AN INVESTIGATION OF THE ($\pi$,NN) REACTION

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by

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The emission of two nucleons from a nucleus following the absorption of a low energy pi-meson (10 MeV < $T_\pi$ < 120 MeV) is studied in an impulse approximation. Some of the simplifications which are usually made in such studies are investigated and calculations are presented for the Li$^6(\pi^+, 2p)$He$^4$ reaction using cluster model wave functions for the target nucleus.

The matrix element of the transition Hamiltonian is found to be sensitive to the long-range behaviour of the overlap integral between the wave functions of the target and the residual nucleus. The overlap integral must therefore be given the correct asymptotic form as required by the general theory of transfer reactions. An eikonal approximation is used to treat the distortions of the pi-meson field by the target nucleus and of the outgoing nucleon wave functions by the residual nucleus. The results obtained with distorted waves are compared with those obtained using plane waves and distortion is found to play an important part.

It is concluded that one must treat with caution results which have been obtained in calculations which do not take into account the above corrections, in view of the significant errors involved. In particular, it must be considered doubtful whether the ($\pi$,NN) reaction can be used to yield information about nuclear pair correlations without a detailed analysis.
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"All the theoreticians are in agreement that the experimentalists should work harder".

M. Gell-Mann.
1.1 Why Use Pi-Mesons?

The use of pi-mesons as probes for nuclear structure has evolved in an interesting manner. The early studies of production (and absorption) of pi-mesons by a two-nucleon system led to the establishment of basic properties of the pi-meson (spin = 0, parity = -1) using the known properties of the two-nucleon system (Gell-Mann and Watson 1954, Lock 1960). With an improved quantitative understanding of the pi-meson nucleon interaction it became possible to study more closely pi-meson production by a pair of nucleons. An understanding of the mechanism of absorption and production of pi-mesons from free two-nucleon systems has been reached which enables us to give an adequate description of these processes. By making use of this description we may, by extrapolation, describe pi-meson absorption on a nucleus containing more than two nucleons, thus providing a tool for examining some aspects of nuclear structure.

The possibility of absorbing or producing a pi-meson in a nuclear process depends on the fact that it is a boson. We are interested in absorption of a pi-meson by a nucleus, because this releases an enormous amount of energy (at least 140 MeV = the rest mass $m_\pi$ of a pi-meson) with comparatively little momentum, which must, by conservation of energy, be distributed to one or more of the nucleons in the nucleus. Momentum conservation, on the other hand, severely restricts the distribution.

The simplest model is absorption of a meson on a single nucleon, but to satisfy the conservation laws it is necessary to make good a
momentum deficiency which, for a pi-meson at rest, is \( (m^2 + 2m \cdot m)c \)
\( (m = \text{rest mass of a nucleon, } c = \text{velocity of electromagnetic radiation}) \).
This is more than 500 MeV/c and is unlikely to be available to a single nucleon in a nucleus, whose typical Fermi momentum is about 250 MeV/c.
This is supported by evidence from the \((p,2p)\) reaction on p-shell nuclei which indicate a "most probable" momentum range for protons of 80-120 MeV/c (Garron et al 1962, Jain and Jackson 1967, Jackson and Jain 1968). This evidence suggests that the high-momentum components (> 500 MeV/c) have a probability of about 1%. Calculations of single-nucleon absorption show, in fact, that the process \( \pi^+ N \rightarrow N \) in nuclei is suppressed by a factor of the order \( 10^{-3} \) (Letourneux 1966, Ericson 1967b). For pi-mesons incident with about 300 MeV kinetic energy the momentum deficiency is reduced by half. We may safely conclude that the absorption of low energy (<200 MeV) pi-mesons involves at least two other particles.

The \((\pi,\gamma N)\) reaction (radiative capture of a pi-meson) is one possibility involving two other particles. However this reaction accounts for less than 3% of the absorption rate (Davies et al 1968), so we may expect the dominant process to involve more than one nucleon. Clearly two nucleons are more likely to be available for absorbing a meson than a larger number. This is borne out by a recent experiment (Bellotti et al 1970) on the absorption of positively charged pi-mesons incident with kinetic energies in the range 100-150 MeV on \( ^{12} \text{C} \) nuclei. It was found that 50% of all absorptions were accounted for by the \((\pi^+,2p)\) reaction. When one also considers the absorptions accounted for by the \((\pi^+,pn)\) reaction it is seen that about 60% of all absorptions lead to two-nucleon emission. Thus pi-mesons are a practicable probe for observing two-nucleon processes inside nuclei.
This was originally suggested by Brueckner et al (1951).
When a pair of nucleons at rest absorb a static pi-meson they acquire a relative momentum of \( (m^2 + 4m^2 \text{m})^{1/2} \approx 750 \text{ MeV/c} \), corresponding to a wave number of almost \( 2 \text{ fm}^{-1} \), since the momentum conjugate to the internucleon coordinate is half of the relative momentum (Guy et al. 1968). By the uncertainty principle an interaction involving such a high nuclear momentum component will experience the nucleon-nucleon force at distances of the order of 0.5 fm. The \((\pi, \text{NN})\) reaction with a pi-meson at rest should therefore be sensitive to the behaviour of the nuclear forces at a range not accessible to nucleons incident with an energy below several hundred MeV.

Of course, similar arguments (Gottfried 1963) lead us to believe that the same property will be exhibited by nuclear absorption of K-mesons (Rook 1962, 1968) and the high-energy photo-effect \((\gamma, \text{NN})\) (Weise, Huber and Dąnos 1970). We must therefore justify our choice of studying the absorption of pi-mesons. Mesons have an advantage over photons since they exist in electrically charged as well as neutral states; so they can also be used to investigate isospin dependence of nuclear forces through charge exchange and double charge exchange reactions. The latter however have only been observed with small cross-sections (Zupancic 1967). Pi-mesons have a number of additional advantages as nuclear probes. The \(\pi-N\) scattering length is one order of magnitude smaller than the \(K-N\) scattering length and two orders of magnitude smaller than the \(N-N\) scattering length. This means that low energy pi-meson interactions with nuclei are weaker (and hence easier to deal with) than either K-meson or nucleon in interactions at similar energies (Ericson 1967a). Furthermore pi-mesons are the lightest hadrons (strongly interacting particles). Not only does this lead to an additional simplification, since as a
result of the low pi-meson mass few partial waves enter the $\pi$-N interaction, but it also means that the pi-meson is the dominant field quantum at a range of $\geq 0.5$ fm (Ericson 1969).

In common with other bosons, but unlike nuclear projectiles which must be antisymmetrised with target nucleons, pi-mesons are distinguishable from the other participants in the reaction, with a consequent simplification of the algebra. Pi-mesons are also zero spin particles, obviating another source of complexity.

1.2 Nature of Expected Information

We have explained in the previous section that the special properties of the pi-meson make it a highly suitable probe for investigating nuclear forces. We pointed out that, in particular, pi-meson absorption is likely to yield information on the short-range effect of the nucleon-nucleon potential.

Pi-meson absorption is just one particular example of a whole class of reactions (including transfer reactions such as $(p,2p)$, $(p,d)$, $(p,pd)$ and absorption of photons and $K$-mesons) which, although they differ in the details of their interaction mechanisms and also the regions of configuration and momentum space which they involve, display a uniform pattern in the manner in which they yield nuclear structure information. In a nutshell, a reaction in which a group of nucleons is removed from a nucleus tells us about their wave function in the nucleus and about the state of the residual nucleus. It is the former which interests us most in a study of the nucleon-nucleon potential. The latter, which we do not investigate here, is of considerable interest and the study of two-hole states in nuclei has provided much of the incentive for analysing the $(\pi,NN)$ reaction (Zupancic 1967).
In a discussion of reactions in which a pair of nucleons is removed from the target, the wave function of the nucleons in the target nucleus describes their relative motion and the motion of their centre of mass. The relative motion depends on the nucleon-nucleon potential and thus, as a result of what we have said about pi-meson absorption, the sort of reaction we ought to study is \((\pi, \text{NN})\) where \(N\) represents a neutron or a proton. Some comparisons between \((\pi, \text{NN})\) and knock-out reactions will be made.

The state of the residual nucleus created by the removal of the two nucleons will not necessarily be a pure eigenstate but will be spread over several eigenstates. The contributions given by the various eigenstates will depend on the model used to describe the target nucleus. In this way we may obtain information on the relevance of the nuclear model used.

So far, we have made the tacit assumption that the nucleus is a fairly unsophisticated beast. However, in order to ensure that the information we extract is meaningful we must be aware that particles interact, even when they are not bound together. In the case when one of the particles is composite (i.e. a nucleus) the interaction may be described in an average way by an optical potential. Thus the motion of the projectile is distorted, due to its interaction with the target nucleus, and that of the outgoing nucleons by the residual nucleus. These points will be taken up in Chapter III, where we shall also mention one or two other effects, such as the mutual scattering of the outgoing nucleons, which we do not take into account explicitly in our calculations.

Thus it is seen that the \((\pi, \text{NN})\) reaction provides information about the model used to describe the target nucleus. In particular one could expect to discover something about two-body correlations within the nucleus.
The pair correlation function in a nucleus of mass number $A$ is defined by Gottfried (1963) as

$$C(r_1, r_2) = \int |\Psi(r_1, \ldots, r_A)|^2 dr_3 \ldots dr_A$$

where $\Psi$ is the nuclear wave function.

In the shell model of the nucleus it is assumed that the intranuclear forces can be described in terms of a single potential well in which all the nucleons move independently of each other. This means that the nucleons fill up the lowest eigenstates of the Hamiltonian which includes this potential, forming "shells" analogous to the shell structure of the electrons in an atom. For a pair of nucleons in a nucleus under these conditions

$$C(r_1, r_2) = \rho(r_1) \rho(r_2) + \text{Pauli correlations}$$

where

$$\rho(r_1) = \int |\Psi(r_1, \ldots, r_A)|^2 dr_3 \ldots dr_A$$

is the density function. Such a pair is often said to be uncorrelated because there are no correlations between the nucleons other than that implicit in the shell model.

It has been suggested by Brueckner, Eden and Francis (1955) that high-energy nuclear reactions cannot be adequately explained by the "uncorrelated" shell model wave functions. They therefore postulate the introduction of (additional) pair correlations which have the effect of making

$$C(r_1, r_2) = \rho(r_1) \rho(r_2) + D(|r_1 - r_2|),$$

where $D$ differs from the Pauli term.
The function D is discussed by Gottfried (1963). The way in which pair correlations are grafted on to a shell model wave function will be explained in Section 3.5.

Another consideration is the question of localisation of the reaction in the nucleus, viz. is \((\pi , \text{NN})\) particularly sensitive to the surface of the nucleus or to its interior? Before answering this question we must first consider more carefully the nature of the pi-meson. The pi-meson has isospin 1 and can exist in three charge states which are denoted by \(\pi^- \), \(\pi^0 \) and \(\pi^+ \). The superscript indicates the charge on the meson in units of the electronic charge.

A neutral pi-meson disintegrates within a typical time of \(\sim 10^{-16} \) seconds by the process

\[
\pi^0 \rightarrow 2\gamma
\]

so that there are no \(\pi^0 \) beams. This severely reduces the scope for nuclear reactions with a \(\pi^0 \) in the initial state. (A neutral pi-meson is nevertheless a stable particle on the nuclear time scale; direct nuclear reactions have a typical lifetime of \(10^{-21} \) seconds.) The charged pi-mesons are longer lived with lifetimes of \(10^{-8} \) seconds and fairly intense beams are available (Rosen 1967, Nagle 1967). A \(\pi^- \) meson can replace an electron in an atomic orbit to form a bound system known as a (pi-) mesic atom (Ericson 1969).

The \(\pi N \) interaction is dominated by the p-wave resonance at 180 MeV kinetic energy with the \(\pi N \) system in a state of isospin \(3/2 \) and total angular momentum \(3/2 \). In the conventional \(l_{2T, 2J} \) notation this
is known as the $p_{33}$ resonance. The strong p-wave interaction may be assumed to have a minimal effect for very low momentum (kinetic energy $T < 10 \text{ MeV}$) pi-mesons which therefore have a long mean free path in nuclear matter. As a result of this the capture of pi-mesons from atomic orbits is considered to be a volume effect (Sternheimer 1956). For pi-mesons captured in flight with kinetic energy $> 10-20 \text{ MeV}$ the mean free path should be small and absorption is likely to be a surface phenomenon. We should not be surprised therefore, to find that the $(\pi,\text{NN})$ reaction is sensitive to the long-range part of the wave-function of the ejected nucleons in the target nucleus. This point will be examined in Sections 3.3 and 3.4.

We shall be mainly concerned with positively charged pi-mesons, although our formulation is equally valid for negatively charged mesons, provided that they are captured in flight. One advantage of in flight capture is that a range of initial states is available; we shall present results for more than one value of the incident projectile energy. The use of $\pi^+$ as a probe has an additional advantage from the experimental point of view in ensuring that at least one (and possibly both) emitted particle is a proton, and hence relatively easy to detect and measure with accuracy. There is a corresponding computational disadvantage, since we shall not always find it an easy matter to include Coulomb effects in our calculations.

1.3 Experimental Evidence

Over the past decade or so the absorption of pi-mesons by nuclei containing three or more nucleons has attracted considerable interest amongst experimentalists, who have reported numerous results. Most of these, however, relate to absorption of stopped pi-mesons. The review article by Koltun (1970) contains a large collection of references to $\pi$ absorption experiments, only eight of which relate to two-nucleon
emission following absorption of an energetic pi-meson.

