DYNAMIC CHARACTERISTICS OF ROTATING
SHROUDED-BLADED-DISC

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by

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TO THE MEMORY OF

MY FATHER
SUMMARY

This investigation deals with the vibration problem of a rotating shrouded bladed disc and with the dynamic stresses of blades and packets of turbine blades. The wave propagation technique in periodic structures is introduced into the finite element method to reduce the overall number of degrees of freedom.

The application of wave propagation technique to discs, disc-blades and shrouded bladed disc results in a small size eigenvalue problem which is easily solved to obtain the frequencies and mode shapes of these assemblies under the effects of rotation, disc thickness variations, disc flexibility and other design parameters.

The numerical integration is used to evaluate the integrals of the stiffness and mass matrices of the variable thickness discs.

The influence of rotational speeds and other design parameters such as; pretwist angles and stagger angles on the free vibration characteristics of blades, curved beams and packets of blades is studied using finite element method.

The results of the natural frequencies are compared with the experimental results and with the results of other investigators.

The dynamic stresses of blades and packets of blades are derived from the eigenvectors obtained in the vibration study of these components. The effects of rotational speeds, pretwist angles, stagger angles and shrouding on the dynamic stresses are studied.
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NOMENCLATURE

A  Area of a cross-section of blade
Cc  Torsional stiffness
Cg  Centroid
E  Modulus of elasticity
\( x_x, y_y \)  Co-ordinate axes through the centroid
\( x_x, y_y \)  Principle axes through the centroid
\( \eta, \delta \)  Co-ordinate axis through centroid of blade cross-section parallel to plane of disc
\( \xi \)  Co-ordinate axis through centroid of blade cross-section perpendicular to plane of disc
ZZ  Longitudinal axis
Z  Co-ordinate distance measured along undeflected blade from the root
h  Thickness of the plate
\( \phi \)  Stagger angle
E  Modulus of elasticity
\( \theta \)  Torsional displacement
D  Plate elasticity
\( D = \frac{E h^3}{12(1-\nu^2)} \)
g  Gravitational acceleration
a  Length of the plate
b  Width of the plate

\( I_{xx} \)  Second moment of area of cross-section about \( x_1x_1 \)
\( I_{yy} \)  Second moment of area of cross-section about \( y_1y_1 \)
\( I_{xy} \)  Product moment of area of cross-section about \( x_1x_1 \) and \( y_1y_1 \)
\( I_{xx} \)  Second moment of area of cross-section about \( xx \)
\( I_{yy} \)  Second moment of area of cross-section about \( yy \)
\( I_{cg} \)  Polar moment of inertia per unit length about centroid
\( I_{cf} \)  Polar moment of inertia per unit length about centre of flection
\[ l \] length of blade element  
\[ L \] Blade length  
\[ p \] Frequency in radians  
\[ T \] Kinetic energy  
\[ V_s \] Potential energy due to static vibration  
\[ V_c \] Potential energy due to centrifugal force effect  
\[ t \] Time, thickness  
\[ U \] Displacement of centre of flecture in \( x \) direction  
\[ U_1 \] Displacement of centroid in \( x_1 \) direction  
\[ V_1 \] Displacement of centroid in \( y_1 \) direction  
\[ V \] Displacement of centre of flecture in \( y \) direction  
\[ W \] Displacement of centre of flecture in \( z \) direction  
\[ \alpha \] Angle of pretwist; Angle between co-ordinate axes  
\[ \beta \] Pre-twist angle along beam element, element subtended angle, variable in the formula \( h(r) = \frac{C}{r^\beta} \)  
\[ \omega \] Angular velocity of rotation of disc in radians  
\[ i, i+1 \] Node number  
\[ R \] Distance between centroid and centre of flecture, Radius of curvature  
\[ d_x, d_y \] Co-ordinates of centre of flecture with respect to fixed axes system  
\[ d_x', d_y' \] Coordinates of centre of flexure with respect to principle axes system  
\[ \lambda \] Frequency parameter of the beam \( \rho A L^4 p^2/E I \)  
\[ \rho_d \] Density of blade material  
\[ k \] Constant coefficient of shear deformation  
\[ I_{cf_i} \] Polar moment of inertia about centre of flecture at the starting point  
\[ V_{sh} \] Potential energy of the shroud
\( v \) Poisson's ratio  \\
\( T_{sh} \) Kinetic energy of the shroud  \\
\( l_{sh} \) Curved beam element length  \\
\( V_{csh} \) Potential energy of the shroud due to rotation  \\
\( q_I \) Generalized displacements for an in-plane  \\
\( q_O \) Generalized displacements for an out-of-plane  \\
\( \kappa \) Curvature  \\
\( \varepsilon_T \) Torsional strain  \\
\( \varepsilon_s \) Circumferential strain  \\
\( w_s \) Weight of shroud on each blade  \\
\( V \) Volume  \\
\( I_{XX} \) Second moment of area of shroud cross-section about XX  \\
\( i \) Rotation of the tangent of a curved beam  
\[
  i = \frac{\partial w}{\partial y} - \frac{V}{R}
\]  \\
\( G_s \) Shear modulus of shroud material  \\
\( A_s \) Area of the curved beam (shroud)  \\
\( E_s \) Modulus of elasticity of the curved beam (shroud) material  \\
\( N_B \) Number of blades  \\
\( L_{sh} \) Total length of curved beam  \\
\( M \) Number of elements along the curved beam  \\
\( N \) Number of elements along the blade  \\
\( b \) Axial thickness of the curved beam  \\
\( J \) Slope of U deflection of a curved beam.  
\[
  J = -\frac{du}{dy}
\]  \\
\( \rho_s \) Density of shroud  \\
\( V* \) Displacement in matrix form  \\
\( V^1 \) First Derivative of Displacement matrix  \\
\( V^{11} \) Second Derivative of Displacement matrix  \\
\( A_s \) Area of shroud cross-section
\( J_{xxs} \)  
St. Venant torsion constant of shroud

\( I_p \)  
Polar moment of area of a cross-section

\( L_r \)  
Shroud length to blade length ration (\( L_{sn}/L \))

\( h(r) \)  
Thickness of disc

\( c \)  
Constant in the formula

\[
h(r) = \frac{c}{r^3}
\]

\( r_1 \)  
Inner radius of disc

\( r \)  
Radius of disc

\( r_2 \)  
Outer radius of disc

\( \sigma_{rr} \)  
Radial stress at the mid-surface of the disc

\( \sigma_{\theta\theta} \)  
Tangential stress at the mid-surface of disc

\( N_T \)  
Tangential number of element of disc

\( N_R \)  
Radial number of element of disc

\( SCC \)  
Shroud Clamped-Clamped Frequencies

\( BCF \)  
Blade Clamped-Free Frequencies

\( BCP \)  
Blade Clamped-Pinned Frequencies
CHAPTER 1

GENERAL INTRODUCTION

In the design of reliable turbomachinery structures, longer fatigue life is a substantial requirement. The problem is serious because many blade failures are associated with material fatigue as a result of blade vibration which gives rise to complicated high stress conditions. Turbomachinery structures require freedom from aeroelastic instabilities and sufficient separation of shrouded-bladed-disc natural frequencies from resonance with principal periodic excitation. Accurate knowledge of natural frequencies and the corresponding mode shapes are essential in design calculations to make sure that both criteria are met.

It was found that for a given blade frequency the more flexible blades frequently experience a longer service life. This can be achieved by increasing the blade length at certain positions provided that a compromise can be found between building flexibility into the blade and keeping steady-state stresses at a safe level. This can be done by incorporating lacing wires, which provides additional mechanical damping, besides tying the blades together to suppress inputs of an order of excitation equal to the number of tied blade groups.

Although much work has been done on turbomachinery flutter, cases of aeroelastic instability are still being encountered as shown by Jeffers and Meece(1.1). The inability to accurately predict flutter boundaries in turbomachinery engines is attributed to difficulties in predicting accurate mode shapes and frequencies of the rotor. Although reasonable agreement
between theory and test has been achieved in determining natural frequencies for most cases, resonance problems occasionally occur which are almost difficult to correct from the point of view of time and cost.

This work deals with finite element prediction of natural frequencies and mode shapes of rotating Shrouded-Bladed-Disc Assembly in addition to dynamic stresses of rotating and non-rotating turbine blades and a packet of two blades.

To model such turbine stage, the following aspects have been analysed individually:

1. The blade as a pretwisted asymmetrical aerofoil cross-section was analysed using beam theory. Six motions of the blade, three translations and three rotations, are introduced in the analysis to account for the coupled bending-bending-torsional vibration. The effect of centrifugal force is taken into account. For each displacement, an appropriate polynomial shape function is assumed.

2. The shroud is analysed as a rotating curved beam. Appropriate shape functions are also assumed for the various motions of the shroud. The in-plane and out-of-plane vibration of the curved beam were dealt with separately. They are then combined together to give the vibration characteristics of the curved beam in three directions with six motions at each node; namely three translations and three rotations.
3. The disc is developed by finite elements of annular sector geometry, with one translational and two rotational motions at each of its four nodes, in which radial thickness variation and the effects of in-plane stresses due to rotation is introduced using thin plate theory.

Use was made of the rotational periodicity of the structure so that only one substructure consisting of; one blade, annular disc sector and an appropriate length of the shroud segment subtending an angle \( \frac{2\pi}{N_B} \), where \( N_B \) number of blades, needed to be analysed in the finite element method of analysis.

The work has been categorized into nine chapters, in Chapter 1, general introduction and literature survey of the whole work have been cited.

In Chapter 2, under methods of solution heading, finite element method, Gauss quadrature method and wave propagation technique of period structures were summarised.

In Chapter 3, vibration of a pretwisted asymmetrical aerofoil cross-section blade is studied using beam theory and finite element method. The effects of rotation speed, stagger angle, pretwist angle and the distance between the shear centre and centroid are investigated. These results are compared with the results of other investigators.

In Chapter 4, in-plane and out-of-plane vibrations of a rectangular curved beam are examined separately and then combined together to obtain the vibration characteristics of a curved beam in three directions. Stiffening of the beam element due to the centrifugal force effect is taken into account. The results of analysis were compared with other available results.
In Chapter 5, a finite element model is presented for the analysis of coupled bending-bending-torsional vibrations of rotating pretwisted asymmetrical aerofoil cross-section blade packets. The effects of locations of the shroud along the span of the blade, subtended angle, pretwist angle, rotational speed and the ratio of blade length to disc radius on the frequencies of vibration of the blade packet are investigated.

In Chapter 6, finite elements of annular sector geometry are developed for the disc vibration, in which radial thickness variation and the effects of in-plane stresses due to rotation can easily be introduced. The behaviour of these elements in the analysis of simple and complex discs has been examined. Thin plate theory is used for such analysis.

In Chapter 7, the finite element method in conjunction with the wave propagation technique of a rotationally periodic structure is used to study the free vibration characteristics of rotating pretwisted aerofoil cross-section bladed discs, with the disc having a hyperbolic thickness profile of the form \( h(r) = \frac{c}{r^\beta} \). The effects of pretwist angle, stagger angle, rotational speed and disc thickness variation are investigated. The existence of coupling between the blades and disc and its influence on the natural frequencies has been studied and compared with both theoretical and experimental results.

In Chapter 8, vibration characteristics of rotating Shrouded-Bladed-Disc Assembly is investigated using finite element method and the cyclic symmetry approach. The blade-shroud coupling effect on the natural frequencies has been investigated in addition to the effect of the coupling between the blades and disc. The effects of pretwist angle, stagger angle, rotational speed and disc thickness variations are investigated.
In Chapter 9, dynamic stresses of the turbine blade of Chapter 3 have been obtained theoretically for both rotating and non-rotating blades. Stresses of two bladed packet are also studied and the effect of shrouding on the dynamic stresses of the blades is investigated.

Vibration problems of turbomachinery have been dealt with over a considerable period of time with varying degrees of completeness, depending upon assumptions and simplifications made in the course of the analysis.

The vibration analysis of a single blade, using classical beam theory, has undergone steady development with the mathematical modeling generally becoming increasingly realistic. Houbolt and Anderson(1.2) have attempted to model turbine and compressor blades as non-uniform cantilever beams, calculating only their uncoupled, free bending and torsional vibrations using the Stodola method. Wong, Lane and Vaccaro(1.3) studied relatively flexible blades mounted on the periphery of an infinitely rigid disc. The complete system was treated by studying one blade, and in the complete assembly, all the blades were assumed to oscillate with identical amplitude, in the same mode shape, with the motion of each blade leading that of following one by some appropriate phase angle. Mendelson(1.4) presented a method which combines some of the advantages of both the Stodola and Rayleigh-Ritz methods as used in the coupled bending-torsional vibrations of nonuniform pretwisted blades using beam theory. Rosard(1.5), Jarret and Warner(1.6), Carnegie(1.7) and Isakson and Eisley(1.8) also considered the effect of pretwist on blade frequencies and mode shapes which is based on classical beam theory. One of the more realistic and comprehensive treatment of blade vibration problems was given by Houbolt and Brooks(1.9) where coupled flapwise bending, chordwise bending and torsion of a twisted, rotating non-uniform blade were considered. Further surveys of the work of other investigators concerning the blade vibrations are given in Chapter 3, references (3.1 to 3.40).
In practice it is well known that in turbomachinery, blades are the most seriously affected component by vibrations. For this reason almost all early investigators studied the vibration problems of an isolated blade attached to an infinitely rigid disc. The results of such theoretical work, however, could not explain the complex character of many of the experimentally obtained results.

Turbine blades are usually connected together by lacing wires or shrouds. The connections may be continuous, resulting in all blades being connected in a ring, or segmented, resulting in a finite number of blade groups, (referred to as packets of blades) where blades are interconnected in each packet.

Publications related to the investigation of the in-plane and out-of-plane vibrations of curved beams and rings by many authors are listed in Chapter 4, references (4.1 to 4.37).

To make the model of analysis closer to the real structure, some researchers studied the packet of blades clamped to an infinitely rigid disc. Prohl (1.10) used a modified Holzer method to predict the frequencies and mode shapes for a packet of blades. He also evaluated vibration amplitude and stresses at resonance. Deak and Baird (1.11) applied the fundamental solution technique to obtain all the natural frequencies and mode shapes of a packet of blades connected by lacing wires. The finite element method was used by Salama and Petyt (1.12) to predict frequencies and mode shapes of a blade packet. They also considered the dynamic response of a packet to periodic excitation. Free and forced vibrations of a packet of blades were treated by Huang (1.13) using the transfer matrix method. Armstrong and Hall (1.14) studied the dynamic characteristics of an assembly of shrouded, double-symmetric blades of uniform cross-section, attached to an infinitely
rigid disc. Their study was based on a finite difference technique using thin classical beam theory. Srinivasan, Lionberger and Brown(1.15) examined the dynamic behaviour of an assembly of shrouded blades by considering the modal characteristics of its components. The component modes were calculated for a single blade and shroud, modelled with triangular plate elements. Damping was included in the analysis, to account for any energy dissipation due to vibratory rubbing action at the shroud interfaces.

Further survey of the work of other investigators about the blade packet vibrations are given in Chapter 5, references (5.1 to 5.21).

Turbine blades are usually attached to a flexible disc. This introduces additional coupling between blades through the disc, and it reduces stiffness at the blade root as well, compared to the case of blades clamped to an infinitely rigid disc. Thus, it is clear that to accurately predict the vibration characteristics of turbomachinery blading, it is necessary to consider in the analysis, the whole bladed-disc assembly, with disc flexibility and the inter-blade coupling.

A detailed survey of the previous research concerning the disc vibration is given in Chapter 6, references (6.1 to 6.53).

One of the early works accounting for disc flexibility is that of Carta(1.16), where frequencies and mode shapes of rotating bladed-disc assemblies were used to investigate flutter instability of turbojet engines using energy method.

A number of methods have been developed for the structural dynamics analysis of bladed-disc. Armstrong, Christie and Hague(1.17) developed a method which uses a receptance coupling technique to predict nature(
frequencies of bladed discs of simple geometry. In their analysis, blades were assumed to be identical and handled as one component. This method has been further developed and used by Ewins\(^{(1.18)}\), who relaxed the assumption of identical blades. Individual components were studied and then joined by matching forces and moments at junction points. This method is restricted to bladed discs of simple geometry with a large number of blades.

Blades with more complicated geometry and discs with varying thickness have been used to meet both weight and performance requirements in modern gas turbine engines. All these features add to the complexity of the structural dynamics problem of bladed discs, and add the need for devising efficient numerical techniques. The growth of large scale computing and its favourable economic development have made the finite element method one of the most attractive of the available numerical techniques. Among many authors in this field were Mota Soares and Petyt\(^{(1.19)}\) who applied the finite element method to predict the vibration characteristics of a bladed disc. Sector elements were used to model the disc with blades represented by thick shell elements. Loose shrouds were modelled by lumped masses and moments of inertia while tight shrouds were modelled by two straight beam elements with six degrees of freedom at each node. Dugundji and Chen\(^{(1.20)}\) studied the static deformation and dynamic behaviour of shrouded bladed discs using the finite element method. Blades were modelled as beam elements with uniform pretwist, shrouds as curved beam elements, and the disc as a circular plate clamped at the centre.

Application of any of the aforementioned numerical techniques to shrouded bladed discs will result in a large eigenvalue problem which requires large computer storage and is likely to be costly to run. This difficulty can be overcome by making use of the wave propagation technique of a rotationally periodic structure. In a rotationally periodic structure, displacements of all substructures can be expressed in terms of the
displacements of one substructure and this result in an eigenvalue problem of reduced order. Thomas(1.21) applied this procedure in the vibration analysis of a stationary 151-bladed turbine wheel in which a finite element method was used to model the fundamental substructure. Wildheim(1.22,1.23) applied the same method to reduce equations of motion of the complete rotationally periodic structure as a set of equations of motion pertaining to a single substructure. Free undamped, and forced vibrations were considered, taking into account effects due to rotation.

Further surveys of the work of other investigators concerning the bladed disc and shrouded bladed disc vibrations are given in Chapters 7 and 8, references (7.1 to 7.17) and (8.1 to 8.11) respectively.
CHAPTER 2

METHODS OF SOLUTION

2.1 INTRODUCTION

Exact solutions to the turbomachinery vibration problems are valuable for the checks they provide for approximate methods of analysis. They are rarely used because they are inflexible, restrictive and tedious to evaluate.

Approximate methods are widely used. Classically it has been the Rayleigh-Ritz method which has been most used. It is an energy method providing an upper bound to the true solution and starting from the expressions for strain energy and kinetic energy of the vibrating element.

The finite difference method begins from the differential equation of motion and solves this equation at a number of points in the vibrating element. It is similar to the finite element method in being a matrix formulation of the problem.

A number of approximate numerical methods such as Rayleigh, Rayleigh-Ritz, Myklestad, Stodola and Finite Difference methods were used by several investigators. The details of these methods are given in several literatures and are not given here.

Two of the recently available methods are Boundary Element Methods (BEM) and Finite Element Method (FEM).
Boundary Element Methods are based on a study of the equations governing field problems in the form of boundary integral equations rather than the more usual differential equations. In principle, this method involves only the discretization of the boundaries of the region under investigation. This can be done by integrating the differential equations which would involve only values of the variables at the extremes of the range of integration. This would imply that any discretization scheme needed would only involve subdivisions of a bounding surface of a body as any homogeneous region requires only surface discretization.

Although all BEM have a common origin they divide naturally into three different but closely related categories.

1. The direct formulation of BEM

In this formulation the unknown functions appearing in the integral equations are the actual physical variables of the problem. Thus, for example, in an elasticity problem such an integral equation solution would yield all the tractions and displacements on the system boundary directly and those within the body can be derived from the boundary values by numerical integrations.
2. Semi-direct formulations of BEM

Alternatively, the integral equations can be formulated in terms of unknown functions analogous to stress functions in elasticity or stream functions in potential flow. When the solution has been obtained in these terms simple differentiation will yield, for example, the internal stress distribution.

3. Indirect formulations of BEM

In the indirect formulation the integral equations are expressed entirely in terms of a unit singular solution of the original differential equations distributed at a specific density over the boundaries of the region of interest.
2.3 FINITE ELEMENT METHOD

2.3.1 INTRODUCTION

In practice, most problems are too complicated for a closed-form mathematical solution. When a numerical solution is required, the most versatile method that provides it is the finite element method.

The finite element method, like the Rayleigh-Ritz method, starts from the expressions for strain and kinetic energy of the vibrating object. It is preferred to be used in analysing the vibration problem of the individual and coupled parts of turbomachinery system for several reasons; it is more flexible than the energy method of solution, it possesses the great advantage of being easily able to deal with variable boundary conditions, changes of section, variations in thickness and the material properties. It is not necessary to re-write the deflection function each time and extra degrees of freedom can be introduced simply by subdividing the element in the model.

Since the original development of the concept of discretisation of continua by Turner, Clough, Martin and Topp\(^2.1\) in 1956 much progress has been made and the method has received wide application in a variety of problems, such as small deflection elastic problems, field problems, eigenvalue problems and non-linear problems. Many books and articles have been published dealing with the finite element approach. Mention could be made of the recently published books by; Zienkiewicz\(^2.2\), Przemieniecki\(^2.3\), Rao\(^2.4\), Nath\(^2.5\), Hinton and Owen\(^2.6\) and Cook\(^2.7\), which also contains an extensive bibliography.
The analysis of a structure by the finite element method has three basic steps: structure idealisation which is the subdivision of the actual continuum into an assemblage of discrete structure elements, evaluation of element characteristics such as dynamic stiffness matrix, and structural analysis of the element assemblage.
The behaviour of a finite element is approximated by the most widely used approximated assumption of displacement functions to ensure compatibility conditions of deformations both within the elements and across their boundaries. This is known as the displacement model.

The simplest and most direct way of assuming a displacement function is to select a polynomial having as many unknown coefficients as there are generalised displacement components. Thus it is possible to assume a displacement function that will ensure compatibility of deformations both within the elements and between the adjacent elements.

The general displacement function can be written in matrix form as

\[
\{a\} = [N]\{U\} \tag{2.1}
\]

where

\(\{a\}\) is the assumed displacement vector

\(\{U\}\) is the nodal displacement vector

\([N]\) is the shape function matrix
2.3.2b STRAIN–DISPLACEMENT RELATIONS

To represent fully three dimensional deformed shape of an elastic structure under a given system of loads, three displacements, in three perpendicular directions are assumed. These three displacements can sometimes be time-dependent, depending on the external load. If the external load is not changing with time or not existing, as in the case of free vibration, the three displacement functions $u$, $v$, $w$ are independent of time and can be written as

$$
U = u(x, y, z) \\
V = v(x, y, z) \\
W = w(x, y, z)
$$

(2.2)

The positive directions of the vectors which are representing the three displacements at a point in the structure are perpendicular to each other and correspond to the positive directions of the coordinate axes $x$, $y$ and $z$.

The partial derivatives of the displacements $U$, $V$ and $W$ can express the strain in the deformed structure.
If the strain-displacement relations are linear for a small deformation the strain components can be written in matrix form as

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{zx}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{bmatrix}
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
\]

(2.3)

where

\( \varepsilon_{xx}, \varepsilon_{yy}, \text{ and } \varepsilon_{zz} \) are the normal strains

\( \varepsilon_{xy}, \varepsilon_{yz}, \text{ and } \varepsilon_{zx} \) are the shearing strains

Equation (2.3) can be written as

\[
\{\varepsilon\} = [\chi]\{U\}
\]

(2.4)

where

\( \{\varepsilon\} \) is the strain vector

\( \{U\} \) is the displacement vector

\([\chi]\) is a transforming operator matrix
2.3.2c STRESS-STRAIN RELATIONS

Since the relationships between the elastic strains $\varepsilon_{ij}$ and stresses $\sigma_{ij}$ are related by means of the Hooke's law, they can be written as

\[
\varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{yy} + \sigma_{zz} \right) \right]
\]
\[
\varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu \left( \sigma_{zz} + \sigma_{xx} \right) \right]
\]
\[
\varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu \left( \sigma_{xx} + \sigma_{yy} \right) \right]
\]
\[
\varepsilon_{xy} = \frac{1}{G} \sigma_{xy}
\]
\[
\varepsilon_{yz} = \frac{1}{G} \sigma_{yz}
\]
\[
\varepsilon_{zx} = \frac{1}{G} \sigma_{zx}
\]

where $E$ is the modulus of elasticity, $\nu$ is the Poisson's Ratio and $G = E/2(1+\nu)$ is the modulus of rigidity.

Equation (2.5) can be solved for the stresses and the following stress-strain relationships are then obtained:

\[
\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \varepsilon_{xx} + \nu (\varepsilon_{yy} + \varepsilon_{zz}) \right]
\]
\[
\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \varepsilon_{yy} + \nu (\varepsilon_{zz} + \varepsilon_{xx}) \right]
\]
\[
\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \varepsilon_{zz} + \nu (\varepsilon_{xx} + \varepsilon_{yy}) \right]
\]
\[
\sigma_{xy} = \frac{E}{2(1+\nu)} \varepsilon_{xy}
\]
\[
\sigma_{yz} = \frac{E}{2(1+\nu)} \varepsilon_{yz}
\]
\[
\sigma_{zx} = \frac{E}{2(1+\nu)} \varepsilon_{zx}
\]
Equation (2.6) can be written in matrix form as

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{zx}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{zx}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\sigma \\
\varepsilon
\end{bmatrix} = \mathbf{E} \begin{bmatrix}
\varepsilon \\
\sigma
\end{bmatrix}
\]

(2.7)

where

\[
\begin{align*}
\{\sigma\} & \text{ is the stress vector} \\
\{\varepsilon\} & \text{ is the strain vector} \\
\mathbf{E} & \text{ is the elasticity matrix}
\end{align*}
\]

2.3.2d STRAIN ENERGY

\[V_i = \frac{1}{2} \int_{V} \{\varepsilon\}^T \sigma \, dv \]

(2.9)

For linearly elastic structure, the strain energy \(V_i\) of an element in the structure, can be represented in terms of the strains and stresses as
Substituting equation (2.8) into equation (2.9), the strain energy expression can be written as

\[ V_i = \frac{1}{2} \int \{\varepsilon\}^T [E]\{\varepsilon\} \, dv \]  

(2.10)

Substituting equation (2.4) into equation (2.10), the expression becomes

\[ V_i = \frac{1}{2} \int \{U\}^T [X]^T [E][X]\{U\} \, dv \]  

(2.11)

or

\[ V_i = \frac{1}{2} \{U\}^T [k_i] \{U\} \]  

(2.12)

where the matrix \([k_i]\) is known as the stiffness matrix of the element and is given by

\[ [k_i] = \int [X]^T [E] [X] \, dv \]  

(2.13)

2.3.2e KINETIC ENERGY

The kinetic energy \(T_1\) of an element is given as follows

\[ T_1 = \frac{1}{2} \int \rho \{\dot{\alpha}\}^T \{\dot{\alpha}\} \, dv \]

(2.14)

where dots denote partial differentiation with respect to time and \(\rho\) is the mass density of the material, and \(\alpha\) is the generalized displacement.
Substituting equation (2.1) into equation (2.14), then the kinetic energy expression becomes

\[ T_i = \frac{1}{2} \int (U)^T [\ddot{N}]^T \rho [N] [U] \, dv \]  

which can be written as

\[ T_i = \frac{1}{2} (U)^T [m_1] (U) \]  

(2.16)

where the matrix \( [m_1] \) is the mass matrix of the element and is given by

\[ [m_1] = \int [N]^T \rho [N] \, dv \]  

(2.17)

2.3.3 STRAIN AND KINETIC ENERGIES OF A COMPLETE SYSTEM

To ease the computation, a different coordinate system may, in fact, be used for every element. Then the whole local coordinate system is transformed to one global coordinate system before an assembly of the whole structure can be attempted.

The local displacement components \( \{q_L\} \) can be transformed to the global displacement components \( \{q\} \) by a suitable matrix of direction cosines \( [L] \)

\[ \{q\} = [L] \{q_L\} \]  

(2.18)
The total strain and kinetic energies of the system are equal to the sum of the strain and kinetic energies of each element.

The total strain and kinetic energies of the system can be written as

\[ V = \frac{1}{2} (q)^T[K](q) \]
\[ T = \frac{1}{2} (q)^T[M](q) \]

where \([K]\) and \([M]\) are the global stiffness and mass matrices of the whole system.

2.3.4 EQUATION OF MOTION

Hamilton's principle is easily applied to a conservative system. In this case \( \delta W = \delta V \), where \( V \) is the potential energy of the system. Consequently it can be written in the following form

\[ t_2 \]
\[ \delta A = 0 \text{ , } A = \int_{t_0}^{t_1} Ldt \text{ , } L = T-V \]

The quantity \( L \) is known as the "Lagrangian Function". The variational equation, \( \delta A = 0 \), ensures that, for all motions that will carry a conservative system from a given initial configuration \( x_1 \) in a given time interval \( (t_0 \text{ , } t_1) \), the integral \( A \) has a stationary value.
If the system has a finite number of generalised coordinates $q_i$, the differential equations of motion are accordingly

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial q_i} \right] - \frac{\partial L}{\partial q_i} = \mathbf{Q}$$  \hspace{1cm} (2.22)

Equation (2.22) yields Lagrange's equation

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{q}} \right] - \frac{d}{dt} \left[ \frac{\partial V}{\partial q} \right] - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = \mathbf{Q}$$ \hspace{1cm} (2.23)

where $\mathbf{Q}$ is the general nodal force corresponding to $\mathbf{q}$.

