Investigation of Nuclear Structure through
the Analysis of
Alpha-Particle Scattering

by

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for the Degree
Doctor of Philosophy
ABSTRACT

This thesis is concerned for the most part with a microscopic description of alpha-particle scattering as a means of studying nuclear structure. Both elastic and inelastic scattering are considered. Elastic scattering is analysed in terms of the usual optical model and inelastic scattering by both coupled channels and distorted wave Born approximation (DWBA) methods.

The target nuclei are described in terms of the simple shell model and an effective alpha-nucleon interaction related to the free proton-alpha elastic scattering. The nuclei considered are $^{42}\text{Ca}$ and $^{50}\text{Ti}$ which in terms of the simple shell model are described as a closed core plus two identical extra core nucleons. Differences found between the structure of the two nuclei are noted.

The results of the microscopic description of elastic scattering are compared with results found in a conventional phenomenological optical model analysis. In the phenomenological analysis the criteria for the selection of optical potentials are studied in some detail and new criteria proposed.

The results of the microscopic description of inelastic scattering are compared with rotational model calculations. Differences between the results found by coupled channels and DWBA methods are noted together with their relative merits. The microscopic description of inelastic scattering is considered in conjunction with both microscopic and phenomenological optical potentials. The effects that the choice of optical potential has on inelastic scattering are examined.

The extent to which the microscopic model can yield nuclear structure information, its sensitivity and its limitations are discussed. Further extensions of this work are suggested.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>6</td>
</tr>
<tr>
<td>2. SCATTERING THEORY</td>
<td>12</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>12</td>
</tr>
<tr>
<td>2. The General form of the Coupled Equations</td>
<td>15</td>
</tr>
<tr>
<td>3. Cross-sections</td>
<td>19</td>
</tr>
<tr>
<td>4. DWBA and the Optical Model</td>
<td>21</td>
</tr>
<tr>
<td>3. THE COUPLING MATRIX ELEMENTS</td>
<td>24</td>
</tr>
<tr>
<td>1. Microscopic Model</td>
<td>24</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>24</td>
</tr>
<tr>
<td>2. Form of the Coupling Matrix Elements</td>
<td>26</td>
</tr>
<tr>
<td>3. Nuclear Transition Density for Shell Model States</td>
<td>28</td>
</tr>
<tr>
<td>4. Transition Density for states of Generalised Configuration Mixtures</td>
<td>33</td>
</tr>
<tr>
<td>5. Effective Interaction</td>
<td>34</td>
</tr>
<tr>
<td>6. Optical Model</td>
<td>36</td>
</tr>
<tr>
<td>2. Rotational Model</td>
<td>37</td>
</tr>
<tr>
<td>4. ELASTIC SCATTERING ANALYSIS</td>
<td>40</td>
</tr>
<tr>
<td>1. Experimental Data</td>
<td>41</td>
</tr>
<tr>
<td>2. Phenomenological Analysis</td>
<td>42</td>
</tr>
<tr>
<td>3. Microscopic Analysis</td>
<td>72</td>
</tr>
<tr>
<td>5. INELASTIC SCATTERING ANALYSIS</td>
<td>89</td>
</tr>
<tr>
<td>1. Experimental Data</td>
<td>90</td>
</tr>
<tr>
<td>2. Rotational Model</td>
<td>91</td>
</tr>
<tr>
<td>3. Microscopic Model</td>
<td>101</td>
</tr>
<tr>
<td>4. Summary</td>
<td>125</td>
</tr>
</tbody>
</table>
Table of Contents (Continued)

6. SUMMARY AND CONCLUSIONS 128

APPENDIX 3.1 Nuclear Transition Density 131

REFERENCES 133
Chapter 1

Introduction

The elastic and inelastic scattering of alpha-particles has proved to be a very useful means of investigating the structure of nuclei. Direct reaction theories for the scattering mechanism have given good agreement with the observed angular distributions. Considering the inelastic scattering mechanism as known, we can extract detailed information on the nature of the nuclear states which are excited. The values found for spin, parity and transition rates are generally in good agreement with values determined by other means.

It must be made clear that in all methods of investigating nuclei only the simplest nuclear properties can be inferred essentially directly from experiment, for example the position of energy levels. For more complex properties such as nuclear structure, which is of particular interest in this work, such direct analysis is not possible. The nuclear structure is not an observable, but it influences and reflects itself through observables. The approach is therefore to construct models of the nucleus from which observables can be calculated and compared with experiment.

At medium energies elastic scattering is generally analysed by means of the Optical Model (Ho63), here the complex interactions between the nucleons of the target and those of the projectile are approximated by an attractive spherically symmetric potential. To allow for inelastic processes an imaginary component is introduced into the potential which has the effect of absorbing particles from the elastic channel. Direct inelastic scattering is generally analysed by the Extended Optical Model (To65). Here the target nucleus is considered as a non-spherical potential and the scattering analysed in terms of rotational and vibrational excitations.

A natural extension of these models is to replace the phenomenological potentials by a microscopic description of the scattering mechanism. The target nucleus can for example, be
described by the nuclear shell model and the interaction with the alpha-particle treated as a sum of two-body interactions with each of the target nucleons. The nature of the alpha-nucleon interaction is then taken from the analysis of proton-alpha elastic scattering, making the assumption that the proton-alpha and the neutron-alpha interactions are the same. If such a model proved to be successful, it could, with its direct test of microscopic ideas, be expected to yield more detailed information of the nuclear structure than models employing phenomenological potentials. Work along these lines for alpha-particle and also proton scattering has been reported by several authors (Gl65, Al66, Ma65, Ba65, Fu64, Sa66, Yn67) and it is with this approach that this thesis is concerned.

Such a microscopic approach, in the case of elastic electron scattering, is well established (Ho57, El61) where the formalism allows a direct connection to be made between the cross-section and the charge density distribution in the ground state. For the electron scattering case the effects of intermediate excitation of excited states are very small and may safely be neglected. This is no longer the case where the projectiles are alpha-particles. It was pointed out by Satchler (Sa66) that such terms could lead to appreciable corrections to the form of the predicted optical potentials. In the simplest optical model or distorted wave Born approximation (DWBA) analyses such terms are neglected. Since in the DWBA formalism it is necessary to have an accurate description of the wavefunctions for elastic scattering, it has been the general practice in the microscopic description of inelastic alpha-particle scattering to use a phenomenological optical potential to describe the elastic scattering. This procedure is formally correct and implies there are no free parameters in the description of elastic scattering. It also allows an accurate treatment of multiple scattering in the description of the elastic scattering (Sa66), while the error in the treatment of multiple scattering in the off-diagonal or inelastic part of the interaction may be compensated by modification
of the effective two body interaction (Ja 69). It has been argued, however, (Ja 65, Ja 69) that this arbitrary adjustment of the effective interaction for the inelastic scattering together with the ambiguities in the optical potential leads to a certain inconsistency in the treatment of elastic and inelastic scattering. A microscopic description of elastic scattering could help to remove the uncertainties in the parameters of the two-body interaction, and the self-consistency criterion could be useful for the description of scattering of strongly absorbed projectiles, for which ambiguities in the strength of the optical potential are well known. In this work we approach the problem both ways. A wide range of phenomenological optical potentials are considered varying in depth from very shallow to very deep. The criteria which optical potentials need to satisfy are also investigated in some detail. Optical potentials based upon a microscopic description are also considered. As the model neglects multiple scattering, the optical potential found is purely real. In order to use these potentials to calculate scattering, it is necessary to introduce an imaginary component and the manner in which this is done is discussed in chapter 3.

In addition to the usual DWBA, inelastic scattering is also calculated by means of the coupled channels method. Though first formulated in 1933 by Mott and Massey (Mo 65), this latter method has only in recent years become a practical method of solution with the advent of high speed computers. The coupled channels method has the advantage that it automatically includes some of the multiple scattering terms (those for the coupled states) neglected by DWBA and optical model calculations, and thus overcomes some of the theoretical objections to the full microscopic description of the process already discussed. A comparison of the results obtained by the two methods should give some indication of the extent of the effect produced by the coupling.

The nuclei considered are \( ^{42}\text{Ca} \) and \( ^{50}\text{Ti} \). In the simple shell model both these nuclei are described as two identical nucleons
in the $1f_{7/2}^2$ configuration outside a closed core.

\[
\begin{align*}
\text{Ca}^{42} &= \text{Ca}^{40} \text{ (core) + 2 neutrons} \\
\text{Ti}^{50} &= \text{Ca}^{48} \text{ (core) + 2 protons}
\end{align*}
\]

The experimental spectra for the low lying states are shown in figure 1.1.

<table>
<thead>
<tr>
<th>Energy Level</th>
<th>Ca$^{42}$</th>
<th>Ti$^{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.191 MeV</td>
<td>$6^+$</td>
<td>3.215 MeV</td>
</tr>
<tr>
<td>2.752</td>
<td>$4^+$</td>
<td>2.695</td>
</tr>
<tr>
<td>2.422</td>
<td>$2^+$</td>
<td></td>
</tr>
<tr>
<td>1.838</td>
<td>$0^+$</td>
<td></td>
</tr>
<tr>
<td>1.524</td>
<td>$2^+$</td>
<td>1.570</td>
</tr>
<tr>
<td>0</td>
<td>$0^+$</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1.1

Experimental energy levels for Ca$^{42}$ and Ti$^{50}$.

The simple energy level structure of Ti$^{50}$ can be understood in terms of the pure shell model. In the case of Ca$^{42}$ we note corresponding levels of the same spin and parity as for Ti$^{50}$ at extremely close excitation energies. In addition we note the presence of $0^+$ and $2^+$ levels not found in Ti$^{50}$. Federman (Fe 66) showed that the extra levels can be thought of as forming a rotational band based upon the second $0^+$ level. He took two protons from the $1d_{3/2}$ state and raised them into deformed $1f_{7/2}$ states and was able to reproduce the experimental energy levels. The corresponding situation in Ti$^{50}$ is not possible due to the filled $1f_{7/2}$ neutron shell. Also for Ca$^{42}$ the experimental values of the electromagnetic transition probabilities from an excited state to the ground state or between two excited states is far larger than the shell model values, indicating the presence of collective enhancements. We thus see that the evidence suggests that the Ca$^{48}$ core is more stable than the Ca$^{40}$ core and thus it can be expected the
model should be more successful for \( ^{50}_{10} \text{Ti} \) than for \( ^{42}_{20} \text{Ca} \). Experimental alpha-particle scattering data is available for both nuclei at two bombarding energies. Another reason for the choice of nuclei is the ease with which the wavefunctions may be constructed with only two extra core nucleons.

The work follows on from that reported by Jackson (Ja 65) on \( ^{42}_{20} \text{Ca} \) but with several notable improvements. First, the oscillator wavefunctions used in that work to describe the target nucleus, and which have an incorrect asymptotic form are replaced by wavefunctions generated in a Saxon Woods potential well. Also in the case of \( ^{42}_{20} \text{Ca} \) configuration mixture wavefunctions are considered, using admixture coefficients calculated by Law (La 68).

The model assumes that the form of the alpha-nucleon interaction for a nucleon in the nucleus is similar to that for a free nucleon and that the complex effects of many body forces can be compensated by small changes in the potential parameters. It is questionable that the presence of the other nucleons can so simply be treated. A calculation is done in which the form of the alpha-nucleon interaction is made dependent upon the nuclear density, thus simulating very simply the presence of other nucleons. In this way the influence of nucleons deep in the nuclear interior is reduced, and the sensitivity of the results to this parameterisation is studied.

Another problem with the model is the fact that alpha-particles being strongly absorbed by nuclei tend to be only sensitive to the magnitude of the form factor in the region of the nuclear surface but not to any extent sensitive to the shape of it. This limits the amount of fine structural detail that can be found.

The general results of other peoples work tend to suggest that the magnitude of the predicted inelastic scattering cross-sections are far lower than the experimental values. The same is also true of similar analyses of inelastic proton scattering. Agreement in magnitude of cross-sections with data can be obtained by enhancing the projectile-nucleon interaction. A justification for such an
enhancement has been discussed by Satchler (Sa 66), who suggests that the model may neglect important collective correlations, virtual transitions may lead to core polarisation with a resulting renormalisation of the interaction. A calculation based upon these ideas has been reported by Love and Satchler (Lo 67).

In chapter 2 the theory of scattering is discussed. The coupled equations are derived and their relation to the DWBA shown. In chapter 3 the coupling matrix elements for the microscopic model are derived and the form of the optical potentials both phenomenological and due to the microscopic model is examined. For a comparison the rotational model is also included. The formalism of chapters 2 and 3 form the basis for the computer programmes coding with which the calculations were performed.

In chapter 4 the results of the analysis of elastic scattering are presented. Optical potentials based on the microscopic model are found and the resulting scattering compared with that found by using phenomenological potentials. Chapter 5 gives the results of the inelastic scattering analysis. The microscopic model is considered in conjunction with both phenomenological and microscopic optical potentials and compared with results of a rotational model analysis. Results of coupled channels and DWBA calculations are also compared.

Chapter 6 presents the conclusions to be drawn from these calculations and suggestions for possible extensions to the work are made.
Chapter 2

Scattering Theory

2.1 Introduction

The subject of scattering is dealt with in almost any textbook on Quantum Mechanics. In this work we are interested in alpha-particle scattering, both elastic and inelastic to the low lying states of the target nucleus. The intrinsic structure of the low lying states can be thought of as not differing very much from that of the ground state. Those that do, lie at a much higher energy. For example, on the shell model picture one may have a nucleus consisting of a small number of nucleons outside a closed core. The lowest states may then differ from one another essentially only in the states of these extra core nucleons. Alternatively one may be thinking in terms of a rotational model where one has a permanently deformed nucleus, the lowest states can then be thought of as corresponding simply to different states of rotational motion of the system. In either case when considering scattering involving transitions from the ground state to the low lying states one can think in terms of a "direct reaction", characterised by a short period of interaction involving a small number of degrees of freedom. For the shell model case these are the degrees of freedom of the extra core nucleons. For the rotational model one is considering the degrees of freedom describing the rotation.

It is this process of a direct reaction involving a minimum of rearrangement which is of interest, the theory of which will be developed in the following sections. An alternative mechanism which competes with the direct reaction is where the projectile combines with the target nucleus to form an intermediate quasi-stationary state, or compound nucleus. The subsequent decay of the compound nucleus can occur in many ways and the contribution to any one channel is generally small. The total cross-section for compound nucleus formation shows pronounced resonances with
changing bombarding energy. The distinctive feature of direct reactions, on the other hand, is that the differential cross-sections show a strong dependence on the direction of emission of scattered particles and a smooth dependence upon bombarding energy. As the energy increases the width of the compound nucleus resonances also increase. Eventually the width increases to the point where the lifetime has decreased sufficiently to be comparable with the nuclear transit time, and the concept of compound nucleus formation breaks down. Before this stage is reached however, the ever increasing overlap between the various compound nucleus states results in a smoothing over of the resonance effects which complicate direct reaction analysis. There is however evidence (Gr 65) that resonance effects can still be important at energies as high as 30.5 MeV.

In the Optical Model of elastic scattering, the complex interactions between the nucleons of the target and those of the projectile are approximated by a phenomenological attractive spherically symmetric potential. This potential is usually taken as local and energy dependent. The only allowance for inelastic and other processes is the inclusion of a complex component in the potential which has the effect of absorbing particles from the elastic channel. The direct reaction theory can be thought of as an extension of the Optical Model to include inelastic processes. This was first done in the case of inelastic scattering by Yoshida (Yo 56) and by Chase, Wilets and Edmonds (Ch 58). The model takes the Optical Model as the first approximation but introduces an additional interaction affecting some simple internal degrees of freedom of the target nucleus, thus giving rise to inelastic processes.

Once a choice is made of models to describe the target nucleus and projectile, the procedure for solving the resulting scattering problem is, in principle, determined. In practice however one is not able to solve the partial differential equations resulting from the Schrödinger equation directly. This is due
Essentially to the existence of many bound state solutions as well as scattering solutions to the equations. Some approximation is necessary. The most popular is the DWBA, this represents a first order perturbation solution to the equations. An alternative approximation, as mentioned in chapter 1, is the coupled channels method (Bu 63, Ta 65, Wi 63, Gl 67). In this method the total wavefunction is expanded in terms of the eigen states of the nuclear Hamiltonian and in this manner an infinite set of coupled ordinary differential equations is obtained. The approximation then consists of terminating the coupling at some finite number of states, that is, to ignore higher excited states. The now finite number of coupled equations is then solved exactly. This method has a major advantage over the DWBA for scattering in which multiple excitation is important. In coupled channels, the relevant terms are automatically included. DWBA on the other hand, being only a first order approximation, neglects such terms, the corresponding higher order DWBA methods which become necessary in such a case are extremely cumbersome.

In the following section, in which the general form of the coupled equations is derived, we follow the approaches given by Wills (Wi 63) and Glendenning (Gl 67). In section 3, the formulae relating to cross-sections are formed. Section 4 deals with the relation of coupled channels to DWBA and to the optical model. In particular it shows how it is possible that once the coupled channels method is formulated for a computer programme, it is a simple modification to perform DWBA calculations also. This is important as direct comparison between the two methods is then possible.

In the theory we take advantage of the fact that the alpha-particle has no spin or iso-spin. Also we assume that the alpha-particle undergoes no internal change or polarisation during the interaction.
2.2 The General Form of the Coupled Equations

We have available a model of the nucleus which provides wavefunctions for the nuclear states. We wish to compare the description of scattering with experiment that this model provides.

Let \( H(r) \) denote the model Hamiltonian of the isolated target nucleus, whose internal coordinates are denoted by \( (r) \). The nuclear eigenfunctions \( \phi_{IM}^j(r) \) satisfy the Schrödinger eigenvalue equation.

\[
(H(r) - \varepsilon_{lj}) \phi_{IM}^j(r) = 0 
\]

where \( I \) represents the nuclear spin with z-component \( M \). The superscript \( j \) denotes any other quantum numbers necessary to describe the nuclear state. These wavefunctions satisfy the orthogonality condition.

\[
\langle \phi_{IM}^j(r) | \phi_{IM'}^{j'}(r) \rangle = \delta_{II'} \delta_{MM'} \delta_{jj'} \quad (2.2) 
\]

For the system of nucleus and alpha particle the Hamiltonian describing the relative motion can be written

\[
H = H(r) + T + V(R, r) 
\]

where \( T \) is the relative kinetic energy operator and \( V(R, r) \) is the potential energy of the interaction of the alpha-particle with the nucleus. The vector \( R \) is the distance between the centre of masses of the nucleus and alpha-particle. Also involved are the reduced mass \( \mu \) of the system and the centre of mass energy \( E \). The alpha particle is assumed to undergo no internal change during the scattering and thus no term is necessary in the Hamiltonian to describe its internal structure.

The time independent Schrödinger equation for the system is

\[
\left[ H(r) + T + V(R, r) \right] \psi(R, r) = E \psi(R, r) \quad (2.4) 
\]

The quantization axis (the z axis) is chosen to lie along the initial direction of the incoming alpha particles. As only even-even
nuclei, initially in their ground state, and spinless projectiles are considered, this choice of the $z$ axis implies that the $z$ component of the total angular momentum is zero.

The nuclear eigenfunctions $\phi_{IM}^j(r)$ form a complete set of functions and thus provide a basis for the expansion of the wavefunction $\Psi(R, r)$. A partial wave expansion of the wavefunction can now be performed. It is convenient to couple the angular part of the alpha-particle wavefunction with the nuclear wavefunctions to form eigenstates of the total angular momentum $L$.

$$\Psi(R, r) = \sum_{l,l'jL} \Psi_{l,l'}^{Lj}(R, r) \quad \ldots (2.5)$$

with

$$\psi_{l,l'}^{Lj}(R, r) = \frac{1}{R} U_{l,l'}^{Lj}(R) \left[ l l' I L, j \right] \quad \ldots (2.6)$$

and where

$$\left[ l l' I L, j \right] = \sum_{mM} C(l I L, mMO) Y_{l}^{m}(\theta, \phi) \phi_{IM}^{j}(r) \quad \ldots (2.7)$$

In Eq. 7. $C(l l' I L, mMO)$ are the usual vector coupling or Clebsch-Gordan coefficients as defined by Rose (Ro 63). The quantum numbers $l$ and $m$ represent the angular momentum and its projection, of the alpha-particle relative to the nucleus. The term $\frac{1}{R} U_{l}^{Lj}(R)$ is the radial part of the scattering wavefunction.

