SANDWICH BEAMS SUBJECTED TO TRANSVERSE LOAD

by

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SUMMARY

The recent development of new forms of composite and hybrid materials together with the need for stronger but lighter structures for use in air and space craft has lead to a rapid rise in the use of sandwich construction. However, the deformation behaviour characteristics of sandwich structures are very complex, even under the simplest of loading arrangements, when compared to the simple homogeneous case. This additional complexity increases the number of possible modes of failure and if sandwich construction is to be used most efficiently then a simple analysis is required to calculate the stresses induced in sandwich structures and hence the expected mode and load at failure.

Two such analyses are developed within this region to predict the stress distributions for both symmetrical and non-symmetrical sandwich beams subjected to a transverse load. To provide the simplest possible solution to the problem it was assumed that the elastic deformation behaviour of the facings conformed with the Bernoulli-Euler Theory. The core is assumed to act strictly as an elastic connection between the facings, sustaining the interfacial shear stresses and supporting much of the shear force.

The theoretical results obtained from the simple solutions developed were verified by a series of experimental studies involving the testing of two-dimensional photoelastic models. This provided a method for making direct measurements of those stresses of interest, for it is possible to determine the state of stress at any point within the model.

Thus despite the complex deformation behaviour characteristics of sandwich structures two simple analyses have been developed which have been found to be effective in predicting the stress distributions associated with symmetrical and non-symmetrical sandwich beams subjected to transverse load.
CHAPTER 1

GENERAL INTRODUCTION

1.1 Sandwich Construction

Sandwich Construction consists of thin facings of strong stiff material bonded either side of a much thicker core of lower modulus. This idea of using strong facings spaced with a weaker core was introduced in about 1820 by a Frenchman named Duleau. However, sandwich construction was rarely used until World War II, when the increased speeds of aircraft resulted in the need for laminar flow over aeroplane wings. In the Mosquito aircraft this problem was solved by the introduction of sandwich construction which being lighter also increased the load capacity. This increase in load capacity resulted in sandwich construction becoming more widely used throughout many aspects of aircraft construction.

More recently the development of new forms of composite and hybrid materials and the need for stronger but lighter structures for use in air and spacecraft construction. At present the choice of facing material is almost unlimited with the possibilities ranging from paper or wood through hardboards and plastics to metals or reinforced plastic laminates.

In contrast however the available cores generally have different inherent characteristics which make them useful in specific applications. Some examples of the core types available are:-

Ribbed or Corrugated Cores

Ribbed or corrugated cores are of similar design to the honeycomb cores mentioned later. They do however possess significantly less strength although they have the advantage of being much cheaper to construct. Thus their use is restricted to providing low cost but low load-bearing panels. Various forms of such panels are widely used throughout many aspects of the building industry, for example internal partitions are often made with plasterboard faces and with resin/fibre ribbed cores.
Cellular Plastics

Foam or cellular plastics are well known for their thermal insulation properties. However they do exhibit other characteristics which make them an ideal core material. They are of light weight and have good water resistance, high strength-to-weight ratios, fire resistance, low moisture permeability, buoyancy and corrosion resistance. Sandwich construction with cores of this type are generally in lower load bearing applications where some specific property is required. For example carbon/epoxy laminate facings and a syntactic foam core are used in certain torpedo casings\(^1\). This provides many advantages over an ordinary solid metal casing, including improved noise attenuation and light weight design which gives greater depth and manoeuvrability with a greater payload to endurance ratio.

Honeycombe

Although honeycomb cores can be made of a variety of materials, aluminium foil is probably the most widely used. In this form the sandwich consists of hexagonal cells of aluminium foil running perpendicular to the facing. In some cases where specific properties are required e.g. heat resistance, these cells can be filled with a plastic foam. This type of sandwich construction has many uses, both in primary and secondary structural applications e.g. in the Autek 400 Business Aircraft\(^2\) nearly 70% of the airframe is made of a sandwich of "Nomex" aramid honeycomb core and "Kevlar" aramid facings, the rudder on Concorde\(^3\) is an aluminium foil honeycomb core sandwiched with Aluminium Alloy facings.

With all forms of sandwich construction there is the possibility of shaping the beams or panels to accommodate the stresses and loads to which that member will be subjected, so that each material is stressed to the practical limits of its possibilities. For example where the bending moment is small, shallow beams can be employed and as the bending moment increases so the depth of the beam can be increased. Another important aspect in the use of sandwich construction is the ability to use materials which provide special properties not available using standard construction methods.
Typical examples are radomes or translucent structures. A specific example is AWACS\textsuperscript{(4)} the Airborne Warning and Control System which has a 30ft diameter radome of sandwich construction made from epoxy/fibreglass prepreg facings with a phenolic/fibreglass honeycomb core.

R. M. Jones\textsuperscript{(5)} has noted a particular aspect of the cost effectiveness of composite materials of which sandwich beams are one form: "A significant consideration is the scrappage in fabrication operations". Scrappage is the material that is trimmed or machined from the starting form of the material in achieving the final product. For most conventional materials, scrappage is returned to the manufacturer for reprocessing. However, scrappage should be less for composites than for conventional materials because composites are fabricated in as close to the final configuration as possible. For example, spars and longerons in aeroplanes wings are beam elements that are usually tapered in both depth and width and have holes in their webs to decrease weight. The fabrication of such members from conventional materials such as aluminium or other alloys consists of hogging out (machined) a large large blank of material that sometimes weighs as much as seven times the final spar weight. The scrappage is then 600 percent! On the other hand, spars have been fabricated from composite materials with as little as 10 percent scrappage! This comparison may seem unfair in the light of other examples, but it actually is quite realistic. Composite materials are not claimed to be a cure-all for every application or even competitive with other materials. There are many instances in which composite materials are uniquely suited because of their peculiar fabrication process. Thus, this "special" case of a spar is not really special, but is actually a powerful example of the class of applications where composite materials offer significant advantages over conventional materials.

### 1.2 Deformation Behaviour of Sandwich Beams

The deformation behaviour characteristics of a sandwich beam are very complex, even under the simplest of loading arrangements, when compared to the simple homogeneous case. When a sandwich beam is subjected to a transverse shear load the facings support axial
compressive and tensile forces, forming a couple which opposes the moment produced by the load. The core however carries little if no axial force but a majority of the shear load which is transferred to the core via the facings. The shear load transfer mechanism consists of the axial forces within the facings causing large interfacial shear stresses to be induced at the boundaries between the facings and the core. Thus the core is subjected to almost pure shear. Because of this large shear load carried by the core it is clear that the shear deformation and flexibility of the core must be taken into account if a true understanding of the overall behaviour of sandwich beams is to be obtained.

As mentioned previously the behaviour of sandwich beams is very much more complex than the simple homogeneous beam. As would be expected this increase in complexity also increases the number of possible modes of failure (as shown in Fig [1.1]). Clearly the mode of failure for any particular beam will be governed by both the geometry of the structure and the mechanical properties of the constituent materials.

If sandwich structures are to be used without an analysis which can accurately predict the mode and load at failure it becomes necessary to conduct a comprehensive series of tests for each individual sandwich element. Such a series of tests would not only prove very expensive but would also take valuable time both of which would have to be taken into account during development. Thus production of a simple analysis to calculate the stresses induced in sandwich structures and hence the expected mode and load at failure would be of great benefit in the use of sandwich structures.

1.3 Elastic Stresses in Sandwich Beams

As stated, conventional theory for the bending of a homogeneous beam under transverse load becomes inadequate when the problem of a sandwich beam is considered. It becomes necessary to consider the shear load transfer mechanism by which the shear forces are transmitted from the facings to the core which has mechanical properties quite different from those of the facings.
To provide the simplest possible solution to the problem it is assumed that the elastic deformation behaviour of the facings conform with the Bernoulli-Euler Theory. The core acts strictly as an elastic connection between the facings, sustaining the interfacial shear stresses and supporting much of the shear force. It does not however make a significant contribution towards carrying the bending moment introduced.

From these simple assumptions it is possible to build up a picture of the form that the stress distributions will take. These are shown in Figure [1.2] and involve linear bending stress distributions in both facings. However for compatibility it is clear that the bending stress within the core must be in the reverse direction, although it should also be noted that it is comparatively small in magnitude. The shear stress distribution consists of parabolic shear within the facings and an almost constant shear stress across the core, (this originates from the core supporting almost pure shear as mentioned previously).

Figure [1.2] clearly shows the existence of three neutral axes within the beam, one in each facing and one in the centre of the core. It is also important to note how the axial force within each facing displaces the neutral axis of that facing towards the core. It also displaces the parabolic shear in the facing toward the core producing the interfacial shear stress. This is the actual shear load transfer.

A comparison of Figures [1.1] and [1.2] will demonstrate how once the stress distributions are obtained it would be a relatively simple task to determine the critical mode of failure for that particular sandwich beam.

With the widely used work of Plantema (6) and Allen (7) which developed a strain energy analysis the displacement of the overall sandwich beam is considered to be a combination of two different displacements, the so called primary and secondary. The primary displacement is that due to bending whereas the secondary displacement is that associated with the shear strain of the core of the sandwich. The analysis involves a solution to predict these
primary and secondary displacements and hence the stresses associated with them. A comparison of this plane strain solution with the plane stress solution presented in this thesis yields stress distributions which differ by less than four percent (for the three beams considered), thus confirming the equivalence of the two approaches.

A simple plane stress solution has been developed to obtain more detailed information about the stress distributions in sandwich beams. The solution of such a complex problem can only be verified by experimental studies. The testing of a two-dimensional photoelastic model provides an obvious method for making direct measurements of those stresses of interest, for it is possible to determine the state of stress at any point within the model. However, in order to obtain a reasonable set of experimental data the photoelastic models consisted of thick faces and flexible cores and it should be noted that this is not typical of the most efficient sandwiches in which bending is resisted primarily by axial forces in the faces.
1  Tensile failure of facing material.
2  Shear failure of facing material.
3  Shear failure at facing/core interface.
4  Compressive failure of facing material.
5  Tensile failure of core material.
6  Compressive failure of core material.
7  Shear failure of core material.

