A comparative study of heavy charged particles
and X-rays for CT scanning

by

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A thesis submitted to
the Faculty of Mathematical and Physical Sciences of
the University of Surrey for the degree of
Doctor of Philosophy

June 1981
To the memory of my father

who awoke my interest in science
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ABSTRACT

The use of heavy charged particles, such as protons, α-particles and heavy ions, in computed tomography CT is explored. The technique is based on certain physical properties of tissue, the stopping power, which has never been used by any available diagnostic method. Advantages and limitations of this modality are compared to those of X-ray CT.

In a detailed study of their interaction with matter, the stopping power and range of charged particles are calculated for different materials of biological interest. A correction to the standard Rossi formula, defining the root mean square angle arising from small-angle multiple scattering, has been derived. The correction is more important for proton and α-particles, particularly when they traverse thick targets.

The dose advantage and mass resolution improvement for particles over X-rays are presented quantitatively. The object surface dose, and to a lesser extent the dose at the centre, are remarkably reduced when using charged particles. The effect of small angle scattering on spatial resolution is examined and shown to be improved by employing an exit position detection system. The potential of several particles for computed tomography have been compared. The particles performance together with design considerations indicate the feasibility of a clinically useful particle accelerator which could be implemented for tomography.

Simulated data for reconstructed tomography using particles were generated for a head phantom using the general purpose SNARK reconstruction programme, basically designed for X-rays. The programme was implemented to accommodate the way particles convey information about the sample traversed and to account for the influence of the beam divergence. Reconstructions obtained for different particles demonstrate the possibility of producing images of comparable spatial resolution to X-rays, especially with the heavier ions.
Charged particle tomography would add a new dimension to the practice of diagnostic radiology for differentiating and imaging the body structure sensitively and relatively safely.
Acknowledgements

I would like to express my gratitude to my supervisor, Professor Daphne F. Jackson, for the unstinting assistance and academic guidance she offered me throughout this project. I sincerely thank her for the truthful interest regarding my career.

I am grateful to my colleagues, K. Kouris, J. Foster and M. Jones for their helpful discussions and suggestions.

I would like to thank members of the technical and support staff in the Computing Unit, and Miss S. Deane for her patience and friendship.

For financial support I am indebted to my family and relatives for the invaluable help they offered me. I particularly thank my brother Mohammed for his continual support.

Finally, I would like to thank my wife for her support, encouragement and tolerance during the past years of my study.
Introduction

Computed tomography has been the subject of numerous investigations in recent years. With such a sensitive technique, tissue abnormalities that possess small density variation from the surrounding tissue were detected. However, the density resolution is limited by the dose received from X-rays which increases with increasing the number of photons in the incident beam.

Several years ago it was realised that charged particles have some considerable advantages in providing improved density resolution per unit of dose. A variety of medical imaging techniques and diagnostic schemes using this modality were developed.

The advantages of charged particles have provoked the interest of many institutions to utilise the recently developed accelerator technology to provide particle beams for medical diagnosis at a cost competitive with recent X-ray CT scanners.

The ultimate aim of this study is to compare the advantages, limitations, and practicalities of charged particles and X-ray for medical computed tomography.

Chapter 1 includes a general description of CT scanning with emphasis on the superiority of the technique as compared to conventional radiography in many clinical applications. The method of reconstruction from projected data together with the well known mathematical algorithms are examined. The relationship between patient dose and image quality, as a significant ingredient of this work, indicates that there is a trade-off between the dose, spatial resolution, and density resolution that would be achieved in a single scan. The chapter also includes a brief review of the diagnostic performance of charged particles, using biological samples and the human body.
Chapter 2 is the "backbone" of this study for particles. It contains a theoretical investigation of the interaction of charged particles, with energies of diagnostic interest, with matter. Analysis of the energy loss mechanism, the stopping power formula, range-energy relationship, and the Bragg additivity rule, indicates that although the interaction of particles with target atoms is a complicated process, it is still more straightforward than that of a diagnostic X-ray spectrum.

The measured stopping power of the particles lacks the dependence on the atomic structure of the material and has little dependence on the energy of the beam. It measures a quantity that is proportional to the density of the sample material. Due to range straggling, there will be a distribution of stopping points which has a mean and an rms deviation which can be reduced by increasing the particle charge-to-mass ratio. In this chapter we also present a correction to the small-angle scattering formula which will influence the calculation of beam lateral displacement from the zero divergence position. The correction is dependent on the mass and energy of the particle and the thickness of the target.

Chapter 3 gives an analysis of the potential of charged particles for medical imaging based on their interaction mechanism with matter. It shows the high density resolution achieved per unit dose relative to X-rays. Mathematical formulae relating dose and mass resolution indicates that about 200 X-ray photons, of energy equal to 70 kev, are needed to produce the same density resolution obtained by a single proton of energy 200 Mev/a.m.u. The effect of small angle scattering, however, is to spatially average the actual stopping distribution to produce a blurred image. A partial solution to this problem is to associate every particle with its exit position.
In Chapter 4, the SNARK reconstruction programme is implemented to be applicable for charged particles. A projected density, calculated for X-rays as the exponential of the integrated absorption coefficient, is calculated from the integrated energy loss of a particle along the ray. To account for beam divergence, every ray is divided into \( N \) sub-rays where each sub-ray is considered to contain all particles scattered with an angle \( \alpha \). The calculated ray sums for all sub-rays are averaged and assigned to the ray they all belong. Images produced for a head phantom using protons, \( \alpha \)-particles, carbon ions, and X-rays show comparable spatial resolution but higher contrast between objects and their surroundings in the phantom when heavy particles are used, and different behaviour at the interface between high and low density objects.

The Conclusion includes a discussion of the results obtained, ideas presented, and suggestions on those areas of charged particle tomography which need more investigation.
CHAPTER 1

Computed Tomography with X-rays and charged particles

1.1 Introduction

With the progress in computation techniques in the early 1960s, several authors (Kuhl et al 1963, Cormack 1963 and 1964) began to explore the possibility of improving the quality of ordinary X-ray tomography using exact mathematical reconstruction techniques. The fundamental idea is to measure photon beam transmission along sets of parallel lines directed at different angles, with all lines covering the same transverse slice taken across the body. The idea has been developed (Hounsfield 1973) such that measurements could be obtained for various adjacent thin slices using a highly collimated photon beam to minimise the scattering contribution to the detected signal. In the original CT scanner, the collimated photon beam is directed towards a detector. The X-ray source and the detector are moved together across the object being visualised and the number of photons emerging at each position measured and stored. A frame holding the X-ray source and detector is then rotated through a small angle and a further set of transmission measurements recorded, Fig.1. A map of the distribution of X-ray transmission or attenuation through the slice is displayed.

Developments in computing power, detector technology, and reconstruction techniques have permitted highly improved designs in terms of short scanning times, reduced processing time and better image display matrices. Most developed scanners employ a fan of X-ray beams with different types of scan motion. However, more recent systems have more complicated arrangements of source and detector.
1.2 Comparison with conventional radiography

In conventional radiography a large proportion of the available information is lost in attempting to superimpose all the morphological information from a three dimensional body on a two-dimensional film. CT is able to present such information in the form of a series of thin slices. Such techniques have proved to be very sensitive to small
density variations of different soft tissues. It also can measure the values of X-ray attenuation of these tissues very accurately thus enabling the nature of the tissue to be studied.

After less than a decade, CT scanning techniques had not only supplemented or displaced some radiography methods, especially the invasive ones, but were expanding to become an important and efficient tool in other medical fields like radiotherapy and oncology. In radiotherapy treatment planning, CT is at the moment undoubtedly the method of choice as the images are produced in a way allowing for precise tumour localisation (Parker & Hobday 1980, Nusslin 1980). Tumour staging is also superior using this technique because it has the comprehensive capability for displaying the extent of tumour spread (Best 1980). However, tissue characterisation is still a major unsolved problem in diagnostic imaging, and CT has not offered much in this field where, occasionally the technique cannot differentiate between solid and fluid lesions, and cannot be certain about signs of malignancy (Ritchings et al 1980, Kreel 1980).

The ability of X-rays to discriminate between different types of tissue depends on their absorption coefficients. The strong dependence of the attenuation coefficient for each element on the atomic number and, to a less extent, the density of the object is the prime reason for tissue differentiation.

The absorption coefficient is also dependent upon the X-ray energy in a complicated way. It becomes more difficult when the test object is made of a chemical compound or a mixture. Thus, the mixture additivity rule is to be applied to estimate the sample absorption coefficient (Deslattes 1969).

Low energy X-rays will have higher absorption coefficient and greater ability to differentiate between tissues of similar atomic
constituents. However, using low energy X-rays would mean lower beam penetration and higher patient dosage.

1.3 Image reconstruction in computed tomography

1.3.1 The mathematical problem

Many techniques in medical physics depend on being able to identify a certain physical characteristic of tissue in space by projecting that property on to a plane. In computed tomography the section under consideration is defined by a function $f(x,y)$. Multiple views are taken along the projections parallel to the $x$-$y$ plane and the function distribution within the slice is displayed mainly by forming grey scale images. The same procedure is repeated for other thin slices at different depths in the $z$-direction.

Assume that $f(x,y)$ is confined to a circular object $S$ of radius $R$, where $f(x,y)$ is zero outside $S$. It is not possible to find $f(x,y)$ directly, so external measurements are performed and, ideally, presented as integrals of the function along rays. In Fig. 2, the line $L$ represents a ray-path traversing the region $S$ containing the unknown density. Let $(r, \phi)$ be the point along $L$ which lies at the intersection of the ray $L$ and the line, $r$, passing through the centre of $S$ and perpendicular to $L$. The equation of the ray is given by $r = t \cos(\phi-\theta)$, and all points $(x,y)$ along the ray satisfy this relation, with $(t,\theta)$ in the polar coordinates equivalent to $(x,y)$ in the rectangular system.

The measurement $P(r,\phi)$ associated with the ray defined by the parameters $(r,\phi)$ is given by the line integral

$$P(r,\phi) = \int_{L(r,\phi)} f(x,y) \, dS$$  \hspace{1cm} (1.1)
Equation (1.1) defines the measurement in terms of the unknown two-dimensional density to be reconstructed. The mathematical problem is to derive \( f(x,y) \) throughout \( S \) by inverting the measurements taken for a number of rays having different orientations.

![Diagram](image)

**Fig. 2** Projection-measurement geometry. Each projection is an estimate of a particular line integral of \( S \), where the line of integration is specified by the parameters \( r \) and \( \phi \).

In X-ray tomography \( f(x,y) \) represents a two-dimensional distribution of the linear attenuation coefficient while for charged particle tomography, \( f(x,y) \) describes the energy loss distribution, which is related to the integrated density (mass/volume) along \( L \).
For monoenergetic X-rays the logarithmic beam attenuation along L is defined by

\[ \ln \left( \frac{I_o}{I} \right) = \int_L \mu(x,y) \, dL \]  

(1.2)

Using equation (1.1), (with \( f = \mu \)), the ray-sum P is

\[ P = \ln \left( \frac{I_o}{I} \right) \]  

(1.3)

Ideally, the density function \( \mu(x,y) \) is reconstructed from an infinite number of projections. In practice it is calculated at finite points from a finite number of projections (Brooks & Dichiro 1976).

**Terminology:** A number of terms, widely used in reconstruction tomography, are defined as follows:

a) **Picture:** The picture is defined as a function of two variables whose value is zero outside a square whose centre is at the origin of the coordinate system.

b) **Pixel:** An n-element grid subdivides the picture region into \( n^2 \) equal squares. Each of these small squares is called pixel, short for picture element. The length of the pixel side denotes the pixel size.

c) **Ray-sum:** Given a picture and a ray, the real-ray sum is the integral of the picture along the ray.

In this terminology, the reconstruction problem may be stated roughly as: given approximations, based on physical measurements, of the real ray-sums of a density function along a number of rays, estimate \( n^2 \) numbers which describe the \( n \times n \) digitised version of the picture (Herman 1974).
1.3.2 Reconstruction Algorithms

Several algorithms have been proposed for reconstructing the variable density functions of tissues in the body specially from X-ray transmission (Brooks & Dichiro 1976). They all produce as output an estimate of the original structure based on the available projection data. We present a brief review of the algorithms, widely used in CT, with emphasis on the convolution technique as it is used by most commercial X-ray CT scanners and favoured for use in charged particle tomography (Hanson 1978).

A purely descriptive discussion of the nature of the methods under consideration will be given here, while a practical comparison consisting of simulating test objects (phantoms) and calculating their projections using the general-purpose SNARK program (Herman 1975), will be given in Chapter 4.

Reconstruction algorithms can be classified into:

(i) Back-projection, (ii) Iterative reconstruction and (iii) analytical reconstruction.

(i) Back-projection

This is the simplest algorithm for reconstruction where the signal corresponding to a ray-sum is applied, back-projected, to all points that make up that ray. If a large number of projections, spaced at small intervals, is taken then the process can be mathematically represented by the equation (Brooks & Dichiro 1975, Kohl & Edwards 1963)

\[
\hat{f}(x,y) = \sum_{j=1}^{m} P(r_j, \phi_j) \Delta \phi
\]

where \(\hat{f}(x,y)\) is the reconstructed density value, \(r_j\) defined by \(r_j = x \cos \phi_j + y \sin \phi_j\), and \(\phi_j\) is the jth projection angle with \(\Delta \phi\) as the angular distance between projections of total number \(m\).
The accuracy of the back-projection method is limited because each ray-sum is applied to all points along the ray regardless of their real densities. Such an effect blurs out sharp features in the original, so that subtle differences in density cannot be distinguished (Kuhl et al 1963). Improvements in the back-projection technique have been suggested by different authors (Vainshtein 1971, Muehllehner & Wetzer 1971, Bates & Peters 1971, and Smith et al 1973). However, interest has shifted towards other more accurate and basically different techniques.

(ii) Iterative reconstruction

The iterative technique is a method of successive guesses. Starting with a randomly selected image, repeated corrections are applied to achieve satisfactory results that would match the measured projection (Bracewell 1967, Gordon et al 1970). The iterative technique was employed by Hounsfield (1973) in the first generation of CT scanners.

The sample to be reconstructed is approximated by a grid of \( N \) pixels of uniform density with value \( \rho_i (i=1, 2, \ldots n) \). For the sake of comparable resolution, every projection is divided into \( n \) equally spaced rays. The \( j \)th ray-sum (\( j = 1, 2, \ldots n \)) is represented by

\[
P_j = \sum_{i=1}^{n} \rho_i \omega_{ij} \quad (1.5)
\]

where \( \omega_{ij} \) is a weighting factor representing the contribution of the \( i \)th pixel to the \( j \)th ray, and has the value of zero if the pixel is not intersected by the ray.

An iteration is performed by assigning arbitrary selected values for \( \rho_i \). Projections are calculated using equation (1.5), and then compared with the measured values. If there is a difference between
the measured and the calculated ray-sums, each pixel density, within the ray, is altered repeatedly by the necessary amount calculated from the difference. The first iteration is completed when the above procedure is done for all pixels and all rays.

To achieve the required accuracy, the procedure is repeated several times. An estimate of the ith density after the iteration q is given by (Brooks & Dichiro 1975)

$$\rho_i^q = \rho_i^{q-1} \sum_{j=1}^{M} \Delta \rho_{ij}^q$$  

where $\rho_i^{q-1}$ is the ith density before the qth iteration and $\Delta \rho_{ij}^q$ is the correction from the jth ray applied to the ith pixel, and M is the total number of ray-sums in the projection.

The iterative technique is similar to back-projection in the sense that corrections are applied not only to selected pixels of higher density but to all pixels along the ray. The difference is that in equation (1.6) the quantities added are corrections calculated from the projections and not the projections themselves.

(iii) Analytical reconstruction

This class of reconstruction method is based on direct solution of equation (1.1). It reduces the problem into the solution of a one-dimensional integral equation. The technique assumes a maximum frequency, $K_m$, where all spatial frequencies within the image are smaller than $K_m$. The main analytical reconstruction techniques employed by CT scanning are the two-dimensional fourier reconstruction and filtered back-projection.

The problem of blurring in simple back-projection has raised the question of finding a function which if it can operate on the measured projections, and then back-projected, would produce an image which is free of blurring and ideally the same as the original structure.
In filtered back-projection the profiles are modified, or filtered, before being back-projected. There are different mathematical filtering formulae, but the most accurate and widely used one (Ramachandran 1971, Shepp & Logan 1974) is the convolution filtering, first derived by Bracewell and Riddle (1967). It uses the maximum frequency $K_m$, together with a discrete weighting factor ($\sin^2$ function) directly related to the radius vector $r$.

An expression of the above form, for the filtered projection $\hat{p}(r,\phi)$, has been derived by Brooks and Dichiro (1976) and presented as

$$\hat{p}(r,\phi) = \frac{f(x,y) = K_m P(r,\phi)}{\int_{-\infty}^{\infty} P(r',\phi) \frac{\sin^2[\pi K_m (r-r')]}{\pi^2 (r-r')^2} dr'}$$

For sampled data spaced at intervals $a = 1/2K_m$, and because $\sin^2[\pi K_m (r-r')]$ becomes zero when $(r-r')$ is an even multiple of $a$, equation (1.7) will reduce to an evaluation of a finite length summation rather than a numerical evaluation of an integral (Gordon and Herman 1974). The summation form derived by Gordon and Herman is given by

$$P(r_i) = \frac{1}{a} \left[ \frac{P(r_i)}{4 \pi^2} \sum_{j=\text{odd}}^{n} \frac{P(r_j)}{(i-j)^2} \right]$$

The sum is taken over all the odd values of $(i-j)$.

The reconstruction process using the convolution method can be summarised by (i) filtering the measurements (projections) using equation (1.8), (ii) for each fixed angle $\theta$, back project $\hat{p}(r,\phi)$ as a constant, using equation (1.4), along all points $(x,y)$ on each ray defined by $r = t \cos(\phi-\theta)$, and (iii) repeat for all projections. The convolution method provides closer approximations to the original densities, and it has the advantage of performing a large number of the reconstructions during the measurements.

Figure 3 illustrates schematically the convolution back-projection method. The method was first used in a CT scanner by Ledley et al (1974).
1.4 The image quality

1.4.1 Introduction

Pictures of high quality are those which truly represent the body structure under examination. They are obtained by scanners designed to meet all the fundamental mathematical and physical requirements which determine the possibility of obtaining high quality pictures.

In principle all the photons which penetrate the body should be detected to ensure maximum dose efficiency. The required spatial and density resolution could be achieved by taking and efficiently processing a sufficient number of readings avoiding over- or under-determined pictures.

Fig. 3 Flow chart of the filtered back-projection reconstruction process (Barrett 1977)
Proper beam filtering and suitable software corrections could reduce the beam hardening effect of the photons energy spectrum (Herman 1979). Failure to observe one or more of the mentioned conditions will cause the picture quality to deteriorate.

1.4.2 Noise, resolution and dose in X-ray CT

The sensitivity of X-ray CT technique to changes in tissue attenuation is limited by the fluctuation of the individual transmission measurements. That is due to the discrete nature of photon emission from an X-ray tube in a given direction (Barret and Swindell 1977). The uncertainty in single transmission measurements, which leads to a background of phantom noise, is dependent upon the sample attenuation properties (thickness, composition), the initial quality of the photon beam, pixel dimensions, patient dosage, and X-ray beam thickness (Brooks and Dichiro 1976a). An expression relating these parameters for an object immersed in a constant length water bath is

\[ \sigma^2(\mu) \propto CE/a^3h D_o \]  

(1.9)

where \( C \) is the subject attenuation corrects for a number of photons \( N \) for attenuation while traversing a distance of tissue and water, \( E \) is the beam effective energy, \( a \) is the pixel width and \( D_o \) is the maximum surface dose with \( h \) as the slice thickness. The dependence of \( \sigma(\mu) \) upon the photon effective energy becomes less important at energies above 50 keV (Brooks and Dichiro 1976). Equation (1.9) predicts that the smallest density change which can be detected will be equivalent to \( \sigma(\mu) \). The pixel size is obviously related to the spatial resolution which describes the fineness of a detail that can be seen in the reconstruction. Accordingly, the precision formula states that for a given subject dose, and spatial resolution, the nature of the measurement
will limit the sensitivity to tissue variation to a fixed value and an improvement in density resolution can only be achieved at the expense of a worse spatial resolution or an increased dose.

A compromise between density resolution and spatial resolution should be considered in view of the clinical situation. Looking for metastasis in liver would require a wider pixel width (reduced spatial resolution) and a higher sensitivity if the dose is to be kept constant.

Values of $\sigma(\mu)$ obtained from (1.9) are not quite similar to those produced from a real CT scanner (McCullough 1977). For scanners that do not use the water bag, proposed in the derivation of (1.9), a reduction in $\sigma(\mu)$ of roughly a factor of 1.4 is expected (Brooks and Dichiro 1976a). On the other hand, equation (1.9) does not take into account the fact that Compton scattered radiation removes photons from the section being scanned, thereby reducing the photon statistics. If only one slice is being imaged, a correction factor, $f_c$, of about 0.2 is to be applied to $\sigma^2(\mu)$, while a value closer to 1 is probably convenient if many slices are viewed, since each slice receives photons scattered from neighbouring slices (Barrett and Swindell 1977).

The signal-to-noise ratio of a CT system can be defined as the ratio of the mean reconstructed density to its standard deviation. For an X-ray reconstructed image

$$\text{SNR} = \frac{\mu_{av}}{\sigma(\mu_{av})}$$  (1.10)

A general expression for reconstruction accuracy based on equation (1.9) relates the dose at the tomogram centre, $D_c$, to the SNR and the resolution is given by (Barrett et al 1977)

$$D_c = \frac{2.26(I_0/I) E(SNR)^2 f_c}{\rho \mu_{av} a^3 h} = 1.6 \times 10^{-13} \text{ Gray}$$  (1.11)
where \( \mu_{av} \) is the mean energy absorption coefficient, \( \rho \) is the density (g/cm\(^3\)), \( E \) is photon energy (keV), \( (I_o/I) = e^{\mu t} \), with \( t \) equals the object half-thickness, and the factor \( 1.6 \times 10^{-13} \) is to convert from keV/g to Gray. The convolution technique is assumed in equation (1.11) with a rho-filter (Bates and Peters 1971, and Smith et al 1973). For other filters, the numerical value, 2.26, will be slightly different.

The skin dose of a body surrounded by a water bag can be calculated from the dose at the centre by (Barrett et al 1977)

\[
D_{\text{skin}} = D_{\text{centre}} \cdot I_o(\mu t)
\]  

(1.12)

where \( I_o(\mu t) \) is the zero-order modified Bessel function (Abramowitz et al 1970).

Consider a photon beam of effective energy 70 keV and an object of mean density 1 g/cm\(^3\), \( \mu_{av} = 0.0191 \text{ mm}^{-1} \), immersed in water. Figure 4 shows the dependence of dose on spatial resolution and sample thickness for a particular slice thickness, with SNR = 250 (\( \sigma = 4 \) Hounsfield units), \( h = 10 \text{ mm} \), and \( f_c = 0.2 \).

It is obvious that thick objects are likely to receive a higher skin dose. Doubling the sample thickness from 180 mm to 360 mm increases the skin dose by a factor of 4 at the same spatial resolution.

Practical measurements of the dose delivered on X-ray scanners (Villafana et al 1978) shows that maximum doses to the head occur just above and posterior to the right ear and are about 0.038 and 0.045 Gy for 120 and 140 kvp beams, respectively. The average dose over the whole head is of the order of 0.021 and 0.025 Gy for the same kvp settings. They also found that the contribution of the scattered radiation from one slice to the next is about 25% of the main slice dose for the immediately adjacent layer, 15% for the second slice away and 5% for the third slice.
Fig. 4 Variation of dose with spatial resolution at constant standard deviation, in the reconstructed values obtained for a single slice of tissue at photon energy $E=70$keV. Solid lines represent the $\gamma$ dose, while dashed lines represent the centre dose. The figure also indicates the dependence of dose on the sample thickness, where $\alpha$, $\alpha'$ are for a 90mm object radius, $\beta$ and $\beta'$ are for 140mm object radius, and $\gamma, \gamma'$ are for 180mm radius. Dose values are in milligray. 1 Gray = 100 rad.
1.4.3 Beam hardening artifacts

The use of a polyenergetic X-ray source in CT scanning causes averaging of the reconstructed attenuation coefficients. Bear's law for photon beam attenuation only applies for monoenergetic beam. X-ray beams reaching a particular point inside the body from different directions are likely to have different spectra and will be attenuated differently at that point. This will lead to various artifacts in the reconstructed images due to inaccuracies in the attenuation values being obtained (Brooks and Dichiro 1976b). This phenomena would have undetected effect if the photons are traversing a homogeneous object, while image distortion is likely to be observed at the interface between regions of different attenuation properties.

Because of its high calcium content, attenuation in bone is substantially higher than that of soft tissue. Different authors (Ambrose 1973) have demonstrated a region of increased CT values in the periphery of the brain which could be misinterpreted as being abnormality in the grey matter of the cerebral cortex. However, the effects are local, in the region of the attenuation interface, and to a less extent along the area traversed by the polyenergetic beam (Mustafa 1978).

Zatz (1977) has reported a shift in the energy spectrum of the photon beam as it traverses materials with attenuation quite different from that of water. The direction of the shift is towards increasing the average energy of the beam as it traverses higher Z materials. Errors in the CT values, caused by bone hardening, are partly corrected by applying a pre-determined correction factor for up to 80 mm of bone (Hounsfield 1977). Such corrections are less effective when CT scanning of greater non-uniformities, such as the abdomen, is performed, and in the lower parts of the skull where the bone distribution is thicker and less predictable (Mustafa 1978).
The use of the water bag and additional pre-filtering of the beam (leaving Compton scattering as the dominant interaction, but altering the contrast of the image) would minimise the beam hardening effect.

A software correction was suggested by Herman (1979), who investigated how one can estimate from the total attenuation, \( P \), of a polyenergetic X-ray beam, what the total attenuation, \( m \), of a monoenergetic beam would have been along the same ray. But because the correction was obtained under the restrictive condition that there are only two different types of material between the source and detector, the correction could be applied, under favourable conditions, for the brain scanning. Such a method is not adequate in a whole body scanner.

1.5 Particle imaging for medical applications

The first study with monoenergetic particles was performed by Koehler (1968) using a 160 MeV proton beam from the Harvard cyclotron to radiograph an aluminium specimen. Since then a number of different radiographic techniques, using charged particles, have been developed.

West and Sherwood (1972) have reported proton radiography of thin biological and metallic objects. The sharp outlines of the objects were clearly seen. The technique they used utilises the phenomenon of charged particle multiple scattering.

The Harvard cyclotron was also used by Steward and Koehler (1973, 1974) to produce the first human tissue radiographs by the end-of-range technique. Their results indicate the possibility of visualising a non-calcified brain tumour in the skull, for the first time, and
observing intra-cranial haemorrhages and cerebral blockages with an initial surface dose of $5 \times 10^{-3}$Gy. In a comparison with conventional mammography (X-rays), tumour-bearing breast samples have been radiographed with a proton beam. Small secondary tumours were observed with an incident dose of less than $3 \times 10^{-3}$Gy, which is only 3% of an average mammography dose.