A stopped pi-meson must perforce be negatively charged, as a \( \pi^+ \) cannot be caught in an atomic orbit owing to the repulsive nature of both the electromagnetic interaction and the s-wave part of the strong interaction between \( \pi^+ \) and a nucleus. The experiments have provided firm evidence in favour of the theory that the most significant absorption process takes place on a pair of nucleons; the angular distributions show clearly that the transition rates to final states with 180° opening angle between the outgoing nucleons (back-to-back emission) are several times larger than the corresponding transition rates at 90° (e.g. Ozaki et al 1960). A free nucleon pair (i.e. not in a nucleus) will leave back-to-back on absorbing a meson at rest, to conserve momentum.

Because of the numerous inelastic processes initiated by an energetic pi-meson incident on a nucleus, absorption experiments suffer from "background" which is more severe than in the case of stopped mesons. However, it is nevertheless easier to carry out experiments with moving pi-mesons because, as we have already pointed out in the previous section, in flight capture can lead to a two-proton final state. From the point of view of obtaining a wide spectrum of information the in flight reaction is attractive - by varying the incident energy we can investigate a large range of final state momentum distributions and involve the \( p_{33} \) resonance in differing proportions.

Almost all absorption of energetic pi-mesons has been with positively charged beams although one bubble chamber experiment with negatively charged beams has been reported (Balandin et al 1964).
The latter authors have shown that \( \pi^+ \) absorption leads on average to emission of about two protons. This result, in common with the similar information obtained by Bellotti et al (1970) which we discussed earlier, strongly motivates theoretical investigations of the \((\pi^+,2p)\) reaction.

When a free nucleon pair absorbs an energetic meson the result is not, of course, back-to-back emission (as for the case of a stopped meson) but for low incident energies \((< 200 \text{ MeV})\) the opening angle between the outgoing nucleons will preferentially be large.

If the main absorption process in a nucleus is on two nucleons then we can expect this feature to be preserved for \((\pi^+,2p)\) and \((\pi^+,pn)\) on nuclei, as indeed our calculations show (infra, Chapters III and IV). The experiment of Bellotti et al (1970) on the \(^{12}\text{C}(\pi^+,2p)\) reaction confirms the pair absorption hypothesis dramatically, with a pronounced peak for large angle emission. Most of the \((\pi^+,2p)\) experiments have made use of the fact that the emitted nucleons are charged by detecting them with counters or in spark chambers, which can measure the moments of two protons in coincidence and accurately. It is possible in this way to obtain differential cross-sections as a function of energy for particular angular configurations using counters, or spark chambers, and also angular distributions in those experiments which have used bubble chambers. These are less accurate than spark chambers, but more reliable for determining the inelastic "background".

By summing the proton energies Charpak et al (1965, 1967) have also obtained distributions in the "missing mass", i.e. the difference between the relativistic mass of the incident pi-meson and the sum of the kinetic energies of the final state particles plus the separation
energy for the break-up of the target nucleus. The peaks in the "missing mass" distributions correspond to two-hole excitations of the residual nucleus, so that $\left( {\pi}^+, 2\text{p} \right)$ can be used to study two-hole states in nuclei. "Missing mass" distributions have also been obtained in the $\left( {\pi}^-, 2\text{n} \right)$ reaction (e.g. Cheshire and Sobottka 1969) but this requires the measurements of neutron momenta for which it is necessary to resort to time-of-flight techniques.
1.4 Organisation of Thesis

We are concerned in this thesis with a theoretical description of $(\pi,\mathrm{NN})$ reactions on nuclei for energetic pi-mesons, with special reference to $\text{Li}^6(\pi^+,2p)\text{He}^4$.

Chapter II reviews the pi-meson-nucleon strong interaction and develops a Hamiltonian which describes absorption of a pi-meson on a pair of nucleons. The coupling constants for the strong interaction are considered in terms of the reactions $\pi + N + N \rightarrow N + N$ for a free nucleon pair. We discuss the kinematics of the reaction.

$$\pi + A \rightarrow C + N + N$$

and determine the cross-section in terms of the matrix element of the interaction Hamiltonian between initial and final nuclear states. The chapter concludes with a brief discussion of the various theoretical approaches to the $(\pi,\mathrm{NN})$ reaction and the relation to them of our approach.

In Chapter III we evaluate the matrix element for the simplified case in which either the target nucleus or the residual nucleus consists of closed sub-shells only. We discuss particularly the overlap integral between the target and residual nuclei. The incoming meson and the outgoing nucleons are described in Section 3.4 in terms of both plane waves and distorted waves. The question of initial-state correlations and final-state interaction between the two "active" nucleons are looked at in Section 3.5.
The results obtained when applying the calculations described in Chapter III to the Li\(^6\)(\(\pi^+,2p\))He\(^4\) reaction are discussed in Chapter IV. We look especially at the improvements to earlier calculations which are suggested in Chapter III and consider their significance.

We have chosen to perform our calculations for a Li\(^6\) target because it satisfies our criterion in Chapter III (the residual nucleus He\(^4\) is certainly a closed shell!) and because most of the experimental information has been obtained using a Li\(^6\) target (Charpak et al 1965 and 1967, Zupancic 1967, Burman and Nordberg 1968). We do not know of any other theoretical calculations of the Li\(^6\)(\(\pi^+,2p\))He\(^4\) reaction. Indeed there have been only very few calculations of the (\(\pi^+,2p\)) reaction at all.
Chapter II : THE ABSORPTION MODEL

2.1 The (\(\pi, NN\)) Hamiltonian

Before we can begin to write down a Hamiltonian for the (\(\pi, NN\)) reaction we must consider the basic properties of pi-mesons and the way in which they interact with nucleons.

We have mentioned in Chapter I that the pi-meson is a strongly interacting particle (i.e. a hadron) which exists in three charge states; these states form a \(T = 1\) triplet in the isospin formalism. The mass of the \(\pi^-\)-meson has been determined from X-ray transitions in pi-mesic Calcium and Titanium atoms (Shafer et al 1965, Shafer 1967, Rosenfeld et al 1967), and by assuming invariance under charge conjugation the mass of a \(\pi^+\)-meson may be deduced. These results give

\[
m_{\pi} \approx 139.58 \text{ MeV/c}^2.
\]

The mass of a \(\pi^0\) meson is marginally different from that of the charged pi-mesons. For the mass of a nucleon it is convenient to take the average proton-neutron value (Taylor, Parker and Langenberg 1969)

\[
m \approx 938.9 \text{ MeV/c}^2.
\]

Since pi-mesons are the dominant quanta of the nuclear force the Compton pi-meson wavelength \(\hbar/m_{\pi} c \approx 1.4 \text{ fm}\) is the characteristic range of nuclear forces. (Other contributions to the nuclear force are of shorter range.) A quasi-classical
argument immediately yields the result that for pi-mesons incident with energies up to a few hundred MeV only s- and p-waves contribute significantly to the pi-meson-nucleon interaction. At these energies several partial angular momentum waves contribute to the nucleon-nucleon interaction owing to the considerable mass difference \( m / m_N \approx 6.7 \), thereby increasing the complexity of the analysis (Lock 1960, Ericson 1967).

The dominant feature of the \( \pi N \) interaction at positive energies is the \( p_{33} \) resonance at kinetic energy \( \sqrt{\Sigma} \approx 185 \) MeV. (The notation for the \( \pi N \) system was introduced in Chapter I.) The Hamiltonian operator for absorption of a pi-meson by a pair of nucleons is usually developed in terms of the operator for the virtual process

\[
\pi + N \rightarrow N.
\]

The part of this operator which is linear in the pi-meson field \( \phi \) is proportional to

\[
\frac{1}{\Sigma} \sum_{\nu = -1} (-1)^\nu \gamma_5 \partial_\nu \phi \partial_\nu
\]

where \( \gamma \), \( \tau \) are the nucleon spin and isospin operators respectively and \( \partial_\nu \) is chosen to be the gradient operator with respect to the relative \( \pi-N \) co-ordinate, making the operator invariant under Galilean transformations. We note that \( \gamma_5 \partial_\nu \phi \partial_\nu \) is a pseudoscalar, as required to reflect the negative intrinsic parity of the pi-meson, and \( \Sigma (-1)^\nu \tau_\nu \phi \partial_\nu \) is a scalar which is linear in the pi-meson field operator. The annihilation of a pi-meson with momentum \( k \to \infty \) and charge \( \nu' \) leads to the matrix element
\[
\langle \phi_{n_1} | \chi_{n_2} \rangle = (-1)^{\nu_1} \delta_{\nu_1} \chi_{n_2} (\pi, \pi, x_1)
\]

where \(\chi_{n_2}\) satisfies the appropriate Klein-Gordon equation and represents the field function of the incident \(\pi\)-meson. (This will be discussed in detail in Chapter III).

The Hamiltonian density for \(\pi + N \rightarrow N\) is then given (Woodruff 1960) by

\[
\mathcal{H} = -f \sqrt{\frac{2 \pi \hbar^2}{E_{\pi}} \frac{\hbar^2}{m_{\pi}}} \bar{G} \left\{ \left[ i \bar{\nabla} \chi_{n_2} - \frac{E_{\pi}}{2mc} \right] \left[ i \nabla \chi_{n_2} + \chi_{n_2} \nabla \right] + \bar{\nabla} \delta (x_{n_1} - x_N) \right\}
\]

where

\[
E_{\pi} = \left( \frac{\hbar^2}{m_{\pi}} \frac{c^2}{m_{\pi}^2} + m_{\pi}^2 c^4 \right)^{1/2}
\]

is the total energy (relativistic mass) of the \(\pi\)-meson and \(\nabla\) is the charge on the \(\pi\)-meson. The nucleon is located at \(x_N\) and \(\bar{\nabla}, \nabla\) operate respectively on \(x_{n_1}, x_N\). The arrow on \(\nabla\) indicates the direction in which \(\nabla\) operates - the expression in parentheses then represents an average over initial and final nucleon momenta when we evaluate the matrix element of \(\mathcal{H}\) between initial and final nucleon states. The \(\pi N\) Coupling constant \(f\) has been deduced from experiment but the various results (e.g. Barnes et al 1960 a, 1960 b McKingley 1963) do not agree to better than about 10%. We may take the value given by \(f^2/\hbar c \approx 0.09\). (In these terms nuclear forces are strong compared to electromagnetic forces for which \(e^2/\hbar c \approx 0.007\).) Kopaleishvili (1967) has used a value for \(f\) which is about 50% higher.

Although the operator \(\mathcal{H}\) is apparently Galilean invariant we cannot avoid introducing ad hoc relativistic corrections, even for \(\pi\)-mesons with low kinetic energy. (A \(\pi\)-meson incident with kinetic energy as low as 50 MeV has a velocity \(\approx 0.65c\) as may be seen from Figure 1.) This explains the
Fig. 1 Variation of particle velocity (in units of c) with 
η = kinetic energy/rest mass (in absolute units).
appearance of the term $E_\pi$ in the formula for $\mathcal{H}$. The problems involved in establishing a Galilean invariant formula for pi-meson absorption on a nucleon are discussed by Barnhill (1969).

When we come to consider the matrix element of $\mathcal{H}$ between initial and final nuclear states we will find that the initial state of the absorbing nucleons is a bound state whereas the final state is a scattering state. Since the operators $\nabla_\pi$, $\hat{\nabla}$ and $\tilde{\nabla}$ may be related through integration by parts, and it is more convenient (in plane wave approximation) to apply the gradient operator to a scattering state than to a bound state, we shall eliminate $\hat{\nabla}$. If $\psi_i$ and $\psi_f$ are respectively initial and final-state wavefunctions then clearly

$$
 \int_{\Psi_f} \overrightarrow{\nabla_\pi} \overrightarrow{\psi_i} = \int_{\Psi_f} \overrightarrow{\nabla}(\psi_f^* \chi_\pi) \psi_i \\
= \int_{\Psi_f} \overrightarrow{\nabla}^* (-i\nabla) \chi_\pi \psi_i + \int_{\Psi_f} \psi_f^* (-i\overrightarrow{\nabla}) \chi_\pi \psi_i,
$$

i.e.

$$\chi_\pi \overrightarrow{\nabla} = -i\nabla \chi_\pi - i\overrightarrow{\nabla} \chi_\pi .$$

Integrating the Hamiltonian density over $\xi_\pi$, we obtain the result

$$- f(2\pi E_\pi)^{\frac{1}{2}} \left( \frac{n_i^2}{m_\pi mc^2} \right) \int g \left\{ \overrightarrow{\nabla} \chi_\pi(k, \xi_N) + \tilde{\nabla} \chi_\pi(k, \xi_N) \right\}$$

in which

$$\zeta = \frac{1}{2} + \frac{mc^2}{E_\pi}$$

$\approx 7$ for very low energy pi-mesons.

We note that in the zero-range approximation ($r_N=0$) the only contributions to the Hamiltonian are from $\tilde{\nabla} \chi_\pi$ for p-wave pi-mesons and from $\chi_\pi$ for s-wave pi-mesons.
As already indicated, the ($\pi', \text{NN}$) reaction proceeds via a direct reaction mechanism, i.e. the dominant process is absorption of the pi-meson by the emitted nucleons, whilst the rest of the nucleus, is a passive participant. For the free reaction

$$\pi + N + N \rightarrow N + N$$

the Hamiltonian operator is

$$H_{12}(k_\pi) = \mathcal{H}(1) + \mathcal{H}(2)$$

where $\mathcal{H}(i)$ is the (integrated) operator for the absorption of the pi-meson by the $i^{th}$ nucleon, whose coordinate vector, spin and isospin are given by $\xi_i$, $\sigma[i]$ and $\tau[i]$ respectively.

