosh \sqrt{k} \left( t^2 - x^2 \right)^{1/2}}{(t^2 - x^2)^{1/2}} \, dx = -\frac{1}{\sqrt{k}} \left[ \sinh \sqrt{k} \left( t^2 - x^2 \right)^{1/2} \right]_0^t,
\]
and therefore
\[
\frac{d}{dt} \int_0^t \frac{x \cosh \sqrt{k} \left( t^2 - x^2 \right)^{1/2}}{(t^2 - x^2)^{1/2}} \, dx = \cosh \sqrt{k} t. \tag{8.4.3}
\]

We integrate (8.4.3) once, by parts to obtain
\[
\frac{d}{dt} \left( \int_0^t \frac{\cos \sqrt{k} \left( t^2 - l^2 \right)^{1/2}}{(t^2 - l^2)^{1/2}} \cosh \sqrt{k} t \, dt \right) = -\frac{1}{\sqrt{k}} \frac{d}{dt} \left[ \sinh \sqrt{k} \left( t^2 - l^2 \right)^{1/2} \cosh \sqrt{k} t \right]_0^t
\]
\[+ \frac{d}{dt} \int_0^t \sin \sqrt{k} \left( t^2 - l^2 \right)^{1/2} \sinh \sqrt{k} t \, dt,
\]
\[= \cos \sqrt{k} t + \frac{tk}{\sqrt{k}} \int_0^t \cos \sqrt{k} \left( t^2 - l^2 \right)^{1/2} \cosh \sqrt{k} t \, dt.
\]

In (8.4.4) we let \( t^2 = z \); then
\[
\frac{d}{dt} \left( \int_0^t \frac{\cos \sqrt{k} \left( t^2 - l^2 \right)^{1/2}}{(t^2 - l^2)^{1/2}} \cosh \sqrt{k} t \, dt \right) = \frac{1}{\sqrt{k}} \left( \cos \sqrt{k} \left( l^2 - z \right)^{1/2} \sinh \sqrt{k} z^{1/2} \right) \, dz. \tag{8.4.5}
\]

Since \( \sinh(kz^{1/2})/z^{1/2} \) can be regarded as \( \sin[k(-z)^{1/2}] / (-z)^{1/2} \),

(A3) of [26] with \( x = v^2 \), and \( t = 0 \) shows that
\[
\frac{d}{dt} \left( \int_0^t \frac{\cos \sqrt{k} \left( t^2 - l^2 \right)^{1/2}}{(t^2 - l^2)^{1/2}} \cosh \sqrt{k} t \, dt \right) = \cos \sqrt{k} v + \frac{tk}{2} \int_{-v}^v \sin \sqrt{k} t \, dt,
\]
\[= \cos \sqrt{k} v + \sqrt{k} \text{Si}(\sqrt{k} v), \tag{8.4.6}
\]
where
\[
\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt.
\]  
(8.4.7)

In (8.4.1) we let
\[
K^{(0)}(v) = 2i\kappa \left\{ \frac{\kappa c_0 - 2u_0}{\kappa^2 c^2} \right\} F^{(0)}(v)
\]
\[+ \frac{8u_0}{\kappa^2 c^2} \int_0^\infty \frac{t}{(v^2 - t^2)^2} \frac{d}{dt} \int_0^t x \cos h \left\{ \frac{\kappa (t^2 - x^2)^2}{(t^2 - x^2)^2} \right\} J_0(\kappa x) \, dx \, dt.
\]  
(8.4.8)

where
\[
F^{(0)}(v) = \frac{1}{2i\kappa} \left\{ \int \cos \kappa v + \int \kappa v \text{Si}(\kappa v) \right\}.
\]  
(8.4.9)

Also, we let
\[
K(v) = 2i\kappa \left\{ \frac{\kappa c_0 - 2u_0}{\kappa^2 c^2} \right\} F(v) + \frac{8u_0}{\kappa^2 c^2} L(v).
\]  
(8.4.10)

For the time being we assume that \( F^{(0)}(v) \), defined by (8.4.2), is not identical to the \( F^{(0)}(v) \) defined by (8.1.11). Clearly, from (8.4.8), (8.4.10) and (7.1.11), \( F(v) \) satisfies the integral equation
\[
F(v) + \frac{1}{\pi} \left( \frac{v}{1+v} \right)^{1/2} F(t) \frac{\tilde{e}^{-i\kappa(t+v)}}{(t+\nu)} \, dt = F^{(0)}(v), \quad 0 < v < 1.
\]  
(8.4.11)

Following Jones [25], we let
\[
F(v) = \frac{1}{2i\kappa} \left[ 2e^{i\kappa v} + 2i\kappa \int_0^\infty \frac{-e^{i\kappa(t+v)}}{2(t+v)} \, dt \right] + F_1(v),
\]
then \( F_1(v) \) satisfies the integral equation
Integral equation (8.4.12) is of the same form as (7.1.29).

Also the first term on the right-hand side of (8.4.12) is, from (8.2.2), equal to \( v^2 (1 + v)^{-1/2} e^{-ikv} \). From these observations we deduce that integral equation (8.4.12) has already been solved asymptotically for large \( k \) and that definition (8.4.9) is equivalent to definition (8.1.11).

For the remaining double integral in (8.4.8) we use the relationship

\[
\int_0^1 (\frac{l^2 - x^2}{x})^{1/4} d\alpha = \frac{L}{2} \int_0^1 (\frac{l^2 - x^2}{x})^{1/4} d\alpha,
\]

and we consider

\[
\frac{L}{2} \int_0^1 (\frac{l^2 - x^2}{x})^{1/4} d\alpha,
\]

which, on letting \( x = \sin \theta \) is equal to

\[
\frac{L}{2} \int_0^{\pi/2} (\frac{l^2 - \sin^2 \theta}{\sin \theta})^{1/4} d\theta.
\]

This last integral is a particular case of Sonine's second finite integral. Hence from Watson [27], it is equal to
\[
\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}} K^{\frac{1}{2}}_v (1 - \epsilon^2)^{\frac{1}{4}}} \frac{d}{dt} \int_{\frac{1}{2}} t^2 k(1 - \epsilon^2)^{\frac{1}{4}} \cos \alpha t = \frac{1}{\alpha} \frac{d}{dt} \sin \alpha t = \cosh \alpha t,
\]

(8.4.14)

and (8.4.8) can be written as

\[
K^{(0)}(\nu) = 2iK \left\{ K_0 - \frac{2u_0}{k \epsilon^2} L^{(0)}(\nu) + \frac{8u_0}{k^2 \epsilon^2} \int_0^\infty \frac{e^{\nu_2 x^2}}{x^2} \cos \frac{2}{\nu_2} \cos \frac{2}{\nu_2} \cosh \alpha x \, dx \right\}
\]

(8.4.15)

It only remains to evaluate the integral in (8.4.15). We expand \( \cosh \alpha x \) in powers of \( \alpha x \). This step will produce a general term of the form

\[
\frac{\alpha^{2n}}{\Gamma(1 + 2n)} \frac{d}{d\nu} \int_0^\infty e^{\nu_2 x^2} \cos \frac{2}{\nu_2} \frac{\sin \frac{2}{\nu_2} e^{-\epsilon x^2}}{e^{-\epsilon x^2}} \, dx,
\]

which, after expressing the trigonometric function as a Bessel function, is equal to

\[
\frac{\alpha^{2n}}{\Gamma(1 + 2n)} \left( \frac{\nu_2}{2} \right)^{\nu_2} \frac{d}{d\nu} \int_0^\infty e^{\nu_2 x^2} \cos \frac{2}{\nu_2} \frac{\sin \frac{2}{\nu_2} e^{-\epsilon x^2}}{e^{-\epsilon x^2}} \, dx.
\]

We let \( t = \nu \cose \), so that the general term in the series becomes

\[
\frac{\alpha^{2n}}{\Gamma(1 + 2n)} \left( \frac{\nu_2}{2} \right)^{\nu_2} \frac{d}{d\nu} \int_0^{\nu_2} e^\nu \cos \nu \cose \cos \frac{2}{\nu_2} \sin \frac{2}{\nu_2} \cose \, d\cose,
\]

which, from Sonine's first integral, is seen to be equal to

\[
\frac{\alpha^{2n}}{\Gamma(1 + 2n)} \left( \frac{\nu_2}{2} \right)^{\nu_2} 2^n \frac{\Gamma(1 + n)}{k^{n+1}} \frac{d}{d\nu} \frac{\sin \nu x}{x^{n+1}} \int_0^{\nu_2} \cos \frac{2}{\nu_2} \sin \frac{2}{\nu_2} \cose \, d\cose,
\]

Now

\[
\frac{d}{d\nu} \left( \frac{\nu^{n+1/2}}{\nu^{n+1/2}} \right) = \frac{\nu^{n+1/2}}{\nu^{n+1/2}} \int_{\nu^{n+1/2}} \left( \frac{k}{\nu} \right),
\]

(8.4.16)
and from page 3 of [12]:

\[ 2^n \frac{\Gamma(1+n)}{\Gamma(1+2n)} = \frac{(2n+1)}{2^{n+1}} \frac{\Gamma(1/2)}{\Gamma(n+3/2)} \]  \hspace{1cm} (8.4.17)

Therefore the general term in the series has the form

\[ \frac{1}{2} \frac{\Gamma(1/2)}{\Gamma(n+3/2)} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{(\nu^2+\nu)}{2k} J(n,\nu), \]

and we write (8.4.15) as

\[ K_0(\nu) = 2ik \left\{ c_0 \nu - 2c_0 \nu \nu + 8u_0 \frac{\Gamma(1/2)}{\Gamma(1+3/2)} \sum_{m=0}^\infty \frac{J_m(k\nu)}{\Gamma(n+1/2)} \frac{(\nu^2+\nu)}{2k} \right\} \]

We impose the restriction

\[ 1 - \frac{1}{k^3} \leq \varepsilon \leq 1, \]

so that \( \alpha^2/2k \leq 1/k^3 \). Then the infinite series in (8.4.18) appears to behave like an asymptotic series. We approximate \( K_0(\nu) \) by taking \( F(0)(\nu) \) and the first five terms of the infinite series to give

\[ K_0(\nu) \sim 2ik \left\{ c_0 \nu - 2c_0 \nu \nu + 8u_0 \frac{\Gamma(1/2)}{\Gamma(1+3/2)} \sum_{m=0}^\infty \frac{J_m(k\nu)}{\Gamma(n+1/2)} \frac{(\nu^2+\nu)}{2k} \right\} \]

\[ + \frac{\alpha^2}{\Gamma(1/2)(2k)} \left\{ \frac{\nu^2}{k^3} \sin(k\nu) - \cos(k\nu) \right\} + \frac{\alpha^2}{\Gamma(1/2)(2k)} \left\{ \frac{\nu^2}{k^3} \sin(k\nu) - \frac{3}{2} \cos(k\nu) \right\} \]

\[ + \frac{\alpha^2}{\Gamma(1/2)(2k)} \left\{ \cos(k\nu) \right\} + \frac{\alpha^2}{\Gamma(1/2)(2k)} \left\{ \frac{3}{2} \sin(k\nu) \right\}, \]

since the Bessel function of order \( n+\frac{1}{2} \) can be expressed in
finite terms by algebraic, and trigonometric functions of argument kv.

This last line can be written as

\[
K^{(a)}(v) \sim 2i\left\{\frac{k\left(1 - 2u_0\right)}{k^2e^2}\right\} F^{(a)}(v) + \frac{4u_0 \Gamma(\frac{1}{2})}{k^2e^2} \left[ \sum_{s=0}^{\infty} \frac{1}{\Gamma(s+\frac{1}{2})} \left\{ e^{i \alpha^2 s^2} \left( -i \alpha^2 s^2 \right)^{\frac{s+1}{2}} \epsilon + (-1)^s e^{i \alpha^2 s^2} \right\} \right].
\]

We let

\[
K(\omega) = 2i\left\{\frac{k\left(1 - 2u_0\right)}{k^2e^2}\right\} f(\omega) + \frac{4u_0 \Gamma(\frac{1}{2})}{k^2e^2} \psi(\omega),
\]

and

\[
\psi(\omega) = \sum_{s=0}^{\infty} \frac{1}{\Gamma(s+\frac{1}{2})} \left\{ e^{i \alpha^2 s^2} \left( -i \alpha^2 s^2 \right)^{\frac{s+1}{2}} \epsilon + (-1)^s e^{i \alpha^2 s^2} \right\} \psi^{(s+1)}(s).
\]

Then

\[
\int_0^1 \omega^s q_1(\omega) \left[ \frac{e^{i k(\omega-v)}}{(\omega-v)} + \frac{e^{i k(\omega+v)}}{(\omega+v)} \right] d\omega = v \left\{ e^{i \alpha^2 s^2} \left( -i \alpha^2 s^2 \right)^{\frac{s+1}{2}} \epsilon + (-1)^s e^{i \alpha^2 s^2} \right\}, 0 \leq v \leq 1,
\]

and \( f(\omega) \) is the solution of equation (8.1.9).

We let

\[
\int_0^1 \omega q_1(\omega) \frac{e^{i k(\omega-v)}}{(\omega-v)} d\omega = Q^{(s)}(v).
\]

Since \( K, f, \) and \( q^{(s)} \) are bounded at the origin we have

\[
q^{(s)}(v) = -\frac{e^{i k\omega}}{\pi^2 \omega y^2 (1-\omega)^2} \int_0^{\frac{1}{2}} \frac{Q^{(s)}(\omega)}{\sqrt{v}} \frac{e^{i k\omega}}{(v-\omega)} d\omega,
\]
and the equation satisfied by $Q_{(i)}^{(S)}(v)$ is

$$Q_{(i)}^{(S)}(v) + \frac{1}{\pi} \left( \frac{v}{1+v} \right)^{\frac{1}{2}} \int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} Q_{(i)}^{(S)}(t) e^{i v \frac{t}{(1+t)}} dt = \frac{v}{1+v} \left[ \left( e^{-i v} + (-1)^{s-v} e^{-i v} \right) t^{s-v} \right],$$

$$0 \leq v \leq 1. \quad (8.4.26)$$

We note that equations (8.4.23) to (8.4.26) are of similar form to equations (6.4.39), (7.1.1), (7.1.5) and (7.1.11).

We let

$$Q_{(i)}^{(S)}(v) = \frac{v}{1+v} \left[ e^{i v} + (-1)^{s-v} e^{i v} \right] G_{(i)}^{(S)}(v), \quad (8.4.27)$$

so that (8.4.26) becomes

$$G_{(i)}^{(S)}(v) = (-1)^{s-v} \left( \frac{1+v}{v} \right)^{\frac{1}{2}} \frac{1}{\pi} \int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} \left( \frac{v}{1+v} \right)^{\frac{1}{2}} e^{-i v \frac{t}{(1+t)}} G_{(i)}^{(S)}(t) dt. \quad (8.4.28)$$

Now on using

$$\frac{t^{s}}{(1+t)^{v}} = \frac{(-1)^{s} v^{s}}{(v+1)^{s}} - \sum_{r=0}^{s} \frac{(-1)^{r} \Gamma(s-v+1)}{v^{s-r}} \Gamma(s-r), \quad s=0, 1, \ldots \quad (8.4.29)$$

we have

$$\int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} t^{s} dt = (-1)^{s} v^{s} \int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} t^{s} dt + \frac{1}{v} \int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} t^{s-v} dt - \sum_{r=0}^{s} \frac{(-1)^{r} \Gamma(s-v+1)}{v^{s-r}} \Gamma(s-r) \int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} t^{s-v} dt. \quad (8.4.30)$$

The first integral on the right-hand side of (8.4.30) has been evaluated in Appendix B of [25], whereas the others give beta
functions. Hence

\[ \int_0^1 \frac{\left(1-t\right)^{\frac{3}{2}}}{t^{3}} \left(\frac{1}{t+v}\right) dt = -\frac{1}{\pi} \left(\gamma(\frac{3}{2}, \gamma) - \frac{1}{\gamma} \right) \]

and on using this result in (8.4.28) we obtain

\[ G^{(s)}(v) = (-1)^s \gamma^s + \frac{1}{\pi} \left[ \sum_{\gamma=0}^{\infty} (-1)^\gamma \gamma^{-1} \psi(\frac{3}{2}, \gamma) \right] \]

Equation (8.4.32) is analogous to equation (7.1.33). Hence, we form the iteration scheme

\[ G_{n+1}^{(s)}(v) = \frac{1}{\pi} \left[ \sum_{\gamma=0}^{\infty} (-1)^\gamma \gamma^{-1} \psi(\frac{3}{2}, \gamma) \right] G_{n}^{(s)}(v) \]

and

\[ G_{n}^{(s)}(v) = (-1)^s \gamma^s + \frac{1}{\pi} \left[ \sum_{\gamma=0}^{\infty} (-1)^\gamma \gamma^{-1} \psi(\frac{3}{2}, \gamma) \right] \]

and

\[ L_{n}^{(s)}(y) = \frac{i k}{4 \pi} e^{i k y} \int_{0}^{\infty} \left(\frac{x}{1+x}\right)^{\gamma} M^{(2)}(x,y) dx, \quad n \geq 1. \]

and

\[ L_{n}^{(s)}(x) = \left(\frac{t}{k}\right)^{\gamma} \int_{0}^{\infty} \left(\frac{t}{1+t}\right)^{\gamma} M^{(2)}(x,y) dx, \quad n \geq 1. \]
where $n^{(2)}(y, y)$ is given by (7.2.16). Expressions (8.4.33) to (8.4.36) are of analogous form to expressions (7.2.21) to (7.2.24).