Operating on $\psi_{l,l'}^{Lj}(R, r)$ with $H(r)$ it is easily seen that

$$H(r) \psi_{l,l'}^{Lj}(R, r) = \epsilon_{l,l'}^{Lj} \psi_{l,l'}^{Lj}(R, r) \quad \ldots (2.8)$$

and operating with $T$ one finds

$$T \psi_{l,l'}^{Lj}(R, r) = -\frac{\hbar^2}{2\mu} \left[ \frac{1}{R} \frac{d^2}{dR^2} R - \frac{l(l+1)}{R^2} \right] \psi_{l,l'}^{Lj}(R, r) \quad \ldots (2.9)$$
Substituting Eq. 5 into the Schrödinger equation Eq. 4 and using the results of Eqs. 8 and 9 gives for each value of $L$.

$$\sum_{\ell''I'j'} \left\{ \frac{-\hbar^2}{2\mu} \left[ \frac{d^2}{dR^2} - \frac{l''(l''+1)}{R^2} \right] - E + \epsilon_{I'j'} + V(R,r) \right\} \psi_{\ell''I'j'}^{Lj'}(R, r) = 0$$

...(2.10)

Using Eq. 6 gives

$$\sum_{\ell''I'j'} \left\{ \frac{-\hbar^2}{2\mu} \left[ \frac{d^2}{dR^2} - \frac{l''(l''+1)}{R^2} \right] - E + \epsilon_{I'j'} + V(R,r) \right\} U_{\ell''I'}^{Lj'}(R) |\ell''I'j'> = 0$$

...(2.11)

Operating on Eq. 11 with $\langle \ell I L, j |$ gives the coupled radial equations

$$\left\{ \frac{-\hbar^2}{2\mu} \left[ \frac{d^2}{dR^2} - \frac{l(l+1)}{R^2} \right] - E + \epsilon_{Ij} \right\} U_{\ell I}^{Lj}(R)$$

$$+ \sum_{\ell''I'j'} \langle \ell''I', j | V(R, r) |\ell''I'j' > U_{\ell''I'}^{Lj'}(R) = 0$$

...(2.12)

In Eq. 12, integration over polar angles and all internal coordinates is implied by the coupling matrix element

$$\langle \ell I L, j | V(R, r) |\ell' I' L, j' >$$

Equation 12 represents the general form of the coupled radial Schrödinger equations. The form of the coupling matrix elements depends upon the nuclear model used to generate the nuclear wavefunctions and upon the method of treating the interaction potential $V(R, r)$. The following chapter will deal with the formation of these matrix elements for the nuclear models used here.

Two important approximations are implied by the form of this expansion. Since it always contains the alpha-particle as the free particle, it neglects those channels such as pickup or knock out. The expansion is in terms only of the elastic and inelastic
channels containing the alpha-particle. Also neglected are effects of exchange between nucleons of the target nucleus and the alpha-particle which may occur due to the indistinguishable nature of identical particles or by the exchange nature of the interaction. These approximations are discussed by Glendenning (Gl 67) who argues as follows. Regarding the first approximation, as the elastic and inelastic processes typically represent the largest part of the direct cross section, we may treat the other channels implicitly through the use of an absorptive component in the diagonal matrix elements of the interaction. For the second approximation, the neglect of exchange is less well founded but it is presumed on the strength of overlap arguments to introduce little error particularly for sufficiently high bombarding energies.

The set of coupled equations derived is infinite in number and thus before any attempt is made at solution some further approximation is necessary. In the coupled channels method this is done simply by terminating the series so as to include only those low lying states strongly coupled to the ground state plus any other states of interest. The equations are then solved exactly for the resulting sub-space of states considered. In field theory this approach is known as the Tamm-Dancoff approximation.

It is necessary to make some allowance for the effects of the neglected channels. Formal scattering theory shows (Gl 67) that by replacing the interaction \( V(R, r) \) which is real by an effective interaction which depends upon the states considered, it is formally possible to simulate the effect of the eliminated channels and thus describe correctly the scattering associated with the channels considered. The effective interaction resulting is generally complex, explicitly dependent on energy and non-local. Formal theory also shows (Gl 67) that this effect of truncation is mainly confined to the diagonal matrix elements, the effect on the off-diagonal matrix elements being small. In this work on the microscopic model the effects of truncation on off-diagonal elements will be neglected. The diagonal matrix
elements will be treated as equivalent local potentials (Pe 62) in two ways as mentioned in chapter 1, as a phenomenological potential as in the optical model or else in the microscopic model with some simple parameterization of the imaginary component. This will be discussed in detail later.

A further approximation in the microscopic model is that all diagonal matrix elements will be taken to be the same and equal to that for the ground state. While this is not strictly correct it is a good approximation for low lying states which are expected to have basically similar structure.

A simplification is possible in this calculation due to the fact that all the lowest lying levels in the nuclei considered have even spin and are of positive purity. By restricting the states considered to even spin and positive parity we note the following. The parity of each partial wave is \((-1)^L\). From Eq. 7 the parity is seen to be \((-1)^\ell\), thus to conserve parity \((L - \ell)\) must take on only even values in Eq. 5.

In the following sections the superscript \(j\) will be dropped where no confusion is likely to occur.

2.3 Cross Sections

When the separation of the alpha-particle and target nucleus is sufficiently large that the nuclear potential is negligible, the equations become uncoupled. In this asymptotic region the radial wavefunctions influenced only by the long range coulomb interaction can be written in the form (B1.52).

\[
U^{L}_{\ell I}(R) = \frac{1}{\sqrt{V_I}} \left[ A^{L}_{\ell I} H^{*}_I(kR) - B^{L}_{\ell I} H^{L}_{\ell}(kR) \right] \quad \ldots(2.13)
\]

Here \(V_I\) is the asymptotic velocity of the alpha particle in channel \(I\), \(A^{L}_{\ell I}\) and \(B^{L}_{\ell I}\) are the amplitudes of the incoming and outgoing waves respectively and \(H^{L}_{\ell}(kR)\) is given by

\[
H^{L}_{\ell}(kR) = G^{L}_{\ell}(kR) + j F^{L}_{\ell}(kR) \quad \ldots(2.14)
\]
where $F^R(kR)$ and $G^R(kR)$ are the regular and irregular Coulomb wavefunctions (Bl 52). These Coulomb wavefunctions are normalised so that at large $R$,
\[ F^R(kR) \to \sin \theta^R(kR) \]
\[ G^R(kR) \to \cos \theta^R(kR) \]

with
\[ \theta^R(kR) = kR - n(k) \ln (2kR) - \frac{\pi}{2} + \sigma^R(k, n). \]

The Coulomb parameter $n$ is given by
\[ n(k) = \frac{\pi}{4} \frac{\mu}{\hbar^2 k} \]

and the Coulomb phase shift is given by
\[ \sigma^R(k, n) = \arg \Gamma (\ell + 1 + in(k)) \]

We require the following asymptotic boundary conditions.

In the channel corresponding to the nuclear ground state there are both incoming and outgoing spherical waves at infinity, corresponding to the fact that there is an incident wave in this channel. However in all other channels, there are outgoing waves only. Thus for the entrance or elastic channel, as the effect of the nuclear potential can influence only the outgoing spherical waves one must have
\[ U^L_{LO}(R) = \frac{1}{2k_o} \sqrt{4\pi(2L+1)} e^{i\sigma_L(k_o)} \left[ H^*_L(k_o R) - B^L_{LO} H_L(k_o R) \right] \]
\[ ... (2.15) \]

Thus from Eqs. 13 and 15, and using the above boundary conditions $A^L_{\ell I}$ is found to be
\[ A^L_{\ell I} = \frac{1}{2k_o} \sqrt{4\pi V_o(2L+1)} e^{i\sigma_L(k_o)} \delta_{I,O} \delta_{\ell, L} ... (2.16) \]

The scattering matrix $S^L_{\ell I, \ell' I'}$ is defined by
\[ B^L_{\ell I} = \sum_{\ell' I'} S^L_{\ell I, \ell' I'} A^L_{\ell' I'} \]
Using Eq. 16 one sees that this reduces to

$$B^L_{II} = S^L_{II, LO} A^L_{LO} \quad \ldots (2.17)$$

For the further development $S^L_{II, LO}$ will be shortened to $S^L_{II}$.

By computing the outgoing flux for any channel the differential cross-section can easily be obtained. Using Eqs. 13, 15, 16 and 17 and substituting the form of Eqs. 5, 6 and 7 gives

$$\frac{d\sigma_I}{d\Omega} = \left| f_c(\theta) \delta_{I, O} + f_N(\theta) \right|^2 \quad \ldots (2.18)$$

where $f_c(\theta)$ is the usual Coulomb scattering amplitude (Mo 65) given by

$$f_c(\theta) = \frac{-n(k_o)\exp \left[ -\text{in}(k_o)\ln(\sin^2 \theta/2) + 2 \sigma_o(k_o, n) \right]}{2k_o \sin^2 \theta/2} \quad \ldots (2.19)$$

and

$$f_N(\theta) = \frac{i}{2k_o} \sum_{L,M} \sqrt{4\pi(2L+1)} \exp \left[ i \left( \sigma_L(k_o) + \sigma_L(k_I) \right) \right]$$

$$\left( \delta_{I, O} - S^L_{II} \right) C(II, M-MO) Y^M_{\ell} (\theta, 0) \quad \ldots (2.20)$$

Here we have made use of the azimuthal symmetry to set $\theta = 0$ in the spherical harmonics. The reaction cross-section, which is defined as the number of particles removed from the incident beam per second divided by the incident flux is given by

$$\sigma_R = \frac{\pi}{k_o^2} \sum_L (2L+1)(1 - |S^L_{II, LO}|^2) \quad \ldots (2.21)$$

2.4 Distorted Wave Born Approximation and the Optical Model

Consider the various types of transition shown in figure 2.1.
Transitions which lead from an excited state back to the ground state (type 2) or which connect excited states (type 3) can only occur after the excited states have been initially populated and thus must correspond to second or higher order in the interaction $V$. The first order perturbation solution to the coupled equations Eq. 12 thus contains transitions of type 1 only. That is one neglects all off-diagonal matrix elements except those which connect directly an excited state to the ground state. The simplified form of Eq. 12 thus becomes

\[
(T - E + V_{\text{oo}}) U_{\text{LO}}^L = 0 \quad \text{Ground state channel} \quad \ldots(2.22a)
\]

\[
(T' - E + \epsilon_I + V_{\text{IO}}^L + V_{\text{II}} U_{\text{LO}}^L = 0 \quad \text{Excited state channels} \quad \ldots(2.22b)
\]

where

\[
T = -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right)
\]

\[
T' = -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dR^2} - \frac{\ell(\ell+1)}{R^2} \right)
\]

\[
V_{\text{oo}} = \langle \text{LOL} | V(R, r) | \text{LOL} \rangle
\]

\[
V_{\text{IO}} = \langle \ell\ell | V(R, r) | \text{LOL} \rangle
\]

\[
V_{\text{II}} = \langle \ell\ell | V(R, r) | \ell\ell \rangle
\]

The system is now much simpler than before, each excited state is coupled only to the ground state channel. Hence each excited state can be considered separately from each other. Also as stated in section 2, in this work for the microscopic model all diagonal matrix elements are treated as identical, thus $V_{\text{oo}} = V_{\text{II}}$. Note that these diagonal matrix elements are not identical with the corresponding elements for the coupled equations case. This is because the neglect of the coupling terms, just as the effect of truncation of the coupled equations discussed in section 2 necessitates a modification of the interaction. The potential $V_{\text{oo}}$ has now become the optical potential, and Eq. 22a is simply the Schrodinger radial
equation for the elastic channel. Eq. 22b describes inelastic scattering to channel I, the wavefunction $U_{LO}^L$ having been found using Eq. 22a. The Eqs. 22a and 22b constitute the DWBA. The exact equivalence between this approach and the more usual DWBA method of integrating the transition matrix is given by Glendenning (G1 67).

The corresponding cross-section for the elastic channel or optical model in this case simplifies to the following. From Eqs. 18, 19 and 20.

$$\frac{d\sigma_o}{d\Omega} = \left| f_c(0) + f_N(0) \right|^2$$

where $f_c(0)$ is the Coulomb scattering amplitude as in Eq.19 and

$$f_N(0) = \frac{i}{2k_0} \sum_L (2L+1) e^{2i\sigma_L(k_0)} (1-\eta_L)^P_L(\cos\theta) \quad \ldots (2.23)$$

here we have put $\eta_L = \frac{S_L}{S_{LO}}$, which is the reflection coefficient for the $L$'th partial wave.
CHAPTER 3

The Coupling Matrix Elements

3.1 Microscopic Model

3.1.1. Introduction

Recent reviews (Sa 66, Ma 66) on the microscopic description of direct reactions have presented the general theory for the form of the coupling matrix elements. In particular, these papers, being concerned with nucleon scattering, include in the formalism spin-orbit and iso-spin dependence. The effect of antisymmetrisation between target and projectile is considered and selection rules for transitions discussed. In this work, the projectile, an alpha-particle, has zero spin and iso-spin.

The form of the alpha-nucleon interaction considered is of a simple central potential. The actual form of the interaction is based on the analysis of proton-alpha elastic scattering. The experimental study of this process is extensive. Early studies were confined to bombarding energies below about 10 MeV and the results analysed in terms of $l=0$ and $l=1$ phase shifts. Sack, Biedenharn and Breit (Sa 54) fitted the phase shifts with phenomenological two-body, local, energy independent potentials having central and spin-orbit terms. They considered potentials of the form (1) square well, (2) exponential well and (3) Gaussian well, with spin-orbit terms as derivatives of the central well. Their results showed clearly the Gaussian potential to be the best of the three and it is this form that will be used here. The spin-orbit term is neglected, its contribution when averaged over a spin-zero nucleus is zero. Thus

$$V(r) = -V_o e^{-Kr^2}$$

Here $r$ is the proton-alpha separation and $K$ the range parameter. The parameters found by Sack et al. are

$$V_o = 47.32 \text{ MeV}, \quad K = 0.435 \text{ fm}^{-1}$$
A later analysis by Gammel and Thaler (Ga 58) studied data up to about 40 MeV and included $l = 2$ phase shifts. Their potential was however, energy dependent, and used different parameters for even and odd $l$ values. However, a Gaussian well with a range parameter $K = 0.565 \text{ fm}^{-1}$ gives a reasonable fit to the Gammel-Thaler potential for even $l$. A Gaussian form for the interaction has also been used in earlier work on the microscopic model (Ma 64, Sa 64, Ja 65). We thus have a potential of depth about $50,000 \text{ eV}$, and a range parameter of the order $0.435 \text{ fm}^{-1}$ to $0.565 \text{ fm}^{-1}$. Although it is to be expected that the interaction of an alpha-particle with a nucleon in the nucleus is not the same as that with a free nucleon, it should, however, represent a reasonable starting point. The finite size of the alpha-particle is taken into account through the alpha-nucleon potential but its polarisability is neglected. The effects of antisymmetrisation between the nucleons of the target and those of the projectile are also neglected as discussed in chapter 2.

The target nucleus is described in terms of the shell model. The predictions and limitations of the shell model have been extensively discussed elsewhere (de 65, Ma 55) and will only briefly be discussed here. The basic assumptions are:-

1) A nucleon is considered to move in a central "shell model potential" $U(r)$, which represents the average of all the complex internucleon interactions.

2) There exist single-particle orbits, each characterised by a radial quantum number $n$ and an orbital angular momentum $\ell$.

3) A strong spin-orbit interaction depresses each $j = \ell + \frac{1}{2}$ level relative to the corresponding $j = \ell - \frac{1}{2}$ level where $\ell = J + \frac{1}{2}$.

The nuclear shell model Hamiltonian can then be written as

$$H = \sum_{i=1}^{A} H_i + a \sum_{i=1}^{A} \ell_i \cdot s_i + \sum_{i>j} V_{ij}$$
where $H_1$ is the single particle Hamiltonian for the particle in the central shell model potential $U(r)$. The second term is a one-body spin-orbit potential with strength parameter $a$. The third term represents a residual two-body interaction. Where this residual interaction can be neglected we have the $j$-$j$ coupling limit. The model Hamiltonian $H$ can then be written directly as a sum of single particle Hamiltonians, consequently the total wavefunction for the nucleus can be expressed as a product of single particle wavefunctions. More generally any linear combination of these products satisfies the Schrodinger equation for the model Hamiltonian $H$. However, since the nucleus is a system of fermions, we need to construct the $n$-particle wavefunction such that it is antisymmetric with respect to exchange of any two particles. The extent to which this problem occurs in this work will be discussed in section 3.1.3. Where the residual two-body interaction is no longer negligible, the resultant wavefunction needed to satisfy the model Hamiltonian can be written as a linear combination of the wavefunctions for the $j$-$j$ coupling scheme. This will be dealt with in section 3.1.4.

For convenience, the derivation of the form of the coupling matrix elements is divided into several sections. The following section derives the relation of the matrix elements to the effective interaction, and in turn its relation to the nuclear transition density. Section 3 deals with the form of the nuclear transition density for shell model states. Section 4 generalises the results to include states with configuration mixing. Section 5 combines the results of the previous sections to form the coupling matrix elements. The final section is concerned with the form of the optical potential.

3.1.2 Form of the coupling matrix elements

The general form of the coupling matrix elements found in chapter 2 can be written
Here we have made use of Eq. 2.7. The new matrix element, which contains the dependence on the target nuclear coordinates is defined to be the effective interaction \( P_{\text{IM}}(R) \)

\[
P_{\text{IM}}(R) = \langle \phi_{\text{IM}}(r) | V(R, r) | \phi_{\text{I'M'}}(r) \rangle \quad \ldots (3.2)
\]

The interaction between the alpha-particle and the target nucleus is taken as a sum of two-body alpha-nucleon interactions. If the position of nucleon \( k \) in the target is specified by the vector \( r_k' \), then the potential between the alpha-particle and the nucleon is simply

\[
V_k = V(|d_k|) = V(|R - r_k|) \quad \ldots (3.3)
\]

This is shown in figure 3.1, \( r_k \) joins the centre of mass of the target nucleus to nucleon \( k \).

The total potential is thus a sum over all nucleons of such terms, i.e.

\[
V(R, r) = \sum_{k=1}^{A} V(|R - r_k|) \quad \ldots (3.4)
\]
The effective interaction can therefore be written

\[ P_{IM} (R) = \sum_{I'M'} A \langle \phi_{IM} (r_1, \ldots, r_A) | V(|R-r_A|) | \phi_{I'M'} (r_1, \ldots, r_A) \rangle \]

\[ \ldots (3.5) \]

Rearranging this gives

\[ P_{IM} (R) = \sum_{I'M'} A \int \left[ \int \phi^*_{IM} (r_1, \ldots, r_A) \phi_{I'M'} (r_1, \ldots, r_A) \right. \]

\[ dr_1 \ldots dr_{k-1} dr_{k+1} \ldots dr_A \left. \right] \cdot V(|R-r_k|) \cdot dr_k \]

\[ \ldots (3.6) \]

which can be written

\[ P_{IM} (R) = \int_{I'M'} \rho (r) \cdot V(|R-r|) \cdot dr \]

\[ \ldots (3.7) \]

where \( \rho (r) \) is the nuclear transition density given by

\[ \rho_{IM} (r) = \left[ \int \sum_{k=1}^A \phi^*_{IM} (r_1, \ldots, r_A) \phi_{I'M'} (r_1, \ldots, r_A) \right. \]

\[ dr_1 \ldots dr_{k-1} dr_{k+1} \ldots dr_A \left. \right] \cdot \frac{r_k}{r} \]

\[ \ldots (3.8) \]

The nuclear transition density represents the overlap of the initial and final state wavefunctions.

3.1.3. Nuclear Transition density for shell model states

For the nuclei under consideration, the simple shell model picture is of a closed core plus two identical extra core nucleons. For Ca\(^{42}\) we have the Ca\(^{40}\) core plus two neutrons. For Ti\(^{50}\) the core is Ca\(^{48}\), and there are two extra core protons. The core is assumed to be unaffected by the transition. We write the wavefunction for the nucleus as a product of terms.

\[ \phi_{IM} (r_1, \ldots, r_A) = \phi_{IM} (12) \phi_c (3, \ldots, A) \]

\[ \ldots (3.9) \]

Here \( \phi_{IM} (12) \) is the wavefunction of the extra core nucleons and
\( \phi_c(3\ldots A) \) that of the core. Each wavefunction is normalised and fully antisymmetric within itself. As this stands however, the wavefunction is not antisymmetric with respect to exchange of core and extra core nucleons. The correct wavefunction that does satisfy this requirement can be written:

\[
\phi_{IM}(r_1\ldots r_A) = \sqrt{\frac{2}{A(A-1)}} \left\{ 1 - \sum_{i=3}^{A} P_{1i} - \sum_{j=3}^{A} P_{2j} + \sum_{i=3}^{A} \sum_{j>i}^{A} P_{1i} P_{2j} \right\} \phi_{IM}(12) \phi_c(3\ldots A)
\]

\( \phi_c(3\ldots A) \)

where \( P_{1i} \) is the usual permutation operator, exchanging the coordinates of particles \( 1 \) and \( i \).

It can easily be shown that as the core plays no part in the transition, the final form obtained for the transition density is the same, whether the form of the wavefunction is taken as Eq. 9 or Eq. 10. To avoid unnecessary confusion, we use the simpler form given by Eq. 9.