Reduction in dose could still be achieved by using a scanning proton beam and an electronic detection system (Steward 1976). Images of a biological sample have been obtained, with this technique, with an initial dose of $10^{-6}$Gy. This is the dose that one would receive from natural background radiation over 8 hours.

The potential usefulness of heavy ion radiography for detection of low contrast structures was studied by Tobias et al (1977). The beam passes through the object and stops in a stack of thin nuclear detectors. Carbon radiograph of a fresh sample of liver containing metastatic oat-cell carcinoma was obtained with abnormalities easily recognised.

The same technique was used by Sommer et al (1978), where the results of imaging a brain slice, that contained a denocarcinoma immersed in a lateral ventricle, were remarkable. Clear images were also obtained for a breast containing ductal carcinoma using an oxygen ion beam.

Recently a feasibility study of proton computed tomography, reported by Hanson (1978), has opened the door for more studies with charged particles. The technique claims better spatial resolution and contrast, and lower dosage as compared to conventional radiography.

The studies reported here suggest the superior density resolution per unit dose obtained from charged particles imaging. In the last few years many institutions and laboratories have initiated studies on
the fundamental principles of the interaction of charged particles with matter, and their potential for diagnostic radiology.

The use of charged particles for diagnostic radiology is different from that of X-rays due to the difference in their interaction with matter. Owing to their charge, particles would interact with the atoms in the medium losing a small amount of their energy. Figure 5 compares the way protons and X-rays characterise the sample atoms, where the measurement of the total energy loss of the particle provides information about the sample traversed.

In X-ray radiography, however, the information is conveyed by those X-rays which make no interaction (scattering, absorption) with the target atoms.

![Schematic comparison between the interaction of heavy charged particles and X-rays with atoms](image-url)
The physical principles of the interaction of charged particles with matter are of great significance for developing our appreciation to the potential and limits of particles for medical imaging. In the next chapter a study of the interaction mechanisms, energy loss process, range-energy relation, and multiple scattering of charged particles will be presented.
Interaction of charged particles with matter

2.1 Introduction

In passing through a medium, composed of atoms of atomic number \( Z \) and mass number \( A \), high energy particles, of mass \( M \) and electric charge \( z e \), experience a random number of collisions with the electrons and nuclei of the medium atoms.

For particles with energies of diagnostic interest (100-300 MeV for protons) energy loss is mainly due to elastic and inelastic collisions with atomic electrons, with inelastic collisions playing the major role (Bethe 1930). These types of interactions are responsible for excitation and ionisation of the atoms.

Interactions with the target nuclei through the strong nuclear force, in the diagnostic energy range, are relatively rare but their effect on beam attenuation is likely to cause higher patient dose.

Small-angle multiple scattering from nuclei and electrons experienced by particles traversing a medium will have a small effect on the energy loss mechanism. However, deviation of the particles from their original trajectory will cause the beam spread with lateral displacement at the exit point.

The stopping power defined as the energy loss \((-\text{d}E)\) per particle per unit path length \((\text{dx})\) is to be calculated for protons assuming that the only source of energy loss is the ionisation and excitation of the atoms.

Since the behaviour of the interaction of heavy charged particles with matter, at high energies, can generally be obtained by a scaling procedure from the corresponding proton interaction, most of this work will be concentrated on the proton. Whenever there is a discrepancy between their characteristics and those of protons, heavy ions will be
studied separately. We will be mainly concerned with very massive, high energy particles, which can be used for medical diagnosis with minimum dose, comparable image resolution, and reasonable production cost.

2.2 The stopping power of charged particles

The theory of energy loss of charged particles was established by Bohr (1913) through a semiclassical technique. According to this technique, the distance of closest approach of the incident projectile to the centre of an atom, "impact parameter", is the basis of the classification of the collisions.

Bethe (1930) solved the problem quantum mechanically. The significant difference between Bethe's technique and that of Bohr is the use by Bethe of momentum transfer rather than impact parameter to characterise collisions. The wave nature of the projectile with the uncertainty principle, and the discreteness of the energy transfers put a severe limitation on the validity of the classical technique, particularly for lighter particles (muons, pions, protons) and fast ions.

Using the quantum mechanical approach, Fano (1963) showed that the average energy loss per unit path length (-dE/dx) of a particle is related to the inelastic cross-section by

\[-dE/dx = N_a \sum_n E_n \sigma_n, \ldots (2.1)\]

where \(\sigma_n\) is the inelastic collision cross-section for excitation to the atomic state \(E_n\) above the ground state. \(N_a\) is the number of atoms per unit volume of the target material.

Equation (2.1) was evaluated by Fano (1963) using the concept of energy transferred (Q) to an unbound electron for momentum transfer (q), to obtain the relativistic Bethe formula for the stopping power S,
where $I$ is the mean excitation energy per electron, $\beta c$ is the particle velocity, $m$ and $e$ are the mass and charge of the electron, $z$ is the particle mass number. The term $Z \left[ \ln \frac{2mc^2\beta^2}{I(1-\beta^2)} - \beta^2 \right]$ is called the stopping number. Its value is important especially at low particle energy.

Assumptions and approximations made in the derivation of the above equation for stopping power, by different authors, would have some effect on the stopping power results. The most important of the assumptions made are:

(i) The projectile velocity $V$ does greatly exceed the orbital velocities of the orbital electrons in the K-shell, and hence of any atomic electron.

The contribution of the electrons to the stopping power will be minimum if their velocities in their orbits are comparable to that of the incident particle, Bichsel (1972). For more precise stopping power results at low particle energy, the total stopping number $R$ is given by (Bichsel 1972)

$$R = Z \left[ \ln \frac{2mc^2\beta^2}{I(1-\beta^2)} - \beta^2 - \frac{\sum C_i}{Z} \right]$$

where, $C_i/Z$ is referred to as the shell correction term with $Z$ as the target atomic number. Being proportional to $1/\beta^2$, shell corrections proposed by Fano (1963) tend to be large for small velocities, but get remarkably small for large velocities. Bichsel (1968) has calculated the K- and L-shell corrections. He concluded that such corrections make not more than 10% of the total stopping number, $R$, for lead.

Figure 1 plotted for protons in several elements at different energies.
shows the percentage of the shell correction contribution to R. Other particles (ions) of the same velocity would have the same corrections.

For the purpose of this work, where interaction of particles with tissue is our main concern, shell corrections amount to less than 3% of the stopping number, or less than 0.7% of the proton (ion) stopping power above 5 MeV/a.m.u. One should point out that particle radiography using the end of range technique should avoid such small energies to minimise the shell-correction effect.

(ii) The density effect

The stopping power formula for heavy particles, as derived by classical or quantum-mechanical theories, varies with kinetic energy in the way shown in Fig. 2 for all particles. In the medium energy range \(1 < E(\text{MeV}/\text{a.m.u.}) \ll M c^2\), the main energy loss variation is as \(1/E\) since the logarithm in the stopping power formula changes slowly. At higher energies where \(\beta \to 1\), stopping power rises very gently with increasing \(E\).

Equation (2.2) for calculating \(S\) is valid for not too relativistic particles in all types of media. For particles with kinetic energies/a.m.u. exceeding their rest mass \((M c^2)\), a considerable rise in the stopping power was observed. Thus, eqn. (2.2) does no longer apply. The stopping power formula was corrected by introducing a "density effect factor" which becomes only important at such high energies. Density effect corrections apply only for solids and liquids, Fermi (1940). For dense material, the interatomic spacing is shorter by a factor of 10, and the assumption of a particle field influencing one electron in one atom at a time is not accurate (Jackson 1975). In the literature this factor appeared in the stopping power formula as \(\delta\). The value of \(\delta\) for particles used in clinical diagnosis can be safely neglected, and equation (2.2) will remain
Fig. 1  Percentage contribution of shell corrections to the stopping number $R$ for heavy particles of the same velocity in different elements.
2.2.1 Stopping power calculation

The stopping power for heavy particles (ions) in different materials can be computed using eqn.(2.2), which can be put in a suitable calculation form as follows,

\[
S = \frac{4\pi e^2 Z^2 N}{m c^2 \beta^2} \left[ \ln \frac{2 m^2 c^2 \beta^2}{I(1-\beta^2)} - \beta^2 \right],
\]

and \( mc^2 = 5.11 \times 10^5 \text{eV} = 8.176 \times 10^{-14} \text{Joule}, \)

\[ I = \text{mean ionization potential, eV}. \]

\[ 4\pi e^2/mc^2 = 4\pi m c^2 r_0^2 = 0.509823 \times 10^{-28} \text{MeV m}^2 \]

\[ r_0 = \text{classical electron radius} = 2.8177 \times 10^{-15} \text{m}. \]

\[ N = N_0 \rho/A, N_0 = \text{Avogadro's number} = 6.022 \times 10^{26} \text{atom/Kgm}. \]

\[ \rho, A = \text{density (kgm/m}^3\text{), and target mass number,} \]

then the mass stopping power \((S/\rho)\) is given by

\[
(S/\rho) = z^2 \left( \frac{Z}{A} \right) \left( \frac{0.030706}{\beta^2} \right) [F(\beta) - \ln I] \tag{2.3}
\]

where, \( S/\rho \) is in \( \text{MeV.m}^2/\text{Kgm} \)

\[ F(\beta) = \ln(2m c^2 \beta^2) - \ln(1-\beta^2) - \beta^2. \]

A FORTRAN program written for the calculation of \((S/\rho)\) for any energetic particle (above 2 MeV/a.m.u.), in any target is listed in Appendix 1. Shell corrections and the density effect are neglected.

Bragg's rule was employed to obtain stopping power values of compounds by adding the stopping powers of the individual component elements. A compound of different elements with mass numbers \( A_1, A_2, \ldots, A_n \),
has a mass stopping power $S_c$ given by

$$S_c = \sum_{i=1}^{n} S_i \gamma_i$$

(2.4)

where $\gamma_i$ is the relative mass abundance of constituent $i$, of mass number $A_i$, in the compound, which is made of $n$ elements, i.e.

$$\gamma_i = a_i A_i / \sum_{i=1}^{n} a_i A_i$$

(2.5)

The validity of the Bragg rule is to be discussed later on in this chapter.

Mass stopping power is calculated for proton in various elements and compounds which are of particular interest for diagnostic radiology. Elements like oxygen, carbon, hydrogen, nitrogen, calcium and compounds like bone, muscle, NaI, and water are involved. Proton energies are between 2 to 600 MeV/a.m.u. The low energy shell correction effect and the high energy density correction are neglected.

The composition of bone, muscle, water are taken from ICRU report 28, and the fractional abundances for their constituents are shown in Table II. The mean ionisation potential, $I$ for various elements is taken from the recommended averages of Ahlen (1980).

Results for proton in single elements are shown in Fig.3, and those for compounds are plotted in Fig.4. As can be expected from eqn.(2.2), the stopping power decreases with increasing kinetic energy of the particle. The weak dependence of stopping power on the absorber charge-to-mass ratio ($Z/A$) is also obvious.

This ratio is relatively constant for most elements of biological importance ($\sim 0.5$). The exception is hydrogen for which $Z/A = 1$. For this reason, proton range-radiography is expected to be sensitive to the hydrogen content of the material. But, because of the direct dependence of stopping power on the absorber density, higher hydrogen
Table II  The composition of some biological materials, and the fractional abundance of their constituents

(ICRU 28)

<table>
<thead>
<tr>
<th>Components</th>
<th>H</th>
<th>C</th>
<th>N</th>
<th>O</th>
<th>Na</th>
<th>Mg</th>
<th>P</th>
<th>S</th>
<th>K</th>
<th>Ca</th>
<th>I</th>
<th>Ar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compound</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Z, I&lt;sub&gt;av&lt;/sub&gt;)</td>
<td>18.5</td>
<td>79.0</td>
<td>82.0</td>
<td>98.5</td>
<td>148.0</td>
<td>156.0</td>
<td>172.0</td>
<td>180.0</td>
<td>193.0</td>
<td>191.0</td>
<td>498.0</td>
<td>188.0</td>
</tr>
<tr>
<td>% by weight</td>
<td>6.4</td>
<td>27.8</td>
<td>2.7</td>
<td>41.0</td>
<td>-</td>
<td>0.2</td>
<td>7.0</td>
<td>0.2</td>
<td>-</td>
<td>14.7</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Bone</td>
<td>10.2</td>
<td>12.3</td>
<td>3.5</td>
<td>72.89</td>
<td>0.08</td>
<td>0.02</td>
<td>0.2</td>
<td>0.5</td>
<td>0.3</td>
<td>0.01</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Muscle</td>
<td>11.21</td>
<td>-</td>
<td>-</td>
<td>88.79</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Water</td>
<td>10.5</td>
<td>23.3</td>
<td>2.3</td>
<td>63.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Soft tissue</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>15.34</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>84.66</td>
</tr>
<tr>
<td>NaI</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.57</td>
</tr>
<tr>
<td>Air</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>74.53</td>
<td>22.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
concentration in a medium would lead to a lower density of that medium, and the effect of density on linear stopping power is usually dominant (Steward and Koehler 1974).

The stopping power for projectiles heavier than the proton can be calculated from the same stopping formula, eqn.(2.2). The energy-loss mechanism of these projectiles is basically the same as that of protons and alpha particles. However, at low energies (< 1 MeV) atomic electrons become attached to slowing nuclei until they become neutral atoms. At such a stage, nuclear stopping, due to elastic Coulomb collisions with the target nuclei, becomes the dominant energy-loss process. Charge capture and loss is not important for protons and alpha particles except at very low energies (< 0.2 MeV/a.m.u.).

In the intermediate energy region, the stopping power for different ions of the same velocity as the proton, passing through the same material can be calculated by a scaling procedure. The use of scaling laws appeared to be a convenient method to match the experimental measurements (Ziegler 1977).

If $S_p(V,Z)$ is the electronic stopping power of the proton of electronic charge $z_p$, at some velocity $V$, traversing an absorber of atomic number $Z$, then the electronic stopping power of a heavy ion, $S_{HI}$, of atomic number $Z_{HI}$ traversing the same medium at the same velocity is given by

$$S_{HI}(V,Z) = \left(\frac{Z_{HI}}{z_p}\right)^2 \cdot S_p(V,Z) \quad (2.4)$$

If the charge exchange is to be taken into account, which will be the case at low energies, the heavy ion and proton charge are to be replaced by their effective charge $z_{eff}$. The significance of this expression is to be discussed in the coming sections.

As can be appreciated from Fig.5, the general behaviour of energetic heavy ions in water is the same as of that for protons. They
Mass stopping power (MeV·m²/kg) in water

Heavy ion energy (MeV/a.m.u.)

Ca 20
Ar 18
Ne 10
Oxygen 8
Carbon 6
α-particle
proton

Fig. 5
will produce stopping power values proportional to their charge squared. The \( z^2 \) dependence of stopping power makes the energy loss of heavy ions considerably higher than protons. As a result they would travel shorter distances, and if to be used in medicine they will deliver a higher dose compared to that of proton.

2.2.2 The mean excitation potential and Bragg's rule

The mean excitation potential, \( I \), is the most important parameter in the development of the stopping-power theory. It is a logarithmic mean over the excitation potentials \( E_n \), defined by

\[
\ln I = \sum_n f_n \ln E_n \tag{2.5}
\]

where, \( f_n \) is the oscillator strength for the \( n^{th} \) energy level and it is dependent upon the target atomic number. The Thomas-Kuhn sum rule gives \( \sum_n f_n = 1 \), Ahlen (1980).

Experimental determination of \( I \) by different authors, Barkas (1964), Zrelov (1974) and others, are consistent with the conclusion of Bethe's theory that \( I \) is independent of projectile charge or energy. A summary of the theoretical calculations based on real atomic wave functions (Dehmer 1975), supports the above conclusion.

Bragg's additivity rule (Bragg 1905), suggests that the stopping power of a compound is equal to the sum of the stopping power of its components. Application of this rule in the calculation of energy loss does not take into consideration the chemical and/or phase charge effects on the stopping power evaluation. However, the energy level \( E_n \), for the outer shell electrons can change remarkably as well as the atomic wavefunction for state \( n \), when several atoms bind together to form a molecule, and rearrangement of valence electrons in the condensed phase
is likely to occur, Brandt (1956).

Experimental data from Platzman (1952), and Dehmer (1975) shows a discrepancy between the mean ionisation potential $I$ for some elements in their atomic and molecular form. Chemical binding was given as the reason for these differences. The effects of these bindings decreases considerably with increasing the target atomic number. This is due to the increased dependence of $I$ on electrons in the inner-shells, such electrons are not sensitive to chemical binding. Table III compares the values of the mean ionisation potential in (eV), obtained experimentally for different elements, by different authors, using various techniques. Some of the techniques used are (i) the measurement of the energy lost by calorimetry, Anderson (1966), (ii) measurement of the relative stopping power using a reference material, (iii) by measuring the ranges at different energies, from which $I$ can be found using the energy-range curve, Fano (1963), Turner (1970).

From their theoretical calculations for gaseous atoms and molecules, Zeiss et al (1977) concluded that Bragg's rule is accurate to within 1.5% at 0.5 MeV/a.m.u. for all compounds. The presence of hydrogen in these compounds would introduce up to 6% deviations in the stopping power at these energies.

Phase effects were also examined theoretically and experimentally by many authors. A detailed survey of the published data in phase effects in stopping power can be found in Thwaites et al (1978). Almost all low-energy experiments (< 1 MeV/a.m.u.) agree that phase effects would cause a reduction in energy loss values obtained for condensed materials, as compared to those in their gaseous state. In spite of this there is no general agreement on how much that effect would be. Depending on the constituents of the medium a 2 - 20% deviation from Bragg's rule was reported. The cause of these effects would involve different mechanisms.
### Table III  Values of the mean ionisation potential (eV) for various materials recommended by different sources

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>H₂ (gas)</td>
<td>1</td>
<td>-</td>
<td>18.3±2.6</td>
<td>18.2</td>
<td>19.2</td>
<td>18.8</td>
</tr>
<tr>
<td>H₂ (in compounds)</td>
<td>1</td>
<td>17.6</td>
<td>15-18</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Be (solid)</td>
<td>4</td>
<td>67</td>
<td>64</td>
<td>61.7</td>
<td>64.0</td>
<td>62.7</td>
</tr>
<tr>
<td>C (graphite)</td>
<td>6</td>
<td>78.4</td>
<td>81</td>
<td>81.2</td>
<td>78</td>
<td>77.3</td>
</tr>
<tr>
<td>C (in compounds)</td>
<td>6</td>
<td>77.3</td>
<td>77-80</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>O₂ (gas)</td>
<td>8</td>
<td>-</td>
<td>101</td>
<td>-</td>
<td>-</td>
<td>97.7</td>
</tr>
<tr>
<td>O (in compounds)</td>
<td>8</td>
<td>98.5</td>
<td>91-101</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Al (solid)</td>
<td>13</td>
<td>164</td>
<td>163</td>
<td>163</td>
<td>166</td>
<td>162</td>
</tr>
<tr>
<td>Ar (gas)</td>
<td>18</td>
<td>-</td>
<td>190</td>
<td>189</td>
<td>182</td>
<td>194</td>
</tr>
<tr>
<td>Ca (solid)</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>187</td>
<td>191</td>
<td>196</td>
</tr>
<tr>
<td>Fe (solid)</td>
<td>26</td>
<td>264</td>
<td>273</td>
<td>277</td>
<td>282</td>
<td>280</td>
</tr>
<tr>
<td>Cu (solid)</td>
<td>29</td>
<td>306</td>
<td>315</td>
<td>316</td>
<td>319</td>
<td>322</td>
</tr>
<tr>
<td>Ag (solid)</td>
<td>47</td>
<td>462</td>
<td>471</td>
<td>466</td>
<td>475</td>
<td>466</td>
</tr>
<tr>
<td>W (solid)</td>
<td>74</td>
<td>750</td>
<td>-</td>
<td>704</td>
<td>-</td>
<td>693</td>
</tr>
<tr>
<td>Au (solid)</td>
<td>79</td>
<td>-</td>
<td>761</td>
<td>760</td>
<td>784</td>
<td>755</td>
</tr>
<tr>
<td>Pb (solid)</td>
<td>82</td>
<td>812</td>
<td>788</td>
<td>767</td>
<td>813</td>
<td>759</td>
</tr>
<tr>
<td>U (solid)</td>
<td>92</td>
<td>945</td>
<td>872</td>
<td>856</td>
<td>-</td>
<td>847</td>
</tr>
</tbody>
</table>

* National Committee on Radiation Protection and Measurements, NCRP Report No.25
Polarisation effects (dielectric screening) and valence electron rearrangements, Thwaites (1978) are associated with the reduction in stopping power for condensed materials.

In the light of this analysis, values of $I$ obtained experimentally can be relied upon to produce accurate calculation of energy loss for particles carrying single charge, and travelling with relative velocity $\beta$, between 0.15 and about 0.88. In this region, Bragg's rule can be applied, safely, for compounds, assuming that chemical binding and the physical state of the material have negligible effect. With such assumptions, less than 1% error in stopping power is expected (Ahlen 1980).

Sternheimer (1966) has suggested a formula which relates the adjusted mean ionisation potential $I_{\text{adj}}$ to the target atomic number as follows:

$$I_{\text{adj}} = \begin{cases} 12Z + 7 \text{ eV}, & Z < 13 \\ 9.76Z + 58.8Z^{-0.19} \text{ eV}, & Z > 13 \end{cases}$$

(2.6)

There is a negligible difference between $I$ and $I_{\text{adj}}$, which will have little implication on the stopping number except for heavy elements.

$$I_{\text{adj}} = I \exp \left( \frac{C}{Z} \right)_{\beta=1}$$

(2.7)

where $\left( \frac{C}{Z} \right)_{\beta=1}$ is the high energy limit of the shell correction term, Fano & Turner (1963).

The mean ionisation potential $I$, can be calculated from a formula suggested by Dalton (1968) which is produced from a fit to the experimental data, and given by:

$$I = \begin{cases} 11.7Z + 11.2, & Z \leq 13 \\ 8.71Z + 52.8, & Z > 13 \end{cases}$$

(2.8)

A comparison between $I$, $I_{\text{adj}}$ evaluated for some materials using the above equations (2.6, 2.8) to those obtained experimentally by Fano (1963),
Turner (1970), Bichsel (1972) and Andersen (1977) are presented in Table IV. The last column is an average recommended by Ahlen (1980). Deviations from those in Table III, especially for Z < 10, can be easily recognised.

In diagnostic radiology, materials of Z ≤ 20, and projectiles travelling with energies between 10-500 MeV/a.m.u. are of particular concern. We are also dealing with materials in all their forms, gaseous, liquid and solid. Since one cannot consider a single value of the mean ionisation potential of a material as its general I, an average over all the recommended values is being used in this work to produce stopping cross-sections. However, the accuracy of such averaging is tested by calculating the effect of the variation in I on the mass stopping power (S/ρ) for several elements, and different projectile energies, Fig.6.

A 10% variation in I for elements which are mainly involved in any tissue equivalent compound (H, C, O, N) makes less than 1% difference in (S/ρ) at E = 2 MeV/a.m.u. and about 2% with E = 1000 MeV/a.m.u.

2.2.3 Electron capture and loss

The use of scaling laws for the calculation of electronic stopping power for ions heavier than the proton could be highly accurate providing that

(i) the interaction of protons and heavy ions with the target is the same,

(ii) the effective charge of an energetic ion is only a function of the ion velocity, and is independent of the physical state of the target.

At low ion energies the possibility that the ion will maintain its charge during interaction decreases as the particle slows down in matter. At specific energies, < 1 MeV/a.m.u., the relative change in the charge of an ion becomes a maximum and the ion could be neutralised, Northcliffe (1963).
Fig 6
Table IV  Values of $I$, $I_{adj}$ calculated from eqn.(2.6) and (2.8), compared to an average proposed by Ahlen (1980)

<table>
<thead>
<tr>
<th>Target Material</th>
<th>$Z$</th>
<th>$I$(eV)$_{Dalton}$ (1968)</th>
<th>$I_{adj}$(eV)$_{Sternheimer}$ (1963)</th>
<th>$I$(eV)$_{recommended}$ $Ahlen$ (1980)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_2$(gas)</td>
<td>1</td>
<td>23</td>
<td>19</td>
<td>18.5 ± 0.2</td>
</tr>
<tr>
<td>H in compounds</td>
<td>1</td>
<td>23</td>
<td>19</td>
<td>17.6</td>
</tr>
<tr>
<td>He(gas)</td>
<td>2</td>
<td>35</td>
<td>31</td>
<td>42.3 ± 0.5</td>
</tr>
<tr>
<td>Li</td>
<td>3</td>
<td>46</td>
<td>43</td>
<td>40.0 ± 1.6</td>
</tr>
<tr>
<td>Be</td>
<td>4</td>
<td>58</td>
<td>55</td>
<td>63.9 ± 0.7</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>70</td>
<td>67</td>
<td>76</td>
</tr>
<tr>
<td>C (graphite)</td>
<td>6</td>
<td>81</td>
<td>79</td>
<td>79.0 ± 0.7</td>
</tr>
<tr>
<td>C in compounds</td>
<td>6</td>
<td>81</td>
<td>79</td>
<td>77.3</td>
</tr>
<tr>
<td>$N_2$(gas)</td>
<td>7</td>
<td>93</td>
<td>91</td>
<td>82 ± 4</td>
</tr>
<tr>
<td>N in compounds</td>
<td>7</td>
<td>93</td>
<td>91</td>
<td>99.5</td>
</tr>
<tr>
<td>$O_2$(gas)</td>
<td>8</td>
<td>105</td>
<td>103</td>
<td>98.5 ± 1.5</td>
</tr>
<tr>
<td>O in compounds</td>
<td>8</td>
<td>105</td>
<td>103</td>
<td>98.5</td>
</tr>
<tr>
<td>Al</td>
<td>13</td>
<td>163</td>
<td>163</td>
<td>164 ± 1</td>
</tr>
<tr>
<td>Ar</td>
<td>18</td>
<td>210</td>
<td>210</td>
<td>188 ± 2</td>
</tr>
<tr>
<td>Fe</td>
<td>26</td>
<td>279</td>
<td>285</td>
<td>275 ± 3</td>
</tr>
<tr>
<td>Cu</td>
<td>29</td>
<td>305</td>
<td>314</td>
<td>317 ± 2</td>
</tr>
<tr>
<td>Kr</td>
<td>36</td>
<td>366</td>
<td>381</td>
<td>357 ± 2</td>
</tr>
<tr>
<td>Pb</td>
<td>82</td>
<td>767</td>
<td>826</td>
<td>793 ± 11</td>
</tr>
</tbody>
</table>
Any further interactions would lead to nearly neutral systems colliding with each other. Thus, the contribution of strongly bound atomic electrons to energy loss is no longer dominant, and electronic stopping power will become negligible. Owing to electron capture by the ion, the charge $z_{\text{HI}}^*$ will not be its nuclear charge, but a smaller 'effective charge' $z_{\text{HI}}^*$, Northcliffe & Schilling (1970), where

$$z_{\text{HI}}^* = z_{\text{HI}} \cdot \gamma$$  \hspace{1cm} (2.9)

where $\gamma$ is the relative charge change ($0 < \gamma < 1$).