To determine $q^{(s)}(w)$ we insert (8.4.27) into (8.4.25) to obtain

$$
\omega_q^{(s)}(w) = -\frac{1}{\pi^2} \left( \frac{\omega}{1-\omega} \right)^{\frac{3}{2}} e^{i k \omega} \int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{v^s}{(v-\omega)} dv
$$

$$
- \frac{1}{\pi^2} \left( \frac{\omega}{1-\omega} \right)^{\frac{3}{2}} e^{i k \omega} \int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)} \sum_{n=1}^\infty \int_0^\infty L_n(l) \frac{e^{i k t}}{(t-v)} dt.
$$

(8.4.37)

Now

$$
\frac{\omega^S}{(v-\omega)} = \sum_{r=0}^S \omega^{S-r} v^{-r} + \frac{\omega^S}{(v-\omega)^r} v^r,
$$

(8.4.38)

and we write

$$
\int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)} = \frac{1}{\omega} \int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)} - \frac{1}{\omega} \int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)} + \frac{S}{S-1} \int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)}
$$

(8.4.39)

The first integral on the right-hand side of (8.4.39) is just

$$
\omega^S \chi(35) \text{ of [25], whereas the others give beta functions. Hence}
$$

$$
\int_0^1 \left( \frac{1-v}{v} \right)^{\frac{3}{2}} \frac{dv}{(v-\omega)} = -\pi \omega^S - \frac{1}{\omega} \int_0^\infty L_1(l) \frac{e^{i k t}}{(t-v)} dt + \sum_{r=0}^S \omega^{S-r} \int_0^\infty L_1(l) \frac{e^{i k t}}{(t-v)} dt.
$$

(8.4.40)

The second quantity on the right-hand side of (8.4.37) has been considered in subsection 7.4. Therefore from (7.4.7)
and (8.4.40), we have

$$
\omega q^{(s)}(\omega) = -\frac{1}{\pi^2} \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{2}} e^{i k \omega} \left\{ \sum_{y=0}^{\infty} \omega^{y+1} \beta(y,2,2+\omega y) - \frac{1}{\omega} \beta(y,2,2+\omega y) - \pi \omega^{s} \right\}
$$

$$
+ \sum_{n=1}^{\infty} \left\{ L_n^{(s)}(\omega) e^{-\frac{i k \omega}{\pi}} + \frac{1}{\pi} \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{2}} e^{i k \omega} \int_{1}^{\infty} \frac{L_n^{(s)}(t) e^{-2 i k t}}{(t-\omega)^{\frac{1}{2}}} dt \right\}.
$$

(8.4.41)

It can be shown from [25] and (8.4.41) that

$$
q^{(s)}(\omega) = -\frac{e^{i k}}{\pi^2 (1-\omega)^{\frac{3}{2}}} \sum_{y=0}^{\infty} \beta(y,2,2+\omega y) - \beta(y,2,2+\omega y) - \pi \omega^{s} \right\}
$$

$$
+ \frac{e^{i k}}{\pi} \sum_{n=1}^{\infty} \int_{1}^{\infty} \frac{L_n^{(s)}(t) e^{-2 i k t}}{t^{\frac{3}{2}} (t-\omega)^{\frac{1}{2}}} dt + O(1-\omega)^{\frac{3}{2}}.
$$

(8.4.42)

From (8.4.5) and (8.4.21)

$$
M = -4 \pi \mu \rho \int_{0}^{\infty} \left\{ 2 i k \left[ \frac{2 u_{0}}{i k} \right] \right\} f(\rho) + \frac{2 u_{0}}{i k} \Gamma(\rho) q(\rho) \right\} d\rho.
$$

(8.4.43)

To evaluate the constant \( C_0 \) we assume

$$
q(\omega) = \gamma (1-\omega)^{\frac{1}{2}} + O (1-\omega)^{\frac{3}{2}}.
$$

(8.4.44)

where

$$
\gamma = \sum_{s=0}^{\infty} \frac{1}{\Gamma(s+\frac{1}{2})} \left( \frac{-i k^2}{2 \pi} \right)^{s+1} \sum_{s=0}^{\infty} \frac{(s+2 \lambda)^s}{\Gamma(s+\frac{1}{2})} \left( \frac{-i k^2}{2 \pi} \right)^s \gamma^{(s+1)}.
$$

(8.4.45)

$$
\gamma^{(s)} = \begin{cases} \frac{e^{i k}}{\pi} + \lambda, & s = 0, \\
- \frac{e^{i k}}{\pi} \sum_{y=0}^{\infty} \beta(y,2,2+\omega y) \lambda^{y-1}, & s > 1.
\end{cases}
$$

(8.4.46)

(8.4.47)
and

\[ \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} L_n^{(s)}(t)e^{-i\xi t}dt = \begin{cases} \lambda^{(s)}, & s = 0, \\ (-1)^S \lambda^{(s)} + \frac{1}{\pi} \sum_{r=1}^{S} (-1)^r \Re \left( \frac{1}{2}, s - \gamma + \frac{1}{2} \right) \lambda^{(s-r)}(t), & s \geq 1. \end{cases} \] (8.4.49)

In order to satisfy the edge condition we must have

\[ 2i\xi \left\{ \left( \frac{\Phi_0}{\xi^2} - 2u_0 \right) \right\} \mu + \frac{4u_0}{\eta^2} \Gamma(\frac{1}{2}) \gamma = 0, \] (8.4.50)

and (8.4.43) becomes

\[ M = \frac{16\pi \mu \xi u_0}{\eta^2} \Gamma(\frac{1}{2}) \int_{0}^{1} \left( \frac{\mu}{\mu} \right) \left\{ \left( \frac{\mu}{\mu} \right) \right\} d\omega, \] (6.4.51)

where \( \mu \) is given by (8.2.36).

We let

\[ \lambda^{(s)}(\nu) = \begin{cases} b^{(s)}_n(\nu), & s = 0, \\ (-1)^S b^{(s)}_n(\nu) + \frac{1}{\pi} \sum_{r=1}^{S} (-1)^r \Re \left( \frac{1}{2}, s - \gamma + \frac{1}{2} \right) b^{(s-r)}_n(\nu), & s \geq 1, \end{cases} \] (8.4.55)

where

\[ b^{(s)}_1(\nu) = \nu^S, \] (8.4.54)

and

\[ \lambda^{(s)} = \sum_{n=1}^{\infty} \lambda^{(s)}_n, \] (8.4.55)
so that (8.4.48) and (8.4.49) can be written in the form

\[
\frac{\text{e}}{2}\left[ 2\int_{0}^{\infty} L_{n}(t) e^{2i\frac{\text{K}}{t}} dt \right] = \begin{cases} 
\chi_{n}^{(0)}, & s = 0, n > 1, \\
-\sum_{r=1}^{s} (-1)^{r} x_{n}^{(r)} \bar{f}(\frac{3}{2} s - r + \frac{1}{2}) \chi_{n}^{(s-r)}, & s > 1, n > 1.
\end{cases}
\] (8.4.56)

In order to evaluate \( \chi_{n}^{(s)} \), we replace \( L_{n}, e_{n}(1), e_{n}(1), e_{n}(0) \) and \( e_{n}(0) \) in (8.2.32) by \( L_{n}^{(s)}, L_{n}^{(s)}(1), L_{n}^{(s)}(1), L_{n}^{(s)}(1) \) and \( L_{n}^{(s)}(0) \) respectively, where \( L_{n}^{(s)} \) is given by

\[
L_{n}^{(s)} = -\frac{1}{8} \left[ L_{n}^{(s)}(1) + \frac{q}{16} \left( \frac{L_{n}^{(s)}(1) - L_{n}^{(s)}(1)}{\text{K}} \right) \right].
\] (8.4.58)

Hence with this in mind, and by using the above definitions, we show that

\[
\chi_{n}^{(s)} = -\frac{e^{i\frac{3}{2}s - \frac{1}{2}}}{2\sqrt{\pi} \text{K}^{1/2}} \left[ b_{n}^{(s)}(1) + \frac{i}{4} \left( b_{n}^{(s)}(1) - b_{n}^{(s)}(1) \right) \right] - \frac{3}{32 \text{K}^{1/2}} \left[ b_{n}^{(s)}(1) - b_{n}^{(s)}(1) + \frac{1}{8} \frac{b_{n}^{(s)}(1)}{\text{K}} \right]
\]

\[
+ \frac{1}{2} b_{n}^{(s)}(1) - \frac{q}{32 \text{K}^{1/2}} b_{n}^{(s)}(1) \right] - \frac{e^{i\frac{3}{2}s - \frac{1}{2}}}{\text{K}} \left[ b_{n}^{(s)}(1) + \frac{q}{16} \frac{b_{n}^{(s)}(1)}{\text{K}} \right] + O \left( \frac{b_{n}^{(s)}}{\text{K}^{1/2}} \right),
\] (8.4.59)

where

\[
b_{n}^{(s)}(1) = \frac{1}{8} \left[ b_{n}^{(s)}(1) + \frac{q}{16} \left( \frac{b_{n}^{(s)}(1) - b_{n}^{(s)}(1)}{\text{K}} \right) \right].
\] (8.4.60)

For \( s = 0 \), we have from (8.4.54)

\[
b_{1}^{(0)}(1) = 1.
\] (8.4.61)
and from (8.4.60)

\[ b^{(o)}_{i,0} = \frac{i}{8} \left( 1 + \frac{q_i}{32k} \right). \tag{8.4.62} \]

From (8.4.55) and (8.4.59), we obtain

\[
\lambda = \frac{e^{i k + \frac{\pi}{4}}}{2^{\frac{3}{2}k + \frac{1}{4}}} \left[ b^{(o)}_{i,1} (1) + \frac{i}{16k} b^{(o)}_{i,2} (1) - \frac{q_i}{512k^2} b^{(o)}_{i,1} (1) + \frac{2 \sqrt{2}}{k} b^{(o)}_{i,0} \left( 1 + \frac{i}{4k} \right) \right.

- \frac{q_i 2 \sqrt{2}}{128k^2} b^{(o)}_{i,0} + b^{(o)}_{i,2} (1) - \frac{i}{k} b^{(o)}_{i,2} (1) + \frac{2 \sqrt{2}}{k} b^{(o)}_{i,2} (1) + b^{(o)}_{i,2} (1) \right]

- \frac{e^{i k}}{\pi k} \left[ b^{(o)}_{i,0} + \frac{q_i}{256k} b^{(o)}_{i,0} + b^{(o)}_{i,2} (1) \right] + O \left( \frac{1}{k^2} \right). \tag{8.4.63} \]

In (8.2.17) to (8.2.21) we can replace \( e_n \) by \( e^{(s)}_n \), and therefore by \( b^{(s)}_n \). Hence with this in mind it can be shown that

\[ b^{(o)}_{2,1} = \frac{i}{8k} \left( 1 - 2 \sqrt{2} \right) \left( 1 + \frac{q_i}{32k} \right) + \frac{2e^{i k - \frac{\pi}{4}}}{16k^{\frac{3}{2}k + \frac{1}{2}}} - \frac{q_i}{128k^2} \left( \frac{3}{2} - 2 \sqrt{2} \right) + O \left( \frac{1}{k^2} \right). \tag{8.4.64} \]

\[ b^{(o)}_{2,1} = \frac{i}{8k} \left( \frac{3}{2} - 2 \sqrt{2} \right) + O \left( \frac{1}{k^2} \right). \tag{8.4.65} \]

\[ b^{(o)}_{3,1} = \frac{1}{128k^2} \left( 1 - 2 \sqrt{2} \right) + O \left( \frac{1}{k^2} \right). \tag{8.4.66} \]

and

\[ \tag{8.4.67} \]
On substituting these last few results into (8.4.63) we arrive at

\[ \chi^{(0)} = \frac{-i k + \frac{\mu_i}{\mu}}{2 \pi^{3/2} \sqrt{\kappa}} \left[ 1 + \frac{3i}{16 \kappa} - i \frac{2i k + \frac{\mu_i}{\mu}}{16 \pi^{3/2} \kappa^{3/2}} - \frac{113}{512 \kappa} \right] - \frac{i k}{\pi \kappa} \left[ \frac{3}{8} + \frac{1}{128 \kappa} \right] + O \left( \frac{1}{\kappa^3} \right), \]

\[ (8.4.65) \]

and hence from (8.4.65) and (8.4.68)

\[ \chi^{(0)} = \frac{e^{i k} - 2 i \frac{\mu_i}{\mu}}{2 \pi^{1/2} \kappa^{3/2}} \left[ 1 - i \frac{3 i \frac{2i k + \frac{\mu_i}{\mu}}{16 \pi^{3/2} \kappa^{3/2}}}{128 \pi^{3/2} \kappa^{3/2}} - \frac{113 e^{i k - \frac{\mu_i}{\mu}}}{1024 \pi^{3/2} \kappa^{3/2}} + O \left( \frac{1}{\kappa^3} \right) \right]. \]

\[ (8.4.68) \]

In a similar manner we have, for \( s = 1 \)

\[ \chi^{(1)} = \frac{e^{i k}}{2\pi} - \chi^{(0)} \left( - \frac{1}{2} \right), \]

\[ (8.4.70) \]

\[ b_1^{(1)}(1) = 1, \quad b_1^{(1)}(0) = 0, \]

\[ b_1^{(n)}(0) = \frac{q}{128 \kappa}, \]

\[ \chi^{(1)} = \frac{-i k + \frac{\mu_i}{\mu}}{2 \pi^{3/2} \kappa^{3/2}} \left[ b_1^{(1)}(1) + 4 \left( b_1^{(1)}(1) - b_1^{(1)}(0) \right) + b_1^{(n)}(1) \right] - \frac{e^{i k}}{\pi \kappa} b_1^{(n)}(0) + O \left( \frac{1}{\kappa^{3/2}} \right), \]

\[ (8.4.71) \]

and

\[ b_2^{(n)}(1) = \frac{-2 i \frac{\mu_i}{\mu}}{16 \pi^{3/2} \kappa^{3/2}} + O \left( \frac{1}{\kappa^2} \right), \]

\[ (8.4.72) \]

so that

\[ \chi^{(n)} = \frac{e^{i k + \frac{\mu_i}{\mu}}}{2 \pi^{3/2} \kappa^{3/2}} \left[ 1 - \frac{3i}{16 \kappa} - i \frac{2i k + \frac{\mu_i}{\mu}}{16 \pi^{3/2} \kappa^{3/2}} - \frac{9 e^{i k}}{128 \pi^{3/2} \kappa^{3/2}} + O \left( \frac{1}{\kappa^{3/2}} \right) \right]. \]

\[ (8.4.73) \]
Hence from (8.4.63), (8.4.70) and (8.4.73)

\[
\gamma = \frac{e^{ik}}{2\pi} \left\{ 1 - \frac{3e^{-2ik + \frac{E}{2}}}{2\pi^{2} 2^{3/2} \frac{V}{k^{3/2}}} + \frac{i}{8k} + \frac{3i e^{-2ik + \frac{E}{2}}}{32\pi^{2} \frac{V}{k^{3/2}}} + \frac{1}{16} \frac{e^{-2ik + \frac{E}{2}}}{32\pi^{2} \frac{V}{k^{3/2}}} + O\left(\frac{1}{k^{3/2}}\right) \right\}.
\]