Figure [1.1] Modes of Failure
Figure [1.2] Typical Stress Distributions For

A Symmetrical Sandwich Beam.
CHAPTER 2

REVIEW OF RELATED LITERATURE

2.1 Introduction

In reviewing available publications it was apparent that there was a relative lack of literature on sandwich structures, so it was decided to increase the field of interest to incorporate laminated structures. Laminates are another form of composite materials and closely related to sandwich structures, thus reviewing the literature within this field could provide important information in understanding sandwich construction. Laminates are also widely used as a facings materials within the sandwich industry and so are of interest here.

2.2 Three-Point Bending of Rectangular Plates

The earliest elastic analysis of laminated or sandwich beams or plates was developed in 1936 by H. W. March(8) who derived a differential equation for the transverse deflection of a centrally loaded rectangular strip of plywood, shown in Figure [2.1]. The deflection is calculated at the position where the concentrated central load is applied to the strip.

In his calculation of the deflection H. W. March uses the simple formula \( w = \frac{P_0^3}{48EI} \) for the deflection of a centrally loaded beam, as a first approximation. It is then noted that the thickness of the strips contemplated is small in comparison with the width and they must be considered as flat plates rather than beams. A correction is then made to the foregoing formula to take into account the effect of anti-clastic curvature (the transverse contraction or extension accompanying a longitudinal tension or compression, respectively). This correction is found by treating the strip as a thin plate simply supported on two opposite edges and free on the other two edges. It is found that this correction for anti-clastic curvature is small.
A further correction is then made to take into account the effect of shear deformation due to shearing stresses in planes perpendicular to the surface of the plate. The correction is found as a function of the span-depth ratio by treating the strip as a double cantilever in a state of plane strain. Again this correction was found to be normally small.

In deriving the overall differential equation for the deflection it was assumed that the components of displacement $u$ and $w$ were functions of $x$ and $z$ only and that $v=0$. Thus all components of stress and strain are independent of $y$. At the interface between adjacent plies, the following two conditions applied:

(i) the components of stress in the directions of $x$ and $z$ co-ordinates were continuous;

(ii) the components of displacement $u$ and $w$ were continuous.

There was a further condition that at the interface between each ply the components of stress and strain were connected by compatibility conditions.

The deflection expression was derived from a stress function in the form of a polynomial (equation 26 of reference [8]). This solution is then used to determine the central deflection in a range of plywood strips, containing three, five, seven and nine plies.

A similar technique was used in 1944 by H. W. March and C. B. Smith\(^9\) to calculate the effective flexural rigidity of an unbalanced sandwich plate subjected to three-point bending as shown in Figure [2.2]. The sandwich plate consisted of plywood facings with a balsa core. The method was developed further in a revised paper of 1955\(^1\). Within this paper the stiffness equation is calculated as (equation 65),

$$\text{Effective Stiffness} = \frac{D}{1+\eta(a)^2}$$

(effective flexural rigidity)
where $D$ refers to the flexural rigidity of the construction, neglecting the effect of shear deformation of the core and $n$ refers to a factor showing the influence of the shear deformation of the core.

Norris et al\(^{(11)}\) pointed out that with the form of the equation for the effective stiffness when the modulus of rigidity of the core is reduced to zero the effective stiffness is also zero. This is clearly incorrect because the stiffness of the individual facings are still present. Thus the theory would be less accurate when considering sandwich construction having thick facings or cores of small moduli of rigidity, which is a typical configuration.

### 2.3 Four-Point Bending of Sandwich Beams

An early elastic analysis of an unbalanced sandwich beam is presented in report No. 1505-A, published by the Forest Products Laboratory in 1952, by Norris et al\(^{(11)}\). The unbalanced sandwich beam is subjected to four-point bending as shown in Figure [2.3]. In the analysis the beam is assumed to carry normal loads with the facings being treated as cylindrically bent plates. The strains in the facings are taken to consist solely of those associated with bending and stretching, and it is also assumed that the component of normal displacement is identical in the two facings. The core material is considered to be weak in shear as compared with the facing material, the bending stiffness of the core is neglected entirely. In the shear analysis it is assumed that the shear deformations are constant over the thickness of the core.

From these assumptions and the governing equilibrium equations an interfacial shear stress function is obtained which takes the form

$$\tau = A \cosh(\alpha x) + B \sinh(\alpha x) + \frac{\beta}{\alpha^2}$$

from this all other relevant stresses and strains are obtained.
The following conclusions were drawn by the authors, Norris et al:–

(i) The results of the new analysis agree with that of Report No. 1505 (by H.W. March and C.B. Smith reference [9]) for usual sandwich construction. For extreme sandwich constructions having thick facings and cores of very small moduli of rigidity, the new analysis may yield values closer to those existing in the specimens. Neither analysis is suitable for very short specimens, nor for the determinations of the stresses near the loads or reactions.

(ii) In general, the values obtained from tests agree reasonably well with those computed by the method of Report No. 1505 by H.W. March and C.B. Smith reference [9].

A more recent analysis was produced by K. Kemmochi and M. Clemura(12) in 1980. Under the assumption that the core was of low modulus material in comparison with the faces of the sandwich beam and that the bending moment was resisted by the faces alone the "multi-layer built-up theory" was developed. The theory leads to the forming of a second order differential equation,

\[
\frac{d^2 P}{dx^2} - Q^2 \rho = -\beta \gamma_0 M
\]

which by considering the boundary conditions the axial forces in the two regions (i) and (ii), see Figure [2.4], were determined. Tests were carried out on four models with varying ratios of the face and core Young's modulii (Ef/Ec), the loads applied were kept well below the elastic limit. The conclusion drawn by the authors was that as the modulus ratio was increased, the location of the neutral axis changed position towards the tensile edge. Further increase caused neutral axes to appear within the faces. Points to note are that a comparison between theoretical and experimental results is only shown for a section in region (ii) and then only \( \sigma_x \) plots are shown. This is despite the fact that experimental plots of \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) are drawn for a section in region (i). Also there is not an explanation or diagram of how the four-point loading is actually applied.
2.4 Composite Laminates Under Cylindrical Bending

In 1969 N.J. Pagano\textsuperscript{13} produced a series of exact solutions (within the linear theory of elasticity) for unidirectional and bidirectional (0\textdegree-90\textdegree) layered anisotropic systems under cylindrical bending.

Within the analysis a state of plane strain is assumed with respect to the xy plane, see Figure [2.5], and the model is also assumed to be simply supported at its ends. The plate under consideration is taken to undergo normal traction with $\sigma_y=q(x)$ applied on the upper surface. This load $q(x)$ takes the form of a Fourier series. All stress, strain and displacement components are taken as functions of $x$ and $y$ only with continuity of traction and displacement being established at the interfaces between layers. The following three separate geometrical configurations were considered, namely:

(i) a unidirectional plate with the fibres orientated in the $x$-direction,

(ii) a bidirectional (coupled) laminate with the transverse and longitudinal directions aligned parallel to $x$ in the top and bottom layers, respectively, the layers being of equal thickness,

(iii) a symmetric 3-ply laminate with layers of equal thickness - the longitudinal direction coincides with $x$ in the outer layers while the transverse direction is parallel to $x$ in the central layers.

Theoretical results are computed for the longitudinal stress, transverse stress, and plate deflection, those are then compared to the classical laminated plate theory summarised in reference [14]. It should be noted that no experimental evidence is presented with which to compare the results of the analysis presented.
2.5 Bending of Laminated Plates Under Transverse Load

A paper was published in 1969 by J.M. Whitney\textsuperscript{(15)} in which a bending theory has been developed for anisotropic laminated plates subjected to transverse loading. The work deals with symmetric laminates in which the material axes of each layer have arbitrary orientation with respect to the plate axes, it was intended to show the effect of shear deformation on the gross behaviour of fibrous composites.

The plate surfaces were assumed to be free of shear tractions and the shear stresses were to be continuous at the interface of adjacent layers.

By assuming a static load in the form of a Fourier series the equations of motion are obtained which are only solveable if certain conditions are assumed, i.e. if the laminates are composed of a large number of layers such that the plate becomes quasi-homogeneous, special orthotropic in which the reduced solution is solveable.

It has been shown that under the transverse shear load, the shear deformation can significantly affect the deflection of anisotropic laminates. For the bending of a four layer symmetric crossply square plate under transverse load, it has been shown (Figure 1 of reference [15]) that the shear deformation reduces the deflection above 15% (in this case the width to thickness ratio was 20). Figure 2 of reference [15] shows that the effect is much less pronounced for low modulus ratios.

It has also been pointed out that the magnitude of the discontinuity in the slope of the shear stress at the interface of adjacent layers of the crossply plate, which appears as a function of the shear modulus ratio, is not as large as the discontinuity which appears in Pagano's\textsuperscript{(13),\textsuperscript{14}) work.\textsuperscript{13},\textsuperscript{14}

The report by J.M. Whitney concludes that despite the quantitative inaccuracy of the assumed shear stress functions it appears that the theory presented can accurately predict gross response characteristics (plate deflections, buckling loads etc).
It shows that careful consideration should be given to plate dimensions and material properties before neglecting transverse shear deformations.

2.6 Photoelastic Studies

An early example of the use of photoelasticity in experimental studies can be found in a paper published in 1931 by M.M. Frocht\textsuperscript{(16)}, who investigated the optical properties of various materials. It is mentioned that under pure bending the fringes in Bakelite and celluloid beams appeared as parallel, straight and equidistant lines, this being due to the linear distribution of the bending stress. From the known loading arrangement and beam dimensions the stress value of each fringe can be calculated. A further investigation of different loading systems such as a cantilever with transverse end load and four-point bending was undertaken using rectangular beams of homogeneous material. As a result it was shown that the Bakelite material was optically more sensitive to stress variations than the celluloid material.