With this modification, eqn.(2.4) for the calculation of heavy ion electronic stopping power will read

$$S_{\text{HI}}(V,Z) = \frac{(\gamma z)^2_{\text{HI}}}{(\gamma z)^2 P} \cdot \text{Sp}(V,Z)$$  \hspace{1cm} (2.10)

Bohr (1941, 1948) has suggested that the fractional change in the charge of an energetic highly stripped heavy ion is determined by the ratio of the ion velocity in the material medium $V$, to the orbital velocity of the electron in the ion $V_0$ (Bohr velocity), i.e.

$$\left( \frac{z_{\text{HI}}^*}{z_{\text{HI}}} \right) = \left( \frac{V}{V_0} \right) K$$  \hspace{1cm} (2.11)

where $K$ is the proportionality constant. The value of $K$ suggested by Bohr is $K = z_{\text{HI}}^{-2/3}$, so equation (2.11) becomes

$$\left( \frac{z_{\text{HI}}^*}{z_{\text{HI}}} \right) = \left( \frac{V}{V_0} \right) z_{\text{HI}}^{-2/3}$$  \hspace{1cm} (2.12)

Bohr also suggests that the effective charge does not differ practically from the r.m.s. value of the charges actually carried by the ions, and that the width of the actual charge distribution is small.

Pierce and Blann (1968) have studied $z^*$ for ions with $Z \leq 53$ in gaseous and solid stopping materials. They found that the effective charge of the ion doesn't depend significantly on whether the target material is gas or...
solid (also reported by Betz 1972).

Equation (2.12) has been modified by many authors, Northcliffe (1963), to be valid at lower energies giving the form

\[
\left( \frac{z_{HI}^*}{z_{HI}} \right) = \gamma = 1 - \exp\left[ a \left( \frac{V}{V_0} \right) z_{HI}^{-2.3} \right] \tag{2.13}
\]

where, \(a\) is a fitting constant, \(V_0\) is Bohr velocity given by \(V_0 = e^2/h\), (\(e\) is the elementary charge, \(h\) is plank constant).

Forster et al (1976) used a derived parametrisation for the effective charge in terms of the He\(^4\) data given by Ward et al (1976). The result for \(V/V_0 > 2\) is given by

\[
\gamma = 1 - A(z_{HI}) \exp\left[ -0.879 \left( \frac{V}{V_0} \right) z^{-0.65} \right] \tag{2.14}
\]

where, \(A(z_{HI})\) is a secondary factor produced from fitting the data,

\[
A(z_{HI}) = 1.035 - 0.4 \exp(-0.16 z_{HI}) \tag{2.14a}
\]

This term is of small importance for heavy ions, but of higher importance for lighter ions.

A combination of this proposed secondary factor and the primary expression for the effective charge, eqn.(2.14), would produce

\[
\gamma = [1 - (1.035 - 9.4 \exp(-0.16 z_{HI}))\{\exp(-0.879 V/V_0 z^{-0.65})\}] \tag{2.15}
\]

A similar formula was obtained by Ziegler (1977) using the term "reduced stopping power, \(S\)", which is equal to \(\gamma^2\). His formula given in terms of those in (2.15) is

\[
\gamma = 1 - \exp(-V_2)\{1.034 - 0.1777 \exp(-0.08114 z_{HI})\} \tag{2.16}
\]

where, \(V_2 = V_1 + 0.0378 \sin \frac{n}{2} V_1 \)

\(V_1 = 0.886 \left( \frac{V}{V_0} \right) z_{HI}^{-0.65} \)

and \(\left( \frac{V}{V_0} \right) = (4 \times 10^{-2} E)^{1/2}\)

\(E\) is the ion kinetic energy in (keV/a.m.u.)
Figure 7 shows the dependence of the ion effective charge on its original charge and its energy for carbon and oxygen in the energy range 0.2 - 20 MeV/a.m.u. The fractional change in ion charge, \( \gamma \), and the reduced stopping power (electronic), \( \gamma^2 \), can be calculated using either (2.15) or (2.16); they give very close results.

The value of \( \gamma \) approaches 1 at energies equal to 4 and 8 MeV/a.m.u. for carbon and oxygen ions respectively. For lighter ions or particles, like alpha particles and the proton, the effect of the charge exchange is much less important even at energies below 0.5 MeV/a.m.u. For other heavier ions, with \( z_{HI} \leq 20 \), at energies above 10 MeV/a.m.u., the electron capture and loss mechanism can have little or negligible effect on the electronic stopping power dominance in the energy loss process. Finally one should notice that expressions used to calculate \( \gamma \) are independent of the target identity. Although this is not basically the case, it is a remarkable approximation which can be accepted in practice.

2.3 Nuclear stopping power

At low particle velocities where \( \left( \frac{V}{V_0} \right) < \frac{1}{3} \), the fractional change of the ion charge, \( \gamma \), is small and we are concerned with nearly neutral systems colliding with each other. Nuclear stopping can be understood as the transfer of energy to translatory motion of an atom as a whole, through the elastic Coulomb collisions of ion and nucleus of the stopping atom (Steward 1968).

Lindhard et al (1963), have developed a theory to describe the nuclear stopping power process of heavy ions which is valid at low ion velocities. They used the statistical model of interatomic interaction proposed by Thomas-Fermi to give a formula for the stopping power due, not only to ionisation and excitation of the stopping atoms but also to
the elastic collisions of partially or fully stripped ions with the nuclei of the stopping medium. They have expressed their theory in dimensionless units. Their units corresponding to range and energy are

$$\rho = \frac{R N_a M_2 4\pi a^2 M_1}{(M_1 + M_2)^2}$$  \hspace{1cm} (2.17)

$$\varepsilon = \frac{E a M_2}{z Z e^2 (M_1 + M_2)} \times 1.6 \times 10^{-6}$$  \hspace{1cm} (2.18)

where \( e \) is the electronic charge in e.s.u.,

\( N_a \) is Avogadro's number (mole\(^{-1}\)),

\( a \) is the screening constant, \( a = 0.885 a_o (z^{2/3} + Z^{2/3})^{-1/2} \)

\( a_o \) is the first Bohr radius

\( M_1 \) and \( M_2 \) are the mass numbers of the ion and the stopping medium,

\( z, Z \) are their atomic numbers.

Lindhard's expression for the electronic stopping power is

$$S_{\text{ele}} = K \varepsilon^{1/2}$$  \hspace{1cm} (2.19)

where \( K \) is a constant various only slowly with \( z \) and \( Z \). Its value is normally of the order of 0.1 to 0.2.

The contribution of nuclear stopping will be dominant to the total stopping power for ions of specific energies under 1.0 MeV/a.m.u. For specific energies above this, electronic stopping is usually more than two orders of magnitude greater than any other contribution (Steward 1968). In his work Steward (1968) mentioned that the contribution of the nuclear component to the total stopping is significant only in the region identified by the inequality

$$\begin{align*}
\frac{1}{z_{HI}} &\geq 137 \beta, \\
z_{HI} \text{ and } Z &\gg 1
\end{align*}$$  \hspace{1cm} (2.19)

where \( \beta \) is the ion relative velocity.
Electronic and nuclear stopping are presented in Fig. 8, (Lindhard et al 1963).

The Thomas-Fermi model of interatomic interaction implemented by Lindhard et al is a statistical one expressed as a Coulombic term, due to the nucleus, multiplied by a function to correct for the screening effect of the atomic electrons. Because of its easy use this model and many other approximations based on it (Sommerfeld 1932, Molière 1947, Bohr 1948a) have been employed widely in the treatment of nuclear stopping power (Wilson 1977). The main drawback of statistical models is the lack of accuracy when they are to be applied for materials of small atomic number, low Z, such as those constituting tissue.

Wilson and Haggmark (1976) have employed the free-electron model of interatomic interactions. In this model the atomic charge densities are calculated using the free-electron approximation, and the electromagnetic interaction of the two atoms is obtained classically.

Similar experimental studies (Kalbitzer et al 1975) were made to improve the calculation of nuclear stopping. Analysis of these studies made by Kalbitzer et al (1976) proposed a series of nuclear stopping expressions, $S_n$, as a function of reduced energy of the ion $\varepsilon$ defined in (2.18). This formula can be written again as

$$\varepsilon = \frac{3.253 \times 10^4 M_2 E}{z^2 Z (M_1+M_2)(z^{2\alpha}+z^{2\beta})^{1/2}}$$  \hspace{1cm} (2.19)

where E is the ion kinetic energy in MeV.

Kablitzer proposed formulae, expressed in terms of Lindhard reduced stopping units, are

$$S_n = \begin{cases} 
1.593 \varepsilon^{1/2} & \varepsilon < 0.01; \\
1.7 \varepsilon^{1/2} \ln(\varepsilon+2.72) & 0.01 \leq \varepsilon \leq 10; \\
\ln(0.47\varepsilon)/2\varepsilon & \varepsilon > 10.
\end{cases}$$  \hspace{1cm} (2.20)
To convert to units of MeV/(10^{25} \text{ atoms/m}^2) multiply by
\[
[(8.462 z_{HI} Z_{M_1})/(M_1+M_2)(z_{HI}^2+Z_{M_2}^2)]^{\frac{1}{2}}
\]

Results obtained from using eqn.(2.20) are in good agreement with those obtained experimentally, the differences are less than 10%, (Ziegler 1977). Compared to Lindhard's curve for nuclear stopping, these formulae tend to underestimate \( S_n \). For reduced energies \( \varepsilon \) between 0.01 and 10, Table V compares the results obtained using equation (2.20) and those obtained from Fig.8.

The new proposed nuclear stopping brings the experimental electronic stopping values closer to the calculated values by an amount directly related to the difference between the nuclear stopping given by theory of Lindhard et al and that of eqn.(2.20).

2.4 The range of charged particles

2.4.1 Definition

The distance that a charged particle of energy \( E_0 \) and mass \( M \), goes in a medium until it comes to rest can be evaluated theoretically by the integration

\[
R_o = \int_0^{E_0} \frac{dE}{(S/\rho)} \quad (2.21)
\]

where \( (S/\rho) \) is the specific energy loss to the target atomic electrons through excitation and/or ionisation. Owing to the random nature of the energy loss process, particles having the same initial velocity will not have exactly the same \( R_o \). Consequently, a large number of monoenergetic particles will have an average path length, which can be considered as their mean range. Heavy particles which have undergone
Fig. 8  Theoretical nuclear and electronic stopping in \( \rho - \varepsilon \) variables, \( K = 0.15 \). From Lindhard et al. (1963)

Table V  A comparison between \( S_n \) produced from equation (2.20) to that obtained from Fig. 8. \( 0.01 \leq \varepsilon \leq 10 \).

<table>
<thead>
<tr>
<th>( \varepsilon^3 )</th>
<th>( S_n ) (from Eqn. 2.20)</th>
<th>( S_n ) (Fig. 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.26</td>
<td>0.32</td>
</tr>
<tr>
<td>0.4</td>
<td>0.31</td>
<td>0.42</td>
</tr>
<tr>
<td>0.6</td>
<td>0.22</td>
<td>0.41</td>
</tr>
<tr>
<td>1</td>
<td>0.20</td>
<td>0.37</td>
</tr>
<tr>
<td>2</td>
<td>0.11</td>
<td>0.22</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>0.14</td>
</tr>
<tr>
<td>4</td>
<td>0.06</td>
<td>0.09</td>
</tr>
</tbody>
</table>
catastrophic energy losses, such as nuclear reactions, are to be excluded from such averaging. This could only happen at fairly high particle energies, where complete absorption of the particle is likely to occur.

Since there is a considerable uncertainty in the stopping power calculation using Bethe formula at low energies (< 1 MeV/a.m.u.), the integration for the range should start at some energy $E_1$, and an experimental value $R_1$ should be added to the integral (Sternheimer 1959a),

$$ R_o = R_1(E_1) + \int_{E_1}^{E_0} \frac{dE}{s/\rho} $$

(2.22)

The Stochastic nature of the energy loss process, (small, and discrete), leads to a discrete slowing down of the particle, causing "range straggling". Therefore $R_o$ given in (2.22) will not be exactly the range but a very close quantity, called the continuous slowing approximation range (CSDA) (Fano 1963). It differs from $R_o$ by only 0.2% for protons (Bichsel 1972).

2.4.2 Range - Energy data

At energies above 10 MeV/a.m.u., range-energy tabulations done by many authors are based on the Bethe theory. Theoretical and semi-empirical tabulations of range-energy for protons are available from different sources. Steward and Wallace (1970) have produced the range of protons with kinetic energies between 10-1000 MeV. At lower energies, Fleischer et al (1975), Benton and Henke (1969) have obtained ranges corresponding to energies down to $E < 0.1$ MeV. The tabulations of Northcliffe et al (1970) at low energies seem to be more accurate. Barkas and Berger (1964) have produced data based on the exact solution of the Bethe formula down to 8 MeV, with shell corrections and density effect factor consideration. They also fit a polynomial in $\log (\lambda) \ VS \ log(E)$ which gives results close to the integrated data.
The range of protons between 7 and 1200 MeV can be obtained from the approximate formula of Barkas (1964)

\[
\log \lambda = \log \frac{\Delta}{Z} + \sum_{n=0}^{3} \sum_{m=0}^{3} \alpha_{mn} \left( \log I_{\text{adj}} \right)^m \left( \log E \right)^n \tag{2.23}
\]

\(\lambda\) is in \(g/cm^2\), \(\alpha_{mn}\) values were obtained using a least-square adjustment based on 600 range values, at different energies, and for several values of \(I_{\text{adj}}\). For energies \(1 \leq E \leq 9\) MeV, Barkas fit formula reads

\[
\log \lambda = \log \frac{\Delta}{Z} + \sum_{n=0}^{2} \sum_{m=0}^{2} \alpha_{mn} \left( \log I_{\text{adj}} \right)^m \left( \log E \right)^n \tag{2.24}
\]

values of \(\alpha_{mn}\) and \(\alpha_{mn}\) can be obtained from Barkas (1964). Using equation (2.22) we calculated \(R_o\) for protons in soft tissue, made up of the elements shown in Table II. The log formula was also examined for proton energies in the range 2-2000 MeV. The following approximations are made

(i) Materials and compounds of biological interest are light elements for which shell corrections can be neglected.

(ii) Initial ranges, \(R(2)\), are taken from the experimental work of Steward (1968).

A FORTRAN program to carry out these range calculations can be found in Appendix 1.

The range in a multi component material of \(n\)-elements is calculated from the formula

\[
\frac{1}{R} = \sum_{i=1}^{n} \frac{w_i}{R_i} \tag{2.25}
\]

where \(R_i\) is the range calculated for the \(i^{th}\) element, and \(w_i\) is the fraction by weight of that component of the compound, provided that \(R\) is in units of \((kgm/m^2)\).

Table VI shows a comparison between proton range in soft tissue at certain energies obtained by using eqns. (2.22) and (2.23). The third
column represents Barkas (1968) results for proton in water. The analytical solution of the integral form using Simpson's rule provides results which are very close to those of Barkas. The log formula produces the same values with certain deviations, the minimum of which is about 5%. Maximum discrepancies are obtained from both formulae at the low energy-range relationship.

Table VI Proton range in tissue and water using different formulae

<table>
<thead>
<tr>
<th>Kinetic Energy (MeV/a.m.u.)</th>
<th>RANGE (MM)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(log formula)</td>
<td>(Integral)</td>
<td>Barkas (1968)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(soft tissue)</td>
<td>(soft tissue)</td>
<td>(water)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.30</td>
<td>1.24</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>23.8</td>
<td>20.1</td>
<td>21.8</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>168.1</td>
<td>155.7</td>
<td>154.9</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>866.9</td>
<td>803.04</td>
<td>808.9</td>
<td></td>
</tr>
<tr>
<td>700</td>
<td>2062.1</td>
<td>1867.8</td>
<td>1920.8</td>
<td></td>
</tr>
</tbody>
</table>

2.4.3 Range-Energy relationship for particles (ions) heavier than the proton

Ranges are mostly calculated for protons, but when both range and energy are normalised by the appropriate mass ratio, the energy-range relation could be used for all heavy singly-charged positive particles. Ranges of heavy hydrogen nuclei, and multiply charged nuclei can also be considered in this way.

The relation between a heavy ion range and that of a proton with the same velocity, \( \beta \), is given by (Barkas and Berger 1964)

\[
R(\beta)_{HI} = \frac{M}{z^2} \left[ R(\beta)_{P} + B_z(\beta) \right]
\]  

(2.26)

where
the quantity $R(β)_p$ is the range of an ideal proton, which does not capture electrons or interact strongly with nuclei, as a function of its relative velocity ($β$). $M$ and $z$ are the particles' mass and charge relative to those of proton. The term $B_z(β)$ is added to evaluate the range extension caused by electron capture.

For multiply-charged ions, $B_z(β)$ is derived from the emulsion measurements done by Barkas (1964), and the charge state formula, eqn(2.13). A crude formula which approximates $B_z(I, β)$, for $β < \frac{2z}{137}$ (about 0.4 Mev/a.m.u. for oxygen ions), is

$$B_z(I, β) = (48.0+5.8I^{5/8}) \frac{A}{Z} 10^{-5} z^{5/3} \beta \text{ gm/cm}^2$$

where, $Z$, $A$ are the atomic and mass numbers of the target, and $I$ is the adjusted ionisation potential. For $β > 2z/137$, the range extension is assumed to be constant, and the stopping power is assumed to scale as $z^2_{HI}$.

In this case, $B_z(I)$ is given by

$$B_z(I) = (7.0 + 0.85 I^{5/8})(A/Z) 10^{-6} z^{8/3} \text{ gm/cm}^2$$

As far as this work is concerned, $β$ is always above $2z/137$, as shown in Table VII. In heavy ion radiology all specific energies are recommended to be well above the values presented in Table VII.

Accordingly, the quantity $B_z$, for $β < 2z/137$, is found to be very small even for the heaviest ions, such as neon. Therefore, and within the accuracy wanted, we can neglect $B_z(I)$, and equation (2.26) will read

$$R(β)_{HI} = \frac{M_z}{z^2} R_p(β) \quad (2.27)$$

The K.E. of the ion (MeV/a.m.u.) is calculated from the formula

$$E_{\text{tot}} = \frac{M_o}{\sqrt{1-β^2}} , \quad (2.28)$$

where $M_o$ is the ion rest mass in MeV, and

$$E_{\text{tot}} = \text{K.E.} + M_o .$$

$M/z^2$ is calculated from:

$$M/z^2_{HI} = \frac{\text{Mass of the heavy ion}}{\text{Mass of the proton}} \times \left( \frac{\text{charge of proton}}{\text{charge of the ion}} \right)^2$$
Table VII  \((\text{M}/z^2)\) for some ions, and the kinetic energy/\(\text{a.m.u.}\) equivalent to the velocity, \(\beta = \frac{2z}{137}\)

<table>
<thead>
<tr>
<th>Particle (ion)</th>
<th>Symbol</th>
<th>(z)</th>
<th>(\text{M}/z^2)</th>
<th>(2z/137)</th>
<th>Equivalent K.E. (MeV/(\text{a.m.u.}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positrons</td>
<td>(e^+)</td>
<td>1</td>
<td>0.000544</td>
<td>0.0146</td>
<td>0.000054</td>
</tr>
<tr>
<td>Pions</td>
<td>(\pi^+)</td>
<td>1</td>
<td>0.1488</td>
<td>0.0146</td>
<td>0.0148</td>
</tr>
<tr>
<td>Protons</td>
<td>(p)</td>
<td>1</td>
<td>1.00</td>
<td>0.0146</td>
<td>0.099</td>
</tr>
<tr>
<td>(\alpha)-particles</td>
<td>(\alpha)</td>
<td>2</td>
<td>0.99226</td>
<td>0.0292</td>
<td>0.397</td>
</tr>
<tr>
<td>Carbon-6</td>
<td>(C^6)</td>
<td>6</td>
<td>0.3308</td>
<td>0.0876</td>
<td>3.59</td>
</tr>
<tr>
<td>Oxygen-8</td>
<td>(O^8)</td>
<td>8</td>
<td>0.24812</td>
<td>0.01168</td>
<td>6.42</td>
</tr>
<tr>
<td>Neon-10</td>
<td>(Ne^{10})</td>
<td>10</td>
<td>0.19849</td>
<td>0.1459</td>
<td>10.1</td>
</tr>
</tbody>
</table>

Using eqn. (2.27) we obtained the range of heavy ions in water at energies above 10 MeV/\(\text{a.m.u.}\). Those for protons are taken from Barkas (1964).

Since water is a tissue equivalent material one can estimate the energies to which these ions can be accelerated to penetrate a certain amount of homogeneous tissue. If we accept 25 cm as a desirable range, the beam energies will be \(\sim 198\) MeV/\(\text{a.m.u.}\) for protons, \(\sim 390\) MeV/\(\text{a.m.u.}\) for carbon ions, and \(\sim 530\) MeV/\(\text{a.m.u.}\) for neon ions.

Figure 9 shows a graphical representation of the range of data obtained.

2.4.4 The energy loss and range fluctuations

In previous sections we discussed the energy loss mechanism and pointed out that the energy loss is mainly due to discrete, random collisions with atomic electrons.

According to the central-limit theorem, Feller (1968), the probability density function of the sum of a set of commonly distributed random variables approaches a Gaussian distribution providing one has an infinite number of random variables. If the energy lost in a very
Fig. 9
thin slab of an absorber is assigned as the random variable, then the sum of the energies lost in all the slabs, which makes an absorber of finite thickness, should be determined by a Gaussian distribution.

To ensure the validity of this theory the absorber should be thick enough to allow for such a large number of collisions to occur. Bohr (1915) has obtained the standard deviation, $\sigma_R$ of the distribution for homogeneous absorbers by adding the square of the standard deviations of the distributions of thin slabs. The non-relativistic formula of Livingstone and Bethe (1937) for $\sigma_E$ in absorbers of thickness $t$ based on Bohr's approach is given by

$$\sigma_E^2 = 4\pi e^4 z^2 Z N t$$

(2.29)

For relativistic particles eqn.(2.29) reads (Ahlen 1980)

$$\sigma_E^2 = 4\pi e^4 z^2 Z N t \left(1 - \frac{\beta^2}{2}\right)/(1 - \beta^2)$$

(2.30)

The above equations are only valid for thin absorbers where energy lost is a small fraction of the projectile initial kinetic energy/a.m.u. (Ahlen 1980). For moderate energy losses, where $E$, the residual kinetic energy/a.m.u., is between $40\%-60\%$ of the initial kinetic energy/a.m.u., $E_0$. Bichsel (1972) multiplied $\sigma_E^2$ by a factor $Q$ calculated from Tschalär's experimental results (1970). The value of $Q$ is dependent upon $(E_0/E)$ and the stopping number of the stopping power formula defined in eqn.(2.3).

In thick absorbers, where energy losses of up to $80\%$ of the initial kinetic energy/a.m.u. could occur, $\sigma_E$ could be obtained from the results of Tschalär (1967, 1968a, b) and Payne (1969) which are presented graphically.

As a consequence of fluctuations in energy loss, particles having the same incident energy/a.m.u. will travel different distances in the
absorber before they completely stop. There will be a probability distribution of the ranges about their mean \( R_q \), called the range straggling. The distribution is given by a Gaussian function (Rossi 1952), where

\[
P(R)dR = \left(2\pi\sigma_R^2\right)^{-\frac{1}{2}} \exp\left[-\frac{(R-R_o)^2}{2\sigma_R^2}\right] dR
\]

(2.31)

where \( R_o \) is the mean range, and \( 2\sigma_R \) is the full width of the distribution curve between the points of maximum slope. The full width at half maximum is \( 2.35\sigma_R \) and the full width at \((1/e)\) of the maximum height is \( 2.82\sigma_R \).

The root mean square fractional straggling \( \left(\frac{\sigma_R}{R_o}\right) \) for particles of rest mass \( M_c^2 \) (MeV), is given by (Rossi 1952)

\[
\frac{\sigma_R}{R_o} = \frac{\sqrt{102.2}}{M_c^2} f(\beta)
\]

(2.32)

with \( f(\beta) \) given in fig.10, and valid for particles stopping in iron.

The fractional straggling \( (\sigma_R/R_o) \) for protons in other elements relative to that in iron are calculated by Bichsel (1972) and can be obtained from Table VIII. An approximate formula for \( (\sigma_R/R_o) \) for protons in most materials and beam energies of interest was suggested by Curry and Steward (1978), where \( (\sigma_R/R_o) = 0.012 \), i.e. constant value.

For other particles of mass \( M_o \), having the same initial kinetic energy/a.m.u. as of the proton, the relative straggling can be estimated from (Bichsel 1972)

\[
(\sigma_R/R_o)(E)M_o = \left(\frac{M}{M_p}\right)^{\frac{1}{2}} \left(\frac{\sigma_R}{R_o}\right)(E)M_p
\]

(2.33)
Fig. 10  Range straggling parameter $f$ for particle of mass $M$, in iron
(from nuclear data Analysis tables and graphs 1968)
2.4.4 Calculation of range straggling of particles

We used Table VIII to deduce values of $\sigma_R/R_o$ for protons in tissue relative to that in iron by extrapolation. From these numbers and eqn.(2.32) we calculated $(\sigma_R/R_o)$ for protons in soft tissue as a function of $f(\beta)$, and the results are plotted in Fig.11. We used eqn.(2.33) to calculate $(\sigma_R/R_o)$ for alpha particles, carbon, oxygen and neon ions, all having the same initial velocity as for the proton in tissue.

Because there is no general expression for the evaluation of $(\sigma_R/R_o)$ for protons, where one should use tables and graphs, estimation of that value for other ions is not a straight forward process. We started by calculating the proton mean range $R_p$ corresponding to that of the particle in the same medium using eqn.(2.27). We looked up the proton kinetic energy corresponding to this range from tabulated values and Fig.9 for water. Then $f(\beta)$ for the proton in iron is taken from fig.10. The fractional straggling $(\sigma_R/R_o)$ for protons in tissue is obtained from fig.11 and used in eqn.(2.33) to obtain $(\sigma_R/R_o)$ for other particles.