(8.4.74)

Similarly for \( s = 2 \), we obtain

\[
\gamma^{(2)} = \frac{3e^{ik}}{8\pi} + \frac{\lambda^{(2)} - \frac{1}{2} \lambda^{(1)} - \frac{1}{8} \lambda^{(0)}}{8},
\]

(8.4.75)

\[
b_{1}^{(2)}(1) = 1, \quad b_{1}^{(2)}(0) = 0,
\]

\[
b_{1}^{(2)}(1) = 2, \quad b_{1}^{(2)}(0) = 0,
\]

\[
b_{1}^{(2)}(v) = 2, \quad b_{1,0}^{(2)} = 0,
\]

\[
\lambda^{(2)} = \frac{e^{i \frac{k}{16} + \frac{E}{2}}}{2\pi^{3/2} 2^{3/2} \frac{V}{k^{3/2}}} \left[ b_{1}^{(2)}(1) + \frac{i}{4\pi k} \left( b_{1}^{(2)}(1) - b_{1}^{(2)}(1) \right) \right] + O\left(\frac{1}{k^{3/2}}\right),
\]

\[
= \frac{-e^{i \frac{k}{16} + \frac{E}{2}}}{2\pi^{3/2} 2^{3/2} \frac{V}{k^{3/2}}} \left( 1 - \frac{7i}{16k} \right) + O\left(\frac{1}{k^{3/2}}\right),
\]

(8.4.76)

and from (8.4.63), (8.4.73), (8.4.75) and (8.4.76) it is seen that

\[
\gamma^{(2)} = \frac{e^{ik}}{8\pi} \left\{ 3 + \frac{i e^{-2ik + \frac{E}{2}}}{2\pi^{2} 2^{3/2} \frac{V}{k^{3/2}}} + \frac{1}{8k} - \frac{7i e^{-2ik + \frac{E}{2}}}{32\pi^{2} \frac{V}{k^{3/2}}} + O\left(\frac{1}{k^{3/2}}\right) \right\}.
\]

(8.4.77)
For \( s = 3 \), we have from (8.4.47)

\[
\gamma^{(3)} = \frac{5 e^{i^{(1)}}}{16\pi} - \lambda^{(3)} - \frac{1}{2} \lambda^{(2)} + \frac{1}{8} \lambda^{(1)} - \frac{1}{16} \lambda^{(0)},
\]

and in a like manner to the above we deduce that

\[
b^{(3)}_1(1) = 1, \quad b^{(3)}_{1,0} = 0,
\]

\[
\lambda^{(3)} = \frac{e^{-i^{(1)}}}{2^{3/2} K^{1/2}} b^{(3)}_0(1) + O\left(\frac{1}{K^{3/2}}\right),
\]

\[
= \frac{e^{-i^{(1)}}}{2^{3/2} K^{1/2}} + O\left(\frac{1}{K^{3/2}}\right),
\]

and

\[
\gamma^{(3)} = \frac{e^{i^{(1)}}}{16\pi} \left\{ 5 - \frac{23}{2} \frac{e^{i^{(1)}}}{K^{1/2}} + \frac{i^{(1)}}{8K} + O\left(\frac{1}{K^{3/2}}\right) \right\}.
\]

For \( s = 4 \), we have

\[
\gamma^{(4)} = \frac{35 e^{i^{(1)}} + \lambda^{(4)} + \frac{1}{2} \lambda^{(3)} + \frac{1}{8} \lambda^{(2)} - \frac{1}{16} \lambda^{(1)} - \frac{5}{128} \lambda^{(0)}}{128\pi},
\]

\[
b^{(4)}_1(1) = 1, \quad b^{(4)}_{1,0} = 0,
\]

\[
\lambda^{(4)} = \frac{e^{i^{(1)}}}{2^{3/2} K^{1/2}} + O\left(\frac{1}{K^{3/2}}\right),
\]

so that from (8.4.63), (8.4.75), (8.4.76), (8.4.79) and (8.4.82)

\[
\gamma^{(4)} = \frac{e^{i^{(1)}}}{128\pi} \left\{ 35 + 179 \frac{e^{i^{(1)}}}{2^{3/2} K^{1/2}} + O\left(\frac{1}{K^{3/2}}\right) \right\}.
\]
Finally, for \( s = 5 \) we have from (8.4.47)

\[
\gamma^{(s)} = \frac{63}{256}\frac{e^{i k}}{\pi} + O\left(\frac{1}{k^{1/2}}\right). \tag{8.4.84}
\]

We substitute (8.4.60), (8.4.74), (8.4.77), (8.4.80), (8.4.83) and (8.4.84) into (8.4.45), to obtain

\[
\gamma = \frac{e^{i k}}{\pi} \left\{ 1 + \frac{1}{2k^{1/2}} \left( -\frac{2i e^{i k + \frac{\pi}{2}}}{\pi^{1/2}} - \frac{i}{k^{1/2}} \left( 1 - \varepsilon^2 \right)^2 \right) - \frac{i}{8k} \left( 1 - \frac{6e^{-2i (k + \frac{\pi}{2})}}{\pi^{1/2}} \right) \right\} (1 - \varepsilon^2)^2
\]

\[
+ \frac{2k^{1/2}}{q} (1 - \varepsilon^2)^3 \left\{ \frac{1}{4} + \frac{i e^{-i k + \frac{\pi}{2}}}{\pi} - \frac{3e^{-2i (k + \frac{\pi}{2})}}{2\pi^{1/2}} \right\}
\]

\[
+ \frac{113}{2} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^2 + \frac{23}{15} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^3 - \frac{1}{12} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^4 \right\}
\]

\[
\right. \quad \frac{1}{64k^{1/2}} \left( 1 - \frac{6e^{-2i (k + \frac{\pi}{2})}}{\pi^{1/2}} \right)
\]

\[
+ \frac{19}{4} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^2 - \frac{263}{12} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^3 - \frac{3e^{-2i (k + \frac{\pi}{2})}}{\pi} \right\}
\]

\[
+ \frac{2q}{6} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^3 \left\{ \frac{1}{4} - \frac{2i e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^4 + \frac{i}{6q} \frac{e^{i k + \frac{3\pi}{2}}}{\pi^{1/2}} (1 - \varepsilon^2)^5 \right\} + O\left(\frac{1}{k^{1/2}}\right). \tag{8.4.85}
\]

We next consider the asymptotic evaluation of \( \int_0^1 \omega \psi^{(s)}(\omega) d\omega \).

By using (8.4.41) we write
\[
\int_0^1 \omega q^{(s)}(\omega) d\omega = \left\{ \begin{array}{l}
\frac{1}{\pi} \int_0^1 \left( \frac{\omega}{1 - \omega} \right)^{\nu} e^{iKw} + d^{(\nu)}, \quad s = 0, \\
- \frac{1}{\pi^2} \int_0^1 \left( \frac{\omega}{1 - \omega} \right)^{\nu} e^{iKw} \sum_{r=1}^{\infty} \omega^{r-1} b(\nu_2, s - r + \frac{1}{2}) - \pi w^{\nu} d\omega + (-1)^{\nu} d^{(s)} \\
+ \frac{1}{\pi^2} \sum_{r=1}^{\infty} (-1)^r b(\nu_2, s - r + \frac{1}{2}) d^{(s - r)}, \quad s \geq 1.
\end{array} \right.
\] (8.4.87)

We let
\[
d^{(s)} = \sum_{n=1}^{\infty} d_n^{(s)};
\] (8.4.88)

then, by using reasoning similar to that employed in obtaining (8.4.151), we show that
\[
d_n^{(s)} = \frac{e^{-\nu/4}}{2\nu^{3/2}} b_n^{(s)}(0) - \frac{e^{iK}}{2\pi iK} \left\{ b_n^{(s)}(1) - \frac{7}{6} b_n^{(s)}(1) - \frac{5}{4} b_n^{(s)}(1) + \frac{2}{8} b_n^{(s)}(0) \right\}
\]
\[
- \frac{e^{iK + \pi i}}{8\pi^{1/2} K^{3/2}} b_n^{(s)}(0) + O\left( \frac{b_n^{(s)}}{K^{3/2}} \right). 
\] (8.4.89)

It is convenient to consider the asymptotic evaluation of
\[
\int_0^1 \frac{e^{iKw} \omega \nu/2}{(1 - \omega)^{\nu/2}} d\omega.
\]

On deforming the path of integration in the, by now, familiar way we have
\[
\int_0^1 \frac{e^{iKw} \omega \nu/2}{(1 - \omega)^{\nu/2}} d\omega = e^{iK - \pi i/4} \int_0^\infty \frac{e^{-Kw}}{(1 + i\omega)^{\nu/2}} d\omega + e^{iK - \pi i/4} \int_0^\infty \frac{e^{-Kw}}{(1 - i\omega)^{\nu/2}} d\omega,
\]
\[
\sim e^{iK - \pi i/4} \int_0^\infty \frac{e^{-Kw}}{(1 + (\nu + 1/2)i\omega)^{\nu/2}} d\omega + e^{iK - \pi i/4} \int_0^\infty \frac{e^{-Kw}}{(1 - (\nu + 1/2)i\omega)^{\nu/2}} d\omega,
\]
For \( s = 0 \), we have from (8.4.88) and (8.4.89)

\[
d^{(s)} = \frac{-e^{-\frac{i kr}{\sqrt{2}}} b_s^{(0)}(0)}{2 \sqrt{2} R^{\sqrt{2}}} - \frac{-e^{-\frac{i kr}{\sqrt{2}}} \left\{ b_s^{(0)}(0) - \frac{7}{16} b_s^{(0)}(0) - \frac{5}{4} b_s^{(0)}(0) + 2 \frac{1}{4} b_s^{(0)}(0) + b_s^{(0)}(0) \right\}}{8 \sqrt{2} R^{\sqrt{2}}} + \frac{1}{k R^{\sqrt{2}}},
\]

and therefore from (8.4.91) and previous results we obtain

\[
d^{(s)} = \frac{-e^{-\frac{i kr}{\sqrt{2}}} b_s^{(0)}(0)}{2 \sqrt{2} R^{\sqrt{2}}} - \frac{-e^{-\frac{i kr}{\sqrt{2}}} \left\{ b_s^{(0)}(0) - \frac{7}{16} b_s^{(0)}(0) - \frac{5}{4} b_s^{(0)}(0) + 2 \frac{1}{4} b_s^{(0)}(0) + b_s^{(0)}(0) \right\}}{8 \sqrt{2} R^{\sqrt{2}}} + \frac{1}{k R^{\sqrt{2}}},
\]

(8.4.92) gives

\[
\int \frac{e^{ikx y/2}}{\omega^{1/2}} d\omega = -\frac{e^{-\frac{i kr}{\sqrt{2}}} \left\{ b_s^{(0)}(0) - \frac{7}{16} b_s^{(0)}(0) - \frac{5}{4} b_s^{(0)}(0) + 2 \frac{1}{4} b_s^{(0)}(0) + b_s^{(0)}(0) \right\}}{8 \sqrt{2} R^{\sqrt{2}}} + \frac{1}{k R^{\sqrt{2}}},
\]

and hence from (8.4.88), (8.4.92) and (8.4.93) it is seen that

\[
\int \omega q_s^{(0)}(\omega) d\omega = -\frac{e^{-\frac{i kr}{\sqrt{2}}} \left\{ b_s^{(0)}(0) - \frac{7}{16} b_s^{(0)}(0) - \frac{5}{4} b_s^{(0)}(0) + 2 \frac{1}{4} b_s^{(0)}(0) + b_s^{(0)}(0) \right\}}{8 \sqrt{2} R^{\sqrt{2}}} + \frac{1}{k R^{\sqrt{2}}},
\]

and from (8.4.93) and (8.4.94) we obtain

\[
\int e^{ikx y/2} d\omega = \frac{-e^{-\frac{i kr}{\sqrt{2}}} \left\{ b_s^{(0)}(0) - \frac{7}{16} b_s^{(0)}(0) - \frac{5}{4} b_s^{(0)}(0) + 2 \frac{1}{4} b_s^{(0)}(0) + b_s^{(0)}(0) \right\}}{8 \sqrt{2} R^{\sqrt{2}}} + \frac{1}{k R^{\sqrt{2}}},
\]

so that by using the above results with (8.4.93) and (8.4.92) we
arrive at
\[
\int_0^{\infty} w q_1^{(1)}(w) dw = \frac{e^{i k - \frac{\pi}{4}}}{2\pi i^{\frac{1}{2}} \frac{\sqrt{2}}{R}} \left\{ 1 + 3 e^{-2i k + \frac{\pi}{4}} + 3 i + O(\frac{1}{k^{\frac{3}{2}}}) \right\}. \quad (8.4.96)
\]

In a similar manner we obtain, for \( s = 2 \)
\[
\int_0^{\infty} w q_1^{(2)}(w) dw = \frac{1}{\kappa} \int_0^{\frac{\sqrt{2}}{\kappa}} e^{i k w} e^{\frac{\pi}{4}} dw - \frac{1}{8\pi} \int_0^{\infty} e^{i k w} \frac{\sqrt{2}}{\kappa} dw - \frac{1}{8\pi} \int_0^{\infty} e^{i k w} \frac{\sqrt{2}}{\kappa} dw
\]
\[
+ d^{(2)} + \frac{1}{2} d^{(1)} - \frac{1}{8} d^{(0)}, \quad (8.4.99)
\]

\[
\int_0^{\frac{\sqrt{2}}{\kappa}} e^{i k w} \frac{\sqrt{2}}{\kappa} dw = \frac{\sqrt{2}}{\kappa} e^{i k - \frac{\pi}{4}} + O(\frac{1}{k^{\frac{3}{2}}}), \quad (8.4.100)
\]

\[
d^{(2)} = \frac{-e^{i k}}{2\pi k} + O(\frac{1}{k^2}), \quad (8.4.101)
\]

and
\[
\int_0^{\infty} w q_1^{(2)}(w) dw = \frac{e^{i k - \frac{\pi}{4}}}{8\pi i^{\frac{1}{2}} \frac{\sqrt{2}}{R}} \left\{ 3 - \frac{1}{16} e^{2i k + \frac{\pi}{4}} + O(\frac{1}{k^4}) \right\}. \quad (8.4.102)
\]

Finally, we have for \( s = 3 \)
\[
\int_0^{\infty} w q_1^{(3)}(w) dw = \frac{1}{\kappa} \int_0^{\frac{\sqrt{2}}{\kappa}} e^{i k w} \frac{\sqrt{2}}{\kappa} dw - \frac{1}{2\pi} \int_0^{\infty} e^{i k w} \frac{\sqrt{2}}{\kappa} dw - \frac{1}{8\pi} \int_0^{\infty} e^{i k w} \frac{\sqrt{2}}{\kappa} dw
\]
\[
- \frac{1}{16\pi} \int_0^{\infty} e^{i k w} \frac{\sqrt{2}}{\kappa} dw + O(\frac{1}{k^4}), \quad (8.4.103)
\]

\[
\int_0^{\frac{\sqrt{2}}{\kappa}} e^{i k w} \frac{\sqrt{2}}{\kappa} dw = \frac{\sqrt{2}}{\kappa} e^{i k - \frac{\pi}{4}} + O(\frac{1}{k^{\frac{3}{2}}}), \quad (8.4.104)
\]
and
\[ \int_0^1 \omega q^{(1)}(\omega) d\omega = \frac{e^{\frac{iK}{4} - \frac{\Omega}{4}}}{16 \pi^2 \frac{K}{R}} \left\{ 5 + O \left( \frac{1}{K^2} \right) \right\}. \quad (8.4.105) \]

We further approximate (8.4.99) to
\[ \Gamma \left( \frac{1}{2} \right) q^{(0)}(\omega) \sim q^{(0)}(\omega) - \frac{i}{\sqrt{K}} \left( q^{(3/2)}(1 - \epsilon^2) q^{(0)}(\omega) - \frac{1}{3} K^3 (1 - \epsilon^2)^2 q^{(2)}(\omega) \right) \]
\[ + \frac{i}{\sqrt{K}} \left( q^{(3/2)}(1 - \epsilon^2)^2 q^{(0)}(\omega) \right). \quad (8.4.106) \]

