OPTICAL MODEL POTENTIAL FOR PION-NUCLEUS
SCATTERING

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Abstract

Our object is to carry out an optical model study of the elastic scattering of pions from nuclei using the multiple scattering formalism of Watson. This treatment includes only the 1st order and 2nd order contributions. The second order optical potential involves two-body correlations of which we are primarily interested in those due to Pauli principle. We have obtained results for the two-body correlation functions for $^6$Li, $^{12}$C, $^{16}$O, and $^{40}$Ca based on the shell model, using single-particle wavefunctions generated from a potential of the Saxon-Wood type or from the Harmonic Oscillator potential. The extent to which these differ is discussed. We have also considered the effect of the Fermi-motion of the target nucleons on the $\pi$-$N$ forward scattering amplitudes. Using the optical model potential so derived, we have obtained the total reaction cross-sections for $^{208}$Pb and $^{12}$C within the momentum range $0.7 < P_\pi < 2$ GeV/c. The theoretical predictions have been compared with the available data to obtain information on the neutron distributions in nuclei.
Introduction

The scattering of pions by nucleons has played a particularly significant role in the development of nuclear physics. On the one hand, it is readily accessible to experimental investigations because of the availability of high intensity pion beams over a wide range of energies. On the other hand, it provides a testing ground for many theoretical ideas. For example, single and double elastic charge exchange scattering of pions can be used to study isobaric analogue states of nuclei. Because of its strong interaction with the nucleus, the pion can be used to probe nuclear density and energy levels.

It is interesting to note that the results obtained by the study of pion-nucleus scattering could be used to make comparisons with information obtained from nucleon-nucleus scattering.

The study of a system of strongly interacting particles is a difficult subject. If the number of particles are large one may be able to use the general statistical methods to give excellent results. In some cases, e.g. the atom, there is a centre of force and the residual interactions can be treated as perturbations. But the situation is not the same with the nucleus. Here we have too few particles for a statistical treatment, and there is no overall centre of force which would enable us to treat the forces between nucleons as perturbations. However, there is also the difficulty that if the nucleon force has
a hard core it cannot be treated by ordinary methods of perturbation theory. Hence various methods of looking into the problem have been designed, of which the method of nuclear models the most prominent. The method consists of studying the 'model' with properties with which we are familiar and which at the same time resembles the actual nucleus. The physics of the model is then investigated and it is hoped that any properties discovered will also be properties of the nucleus. In this manner the nucleus has been treated as a liquid drop, a gas, shell model etc. However none of these are capable of giving all the known facts about the nucleus.

The construction of a nuclear model has been approached in two ways. The first takes account of the strong nuclear interaction and considers the nucleus as a compact, rigidly bound body, while in the second, nuclei are considered as systems of particles directly independent of each other and moving in a certain average nuclear field.

The first of these models which was helpful in explaining the nuclear structure and in calculating the electric and magnetic nuclear moments was the shell model. The nucleons were assumed to move in a simple average real potential field. As in the case of electrons in atoms, the nucleons fill up levels according to Pauli's principle.

In order to understand the properties of the nucleus more clearly, complexities in the potentials were introduced, for e.g. from the square well or the oscillator type to potentials
including spin and isospin dependence.

A modified form of this model ('the unified model') was proposed in order to take into account the strong coupling between nucleons.

The single-particle model \(^1\) with a real potential well was however unable to give a good description of the scattering of nucleons on nuclei or nuclear reactions. The discrepancy of resonance capture \(^2\) of slow neutrons by nuclei disproved the possibility of using this model for the explanation of a nuclear reaction. The resonance levels were found to possess inelastic scattering components.

In order to account for this discrepancy Neils Bohr \(^3\) proposed the compound nucleus model which assumes that the interaction between the nucleus and the incident nucleon is so strong that the nucleon is absorbed by the nucleus. The captured nucleon together with the original nucleus of \(A\) particles form a compound system of \(A + 1\) particles. Due to instability the \(A + 1\) system decays, in a manner which is independent of the original capture process but depends only on the properties of the compound nucleus itself. All process including elastic undergoes through the compound nucleus. The energy dependence of the cross-sections in the neighbourhood of the resonance is described by Breit and Wigner's \(^4\) formula. It was however impossible to determine precisely the positions of the resonance maxima, which correspond to the quasi-stationary levels of the compound nucleus. Moreover it was necessary to solve the problem of \(A + 1\) particles
in strong interaction. Even before theoretical investigations were done, experiments showed that the model of a compound nucleus was not satisfactory.

Fernbach, Serber and Taylor proposed a model, in which the nucleus was not considered to be totally absorbing, but only of the partly absorbing kind. Such a model was successful in explaining the dependence of cross-sections for neutron scattering on light nuclei and nuclear reactions in range of 100 MeV. This is the optical model which like the shell model is a model of independent particles. There is however one difference in that the average potential in which the nucleons move is complex.

1.1. Optical Model for Pion-nucleus Scattering

The optical model is based on the assumption that the interaction of a nucleon can be described by a single-particle potential. The complex potential is a function of the variables which characterises the particle: the real part describes the scattering of the incident particle and the imaginary part represents their absorption.

When there is a strong absorption, the nucleus is referred to as "black". The interaction cross-section is then mainly determined by the nuclear radius and, to a lesser extent, by the energy of the incident particle, except when \( \lambda \gg R \). Both functions vary slowly. On the other hand, where the absorption coefficient is small the cross-section depends on both the real and imaginary parts of the potential and this gives rise to the
usual characteristic interference effects.

The optical model approach can be classified in three different ways.

1) Phenomenological optical models, with parameters to fit the data.

2) Optical models derived from multiple scattering theory.

3) Impulse approximation with multiple scattering correction.

1.2. Impulse Approximation

The impulse approximation has generally been applied to pion-nucleus scattering at high energies. The basic assumption in this approximation is that the two-body t-matrix can be replaced by the free π-N two body t-matrix. The full π-nucleus T-matrix can then be factored into the scattering components and nuclear structure aspects \(^{11}\). This process is brought about under the assumption that the target nucleon is stationary. This assumption however, is not expected to hold at intermediate energies, which is verified by calculations done on pion-deuteron \(^{12}\) scattering where the impulse theory is used. Here it is observed that the particular choice of the target nucleon momentum has a substantial effect in the differential cross-section at back angles. Although calculations \(^{13}\) performed using this approximation were successful in giving the general trend of the shape of the diffraction pattern, but they were unable to predict the position of the diffraction minima.
1.3. Microscopic Optical Model

Watson \textsuperscript{14} had derived an optical model potential based on the multiple scattering theory. He showed that if one started with a series of two-body scatterings, but restricted the nuclear intermediate states to the ground state only, the full nuclear T-matrix could be calculated from a two-body potential. If we now replace the two-body t-matrix by the free two-body t-matrix (i.e. the impulse approximation) then the optical potential could be expressed in terms of the $\pi$-N forward scattering amplitudes. This form of approximation is not capable of predicting the back angle scattering for the diffraction pattern \textsuperscript{13}. Kisslinger \textsuperscript{15} using the multiple scattering approach of Watson was able to fit the pion-nucleus data on $^{12}$C. He took into account of both s and p wave interactions, the p term of which gave rise to a velocity dependent term in the pion-nucleus optical potential. By varying the parameters of the potential he was able to obtain good fit to the measured diffraction cross-section for 62 MeV $^{12}$C. His calculation was quite successful at back angles, indicating that the gradient terms derived from the p-wave $\pi$-N scattering were probably necessary to explain the shape of the angular distribution.

The Kisslinger model was also applied in a modified form by Baker et al \textsuperscript{16} who analysed several of their experiments around 80 MeV, and were able to fit their data quite well.

Auerbach et al \textsuperscript{17} applied the Kisslinger model on several nuclei with parameters derived from $\pi$-N phase shifts. They were
able to obtain resonably good fits for elastic scattering below $E_{\pi} = 100$ MeV.

The Ericsons\textsuperscript{18)} however have introduced modifications of the Kisslinger potential. Besides the ordinary gradient term, they found that it was a better approximation to replace this term in a modified form, which takes into account of the Lorentz-Lorenz effect. The result of these, gives rise to a potential that depends both on $\rho(r)$ and $\rho^2(r)$. No calculations using this potential have been performed so far, but calculations done by Ericsons on the potential for the interior of a uniform density nucleus gives roughly empirical shape of the variation of $W$ (the imaginary part of the optical potential) with $k$, but not the approximate constancy (with $k$) of $V_0$.

1.4. Phenomenological Optical Models

An optical model for the scattering of high energy particles by nuclei was first introduced by Fernbach, Serber and Taylor\textsuperscript{6)}. They suggested that the propagation of the projectile (in their case a nucleon) in the interior of a nucleus could be given by a complex potential.

The first application of this approach to pions was by Bethe and Wilson\textsuperscript{19)}. Due to insufficient data their results were not conclusive.

Other attempts to fit scattering experiments around $60 < E_{\pi} < 150$ MeV with a complex potential were found to have the same difficulties. One could fit the differential cross-
sections for smaller angles $\theta < 60^\circ$, but at large angles predictions were not realistic.

Williams, Baker and Rainwater $^{20}$) tried to improve the calculations of large angle scattering by the addition of gradient terms at the nuclear surface. Their calculations were able to alter the back angle scattering, but produced more diffraction oscillation than were observed in experiments on 80 MeV pions on Cu.

In a later paper Baker et al $^{16}$) analyzed new data on scattering of 80 MeV pions by Li, C, and Al using variety of local complex potentials, of the Wood-Saxon type. They were unable to improve the calculations of angular distributions for angles $\theta > 60^\circ$. But they found that a non local potential of the Kisslinger type with adjustable parameters was able to give better differential cross-sections at larger angles.

The work of Edelstein et al $^{21}$) and Valckx et al $^{22}$) seem to demonstrate that a potential of the Kisslinger type is able to reproduce angular distribution to about $\theta = 100^\circ$, giving the experimental second maximum and dip.

Frank et al $^{23}$) using a square well potential together with an additional term representing pion absorption found the potential to exhibit a strong momentum dependence with attraction at lower energies, changing into repulsion above 180 MeV through the influence of the (33), (31) resonance. Their results were not very suggestive owing to the neglect
of nuclear correlations as well as finite size of the nucleus.

In early work on the optical model potential the effect of correlations were not taken into account. In a paper by Cronin et al. on high energy scattering of pions on nuclei, they observed that an optical model potential having the shape of the Fermi distribution was able to fit the experimental measurements on total absorption cross-sections. The measured diffraction cross-sections were 20-30% larger than those predicted by the optical model. Their results indicate that for pions the real potential of the nucleus may be larger than the value derived from the optical model and dispersion theory. It is difficult to say whether this discrepancy was due to unknown errors in measurements or if it is in the inadequacy of the optical model potential to predict diffraction cross-section.

However Beg demonstrated that the discrepancies could have risen in the action of the Pauli principle effect at high energies. His calculations based on the Fermi-gas model indicate that Pauli principle effect does enhance the cross-sections.

It has also been shown (in the case of nucleon-nucleus scattering) by Kerman et al. who using Watson's approach to the optical potential, that the effect of pair correlations in the nucleus is to decrease the magnitude of the imaginary part of the potential at low energies and to increase it at energies $\geq 300$ MeV.
In the case of deuteron scattering Glauber 27) has indicated that correlation corrections may in fact have the opposite sign to that in other nuclei.

Johnston and Watson 28) estimated corrections to the optical potential due to nuclear excitation, using the high energy approximation. For two simple models of the nuclear target, i.e. one for a Fermi gas model of the nucleus and the other, Brueckner model of the nucleus, they found corrections to the real potential strength of 20% and smaller effects on the imaginary part, for pions with $1 < E_\pi < 5$ GeV.

All calculations on π-nucleus scattering were based on the assumption that the target nucleon was stationary or a somewhat similar approximation. Earlier workers were unable to fit their experimental total cross-sections on π-deuteron scattering using the high energy approximation of Glauber 27). Faldt and Ericson 28) were able to show that the discrepancy between theory and experiment was due to the neglect of the Fermi motion of target nucleons. Reeder and Makowitz 30) found that the full width at half maximum (FWHM) of the $^{12}$C ($\pi^-$, $\pi^-N$) $^{11}$C peak is about 270 MeV, in comparison, the FWHM of the $\pi^-N$ peak is about 145 MeV. The greater width of the ($\pi^-$, $\pi^-N$) peak is probably due to the fact that the struck nucleon is moving rapidly within the potential well of the $^{12}$C nucleus, thus producing a smearing effect of the resonance peak.

The present thesis is divided into eight chapters. In chapter 2 we shall go through the derivation of the optical
model potential. Chapter 3 will be concerned with the
deduction of the total reaction cross-sections using the
derived optical potential. We shall obtain a formalism for
the two-body density functions and correlation functions in
Chapter 4. The use of various single particle wavefunctions
in calculating the two-body correlations, and the application
of these to a few light nuclei will be discussed in Chapter 5.
Chapter 6 deals with the effect of Fermi motion of target nucleon
on the \( \pi-N \) forward scattering amplitudes. Discrepancies if
any, in using different momentum distributions are discussed.
Chapter 7 concerns mainly with application of the results of
earlier chapters in obtaining total reaction cross-sections.
Comparison with experimental data are made. Finally in Chapter
8 we have briefly summarized the results and conclusions.
Chapter 2

Optical Model Potential for π-nucleus Scattering

In Chapter 1 we discussed the various forms of optical potential that have been used to study the interactions of pions with nuclei. We noted that all earlier potentials of the square well or Kisslinger type gave reasonably good fit to experimental data on diffraction cross-sections for low energy pion-nucleus scattering. However we observed that certain discrepancies arose in high energy limit when such potentials were used. This was accounted for by suggesting that the effect of correlations became prominent at these energies and that the complex potential which was used to describe absorption processes would still be valid if we take into account of correlations. We also noted that in the case of π-nucleus scattering the effect of Fermi motion of the target nucleons was significant and these effects should be taken into consideration.

In the present thesis we shall not use a simple complex square potential but obtain an energy dependent potential using the multiple scattering formalism of Watson. The derivation here is based on the high energy limit where the impulse approximation is valid. This is a good approximation, since at high energies of the incident particle, the binding potential of the target nucleon is much less than the kinetic energy of the projectile. It has been shown by Takeda and Watson that the impulse approximation is valid even under less strong assumptions, i.e. $\Delta U/E < 1$ (where $\Delta U$ is the change in binding
potential during the scattering process and \( E \) is the energy of the incident particle).

Unlike the case of nucleon-nucleus scattering the problem of antisymmetrization of the wavefunctions of the \( A + 1 \) system does not arise here, since in our case the projectile is a pion.

2.1. Theory

The treatment here follows closely to that of Jones 32). Let \( H_1 \) be the Hamiltonian of the target nucleus and \( E_1 \) the energy eigenvalue. If the eigenfunction of the Hamiltonian is denoted by the state vector \( |\xi> \) then

\[
H_1 |\xi> = E_1 (\xi) |\xi>
\]

(2.1)

Similarly the eigenvalue equation for the incident pion is

\[
H_2 |k> = E_2 (k) |k>
\]

(2.2)

where \( k \) is the momentum of the pion.

Now the states \( |\xi> \) and \( |k> \) form a complete set of linearly independent orthogonal basis states. Since the product of these sets is itself complete and orthogonal we can use this as a basis for the whole system. The corresponding eigenvalue equation is then

\[
H |\psi> = E |\psi>
\]

(2.3)

where

\[
H = H_1 + H_2
\]

(2.4)
and

\[ E = E_1 + E_2 \tag{2.5} \]

The Schrödinger equation (2.3) can be written in integral form

\[ |\psi\rangle = |\zeta_0, k\rangle + G(E) V |\psi\rangle \tag{2.6} \]

where \( V \) is the interaction between the incident pion and the target nucleus. \( |\zeta_0\rangle \) represents the groundstate of the nucleus. The propagator \( G(E) \) is defined as

\[ G(E) = \frac{1}{E - H + i\epsilon} \tag{2.7} \]

The vector \( |\psi\rangle \) represents the total state of the system and hence consists diagonal and off diagonal scattering components. Since our aim is to obtain an optical potential that represents elastic scattering, we have therefore to separate the elastic components.

We do this by defining an elastic scattering state vector

\[ |\phi\rangle = \frac{1}{(2\pi)^3} \int dk'| \zeta_{0}', k'\rangle \langle \zeta_{0}, k |\psi\rangle \tag{2.8} \]

The optical model operator which gives rise to this elastic state vector can be defined via the equation

\[ |\phi\rangle = |\zeta_0, k\rangle + G(E) U |\phi\rangle \tag{2.9} \]
Since $|\phi>\text{ contains only elastic components the scalar product of } |\phi> \text{ with } |\zeta_0, k'> \text{ is zero i.e.}$

$$\langle \zeta, k' | \phi > = 0 \text{ for all } \zeta \neq \zeta_0$$  \hspace{1cm} (2.10)

If we introduce an operator $F$ such that

$$|\psi> = F |\phi>$$  \hspace{1cm} (2.11)

then by equation (2.8) and (2.9) $F$ satisfies the relation

$$F = 1 + g(E) (V - U) F$$  \hspace{1cm} (2.12)

where

$$g(E) = \frac{1}{E - H - U + i\varepsilon}$$  \hspace{1cm} (2.13)

The elastic transition amplitude between the initial and final states is then

$$\langle \xi_0, k' | U | \phi > = \langle \xi_0, k' | V | \psi >$$  \hspace{1cm} (2.14)

$$\approx \langle \xi_0, k' | VF | \phi >$$  \hspace{1cm} (2.15)

Using the definitions of $G$ and $g$ in equations (2.7) and (2.13) we obtain the series expansion

$$G = g - gUg + gUgUg + \ldots .$$  \hspace{1cm} (2.16)
(where we have used the series

\[
\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \ldots \tag{2.17}
\]

see for e.g. Chapter 1, ref. 33). From equations (2.12) and (2.15) we deduce the series

\[
\langle U \rangle = \langle V + V_g (V - U) + V_g (V - U) g (V - U) + \ldots \rangle \tag{2.18}
\]

The brackets here and throughout this section shall henceforth denote that both sides of the equation are taken between states of the form in eq'n (2.15). Rearranging the right hand side of equation (2.18) and collecting all terms in which V appears a certain number of times into a single product term which is of the form

\[
V(g - gUg + \ldots) V + \ldots (g - gUg + \ldots) V (1 - gU + \ldots) \tag{2.19}
\]

using equation (2.16) this becomes

\[
VGV \ldots VGV (1 - gU) . \tag{2.20}
\]

Hence equation (2.18) can be written as

\[
\langle U \rangle = \langle (V + VGV + \ldots) (1 - gU) \rangle \tag{2.21}
\]

Equation (2.21) contains excited intermediate nuclear states. The separation of these excited states can be carried out by
defining a projection operator

\[ Q = 1 - \frac{1}{(2\pi)^3} \int \frac{dk'}{\zeta_0, k'} < k', \zeta_0 | (2.22) \]

and

\[ P = 1 - Q \] (2.23)

The operator P projects out the target ground state. If we define an operator K

\[ K = V + VGV + VGVGV + \ldots. \] (2.24)

equation (2.21) reduces to

\[ <U> = <K (1 - GU)> \] (2.25)

Iterating equation (2.25) and noting that U is diagonal in nuclear states we obtain

\[ <U> = <K - kPGK + KPGKPGK + \ldots.> \] (2.26)

Inserting \( P + Q = 1 \) between the V's and the G's in equation (2.24), each term separates into a sum of terms of the form

\[ VPGVPGVQGV \ldots \ldots. \] V (2.27)

If we collect terms in which PG appears a certain number of times we can write

\[ K = L + LPGL + LPGLPGL + \ldots. \] (2.28)
where

\[ L = V + VQGV + \ldots \]  

(2.29)

Inserting the expansion for \( K \) i.e. equation (2.28) in equation (2.26) we obtain

\[ <U> = <L> \]  

(2.30)

Let us define an operator \( t_i \) by

\[
t_i = v_i + v_i Q G v_i + v_i Q G v_i Q G v_i + \ldots
\]

\[ = v_i + v_i Q G t_i \]  

(2.31)

where \( v_i \) represents the two-body interaction i.e.

\[ V = \sum_i v_i \]  

(2.32)

then using equations (2.29) and (2.31) we have

\[
<U> = \left\langle \sum_i t_i + \sum_i \sum_{j \neq i} t_i Q G t_j + \ldots \right\rangle
\]  

(2.33)

Equation (2.33) gives a series expansion of the optical model operator in terms of the two-body t-matrices. The first term represents a sum of two-body interactions. The second represents a process in which the incident pion is first scattered by the \( j^{\text{th}} \) nucleon into an intermediate state in which the whole system is excited. De-excitation of the nucleus takes place when the incident pion strikes
particle $i$. This is achieved by the Green's function $G$, which propagates the pion from $j$ to $i$ through the intermediate excited state. The process of de-excitation can only take place when the particles labelled $(i,j)$ are closely correlated. Higher order terms in the series (2.33) gives rise to three-body correlations, four etc. However we would expect such terms to give negligible contribution to the optical potential. For the series (2.33) to have any meaning we must expect it to converge rapidly, otherwise the expansion is fruitless.