It must be pointed out that in the shell model, the coordinates of a nucleon are measured with respect to the centre of potential and not to the centre of mass of the nucleus. This means that the Hamiltonian is not translationally invariant and in addition to describing the motion of the nucleon with respect to the centre of mass, the wavefunction also contains a motion of the centre of mass relative to the origin of coordinates. This spurious motion of the centre of mass can only be corrected exactly (Ga67) for harmonic oscillator wavefunctions. The correction term is of the order \( 1/A \) and will be neglected in this work.

Substituting Eq. 9 for the wavefunction into Eq. 8 for the nuclear transition density gives:

\[
\rho_{IM}(r) = \langle \phi_c(3\ldots A) | \phi_c(3\ldots A) \rangle \left[ 2\int \phi_{IM}^*(12) \phi_{I'M'}^*(12) \, dr_1 \right] r_2=r

+ \langle \phi_{IM}(12) | \phi_{I'M'}(12) \rangle \left[ \int_{k=3}^{A} \phi_{c}^*(3\ldots A) \phi_{c}(3\ldots A) \, dr_3 \cdots dr_{k-1} \, dr_{k+1} \cdots dr_A \right] r_k=r
\]

... (3.11)
The diagonal elements, i.e., where \( \psi_{\text{IM}}(12) = \psi_{\text{IM}'}(12) \) give rise to the optical potential. These will be dealt with later, here we consider the off-diagonal elements. In this case, as the wavefunctions are orthogonal, the second term in Eq.11 vanishes. Also the normalisation of the core wave-functions gives

\[
\langle \psi_c(3\ldots A) | \psi_c(3\ldots A) \rangle = 1
\]

Eq.11 also makes use of the antisymmetric nature of \( \psi(12) \) in order to replace the summation over \( r_1 \) and \( r_2 \) by twice the contribution of one of the terms. The nuclear transition density thus becomes:

\[
\rho_{\text{IM}}(r) = 2\left[ \int \psi_{\text{IM}}^*(12) \psi_{\text{IM}'}(12) \, dr_1 \right]_{r_2 = r}
\]  

...(3.12)

We now consider the form of \( \psi(12) \). The shell model single particle wavefunctions are characterised by the quantum numbers \( n \ell j \) using the usual notation (de 65). Writing this to show the coupling between the spin and orbital motion, we have the single particle wavefunction:

\[
\psi_{j m}(1) = \sum_{\ell v w} C(\ell s j, v w m) \psi_{n \ell s j}^{v m}(1) \chi_{n \ell s j}^{w m}(1)
\]  

...(3.13)

Here \( \psi_{n \ell s j}^{v}(1) \) is the orbital and \( \chi_{n \ell s j}^{w}(1) \) the spin wavefunction.

These single particle wavefunctions of the extra core nucleons are coupled to form the wavefunctions \( \psi_{\text{IM}}(12) \). The simplest case is where both nucleons are characterised by the same \( n \ell j \) quantum numbers, and in this case,

\[
\psi_{\text{IM}}(12) = \sum_{v w x y m'} C(j j', m m' M) C(\ell s j, v w m) C(\ell s j', x y m')
\]

\[
\times \psi_{n \ell s j}^{v}(1) \chi_{n \ell s j}^{w}(1) \psi_{n \ell s j}^{x}(2) \chi_{n \ell s j}^{y}(2)
\]  

...(3.14)

Where \( I \) is even, the antisymmetric requirement on \( \psi_{\text{IM}}(12) \) is guaranteed by the properties of the vector coupling coefficients (Ro 63).
More generally, where the nucleons are in different states, the
antisymmetric wavefunction is written as

\[
\theta_{\text{IM}}(12) = \frac{1}{\sqrt{2}} \sum_{vwxym} C(jj'I, mm'M)C(\ell s j', vwm)C(\ell' s' j', xym')
\]

Further, the orbital and radial components of \( \psi^V(1) \) can be separated, i.e.

\[
\psi_{nl}(1) = R_{nl}(r_1) Y^V_{\ell}(\Omega r_1)
\]

Substituting the wavefunctions into Eq. 12 for the transition
density, we sum over spin coordinates and integrate over the spatial
coordinates of particle 1. The algebraic details are straightforward
and given in Appendix 3.1. Because the effective interaction is a one
body operator in the space of the target nucleons, that is it operates on
one nucleon at a time, it follows that the transition density will
vanish if the initial and final states differ in more than the coordinates of one particle. The initial and final state wavefunctions of
interest can therefore be written:

\[
\theta_{\text{IM}}(12) = \theta_{\text{IM}'}(n \ell j, n'\ell' j')
\]

Now \( \theta_{\text{IM}}(12) \) and \( \theta_{\text{IM}'}(12) \) have at least one set of quantum numbers in common, also as the wavefunctions are antisymmetric we have

\[
\theta_{\text{IM}}(n \ell j, n'\ell' j') = -\theta_{\text{IM}}(n'\ell' j', n \ell j)
\]
\[ \rho_{IM}(r) = S(-1)^{j-s+I'} R_{n'\ell'}(r) R_{n''\ell''}(r) \sum_{q} Y_{M-M'}^{*}(\Omega) r \]

\[ C(\ell''\ell'q, 000) C(I'qI, M', M-M', M) \]

\[ W(j''j'I', jq) W(\ell'j'\ell'', sq) \]

\[ \frac{[(2\ell'+1)(2\ell''+1)(2j'+1)(2j''+1)(2I'+1)]^{1/2}}{4\pi} \]

...(3.19)

Where \( W(\ell'j'\ell'', sq) \) are the usual Racah coefficients as defined by Rose (Ro 63), and \( S = 2 \) if all states have the same quantum numbers \( n \ell j = n'\ell'j' = n''\ell''j'' \)

\( S = \sqrt{2} \) if either \( n \ell j = n'\ell'j' \) or \( n \ell j = n''\ell''j'' \)

\( S = 1 \) if \( n \ell j \neq n'\ell'j' \) and \( n \ell j \neq n''\ell''j'' \)

For the case in which \( S = 1 \), if \( n'\ell'j' = n''\ell''j'' \) there is a second term present for the transition density. This is obtained by exchanging the roles of \( n \ell j \) and \( n'\ell'j' \) by means of Eq. 18.

As discussed in chapter 1, the parity of the initial and final states of interest is always positive, thus \( I+I' \) and \( I+I'' \) are both even, and therefore \( \ell'+\ell'' \) is also even. The properties of the vector coupling coefficients are such that the term \( C(\ell''\ell'q 000) \), in Eq. 19 vanishes unless \( \ell''+\ell' - q \) is even. Thus only even values of \( q \) contribute in the summation. The value of \( q \) is also bounded by the triangular conditions.

\[ |\ell'-\ell''| \leq q \leq \ell'+\ell'' \]

\[ |j'-j''| \leq q \leq j'+j'' \]

\[ |I'-I| \leq q \leq I'+I \]

For those terms where the initial state is the ground state, which are the only terms that occur in DWBA, the equation takes on a particularly simple form. We have \( I'=0 \), thus \( n''\ell''j'' = n \ell j \)

\[ \rho_{IM}(r) = S(-1)^{j-S} R_{n\ell}(r) R_{n'\ell'}(r) Y^{*M}(\Omega) r \]

\[ C(\ell \ell'I, 000) W(\ell'j'\ell j, s I) \]

\[ \frac{[(2\ell+1)(2\ell'+1)(2j'+1)]^{1/2}}{4\pi(2I'+1)} \]

...(3.20)
where the value of $S$ is the same as in the previous case.

3.1.4. **Transition density for states of generalised configuration mixtures**

The simple shell model assumes that the complex interaction between nucleons can be approximated by a central potential well. This results in simple single particle wavefunctions. A more realistic picture is to take some account of residual inter-nucleon forces. Such calculations, using various forms for the residual interaction have been performed for $\text{Ca}^{42}$ by Law (La 68). The simple single particle wavefunctions used up to this point are now no longer eigenfunctions of the total Hamiltonian. These wavefunctions do however, form a complete set of functions and thus form a basis in which the actual wavefunctions can be expanded.

If we write the simple shell model wavefunction for the extra core nucleons as:

$$\mathcal{\Omega}_{IM'(12)} = \mathcal{\Omega}_{IM}(n I j, n'I'j') = \mathcal{\Omega}_{IM}(a, b)$$

where $a$ and $b$ represent the collective single particle quantum numbers of the state.

The generalised configuration mixture states can now be written as (La 68)

$$\mathcal{\Omega}_{IM'(12)} = \sum_{a, b} g_{ab} \mathcal{\Omega}_{IM}(a, b)$$

$$\mathcal{\Omega}_{I'M'(12)} = \sum_{c, d} h_{cd} \mathcal{\Omega}_{I'M}(c, d)$$

The coefficients $g_{ab}$, $h_{cd}$, must satisfy the normalisation condition.

$$\sum_{ab} g_{ab}^2 = 1$$

If we write the form of the transition density Eq. 19 as

$$\rho_{IM}(r) = \sum_{q} Y_{q}^{*M-M'}(\Omega r) f_{q}(r, abcd)C(I'q I, M'M-M', M)$$

$$\ldots(3.21)$$
where \( f_q(r, abcd) \) contains all the dependence on \( r \) and the single particle quantum numbers, the generalised transition density for configuration mixture states can be written as:

\[
\rho_{IM}^q(r) = \sum_q Y_q^{*M-M'}(r)C(I'q I, M', M-M', M)
\]

\[
\cdots \sum_{ab} f_q(r, abcd) g_{ab} h_{cd}
\]

... (3.22)

3.1.5. Effective interaction

We have that the effective interaction is given by Eq. 7.

\[
P_{IM}(R) = \int \rho_{IM}^q(r) V(|R-r|) dr
\]

and that the general form of the transition density can be written as

\[
\rho_{IM}^q(r) = \sum_q Y_q^{*M-M'}(r)\sum_{abcd} f_q(r, abcd) C(I'q I, M', M-M', M) \cdots (3.23)
\]

We use the Gaussian form for the alpha-nucleon potential

\[
V(|R-r|) = -V_o \ e^{-K^2 |R-r|^2} \cdots (3.24)
\]

This can conveniently be expanded in multipoles (Pe 66)

\[
V(|R-r|) = -V_o \ 4\pi \sum_{km} i^k j_k(-2ikRr) e^{-K^2(R^2+r^2)}
\]

\[
\cdots Y_m^k(\Omega R) Y^m_k(\Omega r) \cdots (3.25)
\]

where \( j_k \) are the spherical Bessel functions (Sn 61).

Substituting Eqs. 23 and 25 into Eq. 7 we get for the effective interaction

\[
P_{IM}(R) = -V_o \ 4\pi \sum_{qlkm} i^k C(I'q I, M', M-M', M) e^{-K^2R^2} Y_m^k(\Omega R)
\]
Integrating over $\Omega r$ gives terms $\delta(m, M-M')$ and $\delta(k, q)$, so that

$$
P_{IM}(R) = -V_o \sum_q i^q C(I'|q, I, M', M-M', M) e^{-K^2 R^2} q_{M-M'}^* (\Omega R) \int \mathcal{f}_q(r) j_k(-2iK^2 R r) e^{-K^2 r^2} dr \quad \ldots (3.26)
$$

Substituting in the form of $j_k$, (Sn 61) gives

$$
P_{IM}(R) = -V_o \sum_q C(I'|q, I, M', M-M', M) q_{M-M'}^* (\Omega R) \frac{e^{-K^2 R^2}}{2\pi (K^2 R)^q} \frac{T(q + \frac{3}{2})}{T(q + \frac{3}{2})} \int \mathcal{f}_q(r) e^{-K^2 r^2} r^{q+2} \, {}_0F_1(q+3; (K^2 R)^2) dr \quad \ldots (3.27)
$$

Here $T$ is the Gamma function and $\, {}_0F_1$ the hypergeometric function as defined by Sneddon (Sn 61).

In this form, the effective interaction can be conveniently evaluated on a computer.

We now wish to substitute this form for the effective interaction into Eq. 1 for the coupling matrix elements. For convenience we write Eq. 28 in the form

$$
P_{IM}(R) = \sum_q C(I'|q, I, M', M-M', M) q_{M-M'}^* (\Omega R) \mathcal{f}_q(I'|I) \quad \ldots (3.28)
$$

Substituting in Eq. 1 gives

$$
<iL|V(R, r)|i'L'\rangle = i^{I'-I} \sum_q C(iL, -M, M, 0) \cdot C(i'I'L', -M'M', 0) C(I'|q, I', M', M-M', M) \mathcal{f}_q(I'|I') \quad \ldots (3.29)
$$
The integration can be carried out using the identity relation of spherical harmonics \( (\text{Ro} \ 63, \ \text{page} \ 62) \) to give

\[
\left< \ell \ I \ L \ | \ V(R, r) \ | \ell' I' L \right> = i^{\ell'-\ell} \sum_{M'q} C(\ell \ I \ L, -M \ M \ 0) \\
\times C(\ell' I' L, -M' M \ 0) C(I'q I, M'M-M', 0) \int \mathcal{F}(R) \\
(\mp 1)^{M+M'} \left[ \frac{(2\ell+1)(2\ell'+1)}{4\pi (2q+1)} \right]^{1/2} C(\ell'q, -M', M, M-M') C(\ell'q 000)
\]

\[\ldots (3.31)\]

Summing over \( M \) and \( M' \) using the relations between the vector coupling coefficients, one can easily obtain

\[
\left< \ell \ I \ L \ | \ V(R, r) \ | \ell' I' L \right> = i^{\ell'-\ell} \sum_{q I' L'} \int \mathcal{F}(R) \ C(\ell'q, 000) \mathcal{W}(\ell' I' L', q L) \\
(\mp 1)^{L} \left[ \frac{(2\ell+1)(2\ell'+1)(2I+1)(2I'+1)}{4\pi (2q+1)} \right]^{1/2} \ldots (3.32)
\]

This is now the general form for the coupling matrix elements.

### 3.1.6 Optical potential

The diagonal coupling matrix elements constitute the optical potential. We note that for such elements, as the initial and final states are the same, the transition density is now simply the nuclear density. The effective interaction is obtained by folding the alpha-nucleon potential into the nuclear density distribution. As stated in Chapter 2, we have made the approximation that all diagonal matrix elements are the same, and equal to that of the ground state. In this case \( q = I' = I = 0, \ \ell = \ell' = L, \ M = M' = 0 \)

The simple form of the Racah algebra in this case allows this matrix element to be written

\[
\left< L0L \ | \ V(R, r) \ | \ L0L \right> = \int \rho(r)V(|R-r|)dr \ldots (3.33)
\]
where \( \rho(r) \) is the nuclear density distribution. Making use of the previously obtained result for the form of this integral Eq. 32 we obtain on performing the necessary vector algebra

\[
\langle L0L|V(R, r)|L0L \rangle = V(R) = -V_o \sqrt{4\pi} \int e^{-\frac{K^2 R^2}{r^2}} e^{-\frac{K^2 r^2}{r^2}} ^{(3/2, (KRr)^2)}dr
\]

...(3.34)

The nuclear density \( \rho(r) \) is easily obtained once the single particle wavefunctions are known. Substituting in Eq. 20 the explicit form for the vector coupling coefficients, we find that the nuclear density can be written

\[
\rho(r) = \frac{1}{4\pi} \sum R^2_{nl} (r)
\]

...(3.35)

where the summation is taken over the occupied states.

The optical potential so far derived is purely real, and as already mentioned some provision must be made regarding the imaginary component, which is necessary in order to absorb particles from the elastic channel, thus allowing for those inelastic processes not explicitly included in the coupling. In this work this is done by a simple parameterisation. An imaginary component is introduced with the same radial form as the real part, but which is allowed to vary in strength in order to produce the best agreement with experiment.

3.2 Rotational Model

Many accounts of the Rotational Model are to be found in the literature, here only a brief outline will be given. We follow the method of Wills (Wi 63).

The nucleus is considered to be permanently deformed and axially symmetric. The normalised wavefunctions for an even-even axially symmetric nucleus in the lowest rotational band is simply (Ch 58, Ke 59)

\[
\varrho_{IM}(r) = Y^M_I(\Omega r)
\]

...(3.36)
Here the internal coordinates, \( r \), are just the Eulerian angles specifying the orientation of the distinct symmetry axis with respect to the quantisation axis. The eigenvalues of \( H(r) = T \) are well known (Ch 58, Ke 59, La 58) to be \( \varepsilon_1 = \hbar^2/2I \). The nucleus is also assumed to be symmetric with respect to reflection, in a plane which is the perpendicular bisector of the distinct symmetry axis, and therefore it takes on only even values (Ke 59). \( J \) is the moment of inertia about a line in this symmetry plane. It is assumed that the interaction potential \( V(R, r) \) is of the form.

\[
V(R, r) = V(R - R_o \left[ 1 + \sum_{\lambda} \beta_{\lambda} Y_{\lambda}^{o}(0') \right]) + iW(R - R_o \left[ 1 + \sum_{\lambda} \beta_{\lambda} Y_{\lambda}^{o}(\theta) \right]) \quad \ldots(3.37)
\]

Where \( R_o \) and \( R^{'}_o \) are the radial parameters for the real and imaginary components, respectively, of the potential well. \( \beta_{\lambda} \) are the usual deformation parameters, where in this case \( \lambda \) takes on even values only. The angle \( 0' \) is the angle between \( R \) and the distinct symmetry axis.

There are two methods of treating the coupling potential which lead to reasonably simple expressions for the coupling matrix elements. These involve expanding the potential in either Legendre polynomials or a Taylor series in powers of the deformation parameters. We will use here the Legendre polynomial expansion. This method has advantages over the Taylor series method for large deformations. It can be shown that the radial function for each term in the Legendre polynomial expansion constitutes an infinite sum of terms in the Taylor expansion (Ta 65, Bl 66). Thus higher terms in powers of \( \beta \), important for large deformations, can be automatically included. Expanding in Legendre polynomials one gets

\[
V(R, r) = \sum_{k} V_k(R) P_k(\cos 0') \quad \ldots(3.38)
\]
The coupling matrix can thus be written

\[ \langle \ell' \ell L | \mathbf{V}(R, r) | \ell' \ell' L' \rangle = \sum_k \langle \ell' \ell L | P_k(\cos \theta') | \ell' \ell' L' \rangle V_k(R) \]

\[ \ldots (3.39) \]

These matrix elements were evaluated by Wills (Wi 63) who shows:

\[ \langle \ell' \ell L | P_k(\cos \theta) | \ell' \ell' L' \rangle = i^{\ell'-\ell} \sqrt{(2\ell+1)(2I'+1)} \frac{C(K \ell \ell', 000)}{C(K I I', 000)} W(I' \ell' \ell, L K) \]

\[ \ldots (3.40) \]
CHAPTER 4
Elastic Scattering Analysis

Here we present the results of the analysis of the elastic scattering data. We have seen from chapter 2, that apart from the value of the results within themselves, a knowledge of the elastic scattering wavefunctions forms an essential basis for the work on inelastic scattering which is to follow. The chapter is divided into three sections. The first describes the experimental data which is used in this work, the second deals with the elastic scattering due to a phenomenological optical potential, and the third with the optical potential derived from the microscopic model.

In the usual phenomenological analysis the ambiguities in the optical model potential for strongly absorbed particles are well known (Dr 63). For alpha-particles, potential depths of the order of 50 MeV are in common use, although by folding the alpha-nucleon potential into the nuclear density distribution, depths of the order of 250 MeV are produced (Ja 65). Such variations in optical model potentials can be expected to influence results for inelastic scattering (Dr 63). For this reason, in section 4.2, optical model potentials covering a wide range of depths are presented. Also studied in this section are the criteria which optical potentials need to satisfy. For alpha-particles, which are strongly absorbed, the scattering is determined by a small number of partial waves. These important partial waves have their turning point in the surface region and are therefore strongly influenced by the potential in this region. Igo (Ig 58) suggested, that provided the correct form for the optical potential in the surface region is maintained, the actual depth of the potential is unimportant. Later Austern (Au 61) showed that the reflection coefficients $\eta_\perp$ could be considered to be made up of two terms. The partial waves being partly reflected from the nuclear surface and partly from the centrifugal barrier. The reflection coefficients for the important partial waves now depend on the phase averaging or
impedance matching between the two terms. Only discrete optical potentials are able to satisfy this criterion. Drisb et. al. (Dr 63) have shown that these two criteria are compatible although at 43 MeV, Igo's condition did not appear to apply to the imaginary part of the potential. An account of the parameter variation for the optical model potentials is given by El-Nadi and Riad (El 65). The applicability of the above criteria will be discussed and others suggested.

In section 4.3 the optical potential calculated using the microscopic model of the nucleus is considered. The choice of the nuclear wavefunctions used to generate the density distribution is discussed. Comparison is made between the quality of the fits to the scattering data so produced and those found in the phenomenological analysis. A study is made of the sensitivity of the results to the strength and range of the alpha-nucleon potential. The extent to which nuclear structure information can be extracted is discussed.