We substitute (8.4.96), (8.4.98), (8.4.102) and (8.4.105) in (8.4.106) to give
\[ \int_0^1 \omega q^{(1)}(\omega) d\omega = \frac{e^{\frac{iK}{4} - \frac{\Omega}{4}}}{\pi^2 \frac{K}{R}} \left\{ 1 - \frac{1}{2 \pi^2} \left\{ -\frac{2iK + \Omega}{\pi^2} + i K^3 (1 - \epsilon^2) \right\} \right\} \]
\[ + \frac{i}{8K} \left\{ 1 - \frac{6e^{2iK + \Omega}}{\pi^2} K^3 (1 - \epsilon^2) + i K^3 (1 - \epsilon^2)^2 \right\} \]
\[ + \frac{i}{16 K^{3/2}} \left\{ \frac{15i e^{2iK + \Omega}}{\pi^2} - 661 K^2 (1 - \epsilon^2) - 221 e^{2iK + \Omega} K^3 (1 - \epsilon^2)^2 \right\} \]
\[ + 2 K^{3/2} (1 - \epsilon^2)^3 \right\} + O \left( \frac{1}{K^2} \right) \right\}. \quad (8.4.107) \]

It is seen from (8.4.51), (8.2.155), (8.4.85) and (8.4.107) that we can now write down \( \Xi \). However, once again there is some lengthy algebra involved and only the final result will
be given, i.e.

\[ M = -i \frac{8 \sqrt{\pi}}{k^2 \varepsilon^2} \mu c u \varepsilon^{1/2} e^{i \frac{k \xi}{\varepsilon}} \left[ 1 + \frac{1}{2 \varepsilon^{1/2}} \left\{ e^{-2i \frac{k \xi}{\varepsilon}} + \frac{e^{-2i \frac{k \xi}{\varepsilon}} + e^{2i \frac{k \xi}{\varepsilon}}}{2} - i \frac{k}{\varepsilon^{1/2}} (1 - \varepsilon^2) \right\} \right] \]

\[ + \frac{1}{8k} \left\{ q + \frac{6 e^{-2i k \xi + i \frac{k \xi}{\varepsilon}} k^{3/2} (1 - \varepsilon^2) + i k^3 (1 - \varepsilon^2)^2}{\pi^{1/2}} \right\} - \frac{i}{8k} \left\{ 3 e^{-2i \frac{k \xi}{\varepsilon}} + \frac{e^{-2i \frac{k \xi}{\varepsilon}} + e^{2i \frac{k \xi}{\varepsilon}}}{2} - \frac{2}{128k^2} \right\} \]

\[ + \frac{1}{278} e^{-i k \xi + i \frac{k \xi}{\varepsilon}} k^{3/2} (1 - \varepsilon^2) + \frac{1}{15} e^{-2i \frac{k \xi}{\varepsilon}} k^3 (1 - \varepsilon^2)^2 + \frac{1}{18} e^{-i k \xi + i \frac{k \xi}{\varepsilon}} k^{3/2} (1 - \varepsilon^2)^3 \]

\[ - \frac{1}{3} k^3 (1 - \varepsilon^2)^4 \left\{ \frac{387 e^{-2i k \xi + i \frac{k \xi}{\varepsilon}}}{8 \pi^{1/2}} - \frac{9 \pi i}{2} e^{-i k \xi + i \frac{k \xi}{\varepsilon}} k^{3/2} (1 - \varepsilon^2) + \frac{18}{15} e^{-2i \frac{k \xi}{\varepsilon}} k^3 (1 - \varepsilon^2)^3 \right\} \]

\[ - \frac{735 \pi}{2} e^{-2i k \xi + i \frac{k \xi}{\varepsilon}} k^{3/2} (1 - \varepsilon^2)^2 + \frac{3 \pi}{3} e^{-2i \frac{k \xi}{\varepsilon}} k^3 (1 - \varepsilon^2)^3 \right\} \]

\[ + \frac{1}{10} k^{15/2} (1 - \varepsilon^2)^5 \right\} + O \left( \frac{1}{k^3} \right) \right] \]

(8.4.108)

From (8.1.52) and (8.4.108) it is easily shown that

\[ h_{1z} = -\frac{8 \sqrt{\pi} \xi^{1/2} k}{k^2 \varepsilon^2} \left[ \sin \left( k \xi - \frac{\pi}{4} \right) - \frac{1}{2 \varepsilon^{1/2}} \left\{ \sin \frac{k \xi}{\varepsilon} + k^{3/2} (1 - \varepsilon^2) \cos \left( k \xi - \frac{\pi}{4} \right) \right\} \right] \]

\[ + \frac{1}{8k} \left\{ q \cos \left( k \xi - \frac{\pi}{4} \right) + \frac{6 k^{3/2} (1 - \varepsilon^2) \cos k \xi}{\pi^{1/2}} - k^3 (1 - \varepsilon^2) \sin \left( k \xi - \frac{\pi}{4} \right) \right\} \]
\[
- \frac{1}{96 \pi^{\gamma/2}} \left[ \frac{\pi^{\gamma/2}}{\pi^{\gamma/2}} + 6 \pi^{\gamma/2} (1-\varepsilon^2) \sin (k-\pi/14) - \frac{22}{\pi^{\gamma/2}} \pi^{\gamma/2} (1-\varepsilon^2)^2 \sin k \right] - 2 \pi^{\gamma/2} \left[ (1-\varepsilon^2)^3 \cos (k-\pi/14) \right] - \frac{1}{128 \pi^{\gamma/2}} \left[ 57 \sin (k-\pi/14) - \frac{1}{\pi^{\gamma/2}} \cos (3 \pi-\pi/14) \right] \\
+ \frac{278}{\pi^{\gamma/2}} \left[ (1-\varepsilon^2)^3 \sin k + 4 \pi^{\gamma/2} (1-\varepsilon^2)^2 \cos (k-\pi/14) - \frac{9}{\pi^{\gamma/2}} \pi^{\gamma/2} (1-\varepsilon^2)^3 \sin k \right] \\
+ \frac{1}{3} \left[ (1-\varepsilon^2)^4 \sin (k-\pi/14) \right] + \frac{1}{384 \pi^{\gamma/2}} \left[ \frac{387}{8} \sin k + 9 \left( \frac{\pi^{\gamma/2}}{\pi^{\gamma/2}} + 3 \pi^{\gamma/2} (1-\varepsilon^2)^2 \sin (k-\pi/14) \right) \right] \\
+ \frac{2}{\pi} \sin (3 \pi-\pi/14) \left[ (1-\varepsilon^2)^3 \sin k - \frac{735}{2} \pi^{\gamma/2} (1-\varepsilon^2)^2 \sin (k-\pi/14) + 39 \pi^{\gamma/2} (1-\varepsilon^2)^3 \sin (k-\pi/14) \right] \\
- \frac{179}{70} \pi^{\gamma/2} (1-\varepsilon^2)^4 \sin k - \frac{1}{10} \pi^{\gamma/2} (1-\varepsilon^2)^5 \cos (k-\pi/14) \right] + O \left( \frac{1}{\pi^{\gamma/2}} \right),
\]

\[ (8.4.109) \]
\[- \frac{278}{\pi^{3/2}} k^{3/2} (1 - \varepsilon^2) \cos k - \frac{4 \pi^{3/2}}{5} (1 - \varepsilon^2)^2 \sin (k - \pi/4) + \frac{12}{25 \pi^{3/2}} k^{3/2} (1 - \varepsilon^2)^3 \cos k \]

\[- \frac{1}{3} k^{6} (1 - \varepsilon^2)^4 \cos^3 (k - \pi/4) \right] - \frac{1}{384 k^{3/2}} \left[ \frac{387}{8 \pi^{3/2}} \cos k + 9 \left( \frac{111}{2} \sin (k - \pi/4) \right) \right] \]

\[+ \frac{735}{2} k^{2} (1 - \varepsilon^2) ^2 \cos k + 39 k^{3/2} (1 - \varepsilon^2)^3 \cos (k - \pi/4) \]

\[- \frac{179}{70} k^{6} (1 - \varepsilon^2)^4 \cos^3 k - \frac{1}{10} k^{5/2} (1 - \varepsilon^2)^5 \sin (k - \pi/4) \right] + O\left( \frac{1}{k^3} \right) \]

(2.4.110)

From the last two expressions we can calculate $g_1$, $g_2$ and $\Delta$. 
6.3 Numerical results and discussion

The variation of \( \sin \Lambda \) and \( \cos \Lambda \), for \( \varepsilon = 0.0(0.1)1.0 \) and \( \varepsilon = 0.32 \), and for \( 2 \leq k \leq 10 \) is shown in figures 32 to 55.

It was decided not to consider \( \tan \Lambda \), unlike Thomas [23], since for even fairly small values of \( k \), \( \Lambda \) can be an odd multiple of \( \pi/2 \). For all the values of \( \varepsilon \) and \( k \) considered, it appears that the relevant cases give numerical values for \( \sin \Lambda \) and \( \cos \Lambda \) which satisfy \( \sin^2 \Lambda + \cos^2 \Lambda = 1 \), to within 1%.

The quantities \( \sin \Lambda \) and \( \cos \Lambda \) were calculated numerically by, first of all obtaining numerical values for \( h_1 \) and \( h_2 \), and then inserting these numerical values in

\[
\sin \Lambda = -\frac{h_2}{\sqrt{h_1^2 + h_2^2}},
\]

and

\[
\cos \Lambda = \frac{h_1}{\sqrt{h_1^2 + h_2^2}}.
\]

Analytic expressions were not obtained for \( \sin \Lambda \) and \( \cos \Lambda \), because of the laborious nature of the algebra.

\( \varepsilon = 0.0 \) and \( \varepsilon = 0.1 \), figures 32, 33, 34 and 35

These are all examples of Case II.
$\varepsilon = 0.2$ and $\varepsilon = 0.5$, figures 36, 37, 38 and 39

Case II should give accurate results for $2 \leq k \leq 10$.

$\varepsilon = 0.4$, figures 40 and 41

For $2 \leq k \leq 0.5$ we consider Case II and for $k > 0.5$ we consider Case I.

$\varepsilon = 0.5$, figures 42 and 43

For $2 \leq k \leq 7$ we consider Case II and for $k > 7$ we consider Case I.

$\varepsilon = 0.6$, figures 44 and 45

For $2 \leq k \leq 6.5$ we consider Case II and for $k > 6.5$ we consider Case I.

$\varepsilon = 0.7$, figures 46 and 47

For $2 \leq k \leq 5.3$ we consider Case II and for $k > 5.3$ we consider Case I.

$\varepsilon = 0.8$, figures 48 and 49

When we considered the dynamic stress-intensity factor, for $\varepsilon = 0.8$, there appeared to be significant "gaps" between the ranges of validity of the different cases and, we see that a
similar situation arises here. However, this time it seems that, since we have considered more terms when evaluating Case II, (c.f. (3.3.26), (8.3.27) and (3.2.50)), we can use this Case for certain k. For k > 8, the coloured curves should give reasonably accurate results, but for 4 ≤ k ≤ 8, there is a certain amount of doubt.

ε = 0.9, figures 50 and 51

Case III should give accurate results for 2 ≤ k ≤ 10.

ε = 0.92 and ε = 1.0, figures 52, 53, 54 and 55

These are examples of Case III.
Case II

Epsilon = 0.0
Case II

EPSILON = 0.0
Epsilon = 0.7

Case I

Normalized Wave Number

Case II
Section 9

Scattering coefficient

9.1 Introduction

The definition of the scattering coefficient $Q$ and the spherical polar coordinate system adopted in subsection 5.1, are used here. Since the incident field used in Section 5 is the incident field used here, we can use expressions (5.1.2), (5.1.3) and (5.1.16) for $u^{(1)}(R, \phi)$, $\sigma^{(1)}(R, \phi)$ and $E_1$ respectively. However, we must derive different expressions for $u^{(s)}(R, \phi)$, $\sigma^{(s)}_{R\theta}(R, \phi)$ and $E_s$.

From (6.3.2) and (6.3.6) we have

$$u^{(s)}_{e}(R, \phi) = -\frac{1}{4\pi} \int_{0}^{2\pi} t \sigma(t) \psi_{1}(R, \phi; t) dt,$$  \hspace{1cm} (9.1.1)

where

$$\psi_{1}(R, \phi; t) = \frac{2\pi}{\exp \left\{ -ik(R^2 + t^2 - 2Rt \sin \phi \cos \theta') \right\} \sqrt{2} \cos \theta' d\theta' \left( R^2 + t^2 - 2Rt \sin \phi \cos \theta' \right)^{1/2}} \cos \theta' d\theta'.$$  \hspace{1cm} (9.1.2)

For large $R$ we write (9.1.2) as

$$\psi_{1}(R, \phi; t) \sim \frac{e^{ikR}}{R} \int_{0}^{2\pi} e^{ikt \sin \phi \cos \theta'} \cos \theta' d\theta',$$  \hspace{1cm} (9.1.3)

and on using (8.2.63) this becomes

$$\psi_{1}(R, \phi; t) \sim 2\pi \frac{e^{ikR}}{R} J_i(k \ell \sin \phi).$$  \hspace{1cm} (9.1.4)
We let
\[ \Psi(\sin \phi) = \int_0^1 \sigma(t) J_1(kt \sin \phi) \, dt; \]  
(9.1.5)

then (9.1.4) can be written in the form
\[ u_{(s)}(R, \phi) \sim -\frac{1}{2} \frac{e^{iKR}}{R} \Psi(\sin \phi), \quad \text{as } R \to \infty. \]  
(9.1.6)

We can show, from (9.1.6), that
\[ a^{(s)}(R, \phi) \sim -\frac{\mu R}{2} \frac{e^{iKR}}{R} \Psi(\sin \phi). \]  
(9.1.7)

We obtain a different formula for \( \Psi(\sin \phi) \) by integrating
(9.1.5) once, by parts, and then using (6.4.9) to give
\[ \Psi(\sin \phi) = \int_0^1 \left\{ \frac{d}{\frac{d}{dt}} \int_0^t J_1(kt \sin \phi) \right\} \kappa(t) \, dt. \]  
(9.1.8)

However,
\[ \frac{d}{dt} J_1(kt \sin \phi) = \tan \phi \frac{d}{d \phi} J_1(kt \sin \phi), \]  
(9.1.9)

and therefore we have
\[ \Psi(\sin \phi) = \left(1 + \tan \phi \frac{d}{d \phi}\right) \int_0^1 k(t) J_1(kt \sin \phi) \, dt. \]  
(9.1.10)

It is possible to use (5.1.20) to evaluate \( I_1; J_1 \) and \( J_2 \)
being given by (5.1.21) and (5.1.22) respectively. After

substituting from (9.1.7) and (5.1.2) we have, for (5.1.21)
\[ J_1 = \lim_{R \to \infty} \int_0^{2\pi} \int_0^1 \left( -\frac{\mu \sigma k}{2} \frac{e^{iKR}}{R} \Psi(\sin \phi) \right) \frac{2u_0}{\lambda} J_1(\lambda R \sin \phi) e^{iKR \cos \phi} R^2 \sin \phi \, d \phi \, d \theta, \]  
(9.1.11)
which, on replacing the Bessel function by the first term in its asymptotic expansion, and on carrying out the integration with respect to $\Theta$, can be written as

\[
J_1 = -\frac{2^{1/2} \mu_c u_o \kappa^2}{\lambda^{3/2}} \lim_{\kappa \to \infty} R^{1/2} e^{iKR} \left( e^{\frac{i}{2\kappa} \int_0^\pi \Psi(\sin\varphi) \sin\varphi e^{iR_l\sin \varphi + x \cos \varphi} d\phi} + e^{\frac{i}{2\kappa} \int_0^\pi \Psi(\sin\varphi) \sin\varphi e^{iR_l\sin \varphi + x \cos \varphi} d\phi} \right).
\]

(9.1.12)

Hence, as in subsection 5.1, from the principle of stationary phase we obtain

\[
J_1 = \frac{2 \mu_c u_o \kappa}{\lambda} \left( 1 + e^{2iKR} \right) \Psi(\xi).
\]

(9.1.13)

Likewise, we substitute from (5.1.4) and (9.1.6), into (5.1.22), to give

\[
J_2 = \frac{2^{1/2} \mu_c u_o \kappa}{\lambda} \lim_{\kappa \to \infty} \int_0^\pi \int_0^\pi \left\{ -i \alpha \cos \varphi \frac{J_1(\lambda R \sin \varphi) + \lambda \sin \varphi J_1'(\lambda R \sin \varphi)}{2} \right\} e^{-i\alpha R \cos \varphi} \times \frac{e^{iKR}}{R} \Psi(\sin\varphi) R^2 \sin \varphi d\phi d\Theta,
\]