Table VIII Ratio of projectile range straggling in a given target material to that in iron (from Bichsel 1972)

<table>
<thead>
<tr>
<th>Target Material</th>
<th>Atomic Number</th>
<th>Relative Path length straggling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>15 MeV</td>
</tr>
<tr>
<td>Be</td>
<td>4</td>
<td>0.85</td>
</tr>
<tr>
<td>Al</td>
<td>13</td>
<td>0.93</td>
</tr>
<tr>
<td>Fe</td>
<td>26</td>
<td>1.00</td>
</tr>
<tr>
<td>Cu</td>
<td>29</td>
<td>1.00</td>
</tr>
<tr>
<td>Ag</td>
<td>47</td>
<td>1.00</td>
</tr>
<tr>
<td>Pb</td>
<td>82</td>
<td>1.13</td>
</tr>
<tr>
<td>soft tissue</td>
<td></td>
<td>0.87</td>
</tr>
</tbody>
</table>
Fig. 11 The proton r.m.s. fractional straggling as a function of \( f(\beta) \) calculated for soft tissue
Table IX gives a detailed idea about the procedure followed, and the fractional straggling for different particles in soft tissue. Figure 12 shows $\sigma_R$ plotted against the mean range of the ions mentioned above in soft tissue.

The approximate formula of Curry and Steward of $(\sigma/R) = 0.012$, is independent of the projectile energy and target material. Comparison of this value to our results in Table IX for the proton can be considered as an average of $(\sigma_R/R_0)$ in the proton kinetic energy range 100-200 MeV.

The validity of our results for the range straggling are associated with the assumptions and approximations made in the evaluation of the mean range of particles, in section 2.3. Therefore, previous calculations are limited to particle kinetic energies between 10-1000 MeV/a.m.u. At lower energies, shell corrections should be used, while at very large energies, the density effect correction should be taken into consideration.

2.4.42 The number-distance curve

A plot of the number of particles $P(R)$ against the distance they reach $R$, forms what is known as the number-distance curve shown in Fig. 13, with $R_0$ as the mean range. The mean range $R_0$ found experimentally is defined by Bethe (1953), as that distance which is reached by just 50% of the incident particles. For an inhomogeneous beam this definition does not hold. A spread in the incident beam energy makes the slope of the range curve near the mean range less steep than it is for a mono-energetic beam. The energy-spread component must be added to the range straggling for a monoenergetic beam at the mean energy $\bar{E}$ to give the range straggling actually observed (Curry and Steward 1978).
Fig. 12 The root mean square range straggling of several particles in soft tissue having the same range in water.
Table IX  Heavy ions r.m.s. fractional straggling calculation in water

<table>
<thead>
<tr>
<th>Projectile</th>
<th>Projectile Range, ( R_i ) (mm)</th>
<th>Proton range, ( R ) (mm)</th>
<th>Proton K.E. (MeV)</th>
<th>( f(\beta) )%</th>
<th>( \left{ \frac{R}{R} \right}_{\text{particle}} )</th>
<th>( \left{ \frac{R}{R} \right}_{\text{proton}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALPHA</td>
<td>120</td>
<td>120.1</td>
<td>130</td>
<td>3.68</td>
<td>1.110</td>
<td>0.322</td>
</tr>
<tr>
<td>( M_o = \frac{M}{M_p} = 3.971 )</td>
<td>160</td>
<td>161.1</td>
<td>153</td>
<td>3.62</td>
<td>1.100</td>
<td>0.550</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>201.4</td>
<td>174</td>
<td>3.51</td>
<td>1.070</td>
<td>0.537</td>
</tr>
<tr>
<td></td>
<td>240</td>
<td>241.7</td>
<td>194</td>
<td>3.49</td>
<td>1.060</td>
<td>0.532</td>
</tr>
<tr>
<td></td>
<td>280</td>
<td>282.0</td>
<td>212</td>
<td>3.47</td>
<td>1.055</td>
<td>0.530</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>322.0</td>
<td>229</td>
<td>3.44</td>
<td>1.045</td>
<td>0.520</td>
</tr>
<tr>
<td>Carbon, 12</td>
<td>40</td>
<td>121.0</td>
<td>130</td>
<td>3.68</td>
<td>1.110</td>
<td>0.322</td>
</tr>
<tr>
<td>( M_o = 11.9 )</td>
<td>80</td>
<td>242.0</td>
<td>195</td>
<td>3.50</td>
<td>1.060</td>
<td>0.307</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>363.0</td>
<td>243</td>
<td>3.40</td>
<td>1.035</td>
<td>0.300</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>484.0</td>
<td>292</td>
<td>3.37</td>
<td>1.025</td>
<td>0.297</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>605.0</td>
<td>333</td>
<td>3.31</td>
<td>1.012</td>
<td>0.293</td>
</tr>
<tr>
<td></td>
<td>240</td>
<td>726.0</td>
<td>373</td>
<td>3.22</td>
<td>0.980</td>
<td>0.284</td>
</tr>
<tr>
<td></td>
<td>280</td>
<td>847.0</td>
<td>411</td>
<td>3.18</td>
<td>0.970</td>
<td>0.281</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>968.0</td>
<td>446</td>
<td>3.14</td>
<td>0.955</td>
<td>0.277</td>
</tr>
<tr>
<td>Oxygen, 16</td>
<td>40</td>
<td>161.1</td>
<td>155</td>
<td>3.63</td>
<td>1.082</td>
<td>0.254</td>
</tr>
<tr>
<td>( M_o = 15.89 )</td>
<td>80</td>
<td>322.2</td>
<td>229</td>
<td>3.44</td>
<td>1.050</td>
<td>0.263</td>
</tr>
<tr>
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<td></td>
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<td>0.970</td>
<td>0.243</td>
</tr>
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<td>966.6</td>
<td>446</td>
<td>3.14</td>
<td>0.955</td>
<td>0.239</td>
</tr>
<tr>
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<td>1128.0</td>
<td>494</td>
<td>3.10</td>
<td>0.940</td>
<td>0.236</td>
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<tr>
<td></td>
<td>320</td>
<td>1289.0</td>
<td>538</td>
<td>3.08</td>
<td>0.938</td>
<td>0.235</td>
</tr>
<tr>
<td>Neon, 20</td>
<td>40</td>
<td>202.0</td>
<td>175</td>
<td>3.51</td>
<td>1.065</td>
<td>0.239</td>
</tr>
<tr>
<td>( M_o = 19.8 )</td>
<td>80</td>
<td>404.0</td>
<td>262</td>
<td>3.59</td>
<td>1.050</td>
<td>0.230</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>606.0</td>
<td>333</td>
<td>3.31</td>
<td>1.005</td>
<td>0.226</td>
</tr>
<tr>
<td></td>
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<td>808.0</td>
<td>400</td>
<td>3.2</td>
<td>0.974</td>
<td>0.219</td>
</tr>
<tr>
<td></td>
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<td>460</td>
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<td>0.974</td>
<td>0.219</td>
</tr>
<tr>
<td></td>
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<td>478</td>
<td>3.18</td>
<td>0.970</td>
<td>0.218</td>
</tr>
<tr>
<td></td>
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<td>525</td>
<td>3.14</td>
<td>0.958</td>
<td>0.211</td>
</tr>
<tr>
<td></td>
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<td>1613.0</td>
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<td>0.94</td>
<td>0.209</td>
</tr>
<tr>
<td>Proton</td>
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<td>3.88</td>
<td>1.15</td>
<td>1.15</td>
</tr>
<tr>
<td>( M_o = 1.0 )</td>
<td>80</td>
<td>80.0</td>
<td>98</td>
<td>3.80</td>
<td>1.14</td>
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<td>120.0</td>
<td>123</td>
<td>3.68</td>
<td>1.12</td>
<td>1.12</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>160.0</td>
<td>144</td>
<td>3.62</td>
<td>1.10</td>
<td>1.10</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>200.0</td>
<td>165</td>
<td>3.51</td>
<td>1.07</td>
<td>1.07</td>
</tr>
<tr>
<td></td>
<td>240</td>
<td>240.0</td>
<td>183</td>
<td>3.49</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
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<td>280.0</td>
<td>200</td>
<td>3.47</td>
<td>1.055</td>
<td>1.055</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>320.0</td>
<td>216</td>
<td>3.44</td>
<td>1.045</td>
<td>1.045</td>
</tr>
</tbody>
</table>
If $\sigma_R$ is the monoenergetic beam r.m.s. straggling at energy $E^*$, $\sigma_{EN}$ is the r.m.s. straggling due to initial energy spread given by $\sigma_{IN} = 0.0034 E^{0.8} \Delta E$ for protons in water (Curry 1978), and $\sigma_S$ is the actual r.m.s. straggling for non monoenergetic beam, then

$$\sigma^2_S = \sigma^2_{EN} + \sigma^2_R$$  \hspace{1cm} (2.33)

Figure 13 is the cornerstone of the end-of-range radiography technique. Before they reach the end of their range absorption of particles could only happen by nuclear interactions. Once the mean range of the surviving particles is reached, a rapid beam attenuation occurs. A slight change in the density content of the absorber along the beam path shifts the position of this absorption region in space.

Range straggling puts a limit on the technique enhanced sensitivity to small density variations. This subject and other relevant matters like the Bragg peak will be treated in the next chapter.

2.5 The nuclear interactions of heavy particles

High energy heavy ions traversing an absorbing medium produce ionisation by atomic and molecular collisions, in addition they interact with nuclei of the medium.

At low kinetic energies ($< 20$ MeV/a.m.u.) the projectile comes into approximately tangential contact with the target (Bock 1979). The projectile may then move along the surface of the target until it reaches a point at which its forward momentum is sufficient to break the nuclear attraction. If this process is completed before fusion to a compound system is possible, a "grazing" contact results. This reaction may result in single nucleon as well as multi-nucleon transfer and the production of lighter fragments (Tobias et al 1971). At high energies nuclear reactions in the central region may occur. These reactions
involves "head-on" collisions in which large amounts of energy are transferred, as opposed to grazing collisions.

The nucleons in the nuclei are likely to be heated by the energy transferred leading to a quasi-equilibrated "fireball" which cools by emission of light fragments and even \( \pi \) mesons of low energy. The angular distribution of these fragments is peaked in the forward direction (Chatterjee et al. 1977).

2.5.1 Attenuation of a particle beam

The significance of nuclear interactions in an absorber can be estimated from the attenuation of the primary beam using the total nuclear absorption cross-section per target nucleus, \( \sigma_t \). If the number of particles initially in the beam is \( N_0 \), then the number that traverses a distance \( t \) without making a nuclear collision is given by

\[
N = N_0 \exp[-n \sigma_t t] = N_0 \exp[-t/\lambda]
\]

(2.34)

where, \( n \) is the number of target nuclei per unit volume = \( N_A \rho/A \),

\( N_A \) is Avogadro's number, \( \rho \) is the absorber density,

\( A \) is its mass number and \( \lambda \) is the mean free path = \( 1/n \sigma_t \).

The total reaction cross section per target nucleus derived by Bradt et al. (1950) using an overlap model obtained from the geometrical cross-section of the collisions is given by

\[
\sigma_t = \pi r^2 [A_p^{1/2} + A_T^{1/2} - 2(\Delta r/T)]^2
\]

(2.35)

where \( A_p, A_T \) are the projectile and the target mass numbers, \( r \) is the radius of a nucleon in the nucleus, and \( \Delta r \) is the overlapping parameter.

Attempts to assign a unique value to the overlap parameter, \( \Delta r \), resulted in having different versions of eqn. (2.35) Bradt (1950) value for \( \Delta r \) is 0.85 fm and for \( r \) is 1.45 fm. Cleghorn et al. (1968) have
attempted to fit their experimental data to the above model using
\( \Delta r = 0.25 \text{ fm} \) and \( r = 1.2 \text{ fm} \). Litton et al (1968) have suggested \( 0.4 \text{ fm} \)
for \( \Delta r \) and \( 1.4 \text{ fm} \) for \( r \).

Because of the lack of agreement on what value to use for the
overlap parameter and because of the absence of energy dependence,
however small it is at high energies, the above expression has an
inherent inaccuracy which would affect the conclusions drawn from the
beam attenuation.

A more accurate analytical energy-dependent expression is derived
by Karol (1975) for high energy heavy ions. Karol's expression for the
total reaction cross-section per target nuclei in \( \text{mb} \) at any kinetic
energy/nucleon, \( E \), is

\[
\sigma_t(E) = 10\pi\left(\alpha_T^2 + \alpha_p^2\right)\left[E_1(\chi) + \ln \chi + 0.557\right](\text{mb}) \tag{2.36}
\]

where, \( \alpha_{T,p} = (1.596 A_T^{1/3} + 0.89)^{1/3} \text{ fm} \), \( E_1(\chi) \) is a function of negligible
magnitude (Karol 1975), and \( A_{T,p} \) is the target or projectile mass number.
\( \chi \) is an energy dependent function defined with its parameters in Appendix B.
Moreover, Karol's expression at any energy \( E \) can be approximated from the
2.1 GeV/nucleon total cross-section, using the expression

\[
\sigma_t(E) = \sigma_t(2.1 \text{ GeV}) + 16\pi[A_T^{1/3} + A_p^{1/3} + 1.125]\ln\frac{\tilde{\sigma}(E)}{\tilde{\sigma}(2.1 \text{ GeV/n})} \tag{2.37}
\]

where \( \tilde{\sigma}(E) \) is dependent upon the proton-proton, and neutron-proton
interaction cross-sections, \( \tilde{\sigma}(E) \) is also defined in Appendix B.

We calculated \( \sigma_t(E) \) for a monoenergetic proton beam in oxygen
target using (2.36) and (2.37) for proton kinetic energies between
100-1200 MeV. Figure 14 shows the dependence of \( \sigma_t(E) \) on the proton
energy. It becomes constant at kinetic energies above 600 MeV/a.m.u.

Minimum values for \( \sigma_t(E) \) (minimum absorption) could be obtained
for the proton in oxygen gas in the energy range 200-400 MeV. Therefore
Fig. 13  Beam Attenuation

Fig. 14  Total interaction nuclear cross-section for proton beam in oxygen
in particle tomography, where the patient dose is to be kept to a minimum, one should avoid working outside this energy limit. In Table X, we compare $\sigma_t(E)$ calculated from (2.36) for different ions penetrating 255 mm of water. The percentages of ions transmitted are also presented. Heavy ions suffer more attenuation in penetrating thick absorbers causing larger energy depositions.

2.5.2 Nuclear fragmentation

Knowledge of the nuclear fragmentation process is important in obtaining a better understanding of the specific biological effects of these beams and their optimal application for diagnostic radiology. Projectile fragmentation is the most important end product that would produce an effective change in the radiation beam quality (Curtis et al 1977). The projectile fragments emerge from the interaction that produced them in a direction very close to that of the incoming heavy ion and with little or no change in momentum from that of the primary beam (Greiner et al 1975).

Table X. The total interaction cross section for different ions penetrating 255 mm of water. The last column represents the fraction of particles transmitted. Ion K.E. is the energy required to penetrate this depth.

<table>
<thead>
<tr>
<th>Ion</th>
<th>Ion kinetic energy (MeV/a.m.u.)</th>
<th>$\sigma_t(E)$ (mb)</th>
<th>$n \sigma_t(E) \times 10^{-4}$ mm$^{-1}$</th>
<th>% transmitted</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$^1$</td>
<td>213</td>
<td>302.5</td>
<td>10.0</td>
<td>77</td>
</tr>
<tr>
<td>He$^4$</td>
<td>213</td>
<td>536.4</td>
<td>17.9</td>
<td>63</td>
</tr>
<tr>
<td>C$^{12}$</td>
<td>415</td>
<td>1071.7</td>
<td>35.8</td>
<td>40</td>
</tr>
<tr>
<td>Ne$^{20}$</td>
<td>568</td>
<td>1342.2</td>
<td>44.9</td>
<td>32</td>
</tr>
</tbody>
</table>
Schimmerling et al (1977) have calculated the fragmentation cross-sections, which determine the fraction of the total cross-section resulting in the production of a specific fragment. Their calculations are based on the semi-empirical model of Silberberg-Tsao (1973) which predicts proton-nucleus cross-sections based on a fit to experimental data. If the mass number of the projectile, $A_p$, and that of the target $A_t$, are both greater than 1, then the nucleus-nucleus cross-section produced from a scaling of the proton-nucleus cross-section for a proton incident on the larger of $(A_p, A_t)$ is given by (Schimmerling et al 1977)

$$\sigma_{A_p, A_t} = \frac{\frac{A_p}{A_t}^{\frac{1}{3} + \frac{1}{2} - 2(\frac{\Delta r}{r})}}{A_t^{\frac{1}{3} + 1 - 2(\frac{\Delta r}{r})}} \sigma_{p, A}$$ (2.38)

where, $\sigma_{A_p, A_t}$ = the partial cross section for a specific secondary to be produced in a nucleus-nucleus collision,

$A = \text{the larger of } (A_p, A_t)$,

$\sigma_{p, A}$ = the cross-section for the specific secondary particle to be produced in a proton-nucleus collision.

To appreciate the effect of these secondaries on our study, we need to know their fluence distribution at different depths within the absorber. The fluence of the primary heavy ions at a depth $x$ is given by an exponential attenuation of the form (Macabee-Ritter 1974)

$$\phi_p(x) = \phi_p(0) \exp (-\Sigma_p x)$$ (2.39)

where, $\phi_p(x) = \text{fluence of ions at depth } x$, $\phi_p(0) = \text{incident fluence}$, and $\Sigma_p$ is the total interaction cross-section for removal of primaries from the incident beam. Owing to the loss of primaries, secondary fragments are produced. If nuclear interaction of these secondaries can be neglected, then their fluence at depth $x$, $\phi_s(x)$, where $x \ll R$ (the mean range) is given as (Chatterjee and Tobias 1977)
\[ \phi_f(x) = \phi_p(0) [1 - \exp(-E_p x)] \frac{\Sigma_{pf}}{\Sigma_p} \]  

(2.40)

where, \( \Sigma_{pf} \) is the total cross-section for production of secondary fragments (equation 2.38).

Fragments retain approximately the same velocity at high energies as the parent ion and continue to move in with the unfragmented ions of the beam (Tobias et al 1971). They will, generally, have a greater range than the primary because ranges of fast heavy charged particles scale as \((M/z^2)\), where \(M\) is the particle mass and \(z\) is the nuclear charge. The ratio of the range of the fragment particles to that of the primary calculated from eqn.(2.26) is

\[ \delta \frac{R_f}{R_p} = \frac{M_p}{M_f} \frac{z_f^2}{z_p^2} \]  

(2.41)

If one is to take such considerations more seriously one would conclude that while there is an attenuation in the primary fluence there is a build up in the secondary fragments up to range \( R_p \) where all primaries stop. Then, there is a continuous drop in the secondaries as they are removed at the end of their respective ranges.

We used the number-distance curve of a 233 MeV/nucleon oxygen-ion beam in water of Maccabee et al (1974), Fig.15, to calculate the contribution of these fragments to the total absorbed dose at different penetration depths, Table XI. The main constituents of the secondary beam are nitrogen, carbon and boron. The dose due to the secondaries is less than 12% of the total dose at the end of the mean range of the ions which in this case is about 80mm.

<table>
<thead>
<tr>
<th>Depth (mm)</th>
<th>Dose in Gy/particle/mm² ( \times 10^{-8} )</th>
<th>% of secondaries to total dose</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.166</td>
<td>3.83</td>
</tr>
<tr>
<td>40</td>
<td>0.500</td>
<td>10.4</td>
</tr>
<tr>
<td>60</td>
<td>0.657</td>
<td>11.7</td>
</tr>
<tr>
<td>80</td>
<td>1.590</td>
<td>11.1</td>
</tr>
</tbody>
</table>
Fig. 15 A 233 MeV/a.m.u. oxygen-ion beam traversing water (MacCabe and Ritter 1974)
2.5.3 Implications of nuclear interactions on particle tomography

The slowing down process of the primaries, not undergoing nuclear interactions, will not be affected by the fragments produced, and conclusions drawn from these primaries will not be altered. Beam attenuation illustrated in Table XI shows that nuclear reactions would not "spoil" high energy ion beams used for diagnostic purposes. Nevertheless, a reduction in the number of ions "carrying information" about electron density in the target material would reduce the advantageous high contrast expected from these particles.

The secondary particles produced would have a lower stopping power compared to the primaries. Because of their lower charge they would also have an energy distribution similar to the primaries, but with less energy straggling. Thus, the root mean square deviation of the mean residual energy, and consequently the mean residual range, would increase causing a deterioration in the spatial resolution, and inaccuracy in the information obtained, especially when high Z materials are involved.

As to the dose produced by the secondaries, calculations performed by Curtis (1968) support our early conclusions about their small contribution to the total dose. The dose due to secondaries, from a 300 MeV/nucleon neon beam penetrating water, is only 15% of the primary dose at 10 cm depth, which is the position of the Bragg peak.

In addition to the fragmentation processes, heavy ions are scattered slightly due to multiple collisions with the electric fields of the absorber nuclei (multiple Coulomb scattering). This will result in beam broadening and contribute to the dose received by lateral parts of the absorber. Because of its importance, MCS will be treated thoroughly in the coming section. However, lighter particles and protons, at energies above 500 MeV, will have strong interactions with
the nuclei in the object and will be scattered to wide angles. Although there is a complete radiography technique which relies upon this idea (Saudinos 1975), we are not going to consider such high energies for charged particle tomography.

2.6 Small-angle multiple scattering and beam deflection

Although charged particles have relatively large masses and are associated with linear trajectories, they do undergo frequent small deflections from close collisions with the charge field of the nuclei of the target material (MCS). The effect of these casual deflections is to produce an angular distribution of the particles, and a beam of small diameter will spread out as it passes through a sample. The lateral displacement of the beam at the end of its range can be related to the total rms angle of deflection, $\theta_{\text{rms}}$.

A simple formula for $\theta_{\text{rms}}$ has been given by Rossi and Greisen (1941) but it has been suggested that this formula needs a correction factor (Highland 1975). Hanson (1978) has plotted such a correction factor for protons and has shown that it is not unimportant for proton tomography. In this section we present the key formulae and derive a correction factor which we use to calculate the lateral resolution of protons passing through water.

The theory of small-angle multiple scattering (Scott 1963, Bethe 1953) yields two distribution functions - the function $F(\theta, \phi, t)$ which represents the number of particles scattered into the direction $(\theta, \phi)$ after the beam has passed through a thickness $t$ of material, and the projected-angle function $F_p(\phi, t)$ which represents the projection of $F(\theta, \phi, t)$ on to the $x$-$z$ plane. The initial beam direction is taken to define the $z$-direction, Fig.16. The calculation of these distribution
functions requires the probability $W(\theta,t)$ for a single scattering into a solid angle $d\Omega = 2\pi \sin \theta \, d\theta$ in a thickness $dt$ of material.

Fig. 16 Multiple scattering geometry, used to define the scattering angles and the beam direction before and after scattering

2.6.1 The single scattering process

The simplest assumption is that the single scattering probability can be represented by the Rutherford scattering formula

$$W(\theta,t) = N(t) \sigma_R(\theta)$$

(2.42)

where $\sigma_R(\theta)$ is the differential cross-section for Rutherford scattering,

$N(t) = N_A \rho/A$ is the number of atoms per unit volume, and $\rho$ is the density,
A is the relative atomic mass and $N_A$ is Avogadro's number. With $eta = v/c$, $p = \hbar k$, and $r_e = e^2/m_e c^2$, we have (Scott 1963),

$$W(\theta,t) \, d\Omega = \frac{1}{4} N(t) \, z^2 \, Z^2 \, r_e^2 \, \left( \frac{m_e c}{p} \right)^2 \, \frac{1}{\sin^4(\theta/2)} \, d\Omega$$

(2.43)

where $z$ is the atomic number of the projectile and $Z$ is the atomic number of the target nucleus. For small angles this becomes

$$W(\theta,t) d\Omega = 4N(t) \, z^2 \, Z^2 \, r_e^2 \left( \frac{m_e c}{p} \right)^2 \frac{2\pi \, \theta \, d\theta}{\theta^4}$$

(2.44)

$$= 4N(t) \, \frac{\eta^2}{k^2 \theta^3}$$

(2.45)

where

$$\eta = z \, \frac{Ze^2}{\hbar v} \quad (2.46)$$

The effect of screening of the nuclear Coulomb potential by the atomic electrons can be taken into account using a screened potential of the form (Scott 1963)

$$V(r) = \pm \frac{Z}{r} \frac{e^2}{f(r/r_0)}$$

(2.47)

where the screening radius $r_0$ is frequently taken to be the Thomas-Fermi radius

$$r_0 = 0.885 \, a_0 \, Z^{-\frac{1}{3}} = 0.885 \, (\hbar/m_e c \alpha \, Z^{-\frac{1}{3}}) \quad (2.48)$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant and $a_0$ is the Bohr radius. For the Yukawa potential, which has $f(r/r_0) = \exp(-r/r_0)$, the first Born approximation yields (Scott 1963)

$$W(\theta,t) = 4N(t) \, \frac{\eta^2}{k^2 (\theta^2 + \chi_0^2)^2} \quad (2.49)$$

where $\chi_0$ is the Born screening angle.
\[ \chi_0 = \frac{\hbar}{\mathcal{p}r_0} = 1.13 \alpha \left( \frac{m_e c}{\mathcal{p}} \right) Z^{1/3}. \]  

(2.50)

In these formulae the nucleus has been treated as a point charge. The finite size of the nucleus is expected to be significant at angles

\[ \theta_N \sim \frac{\hbar}{p r_N} \]

where \( r_N \) is an estimate of the nuclear radius. For \( r_N = 1.2 A_N^{1/6} \), where \( A_N \) is the mass number of the nucleus, we have

\[ \theta_N \sim \left( \frac{m_e c}{p} \right) A_N^{-1/3} 322. \]  

(2.51)

The numerical coefficient derived by Williams (1940) corresponds to \( r_N = 1.38 A_N^{1/3} \).

Equation (2.45) can also be written in the form (Molière 1947, 1948)

\[ W(\theta,t) \ d\Omega = 2 \chi_c^2 \frac{\theta d\theta}{\theta^4} q(\theta) \]  

(2.52)

where \( q(\theta) \) represents the departure from Rutherford scattering due to screening.

The characteristic angle \( \chi_c \) is chosen so that the total probability of scattering through an angle greater than \( \chi_c \) is unity. For a mixture of scatterers an average scattering probability can be defined as

\[ t \tilde{W}(\theta,t) = \int_0^t W(\theta,t') \ dt' \]  

(2.53)

and the characteristic angle then becomes

\[ \chi_c = 4\pi \int_0^t \ dt' \sum_i N_i(t') \frac{n_i^2(t')}{k^2(t')} \]

\[ = 4\pi e^4 z^2 \int_0^t \ dt' \sum_i N_i(t') \frac{Z_i^2}{(pv)^2}. \]

(2.54)
For a homogeneous sample and zero energy loss this reduces to

\[ \chi_C^2 = 4\pi e^4 z^2 Z^2 N t/p^2 v^2 \]

\[ \times 4\pi e^4 z^2 Z^2 N_A \rho t/A p^2 v^2. \]  

(2.55)

In Molière's method the screening angle is defined as

\[ \log \chi = -\frac{1}{2} - \lim_{\chi_m \to \infty} \int_\theta^\infty \frac{d\theta}{\theta} \frac{q(\theta)}{\log \chi_m}. \]  

(2.56)

Using a function \( f(r/r_0) \) derived from a fit to the Thomas-Fermi model for the atomic potential and evaluating \( q(\theta) \) and eqn.(2.56) Molière derived an approximate expression for the screening angle of the form

\[ \chi_a^2 = \chi_0^2 (R + 3.76n^2) \]  

(2.57)

where \( R = 1.13 \) and is a constant for all \( Z \). The Thomas-Fermi model is not accurate for light atoms because it is a statistical model.