A more recent example of the use of the photoelastic technique in the experimental study of sandwich beams in the work of K. Kemmochi and M. \textsuperscript{12}Uemura mentioned previously, see under Four-Point Bending of Sandwich Beams. Within their paper, published in 1980 they used the shear difference method of stress separation to determine the stress distributions across four different photoelastic models.
Figure [2.1] H.W. March's Model of a Centrally Loaded Strip of Plywood.
Figure [2.2] H.W. March and C.B. Smith's Model

Subjected to 3-point-bending.
Figure [2.3] Norris et al. Unbalanced Sandwich Beam

Subjected to Four Point Bending.
Figure [2.4] Kemmochi and Uemura's Model of a Sandwich Beam Subject to Four Point Bending.
Figure [2.5] N.J. Pagano's Model Arrangement.
CHAPTER 3

THEORETICAL ANALYSIS OF A SYMMETRICAL SANDWICH BEAM

3.1 Introduction

Here a theory is developed to predict the elastic bending and shear stress distributions for a sandwich beam of symmetrical construction. For any sandwich beam the value of the stresses at the critical point, the point from which failure is likely to originate, will depend upon both the geometrical configuration and the loading arrangement. However, sandwich-structures have an additional problem of possible interfacial shear failure thus the most critical case is transverse loading which induces high interfacial shear stresses.

By assuming a relatively simple displacement condition it is possible to develop, from first principles, a shear transfer mechanism to explain the method by which shear loads applied via the facings are transferred to the core material of the sandwich. The resulting theory which is developed here can be used to predict not only the bending and shear stress distributions across the facings and core, but also the variation in the stresses along the length of the sandwich.

Further within this chapter there will be a discussion of the results presented and in a later chapter a comparison will also be made with results obtained from photoelastic experimentation.

3.2 Details of the Theoretical Model Arrangement

The theoretical analysis involves the consideration of a cantilever subjected to a transverse end load as shown in Figure [3.1]. This configuration was chosen because it is the simplest case involving both bending and shear stress.

Figure [3.1] also shows the system of rectangular coordinates and typical geometric parameters. It should be noted that in order to simplify the calculations all loads are per unit thickness.
3.3 Assumptions

(i) Each face of the beam behaves as a simple elastic beam subjected to a combination of bending and transverse shear forces.

(ii) The depth to width ratio of the whole beam is large and the components can be considered to be in a state of plane stress.

(iii) The bending stress distribution varies linearly through the depth of each facing and the core.

(iv) Both interface materials are assumed to exhibit linear elastic properties.

(v) Complete bonding is assumed at the interface of each facing and the core.

3.4 Notation

A, B  constants of integration.

x, y  cartesian frame with x parallel to the neutral axis of the beam.

L  span of the cantilever.

d  thickness of the facings.

c  thickness of the core.

W  transverse load per unit width.

S  shear force per unit width in facing.

S₀  shear force per unit width in core.

M  bending moment per unit width in facing.

M₀  bending moment per unit width in core.

T  axial tension per unit width in the facing.

u  mean longitudinal displacement of the facing.

v  transverse displacement of both facing and core.

δ  shear displacement in facing.

δ₀  shear displacement in core.

E, G  elastic constants of facings.

E₀, G₀  elastic constants of core.

m=E/E₀  modular ratio.

n, k  geometric constants.

τ  interface shear stress.

σₓₓ, σᵧᵧ  cartesian stress components.

τ₀=W  nominal shear stress.

\[ \sigma₀=\frac{6WL}{(2d+c)^2} \]  nominal bending stress.
Consider an element, $\delta x$, of the sandwich at a distance $x$ from the fixed origin. Figure [3.2] shows such an element together with the system of forces which are assumed to act upon the element.

For longitudinal force equilibrium within each facing,

$$\frac{dT + \tau}{dx} = 0 \quad (1)$$

Again for the facing, the bending moment equilibrium gives,

$$\frac{dM + S - d/2 \tau}{dx} = 0 \quad (2)$$

Similarly the bending moment equilibrium for the core can be written as,

$$\frac{dM_0 + S_0 - c \tau}{dx} = 0 \quad (3)$$

Now at the interface between the core and each facing the longitudinal strain in both core and facing is the same. Thus the stresses on either side of the interface are related by their Young's moduli. This relationship can be written thus (for the lower facing),

$$\frac{T - 6M}{d} = m \frac{6M_0}{d} \quad (4)$$

where $m$ is the modular ratio of the facing to core, $(E/E_0)$.

Differentiating equation (4) and rearranging gives,

$$\frac{d^2 M + dM_0}{dx^2} = \frac{m}{(E/E_0)} \frac{dM_0}{dx} \quad (5)$$
where upon substitution for \( \frac{dT}{dx}, \frac{dM}{dx} \) and \( \frac{dM_0}{dx} \) from equations (1), (2) and (3) respectively leads to,

\[
S + m_s \frac{d^2}{dx^2} S_0 = T \cdot d_s \left( \frac{m_s \cdot d + 2}{3} \right)
\]  

However, the total shear force relationship is given by

\[
S_0 + 2 \cdot S = W
\]

and on combining this with equation (6) the shear forces within the facings and core can be expressed in terms of the interface shear stress, the applied load and various model parameters as shown below.

\[
S = T \cdot d_s \cdot \left( \frac{2 + 3 \cdot m_s \cdot d}{c} \right) - m_s \left( \frac{d}{c} \right)^2 \cdot W
\]

\[
1 - 2 \cdot m_s \left( \frac{d}{c} \right)^2
\]

and

\[
S_0 = W - T \cdot 2 \cdot d_s \cdot \left( \frac{2 + 3 \cdot m_s \cdot d}{c} \right)
\]

\[
1 - 2 \cdot m_s \left( \frac{d}{c} \right)^2
\]

Figure [3.3] shows the longitudinal displacement conditions for the sandwich beam. These can be equated to give,

\[
u = \frac{d \cdot dv}{2 \cdot dx} \cdot \frac{\delta}{2 \cdot dx} - \frac{S_0}{2 \cdot dx}
\]

Rearranging and differentiating twice with respect to \( x \), gives,

\[
\frac{d^4}{dx^4}(S_0 + \delta) = (c + d) \cdot \frac{d^2 v}{dx^2} - \frac{d^2 u}{dx^2}
\]

Now the bending moment curvature relationship is

\[
\frac{d^2 v}{dx^2} = \frac{M}{EI}
\]
where \( E \) refers to the flexural rigidity of each facing in which the moment of inertia \( I \) is given by \( \frac{d^3}{12} \). On differentiation w.r.t \( x \) the relationship becomes,

\[
\frac{d^3v}{dx^3} = \frac{12 \cdot dM}{E \cdot d^3 dx} \tag{13}
\]

Combining equations (2) and (8) to eliminate the facing shear force \( S \) and substituting for \( \frac{dM}{dx} \) in equation (13) gives,

\[
\frac{d^3v}{dx^3} = \frac{12 \cdot m \cdot (d/c)^2}{E \cdot d^3} \left[ \frac{W - \frac{1}{2} \cdot d^4 \cdot (1 + 4 \cdot m \cdot (d/c)^3)}{1 - 2 \cdot m \cdot (d/c)^2} \right] \tag{14}
\]

Application of Hooke's Law to the facing gives the simple relationship,

\[
\frac{du}{dx} = \frac{T}{E \cdot d} \tag{15}
\]

which on differentiating w.r.t \( x \) and substituting for \( \frac{dT}{dx} \) from equation (1) becomes,

\[
\frac{d^2u}{dx^2} = \frac{-T}{E \cdot d} \tag{16}
\]

The total shear displacement, \( \delta \), of one of the facings can be found from the expression,

\[
\delta = \frac{S}{G} \tag{17}
\]

differentiating twice w.r.t \( x \) and substituting for \( S \), the shear force, from equation (8) gives,

\[
\frac{d^2 \delta}{dx^2} = \frac{d^3(2 \cdot 3 \cdot m \cdot (d/c))}{G \cdot (1 - 2 \cdot m \cdot (d/c)^2)^2} \cdot \frac{dT}{dx} \tag{18}
\]
Similarly the core shear displacement, $\delta_0$, which in this case represents only half the total core shear displacement is given by,

$$\delta_0 = \frac{S_0}{2G_0} \quad (19)$$

which on differentiation and substitution for, $s_0$, the core shear force, from equation (9) gives,

$$\frac{d^2\delta_0}{dx^2} = \frac{-d/3(2+3m\frac{d}{c})}{G_0(1-2m[\frac{d}{c}]^2)} \frac{d^2T}{dx^2} \quad (20)$$

Substituting equations (14), (16), (18) and (20) back into (11) yields a second order differential equation in terms of $\tau$, the interface shear stress. This fundamental differential equation is of the form,

$$\frac{d^2\tau}{dx^2} - n^2\tau = -n^2K\tau W \quad (21)$$

where

$$n^2 = \frac{3G_GG_0}{G-G_0} \frac{c_d+2m[\frac{d}{c}]^2(4c-d^2+6c^2d+3c^3)}{E_d^3(2+3m\frac{d}{c})}$$

and

$$K = \frac{6d(c+d)m[\frac{d}{c}]^2}{c_d+2m[\frac{d}{c}]^2(4c^2+6c^2d+3c^3)}$$

### 3.5.2 Determination of the Interfacial Shear Stress Distributions

The solution of the fundamental differential equation, equation (21), is found to be of the form,

$$\tau = A\sinh(nx) + B\cosh(nx) + K\tau W \quad (22)$$

where $A$ and $B$ are constants.
To find values for A and B requires two boundary conditions. One arises from the fact that at the fixed end of the cantilever, that is \( x=0 \), the interfacial shear stress will be zero. This gives,

\[
B = -K \cdot W
\]

therefore,

\[
T = A \cdot \sinh(nx) + K \cdot W \cdot (1 - \cosh(nx)) \tag{23}
\]

The second boundary condition is less obvious, but arises from the facts that the free end of the beam acts as a point of inflection and therefore at this point, \( x=L \), the rate of change of the interfacial shear stress along the beam will be zero, that is \( \frac{dT}{dx} = 0 \). This gives,

\[
A = K \cdot W \cdot \tanh(nL)
\]

and therefore the interfacial shear stress distribution is given by,

\[
T = K \cdot W \cdot [1 - \cosh(nx) + \tanh(nL) \cdot \sinh(nx)] \tag{24}
\]

or in a nominalised form,

\[
\frac{T}{\tau_0} = K \cdot (2d + c) \cdot [1 - \cosh(nx) + \tanh(nL) \cdot \sinh(nx)] \tag{25}
\]

where \( \tau_0 \) is the nominal shear stress, \( \tau_0 = \frac{W}{2d + c} \).