(9.1.14)

whence, on once again replacing the Bessel functions by the first terms in their respective asymptotic expansions, and then carrying out the integration with respect to $\Theta$, we obtain
An application of the principle of stationary phase gives
\[
J_2 = -\frac{2\mu\nu_0}{\lambda} \left( 1 - 2e^{iKR} \right) \Psi(t).
\] (9.1.16)

Hence from (5.1.20), (9.1.13) and (9.1.16) we have
\[
E_s = -\frac{\pi\omega^2\mu\nu_0}{\lambda} \left[ \Psi(t) - \Psi(t) + e^{2iKR}\Psi(t) + e^{2iKR}\Psi(t) \right].
\] (9.1.17)

and since the last two terms in the square brackets combine

to give a quantity which is wholly real we write
\[
E_s = \frac{2\pi\omega^2\mu\nu_0}{\lambda} \Psi(t).
\] (9.1.18)

Therefore, from (9.1.10)
\[
E_s = \frac{2\pi\omega^2\mu\nu_0}{\lambda} \left[ \left( 1 + \tan\phi \frac{d}{d\phi} \right)^{1/2} K(t) J_1(kt\sin\phi) dt \right] \quad \Psi = \phi_\alpha
\] (9.1.19)

where
\[
\phi_\alpha = \sin^{-1}(\epsilon).
\] (9.1.20)

On using (1.3.11), (8.1.8), (8.1.28) and (8.1.29), in (9.1.19),
it is seen that
\[
E_s = \frac{8\pi\omega^2\mu_0}{k^3\epsilon^2} \Re \left[ \left( 1 + \tan\phi \frac{d}{d\phi} \right) \frac{\mu}{\mu} \Omega(\phi) - \overline{\Omega}(\phi) \right] \quad \Psi = \phi_\alpha
\] (9.1.21)
In this subsection we derive an asymptotic formula for (9.1.21). From (8.2.66) we have, on replacing \( J_0(ksin\phi) \) by the first few terms in its asymptotic expansion

\[
\mathcal{U}_0(\phi) \sim \frac{1}{k sin\phi} - \frac{2^{1/2}}{\pi^{1/2}(ksin\phi)^{3/2}} \left[ \left(1 - \frac{9}{128k^2 sin^2\phi}\right) \cos k sin\phi - \frac{1}{4} \right] \\
+ \frac{1}{8ksin\phi} \sin \{ksin\phi - \pi/4\} + \frac{e^{it} + e^{-it}}{2k^{1/2} t^{1/2}} \left[ \frac{1}{4} - \frac{15}{32t^2} \right] \int_0^{\infty} e^{iwt} J_1(kwsin\phi)dw.
\]

(8.2.1)

We deform the path of integration of the inner integral in the usual way, so that on replacing the Bessel function \( J_1(k(1+iw)sin\phi) \) by its asymptotic series, and expanding the resulting integrand, we have

\[
\int_0^{\infty} e^{iwt} J_1(kwsin\phi)dw = e^{it + \frac{3\pi i}{4}} \int_0^{\infty} e^{iwt} J_1(kwsin\phi)dw

\]

\[= e^{it + \frac{3\pi i}{4}} \int_0^{\infty} e^{iwt} J_1(k(w+1)sin\phi)dw,
\]

\[\approx \int_0^{\infty} e^{iwt} J_1(kwsin\phi)dw - \frac{2}{\pi k sin\phi} \int_0^{\infty} e^{iwt} \left(1-iw-w^2\right) cos k(1+iw)sin\phi \frac{3\pi i}{4} dw
\]

\[= \frac{3}{8ksin\phi} \int_0^{\infty} e^{iwt} \left(1-2iwsin\phi\right)k(1+iw)sin\phi \frac{3\pi i}{4} dw + \frac{15}{128k^2 sin^2\phi} \int_0^{\infty} e^{iwt} \cos k(1+iw)sin\phi \frac{3\pi i}{4} dw.
\]

(9.2.2)
We next express the trigonometric functions in terms of exponential functions, so that

\[
\int_{-\infty}^{\infty} e^{i \omega t} \frac{J_1(\kappa \omega \sin \phi)}{\omega^{1/2} (1 - i \omega)^{1/2}} d\omega = e^{\frac{3i \pi}{4}} \int_{-\infty}^{\infty} e^{i \omega t} \frac{J_1(\kappa \omega \sin \phi)}{\omega^{1/2} (1 - i \omega)^{1/2}} d\omega - \frac{e^{i \kappa \sin \phi - \frac{3i \pi}{4}}}{(2 \pi \kappa \sin \phi)^{1/2}} \int_{-\infty}^{\infty} e^{i \kappa \sin \phi + \frac{3i \pi}{4}} \frac{J_1(\kappa \omega \sin \phi)}{\omega^{1/2} (1 - i \omega)^{1/2}} d\omega
\]

\[
\times \left\{ \int_{0}^{\infty} \frac{(t + k \sin \phi \omega)(1 - i \omega - \omega^2)}{\omega^2} d\omega + \frac{3i}{8 k \sin \phi} \int_{0}^{\infty} \frac{(t + k \sin \phi \omega)(1 - 2i \omega)}{\omega^2} d\omega + \frac{15}{128 k^2 \sin^2 \phi} \int_{0}^{\infty} \frac{(t + k \sin \phi \omega)(1 - i \omega - \omega^2)}{\omega^2} d\omega \right\}
\]

\[
= e^{\frac{3i \pi}{4}} \int_{-\infty}^{\infty} e^{i \omega t} \frac{J_1(\kappa \omega \sin \phi)}{\omega^{1/2} (1 - i \omega)^{1/2}} d\omega - \frac{e^{i \kappa \sin \phi - \frac{3i \pi}{4}}}{(2 \pi \kappa \sin \phi)^{1/2}} \left\{ \frac{1}{2(t + k \sin \phi)} \right\}
\]

\[
- \frac{3}{4(t + k \sin \phi)^2} + \frac{3i}{8 k \sin \phi} \left(1 - \frac{1}{t + k \sin \phi} \right) + \frac{15}{128 k^2 \sin^2 \phi} \right\}
\]

\[
+ \frac{e^{i \kappa \sin \phi + \frac{3i \pi}{4}}}{(1 - k \sin \phi)^{1/2}} \left\{ 1 - \frac{i}{2(t - k \sin \phi)} - \frac{3}{4(t - k \sin \phi)^2} - \frac{3i}{8 k \sin \phi} \left(1 - \frac{1}{t - k \sin \phi} \right) \right\}
\]

\[
+ \frac{15}{128 k^2 \sin^2 \phi} \right\}
\]

(9.2.3)

We substitute (9.2.3) into the repeated integral in (9.2.1),
which, after interchanging the order of integration and on
retaining only the dominant terms, is asymptotic to
\[
\frac{e^{\pi/2}}{2\pi^2} \int_0^\infty \frac{t^3}{\omega^2(1+i \omega)^2} e^i t (1-\omega) \frac{1}{t} dt
\]
\[
- \frac{e^{\pi/2}}{2^{3/2} \pi^{1/2}} (\frac{e^{i k \sin \phi}}{4 k \sin \phi}) \left\{ (1 + \frac{3i}{8 k \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi}) \int_k^{\infty} \frac{dt}{t^{3/2}(t + k \sin \phi)^{1/2}} \right\}
\]
\[+ \frac{15}{32} \int_k^{\infty} \frac{dt}{t^{3/2}(t + k \sin \phi)^{1/2}} - \frac{3}{4} \int_k^{\infty} \frac{dt}{t^{3/2}(t + k \sin \phi)^{1/2}} - \frac{3}{8} \int_k^{\infty} \frac{dt}{t^{3/2}(t + k \sin \phi)^{1/2}} \left\{ (1 - \frac{3i}{8 k \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi}) \right\}
\]
\[+ e^{i k \sin \phi + \frac{3i}{4 k}} \left\{ (1 - \frac{3i}{8 k \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi}) \right\}
\]
\[- \frac{1}{2} (1 - \frac{3i}{4 k \sin \phi}) \int_k^{\infty} \frac{dt}{t^{3/2}(t - k \sin \phi)^{1/2}} - \frac{3i}{4} (1 - \frac{3i}{8 k \sin \phi}) \int_k^{\infty} \frac{dt}{t^{3/2}(t - k \sin \phi)^{1/2}} \]
\[+ \frac{15}{32} \int_k^{\infty} \frac{dt}{t^{3/2}(t - k \sin \phi)^{1/2}} - \frac{3}{4} \int_k^{\infty} \frac{dt}{t^{3/2}(t - k \sin \phi)^{1/2}} - \frac{3}{8} \int_k^{\infty} \frac{dt}{t^{3/2}(t - k \sin \phi)^{1/2}} \left\{ (1 - \frac{3i}{8 k \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi}) \right\}.
\]
\[(9.2.4)\]

We deform the path of integration into the straight line
joining \(k\) to \(k - i k \omega\) for the first of the integrals in (9.2.4),
to give
\[
\int_k^{\infty} \frac{e^i t (1 - \omega)}{t^{3/2}} \left\{ (1 - \frac{3i}{4 \omega}) \right\} dt = - \frac{e^{i(k(1 - i) \omega)}}{k^{1/2}} \int_0^{(k(1-i)\omega)} \left\{ (1 - \frac{3i}{4 k(1-i) \omega}) \right\} dt,
\]
\[(9.2.5)\]
since $\Re(1-i\omega) > 0$. In order to evaluate the integral (9.2.5) for large $k$, we consider
\[
\int_0^\infty \frac{e^{-k(1+\omega)^{1/2}e^{i\theta}t}}{(1-it)^{n/2}} dt
\]
where $n = 3, 5, \ldots$ and $\Theta = \tan^{-1}\omega$. Also, let us consider
\[
e^{\theta} \int_0^\infty \frac{e^{-k(1+\omega)^{1/2}e^{i\Theta}t}}{(1-i e^{i\Theta}t)^{n/2}} dt
\]
where $\nabla$, is shown in figure 56.

For $0 < t < 1$
\[
(1-i e^{i\Theta}t)^{-n/2} = 1 + \frac{n}{2} i e^{i\Theta}t + \ldots
\]
and hence
\[
\int_0^\infty \frac{e^{-k(1+\omega)^{1/2}e^{i\Theta}t}}{0 (1-it)^{n/2}} dt \sim e^{i\Theta} \left\{ \frac{1}{k(1+\omega)^{1/2}} + \frac{n}{2k^2(1+\omega)} \right\}.
\]
(9.2.7)

However
\[
\frac{1}{(1+\omega^2)^{1/2}} = \frac{e^{i\Theta}}{(1-i\omega)},
\]
so from (9.2.7) we have
\[
\int_0^\infty \frac{e^{-k(1-i\omega)t}}{(1-it)^{n/2}} dt \sim \frac{1}{k(1-i\omega)} + \frac{i n}{2k^2(1-i\omega)^2}.
\]
(9.2.8)

By using this last result it can be shown that
\[
\int_0^\infty \frac{e^{-k(1-i\omega)t}}{(1-it)^{n/2}} dt \sim \frac{1}{4k} \left\{ \frac{3i}{k} \right\} + \frac{3i}{2k^2(1-i\omega)}.
\]
(9.2.9)

Therefore the double integral in (9.2.4) is asymptotic to
\[
\frac{e^{i\phi}}{2\sqrt{k} W^{1/2}} \left[ \frac{1}{4k} \int_0^\infty \frac{e^{i\omega}}{W^{1/2}(1-i\omega)^{3/2}} d\omega + \frac{3i}{2k} \int_0^\infty \frac{e^{i\omega}}{W^{1/2}(1-i\omega)^{3/2}} d\omega \right]
\]
(9.2.9)
\begin{align*}
\int_0^\infty e^{it\mathbf{I}_\nu(at)} dt &= \frac{\Gamma(\mu+\nu+1)}{\mu+\nu} P_\mu^\nu \left( \frac{t}{s} \right), \\
(9.2.11)
\end{align*}

\( s = \left( p^2 - a^2 \right)^{1/2}, \quad \Re \mu + \nu > -1, \quad \Re p > |R a|, \)

so that with \( \nu = 1, \mu = k, a = \kappa \sin \phi \) and \( s = \kappa \cos \phi \) we obtain

\begin{equation}
\int_0^\infty e^{i\kappa \omega} \mathbf{I}_1(\kappa \omega \sin \phi) d\omega = \frac{\Gamma(n+2)}{\kappa^{n+1} \cos^{n+1} \phi} P_n^{\nu}(\sec \phi). \tag{9.2.12}
\end{equation}

Also, from section 3.3.1 of [36], it is seen that

\begin{equation}
P_n^{\nu}(\sec \phi) = \frac{\Gamma(n)}{\Gamma(n+2)} P_n^{\nu}(\sec \phi), \tag{9.2.13}
\end{equation}

and hence we write (9.2.12) as

\begin{equation}
\int_0^\infty e^{i\kappa \omega} \mathbf{I}_1(\kappa \omega \sin \phi) d\omega = \frac{\Gamma(n)}{\kappa^{n+1} \cos^{n+1} \phi} P_n^{\nu}(\sec \phi). \tag{9.2.14}
\end{equation}

Result (9.2.14) enables us to evaluate (9.2.10) and the rest of the integrals in (9.2.4) are of similar form to integrals in (8.2.71). Hence for (9.2.1) we obtain

\begin{align*}
\Omega_0(\phi) &= \frac{1}{i \kappa \cos \phi} \left[ \left( 1 + \frac{3i}{4} \right) + \frac{2i}{4 \kappa \cos \phi} \mathbf{I}_1(\kappa \sin \phi) \mathbf{I}_1(\kappa \sin \phi) \right] \\
&\quad - \frac{e^{i\kappa \sin \phi / 2}}{(2\pi)^{1/2} (\kappa \sin \phi)^{1/2}} \left[ \frac{e^{i\kappa \sin \phi - \frac{3i}{4}}}{\kappa \sin \phi} \right] \left( 1 \sin \phi \right)^{3/2} - \frac{i}{8 \kappa \sin \phi} \left( 2 \left( 1 \sin \phi \right)^{3/2} - 5 \left( 1 + \sin \phi \right)^{3/2} \right) \\
&\quad + \frac{4}{\left( 1 + \sin \phi \right)^{1/2}} + \frac{1}{128 \kappa^2 \sin \phi} \left( 12 \left( 1 + \sin \phi \right)^{3/2} - 4 \left( 1 + \sin \phi \right)^{3/2} + 87 \left( 1 + \sin \phi \right)^{3/2} \right)
\end{align*}
We consider next the asymptotic evaluation of $\Omega_n(\phi)$, for $n \geq 1$. We have, from (8.2.97), and on deforming the path of integration of the first integral on the right-hand side

$$\int_0^\infty E_\omega(\omega) e^{ikx} \frac{I_1(kx \sin \phi)}{\omega} d\omega = \frac{e^{ikx}}{\pi} \int_0^\infty \frac{I_1(kx \sin \phi) e_\omega(-ix)}{x^{3/2} (1 - ix)^{3/2}} dx$$

$$- \frac{e^{ikx}}{\pi} \int_0^\infty \frac{I_1(kx (1 - ix) \sin \phi) e_\omega(1 - ix)}{(1 - ix)^{3/2} (2 - ix)^{3/2}} dx = \frac{e^{ikx}}{\pi} \int_0^\infty \frac{I_1(kx \sin \phi)}{x^{3/2}} dx.$$

Now

$$\int_0^\infty \frac{e^{ikx} I_1(kx \sin \phi) e_\omega(-ix)}{x^{3/2} (1 - ix)^{3/2}} dx \sim e_\omega(0) \int_0^\infty \frac{e^{ikx} I_1(kx \sin \phi) dx}{x^{3/2}}$$

$$- i \left[ e_\omega'(0) \frac{1}{2} e_\omega(0) \right] \int_0^\infty \frac{e^{ikx} I_1(kx \sin \phi) x^{3/2}}{dx}$$

$$= - \frac{2x^{3/2}}{k^{3/2} \cos x^{3/2}} \left[ e_\omega(0) P_\omega'(\sec \phi) + \frac{i}{2} \frac{e_\omega'(0) - \frac{1}{2} e_\omega(0)}{x^{3/2} P_\omega'(\sec \phi)} \right].$$