The possibility of inelastic collisions with electrons must also be taken into account (Fano 1954). For this purpose it is convenient to separate the elastic and inelastic contributions to the cross-sections,

\[ \sigma(\theta) = \sigma_{el}(\theta) + \sigma_{inel}(\theta) \]  

(2.58)

so that

\[ N t \sigma_{el}(\theta) \sin\theta d\theta = 2 Z^2 \chi_C^2 q_{el}(\theta) \sin\theta d\theta \frac{1}{4(1-\cos\theta)^2} \]  

(2.59)

where \( \chi_C^2 \) is as given previously in equations (2.54) or (2.55).

For scattering of incident heavy particles the inelastic contribution can be written as (Fano 1954)

\[ N t \sigma_{inel}(Q)dQ = Z \chi_C^2 S(b) \left( \frac{p^2}{2m} \right) \frac{dQ}{Q^2} \left[ 1 - \frac{Q\theta^2}{Q_{max}} \right]. \]  

(2.60)
where, \( S(b) \) is the incoherent scattering function and
\[
\begin{align*}
b &= 0.333 \frac{Z^{2/3}}{\hbar} \frac{P}{a_o} [2(1-\cos \theta)]^{1/2} \\
&= 0.333 \frac{Z^{2/3}}{\hbar} \frac{P}{a_o} \theta
\end{align*}
\]
and \( Q \) is the particle recoil energy.

Equation (2.60) yields
\[
\chi_c^2 \log \chi_{\alpha} = 4\pi \int_0^t \frac{dt'}{k^2(t')} \sum \frac{N_i(t')}{\eta^2(t')} \left[ \log \chi_{\alpha i}^{el} - \frac{1}{2z_i} D_i \right]
\]
where
\[
D = \int_{Q_{\min}}^{Q_{\max}} \frac{dQ}{Q} \left[ \frac{Q}{Q_{\max}} \right] \frac{1-2\beta^2}{Q_{\max}}
\]
(2.63)
(2.64)

where \( \gamma = (1-\beta^2)^{-1/2} \). Estimates of \( u_{inel} \) in the Thomas-Fermi model
yield -5.8 for all \( Z \) while exact calculations for hydrogen yields -3.6 (Fano 1954). For a homogeneous sample with no energy loss, \( \chi_c^2 \) is again
given by equation (2.55) and \( \log \chi_{\alpha} \) is given by
\[
\log \chi_{\alpha} \sum \frac{N_i}{\eta_i} Z_i^2 = \sum \frac{N_i}{\eta_i} Z_i^2 \left[ \log \chi_{\alpha i}^{el} - \frac{1}{2z_i} D_i \right]
\]
(2.65)

For thick targets energy loss should be taken into account by
using equations (2.54) and (2.62) instead of equations (2.55) and (2.65).

Equations (2.54) and (2.62) can be rewritten as
\[
\begin{align*}
\chi_c^2 &= \int_0^t dt' \chi_c^2(t') \\
\chi_c^2 \log \chi_{\alpha}^2 &= \int_0^t dt' \chi_c^2(t') \log \chi_{\alpha}^2(t')
\end{align*}
\]
(2.66)
(2.67)

These equations can be converted to integrals over the kinetic energy \( T \)
of the form (Berger and Seltzer 1964)
\[ x_c^2 = \int_T^{T_0} dT' x_c^2(T') \left[ -\frac{1}{\rho} \frac{dE}{dx}(T') \right]^{-1} \]  

\[ x_c^2 \log x_c^2 = \int_T^{T_0} dT' x_c^2(T') \log x_c^2(T') \left[ -\frac{1}{\rho} \frac{dE}{dx}(T') \right]^{-1} \]  

where \( dE/dx \) is the stopping power, \( T_0 \) is the initial kinetic energy and \( T \) is the kinetic energy at thickness \( t \). These formulae are reliable when the final energy is not near to zero. This condition is satisfied in all our examples.

### 2.6.2 Multiple Scattering

Rossi and Greisen (1941) give the rms scattering angle for multiple scattering when the energy loss can be neglected as

\[ \theta_{\text{rms}} = \frac{E_s^z}{p \beta c} \left( \frac{\omega t}{X_0} \right)^{\frac{3}{2}} \]  

where \( E_s = (4\pi/\alpha)^{\frac{1}{2}} m_e c^2 \) and \( X_0 \) is the radiation length (in units of mass/length^2). The definition of \( X_0 \) used by Rossi is

\[ X_0 = \frac{1}{4\alpha} \frac{A}{N_A Z^2} \frac{1}{r_e^2} \frac{1}{\log(183 Z^{-1/3})} \]  

and hence

\[ \theta_{\text{rms}}^2 (\text{Rossi}) = 4\pi e^4 \left( \frac{Z^2}{p \beta c} \right)^2 \frac{N_A \rho t}{A^2} \frac{1}{4 \log(183 Z^{-1/3})} \]  

A more accurate expression for the radiation length has been given by Tsai (1974) in the form

\[ X_0 \propto A \frac{1}{Z^2 (L_{\text{rad}} + f) + Z L_{\text{rad}}} \]
where \( L'_{\text{rad}} \) is obtained from the incoherent scattering function and \( f \) is the Coulomb correction (Davies, Bethe and Maximon 1954)

\[
f = (aZ)^2 \left[ \{1+(aZ)^2\}^{-1} + 0.20206 - 0.0369(aZ)^2 + 0.0083(aZ)^4 \ldots \right]
\]  

(2.74)

From Molière's fit to the Thomas-Fermi potential, Tsai obtained

\[
L_{\text{rad}} = \log(184.15 Z^{-1/3}) 
\]  

(2.75)

\[
L'_{\text{rad}} = \log(1194 Z^{-2/3}) 
\]  

(2.76)

In Molière's model of multiple scattering the rms angle of the Gaussian part of the distribution is given by

\[
\theta_{\text{rms}} = \chi_c B^{3/2}
\]  

(2.77)

where (Scott 1963)

\[
B = 1.153 + 2.583 \log \left( \frac{x_c}{x_0} \right)^2
\]  

(2.78)

Hence, using equation (2.55) we have

\[
\theta_{\text{rms}} (\text{Molière}) = 4\pi e^4 \left( \frac{zZ}{Bpc} \right)^2 \frac{N_A}{A} \rho t B
\]  

(2.79)

We now follow Hanson (1978, 1979) and define a correction to the Rossi rms angle of the form

\[
\theta_{\text{rms}} (C) = \theta_{\text{rms}} \text{(Rossi)} \ (1+\epsilon)
\]  

(2.80)

and setting

\[
\theta_{\text{rms}} (C) = \theta_{\text{rms}} (M) \text{ we have}
\]

\[
\epsilon = \left[ \frac{0.25 Z^2 B}{Z^2 (L_{\text{rad}} - f) + Z L'_{\text{rad}}} \right]^{1/2} - 1,
\]  

(2.81)
and

\[ \theta_{\text{rms}} (C) = \left( 4\pi e^4 N A z^2 \frac{pt}{A} \frac{4}{(p8c)^2} \left[ Z^2 \left( L_{\text{rad}} - f \right) + Z L_{\text{rad}} \right] \right)^{\frac{1}{2}} \left( 1 + \varepsilon \right). \]

(2.82)

Highland (1975) has investigated the angle \( \theta_{1/e} \), which is the angle at which the measured distribution falls to \( 1/e \) of its value at \( \theta = 0 \), and is given by (Scott 1963)

\[ \theta_{1/e} = x_c (1.007B - 1.33)^{\frac{1}{2}}. \]

(2.83)

This angle is preferred for examination of experimental data because it is least affected by the unmeasured tail of the distribution, and Highland (1975) suggests that it should be calculated from the formula

\[ \theta_{1/e} = \frac{17.5}{p8c \times X_0} \left( \frac{pt}{X_0} \right)^{\frac{1}{2}} \left( 1 + \varepsilon' \right) \]

(2.84)

\[ \varepsilon' = a \log_{10} \left( \frac{pt}{bX_0} \right) \]

(2.85)

where \( a \) and \( b \) are constants, independent of \( p \) and \( Z \).

Hungerford et al (1972) have derived a correction to \( \theta_{\text{rms}} \) by applying a conversion factor from the lab to the centre-of-mass frame of reference. However, since all integrals in multiple scattering theory are of the form

\[ \int \sin \theta \, d\theta \int W(\theta, t) \, dt \]

and \( W(\theta, t) \sin \theta \, d\theta \) is invariant with respect to the frame of reference, it is not easy to see the justification for their procedure.
Calculations

Our ultimate aim is to calculate the root-mean square lateral displacement of the particles at the end of their range in a plane perpendicular to their initial direction of motion, Fig.17.

Consider the projection of motion of the particles on the (Z,y) plane and let \( P(Z,Y,\theta_y)dYd\theta_y \) be the number of particles at the thickness \( t \), having a lateral displacement in \( dy \) at \( Y \) and travelling at angle in \( d\theta_y \) at \( \theta_y \) with the Z-axis. The total scattering angle \( \theta_{\text{rms}} \) defined by Rossi (1952)

\[
\theta_{\text{rms}}^2 = \theta_y^2 + \theta_x^2
\]

If the energy loss is neglected, then \( \theta_y = \theta_x \) (Rossi 1952) and

\[
\theta_y^2 = \frac{1}{2} \theta_{\text{rms}}^2
\]  
(2.86)

The root mean-square displacement, \( Y_{\text{rms}} \), is given by (Rossi 1940)

\[
Y_{\text{rms}} = \frac{1}{\sqrt{3}} \theta_y(Y)
\]  
(2.87)

We first carried out a complete calculation in the Molière model, disregarding energy loss, i.e. calculating \( \chi_C^2 \) from eqn.(2.55), \( \chi_{\alpha\text{I}}^{\text{el}} \) from eqn.(2.57) with \( R = 1.13 \) for all \( Z \), \( \log \chi_{\alpha} \) from eqns. (2.64) and (2.65) with \( u_{\text{inel}} = -5.8 \) for all \( Z \), \( B \) from eqn.(2.78) and finally \( \theta_{\text{rms}}(M) \) from equation (2.79). Then we calculated \( \varepsilon \) from eqn.(2.81). Using \( L_{\text{rad}} \) and \( L_{\text{rad}}' \) given by eqns. (2.75) and (2.76). The value of \( X_0 \) was calculated from eqn.(2.73).

Hanson (1979) has recalculated \( \chi_{\alpha}^{\text{el}} \) from equation (2.54) using the atomic form factors tabulated by Hubbell et al (1975) and has deduced values for \( R_i \) for each \( Z \). He has also recalculated \( L_{\text{rad}}, L_{\text{rad}}', X_0 \) and \( u_{\text{inel}} \) using the atomic form factors and incoherent scattering factors.
Fig. 17 The beam trajectory due to the scatterer, with Y representing the lateral displacement in the Y-direction perpendicular to the direction of motion.

Tabulated by Hubbell et al (1975). These tabulations are based on the best available atomic wavefunctions and the parameters derived from them should represent an improvement on the values derived by Molière and others from the Thomas-Fermi model, particularly for light elements. We have used Hanson's values of $R_i$, $L_{\text{rad}}$, $L_{\text{rad}}$ and $u_{\text{inel}}$ in a second set of calculations using the same equations as listed above.

The results obtained for $\varepsilon$ for incident protons are given in Tables XII-XIV for oxygen, calcium and water. It can be seen that the magnitude and variation of $\varepsilon$ is such that $\theta_{\text{rms}}$ should always be calculated from a formula which takes this correction into account. The discrepancies between the values of $\varepsilon$ calculated from Molière's model and those calculated with Hanson's parameters are small and arise mainly from differences in $X_0$. We have investigated the effect of replacing eqn. (2.57)
Table XII  (a) Values of $\epsilon$ for incident protons calculated with the Moïlire model for oxygen ($Z=8$, $A=15.99$) with $R=1.13$, $X_0=34.34$,
  (b) values of $\epsilon$ calculated with Hanson's parameters $R=1.263$, $X_0=34.845$, $U_{inel}=-4.62$
  (The units of $X$, $X_0$ are g cm$^{-2}$ = 10 Kg m$^{-2}$)

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$X$ = pt</th>
<th>$\beta$ = 0.3</th>
<th>$\beta$ = 0.5</th>
<th>$\beta$ = 0.7</th>
<th>$\beta$ = 0.9</th>
</tr>
</thead>
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<td>0.001</td>
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</tr>
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Table XIV  (a) Values of $\varepsilon$ for incident protons calculated with the Molibre model for water (mean $Z=7.30$, $A=18.02$) with $R=1.13$, $\chi_0=36.70$, (b) Values of $\varepsilon$ calculated with Hanson's parameters $R=1.221$, $\chi_0=36.70$, $u_{inel}=-4.64$ (The units for $X$ and $\chi_0$ are g cm$^{-2} \equiv 10$ kg m$^{-2}$)

(a) $X = \rho t$

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(b) $X = \rho t$

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<td>-0.1698</td>
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<td>-0.1479</td>
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<td>30.000</td>
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<tr>
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<td>-0.0989</td>
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by the approximate expression
\[ x_\alpha^2 = x_\alpha^2 R(1 + 3.33\eta^2) \] (2.88)
in the calculation with Hanson's parameters. The effect is negligible
for \( \beta \geq 0.6 \), while for small \( \beta \) and large values of \( X = \rho t \) the maximum
change is \( \sim 9\% \). Since \( B \sim 15 \) the difference between \( \theta_{\text{rms}} \) and \( \theta_{1/e} \) is
small in all cases.

Typical behaviour of \( \varepsilon \) as a function of \( \rho t/X_\alpha \) is shown in Figs. 18
and 19. It is clear that the simple formula (2.85) for \( \varepsilon \) could not
reproduce this behaviour over the whole range of \( \beta \) and for different
absorbing materials.

Figure 20 shows the behaviour of \( \varepsilon \) for different ions calculated
at \( \beta = 0.65 \) as a function of \( X = \rho t \) while Fig. 21 shows the behaviour
of \( \varepsilon \) as a function of \( \beta \) for a fixed depth \( t = 200\text{mm} \) in water. For the
same target material, \( \varepsilon \) varies with \( (\beta/z) \).

It is interesting to note that the correction to the Rossi
formula (2.70) for \( \theta_{\text{rms}} \) is negligible for heavy ions with mass number \( \geq 12 \).

The rms lateral displacement \( y_{\text{rms}} \) of a proton beam as it passes
through water has been calculated using eqn. (2.87) with \( \theta_{(Y)_{\text{rms}}} \) given
by eqn. (2.86). Results are given in Table XV and show that inclusion
of the correction \( \varepsilon \) gives an apparent improvement in the lateral
resolution of \( \sim 10\% \). Similar calculations are performed for heavier
ions, as \( \alpha \)-particle, carbon ion and oxygen ion, Table XVa.

From these results we conclude that it is necessary to include
the correction \( \varepsilon \), i.e. to correct the Rossi formula for \( \theta_{\text{rms}} \), for
protons and \( \alpha \)-particles but not for heavy ions. The Molière model is
sufficiently accurate for the calculation of \( \varepsilon \) and \( \theta_{\text{rms}} \).

Results for a proton beam at various energies passing through
various thicknesses in water with energy loss taken into account are
given in Table XVI. Both $\chi_c$ and $\chi_a$ increase quite substantially compared with the results for no energy loss, leading to an increase in $B$ of only a few percent and to a decrease in $e$ of 10-30% depending on the energy and thickness, Table XVI. The large increase in $\chi_c$ is reflected in the large increase in the values of $\theta_{\text{rms}}$ and $y$ given in Table XVII compared with those given in Table XV. Thus for protons passing through thick targets it is essential to take energy loss into account rather accurately. It is also evident from Tables XV and XVII that spatial resolution in proton tomography will be poor for thick targets, as already noted by Hanson (1978).

The uncertainties in the values given in Tables XV and XVII are small. We estimate that, in the energy region of interest, uncertainties due to errors in measured values of stopping power of a single element and to departures from the Bragg rule should be less than 1%. This will have a negligible effect on our calculations. Uncertainties of $\approx$ 10% in the mean residual energy $T$ lead to a change in $y_{\text{rms}}$ of $\approx$ 0.5%. Hence, provided that $T$ does not fall below about 10 MeV, the results given by such calculations should be very reliable.
Fig. 18 Behaviour of ε for protons passing through oxygen as function of $X/X_0$ where $x=\rho t$ and $X_0$ is the radiation length.

Fig. 19 Behaviour of ε for protons passing through water as a function of $X/X_0$ where $x=\rho t$ and $X_0$ is the radiation length.
Fig. 20 Behaviour of $\epsilon$ as a function of $X=pt(gm/cm^2)$ for different ions with $\delta=0.65$ (The units of $X$ are $gm cm^{-2}10 Kgm m^{-2}$)
Fig. 21  Behaviour of $\epsilon$ as a function of $\beta$ for different ions at depth 200 mm in water
Table XV  The rms scattering angle and lateral displacement for a proton beam passing through water.  The last two columns give the values for $\epsilon = 0$.

<table>
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<tr>
<th>Proton Energy (MeV)</th>
<th>$\beta$</th>
<th>Depth $t$ (mm)</th>
<th>$\epsilon$</th>
<th>$\theta_{Y_{rms}}$ (rad)</th>
<th>$Y_{rms}$ (mm)</th>
<th>Uncorr $\theta_{Y_{rms}}$ (rad)</th>
<th>Uncorr $Y_{rms}$ (mm)</th>
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<td>0.0222</td>
<td>1.28</td>
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Table XVa: The rms scattering angle and lateral displacement for different ion beams passing through water ($\beta = 0.65$). The last column gives the values for $\epsilon = 0$. The formula for calculating $\theta_{\text{projectile}}$ in terms of that of proton is derived in Appendix C.

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<th>Depth (mm)</th>
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<th>$\epsilon$</th>
<th>$\theta$ (Y)$_{\text{rms}}$ $10^{-3}$</th>
<th>$\theta$ (Y)$_{\text{rms}}$ $10^{-3}$</th>
<th>$\theta$ (Y)$_{\text{rms}}$ $10^{-3}$</th>
<th>Carbon-ion</th>
<th>Oxygen-ion</th>
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<td>Uncorr (rad)</td>
<td></td>
<td>Uncorr (rad)</td>
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<td>Uncorr (rad)</td>
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Table XVI  Results of $x_c$, $x_\alpha$, B and $\varepsilon$ for a proton beam passing through water when energy loss is taken into account, and using Molière's model. The values underlined are obtained with energy loss neglected. The values in brackets are the percentage difference between the two cases.

<table>
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<tr>
<th>Proton Energy (MeV)</th>
<th>Depth $t$ (mm)</th>
<th>$x_c$ (rad)</th>
<th>$x_\alpha$ (rad)</th>
<th>B</th>
<th>$\varepsilon$</th>
<th>$\theta_{\text{rms}}$ (rad)</th>
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<td>7.32 $10^{-6}$</td>
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<td></td>
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<td>(44%)</td>
<td>(18%)</td>
<td>(5.4%)</td>
<td>(29%)</td>
<td>(46%)</td>
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<td>240</td>
<td>300</td>
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<td>9.18 $10^{-6}$</td>
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<td>6.86 $10^{-6}$</td>
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<tr>
<td></td>
<td></td>
<td>(37%)</td>
<td>(25%)</td>
<td>(2.8%)</td>
<td>(15%)</td>
<td>(38%)</td>
</tr>
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</table>

Table XVII  Results for a proton beam passing through water when energy loss is taken into account and using the Molière model.

<table>
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<th>Proton energy (MeV)</th>
<th>Depth $t$ (mm)</th>
<th>$\varepsilon$</th>
<th>$\theta_{\text{rms}}$ (rad)</th>
<th>$y_{\text{rms}}$ (mm)</th>
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<td>7.69</td>
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<tr>
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<td>300</td>
<td>-0.086</td>
<td>0.0467</td>
<td>8.08</td>
</tr>
<tr>
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<td>200</td>
<td>-0.101</td>
<td>0.0281</td>
<td>3.20</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>-0.087</td>
<td>0.0427</td>
<td>7.40</td>
</tr>
<tr>
<td>350</td>
<td>200</td>
<td>-0.107</td>
<td>0.0188</td>
<td>2.17</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>-0.095</td>
<td>0.0246</td>
<td>4.26</td>
</tr>
</tbody>
</table>
CHAPTER 3

Charged particle computed tomography: diagnostic potential of heavy particles

3.1 Introduction

The basis for interest in charged particle imaging can be understood from the way monoenergetic particles behave when they traverse a sample, Fig.1. Most of the attenuation of a beam of monoenergetic particles in a homogeneous medium occurs after the beam has penetrated a thickness, approximately equal to the particles mean range. At such stage of their travel, particles run out of energy and a massive energy deposition occurs. It is the region of rapid fall-off of the transmission curve which made use of to achieve a greatly enhanced sensitivity to small changes in the total areal mass traversed.

A comparison of ionisation energy deposited within a sample and detector system irradiated by an X-ray beam and a heavy particle beam is presented in Fig.2. For charged particle imaging, the variation of specific energy loss is relatively constant through the imaged sample, and the Bragg peak of ionisation is at the end of the particle range with a sharp terminus of dose. With X-rays an exponential decrease in the energy deposition is seen along the object imaged with maximum dose at the sample entrance. It is important to note that the Bragg peak of ionisation may be adjusted so that the dose delivered at the exit point of the beam is approximately equal to the, relatively, uniform dose distributed along the particle track in the sample. Supplementary attenuators and careful positioning are required for this purpose. This phenomenon is useful, for radiotherapy treatment, where dose localisation within the tumour could be achieved, and for diagnosis, where detector positioning is an important factor in finding optimum results.
Fig. 1 Transmitted intensity curves for monoenergetic protons and X-rays. A small change in object mass $\Delta m$, shows a significant difference in available contrast between the two rays.

Fig. 2 An exponential decrease of energy deposition is seen in the sample image with X-rays. With charged particles the energy deposition is small and uniform until the end of the particle range where maximum deposition occurs.
3.2 Bragg curves for monoenergetic beams

In the study of the characteristic Bragg peak there are certain quantities which have an effect on the diagnostic potential of the particles. These are (i) the relative peak dose, defined as the dose at the peak to that at the surface (peak-to-plateau ratio), (ii) the full-width at half maximum of the peak and (iii) the shape of the energy spectrum at the end of the range. The values of these quantities will depend on the nature of the particle (ion), the incident energy of the beam, the energy straggling and the characteristic of the target material (Litton et al 1968).

For monoenergetic particles, the main contributors to the uncertainty in the path-length distribution are energy straggling and small angle multiple scattering processes. Particles (ions) with heavier mass, and higher charge are likely to suffer less Coulomb scattering and lower range straggling (equation (2.83) and Fig.12, chapter 2). Consequently, a change in the character of the bombarding ion would affect all the parameters that are dependent on the fluctuation in the path-length distribution. In Fig.3a (Litton et al 1968), it is shown that the Bragg peak width decreases sharply with increasing particle atomic number \( z \), up to \( z = 20 \). The width starts to increase very gently for projectiles with higher \( z \) (high charge-to-mass ratio). Partly, because the charge exchange becomes an important factor and the effective charge will not be proportional, any more, to the atomic number.

Figure 3b illustrates the peak-to-plateau dose ratio which becomes maximum for projectiles of atomic number around 7. This occurs mainly because high \( z \) projectiles are characterised with higher stopping power (Eqn.2.2) and narrower Bragg peak, and therefore they are likely to deposit more energy in a smaller distance. However, at projectile atomic numbers greater than 10, particles are rapidly removed from the primary beam due to the increase in the nuclear reaction cross-section. Thus, lower doses,
Fig. 3 Various ions penetrating water, with the peak at 5 gm/cm$^2$,
(a) Variation of the Bragg peak width $z_p$, (b) Dependence of peak-to-plateau dose ratio on $z$. (Litton 1968)
especially near the end of the range are expected. This effect will be enhanced by the charge exchange process to reduce the peak-to-plateau ratio for high z projectiles.

The initial energy of the incident beam would also have an effect on the shape and behaviour of the Bragg curve. Conclusions by West (1980) indicate that protons with high initial kinetic energy observed at the end of their range are expected to have wider Bragg peaks, lower peak-plateau dose ratio and less steepening of the distal part of the curve. In addition, the lateral displacement of the beam due to Coulomb scattering will significantly increase (Table XV, Chapter 2), because of the larger depths they are able to travel within the sample before they stop. These factors have the effect of degrading the attainable spatial resolution, and the image contrast per unit dose. It will be a great diagnostic advantage if the energy of the incident particle, and consequently its range, are matched to the object thickness.

For high-energy heavy ions, secondary fragments of approximately the same forward velocity as that of the primary beam (Eqn.2.4) would be generated, mainly in the initial part of the ion path length (Curtis 1977, Greiner et al 1975, Schimmerling 1977). Owing to their lower charge and longer range, the fragments contribution to the Bragg peak will be minimum and they will deliver most of their energy at depths beyond the distal side of the peak (Raju et al 1978, Lyman 1977).

Particles with non-homogeneous initial energy, are expected to have more uncertainty in the particle range distribution (Eqn.2.33).

3.3 The density resolution

In heavy particle radiography (conventional and CT) the wealth
of information available in the beam cannot be extracted by simply measuring the particle transmission alone, as for X-rays radiography. For charged particle tomography the most important measurement is the total energy lost in the medium rather than its attenuation. Most of the information on the total areal density penetrated will be conveyed by the residual energy of the particles or the residual range.

Since the range of the particle is proportional to the stopping power of the medium, which in turn is directly related to the electron density, uncertainty in the measured mass, \( \delta m \), can be obtained from the range straggling of the transmitted beam. Referring to Table IX, Chapter 2, a 200 MeV proton, with \( \sigma_R/R = 0.01055 \), and a range of 280 mm in water, can provide mass measurement to an accuracy of 0.295 g/cm\(^2\). Other likely sources of inaccuracy in the measurements are the statistical fluctuation of the beam and the detection efficiency.

Within this background the mass uncertainty \( \delta m \) will be an estimate of the minimum density difference that can be detected with the modality and conditions of the measurements (Kramer et al 1980a).

Motz and Danos (1978) have shown that for the eye to detect a discontinuity in a homogeneous background, there is a minimum signal-to-noise ratio (SNR) which the discontinuity should exceed. This threshold (SNR) has a common value of 5.