Applying Lagrange's principle to the energies yields the equation of dynamic equilibrium of the system.

$$[\mathbf{M}] \{\ddot{\mathbf{q}}\} + [\mathbf{K}] \{\mathbf{q}\} = \mathbf{Q}$$ \hspace{1cm} (2.24)

For free vibration, equation (2.24) becomes

$$[\mathbf{K}] - \lambda [\mathbf{M}] \{\mathbf{q}\} = 0$$ \hspace{1cm} (2.25)

where $\lambda$ is the frequency parameter, $[\mathbf{K}]$ is the stiffness matrix and $[\mathbf{M}]$ is the mass matrix of the system, and both the two matrices are real, symmetric and positive definite.
Whatever the Finite Element model may be, the approximate element characteristics must ensure certain vital conditions for convergence of the results to the exact solution with finer subdivision. This requires that the displacement of a subelement can be identical to those generated in it by displacements on the element before subdivision. For the convergence to the actual results the assumed displacements must be continuous over each element and across their boundaries and they should include rigid body movements. As elements become smaller nearly constant strain conditions will arise.

The convergence of the finite element method has been dealt with more by Tong and Pian (2.8).
2.4 NUMERICAL INTEGRATION

2.4.1 INTRODUCTION

Throughout the course of this thesis both algebraic integration and numerical integration are used. Algebraic integration is used to find out the dynamic stiffness matrices of shroud element and blade element using beam theory. Whereas for disc element numerical integration is used as the disc configuration becomes complex when the radial thickness variation together with the effects of in-plane stresses due to rotation are introduced. Thus the lengthy algebraic manipulation needed for the derivation of the stiffness and the mass matrices of the disc can be avoided entirely.

There are several methods to evaluate any definite integral numerically. The most widely used and easily applicable to finite element method is the Gauss method which has been described fully by Kopal (2.9).

The Gauss quadrature method is utilised in this analysis and can be categorized into three sections.

2.4.2 ONE DIMENSIONAL ANALYSIS

Assume the following integral to be evaluated

$$ I = \int_{-1}^{1} \phi \, d\xi \quad \text{where} \quad \phi = \phi (\xi) $$

(2.26)
We can evaluate $\phi$ at the midpoint of the interval and multiply by the length of the interval, as shown in Figure (2.1). Thus we find $I \approx 2\phi l$. This result is exact if the function $\phi$ happens to be a straight line of any slope. Generalization of equation (2.26) gives

$$I = \int_{-1}^{1} \phi(\xi) d\xi = \sum_{i=1}^{n} W_i \phi(\xi_i)$$  \hspace{1cm} (2.27)

where

- $n$ total number of integration points
- $W_i$ weighting factor
- $\xi_i$ coordinates of the $i$th integration point.

A list of these is given in tabular form in many finite element and numerical analysis books.

In order to evaluate the integral in equation (2.27), $\phi = \phi(\xi)$ is evaluated at each of several locations $\xi_i$, and multiplied by an appropriate weight $W_i$, and summed thus

$$I = W_1 \phi_1 + W_2 \phi_2 + \cdots + W_n \phi_n$$ \hspace{1cm} (2.28)

Gauss's method locates the sampling points so that for a given number of points, greatest accuracy can be obtained. The sampling points are located symmetrically about the centre of the interval. Symmetrically paired points have the same weighting coefficient $W_i$.

It should be noted that an $n$-point rule integrates any polynomial of degree $x^{2n-1}$, or less, exactly.
2.4.3 TWO DIMENSIONAL ANALYSIS

In two dimensions the quadrature formula for $\phi = \phi(\xi, \eta)$ can be obtained by integrating with respect to $\xi$ and then with respect to $\eta$.

$$I = \int_{-1}^{1} \int_{-1}^{1} \phi(\xi, \eta) \, d\xi \, d\eta = \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j \phi(\xi_i, \eta_j)$$  \hspace{1cm} (2.29)

where $\phi(\xi_i, \eta_i)$ is the numerical value of the function at the $i$th Gauss point.

2.4.4 THREE DIMENSIONAL ANALYSIS

In three dimensions we find the quadrature formula for $\phi = \phi(\xi, \eta, \zeta)$ by integrating with respect to $\xi$, $\eta$ and then with respect to $\zeta$.

$$I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \phi(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} W_i W_j W_k \phi(\xi_i, \eta_j, \zeta_k)$$  \hspace{1cm} (2.30)
2.5 WAVE PROPAGATION TECHNIQUE

2.5.1 INTRODUCTION

Turbomachinery bladings form a rotationally periodic structure which consists of a finite number of identical substructures forming a closed ring. In this analysis, this technique has been utilized, by taking a substructure consisting of one blade, an annular sector of a disc and an appropriate shroud segment.

A periodic structure, whether it is rotational or not, consists of identical elements coupled together in identical ways (end-to-end or side-by-side) to form the whole structure. Then, instead of finding the normal modes and natural frequencies of such a structure, it is easier to describe the behaviour of the periodic structure by the properties of its characteristic waves. Each of these waves is associated with a "Propagation constant", $\mu$. When the system vibrates in just one of the free waves of frequency $\omega$, the harmonic motion at any point in one element is equal to $e^{i\mu}$ times the motion at the corresponding point in the next element. The values of the complex propagation constant, $\mu$, correspond to identical, but opposite-going waves. The real part of $\mu$ is known as the "attenuation constant", and the imaginary part as the "phase constant".

Many articles have been published, using the wave propagation technique, to study the vibration characteristics of different structures (2.10, 2.11, 2.12). Later Orris and Petyt (2.13) applied the technique, using the finite element method, to evaluate the phase.
THEORETICAL CONSIDERATION

Consider a periodic substructure which consists of one blade together with an annular sector of the disc and shroud segment with an angle of curvature $\beta = \frac{2\pi}{N_B}$ as shown in Figure (2.2).

When the periodic substructure vibrates harmonically the following equation of motion can be obtained

$$\left[ [K] - p^2 [M] \right] \{ q \} = \{ Q \}$$

(2.31)

where $[K]$ and $[M]$ are the stiffness and mass matrices of the substructure respectively, $p$ is the natural frequency, $\{ q \}$ is the generalized displacements and $\{ Q \}$ is the generalized forces.

The column matrix of nodal degrees of freedom of a substructure can be classified into three groups, namely, those on the left hand boundary $\{ q_L \}$, those on the right hand boundary $\{ q_R \}$, and all other nodes $\{ q_I \}$. Similarly the generalized forces corresponding to each nodal displacement are classified into $\{ Q_L \}$, $\{ Q_R \}$, $\{ Q_I \}$ respectively.

The propagation of free wave motion is affected by $\{ Q_L \}$ the left hand and $\{ Q_R \}$ the right hand boundary generalized forces.
while \( Q_i \) is zero.

Thus, if the node numbering sequence of the substructure is identical at its right and left boundaries, the nodal displacement and generalized forces are related by

\[
\{q_R\} = e^{i\mu} \{q_L\}
\]

\[
\{Q_R\} = -e^{i\mu} \{Q_L\}
\]

Substituting equation (2.31) into equation (2.32) and eliminating \( \{Q_L\} \) yields

\[
\left[ [K(\mu)] - p^2 [M(\mu)] \right] \begin{bmatrix} q_R \\ q_I \end{bmatrix} = 0
\]

(2.33)

where

\[
[K(\mu)] = \begin{bmatrix}
K_{LL} + K_{RR} + e^{i\mu} K_{LR} + e^{-i\mu} K_{RL} & K_{LI} + e^{i\mu} K_{RI} \\
K_{IL} + e^{i\mu} K_{IR} & K_{II}
\end{bmatrix}
\]

(2.34)

\[
[M(\mu)] = \begin{bmatrix}
M_{LL} + M_{RR} + e^{i\mu} M_{LR} + e^{-i\mu} M_{RL} & M_{LI} + e^{i\mu} M_{RI} \\
M_{IL} + e^{i\mu} M_{IR} & M_{II}
\end{bmatrix}
\]

(2.35)

The matrices \([K(\mu)]\) and \([M(\mu)]\) are real functions of the complex variable \( \mu \).
Equation (2.33) represents an eigenvalue problem for $p^2$ corresponding to a given value of $\mu$.

Since the propagation constant $\mu$ is, in general, complex it can be written as

$$\mu = \mu^r + j \mu^i$$  \hspace{1cm} (2.36)

where $\mu^r$ represents the amplitude ratio and $\mu^i$ the phase difference of the motion in adjacent substructure.

By plotting $\mu^r$ and $\mu^i$ against frequency Sengupta(2.16) showed alternative pass bands ($\mu^r \neq 0$, $\mu^i \neq 0$) and stop bands ($\mu^r \neq 0$, $\mu^i = \pm \pi$, $\pi = 1, 2, \ldots$). The pass band waves propagate with constant amplitude whereas for the stop band only non-propagation motion is possible which decays from one substructure to the next. If in a pass band $\mu$ is purely imaginary, then $[K(\mu)]$ and $[M(\mu)]$ become Hermitian matrices.

As described by Wilkinson(2.17) the solution of the complex eigenvalue problem is reduced to that of a real symmetric matrix.

Let

$$[K(\mu)] = [K^r + jK^i]$$
$$[M(\mu)] = [M^r + jM^i]$$  \hspace{1cm} (2.37)

$$\begin{bmatrix} q_r \\ q_i \end{bmatrix} = \begin{bmatrix} q_r \\ q_i \end{bmatrix}$$

Substituting equations (2.37) into equation (2.33) and separating real and imaginary parts gives

$$\begin{bmatrix} K^r & -K^i \\ K^i & K^r \end{bmatrix} - p^2 \begin{bmatrix} M^r & -M^i \\ M^i & M^r \end{bmatrix} \begin{bmatrix} q_r \\ q_i \end{bmatrix} = 0$$  \hspace{1cm} (2.38)
where

\[
[K^r] = \begin{bmatrix}
K_{LL} + K_{RR} + \cos \mu (K_{LR} + K_{RL}) & K_{LL} + K_{RI} \cos \mu \\
K_{IL} + K_{IR} \cos \mu & K_{II}
\end{bmatrix}
\]

\[
[K^i] = \begin{bmatrix}
\sin \mu (K_{LR} - K_{RL}) & -K_{RI} \sin \mu \\
K_{IR} \sin \mu & 0
\end{bmatrix}
\]

(2.39)

(2.40)

and \([M^r]\) , \([M^i]\) are given similarly.

If \([K(\mu)]\) and \([M(\mu)]\) are Hermitian matrices in equation (2.33) then \([K^i] = -[K^i]^T\) and \([M^i] = -[M^i]^T\). Thus equation (2.38) becomes a real symmetric eigenvalue problem.

For any value of \(\mu\) there will be a set of frequencies with associated eigenvalues defining the wave motion in the periodic section at that frequency. Any normal mode of vibration of a rotationally periodic structure can be regarded as a standing wave, the imaginary propagation constant of which must satisfy,

\[
e^{\Pi \mu} = 1
\]

(2.41)

where \(N\) is the number of identical substructure forming a closed ring. There are \(N\) independent values of the imaginary part of \(\mu\) between \(-\pi\) and \(\pi\). For even \(N\) these are;

\(-\pi /2, -2\pi(N/2-1)/N, \ldots, -4\pi/N, -2\pi/N, 0, 2\pi/N, 4\pi/N, \ldots, 2\pi(N/2-1), \pi\)
and for odd $N$ they are

$$-\pi, -\frac{2\pi(N-1)}{2N}, \ldots, -2\pi/N, 0, 2\pi/N, 4\pi/N, \ldots, 2\pi(N-1)/2N, \pi$$

By considering a single substructure it is possible to obtain all the natural frequencies of a rotationally periodic structure and it is sufficient to consider the positive value of $\mu$ including zero.
(a) Using one point

(b) Using two points

(c) Using three points

FIG. 2.1: SAMPLING POINTS OF ONE-DIMENSIONAL NUMERICAL INTEGRAL
FIG. 2.2: GEOMETRY OF THE SUBSTRUCTURE OF SHROUDED-BLADED-DISC

- nodes on the left boundary
- interior nodes
- nodes on the right boundary
CHAPTER 3

VIBRATION CHARACTERISTICS OF ROTATING CANTILEVER BLADING

3.1 INTRODUCTION

The blades of turbomachinery are typically the most critical parts of the design and its vibrational behaviour is one of the most serious problems faced in engine design. The magnitude of the problem lies in the fact that in some turbines there can be as many as hundreds of fixed and rotating blades of different characteristics and the failure of even one of them may force a shutdown. The solution of a vibration problem involves a number of topic areas; such as mathematical modelling of elastic deformations during vibration, effects of various acceleration components of an elastic rotating body, methods of analysis for eigenvalue problems, and experimental vibration measurement techniques for bodies rotating at high speeds.

For reasons of efficiency and weight turbine blades must be thin, yet they must operate in severe thermal environments and at high rotational speeds causing large centrifugal body forces.

Practical turbine blades have an aerofoil cross-section and possess, in addition to camber and longitudinal taper, a pretwist to allow for the variation in tangential velocity along the span of the blade. Since all these factors complicate the analysis, in practice many simplifying assumptions are usually made in the analysis. In most of the analytical methods suggested for the analysis, the blades
are idealized to behave as beams having radial variation in cross-sectional properties and pretwist provided that it is relatively long with respect to its width, the blade is relatively thick, and only the first few vibration frequencies and mode shapes are needed. This type of analysis assumes that each chordwise segment of the blade moves as a rigid body either translating, rotating, or some linear combination of both.

When a vibrating blade has two-fold symmetry, then three types of free vibration modes are possible, all uncoupled from each other. These are; radial modes, bending modes about each of the two principal axes, and torsional modes of vibration. With only one axis of cross-sectional symmetry present, the bending displacement in the direction of the symmetry axis is uncoupled, but the transverse bending and torsional modes are coupled. With no symmetry, all three types of modes are coupled. Further coupling is introduced into the problem if the blade has initial pretwist. Cross-sections having two or more axes of symmetry exhibit coupling between the bending modes if pre-twist is present, although the torsional mode is uncoupled. With less symmetry and pretwist, all three modes are coupled.

Beam type models have been successfully used for high aspect ratio thin compressor blades and somewhat less successfully for high aspect ratio turbine blades. For low aspect ratio blading, which is increasingly used, a curved shell of varying thickness and curvature probably will be required for accurate modelling.

There are hundreds of references in the literature which incorporate most of the considerations, needed in blade vibration analysis by means of beam models, such as coupling between bending
and torsion, taper, shear deformation, rotary inertia, pretwist, and rotational effects. Many of the references are described by the survey articles by Rao(3.1-3.3), in which he gave hundreds of references dealing with various methods to solve the problem of turbine blades vibrations.

In its simplest form the turbine blade is considered to be a beam of rectangular cross-section, which can be uniform or tapered, with or without pretwist. Mabie and Rogers(3.4) have investigated the free vibrations of a cantilever beam with constant width and linearly variable thickness and constant thickness and linearly variable width using Bernoulli-Euler equations. Carnegie and Thomas(3.5) have given a method of analysis of cantilever beams of constant thickness and linear taper in breadth by reducing the Euler-Bernoulli equation of a beam to an eigenvalue problem. Thomas and Carnegie (3.6) have used the finite difference method to study the effect of taper on the torsional vibration characteristics of slender beams. Carnegie(3.7) has used Rayleigh's energy method to calculate the first frequency in bending of a pretwisted cantilever beam. The static deflection curve was used in the analysis. Dawson(3.8) presented a solution for pretwisted rectangular cross-section beams executing lateral vibrations by the Rayleigh-Ritz energy method. Rao(3.9) has used the Ritz-Galerkin process to solve the two coupled differential equations of motion of a pretwisted tapered cantilever blade vibrating in flexure to obtain the first five natural frequencies. Carnegie and Thomas(3.10) have used the finite difference method to consider the pretwisted tapered blading and showed good agreement between theoretical and experimental results. Thomas and Ducumaci(3.11) have used the finite element method to study the vibration analysis of a tapered beam using
quintic polynomial displacement functions. To use the finite element method to study the vibration characteristics of two higher order tapered beam elements. Subrahmanyan and Rao used the Reissner method to find natural frequencies and mode shapes of the first four modes of a uniform pretwisted cantilever blade and the first five flexural frequencies of a pretwisted tapered blade.

When the aerofoil section of the blade is considered the torsional vibration is also coupled with the bending vibration of the blade. Vaghti used the finite element method to study the vibration characteristics of non-rotating pretwisted asymmetrical aerofoil cross-section blades using beam theory. Carnegie, Dawson and Thomas determined the natural frequencies and mode shapes for the first five modes of coupled bending-bending vibration of a pretwisted double tapered beam using a finite difference method. Montoya has derived the governing differential equations for the vibration analysis of pretwisted blades of aerofoil section, including coupling between bending and torsion. Runge-Kutta numerical procedure is followed to solve the problem and the differential equations are transformed to first order differential equations. Dawson used the transformation method to solve the equations of motion by transforming the original differential equations of motion into a set of ten simultaneous first order differential equations and solving these by a step-by-step procedure to determine the natural frequencies and mode shapes for a pretwisted blade of asymmetrical aerofoil cross-section.

When a rotating blade is considered, the additional stiffness due to the centrifugal forces is to be considered. The centrifugal forces induce several additional coupling terms in the already
complicated equations of motion. The effect of rotation on the blading frequencies has been considered by many investigators. More than a hundred references are described faithfully in the survey article, concerning rotating blades, by Leissa(3.19). Houbolt and Brooks(3.20) developed the differential equation of deformation under the action of various applied loads of a coupled bending-bending and torsion of a twisted rotating helicopter rotor and propeller blades using beam theory. The elastic axis, mass axis, and tension axis are not coincident in their analysis.

Carnegie(3.21) derived a theoretical expression for the work done due to centrifugal effects for small vibrations of rotating cantilever beams and established an equation for the fundamental frequency of vibration by the use of Rayleigh’s method. Carnegie(3.22) used Holzer method to solve the equations of motion for the coupled bending-bending vibration of a rotating blade. By making use of the eight equations derived, which connect the shearing force, bending moments, slopes and deflections at each end of the blade, the blade frequencies and corresponding mode shapes can be determined. Rao and Carnegie(2.23) used the equations derived by Carnegie(3.22) to determine the natural frequencies and mode shapes of a uniform cantilever blade, allowing for the effect of pretwist and rotation. Murthy(3.24) has used the transfer matrix method to determine the natural frequencies and mode shapes of pretwisted non-uniform rotor blades. Swaminadham(3.25) has applied Rayleigh-Ritz method to investigate the vibration characteristics of rotating, twisted and tapered blades. Hoa(3.26) has used the finite element method to investigate the effect of the root radius, the stagger angle and the tip mass on the vibration frequency of a rotating beam. Thomas and Sabuncu(3.27) have used the finite element method to study
the vibrations of pretwisted aerofoil asymmetric cross-section blades under the effect of rotation. The effects of various setting angles, pretwist angles and different rotational speeds have been investigated and compared with other works. Wright, Smith, Thresher and Wang\(^{(3.28)}\) have found the exact frequencies and mode shapes for rotating beams. Uniform and tapered beams have been investigated, with root offset and tip mass, and for both hinged root and fixed root boundary conditions.

When the beams are stubby and when higher modes are needed, the classical Bernoulli-Euler equation of motion for bending vibrations is known to give higher values of frequencies. In such cases, transverse shear and rotary inertia should be included in the analysis. Rayleigh improved the classical theory considering rotary inertia of the cross-section of the beam. Timoshenko extended the theory to include the effects of transverse shear deformation. Huang\(^{(3.29)}\) has derived new frequency and normal mode equations of free flexural vibrations of finite beams including the effect of shear deflection and rotary inertia. Carnegie\(^{(3.30)}\) studied the effects of shear deflection and rotary inertia for straight and pretwisted cantilever beams. Krupka and Baumanis\(^{(3.31)}\) considered the bending-bending mode of a rotating tapered pretwisted turbomachine blade including the effects of shear deformation and rotary inertia. Carnegie and Thomas\(^{(3.32)}\) have used the finite difference method for the bending vibration analysis of pretwisted cantilevers including the effects of transverse shear and rotary inertia. The effects of various taper, depth to length ratios and pretwist angles on the frequencies of vibration are investigated for the first five modes. Thomas and Abbas\(^{(3.33)}\) have used the finite element method to analyse uniform Timoshenko beams by taking total
deflection, total slope, bending slope, and the derivative of the bending slope as nodal degrees of freedom. Bishop and Price(3.34) have derived the equations of motion of a non-uniform beam vibrating in a coupled bending and twisting mode. Allowance has been made for shear deformation and for rotary inertia. Gupta and Rao(3.35) have used the finite element technique to investigate the dynamic analysis of doubly tapered and twisted beams with the effect of shear deflection and rotary inertia. Rouch and Kao(3.36) have used the finite element method for the dynamic analysis of rotating beams. The effects of shear deformation and rotary inertia have been taken into account in their analysis. Subrahamanyam, Kulkarni and Rao(3.37) have investigated the vibration characteristics of rotating straight blades of asymmetric aerofoil cross-section with the allowance for shear deflection and rotary inertia by use of the Reissner method. Subrahamanyam, Kulkarni and Rao(3.38) again used a Reissner method to study the vibrations of pretwisted cantilever blading allowing for shear deformation and rotary inertia. Lees and Thomas(3.39) have applied the finite element method to study the vibrations of a Timoshenko beam by using a complex element of four degrees of freedom. These are; the transverse displacement, cross-section rotation and the coefficient of polynomial expansions of the transverse displacement and the shear deformation.
3.2 ANALYSIS

3.2.1 INTRODUCTION

When a straight cantilever blade is pretwisted, flexural displacements take place simultaneously in two planes because of the unequal flexural rigidities. An additional torsional displacement becomes coupled with these flexural displacements when the cross-section of the beam is asymmetric.

The blade is considered mounted on the periphery of a rotating disc in such a manner that the mid-plane of the blade is inclined to the plane of rotation of the disc at an angle φ, called the setting angle.

Under the effect of the rotational speed, the blade is stiffened due to additional stresses created by centrifugal forces. These forces create stresses in the neutral surface, and the strain energy stored in the element is higher than the bending strain energy by a certain amount.

In this analysis, a finite element beam model is developed for the vibration analysis of rotating pretwisted asymmetric aerofoil cross-section turbine blade. The effect of pretwist angle, setting angle, rotational speed, and the distances of shear centre from centroid on the vibration characteristics are investigated.

Twelve degrees of freedom, six degrees at each node of the beam element, namely; two transverse displacements, longitudinal displacement, torsional displacement, and two slopes of transverse displacements, are used in this analysis.
3.2.2 POTENTIAL ENERGY

The total potential energy of a rotating blade, when the gravitational effects are ignored, will be equal to the summation of the potential energies due to static vibration and that due to the effects of centrifugal force:

\[ V = V_s + V_c \]  

(3.1)

3.2.2a POTENTIAL ENERGY OF A STATIONARY BLADE

When a pretwisted blade of asymmetrical aerofoil cross-section is subjected to vibrating movement, the motion consists of simultaneous displacements in two perpendicular directions coupled with torsion. The potential energy expression for such blades is given by Carnegie (3.30),

\[ V_s = \int_0^L \left[ \frac{1}{2} E I_{xx} \left( \frac{\partial^2 w}{\partial z^2} \right)^2 + E I_{xy} \left( \frac{\partial^2 u}{\partial z^2} \right) \left( \frac{\partial^2 v}{\partial z^2} \right) + \frac{1}{2} E I_{yy} \left( \frac{\partial^2 u}{\partial z^2} \right)^2 + C \left( \frac{\partial \theta}{\partial z} \right)^2 + \frac{E A}{2} \left( \frac{\partial w}{\partial z} \right)^2 \right] \, dz \]  

(3.2)
3.2.2b POTENTIAL ENERGY DUE TO ROTATION

When the blade is rotating, the additional stiffness due to centrifugal forces should be considered. The centrifugal forces induce several additional coupling terms in the already complicated equation of motion. These terms can be developed in the form of rotational potential energy and is given by Carnegie (3.30).

\[ V_c = \frac{1}{2} \rho A \omega^2 \int_0^L \left( (r+z) \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right) dz - \frac{W}{2} \left( u^2 \cos^2 \phi + v^2 \sin^2 \phi \right) \]

\[ z \quad (3.3) \]

3.2.3 RELATIONSHIP BETWEEN COORDINATE AXES

The relationship between \( u_l \) and \( u \), \( v_l \) and \( v \), being given by

\[ u_l = u + d_y \theta \quad (3.4) \]

\[ v_l = v + d_x \theta \quad (3.5) \]

For the sign convention of Figure (3.1)

\[ d_x = d_x \cos (\beta z + \phi_1) - d_y \sin (\beta z + \phi_1) \quad (3.6) \]

\[ d_y = d_x \sin (\beta z + \phi_1) + d_y \cos (\beta z + \phi_1) \quad (3.7) \]
The pretwist along the blade length can be written as

$$\gamma = (i-1) \frac{\alpha}{L} l + \frac{\alpha}{l} z$$  \hspace{1cm} (3.8)$$

where $i$ is the order number of element.

The relationship between $I_{xx}$, $I_{xy}$, $I_{yy}$ and $I_{xx}$ and $I_{yy}$ are given by

$$I_{xx} = (S - H \cos 2\gamma) I_{xx} \quad (3.9)$$
$$I_{yy} = (S + H \cos 2\gamma) I_{xx} \quad (3.10)$$
$$I_{xy} = H \sin 2\gamma I_{xx} \quad (3.11)$$

where

$$S = \frac{1}{2} \left[ \frac{I_{yy}}{I_{xx}} + 1 \right]$$
$$H = \frac{1}{2} \left[ \frac{I_{yy}}{I_{xx}} - 1 \right]$$

3.2.4 KINETIC ENERGY

The kinetic energy, $T$, of a blade vibrating in combined bending-bending-torsion is given by Carnegie$^{(3.30)}$,

$$T = \frac{\rho}{2} \int_{0}^{L} \left\{ I_{cg} \dot{e}^2 + A \left( \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) \right\} \, dz \quad (3.12)$$
By writing kinetic energy in terms of centre of flexure coordinates, from Figure (3.2)

\[ u_1 = u - R \cos(\gamma + \alpha' + \theta) \]  \hspace{1cm} (3.13)

\[ v_1 = v + R \sin(\gamma + \alpha' + \theta) \]  \hspace{1cm} (3.14)

Taking \( \theta \) to be very small then,

\[ R \cos(\gamma + \alpha') = R \cos(\gamma + \alpha' + \theta) = dx \]  \hspace{1cm} (3.15)

\[ R \sin(\gamma + \alpha') = R \sin(\gamma + \alpha' + \theta) = dy \]  \hspace{1cm} (3.16)

By differentiating equations (3.15) and (3.16) with respect to time and inserting equations (3.4) and (3.5), we get

\[ \dot{u}_1 = \dot{u} + dy \dot{\theta} \]  \hspace{1cm} (3.17)

\[ \dot{v}_1 = \dot{v} + dx \dot{\theta} \]  \hspace{1cm} (3.18)

\[ T = \frac{1}{2} \int_0^L \rho \left[ I_{cg} \dot{\theta}^2 + A(\dot{\omega}^2 + \dot{u}^2 + \dot{v}^2) + A(dx^2 + dy^2)(\dot{\theta}^2 + \dot{\phi}^2) + 2A(dx \dot{u} \dot{\phi}) + 2A(dy \dot{v} \dot{\phi}) \right] \, dz \]  \hspace{1cm} (3.19)

where

\[ R^2 = dx^2 + dy^2 \]  \hspace{1cm} (3.20)

\[ I_{cf} = I_{cg} + AR^2 \]  \hspace{1cm} (3.21)
The approximate displacement functions, along the directions which will be assumed that describe the displacement shape close enough to the true shape, can be express in general form as

\[ y = \sum_{i=1}^{r} a_i \eta_i \]  

(3.22)

Equation (3.22) can be written, for known degrees of freedom, as

\[ u = a_1 + a_2 z + a_3 z^2 + a_4 z^3 \]  

(3.23)

\[ v = a_5 + a_6 z + a_7 z^2 + a_8 z^3 \]  

(3.24)

\[ w = a_9 + a_{10} z \]  

(3.25)

\[ \theta = a_{11} + a_{12} z \]  

(3.26)

Equation (3.22) represents the relationship between the general displacement \( y \) and the generalised displacement amplitudes \( a \)'s and can be written in matrix notation as

\[ y = [\phi][a] \]  

(3.27)

The displacement of the nodes are determined by substituting the
modal coordinates into the model. Hence the following matrix equation can be obtained

\[ (q) = [C](a) \]  \hspace{1cm} (3.28)

or \( (a) = [C]^{-1}(q) \) \hspace{1cm} (3.29)

where \( (q) \) is the vector of nodal displacements.

From equations (3.27) and (3.29)

\[ y = [\phi][C]^{-1}(q) \]  \hspace{1cm} (3.30)

\( [C] \) is a 12 x 12 square matrix and is called the coefficient matrix which is given by Sabuncu (3.40).

Now all the displacement functions can be expressed in matrix form, from equations (3.23-3.26) as follows

\[ u = [\phi_1](a) \]  \hspace{1cm} (3.31)

\[ v = [\phi_2](a) \]  \hspace{1cm} (3.32)

\[ w = [\phi_3](a) \]  \hspace{1cm} (3.33)

\[ \theta = [\phi_4](a) \]  \hspace{1cm} (3.34)

Substituting equation (3.29) into equations (3.31-3.34) thus

\[ u = [\phi_1][C]^{-1}(q) \]  \hspace{1cm} (3.35)
3.2.6 FORMULATION OF THE STIFFNESS AND MASS MATRICES OF A BEAM ELEMENT

3.2.6a STIFFNESS MATRIX

Making use of sections (3.2.3) and (3.2.5) and substituting the convenient variables into equations (3.2) and 3.3, the potential energy of a stationary blade in matrix form can be expressed as

\[
V_s = \frac{1}{2} \frac{EI}{L^3} [q]^T [C]^{-T} [k_s][C^{-1}][q] \tag{3.39}
\]

and the potential energy due to rotation can be expressed in matrix form as

\[
V_c = \frac{1}{2} \frac{EI}{L^3} [q]^T [C]^{-T} [k_r][C^{-1}][q] \tag{3.40}
\]

Substituting equations (3.39) and (3.40) into equation (3.1), the total stiffness matrix of a rotating pretwisted asymmetrical aerofoil cross-section blade can be obtained.