4.1 Experimental Data

Gruhn and Wall (Gr 65) measured the differential cross-sections for Ca$^{42}$ and Ti$^{50}$ at an energy (laboratory) of 30.5 MeV. The angular range of their measurements in the centre of mass system being $30^0$ to $170^0$. Percentage experimental errors on each point are quoted. These vary from less than 1% at forward angles to of the order of 20 to 30% at back angles. For Ti$^{50}$ the results are quoted in the usual units (mb/sr) but no value is given for the overall uncertainty in normalisation of the data. For Ca$^{42}$ the overall uncertainty in normalisation was such that the results are quoted in arbitrary units. In this case therefore, an additional problem in the analysis involved making an estimate of some normalisation factor.

Peterson (Pe 66) measured the differential cross-section for Ca$^{42}$ at 42 MeV. The angular range of the data being $12^0$ to
70° and the uncertainty in overall normalisation quoted to be 10%. The experimental errors are of the order of 1% at forward angles to about 10% at the backward angle minima.

Bruge (Br 67), in the most recent results used in this work, gives the differential cross-section for Ti at 44 MeV. Here the angular range is 11° to 55° and the overall uncertainty in normalisation 10%. The errors on the experimental points are, in the worst cases, 5%.

4.2 Phenomenological Analysis.

We choose the nuclear optical potential to be of the Saxon-Woods form

\[ V_o(R) = -V(1 + e^x)^{-1} - iW(1 + e^{x'})^{-1} \]  \(\ldots(4.1)\)

with

\[ x = (R - R_0)/a \quad \text{and} \quad x' = (R - R_0)/a' \]

The real part of the potential is today almost universally taken to be of this form. It has the flat form near the centre of the nucleus one expects from the short range nature of the nuclear force. The edge is diffuse as one would expect, as the nuclear density is known to be diffuse. The form of the imaginary part of the potential needed is less clear. At low bombarding energies, the Pauli exclusion principle inhibits excitations of target nucleons in the nuclear interior (Pr 62) since few unfilled states are available. Thus a surface peaked potential is to be expected. On increasing the energy, more excitations become possible, thus one could expect a change towards volume absorption. It is known that in the case of alpha-particles which are strongly absorbed, whether one uses volume or surface peaked absorption makes little difference to the quality of fit one can obtain. On this basis we restrict ourselves to volume absorption.

The parameter \(R_0\) characterises the radius at which the
potential falls to half its maximum depth. The parameter 'a' is the diffuseness, the potential falls from 98% to 2% of its maximum value in the range $R_0 - 4a$ to $R_0 + 4a$. Usually $R_0$ and $R_0'$ are expressed in the form:

$$R_0 = r_0 A^{1/3}; \quad R_0' = r_0' A^{1/3}$$

Typically $r_0, r_0' \approx 1.3 - 1.6 F$ and $a \approx 0.6 F$. The Coulomb potential is taken to be that due to a uniformly charged sphere of radius $R_c$, again we take $R_c = r_c A^{1/3}$ and where $r_c \approx 1.4 F$. This gives a Coulomb potential

$$V_c(R) = \frac{Z_1 Z_T e^2}{2} \left[ \frac{3}{R_c} - \frac{R^2}{3R_c^3} \right] \quad R < R_c$$

$$V_c(R) = \frac{Z_1 Z_T e^2}{R^2} \quad R \geq R_c$$

where $Z_1$ is the proton number of the projectile and $Z_T$ that of the target.

The numerical calculations of differential cross-sections were performed on the University's Elliott 503 computer. The computer programme is based on an Optical Model search code due to Smith (Sm 67). The original code was written in Fortran and as the University computer has no Fortran compiler it was necessary to translate into Algol. A modification made to the structure of the programme was to separate as a procedure, the formation of the nuclear potential. This has the advantage that the Saxon-Woods form for the potential used here can be simply replaced by any other form of nuclear potential, and in particular by that of the microscopic model, the results of which we consider in the next section. The numerical details of the programme are fully discussed by Smith (Sm 67).

The automatic search method employed represents an efficient and powerful method for determining the potential parameters which give best agreement with experimental data. We
choose as the criterion for the quality of fit, a least squares comparison between experimental and calculated differential cross-sections given by the expression

\[ X^2 = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\sigma_{\text{th}}(\theta_i) - \sigma_{\text{exp}}(\theta_i)}{\Delta \sigma_{\text{exp}}(\theta_i)} \right]^2 \]

Here \( N \) is the number of experimental points, \( \sigma_{\text{exp}}(\theta_i) \) is the experimental differential cross-section at angle \( \theta_i \) and \( \sigma_{\text{th}}(\theta_i) \) the corresponding theoretical value. The term \( \Delta \sigma_{\text{exp}}(\theta_i) \) is the associated experimental error.

In an actual calculation, initial estimates for the potential parameters are given, the programme then automatically makes systematic variation in selected parameters in a manner which minimises \( X^2 \). We know that generally the data can be fitted by several discrete sets of potential parameters, although these would not all be equally good fits, the automatic search procedure will generally converge upon one of these sets of parameters nearest to the starting point of the search in parameter space. The efficient use of a search code is largely a matter of experience. Allowing many parameters to vary freely tends to result in their uncontrolled wandering which is wasteful of computer time and produces results of limited usefulness. A better approach is to vary a small number of parameters (one or two) at a time, changing parameters throughout the search. By this means, knowing when to vary any given parameter, one can control fairly predictably the progress of the search. For example, if we wished to look for a set of potential parameters for which \( V \approx 100 \text{ MeV} \) and \( a \approx 0.6 \text{ F} \). Then considering only the real part of the potential and giving the above values as the initial estimates, together with some suitable value for the radial parameter, we would first allow only the radial parameter to vary. Its value would be adjusted to give the minimum value of \( X^2 \) compatible with the above constraints. Only now will the potential and diffuseness parameters
be allowed to vary and a few search cycles should suffice to accurately locate the minimum $X^2$ in this region of parameter space. Thus we will have found the optical potential parameters which lie closest to the selected region of interest.

In all the results here the Coulomb radius was fixed at $r_c = 1.4 F$. This was done because it is known $r_c$ is not a very sensitive parameter (Ho 63). Changes in $r_c$ can usually be compensated by small changes in other parameters.

For the definition of $X^2$, the errors on all experimental points were taken to be 10% of their magnitude. This was done to simplify data preparation.

Table 1 presents optical model parameters found to fit the experimental data for $^{42}\text{Ca}$ at 30.5 MeV and 42 MeV and for $^{50}\text{Ti}$ at 30.5 MeV and 44 MeV. Figures 1 to 4 give corresponding examples of the fits to the scattering data for several sets of potential parameters. The potentials cover a wide range of potential depths but the diffuseness parameters $a$ and $a'$ were kept, for the most part, in the region of 0.6 F. The first section of each table contains potential parameters in which the six parameters $V, W, r, r', a, a'$ were all allowed to vary independently during the search. In the final section the parameters were restricted so that $r = r'$ and $a = a'$, thus leaving only four free parameters.

The optical model potentials for $^{42}\text{Ca}$ at 42 MeV contain two additional sections. One of these contains parameters where the diffuseness was free to vary far from the region of 0.6 F. The other contains a very deep potential $V = 1336$ MeV. The significance of these additional potentials will be discussed in detail later. Each of the four cases will now be discussed.

$^{42}\text{Ca}$, 30.5 MeV

The optical model for strongly absorbed particles generally produces an oscillatory angular distribution with a decreasing envelope at large angles. The experimental data in this case however shows a plateau region around 105°. It was not found possible
Table 4.1  Optical Model Potential Parameters

Where gaps exist in the a' and r' columns it indicates that the real and imaginary components are equal. The values of $X^2$ are based on a uniform experimental error of 10%. In the case of Ca$^{42}$ at 30.5 MeV an additional column (NORM) gives the factor by which the experimental data was multiplied as explained in the text.

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</table>
Figures 4.1 to 4.4

Cross-sections for elastic scattering obtained from some of the optical potentials given in table 1.

- Figure 4.1: Ca\textsuperscript{42} at 30.5 MeV
- Figure 4.2: Ca\textsuperscript{42} at 42 MeV
- Figure 4.3: Ti\textsuperscript{50} at 30.5 MeV
- Figure 4.4: Ti\textsuperscript{50} at 44 MeV
Figure 4.1

$d\sigma/d\Omega$

mb/sr

$\theta_{c.m.}$ (DEGREES)

$^{42}$Ca 30.5 MeV

POTENTIAL 1

POTENTIAL $3 \times 10^2$
Figure 4.2
Figure 4.3
Figure 4.4

$^{50}\text{Ti} \ 44 \text{ MeV}$

POTENTIAL 1
POTENTIAL 2 $\times 10$
POTENTIAL 3 $\times 100$

$\frac{d\sigma}{d\Omega}$
mb/sr

$\theta_{\text{c.m.}}$ (DEGREES)
to fit the data over the entire angular range of 30° to 175°. The same conclusion was reached by Gruhn and Wall (Gr 65) who discuss the problem in some detail. They suggest that a potential shape resonance occurs which attenuates a particular partial wave. They plotted the total effective potential (Coulomb + Nuclear + centrifugal barrier) for various partial waves. In the nuclear surface this function for certain L has a minimum. If the effective wavelength is of the same size as the width of the valley then a standing wave or type of potential shape resonance can occur. They then investigated the effect such a resonance would have by considering a modified Blair Smooth cut off model where a particular partial wave could be attenuated. They found the differential cross-section at back angles extremely sensitive to such an attenuation but the effects at forward angles to be slight. Thus they showed that extreme difficulty would be encountered in obtaining optical model fits over the entire angular distribution, whereas forward angle data may be fitted with relative ease. A discussion of the limitations of the optical model is also given by Carter et al (Ca 64). They show that the optical model cannot reproduce anomalies and that it is generally more reliable at forward angles rather than backward angles.

In order to get a reasonable fit to the data, only points out as far as 68° were included, since even the addition of one or two points resulted in a large increase in $X^2$. As already discussed, the experimental data was quoted in arbitrary units and an estimate of a normalisation factor had to be made. This normalisation factor occurred as an additional parameter in the automatic search, and its values are shown in table 1. It was found in all cases to be close to 5. This value for the normalisation led to a reaction cross-section of about 1500 mb., which is about the same as was found from the analysis at 42 MeV. This suggests that such a normalisation is realistic in value. Limiting the angular range of the data and allowing the normalisation factor to assume a best value, satisfactory fits could then be obtained for a large range of well depths.
The quality of fit does not vary greatly with the various potential depths, although the shallow potentials have a somewhat smaller value of $X^2$. As is to be expected the fits with six parameters are somewhat better than those with only four.

Ca$^{42}$ 42 MeV

Unlike the data at 30.5 MeV, here the strong diffraction pattern extends over the entire angular range of the data. Good fits could be found as can be seen from Figure 2. The quality of the fit does not seem to depend to any extent on the potential depth. As before the six parameter fits are a little better than the four parameter fits. The potentials with a large variation in diffuseness, produced fits on average of much the same quality as the four parameter fits. The quality of the very deep potential is definitely worse than that of the rest, but in this case the purpose was rather different as will be discussed later. A general characteristic found for all fits was that the predicted cross-section at the first diffraction peak is slightly greater than the data points give. A reason for this, suggested by Blair (Bl 68), was that this was probably due to experimental errors on the normalisation of the forward angle data.

Ti$^{50}$ 30.5 MeV

Again unlike the data for Ca$^{42}$ at 30.5 MeV we have here data showing a strong uniform diffraction pattern over the entire angular range. Reasonable quality fits could be obtained covering the entire angular range. As is to be expected, especially in a case such as this where the data covers a wide range (30° to 175°), improvements in $X^2$ can be made by restricting the range of data. In general, only data out to 110° or 130° was used. A second reason for the limitation on data was the amount the computer could handle conveniently. The values of $X^2$ are rather larger than in the previous cases but this is to be balanced against the fact that more data are being fitted. It was found that when the amount of data in the search was limited, the optical potentials found did not
necessarily give good agreement to data beyond the range used. This can be seen in figure 3 where potential 2 was obtained by fitting data only out as far as $110^\circ$, and does not reproduce the data over the final $20^\circ$ shown on the graph. Potential 4 was obtained using the data out to $130^\circ$ so that the data over the final $20^\circ$ are correctly reproduced.

$\text{Ti}^{50}_{44} \text{MeV}$

As can be seen from figure 4 good quality fits were found over the entire angular range. The fits with deep or shallow potentials are of basically the same quality. Again the six parameter fits are somewhat better than those of four parameters.

Comparing the four sets of optical model parameters we note that the values of the depth of the imaginary potential in the case of $\text{Ca}^{42}$ at 30.5 MeV tend to be larger than in the other cases. This can be seen most clearly by comparing those parameters for which $r = r'$ and $a = a'$. In this case there is an almost linear relation between $V$ and $W$. The magnitude of the wavefunction in the nuclear interior, in the case of $\text{Ca}^{42}$ at 30.5 MeV, can thus be expected to be smaller than the other three cases. This may have some influence on the inelastic scattering results. Where six parameters were free to vary it was found that $W$ was largely independent of $V$. It was possible to obtain fits using imaginary potentials of varying discrete depths by allowing small changes in the real well parameters. This can be seen best by comparing corresponding potentials of around 200 MeV depth for the four and six parameter cases.

Generally, it was found to be possible to obtain fits to the data with potentials as shallow as about 25 MeV. There appeared to be no clear cut upper limit to the possible potential depths but a general fall in quality as $V$ became very large.

The Igo criterion (Ig 58) that the nuclear surface region must be unchanged can be summarised by the expression
\[
\frac{(R_o - R)}{a} e^{-V_o R/a} \quad R \gg R_o
\]

which must be the same for all potentials, and similarly for the imaginary component. This criterion is normally reduced to

\[
V_o \frac{R_o/a}{a} = \text{constant} \quad \ldots (4.2)
\]

which is a valid statement of Igo's criterion only if the diffuseness parameter \( a \) is varied very little.

An alternative phenomenological description of elastic scattering is obtained through direct parameterisation of the phase shifts \( \delta_L \) or the reflection coefficients \( \eta_L = e^{2i\delta_L} \). A useful size parameter obtained from this form of analysis is the strong absorption radius \( R_{1/2} \) which is defined as \( R_{1/2} = 1.446 A^{1/3} + 2.29 \text{ fm} \) \ldots (4.3)

where \( n = \left( \frac{Z_1 Z_2 e^2 \mu}{\hbar^2 k} \right) \) is the Coulomb parameter and \( L_o \) is the angular momentum for which \( R_e(\eta_L) = \frac{1}{2} \), i.e. \( R_{1/2} \) is the classical turning point for a particle of orbital angular momentum \( L_o \). As pointed out by Alster and Conzett (Al 64) there is no reason to expect \( R_{1/2} \) obtained from a direct phase analysis to be equal to the value of \( R_o \) obtained from an optical model analysis of the same data, and in general it is found \( R_{1/2} > R_o \). A survey of the data over a range of incident energies has given the formula (Fr 63).

\[
R_{1/2} = 1.446 A^{1/3} + 2.29 \text{ fm} \quad \ldots (4.4)
\]

and a further survey at 44 MeV has given (Fa 68)

\[
R_{1/2} = 1.52 A^{1/3} + 2.14 \text{ fm} \quad \ldots (4.5)
\]

with individual nuclei showing substantial deviations from this standard formula. Recently Blair and Fernandez (Bl 67) have shown that it is possible to connect this method of phase shift analysis and optical model analysis by using an optical potential to generate the reflection coefficients and then defining the corresponding strong absorption radius, which we shall denote by \( R_{1/2}^{OM} \).
The results for $R_1^\frac{1}{2}$ and $R_1^{0M}$ obtained by Blair and Fernandez differ by less than 1%.

From the reflection coefficients $\eta_L$ calculated in the optical model analysis, values of $L_0$ (interpolated to non-integer values) and thus strong absorption radii $R_1^{0M}$ were calculated for some potentials. Also calculated were the values of the real and imaginary parts of the nuclear potential at the strong absorption radius $V(R_1^{0M})$ and $W(R_1^{0M})$, and the Igo terms $V^R_{o/a}$ and $W^R_{o/a}$. These are shown in table 2. Figures 5 to 7 give typical plots of the reflection coefficients as a function of $L$. Note the basic similarity in the plots of $\eta_L$ for all potentials corresponding to a given nucleus and energy. The actual variation of $\eta_L$ with $L$ is not a very smooth function. This effect was noted by McFadden and Satchler (Mc 66) who questioned the accuracy of smooth parameterization of $\eta_L$ as a function of $L$. In contrast, the plots of $1 - |\eta_L|^2$ also shown are much smoother varying functions of $L$. It may thus be more accurate, as suggested by Blair and Fernandez (Bl 67), to use an alternative definition of the strong absorption radius in terms of the halfway value of the function $1 - |\eta_L|^2$. The plots also help to emphasize the strength of the absorption. Note in all cases the reflection goes from almost complete absorption to complete reflection within about 8 partial waves.

For the first group of potentials for Ca$^{42}$ at 42 MeV given in table 2, the diffuseness parameters $a$ and $a'$ were kept close to 0.6 F. For these it can be seen that the Igo criterion is approximately satisfied for both the real and imaginary parts of the potential. The second group in table 2 corresponds to parameters where the diffuseness parameters were allowed to vary quite freely. In this case the Igo criterion, in the form of Eq. 2, cannot be satisfied because of the large variation in the diffuseness. On the other hand, the strong absorption radius, and the value of the real potential at the strong absorption radius are remarkably constant for all potentials. The same also holds for the very deep potential also
Table 4.2. Strong Absorption Radii and Igo Terms

<table>
<thead>
<tr>
<th>Potential</th>
<th>$V_o/a$ (MeV)</th>
<th>$W_o/a$ (MeV)</th>
<th>$R'/t$ (F)</th>
<th>$V(\frac{R}{\sqrt{2}})$ (MeV)</th>
<th>$W(\frac{R}{\sqrt{2}})$ (MeV)</th>
</tr>
</thead>
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<tr>
<td>Ca</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>1 4.72$x 10^5$</td>
<td>1.6$x 10^5$</td>
<td>7.323</td>
<td>- 2.34</td>
<td>- 0.83</td>
</tr>
<tr>
<td></td>
<td>2 4.98</td>
<td>1.61</td>
<td>7.313</td>
<td>- 2.44</td>
<td>- 0.74</td>
</tr>
<tr>
<td></td>
<td>3 5.6</td>
<td>1.85</td>
<td>7.318</td>
<td>- 2.44</td>
<td>- 0.72</td>
</tr>
<tr>
<td></td>
<td>4 6.27</td>
<td>1.84</td>
<td>7.327</td>
<td>- 2.45</td>
<td>- 0.66</td>
</tr>
<tr>
<td>MeV</td>
<td>5 2.3</td>
<td>11.68</td>
<td>7.332</td>
<td>- 2.58</td>
<td>- 0.93</td>
</tr>
<tr>
<td></td>
<td>6 1.33</td>
<td>3.55</td>
<td>7.324</td>
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<td>- 0.86</td>
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<td></td>
<td>7 0.44</td>
<td>10.43</td>
<td>7.367</td>
<td>- 2.63</td>
<td>- 0.84</td>
</tr>
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<td>8.13</td>
<td>7.377</td>
<td>- 2.55</td>
<td>- 1.0</td>
</tr>
<tr>
<td></td>
<td>9 5.3</td>
<td>1.5</td>
<td>7.367</td>
<td>- 2.62</td>
<td>- 0.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ca</td>
<td>52.3</td>
<td>3.55</td>
<td>7.601</td>
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<td>- 0.67</td>
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<td>42</td>
<td>48.74</td>
<td>0.83</td>
<td>7.788</td>
<td>- 1.87</td>
<td>- 0.16</td>
</tr>
<tr>
<td>MeV</td>
<td>44</td>
<td>2.54</td>
<td>7.616</td>
<td>- 2.51</td>
<td>- 0.51</td>
</tr>
<tr>
<td></td>
<td>3 8.0</td>
<td>1.81</td>
<td>7.594</td>
<td>- 2.52</td>
<td>- 0.56</td>
</tr>
</tbody>
</table>
Figures 4.5 to 4.7

Examples of the real and imaginary parts of the reflection coefficients $\eta_L$ and the absorption coefficients $1 - |\eta_L|^2$ given by some of the potentials in table 1.

Figure 4.5  Ca$^{42}$ at 30.5 MeV
Figure 4.6  Ca$^{42}$ at 42 MeV
Figure 4.7  Ti$^{50}$ at 30.5 MeV
Figure 4.5
Figure 4.6
Figure 4.7
quoted in the final section of table 2. The values of the imaginary potential at the strong absorption radius are small and approximately equal. The same conclusions also hold, for the results shown in table 2, for Ca at 30.5 MeV and for Ti at 30.5 and 44 MeV. A proposed criterion is thus that for equivalence of optical potentials for strongly absorbed projectiles, the real part of the potentials should be equal at the strong absorption radius.