3.5.3 Determination of the Bending Stress Distributions

Within the sandwich each facing is subjected to a combined axial force and bending moment. The resultant bending stress in the bottom facing is given by,

\[
\sigma_{xx} = \frac{T - 12M_y}{d^3} \tag{26}
\]
To obtain the axial force $T$ it is necessary to substitute equation (24) into equation (1) and then integrate with respect to $x$. This gives,

$$T = K \cdot W \left[ \frac{\sinh(nx) - \tanh(nL) \cosh(nx) - x}{n} \right] + A$$  \hspace{1cm} (27)$$

where $A$ is a constant of integration. But at $x=L$ the axial force $T=0$. Thus,

$$A = L \cdot K \cdot W$$

which gives the axial force as,

$$T = K \cdot W \left[ \frac{L - x - \frac{1}{n} \left( \tanh(nL) \cosh(nx) - \sinh(nx) \right)}{n} \right]$$  \hspace{1cm} (28)$$

The bending moment distribution can be found by substituting the shear force, equation (8), into the bending equilibrium equation (2), and then integrating with respect to $x$, to give,

$$M = \frac{d}{6} \left[ 1 + 6 \cdot m \cdot \frac{d_c}{c} \cdot (1 + d_c^2) \right] K \cdot W \cdot \frac{\sinh(nx) - \tanh(nL) \cosh(nx) - x}{n}$$  

$$+ \frac{m \cdot (d_c^2)}{1 - 2 \cdot m \cdot (d_c^2)^2} W x + A$$  \hspace{1cm} (29)$$

where $A$ is the constant of integration. Now at $x=L$ the bending moment, $M$, is zero thus,

$$A = \frac{d}{6} \left[ 1 + 6 \cdot m \cdot \frac{d_c}{c} \cdot (1 + d_c^2) \right] K \cdot W \cdot L - \frac{m \cdot (d_c^2) \cdot W \cdot L}{1 - 2 \cdot m \cdot (d_c^2)^2}$$

and thus the bending moment distribution becomes,

$$M = \frac{d}{6} \left[ 1 + 6 \cdot m \cdot \frac{d_c}{c} \cdot (1 + d_c^2) \right] K \cdot L \cdot W \left[ 1 - x - \frac{\tanh(nL) \cosh(nx) - \sinh(nx)}{n \cdot L} \right]$$  

$$- \frac{m \cdot (d_c^2) \cdot W \cdot L \cdot (1 - \frac{x}{L})}{1 - 2 \cdot m \cdot (d_c^2)^2}$$  \hspace{1cm} (30)$$
Substituting equation (30) and (28) back into equation (26) gives the bending stress in the bottom facing as,

$$\sigma_{xx} = \frac{(2d+c)^2-2mL^2}{6d[1-2mL]}[1+4mL(1+4L)]K[1-x-\tanh(nL)\cosh(nx)+\sinh(nx)]$$

where $\sigma_0$ is the nominal bending stress and is given by $\sigma_0 = \frac{6WL}{2d+c}$.

Clearly due to the symmetric nature of the sandwich under investigation the bending stress in the top facing will be symmetric to that obtained for the bottom facing. Thus there is no need to carry out further analysis to obtain this other bending stress distribution.

### 3.5.4 Determination of the Transverse Shear Stress Distributions

The transverse shear stress distributions through the depth of each facing can be found by applying the following standard expression, in this case applied to the top facing,

$$\sigma_{xy} = \int \frac{d}{2} \frac{d\sigma_{xx}}{dx} dy$$

For the top facing the bending stress, $\sigma_{xx}$, is given by,

$$\sigma_{xx} = -\frac{I}{d} - 12\frac{M_y}{d^3}$$

which on differentiation w.r.t $x$ becomes,

$$\frac{d\sigma_{xx}}{dx} = -\frac{1}{d^3} \left[ \frac{dI}{dx} + 12\frac{M_y}{d} \frac{dM}{dx} \right]$$
Thus,

\[ \sigma_{xy} = \left( \frac{y}{d} - \frac{1}{2} \right) \left[ \frac{dT}{dx} + \frac{6y}{d} \left( \frac{y}{d} + \frac{1}{2} \right) \frac{dM}{dx} \right] \] (35)

on substitution for \( \frac{dT}{dx} \) from equation (1), and for \( \frac{dM}{dx} \) by substitution of equation (8) into equation (2) this gives,

\[ \sigma_{xy} = \left( \frac{y}{d} - \frac{1}{2} \right) \left[ -\frac{y}{d} \times \frac{1}{2} + \frac{y}{d} \times \frac{1}{2} \left( \frac{1 + 6m_d/c_s(1+1/d/c)}{2m_d/c_s} \right) \frac{d}{d} - \frac{6m_d(1/d/c)}{d} \right] \] (36)

or in non-dimensional form, the shear stress in the top facing is given by

\[ \frac{\sigma_{xy}}{\tau_0} = \left( \frac{y}{d} - \frac{1}{2} \right) \left[ -\frac{y}{d} \times \frac{1}{2} + \frac{y}{d} \times \frac{1}{2} \left( \frac{1 + 6m_d/c_s(1+1/d/c)}{2m_d/c_s} \right) \frac{d}{d} - \frac{6m_d(1/d/c)}{d} \right] \] (37)

Similarly for the bottom facing, to give

\[ \frac{\sigma_{xy}}{\tau_0} = \left( \frac{y}{d} + \frac{1}{2} \right) \left[ -\frac{y}{d} \times \frac{1}{2} + \frac{y}{d} \times \frac{1}{2} \left( \frac{1 + 6m_d/c_s(1+1/d/c)}{2m_d/c_s} \right) \frac{d}{d} - \frac{6m_d(1/d/c)}{d} \right] \] (38)

3.6 Discussion of the Theoretical Results

3.6.1 Introduction

The relative complexity of the sandwich beam problem in comparison with the simple homogeneous case is clearly demonstrated in the theoretical distributions presented. It should be noted that in the bending stress distributions shown in Figure [3.4] the core bending stress is not present because it's value is so small that it is indistinguishable from the y-axis and thus it's conclusion could cause some confusion.
3.6.2 Bending Stress Distribution

The bending stress distributions across three sections of a typical sandwich beam are shown in Figure [3.4] and demonstrate clearly the existence of neutral axes within each of the facings. Further for reasons of compatibility and symmetry there must be a neutral axis at the centre of the core. The analysis does in fact confirm this. Another important feature which shows in Figure [3.4] is the fact that the neutral axes in the facings are displaced towards the core of the sandwich. This is as a result of the longitudinal force in each of the facings. Because of the lower value of the bending moment this becomes more noticeable at section three. Obviously if the geometric and material parameters were adjusted the situation could arise whereby the neutral axes in the facings could be made to disappear, leaving just one neutral axis at the centre of the core.

From these plots it is apparent that should tensile or compressive failure of the facing material occur this would originate at the top or bottom surface respectively of the sandwich beam at the point of greatest bending moment, in this case the built-in end of the cantilever.

3.6.3 Shear Stress Distributions

The shear stress distributions for the same three sections as for the bending stress are shown in Figure [3.5]. From these distributions it is evident that despite the fact that the core shear stress was assumed to be parabolic it is, in this case, almost constant. However, since there are clearly two maxima, one in each facing, there must also be a minimum in the core but the curve is so shallow that it cannot be seen. As with the neutral axes in the bending stress distributions the maxima are displaced towards the core of the sandwich and adjustments of geometric and material parameters could lead to the maxima disappearing from the faces. In this case the minimum in the core would change to a maximum. A further point which shows from Figure [3.5] is that the maximum values of the shear stress distributions in the facings reduce as you move along the beam away from the built-in end, whereas the maximum
value of the shear stress in the core increases as you move along the beam away from the built-in end. Thus in terms of shear failure the facings are most susceptible at the built-in end and the core at the outer extremity of the cantilever. It is also important to again note that the maximum shear stress in the core, in the case considered, is at the core/facing interfaces and shear failure in the core would hence originate from here.

3.6.4 Interfacial Shear Stress Distributions

Figures [3.6] and [3.7] show the effects of varying the geometric parameters and material parameter \((E/E_0)\) respectively, on the interfacial shear stress distributions. Both figures show the relatively high value of the interfacial shear stress over a substantial part of the outer extremities of the cantilever. They also show that increasing either of the ratios or \((E/E_0)\) has the same effect of decreasing the interfacial shear stress, though in both cases such an increase does not cause a proportionate decrease in the interfacial shear stress.

It is possible to combine the two graphs shown in Figures [3.6] and [3.7] by plotting the interfacial shear stress for varying values of the function \(n^2k\). Such a graph is illustrated in Figure [3.8] and together with the previous two figures demonstrates that failure of the sandwich at either facing/core interface will originate at the free end of the cantilever. Also because of the nature of the interface shear stress distribution described, such a failure will quickly spread along the interface.

3.6.5 Shear Force Variation

Figure [3.9] shows the proportion of the shear force carried by the facings and core. As implied by the assumptions, at the built-in end of the beam the shear force is carried by the facings alone and as you move along the beam to the free end so the proportion of the shear force carried by the core increases.
Figure 3.1 Cantilever Subjected to Transverse End Load.
Figure [3.2] System of Forces on an Element Distance $x$ From the Fixed Origin.
Figure 3.3] Longitudinal Displacement Conditions.
Theoretical bending stress distributions for a symmetrical sandwich.