For \([k_s]\) and \([k_r]\) as given by Sabuncu (3.40).
Again making use of sections (3.2.3) and (3.2.5) and substituting the convenient variables into equation (3.19), the kinetic energy of a beam element becomes

$$T = \frac{\rho A L p^2}{2} (q)^T [C]^{-T} [m] [C^{-1}] (q)$$  \hspace{1cm} (3.41)

For $[m]$ as given by Sabuncu (3.40)

**3.2.7 SOLUTION OF THE EIGENVALUE PROBLEM**

The aim of this analysis is to find the eigenvalues and corresponding eigenvectors of the blade under consideration.

Noting from previous sections that the total potential energy is a function of the generalised displacements, and the kinetic energy is a function of the derivative with respect to time of the generalised displacements, from Lagrange's equation

$$\frac{dV}{dq} - \frac{d}{dt} \frac{dT}{dq} = 0$$  \hspace{1cm} (3.42)

Equation (3.42) gives rise to the eigenvalue problem

$$[K] - p^2 [M] [q] = 0$$  \hspace{1cm} (3.43)

Equation (3.43) is the equation of the free vibration of the beam. Where $[K]$ and $[M]$ are the assembled stiffness and mass matrices of the beam.
The assembly procedure can be carried out very easily by a simple computer program to form the above mentioned matrices to get the necessary convergence.

The number of degrees of freedom will be increased during the assembly procedure as the number of the discretized elements increases.
In this chapter the vibrations of cantilever blades having asymmetric pretwisted aerofoil cross-section are investigated, which show the effects of rotation and pretwist on the natural frequencies together with different mode shapes of the vibrating blade.

The coordinates of the shear centre, centroid and area of cross-section were calculated with the aid of numerical integration of a given cross-section of the blade bounded by a number of points in the x and y directions.

The validity and accuracy of the analysis developed here have been assessed by comparing these results with the results of other investigators.

A comparison has been made in Table (3.1) for the frequencies of the first four modes of a uniform pretwisted blade experiencing coupled bending-bending vibration for which experimental results has been obtained by Carnegie(3.7) and theoretical results by Subrahmanyam(3.13) using Reissner methods. Good agreement has been shown between the present finite element method and the other two methods.

Table (3.2) shows a comparison between the present Finite Element Method and Galerkin and experimental methods of Subrahmanyam(3.37) of a rotating straight blade. Good agreement between the results is found. This table also shows the change of frequencies of five modes during the rotation of the blade. An increase in the frequencies at higher speed is significant due to the
stiffening of the blade by the centrifugal force except for the
torsional frequencies which are not affected by the rotation. Thus
it can be concluded that the effect of rotation is more significant
on the first uncoupled or coupled bending mode and that the effect on
higher modes decreases. The uncoupled and coupled torsional modes
are not greatly affected by rotation.

If the blade is pretwisted more coupling would be introduced.
Tables (3.3-3.6) show the effect of pretwist on the non-dimensional
frequency parameter $\omega a^2/\rho t/D$, which is a traditional one used for
flat plates, of vibrating blade having different aspect ratio and
different width to thickness ratio. The most important effect of
pretwist is to increase the frequencies of the torsional modes
considerably as shown in the fifth mode of Table (3.3). Less
significant changes occur in the bending frequencies. A comparison
between Tables (3.3 and 3.4) and (3.5 and 3.6) indicates that as the
thickness increases the frequency parameter of the higher modes
decrease very rapidly. This is probably because vibration modes
which involve predominantly chordwise bending are completely missed
for small thickness blades. Moreover as the thickness decreases the
difference between the flexural rigidities increases and the coupling
becomes stronger in addition to the coupling due to the pretwist
angle.

With low aspect ratio (a/b), a comparison between Tables (3.3
and 3.5) and Tables (3.4 and 3.6) one can see that another second
mode frequency appears for (a/b) equals one and shifts the
frequencies of higher modes when (a/b) equals three due to plate
effects and because the frequency parameter used contains (a), so
when (a) is high for high aspect ratio higher frequency parameters
are obtained.
A general study of Tables (3.3-3.6) indicates a very little increase in frequency parameter of the first mode with higher pretwist angle. This is due to coupling between bending in the yy and xx directions with a torsional movement. The frequency parameter of the second mode is decreased largely with the increase of pretwist angle. Therefore, at zero pretwist, the vibrational movement can be considered as pure bending in the yy direction. The decrease in frequency with pretwist is also caused by coupling between bending in the yy and xx directions with no torsional displacement. The third mode has exactly identical behaviour to that of the fundamental bending mode. A large decrease in frequency parameter of the fourth mode can be noticed which is similar to that of the second mode of vibration. The torsional frequencies are typically low for small pretwist angles and too large for the higher modes at large pretwist angles as can be seen in mode five of Table (3.3).

Effect of rotation on the natural frequencies of vibration of the first five modes of pretwisted aerofoil cross-section cantilever blade is shown in Figure (3.3). The pretwist angle and stagger angle used were 45 and 60 degrees respectively. Other physical properties are given in Table (3.1). A general increase in the natural frequency of vibration of the coupled and uncoupled bending modes is obvious. It can be seen that the coupling effect is increased at higher speeds. This is natural due to the stiffening effect of the higher centrifugal force, the strongest of which is between the second and the third mode. The fourth mode here is a torsional mode and it is not greatly affected by the rotational conditions.
In Figure (3.4) a further study of the effect of pretwist angle on the natural frequency of vibration of a cantilever blade of pretwisted aerofoil cross-section. Other physical properties are given in Table (3.1). As explained earlier there is a slight change in natural frequencies of first mode although it is not clear on the graph. For the second mode of vibration the frequency is decreasing with the increase of pretwist, this is caused by the coupling between bending in the \( yy \) and \( xx \) directions with no torsional movement. A strong coupling between the third mode and the fourth mode is clear in Figure (3.4) at 40 pretwist angle. As the pretwist angle increases a considerable increase in the natural frequency of the third mode is evident. The motion consists of bending in the \( xx \) and \( yy \) directions coupled with torsion. As the pretwist angle increases the natural frequency of the fourth mode increases. The motion consists of bending in the \( yy \) and \( xx \) directions coupled with torsion. For the fifth mode, with the increase in pretwist angle a change in natural frequency occurs. The motion consists of bending in both \( yy \) and \( xx \) directions coupled with torsion and bending in the \( \chi \) direction is predominant.

The coupled bending-bending-torsional mode shapes obtained for 0 rpm and 5000 rpm speeds of rotation are presented in Figures (3.5–3.14) for 45 degrees pretwisted blade. Other physical properties are given in Table (3.1). A comparison between rotating and non-rotating cases indicates that there are no large changes in the shapes of the displacements and slopes. There are slight changes in the amount of displacement due to the increase of the stiffness properties due to centrifugal effects. Inspection of Figures (3.5) and (3.10), for the first mode indicates that the motion in the \( yy \) direction is predominant whereas the torsional displacement \( \theta \) and the longitudinal
displacement \( w \) are almost zero. The motion in the \( xx \) direction is small. The large displacement of the mode shape of the second mode is in the \( xx \) direction as can be seen from Figures (3.6) and (3.11). The component in the \( yy \) direction is also large compared with the component in the \( xx \) direction of the first mode. The torsional displacement this time is much larger whereas the longitudinal displacement is also zero. For the third and fourth modes as can be seen from Figures (3.7 and 3.12) and (3.8 and 3.13), the predominant relative amplitude is the torsional displacement whereas the displacements in the \( xx \) and \( yy \) directions share the same magnitude with different directions. From Figures (3.9) and (3.14) it can be seen that the predominant motion is in the \( xx \) direction where the bending displacement \( u \) is maximum. The relative amplitude \( v \) of the \( yy \) direction motion is also high compared with that of the lower modes.

It can be concluded that the modes of vibration of pretwisted asymmetrical aerofoil cross-section blades are of the coupled bending-bending-torsional type. The coupling with torsion is very small in the first and second modes of vibration and could be neglected. However, the third and forth modes in particular, and the fifth mode to a lesser extent, are complex modes of vibration with bending motion in the \( yy \) and \( xx \) directions coupled with torsion. When the beam vibrates in the plane of rotation, a destabilizing (i.e., frequency reducing) body force arises due to the local component of circumferential displacement. However, inasmuch as the radially directed component must also exist simultaneously, and is more significant than the circumferential component, the resulting frequencies are: (a) \( A \) for the nonrotating case, but (b) less than for vibrations perpendicular to the plane of rotation.
Frequencies of all the bending and torsional modes are increased by the radial body force components and decreased by the circumferential components, the lower frequencies more than the higher ones, and the torsional ones more than the bending ones. These results were demonstrated by Carnegie, Stirling and Fleming(3.41).
TABLE 3.1

Comparison of Coupled Bending–Bending Frequencies (Hz):

Pretwisted Uniform Blade

The following numerical data has been used:

\[ L = 6.0 \text{ in} \]
\[ A = 0.068 \text{ in}^2 \]
\[ I_{xx} = 26.2027 \times 10^{-6} \text{ in}^4 \]
\[ I_{yy} = 56.667 \times 10^{-4} \text{ in}^4 \]
\[ \text{pretwist angle} = 45^\circ \]
\[ E = 30 \times 10^6 \text{ psi} \]
\[ \rho = 0.284 \text{ lb/in}^3 \]
\[ dx = 0.0 \]
\[ dy = 0.0 \]

<table>
<thead>
<tr>
<th>MODE NUMBER</th>
<th>NATURAL FREQUENCY (Hz)</th>
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<td>Present Finite Element (8 Elements)</td>
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<tr>
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<td>305.70</td>
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<td>3</td>
<td>951.20</td>
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<tr>
<td>4</td>
<td>1222.40</td>
</tr>
</tbody>
</table>
The following numerical data has been used:

\[
\begin{align*}
L &= 8.0 \text{ in} \\
A &= 0.36 \text{ in}^2 \\
d_x &= 0.117 \text{ in} \\
d_y &= 0.0 \\
I_{xx} &= 2.408 \times 10^{-3} \text{ in}^4 \\
I_{yy} &= 4.887 \times 10^{-2} \text{ in}^4 \\
\rho &= 0.283 \text{ lb/in}^3 \\
C &= 1.108 \times 10^5 \text{ lb in}^2/\text{rad} \\
E &= 30 \times 10^6 \text{ psi} \\
R &= 6.0 \text{ in} \\
\text{Stagger Angle} &= 90^\circ
\end{align*}
\]

### TABLE 3.2

Comparison of Coupled Bending-Torsion Frequencies of Rotating Blade

<table>
<thead>
<tr>
<th>SPEED OF ROTATION (RPM)</th>
<th>MODE NUMBER</th>
<th>NATURAL FREQUENCY (HZ)</th>
<th>Present Finite Element (8 Element)</th>
<th>Galerkin Process Subrham (3.37)</th>
<th>Experimental Results Subrham(3.37)</th>
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<td>144.7</td>
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<td>1700.0</td>
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</tbody>
</table>
Effect of pretwist angle on frequency parameter of a cantilever beam having the dimensions $a = 3$, $b = 20$, $t = 20b$.

<table>
<thead>
<tr>
<th>MODE NUMBER</th>
<th>$\omega^2 a / \sqrt{pt/D}$ (Pretwist Angle)</th>
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</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>33.454</td>
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<tr>
<td>4</td>
<td>58.85</td>
</tr>
<tr>
<td>5</td>
<td>67.08</td>
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<tr>
<td>6</td>
<td>100.37</td>
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<tr>
<td>7</td>
<td>115.33</td>
</tr>
<tr>
<td>8</td>
<td>167.37</td>
</tr>
<tr>
<td>9</td>
<td>190.65</td>
</tr>
<tr>
<td>10</td>
<td>234.7</td>
</tr>
</tbody>
</table>
TABLE 3.4

Effect of pretwist angle on frequency parameter of a cantilever beam having the dimensions

\[
\frac{a}{D} = 3 \quad \frac{b}{t} = 5
\]

<table>
<thead>
<tr>
<th>MODE NUMBER</th>
<th>( \omega^2 \gamma pt/D )</th>
</tr>
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<tbody>
<tr>
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<td>0.0</td>
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<td>5</td>
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<td>7</td>
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<td>8</td>
<td>87.1062</td>
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<td>105.0965</td>
</tr>
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<td>10</td>
<td>115.3327</td>
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</tbody>
</table>
TABLE 3.5

Effect of pretwist angle on frequency parameter of a cantilever beam having the dimensions:

\[
\frac{a}{b} = 1 \quad \frac{b}{t} = 20
\]

<table>
<thead>
<tr>
<th>MODE NUMBER</th>
<th>( \omega a^2 \sqrt{p/t/D} )</th>
<th>( \text{Pretwist Angle (degree)} )</th>
</tr>
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<td>103.8153</td>
<td>113.2</td>
</tr>
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</table>
TABLE 3.6

Effect of pretwist angle on frequency parameter
of a cantilever beam having the dimensions

\[
\begin{align*}
\frac{a}{b} &= 1 \\
\frac{b}{t} &= 5
\end{align*}
\]

<table>
<thead>
<tr>
<th>MODE NUMBER</th>
<th>( \omega a^2 \sqrt{\rho t/D} )</th>
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</thead>
<tbody>
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<td>8</td>
<td>40.7159</td>
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<tr>
<td>9</td>
<td>52.536</td>
</tr>
<tr>
<td>10</td>
<td>58.8553</td>
</tr>
</tbody>
</table>
Centroid

Section at distance $z$ from root

At root, axes $x, y$ and $XX, YY$ coincide

FIG. 3.1: GEOMETRICAL CONFIGURATION OF A BLADE CROSS-SECTION
FIG. 3.2: AXES SYSTEM
FIG. 3.3: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR THE FIRST FIVE MODES OF A CANTILEVER BLADE
FIG. 3.4: VARIATION OF NATURAL FREQUENCY AGAINST PRETWIST ANGLE FOR THE FIRST FIVE MODES OF A CANTILEVER BLADE
FIG. 3.5: FIRST MODE SHAPES OF A NON-ROTATING CANTILEVER BLADE
FIG. 3.6: SECOND MODE SHAPES OF A NON-ROTATING CANTILEVER BLADE
FIG. 3.7: THIRD MODE SHAPES OF A NON-ROTATING CANTILEVER BLADE
FIG. 3.8: FOURTH MODE SHAPES OF A NON-ROTATING CANTILEVER BLADE
FIG. 3.9: FIFTH MODE SHAPES OF A NON-ROTATING CANTILEVER BLADE
FIG. 3.10: FIRST MODE SHAPES OF A CANTILEVER BLADE ROTATING AT 5000 RPM
FIG. 3.11: SECOND MODE SHAPES OF A CANTILEVER BLADE
ROTATING AT 5000 RPM
FIG. 3.12: THIRD MODE SHAPES OF A CANTILEVER BLADE
ROTATING AT 5000 RPM
FIG. 3.13: FOURTH MODE SHAPES OF A CANTILEVER BLADE
ROTATING AT 5000 RPM
FIG. 3.14: FIFTH MODE SHAPES OF A CANTILEVER BLADE ROTATING AT 5000 RPM
CHAPTER 4

VIBRATION OF ROTATING RECTANGULAR CROSS-SECTION CURVED BEAM

4.1 INTRODUCTION

One of the ways of limiting vibration amplitude to meet life expectancy of turbomachinery is the incorporation of shrouds on the tips of the turbine blades, although they also provide a sealing function. This phenomena alters the vibration characteristics of the blades; therefore avoids some of the dangerous resonant frequencies of single blades and reduce the stresses induced in the blades by excitation under gas and dynamic forces. In some cases a certain number of blades are grouped together by a shroud to form a packet of blades (also known as bucket of blades or cascade of blades). To increase the damping capacity a lacing wire is used around the blades and the blades are mounted somewhat loosely on the disc.

In the flexural vibrations of a curved beam the central line of the undeformed beam is assumed to be a plane curve and its plane a principal plane of the beam at each point. Hence the uncoupling of the in-plane and the out-of-plane vibrations of a complete or incomplete ring, would be obtained. These two types of vibration problem of complete and incomplete rings have been the subject of interest of several investigators, due to its importance in many practical applications. Archer(4.1) used the classical equations of motion to study the in-plane vibrations of an incomplete circular ring of small cross-section with subtended angles between $\pi$ and $2\pi$. Volterra and Morell(4.2) applied a Rayleigh-Ritz method to find the
lowest natural frequencies of hinged elastic arcs having the centre lines in
the forms of a circle, a cycloid, a catenary and a parabola and vibrating
either in the plane of the initial curvature or in the plane perpendicular to
the plane of the initial curvature of the arcs. Volterra and Morell\(^{(4.3)}\)
again used Rayleigh-Ritz method to obtain the lowest natural frequencies of
the same arc in reference\(^{(4.2)}\) but with clamped ends and vibrating outside
the plane of initial curvature of the arc. Ojalvo\(^{(4.4)}\) used the classical
beam theory to study the vibrations of coupled out-of-plane bending and
twisting incomplete rings with clamped ends. Using Hamilton’s principle, he
developed the dynamic equations of an element of a circular arc including the
effects of linear viscous damping. Nelson\(^{(4.5)}\) used Rayleigh-Ritz method to
study the in-plane vibration of a simply supported circular ring segment.
Frequency equations are obtained in the form of infinite series. Washizu\(^{(4.6)}\)
developed an approximate theory for the vibrations of curved and twisted
slender beams. On the derivation of governing equations the effects of
torsion and transverse shear deformation have been taken into account. Ojalvo
and Newman\(^{(4.7)}\) predicted the inplane and out-of-plane frequencies of
clamped-clamped ring segments and presented design charts to determine the
frequencies for four modes of vibration. Culver\(^{(4.8)}\) studied the exact
lateral vibration of a thin curved beam. For the simply supported beam he
assumed that the normal stresses resulting from bending and torsional warping
are zero at the support. He also used Rayleigh-Ritz method to obtain the
approximate frequencies with other boundary conditions. Webster\(^{(4.9)}\)
obtained the exact lowest natural frequencies of uniform rectangular curved
panels with simply supported edges. He also used a general energy method to
study the problem with other boundary conditions. By using the flexibility
method, Reddy\(^{(4.10)}\) found the frequencies of lateral vibration of plane
curved bars by making a lumped mass approximation. Chang and Volterra\(^{(4.11)}\)
obtained the first four natural frequencies of elastic arcs, with built-in
ends, which vibrate in a plane perpendicular to that of the initial curvature
of the arcs where the centre lines of the arcs were in the forms of circles, cycloids, catenaries and parabolas. The Rayleigh-Ritz method was used by Culver and Oestel\(^{(4.12)}\) to find the approximate natural frequencies of horizontally curved girders used in bridge construction. However, this investigation has not dealt with the problem in its general form. Rao and Sundararajan\(^{(4.13)}\) studied the in-plane inextensional vibrations of a free ring and included the secondary effect of shear deformation and rotary inertia in their analysis. Rao\(^{(4.14)}\) used Hamilton's principle to develop the differential equation for the coupled bending and torsional vibration of a curved beam including the effects of shear deformation and rotary inertia. Petyt and Fleischer\(^{(4.15)}\) used the finite element method to investigate three sets of displacement functions of radial vibrations of a curved beam. A comparison has been made between the three models and the one with the cubic polynomial for the two displacements was found to be superior. Ahmed\(^{(4.16)}\) studied the in-plane vibrations of a curved sandwich beam and applied three different shape functions. He has concluded that even for the prediction of the natural frequencies of a curved sandwich beam Petyt and Fleischer's\(^{(4.15)}\) function was superior among the others. Sabir and Ashwell\(^{(4.17)}\) used a finite element method to compare four shape functions of a curved beam vibrating in its plane. They have found that the shape function based on simple strain functions was better than any other shape function. Coupled bending and torsional out-of-plane vibrations of a curved beam by the method of finite elements have been studied by Davis, Henshell and Warburton\(^{(4.18)}\). The effects of shear deformation and rotary inertia are allowed for in their analysis. Their method can be restricted to thin beams by excluding the effects of shear deformation and rotary inertia from the stiffness and mass matrices. Davis, Henshell and Warburton\(^{(4.19)}\) again used the same procedure as in reference\(^{(4.18)}\) but for the in-plane vibrations of curved beam. Chen\(^{(4.20)}\) used a wave propagation technique to study the vibration characteristics of an out-of-plane multispan curved beam based upon the use of
the dynamic three-moment equation. Thomas and Wilson(4.21) used straight beam finite element to get a satisfactory solution for the analysis of curved beams but required a large number of elements and the convergence was very slow. Petyt and Fleischer(4.22) applied the finite element method to determine frequencies and modes of a curved beam with six equally spaced hinged supports. They also carried out experimental investigation to confirm their previous theoretical work.(4.15). Bickford and Strom(4.23) developed two transfer matrices for in-plane and out-of-plane vibration of a constant curvature curved beam. The formulation of the transfer matrix was done by using equations of motion that incorporated the following quantities: shear deformation, extensional deformation, rotary inertia and the dislocation of the centroidal and neutral axis. Wang(4.24) used a Rayleigh-Ritz method to predict the natural frequency of out-of-plane vibration for a clamped elliptic arc. Kirkhope(4.25) used the energy method to derive the dynamic stiffness matrices which describe the out-of-plane coupled bending-torsion-shear vibration of thick circular rings. He assessed his work by comparison of calculated frequencies with experimental data for rings of both circular and rectangular cross-sections. Suzuki, Aida and Takahashi(4.26) studied the free out-of-plane vibration of arcs and curved beams on the basis of the classical beam theory. A study has been made for symmetric arc bars with clamped ends having the centre lines in the form of ellipses, sines, catenaries, hyperbolas, parabolas and cycloids. Irie, Yamada and Takahashi(4.27) studied the free out-of-plane vibrations of arcs based on the Timoshenko beam theory in which both the rotary inertia and shear deformation are taken into account by use of the transfer matrix approach. They have applied their method to bars of linearly, parabolically and exponentially varying rectangular cross-section, and the effects of the varying cross-section and slenderness ratio are studied. Wang, Nettleton and Keita(4.28) determined the natural frequencies of continuous curved beams vibrating out of their initial plane of curvature using the general equation of
motion. The general dynamic slope-deflection equations for horizontally circular curved members in terms of rotation, angle of twist and vertical displacement, have been derived. Suzuki and Takahashi(4.29) studied the free out-of-plane vibration of arcs based on the Timoshenko beam theory in which both the rotary inertia and shear deformation are taken into account. They obtained the natural frequencies and mode shapes of symmetric catenary, parabola and cycloid curved bars with clamped ends. They solved the equations of vibration by a series solution. Irie, Yamada and Takahashi(4.30) used the transfer matrix approach to determine the steady state out-of-plane response of a Timoshenko curved beam with internal damping. The neutral axis being expressed by an arbitrary function, in response to a sinusoidally varying force or moment. The method is applied to free-clamped non-uniform beams with circular, elliptical, catenary and parabolical neutral axes. Wang and Brannen(4.31) studied the natural out-of-plane vibrations of circular curved beams on elastic foundations. The frequency equation has been derived for clamped-clamped curved beam and the effects of foundation and angle of curvature have been illustrated. Irie, Yamada and Tanaka(4.32) used the transfer matrix technique to study the free out-of-plane vibration of arcs with the effects of rotary inertia and shear deformation and for different boundary conditions. Irie, Yamada and Tanaka(4.33) calculated the natural frequencies of out-of-plane vibration based on the Timoshenko beam theory for uniform arcs of circular cross-section under various combinations of boundary conditions. Irie, Yamada, and Okada(4.34) used the transfer matrix technique to study the free vibration characteristics of a ring elastically supported at several points and vibrating out-of-its plane. The effects of the number and the stiffness of supports and the slenderness ratio of rings on the vibration characteristics were studied. Lecoanet and Piranda(4.35) used Galerkin method to study the in-plane vibrations of circular rings with a radially variable thickness.
4.2 ANALYSIS

4.2.1 INTRODUCTION

This analysis is carried out for the case of vibration of in-plane and out-of-plane of a rectangular cross-section curved beam using a finite element technique and ignoring secondary effects. The shape functions of the finite elements are derived for both planes of vibration.

Stiffening of the beam element due to the centrifugal force effect is taken into account.

It is known that if the mass and shear centres of a symmetric cross-section, i.e., square, rectangular, etc., coincide there is no coupling between torsion and bending modes of vibration. This means that for a curved beam, although torsional and out-of-plane vibrations are coupled, in-plane and out-of-plane vibrations are independent of each other and can be examined separately. The in-plane and out-of-plane vibrations of the curved beam are combined together, later in this analysis, to form the vibrations of the curved beam in three directions. For curved beams with small width/thickness ratio, thin beam theory is applied.

4.2.2 IN-PLANE VIBRATION OF THE CURVED BEAM ELEMENT

4.2.2a DEFLECTION FUNCTIONS

A curved finite element is developed, whose shape functions are
derived from independent polynomial expressions for the general strain rather than displacements. The necessity for the exact representation of the rigid body displacement is then satisfied. Two deflection functions are used, given in reference (4.17), based on the strain energy concept. In order to derive a satisfactory shape function, simple strain functions are assumed and the strain displacement functions are integrated. The rigid body displacements are represented by the displacement function. From Figure (4.1) the nodal degrees of freedom are $V$, $W$, and rotation of the tangent, given by Sabuncu (4.36)

$$i = \frac{\partial W}{\partial y} - \frac{V}{R} \tag{4.1}$$

Each element has six degrees of freedom as follows

$$\{q_i\}^T = [V_i \ W_i \ i_i \ V_{i+1} \ W_{i+1} \ i_{i+1}] \tag{4.2}$$

For a curved beam the circumferential strain $\varepsilon_s$ and the change in curvature $\kappa_{zy}$ in $zy$ plane, are given by Belek (4.37).

$$\frac{\partial V}{\partial y} + \frac{W}{R} = \varepsilon_s \tag{4.3}$$

$$\frac{1}{R} \frac{\partial V}{\partial y} - \frac{\partial^2 W}{\partial y^2} = \kappa_{zy} \tag{4.4}$$

If $\varepsilon_s = \kappa_{zy} = 0$, equations (4.3) and (4.4) can be integrated to
give the complementary functions which represent the rigid body displacements.

If \( \varepsilon_s \neq 0 \) and \( \kappa_{zy} \neq 0 \), the following assumptions can be made,

\[
\varepsilon_s = \frac{a_4}{R} + \frac{a_5}{R} \phi \\
\kappa_{zy} = \frac{a_6}{R^2} + \frac{a_7}{R^2} \phi + \frac{a_8}{R^2} \phi^2 + \frac{a_9}{R^2} \phi^3
\] (4.5) (4.6)

Uniform strain and linear increase in curvature is assumed, hence using equations (4.5) and (4.6) and integrating these equations for \( W \) and \( V \), the following functions are obtained.

\[
W = a_1 \cos \phi + a_2 \sin \phi + a_4 - a_6 \phi = [W^*] (a) \\
V = a_1 \sin \phi + a_2 \cos \phi + a_3 + a_5 \phi + \frac{1}{2} a_6 \phi^2 = [V^*] (a)
\] (4.7) (4.8)

where, \( \phi = \frac{V}{R} \)

**4.2.2b STRAIN ENERGY EXPRESSION**

Strain energy of a curved beam element vibrating in its own plane is given by Belek(4.37)

\[
V_{sh} = \frac{1}{2} \{q_1\}^T [C_I]^{-1} \left\{ \begin{array}{c}
\int_{s} [V_{x/R}^* - \frac{W^*}{R}]^T E_s I_{xx} E_s^T \, dy \\
0 \\
\frac{V_{x/R}}{R} + \frac{W^*}{R}
\end{array} \right\} [C_I]^{-1} \{q_1\}
\] (4.9)
Equation (4.9) can be written as

\[
v_{sh} = \frac{1}{2} \{q_i\}^T [C_i^{-1}]^{-T}[k_i] [C_i^{-1}]^{-1}\{q_i\}
\]  

where

\[
k_i = \begin{bmatrix}
\left[ {v_x - \frac{1}{W_i/R}} \right]^T & 0 & \left[ {v_x/R + \frac{1}{W_i}} \right]
\end{bmatrix}
\]  

\[
[C_i^{-1}] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  

where

\[
Q = 12R^2/t^2
\]

\[
[B_i] = \begin{bmatrix}
-B_4 & B_1 & -RB_4 & -B_5 & -B_1 & -RB_5 \\
-\frac{B_3}{B_2} & B_2 & -RB_3 & B_1 & B_2 & RB_1 \\
B_4 & -B_3 & RB_4 & B_1 & B_2 & RB_5 \\
B_6 & -B_7 & RB_8 & B_6 & B_7 & RB_9 \\
-B_1 & B_2 & -RB_1 & B_1 & B_2 & RB_1 \\
\end{bmatrix}
\]  

where

\[
D = 2\cos\beta - 2 + \beta\sin\beta
\]

\[
B_1 = (\cos\beta-1)/D
\]

\[
B_3 = (1 - \beta\sin\beta - \cos\beta)/D
\]

\[
B_4 = (\sin\beta - \beta\cos\beta)/D
\]
\[ B_5 = (\beta - \sin \beta)/D \]
\[ B_6 = B_1 \beta/2 \]
\[ B_7 = B_2 \beta/2 \]
\[ B_8 = B_6 + 1/\beta \]
\[ B_9 = B_6 - 1/\beta \]

**4.2.2c KINETIC ENERGY EXPRESSION**

The kinetic energy expression of a curved beam under combined bending-bending displacements is given by Belek(4.37)

\[
T_{sh} = \frac{1}{2} \left( q_I \right)^T \left[ C_I \right]^T \rho_s A_s \int_{sh} \left\{ \begin{bmatrix} \dot{v}_x \\ \dot{w}_x \end{bmatrix} \right] \left[ \begin{bmatrix} \dot{v}_x \\ \dot{w}_x \end{bmatrix} \right] \, dy
\]

\[
[C_I]^{-1} \{ q_I \} \tag{4.14}
\]

Equation (4.14) can be written as follows

\[
T_{sh} = \frac{1}{2} \left( q_I \right)^T \left[ C_I \right]^{-T} \left[ m_I \right] \left[ C_I \right]^{-1} \{ q_I \} \tag{4.15}
\]

Where the coefficient matrix \([ C_I ]^{-1}\) and \([ q_I ]^T\) have been given in equations (4.13) and (4.2) respectively.