A comparison of results for the strong absorption radii is given in table 3. For the purpose of comparison it must be noted that the formulae given by Eqs. 4 and 5 represent the general trend for a wide range of nuclei, but individual nuclei are known to show deviations from this trend. It must also be noted that the analysis of Blair and Fernandez has been carried out with much more accurate data than has hitherto been available. In these circumstances the agreement between the results for Ca at 42 MeV seem quite satisfactory, and so is the agreement for Ti. There is some discrepancy between the results obtained at 30.5 MeV and 42-44 MeV which might suggest some energy dependence (Ro) for the strong absorption radius. Lippincott and Bernstein (Li 67) in recent work on elastic alpha-scattering from Ca at 31 MeV quote an optical potential which fits their data. Using this, a value of the strong absorption radius was calculated and found to be in excellent agreement with the value quoted here at 30.5 MeV.

Table 4.3. Comparison of Results for the Strong Absorption Radii.

<table>
<thead>
<tr>
<th></th>
<th>This work E=30.5MeV</th>
<th>This work E=42-44MeV</th>
<th>Ref(Bl 67) E=42 MeV</th>
<th>Eq. 4</th>
<th>Eq. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ca</td>
<td>7.65 ± 0.03</td>
<td>7.35 ± 0.03</td>
<td>7.385</td>
<td>7.32</td>
<td>7.43</td>
</tr>
<tr>
<td>Ti</td>
<td>7.77 ± 0.03</td>
<td>7.61 ± 0.03</td>
<td>7.586</td>
<td>7.62</td>
<td>7.75</td>
</tr>
</tbody>
</table>
Also investigated is the nature of the optical model wavefunction in the vicinity of the nucleus. This is given by the usual formula,

$$\psi = \sum_{L} i^{L(2L+1)} J_{L}(kR) P_{L}(\cos \theta)$$

where \( J_{L}(kR) \) is the solution of the radial Schrödinger equation and \( \theta \) is the angle of scattering. By evaluating \( |\psi| \) as a function of \( R \) for \( \theta = 0^\circ \) and \( \theta = 180^\circ \) we obtain a picture of the variation of the modulus of the wavefunction along the z-axis, which is taken in the normal way to be along the direction of the incident beam.

Results for various potentials are shown in figure 8 and 9. From these results it can be seen that on the illuminated side of the nucleus (i.e. \( Z < 0 \)), the modulus of the wavefunctions given by equivalent potentials shows very similar behaviour, and in particular that the absorption effect on the wavefunction begins in the vicinity of the strong absorption radius. Thus we see that the strong absorption radius has a very real physical significance as a measure of the distance from the origin at which the process of absorption begins to be effective. On the dark side of the nucleus we note the now familiar focus (Au 61, Mc 59) is formed. As the strength of the real part of the potential is increased the focusing effect becomes stronger, i.e. the focus increases in intensity and moves farther inside the nucleus. Using the optical analogy this effect would correspond to increasing the refractive index. For the very deep potential (Ca^{42}, 42 MeV potential 9 of table 1) the focus moves still further inside the nucleus, but in this case the imaginary part of the potential is so large that the intensity on the dark side of the nucleus is drastically reduced. In the work of McCarthy (Mc 59) simple single foci were noted. In this work several potentials are far deeper than those considered by McCarthy, and in these cases the principle focus is associated with a series of secondary foci.

The separation of the secondary foci is about 0.6 times the wavelength of the optical model wavefunction in the nuclear interior.
Figures 4.8 and 4.9

The moduli of the optical model wavefunction for scattering obtained from some of the optical potentials given in table 1. The wavefunctions are calculated along the z-axis with the origin at the centre of the potential. The dotted lines indicate the position of the halfway radius of the potential.

Figure 4.8 \[ ^{42}\text{Ca} \] at 42 MeV
Figure 4.9 \[ ^{42}\text{Ca} \] and \[ ^{50}\text{Ti} \] at 30.5 MeV.
Figure 4.8
Figure 4.9
This indicates that the secondary foci are formed by the interference of the incident beam with that reflected back from the far surface of the potential as only for almost head-on beams is such a short separation of fringes possible. Only in the case of the exceptionally deep potential can it be said that the probability for the alpha-particle to penetrate into the interior of the nucleus is negligibly small. The behaviour of the wavefunction on the dark side of the nucleus does not indicate any special role for the strong absorption radius in this region.

It was suggested by Rawitscher (Ra 68) that a study should be made of the effective potential $V_{\text{eff}}$ for partial waves with a reflection coefficient of $\sqrt{0.5}$. It can be shown that for a potential barrier of parabolic shape, the transmission will be 0.5 if the height of the barrier equals the energy of the beam. It is thus interesting to see if the partial waves for which $|\eta_L|^2 = 0.5$ show a barrier in the corresponding effective potential of height equal to the energy of the beam.

$$V_{\text{eff}} = V_c + V_N + \frac{L(L+1)\hbar^2}{2\mu R^2}$$

$V_c$ = Coulomb potential

$V_N$ = Nuclear potential.

Values of $L$ corresponding to $|\eta_L|^2 = 0.5$ were found from the plots of $1 - |\eta_L|^2$ against $L$, and interpolated to non-integer values. Plots of $V_{\text{eff}}$ against $R$ are shown in figures 10 and 11 for several potentials for Ca$^{42}$ at 30.5 MeV and 42 MeV. For the above argument to be applicable we require that $V_{\text{eff}}$ must be roughly parabolic in form near the region of the peak. Also the imaginary component of the potential should be small outside and large enough inside the peak so that what is transmitted through the barrier is mostly absorbed and not retransmitted. The heights of the barriers should be compared to the relative energy of the alpha-particle and target nucleus in the centre of mass system.
Figures 4.10 and 4.11

The effective potential $V_{\text{eff}}$ for the partial waves for which $|\eta_L|^2 = 0.5$ are shown, obtained from some of the optical potentials in table 1. Also shown is the position of the strong absorption radius $R_{\frac{1}{2}}$ and the beam energy in the centre of mass system.

Figure 4.10  Ca$^{42}$  30.5 MeV
Figure 4.11  Ca$^{42}$  42 MeV
\[
E_{\text{LAB}} \times \frac{M}{M + m},
\]
which for Ca\textsuperscript{42} is

<table>
<thead>
<tr>
<th>E\textsubscript{LAB}</th>
<th>E\textsubscript{CM}</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.5 MeV</td>
<td>27.8 MeV</td>
</tr>
<tr>
<td>42</td>
<td>38.3</td>
</tr>
</tbody>
</table>

We see from figures 10 and 11 that \( V_{\text{eff}} \) does, with the exception of potential 1 of figure 11, show a peak. Potential 1, which is the shallowest potential, shows a plateau region where the others have peaks. The peaks fall about 1fm inside the strong absorption radius and are fairly constant in height. Their height however is about 1.5 MeV above the beam energy. Also the imaginary component of the potential is of the order of 1 MeV deep in this region so it cannot be considered entirely negligible. As a guide, however, one can expect a peak of height slightly greater than the energy of the beam, to be shown in a plot of \( V_{\text{eff}} \), for the partial wave where \( |\eta_l|^2 = 0.5 \).

4.3 Microscopic Analysis

The optical potential is obtained here by folding the alpha-nucleon interaction into the nuclear density distribution. For the form of the nuclear density we refer to the work of Elton, Swift and Shaw (El 67, Sw 66, Sh 65) on elastic electron scattering from the calcium isotopes. Their analysis of the isotopes Ca\textsuperscript{40, 44, 48} gave single particle Saxon-Woods potentials that yielded charge distributions, which fitted the elastic electron scattering data, and gave single-particle energies, for both protons and neutrons, in agreement with the known separation energies. For the nuclei Ca\textsuperscript{42} and Ti\textsuperscript{50} which concern us here, and for which single particle potentials are not available, we do as follows. For Ca\textsuperscript{42} we take the well parameters for Ca\textsuperscript{40} (Sh 65) and allow the potential depth to vary slightly in order to fit the separation energy for the 1f\textsubscript{7/2} neutrons. In this case all nucleons are fitted into the same potential well. In
the case of $^{T_{1}}^{50}$ we take the parameters for Ca$^{48}$ (Sw 66) and similarly allow the depth parameter to vary in order to fit the separation energy of the outer core $1f_{\frac{1}{2}}$ protons. The original parameters in this case for Ca$^{48}$ referred to energy dependent potentials. As in the present work the exact form of the nuclear density is not as critical as in the electron scattering case, we simplify the work slightly by fitting all nucleons into this one potential well.

The calculations were performed using a library computer programme for bound state wavefunctions due to Towner (To 66). Writing the potential in the form:

$$V(r) = -\frac{V_{0}}{r} f(r) - V_{so} \frac{1}{r} \frac{d}{dr} f(r) - \frac{\hbar^{2}}{m_{p}c^{2}} + V_{c}$$

where $f(r) = \left(1 + \exp\left(-\frac{r}{\alpha}\right)\right)^{-1}$

and $R = r_{o}A^{1/3}$.

$V_{c}$ is the Coulomb potential which for neutrons is of course zero, and for protons taken to be due to a uniformly charged sphere of radius $R$. The well parameters were found to be:

<table>
<thead>
<tr>
<th></th>
<th>Ca$^{42}$</th>
<th>$^{T_{1}}^{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{0}$</td>
<td>52.1</td>
<td>58.3 MeV</td>
</tr>
<tr>
<td>$r_{o}$</td>
<td>1.29</td>
<td>1.26 F</td>
</tr>
<tr>
<td>$a$</td>
<td>0.65</td>
<td>0.45 F</td>
</tr>
<tr>
<td>$V_{so}$</td>
<td>11</td>
<td>8.3 MeV</td>
</tr>
</tbody>
</table>

With the single particle wavefunctions corresponding to these well parameters the nuclear density distributions were calculated, using Eq. 3.35, to give those shown in figure 12. For Ca$^{42}$ the proton and neutron density distributions are very similar. The slight excess of neutrons in the tail is due to the extra core neutrons in the $1f_{\frac{1}{2}}$ state. For $^{T_{1}}^{50}$ the additional shell of neutrons shows itself by extending clearly beyond the proton distribution.

Optical potentials formed from these density distributions using Eq. 3.34 are shown in figures 13 and 14. In each case the
The figure shows for Ca\(^{42}\) and Ti\(^{50}\) the neutron, proton and total density distributions found from the shell model wavefunctions used in this work.

Figure 4.12
The real part of the optical potentials derived from the microscopic model. In each case \( V_o = 50 \text{ MeV} \), and the range parameter \( K \) varies from \( 0.435 \text{ F}^{-1} \) to \( 0.565 \text{ F}^{-1} \). Also shown for comparison is a Saxon-Woods phenomenological potential found from the previous section.

**Figure 4.13** Ca\(^{42}\): In addition the optical potential calculated analytically by Jackson (Ja 65) with \( K = 0.5 \text{ F}^{-1} \) and \( V_o = 50 \text{ MeV} \) for oscillator wavefunctions is shown.

**Figure 4.14** Ti\(^{50}\).
strength of the potential $V_0 = 50$ MeV and the range parameter $K = 0.435$ to $0.565 \text{ F}^{-1}$. Also shown is a typical Saxon-Woods optical potential found in the previous section. The logarithmic scale helps to emphasise the important surface region. The depth of the potentials so formed are in the region 230 to 460 MeV. In figure 13 the optical potential calculated analytically by Jackson (Ja 65) using oscillator wavefunctions with $V_0 = 50$ MeV and $K = 0.5 \text{ F}^{-1}$ is also shown. Note that the effect of using Saxon-Woods rather than oscillator wavefunctions is very small. In this case the oscillator potential extends slightly beyond that due to the Saxon-Woods wavefunctions. In the important surface region around the strong absorption radius the diffuseness does not change very much as a function of the range parameter $K$. Also the diffuseness is in good agreement with that of the Saxon-Woods optical potential. Comparing with the Saxon-Woods potential at the strong absorption radius, which is known to be important from the previous section, it is clear that the formed potentials will have to be adjusted in magnitude in order to fit the scattering data. As the range parameter $K$ makes little difference to the diffuseness in the surface region we did the following. The range parameter $K$ was fixed at $K = 0.5 \text{ F}^{-1}$ for all subsequent calculations. The strength parameter $V_0$ was allowed to vary in order to give best agreement to the scattering data. Also, as discussed in chapter 3, the imaginary component of the optical potential was made to have the same radial form as the real part with a strength $W_0$ which was free to vary.

Figure 15 shows a plot of the optical potential superimposed on the nuclear density distribution for Ca$^{42}$. Note that the strong absorption radius $R_1^2$ which is the sensitive region for alpha-particle scattering occurs far outside the nuclear matter distribution. We thus see the limitations that are to be expected on what information can be found from this type of analysis. The form of the optical potential at the strong absorption radius cannot reflect to any extent fine detail of the nuclear matter distribution. The close agreement
For Ca$^{42}$ is shown a plot of the microscopic model optical potential superimposed on the nuclear density distribution. The position of the strong absorption radius $R_1$, which is the sensitive region for elastic alpha-particle scattering can be seen to be well clear of the nuclear density distribution.
between the optical potentials derived from oscillator and Saxon-Wood wavefunctions helps to emphasis this fact. Also the similarity of the optical potentials for various range parameters shows the limitation of the analysis for determining the alpha-nucleon potential.

The effect of a density-dependent alpha-nucleon potential was also investigated as discussed in chapter 1. By simple parameterisation the alpha-nucleon potential is written in the form:

\[ V(d, r) = -V_o e^{-K^2 d^2} \left( 1 - k \frac{\rho(r)}{\rho(o)} \right) \]

\[ 0 \leq k \leq 1 \]

The parameter \( k \) has the effect of damping out contributions from the nuclear interior. Figure 16 shows how this influences the form of the resulting optical potential. All potentials shown have the same value of \( V_o \). As is to be expected the effect is greatest in the centre of the nucleus, the tail being only slightly reduced in magnitude with increasing \( k \).

These potentials were used in a modified version of the optical model search code to calculate the scattering they produced. The potential was characterised by just two parameters \( V_o \) and \( W_o \), which were automatically adjusted to give the best fit to the data. The Coulomb potential was again taken to be that due to a uniformly charged sphere, in each case with \( r_c = 1.4 \, F \).

Table 4 gives the values of the parameters which give the best agreement with the elastic scattering data. Corresponding examples of the quality of the fits are shown in figures 17 to 20. In all cases the results are satisfactory considering the small number of parameters involved. In the case of \( \text{Ca}^{42} \) at 42 MeV the values of \( \chi^2 \) are about the same as those found in the phenomenological analysis. For the other three cases \( \chi^2 \) is typically about twice the corresponding phenomenological values. The least satisfactory in appearance is that for \( \text{T}_1^{50} \) at 30.5 MeV but here the data covers a far larger angular range than the other cases. The quality of the fits is not to any extent sensitive to the damping parameter \( k \). The values of \( V_o \) and \( W_o \) rise somewhat with increasing \( k \), as is to be
The figure shows the influence on the microscopic optical potential of the parameter $k$. The calculation is for Ca$^{42}$. In each case $V_0 = 50$ MeV and $K = 0.5 F^{-1}$. The logarithmic scale helps to show the behaviour in the surface region more clearly.
Table 4.4 Microscopic Model, Alpha-Nucleon Potential Parameters.

In each case $K = 0.5 \text{ F}^{-1}$

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<tr>
<th></th>
<th>$k$</th>
<th>$V_0$ (MeV)</th>
<th>$W_0$ (MeV)</th>
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<th>NORM</th>
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<td>0.9</td>
<td>46.4</td>
<td>18.5</td>
<td>5.7</td>
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</tr>
</tbody>
</table>
Examples of cross-sections for elastic scattering obtained with optical potentials derived from the microscopic model. The parameters of the potentials are given in table 4.

Figure 4.17  Ca$^{42}$ at 30.5 MeV
Figure 4.18  Ca$^{42}$ at 42 MeV
Figure 4.19  Ti$^{50}$ at 30.5 MeV
Figure 4.20  Ti$^{50}$ at 44 MeV
Figure 4.18

$42^{\text{Ca}}$ 42 MeV

$\frac{d\sigma}{d\omega}$ (mb/sr)

$\Theta_{\text{cm}}$ (degrees)

$k = 0.6$

$k = 0$
Figure 4.19
Figure 4.20
expected, so that the surface of the optical potential remains approximately constant. The strong absorption radius for these potentials with $k = 0$ together with the value of the real and imaginary components at this radius are given in table 5. Comparing with the corresponding results for the phenomenological analysis we see that in all cases the difference in the strong absorption radii is less than 1%. The values of the real part of the potential at the strong absorption radius are also in very good agreement. The imaginary parts of the potential are not in such good agreement, the Saxon-Woods case being generally a little smaller.

Table 4.5 Microscopic Model Strong Absorption Radii. The values correspond to the case for which $k = 0$.

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$R_1/2$ (F)</th>
<th>$V_1/2$ (MeV)</th>
<th>$W_1/2$ (MeV)</th>
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<tr>
<td>Ca$^{42}$</td>
<td>30.5</td>
<td>7.59</td>
<td>-1.92</td>
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<tr>
<td></td>
<td>42</td>
<td>7.34</td>
<td>-2.3</td>
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<tr>
<td>Ti$^{50}$</td>
<td>30.5</td>
<td>7.78</td>
<td>-1.83</td>
</tr>
<tr>
<td></td>
<td>44</td>
<td>7.61</td>
<td>-2.47</td>
</tr>
</tbody>
</table>

We see from the results that the model is capable of producing optical potentials which fit the scattering data with alpha-nucleon potentials which are reasonably similar to those found from free proton-alpha elastic scattering. In each case the value of $V_0$ was around 40 MeV. The potentials also produced strong absorption radii in agreement with those of the phenomenological potentials and were also very similar in the important surface region. However, we have also found that the model was not sensitive enough to distinguish between the Saxon-Woods and Oscillator density distributions. Indeed, the calculations employing a density dependent alpha-nucleon potential could alternatively be looked upon as changes in the density distribution. In these cases small changes in the potential parameters sufficed to give optical potentials which gave satisfactory fits to the
data. It was also found that the results were not very sensitive to the
form of the alpha-nucleon potential over the range considered here.
Similar calculations by Jackson and Kembhavi (Ja 69) on the calcium
isotopes however, took as the form of the alpha-nucleon interaction
a Yukawa potential. The long tail of this potential when folded into
the nuclear density distribution produced optical potentials too
diffuse and hence unable to give rise to the required sharp diffraction
pattern in the differential cross-sections. Reducing the range
of the potential led to sharper diffraction patterns but gave incorrect
spacing of peaks. We thus see that the form of the alpha-nucleon
interaction needs to be chosen with some care.
CHAPTER 5

Inelastic Scattering Analysis

In the introduction the scattering problems to be examined were defined. In chapters 2 and 3 the equations which offer the solutions to these scattering problems were given. For the most part the construction of a computer programme to perform the necessary calculations is a straightforward, though time consuming process.

The scattering code used is based on a coupled channels programme written and developed by Wills (Wi 63). As the original code was written in Fortran and as no facilities were available for running or developing in this language, the programme was translated into Algol. The original code was a straightforward rotational and vibrational model programme. Besides producing a translation equivalent to the original, by modification to a few subroutines, a second programme was developed which could handle the inelastic scattering for the microscopic model, either in part where the optical potential was taken as a phenomenological form, or in full where the microscopic model is used entirely. The numerical details involved are given by Wills (Wi 63) and will not be discussed here.

As a test on the precision of the programme, it was run with the same data as used by Wills and produced identical results. The original Wills code has itself been extensively tested by the author. Comparisons were also made with another coupled channels code written by Hill (At). The two codes were found to be in excellent agreement. Because the code of Hill is available as a library programme at the Atlas computer laboratory, it was used for some of the rotational model calculations to be found in the following sections. The predicted elastic channel cross-sections were also found to be consistent with those found using optical model programmes. The programme had the facility that it could also be used as a DWBA code.
A separate shorter programme was written to evaluate the effective interaction or form factor of the microscopic model. It was arranged that the form factors were output from the programme on punched paper tape in a form suitable as data for the coupled channels code. A check on this programme was made by comparing with the effective interactions calculated analytically by Jackson (Ja 65) using oscillator wavefunctions.

The following section gives a review of the data on inelastic alpha-particle scattering used in this analysis. A section on the rotational model is included. The results of the microscopic analysis are presented in section 3. It falls broadly into three parts. The first deals with the form of the inelastic form factors and the effect of the inclusion of configuration mixture states. The second part is concerned with the scattering that results when a phenomenological optical model potential is used and the final part, when a full microscopic model description of the scattering is used. The chapter ends with a summary of the results found.