2.2. Form of Potential at High Energies

Consider the first term in the series (2.33)

$$<\xi_0, k' | U | \phi> = <\xi_0, k' | \sum_i t_i | \phi>$$

(2.34)

If we use the first Born approximation for the state vector $|\phi> \text{ i.e. } |\xi_0, k> \text{ then we can write}$

$$<k' | U | k> = <\xi_0, k' | \sum_i t_i | \xi_0, k>$$

(2.35)

If the ground state wavefunction of the target nucleus is completely antisymmetrized, each term in the series will give equal contribution. Thus if there are $A$ particles in the nucleus we can replace equation (2.35) by

$$<k' | U | k> = A <\xi_0, k' | t | \xi_0, k>$$

(2.36)
i.e.
\[
\langle k' | U | k \rangle = \frac{A}{(2\pi)^{3A+3}} \int d\vec{p} \, d\vec{p}' \, d\Gamma \, \psi^\ast_{\zeta_0} (\Gamma, \vec{p}')
\]
\[
\langle p' , k' | t_0 | p, k \rangle \psi_{\zeta_0} (\Gamma, \vec{p})
\]
(2.37)

where \( p \) and \( p' \) are the initial and final momentum of the target nucleon under consideration. \( \Gamma \) denotes the momentum variables of the remaining \( A-1 \) nucleons. We have also replaced the two-body \( t \) operator by the free \( \pi-N \) \( t \) operator \( t_0 \), which is just the impulse approximation. At high energies the motion of the target nucleon can be neglected and we may replace

\[
\langle p' , k' | t_0 | p, k \rangle = \langle k' | t_0 | k \rangle \delta(p' + k' - p - k)
\]
(2.38)

On substitution of equation (2.38) in equation (2.37) we have

\[
\langle k' | U | k \rangle = \frac{A}{(2\pi)^{3A+3}} \int d\Gamma \, d\vec{p} \, \psi^\ast_{\zeta_0} (\Gamma, \vec{p} - q)
\]
\[
\langle k' | t_0 | k \rangle \psi_{\zeta_0} (\Gamma, \vec{p})
\]
(2.39)

where \( q = k' - k \), the momentum transfer. (2.40)

Introducing the Fourier transform of the nuclear wavefunctions.

\[
\psi^\ast_{\zeta_0} (\Gamma, \vec{p} - q) = \int ds \, d\vec{r} \, \psi^\ast_{\zeta_0} (s, \vec{r}) \, e^{i(\vec{p} \cdot q) \cdot \vec{r} + i\vec{r} \cdot \vec{s}}
\]
(2.41)

\[
\psi_{\zeta_0} (\Gamma, \vec{p}) = \int ds' \, d\vec{r}' \, \psi_{\zeta_0} (s', \vec{r}') \, e^{-i(\vec{p} \cdot \vec{r}' + \vec{r}' \cdot \vec{s}')}
\]
(2.42)
into equation (2.39)

\[
<k' | U | k> = A <k' | t_0 | k> \int ds \ dr \ e^{-i\mathbf{q} \cdot \mathbf{r}} |\psi_0 (s, r)|^2
\]  

(2.43)

where we have used the relation

\[
\delta(s - s') = \frac{1}{(2\pi)^{3A-3}} \int d\mathbf{r} \ e^{i\mathbf{q} \cdot (s - s')}
\]  

(2.44)

The ground state nuclear density is defined as

\[
\rho (\mathbf{r}) = \int ds |\psi_0 (s, r)|^2
\]  

(2.45)

with \[ \int \rho (\mathbf{r}) \ dr = 1 \]  

(2.46)

Hence

\[
<k' | U | k> = A <k' | t_0 | k> F(q)
\]  

(2.47)

where

\[
F(q) = \int \rho (\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} \ dr
\]  

(2.48)

In the co-ordinate representation the optical potential is

\[
<r' | U | r> = \int dk \ dk' <r' | k' > <k' | U | k> <k | r>
\]

(2.49)

\[
= A \int dk \ dq <k + q | t_0 | k> F(q) e^{-i(\mathbf{r} - \mathbf{r}') \cdot \mathbf{k}} e^{i\mathbf{q} \cdot \mathbf{r}'}
\]

(2.50)
If we assume that the t-matrix is a function of momentum transfer only (see for e.g. ref. 26;54) then the integral reduces to

\[
<k'|t|k> = \int_0 <\vec{r} - \vec{r'}|F(q)|\vec{r} - \vec{r'}> dq
\]  

(2.51)

Equation (2.51) implies that at high energies we can replace the optical potential by a local one. The t-matrix 

\[
<k'|t|k> = t(q)\]

is defined in the centre of mass of the A+1 system. Transformation to the two-body centre of mass system can be easily achieved (see Appendix B). The optical potential then becomes

\[
U(r) = \frac{\hbar^2}{(2\pi)^2} \int dm e^{i\vec{q} \cdot \vec{r}} M(q) F(q)
\]  

(2.52)

where \(\omega_1, \omega_2, E_1, E_2\) are the pion and nucleon total energies in the two-body and A+1 centre of mass system respectively. If the scattering amplitude varies slowly within the range in which the form factor F(q) falls off, we can then simplify the expression (2.52)

\[
U(r) = \frac{2\pi A \hbar^2}{E_1 E_2} (\omega_1 + \omega_2) M(0) \rho(r)
\]  

(2.53)

Equation (2.53) is the usual high energy optical potential (see for e.g. Glauber 27) and Foldy and Walecka 34). This form of the potential is extensively used in the scattering of high energy pions on nuclei 24,35,36,37). To establish the connection between equation (2.53) and equation (7.21) of ref 34) we note
that

\[ \langle k' | t_o | k \rangle = t_o(q) = -\frac{\hbar^2}{2\mu_{A+1}} \frac{4\pi f(k',k)}{E_2} \]  (2.54)

where \( \mu_{A+1} \) is the reduced mass of the pion in the centre of mass of the \( A+1 \) system.

\[ \mu_{A+1} = \frac{E_1 E_2}{E_1 + E_2} \]  (2.55)

If \( f(0) \) of equation (7.21) in ref. 34 is replaced by

\[ f(0) = \left( \frac{\omega_1 + \omega_2}{E_1 + E_2} \right) \frac{E_2}{E_1} M(0) \] and noting that a factor of

\[ \frac{2\mu_{A+1}}{\hbar^2} \] is absorbed in \( U(r) \) in this ref., we obtain equation (2.53).

2.3. Second Order Correction to the Optical Potential

From equation (2.33) the second order contribution to the optical potential is given by the term

\[ \sum_i \sum_{j \neq i} t_i Q G t_j \]  (2.56)

Substituting for \( Q \) from equation (2.23) into the above expression,

\[ \sum_i \sum_{j \neq i} t_i G t_j - t_i P G t_j \]  (2.57)

Consider a pair of nucleons labelled by \( (i,j) \), then using the impulse approximation

\[ ... 

...
The matrix element

\[ <\xi_0, k' | t_0 | \xi, k''> = \int dr_i \rho_{\xi_0, \xi}^{*}(r_i) <k' | t_0 | k''> e^{-i \lambda_i \cdot \xi_i} \] (2.59)

(see equation 2.44)

where

\[ \rho_{\xi_0, \xi}^{*}(r_i) = \int ds dr_j \psi_\xi^{*}(s, r_i, r_j) \psi_\xi(s, r_i, r_j) \] (2.60)

Here \( \xi \) is the co-ordinate of the remaining A-2 nucleons and \( q_i = k' - k'' \) is the momentum transfer. Using the completeness relation of the target wavefunctions i.e.

\[ \sum_{\xi} \psi_\xi(s, r_i, r_j) \psi_\xi^{*}(s', r_i, r_j) = \delta(s - s') \delta(r_i - r_i') \delta(r_j - r_j') \] (2.61)

and assuming that the excitation of the nuclear states to be small the integral (2.58) becomes

\[ t_i QG t_j = \frac{1}{(2\pi)^3} \int \frac{dk''}{E - E_1(\xi)} \rho_{\xi_0, \xi}^{*}(r_i, r_j) \rho_{\xi_0, \xi}^{*}(r_i, r_j) \] (2.62)
\[
\frac{1}{(2\pi)^3} \int \hat{\mathbf{k}}' \, d\mathbf{r}_1 \, d\mathbf{r}_j \, e^{-i(\hat{\mathbf{q}} \cdot \mathbf{r}_i + \hat{\mathbf{q}}' \cdot \mathbf{r}_j)} \Delta(\mathbf{r}_i, \mathbf{r}_j) \Delta(\mathbf{r}_i', \mathbf{r}_j')
\]

(2.63)

where we have used equation (2.5), and

\[
\Delta(\mathbf{r}_i, \mathbf{r}_j) = \{\rho_{\xi_0}^{(i)}(\mathbf{r}_i, \mathbf{r}_j) - \rho_{\xi_0}^{(i)}(\mathbf{r}_i) \rho_{\xi_0}^{(j)}(\mathbf{r}_j)\}
\]

is defined to be the two-body correlation function.

Summing over \(i\) and \(j\) the 2nd order contribution to the optical potential can be written as

\[
\Delta U(1)(k', k) = A(A^{-1}) \int \frac{d\hat{\mathbf{k}}''}{(2\pi)^3} \, d\mathbf{r}_i \, d\mathbf{r}_j \, e^{-i(\hat{\mathbf{q}} \cdot \mathbf{r}_i + \hat{\mathbf{q}}' \cdot \mathbf{r}_j)} \Delta(\mathbf{r}_i, \mathbf{r}_j)
\]

(2.64)

Substitute \(k' - k'' = \hat{\mathbf{q}}\) and \(k'' - k = \hat{\mathbf{q}}'\) into equation (2.64)

\[
\Delta U(1)(k', k) = A(A^{-1}) \int \frac{d\hat{\mathbf{k}}''}{(2\pi)^3} \, d\mathbf{r}_i \, d\mathbf{r}_j \, e^{i \hat{\mathbf{k}}'' \cdot (\mathbf{r}_i - \mathbf{r}_j)} \Delta(\mathbf{r}_i, \mathbf{r}_j)
\]

(2.65)

using the fourier transform

\[
\Delta(\hat{k}', k) = \int e^{-i\hat{k}' \cdot \mathbf{r}} \Delta(\mathbf{r}, \mathbf{r}') e^{i\hat{k} \cdot \mathbf{r}'} \, d\mathbf{r} \, d\mathbf{r}'
\]

(2.66)
we have

$$\Delta U^{(1)} (k', k) = -\frac{2\mu_{A+1}}{3\pi^2} A(A-1) \int \frac{dk''}{(2\pi)^3} \Delta(k'' - k'', k - k'')$$

$$\frac{<k' | t_o | k''> <k'' | t_o | k>}{k''^2 - k^2 - i\epsilon}$$

(2.67)

If we substitute equation (2.54) into (2.67) we obtain

$$\Delta U^{(1)} (k', k) = -\frac{\hbar^2}{2\mu_{A+1}} A(A-1) \int \frac{dk''}{(2\pi)^3} \Delta(k'' - k'', k' - k'')$$

$$\frac{[4\pi f(k', k'')] [4\pi f(k'', k)]}{k''^2 - k^2 - i\epsilon},$$

(2.68)

a result equivalent to equation (6.5) of Foldy and Walecka

According to Foldy and Walecka the 1st order correlation contribution to the optical potential arises from multiple scattering graphs of the form

![Diagram](image.png)

Fig. (1)
The first diagram on the right gives the two-body correlations with no intermediate scattering. The second gives two-body correlations with intermediate scattering and so on. The total correlation contribution can be evaluated by summing over all diagrams. This can be achieved in analogy with the expansion (2.16) for G in terms of the propagator g. However, here g is the free two-body propagator.

\[ g(k') = \frac{1}{k'^2 - k^2 + i\epsilon} \]  \hspace{1cm} (2.69)

Hence the total Green's function for scattering is then

\[ \langle k' | G | k \rangle = G(k', k) = g(k') \left(2\pi\right)^3 \delta(k' - k) - \]

\[ g(k) \int \frac{dk''}{(2\pi)^3} U(k', k'') G(k'', k) \]  \hspace{1cm} (2.70)

Graphically we can represent this as follows:

The first diagram on the right represents free propagation, the second represents a scattering with final momentum \( k' \). The third represents double scattering, and so on.

Let the Fourier transform of \( G(k', k) \) be
\[ G(\vec{r}, \vec{r'}) = \int \frac{dk'}{(2\pi)^3} \frac{dk}{(2\pi)^3} \ e^{i\vec{k}' \cdot \vec{r}} \ G(k', k) \ e^{-i\vec{k} \cdot \vec{r'}} \]  

(2.71)

Then

\[ G(\vec{r}, \vec{r'}) = g(\vec{r} - \vec{r'}) - \int g(\vec{r} - \vec{r''}) \ U(\vec{r''}, \vec{r'''}) \]  

\[ G(\vec{r''}, \vec{r'}) \ dr'' \ dr''' \]  

(2.72)

Since \( g(\vec{r} - \vec{r'}) \) satisfies the Green's equation

\[ (\nabla^2 + k^2) \ g(\vec{r} - \vec{r'}) = - \delta(\vec{r} - \vec{r'}) \]  

(2.73)

\( G(\vec{r}, \vec{r'}) \) is a solution of

\[ (\nabla^2 + k^2) \ G(\vec{r}, \vec{r'}) - \int dx'' \ U(\vec{x}, \vec{x''}) \ G(\vec{x}, \vec{x''}) = - \delta(\vec{r} - \vec{r'}) \]  

(2.74)

When the optical potential is local, equation (2.74) reduces to

\[ [\nabla^2 + k^2 - U(\vec{r})] \ G(\vec{r}, \vec{r'}) = - \delta(\vec{r} - \vec{r'}) \]  

(2.75)

The generalization of equation (2.68) is then easily seen to be

\[ \Delta U(k', k) = \frac{-\hbar^2}{2\mu A+1} \ A(A - 1) \int \frac{dk''}{(2\pi)^3} \frac{dk'''}{(2\pi)^3} \ \Delta(k' - k''', k - k''') \ G(\vec{k}'', \vec{k}''') \ \left[ 4\pi \rho(\vec{k}'', \vec{k}''') \right] \ G(\vec{k}'', \vec{k}') \]  

(2.76)

Defining the Fourier transform,

\[ \Delta U(\vec{x}, \vec{x'}) = \int \frac{dk'}{(2\pi)^3} \frac{dk}{(2\pi)^3} \ e^{i\vec{k}' \cdot \vec{x}} \ \Delta U(k', k) \ e^{-i\vec{k} \cdot \vec{x'}} \]  

(2.77)
and substituting for \(\Delta U(k', k)\) from equation (2.76) and equation (2.71) we deduce

\[
\Delta U(r, r') = \frac{-\hbar^2}{2\mu_{A+1}} A(A - 1) \int \frac{dk'}{(2\pi)^3} \frac{dk}{(2\pi)^3} [4\pi f(k')] \\
\Delta(k', k) [4\pi f(-k)] G(r, r') e^{ik' \cdot r} e^{-ik \cdot r'}
\]  

(2.78)

where we have made a change of variable \(k' - k'' + k', k - k'' \rightarrow k\) and we have assumed that the scattering amplitude to be a function of momentum transfer only, i.e. \(f(k'', k) \rightarrow f(k' - k)\).

The integral as it stands is difficult to evaluate, and therefore we make some simplifications. Following the assumptions made by Foldy and Walecka

(1) The scattered wavefunction can be written as

\[
\psi_k^+(\vec{r}) = e^{ik \cdot \vec{r}} \phi(\vec{r})
\]  

(2.79)

which is the Glauber approximation at high energies. \(\phi(\vec{r})\) is a slowly varying function over the nucleus.

(2) When solving for \(G(r, r')\) we shall assume that the medium is uniform.

(3) \(\Delta(r, r') \approx \rho^2(\vec{r}) \delta(|\vec{r} - \vec{r}'|)\)  

(2.80)

We know from equation (2.75) that \(G(r, r')\) satisfies

\[
[V^2 + k^2 - U(\vec{r})] G(r, r') = -\delta(\vec{r} - \vec{r}')
\]  

(2.75)

Substitute for \(U(\vec{r})\) from equation (2.53)

In the centre of mass of the \(A+1\) system this is
\[ U(r) = -4\pi A \frac{\hbar^2}{2\mu_{A+1}} f(0)\rho(r) \]  

(2.81)

If we absorb the factor \( \frac{\hbar^2}{2\mu_{A+1}} \) into \( U(r) \) equation (2.75) then satisfies

\[ \left[ \nabla^2 + k^2 + 4\pi A f(0) \right] G(r, r') = -\delta(r - r') \]  

(2.82)

Note that \( \rho \) is to be treated as constant in the above expression under assumption (2). Under these conditions the solution

\[ G(r, r') = e^{\frac{ik|r-r'|}{4\pi|\vec{r}-\vec{r}'|}} \]  

(2.83)

where

\[ K^2 = k^2 + 4\pi A f(0) \rho \]  

(2.84)

In the limit \( k \to \infty \)

\[ K = k + i \frac{4\pi A f(0)}{2k} = k + i \text{Im} K \]  

(2.85)

Hence

\[ G(r, r') = e^{\frac{ik|r-r'|}{4\pi|\vec{r}-\vec{r}'|}} e^{-|\vec{r}-\vec{r}'|/\lambda} \]  

(2.86)

limit \( k \to \infty \)

From equations (2.66) and (2.80)

\[ \Delta(k', k) = \int e^{-i\vec{k}' \cdot \vec{r}} \rho^2(\vec{r}) \theta(|\vec{r} - \vec{r}'|) e^{i\vec{k} \cdot \vec{r}'} \, d\vec{r} \, d\vec{r}' \]

\[ = \rho^2(\vec{k}' - \vec{k}) \theta(k) \]  

(2.87)
Since the scattering amplitude is a slowly varying function of momentum transfer compared to the Fourier transform of the nuclear density, we can write

$$f(k') \approx f(k) = f(-k) = f(k)$$

in equation (2.78). After a change of variables, equation (2.78) becomes

$$\Delta U(r, r') = -A(A - 1)\rho^2(r)G(r, r') \int \frac{dk'}{(2\pi)^3} [4\pi f(k')]^2$$

$$\theta(k') e^{ik' \cdot (r-r')}$$

(2.89)

In the scattering equation we need

$$\int dr' \Delta U(r, r') \psi_{\bar{k}}(r') \approx \int dr' \Delta U(r, r') e^{ik' \cdot r'} \phi(r')$$

(2.90)

$$= \frac{-\hbar^2}{2\mu A+1} A(A - 1) \rho^2(r) e^{ik \cdot r} \int \frac{dk'}{(2\pi)^3} [4\pi f(k')]^2$$

$$\int dz e^{ikz} e^{ik'z} \frac{e^{-z/\lambda}}{4\pi z} \frac{\sin k'z}{k'z} \phi(r + z)$$

(2.91)

As we have assumed $\phi$ to be slowly varying we can write

$$\phi(r + z) \approx \phi(r)$$

integrating over $z$

$$\int dz e^{ikz} e^{ik'z} \frac{e^{-z/\lambda}}{4\pi z} \frac{\sin k'z}{k'z} e^{-z/\lambda}$$

$$\lim_{k \to \infty} \frac{\pi}{4k'k} \left[ \frac{2\tan^{-1}(\lambda k')}{\pi} \right]$$

(2.92)
Hence limit \( k \to \infty \)

\[
\Delta U(x,x') = \frac{-i\hbar^2}{2k_{\mu}^\hbar + 1} h(A - 1) \rho^2(x) \left\{ \int d(2) \frac{k' f^2(k') \theta(k')}{\sin^2(\lambda k')} \right\} \delta(x - x')
\]  

(2.93)

where \( d(2) \frac{k'}{\pi} = 2\pi k' dk' \)

As in the case of the first order optical potential we see that the 2nd order contribution becomes local under the assumptions made at high energies.

From equations (2.54), (B.21) and (B.23) of appendix B, we obtain

\[
f(k') = \left( \frac{\omega_1 + \omega_2}{E_1 + E_2} \right) \frac{E_2}{E} M(q)
\]  

(2.94)

If in equation (2.87) we write

\[
\Delta(k', k) = \delta(k' - k) \Delta(k')
\]  

(2.95)

Then

\[
\theta(k') = \Delta(k)/\rho^2(0)
\]  

(2.96)

On substitution of equation (2.94) and (2.96) into equation (2.93), we obtain

\[
\Delta U(x) = \frac{-2iA(A - 1)}{k} \rho^2(x) \frac{\hbar^2}{E_1 E_2} \left( \frac{\omega_1 + \omega_2}{E_1 + E_2} \right) \frac{E_2}{E}
\]

\[
\int k' dk' \Delta(k') \tan^{-1}(\lambda k') \frac{M^2(k')}{\rho^2(0)} \left[ \rho^2(0) \right]^{-1}
\]  

(2.97)
At high energies the imaginary part of the forward scattering amplitude can be written as

\[
\text{Im } f(0) = \frac{k \sigma_T}{4\pi},
\]

(2.98)

where \( \sigma_T \) is the \( \pi-N \) total cross-section.

Comparing equations (2.85) and (2.98)

\[
\lambda = \frac{2}{\lambda_0 \sigma_T},
\]

(2.99)

In the limit \( \lambda \to \infty \) i.e. \( \frac{f}{k} \to 0 \) we obtain Glauber's result 27). At high energies we would expect multiple scattering to be negligible and limit \( \lambda \to \infty \) implies just this; i.e. \( \Delta U(\chi) \) reduces to \( \Delta U^{(1)}(\chi) \) as in Glauber's formalism.
Chapter 3

π-Nucleus Total Reaction Cross-sections

The object of obtaining an expression for the optical potential is to deduce some theoretical results which can be compared with experimental data. The success of the theory then depends on how closely the theoretical predictions agree with experimental values. If the predictions give a good agreement, this would justify the approximations made in the calculations. Any discrepancies that may arise would necessitate a modification of the theory or discredit the approximations that have been employed. However, for the present we shall show how, using the optical model potential one could arrive at a closed form for the total cross-sections and total reaction cross-sections. We shall follow two approaches to this problem.