The matrix \([ m_I ]\) can be written as follows
4.2.3 OUT-OF-PLANE VIBRATION OF THE CURVED BEAM ELEMENT

The out-of-plane curved beam element used in this analysis is given by Belek(4.37), and is based on strain functions. The rigid body displacements are represented by the shape functions. The nodal degrees of freedom are \( U, \theta \) and the slope of \( U \), Figure (4.1),

\[
J = - \frac{\partial U}{\partial y} \tag{4.17}
\]
Each element has got six degrees of freedom and they can be written as

$$\{q_i\}^T = [\theta_i \ U_i \ J_i \ \theta_{i+1} \ U_{i+1} \ J_{i+1}]$$  \hspace{1cm} (4.18)

The strain displacement equations for a curved beam element vibrating out of its plane are given by Belek(4.37)

$$\epsilon_T = -\frac{1}{R} \frac{\partial U}{\partial y} + \frac{\partial \theta}{\partial y}$$ \hspace{2cm} (Torsional strain)  \hspace{1cm} (4.19)

$$\kappa_{xy} = -\frac{\partial^2 U}{\partial y^2} - \frac{\theta}{R}$$ \hspace{2cm} (Curvature)  \hspace{1cm} (4.20)

The strain displacement equations are integrated by making an assumption of uniform torsional strain and linear increase in curvature. Using equations (4.19) and (4.20) and integrating for U and $\theta$, in order to satisfy the rigid body displacement $\epsilon_T$ and $\kappa_{xy}$ are equated to zero and the complementary solutions are obtained. Adding these to the particular solutions of equations (4.19) and (4.20), the complete shape functions are given by Sabuncu(4.36),

$$U = a_1 R \cos \phi + a_2 R \sin \phi + a_3 R - a_4 R \phi - a_5 R \phi^2 = [U^*][a] \hspace{1cm} (4.21)$$

$$\theta = a_1 \cos \phi + a_2 \sin \phi - a_5 - a_6 \phi = [\theta^*][a] \hspace{1cm} (4.22)$$

where $\phi = \frac{y}{R}$
The strain energy of a curved beam vibrating in coupled bending and torsional mode, neglecting the secondary effects is given by Belek (4.37)

\[ V_{sh} = \frac{1}{2} \{q_0\}^T [C_0]^{-T} \left\{ \int_0^{l_{sh}} \begin{bmatrix} \frac{\theta_0}{R} + \frac{u_{11}}{R} \\ \frac{\theta_1}{R} - \frac{u_1}{R} \end{bmatrix} \right\} \left[ \begin{bmatrix} E_s I_{yy} \\ 0 \end{bmatrix} \frac{G_s J_{xx}}{E_s I_{yy}} \right] \]

\[ \int_0^{l_{sh}} \frac{\theta_0}{R} + \frac{u_{11}}{R} \right\} \left\{ [C_0]^{-1} \{q_0\} \right. \]

Equation (4.23) can be written as

\[ V_{sh} = \frac{1}{2} \{q_0\}^T [C_0]^{-T}[k_0][C_0]^{-1}\{q_0\} \quad (4.24) \]

where the matrix \([k_0]\) is given by

\[
[k_0] = \frac{E_s I_{yy}}{R} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Symmetric \[
C_2 = \frac{G_s J_{yy}}{E_s I_{yy}}
\]

where

\[
C_2 = \frac{G_s J_{yy}}{E_s I_{yy}}
\]
The coefficient matrix $[C_0]^{-1}$ is given by

$$
[C_0]^{-1} = 
\begin{bmatrix}
0 & -\frac{\beta}{R} & B_2 & 0 & B_1/R & B_3 \\
0 & \frac{B_5}{R} & -B_4 & 0 & \frac{B_5}{R} & -B_1 \\
0 & \frac{B_4}{R} & -B_2 & 0 & -\frac{B_1}{R} & -B_3 \\
\frac{1}{\beta} & 1/R\beta & 0 & 1/\beta & -1/R\beta & 0 \\
-1 & \frac{-B_1}{R} & B_2 & 0 & B_1/R & B_3 \\
\frac{1/\beta}{R\beta} & \frac{2B1}{R\beta} & -B_1 & \frac{-1}{\beta} & \frac{-2B1}{R\beta} & B_1
\end{bmatrix}
$$

(4.26)

where

$$D = 2-2\cos\beta-\beta\sin\beta$$

$$B_1 = (\cos\beta-1)/D$$

$$B_2 = (\beta\cos\beta-\sin\beta)/D$$

$$B_3 = (\sin\beta-\beta)/D$$

$$B_4 = (1-\cos\beta-\beta\sin\beta)/D$$

$$B_5 = \sin\beta/D$$
The kinetic energy of a curved beam undergoing combined bending and torsional displacements neglecting the rotary inertia effect is given by Belek(4.37),

\[
T_{sh} = \frac{1}{2} \{q_0\}^T \{C_0\}^{-1} \left[ \int_0^{l_{sh}} \begin{bmatrix} \dot{U}_x \\ \dot{\theta}_*R \end{bmatrix}^T \begin{bmatrix} \rho_s A_s & 0 \\ 0 & I_p/A_s R^2 \end{bmatrix} \begin{bmatrix} \dot{U}_x \\ \dot{\theta}_*R \end{bmatrix} \right] \right]
\]

\[
[C_0]^{-1}(q_0)
\]

Equation (4.27) can be written as follows

\[
T_{sh} = \frac{p^2}{2} \{q_0\}^T [C_0]^{-1} [m_0] [C_0]^{-1}(q_0)
\]

Where the degrees of freedom \([q_0]^T\) and the coefficient matrix \([C_0]^{-1}\) have been given in equations (4.18) and (4.26) respectively, and the matrix \([m_0]\) is

\[
[m_0] = \rho_s A_s R^3 \\
\begin{bmatrix}
\beta & -\beta^2/2 & 0 & -\beta^2/2 \\
C_1 M_4 & M_5 & M_6 & C_3 M_5 & C_4 M_6 \\
\end{bmatrix}
\]

Symmetric

\[
\begin{bmatrix}
\beta^3/3 & 0 & \beta^3/3 \\
C_3 \beta & C_3 \beta^2/2 & C_4 \beta^3/3 \\
\end{bmatrix}
\]
where

\[ M_1 = \beta/2 + \sin2\beta/4 \]

\[ M_2 = \sin2\beta/2 \]

\[ M_3 = 1 - \cos\beta - \beta\sin\beta \]

\[ M_4 = \beta/2 - \sin2\beta/4 \]

\[ M_5 = 1 - \cos\beta \]

\[ M_6 = \beta\cos\beta - \sin\beta \]

\[ C_3 = \frac{I_p}{A_s R^2} \]

\[ C_4 = 1 + C_3 \]

4.2.4 INCREASE OF STRAIN ENERGY IN THE SHROUD DUE TO THE CENTRIFUGAL FORCE

The strain energy of the curved beam, acting as a shroud attached at the tips of the blades, is increased due to the action of the centrifugal force induced when the system rotating with a constant rotational speed of the disc. The strain energy in the shroud, due to the centrifugal force has been derived by Sabuncu(4.36),

\[ V_{\text{csh}} = \frac{1}{2} \rho_s \omega^2 \int_0^{l_{\text{SH}}} \left[ \frac{1}{A_s} \left\{ -ZW + \frac{R+Z}{R} V \right\}^2 + 2W^2 \right] \, dA_{\text{dy}} \quad (4.30) \]
Where $A_g$ is area of the shroud cross-section. If the width of the shroud is kept constant and integration is carried out along the thickness of the shroud, then this yields the increase in strain energy of the shroud.

Using the deflection functions given in equations (4.7) and (4.8) and substituting in equation (4.30) the strain energy of the shroud due to rotation can be written as follows:

$$\Phi_{csh} = \frac{1}{2} \{q_I\}^T [C_I]^T [k_{sf}] [C_I]^{-1} \{q_I\}$$  \hspace{1cm} (4.31)

where

$$[k_{sf}] = X_1 [W_r] + \frac{X_2}{2} [V_r W_r] + X_3 [V_r]$$  \hspace{1cm} (4.32)

$$X_1 = - \rho_s \omega^2 \left(\frac{bt^3}{12} + 2bt\right)$$

$$X_2 = \omega^2 \rho_s \frac{bt^3}{24R}$$

$$X_3 = - \omega^2 \rho_s (bt + \frac{bt^3}{12R^2})$$

$$[W_r] =$$

$$\begin{bmatrix}
C_{2S} & 0 & 0 & 0 \\
C_{2S} & 0 & S & 0 & -S1 \\
0 & 0 & 0 & 0 \\
\text{Symmetric} & \beta & 0 & -\beta^2/2 \\
0 & 0 & \beta^3/3 \\
\end{bmatrix}$$  \hspace{1cm} (4.33)
The matrix $[C_1]^{-1}$ is given in equation (4.13)

Where

$$C_{2S} = \frac{1}{2} (\beta + \frac{1}{2} \sin 2\beta)$$

$$C_3 = \frac{1}{4} (\cos 2\beta - 1)$$

$$C_1 = \cos \beta - 1 + \frac{1}{3} \sin \beta$$
\[ C_2 = 2\beta \cos \beta + (\beta^2 - 2) \sin \beta \]

\[ S_2 S = \frac{1}{2} (\beta - \frac{1}{2} \sin 2\beta) \]

\[ S = -(\cos \beta - 1) \]

\[ S_1 = \sin \beta - \beta \cos \beta \]

\[ S_2 = 2\beta \sin \beta + 2 (\cos \beta + 1) - \beta^2 \cos \beta \]

The total strain energy of a rotating curved beam vibrating in its own plane is equal to the summation of strain energies of static and rotating cases and can be written, by using equations (4.10) and (4.31), as follows

\[ V = \frac{1}{2} \{q_1\}^T [C_1]\text{T}^{-1} \{k_{IC}] \{C_1\}^{-1}\{q_1\} \]

(4.36)

where

\[ [k_{IC}] = [k_{I}] + \frac{E}{R_s} I \frac{3}{I_{XX}} \]

4.2.5 COMBINATION OF THE STIFFNESS AND THE MASS MATRICES OF THE IN-PLANE AND OUT-OF-PLANE CURVED BEAM AND SOLUTION OF THE EIGENVALUE PROBLEM

To obtain the overall matrices which give the vibration of the curved beam in three directions the in-plane and out-of-plane curved beam element matrices are combined.

In-plane and out-of-plane curved beam stiffness and mass matrices can be obtained from equations (4.36), (4.24), (4.15) and (4.28) respectively.
To form the combination of the matrices, in-plane mass and stiffness matrices can be written as follows

\[ \rho_s A_s R [m_1] = \rho_s A_s \frac{l}{\beta} [m_1] \]  

\[ \frac{E_s I_{xx}}{R^3} [k_{IC}] = \frac{E_s I_{xx}}{\frac{l}{\beta} \beta^3} [k_{IC}] \]  

where, \( \beta = \frac{l}{sh} \)

Similarly, out-of-plane mass and stiffness matrices can be written as follows

\[ \rho_s A_s R^3 [m_0] = \rho_s A_s \frac{l}{sh} \beta [m_0] \]  

\[ \frac{E_s I_{yy}}{R} [k_0] = \frac{E_s I_{yy}}{\frac{L}{sh}} \beta [k_0] \]  

From equations (4.2) and (4.18) for the corresponding degrees of freedom of in-plane and out-of-plane curved beam element, the sequence of generalised coordinates are listed node by node according to the directions of the displacements with respect to the axis system we have

\[ \{q_s\}^T = \{q_i\}^T \{q_0\}^T \]  

or more explicitly

\[ \{q_s\}^T = [W_i V_i U_i \theta_i J_i i_i W_{i+1} V_{i+1} U_{i+1} \theta_{i+1} J_{i+1} i_{i+1}] \]
The following assumptions are made for the normalisation of the stiffness and mass matrices

\[
S_1 = \beta^3, \quad S_2 = R^2 \beta^3 \frac{I_{YY_s}}{I_{XX_s}}
\]

\[
M_1 = \frac{1}{\beta}, \quad M_2 = \frac{R^2}{\beta}
\]

In-plane and out-of-plane matrices are combined as follows

\[
[k_s] = \frac{E_s}{3} \frac{I_{XX_s}}{l_{sh}} S_1 [k_{IC}] \left[ \begin{array}{c|c}
0 & \hline
S_2 & [k_0]
\end{array} \right]
\]

(4.43)

and

\[
[m_s] = \rho_s A_s \frac{l_{sh}}{3} M_1 [m_1] \left[ \begin{array}{c|c}
0 & \hline
M_2 & [m_0]
\end{array} \right]
\]

(4.44)

Using equations (4.43) and (4.44) and multiplying by the coefficient matrices derived earlier the final form of the element stiffness and mass matrices of the rectangular curved beam can be obtained.
Noting from foregoing equations that the strain energies are a function of the generalised displacement and the kinetic energies are a function of the derivative with respect to time of the generalised displacements, then from Lagrange’s equation

\[
\frac{dV}{dq} - \frac{d}{dt} \frac{dT}{dq} = (Q)
\]  

which gives rise to the following eigenvalue problem

\[
\begin{bmatrix}
[k_{s}\end{bmatrix} - p^2 [m_{s}] \\
\end{bmatrix} \{q\} = (Q)
\]

On assembling the elements to form the curved beam and applying the boundary conditions \(Q\) becomes zero. Hence

\[
[K - p^2 M] \{q\} = 0 (4.46)
\]

Equation (4.46) is the equation of the free vibration of the curved beam. Where \([K]\) and \([M]\) are the assembled stiffness and mass matrices of the curved beam vibrating in three directions.
Vibration characteristics of a fixed-fixed rotating rectangular curved beam have been studied in this chapter. The effect of different subtended angle, rotation speed and thickness variation was analysed. A comparison between this investigation and other investigators results was carried out.

In Table (4.1) a convergence study has been investigated with the increasing number of elements for the first nine modes of vibration of a fixed-fixed curved beam. As can be seen from the table, the frequencies of vibration of the fundamental mode have converged for five elements. The frequencies of second and third modes of vibration converged for seven elements. Therefore it can be concluded that for the higher modes more number of elements are required for convergence.

A comparison was carried out in Table (4.2) of frequency parameter of a fixed-fixed curved beam having three different subtended angles. There is good agreement between the present finite element method and the results of the classical theory used by Rao\(^{(4.14)}\), Belek\(^{(4.37)}\) and Sabuncu\(^{(4.36)}\) with one internal node. The effect of rotary inertia and rotary inertia and shear deformation studied by Rao\(^{(4.14)}\) is obvious in columns one and two of Table (4.2). It can be seen from the table that extra modes appear in the higher modes of Belek's\(^{(4.37)}\) analysis, Sabuncu's analysis\(^{(4.36)}\) and present analysis as the subtended angle increases due to the effect of torsional vibration. For the three subtended angles studied (180\(^{\circ}\), 270\(^{\circ}\) and 360\(^{\circ}\)) it can be concluded that higher frequency parameters are associated with the stiff beams.
In Table (4.3) the frequencies of a cantilevered curved beam are given and compared with the results of Sabuncu(4.36). The lowest frequencies of the present analysis might be caused by the higher number of elements in spite of the incorporation of an internal node as in reference (4.36).

The effect of subtended angle on natural frequencies of fixed-fixed curved beam has been studied. Tabulated and graphical results are given in Table (4.4) and Figures (4.2) and (4.3) respectively for two different beam thicknesses. Decreasing the subtended angle tends to increase the stiffness while the mass matrix remains unchanged; this has the effect of increasing natural frequencies. That is why, as can be seen from Table (4.4) and Figures (4.2) and (4.3), as the subtended angle increases there is a decrease in natural frequencies for any mode of vibration. Another phenomenon can be concluded from this study that the natural frequency has a greater variation for higher subtended angles, because stiffer systems are associated with wider spacings between modal frequencies. A comparison between Figures (4.2) and (4.3) indicates that as the thickness increases the natural frequencies increase.

Effect of rotation on the natural frequencies of vibration of the first seven modes of a fixed-fixed rectangular curved beam is shown in Table (4.5) for a beam with subtended angle of 180° and Figures (4.4) and (4.5) which have a subtended angle of 45 degrees and different thicknesses of 0.1 inches and 0.22 inches respectively. It can be seen that there is a general increase in frequency for the first four modes with rotational speed due to stiffening effect. The natural frequency of the fifth mode is almost constant because this mode is a torsional mode. The sixth and seventh modes frequencies tend to increase.
are decreasing with the speed of rotation. Thus it can be concluded that the effect of rotation is more significant on the lower modes and that the effect on beam vibration higher modes decreases. The torsion modes are not greatly affected by rotation for the beam.

From Figures (4.4) and (4.5) it can also be concluded that high frequencies are associated with thicker curved beams.

Variation of curved beam frequency with its thickness is shown in Figures (4.6), (4.7) and (4.8) for the first six modes when the subtended angle is 25, 40 and 55 degrees respectively. Other physical properties are given in Table (4.1). As can be seen from these figures that the higher the subtended angle the less the frequencies and the frequency is increasing with the increase of the thickness.

The coupling between the modes is obvious at certain values of the thickness depending upon the angle of curvature.

In Figure (4.6) when the angle is 25° the coupling between the first two modes occur between 0.12 inches and 0.14 inch thickness whereas a strong coupling between second mode and third mode when the thickness is 0.18 inches. When the beam thickness is between 0.04 and 0.08 inches (numerical instability is obtained when the thickness is less than 0.04 in) the coupling is clear between third and fourth modes and between fourth and fifth modes when the thickness is 0.04 inches. Double coupling is occurring between fifth and sixth modes of vibration when the thickness is 0.08 and 0.22 inches respectively.
When the subtended angle is 40° as shown in Figure (4.7) there are different positions of coupling between the vibrating modes. The strong coupling this time is at beam thickness of 0.18 inches between first and second modes. The coupling between second and third modes occurs when the beam thickness is 0.1 inches. A double coupling between fourth and fifth modes when the thicknesses of the beam are between 0.1 and 0.12 inches and between 0.22 and 0.24 inches respectively. When the thickness is 0.15 inches a coupling between fifth and sixth modes can be noticed in the figure.

The same phenomenon happens between first and second modes and between second and third modes when the subtended angles are 40° and 55° as shown in Figures (4.7) and (4.8) respectively. The coupling between fourth and fifth modes of Figure (4.8) occurs when the thickness of the beam is 0.14 inches. There is double coupling between fifth and sixth modes when the beam thicknesses are 0.1 and 0.2 inches respectively.

It can be concluded from the above discussion that; the natural frequency of vibration of a curved beam is decreasing with the increase of the subtended angle and the rotational speed raise all bending frequencies but the torsional frequencies are not greatly affected, the coupling between the modes of vibration is obvious at certain beam thicknesses and the frequency is increasing with the increase of thickness of the beam.
**TABLE 4.1**

Convergence study of fixed-fixed curved beam.

The following data was used in the analysis:

\[E_s = 30 \times 10^6 \text{ lb/in}^2\]
\[\rho = 0.285 \text{ lb/in}^3\]
\[b = 0.345 \text{ in}\]
\[t = 0.0625 \text{ in}\]
\[R = 14 \text{ in}\]
\[\text{Angle} = 180^\circ\]

<table>
<thead>
<tr>
<th>Mode of Vibration</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>FREQUENCY (Hz)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>No. of Elements</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>31.0</td>
<td>118.0</td>
<td>1802.0</td>
<td>2311.0</td>
<td>3065.0</td>
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</tr>
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<td>62.0</td>
<td>78.0</td>
<td>224.0</td>
<td>457.0</td>
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<td>22.3</td>
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<td>54.0</td>
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<td>93.8</td>
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<td>187.0</td>
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<tr>
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<td>20.7</td>
<td>28.5</td>
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<td>82.7</td>
<td>120.0</td>
<td>144.6</td>
<td>178.5</td>
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</table>
TABLE 4.2
Comparison of Natural Frequencies of Circular arch with Fixed-Fixed End Conditions

The following data was used in the analysis

\[ \frac{E_s I_{yy}}{G J} = 1.0 \quad k = 0.833 \quad b/t = 1.0 \quad b/R = 0.5 \]

<table>
<thead>
<tr>
<th>Subtended Angle (degree)</th>
<th>Mode Number</th>
<th>[ \lambda = \frac{\rho A R^4}{E_s I_{yy}} ]</th>
<th>Belek (4.37) D.o.f. (21)</th>
<th>Sabuncu (4.36) with one internal node 3 elements</th>
<th>Present Analysis 4 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>Classical Theory</td>
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<td></td>
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<tr>
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<td>Rotary Inertia and shear</td>
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<td></td>
<td>Deformation</td>
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<td>-</td>
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</tr>
<tr>
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<td>T1</td>
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<td>-</td>
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<tr>
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</table>
TABLE 4.3
Comparison of out-of-plane Frequencies for a curved Cantilever Beam

The following numerical data was used in the analysis:

- \( A_s = 0.5 \times 0.5 \text{ in}^2 \)
- \( R = 10 \text{ in} \)
- \( \text{Angle} = 270^\circ \)
- \( E_s = 10^7 \text{ psi} \)
- \( G_s = 3.6 \times 10^6 \text{ psi} \)
- \( \rho_s = 0.1 \text{ lb/in}^3 \)

<table>
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<tr>
<th>Mode Number</th>
<th>NATURAL FREQUENCY (Hz)</th>
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<tr>
<td></td>
<td>Sabuncu (4.36) (3 elements)</td>
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<tr>
<td>1</td>
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TABLE 4.4

Variation of the natural Frequencies of fixed-fixed curved beam with angles of curvature

Number of elements = 4
For Numerical data see Table 4.1

<table>
<thead>
<tr>
<th>Angle of Curvature (degree)</th>
<th>Mode of vibration</th>
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<tr>
<td>30</td>
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### TABLE 4.5

Variation of the natural Frequencies of fixed-fixed curved beam with speed of rotation

Angle = 180°  
Number of elements = 7  
For physical properties refer to Table 4.1

<table>
<thead>
<tr>
<th>Mode of Vibration</th>
<th>FREQUENCY (Hz)</th>
<th>SPEED OF ROTATION (rpm)</th>
</tr>
</thead>
<tbody>
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<tr>
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</tr>
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</tr>
<tr>
<td>6000</td>
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</table>
FIG. 4.1: COORDINATE SYSTEM AND THE DISPLACEMENTS OF THE CURVED BEAM FINITE ELEMENT
THICKNESS 0.04, SPEED 0

FIG. 4.2: VARIATION OF THE NATURAL FREQUENCY AGAINST SUBTENDED ANGLE FOR THE FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.3: VARIATION OF THE NATURAL FREQUENCY AGAINST SUBTENDED ANGLE FOR THE FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.4: VARIATION OF NATURAL FREQUENCY AGAINST SPEED OF ROTATION
FOR FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.5: VARIATION OF NATURAL FREQUENCY AGAINST SPEED OF ROTATION FOR FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.6: VARIATION OF NATURAL FREQUENCY AGAINST THICKNESS FOR FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.7: VARIATION OF NATURAL FREQUENCY AGAINST THICKNESS FOR FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
FIG. 4.8: VARIATION OF NATURAL FREQUENCY AGAINST THICKNESS FOR FIRST SIX MODES OF CLAMPED-CLAMPED CURVED BEAM
Practically the turbine blades are formed into packets by steel bands. These bands are called shrouds when they are incorporated on the tips of the turbine blades of many aero-engine gas turbines. The shroud may be continuous, i.e., joining together all the blades in one ring, or segmented such that only a group of adjacent blades are interconnected. In the latter case, the blades are said to be grouped in packets. One or more lacing wires may be inserted at part-span on many fan and compressor stages or together with the shrouds. The effect of such lacing wire on vibration is to tie the blades together against pitch variation at the blades span especially when the blades are considerably long. The lacing wire can also be used to introduce damping into the blade assembly. The flexural rigidity of the wire also increases restraint against angular deflection at that span and the mass of the wire introduces inertia. The lacing wires provide some disadvantages since they act as obstacles to the fluid flow and therefore they reduce the blades thermal efficiency. The shroud is favourable since it reduces the fluid spillage over blade tips. This spillage is considerable, especially at the earliest stages where large heat drops and pressure gradient are usually found. Using shrouding and (or) lacing, changes the free vibration characteristics, thus allowing for some flexibility to the designer in avoiding the dangerous resonant
frequencies of single blades. The dynamic stresses induced in the blades due to the exciting gas-dynamic forces can be reduced to a great extent by means of forming blade packets.

Many researchers have dealt with the problems using different approximate methods. Campbell and Heckman (5.1) studied the tangential vibration of steam turbine buckets experimentally. Later in 1927, Stodola (5.2) and in 1933, Sezawa (5.3) studied the vibration of blade packets, but, due to the algebraic complexity of the equations, exact solutions were not sought. Prohl (5.4) has used A Holzer technique to calculate the nature frequencies and mode shapes of packeted blades. Smith (5.4) introduced a method for calculating the natural frequencies and mode shapes of a packet of parallel blades of uniform section, with the blades rigidly fixed at the roots and connected at the tips by a single shrouding strip attached to each blade. The effect of the shroud was assumed to induce shearing forces and bending moments at the blade tips. The shearing forces and bending moments were expressed in terms of the displacements and slopes of the shroud deflection curves. Both the blades and shroud have flexural rigidity but were assumed to be extensible. Consideration was also given to the effect of the introduction of a lacing wire into the shrouded packet. Reeman (5.6) has studied experimentally the effects of a continuous coverband (shroud) on the vibrations of turbomachinery. Jarrett and Warner (5.7) used the Myklestad's (5.8) adaptation of A Holzer's method and presented a numerical procedure to predict the natural frequencies and mode shapes of a rotating packet of blades. The blades were considered as tapered, twisted beams. Therefore, the coupling between the bending in the odgwise and chordwise directions was considered. This procedure was later extended by Prohl (5.9), who also modified the
process of computation to make it more applicable to digital computers. He applied his procedure to obtain the frequencies of a packet. In his analysis, the blades were considered fixed at their roots and the whole packet was idealized by stations of lumped masses, forces, and moments. Although the shear centre and centroid of the blade cross-section was assumed coincident, the axial and torsional vibrations were coupled through the shroud, whereas the tangential vibrations remained uncoupled. The same procedure has been used by Weaver and Prohl (5.10) to obtain the various natural frequencies (tangential, axial, and torsional) of a banded group of buckets and the corresponding mode shapes and vibration-stress levels of medium height buckets which have high natural frequencies and which operate with full 360-degree-arc diaphragms. Singh and Nandeeswaraiya (5.11) have analysed and obtained the natural frequencies of flexural vibration of shrouded turbine blades in the plane of the turbine disc and perpendicular to it on the basis of a rigidly jointed continuous framework. Deak and Baird (5.12) investigated the natural frequencies and mode shapes of a laced group of rotating steam turbine exhaust blades. In their analysis both flexural and torsional vibrations were considered. The bending moments, torsional moment, shearing force, flexural displacements, flexural slope and torsional displacement were taken as state vectors at the stations and by using the finite difference equations, which are derived from the differential equations of motion, the relationships between the state vectors at two adjacent stations were established. Reiger and McCallion (5.13) used the finite difference method of analysis to obtain the natural frequencies of portal frames, but the modal shapes of the frames were determined experimentally. Makarov (5.14) using a finite difference analysis studied the vibration of an assembly of staggered uniform blades on a
stiff disc with slack interlock shrouds. The interlock face was normal to the direction of flapwise motion and edgewise motion was neglected. Rao(5.15) used the variational principle to derive equations of motion for stationary packeted blades in tangential vibration. Hall and Armstrong(5.16) extended Makarov's work (5.14) to include edgewise bending vibration together with general interlock face angles using thin beam vibration theory. Sagendorph(5.17) used the finite element method to study the vibration characteristics of a mid-span shrouded fan blade. Experimental data obtained using holographic techniques are also presented. Two cases were investigated, shrouds free and shrouds locked up. Beam type modes, as well as modes involving plate-like deformations were studied. Thomas and Belek(5.18) studied the free vibration characteristics of straight shrouded blade packets using the finite element method. The effects of various weight ratios, flexural rigidity ratios, and length ratio's between the blades and shrouds on the frequencies of in-plane vibration of the packet were considered. They also showed that the in-plane vibration characteristics of a symmetric cross-section multibladed packet could be predicted from an interference diagram of a two-bladed packet. Srinivasan, Lionberger and Brown(5.19) investigated the dynamic behaviour of locking type laced fan blades which were assumed to be on a very stiff disc. The component mode procedure was applied to obtain the frequencies of the system. By introducing a particular type of viscous damping mechanism, the slip between the locking faces of the lacing was also accounted for. Salama and Petyt(5.20) have used the finite element method to predict the natural frequencies and mode shapes of packets of blades. Only the tangential vibrations have been considered. They also extended their work to make use of a cyclic symmetry phenomenon. A packet of six shrouded blades was analysed. The
effect of certain parameters such as stiffness ratio, mass ratio, the number of blades in the packet, the effect of rotation and the position of the lacing wires have been studied. Thomas and Sabuncu (5.21) have used the finite element method to study the vibration characteristics of a rotating asymmetric cross-section blade packet. The shroud was represented by a curved beam element and the blade was presented by a beam element. An internal node has been introduced which was found to make the convergence more rapid. The effect of various parameters such as the shroud thickness, shroud width, stagger angle, rotational speed, disc radius, and distance of shear centre from the centroid have been investigated. They have illustrated that from a simple model of a two-bladed packet, the vibration characteristics of a multi-bladed packet can be predicted with reasonable accuracy for the lower modes by drawing the interference diagram.
In the present chapter, in-plane and out-of-plane vibration of a rotating packet is considered. The blade is represented by a pretwisted aerofoil asymmetrical cross-section beam element, which represents the coupled bending-bending-torsion vibration, with six degrees of freedom at each of its two nodes. The shroud is represented by curved beam elements. The in-plane and out-of-plane displacements of the shroud are considered separately and then joined together to form a curved beam element undergoing motions in three directions with twelve degrees of freedom at its two nodes.