I.e., after using (9.2.14)
For the second integral on the right-hand side of (9.2.16) we replace the Bessel function by the first term in its asymptotic expansion, so that after further approximation

we write

\[
\frac{\int e^{ikx} J_0(k(1+ix)\sin \phi)}{(1+ix)^{1/2}} e_n(1-ix) \, dx}{(1-ix)^{1/2}(2-ix)^{1/2}} \sim \frac{e_n(1)}{(2\pi k \sin \phi)^{1/2}} \left[ e^{i k \sin \phi} - \frac{3i}{4} e^{i k \sin \phi + \frac{3\pi}{4}} \right] \int_0^\infty e^{-k(1+\sin \phi) x} \, dx
\]

\[= \frac{e_n(1)}{2\pi^{1/2} (k \sin \phi)^{1/2} k(1+\sin \phi)} \left[ e^{i k \sin \phi} - \frac{3i}{4} e^{i k \sin \phi + \frac{3\pi}{4}} \right]. \tag{9.2.16}
\]

Finally, for the last integral on the right-hand side of (9.2.16) we replace \( e_n(1) \) by \( i e_n(1)/\pi \), so that after suitable deformations of the paths of integration, we obtain

\[
e_n(1) \left[ e^{i k \sin \phi} - \frac{3i}{4} e^{i k \sin \phi + \frac{3\pi}{4}} \right] \int_0^\infty e^{i kx} J_0(k(1+ix)\sin \phi)^3 \, dx \]

\[= \frac{2^{1/2} e_n(1)}{16 \pi^{1/2} k(1-\sin \phi)^{1/2}} \left[ e^{i k \sin \phi} - \frac{3i}{4} e^{i k \sin \phi + \frac{3\pi}{4}} \right] \int_0^\infty e^{i k(1+\sin \phi) x} \, dx \]

\[+ i e \left[ e^{i k \sin \phi} - \frac{3i}{4} e^{i k \sin \phi + \frac{3\pi}{4}} \right] \int_0^\infty e^{i k(1-\sin \phi) x} \, dx = O(e_n(1)/k^{1/2}). \tag{9.2.19}
\]
Hence from (3.2.17), (3.2.18) and (3.2.19) it is seen that

(3.2.18) becomes

\[
\left[\int_0^L (k \sin \phi) e^{ikw} \right] \frac{e^{ikw} I_n(k \sin \phi)}{\omega} = \frac{2e^{i\phi}}{\pi^{1/2} k^{1/2} \cos \frac{\omega}{2}} \left[ e^{i\phi} P_{1/2}^{(1/2)}(\cos \phi) + \frac{i}{2} e^{i\phi} P_{1/2}^{(-1/2)}(\cos \phi) \right] \\
- \frac{e^{i\phi}}{2 \pi^{1/2} (k \sin \phi)^{1/2}} \left[ e^{i(k \sin \phi - 3\pi i/4)} + e^{-i(k \sin \phi + 3\pi i/4)} \right] \right] + O(e^{i\phi}). \tag{3.2.20}
\]

In order to evaluate the remaining quantity on the right-hand side of (3.2.21), we use (3.2.103). We deform the path of integration in the usual way and then use (3.2.54) to obtain

\[
\left[\int_0^L (k \sin \phi) e^{ikw} \right] \frac{e^{ikw} I_n(k \sin \phi)}{\omega} = e^{i\phi} \int_0^L (k \sin \phi) e^{ikw} \frac{3\pi i}{2} \left[ e^{i(k \sin \phi - 3\pi i/4)} + e^{-i(k \sin \phi + 3\pi i/4)} \right] \right] \right] + O(e^{i\phi}). \tag{3.2.21}
\]
Consequently the terms in the square brackets of (8.2.103),

which do not involve $e^{-2ikt}$ are asymptotic to

$$i e^{-2ikt} \int_0^{\infty} \frac{e^{-\frac{kw}{2}}}{\omega} I_1(\frac{kw}{2}) d\omega \int e^{-2ikt} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt$$

$$+ \frac{i e^{\frac{i}{4}}}{\pi^2 (2k \sin \phi)^{\frac{1}{2}}} \left[ \int e^{\frac{i}{4} \frac{x}{\omega}} dx \int e^{-\frac{i}{4} x + \frac{k(1-\sin \phi)}{x}} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt \right]$$

$$+ \frac{e^{\frac{i}{4}}}{\pi^2 (2k \sin \phi)^{\frac{1}{2}}} \left[ \int e^{\frac{i}{4} \frac{x}{\omega}} dx \int e^{-\frac{i}{4} x + \frac{k(1+\sin \phi)}{x}} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt \right].$$

(9.2.22)

Note that some of the terms in (8.2.103) have been discarded.

We deform the path of integration of the inner integrals

into the straight line joining 1 to l-\infty, so that the last

line is equal to

$$i e^{-2ikt} \int_0^{\infty} \frac{e^{-\frac{kw}{2}}}{\omega} I_1(\frac{kw}{2}) d\omega \int e^{-2ikt} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt$$

$$+ \frac{i e^{\frac{i}{4}}}{\pi^2 (2k \sin \phi)^{\frac{1}{2}}} \left[ \int e^{\frac{i}{4} \frac{x}{\omega}} dx \int e^{-\frac{i}{4} x + \frac{k(1-\sin \phi)}{x}} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt \right]$$

$$+ \frac{e^{\frac{i}{4}}}{\pi^2 (2k \sin \phi)^{\frac{1}{2}}} \left[ \int e^{\frac{i}{4} \frac{x}{\omega}} dx \int e^{-\frac{i}{4} x + \frac{k(1+\sin \phi)}{x}} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt \right]$$

$$+ \frac{i e^{-2ikt}}{\pi^2 2^{\frac{3}{2}}} e_n(1) \int_0^{\infty} \frac{e^{-\frac{kw}{2}}}{\omega} I_1(\frac{kw}{2}) d\omega \int e^{-2ikt} \frac{t^{-\frac{1}{2}}}{(l-i\omega)(l+it)^{\frac{1}{2}}} e_n(t) dt.$$
The first integral in the last line can be evaluated by using

\( (9.2.14) \), whereas the other two integrals are of similar form to \( (8.2.110) \). Hence, the last line is equal to

\[
- \frac{i e^{ik} e_{n}(1) e^{i k}}{4 \pi^{1/2} (k \sin \phi)^{1/2}} \left\{ \frac{e^{ik \sin \phi - 3\pi i}}{k(1 + \sin \phi)} \right\} \left( 1 - \frac{1}{2^{1/2}} \left(1 - \sin \phi \right)^{1/2} \right) + \frac{e^{ik \sin \phi + 3\pi i}}{k(1 + \sin \phi)} \left( 1 - \frac{1}{2^{1/2}} \left(1 - \sin \phi \right)^{1/2} \right) \}.
\]  

(9.2.24)

The remaining terms of \( (8.2.103) \) are

\[
- \frac{e_{m p}}{k} \int_{0}^{1} \frac{e^{ikw} I_{1}(k\sin \phi)}{(1 - \omega)^{1/2}} - \frac{1}{2} \omega d\omega
\]

\[
- \frac{9 e_{n}(10)}{128 \pi k^{2}} \int_{0}^{1} \frac{e^{ikw} I_{1}(k\sin \phi)}{(1 - \omega)^{3/2}} \left( 1 - \frac{1}{2} \omega - \frac{(1 - \omega)^{1/2}}{2} \right) d\omega,
\]

which, on deforming the paths of integration are seen to be

equal to
\[
\frac{e_n \omega^{-3/4}}{\pi k} \int_0^\infty \frac{e^{-k w}}{\omega^{3/2}} I_1(k \omega \sin \phi) \left\{ \frac{1}{(1-i\omega)^{1/2}} - 1 \right\} d\omega \\
+ \frac{9 e_n(0) \omega^{-3/4}}{128 \pi k^2} \int_0^\infty \frac{e^{-k w}}{\omega^{3/2}} I_1(k \omega \sin \phi) \left\{ \frac{1-i\omega}{2} (1-i\omega)^{1/2} \right\} d\omega \\
+ \frac{i e_n(0) e^{-3/4}}{\pi k} \int_0^\infty \frac{e^{-k w} I_1(1+i\omega \sin \phi)}{(1+i\omega)^{1/2}} \left\{ \frac{u^2}{\omega^{3/2}} - 1 \right\} d\omega \\
+ \frac{9 e_n(0) e^{-3/4}}{128 \pi k^2} \int_0^\infty \frac{e^{-k w} I_1(1+i\omega \sin \phi)}{(i\omega+1)^{1/2} \omega^{3/2}} \left\{ \frac{1-i\omega}{2} (1-i\omega)^{1/2} \right\} d\omega \\
= -\frac{e^{\frac{3\pi}{4}}}{\sqrt{2} \sqrt{\frac{k}{\omega}} \cos \frac{\sqrt{2} \phi}{2}} \frac{e_n(0) P'(sec \phi)}{8} - \frac{9 e_n(0) e^{-3/4}}{8 \pi k^3 (2k \sin \phi)^{1/2}} \left[ e^{\frac{ik \sin \phi - 3\pi}{4}} \right] \\
+ \frac{e \frac{i k \sin \phi + 3\pi}{4}}{4 \sqrt{2} (1-\sin \phi)^{1/2}}. \\
\] 

(3.2.25)

Therefore, by using (3.2.23), (3.2.24) and (3.2.25), we write down an asymptotic expression for \( \Omega_n(\phi) \), i.e.

\[
\Omega_n(\phi) = -\frac{2 e_n(0)}{\sqrt{2} \sqrt{\frac{k}{\omega}} \cos \frac{\sqrt{2} \phi}{2}} \left[ \left\{ e_n(0) + \frac{i e_n(0)}{16k} + \frac{i e^{2i k \phi \pi} e_n(0)}{8 \pi k^3 (2k \sin \phi)^{1/2}} \right\} P'(sec \phi) \right]. 
\]
\[
+ \frac{i}{2k} \left[ e_n'(0) - \frac{1}{2} e_n(0) \right] + \frac{e_{n+1}}{2k \cos \phi} \left[ e_n(0) \right] - \frac{e_{n-1}}{2k \cos \phi} \left[ e_n(0) \right] \frac{i k \sin \phi - \frac{3\pi}{4}}{1 + \sin \phi} (1 + \sin \phi)^\gamma
\]

\[
+ \frac{e^{i k \sin \phi + \frac{3\pi}{4} \left( 1 - \sin \phi \right)^\gamma}}{k (1 + \sin \phi)} - \frac{2 \gamma}{16 \pi k (k \sin \phi)^{1/2}} \left[ e_n(0) \right] \frac{e^{i k \sin \phi - \frac{3\pi}{4}}}{1 + \sin \phi} + \frac{e^{-i k \sin \phi + \frac{3\pi}{4}}}{k^2 (1 + \sin \phi)^{1/2}}
\]

\[
+ O \left( \frac{e_n_k}{k^2} \right).
\]

(9.2.26)

On combining (9.2.15) with (9.2.23), we obtain an asymptotic expansion for \( \Omega(\phi) \) correct to order \( c_n/k^2 \). First of all, let us consider the coefficient of \( \frac{1}{2} \), which is equal to

\[
- \frac{i}{k^2 \cos \phi} \left[ 1 + \frac{3 \pi}{4} \right] - \frac{2 \gamma}{16 \pi k (k \sin \phi)^{1/2}} \left[ e_n(0) \right] + \frac{2 \gamma}{16 \pi k (k \sin \phi)^{1/2}} e_{n+1}(0)
\]

\[
+ e_n(0) + O \left( \frac{1}{k^2} \right).
\]

(9.2.27)

Now \( e_n(0) \) and \( e_{n+1}(0) \) are given by (8.2.7) and (8.2.5) respectively.

Also

\[
e_n(0) = - \frac{\gamma}{2 \pi k} \left\{ \frac{i e^{i k - \frac{3\pi}{4} \gamma}}{16 \gamma} + \frac{e^{2 i k + \frac{1}{2} \pi}}{16 \gamma} + O \left( \frac{1}{k^2} \right) \right\},
\]

(9.2.28)

and from this it is deduced that the coefficient of \( \frac{1}{2} \) is of order \( 1/k^2 \). Similarly, it can be shown that the coefficient of \( \frac{1}{k} \) is of order \( 1/k^2 \). Therefore from
\[ \Omega_n(\phi) = \omega_n(\phi) - \frac{e^{i \frac{\pi}{4}}}{2^{3/2} \pi^{1/2} (\frac{k \sin \phi}{2})^{1/2}} \left[ \frac{e^{i k \sin \phi - \frac{3 \pi i}{4}}}{k (1 - \sin \phi)} \right] (1 + \sin \phi)^{1/2} \]

\[ + \frac{e^{i k \sin \phi + \frac{3 \pi i}{4}}}{k (1 + \sin \phi)} \left[ - \frac{e^{i k + \frac{3 \pi i}{4}}}{k^{3/2} (k \sin \phi)^{1/2}} \right] (1 - \sin \phi)^{1/2} \]

\[ + \frac{e^{i k \sin \phi + \frac{3 \pi i}{4}}}{k^{3/2} (1 - \sin \phi)^{1/2}} + O \left( \frac{1}{k^4} \right) \]

(9.2.29)

where \( \omega_n(\phi) = \Omega_n(\phi) \) - terms containing associated Legendre functions. Hence from subsection 8.2 and the above, we have

\[ \Omega_n(\phi) = \frac{1}{k \sin \phi} - \frac{\frac{e^{i \pi/2}}{2^{1/2} \pi (k \sin \phi)^{1/2}} \left[ e^{i k \sin \phi - \frac{3 \pi i}{4}} \right] (1 + \sin \phi)^{1/2}}{8 k \sin^2 \phi} \]

\[ - \frac{5 (1 + \sin \phi)^{1/2} + \frac{4}{(1 + \sin \phi)^{1/2}} + \frac{2 e^{i k + \frac{3 \pi i}{4}} (1 + \sin \phi)^{1/2}}{8 \pi^{1/2} k^{3/2} (1 - \sin \phi)^{1/2}} - \frac{1}{128 k^3 \sin^2 \phi} \left( 4 (1 + \sin \phi)^{3/2} \right) \]

\[ - 20 (1 + \sin \phi)^{3/2} (1 - \sin \phi)^{1/2} - \frac{8 \left( 1 + \sin \phi \right)^{1/2}}{(1 - \sin \phi)^{1/2}} + \frac{3 \left( 1 - \sin \phi \right)^{3/2}}{(1 - \sin \phi)^{1/2}} \]

\[ - \frac{i}{8 k^2 \sin^2 \phi} \left[ \frac{2 (1 - \sin \phi)^{3/2} - 5 (1 - \sin \phi)^{3/2} + \frac{4}{(1 - \sin \phi)^{1/2}}}{(1 - \sin \phi)^{1/2}} + \frac{2 e^{i k + \frac{3 \pi i}{4}} (1 - \sin \phi)^{1/2}}{8 \pi^{1/2} k^{3/2} (1 + \sin \phi)^{1/2}} \right] \]

\[ - \frac{1}{128 k^3 \sin^2 \phi} \left( 4 (1 - \sin \phi)^{3/2} + 20 (1 - \sin \phi)^{3/2} + 63 (1 - \sin \phi)^{3/2} - \frac{8 \sin \phi + 3 \sin \phi}{(1 - \sin \phi)^{1/2}} \right) \]

\[ + O \left( \frac{1}{k^4} \right) \]

(9.2.30)
From (9.2.30) it can be shown that

\[
\left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \Omega(\phi) \right] = \left( \frac{K}{2\pi} \right)^{\frac{3}{2}} \left\{ e^{-i\frac{K}{4} (1+\varepsilon)\frac{y_2}{2}} - \frac{1}{8K^2 \varepsilon} \left( 2(1+\varepsilon)y_2^2 - 5(1+\varepsilon)\right) \right\}
+ \frac{-2i(1+\varepsilon)^{\frac{5}{2}}}{8\pi\sqrt{\varepsilon}} \left( \frac{2(1+\varepsilon)}{1+\varepsilon} - \frac{3\varepsilon}{1+\varepsilon} \right)
+ \frac{1}{128K^2 \varepsilon^3} \left( \frac{3\varepsilon}{1+\varepsilon} - \frac{5}{1+\varepsilon} \right)
+ O\left( \frac{1}{K^4} \right).
\]