It is convenient to transform the Hamiltonian $H_{12}$ in terms of relative and centre of mass coordinates. We therefore define

$$R' = \frac{1}{2}(\xi_1 + \xi_2)$$
$$\xi = \xi_1 - \xi_2$$
$$V'_R = V_1 + V_2$$
$$V_x = \frac{1}{2}(V_1 - V_2)$$

$$S = (S[1] + S[2]) / \sqrt{2}$$
$$g = (g[1] - g[2]) / \sqrt{2}$$

$$T_y = (T_y[1] + T_y[2]) / \sqrt{2}$$
$$\tau = (\tau_y[1] - \tau_y[2]) / \sqrt{2}$$

$$\Phi(k_\pi, R', \xi) = \frac{1}{2}[\chi_{\pi}(k_\pi, \xi_1) + \chi_{\pi}(k_\pi, \xi_2)]$$
$$\phi(k_\pi, R', \xi) = \chi_{\pi}(k_\pi, \xi_1) - \chi_{\pi}(k_\pi, \xi_2).$$

A few lines of manipulation yield the result (Eisenberg and Letourneux 1967)

$$H_{12} = -\frac{f(2\pi E_\pi)^{\frac{3}{2}}}{m_\pi mc^2} i \{(T_y S + T_y S)[C_2(V_R', \Phi + V_x \phi) + (\bar{V}_R', \bar{\Phi} + \bar{V}_x \bar{\phi})] + (T_y S + T_y S)[C_2(\frac{1}{2} V_R' \phi + 2 \bar{V}_x \bar{\phi}) + (\frac{1}{2} \bar{V}_R' \Phi + \bar{V}_x \Phi)]]$$

This reduction is merely a coordinate transformation which presents the absorption Hamiltonian in a convenient form (a two-nucleon wavefunction is given in terms of the relative coordinate \( r \)). It does not imply any particular correlation between the nucleons 1 and 2.

Brückner, Serber and Watson (1951) suggested that the primary absorption event in the \((\pi, NN)\) reaction on a complex nucleus (> two nucleons) is the inverse of pi-meson production by the collision of two nucleons. This means that we can use the same operator \( H_{12} \) for the \((\pi, NN)\) reaction on a complex nucleus as for the free absorption reaction

\[
\pi + N + N \rightarrow N + N
\]

This assumption is essentially that which goes by the name of the impulse approximation in nuclear collision theory (Jackson 1970a).

2.2 Kinematics and the Reaction Cross-Section

The kinematics of the \((\pi, NN)\) reaction for moving pi-mesons are complicated by two factors - the high energy/mass ratio of the projectile which necessitates a relativistic formulation, and the three-body description required for the final state. For stopped pi-mesons neither of these problems arises in the kinematics (although the mechanics of the final state cannot ignore the three particles scattering off each other). The reason for this is that the kinematical description of a reaction depends on the total number of participants (initial state + final state). The \((\pi, NN)\) reaction for stopped pi-mesons is

\[
A \rightarrow B + N + N
\]
where $\bar{A}$ is the pi-mesic atom and $B$ the residual nucleus, which is essentially equivalent (as far as kinematics are concerned) to

$$C + X \rightarrow D + Y.$$  

However we are considering moving pi-mesons so that the kinematics must describe a total of five particles.

The transition which concerns us is from an initial state consisting of an incident pi-meson $\pi$ and a nucleus $A$ to a final state of two outgoing nucleons 1 and 2 and a residual nucleus $C$ containing $A-2$ nucleons. The asymptotic momenta of the particles in the centre of mass system are $\hbar k_\pi, -\hbar k_\pi, \hbar k_1, \hbar k_2$ and $-\hbar K$ respectively and the corresponding primed quantities refer to the laboratory system (see Figure 2). We denote the masses of $A$ and $C$ by $m_A$ and $m_C$.

Momentum conservation imposes the requirement that $K = k_1 + k_2$. The relationship between the laboratory and centre-of-mass is simple in the final state for which we may assume non-relativistic kinematics (see Figure 2). The velocity of the centre of mass of the $A$ nucleon system in the laboratory frame of reference in the final state is

$$\gamma = \hbar k'/m_c + 2m.$$


Fig. 2 Kinematics of the ($\pi$,NN) reaction for moving pi-mesons. The various symbols are defined in the text.
In the initial state we must use a relativistic transformation to relate the kinematical variables in the two frames.

The energy-momentum relations for the initial state in the two frames of reference are

\[
\begin{align*}
E_\pi^2 &= (\hbar k_\pi c)^2 + m_{\pi}^2 c^4 \\
E_A^2 &= \left(\hbar k_A c\right)^2 + m_A^2 c^4 \\
E_A' &= m_A^2 c^2 \\
E_A'' &= \left(\hbar k_A' c\right)^2 + m_A^2 c^4
\end{align*}
\]

for the target nucleus. In these equations \(E_\pi\), \(E_A\) are the relativistic energies of the \(\pi\)-meson and the nucleus \(A\) in the centre-of-mass frame and \(E_A', E_A''\) are the corresponding quantities in the laboratory frame.

It is well known that for any system with energy \(E\) and momentum \(p\) the quantity \(E^2 - p^2 c^2\) is a Lorentz invariant. Considering the whole \(\pi + A\) system we have

\[
E_i^2 = (E_\pi' + E_A')^2 - \left(\hbar k_A' c\right)^2 = (E_\pi + E_A)^2
\]

in which \(E_i\) is the total energy of the initial state in the centre of mass frame. A few lines of manipulation yield the transformation formula

\[
k_\pi = \frac{k_A' m_A c^2}{E_i}
\]

The kinetic energy \(T_\pi'\) of the incident \(\pi\)-meson beam may be
measured. We may then obtain the momentum of the beam in
the two frames by

\[ \hbar k' / c = \sqrt{T_{1}^\prime (T_{1}^\prime + 2m_{\pi} c^{2})} \]

and Equation 2, noting that

\[ E_{1}^{2} = m_{\pi}^{2} c^{4} + 2E_{\pi}^\prime m_{A} c^{2} + m_{A}^{2} c^{4} \] (from Equation 1)

and

\[ E_{\pi}^\prime = T_{\pi}^\prime + m_{\pi} c^{2} . \]

A free n-body system is described by 3n variables. In
view of the energy-momentum conservation laws an n-body
final state has 3n-4 degrees of freedom. In our case n = 3
and a full specification requires the measurement of five
quantities. A direction is specified by two angles and so
measurement of the directions of two outgoing nucleons
leaves one quantity to be measured. This fifth measurement
is usually either the kinetic energy of one of the nucleons
or the recoil momentum of the residual nucleus. The angular
measurements may also be described in more than one way but
usually one of the measurements is the opening angle between
the two outgoing nucleons. We shall be concerned exclusively
with coplanar experiments i.e. where the plane defined by
the three final state particles includes the incident beam
direction. This is equivalent to two angular measurements.
We shall also specialise to the case in which the incident
beam direction bisects the opening angle between the outgoing
nucleons (symmetric geometry).

The other variables are obtained from the measured
variables using the conservation laws. Suppose \( \Delta E \) is the
energy released in the reaction:
\[ \Delta E = (m_A - m_e - 2m)c^2 + E_L - m_A c^2 \]

i.e. the energy of the absorbed pi-meson plus the Q-value and the kinetic energy of the target.

Then, since the reduced masses of the D-C and the N-N systems are \(2(A-2)m/A\) and \(m/2\) (approximately),

\[ \frac{Ak^2}{4(A-2)} + k^2 = \frac{m\Delta E}{\hbar^2} \quad (3) \]

where \(\hbar k\) is the momentum of the nucleon pair centre-of-mass (which we may conveniently describe as a dinucleon) in the centre of mass system of the A nucleons and, \(\hbar k = \frac{\hbar}{2}(k_1 - k_2)\) is the momentum conjugate to the coordinate \(r\). This formula determines \(k\) once \(K\) is given. Now, if \(\theta\) is the opening angle between the outgoing nucleons then

\[ k_1^2 + k_2^2 + 2k_1 k_2 \cos \theta = k^2 \quad (4) \]

\[ k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta = 4k^2. \]

Alternatively the kinematic variable \(\epsilon = k_2/k_1\) may be measured. If we restrict consideration to the case \(k_1 > k_2\) (with no loss of generality) then \(\epsilon\) lies in the range \(0 < \epsilon \leq 1\). The variable \(\epsilon\) may be useful for comparing reactions with different values of the total available energy \(\Delta E\). Substitution of Equations 4 into Equation 3 yields

\[ 4k^2 = f(\epsilon, \theta)k^2 \quad (5) \]

where

\[ f(\epsilon, \theta) = \frac{1 - 2\epsilon \cos \theta + \epsilon^2}{1 + 2\epsilon \cos \theta + \epsilon^2} \]

It follows immediately that \(2k\) is less than, equal to or greater than \(K\) according to whether \(\theta\) is less than, equal to or greater than \(\pi/2\). When the opening angle \(\theta = \pi/2\)
then energy conservation (Equation 3) requires that the energy of the dinucleon $T_D(= \hbar^2 k^2/4m)$ is

$$\frac{A-2}{2A-2} \Delta E$$

which is $0.4 \Delta E$ for a Li$^6$ target. The most interesting region for the ($\pi$,NN) reaction is for large opening angles i.e. $\Theta > \pi/2$. For fixed $\varepsilon$, $f(\varepsilon, \Theta)$ is an increasing function of $\Theta (\pi/2 \leq \Theta \leq \pi)$. Since $\Delta E$ is fixed it follows that for a given $\varepsilon$, $T_D$ decreases as $\Theta$ increases from $\pi/2$ to $\pi$. This is illustrated in Figure 3.

Inverting Equation 5 gives us

$$\varepsilon^2 + 2\beta \varepsilon + 1 = 0$$

where

$$\beta = \frac{2T_D - (A-2)\Delta E}{2(A-1)T_D - (A-2)\Delta E} \cos \Theta.$$

For a given $K$ (or $T_D$) we determine $\varepsilon$ by

$$\varepsilon = -\beta - \sqrt{\beta^2 - 1}.$$

The physical region is $\beta \leq -1$. (We can use this condition to determine the range of $\Theta$ for which a given value of $K$ lies in the physical region.)
Fig. 3 Isobars of $f(\varepsilon, \theta) = (1 - 2\varepsilon \cos \theta + \varepsilon^2)/(1 + 2\varepsilon \cos \theta + \varepsilon^2)$, labelled by the corresponding values of $T_D/\Delta E$ for an $A = 6$ nucleus.
Correspondingly we find that

\[ T_1 = \frac{(\hbar k_1)^2}{2m} = \frac{(A - 2) \Delta E}{(1 + \varepsilon^2)(A - 1) + 2 \varepsilon \cos \theta} \]

Thus the bounds on the kinetic energy of the faster of the two nucleons are

\[ \frac{A - 2}{2A} \leq \frac{T_1}{\Delta E} \leq \frac{A - 1}{A} \]

The minimum is achieved when the two nucleons are ejected in the same direction with the same momentum, and the maximum is achieved for back-to-back scattering with \( k_1 = (A - 1)k_2 \). Table I gives some typical values for the bounds on \( T_1 \). By comparing the maximum value of \( T_1 \) with the nucleon mass we see that it is justifiable to treat the nucleons non-relativistically at low incident pi-meson energies.

Table I

<table>
<thead>
<tr>
<th>( T'_\pi )</th>
<th>( \Delta E )</th>
<th>( T_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>187</td>
<td>62-156</td>
</tr>
<tr>
<td>78</td>
<td>214</td>
<td>71-178</td>
</tr>
<tr>
<td>100</td>
<td>235</td>
<td>78-196</td>
</tr>
</tbody>
</table>

We shall now obtain the scattering cross-section for the reaction as a function of the kinematic variables. We shall derive formulas in terms of the energy of one nucleon, \( T_1 \), and in terms of the wave number of the dinucleon, \( K \).
The Hamiltonian operator for the whole system is

\[ H_N + H_{12} \]

where \( H_N \) is the Hamiltonian for the A-nucleon system. In practice we use model Hamiltonians which differ for the initial (bound) state and the final state.

In order to calculate the reaction cross-section in terms of the interaction Hamiltonian \( H_{12} \) which causes the transition from the initial \((K + \text{nucleus})\) state to the final \((\text{nucleus} + \text{two nucleons})\) state it is necessary to treat it as a perturbation in time-dependent perturbation theory. Fermi's "golden rule" then give the transition rate

\[
\frac{d\sigma_{i\ell}}{d\omega_{if}} = \frac{2\pi}{\hbar} \left| \langle f | H_{12} | i \rangle \right|^2 \delta(E_f - E_i) \left< f \right| \tilde{S} \left| i \right> dE_f.
\]

In this expression \( |f\rangle \) and \( |i\rangle \) are superpositions of eigenstates of the "final" and "initial" Hamiltonians, respectively, with corresponding energies \( E_f \) and \( E_i \) respectively. The density of states is given by

\[
\rho(E_f) dE_f = (2\pi)^{-6} d^3 k f d^3 k i.
\]

Since \( \hat{k}_i \) is the asymptotic momentum of the \( i^{th} \) emitted nucleon in the centre-of-mass frame, the transition rate is given in this frame.