The derivation of the mass and stiffness matrices for blade and shroud elements is given in chapters three and four respectively, where these matrices are presented in an explicit form. The effect of rotation on the vibration characteristics is also given in the aforementioned chapters.

To ensure that the compatibility conditions are satisfied at the shroud-blade connections, the displacements in the local coordinates have to be converted to those in the global coordinates;

$$\{q_g\} = [T]\{q\} \quad (5.1)$$

This is done by the use of Transformation matrices which are given as follows:

$$[T] = \begin{bmatrix}
[T_1] & 0 & 0 & 0 \\
0 & [T_1] & 0 & 0 \\
0 & 0 & [T_1] & 0 \\
0 & 0 & 0 & [T_1]
\end{bmatrix} \quad (5.2)$$
where

\[
\begin{bmatrix}
T_{0\xi} \\
T_{0\theta} \\
T_{0z}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
l_{ox} & m_{ox} & n_{ox} \\
l_{oy} & m_{oy} & n_{oy} \\
l_{oz} & m_{oz} & n_{oz}
\end{bmatrix}
\]

(5.3)

(5.4a)

(5.4b)

(5.4c)

They show the matrices of direction cosines of the \(0\xi, 0\theta, 0z\) directions measured in the global axis system as seen from Figure (5.1).

If \(\phi\) is the stagger angle of the blade as shown in Figure 5.1 then the direction cosines of the blade element can be written as follows:

\[
l_{ox} = \cos 0 \\
m_{ox} = \cos 90 \\
n_{ox} = \cos 90
\]

\[
l_{oy} = \cos 90 \\
m_{oy} = \cos \phi \\
n_{oy} = \cos (90 + \phi)
\]

(5.5)

\[
l_{oz} = \cos 90 \\
m_{oz} = \cos (90 - \phi) \\
n_{oz} = \cos \phi
\]
Similarly the direction cosines for the shroud are written as follows:

$$l_{ox} = \cos 90 \quad m_{ox} = \cos 0 \quad n_{ox} = \cos 90$$

$$l_{oy} = \cos 0 \quad m_{oy} = \cos 90 \quad n_{oy} = \cos 90$$

$$l_{oz} = \cos 90 \quad m_{oz} = \cos 90 \quad n_{oz} = \cos 180$$

(5.6)

Therefore, before assembling the blade and shroud matrices the transformation into the global axes system has to be made by using the well known transformation operation:

$$[k_g] = [T]^T [k] [T]$$

(5.7)

$$[m_g] = [T]^T [m] [T]$$

(5.8)

where suffix g stands for the global axes system.

These elements global stiffness matrix ($k_g$) and mass matrix ($m_g$) can be assembled by a convenient method to get the packet system stiffness matrix ($K_g$) and mass matrix ($M_g$) comprising several elements. The dynamic stiffness relation for the packet becomes:

$$[K_g] \{q_G\} - p^2 [M_g] \{q_G\} = 0$$

(5.9)

where $q_G$ is the packet deflection vector.
The solution of equation (5.9), which is the equation of the free vibration of the packet, gives the natural frequency, \( p \), and the mode shapes of the blade packet.
In this chapter vibration characteristics of pretwisted aerofoil blade packet has been investigated. The blade element given in Chapter 3 and curved beam element given in Chapter 4 are used to resemble the present blade packet. A summarised study of the effect of different factors such as location of shroud along the blade length, pretwist angle, speed of rotation, angle of curvature and shroud length to blade length ratio ($L_r$) on the natural frequency of the packet is performed.

The finite element method is used to analyse a packet of two blades. The results obtained are compared with the experimental result of Thomas and Sabuncu (5.21) as shown in Table (5.2). The comparison of experimental results of Thomas and Sabuncu (5.21) and theoretical frequencies of this analysis of the same packet under various rotational speeds are also shown in Table (5.3). Good agreement between experimental and theoretical frequencies for some modes is obtained. Some discrepancies between calculated and measured frequencies of other modes are probably, as suggested by the above mentioned authors, caused by uncertainty over the correct values of Young's modulus and the density. But the major source of error is the effect of residual stresses in the test pieces. These residual stresses are mainly caused by the crude pretwisting procedure of the blades, and results in the change in flexural and torsional rigidity. Due to the interference of external effects, such as root fixing, friction of bearings, heat and random noise, which is picked up through the slip-ring, the experimental results of a rotating packet are slightly affected and not as accurate as the non-rotating values.
of the same packet which is fixed on a rigid base. Hence, some of
the experimental frequencies with weak signal have not be observed as
seen from Table (5.3).

The effect of rotation on the vibration characteristics of a
shrouded packet of two blades is considered in Figure (5.2) for the
first five modes. Also shown on the same figure the variation of the
first four modes natural frequencies of a single blade with
rotational speed by assuming the shroud mass to be acting at the tip
of the blade. The frequencies always increase with the increase of
speed of rotation. The increase is more pronounced at higher angular
velocities. The natural frequencies of the packet are higher than
the corresponding natural frequencies of the single blade. This is
because the stiffening effect of the centrifugal forces on the blade
and shroud is different.

The variation of the natural frequencies of the packet with the
blade pretwist angle is shown in Figure (5.3) for the first five
modes. In the case of the first mode, the natural frequency increases
with the increase of pretwist angle. Therefore an increase of
pretwist angle makes the packet stiffer. For the second and third
modes there is a decrease in natural frequencies with the increase of
pretwist angle this is caused by the coupling of blade bending modes
with no torsional movement. There is a slight increase of natural
frequencies of fourth mode with the increase of pretwist angle. The
fifth mode is highly affected by the increase of pretwist angle. This
increase is due to coupling between the two bending modes and the
torsional mode.
Figure 5.4 shows the effect of varying the position of a shroud along the blade span on the first five modes natural frequencies of a two-blade packet. The frequency is increased considerably if the shroud is fixed at a distance of about 0.7Lb from the blade root. Therefore to increase the natural frequency of a certain mode of vibration of a single blade, the shroud should be located at the position where it is expected to get a maximum amplitude in this particular mode shape. The optimum position to increase all the lower modes is about 0.7Lb for the modes considered.

Figure (5.5) shows the effect of increase of the subtended angle on the natural frequencies of the first five modes of the packet. It is quite clear from the figure that as the angle is low the natural frequencies of the packet are high. This is due to the increase in the stiffness of the system. Also the shroud mass acting at the tip of the blade decreases as the subtended angle decreases.

Figure (5.6) shows the frequency interference diagram for two-blade packet. Blade clamped-free and clamped-pinned frequencies are calculated by assuming the shroud mass to be acting at the tip of the blade. Packet frequencies are shown by broken lines while blade and shroud frequencies are shown in solid lines. At low Lr the frequency curves starts off as a blade mode frequency until the curve meets a shroud mode. At this junction the frequency curve follows the shroud mode frequency until it meets another blade frequency. A change of mode shape takes place and the curve changes direction again.
This curve can be used in synthesising the vibration characteristics of a two-bladed packet from the frequency characteristics of the independent frequencies of the blade and shroud.
<table>
<thead>
<tr>
<th>Physical properties of the blade packet used in this analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BLADE</strong></td>
</tr>
<tr>
<td>A = 0.0914 in²</td>
</tr>
<tr>
<td>L = 6 in</td>
</tr>
<tr>
<td>$\phi = 30^\circ$</td>
</tr>
<tr>
<td>$I_{xx} = 84 \times 10^{-6}$ in⁴</td>
</tr>
<tr>
<td>$C = 3240 \text{ lbf} \cdot \text{in}^2$</td>
</tr>
<tr>
<td>$dx = 76 \times 10^{-4}$ in</td>
</tr>
<tr>
<td>$\gamma = 30^\circ$</td>
</tr>
<tr>
<td>$I_{yy} = 671 \times 10^{-5}$ in⁴</td>
</tr>
<tr>
<td>$dx = 47 \times 10^{-3}$ in</td>
</tr>
<tr>
<td>$C = 3240 \text{ lbf} \cdot \text{in}^2$</td>
</tr>
<tr>
<td>$dy = 47 \times 10^{-3}$ in</td>
</tr>
</tbody>
</table>
TABLE 5.2
Comparison between theoretical and experimental
Frequencies of Thomas and Sabuncu(5.21)
and the present analysis Frequencies
of two-blade packet.

For physical properties see Table 5.1

\[ N = 6 \quad M = 5 \quad r_2 = 9.927 \text{ in} \]

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Experimental Results of Thomas and Sabuncu (5.21)</th>
<th>Present Finite Element Result using 96 D.o.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>97</td>
<td>96.88</td>
</tr>
<tr>
<td>2</td>
<td>476</td>
<td>382.5</td>
</tr>
<tr>
<td>3</td>
<td>501</td>
<td>499.4</td>
</tr>
<tr>
<td>4</td>
<td>585</td>
<td>741.3</td>
</tr>
<tr>
<td>5</td>
<td>829</td>
<td>869.8</td>
</tr>
<tr>
<td>6</td>
<td>910</td>
<td>980</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>1282.8</td>
</tr>
<tr>
<td>8</td>
<td>1606</td>
<td>1437.9</td>
</tr>
<tr>
<td>9</td>
<td>1671</td>
<td>1619.2</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>1910</td>
</tr>
<tr>
<td>11</td>
<td>-</td>
<td>2115.6</td>
</tr>
<tr>
<td>12</td>
<td>2616</td>
<td>2471</td>
</tr>
<tr>
<td>13</td>
<td>2927</td>
<td>2936.7</td>
</tr>
<tr>
<td>14</td>
<td>3298</td>
<td>3195</td>
</tr>
<tr>
<td>15</td>
<td>3867</td>
<td>3837</td>
</tr>
</tbody>
</table>
TABLE 5.3
Comparison between theoretical and experimental Frequencies of Thomas and Sabuncu (5.21) and the present analysis Frequencies of a rotating two-blade packet.
For physical properties see Table 5.1

\[ N = 6 \quad M = 5 \quad r_2 = 9.927 \text{ in} \]

<table>
<thead>
<tr>
<th>SPEED (rpm)</th>
<th>500</th>
<th></th>
<th>1000</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode Number</td>
<td>Experimental Results of Thomas and Sabuncu (5.21)</td>
<td>Present F.E. Results using 96 D.o.f.</td>
<td>Experimental Results of Thomas and Sabuncu (5.21)</td>
<td>Present F.E. Results using 96 D.o.f.</td>
</tr>
<tr>
<td>1</td>
<td>97</td>
<td>99</td>
<td>100</td>
<td>106</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>383</td>
<td>-</td>
<td>387</td>
</tr>
<tr>
<td>3</td>
<td>506</td>
<td>502</td>
<td>507</td>
<td>511</td>
</tr>
<tr>
<td>4</td>
<td>586</td>
<td>742</td>
<td>587</td>
<td>746</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>870</td>
<td>-</td>
<td>873</td>
</tr>
<tr>
<td>6</td>
<td>899</td>
<td>980</td>
<td>900</td>
<td>983</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>1283</td>
<td>-</td>
<td>1284</td>
</tr>
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<td>1603</td>
<td>1438</td>
<td>1608</td>
<td>1443</td>
</tr>
<tr>
<td>9</td>
<td>-</td>
<td>1621.6</td>
<td>-</td>
<td>1628.6</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>1910</td>
<td>-</td>
<td>1910.8</td>
</tr>
<tr>
<td>11</td>
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<td>2151</td>
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<td>14</td>
<td>3270</td>
<td>3197</td>
<td>3272</td>
<td>3204</td>
</tr>
</tbody>
</table>
FIG. 5.1: LOCAL AND GLOBAL COORDINATES OF A SECTION OF A PRETWISTED BLADE PACKET
FIG. 5.2: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR FIRST FIVE MODES OF TWO-BLADED PACKET
FIG. 5.3: VARIATION OF NATURAL FREQUENCY AGAINST PRETWIST ANGLE
FOR FIRST FIVE NODES OF TWO-BLADED ROTOR
FIG. 5.4: VARIATION OF NATURAL FREQUENCY AGAINST LOCATION OF SHROUD ALONG BLADE LENGTH FOR FIRST FIVE MODES OF TWO-BLADED PACKET
FIG. 5.5: VARIATION OF NATURAL FREQUENCY AGAINST SUBTENDED ANGLE FOR FIRST FIVE MODES OF TWO-BLADED PACKET
FIG. 5.6: VARIATION OF FREQUENCY PARAMETER AGAINST $L_r$ OF A PRETWISTED PACKET, (INERENCE DIAGRAM)
6.1 INTRODUCTION

The problem of choosing the disc configuration becomes more apparent with the wide spread use of higher power and light weight gas turbines.

In the design of the turbomachinery some geometrical limitations may be imposed on the profile of the disc by its functional aspects as well as the geometry of other parts of the turbine.

During rotation, very high stresses develop due to the centrifugal forces at high speeds. These stresses are added to the stresses resulting from bending, twisting and thermal gradient, and take a major portion of the total stresses and are not reduced by using thicker discs.

The vibration of turbine discs and of circular or annular plates is characterised by modes having integer numbers of nodal diameters and circumferential nodal circles. Much of the early work on plates and discs is summarised by Prescott(6.1) and Stodola(6.2).

The vibration of rotating discs has been quite well recognised since the classic works of Stodola(6.2) and Campbell(6.3). This vibration is also found to comprise of wave patterns involving integer numbers of nodal diameters and nodal circles, these patterns rotating forwards or backwards in the disc. The angular velocities of these waves in the disc are:
forward wave \( f_n/n \) revolutions/sec.

backward wave \(-f_n/n\) revolutions/sec.

where \( f_n \) is the frequency in cycles per second of the mode with \( n \) nodal diameters. When the disc rotates with angular velocity \( \Omega \) revolutions/sec, the wave velocities become:

forward wave \( \Omega + f_n/n \) revolutions/sec

backward wave \( \Omega - f_n/n \) revolutions/sec

Campbell and Stodola found that the dangerous condition of operation was such that the backward wave is stationary in space,

\[
f_n = n\Omega
\]

Thus a mode with \( n \) nodal diameters is strongly excited by the \( n \)th order of rotational speed. The major task of the designer is to avoid the dangerous resonant condition where the backward wave is stationary in space. This can be done by the accurate prediction of the natural frequencies of the disc; these frequencies while mainly dependent upon thin disc elastic and inertia properties can be essentially modified by in-plane stresses and transverse shear and rotary inertia.

Much of the recent work on plates and discs vibrations dealing with both classical theory and the effects of factors such as anisotropy, in-plane forces, variable thickness, surrounding media, large deflections, shear deformation, rotary inertia and material non-homogeneity has been reviewed by Leissa(6.5-6.8). These four review articles are supplementary to his earlier monograph on the vibration of plates(6.4). The transverse vibration of a circular plate of uniform thickness rotating about its axis with constant
angular velocity has been studied by Lamb and Southwell\(6.9,6.10\). They have separated the effect of rotation and have solved the vibration problem of the membrane disc. When both plate flexural stiffness and membrane forces are operative, the following relationship is used to get the natural frequencies of the disc,

\[ p^2 = p_1^2 + p_2^2 \]  

(6.2)

where \( p \) is the lower bound of the combined frequency of the rotating disc, \( p_1 \) is the frequency of the membrane disc where the plate flexure stiffness is neglected, and \( p_2 \) is the frequency of the stationary disc in which membrane stresses are absent. Aggarwall (6.11-6.13) solved the classical equations of motion to find the frequencies of symmetric and antisymmetric vibrations of a finite isotropic disc. Mindlin and Dereciewicz(6.14) have further developed the early work of Mindlin, by applying it to a circular disc, in which he derived the basic sixth order system of partial differential equations of motion along with potential and kinetic energy functions for plate bending. He has also given a consistent set of equations relating moments and transverse shears to transverse deflection and bending rotations. Conway(6.15) has investigated the transverse vibrations of some variable thickness plates when Poisson’s ratio is given particular values. Narayan Raju(6.16) has made a numerical investigation for uniform annular plates and for various sets of boundary conditions using the classical methods. Mote(6.17) has employed a Rayleigh-Ritz procedure to study the approximate vibration characteristics of variable thickness discs subjected to general membrane stresses. Vogel and Skinner(6.18) have given numerical data for the exact calculation of the natural frequencies of uniform circular and annular plates with various boundary conditions. Harris(6.19) has developed an exact solution for the free vibration
of circular plates with parabolic thickness variations. Ghosh(6.20) has extended the approach used by Lamb and Southwell(6.9,6.10) to discs of variable thickness. Eversman and Dodson(6.21) have employed a Rayleigh–Ritz procedure to study the approximate transverse vibrations of a spinning, centrally clamped circular disc. Olson and Lindberg(6.22) have applied the finite element method to develop and use circular and annular sector elements for the analysis of uniform circular and annular plates. Kirkhope and Wilson(6.23) have used the finite element method to develop and study the vibrations of uniform and variable thickness circular and annular plate elements. The annular element has four degrees of freedom, while the circular element has only two or three. Jain(6.24) solved the differential equation of motion to determine the combined effect of thickness variation and of the hydrostatic inplane force on the natural frequencies of circular thin plates. Barasch and Chen(6.25) solved the equation of motion of a clamped-free rotating disc, and determined the frequencies by reducing the fourth order equation of motion to a set of four first order equations. Gallego Juarez(6.26) solved the differential equation of motion to study the vibration characteristics of circular plates of stepped thickness. Ramakrishnan and Kunukkasseril(6.27) have determined the frequencies and corresponding mode shapes for the first four modes of ring-shaped sector plates with radial edges simply supported and circumferential edges free. Raju, Prakasa Rao and Venkateswara Rao(6.28) used the finite element method to study the vibrations of linearly tapered annular plates for various taper ratio with different boundary conditions. Pulmano(6.29) has developed an annular sector finite element with 16 degrees of freedom containing higher order coordinates at its four nodes, and with arbitrary thickness. Soni and Amba-Rao(6.30) have used Chebyshev collocation method to study the free vibrations of linearly
variable thickness annular plates. Frequencies and mode shapes have been computed for different values of taper constant and radii ratio for different combinations of boundary conditions. Ramaiah and Vijayakumar\textsuperscript{(6.31)} applied the Rayleigh–Ritz method to analyse the vibrations of linearly tapered annular plates for all combinations of boundary conditions. Rao and Prasad\textsuperscript{(6.32)} solved the dynamical equations of motion of isoparametric circular annular plates of uniform thickness to study the flexural vibrations for all combinations of boundary conditions taking into account the effects of rotary inertia and shear deformation. They have found that the effect of shear deformation on the frequencies of vibration was more prominent. Wilson and Kirkhope\textsuperscript{(6.33)} have applied the finite element method to develop and analyse the vibrations of an annular two nodes, eight degrees of freedom element with linear thickness. The degrees of freedom per node were the displacement, radial slope, and the radial and tangential shear rotations. The effect of rotation has been taken into account in their analysis and linear variation in stress within the elements has been assumed. Kirkhope and Wilson\textsuperscript{(6.34)} have used the finite element method to study the vibration and stress analysis of rotating variable thickness discs. The uncoupled radial and tangential stresses in a disc of uniform thickness have been obtained to examine the effect of a radially linear temperature variation across a disc having free inner and outer edge boundary conditions. Reissner's theory has been used, in the development of finite elements for moderately thick plates, by Bapu Rao, Guruswamy and Sampathkumaran\textsuperscript{(6.35)}. A sector plate bending element with 20 degrees of freedom has been used. Five degrees of freedom are considered at each nodal point: a lateral deflection; two bending rotations about the two coordinates axis, respectively, and two shear rotations about the two coordinate axes, respectively.
Kennedy and Gorman\textsuperscript{(6.36)} used a finite element method to investigate the dynamic behaviour of variable thickness discs subjected to rotation and thermal stresses. The variable thickness profile of the disc was represented by a series of annular elements of constant axial and radial thickness with each element having four degrees of freedom. Following Rayleigh-Ritz method for vibrating plate problem, Kunda and Basuli\textsuperscript{(6.37)} developed the method of initial functions in cylindrical coordinates for axially symmetric elasto-dynamic deformation of a circular plate. The governing equations are derived from the three dimensional elastodynamic equations and the free vibration frequencies of thick circular plates were determined. Gupta and Lal\textsuperscript{(6.39)} have solved the fourth order linear differential equation of motion by a power series development of the transverse deflection function to study the vibrations of a circular plate of linearly varying thickness. Mota Soares and Petyt\textsuperscript{(6.40)} have developed an annular element for the dynamic analysis of non-rotating, rotating and pre-stressed arbitrary discs using finite elements. The element is based on the Mindlin thick plate theory with two nodes, twelve degrees of freedom and parabolic thickness. Hutchinson\textsuperscript{(6.41)} used a series solution of the general three-dimensional equations of linear elasticity to find accurate natural frequencies and mode shapes for the flexural vibrations of thick free circular plates, which in turn is compared with the approximate thick plate theory. Gorman and Kennedy\textsuperscript{(6.42)} followed the same numerical method presented by them previously\textsuperscript{(6.36)}, with the disc modelled by annular elements having linearly varying axial thickness profile, to investigate the membrane effects upon the transverse vibration of linearly varying thickness discs. Reissner's theory has been used by Guruswamy and Yang\textsuperscript{(6.43)} to develop a 24 degrees of freedom annular sector plate bending finite element for the sake of bending and
vibration analysis. The six degrees of freedom at each nodal point include lateral deflection, radial rotation, circumferential rotation, radial shear rotation, circumferential shear rotation, and the second derivative of the deflection with respect to the radial and circumferential coordinates. Gupta and Lal (6.44) solved the governing differential equation with variable coefficients using Chebyshev collocation technique to study the vibrations of circular annular plates with linear variation in thickness under the action of a hydrostatic in-plane force on the basis of the classical theory of plates. Irie, Yamada and Kanda (6.45) used spline interpolation technique to study the centrifugal stress distributions and free vibration of rotating discs with radially varying thickness. They applied the method to free-clamped rotating discs with linearly, parabolically and exponentially varying thickness. Irie, Yamada, and Aomura (6.46) used the transfer matrix approach to study the free vibration of a Mindlin annular plate of linearly, parabolically, and exponentially varying thickness. Srinivasan and Ramamurti (6.47) used the finite element method to investigate the axisymmetric and asymmetric in-plane, free vibration of annular discs. Lenox and Conway (6.48) gave an exact, closed form, solution for the flexural vibration of a thin annular plate of parabolically varying thickness. Ramaiah (6.49, 6.50) used a Rayleigh-Ritz method to investigate the flexural vibrations of thin annular plates under the effects of rotation, thermal gradient and internal residual stresses for eight different combinations of clamped, simply supported and free boundary conditions. Nigh and Olson (6.51) developed a triangular finite element for the analysis of rotating discs. The investigation was carried out for the vibrations and steady state response of thin rotating discs with the effects of a viscous type damping.
6.2 ANALYSIS

6.2.1 INTRODUCTION

Since exact solutions of rotating discs are restricted to certain simple geometry and boundary conditions, numerical procedure must be used to analyse the practical turbine discs. The finite element technique which is found to be a powerful method of analysis has superiority over the other numerical procedures and when applied to the vibration analysis results in an algebraic eigenvalue problem.

The use of the circular and annular sector finite elements and even triangular elements in the vibration analysis of circular and annular discs results in an eigenvalue problem of considerable magnitude which can be overcome by the use of wave propagation technique. Inclusion of thickness variation and the effects of rotation in these elements would be quite involved. Hence it is desirable to develop simpler elements, particularly suitable for the vibration analysis of turbine discs, and which take advantage of the nature and geometry of the problem.

The main purpose of this investigation is to develop finite elements of annular sector geometry, in which radial thickness variation and the effects of in-plane stresses due to rotation can be easily introduced, and to examine the behaviour of these elements in the analysis of simple and complex discs profiles. The method of analysis is based on thin plate theory.
Owing to the complexity of the problem, especially when the thickness variation is introduced in the rotating discs, which makes the solution quite different from that when the thickness is constant, Gaussian integration is used for the derivation of the total stiffness and mass matrices of the discs. Zienkiewicz (6.52) has shown that the lengthy algebraic manipulations needed for the derivation can be avoided entirely with no penalty in computing time and with no loss of accuracy provided that the number of the sampling points in the Gaussian integration are used carefully. In addition to the forementioned advantages the Gaussian integration is recommended, for with virtually no additional effort, it allows any desired function to be included in the disc geometry. For example, with the program written for Gaussian integration, only one statement would have to be changed to move from the hyperbolic thickness variation to that for uniform thickness.

6.2.2 ELEMENT GEOMETRY AND DEFORMATION FUNCTION

Figure (6.1) shows the annular sector thin plate bending finite element to model the turbine disc under consideration with its four nodes and associated degrees of freedom. The degrees of freedom, at each node, are bending deflection, radial slope and tangential slope, and they are denoted by $u$, $\frac{\partial u}{\partial r}$, and $\frac{\partial u}{\partial \theta}$ respectively.
The following deflection function is found to be suitable for such analysis,

\[ u(r, \theta) = a_1 + a_2 r + a_3 \theta + a_4 r \theta + a_5 r^2 + a_6 \theta^2 + a_7 r^2 \theta + a_8 r \theta^2 + a_9 r^3 + a_{10} \theta^3 + a_{11} r^3 \theta + a_{12} r \theta^3 \] (6.3)

The deflection vector for the annular sector element is represented by the twelve term column vector as follows:

\[ \{q\}^T = [u_1 \ u_{r1} \ u_{\theta 1} \ u_2 \ u_{r2} \ u_{\theta 2} \ u_3 \ u_{r3} \ u_{\theta 3} \ u_4 \ u_{r4} \ u_{\theta 4}] \] (6.4)

Using equation (6.3) for the specified degrees of freedom and applying the boundary conditions at each corner by following the numbered sequence of Figure(6.1), the following equation can be obtained

\[ \{q\} = [C] \{a\} \] (6.5)

where \{a\} is the column vector of polynomial coefficients of equation (6.3) and [C] is the coefficient matrix.
The stiffness and mass matrices of the annular sector element are obtained by substituting the assumed deflection functions into the strain energy and kinetic energy expressions of the element and integrating over the sectorial area by following the procedure mentioned in Section (6.2.1).

For the thin plate annular element the strain energy is given by Wilson (6.33),

$$ U = \frac{1}{2} \int_0^{2\pi} \int_{r_1}^{r_2} \{x\}^T [V] \{x\} r \, dr \, d\theta $$

(6.6)

where

$$ [V] = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} $$

(6.7)

and

$$ D = \frac{E h^3(r)}{12(1-\nu^2)} $$

(6.8)

$$ \{x\} = \begin{bmatrix} -\frac{\partial^2 u}{\partial r^2} \\ -\frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} \\ \frac{2}{\gamma} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{2}{r} \frac{\partial u}{\partial \theta} \end{bmatrix} $$

(6.9)
Substituting equation (6.3) for $u$ in equation (6.9)

\[ \{x\} = [B] [C] \{q\} \]  

(6.10)

where

\[
[B] = \begin{bmatrix}
0 & 0 & 0 & 0 & -2 & 0 & -6r & 0 & -6r\theta & 0 & 0 \\
0 & \frac{1}{r} & 0 & \frac{9}{r} & -2 & \frac{2}{r^2} & -2\theta & \frac{\theta^2}{r} & -3r & \frac{6\theta}{r^2} & -3r\theta & \frac{\theta^3}{r} \\
0 & 0 & \frac{2}{r^2} & 0 & 0 & \frac{4\theta}{r^2} & 2 & 0 & 0 & \frac{6\theta^2}{r^2} & 4r & 0
\end{bmatrix}
\]  

(6.11)

Substituting equation (6.10) in equation (6.6)

\[
U = \frac{1}{2} \int_0^{2\pi} \int_0^{r_2} \{q\}^T [C]^{-T} [B]^T [V] [B] [C]^{-1} \{q\} \ r \ r \ d\theta 
\]

(6.12)

or

\[
U = \frac{1}{2} \int_0^{2\pi} \int_0^{r_2} \{q\}^T [C]^{-T} [k_d] [C]^{-1} \ r \ r \ d\theta \dfrac{d\phi}{d\theta} 
\]

(6.12)

The integration of equation (6.12) is carried out numerically by using $2 \times 2$ Gaussian points and any desired expression for the thickness $h(r)$ can be assumed and introduced in the equation before performing the integration. Eventually the elastic stiffness matrix is given by,

\[
[k_d] = [C]^{-T} [k_d] [C]^{-1}
\]  

(6.13)
The kinetic energy of the annular sector element is given by

\[
T = \frac{1}{2} \int_{0}^{r_1} \int_{0}^{2\pi} \rho h(r) \left( \frac{\partial u(r)}{\partial t} \right)^2 r dr d\theta
\]  

(6.14)

Substituting equation (6.3) in equation (6.14)

\[
T = \frac{1}{2} \int_{0}^{r_1} \int_{0}^{2\pi} \rho h(r) [q]^T [C]^{-T} [N] [C]^{-1} [q] \ r dr d\theta
\]  

(6.15)

where

\[
[N] = \begin{bmatrix}
1 & r & r^2 & r^3 & \cdots \\
0 & 1 & 2r & 3r^2 & \cdots \\
0 & 0 & 1 & 2r & \cdots \\
0 & 0 & 0 & 1 & \cdots
\end{bmatrix}
\]  

(6.16)

and the dot denotes the time derivative.