5.1 Experimental Data

Gruhn and Wall (Gr 65) measured the inelastic alpha-particle scattering to the lowest lying $2^+$ state for both Ca$^{42}$ and Ti$^{50}$. The laboratory energy was 30.5 MeV and the angular range of the measurements being in the case of Ca$^{42}$ 48° to 106° and for Ti$^{50}$ 30° to 77.5°. The experimental uncertainties quoted for Ca$^{42}$ vary from about 8% for small angles increasing to around 15% for large angles. For Ti$^{50}$ the uncertainties are about 5% at small angles, around 15% in the middle region and as much as 35% at large angles. In the case of Ca$^{42}$, as already discussed in chapter 4, the data are quoted in arbitrary units. However, the results for the elastic and inelastic scattering are consistent relative to each other and thus the absolute magnitude of the cross-sections can be found by multiplying by the normalisation factor found in the optical model analysis. It does, however, introduce an additional uncertainty into the analysis.
Peterson (Pe 66) measured the differential cross-section for inelastic scattering to many low lying states in Ca\textsuperscript{42}. The ones of interest here are the lowest lying 2\textsuperscript{+}, 4\textsuperscript{+} and 6\textsuperscript{+} states. The bombarding energy was 42 MeV and the angular range 14\textdegree{} to 70\textdegree{}. In the case of the 2\textsuperscript{+} state, the experimental uncertainty was mostly less than 10\%. For the 4\textsuperscript{+} state, the uncertainties were somewhat larger than for the 2\textsuperscript{+}, and for the 6\textsuperscript{+}, where the magnitude of the cross-section was down by an order of magnitude, the uncertainties were a good deal larger, of the order of 30\%.

From the results given by Bruge (Br 67) of scattering by Ti\textsuperscript{50} at 44 MeV, the states of interest are the first 2\textsuperscript{+} and 4\textsuperscript{+}. The angular range of measurements of differential cross-section is 11\textdegree{} to 55\textdegree{}. For the 2\textsuperscript{+} state, the experimental uncertainties quoted are all less than 5\% and for the 4\textsuperscript{+} state less than about 10\%.

5.2 Rotational Model

In this section are presented the results obtained for the analysis of Ca\textsuperscript{42} and Ti\textsuperscript{50} in terms of the rotational model. The results form a useful comparison with other work on inelastic alpha-particle scattering and also inelastic proton scattering. The results are also useful to compare with the result obtained from the microscopic model regarding quality of fits.

Figures 1 to 4 show typical differential cross-sections obtained for a selection of the optical model potentials found in the previous chapter for Ca\textsuperscript{42} and Ti\textsuperscript{50} at the various incident energies. Table 1 gives corresponding values of the deformation parameters $\beta$ needed to fit the data. Following Blair (Bl 60), also shown is the deformation length $\beta R_0$, the value of which remains more nearly constant for the various optical potentials. In this definition $R_0 = r_0 A^{1/3}$, the halfway radius of the real part of the optical potential. It is to be remembered however, that the coupling also contains a contribution from the imaginary part of the potential. These results use a coupled channels formulation. Recalling that the optical
Table 5.1. Rotational Model Deformation Parameters.
Coupled Channels Calculations.

<table>
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<tr>
<th>Potential</th>
<th>$V_0$ (MeV)</th>
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<th>$\beta_2^R \sigma$ (F)</th>
<th>$\beta_4$</th>
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</tr>
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<td>46.1</td>
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<td>0.78</td>
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<td></td>
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Cross-sections for inelastic scattering obtained by the rotational model. The potentials are based upon the optical potentials of table 4.1 with deformation parameters given in table 5.1. The calculations use a coupled channels formulation.

**Figures 5.1 to 5.4**

- **Figure 5.1**  Ca$^{42}$ at 30.5 MeV, $2^+$
- **Figure 5.2**  Ca$^{42}$ at 42 MeV, $2^+, 4^+$
- **Figure 5.3**  Ti$^{50}$ at 30.5 MeV, $2^+$
- **Figure 5.4**  Ti$^{50}$ at 44 MeV, $2^+, 4^+$
Figure 5.1
Figure 5.2

\[ 4^{2}\text{Ca} \quad 42 \text{ MeV} \]

- \( V = 212.3 \text{ MeV} \) (Pot. 4)
- \( V = 112.1 \text{ MeV} \) (Pot. 11)

\[ \frac{d\sigma}{d\Omega} \quad (\text{mb/sr}) \]

\( \theta_{\text{cm}} \) (degrees)

2+

4\(^+\) \((x_{10})\)
Figure 5.3

\[ \frac{d\sigma}{d\Omega} \] (mb/sr)

\( \Theta_{\text{c.m.}} \) (degrees)

- --- Potential
- - - -

\( ^{50}\text{Ti} \) 30.5 MeV

2^+
Figure 5.4
potentials found in the previous chapter are not strictly applicable to a coupled channels calculation, but only to DWBA, we expect the effect of coupling to have some influence on the elastic scattering cross-section results. In general, it was found that in this work the extent of the coupling made little difference to the elastic scattering cross-sections. So small, in fact, that it sufficed to leave the optical model parameters unchanged. This is not so surprising when one considers that for the cases in question, the contribution to the reaction cross-section for excitation of the $2^+$ state was of the order of 10 mb whereas the total cross-section for elastic scattering was about 1500 mb. Here the excited channel coupling constitutes a small perturbation on the elastic channel. Such a situation is certainly not general, Wills (Wi 63) for example, quotes cases where the effect of coupling is such as to necessitate modification of the optical potential in order to retain a good fit to the elastic scattering data.

We will now consider each of the four cases in turn.

\[ \text{Ca}^{42} \text{ 30.5 MeV} \text{ (2}^+ \text{ state)} \]

The data in this case are confined to the first $2^+$ state only. Also, it is to be remembered that in this case the data are quoted in arbitrary units and multiplication by a normalisation factor found in the elastic scattering analysis is necessary. It was found that generally the first two diffraction peaks could be fitted, that is out to $73^\circ$, the quality of fit beyond this point is then very poor. Only scattering cross-sections out to $73^\circ$ are shown in figure 1, (in the elastic scattering analysis, only data out to $68^\circ$ could be fitted).

There was very little difference found in the cross-sections produced by the various optical potentials covering a wide range of potential depths. The value of the deformation parameter $\beta_2$ was of the order 0.25, the deeper potentials tending to need a somewhat larger value than the shallow ones. The quantity $\beta_2 R_0$ also shown in table 1 was more nearly constant for the various potentials at about 1.35 F.
The data in this case are for the $2^+$, $4^+$ and $6^+$ states. The quality of fit to the $2^+$ state was generally quite good. There is a slight tendency to the fits being slightly high for the peak at about $22^\circ$ and a bit low for the peak at $57^\circ$. In the case of potential 4 the $4^+$ state is also included producing a reasonable fit to the data with $\beta_4 = 0.049$. In this case the inclusion of the $4^+$ state made little difference to the cross-section for the $2^+$ state, there being a slight increase in magnitude at the large angle end of the data. Further runs in which the $6^+$ state was also coupled were not possible due to the very large computing time that would have been involved. The values of $\beta_2$ and $\beta_2 R_0$ found, are shown in table 1. It can be seen that the value of $\beta_2 R_0$ remains fairly constant for all the potentials but is substantially removed from the value obtained from the analysis at 30.5 MeV. This will be dealt with in more detail later.

Here the data are for the $2^+$ state only. As for Ca$^{42}$ the choice of optical potential made little difference to the quality of fit and all produced consistent values of $\beta_2 R_0$ as shown in table 1. The calculated cross-sections generally gave consistent fits to the three peaks of the data at $39^\circ$, $53^\circ$ and $67^\circ$. On the other hand the minimum at $47^\circ$ was always much deeper than that given by the data and the final minimum at $74^\circ$ always predicted to be shallower than the data. Beyond the angular range of the data, the calculated cross-sections due to the various potentials tended to diverge somewhat from each other.

In this case we have the most accurate experimental data of the four cases considered. For potentials 1, 3, 4 and 6 there was a slight tendency, in the case of the $2^+$ state, for the fitted cross section to be a little high at large angles when the forward angle cross-sections fitted. Potential 5 on the other hand produced an
excellent fit over the entire angular range. For potential 6 the
$4^+$ state was also included in the coupling, the effect on the $2^+$ state
cross-section was found to be small. The fit to the $4^+$ state is fair,
being somewhat low at the first peak and somewhat high for the final
peak of the data. Again all potentials produced consistent deforma-
tion parameters but again there was a substantial deviation from the
results at 30.5 MeV.

Table 2 gives a comparison of the deformation parameters
found in this work with those found by other workers in this field.
Peterson (Pe 66) performed the first analysis of the Ca$^{42}$ data, at
42 MeV, used here. Lippincott and Bernstein (Li 67) studied inelas-
tic alpha-particle scattering from Ca$^{42}$ at 31 MeV and Satchler,
Yntema and Broek (Sa 64, Yn 67) studied Ti$^{50}$ at an energy of 43 MeV.

Table 5.2 Comparison of Results of Deformation Parameters

<table>
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<th>Reference</th>
<th>$E_{LAB}$ (MeV)</th>
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<td>$(\alpha, \alpha')$</td>
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<tr>
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<td>0.049</td>
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<tr>
<td>Li 67</td>
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<td>0.067</td>
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<tr>
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<td>0.71</td>
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<td>Ba 68</td>
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<td>0.105</td>
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<td></td>
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</tr>
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<td>$(\alpha, \alpha')$</td>
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<tr>
<td>This work</td>
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<td>0.78</td>
<td>0.067</td>
<td>&quot;</td>
</tr>
<tr>
<td>Yn 67</td>
<td>43</td>
<td>0.136</td>
<td>0.74</td>
<td>0.073</td>
<td>&quot;</td>
</tr>
<tr>
<td>Sa 64</td>
<td>43</td>
<td>0.15</td>
<td></td>
<td></td>
<td>&quot;</td>
</tr>
<tr>
<td>Fu 64</td>
<td>17.45</td>
<td>0.15</td>
<td>0.69</td>
<td>0.11</td>
<td>$(p, p')$</td>
</tr>
<tr>
<td>Gra 65</td>
<td>18.2</td>
<td>0.15</td>
<td>0.69</td>
<td></td>
<td>&quot;</td>
</tr>
</tbody>
</table>

Also shown are inelastic proton scattering results of Bane et. al.
(Ba 68) on Ca$^{42}$ at 22.9 MeV and for Ti$^{50}$ work of Funsten et. al.
(Fu 64) at 17.45 MeV and Gray et. al. (Gra 65) at 18.2 MeV. Note
that in this analysis on Ca$^{42}$ at 42 MeV the predicted value of
$\beta_2 R_0$ is 0.78 F, whereas in Peterson's analysis the value 0.71 F
was found. The discrepancy can be explained by the way the fit to the data was decided. Peterson fitted the magnitude of the cross-section to the data at the first peak at 22° whereas in this analysis the magnitude of $\beta_2^m$ was chosen so as to give the best overall fit to the data. In this way it was found that the fitted cross-section was always high at this first maximum, by an amount which is sufficient to explain the discrepancy. It can be seen that the results for Ca$^{42}$ and Ti$^{50}$ at 30.5 MeV are substantially different from the general trend of deformation parameters. For Ti$^{50}$ the predicted deformation parameters are low, almost by a factor of 2, whereas for Ca$^{42}$ the values are very large. Comparing the results for Ca$^{42}$ with those of Lippincott and Bernstein (Li 67), which were found on the same accelerator at almost the same energy but with much improved resolution, there are clearly large discrepancies.

Leaving aside the results at 30.5 MeV, from the remainder, for Ca$^{42}$ we see the $2^+$ state deformation length centred about $\beta_2 R_o = 0.83 \text{ fm}$ with a scatter of about 35%. For Ti$^{50}$ the value is $\beta_2 R_o = 0.72 \text{ fm}$ with a smaller scatter.

5.3 Microscopic Model

The shell model wave-functions used were as described in chapter 4, based upon a Saxon-Woods potential well. The parameters of the well gave rise to a charge density distribution which produced elastic electron scattering in agreement with experiment and also fitted the known separation energies (El 67, Sw 66, Sh 65). The inelastic scattering form factors or effective interactions resulting from such a microscopic model have been discussed by several authors (Ma 65, Sa 66, Pe 66). Such form factors are generally broader than their collective model counterparts (which are derivatives of the optical potential). They extend into the nuclear interior and out beyond the nuclear surface. Also they are not positive definite and may fluctuate in sign. They peak close to the peaks of the constituent wave-functions which are well inside the
optical potential radius. Because the integral for the formation of the form factors Eq. 3.28 is weighted by a factor $r^I$, the form factors for transitions to states of high angular momentum peak farther out than those of low I value. Figure 5 shows examples of form factors for various configurations calculated for the ground to $2^+$ transition in Ca$^{42}$. Also shown for comparison is a typical derivative Saxon-Woods term used in the rotational model analysis.

Figure 6 shows examples of the form factors for transitions from the ground to $2^+$, $4^+$ and $6^+$ states for Ca$^{42}$.

Law (La 68) produced, for Ca$^{42}$, wave-functions including configuration mixing for various residual two-body interactions. He retained the Ca$^{40}$ core as inert and solved for the configuration states of the two extra core neutrons. He found he could fit the energy levels for the ground and lowest lying $2^+$, $4^+$ and $6^+$ states but was unable to produce the extra low lying $0^+$ and $2^+$ states anywhere near the known excitation energies. He concluded, with reference also to work on transition rates, that the extra $0^+$ and $2^+$ states could not be explained without core excitation. Table 3 shows his wavefunctions for the ground, $2^+$, $4^+$ and $6^+$ states with a Tabakin potential (Ta 64) as the residual interaction, which is typical of all the potentials he considered. Note that for all states the $(1f_{\frac{5}{2}})^2$ configuration is by far the most dominant one.

Figure 7 shows the form factor for the ground to $2^+$ state transition for these configuration mixture states as compared to the pure $(1f_{\frac{5}{2}})^2$ states. It was found in this analysis, with Saxon-Woods shell model wavefunctions, that the $2d_{5/2}$ state was not bound (as against Law who used oscillator wavefunctions). It was thus necessary to delete those configurations from Law's wavefunction involving the $2d_{5/2}$ state and make appropriate re-normalisation to the remaining configurations. In all cases the $2d_{5/2}$ terms were very small. As can be clearly seen from figure 7, the difference between the simple and configuration mixed form factors is very small and thus can be expected, as indeed was found, to give rise to very similar scattering.
Figure 5.5  Examples of microscopic model form factors for the transition from the ground to $2^+$ state in Ca$^{42}$ for various configurations. Also shown, for comparison, is an example of a derivative Saxon-Woods potential (which constitutes the form factor for the collective model).

Figure 5.6  The figure shows for Ca$^{42}$, microscopic model form factors for the transitions from the ground state to final states $2^+$, $4^+$ and $6^+$. The form factors which are due to the $(1f_{7/2})^2$ configuration are shown to arbitrary scale.
Form Factors (arbitrary units)

\[ \text{Ca}^{42} \]
\[ O^+ \rightarrow 2^+ \]

\[ \langle 1f_{7/2}, 0 \rangle \rightarrow \langle 1f_{7/2}, 2 \rangle \]
\[ \langle 1f_{5/2}, 0 \rangle \rightarrow \langle 1f_{5/2}, 2 \rangle \]
\[ \langle 2p_{3/2}, 0 \rangle \rightarrow \langle 2p_{3/2}, 2 \rangle \]
\[ \langle 1f_{7/2}, 0 \rangle \rightarrow \langle 1f_{7/2} 2p_{3/2}, 2 \rangle \]

○ ○ ○ ○ derivative Saxon-Woods

Figure 5.5
Figure 5.6
Table 5.3  Configuration Mixture Coefficients for Ca\(^{42}\) (La 68)

Employing a Tabakin residual interaction.

<table>
<thead>
<tr>
<th>State</th>
<th>Configuration Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground State</td>
<td>( \phi_0(12) = 0.9597 (1f_{\frac{3}{2}})^2 + 0.1657 (2p_{3/2})^2 + 0.0925 (2P_{\frac{1}{2}})^2 - 0.0663 (1g_{9/2})^2 - 0.0273 (2d_{5/2})^2 + 0.1943(1f_{5/2})^2 )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \phi_2(12) = 0.9518 (1f_{\frac{3}{2}})^2 + 0.272 (1f_{\frac{3}{2}} 2p_{3/2}) - 0.0423 (1f_{\frac{3}{2}} 1f_{5/2}) + 0.061 (2p_{3/2})^2 - 0.0641(2p_{3/2} 2P_{\frac{1}{2}}) - 0.0321 (2p_{3/2} 1f_{5/2}) + 0.0523(2p_{\frac{1}{2}} 1f_{5/2}) - 0.0543 (1g_{9/2})^2 - 0.0269(1g_{9/2} 2d_{5/2})^2 - 0.0136(2d_{5/2})^2 + 0.0529(1f_{5/2})^2 )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \phi_4(12) = 0.9873 (1f_{\frac{3}{2}})^2 + 0.1106 (1f_{\frac{3}{2}} 2p_{3/2}) - 0.0669 (1f_{\frac{3}{2}} 2p_{1/2}) - 0.0667 (1f_{\frac{3}{2}} 1f_{5/2}) - 0.0529 (2p_{3/2} 1f_{5/2}) - 0.0211 (1g_{9/2})^2 - 0.008 (1g_{9/2} 2d_{5/2}) - 0.0049 (2d_{5/2})^2 + 0.0265 (1f_{5/2})^2 )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \phi_6(12) = 0.9941 (1f_{\frac{3}{2}})^2 - 0.1079 (1f_{\frac{3}{2}} 1f_{5/2}) - 0.0113 (1g_{9/2})^2 - 0.0021 (1g_{9/2} 2d_{5/2}) )</td>
</tr>
</tbody>
</table>
Figure 5.7  Microscopic model form factors for the transition from the ground to $2^+$ state in Ca$^{42}$. The figure shows the form factor calculated using Law's (La 68) generalised configuration mixture wavefunctions as compared to the pure $(1f_{7/2})^2$ configuration.
The same also holds true for the \(4^+\) and \(6^+\) states. For this reason the use of configuration mixture states was not pursued.

Blair (Bl 59) showed that for strongly absorbed particles such as alpha-particles, where the interaction is confined to the nuclear surface, the general characteristics of scattering cross-sections can be predicted from a knowledge of the spin and parity of the final state. The exact shape of the form factor is known to have little effect on the form of the scattering cross-section, only its magnitude, which determines the magnitude of the cross-section, is important (Ba 62). Bassel et al (Ba 62) showed that the position of the form factor could be moved radially by several fermi, with little effect upon the resulting cross-section, provided one keeps it clear of the nuclear interior. Only Madsen and Tobocman (Ma 65) have found evidence that the shape of the cross-section is at all sensitive to the shape of the form factor. They investigated the influence of variations to both the nuclear shell model well parameters and also the range of the alpha-nucleon potential. They found no substantial sensitivity to the former, for reasonable values of the well parameters, but did find that the alpha-nucleon potential range had some influence on the frequency of the diffraction pattern. They showed this to be due to the large \(r\) behaviour of the form factors. Their wavefunctions did, however, include considerable collective correlations which probably influenced the sensitivity to the shape of the form factors. In the present analysis variation of the range parameter of the alpha-nucleon potential from \(K = 0.9435 \text{ F}^{-1}\) to \(K = 0.565 \text{ F}^{-1}\) had little, if any, effect upon the shape of the resulting inelastic cross-sections. For this reason the bulk of calculations were done with \(K = 0.5 \text{ F}^{-1}\). The strength \(V_0\) was considered as a free parameter to be adjusted in order to fit the absolute magnitude of the scattering cross-sections.

As already discussed, the two approaches used here are as follows. Firstly the inelastic form factors are used in conjunction with a phenomenological optical potential; in this way one can be
confident that at least the elastic scattering is well described. This method also allows a comparison to be made of the effect that the various optical potentials have on the scattering. In each case ranges of optical potentials are employed that vary in depth from around 50 MeV to 200 MeV. Secondly the optical potentials are themselves derived using the microscopic model as described in chapter 4. This allows one to see if it is possible to obtain a consistent description of elastic and inelastic scattering. In this case the strength $V_o$ of the alpha-nucleon interaction needed for the elastic scattering optical potential and that for the inelastic form factors are considered to be independent. A comparison between the coupled channels and DWBA formulations was also made.

5.3.1. Inelastic scattering employing a phenomenological optical potential

All the results in this section are from DWBA calculations. Figures 8 to 11 show examples of the cross-sections for inelastic scattering due to various optical potentials, chosen to cover a wide range of depths. Table 4 gives the corresponding values of the alpha-nucleon potential strength $V_o$ needed in order to reproduce the absolute magnitude of the experimental data. As the results use a DWBA formulation, the various excited states are considered independent and thus the corresponding values of $V_o$ needed for each state can also be chosen independently.

The four cases of data analysed will now be discussed in turn.

$^{42} \text{Ca}$ 30.5 MeV. (2$^+$ state)

As for the case of the rotational model, satisfactory fits to the experimental data could only be found for scattering angles less than 75°. The quality of fit found was very much the same for all three optical potentials, and all needed similar alpha-nucleon potential strengths of around 300 MeV, which is far in excess of the free alpha-nucleon potential of about 50 MeV. Compared to the results
Figure 5.4  Alpha-nucleon potential strengths ($V_0$) needed with a phenomenological optical potential. The calculations are DWBA.