It is well known that at high energies (see for e.g. ref. 29, 38, 39, 40) the Glauber theory gives a very good description of the scattering process. On the otherhand, since we are dealing with relativistic pion-nucleus scattering, it would be more appropriate to describe the scattering mechanism by means of the Klein-Gordon equation. We would expect the latter to give a better prediction as we are dealing with pions with energy $0.5 < E_\pi < 2$ Gev. Nevertheless, it would be interesting to see how these two theoretical models agree.

3.1. The Klein-Gordon Equation for Meson Scattering

The wave equation for meson scattering in a potential $V$
is the Klein-Gordon equation.

\[ [\nabla^2 + \frac{(E-V)^2}{\hbar^2 c^2} - \frac{m^2 c^4}{\hbar^2 c^2}] \psi = 0 \quad (3.1) \]

where \( E \) is the total energy of the meson and \( m \) is its rest mass.

If the potential \( V \) is the sum of the Coulomb potential \( V_c \) and the nuclear potential \( V_N \), then,

\[ [-i\hbar c^2 \nabla^2 + m^2 c^4] \psi = \left[(E - V_c)^2 - 2(E - V_c)V_N + V_N^2\right] \psi \quad (3.2) \]

In spherical co-ordinates the laplacian operator \( \nabla^2 \) may be written as

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{r^2} \hat{L}^2(\theta, \phi) \quad (3.3) \]

where

\[ \hat{L}^2(\theta, \phi) = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \quad (3.4) \]

The operator \( \hat{L}^2 \) given in equation (3.4) is just

\[ \hat{L}^2 = (\mathbf{r} \times \mathbf{p}) \cdot (\mathbf{r} \times \mathbf{p}) = -\hbar^2 \mathbf{r} \times \nabla \cdot (\mathbf{r} \times \nabla) \quad (3.5) \]

and represents the square of the orbital angular momentum.

The spherical harmonics \( Y_{\lambda m}(\theta, \phi) \) are defined to be eigen functions of \( \hat{L}^2 \) and the \( z \) component of \( \hat{L} \),

\[ \hat{L}_z = (\mathbf{r} \times \mathbf{p}) \cdot \hat{z} = -i\hbar \frac{\partial}{\partial \phi} \quad (3.6) \]

such that
\[ \hat{L}^2 Y_{\ell m}(\theta, \phi) = \ell(\ell + 1) \hbar^2 Y_{\ell m}(\theta, \phi) \]  

and

\[ \hat{L}_z Y_{\ell m}(\theta, \phi) = m \hbar Y_{\ell m}(\theta, \phi) \]  

The potential \( V_N(r) \) will be assumed to vanish sufficiently rapidly with increasing \( r \) that it may be neglected beyond some radius \( r = R \). This assumption does not hold for the Coulomb potential which vanishes at infinity as \( r^{-1} \). We shall assume that the wavefunction is finite at the origin, and that its gradient be finite everywhere, in particular at \( r = 0 \). This condition is sufficient to determine the radial wavefunction, to within an overall normalization constant. The physical situation is symmetric about the \( z \) axis, so we shall seek a solution of equation (3.2) which has this symmetry and is therefore independent of \( \phi \). We may write such a solution as

\[ \psi(r) = \frac{1}{kr} \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos \theta) U_\ell(r) \]  

where the radial function \( U_\ell(r) \) satisfies the radial equation

\[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} \frac{1}{\rho} + \left( \frac{2E}{\hbar^2 c^2} + \frac{V_c}{\hbar^2 c^2} - \frac{2E_N}{\hbar^2 c^2} + \frac{V_N}{\hbar^2 c^2} \right) \rho = 0 \]  

\[ + \frac{2 \frac{V_c V_N}{\hbar^2 c^2}}{\rho^2} \frac{\ell(\ell + 1)}{\rho^2} \]  

and \( \rho = kr \). \( k, E \) is defined to be the momentum and total energy of the meson respectively, in the centre of mass of the meson-nucleus system.
3.2. Asymptotic Behaviour of the Radial Wave Equation

We shall follow the procedure of solving equation (3.10) with the nuclear potential \( V_N = 0 \) and the Coulomb potential \( V_c \) represented by that due to a point charge \( V_{cp} \), and compare the solutions in the asymptotic region when \( V_N \neq 0 \) and \( V_c \) for an extended charge distribution. Let us then examine the form of the solution for a point charge. If \( z_1 \) be the charge of the pion and \( z_2 \) be that of the nucleus, then, for a point charge

\[
\frac{V_{cp}}{\hbar c k} = \frac{z_1 z_2 e^2}{\rho^2} \quad \text{for all } \rho. \tag{3.11}
\]

Let

\[
\beta = \frac{z_1 z_2 e^2}{\hbar c} \tag{3.12}
\]

\[
\gamma = \frac{E z_1 z_2 e^2}{k \hbar^2 c^2} \tag{3.13}
\]

The differential equation in this case is

\[
\left\{ \frac{d^2}{d\rho^2} + 1 - \frac{2\gamma}{\rho} + \frac{\beta^2}{\rho^2} - \frac{\ell(\ell + 1)}{\rho^2} \right\} P_\ell (\rho) = 0 \tag{3.14}
\]

The solution of this equation is given by Elton \(^1\). Let the solution near the origin be

\[
P_\ell (\rho) = \rho^\beta \sum_{n=0}^{\infty} a_n \rho^n \tag{3.15}
\]

The indicial equation is then

\[
\mu (\rho - 1) - \ell (\ell + 1) + \beta^2 = 0 \tag{3.16}
\]
from which

\[ p = \frac{1}{2} \pm \sqrt{(\ell + \frac{1}{2})^2 - \beta^2} \quad (3.17) \]

If \( \ell \neq 0 \) and \( \beta \neq 0 \) both values of \( p \) lead to singularities at the origin. But for \( \beta^2 < \frac{1}{4} \), the + sign gives a less singular solution than the - sign and hence we shall choose this as the regular solution as in Elton \(^{41}\). For \( \beta^2 > \frac{1}{4} \), both solutions lead to singularities, and a solution cannot be obtained. If we denote

\[ p = \frac{1}{2} + \sqrt{(\ell + \frac{1}{2})^2 - \beta^2} \quad (3.18) \]

\[ \bar{p} = \frac{1}{2} - \sqrt{(\ell + \frac{1}{2})^2 - \beta^2} \quad (3.19) \]

then the regular solution is given by Mott and Massey \(^{42}\) p.52 as

\[ \rho F_\ell(p) = e^{-\frac{1}{2}y} \left| \frac{\Gamma(p + iy)}{\Gamma(2p)} \right| (2p)^{p-1} e^{ip} \]

\[ W_{1,2}(p + iy, 2p, -2ip) \quad (3.20) \]

where \( W_{1,2} \) is the usual hypergeometric function.

Asymptotically

\[ F_\ell(p) \sim \sin \left( p - \frac{1}{2} \pi + \gamma - \gamma \log 2p \right) \quad (3.21) \]

The independent irregular solution is given by
\[ pG_\lambda(\rho) = e^{-\frac{1}{2} \pi \gamma} \frac{|\Gamma(\bar{p} + i\gamma)|}{\Gamma(2\bar{p})} (2\rho)^{\bar{p}-1} e^{i\rho} \]

\[ W_{1,2}(\bar{p} + i\gamma, 2\bar{p}, -2i\rho) \tag{3.22} \]

where asymptotically,

\[ G_\lambda(\rho) \sim \sin(\rho - \frac{1}{2} \pi + \sigma_\lambda + \phi_\lambda - \gamma \log 2\rho) \tag{3.23} \]

and

\[ \phi_\lambda = \text{arg}\Gamma(\bar{p} + i\gamma) - \text{arg}\Gamma(p + i\gamma) - \frac{\pi}{2} (\bar{p} - p) \tag{3.24} \]

\[ G_\lambda(\rho) \sim \sin(\rho - \frac{1}{2} \pi / 2 + \sigma_\lambda - \gamma \log 2\rho) \cos \phi_\lambda + \]

\[ \cos(\rho - \frac{1}{2} \pi / 2 + \sigma_\lambda - \gamma \log 2\rho) \sin \phi_\lambda \tag{3.25} \]

\[ \therefore G_\lambda(\rho) = \cos \phi_\lambda F_\lambda(\rho) + \sin \phi_\lambda \bar{G}_\lambda(\rho) \tag{3.26} \]

where

\[ \bar{G}_\lambda(\rho) = \cos(\rho - \frac{1}{2} \pi / 2 + \sigma_\lambda - \gamma \log 2\rho) \tag{3.27} \]

The function \( \bar{G}_\lambda(\rho) \) defined in equation (3.27) is conveniently defined as the irregular free solution, but of course any solution of equation (3.14) that is independent of \( F_\lambda(\rho) \) is irregular. Thus any linear combination of \( F_\lambda(\rho) \) and \( G_\lambda(\rho) \) can be taken as irregular solution to equation (3.14).

### 3.3. Extended Coulomb Charge in the Presence of the Nuclear Potential

The point charge representation of the Coulomb field is not realistic at small distances. In this case the form of the potential
has to be modified, since we know that the Coulomb potential arises from a uniform, spherical charge distribution of radius $R_c$.

\[ V_c(r) = \frac{z_1 z_2 e^2}{2 R_c^3} (3 - r^2/R_c^2) \quad r < R_c \]  \hspace{1cm} (3.28) \\
= \frac{z_1 z_2 e^2}{r} \quad r > R_c \]  \hspace{1cm} (3.29)

\[ \frac{V_c(p)}{\hbar c} = \frac{z_1 z_2 e^2}{2 \rho_c^3 \hbar c} (3 - \rho^2/\rho_c^2) \quad \rho < \rho_c \]  \hspace{1cm} (3.30) \\
= \frac{z_1 z_2 e^2}{\rho \hbar c} \quad \rho \geq \rho_c \]  \hspace{1cm} (3.31)

For $\rho > \rho_c$ the solution is similar to that which was obtained for a point charge distribution, except for a phase factor due to the presence of the nuclear potential $V_N$. Following ref. 43, we can write the complete solution at large $\rho$ as

\[ U_\lambda(\rho) \sim e^{i \delta_\lambda} \sin(\rho - \lambda \pi/2 + \sigma_\lambda + \delta_\lambda - \gamma \log 2 \rho) \]  \hspace{1cm} (3.32) \\
= e^{i \delta_\lambda} (\cos \delta_\lambda F_\lambda(\rho) + \sin \delta_\lambda \tilde{G}_\lambda(\rho)) \]  \hspace{1cm} (3.33) \\
\equiv e^{i \delta_\lambda} \{A_\lambda F_\lambda(\rho) + B_\lambda \tilde{G}_\lambda(\rho)\} \]  \hspace{1cm} (3.34)

from which

\[ \tan \delta_\lambda = B_\lambda/A_\lambda \]  \hspace{1cm} (3.35) \\
\[ = \frac{(\eta_\lambda - 1)}{i(\eta_\lambda + 1)} \]  \hspace{1cm} (3.36)

where $\eta_\lambda = e^{2 i \delta_\lambda}$

$\delta_\lambda$ is the phase shift due to the nuclear potential in the presence
of the Coulomb field.

3.4. Solution Near the Origin

The treatment here follows that of Kembhavi. Near the origin let

\[
- \frac{2EV_N}{\hbar^2 c^2 k^2} = \frac{2EV_N(o)}{\hbar^2 c^2 k^2} \tag{3.37}
\]

We can then write the differential equation (3.10) near the origin as

\[
\frac{d^2}{d\rho^2} + \left(1 - \frac{3\gamma}{\rho} + \frac{9\beta^2}{4\rho^4} - \frac{2EV_N(o)}{\hbar^2 c^2 k^2} + \frac{V_N^2(o)}{\hbar^2 c^2 k^2} + \frac{3\beta V_N(o)}{\rho \hbar c k}\right)\rho^2 + \left(\beta^2/4\rho^6\right)\rho^4
\]

\[- \frac{l(l + 1)}{\rho^2 U_L(\rho)} = 0 \tag{3.38}
\]

Let

\[
A_1 = 1 - \frac{3\gamma}{\rho} + \frac{9\beta^2}{4\rho^4} - \frac{2EV_N(o)}{\hbar^2 c^2 k^2} + \frac{3\beta V_N(o)}{\rho \hbar c k} + \frac{V_N^2(o)}{\hbar^2 c^2 k^2} \tag{3.39}
\]

\[
A_2 = \frac{\gamma}{\rho^3} - \frac{6\beta^2}{4\rho^4} - \frac{\beta V_N(o)}{\rho \hbar c k} \tag{3.40}
\]

\[
A_3 = \beta^2/4\rho^6 \tag{3.41}
\]

Real \(A_1 = 1 - \frac{3\gamma}{\rho} + \frac{9\beta^2}{4\rho^4} + \left(\frac{3\beta}{\rho} - \frac{2E}{\hbar c k}\right) \frac{\text{Real } V_N(o)}{\hbar c k} \]

\[+ \frac{1}{\hbar^2 c^2 k^2} \left[ (\text{Real } V_N(o))^2 - (\text{Im } V_N(o))^2 \right] \tag{3.42} \]
\[ \text{Im } A_1 = \left( \frac{3E}{\rho_c} - \frac{2E}{\hbar c} \right) \text{Im } V_N(o) + \frac{2\text{Real } V_N(o) \text{ Im } V_N(o)}{\hbar^2 c^2 k^2} \]  
(3.43)

\[ \text{Real } A_2 = \frac{V}{\rho_c} - \frac{6E}{4\rho_c} - \frac{8\text{Real } V_N(o)}{\rho_c^3 \hbar c^2} \]  
(3.44)

\[ \text{Im } A_2 = \frac{-8\text{Im } V_N(o)}{\rho_c^3 \hbar c^2} \]  
(3.45)

Let solution be \( U_\lambda(\rho) = \rho^{\lambda+1} \sum_{n=0}^{\infty} c_n \rho^n \)  
(3.46)

then the recurrence relation for the coefficients is

\[ [n(n - 1) + 2n(\lambda + 1)]c_n + A_1 c_{n-2} + A_2 c_{n-4} + A_3 c_{n-6} = 0 \]  
(3.47)

Using this series for the starting solution, one integrates numerically outwards and matches on to the Coulomb solution.

3.5. Calculation of the Phase Shift

The condition that the wavefunction and its derivative be continuous at \( \rho = \rho_c \), the point at which the potential is discontinuous is

\[ \text{limit } U_\lambda(\rho = \rho_c - \epsilon) = \text{limit } U_\lambda(\rho = \rho_c + \epsilon) \]  
(3.48)

\[ \text{lim}_{\epsilon \to 0} U_\lambda(\rho = \rho_c - \epsilon) = \text{lim}_{\epsilon \to 0} U_\lambda(\rho = \rho_c + \epsilon) \]

\[ \text{limit } L(\rho = \rho_c - \epsilon) = \text{limit } L(\rho = \rho_c + \epsilon) \]  
(3.49)

\[ \text{lim}_{\epsilon \to 0} L(\rho = \rho_c - \epsilon) = \text{lim}_{\epsilon \to 0} L(\rho = \rho_c + \epsilon) \]

where the logarithmic derivative \( L \) is defined by
The continuity of the logarithmic derivative at $\rho = \rho_c$ gives

$$L(\rho_c) = \frac{F_\lambda'(\rho_c) \pm (B_\lambda/A_\lambda) \bar{G}_\lambda'(\rho_c)}{F_\lambda(\rho_c) + (B_\lambda/A_\lambda) \bar{G}_\lambda(\rho_c)}$$

and the phase shift is obtained by solving

$$\tan \delta_\lambda = \frac{-F_\lambda'(\rho_c) - L(\rho_c) F_\lambda(\rho_c)}{\bar{G}_\lambda'(\rho_c) - L(\rho_c) \bar{G}_\lambda(\rho_c)}$$

where $L(\rho_c)$ is taken to be

$$L(\rho_c) = \frac{U_\lambda'(\rho_c)}{U_\lambda(\rho_c)}$$

This indicates that in order to obtain the phase shift $\delta_\lambda$ we need only know the logarithmic derivative; which means that the procedure of obtaining a series solution (3.46) near the origin is unnecessary. We can always choose an arbitrary small value for the derivative of $U_\lambda(\rho)$ at the origin and integrate outwards to match with the Coulomb solution. The arbitrary constant that comes into $U_\lambda(\rho)$ is automatically got rid of when we take the ratio $U_\lambda'/U_\lambda$.

### 3.6. Cross-sections

The relativistic Coulomb scattering amplitude for point charge is given by (see Elton 41).
where the subscript \( p \) denotes point charge.

\[ G = 0 \text{ when } \beta = 0. \text{ But } \beta = 0 \text{ equation (3.54) reduces to the scattering equation of a non-relativistic pion in a Coulomb field and hence (see Mott and Massey 42) p.48).} \]

\[ F_p = 2\gamma i \cot \theta /2 \exp[-i \gamma \log(1 - \cos \theta) + i \pi + 2i \text{ arg}(1 + i \gamma)] \] \hspace{1cm} (3.55)

when \( \beta \neq 0 \)

\[ G_p = \sum_{\lambda=0}^{\infty} (2\lambda + 1) \left( e^{2i\sigma_\lambda} - e^{2i\sigma_\lambda^0} \right) P_\lambda(\cos \theta) \] \hspace{1cm} (3.56)

where \( \sigma_\lambda \) is the Coulomb phase shift due to relativistic meson scattering, and \( \sigma_\lambda^0 \) is the non-relativistic Coulomb phase shift. Hence, we see that \( G_p \) gives a measure of deviation from the non-relativistic amplitude. If the scattering is due to an extended charge distribution in the presence of the nuclear potential then the scattering amplitude becomes

\[ f_p(\theta) = \frac{1}{2ik} (F_p + G_p) \] \hspace{1cm} (3.57)

Where the subscript \( E \) denotes extended charge distribution,

\[ f_p(\theta) = F_p + G_p + \frac{1}{2ik} \sum_{\lambda=0}^{\infty} (2\lambda + 1) \left( e^{2i\gamma} - e^{2i\gamma} \right) P_\lambda(\cos \theta) \] \hspace{1cm} (3.58)
where

\[ n_\lambda^P = \sigma_\lambda \]  
(3.59)

\[ n_\lambda^E = \delta_\lambda^E + \sigma_\lambda \]  
(3.60)

On substitution of \( G_p \) from equation (3.56) into equation (3.58) we obtain

\[ f_E(\theta) = \frac{F_p}{2\pi k} + \frac{1}{2\pi} \sum_{\lambda=0}^{\infty} (2\lambda + 1) e^{2i\sigma_\lambda^0} (e^{2i\Delta\sigma_\lambda^0} n_\lambda - 1) \]  
(3.61)

where

\[ \Delta\sigma_\lambda = \sigma_\lambda - \sigma_\lambda^0 \]

The logarithmic terms modify the "plane" wave and the "scattered" wave. Their presence is a direct consequence of the infinite range of the Coulomb force. No matter how far away from the scattering centre we may be, we can never consider the Coulomb force to be negligible, and can never write the asymptotic solution as a linear combination of solutions of the force free equations. Hence we can never get a true plane wave or a true spherical outgoing wave.

For simplicity if we consider the non-relativistic scattering of a meson, then the scattering amplitude is given by equation (3.55) when \( V_N = 0 \). In this case if we write

\[ \sigma_{\text{coul}}(0) = |f_{\text{coul}}(0)|^2 \]

then, \( \sigma_{\text{coul}}(0) \) is infinite. If we consider the total cross-section to be \( \sigma_{\text{tot}} \), i.e.
the total cross-section is also infinite in the forward direction. This is to be expected from the infinite range Coulomb force. No particle can pass so far from the scattering centre that it escapes some detection. The differential cross-section for relativistic scattering of a meson is then

$$\frac{d\sigma}{d\Omega} = |f_E(\theta)|^2$$  \hspace{1cm} (3.62)

Total reaction cross-section is

$$\sigma_R = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) [1 - |\eta_\ell|^2]$$  \hspace{1cm} (3.63)

3.7. *High Energy Approximation of Glauber*

The integral equation for the wavefunction can be written as (see for e.g. ref. 43, Chapter 6).

$$\psi_k^+(r) = \phi_k(r) + \int dr' G^+(r, r') V(r') \psi_{k'}^+(r')$$  \hspace{1cm} (3.64)

with

$$G^+(r, r') = - \frac{(2m/4\pi\hbar^2) \exp(ik|r - r'|)}{4\pi |r - r'|}$$  \hspace{1cm} (3.65)

and

$$\phi_k(r) = e^{ik \cdot r}$$  \hspace{1cm} (3.66)

The scattering amplitude is defined as
Under the condition that \( V/E \ll 1 \) we are justified in assuming that the scattering will take place mostly in the forward direction. In this case it may be a good approximation for the wavefunction to be written in the form

\[
\psi_k^+(r) = e^{i k \cdot r} \phi(r)
\]  

(3.68)

where \( \phi(r) \) is a function which varies slowly over the wavelength of the pion. On substitution of equation (3.68) into equation (3.64) we obtain an equation for \( \phi \),

\[
\phi(r) = 1 - \frac{2m}{\pi \hbar^2} \int e^{i k |r - r'|} \left[ - i \hbar \frac{v(r') \phi(r')}{r - r'} \right] \, dr'
\]  

(3.69)

we shall not go through the derivation, but simply quote from Glauber \(^{27}\) for the scattering amplitude,

\[
f(k',k) = \frac{k}{2\pi} \int e^{i(k-k') \cdot b} \int_{-1}^{1} \left\{ -i / \hbar \int_{-\infty}^{\infty} V(b + \zeta z') \, dz' \right\} \, d(2)_b
\]  

(3.70)

where \( d^{(2)}_b \) denotes integration over a plane of impact vectors.