Following the same procedure as for the stiffness matrix but using 3 x 3 Gaussian points the mass matrix is given by

\[
[M_d] = [C]^{-T} [m_d] [C]^{-1}
\]  

(6.17)

### 6.2.4 STIFFNESS MATRIX DUE TO ROTATION

In practice the vibration of turbomachinery is significantly affected by the centrifugal force of rotation which must be taken into account in any realistic investigation. The stresses produced by these forces are proportional to the square of the rotational speed. In this section a stiffness matrix is derived which is dependent on the radial and tangential stresses present in the disc. This matrix would be added to the basic elastic matrix given in equation (6.13) to give the total stiffness matrix of the element.
When in-plane radial stress $\sigma_r$ and tangential stress $\sigma_\theta$, due to rotation, are present at the middle plane of the annular sector thin plate element, the following expression of the potential energy due to these stresses is given by Wilson\(^{(6.33)}\),

$$ U_\tau = \frac{1}{2} \int_0^{2\pi} \int_{r_1}^{r_2} \left[ \frac{\sigma_r}{r^2} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{\sigma_\theta}{r^2} \left( \frac{\partial u}{\partial \theta} \right)^2 \right] h(r) \, r \, dr \, d\theta \quad (6.18) $$

Radial and tangential stress expressions $\sigma_r$ and $\sigma_\theta$ have been derived by Biezeno\(^{(6.53)}\) for discs having the following profile,

$$ h(r) = \frac{c}{r^\beta} \quad (6.19) $$

where $c$ and $\beta$ are any real numbers, and $h(r)$ is the thickness at any radius $r$.

$$ \sigma_r = A_1 \, r^{\alpha_1} + A_2 \, r^{\alpha_2} - \alpha_1 \omega^2 \, r^2 \quad (6.20) $$

$$ \sigma_\theta = (\alpha_1 + 1 - \beta) A_1 \, r^{\alpha_1} + (\alpha_2 + 1 - \beta) A_2 \, r^{\alpha_2} - \beta \omega^2 \, r^2 \quad (6.21) $$

where

$$ \alpha_{1,2} = \frac{\beta}{2} - 1 + \gamma (1 + \beta \nu + \beta^2/4) \quad (6.22) $$

$$ \alpha = \frac{3 + \nu}{8 - 3 \beta - \nu \beta} \frac{\rho}{g} \quad (6.23) $$

$$ \beta = \frac{1 + 3 \nu}{8 - 3 \beta - \nu \beta} \frac{\rho}{g} \quad (6.24) $$

$$ A_1 = \frac{\sigma_0}{r_0^{\alpha_1}} - \frac{\sigma_1}{r_0^{\alpha_2}} + \frac{1}{\alpha_2} \left[ r_0^{\alpha_2} - \frac{r_0^{\alpha_1}}{\sigma_1} \right] \quad (6.25) $$
\[
A2 = \frac{\sigma \alpha_0 r^\alpha_0 - \sigma_0 r^\alpha_1}{\alpha_1 - \alpha_2} \left[ \frac{r^2}{r^2 - \frac{r^2}{r^2}} - \frac{r^2}{r^2} \frac{r^2}{r^2} \right]
\]

\[\omega : \text{angular velocity (rad/sec)}\]
\[\nu : \text{Poisson's ratio}\]
\[\rho : \text{density (lb/in}^3\text{)}\]
\[g : \text{gravitational acceleration (in/sec}^2\text{)}\]
\[\sigma_0 : \text{shrinkage radial stress (lbf/in}^2\text{)}\]
\[\sigma_a : \text{centrifugal radial stress (lbf/in}^2\text{)}\]
\[r_0 : \text{inner radius of disc (in)}\]
\[r_a : \text{outer radius of disc (in)}\]

The stresses \(\sigma_0\) and \(\sigma_a\) are calculated as follows:

a. The shrinkage stress at the hub at the inner radius \(r_0\) is given by \(2\pi r_0 p_0 h(r_0)\), where \(p_0\) is the shrinkfit pressure, which is usually negative, and \(h(r_0)\) the thickness at radius \(r_0\).

b. The centrifugal stress at the root of the blades at the outer radius \(r_a\) is given by \(2\pi m_b \omega^2 R\), where \(z\) is the number of blades, \(m_b\) mass of each blade and \(R\) is the distance of the centre of gravity of blade.

Assuming the deflection function, equation (6.3), as before, and substituting this equation together with equations (6.19), (6.20) and (6.21) in equation (6.18), additional stiffness coefficients for the annular sector element are derived, the additional stiffness matrix of which is,

\[ [K_d^T] = [C]^{-T} [k_d] [C]^{-1} \]
The integration of equation (6.18) is carried out following the same steps of that of equation (6.6).

The overall stiffness matrix of a rotating annular sector element is equal to the summation of equations (6.13) and (6.27),

\[
[K_d^T] = [K_d] + [K_d^r] \tag{6.28}
\]

### 6.2.5 ASSEMBLY PROCEDURE AND SOLUTION OF THE EIGENVALUE PROBLEM

The element stiffness matrix \((K_d^T)\) and mass matrix \((M_d)\) of equations (6.28) and (6.17) can be assembled by a conventional method together with the cyclic symmetry procedure, which was discussed previously in a separate section, to get the disc system stiffness matrix \(K_D\) and mass matrix \(M_D\), for a model of the disc comprising several elements. The dynamic stiffness relation for the disc becomes,

\[
[K_D] - p^2 [M_D] \{q_D\} = 0 \tag{6.29}
\]

where

\[
\{q_D\} \text{ is the disc deflection vector.}
\]

Equation (6.29) is the equation of the free vibration of the disc which is solved as an algebraic eigenvalue problem to give the natural frequencies and mode shapes of the disc.
In this chapter the behaviour of Finite Element model for the vibration characteristics of a rotating disc of variable thickness profile is investigated. The accuracy of this study is demonstrated by numerical comparison with other investigators. Any specified thickness variation in the radial direction can be easily incorporated. The variation of the thickness used in this analysis is in the form of \( h(r) = \frac{c}{r^\beta} \). The effects of different factors, such as; inner to outer radius ratio, nodal diameter, rotational speed and thickness variation due to different values of \( \beta \) on the natural frequencies are investigated. The mode shapes of the disc as the nodal diameter increases are also investigated.

For physical properties refer to Table (6.3) unless specified otherwise.

A comparison of the present analysis with the exact solution of Vogel and Skinner (6.18) of a clamped-free annular disc in terms of non-dimensional frequency parameter are given in Table (6.1), where \( m \) and \( n \) are the diametral and circumferential node numbers respectively, by setting \( \beta \) of the thickness formula equal to zero for different values of inner to outer radius ratios. A six element model with 36 degrees of freedom is used in this analysis in the radial direction. Good agreement between the present Finite Element and the exact solution is obtained. As can be seen from the table, the general tendency of the value of the non-dimensional frequency parameter in all modes is that it increases with increase of \( \frac{r_1}{r_2} \) ratio, i.e., as the clamped radius increases.
To examine the effect of the variation of the disc thickness on the natural frequency, a comparison between a Finite Element and the Rayleigh-Ritz method of Mote [6, 17] is given in Table (6.2) in which the disc has a thickness profile of the form \( h(r) = h_0 \left( \frac{r}{r_0} \right)^{-0.9} \). The agreement is good, with the finite element solution being generally the better.

When the disc is rotating at speed the centrifugal stresses developed increase the stiffness of the disc and the natural frequencies are consequently increased. Figures (6.2) and (6.3) show the relationship between the disc frequencies and number of nodal diameter for the first five modes of vibration when rotational speeds are 0 and 6000 rpm. These frequencies increase in their values with increasing number of diametral nodes and circular nodes.

From Figures (6.4-6.7) of the first four modes of vibration it is seen that when the mode of vibration increases the effect of rotation decreases substantially. Explanation of this phenomenon is consistent with the fact that concentration of the nodal lines around the centre of the disc makes the disc more rigid. In each of these figures the curves represent the rotation speeds of 0, 2000, 4000, 6000 and 8000 rpm respectively.

The effect of thickness variation of the form \( h(r) = \frac{0.5}{r^\beta} \) on the natural frequencies of 0, 2000, 40000 and 8000 rpm are presented in Figures (6.8-6.10) for different modes of vibration respectively. From all these figures it is seen that an increase in \( \beta \) gives a decrease in the natural frequency in all modes. The expected influence of the rotational speed on the
natural frequencies is greater in the lowest modes than in the case of the highest modes as it is seen from Figures (6.8), (6.9) and (6.10).

Mode shapes of the vibrating clamped-free disc for 0, 2 and 4 diametral nodes are shown in Figures (6.11), (6.12) and (6.13) respectively. Four modes of vibration are shown on each figure. It is seen from the figures that as the nodal diameters increase, the disc becomes more rigid. This can be observed by the marginal drop of axial displacements of the disc as shown in the mode shapes at higher nodal diameters. It is also seen that the circular nodes are shifted radially outwards at higher nodal diameters.


<table>
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<tr>
<th>MODE</th>
<th>( r_1/r_2 )</th>
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<tbody>
<tr>
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<tr>
<td></td>
<td>Exact Solution</td>
</tr>
<tr>
<td>m n</td>
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<td>7.53</td>
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<td>1 1</td>
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TABLE 6.2
Comparison between Rayleigh-Ritz method used by Mote(6.17) in terms of non-dimensional frequency $\omega/\rho r^2/Eh^2$, and Finite Element method of a variable thickness disc $h(r) = h_2 \left( \frac{r}{r_2} \right)^{-0.9}$.

<table>
<thead>
<tr>
<th>$\frac{r_1}{r_2}$</th>
<th>Nodal Diameter</th>
<th>Mote(6.17) Results</th>
<th>Present Finite Element (6 elements)</th>
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<td>3</td>
<td>23.289</td>
<td>22.921</td>
</tr>
<tr>
<td>0.7</td>
<td>3</td>
<td>34.912</td>
<td>34.350</td>
</tr>
<tr>
<td>Property</td>
<td>Value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------------------------------------</td>
<td>------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thickness at any radius $r$</td>
<td>$h(r) = \frac{c}{r^\beta}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Disc thickness constant</td>
<td>$c = 0.5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Disc outer radius</td>
<td>$r_2 = 8.0$ in</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Disc inner radius</td>
<td>$r_1 = 1.3$ in</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Young modulus</td>
<td>$E = 30 \times 10^6$ lbf/in$^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>$\rho_g = 0.285$ lbf/in$^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of elements</td>
<td>$N_R = 6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
FIG. 6.1: ANNULAR SECTOR FINITE ELEMENT
FIG. 6.2: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS FOR FIRST FIVE MODES OF NON-ROTATING CLAMPED-FREE DISC
FIG. 6.3: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS FOR THE FIRST FIVE MODES OF CLAMPED-FREE DISC ROTATING AT 6000 RPM
FIG. 6.4: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SPEEDS
FIG. 6.5: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SPEEDS
Fig. 6.6: Variation of natural frequency against number of nodal diameters of clamped-free disc rotating at different speeds.
3 NODAL CIRCLES: DIFFERENT SPEEDS

FIG. 6.7: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SPEEDS
FIRST MODE

FIG. 6.8: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF FORMULA

\[ h(r) = \frac{0.5}{r^\beta} \]

OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SPEEDS
FIG. 6.9: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE FORMULA $h(r) = \frac{0.5}{r^\beta}$ OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SEDS
FIG. 6.10: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE FORMULA $h(r) = \frac{0.5}{r^\beta}$ OF CLAMPED-FREE DISC ROTATING AT DIFFERENT SPEEDS
FIG. 6.11: MODE SHAPES FOR 0 NODAL DIAMETER CLAMPED-FREE NON-ROTATING DISC

FIRST MODE

THIRD MODE

FOURTH MODE

SECOND MODE

Distance along disc radius
FIG. 6.12: MODE SHAPES FOR 2 NODAL DIAMETERS CLAMPED-FREE
NON-ROTATING DISC

FIG. 6.13: MODE SHAPES FOR 4 NODAL DIAMETERS CLAMPED-FREE
NON-ROTATING DISC
INTRODUCTION

Studying the vibratory behaviour of the bladed-disc is of paramount importance to the designer of turbomachinery since this is in fact constitutes an important part of the turbine for it transmits torque from the blades to the shaft of the engine.

Due to the continuing emphasis on longer life, higher power and lighter weight together with reliable and safe operation in severe environments, it is now recognised that for accurate prediction of the vibration characteristics of turbine blades, the whole assembly of the bladed-disc should be considered.

Both experimental and theoretical studies indicate that coupling between the blades and the disc cannot be neglected.

The complexity of the system makes it very difficult to study the entire system with all its generalities. In general the components of the assembly are analysed separately and making several simplifying assumptions to facilitate the analysis.
In this chapter, the finite element method in conjunction with the wave propagation technique is used to study the free vibration characteristics of rotating, pretwisted, asymmetrical aerofoil cross-section bladed-disc, with the disc having a thickness profile of the form, \( h(r) = \frac{c}{r^\alpha} \). The effects of pretwist angle, staggering angle, rotational speed, and disc thickness variation are investigated. The finite elements of the blade and the disc of the previous chapters have been used to model the present bladed-disc.

The existence of coupling between the blades and disc and its influence on the natural frequencies has been studied and comparison is made with both theoretical and experimental results.

The vibration of turbine bladed-disc is found to be, similarly as the unbladed disc, characterised by diametral and circular nodes.

The coupling between blades and disc can lead in some cases to a fatigue failure. A recent example of fatigue failure of turbine rotor blades resulting from coupling between blade and disc vibration is described by Morgan, Lamport and Smith(7.1). Fatigue cracks were found either in the top serration of the fir tree roots or in the blade form starting at the trailing edge near the root. The resonance of the first flapwise mode with sixth order excitation was thought to be the most probable cause. Various modifications were made both to the blade fixings and to the disc which proved to be successful.
The early work reported on the problem of bladed-disc vibration was by Stodola(7.2). He attempted to solve it theoretically. In his approach, the differential equation governing the free vibration of discs was considered and the blades were assumed to impose distributed bending moment and shearing forces around the outer edge of the disc. Large numbers of very stiff blades were considered to derive the bending moment and the shearing force expressions. Having made these assumptions, the deflection shape of the blades in any mode was assumed to be a straight line having a slope at the outer edge of the disc. Rayleigh's method was applied by selecting the different deflection curves for different modes. Johnson and Bishop(7.3) have examined an idealised bladed rotor consisting of identical mass-spring elements to represent the blades, connected to a rigid free mass which represents the disc. They examined the principal modes of such a model and outlined methods for determining the receptances of the system. Ellington and McCallion(7.4) investigated the effect of elastic coupling, through the rim of the disc, on the frequencies of bending vibration using a simplified model. In this model the effect of twist, taper and obliquity is neglected and the blades are replaced by uniform blades fixed to the rim at their roots and vibrating in a plane parallel to the plane of the disc. For the analysis three adjacent blades are assumed to be parallel to each other and the portion of the rim joining them is taken as a straight continuous beam. A relationship between three slopes of the beam at the root of three adjacent blades are established and is used in the calculation of the natural frequencies. Armstrong, Christie and Hague(7.5) and Armstrong(7.6) have investigated bladed-disc frequencies of axial flow rotors. They carried out experimental tests on model rotors comprising of uniform thickness discs and uniform untwisted blades set at varying stagger
angles. They developed an analysis of the coupled system based on approximate receptance relations for the disc and blades, and were able to predict satisfactorily the frequencies of the lower coupled modes of the models. The analysis was restricted to simple model configurations for which receptance relations could be easily obtained. The application to practical rotors was outlined. Wagner (7.7) extended the simplified model of Johnson and Bishop (7.3), representing each blade by a single degree of freedom system which has the same natural frequency and damping factor as that of a particular mode of the blade. These subsystems are attached to a common ring representing the disc. Ewins (7.8), Dye and Henry (7.9), and Ewins (7.10) have studied the effects of detuning upon the vibration characteristics of bladed-discs, in particular the variation of blade stresses which can result when the blades do not have identical frequencies. They concluded that this effect can result in a variation of vibratory stress from blade to blade by a factor as high as 1.25 approximately. Kirkhope and Wilson (7.11) applied the finite element method for the dynamic analysis of the bladed-disc by using an annular element for the disc and a simple rectangular cross-section beam element to represent the blading. The semi-analytical method, presented by those authors, gave good results compared with the experimental values but some limitations exist for the accommodation of the phase angle due to the separation of the flexural and centroidal axes of the blades and the model must have large numbers of blades. Only the frequencies of vibration of modes with low numbers of nodal diameters were obtainable. Wilson and Kirkhope (7.12) have studied the vibration of a disc to which are attached a large number of identical blades. Exact solutions for the coupled blade-disc motion are presented for non-rotating configurations of simple geometry, and assuming that blade loadings
can be considered continuously distributed around the rim of the
disc. Exact frequencies have been obtained for simple models, and
comparison with experimental values indicates that this assumption is
valid for modes involving low numbers of nodal diameters. Kirkhope
and Wilson(7.13) later used the finite element method to study the
coupled vibration modes of a rotating bladed-disc assembly. Their
model is particularly suitable for efficient calculation of practical
configurations, readily accounting for tapered profiles, rotational
and thermal effects, and transverse shear and rotary inertia in
moderately thick discs. Salama, Petyt and Mota Soares(7.14) applied
the finite element method and developed an annular model and a sector
element for the disc and a rectangular beam model for the blade and
used it in conjunction with the wave propagation technique to study
the vibration characteristics of the bladed-disc assembly. Eswaran,
Ganapathi and Srinath(7.15) have analysed the vibration of
a bladed-disc by solving the differential equations of motion governing
the problem using the Runge-Kutta method. They assumed that the
bladed portion of the disc is replaced by an annular disc, the
difference in radii being equal to the length of the blade and the
thickness is so adjusted that the weight of the annular disc is equal
to the weight of all the blades. Hence the idealised bladed-disc
will be two concentric discs of different thicknesses. Irretier(7.16)
has studied the free vibrations of a turbine bladed-disc for bending
and torsional motions of the blades and in-plane and out-of-plane
motion of the disc. The effects of shear deformation, rotary
inertia, rotation speed, and stagger angle were studied. He has
solved the differential equations of motion of the system by direct
integration techniques. Thomas and Sabuncu(7.17) have studied the
free vibration characteristics of blades attached to an elastic disc
under rotating and non-rotating conditions. They have used a finite
sector element to idealise the disc and an aerofoil cross-section beam finite element to idealise the blades. The effects of stagger angle, pretwist angle, the distance between shear centre and centroid, and the speed of rotation were investigated. The wave propagation technique for cyclically symmetric structure was used to obtain the frequencies of vibration and the corresponding mode shapes and the phase angle of the various components of the mode shapes for the bladed disc. They have drawn interference diagrams to investigate the effect of coupling on the frequencies of the unbladed disc and the cantilever blades on a rigid disc.
The dynamic behaviour of blades assembled on a disc is highly affected by the coupling between these components. In this chapter the finite element models adopted in the foregoing chapters are used here to formulate the bladed-disc assembly. One blade attached to the corresponding annular sector of the disc is used to assemble the sub-structure by the finite element method. This sub-structure is then used in conjunction with the wave propagation technique for cyclically symmetric structure to develop the whole structure of the bladed-disc.

In this investigation the blade is idealised to behave as a beam having asymmetric aerofoil cross-section and uniform pretwist along its span with two nodes and twelve degrees of freedom. The disc is idealised using thin plate theory with four nodes and twelve degrees of freedom and has a hyperbolic thickness variation along its radius. However, any general radial thickness profile can be satisfactorily described by the model. The effects of centrifugal force, pretwist angle, stagger angle, disc thickness variation as well as coupling between the blades and the disc are investigated in this chapter.
7.2b FORMULATION OF THE EIGENVALUE EQUATION

By using the following relationship, the compatibility conditions of the blade root with the disc can be ensured.

The torsional displacement of the blade at its root, $\theta_k$, is expressed as a function of the radial rotation $U_\theta$ by

$$\theta_k = \frac{U_\theta}{r_2} \quad (7.1)$$

where $r_2$ is the outer radius of the disc. Therefore the compatibility of the blade root displacements and rotation can be shown in matrix form as follows:

$$\begin{bmatrix}
W_k & V_k & U_k & \theta_k & U^1_k & V^1_k & W_{k+1} & V_{k+1} & U_{k+1} & \theta_{k+1} & U^1_{k+1} & V^1_{k+1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
U_k \\
U_{\theta} \\
U_{k+1} \\
U^1_{k+1} \\
V_{k+1} \\
V^1_{k+1}
\end{bmatrix}

(7.2)
Equation (7.2) satisfies the continuity condition at the blade disc attachment. Hence this relationship is sufficient to allow assembly of the dynamic stiffness matrix of the coupled bladed-disc sub-structure of Figure (7.1), by combining the individual matrices for the disc and blade. The number of degrees of freedom in this sub-structure depends on the number of elements used.

By making use of the wave propagation technique the eigenvalue equation of the whole system is obtained as follows:

\[
\begin{bmatrix}
K_s^r & -K_s^i \\
K_s^i & K_s^r
\end{bmatrix}
- \begin{bmatrix}
M_s^r & -M_s^i \\
M_s^i & M_s^r
\end{bmatrix}
\begin{bmatrix}
q_s^r \\
q_s^i
\end{bmatrix} = 0
\]  

(7.3)

where \( r \) stands for real, \( i \) stands for imaginary, and \( s \) for the whole system matrix.

Equation (7.3) is the equation of the free vibration of the structure which may be solved by any of the standard procedures as an algebraic eigenvalue problem.

The number of degrees of freedom of the system is given as

\[ NDF = 2 (NDFS - NDLB) \]  

(7.4)
where

\[ \text{NDF} : \text{Number of degrees of freedom of the system.} \]
\[ \text{NDFS} : \text{Number of degrees of freedom of the substructure.} \]
\[ \text{NDLB} : \text{Number of degrees of freedom on the left boundary.} \]

The deflection vector consists of the deflections at the left and inner nodes, the magnitude of the deflection being resolved into real and imaginary axes. The magnitude of the deformation is obtained by the vector summation of these components. The magnitude ratio of these components (imaginary/real) gives two different phase angles, one for the torsional and one for the bending deformations. The phase angle difference between these two deformations depends on the configuration of the blade cross-section. If a blade has uniform straight asymmetrical cross-section, where shear centre and centroid do not coincide, and as it executes coupled bending-torsion vibration, the variation of phase angle along the blade length depends on the distance of the shear centre from the centroid. For the case of pretwisted asymmetrical cross-section blades which execute coupled bending-bending-torsion vibrations, the phase angle variation depends on both the distance of the shear centre from the centroid and the pretwist angle. If symmetrical cross-section straight or pretwisted blades are used, the phase difference is always 90 degrees.
The finite element method is used to investigate the vibration characteristics of rotating bladed disc. The Cyclic symmetry of the rotationally periodic structure is used to advantage. The method is based upon a substructuring technique, whereby one blade and an annular sector of the disc form one substructure. The disc used in this analysis has a variable thickness of the form $h(r) = \frac{c}{r^\beta}$. The accuracy of this study is demonstrated by numerical comparison with the theoretical and experimental work of Thomas and Sabuncu (7.17) when the disc has a constant thickness profile by assuming $\beta$ of the above formula equal to zero. As it is seen in Table (7.2), good agreement between the present analysis and their theoretical and experimental investigation was achieved. Some discrepancies between the results of Thomas and Sabuncu (7.17) and the results from the present analysis are due to the increase of number of degrees of freedom used in this analysis.

The effects of different factors on the natural frequencies of the bladed disc assembly such as; number of nodal diameters, rotational speed, $\beta$ and stagger angle are displayed graphically in Figure (7.2a) to Figure (7.8d). The data of the blades and disc required for calculation of these graphs are given in Table (7.3) unless specified otherwise. The sector angle chosen for this analysis is five degrees which is small enough to give the accurate results. However, numerical instability can occur with very small angles.
The mode of vibration of bladed disc occur in a similar manner to those of the disc alone. Figures (7.2a - 7.2f) show the effect of increasing the number of nodal diameters on the natural frequencies of the bladed disc for the first ten modes of vibration. The solid line curves represent the bladed disc frequencies. The dotted line curves are for the disc frequencies alone. The natural frequencies of the cantilever blade are shown on the right hand side of each graph and denoted by BCF. As it is seen from the figures, the natural frequencies increase as the number of nodal lines increases. The frequencies of the bladed disc for large number of nodal diameters instead of increasing as for the disc modes, they approach one of the blade cantilever frequencies and become asymptotic to it. Therefore the disc has little or no influence on the higher modes. This is because the combined system frequencies are only a small fraction of that of the disc alone. It is thus almost infinitely stiff and so the vibration is predominantly pure blade vibration. Frequencies of vibration for two different speeds of rotation are presented in Figure (7.2), namely, 0 and 6000 rpm. Comparison between Figures (7.2a - 7.2c) and (7.2d - 7.2f) indicates that as the speed of rotation increases the frequencies increase and the coupling between the lower modes increase considerably while decreasing the coupling between the higher modes of the system. Figure (7.2) also shows the effect of variation of $\beta$ on the natural frequencies of the bladed disc for both rotating and non-rotating conditions. It is seen that as $\beta$ increases the natural frequencies of the system decrease sharply and the frequency curves of the bladed disc for higher modes become far from being pure blade bending.
The first five modes of frequencies of the system against $\beta$ are presented in Figures (7.3a - 7.3d) and (7.4a - 7.4d) for 0 and 6000 rpm respectively. As it is seen from these figures the coupling between the modes increases as the number of nodal diameters increases. Also it is quite clear from the figures that an increase in $\beta$ leads to a decrease in the natural frequencies in both modes. As the bladed disc rotates at a speed of 6000 (rpm) the natural frequencies rise rapidly especially in the lowest modes. Comparison between Figures (7.3 and 7.4) indicates that the natural frequencies of the bladed disc are substantially modified and the coupling between the individual modes is more noticed in rotating than in non-rotating cases for each nodal diameter.

Variations of natural frequencies of the first five modes with increasing stagger angle from 0 to 90 degrees under rotating conditions are shown in Figures (7.5 - 7.8) for different nodal diameters. It is well known that when the bladed-disc is rotating, the centrifugal stresses developed both in the disc and the blades increase the stiffness of the entire system and the natural frequencies of the bladed disc are increased as a consequence. It is seen from these figures that variations in the frequencies are considerable at lower modes of vibration for each nodal diameter, whereas frequencies of higher modes are not affected much.