<table>
<thead>
<tr>
<th>Potential</th>
<th>$V$ (MeV)</th>
<th>$V_0^{(2^+)}$ (MeV)</th>
<th>$V_0^{(4^+)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ca$_{42}$</td>
<td>4</td>
<td>55.1</td>
<td>300</td>
</tr>
<tr>
<td>30.5 MeV</td>
<td>5</td>
<td>130.7</td>
<td>300</td>
</tr>
<tr>
<td>MeV</td>
<td>6</td>
<td>217.2</td>
<td>300</td>
</tr>
<tr>
<td>Ca$_{42}$</td>
<td>10</td>
<td>46.1</td>
<td>120</td>
</tr>
<tr>
<td>42 MeV</td>
<td>11</td>
<td>112.1</td>
<td>160</td>
</tr>
<tr>
<td>MeV</td>
<td>12</td>
<td>187.7</td>
<td>150</td>
</tr>
<tr>
<td>Ti$_{50}$</td>
<td>5</td>
<td>61.1</td>
<td>100</td>
</tr>
<tr>
<td>30.5 MeV</td>
<td>6</td>
<td>100.0</td>
<td>100</td>
</tr>
<tr>
<td>MeV</td>
<td>7</td>
<td>241.7</td>
<td>100</td>
</tr>
<tr>
<td>Ti$_{50}$</td>
<td>4</td>
<td>42.2</td>
<td>210</td>
</tr>
<tr>
<td>44 MeV</td>
<td>5</td>
<td>88.5</td>
<td>220</td>
</tr>
<tr>
<td>MeV</td>
<td>6</td>
<td>207.2</td>
<td>205</td>
</tr>
</tbody>
</table>

found in the rotational model analysis the quality of fit was very much the same.

Ca$^{42}$ 42 MeV ($2^+$, $4^+$ states)

The two deeper optical potentials gave good fits to the $2^+$ state scattering data, as can be seen from figure 9, almost as good in fact, as was found in the rotational model analysis. The shallower potential on the other hand gave a very poor fit. Where the alphaparticle potential strength was chosen to fit the peak at $22^\circ$, the magnitude of the cross-section predicted at subsequent maxima is far too low. All three potentials gave fits of similar quality to the
Figures 5.8 to 5.11

Cross-sections for inelastic scattering with microscopic model form factors and phenomenological optical potentials. The optical potentials are chosen to cover a wide range of depths. The calculations are all DWBA.

Figure 5.8  \( {\text{Ca}}^{42} \) at 30.5 MeV, \( 2^+ \)
Figure 5.9  \( {\text{Ca}}^{42} \) at 42 MeV, \( 2^+, 4^+ \)
Figure 5.10  \( {\text{Ti}}^{50} \) at 30.5 MeV, \( 2^+ \)
Figure 5.11  \( {\text{Ti}}^{50} \) at 44 MeV, \( 2^+, 4^+ \)
Figure 5.8

\[ \frac{d\sigma}{d\Omega} \] (mb/sr)

\( \theta_{\text{c.m.}} \) (degrees)

Ca\(^{42} \) 30.5 MeV

--- potential 4
--- ... 5
--- ... 6
Figure 5.9

The graph shows the differential cross section $d\sigma/d\Omega$ (in mb/sr) as a function of the center-of-mass angle $\theta_{cm}$ (in degrees) for $^{42}\text{Ca}$ at 42 MeV. The curves represent different potentials:

- $V=187.7$ MeV (Pot. 12)
- $V=112.1$ MeV (Pot. 11)
- $V=46.2$ MeV (Pot. 10)

The data points are shown with markers, and the theoretical curves are depicted with lines. The graph includes levels $2^+$ and $4^+$ (at $x/10$).
Figure 5.10

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta_{\text{c.m.}}$ (degrees)

$^{50}\text{Ti}$ 30.5 MeV

Potential

$\cdot\cdot\cdot\cdot$ 5
$\cdot\cdot\cdot\cdot$ 6
$\cdot\cdot\cdot\cdot$ 7

$2^+$

$10^{-1}$
$10^{-2}$
Figure 5.11
$4^+$ state scattering data, which were almost of the same standard as those due to the rotational model. In all cases an inconsistency exists between the $2^+$ and $4^+$ states as to the strength of the alpha-nucleon potential needed. For the $2^+$ state this is always close to 150 MeV while for the $4^+$ the value is about 100 MeV. Both are far removed from the free alpha-nucleon strength and also that found from the analysis at 30.5 MeV.

$^{40}_30.5\text{ MeV (2+ state)}$

Here, as in the case of $^{42}_5$ at 42 MeV, the quality of the fit to the $2^+$ state due to the shallower optical potential is very poor. Where the magnitude has been chosen to fit the data at forward angles it is too large at back angles. The intermediate optical potential, of depth 100 MeV, produced a moderate fit. The best fit was due to the deepest potential, the quality of which was comparable with those of the rotational model. In each case the strength the alpha-nucleon potential needed was around 100 MeV.

$^{40}_5$ 44 MeV (2+, 4+ states)

Again the shallow optical potential gives a poor fit to the $2^+$ state data, as shown in figure 11. Choosing the strength of $V_0$ such that the cross-section fitted the experimental data at the first maxima, the cross-section subsequently predicted is far too low except for the final maximum, for which it is far too large. The intermediate potential produced a reasonable fit to the data and the deep potential a good fit. The shallow potential also gave a poor fit to the $4^+$ state, being too large at back angles. The two deeper potentials each gave reasonable fits to the $4^+$ state data. The quality of the better fits is quite comparable with those due to the rotational model. In this case the strength of $V_0$ needed is consistent for the $2^+$ and $4^+$ states, in each case being about 200 MeV.

5.3.2 Full Microscopic Model

Here we use as the optical potentials those of the microscopic
model found in chapter 4. Figures 12 to 16 give examples of the inelastic scattering cross-sections found and table 5 gives a summary of the alpha-nucleon potential strengths required. Both coupled channels and DWBA calculations are presented. Each of the four cases will again be considered in turn.

**Table 5.5  Alpha-Nucleon potential strengths for full microscopic calculation.**

<table>
<thead>
<tr>
<th></th>
<th>$V$ (Optical Potential)</th>
<th>$k$</th>
<th>$V_o(2^+)$ (MeV)</th>
<th>$V_o(4^+)$ (MeV)</th>
<th>$V_o(6^+)$ (MeV)</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{42}$Ca</td>
<td>43.6</td>
<td>0</td>
<td>315</td>
<td></td>
<td></td>
<td>DWBA</td>
</tr>
<tr>
<td>30.5 MeV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{42}$Ca</td>
<td>35.6</td>
<td>0</td>
<td>140</td>
<td>90</td>
<td>~140</td>
<td>DWBA</td>
</tr>
<tr>
<td>35.6 MeV</td>
<td></td>
<td></td>
<td></td>
<td>(90)*</td>
<td></td>
<td>CC</td>
</tr>
<tr>
<td>$^{50}$Ti</td>
<td>34.9</td>
<td>0</td>
<td>110</td>
<td></td>
<td></td>
<td>DWBA</td>
</tr>
<tr>
<td>30.5 MeV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>CC</td>
</tr>
<tr>
<td>$^{50}$Ti</td>
<td>36.3</td>
<td>0</td>
<td>190</td>
<td>190</td>
<td></td>
<td>DWBA</td>
</tr>
<tr>
<td>44 MeV</td>
<td></td>
<td>0.6</td>
<td>190</td>
<td>190</td>
<td></td>
<td>CC</td>
</tr>
</tbody>
</table>

+ CC = coupled channels

* Renormalised value needed to fit the data.

As can be seen in figure 12 the quality of fit to the data is about the same as was found in the previous sections using a phenomenological optical potential. The strength of the alpha-nucleon potential $V_o$ is also about the same as previously found, being about 316 MeV. This particular result shown is due to a DWBA calculation.
Figures 5.12 to 5.16

Cross-sections for inelastic scattering for the fully microscopic calculation. Examples are shown of both coupled channels and DWBA calculations.

**Figure 5.12**  
Ca$^{42}$ at 30.5 MeV, $2^+$

**Figure 5.13**  
Ca$^{42}$ at 42 MeV, $2^+, 4^+$

**Figure 5.14**  
Ca$^{42}$ at 42 MeV, $6^+$

**Figure 5.15**  
Ti$^{50}$ at 30.5 MeV, $2^+$

**Figure 5.16**  
Ti$^{50}$ at 44 MeV, $2^+, 4^+$. 
$^{42}$Ca $^{30.5}$ MeV

DWBA

$K = 0.5 \text{ F}^{-1}$

$k = 0$

$d\sigma/d\Omega$ (mb/sr)

Figure 5.12
Figure 5.13

$^{42}\text{Ca}$

- DWBA
- Coupled Channels, $0^+, 2^+, 4^+$

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta_{cm}$ (degrees)

Figure 5.13
Figure 5.14
Figure 5.15
Figure 5.16
Ca$^{42}$ 42 MeV (2$^+$, 4$^+$ and 6$^+$ states).

In the case of the DWBA calculation shown in figure 13, the quality of fit for both the 2$^+$ and 4$^+$ states is almost as good as for the deep phenomenological optical potential, and needs about the same alpha-nucleon potential strengths, the 2$^+$ and 4$^+$ strengths again being inconsistent. Also shown are the coupled channels calculations where only the ground and 2$^+$ states are coupled and that where the ground, 2$^+$ and 4$^+$ states are coupled. In the case of the 2$^+$ state, the three cross-sections shown are all very similar. The DWBA and two level coupled channels cross-sections are almost identical. The three level calculation shows a slightly more pronounced diffraction pattern and a slight shift towards the small angle end of the scale. Due to the inconsistency in the required strength of $V_0$ needed to fit the 2$^+$ and 4$^+$ levels, the cross-section shown for the 4$^+$ state from the three level coupled channels calculation is multiplied by a normalisation constant to bring it into line with the DWBA result. Comparing the predicted cross-sections for the 4$^+$ state due to the DWBA and coupled channels methods it can be seen the coupled channels cross-section falls off more slowly with increasing angle. With regard to the 6$^+$ level, shown in figure 14, the fit is poor and the experimental errors large. The cross-section falls off much too rapidly with the angle. The strength $V_0$ needed is of the same order as for the 2$^+$ state.

Ti$^{50}$ 30.5 MeV (2$^+$ state)

As shown in figure 15, both DWBA and coupled channels methods produce almost identical cross-sections, which are of a quality comparable with those of the rotational model. The DWBA cross-section tended to be a little greater in magnitude at the large angle end of the data. Both calculations need about the same alpha-nucleon potential strength, $V_0$, of about 110 MeV.
Here, with the most accurate data, we also get the best fits to the cross-sections. We have consistency between the $2^+$ and $4^+$ states in magnitude of cross-sections with an alpha-nucleon potential strength of 190 MeV. The coupled channels and DWBA results are illustrated in figure 16. The DWBA, the two level and also the three level coupled channels calculations produce very similar fits to the $2^+$ state experimental data. Also, both the DWBA and coupled channels methods produce very similar results for the $4^+$ state. The quality of fits is quite comparable with those obtained by the rotational model. Also shown is a coupled channels calculation of the case where the alpha-nucleon interaction was made density dependent with $\kappa = 0.6$, as described in chapter 4. As can be seen the quality of fit so produced is much the same as in the other cases.

5.4 Summary

In general it was found that the differences between the cross-sections due to the coupled channels and DWBA calculations were quite small. The use of coupled channels made little difference to the form of the elastic scattering cross-sections, and the inclusion of the $4^+$ state had little effect upon the cross-section of the $2^+$ state.

The DWBA method has the advantage that the computer time needed to evaluate cross-sections is more or less linear with the number of states included. On the other hand with coupled channels the computing time increases very rapidly with the number of coupled states. For example, on the Atlas computer, a two level calculation takes of the order of one minute and a three level calculation twenty minutes. This represents the upper limit possible with the available system. A four level calculation would have run for many hours and was out of the question. The experience gained in this work suggests that the use of coupled channels calculations
should be confined to those cases where the coupling between states is known to be important.

It was found that the choice of a phenomenological optical potential had little influence upon the strength of the alpha-nucleon potential needed, although very definite evidence was found to suggest that the deeper potentials are more satisfactory in regard to the shape of the resulting cross-sections. Only in the case of Ca\textsuperscript{42} at 30.5 MeV was there no improvement in quality of fit in going from a shallow to a deep potential, but in this case it is to be remembered that only a small range of the available data is being fitted. The superiority of the deeper potentials is encouraging as these correspond more closely to the microscopic model optical potentials, and may in fact be used as evidence in helping to reduce the ambiguity in the choice of optical potential.

The fully microscopic description of the nucleus has been shown to be capable of producing cross-section predictions, for both the elastic and inelastic scattering, of a quality comparable with that of the collective model.

The values of $V_\theta$ are, as one would expect, more or less proportional to the corresponding deformation parameters found in the rotational model analysis.

The strength of $V_\theta$ was in all cases far stronger than the free alpha-nucleon value of about 50 MeV. It follows that it was not found possible to give a consistent description of both elastic and inelastic scattering with the same alpha-nucleon interaction for both.

The older data analysed, that at 30.5 MeV for Ca\textsuperscript{42} and Ti\textsuperscript{50} suggested big differences between the two nuclei in regard to deformation parameters $\beta_2$, which is not found in the later higher energy data, or in the general trend of values shown in table 2 found by other workers. For this reason results obtained with this early data should not be given much weight. We are left with the fact that an enhancement, for both Ca\textsuperscript{42} and Ti\textsuperscript{50} in the alpha-
nucleon potential, of about three to four times the free value is required.

In the microscopic description of Ti\(^{50}\) it was found that with a suitable choice of the alpha-nucleon potential, consistency between the \(2^+\) and \(4^+\) states existed. This gives some encouragement to the model of treating the nucleus as a closed core of Ca\(^{48}\) plus two extra core protons. It was found for Ca\(^{42}\) however that it was not possible to describe both the \(2^+\) and \(4^+\) states consistently in this manner, a different value of \(V_o\) being needed for each. This suggests the situation for this nucleus, as expected, is more complex than the simple model used. The inclusion of configuration mixing of the two extra core neutrons had little effect, although more realistic wave-functions, including core excitation may well improve the situation.

The factor three to four in the alpha-nucleon potential strength needed is in line with many of the other results published in this field and described in the introductory chapter. Only Madsen and Tobocman (Ma 65) have produced results in which a realistic alpha-nucleon potential strength has been used. In their work on Ni\(^{58}\), the nuclear wavefunctions included considerable configuration mixing, they found the various terms gave constructive interference in the tails of the wavefunctions leading to a strong surface peaked form factor.
CHAPTER 6

Summary and Conclusions

In this section the results which were given or implied in the previous chapters are briefly restated in order to see what conclusions can be drawn. Towards the end of the section some suggestions are made regarding possible extensions to this work.

From the phenomenological optical model analysis the importance of the role played by the strong absorption radius was shown. It was seen that all optical potentials which gave satisfactory agreement with the experimental data gave rise to more or less identical strong absorption radii. It was also shown that the potential at this radius was remarkably constant and that this formed the basis of a criterion for optical potentials which was more general than the Igo criterion (which held only for small changes in diffuseness).

From the microscopic description of elastic scattering it was shown that with an alpha-nucleon interaction similar to the free form, the model led to optical potentials which agreed with the phenomenological potentials in the important surface region. The quality of the fits to the experimental cross-sections were remarkably good considering the small number of parameters involved. The strong absorption radii were very close to the corresponding values found from the conventional analysis and the potentials at these radii were also very similar. However it was also shown in chapter 4 that the model allowed a considerable latitude in the form of the alpha-nucleon potential and also in the nuclear density distribution. Changes made in one parameter could be compensated by changes elsewhere with little influence on the resultant scattering. In particular it was shown that the contribution to the optical potential from the nuclear interior could be modified considerably. This large latitude in the choice of the various parameters is because the sensitive region for scattering is around the strong absorption radius, which is well clear of the nuclear density distribution. Any reasonable
form for the nuclear density which is broadly correct in the surface region is likely to lead to a satisfactory optical potential. The model in its present form will therefore not yield new information on nuclear density distributions. This position will remain unchanged unless substantial improvements are made in determining the form of the two-body interaction. With such an improvement the model may well prove a useful way of studying the density distribution of the nuclear surface.

It was shown that the microscopic description of inelastic scattering could yield fits to the scattering data comparable with those found by the rotational model. It was encouraging to find that when used in conjunction with phenomenological optical potentials, the deeper ones, which resembled more closely those found by the microscopic approach, proved to be the more satisfactory. This fact can be used as evidence in helping to decide which of the many phenomenological potentials should be used in calculations. The major problem found was that a large enhancement of about 3 to 4 times the free value of the alpha-nucleon interaction was needed in order to fit the magnitude of the scattering data. This prevented any idea of producing a consistent description of elastic and inelastic scattering. Such an enhancement has been found necessary by most other workers in this field and unless a satisfactory explanation is found it will seriously limit the usefulness of the approach. The model may, however, be used to predict the relative strengths of scattering to the various final states. Being able to consistently describe the scattering for many states would give confidence in the description of the nucleus. It was shown that a consistent description of scattering to the $2^+$ and $4^+$ states in Ti$^{50}$ was possible whereas in Ca$^{42}$ this was not so. Thus some additional evidence was found to support the belief that Ti$^{50}$ is closer to the simple shell model idea of a core plus two nucleons than is Ca$^{42}$. The inelastic scattering cross-sections were not found to be very sensitive to the range of the alpha-nucleon potential. For alpha-
particles which are strongly absorbed, the scattering cross-sections can largely be predicted from the diffraction model, the form factor for the most part determining the magnitude only. The use of protons as the projectile in place of alpha-particles may prove a more sensitive means of studying structure as being less strongly absorbed are therefore more strongly influenced by the nuclear interior. This should however be weighed against the additional complexity in the formalism for protons, as spin would be involved, and also the fact that cross-sections for proton scattering generally show less structural detail than in the alpha-particle case.

In these calculations it was found that the simpler DWBA was as satisfactory as the coupled channels method and as long as the coupling between excited states is weak this former method allows simpler and quicker calculations.

It would be interesting if a calculation for Ca$^{42}$ were performed with more realistic wavefunctions, including core excitation, to see if it is then possible to obtain a consistent description of scattering to the various states. It would also be interesting if a calculation were done, involving several nuclei, having simple well understood structure, each with experimental data covering scattering to several states to see how far the idea of predicting the relative strengths of scattering is useful.
APPENDIX 3.1

Nuclear Transition Density

The nuclear transition density is given by Eq. 3.12 as

\[ \rho_{IM}(r) = 2 \left[ \int \frac{\phi_{IM}^{*}(12)}{I'M'} \phi_{I'M'}(12) \, dr \right] \]

The form of the wavefunctions for the extra core nucleons \( \phi_{IM}(12) \) is given by either Eq. 3.14 or Eq. 3.15 depending as to whether or not the two nucleons have the same \( n\ell j \) quantum numbers. We shall consider here the more general case where both initial and final states are given by Eq. 3.15, that is

\[ \phi_{IM}(12) = \frac{1}{\sqrt{2}} \sum \sum_{vwxy} \sum_{mm'} C(jj'I, mm'M'C(\ell s_j, vw m)C(\ell' s_j', x y m')) \]

\[ \phi_{I'M'}(12) = \frac{1}{\sqrt{2}} \sum \sum_{vwxy} \sum_{m'm} C(jj''I, m m'M'C(\ell s_j, \bar{v}w \bar{m})C(\ell'' s_j', \bar{x}y m')) \]

The algebra for the other cases where the initial, final or both states are given by Eq. 3.14 is exactly the same, the answer differing only in the normalisation factor associated with the wavefunctions. We will also take \( n'\ell''j'' \neq n''\ell'j' \) to simplify matters slightly. For the case where \( n'\ell'j' = n''\ell''j'' \) a second term occurs of the same form with the roles of the quantum numbers reversed.

Substituting the form of the wavefunctions into the above equation for \( \rho(r) \) we see that of the four terms in the integral, three will vanish due to orthogonality. For the remaining term, on integrating and summing over spin coordinates, we get terms:
\( \delta(v, v'); \delta(w, w'); \delta(y, y') \)

Summing over \( v \) and \( w \) gives

\[
\rho_{IM}(r) = \sum_{xy} C(jj'I, mm'M) C(l'sj', xym') \]

Writing \( \psi(r) = R(r) Y(\Omega r) \), we may make use of the coupling rule for spherical harmonics (Ro 63, page 61).

\[
\rho_{IM}(r) = R(r) R(r) \sum_{n'l' n''l''} C(jj'I, mm'M) C(l'sj', xym')
\]

From the vector coupling coefficients we see that for non-zero terms

\[
m + m' = M; x + y = m'; m + \tilde{m}' = M'; \tilde{x} + y = \tilde{m}'
\]

Solving we find \( x - \tilde{x} = M - M' \) which is substituted into the spherical harmonic. With slight rearrangement of the terms we can proceed to replace the summations over the magnetic quantum numbers using the summation properties of the Racah coefficients (Ro 63, page 110) to give:

\[
\rho_{IM}(r) = R(r) R(r)(-1)^{j'-s-I'} \sum_{q} Y_{q}^{*}(\Omega r)
\]

Which is the required form.
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III. Analysis of Elastic and Inelastic Scattering
from $^{42}$Ca and $^{50}$Ti

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Abstract

A combined analysis is made of elastic and inelastic scattering of medium energy α-particles from $^{42}$Ca and $^{50}$Ti. This is first done in terms of the generalized optical model. A conventional DWBA analysis of the inelastic scattering is then made using phenomenological optical potentials to generate the distorted waves but using a microscopic description of the nuclear excitation constructed from an effective two-body interaction and shell model wavefunctions for the nucleus. Finally, a fully microscopic calculation is made to give a consistent description of the elastic and inelastic scattering, and it is shown that this model is capable of producing results which are comparable, in terms of quality of fit to the data, to those given by the generalized optical model.