(see figure 3)
If we let

\[ x(b) = - \frac{1}{2\pi i} \int_{0}^{\infty} V(b + \hat{k}z) \, dz \]  \hspace{1cm} (3.71)

Then

\[ f(k',k) = \frac{k}{2\pi i} \int e^{i(k-k') \cdot z} \left[ e^{ix(b)} - 1 \right] \, d\bar{z} \] \hspace{1cm} (3.72)

The optical theorem says that the total scattering cross-section is related to the imaginary part of the forward scattering amplitude; i.e.

\[ \sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im} f(k,k) \] \hspace{1cm} (3.73)

using equation (3.72) this gives
\[ \sigma_{\text{tot}} = 2 \int (1 - \text{Re} e^{i\chi(b)}) \, d(2)b \]  

(3.74)

The total scattering cross-section is also given by

\[ \int |f(k', k)|^2 \, d\Omega_{k'} = (k/2\pi)^2 \int e^{i(k - k') \cdot (b - b')} 
\]

\[ [e^{i\chi(b)} - 1] [e^{-i\chi(b')} - 1] \, d(2)b \, d(2)b' \, d\Omega_{k'}, \]  

(3.75)

If we make the assumption that the scattering is concentrated in the forward direction, we can replace the integration over the sphere \(|k'| = |k|\) by an integration over the plane in \(k'\) space which is tangent to the sphere at \(k' = k\) i.e. in the forward direction. In this case we can write.

\[ d\Omega_{k'} = \frac{d(2)k'}{k'^2} \]  

(3.76)

where the differential element \(d(2)k'\) lies in a plane perpendicular to \(k'\).

Noting that

\[ \int e^{i(k - k') \cdot (b - b')} \, d(2)k' = (2\pi)^2 \delta(2)(b - b') \]  

(3.77)

where \(d(2)(b - b')\) is a two dimensional delta function and carrying out the angular integration we find

\[ \sigma_{\text{scatt}} = \int |e^{i\chi(b)} - 1|^2 \, d(2)b \]  

(3.78)

For real \(\chi\), i.e. in the absence of absorption, we have
\[ \sigma_{\text{scatt}} = 2 \int (1 - \text{Re} e^{i \chi(b)}) \ d(2)_b \]  

(3.79)

which agrees with equation (3.74) obtained from conservation theorem. The absorption cross-section is obtained by subtracting equation (3.78) from equation (3.74).

\[ \sigma_{\text{abs}} = \sigma_{\text{tot}} - \sigma_{\text{scatt}}. \]  

(3.80)

\[ = \int (1 - |e^{i \chi(b)}|^2) \ d(2)_b \]  

(3.81)

If the optical potential is assumed to be spherically symmetric then the phase shift function

\[ \chi(b) = -\frac{1}{h \nu} \int_0^\infty V(\sqrt{b^2 + z^2}) \ dz \]  

(3.82)

As \( \chi(b) \) may be complex depending whether the potential is complex, we may write

\[ \chi = \chi_1 + i \chi_2 \]  

(3.83)

in which case equation (3.81) becomes

\[ \sigma_{\text{abs}} = \int (1 - e^{-2 \chi_2(b)}) \ d(2)_b \]  

(3.84)

If we let \( V = U + iW \) then

\[ \sigma_{\text{abs}} = \int \left( 1 - e^{-\frac{2}{h \nu} \int_0^\infty W(\sqrt{b^2 + z^2}) \ dz} \right) \ d(2)_b \]  

(3.85)
This means that the real part of the optical potential does not contribute to the cross-sections in the Glauber formalism. As we know, the Coulomb potential is real, and in Glauber's treatment the phase shifts are additive; hence the Coulomb contribution to the cross-sections is null. This is the main difference between the two scattering models. The contribution due to the Coulomb field is essential, as the pion is charged. The magnitude contribution to the cross-section depends on the sign of charge. Hence for $\pi^+,\pi^-$ reactions we would expect a difference in the cross-sections. But in the Glauber formalism these are the same for an even-even nucleus, with the assumption that neutron, proton density distributions to be identical. However, as far as the real part of the nuclear field is concerned, the effect of this is very small at high energies, since then, the magnitude of the real part of the optical potential is very small. The Glauber formalism for an uncharged particle should hold good at high energies.
Chapter 4

Two body Density Functions and Correlation Functions

Studies of high energy elastic \(^45\) and inelastic \(^46\) electron scattering, elastic \(^47\) and inelastic \(^48\) nucleon scattering, and pion absorption \(^49\) have stimulated interest in the two nucleon density functions and short range nucleon-nucleon correlations. In the scattering processes the effects of short range correlations may appear in the high momentum components of the nuclear momentum distribution (form factor) or in the second and higher order terms in a multiple scattering approximation. In the case of pion absorption, the energy momentum conservation requires that the absorption occurs on a single nucleon with very high momentum or on a closely correlated pair of nucleons.

The short-range nuclear correlations arise from two sources. The Pauli correlations arise simply from the particle statistics and are introduced through the use of a properly anti-symmetrized nuclear wavefunctions. The dynamical correlations arise from the short-range behaviour of the nucleon-nucleon interactions. The later can be introduced through the use of a Jastrow-type wavefunction \(^50,51\), a Bethe-Goldstone wavefunction calculated with a suitable nucleon-nucleon force \(^52\), or through the use of a Unitary Model operator \(^53\).

In majority of calculations the correlations are introduced into a shell-model wavefunction constructed from harmonic oscillator single-particle functions, and certain of the methods \(^52,53\) are feasible only when an oscillator basis is used.
It is well known (see for e.g. Gerace and Sparrow ref. 45), and references 54,55) that these shell-model wavefunctions cannot give a satisfactory description of the one-particle density function for nuclei unless correlations are included, whereas shell-model wavefunctions constructed from single-particle functions generated in a finite nuclear potential do give a very satisfactory description of the one-particle density function 56,57,58) without invoking correlations.

4.1. Derivation of the Two-body Correlation Functions

We shall be concerned only with the Pauli correlations. The one-particle density function and the two-particle density function \( \rho(r,r') \) are defined as 59)

\[
\rho(\mathbf{r}) = \frac{1}{A} \langle 0 | \sum_{j=1}^{A} \delta(\mathbf{r} - \mathbf{r}_j) | 0 \rangle \quad (4.1)
\]

\[
\rho(\mathbf{r},\mathbf{r}') = \frac{1}{A(A-1)} \langle 0 | \sum_{j,k \neq j} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_k) | 0 \rangle \quad (4.2)
\]

where \( | 0 \rangle \) represents a normalized, fully antisymmetrized wavefunction for the nuclear ground state, and \( A \) is the total number of nucleons. If we construct the nuclear wavefunctions from a single Slater determinant of orthonormal single-particle basis states \( \psi_\mu \), where \( \mu \) represents the appropriate set of quantum numbers, these functions become

\[
A \rho(\mathbf{r}) = \sum_\mu |\psi_\mu(\mathbf{r})|^2 \quad (4.3)
\]
\[ A(A-1)\rho(\vec{r},\vec{r}') = \sum \sum_{\mu \neq \nu} \langle \psi_\mu(\vec{r}') \psi_\nu(\vec{r}) | [\psi_\mu(\vec{r}') \psi_\nu(\vec{r})] \rangle - \psi_\nu(\vec{r}')\psi_\mu(\vec{r}) \] 

\[ \sum \sum_{\mu \neq \nu} |\psi_\mu(\vec{r}')|^2 |\psi_\nu(\vec{r})|^2 - \psi_\mu^*(\vec{r}')\psi_\mu(\vec{r})\psi_\nu^*(\vec{r})\psi_\nu(\vec{r}') \]  

(4.4)

where we have added and subtracted diagonal terms in order to obtain the last line. Hence we can write the two-particle density function in the form

\[ A(A-1)\rho(\vec{r},\vec{r}') = A^2 \rho(\vec{r})\rho(\vec{r}') - A \rho_{\text{ex}}(\vec{r},\vec{r}') \]  

(4.5)

with

\[ A \rho_{\text{ex}}(\vec{r},\vec{r}') = \sum \sum_{\mu \neq \nu} \psi_\mu^*(\vec{r}')\psi_\mu(\vec{r})\psi_\nu^*(\vec{r})\psi_\nu(\vec{r}') \]  

(4.6)

McVoy and Van Hove call \( \rho(\vec{r},\vec{r}') \) the nucleon-nucleon correlation function and \( \rho_{\text{ex}}(\vec{r},\vec{r}') \) the exchange sum. Wong defines the nucleon-nucleon correlation function as

\[ C(\vec{r},\vec{r}') = \rho(\vec{r},\vec{r}') - \rho(\vec{r})\rho(\vec{r}') \]  

(4.7)

which using equations (4.3) and (4.4), is given by

\[ (A-1)C(\vec{r},\vec{r}') = \rho(\vec{r})\rho(\vec{r}') - \rho_{\text{ex}}(\vec{r},\vec{r}') \]  

(4.8)

From this expression and from the definition of \( \rho_{\text{ex}}(\vec{r},\vec{r}') \) we see that if there are no Pauli correlations,

\[ \rho(\vec{r},\vec{r}') = \rho_{\text{ex}}(\vec{r},\vec{r}') = \rho(\vec{r})\rho(\vec{r}') \]
and hence
\[ C(r, r') = 0 \]

Gottfried \(^{59}\) denotes our \( \rho(r, r') \) by \( C(r, r') \), our \( C(r, r') \) by \( D(r, r') \) and calls both pair correlation functions. Introducing the Fourier transforms of the functions \( C(r, r') \) and \( \rho_{\text{ex}}(r, r') \),

\[
C(q) = \iint e^{-iq \cdot (r - r')} C(r, r') \, dr \, dr' \\
D(q) = \iint e^{-iq \cdot (r - r')} \rho_{\text{ex}}(r, r') \, dr \, dr'
\]

we can write equation (4.8) as

\[
A(A-1) C(q) = F^2(q) - D(q)
\]

where \( F(q) \) is the Fourier transform of the matter distribution, i.e.

\[
F(q) = \int \rho(r) e^{iq \cdot r} \, dr
\]

We denote \( C(q) \) as the correlation function and \( D(q) \) the exchange contribution.

This terminology is not entirely satisfactory since, as noted above, the exchange term is not zero when there are no Pauli correlations. The function \( D(q) \) contains terms with \( \mu = \nu \) which cancel out the contribution to \( C(q) \) from the uncorrelated density; thus \(^4\)He where there is only one filled shell and no Pauli correlations it
follows that \( D(q) = F^2(q) \) and \( C(q) = 0 \).

If the single particle states \( \psi_\mu \) are described in terms of \( jj \)-coupling the symbol \( \mu \) represents the quantum numbers \( n, \ell, j, m_j \) and \( m_t \) and the single-particle wavefunctions are given by

\[
\psi_\mu(r) = \sum_{m_\ell} \left( \ell \frac{1}{2} j; m_\ell, m_j, m_j \right) \phi_{n\ell j}^{m_t}(r)
\]

\[
\chi_{\frac{1}{2}}^{m_j - m_\ell} \chi_{\frac{1}{2}}^{m_t}
\]

(4.13)

where \( \chi_{\frac{1}{2}}^{m_j - m_\ell} \) is the usual spin function and \( \chi_{\frac{1}{2}}^{m_t} \) is the isospin function and,

\[
\phi_{n\ell j}^{m_t}(r) = R_{n\ell j}^{m_t}(r) Y_{\ell}^{m_t}(\hat{r})
\]

(4.14)

Hence

\[
\rho_{\text{ex}}(r, r') = \sum_{n\ell j m_j} \sum_{m_\ell} \left( \ell \frac{1}{2} j; m_\ell, m_j, m_j \right) \phi_{n\ell j}^{m_t}(r) \chi_{\frac{1}{2}}^{m_j - m_\ell} \chi_{\frac{1}{2}}^{m_t} \sum_{m_\ell'} \left( \ell' \frac{1}{2} j'; m_\ell', m_j, m_\ell', m_j \right) \phi_{n'\ell' j'}^{m_t'}(r') \chi_{\frac{1}{2}}^{m_j' - m_\ell'} \chi_{\frac{1}{2}}^{m_t'}
\]

(4.15)

now

\[
\sum_{m_s} \chi_{\frac{1}{2}}^{m_s} \chi_{\frac{1}{2}}^{m_s'} = \delta_{m_s m_s'}
\]

(4.16)
Expanding $e^{iq\cdot z}$ we have

$$e^{iq\cdot z} = \sum_{L} i^L [4\pi(2L+1)]^{1/2} Y_{L}^{O}(\hat{z}) j_{L}(qr)$$ \hspace{1cm} (4.18)$$

We make an assumption in evaluating the Fourier transform of $\rho_{ex}(r,r')$, i.e. $\phi_{n'j'm'}^{m'}(r)$ is assumed to be independent of $m_t$, then using the relations (4.15 - 4.17) above we have,

$$AD(q) = \sum_{n'j'} \sum_{j'm'} \int \frac{i^L [4\pi(2L+1)]^{1/2}}{r^2} Y_{L}^{O}(\hat{z}) Y_{L}^{O'}(\hat{z}') j_{L}(qr) \frac{1}{r'^2}$$

$$\left\{ \sum_{m_{L}''} (\lambda' \frac{1}{2} j' ; m_{L}'' , m_{j'} ; m_{j''} , m_{j''} \right\}$$

$$\sum_{m_{L}''} i^L' [4\pi(2L'+1)]^{1/2} \int Y_{L}^{O'} Y_{L}^{O''} Y_{L}^{O} d\Omega_{r},$$

$$\int R_{n'j'}^{m'}(r') R_{n'j'}^{m'}(r') j (qr') r'^2 \frac{1}{r'^2}$$

$$\left\{ \sum_{m_{L}''} (\lambda' \frac{1}{2} j' ; m_{L}'' , m_{j'} ; m_{j''} , m_{j''} \right\}$$

$$\left( \lambda' \frac{1}{2} j' ; m_{L}'' , m_{j'} ; m_{j''} , m_{j''} \right\}$$

Strictly speaking the radial wavefunction $R_{n'j'}^{m'}(r)$ should read as $R_{n'j'}(r)$, since the summation over the isospin functions were carried under the assumption that these radial wavefunctions were independent of $m_t$. Nevertheless, we shall retain this terminology to differentiate protons and neutrons. Summation over $m_t$ here, implies then, the sum
over neutron and proton radial wavefunctions. In order to carry out the angular integration of the spherical harmonics we note (see for e.g. ref. 60)

\[
\int d\Omega \, Y_{l_3}^{m_3} Y_{l_2}^{m_2} Y_{l_1}^{m_1} = \sum_{l} \frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)} (l_1 l_2 l_3 ; m_3 m_2 m_1)
\]

The spherical harmonics are orthonormal and, therefore, only the $l = l_3$ term contributes.

\[
\int d\Omega \, Y_{l_3}^{m_3} Y_{l_2}^{m_2} Y_{l_1}^{m_1} = \frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)} (l_1 l_2 l_3 ; m_3 m_2 m_1)
\]

(4.20)

Using the relation (4.20) we obtain $m'_l = m''_l$ and $m''_l = m'''_l$.

\[
(m_l = m'_l)
\]

\[
\int d\Omega \, Y_{l}^{m_l} Y_{L}^{0} Y_{l'}^{m_l''} = \frac{(2l'+1)(2L+1)}{4\pi}
\]

(4.20)

Applying the symmetry relations for the Clebch-Gordon coefficients, equation on the right becomes

\[
(-)^{2l'-m_l} \frac{(2l'+1)(2L+1)}{4\pi(2L+1)} (l l' L ; 0 0 0) (l l' L ; m_l' , -m_l 0)
\]
Also

\[(\lambda \ j; m_\lambda, m_j, -m_\lambda, m_j)(\lambda' \ j'; m_\lambda, m_j, -m_\lambda, m_j)\]

\[= (-)^{\lambda-m_\lambda}\left(\frac{2j+1}{2}\right)^{\frac{1}{2}} (\lambda \ j; \frac{1}{2}; m_\lambda, -m_j, m_\lambda, -m_j)\]

\[(-)^{\frac{1}{2}+m_j-m_\lambda}\left(\frac{2j'+1}{2}\right)^{\frac{1}{2}} (\lambda' \ j'; m_\lambda, m_j, -m_\lambda, m_j)\]

\[= (-)^{\lambda+m_j+m_\lambda}\left[\frac{(2j+1)(2j'+1)}{2(2\lambda'+1)}\right]^{\frac{1}{2}} \sum_{f} [2(2f+1)]^{\frac{1}{2}}\]

\[W(\lambda \ j \ \lambda' \ j'; \frac{1}{2} f)(j \ j'; -m_j \ m_j 0) (\lambda \ f \ \lambda' \ m_\lambda 0 m_\lambda) \]

(4.21)

where \(W(\lambda \ j \ \lambda' \ j'; \frac{1}{2} f)\) are the Racah coefficients.

Introducing the above quantities in equation (4.19) yields

\[A_D(q) = \sum_{n \lambda j m_\lambda m_j} \sum_{n' \lambda' j' m_\lambda' m_j'} i^{L}(-)^{\lambda+\lambda'+m_j-m_\lambda} W(\lambda \ j \ \lambda' \ j'; \frac{1}{2} f)\]

\[\left[\frac{(2\lambda+1)(2j+1)(2j'+1)(2f+1)}{2(2\lambda'+1)}\right]^{\frac{1}{2}} W(\lambda \ j \ \lambda' \ j'; \frac{1}{2} f)\]

\[(\lambda \ L; 0 \ 0 \ 0) (\lambda \ L; m_\lambda, -m_\lambda, 0)(j \ j'; f; -m_j \ m_j 0)\]

\[(\lambda \ f \ \lambda' \ m_\lambda 0 m_\lambda) \Psi(jj'; \lambda L m_\lambda m_\lambda'; q)\]

\[\sum_{m_\lambda'} i^{L'}(-)^{\lambda+m_j-m_\lambda} \left[\frac{(2\lambda+1)(2j+1)(2j'+1)(2f+1)}{2(2\lambda'+1)}\right]^{\frac{1}{2}}\]

\[L']\]
\[ W(\ell \, j \, j'; L; \frac{1}{2} f') (\& \, \ell' \, L'; 0 \, 0 \, 0) (\& \, \ell' \, L'; m_{\ell'}, -m_{\ell'}, 0) \]

\[ (j \, j'; -m_j \, m_j \, 0) (\& \, f' \, \ell'; m_{\ell'}, 0 \, m_{\ell'}) \]

\[ I(jj' \& \ell' L_{m_t} m_{t'}; -q) \]  \hspace{1cm} (4.22)

where

\[ I(jj' \& \ell' L_{m_t} m_{t'}; q) = \int R_{n \& j}^{m_t(r)} R_{n' \ell' j'}^{m_{t'}(r)} j_L(qr) r^2 \, dr \]  \hspace{1cm} (4.23)

Now

\[ \sum_{m_{\ell}} (\& \, \ell' \, L; m_{\ell'}, -m_{\ell'}, 0) (\& \, f' \, \ell'; m_{\ell'}, 0 \, m_{\ell'}) \]

\[ = \sum_{m_{\ell}} (-)^{\ell-m_{\ell}} \left( \frac{2\ell'+1}{2f+1} \right) (\& \, \ell' \, f; m_{\ell'}, -m_{\ell'}, 0) \]

\[ (\& \, \ell' \, L; m_{\ell'}, -m_{\ell'}, 0) \]

\[ = (-)^{\ell-m_{\ell}} \left( \frac{2\ell'+1}{2f+1} \right) \delta_{\ell L} \]  \hspace{1cm} (4.24)

using this relation in equation (4.22) we can write,

\[ AD(q) = \sum_{n \& j} \sum_{m_{m_t}} \sum_{L} \sum_{L'} (2\ell+1)(2\ell'+1)(2j+1)(2j'+1) \]

\[ W(\ell \, \ell' \, j \, j'; L; \frac{1}{2}) W(\ell \, j \, j'; L'; \frac{1}{2}) (\& \, \ell' \, L; 0 \, 0 \, 0) \]

\[ (\& \, \ell' \, L'; 0 \, 0 \, 0) (j \, j'; -m_j \, m_j \, 0) (j \, j'; -m_j \, m_j \, 0) I(jj' \& \ell' L_{m_t} m_{t'}; q) \]

\[ I(jj' \& \ell' L_{m_t} m_{t'}; -q) \]  \hspace{1cm} (4.25)
If we introduce the functions $P(n \ell j m_t)$ where $P(n \ell j m_t)$ is the occupation probability of the state $n \ell j$ for protons or neutrons (see Appendix A) and noting that

$$\sum_{m_j} (j' j' L; -m_j m_j 0)(j' j' L'; -m_j m_j 0) = \delta_{LL'}$$  \hspace{1cm} (4.26)

we finally arrive at

$$A D(q) = \sum_{n \ell j m_t} \sum_{L} i^{2L}(2\ell+1)(2\ell'+1)(2j+1)(2j'+1) W^2(\ell \ell' j j'; L \frac{1}{2})(\ell \ell' L; 0 0 0)^2 P(n \ell j m_t)$$

$$P(n' \ell' j' m'_t)I(jj' \ell \ell' L m_t m'_t; q)I(jj' \ell \ell' L m_t m'_t; -q)$$  \hspace{1cm} (4.27)

a result similar to that previously obtained by Sitenko and Simenog \(^{46}\) for protons.