The nuclear structure information is contained in the matrix element \( \langle f | H_{12} | i \rangle \) which will be discussed fully in Chapter III. Here we are only considering the kinematical aspects. Since \( d^3 k = k^2 dk d\hat{k} \) where \( k = |\vec{k}| \) and \( \hat{k} = \vec{k}/k \) and \( dT_i = \hbar^2 k_i dk_i/m \) (where \( T_i = \hbar^2 k_i^2/2m \) is the energy of nucleon \( i \)) we have
\[ e(E_f) dE_f = (2\pi)^{-6} \hbar^{-4} m^{-2} k_1 k_2 dT_1 dT_2 d^2 k_1 d^2 k_2 \]

and so

\[ dw_{if} = \frac{f^2 E_n k_1 k_2}{(2\pi c)^4 \hbar m_n^2} |M_{fi}|^2 \delta(E_f - E_i) dT_1 dT_2 d^2 k_1 d^2 k_2 \]

where the operator \( M \) is defined by

\[ H_{12} = -f(2\pi E_n)^{\frac{1}{2}} \hbar^2 (m_n c^2)^{-1} M \]

and we have written \( M_{fi} \) for the matrix element \( \langle f|M|i\rangle \).

Now the total energy of the final state is

\[ E_f = (2m + m_C)c^2 + T_1 + T_2 + T_C, \]

where

\[ T_C = \frac{\hbar^2 k^2}{2(A - 2)m} \]

is the kinetic energy of the recoiling nucleus \( C \) in the centre-of-mass frame (determined in terms of \( T_1 \) and \( T_2 \) by momentum conservation). We can therefore integrate over \( T_2 \) to obtain the formula

\[ dw_{if} = \frac{f^2 E_n k_1 k_2}{(2\pi c)^4 \hbar m_n^2} |M_{fi}|^2 \frac{A-2}{(A-1) \hat{k}_2 + \hat{k}_1 \cdot \hat{k}_2} d\hat{\mathbf{n}}_1 d^2 \hat{k}_1 d^2 \hat{k}_2 \]

in which the matrix element \( M_{fi} \) and the momentum \( \hbar k_2 \) are restricted to be on the energy shell for the reaction.
The scattering cross-section is the outgoing flux into the phase space solid angle element $d^2k_1 d^2k_2$ per unit incident flux, i.e. the transition rate divided by the incident velocity (provided we assume all scattering wave functions to be normalised to one particle per unit volume).

The incident pi-meson velocity is

$$\nu_\pi = \hbar k_\pi c \left( \frac{1}{E_\pi} + \frac{1}{E_A} \right)$$

in the centre of mass frame, giving the differential cross-section

$$\frac{d^5\sigma}{dT_1 d^2k_1 d^2k_2} = \frac{\varepsilon^2 E_\pi^2 k_1 k_2}{(2\pi)^5 m^2 \hbar c^5} \frac{E_A |M_R|^2}{E_\pi + E_A} \frac{A-2}{(A-1)k_\pi + k_1 \cdot k_2}$$

in terms of a complete set of five kinematic variables. However, we see immediately that the differential cross-section must be symmetric with respect to rotation about the incident beam direction (azimuthal symmetry). Let the polar and azimuthal angles of $\hat{k}_1$ and $\hat{k}_2$ be ($\Theta_1$, $\varphi_1$) and ($\Theta_2$, $\varphi_2$) respectively, with the polar axis taken along the incident beam direction $\hat{k}_\pi$.

We may now integrate over one of the azimuthal angles to obtain

$$\frac{d^4\sigma}{dT_1 d\cos\Theta_1 d\cos\Theta_2 d\varphi_2} = 2\pi \frac{d^5\sigma}{dT_1 d^2k_1 d^2k_2} \left| \varphi_1 = 0 \right.$$

In the case of symmetric coplanar geometry we have the additional conditions
\[ \Theta_2 = \Theta_1 \]
\[ \gamma_2 = \pi \]

and the opening angle between the outgoing nucleon directions is \( \Theta = 2\Theta_1 \). The matrix element is then a function of three variables \( M_{fi} (T_x, T_1, \cos \Theta) \).

To obtain the scattering cross-section in terms of the dinucleon momentum (or equivalently the recoil momentum) we require the Jacobian of the transformation from \( k_1, k_2 \) to \( k, K \), which is given by

\[ \frac{\partial (k_1, k_2)}{\partial (k, K)} = 1 \]

which yields the formula for the phase space density

\[ \mathcal{P}(E_f) dE_f = (2\pi)^{-6} d^3 k d^3 K. \]

By a reduction similar to the one above we obtain the differential cross-section in the form

\[ \frac{d^5 \sigma}{dK d^2 k} = \frac{f^2 E^2_\pi k k^2}{(2\pi)^4 2m^2_m k^2_{\pi} c^6} |M_{fi}|^2 \frac{E_n}{E_\pi + E_n} \]

and

\[ \frac{d^4 \sigma}{dK d^2 k d\cos \Theta_k} = 2\pi \left. \frac{d^5 \sigma}{dK d^2 k d^2 \Theta_k} \right|_{\Theta_k = 0} \]

where \( k = (\Theta_k, \gamma_k) \). For the symmetric coplanar geometry we obtain the matrix element as a function \( M_{fi}(T_x, K, \cos \Theta) \) with the opening angle between the outgoing nucleons given by
\[
\cos \theta = \left( \frac{k_1^2}{4} - k_2 \right) \sqrt{\left( \frac{k_1^2}{4} + k_2^2 \right)^2 - (k_1 k_2)^2}
\]

and the azimuthal angle \( \phi \) = 0 (if \( k_1 \geq k_2 \)).

2.3 Review of Theoretical Analyses

The considerable complexity of the (\( \pi,NN \)) reaction has inevitably led to the need for making suitable approximations in any theoretical discussion. The choice of approximations which can be made is large and there are a variety of approaches in the literature stressing different aspects.

The simplest case to discuss is pi-meson absorption by a deuteron. In this case there is no recoiling nucleus and the cross-section may be determined by a simplified version of the theory which we shall develop in Chapter III. Indeed the coupling constant \( f \) in the absorption Hamiltonian is determined not from a microscopic theory but phenomenologically to fit the total cross-section for a reaction such as \( \pi^+ + d \rightarrow p + p \).

Koltun and Reitan (1966) have studied the inverse reaction

\[ p + p \rightarrow \pi^+ + d \]

taking into account second-order terms in the pi-meson field as well as the linear Hamiltonian which we introduced in Section 2.1. One of the second order terms corresponds to charge exchange scattering by one of the nucleons, i.e.

\[ p + p \rightarrow \pi^0 + p + p \rightarrow \pi^+ + d. \]
Koltun and Reitan find that this rescattering makes a significant contribution to the matrix element. (They also find that the results are sensitive to the introduction of the D-state of the deuteron). They suggest that rescattering is an important factor in the (π,NN) reaction on nuclei, resulting in a greater amplitude for absorption on a nucleon pair in a triplet spin state than in a singlet spin state. Since the Pauli exclusion principle prevents two identical nucleons being in a triplet spin state this implies that the ratio of absorption by a neutron-proton pair to absorption by a pair of identical nucleons is greater than the statistical 4:1 ratio. Experiments on the absorption of stopped π⁻ mesons by C¹² (Ozaki et al 1960) have found the ratio of emitted neutron-neutron pairs to emitted neutron-proton pairs to be 5:1 which is accounted for by rescattering, assuming that the absorbing pair of nucleons is in an S-state.

An early theoretical discussion of π⁻ meson absorption in complex nuclei (A > 3) was that of Brueckner, Serber and Watson (1951). On the basis of the two-nucleon absorption model they deduced from the absorption cross-sections that there is a strong degree of nucleon-nucleon correlation in the nucleus, compared with the shell model distribution. This result was developed by Brueckner, Eden and Francis (1955) based on (π,NN) and other high-energy nuclear reactions. They concluded that the independent particle shell model is inadequate to account for the high-energy phenomena and that an admixture of high-momentum components into the nuclear wave function is required. The work of Gottfried (1963) on nuclear pair correlations, however, threw some doubt on the validity of such conclusions drawn from general arguments based on a consideration of total cross-sections. In a similar situation the early analysis of the (p,pd) reaction predicted a need for additional high-momentum components which was not
A phenomenological approach to the \( (\pi, NN) \) reaction on complex nuclei has been adopted by Eckstein (1963). This approach essentially treats the reaction as a quasi-free absorption, i.e. relating the transition rate for absorption by a nucleus to that for absorption by a deuteron and assuming that the absorbing pair of nucleons is correlated in a nucleus as in a deuteron. In addition, Eckstein assumes a zero-range approximation to simplify the calculations. This has the effect of multiplying our Hamiltonian by \( \delta(x) \) so that the deuteron absorbs a \( \pi \)-meson only when the neutron-proton separation distance is zero. In the phenomenological method the coupling constant \( f \) in the absorption Hamiltonian is replaced by a set of constants \( g_0^+, g_0^-, g_1^+, g_1^- \) which depend on the transition

\[
LST \rightarrow L'S'T'
\]

of the two-nucleon system and are recalculate from the reaction

\[
p + p \rightarrow \pi^+ + d.
\]

Since a pair of nucleons in a nucleus can be involved in transitions which are not open to the above reaction it is also necessary to consider reactions which do not involve a deuteron in order to calculate the constants \( g \).

Using this approach Eckstein was able to reproduce various experimental results without the need to introduce explicit correlations. However, the values of both the magnitudes of the \( g \) and
their relative phase are in dispute and the differences may have quite an effect on the transition rates (Divakaran 1965, Figureau and Ericson 1969, Jain 1972). In addition, the use of a zero-range approximation between the absorbing pair of nucleons shrouds the true position regarding pair correlations.

Besides including effects due to the linear term in the Hamiltonian, Eckstein has shown that the second-order terms have matrix elements of the same form so that their effect is included in the constants $g$. Her values give $|g_0^-|^2 \approx |g_1^-|^2$ which implies that the rescattering term is not important since it gives different contributions to $g_0^-$ and $g_1^-$. Consequently her results cannot account for the discrepancy in the n-n to n-p production ratio. On the other hand Figureau and Ericson (1969) obtain the result $|g_1^-|^2 / |g_0^-|^2 \approx 4$, implying (according to Eckstein's analysis) a large contribution from charge exchange rescattering which seems to agree with the results of Koltun and Reitan (1966).

Another approximation which has been invoked by those authors who use the phenomenological approach is to assume that the pi-meson field is constant throughout the nuclear volume. Even if this is valid for bound mesons it is unlikely to be valid for moving mesons. (Figureau and Ericson have, however, used this approximation for moving mesons). Other authors use a hydrogenic wave function (or a plane wave for moving mesons – Koltun and Reitan 1966, Kopaleishvili 1967), while Eisenberg and collaborators (see Table II) have included the effects of distortion of the meson field by the strong interaction with the nucleus.

Most authors have considered modifications to the nucleon-nucleon relative wavefunction in both the initial and final states of the system.
The usual conclusion (Jibuti and Kopaleish vili 1964, Eisenberg and Letourneux 1967) has been that the final-state interaction is more significant than short-range correlations in the bound state wavefunction. It has been pointed out however (Eisenberg and Letourneux 1967, Guy et al 1968) that since the effects of hard-core correlations have "healed" by about 0.5fm we should not expect their inclusion to affect the transition rates. The zero-range approximation of Eckstein and other authors is of course inconsistent with the introduction of hard core correlations.

Kohmura (1965), following a suggestion by Gottfried (1963), has used correlations which are stronger for absorption by a pair of nucleons in a triplet spin state than in a singlet spin state. This can account for the anomalous ratio of n-n to n-p emission in the same way as rescattering, which we mentioned earlier. Indeed it may well be that both rescattering and short-range correlations are manifestations of the same effect, and that the introduction of correlations by a modification in the wave function merely reflects the exchange of a virtual meson and is thus equivalent to rescattering of a real meson. In an analogous manner Weber (1969a, 1970) and Kopaleish ili and Machabeli (1971) suggest that one should not explicitly include both final-state interactions and pi-meson rescattering.

It should be stressed that there is no real physical distinction between correlations due to initial and final-state pair interactions. In view of this fact we should be wary of reading too much into the results achieved by introducing initial-state correlations and interpreting them as proving the shell-model to be an inadequate description of the nucleus. (Weise, Huber and Danos have discussed
Table II

Summary of theoretical analyses of the ($\pi$,NN) reaction.

<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
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<td>-</td>
<td>-</td>
<td>✓</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

(a) in the asymptotic approximation  
(b) not applicable  
(c) phenomenologically  
(d) Figureau and Ericson include distortion

Key:
1. Koltun and Reitan 1966
3. Jibuti and Kopaleishvili 1964
4. Kopaleishvili 1967
5. Kopaleishvili and Machabeli 1969
6. Kopaleishvili and Machabeli 1971
8. Elsaesser and Eisenberg 1970
10. This thesis
correlations in the context of the $\gamma, np$ reaction and they use an identical form in both the initial and final states.

There are a few effects which have been ignored in most of the calculations. One is distortion of wave function of the light reaction products by the residual nucleus, which is usually considered to be important in nuclear reactions (Jackson and Berggren 1965; Jain, Sarma and Banerjee 1970). However Weber has, in a series of papers (1969a, 1969b, 1970) developed a more sophisticated approach in which all the interactions between particles in the final state are included in a coupled channels formulation. Another point which has not been taken into account by a number of authors is that the usual shell model includes the translational motion of the centre-of-mass of the nucleus so that unless special care is taken spurious states may be introduced into the calculations.