$L = 6''$

$E/E_o = 1318$

$\frac{d}{2d+c} = .13$

$\sigma_o = \frac{6WL}{(2d+c)^2}$

Figure [3.4]
Theoretical shear stress distributions for a symmetrical sandwich.

\[ L = 6'' \]

\[ \frac{d}{2d+c} = 0.13 \]

\[ \tau_0 = \frac{W}{(2d+c)} \]

Figure [3.5]
Theoretical interfacial shear stress distributions for a symmetrical sandwich.

\[ L = 6'' \]

\[ \frac{E}{E_0} = 1318 \]

\[ \tau = \frac{W}{(2d+c)} \]

![Diagram showing interfacial shear stress distribution](image)

Figure [3.6]
Theoretical interfacial shear stress distributions for a symmetrical sandwich.

Effect of varying the ratio $E/E_0$.

$L = 6''$

$E_0 = 348$

$\frac{d}{2d+c} = 0.17$

$\tau_0 = \frac{W}{(2d+c)}$

Figure [3.7]
Theoretical interfacial shear stress distributions for a symmetrical sandwich.

Variation in terms of the parameter \( n^2K \)

\[
\tau_0 = \frac{W}{(2d+c)}
\]

\( L = 6'' \)

\( n^2K \)

Figure [3.8]
Theoretical shear force distributions

for a symmetrical sandwich.

\[ \frac{E}{E_0} = 1318 \]

\[ L = 6'' \]

\[ \frac{d}{2d + c} = 0.2 \]

Figure [3.9]
CHAPTER 4

EXPERIMENTAL STUDY OF A SYMMETRICAL SANDWICH BEAM

4.1 Design and Manufacture of the Photoelastic Models

The dimensions of the symmetrical sandwich beam test models are given in Figure [4.1]. The materials chosen for these photoelastic models are Araldite CT200 (formerly known as Araldite B) for the facings and a Urethane rubber, Stycast CPC-41 for the cores. These materials were used mainly to provide the correct ratio of Young’s modulii which would enable a comparison to be made with the postulated theory. There were however other advantages, namely, the good photoelastic properties possessed by both materials and the ease with which the araldite facings could be machined.

4.2 Casting and Curing Procedures

The hot setting resin, CT200, is supplied in the form of small yellow brown platelets and a separate hardener, HT901, as a white powder. Four parts of the resin by weight are heated to a temperature of 120°C, then one part of the hardener is added. The resulting solution is then thoroughly mixed while maintaining the 120°C temperature and then evacuated to remove any air bubbles introduced during the mixing. Before pouring the mould must be properly prepared. Due to the strong adhesion properties of Araldite all the surface which will contact the Araldite are coated with a release agent, Redeasil No. 14. The various parts of the mould are then heated in an oven to a temperature of 140°C and then left for half an hour, after which it is allowed to cool to ambient temperature. The mould is then assembled and preheated to a temperature of 120°C at which point the resin mixture is poured slowly and carefully to avoid the introduction of air bubbles. After curing the casting for 16 hours at 120°C the temperature is reduced to room temperature at a rate of 3°C per hour. It is necessary to anneal the casting after it has been cured. The casting is placed in an oven and the temperature raised to 150°C at a rate of 20°C per hour. Once this maximum temperature is reached it is reduced to room temperature at the same rate, 20°C per hour. The resultant cast Araldite has a Young's Modulus (E) of 459 000 lb ins⁻², with a Shearing Modulus (G) of 170 000 lb ins⁻².
The cast Araldite is then cut and machined to the required dimensions for the model sandwich beam. It is then fixed within the core casting jig which is previously cleaned and coated with release agent, see Figure [4.2]. The jig consists of two perspex sheets which are screwed together sandwiching the Araldite facings between them. The facings themselves are pinned the correct distance apart to be certain that during the casting of the Urethane they do not move.

The Urethane, CPC-41, is then prepared by mixing part A and part B in the ratio of 100 to 120 by weight. The mixture must then be thoroughly evacuated to remove all the air introduced during the mixing. Once this is complete the mixture is carefully poured into the opening at the end of the jig being sure not to introduce any air bubbles. The model is removed from the jig after 24 hours and placed in a hot oven. This serves two purposes. The first is to speed up the curing of the Urethane which would otherwise take a week. The second, more important, reason is the tendency of the model materials to absorb moisture. By storing them in hot ovens the amount of moisture absorption is greatly reduced.

At the same time as the Urethane core is cast a small rectangular test specimen is also produced from the same Urethane CPC-41 mixture, see Figure [4.3]. This specimen is then used to determine the mechanical and optical properties of that particular Urethane. This process is necessary because in the mixing and casting of such small quantities the final properties may show some variation from one batch to another. Average values of Young's Modulus \( E_0 \) and Shearing Modulus \( G_0 \) were 348 \( \text{lb ins}^{-2} \) and 117 \( \text{lb ins}^{-2} \) respectively.

### 4.3 The Loading System and Testing

In order to apply three-point bending to the test models within their own planes it is necessary to construct a loading rig. A suitable loading rig is illustrated in Figure [4.4] and was made from Aluminium Alloy B.S.1474.

The design of the rig is such that it is attached to the model by small pins at both outer extremities of the top and bottom
The design also ensures that equal loads of $W/4$ are applied at these four points. $W$ is the total load carried by the test model and is attached via two plates bolted along the centre line, see Figure [4.5].

Before testing can begin each model must be removed from the storage oven and allowed to cool for approximately 20 minutes to reach ambient temperature. It is then pinned within the loading rig and the complete assembly suspended in the standard photoelastic straining frame. A vertical static load of 41b weight was applied to the central loading attachment.

The applied load was chosen such that a number of distinct fringe orders could be observed while carefully ensuring that the model strains were well below the elastic limit. After a further period of approximately 30 minutes the photoelastic observations could be taken and the model returned to the storage oven.

4.4 The Optical System and Photoelastic Observations

The photoelastic model analysis was carried out on a conventional polariscope, Figure [4.6] showing the relative positions of the optical elements. The light source is a 250 watt high pressure mercury vapour lamp with an infrared heat cut off. After passing through an ultra violet filter the light rays are brought to focus at an iris diaphragm by a small lens. A green filter may be interposed at this point to give a reasonably monochromatic green light of mean wavelength $5461\text{Å}$. A second much larger lens is then used to produce a 4 inch diameter beam of collimated light, a sheet of translucent paper converts this to diffuse light. This beam of diffuse light passes in turn through the polariser, the test model, a quarter wave plate when required, and the analyser. The model is viewed through a travelling microscope and for each observation the cross-wire is positioned over the point of interest.

With the green filter and the quarter wave plate removed and the polariser and analyser crossed and coupled, the isoclinic parameter was noted. Then with the polariscope set up for the Sénarmount
method of measuring fractional fringe orders the isochromatic fringe number was measured. The process was then repeated to obtain the isoclinics and isochromatics for all the test locations.

4.5 Analysis of the Test Data

Within each of the experimental test models three different transverse sections were selected along which photoelastic observations were taken. These sections namely 1, 2 and 3 are illustrated in Figure [4.1]. The three models used had ratios of facing to core thickness (d/c) of 0.33, 0.25 and 0.18. The variation in this ratio was used to determine the effect such geometrical parameters on the behaviour of sandwich beams under a given loading condition.

During the manufacturing procedure of each of the models stresses are induced which cannot be removed by annealing. These stresses, especially those induced at the facing/core interface will clearly show in the final experimental results.

The effect of these initial stresses can be removed by employing the correct experimental technique which is to take two sets of readings the second set with the loading arrangement reversed from the first. In this way the stresses due to the loading arrangement change sign from the first case to the second whereas the initial stresses remain the same. Thus subtracting the two results removes the effect of the initial stresses.

In mathematical terms this becomes,

\[ \sigma_{xx} = \frac{f}{2t} \left( N_1 \cos 2\theta_1 - N_2 \cos 2\theta_2 \right) \]

where \( \sigma_{xx} \) is the bending stress, \( f \) is the stress-optic coefficient, \( t \) the model thickness, \( N_1 \) the isochromatic fringe number for the first loading and \( \theta_1 \) the isoclinic angle for the first loading. \( N_2 \) and \( \theta_2 \) are again the isochromatic fringe number and isoclinic angle but for the reversed loading.
Similarly for the shear stress,

\[ \sigma_{xy} = \frac{f_1(N_1 \sin 2\theta_1 - N_2 \sin 2\theta_2)}{4} \]

In this way, by using the photoelastic observations of isoclinic angles and isochromatic fringe number it is possible to accurately calculate the shear stresses and bending stresses at any of the test points in the three experimental beams.

4.6 Discussion of the Experimental Results

Figures [4.8] through to [4.17] show the experimental results plotted against the theoretical distributions. The comparison shows the good agreement which exists between the results though two interesting aspects need to be discussed. The bending stress, which is not plotted for the core because of its low value, shows excellent agreement for sections 1 and 2 but drifts slightly off at section 3 for all the test beams. This drift is only slight but can be explained by the work of R.S. Alwar\(^{(17)}\), C.L. Amba-Rao and S.K. Bansal\(^{(18)}\) and C.L. Amba-Rao\(^{(19)}\).

R.S. Alwar\(^{(17)}\) considered the effect of having two statically equivalent loading systems which gave different experimental results. In the paper he published in 1970 he carried out a series of photoelastic experiments on a simple sandwich beam subjected to transverse end load could be applied either of two locations A or B, forming two statically equivalent systems. The models tested were all of the same dimensions, see Figure [4.7], although they were constructed of different materials, including Rubber, Araldite D and Plasticizer. This gives a maximum ratio of Young's Modulii \(E_{\text{face}}/E_{\text{core}}\) of about 3600. Since the interest was in finding to what extent the loading effect was carried along the beam a full and complete analysis of the principal stresses was not attempted. Instead the fringe order distributions were plotted at various points along the beam. The following important conclusions were drawn by the author Alwar about the variation caused by loading
at A or B, these are for the worst case then the Young's Modulus ratio was at a maximum:-

(i) At a distance "b" from the load point the maximum difference was found as high as 40%.