(9.2.31)

We use result (8.2.134) in (8.3.14) so that, on interchanging the order of integration, we have

\[
\left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \Omega(\phi) \right] = \left( \frac{K}{2\pi} \right)^{\frac{3}{2}} \left\{ e^{-i\frac{K}{4} (1+\varepsilon)\frac{y_2}{2}} - \frac{1}{8K^2 \varepsilon} \left( 2(1+\varepsilon)y_2^2 - 5(1+\varepsilon)\right) \right\}
+ \frac{-2i(1+\varepsilon)^{\frac{5}{2}}}{8\pi\sqrt{\varepsilon}} \left( \frac{2(1+\varepsilon)}{1+\varepsilon} - \frac{3\varepsilon}{1+\varepsilon} \right)
+ \frac{1}{128K^2 \varepsilon^3} \left( \frac{3\varepsilon}{1+\varepsilon} - \frac{5}{1+\varepsilon} \right)
+ O\left( \frac{1}{K^4} \right).
\]

(9.2.32)

The inner integrals in (9.2.32) are of similar form to the inner integral in (9.2.1), and, as for (9.2.3) we can show that

\[
\int_0^\infty \int_{\omega_1}^{\omega_2} e^{-i(t+K\varepsilon)^2} \frac{\varepsilon}{(1+\varepsilon)^{\frac{3}{2}}} \left\{ \left[ \frac{i}{2} \right] - \frac{3}{4} \right\} \left( \frac{K}{2\pi} \right)^{\frac{3}{2}} \Omega(\phi) d\omega d\varepsilon 
\]

\[
\int_0^\infty \int_{\omega_1}^{\omega_2} e^{-i(t+K\varepsilon)^2} \frac{\varepsilon}{(1+\varepsilon)^{\frac{3}{2}}} \left\{ \left[ \frac{i}{2} \right] - \frac{3}{4} \right\} \left( \frac{K}{2\pi} \right)^{\frac{3}{2}} \Omega(\phi) d\omega d\varepsilon 
\]
\[
+ \frac{3i}{8k \sin \phi} \left( 1 - \frac{i}{t + k(\epsilon \cos \phi)} \right) + \frac{15}{128 k^2 \sin^2 \phi} \left[ e^{\frac{i k \sin \phi + \frac{3i}{4}}{t + k(\epsilon \sin \phi)} - \frac{1}{2} t + k(\epsilon - \sin \phi)} \right] + \frac{3i}{8k \sin \phi} \left( 1 - \frac{i}{t + k(\epsilon - \sin \phi)} \right) + \frac{15}{128 k^2 \sin^2 \phi} \left[ e^{\frac{i k \sin \phi + \frac{3i}{4}}{t + k(\epsilon - \sin \phi)} - \frac{1}{2} t + k(\epsilon - \sin \phi)} \right]\]

\[
\frac{3}{4 (t + k(\epsilon - \sin \phi))^2} - \frac{3i}{8k \sin \phi} \left( 1 - \frac{i}{t + k(\epsilon - \sin \phi)} \right) + \frac{15}{128 k^2 \sin^2 \phi} \right] \right), t \geq k(1-\epsilon).
\]

(9.2.33) is still valid if we change \( \epsilon \) to \(-\epsilon\), and take \( t \geq k(1+\epsilon) \). Hence, with the aid of (9.2.33), (9.2.32) becomes

\[
\frac{1}{k} \left[ \frac{e^{\frac{i k \epsilon}{2}}}{2\sqrt{\pi k \epsilon}} \left[ \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 - \frac{i}{k \epsilon} \right) + \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 + \frac{i}{k \epsilon} \right) \right] \right],
\]

\[
+ e^{\frac{i k \epsilon}{2}} \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 - \frac{i}{k \epsilon} \right) + \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 + \frac{i}{k \epsilon} \right) \right], k(1-\epsilon)
\]

\[
- \frac{e^{\frac{i k \epsilon}{2}}}{4\pi(k \sin \phi)^2/k \epsilon} \left[ \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 - \frac{i}{k \epsilon} \right) + \int_0^\infty e^{\frac{y^2}{4k \epsilon}} \frac{k \epsilon}{y^2} \left( 1 + \frac{i}{k \epsilon} \right) \right], k(1+\epsilon)
\]

\[
\frac{15}{128 k^2 \sin^2 \phi} \int_0^\infty \frac{dt}{k(1-\epsilon)} - i \left( 1 - \frac{i}{k \epsilon} \right) + \int_0^\infty \frac{dt}{k(1-\epsilon)} - i \left( 1 - \frac{i}{k \epsilon} \right)
\]

\[
- \frac{3i}{8k \sin \phi} - \frac{3}{16k \epsilon^2 \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi} \int_0^\infty \frac{dt}{k(1-\epsilon)} - i \left( 1 - \frac{i}{k \epsilon} \right)
\]

\[
- \frac{3i}{4k \sin \phi} \int_0^\infty \frac{dt}{k(1-\epsilon)} + \frac{3}{16k \epsilon^2 \sin \phi} + \frac{15}{128 k^2 \sin^2 \phi} \int_0^\infty \frac{dt}{k(1-\epsilon)} - i \left( 1 - \frac{i}{k \epsilon} \right)
\]
after discarding surplus terms.

We deform the path of integration of the inner integral of the first double integral in (9.2.34) into the straight line joining \( k(1-\varepsilon) \) to \( k(1-\varepsilon)(1-\infty) \), to give

\[
\left\{1 - \frac{i}{8k^2 t} \right\} \int_0^{\infty} e^{(1-i) \omega t} dt = \frac{1}{8k^2} \left\{1 - \frac{i}{k(1-\varepsilon)(1-i)} \right\} \int_0^{\infty} e^{k(1-\varepsilon)(1-i) \omega t} dt
\]

\[
\sim \frac{-i}{k^{3/2} (1-\varepsilon)^{3/2} (1-i)^{3/2}} \left\{1 - \frac{i}{8k^2} \frac{3i}{2k^2} \right\}
\]

i.e. after using (9.2.8). This last result still holds if we replace \( \varepsilon \) by \(-\varepsilon\). Therefore, the first double integral in
(9.2.54) is asymptotic to
\[
\frac{e^{ik(1-\varepsilon)-\pi/4}}{2^{3/2} \pi (k \xi)^{3/2} k^{3/2} (1-\varepsilon)^{3/2}} \left[ \left( 1 - \frac{i}{8k \xi} \right) \int_0^{\infty} e^{kw} I_1(k \xi \sin \phi) \, dw + \frac{3i}{2k(1-\varepsilon)} \right] \int_0^{\infty} \frac{e^{kw} I_1(k \xi \sin \phi) \, dw}{\sqrt{1-\varepsilon}}
\]
\[
\sim \frac{e^{ik(1-\varepsilon)-\pi/4}}{2^{3/2} \pi (k \xi)^{3/2} k^{3/2} (1-\varepsilon)^{3/2}} \left[ \left( 1 - \frac{i}{8k \xi} + \frac{3i}{2k(1-\varepsilon)} \right) \int_0^{\infty} e^{kw} I_1(k \xi \sin \phi) \, dw \right]
\]
\[
+ \frac{3i}{2} \int_0^{\infty} \frac{e^{kw} I_1(k \xi \sin \phi) \, dw}{\sqrt{1-\varepsilon}}
\]
\[
= - \frac{e^{ik(1-\varepsilon)-\pi/4}}{(2\pi)^{3/2} (k \xi)^{3/2} k^{3/2} (1-\varepsilon)^{3/2}} \int_0^{\infty} \frac{e^{kw} I_1(k \xi \sin \phi) \, dw}{\sqrt{1-\varepsilon}} \left[ \left( 1 - \frac{i}{8k \xi} + \frac{3i}{2k(1-\varepsilon)} \right) \right] p'_{1/2} (\sec \phi)
\]
\[
- \frac{3i}{4k \cos \phi} p'_{1/2} (\sec \phi)
\]
(9.2.36)

after using (9.2.14).

Also
\[
\int_0^{\infty} \frac{dt}{k(1-\varepsilon)^{3/2} \sqrt{t + k(\xi + \sin \phi)}} = \frac{2}{k \xi (\xi + \sin \phi)} \left[ \frac{(1+\sin \phi)^{3/2}}{(1-\varepsilon)^{3/2}} - 1 \right], \quad (9.2.37)
\]
\[
\int_0^{\infty} \frac{dt}{k(1-\varepsilon)^{3/2} \sqrt{t + k(\xi + \sin \phi)}} = \frac{2}{k \xi (\xi + \sin \phi)^2} \left[ \frac{(1+\sin \phi)^{3/2}}{(1-\varepsilon)^{3/2}} + \frac{(1-\varepsilon)^{3/2}}{(1+\sin \phi)^{3/2}} - 2 \right], \quad (9.2.38)
\]
\[
\int_{1/2}^{1} \frac{dt}{t + i(1 + \sin \phi)} = \frac{2}{3} \frac{\eta - 1}{\eta^2} \left\{ \frac{(1 + \sin \phi)}{(1 - \eta)^{3/2}} - \frac{3(1 + \sin \phi)}{(1 - \eta)^{5/2}} \right\},
\]

\[
\int_{1/2}^{1} \frac{dt}{t + i(1 - \sin \phi)} = \frac{2}{3} \frac{\eta - 1}{\eta^2} \left\{ \frac{3(1 + \sin \phi)}{(1 - \eta)^{3/2}} + \frac{6(1 - \eta)}{(1 + \sin \phi)^{3/2}} - \frac{(1 - \eta)^{3/2}}{(1 + \sin \phi)^{5/2}} - 8 \right\},
\]

and similarly for the rest of the integrals in (9.2.34).

By using the last few results we write down \( \Phi_0(\phi) \), i.e.

\[
\Phi_0(\phi) = \phi(\phi) - \frac{e^{iK}}{(2\pi)^2 (K\eta)^2} \rho_{K\phi} \left\{ \frac{e^{iK(1+\sin \phi)}}{(1+\eta)^{3/2}} \left\{ \frac{1 - i}{8K} + \frac{3i}{2K(1+\eta)} \right\} \frac{p_1'(\sec \phi)}{\sqrt{2}} \right. \\
- \frac{3i}{4K \cos \phi} \frac{\sqrt{2}}{p_1'(\sec \phi)} \left. \left( \frac{1 + i}{8K} + \frac{3i}{2K(1+\eta)} \right) \frac{p_1'(\sec \phi)}{\sqrt{2}} - \frac{3i}{4K \cos \phi} \right\}
\]

\[
- \frac{e^{iK(1+\sin \phi)}}{(2\pi)^2} \rho_{K\phi} \left\{ \frac{1 - i}{8K} + \frac{3i}{8K \sin \phi} - \frac{3}{128K^2 \sin^2 \phi} - \frac{3}{64K \cos \phi} \right\}
\]

\[
+ \frac{15}{128K^2 \sin^2 \phi} \left\{ \frac{1 + \sin \phi}{(1 - \eta)^{3/2}} - \frac{i}{2K} \left( \frac{1 - i}{8K} + \frac{3i}{4K \cos \phi} \right) \left( \frac{1 + \sin \phi}{(1 - \eta)^{3/2}} - \frac{3}{128K^2 \sin^2 \phi} + \frac{3}{16K \cos \phi} \right) \right\}
\]

\[
+ \frac{1}{16K \cos \phi} \left( \frac{1 + \sin \phi}{(1 - \eta)^{3/2}} - \frac{3}{128K^2 \sin^2 \phi} \right) \frac{1}{2K} \left( \frac{1 - i}{8K} + \frac{3i}{4K \cos \phi} \right) \left( \frac{1 + \sin \phi}{(1 - \eta)^{3/2}} - \frac{3}{128K^2 \sin^2 \phi} + \frac{3}{16K \cos \phi} \right)
\]

\[
- \frac{1}{4K^2 \sin \phi} \left\{ \frac{3(1 + \sin \phi)}{(1 - \eta)^{3/2}} + \frac{6(1 - \eta)}{(1 + \sin \phi)^{3/2}} - \frac{(1 - \eta)_{3/2}}{(1 + \sin \phi)^{5/2}} - 8 \right\} + \frac{e^{iK(1-\sin \phi)}}{(2\pi)^2} \rho_{K\phi} \left\{ \frac{1 - i}{8K \cos \phi} \right\}
\]

\[
- \frac{3i}{8K \sin \phi} - \frac{3}{128K^2 \sin^2 \phi} - \frac{15}{64K \cos \phi} \sin \phi \left( \frac{1 - i}{8K} \right)
\]
\[
- \frac{3i}{4K \sin \phi} \left(\frac{(1-\sin \phi)^2 + (1+\epsilon)^2}{(1-\epsilon)^2} - 2\right) + \frac{1}{16K \epsilon \bar{K} (\epsilon - \sin \phi)} \left(\frac{(1-\sin \phi)^2}{(1-\epsilon)^2} - 3\frac{(1-\sin \phi)^2}{(1-\epsilon)^2} + 2\right)
\]

\[
- \frac{1}{4K^2 (\epsilon - \sin \phi)^2} \left(\frac{3(1-\sin \phi)^2}{(1-\epsilon)^2} + 6\frac{(1+\epsilon)^2}{(1-\sin \phi)^2} - \frac{(1-\epsilon)^2}{(1-\sin \phi)^2} - 8\right) \right\} + \frac{i e^{iK(\epsilon + \sin \phi) + \pi}}{\bar{K} (\epsilon + \sin \phi)} \left\{1 + i + \frac{3i}{8K \epsilon} \right\}
\]

\[
X \left(\frac{(1+\sin \phi)^2}{(1+\epsilon)^2} - \frac{(1+\sin \phi)^2}{(1+\epsilon)^2} - 2\right) - \frac{1}{16K \epsilon \bar{K} (\sin \phi - \epsilon)} \left(\frac{(1+\sin \phi)^2}{(1+\epsilon)^2} - 3\frac{(1+\sin \phi)^2}{(1+\epsilon)^2} + 2\right)
\]

\[
- \frac{1}{4K^2 (\sin \phi - \epsilon)^2} \left(\frac{3(1+\sin \phi)^2}{(1+\epsilon)^2} + 6\frac{(1+\epsilon)^2}{(1+\sin \phi)^2} - \frac{(1+\epsilon)^2}{(1+\sin \phi)^2} - 8\right) \right\} - \frac{i e^{iK(\epsilon + \sin \phi) + \pi}}{\bar{K} (\epsilon + \sin \phi)} \left\{1 + i + \frac{3i}{8K \epsilon} \right\}
\]

\[
- \frac{3i}{8K \sin \phi} \left(\frac{(1-\sin \phi)^2}{(1+\epsilon)^2} - \frac{(1+\epsilon)^2}{(1-\sin \phi)^2} - 2\right) + \frac{1}{16K \epsilon \bar{K} (\epsilon + \sin \phi)} \left(\frac{(1-\sin \phi)^2}{(1+\epsilon)^2} - 3\frac{(1-\sin \phi)^2}{(1+\epsilon)^2} + 2\right)
\]

\[
- \frac{1}{4K^2 (\epsilon + \sin \phi)^2} \left(\frac{3(1-\sin \phi)^2}{(1+\epsilon)^2} + 6\frac{(1+\epsilon)^2}{(1-\sin \phi)^2} - \frac{(1+\epsilon)^2}{(1-\sin \phi)^2} - 8\right) \right\} + O\left(\frac{1}{K^2}\right)
\]

(9.2.40)

In order to evaluate \( \Phi(\phi) \) it only remains to obtain an expression for \( \Phi_n(\phi) \), valid for \( n \geq 1 \). However, in (9.2.28) we can replace \( \Omega_n \) and \( e_n \) by \( \Phi_n \) and \( s_n \) respectively. Since
we can show that the coefficients of the associated Legendre functions are of order \(1/k^{3/2}\). Therefore from (9.2.26) and (9.2.40) we write

\[
\Phi(\phi) = \phi_r(\phi) - \frac{e^{-i k}}{2^{3/2} \pi^{3/2} (k^2 \sin^2 \phi)^{1/2}} \left[ \frac{\cosh(\frac{3\pi}{2})}{2} \left( \frac{2 i k \sin \phi + \frac{3\pi}{4}}{k^2 (1 - \sin^2 \phi)^{1/2}} \right)^2 + \frac{1}{2} \left( \frac{2 i k \sin \phi + \frac{3\pi}{4}}{k^2 (1 - \sin^2 \phi)^{1/2}} \right)^2 \right] + O\left(\frac{1}{k^{3/2}}\right),
\]

where \(\phi_r(\phi)\) is defined in a similar manner to \(\omega_r(\phi)\).