Finally, the wave function appearing in the evaluation of the matrix element which describes the relative motion of the dinucleon and the residual nucleus must have certain long-range properties. Jackson (1967) has pointed out that the use of an incorrect form (in almost all the published work on pi-meson absorption) makes it difficult to draw any conclusions about which effects should be ascribed to pair correlations.

In our approach we look at distortion of both the pi-meson and dinucleon wave functions by the nucleus, and we describe the reactions using a cluster model (in which the centre-of-mass motion separates out) with the correct long-range behaviour for the relative wave function. Although our approach formally resembles some of the
features of the approach used by Eisenberg's group (who use a shell model with an oscillator potential transformed to relative coordinates) the use of a different asymptotic form for the nucleus-dinucleon relative wave function is a major difference in terms of the physics of the reaction.

Table II summarises some of the differences between the various approaches and the relation to them of this thesis. We mention here that almost all the papers quoted are concerned with absorption of bound pi-mesons. There is very little theoretical discussion in the literature of absorption of energetic pi-mesons, with which we are concerned here.
Chapter III EVALUATION OF THE MATRIX ELEMENT

3.1 Construction

We now begin to discuss the matrix element $M_{f_1} = \langle f|M|i \rangle$ introduced in Chapter II.

The initial state vector $|i\rangle$ describes the target nucleus $A$. We set

$$|i\rangle = |{-k}_n, A^i\rangle$$

where $A^i$ represents the initial (intrinsic) state of the target nucleus which has momentum $-k_n$ in the centre of mass system. In order to write $|i\rangle$ in this form we hived off the absorption interaction $H_{j2}$ which couples the pi-meson and nuclear fields; this has already been dealt with in Section 2.1.

The final state consists of the residual nucleus and two free nucleons; its Hamiltonian is

$$H_N = H_C + T_1 + T_2 + T_C + V_{1C} + V_{2C} + V_{12}$$

$H_C$ is the intrinsic Hamiltonian for the residual nucleus $C$; $T_1$, $T_2$ and $T_C$ are the kinetic energy operators for the outgoing nucleons and the centre-of-mass of $C$; and $V_{\alpha\beta}$ gives the interaction between $\alpha$ and $\beta$.

In order to describe the states of the $A$ nucleon system in the matrix element we shall use the dinucleon coordinates introduced by Jackson (1967) for the $(p,2p)$ reaction (see figure 4).
Figure 4: Dinucleon coordinates for the system of $A$ nucleons.
The relative co-ordinate vector of the two nucleons is \( \xi \) and the co-ordinate vector of the dinucleon relative to the centre of mass of the core (or residual nucleus) C is \( R \). The internal co-ordinates of C are \( \xi_i \) where \( i = 1, 2, \ldots, A - 3 \). This gives \( A - 1 \) coordinate vectors; a complete description requires in addition the position of some point relative to an origin. We can choose \( \sim_{A} \), the coordinate vector the centre of mass of the A nucleon system.

In terms of \( \sim_{1}, \sim_{2} \) (the co-ordinate vectors of the two "active" nucleons) and \( \sim_{A} \), the dinucleon co-ordinates \( \sim \) and \( \sim_{A} \) are given by

\[
\xi = \sim_{1} - \sim_{2}
\]

\[
R = \frac{A}{A - 2} \left[ \frac{1}{2} (\sim_{1} + \sim_{2}) - \sim_{A} \right] = \frac{A}{A - 2} \left[ R' - \sim_{A} \right].
\]

If \( \sim_{p_{1}}, \sim_{p_{2}} \) and \( \sim_{p_{A}} \) denote the momenta conjugate to \( \sim_{1}, \sim_{2} \) and \( \sim_{A} \) then the momenta \( \sim{k} \) and \( \sim{K} \) conjugate to \( \sim \) and \( \sim_{A} \) will be given by

\[
k = \frac{1}{2} (p_{1} - p_{2}) = \frac{1}{2} (\sim_{1} - \sim_{2})
\]

\[
K = p_{1} + p_{2} - \frac{2p}{A - A_{A}} = \sim_{1} + \sim_{2}
\]

where, as in Chapter II, \( \sim{k} \) and \( \sim{K} \) are the nucleon momenta in the centre of mass system (i.e. the momenta conjugate to \( \xi_{1} - \sim_{A} \), \( \xi_{2} - \sim_{A} \)).

We must now transform the Hamiltonian \( H_{N} \) to the new co-ordinate system. Following Jackson (1970a, page 200) we obtain
\[ H_C + T_{12} + T_{DC} + V_{DC}(R) + V_{12}(\vec{r}) + V_{\text{coup}}(\vec{r}, \vec{R}) + T_A, \]

in which \( T_{12}, T_{DC} \) and \( T_A \) are the kinetic energy operators corresponding respectively to \( \vec{r}, \vec{R} \) and \( \vec{r}_A \) and \( V_{DC} \) is the spherically symmetric part of \( V_{1C} + V_{2C} \).

We note that using these coordinates the centre-of-mass motion is separated out. This avoids the possibility of introducing spurious excited states of the centre-of-mass motion. In the shell model the co-ordinates \( \vec{r}_1 \) and \( \vec{r}_2 \) are used (or equivalently the transformed coordinates \( \vec{r} \) and \( \vec{R}' \)) and such spurious states may arise. Extracting the centre-of-mass motion from a shell-model wavefunction is not easy (Kopaleishvili and Machabeli 1969). In the dinucleon coordinates system the wavefunction of the centre of mass motion factorises out; we conveniently neglect \( T_A \) altogether by working in the centre-of-mass (of A) frame of reference.

We write the final state vector as

\[ |f\rangle = |k, K, \xi^f\rangle \]

i.e. an intrinsic state \( f \) of C and states of the \( \vec{r} \) and \( \vec{R} \) channels with asymptotic momenta \( \hbar k \) and \( \hbar K \). Thus

\[ M_{fi} = \langle k, K, \xi^f | M | -k, -K, \xi^i \rangle. \]

Now \( H_{12} \) (and so \( M \)) does not depend at all on the internal co-ordinates \( \xi \) of C — this is the direct reaction assumption that the nucleons in C are passive. So
where

\[ M_{fi} = \langle k, k| M |-k, -k \rangle, \psi \rangle \] (1)

is the overlap between the wavefunctions of the target nucleus and the residual nucleus and is referred to in the literature as the overlap integral (Berggren 1965) or form factor for the reaction.

So far we have ignored any discussion of spin and isospin quantum numbers. We shall be concerned only with reactions in which the residual nucleus C is a closed subshell \((J = S = T = 0)\); in this case \(\psi\) has the same spin and isospin as A. We are not considering cases in which C is in an excited state. Such two-hole states may however be expected to appear (Brueckner et al 1951); they show up clearly in the experimental analysis (Charpak et al 1965, 1967, Burman and Nordberg 1968 and others) and pi-meson absorption is an important tool for studying them. Our formalism also applies to reactions in which the target nucleus A is a closed subshell in which case \(\psi\) has the same spin and isospin as C. Once \(\psi\) has been determined the way it appears in \(M_{fi}\) is as in Equation 1.

The matrix element can be said to be composed of three components (Jackson 1970b):

(a) the interaction \(M\) between the projectile and the active nucleons, which we described in Chapter II, and will
develop in Section 3.2;

(b) the overlap integral $\Psi$ which contains the nuclear structure information that we may hope to derive from a study of the reaction - this will be discussed in Sections 3.3 and 3.5;

(c) the wavefunction $\langle x,R|k,K \rangle$ describing the motion of the free particles, which we shall consider in detail in Sections 3.4 and 3.5.

3.2 The Direct Interaction

The absorption Hamiltonian for the $(\pi,NN)$ reaction was obtained in Section 2.1 in the form

$$H_{12} = -i\hbar \left( \frac{2\pi}{\hbar^2} \frac{\hbar^2}{m_\pi m_e^2} \right) M$$

with

$$M = \{ (\tau_\omega S + \tau_\omega T) [\zeta (\nabla_\omega R_\phi + \nabla_\omega R_\phi) - (\nabla_\omega R_\phi + \nabla_\omega R_\phi)]$$

$$+ (\tau_\omega S + \tau_\omega T) [\zeta (\frac{1}{2} \nabla_\omega R_\phi + 2 \nabla_\omega R_\phi) - (\frac{1}{2} \nabla_\omega R_\phi + 2 \nabla_\omega R_\phi)] \}.$$

The various symbols in the above expressions were defined in Chapter II.

The spin and isospin operators cause the transitions between initial and final spin and isospin states.

Let $S, T, M_S$ and $M_T$ be, respectively, the quantum numbers of the dinucleon pair in the initial state corresponding to spin, isospin and their projections along the z-axis in their
appropriate quantum numbers in the final state. If, in addition, the orbital angular momentum and total angular-momentum quantum numbers in the initial state are \( L \) and \( J \) then, taking space, spin and isospin into account,

\[
|\psi_{LSJM}\rangle = \sum_{M_S} (L_{M_L} S_{M_S} J M |J M\rangle |\psi_{LM}\rangle |SM_S\rangle |TM_T\rangle .
\]

In this expression we have omitted the explicit (redundant) summation over the magnetic quantum number \( M_L = M - M_S \).

The notation \((a\alpha b\beta|c\gamma)\) denotes the Clebsh-Gordan coefficient for coupling the states \(|a\alpha\rangle\) and \(|b\beta\rangle\) to \(|c\gamma\rangle\).

For the final state we have

\[
|k,K\rangle |S'M'_S\rangle |T'M'_T\rangle .
\]

The two-particle spin and isospin states are defined as follows. If \( \chi^{\sigma}_S \) denotes a single particle spin function with quantum number \( S \) and projection \( \sigma \) then

\[
|SM_S\rangle = \sum_{\sigma_1\sigma_2} \left( \frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2 |SM_S\rangle \right) \chi^{\sigma_1} \chi^{\sigma_2} .
\]

Similarly if \( \zeta^T_T \) denotes a single particle isospin function with quantum number \( T \) and projection \( \tau \) then
The inner products which appear in the expression for $M$ may be expanded as follows (Rose 1957)

$$S \cdot \nabla = \sum_{\mu=-1}^{1} (-1)^{\mu} S |_{\mu} \nabla_{\mu-\mu} \quad (2)$$

To compute the matrix elements of the components of the spin operators we use the Wigner-Eckart theorem (Eckart 1930, Wigner 1931), obtaining

$$\langle S'M'|S_\mu|S'M_S \rangle = (SM_1 \mu |S'M' \rangle (S'| |S \rangle |S \rangle = \sqrt{2} (SM_1 \mu |S'M' \rangle \alpha$$

$$\langle S'M'|\sigma_\mu|S'M_S \rangle = (SM_1 \mu |S'M' \rangle (S'| |\sigma \rangle |S \rangle = \sqrt{2} (SM_1 \mu |S'M' \rangle \beta$$

and

$$\alpha = \sqrt{S(S+1)} \delta_{S'S'}$$
$$\beta = \sqrt{S'(2S'+1)} (\delta_{S',S+1} - \delta_{S',S-1})$$

We see that $S_\mu$ induces a transition between symmetric spin states, and $\sigma_\mu$ causes a spin flip.

For the isospin operators $T_\nu, T_\nu$, we have $\nu = \pm 1$ according as the incident beam is $\pi^+$ (since we are only considering charged pi-mesons). The matrix elements yield expressions analogous to those above:
\[
\langle T' M'_T | T'_\pm | T M_T \rangle = \sqrt{2} (\delta_{T+1} + | T' M'_T \rangle \sqrt{2} \delta_{T+1} = \sqrt{2} \eta \delta_{M'_T M_{T+1}}
\]

\[
\langle T' M'_T | T'_\pm | T M_T \rangle = \sqrt{2} (\delta_{T+1} + | T' M'_T \rangle \sqrt{3} \sqrt{2} \delta_{T', T+1} = \delta_{T', T-1} - \delta_{T', T-1}
\]

where

\[\eta = \delta_{T', T+1} - \delta_{T', T-1}.\]

Here again \(T_+\) induces a transition between \(T = 1\) states

and \(T_+\) induces an isospin flip. For both spin and isospin

the transition \(0 \to 0\) is forbidden.

Combining our results:

\[
\langle S'M'_S | T'M_{T-1} | T'_\pm S' + \sigma + \mu | SM_{T_M} \rangle = 2 (\delta_{S'M'_S} | S'M'_S \rangle (\eta + \alpha \epsilon)
\]

\[
\langle S'M'_S | T'M_{T+1} | T'_\pm S' + \sigma + \mu | SM_{T_M} \rangle = 2 (\delta_{S'M'_S} | S'M'_S \rangle (\alpha + \beta \epsilon)
\]

in which we have explicitly assumed that the process under

consideration satisfies charge conservation. (The allowed

processes are \(nn \pi^+ \to np\), \(np \pi^+ \to pp\), \(pp \pi^- \to np\), \(np \pi^- \to nn\).)