The SNR for a certain thickness charge \( \Delta t = \ell \rho_0 \) of a background material, of density \( \rho_0 \) and length \( \ell \), can be obtained from \( \text{SNR} = \Delta t/\delta t \), which is equivalent to the SNR defined by equation (1.10) for X-rays. Accordingly the criterion for the detectability of a structure having an equivalent mass difference, \( \Delta t_e \) (g/cm\(^2\) equivalent of background material) is given by

\[
\Delta t_e > 5 \delta t
\]  

(3.1)

where the equivalent mass, \( \Delta t_e \) is equal to the mass difference of the
background material multiplied by the relative stopping power of the discontinuity, $S_r$, defined by $S_r = (S_b/S_o)$ with $S_b$ and $S_o$ as the mass stopping powers for the discontinuity and the background material, respectively (Kramer et al 1979). For X-rays, the SNR of a structure is given in terms of the transmitted photon fluence signal and its statistical fluctuation. The mass difference of the background material should be multiplied by the relative mass attenuation coefficient, $\mu_r = \mu_d/\mu_o$, to yield the equivalent mass difference, $\Delta t_e$ (Kramer et al 1979).

Instead of examining a change in geometrical thickness, it is usually of more interest to determine the criterion for detecting an inclusion of material of length $a$ and density $\rho_d$ plunged in a homogeneous matrix material of density $\rho_o$. This can be achieved by calculating $\Delta t_e$ of matrix material which would have the same mass signal as the inclusion of thickness $\Delta t_I (\Delta t_I = a \rho_d)$. The expression relating these parameters is given by (Koehler and Berger 1973, Kramer et al 1979)

$$\Delta t_e = \Delta t_I (\lambda - 1) \text{ for particles,}$$

$$\Delta t_e = \Delta t_I (\gamma - 1) \text{ for X-rays}$$

(3.2)

where $\lambda = S_r$ and $\gamma = \mu_r$. $(\rho_d/\rho_o)$ is the relative linear stopping power of the inclusion and $\gamma = \mu_r$. $(\rho_d/\rho_o)$ is relative linear attenuation coefficient.

Since the relative mass stopping power, $S_r$, is approximately constant over the energy range 10-1000 MeV/a.m.u. (Koehler et al 1965), the mass signal produced by the inclusion is independent of its depth within the background material (object).

The expression for $\gamma$ is more complicated due to the spectral nature of the X-ray beam employed by most diagnostic systems. Therefore, $\gamma$ would become dependent on where the inclusion is positioned in the object.

Equation (3.2) for charged particles indicates that variations
in thickness due to irregularities in the sample geometry can be suppressed by surrounding the sample with a reference material for which \( \lambda = 1 \). It also shows that the sample can be positioned in the part of the path (the plateau) where the energy loss (dose) and multiple scattering are relatively low (Steward 1978).

3.4 Charged particle dose

The density resolution, that could be obtained from the measurement of the energy deposited by a single particle can be increased by increasing the number of particles in the incident beam. The relationship between the incident particles fluence \( N_0 \), and the mass uncertainty, \( \delta t \), for particles, and X-rays, can be given by (Kramer et al 1980)

\[
\delta t = \left( \frac{P}{N_0} \right)^{\frac{3}{2}} \left( \frac{dP}{dt} \right)^{-1}
\]  

(3.3)

where \( P \) is the value of the transmitted intensity and \( (dP/dt) \) is the rate of change of \( P \) with thickness (slope of the transmission curve, Fig.1). For X-rays with \( N_X \) detected photons, \( P_X = N_X / N_0 \alpha \) = \( \exp(-\mu t) \), where \( \mu \) is the absorption coefficient (length\(^{-1}\)) and \( t \) is the total thickness penetrated.

For particles, \( P \) can be obtained from equation (2.31). In Appendix D we derive \( \delta t \) for X-rays and particles, the results are

\[
\begin{align*}
\delta t_p &= (\pi / 2N_p)^{\frac{3}{2}} \cdot \sigma_R \quad \text{for particles} \\
\delta t_X &= (1/N_X)^{\frac{3}{2}} \cdot \mu \quad \text{for X-rays}
\end{align*}
\]  

(3.4)

where \( \sigma_R \) is the particle range-straggling parameter.

A particle beam and an X-ray beam would have the same density
resolution, same accuracy, if the number of detected X-rays, \( N_x \), is related to that of the particles, \( N_p \), by (proved in Appendix D)

\[
N_p = (1.25 \sigma_R \mu)^2 \cdot N_x
\]  

(3.5)

To illustrate the significance of equation (3.5) consider \( \sigma_R = 2.95 \text{ mm} \), for a 200 Mev/a.m.u. proton beam penetrating 280 mm of water. We find that one proton would have the same density error as that obtained by 201 X-ray photons with \( \mu = 0.0191 \text{ mm}^{-1} \), \( E = 70 \text{ keV} \).

If the value of \( (N_p/N_x) \) is substituted in equation (1.11), the relative doses at the same object centre for equal SNR and spatial resolution can be obtained from

\[
\frac{D_c(\text{particles})}{D_c(\text{X-rays})} = \frac{\mu (1.25 \sigma_R)^2 (dE/dx)_{t=d/2}}{f_c E_x (I_o/I)_{t=d/2}}
\]  

(3.6)

where \( (dE/dx) \) is the linear stopping power at the object centre (kev. length\(^{-1}\)), and the dose at the centre due to a single particle is given by (Barrett 1975)

\[
D_c(\text{particle}) = \frac{2.26(dE/dx)_t (\text{SNR})^2 \cdot 1.6 \times 10^{-13}}{\rho a^3 h} \text{ Gy}
\]  

(3.7)

d is the object diameter, \( a \) is the pixel width and \( h \) is the slice thickness.

Equation (3.6), originally derived for reconstruction tomography, is applicable to conventional radiography.

In the dose comparison, between particles and X-rays, the relative biological effectiveness (RBE), (which is the ratio of the dose required of a standard radiation, such as 250 keV X-rays, to produce a certain effect to the dose required of the radiation under consideration) for diagnostic particles will be considered to be unity (Raju et al 1978).

In Table I we present our calculation of particle dose advantage over X-rays in water using equation (3.6). The values of \( \sigma_R \) are obtained
Table I Calculation of the dose advantage of charged particles over X-rays in the centre and surface of a water phantom. The X-ray energies are the optimum energies at the specified depths (minimum dose) taken from Hanson (1978)

<table>
<thead>
<tr>
<th>X-rays</th>
<th>Particles</th>
<th>d=2t (mm)</th>
<th>K.E. (Mev/a.m.u.)</th>
<th>(dE/dx) Mev.cm⁻¹</th>
<th>δR (cm)</th>
<th>Dose advantage</th>
</tr>
</thead>
<tbody>
<tr>
<td>E (kev)</td>
<td>μ</td>
<td>(Iₒ/I)</td>
<td>μₜ</td>
<td>Iₒ(t)</td>
<td></td>
<td>Dc</td>
</tr>
<tr>
<td>55</td>
<td>0.211</td>
<td>2.87</td>
<td>1.055</td>
<td>1.289</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.185</td>
<td>6.36</td>
<td>1.85</td>
<td>2.052</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.168</td>
<td>12.42</td>
<td>2.52</td>
<td>3.326</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>100</td>
<td>118</td>
<td>9.19</td>
<td>0.113</td>
<td>4.08</td>
</tr>
<tr>
<td></td>
<td>t</td>
<td>100</td>
<td>62</td>
<td>16.13</td>
<td>0.074</td>
<td>5.42</td>
</tr>
<tr>
<td></td>
<td>α</td>
<td>100</td>
<td>118</td>
<td>36.76</td>
<td>0.056</td>
<td>4.09</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>100</td>
<td>220</td>
<td>177.3</td>
<td>0.03</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>200</td>
<td>174</td>
<td>25.76</td>
<td>0.107</td>
<td>5.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>300</td>
<td>224</td>
<td>21.76</td>
<td>0.156</td>
<td>8.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>200</td>
<td>337</td>
<td>141.5</td>
<td>0.058</td>
<td>3.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>300</td>
<td>427</td>
<td>122.7</td>
<td>0.082</td>
<td>5.73</td>
</tr>
</tbody>
</table>
from Table IX, Chapter 2, for several particle ranges, and \((dE/dx)\) at different particle energy is obtained from Fig. 5 (Chapter 2).

It is apparent that protons and \(\alpha\)-particles could provide the same density resolution as monoenergetic X-rays but with reduced doses, at the centre of the object, of a factor of 6 for a 200 mm diameter water phantom, and a factor of about 9 for the 300 mm phantom. However, using equation (1.12), one can conclude that the skin dose for X-rays exceeds that for protons and \(\alpha\)-particles by a factor of 12 for the 300 mm object.

Obviously the dose advantage for particles is remarkably less for thinner objects. That is mainly because the dose distribution for X-rays is non-uniform, where it tends to be maximum in the proximity of the object periphery.

For heavier ions, such as \(C^6\) shown in the table, the dose reduction factor is lower than that for the proton because of the heavy ions larger stopping power (energy loss) which is proportional to the square of the particle charge.

3.5 The effect of small-angle scattering

Multiple scattering of charged particles creates the most serious problem facing particle radiography. The angular divergence of the highly collimated beam, used for tomography, will impose a limit upon the spatial resolution, a problem that does not exist for X-rays. Figure 4 shows the spatial spread of a zero width, 210 Mev proton beam, which will spread to a rms width of about 15.4 mm (with energy loss taken into account) after penetrating 250 mm of water. The spread at the middle of the phantom is about 4 mm.

The effect of the beam spreading is to spatially average the
actual stopping power distribution to produce a blurred image. For proton tomography, such blurring will cover distances larger than 10 mm for compositions near the back surface of the body.

The spatial resolution obtained in X-ray CT, about 2 mm, demonstrates the inferiority of the proton spatial resolution and the need to improve it. This could partly be achieved by measuring the position, as well as the angle, of the exiting protons (Moffat 1974, Hanson et al 1978). Such coincident spatial information could be used by computed tomography to produce images with spatial resolution comparable to X-rays. These predictions were verified by Hanson et al (1978) who reported a good spatial resolution at the entrance and exit of the object with the worst resolution (2 - 2.5 times worse than X-ray CT scanning) occurring near the middle of the object. Further improvements could be reached if the reconstruction algorithm is adapted to handle curved projection paths rather than straight-line trajectories. However, using alpha particles would improve the spatial resolution by a factor of 2, and using heavier ions would provide much better spatial resolution but at the expense of less mass resolution, higher dose, and increased production cost.

The phenomenon of Coulomb scattering, which degrades the image quality in conventional and CT techniques, may be turned to advantage to obtain radiographs where edges of the internal structures are enhanced in the presence of various amounts of overlapping material (West and Sherwood 1972, West 1980). The idea forms the basis of proton scattering radiography for detecting anomalies with clear cut boundaries.

Although the sensitivity of the technique to density variations is relatively low, sharp density changes, which occur between soft tissues and gases (gut) or soft tissues and bone, could be picked and
visualised. Nevertheless, the technique will be of limited advantage for medical diagnosis. But, the high degree of absorption of X-rays used to examine heavy elements demonstrates the superiority of proton scattering radiography for industrial purposes (West 1980).

3.6 The particle of choice for computed tomography

Before discussing which of the heavy charged particles is most applicable to diagnostic radiology in general and to CT in particular, we are going to investigate, briefly, the usefulness of particles other than the proton and heavier ions for diagnostic radiology.

To be of use in medical diagnosis, the particle should satisfy the following criteria.
(i) It should cause a disturbance of some kind within the object it traverses. The quantity and type of such disturbance should be measured via a change in the physical properties of the particle beam.

(ii) The half-life should be enough to carry it through the body (200-300 mm).

(iii) It should possess reasonable dose advantage, low Coulomb scattering, high density resolution, and can be produced easily at low cost.

Particles such as the pion, kaon, and muon fail to meet one or more of the mentioned conditions (Steward 1978). As an example, positive or negative pions would suffer more scattering than the proton and, with absorption, the attenuation curve of a primary pion beam would be close to an X-ray or a neutron beam attenuation curve (exponential). In addition the Bragg peak of the pion would also be wider and the steepness of the useful part of the peak will be less than that for heavier particles. The pion and muon will also decay.

Other particles, not mentioned in this study, could also be useful for diagnostic radiology. McGonnale (1961) has used the electron to provide sensitivity for very thin samples, while the neutron has been used to detect low atomic number structures in a high atomic number matrix (Berger 1970).

For the proton and other heavy ions, the question of "the most appropriate particle for tomography" does not have a straightforward answer. In computed tomography, measurement of the mean residual range of the particle which has traversed different parts of the body will be reconstructed and a display of the object density distribution could be obtained. The ability of a particle to convey the information about the object penetrated with high accuracy will depend on the particle's behaviour through the interaction process. From previous calculation, the summary presented in Table II for a 300 mm diameter water phantom,
one can conclude that heavy ions would have greater depth resolution than proton and α-particles. On the other hand the dose required to detect a density anomaly is considerably higher for heavy ions than it is for protons or α-particles.

Table II  Comparison between characteristics of several heavy ions obtained after traversing 300 mm of water

<table>
<thead>
<tr>
<th>Particle</th>
<th>kinetic energy (Mev/a.m.u.)</th>
<th>$(\sigma_{R}/R)$ $10^{-2}$</th>
<th>Relative spatial Resolution</th>
<th>Relative dose advantage at the centre</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1$p</td>
<td>224</td>
<td>1.05</td>
<td>1.0</td>
<td>8.76</td>
</tr>
<tr>
<td>$^3$t</td>
<td>115</td>
<td>0.70</td>
<td>0.65</td>
<td>11.5</td>
</tr>
<tr>
<td>$^4$He</td>
<td>224</td>
<td>0.52</td>
<td>0.52</td>
<td>8.93</td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>427</td>
<td>0.27</td>
<td>0.29</td>
<td>5.73</td>
</tr>
</tbody>
</table>

However, tritium offers the advantage of the best dose efficiency with moderate spatial resolution, as compared to the proton.

From the economic point of view, the cost and complexity of producing an ion beam are defined by the beam specification; such as the required beam energy, determined from the particle atomic and mass numbers and the required range in tissue. The proton with its unique charge-to-mass ratio of 1, has an advantage from the pure accelerator point of view (Grunder and Leeman 1980).

Within this background it is clear that there is a trade-off between dose and cost, on the one hand, which is in favour of proton and α-particles, and spatial resolution and density resolution which is in favour of heavier ions. The importance of these parameters will depend on the judgment of individual institutions employing, or planning to employ, charged particles and heavy ions for radiography. However,
using computed tomography techniques with appropriate software, which would reduce the beam divergence effect, is likely to optimise the use of the proton for medical diagnosis.

3.7 Practical studies of charged particle CT

Much of the study of charged particle tomography has been done with the proton beam from Los Alamos Meson Physics Facility (LAMPF) by Hanson et al (1978). The residual energy of each transmitted proton was measured with a hyperpure germanium detector with an active diameter of 20 mm, and a thickness of 12.5 mm. The exit position of each proton is measured by a set of proportional counters positioned at the exit of the water bath surrounding the phantom, fig.5, to be scanned. A complete scan was performed by moving the phantom through the water bath across the beam. Before beginning a new traverse, the phantom was rotated (automatically) and 360 projections were taken with 1 degree interval between each. A filtered back-projection algorithm was employed to produce an image of 256 x 256 pixels and 1.25 mm pixel size.

A polyethylene phantom, of 300 mm diameter, with different sizes and different contrast holes, was scanned using the above arrangements for the proton and an EMI CT scanner for X-rays (Hanson et al 1978).

From the comparison between the images produced a conclusion was reached that a 20 s X-ray scan and the proton beam scan produce images of the same density resolution with an average dose advantage of the proton of about 4. The resolving power of the X-ray scanner measured with the high-density resolution part of the phantom is about 0.6 that achieved by the proton. Such advantageous X-ray spatial resolution will be reduced when low-density resolution structures are to be detected. Beam hardening artifacts were also observed in the
Schematic diagram of proton CT experiment performed by Hanson (1978)

X-ray image. Such artifacts are absent in the proton reconstructions.

CT scanning with $\alpha$-particles using biological specimens was achieved by Crowe et al (1975). They compared X-ray CT scanning of a human head with a similar scan obtained with 900 Mev alpha particles. In spite of X-ray CT technical superiority, the alpha scan method was able to obtain comparable images with a dose only 2% of the dose delivered by an X-ray scan.

Saudinos et al (1975) have investigated the possibility of using the nuclear scattering of protons (500-1000 Mev) to obtain with a single exposure, a three-dimensional reconstruction of the object. The resolution and sensitivity are comparable to X-ray CT, but the low rate of data collection and difficulty of optimising the proton beam at such high energies limits the usefulness of the technique.

The possibility of two-dimensional reconstruction using charged
particles (heavy ions) was also mentioned by several authors (Sommer et al 1978, Tobias et al 1977, Curry and Steward 1978, and Kramer et al 1980). They recognised the advantages of the technique and pressed the need for a specially designed medical accelerator capable of producing different ion beams with different energies.

The technical requirements of a hospital-based proton diagnostic system was discussed by Martin et al (1975). A similar study was performed for other heavy ions (Behrsing et al 1979) that can be used for therapy and diagnosis. Both studies have shown encouraging results in terms of defining the criteria for the accelerator design. They concluded that it is possible to design a diagnostic system which will be able to meet the general conditions of being simple, reliable, economical and suitable for hospital setting.

A schematic representation for a future beam delivery system for CT scanning was suggested by Hanson (1978). It should be able to obtain 300 projection measurements in 10 s. To acquire $10^8$ events in 10 s, the detector system envisaged would exclusively use plastic scintillators and high speed photomultipliers which would achieve a time resolution of about 10 ns (Hanson 1978).

Within this background of understanding charged particles interaction with matter, their potential and limits for diagnostic radiology, and the feasibility of being used for CT scanning, we were encouraged to employ them in the general purpose SNARK reconstruction package to produce images and to compare their results to similar X-ray simulations. To achieve this, certain modifications within SNARK should be performed to make it applicable for charged particles. In the following chapter we present the most important of these changes, and the simulations achieved.
Chapter 4

Charged Particle Computed Tomography: Simulations
and Computer Experiments

4.1 Introduction

In previous chapters we have studied the diagnostic potential of charged particles and compared their performance to X-rays, used for the same purpose. In this chapter we are going to investigate what particles can offer for reconstruction tomography by employing them in the general-purpose SNARK programme using simulated data.

The SNARK programme (Herman et al 1975) from Buffalo, New York, has been implemented on this University's PRIME computer (Foster 1981). The programme includes several reconstruction techniques such as filtered back-projection, and algebraic reconstruction methods. The programme is first designed for X-rays but was made flexible so that it can be implemented for other sources of radiation.

Since interactions of X-rays and particles with matter are not alike, the type and the geometry of data collection for reconstruction will depend on the type of radiation used. Therefore, certain modifications within SNARK are necessary to accommodate the characteristics of charged particle beams. With these modifications, we performed several computer experiments simulating a head phantom suggested by Shepp and Logan (1974). The results for proton, α-particle, and carbon ions at different arrangements are compared to those for X-rays. The criteria for comparison were the visual display of the pictures (screen, hard copy) and the quantitative analysis facilities available in SNARK.

4.2 The SNARK Programme

4.2.1 General description.

The SNARK programme has been specifically designed to reconstruct
the distribution of the monoenergetic X-ray attenuation coefficient inside the region of interest in the human body from X-ray projections. The programme assumes that a detector is located at a user-specified distance from the object centre opposite to an X-ray source positioned at a specified distance from the object to be reconstructed.

The system is capable of handling different combinations of data collection (such as various arrangements of source and detectors), and will perform reconstructions using either mathematically generated data describing a test object or measured data really reflecting the characteristics of the object under consideration. The test object is to be presented mathematically as a set of superimposed ellipses and rectangles. SNARK contains different reconstruction methods and routines to compare, statistically, the reconstructions obtained under different conditions. It also contains a variety of output display methods. Additional facilities allowing the display of the image on an ordinary film have been implemented, (Edwards et al 1979), on the London University computing system. However, the new version of SNARK (SNARK 77, Herman et al 1978), contains extra routines which serve certain objectives such as reconstruction for polyenergetic X-rays.

4.2.2 The physical meaning of SNARK terminology

Although SNARK does not employ a particular set of units, consistency must be maintained throughout every single run of reconstruction.

The reconstruction region is a square with its centre at (0,0) of an (x,y) two-dimensional coordinate system. The region is divided in $n^2$ smaller squares called "pixels". The integral of the density along any straight line from the source to a detector is the ray-sum. Pixels that are not intersected by the ray do not contribute to its value.
The number of all rays (along which data is collected) having the same inclination makes one projection. All projections have the same number of rays. For more accurate results, a ray can be divided into smaller sub-rays along which the integrated density can be calculated.

SNARK uses two different methods for data collection, divergent and parallel, Fig.1. In the divergent geometry a projection is made of all rays (sub-rays) which pass through a common point, the source. For all projections, the source to detector and source to origin distances are constant and the projection angle is the angle the central ray makes with the x-axis.

Of more interest to us is the parallel geometry where a projection is made of a set of equally spaced rays. The projection angle is the angle that any one of the rays makes with the x-axis. Detectors are positioned along a line normal to the rays' exits, and detector spacing can either be uniform or variable. The number of rays within a single projection which is required to span a circle enclosing the squared picture is calculated by the programme using the specified detector spacing, pixel size, and number of pixels.

4.2.3 Execution sequence

A complete SNARK run consists of generating the data, initialisation (preparing data for reconstruction) and reconstruction, and finally analysis of results. Each stage needs some input files and produces others as an output.

During the first stage (data generation) test and/or projection data of the phantom are generated. The initialisation stage is to define the reconstruction matrix and the geometry of data collection; the results obtained are used as an input to the reconstruction stage
Fig. 1 Geometry of data collection. (a) Divergent geometry. (b) Parallel geometry.
at which the algorithm specified is initialised and the appropriate parameters are supplied. Evaluation and display of the results obtained occurs in the final stage. Figure 2 gives a schematic representation of the execution of SNARK.

If simulated data are to be produced, the test objects are to be specified as a collection of superimposed tilted ellipses and/or rectangles by giving their desired locations, dimensions, angles of inclination and their densities. The density at any point within the picture is determined from the sum of the densities associated with all the elemental objects within which the point lies.

The actual projection data for X-rays are estimated from the relationship between incident and detected intensities of a beam of monoenergetic X-ray photons. Because of the statistical nature of the measurement, the projections obtained are subjected to quantum noise which can be included in the calculations. The projected densities (data), simulated or otherwise, can be used as input to one of the reconstruction methods, namely, (i) Algebraic reconstruction techniques (Herman 1974, Herman et al 1973, Gordon et al 1970), (ii) Convolution methods (Bracewell and Riddle 1967; Shepp and Logan 1974, Rammachandran and Lakshminarayanan 1971), and (iii) Simultaneous iterative technique (Gilbert 1972).

A general idea on how these techniques work is presented in Chapter 1. The new version of SNARK (SNARK 77, Herman et al 1978) includes more reconstruction methods.

4.2.4 Interpretation of results

To compare a test picture and its reconstruction, one can use one or all of the routines existing for this purpose. Of particular interest is the routine which calculates the following parameters:

(i) Difference, that is the difference between the densities in the
Fig. 2 SNARK. Execution diagram (Herman et al, 1978)
phantom and the reconstructed pictures,

(ii) Relative Error, defined as the relative change between the
reconstructed densities and those in the phantom,

(iii) The variance and standard deviation of the reconstructed densities.

(iv) A comparison between the phantom values and the reconstruction
for every single row of the matrix.

The physical meanings and mathematical representation of these measures
are given in Appendix E.

An additional facility which converts the binary reconstruction
file into a special format, enables the user to produce images on a grey
level film. These pictures (Edwards et al 1979) are superior to any
hard copy or any other display facility available to us.

4.3 Implementation of SNARK for charged particles

Charged particle computed tomography consists of estimating the
integrated density \( \int p \) distributions along a ray from a point source
to a detector by measuring the energy lost by particles which penetrate
the sample at the exit point. Let this be \( P_L(r,\phi) \). Then

\[
P_L(r,\phi) = K \int_0^L p(x,y) ds = K_{(ray-sum)} \tag{4.1}
\]

where \( K \) is constant for a given particle, and can be evaluated from
the stopping power formula (Eqn.2.2) as,

\[
K \alpha \frac{z^2}{\beta^2} \tag{4.2}
\]

\( z \) is the particle charge, and \( \beta \) is its relative velocity.

4.3.1 Particles with neglected scattering

In principle, the parallel scan geometry for particles following
straight line trajectories is similar to that for X-rays where a complete
scan consists of a set of projections taken at desired angles with all projections containing equal numbers of equally spaced rays. For particles whose multiple scattering is neglected. Let \( E_0 \) be the initial energy of the monoenergetic particle beam and \( \bar{E} \) be the mean residual energy of these particles after traversing the object, Fig.3(a). The amount of energy lost along the \( N \)th sub-ray, \( \Delta E(N) \), is given by

\[
\Delta E(N) = K \cdot \text{Ray-sum (N)},
\]

and the projection along the ray is given by the average over all sub-rays in that ray, i.e.

\[
\text{Projection} = \frac{\sum \text{RAYSUM(N)}}{N} = K_{\text{TOT}}
\]

4.3.2 Particles with scattering

With scattering taken into consideration, the data collection geometry for the parallel scan is shown in Fig.3(b). The projection is similarly made of a number of rays, and every ray is divided into \( N \) diverging sub-rays. This approximation is made to include the effect of scattering. The slope of every sub-ray is different from that of the others in the same ray.

Projections are calculated from equation (4.4) for unscattered particles, but the way the summation is estimated is going to be different. The raysum along any sub-ray is the sum of the intersections with the object (ellipse, rectangle) multiplied by the density increments associated with the object. These intersections are calculated in SNARK by the function "RAYLEN". This is a real function which when given as an argument calculates the equation of a line and the equation of either an ellipse or a rectangle providing that the following parameters are available, (i) the type and description of the object(s), (ii) coordinates
Fig. 3 (a) A single ray containing a number of parallel sub-rays. The detector measures the residual energy of the traversing particles. PINC represents the detector width (detector spacing in SNARK).

(b) For particles with scattering, in parallel geometry, every ray is made of N diverging sub-rays. Sub-rays have different slopes but a single common point at the entrance.
of any point \((x', y')\) on the X-axis, (iii) the slope of the line calculated from the equation \((x-x')\sin\theta = (y-y')\cos\theta\), where \(\theta\) is the projection angle. Consequently, in the case of scattering included, different sub-rays will have different slopes where, in addition to the projection angle every sub-ray will have an additional small angle, \(\phi\), the value of which will depend on the rms scattering angle of the beam and the position of the sub-ray within the ray, Fig.4(a). The total rms angle and lateral displacement of the beam, \(Y_{\text{rms}}\), are given in equation (2.85).