(ii) At a distance "2b" the maximum difference was found to be about 18%.

(iii) At a distance "3b" the maximum difference was found to be insignificant.

(The work of C.L. Amba-Rao and S.K. Bansal(18) and C.L. Amba-Rao(19) continues this further).

This effect so described can thus explain the slight variation which occurs at section 3. The lack in variation at section 1 is probably as a result of the solid section through which the central load is applied.

The second point which must be discussed is the accuracy of the experimental shear stress distributions. As can be seen from the plots the core shear stress shows very good agreement across all three sections for all three of the beams, the error is within 5%. The experimental shear stresses in the facings however show much more variation with the difference between the theoretical and experimental results as much as 100-150 percent in some cases. The reason for this variation and apparently large error is purely the accuracy to which the photoelastic observations (namely the isoclinics) can be made. This observation error is greater in the facings than the core due to the nature of the fringe patterns but also the resultant error in the calculated shear stress is greater in facings than the core for the same observation error. This arises because the shear stress is calculated from the general equation,

$$\sigma_{xy} = \frac{f}{4t} (N_1 \sin 2\theta_1 - N_2 \sin 2\theta_2)$$
(as mentioned previously) and in this form errors in the isoclinic angles $\theta_1$ or $\theta_2$ when $\theta$ is around $0^0$ or $90^0$ have greater significance that if $\theta$ is nearer $45^0$. This also explains the reason for the accuracy of the facings bending stress. In this case the bending stress is given by,

$$\sigma_{xx} = \frac{f_\pi}{2\ell} (N_1 \cos2\theta_1 - N_2 \cos2\theta_2)$$

in which the $\cos 2\theta$ terms have replaced the $\sin 2\theta$ terms in the shear stress equation. Thus errors in the isoclinic angle when $\theta$ is around $45^0$ now have greater significance than if $\theta$ is nearer $0^0$ or $90^0$. In general the isoclinic observations in the facings are around $0^0$-$20^0$ and $70^0$-$90^0$ whereas in the core the observations are within the range $40^0$-$50^0$, thus the resultant errors in the experimental shear stress in the facing will be greater than those in the core. The resultant bending stress experimental errors will conversely be greater in the core than the facing, however the core bending stress is not included because its value is so small. The errors in the facing bending stress are of the order of $7\%-8\%$.

Apart from the two situations just discussed the experimental results confirm the theoretical distributions across the three sections of all three beams as well as the interfacial shear stress distributions along the length of the beam.

The use of a photoelastic technique to obtain experimental corroboration of the theoretical distribution enables the stresses, both shear and bending, to be calculated for any position within the test beam. With such an obvious advantage similar model techniques have many additional applications within the further study of sandwich structures. Typically the study of tapered facings, localised loads or the analysis of geometrical discontinuities within sandwich structures could be tackled by employing a similar photoelastic analysis as has been carried out here.
Figure 4.1 Typical Symmetrical Test Model.
Thickness = $\frac{1}{4}$"
Figure [4.4] Loading Frame for a Symmetrical Model.

All dimensions in inches.
Figure [4.5] Loading Arrangement.
Figure [4.6] Disposition of the Optical Elements.

S Light Source
I Lens and Iris
L Collimating Lens
D Diffuser
P Polariser
M Model
Q Quarter Wave Plate
A Analyser
T Travelling Microscope
Figure [4.7] R.S. Alwar's Test Model.
Experimental bending stress distributions

for a symmetrical sandwich.

\[ \frac{d}{2d+c} = 0.13 \]

\[ \sigma_0 = \frac{6WL}{(2d+c)^3} \]

Theoretical

Experimental

Figure [4.8]
Experimental bending stress distributions for a symmetrical sandwich.

\[ \frac{E}{E_0} = 1318 \]

\[ \frac{d}{2d+c} = 0.17 \]

\[ \sigma_0 = \frac{6WL}{(2d+c)^2} \]

Theoretical

Experimental

Figure [4.9]
Experimental bending stress distributions for a symmetrical sandwich.

\[ \frac{d}{2d+c} = 0.2 \]

\[ \sigma_0 = \frac{6WL}{(2d+c)^2} \]

Sections

Experimental

Theoretical

Figure [4.10]
Experimental shear stress distributions
for a symmetrical sandwich.

$E/E_0 = 1318$

$\frac{d}{2d+c} = 0.17$

$\tau_0 = \frac{W}{(2d+c)}$

--- Theoretical

--- Experimental

Figure [4.11]
Experimental shear stress distributions

for a symmetrical sandwich.

\[ \frac{d}{2d+c} = 0.13 \]

\[ \tau_0 = \frac{W}{(2d+c)} \]

---

**Sections**

1. Section 1
2. Section 2
3. Section 3

---

**Theoretical**

**Experimental**

---

Figure [4.12]
Experimental shear stress distributions

for a symmetrical sandwich.

\[ E/E_0 = 1318 \]

\[ \frac{d}{2d+c} = 0.2 \]

\[ \tau_0 = \frac{W}{(2d+c)} \]

---

Theoretical

Experimental

---

Figure [4.13]
CHAPTER 5

THEORETICAL ANALYSIS OF A NON-SYMMETRICAL SANDWICH BEAM

5.1 Introduction

Here a theory is developed to predict the elastic bending and shear stress distributions for a sandwich beam of non-symmetrical construction, in which the facings are of different thickness. The defining of this more difficult problem requires a more complex system of forces and additional assumptions are necessary if a solution is to be produced.

One of these extra assumptions states that the shear stress across the core is constant which in turn leads to the interfacial shear stress being the same at both interfaces and thus the axial force being the same in each facing. This assumption which proved essential in obtaining a solution was justified from experimental observations taken from a series of tests on photoelastic models of varying geometrical configuration.

The theoretical analysis itself follows a very similar course to the previous analysis and involves consideration of a cantilever subjected to a transverse end load as shown in Figure [5.1], which also shows the system of rectangular co-ordinates and typical geometrical parameters. As before it should be noted that in order to simplify the calculations all loads are per unit thickness.

5.2 Assumptions

(i) Each face of the beam behaves as a simple elastic beam subjected to a combination of bending and transverse shear forces.

(ii) The depth to width ratio of the whole beam is large and the components can be considered to be in a state of plane stress.
(iii) The bending stress distribution varies linearly through the depth of each facing.

(iv) No bending stress is carried by the core.

(v) Both interface materials are assumed to exhibit linear elastic properties.

(vi) Complete bonding is assumed at the interface of each facing and the core.

(vii) Shear stress distribution is assumed to be constant across the core.

(viii) Both facings are of the same material.

(xi) The transverse displacement is assumed to be the same in both facings.

5.3 Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B, C</td>
<td>suffix's denote the top facing, bottom facing and core respectively.</td>
</tr>
<tr>
<td>D, F</td>
<td>constants of integraton.</td>
</tr>
<tr>
<td>x, y</td>
<td>cartesian frame with x parallel to the neutral axis of the beam.</td>
</tr>
<tr>
<td>L</td>
<td>span of the cantilever.</td>
</tr>
<tr>
<td>d</td>
<td>thickness of the facing or core.</td>
</tr>
<tr>
<td>W</td>
<td>transverse load per unit width.</td>
</tr>
<tr>
<td>S</td>
<td>shear force per unit width.</td>
</tr>
<tr>
<td>M</td>
<td>bending moment per unit width.</td>
</tr>
<tr>
<td>T</td>
<td>axial tension per unit width.</td>
</tr>
<tr>
<td>u</td>
<td>mean longitudinal displacement.</td>
</tr>
<tr>
<td>v</td>
<td>transverse displacement.</td>
</tr>
<tr>
<td>@C</td>
<td>shear displacement.</td>
</tr>
<tr>
<td>E, G</td>
<td>elastic constants of the facing material.</td>
</tr>
<tr>
<td>E_C, G_C</td>
<td>elastic constants of the core material.</td>
</tr>
<tr>
<td>m=E/E_C</td>
<td>modular ratio.</td>
</tr>
</tbody>
</table>
\[ \eta, K \quad \text{geometric constants.} \]
\[ \tau \quad \text{interface shear stress.} \]
\[ \sigma_{xx}, \sigma_{xy} \quad \text{cartesian stress components.} \]
\[ \tau_0 = \frac{W}{(dA+dB+dC)} \quad \text{nominal shear stress.} \]
\[ \sigma_0 = \frac{6WL}{(dA+dB+dC)^2} \quad \text{nominal bending stress.} \]

5.4 Cantilever with Transverse End Load

5.4.1 Derivation of the Governing Equilibrium Equations

Consider an element, \( \delta x \), of the sandwich at a distance \( x \) from the fixed origin. Figure [5.2] shows such an element together with the system of forces which are assumed to act upon the element.

For longitudinal force equilibrium within each facing,

\[ \frac{dT}{dx} + \tau = 0 \]  \hspace{1cm} (1)

Now for the top facing, the bending moment equilibrium gives,

\[ \frac{dM_A}{dx} + S_A - \frac{dA \cdot \tau}{2} = 0 \]  \hspace{1cm} (2)

Similarly for the bottom facings,

\[ \frac{dM_B}{dx} + S_B - \frac{dB \cdot \tau}{2} = 0 \]  \hspace{1cm} (3)

The core however is assumed to carry no bending moment, thus the moment equilibrium is reduced to the form,

\[ S_C - dC \cdot \tau = 0 \]  \hspace{1cm} (4)

The bending moment curvature relationship is

\[ \frac{d^2v}{dx^2} = \frac{M_A}{E_I_A} = \frac{M_B}{E_I_B} \]  \hspace{1cm} (5)
where $EI$ refers to the flexural rigidity of each facing in which the moment of inertia $I$ is given by $d^3/12$.