We now look at the scattering wavefunctions in the matrix

element. As a first approximation we consider plane-wave

solutions of the Schrödinger equation (for the nucleons) and

the Klein-Gordon equation (for the pi-meson). This approach

goals the effects of distortion by the nucleus and between

the emerging nucleons, but avoids complicated mathematics. Thus
we cannot expect our description to be an adequate representation, and we improve on it in Section 3.4 where we take a closer look at these deviations from plane waves. In terms of the plane wave approximation we have

\[
\langle kR | r, R \rangle = e^{-i(k \cdot r + k \cdot R)}
\]

and the nucleon gradient operators reduce simply to

\[
\begin{align*}
\nabla'_{R} &= -i \cdot k', \\
\nabla_{r} &= -ik.
\end{align*}
\]

For the pi-meson wavefunctions we write

\[
\phi(R, r) = \frac{1}{2} (e^{i(k \cdot r - k \cdot R) + e^{i(k \cdot r - k \cdot R)^2}} = e^{-i(k \cdot R)} \cdot \cos(\frac{1}{2} k \cdot r)
\]

\[
\phi(R, r) = (e^{i(k \cdot r)} - e^{i(k \cdot r)^2}) = 2i e^{i(k \cdot R)} \cdot \sin(\frac{1}{2} k \cdot r)
\]

The pi-meson gradient operators then give

\[
\begin{align*}
\nabla'_{R} \phi &= \nabla_{r} \phi = ik \phi, \\
2 \nabla_{r} \phi &= \frac{1}{2} \nabla'_{R} \phi = \frac{i}{2} k \phi.
\end{align*}
\]

In the centre-of-mass frame the initial state wavefunction of the target nucleus is

\[
\langle R_A | -k \rangle = e^{-i(k \cdot R_A)}.
\]
It is convenient to take this term together with the pi-meson functions since \( R_1^A - R_2^A = (A-2)R/A \). We now define the quantities

\[
q^+ = k + \frac{1}{2}k_\pi
\]

\[
Q = k - (A-2)k_\pi/A
\]

\[
\delta^- = \zeta k_\pi - \frac{K/2}{E_\pi} + k
\]

\[
= \frac{mk_\pi}{E_\pi} - \frac{1}{2}k + q^+
\]

and the momentum transform

\[
G_{LM} (Q, q) = \int d^3r e^{-iQ \cdot r} \int d^3r e^{-iQ \cdot r} \frac{R_{LM}^F (R, \zeta)}{2}
\]

We note that if the relativistic corrections are ignored then \(-hQ\) is the recoil momentum of C in the laboratory frame of reference.

Equations (2) and (3) now yield

\[
M^{\mu} (M, M') = \sum_{LM} M_L M_S \mu \begin{pmatrix} (-1)^L (LM_S |JM)(SM_S |JM') \\ SM_S |JM' \end{pmatrix} \\
\times \{ (\bar{\beta} \gamma^\tau \alpha \epsilon) [\delta^- \mu G_{LM}^L (Q, q^-) + \delta^+ \mu G_{LM}^R (Q, q^+)] + (\bar{\alpha} \tau \beta \epsilon) [\delta^- \mu G_{LM}^L (Q, q^-) - \delta^- \mu G_{LM}^R (Q, q^+)] \}.
\]

In this expression the term with \( \beta \gamma^\tau \alpha \epsilon \) contributes to the following spin-isospin transitions \( ST \rightarrow S'T' \):
\[
\begin{align*}
00 &\rightarrow 11 \\
01 &\rightarrow 10 \\
10 &\rightarrow 01 \\
11 &\rightarrow 00 \\
11 &\rightarrow 11
\end{align*}
\]

and the term with \( \alpha \bar{\nu} + \beta \epsilon \) contributes to the transitions

\[
\begin{align*}
01 &\rightarrow 11 \\
10 &\rightarrow 11 \\
11 &\rightarrow 10 \\
11 &\rightarrow 01.
\end{align*}
\]

This accounts for all possible cases besides the forbidden 0 \( \rightarrow \) 0 transitions.

An initial \( 2T+1, 2S+1 L \) state can then go into a final state with \( S', T' \) given by one of the above and \( L \) determined by the conservation of total angular momentum and parity. The fact that the intrinsic parity of the pi-meson is \(-1\) is reflected by every term in the matrix element containing a gradient operator. In our analysis we have replaced all the gradient operators by wave vectors. Whilst this simplifies the calculations it means of course that the two-particle partial waves have become transformed into each other, as we shall explain presently. For example, consider an initial \( ^{13}S_1 \) state. (This is a good model for both the deuteron and for the Li\(^6\) ground state.) For s- and p-wave pi-mesons the only possible transitions to
properly antisymmetrised states are
\[ (^{13}S_1s)_1 + ^{33}p_1 \]
\[ (^{13}S_1p)_0 + ^{31}S_0 \]
\[ (^{13}S_1p)_2 + ^{31}p_2, \]

where the second suffix on the left-hand side is the total angular momentum quantum number of the initial state. It is seen from the example that s-wave pimeson absorption is given by the \((\omega nT\varepsilon)\) term and p-wave absorption by the \((BnT\varepsilon)\) term. The matrix element for s-wave absorption comes from taking the s-wave part of the pimeson function \(e^{ik_{1/2}R}\) with the terms \(-\frac{1}{2}k_{\perp}+k\) in \(\delta^{\perp}\), and the p-wave part of \(e^{ik_{3/2}R}\) with the term \(k_{\perp}\). This is because if we start with the s-wave part of \(e^{i(k_{1/2}R)}\) and operate with \(\gamma\) we obtain the p-wave part of \(k_{\perp}e^{-i(k_{1/2}R)}\). A similar argument shows that the s-wave part of \(e^{-i(k\cdot r + K\cdot R)}\) contributes to the \(^{33}p_1\) state in our formulation. In the same way absorption of p-wave pimesons has contributions from the s- and d-wave parts of \(e^{-i(k_{1/2}R)}\).

In addition the pimeson partial waves are recoupled by the co-ordinate transformation from \(r_1, r_2\) to \(r, R\). This means that taking the s- and p-wave parts of \(\phi\) and \(\phi\) does not correspond to s- and p-wave pimesons. To obtain these we should have to first take the s- and p-wave parts of the incident wavefunction and then recouple to \(r, R\) coordinates, as is done by Eisenberg and Letourneux (1967). However for pimeson beams this doesn't matter as long as we do not wish to consider
the separate contributions of the individual partial waves. (For bound pi-mesons, however, it is extremely important to know from which orbit the absorption takes place to take into account the different probabilities for occupation of orbits.)

3.3 The Overlap Integral

The question of how best to describe the overlap integral which appears in the six-dimensional momentum transforms \( G_{LM} \) has been critically analysed in the literature (Austern 1964, Berggren 1965, Pinkston and Satchler 1965).

In the case of single nucleon transfer the overlap integral is a function of a single coordinate vector \( R \). It is not an eigenstate of angular momentum but can be expanded in single-particle states as

\[
\phi_{jm}(R) = \sum_{JM} (B_{JM} j_m | J_{A'A} ) \phi_{jm}(R)
\]

where each \( \phi_{jm} \) is not a normalised wavefunction as we have absorbed into it a fractional parentage coefficient. The vital problem is to determine the choice of \( \phi_{jm} \).

One method is to assume an independent particle shell model so that \( \phi_{jm} \) is just a shell model wavefunction (Tobocman 1961). The shell model suffers however from difficulty in separating out the centre of mass motion, which is irrelevant to the process. Another method (Sherr et al 1965) is to obtain the overlap integral as the wavefunction of the transferred nucleon.
in a potential well, adjusting the binding energy to yield the correct angular distributions. This "effective binding energy procedure" is wrong in principle and leads to significant errors (Austern 1964, Pinkston and Satchler 1965). The reason is that it can be shown in a model independent way that the asymptotic form of the overlap integral is determined by the separation energy $\epsilon_B - \epsilon_A$ (Berggren 1965). In particular, the importance of using a finite potential well rather than the usual harmonic oscillator was stressed by Drisko and Rybicki (1966) who suggested modifying the usual overlap integrals by tacking the correct tail on to a single particle oscillator wavefunction. The above discussion also applies when, as in our case, more than one nucleon is transferred (Jackson 1967 1970b). The initial state Hamiltonian for the target nucleus is formally identical with the final state Hamiltonian which we wrote down in Section 3.1, and its intrinsic part is

$$H_A = H_C + T_{DC} + W_{DC} + H_D = H_N - T_A$$

in which we have written $H_D$ for the dinucleon internal Hamiltonian $T_{12} + V_{12}$ and $W_{DC}$ for the potential $V_{DC} + V_{coup}$ between the two "clusters" in the target. The overlap integral $\psi^f = \langle C^f | A^i \rangle$ satisfies

$$\langle C^f | H_A | A^i \rangle = \epsilon_A^i \langle \psi \rangle = \epsilon_C^f \langle \psi \rangle + \langle T_{DC} + W_{DC} + H_D | \psi \rangle$$

since we can write

$$\langle H_A - \epsilon_A^i | A^i \rangle = (H_C - \epsilon_C^f) | C^f \rangle = | 0 \rangle$$
If now \(|k\rangle\) is a particular dinucleon state then the reduced overlap integral

\[
\langle R|k_{fi}\rangle = \int \phi^*_D(r) \psi_{fi}(r,R)d^3r
\]

represents the probability amplitude for finding in the nucleus A a pair of nucleons in state \(|k\rangle\) at a point \(R\) relative to the core \(C\). If an addition \(|k\rangle\) is an eigenstate then

\[
(H_D - \epsilon_D^k) \phi_D^k(r) = 0
\]

and the reduced overlap integral satisfies

\[
\langle T_{DC} + W_{DC}|k_{fi}\rangle = (\epsilon_A^i - \epsilon_D^k - \epsilon_C^f)|k_{fi}\rangle = - S_{AD}^{kf}|k_{fi}\rangle \quad (5)
\]

\(S_{AD}^{kf}\) is the separation energy for the removal of the pair of nucleons \(D\) in state \(k\) from the nucleus \(A\) to leave the nucleus \(C^f\), and it is now assumed (following Berggren) that \(W_{DC}\)

\((V_{DC} + V_{\text{coup}})\) is of short range so that the asymptotic radial behaviour of \(\langle R|k_{fi}\rangle\) will go as \(\exp(-aR)/R\) (or, more precisely, a Hankel function corresponding to the angular momentum state of the reduced overlap integral) where

\[
a^2 = 2\mu S_{AD}/\hbar^2 \quad (\mu = \text{reduced mass of } DC \text{ channel} \approx 2m(A-2)A).
\]

In the event that \(D\) includes a proton the potential \(W_{DC}\) will
have a Coulomb potential term. This does not materially
alter the argument and is simply taken into account; the
radial behaviour of \( \langle R | k f i \rangle \) behaves asymptotically as a
Coulombic Hankel function with attenuation coefficient \( a \).

Jain et al (1970) have discussed the Li\(^6\)\((p,pd)\) reaction and
find that adopting the correct asymptotic radial form for
\( \langle R | k f i \rangle \) is by far the most important consideration in
determining the shape of the angular distribution of the
emerging particles. By comparison complete anti-symmetrisation
of the target wavefunction is only a small effect. A similar
result has been reported by Saito et al (1968) for the Li\(^6\)\((p,2p)\)
reaction.

These results may be developed in terms of a model as follows.
The target nucleus wavefunction \( \psi_A^i \) may be expanded in a complete
set of normalised states (in an obvious notation) as

\[
|A^i\rangle = \sum_{\alpha \beta \gamma} N |c^\alpha\rangle |D^\beta\rangle |\gamma\rangle.
\]

Here \( \alpha, \beta, \gamma \) include all the relevant quantum numbers of their
respective states and the summation sign includes integration
over continuum states. The overlap integral then satisfies

\[
|\psi\rangle = \sum_{\beta \gamma} N \sum_{\alpha \beta \gamma} D^\beta |\gamma\rangle
\]

(6)
since \( \langle C^\beta | c^\alpha \rangle = \delta_{f\alpha} \). It must be pointed out that this
is true in principle but in practice the wavefunctions for the
may not be constructed in the same potential as 
\[ |c^\alpha\rangle \]
invalidating the accuracy of Equation 6. The reduced overlap integral for the nucleon pair state \( k \) becomes

\[ |k\rangle = \sum_{\gamma} N_{\gamma k} |\gamma\rangle \]

which is normalised by the condition \( \sum_{\gamma} |N_{\gamma k}|^2 = 1 \). In order that the expansion be useful it is desirable that only a few terms be significant. The extreme assumption that only one term in the expansion need be taken is known as the cluster model.

We have performed calculations with the target nucleus \( Li^6 \) which is described in the shell model by a \( (1s)^4(1p)^2 \) configuration. The s-shell is closed and the \( (1p)^2 \) couples predominantly to \( ^{13}S_1 \) in the ground state (Lauritsen and Ajzenberg-Selove 1966). In an oscillator potential the shell model wavefunction of two nucleons may always be transformed from \( \psi_1(\xi_1) \psi_2(\xi_2) \) to \( \psi(\xi)\psi(\xi') \) and under such a transformation a \( (1p)^2 \) state transforms to a superposition of a \( (2s)(1s) \) and a \( (1s)(2s) \) state. It is therefore not unreasonable to consider a similar superposition in a cluster expansion although in this case the coordinate vectors are \( \xi,\xi' \) instead of \( \xi,\xi' \) and we are using a different potential (Wildermuth and Kanellopoulos 1959, Wildermuth 1962, Jackson 1967). In the cluster model we may take just the \( (1s)(2s) \) term, i.e. with the dinucleon in an internal \( 1s \) (ground) state and with the intercluster motion described by a \( 2s \) wavefunction. The calculations of Aurdal et al (1969) also use a \( 2s \) state.
for the intercluster wavefunction.