The above expression is correct form for $D(q)$ when the single particle wavefunctions are generated in a realistic single-particle potential with spin-orbit and Coulomb forces. When the radial integrals $I$ are independent of $j$ and $j'$, and $m_t$ and $m'_t$, i.e. when the spin-orbit and Coulomb forces in the single-particle potential are omitted, and both $j$-subshells are fully occupied for a given $\ell$, the sums over $jj'$ and $m_t m'_t$ can be carried out independently to give
\[ \sum_{jj'} (2j+1)(2j'+1) W^2 (LL' jj'; L_{1/2} L_{3/2}) = \]
\[ \sum_{jj'} 2(2L+1)(2j+1)(2j'+1) \left( \begin{array}{c} \frac{1}{2} \; 0 \\ jj' L_L \end{array} \right)^2 \]
\[ = \sum_{jj'} 2(2L+1)(2j+1)(2j'+1) \left( \begin{array}{c} \frac{1}{2} \; 0 \\ jj' L_L \end{array} \right)^2 \]
\[ = \frac{2(2L+1) \delta_{LL}}{(2L+1)} = 2 \] (4.28)

\[ \sum_{m_t'm_t'} P(n_Lj_m_t) P(n_L'j_m_t') = 2 , \] (4.29)

so that

\[ AD(q) = \sum_{L} \sum_{nL} i^{2L} (2L+1)(2L'+1)(LL_L; 0 0 0)^2 \]
\[ I(\frac{1}{2}L; q) I(\frac{1}{2}L; -q) \] (4.30)

This expression for \( D(q) \) is the appropriate form when oscillator wavefunctions are used for nuclei with closed shells in \( LS \) scheme, e.g. \(^4\text{He}, \; ^{16}\text{O}, \; ^{40}\text{Ca}\). For a nucleus such as \(^{12}\text{C}\) where in the \( jj \) coupling model the \( P_{1/2} \) sub-shell is empty and equation (4.28) does not hold, as can be seen from Table 1, and hence even if the radial integrals were independent of \( jj' \) and \( m_t'm_t' \) equations (4.27) and (4.30) would not give identical results. The same is true for \(^6\text{Li}\) in which the \( P_{3/2} \) sub-shell is partially filled. The correlation function for these nuclei in the oscillator basis can be evaluated by labelling the single-particle states \( \nu \) by \( n_L L_m m_t \) only. This gives
Weighting of the contributions to the exchange term $D_{SWO}(q)$. The quantity $X(jj'\ell\ell'L)$ represents the product $(2j+1)(2j'+1)n (\ell \ell' j j'; L \frac{1}{2})$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$j$</th>
<th>$\ell'$</th>
<th>$j'$</th>
<th>$L$</th>
<th>$X(jj'\ell\ell'L)$</th>
<th>$\sum_{jj'} X(jj'\ell\ell'L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{1/2}$</td>
<td>$s_{1/2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
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<td>1</td>
<td>2</td>
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<tr>
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<td>2</td>
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<td>$p_{1/2}$</td>
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<td>$\frac{1}{2}$</td>
<td>$\frac{2}{3}$</td>
<td>2</td>
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<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>$\frac{4}{3}$</td>
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<td>$p_{3/2}$</td>
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<td>$p_{3/2}$</td>
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<td>$\frac{2}{3}$</td>
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<tr>
<td>$p_{1/2}$</td>
<td>$p_{1/2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
AD(q) = \sum_{n \leq m' } \sum_{L}^{L} i^{2L} L_{4}(2L+1)(2L'+1)(L L' L; 0 0 0)^{2}

P(n\ell)P(n'\ell')I(\ell\ell'L; q)I(\ell\ell'L;-q) \quad (4.31)

For spherically symmetric nuclei, the form factor F(q) has terms with L = 0 only and the spin orbit term is in the single-particle potential affects the result only through the behaviour of the radial integrals of the form

\[ I(j\ell 0; q) = \int R_{n}^{2}(r) j_{0}(qr) r^{2} dr \]

For nuclei with non-zero quadrupole moments there are also terms with L = 2. Of the nuclei considered here, only \(^6\)Li has a non-zero quadrupole moment and this is sufficiently small to be neglected.

4.2. Evaluation of the Exchange Sum

A) Harmonic oscillator well.

The equations (4.27) and (4.31) which were derived in the last section give us the general expressions for the exchange contributions in the \( jj \) and \( ls \) coupling schemes respectively. Nevertheless, it would be of interest to obtain explicit formula for the function D(q) when an oscillator basis is used.

The harmonic oscillator radial wavefunction \( R_{n\ell}(r) \) in the reduced mass system is

\[ R_{n\ell}(r) = A_{n\ell}(r/b)^{\ell} L_{n-\ell} L_{\ell+\frac{1}{2}}(r^{2}/b^{2}) e^{-r^{2}/2b^{2}} \quad (4.32) \]
where \( b \) is the oscillator length parameter and is given by
\[ b^2 = \hbar/\mu \omega, \]
where \( \mu \) is the reduced mass of the nucleon. \( L_k^m(r) \) are the generalized Laguerre polynomials defined as
\[
L_k^m(r) = \frac{k!}{s!} \frac{\Gamma(k + m + 1) (-r)^s}{\Gamma(s + m + 1) (k - s)!}
\]
and
\[
|A_{n\ell}|^2 = \frac{2(n - 1)!}{b^3 \Gamma(n + \ell + \frac{3}{2})}
\]

From the above expression for the Laguerre polynomials we obtain for the lowest values of the principal quantum numbers \( n \)

\[
L_0^{\ell+\frac{3}{2}}(r) = 1 \quad L_1^{\ell+\frac{3}{2}}(r) = (\ell + \frac{3}{2})
\]

\[
L_2^{\ell+\frac{3}{2}}(r) = \frac{3}{2} (\ell + \frac{3}{2}) (\ell + \frac{5}{2}) - (\ell + \frac{5}{2}) r + \frac{1}{2} r^2
\]

The oscillator shell-model Pauli correlations and exchange matrix elements can be obtained using equations (4.11), (4.31) and (4.32).

The results are:

**\(^6\)Li:**

\[
(A-1)C(q) = -e^{-\frac{1}{2} q^2 b^2} \left(-\frac{2}{3} q^2 b^2 \sqrt{\frac{q}{6}} - \frac{2 q^4 b^4}{81}\right)
\]

\[
D(q) = e^{-\frac{1}{2} q^2 b^2} \left[1 + \frac{2}{3} q^2 b^2 \left(\sqrt{\frac{q}{6}} - \frac{1}{6}\right) + \frac{q^4 b^4}{36}\right]
\]

**\(^{12}\)C:**

\[
(A-1)C(q) = -e^{-\frac{1}{2} q^2 b^2} \left(\frac{2}{3} \sqrt{\frac{q}{3}} q^2 b^2 + \frac{7}{162} q^4 b^4\right)
\]

\[
D(q) = e^{-\frac{1}{2} q^2 b^2} \left[1 + \frac{q^2 b^2}{3} \left(\sqrt{\frac{q}{3}} - \frac{1}{2}\right) + \frac{q^4 b^4}{18}\right]
\]
\[(\Lambda - 1)C(q) = -e^{-\frac{1}{2}q^2b^2}\left(\frac{1}{4}q^2b^2 + \frac{3}{64}q^4b^4\right)\] (4.39)

\[D(q) = e^{-\frac{1}{2}q^2b^2}\left(1 + \frac{1}{16}q^4b^4\right)\] (4.40)

\[\text{\textsuperscript{40}Ca:}\]
\[(\Lambda - 1)C(q) = -e^{-\frac{1}{2}q^2b^2}\left(\frac{1}{2}q^2b^2 + \frac{3}{80}q^4b^4\right)
- \frac{1}{160}q^6b^6 + \frac{9}{6400}q^8b^8)\] (4.41)

\[D(q) = e^{-\frac{1}{2}q^2b^2}\left(1 + \frac{1}{8}q^4b^4 - \frac{1}{80}q^6b^6
+ \frac{1}{640}q^8b^8\right)\] (4.42)

For \textsuperscript{160}, \textsuperscript{40}Ca, formula for the correlation functions and exchange contributions have already been derived by Wong using the Brody-Moshinsky transformation, and are in agreement with those given here.

B) Saxon-Wood Potential

The wavefunctions in this case are obtained from a potential of the form

\[V(r) = -V_e f(r) + V_{s\epsilon} \left(\frac{\hbar}{\pi mc}\right)^2 \frac{1}{r} \frac{df}{dr} \cdot \mathbf{\hat{r}} + V_c\] (4.43)

where

- \(V_c\) - Coulomb potential
- \(V_e\) - Central potential
- \(V_{s\epsilon}\) - Spin orbit potential
- \(R = r_o (A - 1)\frac{1}{2}\)
- \(f(r) = (1 + \exp\left(\frac{1 - R}{a}\right))^{-1}\)
The potentials $V_s, V_{se}, V_e$ are different for each shell and, $V_e$ is the Coulomb potential taken to be due to an equivalent uniform sphere of the same radius as the actual nucleus. Since the potential strengths are different for each level, the Hamiltonian for each level varies in a similar manner. As we know, the eigenfunctions of the Hamiltonian forms a complete set, if and only if these belong to that particular Hamiltonian. Hence the states of the nucleus do not form a complete set, since these states do not belong to the same Hamiltonian. This leads to slight non-orthogonality of the wavefunctions. It is possible that for nuclei with $A < 12$ this discrepancy would be small, as we would expect the variation in potential strength with level spacing to be small. However, the overlap integral of the $1s$ and $2s$ state wavefunctions for $^{40}$Ca gives rise to $-0.12$ instead of $0$. This means that in obtaining the density distribution for e.g., we introduce an error, since the overlap integrals are non-zero in this case. The problem of the energy dependence of the potential can be avoided if we use a non-local potential \textsuperscript{57}). For the present calculations, we shall assume that this discrepancy is not too severe for nuclei $A < 40$, and hence we shall take for granted that the set of states of the nucleus form a complete and orthogonal set.

Unlike the case of the oscillator well, it is not possible to obtain analytical expressions for the exchange contributions and correlations. But numerical values for these have been calculated by means of the computer, and it was found that the computing time was not excessive.
Some Applications of the Two-body Correlations

5.1. Nuclei Studied:

In the last Chapter we went through the derivation of the two-body density functions and correlation functions. We shall now discuss the results of the calculations on $^6\text{Li}$, $^{12}\text{C}$, $^{16}\text{O}$, and $^{40}\text{Ca}$.

For Saxon-wood potentials we have used parameters given by Elton and Swift $^{56}$. These parameters are listed in Table 2. When the nuclear charge distribution is obtained from the proton wavefunctions generated in these potentials, the cross-section for elastic electron scattering from this charge distribution calculated by an exact phase shift analysis is in agreement with the experimental data for incident electron energies in the range 150 - 400 MeV and momentum transfers up to $q - 2.5\text{fm}^{-1}$. This can be seen from the comparisons with data given in ref. $^{56,61}$. Direct comparison of $F^2(q)$ with experimental data can be quite misleading since

1) even for $^{12}\text{C}$ and $^{16}\text{O}$ there is some difference between Born approximation and the exact phase analysis at and beyond the first Born minimum, and

2) the finite size of the proton has a large effect beyond the first minimum.
Table 2. Well parameters and energy levels.

(Energies in MeV, distances in fm.)

<table>
<thead>
<tr>
<th>Nucleide</th>
<th>Level</th>
<th>$V_\epsilon$</th>
<th>$r_0$</th>
<th>$V_{sc}$</th>
<th>$a$</th>
</tr>
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<tbody>
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<td>$^6$Li</td>
<td>$1s_{1/2}$</td>
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<td>1.42</td>
<td></td>
<td>0.65</td>
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<tr>
<td></td>
<td>$1p_{3/2}$</td>
<td>51</td>
<td>1.45</td>
<td>8.3</td>
<td>0.65</td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>$1s_{1/2}$</td>
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<td>1.36</td>
<td></td>
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<td>$1p_{3/2}$</td>
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<td>1.36</td>
<td>9</td>
<td>0.55</td>
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<tr>
<td>$^{16}$O</td>
<td>$1s_{1/2}$</td>
<td>68</td>
<td>1.41</td>
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<tr>
<td></td>
<td>$1p_{3/2}$</td>
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<td>1.41</td>
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<td>0.65</td>
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<tr>
<td></td>
<td>$1p_{1/2}$</td>
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<td></td>
</tr>
<tr>
<td>$^{40}$Ca</td>
<td>$1s_{1/2}$</td>
<td>85</td>
<td>1.30</td>
<td></td>
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<tr>
<td></td>
<td>$1d_{3/2}$</td>
<td></td>
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</tr>
</tbody>
</table>
These two points, of which the second is the more important in this context, were established in the early literature on electron scattering.

The results obtained using harmonic oscillator functions are denoted by the suffix HO, those obtained using Saxon-woods potential listed in Table 2 are denoted by SWO, and those obtained using the Saxon-woods potentials but without the spin-orbit term are denoted by SW.

Since the correction for the centre-of-mass motion in the shell model is not usually applied to the Saxon-woods wavefunctions, we have not included this correction in the calculations with oscillator functions.

\(^6\text{Li}: \) No satisfactory fit to the data on elastic electron scattering from this nucleus can be obtained using a single oscillator parameter \(^6\text{Li},^6\text{Li}^\) . We have used the oscillator parameter \(b = 1.78\text{fm}\) which gives fair agreement with the data for \(q^2 < 2\text{fm}^{-2}\). The form factors \(F^2(q)\) obtained using HO and SWO wavefunctions are shown in Fig. (4), and it can be seen that there is a large discrepancy for \(q > 1.5\text{fm}^{-1}\). The use of these wavefunctions also leads to a marked difference in the predictions for the \(^6\text{Li}(p,2p)\) reaction \(^6\text{Li}\). The charge form-factor \(\chi^2F^2(q)\) which is the Fourier transform of the nuclear charge distribution is also shown for SWO functions.

\(^{12}\text{C}: \) For this nucleus it is possible to give a good description of the data on elastic electron scattering over a range of momentum transfer up to \(-2.5\text{fm}^{-1}\) using oscillator wavefunctions with a length parameter \(b = 1.64\text{fm}\). Consequently, the form factors
$b = 1.78 \text{ fm}$

$F_{ho}^2(q)$

$F_{sw0}^2(q)$

$F_{sw0}(q)$

$F_{ho}(q)$

$F_{ho}(q)$

$F_{sw0}(q)$

$F_{sw0}(q)$

$\log_{10}^2$ vs $q_{(\text{fm}^{-1})}$

FIGURE (4)
F²(q) obtained using HO and SWO wavefunctions are in reasonable agreement as can be seen from Fig. (5). The HO functions are also relatively successful in describing such processes as the ¹²C(p,d) reaction ⁶⁶) and ¹²C(p,2p) reaction ⁶⁵) at incident energies in the region of 100 - 200 MeV, and this appears to be due to the rather high separation energies of the p-shell nucleons. The difference between F²(q) and CHF²(q) is evident in Fig. (5).

¹⁶O: For this nucleus it is possible to use HO functions to describe elastic electron scattering for q < 2.5fm⁻¹ with a length parameter b = 1.76fm ⁶²,⁶³). We have used a parameter b = 1.75fm in order to make a comparison with the calculations by Wong ⁵²) of the correlation function. The form factors are shown in Fig. (6).

⁴⁰Ca: For this nucleus the HO functions do not give a satisfactory fit to elastic electron scattering. The best fit is obtained with b = 1.95fm ⁶¹) but the agreement is satisfactory only within the first diffraction minimum which falls at about q ~ 1.2fm⁻¹. We have chosen the parameter b = 1.99fm, again to make comparison with calculations by Wong. The SWO functions do not fit the data beyond q ~ 2.5fm⁻¹ without further modification such as the explicit inclusion of configuration mixing ⁵⁷). Also the energy dependence of the Saxon-woods potential leads to a slight non-orthogonality of the 1s and 2s functions which we have neglected. The form factors are compared in Fig. (7), and it can be seen that there is substantial disagreement in the region of the second Born minimum and beyond.
\( b = 1.64 \text{ fm} \)

- \( F_{ho}^2(q) \)
- \( F_{swo}^2(q) \)

FIG(5)
FIG(6)
5.2. Results and Discussion

Results for the correlation function \( C(q) \) and exchange contributions \( D(q) \) calculated using HO functions are shown in Figs. (8-11). For a given nucleus, the maximum value of \( C_{\text{HO}}(q) \) is independent of the length parameter \( b \), but the position of the maximum moves as \( b \) is varied in such a way that \( q_{\text{max}} b = \text{constant} \). This is illustrated in Fig. (7). It can also be seen from Fig. (12) that \( C_{\text{HO}}(q) \) is very insensitive to the actual value of \( b \) for \( q < q_{\text{max}} \) but for \( q > 1.5 \text{fm}^{-1} \) the effect of small variations in \( b \) is quite marked. The results for \(^{160}\text{O}\) and \(^{40}\text{Ca}\) are in excellent agreement with those obtained by Wong \(^{52}\) who used a Brody-Moshinsky transformation to derive expressions for \( C_{\text{HO}}(q) \) and \( D_{\text{HO}}(q) \).

The results obtained for the correlation function with SWO functions are also shown in Figs. (8-11) and the ratio \( C_{\text{HO}}(q)/C_{\text{SWO}}(q) \) is plotted in Fig. (13). We see that there is a considerable disagreement between the HO and SWO results. The best agreement occurs for \(^{160}\text{O}\) as might be expected from the good agreement for the form factor shown in Fig. (6), and similarly the disagreement for \( q > 1\text{fm}^{-1} \) between the correlation function for \(^{40}\text{Ca}\) is to be expected from the poor agreement for the form factors, also shown in Fig. (7). The comparison for \(^{6}\text{Li}\) and \(^{12}\text{C}\) is discussed below. The separate contributions to the SWO exchange term for \(^{160}\text{O}\) are shown in Fig. (14). It can be seen that at small \( q \) the large contributions are those with \( n\ell j = n\ell'j' \) and these cancel out the corresponding contributions to the form factor so that \( C(q) \) is small, but as \( q \) increases the exchange between sub-shells becomes increasingly important and it
$b = 1.78 \text{ fm}$

$^6\text{Li}$

**FIG (8)**

- $D_{ho}(q)$
- $-(A-1)C_{ho}(q)$
- $-(A-1)C_{sw}(q)$

Graph showing functions against $q(\text{fm}^{-1})$. Axes range from 0 to 0.9 on the y-axis and 0 to 3 on the x-axis.
\( b = 1.64 \text{ fm} \)

\[ \frac{12}{C} \]

**FIG(9)**

- \( D_{ho}^{(q)} \)
- \( (A-1)C_{ho}^{(q)} \)
- \( (A-1)C_{sw}^{(q)} \)

**Q \( (\text{fm}^{-1}) \)**

0.9 \( \quad 0.8 \quad 0.7 \quad 0.6 \quad 0.5 \quad 0.4 \quad 0.3 \quad 0.2 \quad 0.1 \quad 0.0 \quad 1 \quad 2 \quad 3 \)
$^{16}_0$ O

FIG (12)

$q_q (fm^{-1})$

$b = 1.85 fm$

$b = 1.75 fm$

$b = 1.65 fm$
FIG(13)
is mainly through these contributions that sensitivity to the high momentum components of the single-particle wavefunctions occurs. Each contribution plotted in Fig. (9) contains terms arising from the various values of the angular momentum transfer $L$ allowed by the Coupling Coefficients in equation (4.27). The weighting of these terms is given in Table 1.

In order to investigate further the results for $^6$Li and $^{12}$C, we first recalculated the radial integrals $I(jj'\ell\ell'\ell_m\ell_{m'};q)$ using such functions and found complete agreement with the SWO results over the range of $q$ considered (the same agreement occurs for $^{16}$O and $^{40}$Ca and is in accordance with the result observed in electron scattering calculations $^{56,61}$) that the spin-orbit term in the single-particle potential is required to fit the nucleon separation energies but does not have any effect in the form factor. We then calculated a correlation function $C_{SW}(q)$ using SW functions and equation (4.31) for $D(q)$ so that the difference between $C_{HO}(q)$ arises only from the difference between the HO and SW single-particle wavefunctions. The ratio of these correlations is shown in Fig. (15). Comparison with Fig. (13) shows that the results for $^{16}$O and $^{40}$Ca are scarcely changed except at large $q$ where the correlation functions are very small. The agreement between the HO and SW calculations for $^{12}$C has improved dramatically; this is due to relatively good agreement of the radial wavefunctions in this case and indicates that the disagreement in Fig. (13) is due to the difference between $\ell s$ and $jj$ coupling. For $^6$Li a discrepancy between HO and SW calculations still remains, as is to be expected from the inadequacy of uncorrelated HO functions for this nucleus.
One could of course extend the present treatment to include centre of mass motion of the nucleus, in addition to using correlated wavefunctions to obtain better fit with experimental data for form factors. However, as we said earlier, the centre of mass motion can only be included if we use harmonic oscillator wavefunctions. Since our later work deals primarily with wavefunctions generated from potentials of the Saxon-woods type, we shall hence neglect the effect of centre of mass motion. The inclusion of correlated wavefunctions is probably necessary for nuclei such as $^6$Li, but as our present work is mainly concerned with "well behaved" nuclei we shall therefore not include such corrections.
Chapter 6

Fermi Motion of the Target Nucleons

So far we have concerned ourselves with the properties of two nucleon correlations. We investigated the Pauli correlations of nucleons using various shell model wavefunctions and find that the effect of Pauli correlations tend to show up at high momentum transfer. Hence, if we are obtaining information about nuclear properties by means of nucleon-nucleus scattering or by any other incident particle, we would expect the correlations to play a significant role in the intermediate energy range of the projectile, or when the momentum transfer is high.

However, when we are considering the kinematics of the reaction process, we note that the effect of Fermi motion of target nucleons have to be taken into account. This has a substantial effect on the π-nucleus total cross-sections. The experimental observations of Reeder and Makowitz 30) have indicated that the Fermi motion tends to broaden the full width at half maximum of the $^{12}\text{C}(\pi^-\nu^-)^{11}\text{C}$ total cross-section. Miller 35) using similar arguments averaged the $\pi-\text{N}$ scattering amplitudes over the nuclear momentum distribution $\rho(p)$, and was able to obtain reasonable agreement with the experimental estimation of the $\pi$-nucleus optical potential for several light nuclei. In his calculation he used a momentum distribution of the Fermi Gas model. It is our interest to estimate the effect of the choice of $\rho(p)$ on the averaging calculations. As we shall be studying $\pi$-nucleus scattering at high energies, i.e. $E_\pi > 500$ MeV, it is necessary that the calculation be treated relativistically. Although
the calculations will be done in the lab frame, transformation to
the two-body centre of mass system can be easily carried out, using
the invariance properties of the total cross-sections.