Now the total shear force is given by,

$$S_A + S_B + S_C = W \tag{6}$$

By manipulation of equation (2) through to (6) the shear forces $S_A$, $S_B$ and $S_C$ can be obtained in terms of the interfacial shear stress, $\tau$, and various model parametric parameters, thus

$$S_A = \frac{(\frac{dA}{dB})^3}{1 + (\frac{dA}{dB})^3} - \frac{1}{2} [d_A - (d_B^3)(d_B + 2d_C).\tau]$$ \tag{7}

$$S_B = \frac{W + \frac{1}{2} [d_A^3](d_B - d_A - 2d_C).\tau}{[1 + (\frac{dA}{dB})^3]}$$ \tag{8}

$$S_C = \tau.d_C \tag{4}$$

Figure [5.3] shows the longitudinal displacement conditions for the non-symmetric sandwich beam. The longitudinal displacements can be equated to give,

$$u_A - \frac{dA}{dx}.\frac{dv}{dx} + \delta_C = u_B + \frac{dB}{dx}.\frac{dv}{dx} + \frac{dC}{dx}.\frac{dv}{dx} \tag{9}$$

By applying Hooke's Law to the facings gives the simple relationships,

$$\frac{du_A}{dx} = \frac{T}{E_A.d_A} \tag{10}$$

and

$$\frac{du_B}{dx} = \frac{T}{E_B.d_B} \tag{11}$$
As before, by rearranging, substitution and differentiation it is possible to obtain a differential equation in terms of the interfacial shear, $\tau$, thus

$$\frac{d^2 \tau}{dx^2} - n^2 \tau = -n^2 K \cdot W$$  \hspace{1cm} (12)

where

$$n^2 = \frac{G_C}{E \cdot d_C} \left[ 3 \left( \frac{d_A + d_B + 2 \cdot d_C}{d_B} \right)^2 + \frac{(d_A + d_B)}{d_A \cdot d_B} \right]$$  \hspace{1cm} (13)

and

$$K \cdot n^2 = \frac{6 \cdot G_C}{E \cdot d_C \cdot d_B} \cdot \left( \frac{d_A + d_B + 2 \cdot d_C}{d_B} \right) \left[ 1 + \left( \frac{d_A}{d_B} \right)^3 \right]$$  \hspace{1cm} (14)

5.4.2 Determination of the Interfacial Shear Stress Distribution

The solution of the fundamental differential equation, equation (12), is found to be of the form,

$$\tau = D \cdot \sinh(nx) + F \cdot \cosh(nx) + K \cdot W$$  \hspace{1cm} (15)

where $D$ and $F$ are constants.

To find values for $D$ and $F$ requires two boundary conditions. One arises from the fact that at the fixed end of the cantilever, that is $x=0$, the interfacial shear stress will be zero. This gives,

$$F = - K \cdot W$$

therefore,

$$\tau = D \cdot \sinh(nx) + K \cdot W \cdot [1 - \cosh(nx)]$$  \hspace{1cm} (16)

The second boundary condition is less obvious, but arises from the fact that the free end of the beam acts as a point of inflection and therefore at this point, $x=L$, the rate of change of the
interfacial shear stress along the beam will be zero, that is \( \frac{dT}{dx} = 0 \). This gives,

\[ D = K_s W_s \tanh(nL) \]

and therefore the interfacial shear stress distribution is given by,

\[ \tau = K_s W_s [1 - \cosh(nx) + \tanh(nL) \sinh(nx)] \]  \( (17) \)

or in a normalised form,

\[ \frac{T}{\tau_0} = K_s (d_A + dB + d_C) [\tanh(nL) \sinh(nx) + 1 - \cosh(nx)] \]  \( (18) \)

where \( \tau_0 \) is the nominal shear stress, \( \tau_0 = \frac{W}{(d_A + dB + d_C)} \).

5.4.3 Determination of the Bending Stress Distribution

Within the sandwich each facing is subjected to a combined axial force and bending moment. The resulting bending stress in the top facing is given by,

\[ \sigma_{xx} = -\frac{T}{d_A} - 12 \frac{M_A y}{d_A^3} \]  \( (19) \)

To obtain the axial force \( T \) it is necessary to substitute equation (17) into equation (1) and integrate with respect to \( x \). This gives,

\[ T = K_s W_s [L - x - \frac{1}{n} \tanh(nL) \cosh(nx) - \sinh(nx)] \]  \( (20) \)

The bending moment distribution can be found by substitution of the shear forces, equations (7) and (8), into the bending equilibrium equations (2) and (3), and then integrating with respect to \( x \), to give for the top facing,

\[ M_A = \frac{(d_A)^3}{dB} (d_A + dB + 2d_C) K_s W_s [\tanh(nL) \cosh(nx) - \sinh(nx) + n x - L] \]

\[ 2 [1 + (\frac{d_A}{dB})^3] \]

\[ + \frac{(d_A)^3}{dB} W_s L_s [1 - \frac{x}{L}] \]

\[ \frac{1}{1 + (\frac{d_A}{dB})^3} \]  \( (21) \)
and for the bottom facing,

\[ M_B = \frac{(d_A + d_B + 2d_C)W_n \tanh(nL) \cosh(nx) - \sinh(nx) + nx - L}{2\left[1 + \left(\frac{d_A}{d_B}\right)^3\right]^n} + \frac{W_n L [1 - x_L]}{1 - \left(\frac{d_A}{d_B}\right)^3} \]  

(22)

Substituting equations (20) and (21) back into equation (19) gives the bending stress in the top facing as,

\[
\sigma_{xxA} = \frac{(d_A + d_B + d_C)^2[1 - \left(\frac{d_A}{d_B}\right)^3 - 6\left(\frac{d_A}{d_B}\right)^3(d_A + d_B + 2d_C)]\left(\frac{y}{d_A}\right)K_n[1 - x_L - \tanh(nL) \cosh(nx) - \sinh(nx)]}{6 \cdot d_A \cdot \left[1 + \left(\frac{d_A}{d_B}\right)^3\right]} \]  

(23)

and for the bottom facing,

\[
\sigma_{xxB} = \frac{(d_A + d_B + d_C)^2[1 - \left(\frac{d_A}{d_B}\right)^3 - 6\left(\frac{d_A}{d_B}\right)^3(d_A + d_B + 2d_C)]\left(\frac{y}{d_A}\right)K_n[1 - x_L - \tanh(nL) \cosh(nx) - \sinh(nx)]}{6 \cdot d_B \cdot \left[1 + \left(\frac{d_A}{d_B}\right)^3\right]} \]  

(24)

where \(\sigma_0\) is the nominal bending stress and is given by \(\sigma_0 = \frac{6 \cdot W_n L}{(d_A + d_B + d_C)^2}\).
5.4.4 Determination of the Transverse Shear Stress Distribution

The transverse shear stress distribution through the depth of each facing can be found by applying the following standard expression, in this case applied to the top facing,

\[ \sigma_{xyA} = \int_{y}^{d_{A}} \frac{d}{dx} \sigma_{xxA} \cdot dy \]

By substituting for \( \sigma_{xxA} \) from equation (19) and by substituting for \( \frac{dT}{dx} \) and \( \frac{dM_{A}}{dx} \) it is possible to obtain an expression for the shear stress \( \sigma_{xyA} \). In nominal form this becomes,

\[ \frac{\sigma_{xyA}}{\tau_{0}} = \left( \frac{y}{d_{A}} - \frac{1}{2} \right) \left[ \frac{6}{d_{B}} \left( \frac{y}{d_{B}} + \frac{1}{2} \right)^{1} \cdot \frac{\left( d_{A} + d_{B} + 2 \cdot d_{C} \right)^{3}}{2} \cdot \frac{I}{\tau_{0}} - \left( d_{A} + d_{B} + d_{C} \right) - \frac{I}{\tau_{0}} \right] \]

Similarly for the bottom facing to give,

\[ \frac{\sigma_{xyB}}{\tau_{0}} = \left( \frac{y}{d_{B}} + \frac{1}{2} \right) \left[ \frac{6}{d_{B}} \left( \frac{y}{d_{B}} - \frac{1}{2} \right)^{1} \cdot \frac{1}{2} \left[ \frac{\left( d_{A} + d_{B} + 2 \cdot d_{C} \right)^{3}}{2} \cdot \frac{I}{\tau_{0}} - \left( d_{A} + d_{B} + d_{C} \right) + \frac{I}{\tau_{0}} \right] \right] \]

where \( \tau_{0} \) is the nominal shear stress and is given by \( \tau_{0} = \frac{W}{(d_{A} + d_{B} + d_{C})} \).

5.5 Discussion of the Theoretical Results

Although the non-symmetrical sandwich beam problem is more complex than the symmetrical case and requires additional assumptions to obtain solutions, the theoretical stress distributions, Figures [5.4] to [5.6], are remarkably similar to the symmetrical distributions previously described in Chapter 3. There are however a couple of differences which ought to be discussed here.
Firstly the bending stress distributions, three sections of a typical non-symmetrical beam are shown in Figure [5.4], are such that the situation is likely to arise whereby a neutral axis no longer exists in the thinner facing. This almost occurs at section 3 shown in Figure [5.4]. It should also be noted that when the neutral axis disappears from the thinner facing the neutral axis which exists in the core will also disappear. (Note: the core bending stress distributions are not included in the Figure for the same reasons as mentioned in Chapter 3). In terms of failure of the non-symmetrical beam as a result of high bending stress this will now originate at the outer (free) surface of the thicker facing at the section of greatest bending moment, in the case considered the built-in end of the beam.

The second point follows from the first and concerns the shear stress distributions which are shown for a typical beam in Figure [5.5]. These distributions which clearly demonstrate the non-symmetric nature of the problem also show that it is possible for the shear stress maxima in the thinner facing to disappear from the facing while still leaving a shear stress maxima in the thicker facing, thus the thicker facing will be more susceptible to shear failure than the thinner facing.