In order to obtain a suitable intercluster wavefunction for the reduced overlap integral we have generated the 2s solution of Equation 5 neglecting the coupling potential $V_{\text{coup}}$, i.e. retaining only the effective one-body spherically symmetric term $V_{\text{DC}}$. The nuclear part of $V_{\text{DC}}$ was given a Saxon-Woods shape

$$V_0 \frac{1}{1 + \exp \left( \frac{(R-R_0)/a_0}{a_0} \right)}$$

with

$$R_0 = r_0 A^{1/3}$$

and the Coulomb part was taken to be the potential due to a uniformly charged sphere with the same radius $R_0$. The term $a_0^k$ was set equal to the binding energy of the deuteron, so that $S_{\text{AD}}$ is the break up energy for the process

$$\text{Li}^6 \rightarrow \alpha + d.$$ 

The radius parameter $r_0$ and the diffuseness $a_0$ were assigned values appropriate to the p-shell proton in $\text{Li}^6$ and the depth $V_0$ was determined by the eigenvalue method of Buck (1960). The proton parameters are derived from electron scattering data to fit the r.m.s. radius of the $\text{Li}^6$ nucleus. We assume that using the same parameters for the intercluster wavefunction is not too much in error but we have not recalculated the r.m.s. radius to check. We refer to our wavefunction as IT.
It is desirable to compare results obtained using this form for the intercluster wavefunction with those obtained using other cluster model wavefunctions which have been used in analysis of reactions involving Li$^6$. Tang, Wildermuth and Pearlstein (1962) fit the binding energy (but not the r.m.s. radius) of Li$^6$ with the oscillator type wavefunction (which we refer to as $G$)

$$N_B R^2 \exp(-\frac{2}{3}B R^2)$$

for the intercluster motion. (Lodhi (1967) has modified the wavefunction of Tang et al to fit the r.m.s. radius of Li$^6$ by adjusting only the parameter of the internal dinucleon wavefunction.) This wavefunction does not fulfil the criterion for the long-range behaviour of the reduced overlap integral. In addition we note that it vanishes for $R = 0$ and has no nodes so that it does not have the form of a $2s$ wavefunction. Thus $G$ and $IT$ differ in the nuclear interior as well as the nuclear surface.

Table III

Parameters for the reduced overlap integral

<table>
<thead>
<tr>
<th></th>
<th>$S_{AD} = 1.47$ MeV</th>
<th>$V_0 = 33.9$ MeV</th>
<th>$r_0 = 1.81$ fm.</th>
<th>$a_0 = 0.80$ fm.</th>
<th>$\beta = 0.329$ fm$^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IT, GT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G, GT$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Jackson 1970b)  
(Elton 1961)  
(Tang et al 1962)
Jain, Sarma and Banerjee (1970), in a manner analogous to the approach suggested by Drisko and Rybicki (1966) for a single particle overlap integrals have used a modified form of $G$ with the correct tail $\exp(-aR)/R$ attached at a radius $R = 3$ fm. ($a = (2\mu S_{AD}^{1/2}/\hbar)$). We refer to this function as $G_T$ in Table III which gives values for the parameters of $G$ and $\Gamma_T$. The function $G_T$ is not normalised to unity; to do this requires an extra factor $0.838$.

Our procedure for determining the intercluster wavefunction yields the correct asymptotic form for the overlap integral, but it does not give the correct form in the interior of the nucleus. Thus $\Gamma_T$ and $G_T$ (see Figure 3) differ in the interior region. Moreover, they differ in magnitude even in the asymptotic region. At $R \approx 10$ fm, $\left|\psi_{\Gamma_T}(R)\psi_{G_T}(R)\right| \approx 1.17$.

Clearly, to obtain a more accurate wavefunction we should take into account the coupling potential. This of course necessitates taking more terms in the (cluster) expansion. If $\tilde{\psi}(R)$ is the product of the radial part of $\left< \beta | \gamma \right>$ with its fractional parentage coefficient $\left< A | \alpha \right>^\dagger$ then we require (Jackson 1967 b, Philpott et al 1968)

$$(T'_{DC} + V_{DC} - S_{AD}^{\alpha \beta} \delta_{\alpha \gamma} + \Sigma U_{\gamma \delta}^\delta = 0$$

In this set of coupled equations $T'_{DC}$ is the radial part of the kinetic energy operator and the coupling potentials $U_{\gamma \delta}$ are derived from $V_{\text{coup}}$. A number of difficulties are raised in the attempt to obtain a solution of these equations and they are discussed in Appendix B.
Fig. 5 Radial intercluster wavefunctions for Li$^6$. The definitions of $G$, $G_T$, and $I_T$ are given in the text.
We now expand equation (6) as follows.

\[ \psi_{LM_{L}}(r,R) = N_{L} R_{L} \sum_{m} \langle \hat{L} m \| L_{L} M_{L} \rangle Y_{LM_{L}}(r) Y_{L_{L}M_{L}}(R) d^{n} \hat{L}(r) X_{L_{L}}(R) \]  

(7)

\( n \) and \( N \) are the principal quantum numbers of the dinucleon and intercluster wavefunctions and as usual \( n-1 \) and \( N-1 \) are the numbers of nodes of their respective radial parts. In the cluster model \( N, L, n \) and \( \hat{L} \) have only one value and there is but a single summation (over \( m \)).

The properties of the spherical harmonies and the phase convention we are using are discussed briefly in Appendix A. Particularly, \( Y_{L}(-r) = (-1)^{L} Y_{L}(r) \) so that the summation is implied to include only those values of \( L \) which make \( L + S + T \) odd. This ensures that \( \psi_{LM_{L}} \) is antisymmetrised between the two "active" nucleons. It is not necessary to explicitly antisymmetrise the two nucleons in the final state as this will follow automatically since our Hamiltonian is a symmetric operator. In an exact calculation we should also include terms representing antisymmetrisation of active nucleons with the core. The neglect of such terms is partly justified by the high energies which the emitted nucleons have relative to the residual nucleus. Table IV shows some typical momenta for the outgoing nucleons relative to the residual nucleus for an incident 
\[ \text{pimeson kinetic energy of 50 MeV (laboratory frame of reference). } \theta \text{ is the angle between the emitted nucleons in the A particle centre-of-mass frame. For higher incident beam energies the outgoing momenta are correspondingly greater.} \]
An additional justification is presented by the result, mentioned earlier, that lack of inclusion of the exchange terms has a minimal effect on the angular distribution by comparison with using an overlap integral with the wrong long range behaviour as is usually done (e.g. Eisenberg and Letourneux 1967, Kopaleishvili and Machabeli 1969 and 1970).

Using the expansion (7) of the overlap integral in equation (4) the transforms $G_{L M L}$ separate out as

$$G_{L M L}(Q,q) = \sum_{N} \sum_{m} \sum_{n} \sum_{M} (L M M_{L} | a_{n L M} (q)) e^{i Q \cdot r}$$

with

$$a_{n L M}(q) = \int d^{3}r e^{-i Q \cdot r} n_{L}^{m}(r) Y_{L}^{m}(r)$$

Using the expansion (7) of the overlap integral in equation (4) the transforms $G_{L M L}$ separate out as

$$G_{L M L}(Q,q) = \sum_{N} \sum_{m} \sum_{n} \sum_{M} (L M M_{L} | a_{n L M} (q)) e^{i Q \cdot r}$$

with

$$a_{n L M}(q) = \int d^{3}r e^{-i Q \cdot r} n_{L}^{m}(r) Y_{L}^{m}(r)$$

(7a)

### Table IV

Typical momenta (in fm$^{-1}$) for protons emitted in the Li$^{6}(\pi^{+}, 2p)$ reaction with $T'_{\pi} = 50$ MeV

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Typical $(k_1, k_2)$ in $k_C = 0$ frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>(2.48, 2.13)  (2.18, 2.63)</td>
</tr>
<tr>
<td>180°</td>
<td>(2.48, 1.71)  (2.18, 2.06)</td>
</tr>
</tbody>
</table>
3.4 Distortion by the Nucleus

The sharply defined momenta $\tilde{h}k^\alpha$, $\tilde{h}K$ and $\tilde{h}K$ which we have been using hitherto are inadequate for a satisfactory description of a reaction process because they assume implicitly the lack of a potential in their respective channels. We shall find it convenient to consider together the distortions in $\tilde{k}$ and $\tilde{K}$; these are both due to interactions with the nucleons in the nucleus and are conveniently described in terms of optical potentials. The distortion in the conjugate relative momentum $\tilde{k}$ due to final state nucleon-nucleon scattering must be considered in conjunction with deviations from our earlier description due to initial-state correlations in the $\gamma$ channel; this is left until the next section.

While our use of plane wave scattering functions provides a general guide to the process, it represents a considerable simplification of the physical situation. In particular it cannot be relied upon to yield correct magnitudes for cross-sections. We shall present here an approximate distorted wave approach, i.e. we shall attempt to improve on the plane-wave matrix element, but we shall use a high-energy approximation instead of solving an equation including the (optical) potential in each case. Our approximations also include neglecting the Coulomb potential due to the nucleus. It has been noted by Philpott et al (1968) that the optical model parameters - as determined from elastic scattering - contain an element of uncertainty which may introduce errors as large as those due to using an overlap integral with the correct long range
behaviour but with an incorrect interior form. Neglect of these errors is therefore consistent with our approximation for the overlap integral. In addition, a given set of parameters determines the optical model wavefunctions properly only in the region beyond the nuclear surface. Again, this is consistent with our approximation that the nuclear core is inert.

The distortion of the pi-meson wavefunctions arises from the interaction between the incoming pion and the target nucleus. The calculations of Eisenberg and Letourneux (1967) includes the effect of the pion-nucleus force in the case of absorption of bound mesons, but for moving mesons the incident beam is usually taken as a plane wave (e.g. Kopaleishvili 1967).

If we denote the π-A Coulomb potential by $V_c$ and the π-A nuclear optical potential by $V_\pi$ then (Auerbach et al 1967) the pi-meson wavefunction $\chi_\pi^+(k_\pi, r_\pi)$ should satisfy the Klein-Gordon equation

\[
(-\hbar^2 c^2 \nabla^2 + \frac{2}{m^2 c^4}) \chi_\pi^+ = (E_\pi - V_c - V_\pi)^2 \chi_\pi^+
\]

\[
\simeq [(E_\pi - V_c)^2 - 2E_\pi V_\pi] \chi_\pi^+
\]

\[
\simeq [E_\pi^2 - 2E_\pi V_\pi] \chi_\pi^+ \quad \text{................} (8)
\]

neglecting the Coulomb potential. The superscript + indicates the boundary condition for incoming particles

\[
\chi_\pi^+(k_\pi, r_\pi) \sim \frac{e^{i k_\pi \cdot r_\pi}}{r_\pi} + \int_{-\infty}^{\infty} \chi_\pi^0(k_\pi) \frac{e^{i k_\pi \cdot r_\pi}}{r_\pi}
\]
The optical potential $V_\pi$ has been derived by Silbar and Sternheim (1972) in the form

$$\frac{2E_\pi V_\pi}{A(\hbar c)^2} = -(b_0 + b_1)k^2_\pi \rho + \im b_1 V^2 \zeta,$$

neglecting the motion of the struck nucleon in the nucleus.

Here $b_\rho$ is related to the forward scattering amplitude for $\pi-N$ scattering

$$b_\rho = \frac{4\pi}{k^2} \overline{[k_f(0)]_\rho} \quad \rho = 0, 1 \quad (9)$$

$$\overline{k} = k_{\pi}^p m_{\pi}/(m_{\pi}^2 + 2mE_{\pi}^\prime/c^2 + m^2)^{1/2} \quad (cp. \ equation \ (2) \ of \ Chapter \ II),$$

and $\zeta$ is the nuclear density which may be given a modified Gaussian form

$$\zeta(r) = \frac{2[1 + (Z-2)r^2/3a^2]}{\pi^{3/2}a^{3}Z} \exp(-r^2/a^2)$$

for a light nucleus of atomic number $Z$. We have taken $a = 1.6 \ fm$ for a Li$^6$ target (Ravenhall 1958).

Our formula (9) gives $b_\rho$ in terms of $k_{\pi}$, as is appropriate for the centre-of-mass frame. In practice we have used values for $b_\rho$ calculated in the laboratory frame (i.e. using $k_{\pi}^l$).

The forward amplitudes for free $\pi$-meson-nucleon are given by

$$[\overline{k} f(0)]_0 = \alpha_3$$

$$[\overline{k} f(0)]_1 = 2\alpha_{33} + \alpha_{31}$$
for $\pi^+ - p$ and $\pi^- - n$ scattering, and by

$$[\tilde{k} f(0)]_0 = (\alpha_3 + 2\alpha_1)/3$$

$$[\tilde{k} f(0)]_1 = (2\alpha_{33} + \alpha_{31} + 4\alpha_{13} + 2\alpha_{11})/3$$

for $\pi^- - p$ and $\pi^+ - n$ scattering (Lock 1960). Here

$$\alpha_{2T,2J} = \exp(i\delta_{2T,2J}) \sin\delta_{2T,2J}$$

where $\delta_{2T,2J}$ is the phase shift for $\pi N$ scattering in the $p$-wave isospin $T$, spin $J$ channel, and we drop the $2J (=1)$ for $s$-wave.

The amplitudes for $\pi - A$ scattering are obtained by averaging over protons and nucleons in the nucleus.