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1. Introduction

In two previous papers\textsuperscript{1,2} we have studied the elastic scattering of medium energy α particles using a phenomenological optical potential (paper I) and a microscopic model for the optical potential (paper II). We have now extended these calculations to give a combined analysis of elastic and inelastic scattering from two selected nuclei, \(^{42}\text{Ca}\) and \(^{50}\text{Ti}\).

We first use the generalized optical model and make a coupled channels analysis of the elastic and inelastic scattering. This allows us to compare the predictions given for inelastic scattering by the various optical potentials obtained in I, and by expressing the nuclear matrix elements in terms of the rotational model we can extract values for the deformation parameters \(\beta_2\) and the products \(\beta_2 R_0\) which can be compared with values obtained from other studies of the same nuclei. We then make a DWBA analysis of the inelastic scattering using the phenomenological optical potentials to generate the distorted waves but using a microscopic description of the nuclear excitation in terms of an effective two-body interaction and shell model wavefunctions for the nucleus. Finally, we make a fully microscopic calculation to give a consistent description of the elastic and inelastic scattering. This is done in both the DWBA and coupled channels approximations.

For the microscopic calculations the nuclear wavefunctions are constructed from the shell model configurations of the two extra-core nucleons, i.e. two neutrons outside \(^{40}\text{Ca}\) and two protons outside \(^{48}\text{Ca}\). It was hoped that this would be a reasonably satisfactory procedure for \(^{50}\text{Ti}\) but was not expected to be satisfactory for \(^{42}\text{Ca}\) in which core excitation is known to be important. The second 0\(^+\) and 2\(^+\) states in \(^{42}\text{Ca}\) have not been included. The effective interaction for inelastic scattering in
DWBA or the coupling potentials for the coupled channels calculation are then constructed as the matrix elements of the effective two-body (α-nucleon) interaction between the appropriate nuclear states, i.e.

\[ U_{nn',}(\mathbf{R}) = \int \rho_{nn',}(\mathbf{r}) V_{\text{eff}}(|\mathbf{r} - \mathbf{R}|) d\mathbf{r} \]

where \( \rho_{nn',} \) is the transition density

\[ \rho_{nn',}(\mathbf{r}) = \langle n' | \sum_{i=1}^{A} \delta(\mathbf{r} - \mathbf{r}_i) | n \rangle . \]

\( V_{\text{eff}} \) is taken to be of gaussian form, instead of the single Yukawa form used in II, and the parameters are determined by fitting the elastic scattering using the optical potential \( U_{\text{oo}} \). The ground state density \( \rho_{oo} \) and the transition densities are calculated from single-particle wavefunctions generated in Saxon-Woods potentials. These wavefunctions have the correct separation energies, as far as these are known, and the proton distributions yield agreement with elastic electron scattering.
2. Generalized optical model

The coupled channel calculations were carried out using a computer code based on the one developed by Wills and also using the Atlas library code developed by Hill$^3$. These codes gave identical results for the same input data and could be used to describe the coupling of up to three levels. We used the rotational model of Wills$^3$, in which the generalized optical potential is expanded in Legendre polynomials. This causes the automatic inclusion of certain higher order terms in the coupling. However, our purpose in these calculations was not to study the relevance of this model but rather to compare the predictions, within the framework of this model, given by the various optical potentials obtained in I, and to obtain fits to the data which could be compared in terms of quality of fit to those obtained by the microscopic method.

Peterson$^4$ has measured the cross-sections for excitation of the $2^+$ and $4^+$ states in Ca at 42 MeV. A variety of parameters for optical potentials of Saxon-Woods form which had previously been obtained from the analysis$^1,5$ of the elastic scattering data alone, were used and gave a generally good fit to the data for the $2^+$ state. Inclusion of the $4^+$ state in the coupled channels calculation made little difference to the cross-section calculated for the $2^+$ state. For $^{50}$Ti, the cross-sections for excitation of the $2^+$ and $4^+$ states have been measured by Bruge et al$^6$ at 44 MeV. In this case also, the inclusion of the $4^+$ state had little effect on the cross-sections for the $2^+$ state. Typical results for the cross-sections are shown in Figures 1 and 2 and the values of $\beta_2$, $\beta_4$ and $\beta_2R_0$ are given in Table I.

It can be seen from Table I that the different optical potentials produce consistent results for $\beta_2$ and $\beta_2R_0$. In Table II we give the values of these parameters found in analyses of other data for the same nuclei, and from these it appears that our results at 42-44 MeV are in satisfactory agreement with all other analyses.
3. Elastic scattering in the microscopic model

In this work the $\alpha$-nucleon interaction is taken to be of gaussian form, i.e.

$$V_{\text{eff}}(r - R) = -V_0 e^{-K^2(r - R)^2}.$$ 

A preliminary estimate of the parameters $V_0$ and $K$ was obtained by requiring that the calculated optical potential has the same shape as the phenomenological optical potential in the important surface region. This gave $V_0 \approx 50$ MeV, $K = 0.5$ F$^{-1}$. It was also found that, in the surface region, variations in $K$ changed the magnitude of the potential but had little effect on the shape. For this reason the parameter $K$ was fixed at $0.5$ F$^{-1}$. The calculated potential $U_{oo}(R)$ was then used in an ordinary optical model search code. The imaginary part of the potential was taken to have the same radial behaviour, and strength determined by an additional parameter $W_0$. The search code then determined the values of the two parameters $V_0$ and $W_0$ which yield a best fit to each set of elastic scattering data. The calculated cross-sections are compared with the data in Figures 3 and 4, and the relevant parameters are given in Table III. Comparison with the results given in I shows that, the quality of fit is comparable to that obtained using a phenomenological potential with many more parameters. The values of the strong absorption radii $R_1$ and the magnitudes of the real and imaginary parts of the potential at the strong absorption radii, $V(R_1)$ and $W(R_1)$, are identical to those obtained in I.

The optical potential has also been calculated with a density-dependent interaction using the expression

$$U_{oo}(R) = \int \rho_{oo}(r) V_{\text{eff}}(r - R)\left\{1 - k \rho_{oo}(r)/\rho_{oo}(0)\right\} dr,$$

where $\rho_{oo}(r)$ is the nuclear density in the ground state and $V_{\text{eff}}$ is still of
gaussian form. The effect of varying the parameter \( k \) between zero and unity is to reduce the contribution from the inner region of the nucleus. However, in order to fit the data the parameters \( V_0, W_0 \) must be increased as \( k \) is increased so that the potential remains the same in the surface region, as can be seen from the potentials shown in Figure 5 and the results are given in Table IV. Some fits to the data with \( k \neq 0 \) are shown in Figures 3 and 4. In contrast, comparison with the results obtained in II using a Yukawa interaction confirms the conclusion reached in II that the form of \( V_{\text{eff}} \) is of crucial importance for a satisfactory fit to the data. The gaussian interaction used here is sufficiently sharply peaked so that it produces an optical potential with the required behaviour in the surface region whereas the Yukawa interaction produced a potential which was too diffuse and hence did not reproduce the pronounced diffraction minima of the elastic scattering data. By folding a gaussian nucleon-nucleon potential into the nuclear density of the \( \alpha \)-particle, Bernstein\(^{14} \) has derived a gaussian \( \alpha \)-nucleon potential with parameters \( V_0 \) and \( K \) which are very close to those determined in this analysis. It is encouraging that there is reasonable agreement between the two sets of parameters since this suggests that there is internal consistency in the microscopic description, but close agreement is probably fortuitous in view of the number of approximations made in both calculations.
4. Inelastic scattering in the microscopic model

The gaussian α-nucleon interaction was also used to construct the effective interaction for excitation of low-lying states in \(^{42}\)Ca and \(^{50}\)Ti. It was found that the effect of configuration mixing of the two extra-core nucleons is negligible and so the nuclear wavefunctions were constructed from the \((1f_7^7)^2\) configuration. Figure 6 shows the behaviour of the radial parts, or form factors, of the effective interactions for the excitation of the \(2^+, 4^+\) and \(6^+\) states in \(^{42}\)Ca. The magnitude of the effective interactions should be fixed by the strength \(V_0\) of the α-nucleon potential which has been determined from the analysis of the elastic scattering data described in section 3. We preferred, however, to treat \(V_0\) as a free parameter in the inelastic scattering and then to use the comparison of the values of \(V_0\) required as a test of the microscopic model.

In the first set of DWBA calculations the distorted waves were generated in the phenomenological optical potentials of Saxon-Woods form which were previously used in the calculations with the generalized optical model described in section 2. For \(^{42}\)Ca at 42 MeV the two deeper optical potentials used gave good fits to the inelastic cross-section for the \(2^+\) state, as can be seen from Figure 7, but the shallow potential gave a very poor fit. All three potentials gave good fits to the inelastic cross-section for the \(4^+\) state, but the values of \(V_0\) required to fit the magnitude of this cross-section differ from the values required to fit the \(2^+\) state or the elastic scattering. These values are given in Table V. For \(^{50}\)Ti at 44 MeV, the deep potential again gives a good fit to the \(2^+\) cross-section while the shallow potential gives a poor fit, as can be seen from Figure 8. The intermediate and the deepest potential both yield reasonable fits to the \(4^+\) cross-section, and in this case the same value of \(V_0\) is required for the \(2^+\) and \(4^+\) states.
Finally, the full microscopic model was used for both DWBA and coupled channels calculations. In the latter, the diagonal coupling potentials were calculated using the effective $\alpha$-nucleon interaction obtained by fitting the elastic scattering data, but the strength of the $\alpha$-nucleon interaction used to calculate the off-diagonal terms was allowed to vary. In a coupled channels calculation with two levels included this procedure leads to two different values of $V_o$ which are associated in Table VI with the $0^+$ and $2^+$ states for ease of comparison with the DWBA results. When three levels are included the same procedure is followed and the strength $V_o$ used for the diagonal terms is associated with the $2^+$ state. In the case of the three level calculation for $^{42}$Ca it was necessary to re-normalise the predicted $4^+$ cross-section to bring it into agreement with the data and the DWBA calculation, and the product of the re-normalisation factor and the strength $V_o$ has been given in Table VI as the strength $V_o$ for the $4^+$ state. In all cases the coupling potentials are real.

For $^{42}$Ca at 42 MeV the DWBA calculations yield fits to the $2^+$ and $4^+$ cross-sections which are quite comparable with those obtained in section 2, as can be seen from Figure 9, but the fit to the $6^+$ cross-section is poor. The values of $V_o$ required are in agreement with those required in the previous DWBA analysis with the deep optical potential but are, unfortunately, not constant for all the levels considered. The two level coupled channels calculation yields almost identical results to the DWBA calculation; there is a slight change in the results obtained when three levels are included and it is necessary to re-normalise the $4^+$ cross-section to obtain a magnitude fit to the data. For $^{50}$Ti at 44 MeV, the DWBA and coupled channels calculations yield good fits to the data, as shown in Figure 10, and the value of $V_o$ used for the off-diagonal potentials is constant, as it should be. One coupled channels calculation was made with the density dependent effective interaction described in section 3, and it was found that the quality of fit and the values of $V_o$ required were comparable with the other calculations.
5. Discussion

We conclude from these results that the full microscopic model is capable of producing cross-sections for elastic and inelastic scattering of medium energy $\alpha$-particles which are comparable in terms of quality of fit to those produced by the generalized optical model. When the microscopic model is used for inelastic scattering only, by far the best results are obtained when the deep optical potentials with $V \approx 200$ MeV are used. Since these deep potentials correspond closely to the potentials calculated from the microscopic model, we take this as evidence for the importance of a consistent treatment of elastic and inelastic scattering.

The description we have given for the two nuclei $^{42}$Ca and $^{50}$Ti in terms of the shell model without core excitation is evidently inadequate and this has shown up in the variations of the $\alpha$-nucleon strength parameters listed in Tables V and VI. It is reassuring, however, that these parameters show a more uniform behaviour for $^{50}$Ti, where we expect the nuclear model to be less defective, than for $^{42}$Ca. Within the framework of this nuclear model the effect of coupling is rather small, so that any uncertainties in our treatment of the coupling should not affect the general conclusions.
References and footnotes


   (unpublished). More recent measurements indicate that the
   normalization of Peterson's data may be about 10% low
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8. J. S. Bane, J. J. Kraushaar, B. W. Ridley, and M. M. Stautberg,


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<th>V (Mev)</th>
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Potential parameters are given in references 1 and 5. Details of the deformation parameters obtained for 42Ca and 50Ti. Full details of the

Table I.
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<th>$g^*$</th>
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Table II. Comparison of results for the deformation parameters for $^{42}$Ca and $^{54}$Ti.
Table III. Results obtained for elastic scattering using the calculated potentials with parameters $k = 0.5, F = 1, \xi = 0$.
Table IV. Parameters of the optical potentials for 42Ca at 42 MeV.

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Table 1: Two-body potential strengths required for the microscopic DWIA calculation with phenomenological optical potentials. The potential numbers refer to the optical potentials which are the same as those listed in Table I.
Table VI. Two-body potentials strengths required for the fully microscopic calculation.

<table>
<thead>
<tr>
<th>Method</th>
<th>Λ(ΛN)</th>
<th>Λ(ΔN)</th>
<th>Λ(ΛN)</th>
<th>Λ(ΔN)</th>
<th>Λ(ΛN)</th>
<th>Λ(ΔN)</th>
<th>Λ(ΛN)</th>
<th>Λ(ΔN)</th>
</tr>
</thead>
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<tr>
<td>Coupled Channels</td>
<td>0.6</td>
<td>42.5</td>
<td>0</td>
<td>190</td>
<td>0</td>
<td>190</td>
<td>190</td>
<td>190</td>
</tr>
<tr>
<td>Coupled Channels</td>
<td>0</td>
<td>190</td>
<td>190</td>
<td>36.3</td>
<td>0</td>
<td>190</td>
<td>0</td>
<td>190</td>
</tr>
<tr>
<td>DMBA</td>
<td>44</td>
<td>42</td>
<td>42</td>
<td>42</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The values quoted for the 0+ ground state are those obtained from elastic scattering and listed in Table III.
Captions for the figures

Figure 1. The cross-sections for inelastic scattering from $^{42}$Ca at 42 MeV obtained using the generalized optical model. The parameters are given in Table I.

Figure 2. The cross-sections for inelastic scattering from $^{50}$Ti at 44 MeV obtained using the generalized optical model. The parameters are given in Table I.

Figure 3. The cross-section for elastic scattering from $^{42}$Ca at 42 MeV obtained using the microscopic model. The parameters are given in Tables III and IV.

Figure 4. The cross-section for elastic scattering from $^{50}$Ti at 44 MeV obtained using the microscopic model. The parameters are given in Table III.

Figure 5. The optical potentials for $^{42}$Ca calculated from the microscopic model, and used to generate the cross-sections shown in Figure 3.

Figure 6. The form factors of the effective interaction for the excitation of the lowest $2^+$, $4^+$ and $6^+$ states in $^{42}$Ca calculated from the microscopic model.

Figure 7. The cross-sections for inelastic scattering from $^{42}$Ca at 42 MeV obtained using the microscopic model in DWBA with phenomenological distorting potentials. The parameters are given in Table V.
Figure 8. The cross-section for inelastic scattering from $^{50}$Ti at 44 MeV obtained using the microscopic model in DWBA with phenomenological distorting potentials. The parameters are given in Table V.

Figure 9. The cross-sections for inelastic scattering from $^{42}$Ca at 42 MeV obtained using the full microscopic model. The parameters are given in Table VI.

Figure 10. The cross-sections for inelastic scattering from $^{50}$Ti at 44 MeV obtained using the full microscopic model. The parameters are given in Table VI.
$^{42}_{\text{Ca}}$

- $V = 212.3 \text{ MeV}$
- $V = 112.1 \text{ MeV}$

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta_{cm}$ (degrees)
$^{50}$Ti

$V = 207.2$ MeV

$V = 47.6$ MeV

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta_{cm}$ (degrees)
The graph shows the angular distribution of the cross-section in the c.m. system for the reaction $^{50}_{\text{Ti}}$. The data points are plotted for two different values of $k$: $k=0.6$ and $k=0$. The $y$-axis represents the differential cross-section $\frac{d\sigma}{d\Omega}$ in mb/sr (meters squared per steradian). The $x$-axis represents the center-of-mass angle $\Theta_{cm}$ in degrees, ranging from 0 to 60 degrees.
$^{42}$Ca

--- $k=0$ $V_0 = 35.6$ MeV

--- $k=0.6$ $V_0 = 43.5$ MeV

× Saxon Woods potential 4

$-\text{Re}U_{oo}(R)$ (MeV)

0  2  4  6  8  10  12  14  16  18  20
$R$ (F)

$R_{1/2}$
\[ \frac{d\sigma}{d\Omega} (\text{m b/sr}) \]

\[ \theta_{\text{cm}} \text{ (degrees)} \]

- \( V = 207.2 \text{ MeV} \)
- \( V = 88.5 \text{ MeV} \)
- \( V = 42.2 \text{ MeV} \)

\( ^{50}\text{Ti} \)

\( 2^+ \)

\( 4^+ (\times \frac{1}{10}) \)
$^{42}\text{Ca}$

- **DWBA**
- **Coupled Channels**

$\frac{d\sigma}{d\Omega} \text{ (mb/sr)}$

$\theta_{cm} \text{ (degrees)}$

$2^+$

$4^+ (\times \frac{1}{10})$
50\textsuperscript{Ti}

- \( k=0 \) DWBA
- \( k=0.6 \) DWBA
- Coupled Channels

\[ \frac{d\sigma}{d\Omega} \] (mb/sr)

\[ \theta_{cm} \] (degrees)
distorted waves which are calculated from phenomenological optical potentials. The form and parameters of the effective interaction are also a source of uncertainty. In order to eliminate these uncertainties we have made a fully microscopic calculation of elastic and inelastic \( \alpha \)-particle scattering from \(^{42}\text{Ca}\) and \(^{50}\text{Ti}\) at 42-44 MeV. We report here the results for elastic scattering and the conclusions reached regarding the two-body interaction.

The real part of the optical potential is calculated from the formula,
\[
U(R) = \int \rho(r) V_{\text{eff}}(r-R) dr,
\]
and the effective \( \alpha \)-nucleon interaction \( V_{\text{eff}} \) is taken to be of gaussian form,
\[
V_{\text{eff}} = -V_0 \exp\left\{-K^2 (r-R)^2\right\},
\]
or of density-dependent form,
\[
V_{\text{eff}} = -V_0 \left\{1 - k \frac{\rho(r)}{\rho(0)}\right\} \exp\left\{-K^2 (r-R)^2\right\},
\]
where \( V_0, K, k \) are real parameters. The imaginary part is taken to have the same radial behaviour as the real part but with strength determined by an additional parameter \( W_0 \). The ground state density \( \rho(r) \) is calculated from single-particle wavefunctions generated from Saxon-Woods potentials; these wavefunctions have the correct separation energies, as far as they are known, and the proton density yields agreement with elastic electron scattering data. Variations in the value of \( K \) around 0.5 fm\(^{-1}\) do not change the shape of the potential in the important surface region. For this reason, \( K \) was fixed at 0.5 fm\(^{-1}\) and the search code then determined values of \( V_0, W_0 \) which gave the best fit to the data. The quality of fit is quite comparable with that obtained by Jackson and Morgan (Phys. Rev. 175 (1968). 1402) using phenomenological potentials and far superior to that obtained by Jackson and Kembhavi (Phys. Rev. 178 (1969) ) using a Yukawa form for \( V_{\text{eff}} \). The values of the strong absorption radii \( R_j \) and the real and imaginary parts of the potentials at \( R_j \) are identical with previous results of Jackson and Morgan. The effect of varying the parameter \( k \) between zero and unity is to reduce the contribution from the interior of the nucleus, but in order to fit the data the parameters \( V_0, W_0 \) must be increased as \( k \) is increased. Two such potentials which fit the data are shown in the figure, and compared with a Saxon-Woods potential which also fits the data. By folding a gaussian nucleon-nucleon potential into the \( \alpha \)-particle density, Bernstein (Advances in Physics, to be published) has derived a gaussian \( \alpha \)-nucleon interaction whose parameters are very close to those obtained here. This suggests that there is internal consistency in the microscopic description and is supported by the fact that there is a difference of only 2\% in the values of \( V_0 \) which give the best fit to the data for \(^{42}\text{Ca}\) and \(^{50}\text{Ti}\).