A further point which follows from Figures [5.5] and [5.6] is the possible shear failure at either facing/core interface. In the analysis presented it was assumed that the shear stress was constant across the core. This can be seen to be justified from the experimental results presented in the next chapter. However in reality when shear maxima exist in both facings a minimum must exist in the core. But due to the non-symmetry the value of the shear stress at each interface is not the same and thus interface shear failure would favour one particular interface. This would happen in a perfect world however the difference between the shear stress at the interfaces is so small that the overriding criteria becomes the degree of bonding between the core and each facing.

Figure [5.7], which shows the proportion of the shear force carried by each facing and the core, has been included to show the high degree of shear force carried by the thicker facing, especially
at the built-in end of the cantilever. This arises because at the built-in end the shear forces in the facings are related by the cube of the ratio of their thicknesses. Thus in the case of Figure [5.7] in which the ratio of the facing thicknesses is 3:1 the shear forces in the facings at the built-in end are in the ratio of 27:1.
Figure [5.1] Cantilever Subjected to Transverse End Load.
Figure [5.2] System of Forces on an Element Distance $x$ from the Fixed Origin.
Figure [5.3] Longitudinal Displacement Conditions.
Theoretical bending stress distributions for a non-symmetrical sandwich.

\[ L = 6'' \]

\[ \frac{d_B}{d_A} = 3 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.1 \]

\[ \sigma_0 = \frac{6WL}{(d_A + d_B + d_C)^3} \]

Figure [5.4]
Theoretical shear stress distributions

for a non-symmetrical sandwich.

\[ L = 6'' \]

\[ \frac{d_B}{d_A} = 3 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.1 \]

\[ \tau_0 = \frac{W}{(d_A + d_B + d_C)} \]

Figure [5.5]
Theoretical interface shear stress distributions

for a non-symmetrical sandwich.

\[ L = 6'' \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.1 \]

\[ \tau_o = \frac{W}{(d_A + d_B + d_C)} \]

**Figure [5.6]**
Theoretical shear force distributions for a non-symmetrical sandwich.

$L = 6''$

$E/E_0 = 1318$

$$\frac{d_A}{(d_A + d_B + d_C)} = 0.1$$

$$\frac{d_B}{d_A} = 3$$

Figure [5.7]
CHAPTER 6

EXPERIMENTAL STUDY OF A NON-SYMMETRICAL SANDWICH BEAM

6.1 Introduction

The dimensions of the non-symmetrical sandwich beam test models are given in Figure [6.1], thus providing ratios of facing thickness of 1.5:1, 2:1 and 3:1.

In order to provide a true comparison between the non-symmetrical and the symmetrical cases the non-symmetrical test beams were made of exactly the same materials with an exactly similar casting, curing and construction technique as used for the symmetrical test models.

The system of loading, however, had to be changed in order to satisfy an additional assumption required by the non-symmetrical theory. This assumption states that the transverse displacements of both facings are the same. Thus the loading rig illustrated in Figure [6.2] which while applying three-point bending to the model also satisfied the new assumption by maintaining a fixed transverse distance between the loading points of each facing.

Once the rig was loaded an exactly similar testing procedure was employed as previously detailed in Chapter 4. The test data thus obtained was then analysed using the same process as described in the aforementioned chapter such that the effect of initial stresses induced during construction could be removed.

6.2 Discussion of the Experimental Results

Figures [6.3] through to [6.8] show the experimental results plotted against the theoretical distributions. Apart from the two situations discussed in Chapter 4, which hold true here as well, the experimental results confirm the theoretical distributions across the three sections of all three of the non-symmetrical test beams.
These results thus justify the assumptions that the shear stress is constant across the core and that the transverse displacements are the same in each facing. This second assumption is shown to be true by the accuracy of the bending stress distributions in each of the facings.
Figure 16.1 Typical Non-Symmetrical Test Model.

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_A$</td>
<td>0.2</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>$d_B$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.45</td>
</tr>
<tr>
<td>$d_C$</td>
<td>1.0</td>
<td>1.05</td>
<td>0.9</td>
</tr>
</tbody>
</table>
Figure [6.2] Non-Symmetrical Loading Arrangement.
Experimental bending stress distributions

for a non-symmetrical sandwich.

\[ E/E_0 = 1318 \]

\[ \frac{d_B}{d_A} = 1.5 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.13 \]

\[ \sigma_0 = \frac{6WL}{(d_A + d_B + d_C)^3} \]

---

**Theoretical**

- **Experimental**

---

![Graph showing stress distributions](image)

**Figure [6.3]**
Experimental bending stress distributions

for a non-symmetrical sandwich.

\[ E/E_0 = 1318 \]

\[ \frac{d_B}{d_A} = 2 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.1 \]

\[ \sigma_0 = \frac{6WL}{(d_A + d_B + d_C)^3} \]

Theoretical

- Experimental

Figure 6.4
Experimental bending stress distributions
for a non-symmetrical sandwich.

$E/E_0 = 1318$

Sections

Section 1
Section 2
Section 3

$\frac{dB}{dA} = 3$

$\frac{dA}{(dA + dB + dC)} = 0.1$

$\sigma_0 = \frac{6WL}{(dA + dB + dC)^3}$

Theoretical

Experimental

Figure [6.5]
Experimental shear stress distributions

for a non-symmetrical sandwich.

\[ E/E_0 = 1318 \]

\[ \frac{d_B}{d_A} = 1.5 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.13 \]

\[ \tau_0 = \frac{W}{(d_A + d_B + d_C)} \]

--- Theoretical

- Experimental

---

\[ \frac{\sigma_{xy}}{\tau_0} \quad \frac{\sigma_{xy}}{\tau_0} \quad \frac{\sigma_{xy}}{\tau_0} \]

Figure [6.6]
Experimental shear stress distributions
for a non-symmetrical sandwich.

\[ E/E_0 = 1318 \]

Sections

\[ \frac{dB}{dA} = 2 \]

\[ \frac{dA}{(dA + dB + dC)} = 0.1 \]

\[ T_0 = \frac{W}{(dA + dB + dC)} \]

- Theoretical
- Experimental

Figure [6.7]
Experimental shear stress distributions
for a non-symmetrical sandwich.

\[ \frac{d_B}{d_A} = 3 \]

\[ \frac{d_A}{(d_A + d_B + d_C)} = 0.1 \]

\[ \tau_o = \frac{W}{(d_A + d_B + d_C)} \]

---

**Theoretical**

**Experimental**

Figure [6.8]
7.1 Analysis of the Sandwich Beam

Sandwich structures have many advantages over other structures, as mentioned previously in the introduction. However it has been shown that they exhibit a much more complex deformation behaviour than normal homogeneous structures, which in turn leads to more numerous possible modes of failure. It is necessary to have a detailed understanding of the stress distributions and behaviour of sandwich beams if the relative importance of these various modes of failure are to be ascertained or for the efficient design of sandwich structures. The two sets of theoretical analyses and experimental studies presented here are an attempt to provide that understanding.

7.2 Deformation Behaviour of Sandwich Beams

Both the symmetrical and non-symmetrical analyses together with their respective experimental test data confirm the expected stress distributions outlined in the introduction. This involves the facings supporting relatively high bending stresses whilst the core carries a high proportion of the shear load (as much as 70% in the symmetrical models tested, see Figure [3.9]). However, the non-symmetrical results show that if the thickness of one of the facings is increased significantly then the proportion of the shear load carried by that facing in relation to the core also increases significantly. This can be seen in Figure [5.7] which shows the theoretical distribution of the shear load carried by a non-symmetrical sandwich beam in which one facing is three times the thickness of the other. In this case the thicker facing supports a greater proportion of the shear load than the core. Obviously, the relative thickness of the core is of vital importance under these circumstances and in the case of the sandwich beam considered the core is twice the thickness of the thicker facing.
If the relative thickness of the thicker facing is increased further it will begin to dominate the sandwich eventually supporting a very high proportion of both the bending stress and the shear load. With these geometric conditions it would be necessary to review and possibly adjust the basic assumptions upon which the analysis presented is based. It should be noted that in the case of the sandwich beam just considered (with one facing three times the thickness of the other and with the core twice the thickness of the thicker facing) the experimental results agree very well with the theoretical distributions. See Figures [6.5] and [6.8] for a comparison of the theoretical and experimental results. It is worth noting that relative stiffness of the facings in this case are in the ratio of 27:1, so the theory has been shown to be very accurate over a very large range.

Also apparent from both analyses and the two sets of experimental studies is the high shear stress sustained at the interface between the core and each facing. This interface shear stress, which is associated with the method by which load is transferred between components of the sandwich, has been shown to have a relatively high value over a large proportion of the span. It has also been shown that changes in geometric parameters do not cause a proportionate change in the value of the interface shear stress.

Further, it has been shown that the shear stress distributions consist of parabolas in the facings with an apparently constant shear stress across the core, this applies to both symmetrical and non-symmetrical sandwich beams. The bending stress distributions are shown to be linear in both facings and core, however the core bending stress has also been shown to be virtually negligible.

7.3 Conclusions

A. Despite the complex deformation behaviour characteristics of sandwich structures two simple analyses have been developed, based upon the Bernoulli-Euler Theory, which have been found to be effective in predicting the stress distributions associated with symmetrical and non-symmetrical sandwich beams subjected to transverse load.
B. The simplistic nature of the analyses provides for a better understanding of the behaviour characteristics of sandwich beams and the shear load transfer associated with such structures.

C. The photoelastic technique was shown to be a very effective method of obtaining test data for use in the analysis of sandwich beams.

7.4 Suggestions for Further Work

A. As experimental studies have been carried out on only a narrow range of models, further photoelastic studies with different materials and different geometric configurations could prove interesting.

B. The analyses could be expanded to take into account more complicated loading configurations.

C. A similar philosophy could be used to predict the stress distributions associated with a localised load, or to take into account the non-linear effects associated with some foam materials.

D. In consideration of "real" sandwich structures it would probably prove a worthwhile task to expand upon the theories developed here and try to analyse the three-dimensional system.
REFERENCES