6.1. A Relativistic Form for the $\pi$-N Relative Momentum

Before we proceed to the relativistic treatment, let us first
consider the non-relativistic limit of the scattering process. Let
$P_\pi$ be the momentum of pion incident on a target nucleon with momen­
tum $P_N$. The relative momentum at which the pion strikes the target
nucleon is

$$K = P_\pi - \frac{\mu}{M_N} P_N$$  \hspace{1cm} (6.1)

where $\mu$, $M_N$ are pion and nucleon rest mass respectively. To carry
out the generalization for a relativistic pion let us consider two
Lorentz frames of reference. Let the pion and nucleon four momenta
be $P_\pi$ and $P_N$ respectively, where

$$P_\pi = (p_\pi, iE_\pi) \hspace{1cm} P_N = (p_N, iE_N)$$  \hspace{1cm} (6.2)

We have taken $c$ to be 1. $p_\pi$ and $p_N$ are the three momenta of pion
and nucleon respectively. Similarly let $P_\pi'$ and $P_N'$ be the four
momenta of the pion and nucleon in a second Lorentz frame. Since
the scalar product of the four momentum is an invariant under a
Lorentz transformation we must have

$$(P_\pi + P_N)^2 = (P_\pi' + P_N')^2$$  \hspace{1cm} (6.3)
from which we deduce

\[ P_\pi P_N = P_\pi' P_N' \]  (6.4)

Since

\[ \frac{P_\pi^2}{N} = \frac{P_\pi'^2}{N} = \mu^2 \]

and

\[ \frac{P_N^2}{N} = \frac{P_N'^2}{N} = M_N^2 \]

are invariants.

We therefore have from equation (6.4)

\[ P_\pi P_N - E_\pi E_N = P_\pi' P_N' - E_\pi' E_N' \]  (6.5)

This gives

\[ P_\pi P_N - \sqrt{M_N^2 + p_N^2} \sqrt{\mu^2 + p_\pi^2} = P_\pi' P_N' - \sqrt{M_N'^2 + p_N'^2} \sqrt{\mu^2 + p_\pi'^2} \]  (6.6)

We are interested in a transformation where the nucleon is at rest

i.e. \[ P_N' = 0 \]. If we let \[ K_\pi = P_\pi' \] then equation (6.6) becomes

\[ P_\pi P_N - \sqrt{M_N^2 + p_N^2} \sqrt{\mu^2 + p_\pi^2} = - \sqrt{\mu^2 + M_N^2} K_\pi M_N \]  (6.7)

Squaring both sides of equation (6.7) and collecting terms we have

\[ |K_\pi| = \frac{1}{M_N} \text{Sqrt}(\mu^2 p_N^2 + M_N^2 p_\pi^2 + p_\pi^2 p_N^2 (1 + \cos^2 \theta)) \]

\[ - 2 p_\pi p_N \cos \theta \sqrt{\mu^2 + p_\pi^2} \sqrt{M_N^2 + p_N^2}) \]  (6.8)
In the non-relativistic limit $p_\pi << \mu, p_N << M_N$ and we have

$$K_\pi^2 = \frac{\mu^2 p_N^2}{M_N^2} + p_\pi^2 - 2p_\pi \frac{p_N \mu}{M_N} \cos \theta$$

which is equivalent to equation (6.1).

The average scattering amplitude is then

$$\mathcal{F}_{n,p}(p_\pi) = \frac{\int f_{n,p}(K_\pi) \rho_{n,p}(p) \, dp}{\int \rho_{n,p}(p) \, dp} \quad (6.9)$$

where the suffix $n$ and $p$ denote neutrons and protons respectively.

If $\rho(p)$ is normalized to 1, we can then write

$$\mathcal{F}_{n,p}(p_\pi) = \int f_{n,p}(K_\pi) \rho_{n,p}(p) \, dp \quad (6.10)$$

$\rho(p)$ is the momentum spectrum of nucleons, which is the probability of finding a nucleon with momentum $p$.

$$\rho(p) = \langle 0 | \sum_{i=1}^{A} \delta(p - p_{i1}) | 0 \rangle \quad (6.11)$$

which is just equation (4.1) except for a change of variables. As before if we construct the wavefunctions from a single slater determinant of orthonormal functions, the above expression simplifies to

$$\rho(p) = \frac{1}{A} \sum_{\mu} |\phi_{\mu}(p)|^2 \quad (6.12)$$

where $\phi_{\mu}(p)$ is the Fourier transform of $\phi_{\mu}(r)$. The state of a
nucleon in the co-ordinate representation is

\[ \phi \mu (r) = \sum_{m_\lambda m_s} (\lambda^2 j; m_\lambda m_s m_j) \gamma_{\lambda m}^\mu (\hat{r}) \chi_2^{m_\lambda} \chi_2^{m_t} \]

\[ R_{n\lambda j}^{m_t}(r) \]  \hspace{1cm} (6.13)

\[ \therefore \phi \mu (p) = \phi_{n\lambda j m_t} (p) \]

\[ = \sum_{m_\lambda m_s} (\lambda^2 j; m_\lambda m_s m_j) \gamma_{\lambda m}^\mu (\hat{p}) \chi_2^{m_\lambda} \chi_2^{m_t} \]

\[ R_{n\lambda j}^{m_t}(p) \]  \hspace{1cm} (6.14)

\[ = \sum_{m_\lambda m_s} (\lambda^2 j; m_\lambda m_s m_j) \chi_2^{m_\lambda} \chi_2^{m_t} \]

\[ \int \frac{dr}{(2\pi)^{3/2}} e^{-ip\cdot r} \gamma_{\lambda m}^\mu (\hat{r}) R_{n\lambda j}^{m_t}(r) \]  \hspace{1cm} (6.15)

now

\[ \int d\hat{r}^2 e^{-ip\cdot \hat{r}} \gamma_{\lambda m}^\mu (\hat{r}) = 4\pi (-i)^{\lambda} j_\lambda (pr) \gamma_{\lambda m}^\mu (\hat{r}) \]  \hspace{1cm} (6.16)

Hence

\[ \phi_{n\lambda j m_t} (p) = \frac{4\pi}{(2\pi)^{3/2}} (-i)^{\lambda} \sum_{m_\lambda m_s} (\lambda^2 j; m_\lambda m_s m_j) \chi_2^{m_\lambda} \chi_2^{m_t} \gamma_{\lambda m}^\mu (\hat{r}) \int r^2 dr j_\lambda (pr) R_{n\lambda j}^{m_t}(r) \]  \hspace{1cm} (6.17)
\[ \rho(p) = \frac{1}{A} \sum_{n,l,j,m_j} \sum_{m_t, m_s} (\lambda \beta; j; m_s, m_j) \]
\[ \times (\lambda \beta; m_s, m_j, m) Y_{l}^{m_s}(\phi) Y_{l}^{m_j}(\theta) \chi_{\lambda}^{m_s} \chi_{\lambda}^{m_j} \chi_{\lambda}^{m_t} \]
\[ R_{n\lambda j}^{m_t}(p) R_{n\lambda j}^{m_t'}(p) \]
\[ \text{where} \]
\[ R_{n\lambda j}^{m_t}(p) = \frac{4\pi(-i)^{\lambda}}{(2\pi)^{3/2}} \int dr r^{2j+1} Y_{\lambda}^{m_t}(p r) R_{n\lambda j}^{m_t}(r) \]
\[ \text{using the orthogonality relations (4.16 and 4.17) of the spin functions, and the property,} \]
\[ \sum_{m_s} (\lambda \beta; m_s, m_j) (\lambda \beta; m_s, m_j) = \left( \frac{2j+1}{2\lambda+1} \right)^{1/2} \delta_{m_\lambda, m_\beta} \]  
\[ P_\lambda(\cos \theta_{1,2}) = \frac{4\pi}{(2\lambda+1)} \sum_{m_\lambda} Y_{\lambda}^{m_\lambda}(\theta_1, \phi_1) Y_{\lambda}^{m_\lambda}(\theta_2, \phi_2) \]
\[ \rho(p) = \frac{2}{A} \sum_{n\lambda j} (2j+1) |R_{n\lambda j}^{m_t}(p)|^2 \]
\[ \int p^2 dp |R_{n\lambda j}^{m_t}(p)|^2 = \int r^2 dr |R_{n\lambda j}^{m_t}(r)|^2 = 1 \]
In obtaining equation (6.22) we have used the same assumptions as in equation (4.19).
6.2. The Average Scattering Amplitude

The radial wavefunctions i.e. $R_{n\ell j}^{m_{\ell}}(r)$ were evaluated using the SWO potential whose parameters were obtained from Elton and Swift 56). Calculations of the average scattering amplitude were done on $^{12}$C, $^{40}$Ca, and $^{208}$Pb. In the case of $^{208}$Pb we used the potential parameters of Batty and Greenlees 67). The data for the free $\pi$-N total cross-sections were obtained from ref. 68). The results of the momentum distribution for the various nuclei are shown in Fig. (16). We have also included a plot of the distribution

$$\rho(p) = \frac{A}{1 + \exp[(p - P_0)/\lambda]} \quad (6.24)$$

quoted in Miller 35), where $P_0 = 100$ MeV/c and $\lambda = 50$ MeV/c. We notice that this distribution matches well with those evaluated using realistic wavefunctions for $p < 1.5$ fm$^{-1}$. For large momentum we would expect the Miller distribution to fall off less rapidly than the realistic ones. This is because the realistic wavefunctions have an exponential tail, and the modulus square of it decays much more rapidly than equation (6.24), which at large $p$ falls off linearly, as indicated in Fig. (16). Although differences occur for $p > 1.5$ fm$^{-1}$, values obtained for the average scattering amplitude using all four momentum distributions give essentially the same results. This is not surprising, since the high momentum components of $\rho(p)$ are small, such that any difference that arise are damped out. Fig. (17) indicates a plot of the unaveraged and averaged free $\pi$-N total cross-sections. Notice that the inter-
FIG(16)

$P(P)$

$^{12}_C$

$^{40}_Ca$

$^{208}_Pb$

MILLER

$P(fm^{-1})$
FIG (17)
section of the $\pi^+\cdot n$ and $\pi^+\cdot p$ total cross-section has had a
displacement in the $+ve \ p_\pi$ direction of the order of 60 MeV/c.
Also the peaks of the total cross-sections have been smoothed out
in agreement with the experimental results of Reeder and
Makowitz 30). In Fig. (21) we have indicated the various $\pi\cdot N$
resonances obtained from ref. 7). We see that the resonances
persist up to high pion energies. The effect of the Fermi-motion
is not only to broaden these resonance peaks but also to displace
them. This suggests that in the case of $\pi$-nucleus scattering the
absorption of a pion is not so pronounced as compared to the case
of free scattering, since now the number of nucleons with momentum
corresponding to the resonance peak is reduced.

From this we can conclude that when we are studying the
kinematics of $\pi$-nucleus scattering below $p_\pi = 2.5$ GeV/c. the effect
of Fermi motion should be taken into account. At energies above
$p_\pi = 2.5$ GeV/c the Fermi motion is negligible since now either the
resonances are very small or negligible. The averaging calculation can
be carried out using a simple distribution of the form in equation
(6.24).
Total Reaction Cross-section: results and discussions

We are at present in a position to apply the theoretical formalism derived in the earlier chapters. To recapitulate, we note that the main physical features that were introduced into the optical model potential were the correlation contributions due to double scattering of the pion and the Fermi-motion of the target nucleons.

Before we proceed to discuss the results of our calculations let us reconsider equations (2.52) and (2.97).

The first order optical potential in our formalism is

\[
U(r) = \frac{\hbar^2}{(2\pi)^2} \frac{(\omega_1 + \omega_2)}{E_1 E_2} \int dq \ e^{i \mathbf{q} \cdot \mathbf{r}} M(q) F(q)
\] (2.52)

\[
= - \frac{\hbar^2}{(2\pi)^2} \frac{(\omega_1 + \omega_2)}{E_1 E_2} \left[ \int dq \ e^{i \mathbf{q} \cdot \mathbf{r}} M_p(q) F_p(q) + N \int dq \ e^{i \mathbf{q} \cdot \mathbf{r}} M_n(q) F_n(q) \right]
\] (7.1)

where the suffices p and n denote proton and neutron respectively.

Similarly the second order contribution is

\[
\Delta U(r) = \frac{2i\hbar (A-1) \rho^2(r) \hbar^2}{k} \frac{(\omega_1 + \omega_2)^2}{(E_1 + E_2) E_2} \int dk' k' \Delta(k') \tan^{-1}(\lambda k') \frac{H^2(k')}{[\rho^2(0)]^{-1}}
\] (2.97)
\[ e^{-iA(A-1)} = \frac{\rho^2(r)\tilde{\rho}^2}{E_1 E_2} \frac{(\omega_1 + \omega_2)^2}{(E_1 + E_2)^2} \int \frac{dk'}{k'} A(k') \]

\[ \{\tan^{-1}(\lambda_{k''}) \rho^2_0(k'') + \tan^{-1}(\lambda_{k''}) \rho^2_0(k'')\} \rho^2(0)^{-1} \]  

(7.2)

where \( \lambda_P = 2/\omega P \sigma^D_T \) and \( \sigma^D_T \) is the average \( \pi^+ \)-proton total cross-section at the particular energy considered.

We have employed the latter form of the equations in our calculations for the total reaction cross-sections. The parameters for the real part of the scattering amplitude were obtained from ref. 72).

We do not have a knowledge of the functional form of \( M(q) \), but at high energies 69) we shall assume \( M(q) \) to be of the form

\[ M(q) = M(0) e^{-\frac{1}{2} \beta^2 q^2} \]  

(7.3)

where \( M(0) = \frac{k_0 \sigma_T}{4\pi} \) and \( k_0 \) is the pion momentum in the pion-nucleon centre of mass system.

We can insert the value \( \beta^2 = 0 \) which would imply scattering with zero momentum transfer i.e., forward angle scattering. The exact value of the parameter \( \beta \) is uncertain. The value \( \beta^2 = 0.3 \) is used to fit the high energy data 69) on differential cross-sections. However for the purpose of theoretical considerations we shall make use of the values \( \beta^2 = 0 \) and 0.3. Only comparisons with experimental results will indicate the appropriate choice of the parameter \( \beta \).

In our present calculations we have restricted the number of nuclei to two, namely \(^{208}\text{Pb} \) and \(^{12}\text{C} \). For \(^{208}\text{Pb} \) we have used two different density distributions; the Batty-Greenlees proton and
neutron distributions (BG) and the Batty-Greenlees proton (BGp) with the Zaidi neutron distribution (Z). In the case of $^{12}$C we have in addition the Elton-Swift density distributions (ES). The main purpose of this distribution is to estimate the 2nd order contributions to the total reaction cross-sections, since our earlier work on Pauli correlations were based on the Elton-Swift potentials.

The results of our calculations are depicted in tables 3-10. If we study these results we are led to the following conclusions.

1) For $^{208}$Pb the BG distributions give larger values of the total reaction cross-sections than the BGp + Z distributions for both $\pi^+$ and $\pi^-$ scattering.

2) The cross-section obtained by using $\beta^2 = 0.3$ is larger than with $\beta^2 = 0$.

3) For $^{12}$C the ES distributions give larger cross-sections than the BGp + Z distributions.

4) The 2nd order contribution to the cross-section are negligible at the energies considered.

5) The Glauber theory, though not applicable for $^{208}$Pb at the energies considered seems to give good agreement with the results obtained by solving the Klein-Gordon (KG) equation for $^{12}$C.

6) The ratio $\sigma_R(\pi^-)/\sigma_R(\pi^+)$ (nucleus) is largest for the BGp + Z distributions.
Table 3

$\pi^+ - ^{208}\text{Pb}$ Absorption cross-sections (mb)

<table>
<thead>
<tr>
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<th>$\beta^2=0.3$</th>
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<tr>
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$P_\pi$ - incident pion momentum in the lab system (MeV/c)

BG - Batty Greenlees proton + neutron distributions

BGp+Z - Batty Greenlees proton + Zaidi neutron distributions

KG - results calculated using the Klein-Gordon equation

G - results calculated using the Glauber approximation.
Table 4

$\pi^- - ^{208}\text{Pb}$ Absorption cross-sections (mb)

<table>
<thead>
<tr>
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Table 5

$\pi^+ - ^{12}$C Absorption cross-sections (mb)

(Elton-Swift distributions).

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</table>
Table 6

$\pi^- - {}^{12}\text{C}$ Absorption cross-sections (mb)

(Elton-Swift distributions).

<table>
<thead>
<tr>
<th></th>
<th>with correlations</th>
<th></th>
<th>without correlations</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta^2 = 0$</td>
<td>$\beta^2 = 0.3$</td>
<td>$\beta^2 = 0$</td>
<td>$\beta^2 = 0.3$</td>
</tr>
<tr>
<td>KG</td>
<td>G</td>
<td>KG</td>
<td>G</td>
<td>KG</td>
</tr>
<tr>
<td>212</td>
<td>214</td>
<td>226</td>
<td>227</td>
<td>710</td>
</tr>
<tr>
<td>226</td>
<td>240</td>
<td>225</td>
<td>224</td>
<td>240</td>
</tr>
<tr>
<td>245</td>
<td>263</td>
<td>244</td>
<td>243</td>
<td>262</td>
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<td>247</td>
<td>264</td>
<td>245</td>
<td>263</td>
<td>263</td>
</tr>
</tbody>
</table>
Table 7

$\pi - ^{12}$C Absorption cross-sections (mb)

(BGp + Z distributions with $\beta^2 = 0$).

<table>
<thead>
<tr>
<th></th>
<th>with correlations</th>
<th>without correlations</th>
<th>with correlations</th>
<th>without correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi^+$</td>
<td>$\pi^-$</td>
<td>$\pi^+$</td>
<td>$\pi^-$</td>
</tr>
<tr>
<td>KG</td>
<td>G</td>
<td>KG</td>
<td>G</td>
<td>KG</td>
</tr>
<tr>
<td>KG</td>
<td>208</td>
<td>201</td>
<td>207</td>
<td>710</td>
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<td>KG</td>
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<td>215</td>
<td>217</td>
<td>840</td>
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<td>KG</td>
<td>238</td>
<td>234</td>
<td>236</td>
<td>1000</td>
</tr>
<tr>
<td>KG</td>
<td>242</td>
<td>238</td>
<td>240</td>
<td>1360</td>
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<td>KG</td>
<td>239</td>
<td>235</td>
<td>238</td>
<td>1580</td>
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<tr>
<td>KG</td>
<td>233</td>
<td>229</td>
<td>231</td>
<td>1900</td>
</tr>
<tr>
<td>KG</td>
<td>209</td>
<td>211</td>
<td>212</td>
<td>242</td>
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<td>KG</td>
<td>222</td>
<td>221</td>
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<td>240</td>
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<td>KG</td>
<td>237</td>
<td>238</td>
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</tr>
<tr>
<td>KG</td>
<td>232</td>
<td>232</td>
<td>234</td>
<td>232</td>
</tr>
</tbody>
</table>
Table 8

\( \sigma_R(\pi^-)/\sigma_R(\pi^+) \) in \(^{208}\text{Pb}\).

<table>
<thead>
<tr>
<th></th>
<th>BG</th>
<th></th>
<th>BGp + Z</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta^2 = 0 )</td>
<td>( \beta^2 = 0.3 )</td>
<td>( \beta^2 = 0 )</td>
</tr>
<tr>
<td>KG G</td>
<td>KG G</td>
<td>KG G</td>
<td>KG G</td>
</tr>
<tr>
<td>0.958</td>
<td>0.845</td>
<td>0.961</td>
<td>0.897</td>
</tr>
<tr>
<td>0.965</td>
<td>0.906</td>
<td>0.967</td>
<td>0.907</td>
</tr>
<tr>
<td>0.981</td>
<td>0.925</td>
<td>0.981</td>
<td>0.926</td>
</tr>
<tr>
<td>1.047</td>
<td>1</td>
<td>1.047</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>P_\pi KG G</td>
<td>KG G</td>
<td>KG G</td>
</tr>
<tr>
<td>710</td>
<td>1.035</td>
<td>0.953</td>
<td>1.033</td>
</tr>
<tr>
<td>840</td>
<td>1.031</td>
<td>0.959</td>
<td>1.030</td>
</tr>
<tr>
<td>1000</td>
<td>1.032</td>
<td>0.967</td>
<td>1.030</td>
</tr>
<tr>
<td>1360</td>
<td>1.050</td>
<td>1</td>
<td>1.050</td>
</tr>
<tr>
<td>1580</td>
<td>1.072</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1900</td>
<td>1.026</td>
<td>0.992</td>
<td></td>
</tr>
</tbody>
</table>
Table 9

$\sigma_R(\pi^-)/\sigma_R(\pi^+)$ in $^{12}\text{C}$

(Elton-Swift distributions)

<table>
<thead>
<tr>
<th>with correlations</th>
<th>without correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^2 = 0$</td>
<td>$\beta^2 = 0.3$</td>
</tr>
<tr>
<td>KG</td>
<td>G</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>1.024</td>
<td>1.005</td>
</tr>
<tr>
<td>1.004</td>
<td>1.004</td>
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<tr>
<td>1.004</td>
<td>1.004</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 10

\[ \sigma_R(\pi^-)/\sigma_R(\pi^+) \text{ in } ^{12}\text{C} \]

(BGp + Z distributions with \(\beta^2 = 0\) and no correlations).