From figure (4) the angle \(\phi\) is given by

\[
\phi_N = \tan^{-1} \frac{A_N C}{t}, \quad \left\{ \begin{array}{l} \hspace{1cm} A_N C = \frac{2 Y_{\text{rms}}}{(N-1)} \cdot M \end{array} \right.
\]

where \(M\) is the order factor of the sub-ray. \(M\) can be positive or negative and could be evaluated for the Nth sub-ray from the equation,

\[
M = N' - (N+1)/2
\]

where \(N'\) is the Nth sub-ray.

As an example consider the number of sub-rays \(N = 9\) in Fig.4, \(Y_{\text{rms}} = 6\text{mm}\) and \(t = 250\text{mm}\), then the 3rd sub-ray factor, \(M_3\) is (from 4.6)

\[
3 - (9+1)/2 = -2, \quad A_3 C = 2(6)(-2)/8 = -3, \quad \text{and} \quad \phi_3 = -0.012 \text{ radians}. \quad \text{The central sub-ray will have} \quad M = 0, \quad \text{and} \quad \phi = 0. \quad \text{The slope of any sub-ray is given by} \quad \sin(\theta+\phi)/\cos(\theta+\phi).
\]

The above considerations are introduced in SNARK to make it applicable for particles.

The second requirement for defining the equation of the ray in SNARK is to calculate the coordinates of any point \((x', y')\) on a sub-ray which is uniformly spaced from the adjacent sub-rays, Fig.4(b). For particles with scattering, Fig.4(c), the sub-rays are not uniformly
Fig. 4 (a) Divergence geometry for the calculation of sub-ray slope. \( \phi_1, \phi_2, \phi_3, \ldots \phi_N \) are the divergence angles of the sub-rays, \( Y_{\text{rms}} \) is the rms lateral displacement of the beam after traversing an object of thickness \( t \). \( C \) is the exit position of the central sub-ray.

(b) Coordinates of a point \((x^\prime, y^\prime)\) for parallel sub-rays without scattering

(c) Coordinates of the beam entrance point \((x^\prime, y^\prime)\) for scattering geometry
spaced and "any point" will not satisfy the conditions of the line equation. We solved the problem by calculating the coordinates of the point \((x', y')\) which is the only point that can be considered as common for all sub-rays in the ray. That is the beam entrance to the object. With this point and the modified slope, we can find the equation for every sub-ray. Unfortunately SNARK does not calculate the coordinates of this particular point. To accommodate this situation we defined some new parameters, and modified a few routines in SNARK. The derivation and the changes needed are presented in Appendix F.

Another complication of small angle scattering is the actual number of particles scattered through angle \(\phi\). In our calculation of ray sum we are assigning a value to the central sub-ray calculated from the average over all the sub-rays within that ray. We assumed that all sub-rays will contribute equally to the ray sum. However, the number of particles along the central sub-ray will be maximum but should decrease with increasing angle of divergence of the sub-ray. Thus, at the edge of the beam the number of particles will be minimum. Within this approach, the contribution of every sub-ray to the ray sum should be weighted according to the number of particles included in that sub-ray.

The angular distribution of the particles can be approximated by an exponential function (Bethe and Ashkin 1953):

\[
P(\phi) = \frac{1}{\pi(\phi_{\text{rms}})^2} \exp(-\phi^2/\phi_{\text{rms}}^2) \quad (4.7)
\]

where \(P(\phi)\) is the probability that a particle scattered with angle \(\phi\), and \(\phi_{\text{rms}}\) is the total rms scattering angle of the beam.

From equation (4.7) the number of particles in sub-ray \(N\) with angle \(\phi\) will be proportional to the exponential term
\[
N(\phi) = N_0 \exp\left(-\frac{\phi^2}{\phi_{\text{rms}}^2}\right)
\]
and
\[
N(\phi) \propto \exp\left(-\frac{\phi^2}{\phi_{\text{rms}}^2}\right)
\]  

(4.8)

To implement this particular modification in SNARK every calculated density is to be multiplied by a weighting factor calculated from (4.8). \(\phi_N\) in the modified SNARK is given by equation (4.5) and \(\phi_{\text{rms}}\) is to be calculated from the geometry in figure 4(a). The effect of such weighting on the ray sum and on the image obtained will depend on the lateral displacement of the beam at the end of the range, which in turn is dependent upon the particle type. However, with these modifications we expect our simulations to be closer to the real situation, especially for protons.

A final modification in the programme is needed to include the effects of the arrangements shown in Fig.5. In the parallel geometry (Fig.5a) the ray sum calculated for a certain source-detector arrangement (projection angle \(\theta\)) will be the same as that obtained from the opposite direction (projection angle \((\theta+180^\circ)\)), because the type and quantity of the phantom material contained in the ray are the same at both directions. For the scattering geometry, Fig.(5b), the average density intersected by the diverging ray will depend on whether the projection is taken at angle \(\theta\) or its opposite \((\theta+\pi)\). Therefore, a non-homogeneous object scanned over the range \((0-\pi)\) is expected to have different reconstruction from a \((\pi-2\pi)\) scan. This problem does not exist in X-ray CT.

A solution to this problem is to scan over the whole range of angles \((0-2\pi)\). The convolution reconstruction algorithm employed by SNARK is designed for the \((0-\pi)\) arrangement.
Fig. 5  (a) Source-detector arrangement for particles with no scattering, Raysum 1 = Raysum 2

(b) Particles with scattering, Raysum 1 ≠ Raysum 2. The projection angle for ray 1 and ray 2 are θ and (θ+π), respectively. All phantoms are assumed to be surrounded with a homogeneous background material
4.4 Computer experiments (simulations)

In order to examine the use of charged particles for CT scanning, a number of test object reconstructions were performed using our version of SNARK which includes all the modifications and ideas suggested in section (4.3). All experiments have been using the head phantom of Shepp and Log (1974). The phantom is surrounded by water and the total thickness (water + phantom) is 255mm. The reconstructions are obtained on a 127 x 127 pixels grid, each pixel is 2 x 2 mm$^2$. The ray width is taken to be equal to the detector spacing of 2mm. Every ray is divided into 5 sub-rays. Fluctuations in the data are neglected. The beam divergence is calculated from the scattering formula for different particles traversing the 255mm of water + phantom. After a few experiments we found the optimum number of projections to be 90 over the angles range (0-π); this number is used for most of the simulations.

4.4.1 The head phantom specifications

Figure (6) shows a schematic diagram of the phantom used, with the origin at (0,0) of an x-y coordinate system. The description of the phantom is summarised in Table I. The skull density is assumed to be twice that of the interior tissue. The grey matter fills the phantom except for the right and left ventricles, and for several tumours of several sizes at different locations.
Fig. 6 A schematic representation of a head section. The interior tissue has about half the density of the skull (a)-(b) which is thicker at the front. The ventricles (c) and (d) filled with spinal fluid have the minimum density (1.0). The grey matter fills the head (1.02) except for several, small and large, tumours, (e)-(j) of densities between (1.03-1.04). In our simulations the unit on the figure is to be multiplied by about 127 to match the 255mm phantom specified in the runs.

| Table I Specifications of the head phantom in a 250 x 250 mm² matrix |
|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Ellipse | x-coord | y-coord | Minor semi-axis Length (mm) | Major semi-axis Length (mm) | Pitch degree | Density |
| a     | 0.00    | 0.00    | 67.5          | 90.0          | 0.00        | 2.00       |
| b     | 0.00    | -1.8    | 64.8          | 85.5          | 0.00        | -0.98      |
| c     | 21.5    | 0.00    | 10.8          | 30.3          | -18.00      | -0.02      |
| d     | -21.5   | 0.00    | 15.6          | 40.1          | 18.00       | -0.02      |
| e     | 0.00    | 34.0    | 20.5          | 24.4          | 0.00        | 0.01       |
| f     | 0.00    | 9.8     | 4.5           | 4.5           | 0.00        | 0.01       |
| g     | 0.00    | -9.8    | 4.5           | 4.5           | 0.00        | 0.01       |
| h     | 7.8     | -59.2   | 4.5           | 2.2           | 0.00        | 0.01       |
| i     | 0.00    | -59.2   | 2.2           | 2.2           | 0.00        | 0.01       |
| j     | 5.8     | -59.2   | 2.2           | 4.5           | 0.00        | 0.01       |
4.4.2 The effect of radiation type

In this experiment we compare the reconstructions obtained from simulated data for X-rays, protons with and without scattering, \(\alpha\)-particles and carbon ions. The scattering distances of the particle beams \((2 Y_{\text{rms}})\) are 12.6, 6.2 and 3.5 mm for the proton, \(\alpha\)-particles and carbon ions, respectively. The initial relative velocity, \(\beta\) of the particles, needed to penetrate the 255 mm phantom and water are 0.57, 0.57 and 0.95 for the mentioned particles.

Table II presents a statistical comparison of the test picture and its reconstructions, using the parameters defined in section 4.2.4 and Appendix E.

<table>
<thead>
<tr>
<th>Test Picture</th>
<th>Average</th>
<th>Difference</th>
<th>R. Error</th>
<th>STD.Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>X-ray</td>
<td>0.325</td>
<td>0.163</td>
<td>0.150</td>
<td>0.518</td>
</tr>
<tr>
<td>proton (without scattering)</td>
<td>0.326</td>
<td>0.161</td>
<td>0.168</td>
<td>0.522</td>
</tr>
<tr>
<td>proton (with scattering)</td>
<td>0.326</td>
<td>0.251</td>
<td>0.122</td>
<td>0.487</td>
</tr>
<tr>
<td>(\alpha)-particle</td>
<td>0.326</td>
<td>0.135</td>
<td>0.070</td>
<td>0.502</td>
</tr>
<tr>
<td>carbon ions</td>
<td>0.326</td>
<td>0.113</td>
<td>0.102</td>
<td>0.512</td>
</tr>
</tbody>
</table>

The table demonstrates the improvement in the difference between the test picture and its reconstructions with increasing the particle charge-to-mass ratio. It also shows how the reconstructed densities with scattered protons (relatively large scattering) are relatively far from the test picture densities. However, the scattering problem brings about less relative error in the reconstructed densities because with such geometry the ray-sum will be averaged over a larger number of pixels. But it looks as if there is an optimum scattering width for
which the relative error is minimum. The Table shows little difference in the relative error for proton with 12.4 mm scattering width and carbon ions, with 3.2 mm scattering width. Best statistical results are obtained with α-particles (2Y = 6.2 mm).

The standard deviation of the mean of the densities from all reconstructions are very close to that of the phantom except for the larger scattering, protons beam.

The quantitative characteristics of the reconstructions are reflected qualitatively in the images obtained, Figs. 7 and 9. Figure 7 compares the phantom (a) with three reconstructions from X-rays(b), proton without scattering (c), and proton with scattering (d). The projected densities are normalised to those of water (ρ = 1.0). Little difference can be seen between X-ray and the proton without scattering pictures. Both pictures are affected by the circular artifact. The X-ray picture is generally lighter, because of the exponential nature of the projected densities calculation. Small tumours between the ventricles are better resolved with X-rays.

Circular artifacts are remarkably reduced in the proton-with-scattering picture due to the averaging process. The picture demonstrates the drawback of this process, where a "shadow" can be seen around the lower density objects in the phantom. The edges of these objects would have intermediate density values between those of the object and the surrounding material. This effect will increase with increasing the scattering width.

Figure (9) compares the performance of α-particles and carbon ions to those of proton with scattering, for the same phantom and similar conditions as in figure (7). As expected, alpha particles (9c) produced the best image compared to the others, with better smoothing and less "shadow" effect. It also resolves the small tumours in the phantom relatively clearly. The reconstruction for carbon ions (with less scattering( 9d), suffers, to a lesser extent, from the circular
artifacts seen in Fig.7(b,c) with comparative resolution but higher contrast between the small tumours and their surroundings.

Figure 8 is a quantitative representation of the original and reconstructed densities, along column 64 of the 128 x 128 matrix, obtained for X-ray and different particles with and without scattering. Small differences can be seen in the plots for X-ray (8a) and proton without scattering (8b). These differences are reduced still further when proton with scattering (8c), alpha particle (8d) and carbon ions (8e) are used. Figure (8f) presents the absolute values of the estimated densities (density x z²/β²) for different particles traversing the phantom.

4.4.3 The selection of scan range of angles

Figure 11 presents images obtained over the range of angles 180°-360° using X-ray and scattering beams of proton, α-particle and carbon ions. Comparison between these images and the corresponding images presented in figures 7 and 9, for 0°-180° scans, shows no difference for X-ray and insignificant difference for carbon ions. Clear differences can be observed in the case of proton and α-particle, where the three small objects at the bottom of the phantom are distinctly resolved and artifacts are shifted towards the lower half of the phantom which is covered by the scan (180°-360°). The differences are demonstrated in plots of the projected densities, Fig.10, obtained for all rays (183 rays) in the 20th projection which makes an angle with x-axis equal to 38 degrees in the (0°-180°) scan and 218 degrees in the (180°-360°) scan.

Discrepancies are observed in those rays which intersect different objects in the phantom, i.e. traverses inhomogeneous parts.
That could be seen between rays 90 to 140 in the 0°-180° scan and 44 to 94 in the 180°-360° scan. The discrepancies are small for the proton reconstruction and even smaller for alpha particle.

4.4.4 The effect of weighted ray-sums on the reconstructions

Figure 12, for proton and α-particle reconstructions, compares the estimated densities along column 64 of the matrix using weighted ray-sums suggested in section 4.3.2. Little differences can be appreciated between weighted and unweighted ray-sums. Nevertheless, reconstructions from weighted ray-sums were able to show the variation of densities across the phantom relatively clearly. That is because of the lower weight given for those sub-rays which are at a distance from the central sub-ray and consequently their reduced effect on the averaging process for the calculation of ray-sums. This is demonstrated in the images obtained for proton with-scattering and weighted ray-sums, Fig.13, where the picture is generally closer to that of proton without-scattering, i.e. less smoothing and more circular artifacts. But it shows the small object within the phantom rather clearly.

Although it looks as if there is little improvement to be gained from taking into consideration weighted ray-sums, it should be included in any simulation with scattered particle beams because it is a true representation of the actual distribution of particles in a ray.
Fig. 7 A head phantom (a) and three reconstructions using X-ray (b), proton without scattering (c), and proton with scattering (d). The projected densities are normalised to water.
Fig. (8b): Proton, without scattering, 0-180° scan, Column = 64

Fig. (8a): X-ray, 90 projections, 0-180° scan, Column = 64
Fig. (8d): Alpha particle, with scattering, 0-180° scan, Column = 64

Fig. (8c): Proton, with scattering, 0-180° scan, Column = 64
Fig.(8f) Comparison between reconstructions for different particles

Fig.(8e): Carbon ions, with scattering, 0-180° scan, Column = 64
Fig. 9  Comparison of the phantom (a) with reconstructions using scattering beams of proton (b), alpha particles (c), and carbon ions (d). Every ray is divided into 5 equally weighted sub-rays.
Fig. 10(a) Proton, Projection No. 20, with scattering

Ray Number
(0°-180°) scan

Fig. 10(b) Alpha-particle, Projection No. 20, with scattering

Ray Number
(0°-180°) scan
Fig. 11 Reconstructions of the head phantom over the range of angles 180-360°, (a) X-ray, (b) protons, $Y_{rms} = 6.25$ mm, (c) Alpha particles, $Y_{rms} = 3.1$ mm, and (d) Carbon ions, $Y_{rms} = 1.75$ mm. The total thickness penetrated (water + phantom) is equal to 255 mm.
Fig. 12(b) Alpha particle, weighted and unweighted ray-sums
Fig. 13 Shows reconstructions obtained for proton with scattering, (a) the head phantom, (b) every sub-ray-sum is multiplied by a weighting factor, the value of which is proportional to the number of particles in the sub-ray, (c) the weighting factor is neglected
Conclusion

There is a strong theoretical and experimental evidence that the application of charged particles to computed tomography provides a powerful new approach for diagnostic radiology, particularly for the differentiation of low contrast objects such as brain tumours.

The absorption coefficient of diagnostic X-rays is dependent upon the atomic structure of the material and the X-ray photons energies. The relative stopping power of charged particles exhibits insignificant dependence on the atomic composition of the material or on the energy of the beam. Accordingly, the signal obtained from particle tomography views different characteristics of the same tissue.

We have demonstrated that the noise-to-signal value per unit of dose to the patient, provided by this new modality, is relatively low compared to that gained from diagnostic X-rays. The reduction in noise is due to the interaction mechanism of the particles and the relatively high transmitted intensity of the incident beam.

Certain approximations have been made in the calculation of stopping powers and ranges of particles passing through an absorber. The accuracy of such measures will depend on the significance and validity of the approximations. For instance, Bragg's additivity rule for compounds does not take into consideration the type of chemical bonding and phase effects on the calculated stopping power of the compound. In view of the conflicting results, obtained by different authors, on deviations from measured stopping powers further physio-chemical work is needed to explore the rule of these chemical parameters on the atomic wavefunction.

Another advantage of charged particle beams over X-rays is their lack of beam hardening artifacts associated with X-ray CT. However,
there are certain disadvantages of particles for diagnostic radiology in general and computed tomography in particular. Clinical accelerators which would produce particles of adequate energy are expensive and more complicated than X-ray sources. Multiple Coulomb scattering limits the spatial resolution of proton scans. The use of heavy ions would reduce the effect of this problem. A compromise is to be made between the dose advantage and lower cost accelerators on the one hand (proton) and better spatial resolution and less range-straggling on the other (heavy ions).

Our correction factor to the standard Rossi formula, for the calculation of root mean square scattering angle of particles, would narrow the gap between practical measurements and theoretical calculations of the spread of charged particle beams. By employing such a factor, better estimation to the spatial resolution could be obtained especially for the proton and when energy loss of particles is taken into consideration.

Because the energy-loss mechanism of the particles lacks the dependence on the material atomic number the use of contrast media (high Z materials), which have an important diagnostic potential, is excluded.

In the simulation and reconstruction programme adapted for this study, images obtained using charged particles are generally comparable to those of X-ray. The modifications introduced to include the particle scattering are generally effective. However, particles should not be credited for the smoother images obtained when they are employed in SNARK. It is the averaging technique we introduced, which brings about a reduction in the circular artifacts in the reconstruction. The technique tends to divide the densities in a ray between a larger number of pixels, and this similarly applies to reconstruction artifacts.

In the case of proton (larger scattering), better approximations could be obtained by reconstructing along every sub-ray in the ray
instead of assigning the average of the sub-ray-sums to the ray including them. That could be done (in SNARK) by taking all parallel sub-rays, with angle $(\theta + \phi_N)$, in a certain projection, with angle $\theta$, to create a new set of projections. The projection angles of the new projections are given by $\theta + \phi_N$, where $N$ is the number of sub-rays (specified by the user) in a ray. Although this would need larger storage and longer processing time it still can be used with algorithms which rely on the exact solution of the integral associated with a ray convolution algorithm.

Another suggestion for future work is to include the influence of energy-straggling on the reconstructed densities. The routines already existing in SNARK for the calculation of the effect of photon statistics on a ray-sum could be implemented to accommodate the approximately Gaussian distribution of the particle ranges.

Furthermore, there is a possibility that the spatial resolution would improve if an algorithm that uses the curved characteristic of the particle trajectories is employed.
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C TO CALCULATE THE PROTON RANGE IN A TISSUE EQUIVALENT MATERIAL USING
C THE LOGARITHMIC & SUMMATION FORMULA
C D= MATERIAL DENSITY, Z= MATERIAL ATOMIC NUMBER, W= MATERIAL MASS NUMBER
C E= AVERAGE IONIZATION AND excitation POTENTIAL
C \( \phi = \alpha(M, N) \), A factor calculated and presented in tables (Barkas, 1964).

DIMENSION D(11), Z(11), W(11), E(11), GAM(11), KT(I15), P(16), R(11, 115),
15(I1, 115)
DO 5 K=1, 11
READ(1, 100)D(K), Z(K), W(K), E(K), GAM(K)
WRITE(2, 200)D(K), Z(K), W(K), E(K), GAM(K)
5 CONTINUE

WRITE(2, 205)
READ(1, 220)(P(M), M=1, 16)
WRITE(2, 230)(P(M), M=1, 16)
READ(1, 330)(KT(I), I=1, 115)
DO 10 K=1, 11
DO 20 I=1, 115
F=ALOGI(E(K))
B=ALOGI(W(K))
REAL(KT(I))
C=ALOGI(D(K))
R(K, I)=S(K, I)
10 CONTINUE
20 CONTINUE
WRITE(2, 500)
READ(1, 450)
WRITE(2, 460)
DO 30 El=1, 15
RT=GM(1)*R(1, 1)+GM(2)*R(2, 1)+GM(3)*R(3, 1)+GM(4)*R(4, 1)+GM(5)*R(5, 1)+GM(6)*R(6, 1)+GM(7)*R(7, 1)+GM(8)*R(8, 1)+GM(9)*R(9, 1)+GM(10)*R(10, 1)+GM(11)*R(11, 1)
30 CONTINUE

C THIS IS AN INPUT DATA SAMPLE

\begin{verbatim}
D  Z  W  E  Gam
0.00008  1.0  1.079  19.0  0.1
2.25  6.0  12.011  79.0  0.23
6.00133  8.0  15.99  103.0  0.61
0.00116  7.0  14.0067  91.0  0.026
1.55  20.0  40.08  228.5  0.014
0.00294  17.0  35.463  202.2  0.0012
0.86  19.0  30.102  219.0  0.002
1.735  120.0  30.312  124.12  0.00027
0.969  11.0  22.98  114.36  0.0014
1.52  15.0  30.97  181.5  0.011
1.953  16.0  32.068  141.0  0.002
\end{verbatim}

\( \alpha_{MN} = -0.0155  \quad 1.8371  \quad -0.005823  \quad -0.005989  \quad 0.36916  \)
-0.01552  -0.00025  -0.0053215  -0.01937  -0.30102
0.0017303  -0.00033802  0.0034718  0.0023603  -0.00006838

\( \kappa = 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad 22 \quad 24 \quad 26 \quad 28 \quad 30 \quad 32 \quad 34 \quad 36 \)
38 40 42 44 46 48 50 52 55 60 65 70 75 80 85 90 95 100 105
110 115 120 125 130 135 140 145 150 155 160 165 170 175 180 185 190 195
290 295 300 305 310 315 320 325 330 335 340 345 350 355 360 365 370 375
380 385 390 395 400 405 410 415 420 425 430 435 440 450 455 460 465
470 475 480 485 490 495 500

Appendix A

Programmes for the calculation of stopping power and range of heavy charged particles
in elements and compounds
C TO CALCULATE THE RANGE OF PROTON IN A TISSUE EQUIVALENT MATERIAL

C AT CERTAIN ENERGIES USING THE ANALYTICAL METHOD (SIMPSON RULE)

C G=2*ELECTRON MASS*0.625*E**2
C F=3.14*(Electron Charge**4)*2*(C)*N(A)/M(E)*(E**2)*0.62566
C ELECTRON Charge = 4.8 E-10 e.s.u.
C H= THE REST MASS OF THE PROTON(MEV)
C D=MATERIAL DENSITY, Z=ATOMIC NUMBER, W=MATERIAL MASS NUMBER.
C E=AVERAGE IONIZATION AND EXCITATION POTENTIAL, R0=INITIAL PROTON RANGE

DIMENSION D(11),Z(11),W(11),E(11),GAM1(11),R0(11),YJ(10),X(50),XL(1150),YH(400),YH(700),R1(11),R2(11),R3(11),R4(11),R5(11),R10(11),R510(11),R8400(11),R7000(11)

H=1.0238E6
P0=3.0656
H=938.256

READ(5,70)IB
IF(IB.EQ.4)GO TO 2
WRITE(6,90)IB
GO TO 1000

DO 5 K=1,IB
READ(5,1000)D,K,Z,K,W,K,E,K,GAM,K,R0(K)
WRITE(6,2000)(K,D(K),Z(K),W(K),E(K),GAM(K),R0(K))
5 CONTINUE

DO 6 K=1,4
DO 10 J=2,10
IF (J=I) **2
QJ=(FI-H)**2
AQ=GFJ
Y(J,1)=0.0/(((PZ(K)**FI)/(W(K)**QJ))*(ALOG(G**QJ/(E(K)**H**2))-(QJ/FJ))
10 CONTINUE

DO 20 J=12,48,2
FJ=(J+H)**2
QJ=(FI-H)**2
AQ=GFJ
Y(J,1)=0.0/(((PZ(K)**FI)/(W(K)**QJ))*(ALOG(G**QJ/(E(K)**H**2))-(QJ/FJ))
20 CONTINUE

DO 30 L=50,150,5
FL=(L+H)**2
QL=(FL-H)**2
AL=GFQ
YL(L,1)=0.0/(((PZ(K)**FL)/(W(K)**QL))*(ALOG(G**QL/(E(K)**H**2))-(QL/FL))
30 CONTINUE

DO 40 N=160,400,10
F=+(N+H)**2
QW=(PZ(H)**2)
AQ=GFQ
YW(N)=1.0/(((PZ(K)**F)/(W(K)**QW))*(ALOG(G**QW/(E(K)**H**2))-(QW/FN))
40 CONTINUE

DO 50 M=20,700,20
F=+(M+H)**2
QW=(PZ(H)**2)
AQ=GFQ
YW(M)=1.0/(((PZ(K)**F)/(W(K)**QW))*(ALOG(G**QW/(E(K)**H**2))-(QW/FN))
50 CONTINUE

DELTX1=1.0
DELTX2=2.0
DELTX3=5.0
DELTX4=10.0
DELTX5=20.0

R1(K)=(DELTX/5.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R2(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R3(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R4(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R5(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R6(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R7(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R8(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R9(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R10(K)=(DELTX/3.0)**(YJ(2)+4*YJ(3)+2*YJ(4)+4*YJ(5)+2*YJ(6)+4*YJ(7)+
12*YJ(8)+4*YJ(9)+YJ(10))
R400TO = R400TO + GAM(K) / R400(K)
R700TO = R700TO + GAM(K) / R700(K)

999 CONTINUE

R10TO = 1.0 / R10TO
R50TO = 1.0 / R50TO
R150TO = 1.0 / R150TO
R400TO = 1.0 / R400TO
R700TO = 1.0 / R700TO

WRITE(6,220)
DO 7 K=1,11
WRITE(6,250)R1(K),R2(K),R3(K),R4(K),R5(K)
7 CONTINUE
WRITE(6,300)
WRITE(6,400)
DO 8 K=1,11
WRITE(6,500)R10(K),R50(K),R150(K),R400(K),R700(K)
8 CONTINUE
WRITE(6,600)
WRITE(6,700)
WRITE(6,800)
WRITE(6,900) R10TO,R50TO,R150TO,R400TO,R700TO