Also from these figures it was found that some mode natural frequencies show a strong dependence on stagger angle even when the rotation speed is zero. It is also seen that the stagger angle controls the degree of coupling between the motion of the blades and disc. At high rotational speeds it is seen that for the first mode
of vibration the bladed-disc frequencies are lower for lower values of stagger angle. But for the higher modes this is reversed and the bladed disc frequencies are higher for lower values of stagger angle.
**TABLE 7.1**  
Physical Properties of Bladed-Disc Assembly

<table>
<thead>
<tr>
<th>DISC</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc outer radius</td>
<td>( r_2 ) = 8.0 \text{ in}</td>
<td></td>
</tr>
<tr>
<td>Disc inner radius</td>
<td>( r_1 ) = 1.28 \text{ in}</td>
<td></td>
</tr>
<tr>
<td>Disc thickness</td>
<td>( h ) = 0.34 \text{ in}</td>
<td></td>
</tr>
<tr>
<td>Young Modulus</td>
<td>( E ) = 30 \times 10^6 \text{ lbf/in}^2</td>
<td></td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>( \nu ) = 0.3</td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>( \rho_g ) = 0.285 \text{ lbf/in}^3</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BLADE</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Blades</td>
<td>( N_B ) = 24</td>
<td></td>
</tr>
<tr>
<td>Area of Cross-section</td>
<td>( A ) = 0.0914 \text{ in}^2</td>
<td></td>
</tr>
<tr>
<td>Second Moment of Area:</td>
<td>( I_{XX} ) = 84 \times 10^{-6} \text{ in}^4</td>
<td>( I_{YY} ) = 671 \times 10^{-5} \text{ in}^4</td>
</tr>
<tr>
<td>Shear Centre Distance:</td>
<td>( d_x ) = 0.0076 \text{ in}</td>
<td>( d_y ) = 0.047 \text{ in}</td>
</tr>
<tr>
<td>Blade length</td>
<td>( L ) = 6 \text{ in}</td>
<td></td>
</tr>
<tr>
<td>Young's Modulus</td>
<td>( E ) = 30 \times 10^6 \text{ lbf/in}^2</td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>( \rho_g ) = 0.285 \text{ lbf/in}^3</td>
<td></td>
</tr>
<tr>
<td>Torsional Rigidity</td>
<td>( C ) = 3240 \text{ lbf.in}^2</td>
<td>( C_C ) = 392 \text{ lbf.in}</td>
</tr>
<tr>
<td>Increment of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Torsional Rigidity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pretwist Angle</td>
<td>( \alpha ) = 30°</td>
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</tr>
<tr>
<td>Stagger Angle</td>
<td>( \phi ) = -74°</td>
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</tr>
</tbody>
</table>
TABLE 7.2
Comparison of Experimental and Theoretical Frequencies
of Free Vibration of a Free-Free Pretwisted,
Aerofoil Cross-Section Bladed Disc Assembly

Physical properties of the assembly are given in Table 7.1

<table>
<thead>
<tr>
<th>Number of Nodal Diameters</th>
<th>NATURAL FREQUENCY (Hz)</th>
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<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>THEORETICAL P.E.</td>
<td>THOMAS AND SUBUNCU(7.17)</td>
<td>EXPERIMENTAL</td>
</tr>
<tr>
<td></td>
<td>Number of D.o.f.(81)</td>
<td></td>
<td>Number of D.o.f.(90)</td>
</tr>
<tr>
<td>-------------------------</td>
<td>---------------------</td>
<td>------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.64</td>
<td>94</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>241.8</td>
<td>262</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>550.1</td>
<td>573</td>
<td></td>
<td></td>
</tr>
<tr>
<td>949.1</td>
<td>903</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1066.7</td>
<td>1045</td>
<td></td>
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</tr>
<tr>
<td>1651</td>
<td>1540</td>
<td></td>
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<td>1790.1</td>
<td>1898</td>
<td></td>
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</tr>
<tr>
<td>3262.5</td>
<td>3177</td>
<td></td>
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</tr>
<tr>
<td>3348.9</td>
<td>3335</td>
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</tr>
<tr>
<td>4025.3</td>
<td>4012</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Nodal Diameters</td>
<td>NATURAL FREQUENCY (Hz)</td>
<td>THOMAS AND SUBUNCU (7.17)</td>
<td>EXPERIMENTAL</td>
</tr>
<tr>
<td>--------------------------</td>
<td>------------------------</td>
<td>---------------------------</td>
<td>--------------</td>
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<td>THEORETICAL</td>
<td>F.E.</td>
<td>EXPERIMENTAL</td>
</tr>
<tr>
<td></td>
<td>Number of D.o.f.(81)</td>
<td></td>
<td>Number of D.o.f.(90)</td>
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<tr>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>96.5</td>
<td></td>
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<td>94.7</td>
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<td>467.8</td>
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<td>433</td>
<td>493</td>
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<tr>
<td>660.3</td>
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<td>750</td>
<td>548</td>
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<td>979.3</td>
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<td>926</td>
<td>1020.3</td>
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<td>1045</td>
<td>1080.7</td>
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<td>5102.3</td>
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<td>96.6</td>
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<td>514.4</td>
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<td>504</td>
<td>497.7</td>
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<td>954.5</td>
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<td>905</td>
<td>891.3</td>
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<td></td>
<td>-</td>
<td>1061.2</td>
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<td></td>
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<td>1815</td>
<td></td>
<td>1939</td>
<td>1850</td>
</tr>
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<td>3264</td>
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<td>3240</td>
<td>3211</td>
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<td></td>
<td>3351</td>
<td>3310</td>
</tr>
<tr>
<td>4577.1</td>
<td></td>
<td>4329</td>
<td>4589</td>
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<tr>
<td>5484.2</td>
<td></td>
<td>5071</td>
<td>5227.9</td>
</tr>
</tbody>
</table>
### TABLE 7.3

Physical Properties of Bladed-Disc Assembly with the disc having hyperbolic thickness variation of the form $h(r) = \frac{c}{r^\beta}$

<table>
<thead>
<tr>
<th>DISC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc outer radius</td>
<td>$r_2 = 8.0$ in</td>
</tr>
<tr>
<td>Disc inner radius</td>
<td>$r_1 = 1.3$ in</td>
</tr>
<tr>
<td>Disc thickness constant</td>
<td>$c = 0.5$</td>
</tr>
<tr>
<td>Young Modulus</td>
<td>$E = 30 \times 10^6 \text{ lbf/in}^2$</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho_g = 0.285 \text{ lbf/in}^3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BLADE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Blades</td>
<td>$N_B = 36$</td>
</tr>
<tr>
<td>Area of Cross-section</td>
<td>$A = 0.0914 \text{ in}^2$</td>
</tr>
<tr>
<td>Second Moment of Area:</td>
<td>$I_{XX} = 84 \times 10^{-6} \text{ in}^4$</td>
</tr>
<tr>
<td></td>
<td>$I_{YY} = 671 \times 10^{-5} \text{ in}^4$</td>
</tr>
<tr>
<td>Shear Centre Distance:</td>
<td>$d_x = 0.076 \text{ in}$</td>
</tr>
<tr>
<td></td>
<td>$d_y = 0.047 \text{ in}$</td>
</tr>
<tr>
<td>Blade length</td>
<td>$L = 6 \text{ in}$</td>
</tr>
<tr>
<td>Young's Modulus</td>
<td>$E = 30 \times 10^6 \text{ lbf/in}^2$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho_g = 0.285 \text{ lbf/in}^3$</td>
</tr>
<tr>
<td>Torsional Rigidity</td>
<td>$C = 3240 \text{ lbf.in}$</td>
</tr>
<tr>
<td>Increment of Torsional Rigidity</td>
<td>$C_c = 392 \text{ lbf.in}$</td>
</tr>
<tr>
<td>Pretwist Angle</td>
<td>$\alpha = 60^\circ$</td>
</tr>
<tr>
<td>Stagger Angle</td>
<td>$\phi = 30^\circ$</td>
</tr>
</tbody>
</table>
FIG. 7.1: GEOMETRY OF THE SUBSTRUCTURE OF BLADED-DISC

- nodes on the left boundary
- interior nodes
- nodes on the right boundary.
FIG. 7.2: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS FOR THE FIRST 10 MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.3: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE RELATION

$$h(r) = \frac{0.5}{r^\beta}$$

FOR THE FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.4: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE FORMULA

$$h(r) = \frac{0.5}{r^\beta}$$

FOR THE FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.5: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.6: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR THE FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.7: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
FIG. 7.8: VARIATION OF NATURAL FREQUENCY AGAINST ROTATIONAL SPEED FOR FIRST FIVE MODES OF CLAMPED-FREE BLADED-DISC
CHAPTER 8

VIBRATIONS OF ROTATING SHROUDED-BLADED-DISC

8.1 INTRODUCTION

Either flutter or resonance can cause intolerable vibratory stresses and strains within the turbomachinery. A failure of the system would occur as a consequence of fatigue of blading. Therefore the determination of natural frequencies and their modes is one of the essential task of the designers; since these are the frequencies at which, for the sake of its safety, the system must not be excited.

During vibration the blades are most seriously affected by the whole assembly of the turbomachinery and the vibrations of a set of blades assembled on a rotor exhibit more complex characteristics than those predicted for the single cantilevered blade. Therefore studying the vibratory behaviour of a single blade cantilevered at its root is not adequate and the importance of considering a complete shrouded-bladed-disc assembly when investigating the vibration characteristics of the blading under practical conditions is highly recommended.

The bladed-disc which transmits torque from the blades to the shaft of the engine constitutes an important part of the turbine. This interaction between the blades and disc can induce vibration problems of major concern, such as, the existence of coupled blade-disc resonance which could increase the dynamic response level and decrease the fatigue life, and the disc elastic effect on the natural frequencies of the blades.
One of the factors that affect the vibration of rotor stages is the blade shroud coupling. Hence, in practice, the blades are fastened together in groups by the use of shrouding along their span. This alters the vibratory behaviour of the system which then alters some of the dangerous resonant frequencies of single blades and reduces the stresses induced in the blades by excitation under gas and dynamic forces. The flexibility of disc, which constitutes an integral part of the shrouded-bladed-disc, would alter the dynamic behaviour of the shrouded-blade.

In a turbomachinery system, very high stresses develop due to the centrifugal forces at high speeds. These stresses constitute the major portion of the total stresses. In order to avoid strong resonant vibration within the operating range of the machine it is essential that the designer should be able to predict accurately the natural frequencies of the rotating system.

In this analysis, vibration of rotating shrouded-bladed-disc is considered. Due to the complexity of the assembly, each individual part has been studied separately in the aforementioned chapters. The blade has a pretwisted aerofoil cross-section with two nodes and six degrees of freedom at each node. It has been analysed using beam theory under rotating conditions. The rotating shroud element situated at the tip of the blade is of curved beam form with two nodes. Its in-plane and out-of-plane energy matrices are formed separately and then they are joined together to form a rotating shroud element vibrating in three directions with six degrees of freedom at each of its two nodes. Thin plate theory has been used for the analysis of the 4-noded non-uniform annular sector rotating disc element with three degrees of freedom at each node.
Many investigators have studied the individual parts of the assembly under rotating and non-rotating conditions. Most of them have been listed in the past chapters. Limited number of articles have been published dealing with the vibration of shrouded-bladed-disc.

Carta\(^{(8.1)}\) used the energy method to describe an aeroelastic instability condition which is governed by strong coupling between bending and torsion of the blades resulting from disc or shroud dynamic coupling. This flutter condition is highly dependent on the coupled bladed-disc-shroud mode shape, which must be accurately determined. Ewins\(^{(8.2)}\) studied the free vibration characteristics of bladed-disc assemblies using the receptance coupling technique. The effects of blade detuning were studied and found to give rise to irregular and complex modes of vibration. The effect of adding a shroud to the blades was considered as also is the response of the system to certain excitation conditions. Later Cottney and Ewins\(^{(8.3)}\) also used the receptance technique to analyse the vibration of shrouded-blade disc assemblies. They discussed the possibility of eliminating some coupling coordinates joining the three different components to reduce the size of the problem without effecting the accuracy of the results. Although the receptance technique has shown a good agreement with the experimental values in those particular models, it is limited by the inability of including in-plane stress effects. The wave propagation and Finite Element method for rotationally periodic structure have been used by Mota Soares, Petyt and Salama\(^{(8.4)}\) and applied to studying the vibration of shrouded-bladed-discs consisting of untwisted blades. This technique is capable of considering the discrete blade mass distribution along the rim radius and can predict more accurately the
higher mode vibrations. Later Mota Soares and Petyt\(^{8.5}\) used the finite element method to study the vibrations of non-rotating shrouded-bladed-discs. The discs are modelled by using both annular and sector elements. The blades are modelled by means of shell elements. The shrouds are represented by both lumped masses and straight beam elements. Their predicted frequencies were compared with experimental results. Chen and Dugundji\(^{8.6}\) used the finite element method to analyse the static deformation and dynamic behavior of a rotating shrouded-bladed-disc. The blades were divided into a finite number of beam elements which were assumed to have a uniform pretwist angle within the element; the shroud, located either at the tip or at the part-span of the blade, was represented by curved beam elements between two adjacent blades, and the disc was modelled as a thin uniform circular plate clamped at the centre. A regular blade element was assumed to have ten degrees of freedom, a disc element has one degree of freedom, and shroud element has ten degrees of freedom. Huang\(^{8.7}\) used the transfer matrix approach to study the free and forced vibration of closely coupled turbomachinery blades on a disc which are connected by elements forming a rotationally periodic structure. Different types of shrouding and lacing wires have been adopted. Kuo\(^{8.8}\) studied the coupled and uncoupled dynamics analysis of bladed-disc assemblies using finite element rotationally symmetric technique. The disc is composed of a solid circular disc of a variable cross-section and a flexible conical shell which is attached to the driven shaft through the central hole. The disc has eight nodes with three translational degrees of freedom at each node. The blade is represented approximately by three quadrilateral plate elements. The shroud, also represented by plate elements, interconnects the fan blades at the base and the corresponding part-span location respectively. Singh and
Schiffer\(^8.9\) studied the vibration characteristics of packeted bladed-disc. Blades and shroud blades are modelled as three-dimensional beams and disc's elements are modelled as thick circular plates. They have used the finite element method to investigate the arrangement of ninety blades which are arranged in fifteen packets of six blades each. They have also studied a simplified model of spring-mass system which simulates the dynamic behaviour of packeted bladed-disc. Wildheim\(^8.10\) studied the vibrations of rotating bladed discs with a shroud extending around the whole circumference to form a rotationally periodic structure. The free modes of the disc were used to describe the dynamics of the disc by a \(4 \times 4\) receptance matrix. The row of blades was described by a dynamic stiffness matrix of order \(4 + 10l\), where \(l\) is the number of lacing-wires. The dynamic stiffness matrix of the blading was formed directly from the modes of one single clamped-free blade using beam theory. The lacing-wires were treated as elastic and massless. His calculation procedure has proved to be very efficient and the only disadvantage was the necessity of having at least two disc modes in order to get a non-singular receptance matrix.
The vibration of shrouded-bladed-disc assembly has the similar modal characteristics as the disc of circular or annular plates and is characterised by both concentric and diametric modes having integer numbers of nodal diameters and circumferential nodal circles.

To represent the whole turbine stage; by making use of the previous chapters, a substructure of one disc sector, one blade and an appropriate shroud length is taken and the general assembly procedure is formed to get the convergence required; Figure (2.2). These sub-structures are joined together by using the cyclic symmetry method of analysis. Hence a complete circumferentially closed structure of; annular disc, a closed curved beam (shroud), and a finite number of blades, depending upon the subtended angle taken to form the annular disc sector, can be obtained and analysed.

The derivation of mass and stiffness matrices for blade, shroud and disc elements is given in chapters three, four and six respectively.

To ensure that the compatibility conditions are satisfied at the shrouded-blade connections and at the bladed-disc connections use is made of equations (5.7) and (5.8) of Chapter 5 and equation (7.2) of Chapter 7. Similar to the eigenvalue equation (7.3) of rotationally periodic structure of Chapter 7 an eigenvalue equation can be obtained and solved for the frequencies and mode shapes of the whole assembly of the shrouded-bladed-disc. Many features have been investigated in this analysis. Natural frequency variation with the number of nodal diameters and with the speed of rotation have been
studied. Diagrams for this study are drawn. Interference diagrams of the individual parts and of the whole structure of the frequencies and nodal diameters are shown. Variation of the frequencies with the shape of the disc and with the blade pretwist and stagger angles under different rotation speeds has been investigated diagramatically. Different mode shapes for each displacement according to their effect on the system have been drawn both for rotating and nonrotating cases.
In this chapter a shroud element at the tip of the blade is introduced to the substructure of the bladed disc of Chapter 7.

The addition of a shroud with 12 degrees of freedom requires the introduction of additional degrees of freedom to the analysis of the bladed disc assembly. The element of the curved beam segment of Chapter 4 is used to ensure the compatibility at the junction between each blade and the shroud.

The effects of various factors such as; number of nodal diameter, variation of disc thickness, pretwist angle of the blade and speed of rotation on the dynamic characteristics of the bladed-disc are investigated in addition to some mode shapes for non-rotating and rotating bladed-disc assembly.

To verify the accuracy of the present analysis a comparison is made between this analysis results and the results of Sabuncu(8.11) with the disc having constant thickness profile by setting $\beta$ of the thickness equation $h(r) = \frac{c}{r^\beta}$ equal to zero as shown in Table (8.3). Good agreement is obtained between the present analysis results and the results of Sabuncu. Some discrepancies might have occurred due to the increase of the number of degrees of freedom of the present investigation.

Unless stated otherwise, all data of the following graphs are given in Table (8.2).
Figures (8.1a-8.1f) show the interference diagram of the shrouded-bladed-disc assembly when rotational speed varies from 0 to 5000 rpm. The dotted lines represent the cantilever blade frequencies, the solid curves are the frequencies of disc modes and the dotted and solid lines represent the shroud frequencies. It is clear from these graphs that the modes of the assembly are grouped into coupled sets. Each set may be associated either with one of the blade cantilever modes, to whose natural frequency those of the assembly modes tend asymptotically in the higher diametral modes due to the progressive stiffening of the disc, or with a particular number of nodal circles. The presence of the shroud constrains the vibration of the blades, therefore, increases the coupling between the modes. Hence the overall pattern of the vibration properties is altered by the shroud, which transmits the axial and circumferential forces at the blade shroud junction. The natural frequencies of the assembly increase with the speed of rotation. This increase is only significant at the natural frequencies of modes with a low number of nodal circles. This effect is more pronounced at the frequencies of the lower nodal diameter modes as shown in Figures (8.1a) and (8.1f).

In general Figures (8.1a-8.1f) indicate the interchange of some adjacent modes due to the influence of disc motion and shroud.

Figures (8.2a-8.2d) and (8.3a-8.3d) show the effect of increasing the value of $\beta$ in disc thickness formula, $h(r) = \frac{c}{r^\beta}$ for 0 and 6000 rpm respectively on the frequencies of first six modes of vibration for various nodal diameter modes. Comparison between Figures (8.2a-8.2d) and Figures (8.3a-8.3d) indicates that rotation speed increases the natural frequencies of all modes together with
the coupling between the adjacent modes. It is also seen from Figures (8.3a-8.3d) that the frequencies of the higher modes are less affected by the change of the disc thickness than the frequencies of Figures (8.2a-8.2d) under non-rotating conditions.

Figures (8.4a-8.4f) show the effect of inclusion of blade pretwist on the vibration of the shrouded-bladed-disc. The frequencies of the first six modes are drawn for different nodal diameters under non-rotating and rotating conditions. The most important effect of pretwist is to increase the frequencies of the torsional modes considerably as shown in the fourth mode of Figures (8.4a,8.4d) for 0 nodal diameter. This effect is increased at lower modes as number of nodal diameter increases. Less significant changes occur due to pretwist in the bending frequencies. Figures (8.4a-8.4f) also shows the effect of rotational speed on the natural frequencies of vibration.

Figures (8.5a-8.5f) and (8.6a-8.6f) show some mode shapes of the transverse displacement $u$ of the vibrating clamped-free bladed-disc for different nodal diameters and when the speeds of rotation are 0 and 6000 respectively. The effect of the disc flexibility on the vibration modes of the bladed-disc is shown. It is clear that for the higher nodal diametral modes, the disc behaves as if it is rigid. It is also clear that when the speed of rotation is 6000 (rpm) the rigidity of the system increases considerably. There are some changes in some mode shapes which indicate that there is coupling. The continuity in most of the modes shapes between the disc and the blades is because the flexibility of the disc and the blades is almost the same.
### TABLE 8.1
Physical Properties of a Free-Free Shrouded-Bladed-Disc Assembly

\( \alpha = 30^\circ \quad \phi = -74^\circ \)

<table>
<thead>
<tr>
<th>DISC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc Outer radius ( r_2 = 8 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>Disc Inner radius ( r_1 = 1.28 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>Disc thickness ( h = 0.34 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( E = 30 \times 10^6 \text{ lb/in}^2 ) ( \nu = 0.3 ) ( \rho_g = 0.285 \text{ lb/in}^3 )</td>
<td></td>
</tr>
<tr>
<td>( N_T = 2 \quad N_R = 5 )</td>
<td></td>
</tr>
<tr>
<td>Number of Blades ( N_B = 24 )</td>
<td></td>
</tr>
<tr>
<td>Area of Blade Cross-Section ( A = 914 \times 10^{-4} \text{ in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Moments of Area of the Blade: ( I_{XX} = 84 \times 10^{-6} \text{ in}^4 ) ( I_{YY} = 671 \times 10^{-5} \text{ in}^4 )</td>
<td></td>
</tr>
<tr>
<td>Length of Blade ( L = 6 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( E = 30 \times 10^6 \text{ lb/in}^2 ) ( \rho = 0.285 \text{ lb/in}^3 )</td>
<td></td>
</tr>
<tr>
<td>Torsional Rigidity ( C = 3270 \text{ lbf in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Increment of Torsional Rigidity ( C_c = 392 \text{ lbf in} )</td>
<td></td>
</tr>
<tr>
<td>Shear Centre distances: ( d_x = 0.0076 \text{ in} ) ( d_y = 0.047 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( N = 6 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BLADE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of Shroud Cross-Section ( A_s = 0.25 \times 0.0625 \text{ in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Young's Modulus of the Shroud ( E_s = 30 \times 10^6 \text{ lbf/in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Density of Shroud Material ( \rho_s g = 0.285 \text{ lb/in}^3 )</td>
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</tr>
<tr>
<td>( M = 4 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SHROUD</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>TABLE 8.2</td>
<td></td>
</tr>
<tr>
<td>-------------------</td>
<td></td>
</tr>
<tr>
<td>Physical Properties of a Clamped-Free Shrouded-Bladed-Disc Assembly</td>
<td></td>
</tr>
<tr>
<td>( \alpha = 60^\circ \quad \phi = 30^\circ )</td>
<td></td>
</tr>
<tr>
<td>Disc thickness ( h(r) = \frac{c}{r^\beta} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>DISC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc Outer radius ( r_2 = 8 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>Disc Inner radius ( r_1 = 1.3 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>Thickness Constant ( C = 0.5 )</td>
<td></td>
</tr>
<tr>
<td>( E = 30 \times 10^6 \text{ lb/in}^2 )</td>
<td></td>
</tr>
<tr>
<td>( \nu = 0.3 )</td>
<td></td>
</tr>
<tr>
<td>( \rho_g = 0.285 \text{ lb/in}^3 )</td>
<td></td>
</tr>
<tr>
<td>( N_T = 2 )</td>
<td></td>
</tr>
<tr>
<td>( N_R = 6 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BLADE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Blades ( N_B = 36 \quad \text{NBE} = 6 )</td>
<td></td>
</tr>
<tr>
<td>Area of Blade Cross-Section ( A = 914 \times 10^{-4} \text{ in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Moments of Area of the Blade: ( I_{XX} = 84 \times 10^{-6} \text{ in}^4 )</td>
<td></td>
</tr>
<tr>
<td>( I_{YY} = 671 \times 10^{-5} \text{ in}^4 )</td>
<td></td>
</tr>
<tr>
<td>Length of Blade ( L = 6 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( E = 30 \times 10^6 \text{ lb/in}^2 )</td>
<td></td>
</tr>
<tr>
<td>( \rho_g = 0.285 \text{ lb/in}^3 )</td>
<td></td>
</tr>
<tr>
<td>Torsional Rigidity ( C = 3270 \text{ lbf} \cdot \text{in}^2 )</td>
<td></td>
</tr>
<tr>
<td>Increment of Torsional Rigidity ( C_c = 392 \text{ lbf} \cdot \text{in} )</td>
<td></td>
</tr>
<tr>
<td>Shear Centre distances: ( d_x = 0.0076 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( d_y = 0.047 \text{ in} )</td>
<td></td>
</tr>
<tr>
<td>( N = 6 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SHROUD</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of Shroud Cross-Section ( A_S = 0.25 \times 0.0625 \text{ in}^{-2} )</td>
<td></td>
</tr>
<tr>
<td>Young's Modulus of the Shroud ( E_s = 30 \times 10^6 \text{ lbf/in}^2 )</td>
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<td>Density of Shroud Material ( \rho_g = 0.285 \text{ lb/in}^3 )</td>
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TABLE 8.3
Comparison of Experimental and Theoretical Frequencies
of Free Vibration of Free-Free Shrouded-pretwisted
aerofoil Cross-section Bladed-Disc assembly
of Sabuncu(8.11) and present analysis

Details of the assembly are given in Table 8.1

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<th>Number of Nodal Diameters</th>
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<th>Experimental</th>
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FIG. 8.1: VARIATION OF NATURAL FREQUENCY AGAINST NUMBER OF NODAL DIAMETERS FOR THE FIRST 15 MODES
FIG. 8.2: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE FORMULA

$$h(r) = \frac{0.5}{r^\beta}$$ FOR THE FIRST SIX MODES OF CLamped-FREE SHROUDED-BLADED-DISC
FIG. 8.3: VARIATION OF NATURAL FREQUENCY AGAINST $\beta$ OF THE FORMULA

$$h(r) = \frac{0.5}{r^\beta}$$

FOR THE FIRST SIX MODES OF CLAMPED-FREE

SHROUDED-BLADED-DISC
FIG. 8.4: VARIATION OF NATURAL FREQUENCY AGAINST ANGLE OF PRETWIST
FOR THE FIRST SIX MODES OF CLAMPED-FREE SHROUDED-BLADED-DISC
Fig. 8.5
FIG. 8.6

(a) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC

(b) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC

(c) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC

(d) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC

(e) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC

(f) BENDING MODE SHAPES FOR 4000 RPM
4 RODAL DIAMETER CLAMPED-FREE SHROUDED-BLADED-DISC
Many blade failures are associated with material fatigue as a result of blade vibration which is frequently complicated by high stresses, so that the control of the alternating stresses becomes imperative. Hence in the development of reliable gas turbine engines, when predicting the natural frequencies and their corresponding mode shapes it is necessary to include stress calculations of the turbine blading which can be obtained by using the eigenvector solution and the rotation speeds.

Once operating conditions of the turbine have been reached the material stress of the turbine blade is the sum of the static or steady-state stress due to the centrifugal force and the dynamic stress due to the blade vibrations.

A turbomachinery blade can be considered to be of asymmetrical cross-section, fixed at its base, and linearly pretwisted from the fixed end to the free end.

This chapter is divided into two sections, in section one the stresses of a pretwisted aerofoil cross-section turbine blade under rotating and non-rotating conditions have been calculated. The
effects of stagger angles, pretwist angles and rotation speeds on the
stresses have been investigated. In section two the dynamic stresses
of a pretwisted aerofoil cross-section two bladed packet under
rotating and non-rotating conditions have been calculated. The
effects of shroud, stagger angles, pretwist angles and rotational
speeds on the dynamic stresses of the blade packet have been
investigated.

In the literature, much emphasis has been laid on the
determination of natural frequencies and their corresponding mode
shapes rather than on the stresses. Very recently Thomas and
Abdulrahman(9.1,9.2) have studied the dynamic stresses of a turbine
blade using finite element shell. They have given the values of the
principal stresses at each of the nodal points obtained from the
eigenvector solutions. They have found that in some modes of
vibration the maximum stress does not occur at the fixed end of the
blade but occurs at a section near the tip which means that the
position of the maximum principal stress is dependent on the mode
order and the mode shape.
To obtain blade stresses due to vibration use is made of beam theory of a foregoing chapter. Referring to Figure 9.1, the displacements in the x, y and z directions are w, v and u respectively and the torsional displacement is θ. Using Chapter 3, these displacements can be written in matrix form as follows:

\[
\begin{align*}
\mathbf{u} &= \left[ 1 \times x^2 \times x^3 \right] \mathbf{C_u}^{-1} \{\text{EVEC}_u\} = [F_u] \mathbf{C_u}^{-1} \{\text{EVEC}_u\} \\
\mathbf{v} &= \left[ 1 \times x^2 \times x^3 \right] \mathbf{C_v}^{-1} \{\text{EVEC}_v\} = [F_v] \mathbf{C_v}^{-1} \{\text{EVEC}_v\} \\
\mathbf{w} &= \left[ 1 \times x^2 \times x^3 \right] \mathbf{C_w}^{-1} \{\text{EVEC}_w\} = [F_w] \mathbf{C_w}^{-1} \{\text{EVEC}_w\} \\
\mathbf{\theta} &= \left[ 1 \times x \times x^3 \right] \mathbf{C_\theta}^{-1} \{\text{EVEC}_\theta\} = [F_\theta] \mathbf{C_\theta}^{-1} \{\text{EVEC}_\theta\}
\end{align*}
\]

(9.1)

where:

The matrices \([\mathbf{C_u}], [\mathbf{C_v}], [\mathbf{C_w}]\) and \([\mathbf{C_\theta}]\) are the coefficient matrices of the displacements \(u, v, w\) and \(\theta\) respectively.

\([\text{EVEC}]\) are the eigenvectors of the displacements of one element.