<table>
<thead>
<tr>
<th>Pion momentum (P_\pi) (MeV/c)</th>
<th>(\sigma_R(\pi^-)/\sigma_R(\pi^+)) KG</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>710</td>
<td>1.039</td>
<td>1.019</td>
</tr>
<tr>
<td>840</td>
<td>1.033</td>
<td>1.018</td>
</tr>
<tr>
<td>1000</td>
<td>1.029</td>
<td>1.017</td>
</tr>
<tr>
<td>1360</td>
<td>1.012</td>
<td>1</td>
</tr>
<tr>
<td>1580</td>
<td>1.009</td>
<td>1</td>
</tr>
<tr>
<td>1900</td>
<td>1.013</td>
<td>1.004</td>
</tr>
</tbody>
</table>
We would expect in the case of $^{208}$Pb the BG distributions to give a larger absorption cross-section, simply because of the larger neutron surface density. If we observe figure (20) we note that the Zaidi distribution gives a larger central neutron density and less in the surface region. A similar situation holds for $^{12}$C. We know $^{17,70,71}$ that the absorption of pions by nuclei mainly take place at the surface region. This is substantiated by the fact that at the surface as the BG neutron density is larger than the Zaidi neutron density, the absorption cross-section is larger in the case of the BG distributions than as compared to the BGp + Z distributions. As most of the absorption takes place near the surface region (i.e. see figure (20) within a region $3 < r < 8$fm) it is possible the absorption of pions would be quite insensitive to slight variations in the neutron and proton central densities. The distributions in figure (20) are normalized to 1.

The result obtained by taking $\beta^2 = 0.3$ is obvious from equation (2.52). The percentage change in the cross-sections when $\beta^2$ is increased from zero to 0.3 is higher for $^{12}$C than for $^{208}$Pb, and this is due to the fact that the formfactor $F(q)$ for $^{12}$C falls less rapidly as compared to $^{208}$Pb. This can be conceived by comparing figures (4) and (18).

At high energies we would expect the 2nd order contribution to the absorption cross-section to be negligible. This is one of the criteria for the basis of our model. The $1/k$ dependence of the 2nd order optical potential confirms our results. Furthermore, at such energies the probability of double scattering would be low.
$F^2(q)$

$^{208}\text{Pb}$

$10^0$

$10^{-1}$

$10^{-2}$

$10^{-3}$

$10^{-4}$

$10^{-5}$

$10^{-6}$

$q_f (fm^{-1})$
and the pion is more likely to undergo single scattering in the forward direction.

It is not surprising to find that the Glauber theory gives a good prediction in the case of $^{12}$C. For this nucleus the Coulomb correction at high energies are small, and hence even if we neglect this term we should not obtain large errors in the absorption cross-sections. One point must be noted; in our calculations we have not considered the relativistic treatment of the Coulomb phase shifts for $^{208}$Pb due to singularities discussed in Chapter (3). However this is not so for $^{12}$C. It would not be completely wrong to infer that at energies above $p_{\pi} = 3$GeV/c the Glauber theory would be able to provide a good description for $^{208}$Pb.

In order to illustrate differences that may occur in the shape of the optical potential due to changes in the parameter $\beta$ we have given a plot of potentials in figure (19). The potential obtained with $\beta^2 = 0$ has a smaller central depth but falls less rapidly than the potential obtained with $\beta^2 = 0.3$. Except for the slight alteration we have observed, the general trend of shape of the potential remains the same.

Some preliminary experimental results $^{67}$ are given in Table 11. The experiments were done at the Rutherford Laboratory by a combined team from Birmingham University, Rutherford Laboratory and Surrey University. The object of this experiment is to measure the total reaction cross-sections for both $\pi^+$ and $\pi^-$ mesons on a range of nuclei in the energy region from 0.5 to 2.0 GeV. The aim of the
$\rho(r)$

- BG proton
- BG neutron
- Z neutron

$^{208}\text{Pb}$

$\Gamma(\text{fm})$
Table 11

Preliminary experimental values for $\sigma_R(\pi^-)/\sigma_R(\pi^+)$

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Momentum (GeV/c)</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{12}_C$</td>
<td>1</td>
<td>1.026 ± 0.008</td>
</tr>
<tr>
<td>$^{12}_C$</td>
<td>1.36</td>
<td>1.019 ± 0.007</td>
</tr>
<tr>
<td>$^{12}_C$</td>
<td>2.0</td>
<td>1.008 ± 0.005</td>
</tr>
<tr>
<td>$^{208}_Pb$</td>
<td>1</td>
<td>1.020 ± 0.004</td>
</tr>
<tr>
<td>$^{208}_Pb$</td>
<td>1.36</td>
<td>1.044 ± 0.004</td>
</tr>
<tr>
<td>$^{208}_Pb$</td>
<td>2</td>
<td>1.023 ± 0.004</td>
</tr>
</tbody>
</table>
measurements is to obtain further information about the density distributions of neutrons in nuclei, particularly in the surface region. The actual momenta chosen were based on consideration of total cross-section for $\pi^+$ and $\pi^-$ on nucleons. The effective pion-nucleon cross-sections obtained after including effects of Fermi-motion in the nucleus is significantly different from the free values. It was after comparison of the Fermi averaged cross-sections that were obtained in Chapter (6) with the free values, that it was decided to choose the values of momenta at resonance regions and at the $\pi^+, \pi^-$ cross-over points of the total cross-sections (see figure (17)). The targets were chosen to cover a wide range of nuclei beginning from $^{12}$C to $^{208}$Pb. The experiments were done to within 1/2 % accuracy on the cross-sections. It is important that one requires such an accuracy, because the ratio $\sigma_R(\pi^-)/\sigma_R(\pi^+)$ (nucleus) might be sensitive to slight changes in the absorption cross-sections.

On comparison with theoretical results, we deduce that the BGp + Z distribution is able to provide a better fit with the experimental values. If we look at Table 12 we note that r.m.s. (root mean square) radius for the BGp proton distribution is larger than the r.m.s. radius for the Zaidi neutron distribution for both $^{208}$Pb and $^{12}$C. This result is illuminating from the physical point of view, since we would expect the proton r.m.s. radius to be larger owing to the Coulomb repulsive force. Hence an important conclusion that can be drawn from these results is for both $^{208}$Pb and $^{12}$C the proton r.m.s. radius is larger than the neutron r.m.s. radius.
Table 12

Root mean square radius for proton and neutron distributions in nuclei (r.m.s.) (fm)

For $^{208}\text{Pb}$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{BGp}$</td>
<td>5.44</td>
</tr>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{BGn}$</td>
<td>6.004</td>
</tr>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{Z}$</td>
<td>4.906</td>
</tr>
</tbody>
</table>

For $^{12}\text{C}$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{BGp}$</td>
<td>2.46</td>
</tr>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{Z}$</td>
<td>2.34</td>
</tr>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{ESP}$</td>
<td>2.46</td>
</tr>
<tr>
<td>$&lt;r^2&gt;^{1/2}_{ESn}$</td>
<td>2.43</td>
</tr>
</tbody>
</table>

BGp - Batty Greenlees proton distribution
BGn - Batty Greenlees neutron distribution
ESP - Elton-Swift proton distribution
ESn - Elton-Swift neutron distribution
Z - Zaidi neutron distribution
The Glauber theory predicts the same values for $\pi^-$, $\pi^+$ absorption cross-sections at $p_\pi = 1.36$ GeV/c and $p_\pi = 1.58$ GeV/c, which are the cross over points of $\pi^+$, $\pi^-$ Fermi averaged cross-sections (see figure (17)). The KG equation gives a larger value for the $\pi^-$ absorption cross-section. This is probably due to the effect of the Coulomb potential.

The appropriate choice of the neutron density for the Zaidi distribution depends on how well it fits the absorption cross-sections for negative pions, particularly in the case of $^{208}$Pb.

Although strictly it may be a bit premature to form any general conclusions as the analysis of the experimental data is incomplete, but nevertheless for the purpose of some understanding we shall briefly form the following conclusions.

The theoretical value of the absorption cross-section obtained for $\pi^+$, $\pi^-$ reactions on $^{12}$C at $p_\pi = 288$ MeV/c are 402 and 407 mbs respectively. The value $\beta^2 = 0.3$ was found to give a better fit to the experimental data. The correlation contribution at this energy i.e. corresponding to the $(3,3)$ resonance of the free $\pi$-$N$ scattering were found to be negligible. The experimental value for $\pi^- - ^{12}$C reaction at the above energy is $423 \pm 13$ mbs. This seems to indicate the validity of the impulse approximation to some extent. But if we were to compare the values of the ratio $\sigma_R(\pi^-)/\sigma_R(\pi^+)$ (nucleus) for both $^{208}$Pb and $^{12}$C at $p_\pi = 1.36$ and 1.9 GeV/c (where we have assumed that the absorption cross-section at $p_\pi = 2$ GeV/c differs very slightly from that at 1.9 GeV/c, which is reasonable
as the $\pi$-N total cross-sections are almost constant at this region) we find that the discrepancy between theory and experiment is more pronounced at $p_\pi = 1.36$ GeV/c than at $p_\pi = 1.9$ GeV/c. Although both energies lie near the resonance regions, the one at $p_\pi = 1.9$ GeV/c is less dominant in resonant behaviour than that at $p_\pi = 1.36$ GeV/c (see figure (17)). Hence we can infer from this that the impulse approximation may not hold very well near and at resonance energies.

In the preceding paragraph we remarked that a discrepancy exists between theory and experiment at $p_\pi = 1.36$ GeV/c for $^{12}$C. Experimental results of Carter et al \cite{74} on $\pi$-deuterium reactions in the energy range 0.5 to 2.5 GeV/c indicate that $\pi^- - d$ absorption cross-section is always greater than the $\pi^+ - d$ absorption cross-sections. A possible reason for this discrepancy is due to the violation of charge symmetry in $\pi$-N reactions. The $^{12}$C nucleus has the same number of protons as neutrons, and particularly at the energy considered the Fermi averaged $\pi^+, \pi^-$ total cross-sections are identical. For this nucleus both the real and imaginary part of the optical potentials for $\pi^-, \pi^+$ reactions at $p_\pi = 1.36$ GeV/c differ only slightly, which is due to the slight difference in the neutron-proton density distributions. Hence the only conclusive evidence that could account for the different $\pi^+, \pi^-$ absorption cross-sections for $^{12}$C is due to the Coulomb effect. But the Coulomb correction for $^{12}$C, is not only small at these energies, but it is an effect that has been correctly taken into account when solving the Klein-Gordon equation. The only other possibility besides the impulse approximation that could account for the discrepancy between theory and experiment is
in the assumption of charge symmetry, i.e. the assumption that the scattering amplitudes for $\pi^+ + n$ and $\pi^- + p$ are identical may not be valid.

It is found that the effect of Fermi-motion at energies above 2 GeV is negligible, but at resonance regions it gives a smaller value for the absorption cross-sections than that would be obtained if the Fermi-motion of the target nucleons were neglected. For example at $p_\pi = 288$ MeV/c, $\pi^+,$ $\pi^-$ absorption cross-section for $^{12}\text{C}$ using the free $\pi$-$\text{N}$ scattering amplitudes are 427, 415 mbs respectively as compared to 410, 399 mbs obtained using the averaged $\pi$-$\text{N}$ scattering amplitudes. The above results were calculated using the Glauber theory with the parameter $\beta^2 = 0.3$. The ratio $\sigma_R(\pi^-)/\sigma_R(\pi^+)$ (nucleus) using the averaged and free $\pi$-$\text{N}$ scattering amplitudes are 0.973, 0.972 respectively in the Glauber approximation. But similar calculations on $^{12}\text{C}$ at $p_\pi = 1$ GeV/c give for the ratio the value 1.022 using the free $\pi$-$\text{N}$ scattering amplitudes. Comparing this result with that given in Table 9 we see that there is a marked difference in the ratios. The experimental value given in ref. 73 for $\pi^- - ^{12}\text{C}$ reaction is $423 \pm 13$ mbs. The value obtained in our calculations using the free and averaged $\pi$-$\text{N}$ scattering amplitudes are 422 and 407 mbs respectively (Note these values were obtained by solving the Klein-Gordon equation). If we take the lower extreme of the experimental value, i.e. 410 mbs, then the value obtained for $\pi^- - ^{12}\text{C}$ absorption cross-section using the averaged $\pi$-$\text{N}$ scattering amplitude provides a better fit to the experimental data. On the other hand if we consider the upper extreme, then neither of our
calculated results provide a fit to the experimental value. Hence on comparing the two possibilities in the experimental result, we can, without being completely wrong conclude that the values obtained for the absorption cross-section using the averaged $\pi$-$N$ scattering amplitudes gives a better fit to the experimental data in ref. 73) for $\pi^-$-$^{12}$C reactions. Basing on the outcome of this conclusion we can generalize by saying that the absorption cross-sections obtained using the Fermi-averaged scattering amplitudes would probably furnish a better fit to the experimental data.
Fig. 21 Total cross sections for $\pi^+p$ and $\pi^-p$ scattering.
Conclusions

In this final Chapter we shall summarize the results of our work. Our treatment of Pauli correlations demonstrated that there were differences in the ls and jj coupling schemes. The assumption that the orthogonality of the wavefunctions generated by the Saxonwood potential was found to be incorrect.

The effect of two body correlations in the pion-nucleus absorption cross-sections were found to be negligible at energies considered.

The most important feature of our work is the effect of Fermi-motion on the pion-nucleon scattering amplitudes. We found that the averaging integral was virtually independent of the particular choice of the momentum spectrum. A simple Fermi distribution was able to furnish the same results as Elton-Swift or BG distributions. The relatively good agreement between theory and experiment provide some justification of our theoretical model employed.
Our definition of the occupation probabilities is as follows:

\[
p^2(n\&j_{m_t} = - \frac{1}{2}) = \frac{\text{Number of protons in the sub-shell } n\&j}{2(2j+1)}
\]

\[
P^2(n\&j) = \sum_{m_t} p^2(n\&j_{m_t}) = \frac{\text{Number of nucleons in the sub-shell } n\&j}{2(2j+1)}
\]

\[
P(n\&) = \frac{\text{Number of nucleons in the sub-shell } n\&}{4(2\ell+1)}
\]
Appendix B

We wish to find a transformation such that the kinematics in the A+l system can be expressed in terms of the two-body system. The treatment here follows that given in Appendix I of ref. 26).

Let \( \omega_1, \omega_2 \) and \( \omega_1', \omega_2' \) be the total energies of the incident and target particle in the two-body centre of mass before and after collision. Similarly let \( E_1, E_2 \) and \( E_1', E_2' \) be the total energies of the projectile and target nucleon in the centre of mass of the A+l system. The invariant under consideration is \( \langle I \rangle \) where

\[
\langle I \rangle = \sqrt{E_1 E_2} \sqrt{E_1' E_2'}
\]

B.1.

and \( \langle T \rangle = \langle \zeta, \zeta' | T | \zeta, \zeta_0 \rangle \)  

B.2.

Since \( I \) is an invariant it should hold for any co-ordinate system.

We therefore have

\[
\sqrt{E_1 E_2} \langle T \rangle_{A+l} = \sqrt{\omega_1 \omega_2} \langle T \rangle_{2B} \langle \omega_1' \omega_2' \rangle
\]

B.3.

If we consider elastic \( \pi-N \) scattering, as in our case, then,

\[
E_1 E_2 \langle T \rangle_{A+l} = \omega_1 \omega_2 \langle T \rangle_{2B}
\]

B.4.

The above expression relates the A+l system with the two-body system. But it would be more convenient to express \( E_1 E_2 \) and \( \omega_1 \omega_2 \) in terms of lab energy. Hence if \( E_L, k_L \) be the total energy and momentum of pion respectively in the lab, we define an invariant \( \Delta \) by
\[ \Delta^2 = (E_L + AM_2)^2 - k_L^2 \]  

where \( M_2 \) is the rest mass of the nucleon and is assumed that the nucleon is at rest. Let \( k \) be the momentum of pion in the \( A+1 \) system. Then,

\[ E_1^2 = M_1^2 + k^2 \]  

\[ E_2^2 = A^2M_2^2 + k^2 \]

where \( M_1 \) is the rest mass of the pion and \( E_2 \) is the total energy of the nucleus in the centre of mass of the \( A+1 \) system.

\[ \therefore E_2 = E_2/A = (M_2^2 + k^2/A^2)^{1/2} \]

As \( \Delta \) is an invariant we must have

\[ E_1^2 + E_2^2 + 2E_1E_2 = 2AM_2E_L + M_2^2 + M_1^2 \]  

also:

\[ E_1^2 - E_2^2 = M_1^2 - A^2M_2^2 \]

from which

\[ E_1 = \frac{AM_2E_L + M_1^2}{\Delta} \]  

\[ E_2 = \frac{AM_2^2 + M_2E_L}{\Delta} \]  

\[ k = \frac{AMk_L}{[2AM_2E_L + A^2M_2^2 + M_1^2]^{1/2}} \]
It is easily seen that in the two-body centre of mass system these take the form

\[ \omega_1 = \frac{E_{1M}M_2 + M_1^2}{\Delta'} \tag{B.14} \]

\[ \omega_2 = \frac{E_{2M}M_2 + M_2^2}{\Delta'} \tag{B.15} \]

where

\[ \Delta' = (M_2^2 + 2E_{L2}M_2 + M_1^2) \tag{B.16} \]

If \( k_o \) is the momentum of the pion in the two-body centre of mass system we have

\[ \omega_1^2 = M_1^2 + k_o^2 \tag{B.17} \]

\[ \omega_2^2 = M_2^2 + k_o^2 \tag{B.18} \]

using equations (B.16), (B.17) and (B.18) we obtain

\[ k_o = \frac{k_2 k_L}{(2M_2 E_L + M_1^2 + M_2^2)^2} \tag{B.19} \]

where

\[ \frac{1}{k_2} = \hbar/M_2 = 0.2103 \]

From equation (B.4) we have

\[ E_1 E_2 \langle t_0 \rangle_{A+1} = \omega_1 \omega_2 \langle t_0 \rangle_{2B} \tag{B.20} \]

\[ \langle t_0 \rangle_{A+1} = \frac{\omega_1 \omega_2}{E_1 E_2} \langle t_0 \rangle_{2B} \tag{B.21} \]
But
\[ <T>_{A+1} = A <k' | t_c | k>_{A+1} F(q) \]  \hspace{1cm} B.22.

and
\[ <t_o>_{2B} = \frac{-\hbar^2}{(2\pi)^2} \frac{(\omega_1 + \omega_2)}{\omega_1 \omega_2} M(q) \]  \hspace{1cm} B.23.

where \( M(q) \) is the scattering amplitude in two-body centre of mass system

\[ <T>_{A+1} = \frac{-A\hbar^2}{(2\pi)^2} \frac{(\omega_1 + \omega_2)}{E_1 E_2} M(q) F(q) \]  \hspace{1cm} B.24.

follows on using equation (B.21).
Figure Captions

**Figure 4.** The square of the form factors and charge form factors for $^6$Li calculated with HO and SWO functions.

**Figure 5.** The square of the form factors and charge form factors for $^{12}$C calculated with HO and SWO functions.

**Figure 6.** The square of the form factors and charge form factors for $^{16}$O calculated with HO and SWO functions.

**Figure 7.** The square of the form factors and charge form factors for $^{40}$Ca calculated with HO and SWO functions.

**Figure 8.** Correlation functions and exchange terms for $^6$Li calculated with HO and SWO functions.

**Figure 9.** Correlation functions and exchange terms for $^{12}$C calculated with HO and SWO functions.

**Figure 10.** Correlation functions and exchange terms for $^{16}$O calculated with HO and SWO functions.

**Figure 11.** Correlation functions and exchange terms for $^{40}$Ca calculated with HO and SWO functions.

**Figure 12.** The effect on the correlation function $C_{HO}(q)$ for $^{16}$O of variation in the oscillator length parameter $\beta$.

**Figure 13.** The ratio of the correlation functions calculated with HO and SWO functions.
Figure 14. The contributions to the exchange term $D_{SWO}(q)$ for $^{16}O$.

Figure 15. The ratio of the correlation functions calculated with HO and SW functions.

Figure 16. Momentum spectrum of nucleons using Miller, $^{12}C$, $^{40}Ca$ and $^{208}Pb$ distributions.

Figure 17. Free and Fermi-averaged $\pi$-N total cross-sections.

Figure 18. The square of the form factor for $^{208}Pb$ using the BGp (proton) and Zaidi neutron distributions.

Figure 19. The effect on the optical potential for $^{12}C$ due to variation in the parameter $\beta^2$.

Figure 20. The density distributions for $^{208}Pb$ calculated using BG and Zaidi potentials.
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TWO-NUCLEON DENSITY FUNCTIONS AND CORRELATION
FUNCTIONS WITH REALISTIC SINGLE-PARTICLE WAVE FUNCTIONS

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Abstract: Two-nucleon density functions and Pauli correlation functions are calculated for \(^{6}\)Li, \(^{12}\)C, \(^{16}\)O and \(^{40}\)Ca using realistic single-particle wave functions. The results are compared with those obtained using oscillator functions. The extent to which uncertainties introduced by the use of different wave functions may obscure investigation of the dynamical correlations is discussed.

1. Introduction

Recent studies of high-energy elastic \(^{1}\) and inelastic \(^{2}\) electron scattering, elastic \(^{3}\) and inelastic \(^{4}\) nucleon scattering, and pion absorption \(^{5}\) have stimulated interest in the two-nucleon density functions and short-range nucleon-nucleon correlations. In the scattering processes the effects of short-range correlations may appear in the high-momentum components of the nuclear momentum distribution (form factor) or in the second- and higher-order terms in a multiple scattering approximation. For pion absorption, the conditions for energy and momentum conservation require that the absorption occurs on a single nucleon with very high Fermi momentum or on a closely correlated pair of nucleons.