From (9.2.42), (8.2.47) and (8.2.49), we have

\[
\left[ 1 + \tan \phi \frac{d}{d\phi} \right] \Phi(\phi) = \left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \phi_r(\phi) \right] + \frac{e^{-2 i k \sin \phi + \frac{3\pi}{4}}}{2\pi i k^3 (1 - \sin^2 \phi)^{1/2} (1 + \sin^2 \phi)^{1/2}} \left[ \frac{2 i k \sin \phi + \frac{3\pi}{4}}{k^2 (1 + \sin^2 \phi)^{1/2}} \right]^2 + \frac{1}{32\pi i k^3} \left[ \frac{2 i k \sin \phi + \frac{3\pi}{4}}{k^2 (1 + \sin^2 \phi)^{1/2} (1 + \sin^2 \phi)^{1/2}} \right] + O\left(\frac{1}{k^{3/2}}\right).
\]

Before we can evaluate \(\phi_r\), we must consider (8.5.15). Now

\[
\left( 1 + \tan \phi \frac{d}{d\phi} \right) \Phi(\phi) = \int_0^1 J_0(k \sin \phi) \left( J_1(k \sin \phi) + k \sin \phi J_1(k \sin \phi) \right) d\omega, \quad (9.2.44)
\]
and

\[ \mathcal{J}(z) + z \mathcal{J}'(z) = z \mathcal{J}_0(z), \]

therefore

\[
\left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \theta(\phi) \right]_{\phi = \phi_0} = k \varepsilon \int_0^\infty \omega \mathcal{J}_0^2(k \varepsilon \omega) d\omega
\]

\[ = \frac{k \varepsilon}{2} \left[ \mathcal{J}_0^2(k \varepsilon) + \mathcal{J}_1^2(k \varepsilon) \right]. \]

\[
\sim \frac{1}{4\pi} \left[ \frac{2i k \varepsilon - \eta}{k \varepsilon} \left( 1 - \frac{i}{4k \varepsilon} - \frac{q}{32k^2 \varepsilon^2} \right) \right] + \frac{1}{2} + \frac{1}{2k^2 \varepsilon^2}
\]

\[ + \frac{2i k \varepsilon + \eta}{k \varepsilon} \left( 1 + \frac{i}{4k \varepsilon} - \frac{q}{32k^2 \varepsilon^2} \right). \quad (9.2.45) \]

From an examination of (9.2.40) it is seen that we now

only have to consider

\[
\left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \theta(\phi) - \phi(\phi) \right]_{\phi = \phi_0}. \]

There are two different types of term to consider. The terms

which contain \((\varepsilon + \sin \phi)\) in the denominator are easily dealt

with; however for the terms which contain \((\varepsilon - \sin \phi)\) in the

denominator, we have to take the limit as \(\phi \to \phi_0\). We use the
following results:

\[
\lim_{\phi \to \phi_0} \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} - 1 \right) = \frac{1}{2(1 - \epsilon)} ,
\]

\[
\lim_{\phi \to \phi_0} \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} + \frac{(1 - \epsilon)^2}{1 - \phi^2} - 2 \right) = \frac{1}{4(1 - \epsilon)} ,
\]

\[
\lim_{\phi \to \phi_0} \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} - 3 \frac{(1 - \sin \phi)^2}{1 - \phi^2} + 2 \right) = \frac{3}{4(1 - \epsilon)} ,
\]

\[
\lim_{\phi \to \phi_0} \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} + \frac{6(1 - \epsilon)^2}{1 - \phi^2} - \frac{(1 - \epsilon)^2}{1 - \phi^2} - 8 \right) = \frac{1}{2(1 - \epsilon)} ,
\]

\[
\lim_{\phi \to \phi_0} \left\{ \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} - 1 \right) - \frac{(1 - \epsilon)^2}{2(1 - \epsilon)^2} \right\} = \frac{1}{8(1 - \epsilon)} ,
\]

\[
\lim_{\phi \to \phi_0} \left\{ \frac{1}{(\epsilon - \sin \phi)^2} \left( \frac{(1 - \sin \phi)^2}{1 - \phi^2} + \frac{(1 - \epsilon)^2}{1 - \phi^2} - 2 \right) - \frac{(1 - \epsilon)^2}{4(1 - \epsilon)^2} \right\} = \frac{1}{8(1 - \epsilon)} ,
\]

\[
\frac{(1 - \epsilon)^2}{4(1 - \epsilon)^2} \frac{(1 - \epsilon)^2}{(1 - \sin \phi)^2} \frac{(1 - \epsilon)^2}{(1 - \sin \phi)^2} = \frac{1}{8(1 - \epsilon)} .
\]

Similar results are obtained if the signs of \( \phi \) and \( \epsilon \) are reversed. Hence after some lengthy algebra, we obtain

\[
\left[ \frac{1}{(1 + \tan \phi) \frac{d}{d\phi}} \Phi(\phi) \right] = \frac{1}{h \pi} \left[ 4 + \frac{1}{2k} e^{2i(k_0 - \hbar)} \right] \frac{1}{k \pi} (1 - \epsilon)^2 \left( \frac{(1 + \epsilon)^2}{1 - \phi^2} - \frac{32}{3k \pi^3} \frac{(1 + \epsilon)^2}{1 - \phi^2} - \frac{(1 + \epsilon)^2}{1 - \phi^2} \right) + \frac{1}{2(1 + \epsilon)^2} \left( \frac{(1 + \epsilon)^2}{1 - \phi^2} - \frac{1}{2(1 + \epsilon)^2} \right) + i(1 + \epsilon)^2 \frac{2i}{k \pi^2} \frac{(1 + \epsilon)^2}{2(1 + \epsilon)^2} + \frac{1}{32 k \pi^3} \frac{(1 + \epsilon)^2}{1 - \phi^2} - \frac{1}{2(1 + \epsilon)^2} \left( \frac{(1 + \epsilon)^2}{1 - \phi^2} - \frac{1}{2(1 + \epsilon)^2} \right) .
\]
\[ + \frac{2}{(1 + \epsilon)^{3/2}} - \frac{4(1 - \epsilon)^2}{(1 + \epsilon)^{3/2}} \left\{ \frac{\epsilon^3}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} - \frac{\epsilon^3}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \left\{ \frac{(1 + \epsilon)^2}{(1 - \epsilon)^2} \right\} - \frac{1}{(1 - \epsilon)^2} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} - \frac{1}{(1 - \epsilon)^2} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} \]

\[ - \frac{1}{8 \sqrt{1 + \epsilon} \sqrt{(1 + \epsilon)^2}} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} - \frac{1}{8 \sqrt{1 + \epsilon} \sqrt{(1 + \epsilon)^2}} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} \]

\[ + 2 i \frac{\epsilon_1 + \epsilon_2}{(1 + \epsilon)^2} \cdot \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} \]

\[ + \frac{1}{32 \sqrt{1 + \epsilon} \sqrt{(1 + \epsilon)^2}} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} \]

\[ + O \left( \frac{1}{(1 - \epsilon)^2} \right) \]  

(9.2.46)

We insert results (9.2.37), (9.2.39), (9.2.31) and (9.2.46) into (9.2.41) to give, on discarding terms which are wholly imaginary.

\[ E = \frac{2 \omega a^3}{R^2 \epsilon^4} \Theta \left[ 4 - \frac{3}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right] - \frac{2 i \epsilon_1 + \epsilon_2}{(1 + \epsilon)^2} \cdot \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} \]

\[ + \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \left\{ \frac{1}{2 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \right\} - \frac{1}{4 \sqrt{(1 - \epsilon)^2 (1 + \epsilon)^2}} \]
\[
- \frac{i}{4 \varepsilon} \left[ (1 - \varepsilon)^{3/2} - 2(1 + \varepsilon) \right] \left( \frac{1}{(1 + \varepsilon)^{1/2}} - \frac{1}{(1 - \varepsilon)^{1/2}} \right)
\]
\[
- \frac{15}{32} \left( \frac{1}{(1 + \varepsilon)^{3/2}} + 5 \frac{1}{(1 + \varepsilon)^{1/2}} \right)
\]
\[
+ 2 \left( \frac{1 + \varepsilon}{(1 - \varepsilon)^{3/2}} - 4 \frac{1 + \varepsilon}{(1 - \varepsilon)^{1/2}} \right)
\]
\[
- \frac{4}{(1 - \varepsilon)^{1/2}} - 4 \left( \frac{1 - \varepsilon}{(1 - \varepsilon)^{1/2}} \right)
\]
\[
+ O \left( \frac{1}{\varepsilon^{1/2}} \right)
\]

From (5.1.18) it can be shown that
\[
E_i = \frac{4 \omega^3}{k^3} \left[ 1 + \frac{1}{2 \varepsilon^2} \left( \cos 2k + \frac{3 \sin 2k}{4 \varepsilon^2} - \frac{3}{4 \varepsilon^2} - \frac{15}{32} \cos 2k + O \left( \frac{1}{k^3} \right) \right) \right]
\]

so that from (5.1.1) and (9.2.47), the scattering coefficient is given by
\[
Q = 2 \left[ 1 - \left( 1 - \frac{1}{(1 - \varepsilon)^{1/2}} \right) \frac{\cos 2k}{2 \varepsilon} + \frac{1}{8 \varepsilon^2} \left( 1 - \frac{1}{(1 - \varepsilon)^{1/2}} \right) \left( 2 \cos 2k + 3 \sin 2k \right) \right]
\]
\[
- \frac{1}{16 \varepsilon^2} \left( \frac{2 \cos (2k - \varepsilon)}{(1 - \varepsilon)^{3/2}} - \frac{(1 - \varepsilon)^{1/2}}{(1 + \varepsilon)^{3/2}} \right) \left( \frac{2 \sin (2k - \varepsilon)}{(1 + \varepsilon)^{3/2}} \right)
\]
\[
- \frac{\cos 2k}{64 \varepsilon^2} \left( 27 + 8 \left( 1 - \frac{1}{(1 - \varepsilon)^{1/2}} \right) \left( \cos 2k + 3 \sin 2k \right) + \frac{32}{(1 - \varepsilon)^{3/2}} - \frac{55}{(1 - \varepsilon)^{1/2}} \right)
\]
\[
- 4 \left( 1 - \varepsilon \right)^{1/2} \right] + O \left( \frac{1}{\varepsilon^{1/2}} \right)
\]

(9.2.49)
In this subsection we consider the asymptotic evaluation of the scattering coefficient, subject to the conditions of Case II. This evaluation is almost trivial since, after some modification, we can use (9.3.22). From (9.3.22) and (9.1.21) we have, for the energy scattered by the rigid disc

\[
E_s = \frac{8\pi \omega}{k^3} u_0^2 \Re \left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \left[ \beta(\phi) - \frac{J_0(\lambda) - J_2(\lambda)}{k \sin \phi} \right] \right] \\
- \frac{2\lambda J_1(\lambda)}{2k} \left\{ \frac{1 - J_0(k \sin \phi)}{k \sin \phi} \right\} - \frac{1}{16k^2} \left\{ (4 \lambda J_1(\lambda) + 3\lambda^2 J_0(\lambda)) \right\} \left\{ \frac{1 - J_0(k \sin \phi)}{k \sin \phi} \right\}
\]

\[
- 2\lambda J_1(\lambda) J_1(k \sin \phi) \left\{ \frac{2 + k \sin \phi}{8\pi \sqrt{4 + k^2}} \right\} \frac{1 - J_0(k \sin \phi)}{k \sin \phi} - \frac{1}{128k^3} J_0(k \sin \phi) \left\{ 16 \lambda J_1(\lambda) \right\}
\]

\[
+ 8\lambda^2 \left\{ J_0(\lambda) - J_2(\lambda) \right\} + \frac{1 - J_0(k \sin \phi)}{k \sin \phi} \left\{ 24 \lambda J_1(\lambda) - 24 \lambda^2 J_0(\lambda) - J_2(\lambda) \right\}
\]

\[
+ 10\lambda^2 \left\{ 3J_1(\lambda) - J_3(\lambda) \right\} - 4k \sin \phi J_1(\lambda) \left\{ J_0(k \sin \phi) - J_2(k \sin \phi) \right\} \left\{ 1 + \frac{1}{k \sin \phi} \right\} \].
\]

Now

\[
\left[ \left( 1 + \tan \phi \frac{d}{d\phi} \right) \left\{ \frac{1 - J_0(k \sin \phi)}{k \sin \phi} \right\} \right]_{\phi = \phi_\infty} = J_1(\lambda),
\]

(9.3.2)

(9.3.31)
\[
\left[(1 + \tan \phi \frac{d}{d\phi}) J_1(k \sin \phi)\right]_{\phi = \phi_0} = \lambda J_0(\lambda),
\]
(9.3.3)

and \(\beta(\phi)\) has been considered in (9.2.15), so (9.3.1) becomes

\[
E_s = \frac{4\pi \omega^3 u_0}{k^2 \ell^2} \left\{ J_0^2(\lambda) + J_1^2(\lambda) - \frac{2}{\lambda} J_0(\lambda) J_1(\lambda) - \frac{1}{8k^2} \left\{ 4 J_1^2(\lambda) 
\right. 
\right. 
\]
\[
+ \lambda J_1(\lambda) \left\{ J_0(\lambda) - 3 J_2(\lambda) \right\} - \frac{2i k^2 + 2}{k \sqrt{2} \psi^{3/2}} J_1^2(\lambda) + O\left(\frac{1}{k^4}\right) \right\}. 
\]
(9.3.4)

Hence from the last line, definition (5.1.1) and (5.1.16) we have

\[
Q = 2\left[1 - \frac{J_0(\lambda)}{8k^2 \left\{ J_0^2(\lambda) - \frac{2}{\lambda} J_0(\lambda) J_1(\lambda) + J_1^2(\lambda) \right\}} \left\{ 4 J_1(\lambda) + \lambda J_0(\lambda) - 3 J_2(\lambda) \right\} 
\right.
\]
\[
+ \frac{2 J_1(\lambda)}{2 \sqrt{2} \psi^{3/2}} \operatorname{csch}(2k^2 \psi^{3/2}) + O\left(\frac{1}{k^4}\right) \right\}, 
\]
(9.3.5)

and for the case of the incident free torsion wave we have,

on taking the limit as \(\lambda \to 0\)

\[
Q = 2\left[1 - \frac{3}{2k^2} - \frac{1}{2 \sqrt{2} \psi^{3/2}} \operatorname{csch}(2k^2 \psi^{3/2}) + O\left(\frac{1}{k^4}\right) \right]. 
\]
(9.3.6)

It is, perhaps, interesting to note that on expanding the Bessel function in (9.1.20) in powers of \(\sin \phi\) and then taking the limit as \(\lambda \to 0\) we have, from (6.1.5)

\[
E_s = -\frac{\omega^3 u_0}{2} \int M, 
\]
(9.3.7)
and hence

$$Q = \frac{2}{\pi \omega} \Re \{ \bar{M} \}$$

when $\lambda = 0$, \hspace{1cm} (9.3.8)

(8.3.25) and (9.3.6) satisfy relationship (9.3.8) which, is

is to be expected.
9.4 Numerical results and discussion

$\varepsilon = 0.0$ and $\varepsilon = 0.1$, figures 57 and 58

These are both examples of Case II.

$\varepsilon = 0.2(0.1)0.6$, figures 59 to 63

It is surprising how the curves, corresponding to Cases I and II, match up. Quite often it is irrelevant which curve we consider.

$\varepsilon = 0.7$, figure 64

For $k < 5$ we consider Case II and for $k > 5$ we consider Case I. There is a certain amount of doubt in whether we should change over from Case II to Case I at $k = 5$, or at $k = 7.6$. However, for $5.0 < k < 7.6$, the two curves give numerical values for the scattering coefficient which are within 1.5% of each other.

$\varepsilon = 0.8$, figure 65

For $k < 7.3$ we consider Case II and for $k > 7.3$ we consider Case I.

$\varepsilon = 0.9$, figure 66

For the larger values of $k$, Case I should give reasonably accurate results.
References


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