The bending moment about y-y axis is

\[
M_{yy} = EI_y \frac{d^2v}{dx^2} + EI_y \frac{d^2u}{dx^2}
\]

(9.2)
Substituting equation (9.1) into equation (9.2) yields

\[ M_{yy} = EI_{yz} \left[ F_{v}^{11} [C_{v}^{-1}] (EVEC_{v}) + EI_{yy} [F_{u}^{11} [C_{u}^{-1}] (EVEC_{u}) \right] \] (9.3)

Then the tangential stress \( \sigma_{x,yy} \) is

\[ \sigma_{x,yy} = \frac{M_{yy}}{I_{yy}} \, \frac{d}{dz} \] (9.4)

From equation (9.3)

\[ \sigma_{x,yy} = \frac{EI_{yz}}{I_{yy}} \, d_{z} \left[ F_{v}^{11} [C_{v}^{-1}] (EVEC_{v}) \right] + \frac{EI_{yy}}{I_{yy}} \, d_{z} \left[ F_{u}^{11} [C_{u}^{-1}] (EVEC_{u}) \right] \] (9.5)

The bending moment about z-z axis is

\[ M_{zz} = EI_{zz} \left[ \frac{d^{2}v}{dx^{2}} + \frac{d^{2}u}{dx^{2}} \right] \] (9.6)

Then the bending stress \( \sigma_{x,zz} \) is

\[ \sigma_{x,zz} = \frac{M_{zz}}{I_{zz}} \, \frac{d}{dy} \] (9.7)

Making use of equations (9.1), (9.6) and (9.7)

\[ \sigma_{x,zz} = \frac{EI_{zz}}{I_{zz}} \, d_{y} \left[ F_{v}^{11} [C_{v}^{-1}] (EVEC_{v}) \right] + \frac{EI_{yz}}{I_{zz}} \, d_{y} \left[ F_{u}^{11} [C_{u}^{-1}] (EVEC_{u}) \right] \] (9.8)

The force acting along the x-axis is

\[ F = EA \left( \frac{dw}{dx} \right) \] (9.9)

Then the stress in the x direction is

\[ \sigma_{x,w} = \frac{F}{A} = E \frac{dw}{dx} \] (9.10)

From equations (9.1) and (9.10)

\[ \sigma_{x,w} = E \left[ F_{w}^{11} [C_{w}^{-1}] (EVEC_{w}) \right] \] (9.11)

Then the total direct stress is equal to the summation of equations (9.5), (9.8) and (9.11)

\[ \sigma_{x} = \sigma_{x,yy} + \sigma_{x,zz} + \sigma_{x,w} \] (9.12)
The shear stress is

\[ \tau = \frac{T}{J} \frac{\partial \theta}{\partial x} \]  

(9.13)

\[ T = \text{torque} = GJ \frac{\partial \theta}{\partial x} \]  

(9.14)

Then

\[ \tau = \frac{GJ}{J} \frac{\partial \theta}{\partial x} r \]  

(9.15)

From equations (9.1) and (9.14)

\[ \tau = G r [F_\theta]^1[C_\theta^{-1}] \{\text{EVEC}_\theta\} \]  

(9.16)

where

\[ G = \frac{E}{2(1+\nu)} \]

Writing equations (9.12) and (9.16) in matrix form yields

\[
\begin{bmatrix}
\sigma_x \\
\tau
\end{bmatrix}
= E
\begin{bmatrix}
FF[QP] & GG[QP] & 0 & 0 & \frac{1}{l} & \frac{1}{l} & \frac{1}{l} \\
0 & 0 & 0 & 0 & 0 & 0 & DD[-\frac{1}{l} \frac{1}{l}] & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\text{EVEC}_v \\
\text{EVEC}_u \\
\text{EVEC}_w \\
\text{EVEC}_\theta
\end{bmatrix}
\]

(9.17)

where:

\[ FF = \frac{I_{zy}}{I_{yy}} d_z + d_y \]

\[ GG = \frac{I_{zy}}{I_{zz}} d_y + d_z \]

\[ DD = \frac{r}{2(1+\nu)} \]

\[ r = \sqrt{\frac{d_z^2}{d_y} + d_y^2} \]

\[ l : \text{length of one blade element} \]

\[ d_z : \text{distance from centre of flexure in the u direction} \]

\[ d_y : \text{distance from centre of flexure in the v direction} \]
The stress acting on a rotating blade due to centrifugal force can be written as follows:

\[
\sigma_{cf} = \int_{r_2+nl}^{r_2+Nl} r \omega^2 \rho \, dx = \rho \omega^2 \int_{r_2+nl}^{r_2+Nl} r \, dx
\]

(9.18)

\[
r = r_2 + x
\]

(9.19)

where \( x \) is any distance along the blade length.
Substituting equation (9.19) into equation (9.18) yields

\[
\sigma = \frac{\rho \omega^2}{r} \left[ \int_{r}^{r+nl} \frac{(r_2 + x) \, dx}{r_2+nl} \right]
\]

\[
= \rho \omega^2 \left[ \frac{r_2^2 x + \frac{x^2}{2}}{r_2+n} \right]_{r_2}^{r_2+nl}
\]

\[
= \rho \omega^2 \left[ r_2 \left((r_2+nl)-(r_2+nl)\right) + \frac{1}{2} \left((r_2+nl)-(r_2+nl)\right)^2 \right]
\]

\[
\sigma_{cf} = \rho \omega^2 \left[ r_2(Nl-nl) + \frac{1}{2} \left(N^2 l^2-2Nnl^2 + n^2 l^2 \right) \right]
\]

(9.20)

where;

- \( N \): number of blade elements
- \( n \): node number
- \( \omega \): rotation speed (rpm)
- \( r_2 \): outer radius of the disc
- \( l \): length of blade element
- \( \rho \): density of blade element

Summation of equations (9.12) and (9.20) gives the total stress acting on the blade due to oscillation and centrifugal force respectively.
9.2b IN-PLANE STRESSES

The shape functions for the in-plane analysis can be written in matrix form as follows:

\[
\{q_i^T\} = \begin{bmatrix}
v_k & w_k & v_{k+1} & w_{k+1}
\end{bmatrix}
\] (9.21)

The direction of these displacements and slopes are given in Figure (4.1).

where

\[v = a_1 \sin \phi + a_2 \cos \phi + a_3 + a_5 \phi + \frac{1}{2} a_6 \phi^2 = [V^*](a)
\] (9.22)

\[w = a_1 \cos \phi + a_2 \sin \phi + a_4 - a_6 \phi = [W^*](a)
\] (9.23)

\[\phi = \frac{v}{R} (9.24)
\]

For a curved beam the circumferential strain \(\epsilon_3\) and the change in curvature \(\kappa_{zy}\) in \(zy\) plane, are given by Belek(9.3) as

\[\frac{\partial v}{\partial y} + \frac{W}{R} = \epsilon_3
\] (9.25)

\[\frac{1}{R} \frac{\partial v}{\partial y} - \frac{\partial^2 W}{\partial y^2} = \kappa_{zy}
\] (9.26)
The relationship between the change in curvature and the magnitude of bending moment is

\[ M_x = -\kappa \cdot EI_{xx} \]  \hspace{1cm} (9.27)

Substituting equation (9.26) into equation (9.27) gives

\[ M_x = EI_{xx} \left[ \frac{\partial^2 w}{\partial y^2} - \frac{1}{R} \frac{\partial v}{\partial y} \right] \]  \hspace{1cm} (9.28)

Equations (9.22) and (9.23) can be written as follows:

\[ v = [F_v] \begin{bmatrix} C_v^{-1} \end{bmatrix} \begin{bmatrix} EVEC_v \end{bmatrix} \]  \hspace{1cm} (9.29)

\[ w = [F_w] \begin{bmatrix} C_w^{-1} \end{bmatrix} \begin{bmatrix} EVEC_w \end{bmatrix} \]  \hspace{1cm} (9.30)

Substituting equations (9.29) and (9.30) into equation (9.28) yields

\[ M_{xx} = EI_{xx} \left[ [F_{w11}][C_w^{-1}] \begin{bmatrix} EVEC_w \end{bmatrix} - \frac{1}{R} [F_{v1}][C_v^{-1}] \begin{bmatrix} EVEC_v \end{bmatrix} \right] \]  \hspace{1cm} (9.31)

The dynamic stress results from this bending moment is given by

\[ \sigma_{xx} = \frac{M_{xx}}{I_{xx}} \cdot d_n \]

where

\[ d_n = \frac{b}{\sqrt{\frac{R + b}{R - b^2}}} \]

\[ R = r_2 + L + \frac{t}{2} \]

Then

\[ \sigma_{xx} = E \left[ [F_{w11}][C_w^{-1}] \begin{bmatrix} EVEC_w \end{bmatrix} - \frac{1}{R} [F_{v1}][C_v^{-1}] \begin{bmatrix} EVEC_v \end{bmatrix} \right] \]  \hspace{1cm} (9.32)
From equations (9.22) and (9.23)

\[
[F'_v] = \begin{bmatrix} \sin \phi & \cos \phi & 1 & 0 \end{bmatrix} F \ 
[F'_w] = \begin{bmatrix} \frac{1}{R} \cos \frac{y}{R} & -\frac{1}{R} \sin \frac{y}{R} & 0 & 0 \end{bmatrix} \ 
(9.33)
\]

\[
[F''_w] = \begin{bmatrix} \cos \phi \sin \phi & 0 & 1 & 0 \end{bmatrix} \ 
[F'''_w] = \begin{bmatrix} -\frac{1}{R} \sin \frac{y}{R} & \frac{1}{R} \cos \frac{y}{R} & 0 & 0 & 0 & -\frac{1}{R} \end{bmatrix} \ 
(9.34)
\]

\[
[F''''_w] = \begin{bmatrix} -\frac{1}{R^2} \cos \frac{y}{R} & -\frac{1}{R^2} \sin \frac{y}{R} & 0 & 0 & 0 & 0 \end{bmatrix} \ 
(9.35)
\]

9.2b.2 OUT-OF-PLANE STRESSES

If \( M_y \) and \( M_z \) are moments acting on the cross-section of Figure (4.1) about \( y \) and \( z \) axes at point 0 (\( M_y \) being the twisting moment and \( M_z \) being the bending moment), \( u \) the displacement of the centroid 0 in the \( x \)-axis, \( \theta \) angle of twist of cross-section, then the moments equations can be derived as follows:

\[
M_y = GJ \left[ \frac{d \phi}{dy} - \frac{1}{R} \frac{du}{dy} \right] \ 
(9.36)
\]

\[
M_z = -EI_{yy} \left[ \frac{d^2 u}{dy^2} + \frac{\theta}{R} \right] \ 
(9.37)
\]

where:

\( GJ \) and \( EI_{yy} \) are the torsional and flexural rigidities respectively.
The shape functions for the out-of-plane analysis can be written in matrix form as follows:

\[ u = a_1 R \cos \phi + a_2 R \sin \phi + a_3 R - a_4 R \phi - a_5 R^2 = [U^*] \{ a \} \] (9.38)

\[ \theta = a_1 \cos \phi + a_2 \sin \phi - a_3 - a_4 \phi = [\Theta^*] \{ a \} \] (9.39)

Equations (9.38) and (9.39) can be written as follows

\[ u = [F_u] [C_0^{-1}] \{ EVEC \} \] (9.40)

\[ \theta = [F_\theta] [C_0^{-1}] \{ EVEC \} \] (9.41)

Substituting equations (9.40) and (9.41) into equations (9.36) and (9.37) yields

\[ M_y = G J \left[ [F_\theta^1] [C_0^{-1}] \{ EVEC \} - \frac{1}{R} [F_u^1] [C_0^{-1}] \{ EVEC \} \right] \]

\[ M_z = EI_{zz} \left[ [F_u^{11}] [C_0^{-1}] \{ EVEC \} + \frac{1}{R} [F_\theta^1] [C_0^{-1}] \{ EVEC \} \right] \]

The dynamic stresses resulting from these moments are

\[ \tau_{xz} = \frac{M_y}{J} \cdot r \] (9.42)

\[ \sigma_z = \frac{M_z}{I_{zz}} \cdot \frac{b}{2} \] (9.43)

where

\[ G = \frac{E}{2(1+\nu)} \]

\[ r = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} \]
From equations (9.38) and (9.39)

\[
\begin{bmatrix}
F_u^1
\end{bmatrix} = \begin{bmatrix}
R\cos\phi & R\sin\phi & R & -y & 0 & -1 & 0 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
F^1_u
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{R} & \cos\frac{Y}{R} & \frac{1}{R} & \sin\frac{Y}{R} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_\theta
\end{bmatrix} = \begin{bmatrix}
\cos\frac{Y}{R} & \sin\frac{Y}{R} & 0 & 0 & -1 & \frac{Y}{R}
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_\theta^1
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{R} & \sin\frac{Y}{R} & \frac{1}{R} & \cos\frac{Y}{R} & 0 & 0 & 0 & 0 & 0 & \frac{1}{R}
\end{bmatrix}
\]

Then

\[
\begin{bmatrix}
\tau_{xz} = G\cdot r \left[ \begin{bmatrix} F_\theta^1 \end{bmatrix} - \frac{1}{R} \begin{bmatrix} F_u^1 \end{bmatrix} \right] \begin{bmatrix} C_0^{-1} \end{bmatrix} \begin{bmatrix} \text{EVEC} \end{bmatrix} \right]
\end{bmatrix}
\]

The in-plane and out-of-plane dynamic stresses in matrix form are as follows

\[
\begin{bmatrix}
\sigma_{xx} \\
\tau_{xz} \\
\sigma_{zz}
\end{bmatrix} = \begin{bmatrix}
[FV] & 0 & \begin{bmatrix} C_0^{-1} \end{bmatrix} \begin{bmatrix} \text{EVEC} \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
[FV] & 0 & \begin{bmatrix} C_0^{-1} \end{bmatrix} \begin{bmatrix} \text{EVEC} \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{zz}
\end{bmatrix} = \begin{bmatrix}
0 & \begin{bmatrix} FUF \end{bmatrix}
\end{bmatrix}
\]

(9.44)

(9.45)

where

From equation (9.32)

\[
[FV] = E\cdot d \left[ \begin{bmatrix} F_W^1 \end{bmatrix} - \frac{1}{R} \begin{bmatrix} F_V^1 \end{bmatrix} \right]
\]

From equation (9.44)

\[
[FU] = G\cdot r \left[ \begin{bmatrix} F_u^1 \end{bmatrix} - \frac{1}{R} \begin{bmatrix} F_u^1 \end{bmatrix} \right]
\]

From equation (9.45)

\[
[FUF] = -E\cdot \frac{b}{d} \left[ \begin{bmatrix} F_u^1 \end{bmatrix} + \frac{1}{R} \begin{bmatrix} F_u^1 \end{bmatrix} \right]
\]

FVF is 3 x 12 matrix

C is 12 x 12 matrix

EVEC is 12 x 1 column matrix eigenvectors
From Figure (4.1) the centrifugal force can be written as follows:

$$\begin{bmatrix} R_0 & \phi \\ R_1 & 0 \end{bmatrix}$$

$$\int_0^\phi \rho b r \, d\theta \, dr \cdot \omega^2 r$$

Then the stress due to centrifugal force is

$$SCF = \frac{CF}{b \cdot r \cdot \theta}$$

These stresses should be added to the bending stresses of equation (9.46) to get the total stress of a vibrated bladed-packet under rotation.
9.3 RESULTS AND DISCUSSION

9.3a BLADE STRESSES

In this section dynamic stresses of turbomachine blades are investigated under non-rotating and rotating conditions. Various factors affecting the vibration characteristics of a blade such as pretwist angle and rotational speed as well as the distribution of the dynamic stresses along the blade length for different modes are studied.

All the data used in this chapter are given in Table (9.1) unless specified otherwise.

Figures (9.2-9.6) show some mode shapes of the cantilever blade for the first five modes. In Figure (9.2) the first mode shape is shown in which the distortion of each blue cross-section increased gradually towards the free end. The mode shape along the length of the blade is increased from zero in the fixed end to the maximum in the free end. Figure (9.3) shows the mode shapes of the second mode in which the displacement is seen to be maximum in the middle of the blade length and in the free end. Figure (9.4) shows the third mode shape of the blade. The rotation of the blade cross-section along the length is maximum at the free end. In Figure (9.5) the fourth mode shape of the whole blade is rotated very significantly and increasing steadily towards the free end. The fifth mode shape of Figure (9.6) is large in two positions along the blade length and reaches zero near the free end.
Figures (9.7-9.11) show the effect of vibratory stresses on the cantilever blade. The principal stresses $\sigma_1$ (blue) and $\sigma_2$ (red) and their variation near the root and at various cross-sections are shown for both top surface and bottom surface of the blade for the first five modes of vibration. In contrast to mode shapes the stresses are maximum near the clamped end and decrease for cross-sections away from the fixed end till they vanish near the free end. For the first mode, Figure (9.7), $\sigma_1$ is predominant near both top and bottom surfaces of the blade near the fixed end while $\sigma_2$ is very small near bottom surface. In the second mode, Figure (9.8), $\sigma_2$ is predominant and $\sigma_1$ is very small near the fixed end of the bottom surface of the blade. $\sigma_1$ increases near the middle of blade length while $\sigma_2$ decreases. In the third mode, Figure (9.9), both $\sigma_1$ and $\sigma_2$ are maximum near the fixed end, then $\sigma_1$ decreases while $\sigma_2$ increases around the middle of blade length. In the fourth mode, Figure (9.10) both $\sigma_1$ and $\sigma_2$ have their maximum value near the fixed end and became smaller and smaller along the blade length towards the free end till they vanish near the free end. In Figure (9.11), stresses in the fifth mode are very small near the mid-section of the blade for both top and bottom surfaces. $\sigma_2$ is very small near the fixed end of the bottom surface while $\sigma_1$ is large for both top and bottom surfaces.

To illustrate the stress distribution near the fixed end of the blade along its width more clearly than Figures (9.7 - 9.11), enlarged graphs are drawn in Figures (9.12a-9.12e).

Stresses due to rotation are also investigated in Figures (9.13a) and (9.13b). Figure (9.13a) shows that the stresses due to centrifugal force alone increase with the increase of speed of
rotation. Figure (9.13b) shows the stress distribution along the blade for the first four modes of vibration. The stresses reach their maximum value near the fixed end of the blade and become zero near the free end for the first four modes of vibration. The stresses increase as the mode order increases. Figures (9.14a) and (9.14b) show variation of stresses with pretwist angle near the root at a certain distance from centre of flexure in the direction of z axis and y axis of the blade cross-section of Figure (9.1). It is seen that the stresses decrease with pretwist angles for the first two modes as shown in Figure (9.14a). For the third and fourth modes the stresses $\sigma_1$ of the third mode and $\sigma_2$ of the fourth mode increase with pretwist angle till they reach the value of 40 degrees pretwist then stay almost steady up to 90 degrees pretwist whereas the stresses $\sigma_2$ of the third mode and $\sigma_1$ of the fourth mode behave exactly the opposite as shown in Figure (9.14b).

BLADE PACKET STRESSES

In this section dynamic stresses of a two bladed packet are investigated under rotating and non-rotating conditions. The effects of shroud, pretwist angles, stagger angles and rotational speeds are studied. Three dimensional mode shapes are drawn for the first four modes of vibration. These mode shapes are compared with the distribution of the normalised principal stresses $\sigma_1$ and $\sigma_2$ around the packet shape for different values of speeds, stagger angles and pretwist angles as shown in Figures (9.15a - 9.18d). The mode shapes are drawn on the left of each figure, whereas the stresses are drawn on the right of each figure. For the stress distribution; $\sigma_1$ is drawn outside the packet whereas $\sigma_2$ is drawn inside the packet. Figures (9.15a - 9.15d) are for zero speed, zero
stagger angle and 30 degrees pretwist. Figure (9.15a) shows that the
displacements are maximum near the free end and the maximum stresses
are concentrated around the shroud segment for the first mode. For
the second mode, Figure (9.15b), the displacements are higher in the
left blade and the maximum stresses are also concentrated around the
shroud segment and there are some stresses affecting the left blade.
Figure (9.15c) shows the third mode shape and the corresponding
stresses which are concentrated around the shroud segment. There are
some stresses around the root of the right blade. For the fourth
mode the displacement is higher in the z direction as seen in
Figure (9.15d). The stresses are concentrated around the shroud and
at the tip of the left blade.

When the speed is high the mode shapes of the packet are
different from the previous ones as shown in Figures (9.16a - 9.16d).
In Figure (9.16a), the first mode stresses are maximum at the left
blade near the connection with the shroud and the displacement is
maximum around the shroud. For the second mode as in Figure (9.16b)
the stresses are maximum at and near the root of the right blade and
the maximum displacement is near its root. Figure (9.16c) shows that
the displacement of the third mode is higher near the connection of the
right blade and the shroud and the stresses are concentrated near the
connection of the left blade and the shroud. Figure (9.16d) shows
that the stresses of the fourth mode are concentrated near the root
of the right blade and around the shroud and near the end of the left
blade away from the disc. The displacement is maximum at the end of
the blades away from the disc. Figures (9.17a - 9.17d) show the mode
shapes and stresses distribution for 6000 rpm speed, 30 degrees
stagger angle and 60 degrees pretwist. Figure (9.17a) shows that the
stresses and displacement of the first mode are maximum at the shroud
and near the connection of the left blade with the shroud. For the second mode, as in Figure (9.17b), the stresses and displacement are concentrated near the root of the right blade. In Figure (9.17c) it is seen that the maximum stress is concentrated near the middle of the left blade for the third mode and also the stress is high around the shroud segment. For the fourth mode, as in Figure (9.17d), the stresses are concentrated at the root of the right blade and also the displacement is maximum along this blade. Figures (9.18a – 9.18d) show the mode shapes and stresses for 6000 rpm speed, 90 degrees stagger angle and 60 degrees pretwist. Figure (9.18a) show that the displacement is maximum along the left blade and the maximum stresses are concentrated near the shroud connection with the left blade. Figure (9.18b) of the second mode shows that the maximum displacement and maximum stresses are concentrated near the root of the right blade. For the third mode, as shown in Figure (9.18c), the maximum stresses are concentrated near the middle of the left blade and near its connection with the shroud and along the length of the shroud as well. Figure (9.18d) shows that the stresses are affecting the entire length of the right blade and concentration of maximum stresses is near its root for the fourth mode.

Variations of maximum stresses of the packet with the pretwist angles, stagger angles and rotational speeds are drawn in Figures (9.19a – 9.21). Figure (9.19a) shows that the stresses are maximum for the first blade and increase with the increase of pretwist angle for both blade and shroud for the first mode and the stresses on the second blade are too small to be shown on the graph. Figure (9.19b) of the second mode shows that the stresses of the two blades drop above 45 pretwist angle whereas the stresses of the shroud rise when the pretwist exceeds 45 degrees. For the third mode, as seen in
Figure (9.19c), the stresses of the first blade drop till the curve reaches 45 degrees then the stresses increase with the pretwist angle. The stresses of second blade and shroud increase when the pretwist angle \( \delta_{pretwist} \) between 30 and 45 degrees then decreases as the pretwist increases. Figure (9.19d) of the fourth mode shows that the stresses increase with the increase in pretwist angle for both blades whereas for the shroud the stresses drop as the angle of pretwist exceed 30 degrees. Figure (9.20a) of the first mode shows that the maximum stresses are on the first blade and they increase with the increase of stagger angle for both blade and shroud. The stresses on the second blade are too small to appear on the graph. Figure (9.20b) shows that for the second mode the stresses are maximum on the first blade when the stagger angle is 60 degrees. For the third mode as in Figure (9.20c) the maximum stresses are on the second blade and increase with the stagger angle. In figure (9.20d) of the fourth mode the stresses of the second blade are too small to be shown and that of the first blade are the highest and increase with the increase of stagger angle. For the first mode as in Figure (9.21) the stresses of the whole structure increase with the increase of rotational speed. The maximum stresses are higher on the first blade when the speeds are between 0 and 3000 (rpm) whereas the stresses are higher on the shroud when the speed of rotation is above 3000 (rpm).
### Table 9.1

Physical properties of the blade packet used in this analysis

<table>
<thead>
<tr>
<th><strong>BLADE</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.0914 in$^2$</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>6 in</td>
<td></td>
</tr>
<tr>
<td>$I_{XX}$</td>
<td>$84 \times 10^{-6}$ in$^4$</td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>3240 lbf$\cdot$in$^2$</td>
<td></td>
</tr>
<tr>
<td>$d_x$</td>
<td>0.02156 in</td>
<td></td>
</tr>
<tr>
<td>$E_{g}$</td>
<td>$30 \times 10^6$ lbf/in$^2$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{g}$</td>
<td>0.285 lb/in$^2$</td>
<td></td>
</tr>
<tr>
<td>$I_{YY}$</td>
<td>$671 \times 10^{-5}$ in$^4$</td>
<td></td>
</tr>
<tr>
<td>$C_c$</td>
<td>392 lbf$\cdot$in</td>
<td></td>
</tr>
<tr>
<td>$d_y$</td>
<td>0.497 in</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>SHROUD</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0.0625 in</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0.345 in</td>
<td></td>
</tr>
<tr>
<td>Subtended angle</td>
<td>$10^\circ$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{g}\cdot R$</td>
<td>0.285 lb/in$^2$</td>
<td></td>
</tr>
<tr>
<td>$E_{g}$</td>
<td>$30 \times 10^6$ lbf/in$^2$</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>14 in</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 9.1: CROSS-SECTION OF THE TURBINE BLADE (Measurements in Millimetres)

Centre of Gravity
BLADE MODE SHAPES FOR MODE NUMBER ONE

FIG. 9.2
BLADE MODE SHAPES FOR MODE NUMBER TWO

FIG. 9.3
BLADE MODE SHAPES FOR MODE NUMBER THREE

FIG. 9.4
BLADE MODE SHAPES FOR MODE NUMBER FOUR

FIG. 9.5
BLADE MODE SHAPES FOR MODE NUMBER FIVE

FIG. 9.6
Principal Stresses on the Blade Sections: First Mode

TOP SURFACE

FREE END

BOTTOM SURFACE

FIXED END

FIG. 9.7
Principal Stresses on the Blade Sections: Second Mode

FIG. 9.2
Principal Stresses on the Blade Sections: Third Mode

FIG. 9.9
Principal Stresses on the Blade Sections: Fourth Mode

FIG. 9.10
Principal Stresses on the Blade Sections: Fifth Mode

FIG. 9.11
FIG. 9.12: VARIATION OF MAXIMUM STRESSES AGAINST DISTANCE ALONG BLADE WIDTH
FIG. 9.14: VARIATION OF BLADE STRESS AGAINST ANGLE OF PRETWIST

(a) Relative Principal Stresses $S_1$ and $S_2$

(b) Relative Principal Stresses $S_1$ and $S_2$

Maximum Principal Stresses at the Fixed End
FIG. 9.15: MODE SHAPES AND STRESSES OF A PACKET OF TWO BLADES, SPEED 0, STAGGER ANGLE 0, PRETWIST ANGLE 30°
FIG. 9.16: MODE SHAPES AND STRESSES OF A PACKET OF TWO BLADES, SPEED 6000 RPM, STAGGER ANGLE 0, PRETWIST ANGLE 90°
FIG. 9.17: MODE SHAPES AND STRESSES OF A PACKET OF TWO BLADES, SPEED 6000 RPM, STAGGER ANGLE 30°, PRETWIST ANGLE 60°
FIG. 9.18: MODE SHAPES AND STRESSES OF A PACKET OF TWO BLADES, SPEED 6000 RPM, STAGGER ANGLE 90°, PRETWIST ANGLE 60°
FIG. 9.19: VARIATION OF MAXIMUM STRESSES AGAINST ANGLE OF PRETWIST
FOR FIRST FOUR MODES OF BLADE PACKET, SPEED = 0, $\phi = 45^\circ$
FIG. 9.20: VARIATION OF MAXIMUM STRESSES AGAINST STAGGER ANGLE
FOR FIRST FOUR MODES OF BLADE PACKET, SPEED = 0, $\alpha = 45^\circ$
FIG. 9.21: VARIATION OF MAXIMUM STRESS AGAINST SPEED OF ROTATION FOR FIRST MODE OF BLADE PACKET, $\alpha = 45^\circ$, $\phi = 45^\circ$
CONCLUSION

This study presents the formulation of the eigenvalue equations of the different components of turbomachinery using a finite element method and wave propagation technique. The resulting equations of the blade and the curved beam are relatively small and easily evaluated. Inclusion of the shroud to study the vibration of the packets complicates the problem and requires more calculations in the shroud-blades connections. To study the effects of the flexibility of the disc on the vibration characteristics of the blades and blade packets, the wave propagation technique is used in conjunction with the finite element method which reduces the size of the eigenvalue problem of the whole structure and saves considerable computer time, but adds complexity to the formulation of the eigenvalue equations.

The results obtained for the vibrations of the blades of Chapter 3 are very accurate when compared with experimental and theoretical results of other investigators. The fundamental frequency of cantilever non-rotating blades is not markedly affected by pretwist whereas the higher modes are very much affected. When the pretwisted blade is rotating the coupling between the modes increases.

In Chapter 4 the curved beam elements whose shape functions are based on simple strain displacement relations represent the free body motion and give fast convergence.

In Chapter 5 vibration characteristics of two bladed packet is investigated. The effect of rotation on the natural frequencies increases with the increase of rotational speed.
Vibration characteristics of a two bladed packet can be synthesised from the inference diagram drawn from the independent modes of the blades and the shroud for the lower modes since the coupling, caused by the difference of shear centre from the centroid, between bending and torsional vibration is very small. When the blades are pretwisted it becomes more complex to predict these frequencies from the inference diagram drawn from the independent modes of the blades and the shroud.

In Chapter 6 numerical integration is used to investigate the vibration characteristics of variable thickness discs. The results obtained are very accurate when compared with the results of other investigators.

When the number of nodal diameters increases the effect of disc flexibility decreases and frequencies of the system converge to blade clamped-free frequencies as shown in Chapter 7.

The introduction of the shroud, as in Chapter 8, alters the vibration characteristics of the system and makes the frequencies converge to clamped-clamped frequencies of the shroud.

The study of the dynamic stresses of various components of the turbomachinery requires the calculation of the eigenvectors of these components using large number of elements. This is shown in Chapter 9.
In the work reported here the vibration characteristics of a rotating
shrouded-bladed-disc and dynamic stresses of the blades and packets of blades
are determined for the case when the effects of pretwist angles, stagger
angles and disc thickness variation are taken into consideration.

To extend this work many features can be studied,

1. The effect of thermal environment and aerodynamic forces can also be
   included in the analysis.

2. Dynamic stresses of other turbomachinery components such as; disc,
   bladed-disc and shrouded-blade-disc.

3. Using shell elements for the vibration of the blade and the shroud and
   a thick disc element for the vibration of the bladed-disc and
   shroud-bladed-disc.

4. The effect of shaft flexibility and bearing flexibility on the
   vibration characteristics of bladed-disc.

5. The gyroscopic and whirling effects associated with the motion of the
   disc.

6. Coriolis forces and their effects on the vibration characteristics at
   high speeds.
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