The short-range nucleon-nucleon correlations arise from two sources. The Pauli correlations arise simply from the particle statistics and are introduced through the use of a properly antisymmetrized nuclear wave function. The dynamical correlations arise from the short-range behaviour of the nucleon-nucleon interaction. The latter can be introduced through the use of a Jastrow-type wave function \(^{6,7}\), a Bethe-Goldstone wave function calculated with a suitable nucleon-nucleon force \(^8\), or through the use of a unitary model operator \(^9\). In the overwhelming majority of calculations these correlations are introduced into a shell-model wave function constructed from harmonic oscillator single-particle functions, and certain of the methods \(^8,9\) are feasible only when an oscillator basis is used. It is well known that these shell-model wave functions cannot give a satisfactory description of the one-particle density function for nuclei unless correlations are included, whereas shell-model wave functions constructed from single-particle functions generated in a finite nuclear potential do give a very satisfactory description of the one-particle density function \(^{11,18,19}\) without any need to invoke correlations.
We have used single-particle wave functions generated in a Saxon-Woods potential. The parameters of this potential are adjusted to give the correct separation energies for the protons and neutrons and to give a proton one-particle density function which yields a charge distribution in agreement with elastic electron scattering up to momentum transfers of approximately 2.5 fm⁻¹ (500 MeV/c). Since the potential parameters are chosen to give agreement with certain experimental data it may be assumed that the corresponding single-particle wave functions are physically realistic at least at low and medium values of momentum transfer and simulate to some extent the effects of any long-range correlations due, for example, to configuration mixing. This means that our single-particle states should be close to the Hartree-Fock states although we are not using a self-consistent procedure in the accepted sense. We have calculated the Fourier transform of the two-particle density function including only the Pauli correlations and have compared the results with those obtained using the appropriate oscillator functions, with the object of investigating the differences arising from the use of oscillator functions or of realistic functions and the extent to which these may obscure any investigation of the dynamical correlations.

The relevant formalism is given in sect. 2. In sect. 3 we discuss the parameters and one-particle density functions of the four nuclei studied, and in sect. 4 we present the results for the two-particle density functions and correlation functions.

2. Formalism

The one-particle density functions \( \rho(r) \) and the two-particle density function \( \rho(r, r') \) are defined as:

\[
\rho(r) = \frac{1}{A} \left< 0 \left| \sum_{j=1}^{A} \delta(r-r_j) \right| 0 \right>,
\]

\[
\rho(r, r') = \frac{1}{A(A-1)} \left< 0 \left| \sum_{j \neq k} \delta(r-r_j)\delta(r'-r_k) \right| 0 \right>,
\]

where \( |0\rangle \) represents a normalized, fully antisymmetrized wave function for the nuclear ground state and \( A \) is the total number of nucleons. If we construct the nuclear wave function from a single Slater determinant of orthonormal single-particle basis states \( \psi_\mu \), where \( \mu \) represents the appropriate set of quantum numbers, these functions become

\[
A \rho(r) = \sum_\mu |\psi_\mu(r)|^2,
\]

\[
A(A-1) \rho(r, r') = \sum_{\mu \neq \nu} \{ \psi_\mu(r')\psi_\nu(r) \}^* \{ \psi_\mu(r')\psi_\nu(r) - \psi_\nu(r')\psi_\mu(r) \}
\]

\[
= \sum_{\mu \nu} |\psi_\mu(r')|^2 |\psi_\nu(r)|^2 - \sum_{\mu \nu} \psi_\mu^*(r')\psi_\nu(r)\psi_\nu^*(r')\psi_\mu(r),
\]
where we have added and subtracted the diagonal terms in order to obtain the last line. Thus the two-particle density function can be written in the form

\[ A(A-1)\rho(r, r') = A^2 \rho(r)\rho(r') - A\rho_{\text{ex}}(r, r'), \]  

with

\[ A\rho_{\text{ex}}(r, r') = \sum \psi_\mu^\dagger(r')\psi_\mu^\dagger(r)\psi_\mu(r). \]

McVoy and Van Hove \(^{10}\) call \( \rho(r, r') \) the nucleon-nucleon correlation function and \( \rho_{\text{ex}}(r, r') \) the exchange sum. Wong \(^{8}\) defines the nucleon-nucleon correlation function as

\[ C(r, r') = \rho(r, r') - \rho(r)\rho(r'), \]

which, using eqs. (3) and (4), is given by

\[ (A-1)C(r, r') = \rho(r)\rho(r') - \rho_{\text{ex}}(r, r'). \]

From this expression and from the definition of \( \rho_{\text{ex}} \) we see that if there are no Pauli correlations

\[ \rho(r, r') = \rho_{\text{ex}}(r, r') = \rho(r)\rho(r'), \]

\[ C(r, r') \equiv 0. \]

Gottfried \(^{10}\) denotes our \( \rho(r, r') \) by \( C(r, r') \), our \( C(r, r') \) by \( D(r, r') \), and calls both pair correlation functions.

The functions of real interest are the Fourier transforms

\[ C(q) = \int \int e^{i\mathbf{q} \cdot (r-r')}C(r, r')drdr', \]

\[ D(q) = \int \int e^{i\mathbf{q} \cdot (r-r')}\rho_{\text{ex}}(r, r')drdr', \]

and from eq. (8) these are connected through the relation

\[ (A-1)C(q) = F(q)^2 - D(q), \]

where \( F(q) \) is the usual form factor defined by

\[ F(q) = \int \rho(r)e^{i\mathbf{q} \cdot r}dr. \]

We choose here to call \( C(q) \) the correlation function\(^1\) and \( D(q) \) the exchange contribution. This terminology is not entirely satisfactory since, as noted above, the exchange term is not zero when there are no Pauli correlations. The function \( D(q) \) contains terms with \( \mu = \nu \) which cancel out the contribution to \( C(q) \) from the uncorrelated density; thus in \(^4\)He where there is only one filled shell and no Pauli correlations it follows that \( D(q) = F(q)^2 \) and \( C(q) \equiv 0. \)

\(^1\) It is emphasised that throughout this paper the only correlation effect considered is that due to the Pauli principle.
If the single-particle states $\psi_\mu$ are described in terms of $jj$-coupling the symbol $\mu$ represents the quantum numbers $n, l, j, m_j, m_t$, and the single-particle wave functions are given by

$$\psi_\mu(r) = \sum_{m_m} (lm_jm_jm_jm_t) \phi_{n l j m t}^\mu(r) \chi_{n l j m t} \chi_{n l j m t},$$

(13)

where $\chi_{n l j m t}$ is the usual spin function, $\chi_{n l j m t}$ is an isospin function, and $\phi_{n l j m t}^\mu(r) = R_{n l j m t}^\mu(r) Y_{n l j m t}^\mu(\hat{r})$.

Using these functions and eqs. (6) and (10) we obtain a result similar to that previously obtained by Sitenko and Simenog for protons

$$AD(q) = \sum_{n'l'm't'} I(jj'jj'; Lk') (10l'0|L0)^2 P(nljm_t) P(n'l'jm_t')$$

$$\times I((jj'll'lm_t; -q),$$

(14)

where $P(nljm_t)$ is the occupation probability of the state $nlj$ for protons or neutrons (see appendix), and

$$I(jj'll'lm_t; q) = \int R_{n l j m t}^\mu(r) R_{n l j m t'}^\mu(r) I(ql; qr) r^2 dr.$$

The expression (14) is the correct form for $D(q)$ when the single-particle wave functions are generated in a realistic single-particle potential with spin-orbit and Coulomb terms.

When the radial integrals $I$ are independent of $j$ and $j'$, and of $m_t$ and $m_t'$, i.e. when the spin-orbit and Coulomb terms in the single-particle potential are omitted, and both $j$-subshells are fully occupied for a given $l$, the sums over $jj'$ and $m_t m_t'$ can be carried out independently to give

$$\sum_{jj'} (2j+1)(2j'+1)W^2(1ljj'; Lk) = 2,$$

(15)

$$\sum_{m_t m_t'} P(nljm_t) P(n'l'jm_t') I(ll'lm_t; q) I(ll'lm_t; -q) = 4I(llLL; q) I(llLL; -q),$$

so that

$$AD(q) = \sum_{n'l'm't'} I^{2l+1}(2l+1)(2l'+1)(10l'0|L0)^2 I(ll'lm_t; q) I(ll'lm_t; -q).$$

(16)

This expression for $D(q)$ is the appropriate form when oscillator functions are used for nuclei with closed shells in the $ls$ sense, e.g. $^4\text{He}, ^{16}\text{O}, ^{40}\text{Ca}$. For a nucleus such as $^{12}\text{C}$ where in the $jj$ coupling model the $p_s$ subshell is empty, eq. (15) does not hold, as can be seen from table I, and hence even if the radial integrals were independent of $jj'$ and $m_t m_t'$ eqs. (14) and (16) would not give identical results. The same is true for $^6\text{Li}$ in which the $p_s$ subshell is partially filled. The correlation function for these nuclei
in the oscillator basis can be evaluated by labelling the single-particle states \( \mu \) by \( n \ell m, m \) only. This gives

\[
AD(q) = \sum_{n\ell m} \sum_{n'\ell'm'} i^{2\ell\ell'}(2\ell+1)(2\ell'+1)(\ell00|L0)P(n\ell)mP(n'\ell'm')I(\ell\ell'; q)L(\ell'\ell'; -q). \quad (17)
\]

For spherically symmetric nuclei, the form factor \( F(q) \) has terms with \( L = 0 \) only and the spin-orbit term in the single-particle potential affects the result only through the behaviour of the radial integrals of the form

\[
I(jl0; q) = \int R_{n\ell}^2(r) j_o(qr) r^2 \, dr.
\]

For nuclei with non-zero quadrupole moments there are also terms with \( L = 2 \). Of the nuclei considered here only \( ^6\text{Li} \) has a non-zero quadrupole and this is sufficiently small to be neglected.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Weighting of the contributions to the exchange term ( D_{\text{two}}(q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>( j )</td>
</tr>
<tr>
<td>( s_\uparrow s_\uparrow )</td>
<td>0 ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( s_\uparrow p_\uparrow )</td>
<td>0 ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( s_\uparrow p_\uparrow )</td>
<td>0 ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( p_\uparrow p_\uparrow )</td>
<td>1 ( \frac{3}{2} )</td>
</tr>
<tr>
<td>( p_\uparrow p_\uparrow )</td>
<td>1 ( \frac{3}{2} )</td>
</tr>
<tr>
<td>( p_\uparrow p_\uparrow )</td>
<td>1 ( \frac{3}{2} )</td>
</tr>
<tr>
<td>( p_\uparrow p_\uparrow )</td>
<td>1 ( \frac{3}{2} )</td>
</tr>
</tbody>
</table>

The quantity \( X(j'j'\ell'\ell) \) represents the product \( (2j+1)(2j'+1)W(l'j'\ell'; L) \).

3. Nuclei studied

Calculations have been carried out for \(^6\text{Li}, ^{12}\text{C}, ^{16}\text{O} \) and \(^{40}\text{Ca} \) using the parameters for the Saxon-Woods potentials given by Elton and Swift. These parameters are listed in table 2. When the nuclear charge distribution is obtained by folding the proton charge distribution into the one-particle density functions constructed from the proton wave functions generated in these potentials, the cross section for elastic electron scattering from this charge distribution calculated by an exact phase shift analysis is in agreement with the experimental data for incident electron energies in the range 150-400 MeV and momentum transfers up to \( q \approx 2.5 \text{ fm}^{-1} \). This can be seen from the comparisons with the data given in refs. \(^{13,14}\). It should be noted that direct comparison of \( F^2(q) \) with the experimental data can be quite misleading since
(i) even for $^{12}$C and $^{16}$O there is some difference between Born approximation and the exact phase analysis at and beyond the first Born minimum, and
(ii) the finite size of the proton has a large effect beyond the first minimum.
These two points, of which the second is the more important in this context, were clearly established in the early literature $^{13}$) on electron scattering.

The results obtained using harmonic oscillator functions are denoted by the suffix HO, those obtained using the Saxon-Woods potentials listed in table 2 are denoted by SWO, and those obtained using Saxon-Woods potentials but with the spin-orbit term omitted are denoted by SW. Since the correction for c.m. motion in the shell model is not usually applied to the Saxon-Woods wave functions we have not included this correction in the calculations with oscillator functions.

### Table 2

<table>
<thead>
<tr>
<th>Nucleide</th>
<th>Level</th>
<th>$V_s$</th>
<th>$r_0$</th>
<th>$V_{ae}$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^6$Li</td>
<td>$1s_1$</td>
<td>56</td>
<td>1.42</td>
<td>8.3</td>
<td>0.65</td>
</tr>
<tr>
<td></td>
<td>$1p_2$</td>
<td>51</td>
<td>1.45</td>
<td>0.65</td>
<td></td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>$1s_1$</td>
<td>60</td>
<td>1.36</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1p_2$</td>
<td>55</td>
<td>1.36</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>$^{16}$O</td>
<td>$1s_1$</td>
<td>68</td>
<td>1.41</td>
<td>0.65</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1p_2$</td>
<td>52</td>
<td>1.41</td>
<td>13</td>
<td>0.65</td>
</tr>
<tr>
<td></td>
<td>$1p_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{40}$Ca</td>
<td>$1s_1$</td>
<td>85</td>
<td>1.30</td>
<td>0.60</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1p_2$</td>
<td>60</td>
<td>1.30</td>
<td>30</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>$1d_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2s_2$</td>
<td>53</td>
<td>1.30</td>
<td>12</td>
<td>0.60</td>
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<tr>
<td></td>
<td>$1d_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Energies in MeV, distances in fm.

$^6$Li. No satisfactory fit to the data on elastic electron scattering from this nucleus can be obtained using a single oscillator parameter $^{13,14})$. We have used the oscillator parameter $b = 1.78$ fm which gives fair agreement with the data for $q^2 < 2$ fm$^{-2}$ [ref. $^{15}$)]. The form factors $F^2(q)$ obtained using HO and SWO wave functions are shown in fig. 1, and it can be seen that there is a large discrepancy for $q > 1.5$ fm$^{-1}$.

The use of these wave functions also leads to a marked difference in the predictions for the $^6$Li(p, 2p) reaction $^{15})$. The charge form factor $^{15}F^2(q)$ which is the transform of the nuclear charge distribution is also shown for SWO functions.

$^{12}$C. For this nucleus it is possible to give a good description of the data on elastic electron scattering over a range of momentum transfer up to $\approx 2.5$ fm$^{-1}$ using oscillator functions with a length parameter $b = 1.64$ fm [refs. $^{13,14}$)]. Consequently, the form factors $F^2(q)$ obtained using HO and SWO wave functions are in reasonable...
agreement as can be seen from fig. 1. The HO functions are also relatively successful in describing such processes at the $^{12}\text{C}(p, d)$ reaction\cite{17} and the $^{12}\text{C}(p, 2p)$ reaction\cite{16} at incident energies in the region of 100-200 MeV, and this appears to be due to the rather high separation energies of the p-shell nucleons. The substantial difference between $F^2(q)$ and $\text{CH}F^2(q)$ is evident from figs. 1 and 2.

Fig. 1. The square of the form factors and charge form factor for $^6\text{Li}$ and $^{12}\text{C}$ calculated with HO and SWO functions.

$^{16}\text{O}$. For this nucleus it is possible to use HO functions to describe elastic electron scattering for $q < 2.5$ fm$^{-1}$ with a length parameter $b = 1.76$ fm [refs. 13,14)]. We have used a parameter $b = 1.75$ fm in order to make a comparison with the calculations by Wong\cite{8} of the correlation function. The form factors are shown in fig. 2.

$^{40}\text{Ca}$. For this nucleus the HO functions do not give a satisfactory fit to elastic electron scattering. The best fit is obtained with $b = 1.95$ fm [ref. 12]) but the agreement is satisfactory only within the first diffraction minimum which falls at about
$q \approx 1.2 \text{ fm}^{-1}$. We have chosen the parameter $b = 1.99 \text{ fm}$, again to make comparison with the calculations by Wong. The SWO functions do not fit the data beyond $q \approx 2.5 \text{ fm}^{-1}$ without further modification such as the explicit inclusion of configuration mixing \cite{18}. Also the energy dependence of the Saxon-Woods potential leads to a slight non-orthogonality of the $1s$ and $2s$ functions which we have neglected. The form factors are compared in fig. 2, and it can be seen that there is substantial disagreement in the region of the second Born minimum and beyond.

Fig. 2. The square of the form factors for $^{16}\text{O}$ and $^{40}\text{Ca}$ calculated with HO and SWO functions.

4. Results and discussion

Results for the correlation function $C(q)$ and exchange contribution $D(q)$ calculated using HO functions are shown in fig. 3. For a given nucleus, the maximum value of $C_{\text{HO}}(q)$ is independent of the length parameter $b$ but the position of the maximum moves as $b$ is varied in such a way that $q_{\text{max}}b = \text{constant}$. This is illustrated in fig. 4.
It can also be seen from fig. 4 that $C_{\text{HO}}(q)$ is very insensitive to the actual value of $b$ for $q < q_{\text{max}}$ but for $q > 1.5 \text{ fm}^{-1}$ the effect of small variations in $b$ is quite marked. The results for $^{16}\text{O}$ and $^{40}\text{Ca}$ are in excellent agreement with those obtained by Wong \cite{Wong} who used a Brody-Moshinsky transformation to derive expressions for $C_{\text{HO}}(q)$ and $D_{\text{HO}}(q)$.

The results obtained for the correlation function with SWO functions are also shown in fig. 3, and the ratio $C_{\text{HO}}(q)/C_{\text{SWO}}(q)$ is plotted in fig. 5. We see that there is considerable disagreement between the HO and SWO results. The best agreement occurs for $^{16}\text{O}$ as might be expected from the good agreement for the form factors shown in fig. 2, and similarly the disagreement for $q > 1 \text{ fm}^{-1}$ between the correlation functions.
for $^{40}$Ca is to be expected from the poor agreement for the form factors, also shown in fig. 2. The comparison for $^6$Li and $^{12}$C is discussed below. The separate contributions to the SWO exchange term for $^{16}$O are shown in fig. 6. It can be seen that at small $q$ the large contributions are those with $nlj = n'l'j'$ and these cancel out the corresponding contributions to the form factor so that $C(q)$ is small, but as $q$ increases the exchange between subshells becomes increasingly important and it is mainly

![Fig. 5. The ratio of the correlation functions calculated with HO and SWO functions.](image)

![Fig. 6. The contributions to the exchange term $D_{SWO}(q)$ for $^{16}$O.](image)

through these contributions that sensitivity to the high-momentum components of the single-particle wave functions occurs. Each contribution plotted in fig. 6 contains terms arising from the various values of the angular momentum transfer $L$ allowed by the coupling coefficients in eq. (14). The weighting of these terms is given in table 1.

In order to investigate further the results for $^6$Li and $^{12}$C, we first recalculated the radial integrals $I(j'lj'lj'l'L,m,m'q)$ using SW functions and found complete agreement
with the SWO results over the range of $q$ considered. (The same agreement occurs for $^{16}\text{O}$ and $^{40}\text{Ca}$ and is in accordance with the result observed in electron scattering calculations$^{11,12}$) that the spin-orbit term in the single-particle potential is required to fit the nucleon separation energies but does not have any effect on the form factor.)

We then calculated a correlation function $C_{\text{SW}}(q)$ using SW functions and eq. (17) for $D(q)$ so that the difference between $C_{\text{HO}}(q)$ and $C_{\text{SW}}(q)$ arises only from the difference between the HO and SW single-particle functions. The ratio of these correlation functions is shown in fig. 7. Comparison with fig. 5 shows that the results for $^{16}\text{O}$ and $^{40}\text{Ca}$ are scarcely changed except at large $q$ where the correlation functions are very small. The agreement between the HO and SW calculations for $^{12}\text{C}$ has improved dramatically; this is due to the relatively good agreement of the radial functions in this case and indicates that the disagreement in fig. 5 is due to the difference between $ls$ and $jj$ coupling. For $^6\text{Li}$ a discrepancy between the HO and SW calculations still remains, as is to be expected from the inadequacy of uncorrelated HO functions for this nucleus.

5. Conclusions

We conclude that calculations of the correlation function must be carried out using the best available nuclear wave functions determined from analyses of electron scattering and nuclear reaction data. There is no particular difficulty in calculating the Pauli correlations using realistic single-particle wave functions and the computing time is not excessive. The expansion of SWO functions in terms of HO functions is a well-established technique so that the use of realistic functions does not preclude the inclusion of dynamical correlations which depend on the relative coordinate and quantum numbers of the two nucleons nor does it preclude the application of a c.m. correction$^\dagger$. However, if realistic functions are not used there is clearly a considerable danger that uncertainties introduced into the $q$-dependence of the Pauli correlations may obscure any conclusions reached concerning the dynamical correlations.

$^\dagger$ Alternatively, the c.m. correction can be applied directly$^{19,20}$ using the method of Gartenhaus and Schwarz$^{21}$.)
We are indebted to Dr. C. W. Wong for providing us with a copy of his report and to Dr. M. E. Grypeos for valuable discussions.

Appendix

Our definition of the occupation probabilities is as follows:

\[
P^2(nlj_m) = -\frac{1}{2} \frac{\text{number of protons in the subshell } nlj}{2(2j+1)},
\]

\[
P^2(nlj) = \sum_{m} P^2(nlj_m) = \frac{\text{number of nucleons in the subshell } nlj}{2(2j+1)},
\]

\[
P^2(nl) = \frac{\text{number of nucleons in the subshell } nl}{4(2I+1)}.
\]

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