70 FORMAT(I4)
90 FORMAT(1X,IR,'IS LARGE')
100 FORMAT(2X,4F9.5,2X,F8.6,2X,F8.5)
200 FORMAT(2X,'D(K)=','F9.5,2X,'Z(K)=','F9.5,2X,'W(K)=','F9.5,2X,'E(K)=')
1F9.5,2X,'GAM(K)=','F8.6,2X,'R0(K)=','F8.5,1F')
220 FORMAT(3X,'R(B/K)',12X,'R2(K)',13X,'R3(K)',12X,'R4(K)',13X,'R5(K)')
250 FORMAT(F12.5,6X,F12.5,6X,F12.5,6X,F12.5,6X,F12.5)
300 FORMAT(3X,'R10(K)',12X,'R50(K)',13X,'R150(K)',12X,'R400(K)',13X,'R
1700(K)')
400 FORMAT(2X,'GH/CH2',9X,'GH/CH2',9X,'GH/CH2',10X,'GH/CH2',10X,'GH/CH
127')
500 FORMAT(F12.5,6X,F12.5,6X,F12.5,6X,F12.5,6X,F12.5)
600 FORMAT(1X,'PROTON RANGE IN TISSUE (GH/
1CM 2)')
700 FORMAT(2X,'TOTAL R AT 10 MEV',7X,'TOTAL R AT 50 MEV',7X,'TOTAL R AT
150 MEV',7X,'TOTAL R AT 400 MEV',7X,'TOTAL R AT 700 MEV',7X)
800 FORMAT(8X,'GH/CH2',15X,'GH/CH2',19X,'GH/CH2',19X,'GH/CH2',19X,'GH/
1CM2')
900 FORMAT(4X,F12.5,14X,F12.5,14X,F12.5,14X,F12.5,14X,F12.5)
1000 STOP
END
TO CALCULATE THE RANGE OF PROTONS IN ANY MATERIAL USING THE ELECTRON Mass (E-2=700 MeV/amu)

\[
G_2 = M_2 \times (\text{Electron Mass}) \times 0.625 \times C#^2
\]

\[
P = M_2 \times (\text{Electron Charge} \times \text{Electron Mass}) \times (\text{C}C^2) \times 0.625 \times E
\]

\[
M_2 = \text{THE MAX RANGE OF THE PROTON (MeV)}
\]

\[
\text{Material Density} = \text{Z} \times \text{ATOMIC MASS NUMBER}
\]

\[
E = \text{AVERAGE IONIZATION AND EXCITATION POTENTIAL}, \text{R} = \text{INITIAL PROTON RANGE}
\]

\[
\text{DIMENSION D(1),Z(1),W(1),E(1),GAM(1),RO(1),Y1(10),TJ(50),YL(150),Y1N(400),YN(700),R1(1),R2(1),R3(1),R4(1),R5(1),R6(1),R7(1),R8(1)}
\]

\[
\text{READ(1,70)IB}
\]

\[
\text{IF(MEQ.1000 TO 2)}
\]

\[
\text{WRITE(2,90)IB}
\]

\[
\text{GO TO 1000}
\]

\[
2 \text{ DO 5 K=1,IB}
\]

\[
\text{READ(1,100)D(K),Z(K),W(K),E(K),GAM(K),RO(K)}
\]

\[
\text{WRITE(2,200)D(K),Z(K),W(K),E(K),GAM(K),RO(K)}
\]

\[
\text{DO 10 I=2,10}
\]

\[
\text{FI(1)} = 1.0
\]

\[
\text{IF(K(1)EQ.12)}
\]

\[
\text{IF((K(1)EQ.12) = (FI(1)EQ.12))}
\]

\[
10 \text{ CONTINUE}
\]

\[
\text{DO 20 J=12,50,2}
\]

\[
\text{FJ(J)} = 1.0
\]

\[
\text{IF(K(J)EQ.12)}
\]

\[
\text{IF((K(J)EQ.12) = (FJ(J)EQ.12))}
\]

\[
20 \text{ CONTINUE}
\]

\[
\text{DO 30 L=55,150,5}
\]

\[
\text{FL(L)} = 1.0
\]

\[
\text{IF(K(L)EQ.12)}
\]

\[
\text{IF((K(L)EQ.12) = (FL(L)EQ.12))}
\]

\[
30 \text{ CONTINUE}
\]

\[
\text{DO 40 N=160,400,10}
\]

\[
\text{FH(N)} = 1.0
\]

\[
\text{IF(K(N)EQ.12)}
\]

\[
\text{IF((K(N)EQ.12) = (FH(N)EQ.12))}
\]

\[
40 \text{ CONTINUE}
\]

\[
\text{DO 50 M=420,700,20}
\]

\[
\text{FM(M)} = 1.0
\]

\[
\text{IF(K(M)EQ.12)}
\]

\[
\text{IF((K(M)EQ.12) = (FM(M)EQ.12))}
\]

\[
50 \text{ CONTINUE}
\]

\[
\text{DELTX1=1.0}
\]

\[
\text{DELTX2=2.0}
\]

\[
\text{DELTX3=5.0}
\]

\[
\text{DELTX4=10.0}
\]

\[
\text{DELTX5=20.0}
\]

\[
R1(K)=(DELTX1/3.0)*(Y1(K)+EY1(K)+4*Y4(K)+2*Y5(K)+4*Y6(K)+4*Y7(K)+14*Y8(K)+2*Y20(K))
\]

\[
R2(K)=(DELTX2/3.0)*(Y1(K)+EY1(K)+4*Y4(K)+2*Y5(K)+4*Y6(K)+4*Y7(K)+14*Y8(K)+2*Y20(K))
\]

\[
R3(K)=(DELTX3/3.0)*(Y1(K)+EY1(K)+4*Y4(K)+2*Y5(K)+4*Y6(K)+4*Y7(K)+14*Y8(K)+2*Y20(K))
\]

\[
R(K)=(DELTX4/3.0)*(Y1(K)+EY1(K)+4*Y4(K)+2*Y5(K)+4*Y6(K)+4*Y7(K)+14*Y8(K)+2*Y20(K))
\]

\[
R(K)=(DELTX5/3.0)*(Y1(K)+EY1(K)+4*Y4(K)+2*Y5(K)+4*Y6(K)+4*Y7(K)+14*Y8(K)+2*Y20(K))
\]

\[
R1(K)=R0(K)+R(K)
\]

\[
R5(K)=R0(K)+R(K)
\]

\[
R7(K)=R0(K)+R(K)
\]

\[
5 \text{ CONTINUE}
\]

\[
\text{WRITE(2,300)}
\]

\[
\text{WRITE(2,400)}
\]

\[
\text{DO 7 K=1,1}
\]

\[
\text{WRITE(2,500)R10(K),R50(K),R150(K),R400(K),R700(K)}
\]

\[
7 \text{ CONTINUE}
\]

\[
1000 \text{ STOP}
\]

\[
70 \text{ FORMAT(98)}
\]

\[
90 \text{ FORMAT(1X,'IS LARGE')}
\]

\[
100 \text{ FORMAT(2X,'A')}
\]

\[
200 \text{ FORMAT(2X,'A')}
\]

\[
300 \text{ FORMAT(2X,'A')}
\]

\[
400 \text{ FORMAT(2X,'A')}
\]

\[
500 \text{ FORMAT(2X,'A')}
\]

\[
\text{END}
\]
TO CALCULATE THE STOPPING POWER OF PROTON IN ANY COMPOUND, USING BRAgg ADDITIVITY RULE.

DIMENSION Z(11), A(1), H(1), G(1), EN(97), BET(97),
1 F(97), C(11), STOP(11,97), SP(11,97)

READ(5,70) N
WRITE(6,90) N
DO 20 J=1,N
READ(5,100) Z(J), A(J), H(J), G(J)
WRITE(6,200) Z(J), A(J), H(J), G(J)
20 CONTINUE

WRITE(6,400)
WRITE(6,450)
READ(5,290,END=299) (EN(I),I=1,97)
READ(5,300) (EN(I),I=1,97)

299 WRITE(6,298)
298 FORMAT(' END OF FILE ')
CH=1.0
DO 30 J=1,N
DO 10 I=1,97
C(J) = Z(J)/A(J)
BET(I) = SORT( 1.0 - (931.478*(931.478*EN(I)))**2)
F(I) = ALOG(10.255*BET(I))**2
F(I) = F(I) - ALOG(1.0-BET(I))**2
F(I) = F(I) - BET(I)**2
STOP(J,1) = C(J) * (CH**2) * (F(I) - ALOG(H(J))) / (BET(I)**2)
STOP(J,1) = STOP(J,1) * 0.030706
SP(J,1) = STOP(J,1)
10 CONTINUE
30 CONTINUE

DO 25 I=1,97
WRITE(6,500) (EN(I), SP(1,I), SP(2,I), SP(3,I), SP(4,I), SP(5,I),
1 SP(6,I), SP(7,I), SP(8,I), SP(9,I), SP(10,I), SP(11,I))
2 SPC
25 CONTINUE

70 FORMAT(14)
90 FORMAT(1X,'THE NO. OF ELEMENTS IN THIS COMPOUND IS = ',I4,1X)
100 FORMAT(2X,3F9.4,2X,8.6)
200 FORMAT(2X,'(J) =',F9.4,2X,'A(J) =',F9.4,2X,'H(J) =',F9.4,2X,'G(J) =',F8.6)
290 FORMAT(6F12.4)
300 FORMAT(6F12.4,1X)
3 6F12.4,/ 
4 6F12.4,/ 
5 6F12.4,/ 
6 6F12.4,/
C TO CALCULATE THE STOPPING POWER OF PROTON IN A TISSUE EQUIVALENT MEDIUM USING THE BETHE-BLOCH FORMULA.

C GELK = 2*M(Electron Mass) * 0.625E12
C PK=PK=3.14*(Electron Charge**2)*Z(C)**2*(K(A)**2)/ME
C B IS A FACTOR TO CONVERT FROM ERG/CM TO HEV/CM
C D=MATERIAL DENSITY, Z = MATERIAL ATOMIC NUMBER, W=MATERIAL MASS NUMBER
C E=AVERAGE IONIZATION AND EXCITATION POTENTIAL

DIMENSION (11),Z(11),E(11),GAM(11),HKEP(70),SP(11,70),EN(11,70)

GELK=1.1375E-15
PK=4.4144E14
C=3.0E10
B=0.625E6
IF(N.LE.11) GO TO 5
WRITE(2,90)N
GO TO 1000

5 DO 20 K=1,N
READ(1,100)(Z(K),W(K),E(K),GAM(K))
WRITE(2,200)(Z(K),W(K),E(K),GAM(K))
20 CONTINUE

WRITE(2,400)
WRITE(2,450)
WRITE(2,460)

READ(1,300)(HKEP(I),I=1,70)

DO 30 K=1,N
DO 10 I=1,70
BETA4=SQR((1-(938./HKEP(I)))**2)
V=BETA4*C
HNUM=GELK*(V**2)
DENM=E(K)**4*(BETA4**2)
EN(K,1)=(PK*K(Z/K)**2)/(W(K)**(V**2))*(ALOG(HNUM/DENM))-(BETA4**2)
SP(K,1)=EN(K,1)**B/D(K)
10 CONTINUE

30 CONTINUE

DO 25 I=1,70
SPC=GAM(1)**SP(1,1)+GAM(2)**SP(2,1)+GAM(3)**SP(3,1)+GAM(4)**SP(4,1)+GA
M(5)**SP(5,1)+GAM(6)**SP(6,2)+GAM(7)**SP(7,1)+GAM(8)**SP(8,1)+GAM(9)**SP
(9,1)+GAM(10)**SP(10,1)+GAM(11)**SP(11,1)
WRITE(2,500)(HKEP(I),SP(1,1),SP(2,1),SP(3,1),SP(4,1),SP(5,1),SP(6,1),SP(7
,1),SP(8,1),SP(9,1),SP(10,1),SP(11,1),SPC)
25 CONTINUE

100 STOP

70 FORMAT(14)
90 FORMAT(12,14,'IS LARGE')
100 FORMAT(2X,F9.5,2X,F8.6)
200 FORMAT(2X,'D(K)=',F9.5,2X,'Z(K)=',F9.5,2X,'W(K)=',F9.5,2X,'E(K)=',
1F9.5,2X,'GAM(K)=',F8.6)
300 FORMAT(12F6.1,/)  
1 12F6.1,/
1 12F6.1,/
1 12F6.1,/
1 12F6.1,/
1 12F6.1,/
1 12F6.1,/
1 10F6.1)

400 FORMAT(H1,1X,'K(EV/HEV)'3X,'(MATERIAL STOPPING POWER)',5X,'(PROTONS)',10X,'(COMPOUND-STOP-POWER)')
450 FORMAT(20X,'(HEV/CM)**2),(0X,'(HEV/CM)**2),8X,'(HEV/CM)**2),6X,'(HEV/CM)**2)
11,12X,'(HEV/CM)**2)'1,)
1,3X,'Z=19',5X,'Z=11',4X,'Z=15',4X,'Z=16',1,)
460 FORMAT(1X,'Z=0',6X,'Z=06',3X,'Z=08',4X,'Z=07',3X,'Z=20',5X,'Z=17
500 FORMAT(1X,F6.1,3X,11(F7.3,1X),4X,F9.4)

END

C .... INPUT DATA SAMPLE ......

11 D Z W E GamK.E
0.00008 1.0 1.0079 19.0 0.1
2.25 6.0 12.011 79.0 0.23
0.00133 8.0 15.99 103.0 0.61
0.00116 7.0 14.0057 91.0 0.026
1.55 20.0 40.08 228.5 0.014
0.00294 17.0 35.953 200.2 0.0012
0.86 19.0 39.102 219.0 0.002
1.735 12.0 34.312 124.12 0.00027
0.959 11.0 22.98 114.36 0.0014
1.82 15.0 30.97 181.5 0.011
1.953 16.0 32.064 191.0 0.002
12.0 4.0 6.0 10.0 14.0 18.0 22.0 26.0 30.0 34.0 38.0
42.0 46.0 50.0 60.0 70.0 80.0 90.0 100.0 110.0 120.0 130.0 140.0
150.0 160.0 170.0 180.0 190.0 200.0 220.0 240.0 260.0 280.0 300.0 320.0
820. 840. 860. 880. 900. 920. 940. 960. 980. 1000. 1020. 1040.
CALCULATION OF STOPPING POWER FOR PROTONS AND DIFFERENT HEAVY IONS IN VARIOUS ELEMENTS AT VARIOUS PROJECTILE ENERGIES.

DIMENSION Z(6), A(6), HI(6), CH(7), EN(89), F(89), BET(89),
1C(6), STOP(6,7,89)

Z, A, HI ARE THE TARGET ATOMIC, MASS NUMBERS, AND THE TARGET ATOMS AVERAGE.

EN IS THE PROJECTILE KINETIC ENERGY (MEV/A.M.U).

CH IS THE PROJECTILE ELECTRONIC CHARGE.

DO 1 J=1,6
READ(5,100) Z(J), A(J), HI(J)
1 CONTINUE
READ(5,200) (CH(K), K=1,7)
READ(5,300) (EN(I), I=1,89)

FOR ELEMENT J,

DO 10 J=1,6
C(J) = Z(J) / A(J)
10 CONTINUE

INTERACTING WITH ION K,

DO 20 K=1,7
C AT ION ENERGY,

DO 30 I=1,89
C PARTICLE RELATIVE VELOCITY BET(I),

BET(I) = SQRT( 1.0 - (931.478/931.478*EN(I)))**2
F(I) = ALOG(10.225*BET(I))**2
F(I) = F(I) - ALOG((1.0-BET(I))**2)
F(I) = F(I) - BET(I)**2

CH(K) = CH(K)*CH(K)
STOP(J,K,I) = C(J)*CH(K)/(F(I)-ALOG(HI(J)))/BET(I)**2
STOP(J,K,I) = STOP(J,K,I) * 0.030706
30 CONTINUE
20 CONTINUE
10 CONTINUE

WRITING THE RESULTS .........

DO 40 K=1,7
WRITE(6,160) CH(K)
WRITE(6,170)
WRITE(6,180) (Z(J), A(J), HI(J), J=1,6)
DO 60 I=1,89
WRITE(6,190) EN(I), STOP(1,K,I), STOP(2,K,I), STOP(3,K,I), STOP(4,K,I), STOP(5,K,I), STOP(6,K,I)
40 CONTINUE
Appendix B  The total reaction cross-section parameters

The total reaction cross-section per target nuclei in mb calculated by Karol (1975) is given by

\[ \sigma_t(E) = 10\pi(a_T^2 + a_p^2)[E_1(x) + \ln x + 0.557] \]  \hspace{1cm} (B.1)

where \( a_{T,p} = (1.596 A_T, p + 0.89)^{1/3} \text{fm} \), \( E_1(x) \) is a function of negligible magnitude and \( A_T, p \) is the target or projectile mass number. \( x \) is an energy dependent function defined by

\[ x = \frac{\pi^2 \bar{o}(E) \rho_T(0) \rho_p(0) a_T^3 a_p^3}{10(a_T^2 + a_p^2)} \]  \hspace{1cm} (B.2)

and

\[ \bar{o}(E), \text{ the average nucleon-nucleon collision cross-section at energy } E, \text{ defined by} \]

\[ \bar{o}(E) = \sigma_{ii}(E) \left[ \left( \frac{Z}{A} \right)_T \left( \frac{Z}{A} \right)_p + \left( \frac{N}{A} \right)_T \left( \frac{N}{A} \right)_p \right] + \sigma_{ij}(E) \left[ \left( \frac{Z}{A} \right)_T \left( \frac{N}{A} \right)_p + \left( \frac{Z}{A} \right)_p \left( \frac{N}{A} \right)_T \right] \]  \hspace{1cm} (B.3)

\( Z_{(T,P)}, N_{(T,P)} \) and \( A_{T,P} \) are the proton, neutron and mass numbers of the target (projectile) respectively; \( \sigma_{ii} \) is the proton-proton (neutron-neutron) total cross section; \( \sigma_{ij} \) is the proton-neutron cross section, all taken from REview of Particle Properties (1976), pp.53, at laboratory kinetic energy \( E = E_{p,T}/A_{p,T} \), where \( E_p \) is the projectile target lab energy.

The parameter \( \rho(0) \) is defined by

\[ \rho(0) = \frac{1}{2} \rho_o \exp(c/a)^2 \]  \hspace{1cm} (B.4)

\[ \rho_o = \frac{3A}{4\pi c^3[1 + (2.936/c^2)]} \]  \hspace{1cm} (B.5)

and

\[ c = 1.07 \frac{A^{1/3}}{\text{fm}} \]  \hspace{1cm} (B.6)
The right hand side of equation (2.36) is related to that of equation (2.37) as can be seen by the following algebraic analysis:

\[ 10\pi (a_T^2 + a_p^2) = 10\pi [(1.596A_p^{1/3} + 0.89) + (1.596A_T^{1/3} + 0.89)] \]

\[ = 10\pi [1.596(A_p^{1/3} + A_T^{1/3} + 1.115)] \]

\[ = 15.96\pi [A_p^{1/3} + A_T^{1/3} + 1.115], \]

\[ = 16\pi [A_p^{1/3} + A_T^{1/3} + 1.115]. \]
Appendix C  The scattering angle of charged particles relative to the proton

The root mean square lateral displacement of charged particles other than the proton, traversing the same medium, can be obtained from the following derivation.

From equation (2.80), the total angle of scattering for particle $i$, $\theta_i^{(\text{rms})}$, relative to that of the proton is given by

$$\frac{\theta_i^{(\text{rms})}}{\theta_p^{(\text{rms})}} = \frac{z_i}{z_p} \frac{(\beta P C)_i}{(\beta P C)_p} \frac{(1+E)_i}{(1+E)_p}$$

(C.1)

$$\frac{(\beta P C)_p}{(E_p^2 - M_p^2)/E_p}$$

(C.2)

where $E_p$ is the total energy of the particle (MeV) and given by $E_p = \text{kinetic energy } (T_p) + \text{Rest mass } (M_p)$.

From (C.2) in (C.1), we get

$$\theta_i = \theta_p \frac{z_i}{z_p} \frac{(T_p + 2M_p)}{(T_i + 2M_i)} \frac{T_i + M_i}{T_p + M_p} \frac{T_p}{T_i} \frac{(1+E)_i}{(1+E)_p}$$

(C.3)

The kinetic energy of the particle $T_i$ is to be obtained from range-energy tables, using equation (2.27) and Table VII, in the following procedure

(i) Find the proton range corresponding to the particle range at energy $T_i$, using (2.27),

(ii) Using range-energy tables (Barkas 1964) for protons in the medium, to find the proton kinetic energy $R_p$,

(iii) This kinetic energy will be equivalent to $T_i \ast (M_p/M_i)$, from which $T_i$ can be obtained.
Appendix D  Radiation fluence and density resolution

Fluence and density Resolution

From equation (3.3) the mass uncertainty for an X-ray beam is given by

$$\delta^2(t)_X = \left. \frac{p_x}{n_0(x)} \right| \frac{dp_x}{dt} \right|^2$$  \hspace{1cm} (D.1)

From Beer's law for X-ray attenuation,

$$p_x = \frac{n_1(x)}{n_0(x)} = e^{-\mu t} \quad \mu = \text{linear attenuation coefficient}$$

and

$$\frac{dp_x}{dt} = -\mu e^{-\mu t} = -\mu p_x$$  \hspace{1cm} (D.2)

From equation (D.2) in (D.1) we obtain

$$\delta^2(t)_X = \left. \frac{p_x}{n_0(x)} \right| \frac{dp_x}{dt} \right|^2 = \frac{1}{n_0(x)} \frac{1}{\mu^2}$$  \hspace{1cm} (D.3)

For charged particles, equation (3.3) reads

$$\delta^2(t)_P = \left. \frac{p_p}{n_0(P)} \right| \frac{dp_p}{dt} \right|^2$$  \hspace{1cm} (D.4)

The beam distribution can be obtained from the gaussian range distribution formula (2.31), where

$$\frac{dp_p}{dt} = (2\pi \sigma_R^2)^{-\frac{3}{2}} \text{EXP} \left[ -\frac{(t-R)^2}{2\sigma_R^2} \right]$$  \hspace{1cm} (D.5)

where $\sigma_R$ is the range straggling parameter.

By substituting from equation (D.5) in (D.4)

$$\delta^2(t)_P = \frac{p_p}{n_0(P)} \cdot 2\pi \sigma_R^2 \text{EXP} \left[ \frac{2(t-R)^2}{2\sigma_R^2} \right]$$  \hspace{1cm} (D.6)

if $\delta^2(t)_P = \delta^2(t)_X$.
\[
\frac{P_P}{N_o(P)} \cdot 2\pi \sigma_R^2 e^{\frac{2(t-R)^2}{2\sigma_R^2}} = \frac{1}{N_o(x)P_x\mu^2}
\]

Substituting \(N_p = N_o(P)P_p\), and \(N_x = N_o(x)P_x\), we get

\[
\frac{N_p}{N_x} = \frac{\frac{2(t-R)^2}{2\sigma_R^2 R}}{2\pi \sigma_R^2 \mu^2 e^{\frac{2(t-R)^2}{2\sigma_R^2 R}}}
\]

For \(\delta(t)\) to be minimum, the thickness traversed by the particles will be equal to the mean range \(R\), and the transmission ratio \(P_p = \frac{1}{2}\) (Chapter 2, section 4.2). Accordingly equation (D.7) becomes

\[
\frac{N_p}{N_x} = 2\pi (0.5)^2 (\sigma_R^2 \mu^2)
\]

\[
\frac{N_p}{N_x} = (1.25 \sigma_R^2 \mu^2)
\]
Appendix E  Measures for the interpretation of a test picture and its reconstruction

The following measures are defined to serve the comparison between a test picture and its reconstruction (Herman et al 1973). But first a set of parameters must be introduced. These are:

\[ \bar{\rho} = \text{the average density of the phantom} \]

\[ = \frac{1}{n^2} \sum_{i,j} \rho_{i,j} \]  \hspace{1cm} (E.1)

\[ \bar{\rho}^* = \text{the average density of the pixels of the reconstruction} \]

\[ = \frac{1}{n^2} \sum_{i,j} \rho^*_{i,j} \]  \hspace{1cm} (E.2)

where, \( n \) is the total number of elements within the picture,

\( \rho_{i,j} \) is the density in the \( i \)th pixel in the \( j \)th column of the test picture, and \( \rho^*_{i,j} \) is the density in the same pixel of the reconstructed picture.

The difference between the densities of the test picture and its reconstruction, \( d \) is given by

\[ d = \left[ \left\{ \sum_{i,j} \left( \rho^*_{i,j} - \rho_{i,j} \right)^2 \right\} \right]^{\frac{1}{2}} \]  \hspace{1cm} (E.3)

The standard deviation (ST.DEV) is defined as

\[ \sigma = \text{ST.DEV} = \frac{1}{n} \sum_{i,j} \left( \rho^*_{i,j} - \bar{\rho}^* \right) \]  \hspace{1cm} (E.4)

and the variance = \( \sigma^2 \).

The relative error, (R.ERR) is given by

\[ \text{R.ERR} = \frac{\sum_{i,j} \left| \rho^*_{i,j} - \rho_{i,j} \right|}{\sum_{i,j} \rho_{i,j}} \]  \hspace{1cm} (E.5)

And finally, the differences between projection data (picture ray sum) and reconstructed ray sum is given by

\[ \tilde{D} \]  \hspace{1cm} (E.6)

where, \( R_i \) is the projection (real) ray sum of the \( i \)th ray, and \( R^*_i \) is the reconstruction (pseudo) ray sum of the same ray.
Appendix F  The equation of a ray in the scattering geometry

Consider the geometry shown in the attached figure. The parameters needed are defined as:

- $W$ = the distance from the central ray entrance to the object origin = half the object thickness ($t/2$).
- $r$ = the distance from the entrance of any ray in the projection to the origin.
- $\theta$ = the projection angle = any ray angle in the projection (0°-360°)
- $\gamma$ = the angle that $r$ makes with the axis, it can be +ve or -ve.

The figure represents the central sub-rays in the scattering geometry for all rays with angle $\theta$. If we take the ray with entrance point $(x,y)$, then

\[
x = r \cos(\theta + \gamma) \quad \text{(F.1)}
\]
\[
y = r \sin(\theta + \gamma)
\]
\[
\tan \gamma = \frac{A}{W} \quad \text{(F.2)}
\]
\[
\gamma = \tan^{-1}(\frac{A}{W})
\]

The value of $(A)$ will depend on the position of the ray in the projection. Every ray will have an "order factor", $B$, and $A = \text{Ray Width} \cdot B$.

$B$ is calculated from the following formula

\[
B = NN - \left(\frac{NN+1}{2}\right) \quad \text{(F.3)}
\]

where $NN$ is the total number of rays in the projection, and $NN^*$ is the $n$th ray. The 30th ray in a projection containing 55 rays will have $B_{30} = 2$.

The parameter $r$ is given by

\[
r = (A^2 + W^2)^{\frac{1}{2}} \quad \text{(F.4)}
\]
The geometry for calculating \((x, y)\). The parallel lines are the central sub-rays of the projection with scattering taken into account.
From (F.4), (F.3), (F.2) in (F.1) one can calculate \((x,y)\) of any ray at the entrance. All these parameters are defined and introduced in SNARK in the POSIT and CREATR subroutines.