For a nucleus with $Z = A/2$ we obtain for both $\pi^+ - A$ and $\pi^- - A$

$$[\tilde{k} f(0)]_0 = (2\alpha_3 + \alpha_1)/3$$

$$[\tilde{k} f(0)]_1 = (4\alpha_{33} + 2\alpha_{31} + 2\alpha_{13} + \alpha_{11})/3.$$  

In view of the complexity of the Laplacian term $\nabla^2 \phi$ in the formula for $V^{\pi}_{\pi}$ it is expedient to ignore it in a distorted wave calculation. This treatment may be justified by reference to the results of Lee and McManus (1971) which show that elastic scattering of pi-mesons by carbon into forward angles is reasonably well described by the term proportional to $\phi$ in the optical potential. The validity of our approximation thus depends on the absorption of pi-mesons by the nucleus before being deflected through large angles.
We have performed calculations for incident pi-mesons at three (laboratory) energies. In table V we have listed the values obtained for $b_0$ and $b_1$ (in the laboratory frame) for a $Z = A/2$ nucleus using phase shifts given by Donnachie et al (1968). The table also gives the depth parameter $V_{\pi}(0) = V_{\pi}(r)\rho(0)/\epsilon(0)$ for a Li$^6$ target and $\lambda_\pi = -\pi V_{\pi}/2\text{Im}V_{\pi}(0)$ which is a measure of the mean free path of pi-mesons in the potential $V_{\pi}$. The relativistic velocity of the pi-meson in the $\pi-A$ centre of mass frame is $v_{\pi} = \frac{\hbar k_{\pi}}{E_{\pi}}$.

We note that $\lambda_\pi$ decreases rapidly with increasing pi-meson energy. This anomaly is a consequence of the strength of the $T = 3/2, J = 3/2$ resonance in the $\pi-N$ system at 180 MeV. The negative sign of $\text{Re} b_0$ indicates the repulsive nature of the s-wave part of the $\pi-N$ potential. We note also that $\text{Im}V_{\pi} < 0$ so that the nucleus acts as a sink for incoming pi-mesons (see equation 13, infra). This loss of flux is usually referred to as absorption in optical model literature. (We have not used the term in this sense, reserving absorption for the phenomenon of a pi-meson being annihilated.)
Table V

<table>
<thead>
<tr>
<th>$T_\pi^*(\text{MeV})$</th>
<th>50</th>
<th>78</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0 (\text{fm}^3)$</td>
<td>$-1.68+0.68i$</td>
<td>$-1.63+0.49i$</td>
<td>$-1.01+0.43i$</td>
</tr>
<tr>
<td>$b_1 (\text{fm}^3)$</td>
<td>$7.37+0.92i$</td>
<td>$8.43+2.38i$</td>
<td>$8.35+3.76i$</td>
</tr>
<tr>
<td>$V_\pi(0) (\text{MeV})$</td>
<td>$-40.9-11.5i$</td>
<td>$-72.0-30.4i$</td>
<td>$-95.9-54.7i$</td>
</tr>
<tr>
<td>$\lambda_\pi (\text{fm})$</td>
<td>5.68</td>
<td>2.45</td>
<td>1.44</td>
</tr>
</tbody>
</table>
Equation (8) for the pi-meson wavefunction can be rewritten as

$$\left(-\nabla^2 + \frac{2E}{\hbar^2 c^2} V_\pi(r) - k^2\right)\chi_\pi^+(k_{\pi}, r_{\pi}) = 0$$

using the relation \( E^2 = \hbar^2 k^2 c^2 + m^2 c^4 \). This is now formally a Schrödinger equation for a particle of mass \( E_\pi/c^2 \) in the potential \( V_\pi \) with momentum \( k_\pi \). We will therefore be able to treat its solution in the same way as we deal with the (non-relativistic) equation for the dinucleon centre-of-mass wavefunction relative to the nucleus. We now proceed to discuss this case before returning to the solution for the scattering functions.

The outgoing nucleons are described by the state \(|k, K\rangle\) which must be an eigenstate of the Hamiltonian (section 3.1)

$$T_{12} + V_{12}(r) + T_{DC} + V_{DC}(R) + V_{coup}(r, R).$$

We neglect the coupling potential so that the scattering wavefunction can be written as a product of eigenfunctions

$$\langle r, R | k, K \rangle = \chi_{DC}^-(K, R) \chi_D^-(k, r)$$

The part which concerns us here is \( \chi_{DC}^- \) which satisfies the R-channel Schrödinger equation

$$\left(-\nabla^2 + \frac{2\mu}{\hbar^2} V_{DC}(R) - K^2\right)\chi_{DC}^- = 0 \quad \ldots \quad (10)$$

with the reduced mass \( \mu \approx 2m(A-2)/A \), subject to the (outgoing) boundary condition

$$\chi_{DC}^-(K, R) \sim e^{-iK \cdot R} + \int_{DC} (-\hat{R}) \frac{e^{iKA}}{R}.$$
The usual approach has been to ignore distortion due to the nucleus. We know of only two calculations for \((\pi, NN)\) which consider this distortion (Kau, shal and Waghmare 1970, Sakamoto 1968). They have both used an approximation of the sort which we use. Weise, Huber and Danos (1970) have solved the Schrodinger equation for the outgoing particles in the case of \((\gamma, pn)\) where they consider the effect to be important.

The optical potential \(V_{DC}(R) = V_{1C}(R) + V_{2C}(R)\) is taken as the sum of the two nucleon-nucleus optical potentials at the centre-of-mass of the dinucleon. The simplicity of this statement belies the difficulties one encounters when attempting to implement it, for the nucleon-nucleus potential is energy dependent. Thus we should write

\[
V_{DC}(R, T_{DC}) = V_{1C}(R, T_{1C}) + V_{2C}(R, T_{2C})
\]  

(11)

Here \(T_{\alpha C}\) is the (channel) kinetic energy,

\[
T_{DC} = \frac{k^2}{2\mu}
\]

\[
T_{iC} = \left[ (A-2)k_i + k \right]^2/2(A-2)(A-1)\mu, \quad i=1,2
\]  

(12)

These yield the result
In the case $k_1 = k_2$ we have

$$T_{1C} = T_{2C} = \frac{A}{2(A - 1)} T_{DC}.$$  

In this event the two nucleons move off with the same momentum and in the same direction i.e. as a composite particle. The formula for $T_{1C} = T_{2C}$ shows that (if we ignore any possible difference between neutron and proton optical potentials)

$$V_{DC}(R,T_{DC}) \approx 2V_{1C}(R, \frac{1}{2} T_{DC})$$

This agrees with the empirical formula derived by Rook (1965) from the formula of Watanabe (1958) for the deuteron optical potential that the (real part of the) potential for deuteron-nucleus scattering at a given energy is twice that for a proton of half that energy. In general, however, $k_1 \neq k_2$ and the empirical formula breaks down because the "dinucleon" is a fictitious particle introduced for the purposes of our calculation and is unbound (Jackson 1967a).

The proton optical potential for scattering by an $A$-particle nucleus is usually taken to have a Saxon-Woods shape

$$-(U + iW)/(1 + \exp \frac{R-R_0}{a_o}) \quad (R_0 = r_o A^{1/3})$$

We ignore spin orbit and Coulomb terms in the distorting potential. The depth $U + iW$ is assumed to vary linearly
with kinetic energy,

\[ U = U_0 - \alpha T \]
\[ W = W_0 + \beta T \]

the parameters \( U_0, W_0, \alpha, \beta \) being characteristic of the scattering nucleus. We therefore obtain the depth parameters of the dinucleon-nucleus optical potential from equation (11) as

\[ U = 2U_0 - \alpha(T_{1C} + T_{2C}) \]
\[ W = 2W_0 + \beta(T_{1C} + T_{2C}) \]

and, from (12),

\[ T_{1C} + T_{2C} = \left[ 1 + \frac{A}{(A-1)(A-2)} \right] \left( T_1 + T_2 \right) + \frac{2}{(A-2)m} \frac{\beta_1 \cdot \beta_2}{\beta_1^2} \]

\[ \approx T_1 + T_2, \] of course.

For \( \text{Li}^6(\pi^+,2p) \) the residual nucleus is \( \text{He}^4 \) (an \( \alpha \)-particle). We have used the parameters given by Jain and Jackson (1967) for \( \text{Li}^6(p,2p)\text{He} \); these are listed in table VII. We have also set \( R_0 = r_0(A-2)^{1/3} \) in our calculations though it would really be more correct to use \( r_0(A-1)^{1/3} \). Since the potential is given by \(-(U+iW)f(R)\), positive values for \( U \) and \( W \) imply an attractive potential and loss of flux.
We show some typical depth parameters in Table VII.

<table>
<thead>
<tr>
<th>$U_0$ (MeV)</th>
<th>$W_0$ (MeV)</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$r_0$ (fm)</th>
<th>$a_0$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.9</td>
<td>2.68</td>
<td>0.098</td>
<td>0.053</td>
<td>1.2</td>
<td>0.91</td>
</tr>
</tbody>
</table>

$\theta = \cos^{-1}(\hat{k}_1 \cdot \hat{k}_2)$ is the angle between the two outgoing nucleons and $K = |\vec{k}_1 + \vec{k}_2|$ is the recoil momentum of the residual nucleus in the centre-of-mass frame. The available kinetic energy is fixed as $E_{1-2}$ minus the binding energy of two nucleons in the target and $K$ is a function of the distribution of final momenta $(\vec{k}_1, \vec{k}_2)$ and the angle $\theta$. For $\theta = \pi/2$, $K = \sqrt{k_1^2 + k_2^2}$ and is uniquely determined independently of the distribution $(\vec{k}_1, \vec{k}_2)$; as $\theta$ ranges away from $\pi/2$ the dependence on the distribution increases. The energy dependence of $U + iW$ is a function of $K$, since $T_1 + T_2$ determines the recoil energy and hence the recoil momentum $-hK$. Table VIII illustrates how $U + iW$ varies with the distribution $(\vec{k}_1, \vec{k}_2)$ for different opening angles $\theta$.

We should now solve the Schrödinger equation (10) and the Klein-Gordon equation (8) with the optical potentials and appropriate boundary conditions to obtain the distorted wavefunctions. This however is a lengthy procedure and, as already mentioned at the beginning of this section, we approximate the wavefunctions. In order to do this we use the eikonal method (Glauber 1959) which assumes that the scattering wavefunctions can be written as
Table VIII

Typical kinematical distributions and dinucleon-nucleus optical potentials for Li\(^{6}(\pi^+,2p)\) with \(T'_\pi = 78\text{MeV}\)

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(T_1) (MeV)</th>
<th>(T_2) (MeV)</th>
<th>(T_D) (MeV)</th>
<th>(K) (fm(^{-1}))</th>
<th>(U) (MeV)</th>
<th>(W) (MeV)</th>
<th>(\lambda_D) (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80°</td>
<td>50</td>
<td>115</td>
<td>93</td>
<td>3.04</td>
<td></td>
<td></td>
<td>2.23</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>95</td>
<td>94</td>
<td>3.05</td>
<td>31.6</td>
<td>14.1</td>
<td>2.24</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>90</td>
<td>94</td>
<td>3.05</td>
<td></td>
<td></td>
<td>2.27</td>
</tr>
<tr>
<td>100°</td>
<td>50</td>
<td>126</td>
<td>73</td>
<td>2.68</td>
<td></td>
<td></td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>107</td>
<td>72</td>
<td>2.66</td>
<td>30.5</td>
<td>14.7</td>
<td>1.88</td>
</tr>
<tr>
<td></td>
<td>87</td>
<td>90</td>
<td>72</td>
<td>2.66</td>
<td></td>
<td></td>
<td>1.87</td>
</tr>
<tr>
<td>120°</td>
<td>50</td>
<td>137</td>
<td>52</td>
<td>2.25</td>
<td>29.4</td>
<td>15.3</td>
<td>1.52</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>119</td>
<td>48</td>
<td>2.17</td>
<td>29.3</td>
<td>15.4</td>
<td>1.46</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>100</td>
<td>47</td>
<td>2.14</td>
<td>29.2</td>
<td>15.4</td>
<td>1.44</td>
</tr>
<tr>
<td>140°</td>
<td>50</td>
<td>147</td>
<td>32</td>
<td>1.78</td>
<td>28.5</td>
<td>15.8</td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>130</td>
<td>27</td>
<td>1.61</td>
<td>28.2</td>
<td>16.0</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>112</td>
<td>24</td>
<td>1.52</td>
<td>28.0</td>
<td>16.0</td>
<td>0.98</td>
</tr>
</tbody>
</table>
\[ \chi_{DC}(\zeta, \dot{\zeta}) = e^{iK \cdot R - \Delta_{DC}(\zeta, \dot{\zeta})} \]

and
\[ \chi^{+}_{\pi}(k, r_{\pi}) = e^{iK \cdot r_{\dot{\pi}}} \Delta^{+}_{\pi}(k, r_{\pi}) \]

where the modulating functions \( \Delta^{-}_{DC} \) and \( \Delta^{+}_{\pi} \) are slowly varying (compared to the optical potentials). This assumption depends on the momenta being large compared with \( 1/d \) where \( d \) is the distance over which the optical potential changes appreciably, and the real part of the depth of the potential should be small compared with the kinetic energy of the dinucleon, denoted by \( T_D \) in Table VIII. ("The high-energy assumption".)

For the Saxon-Woods potential we take \( 4a_0 \) as the range of appreciable variation we see from table VIII that for angles up to about \( 120^\circ \) the high-energy assumption is certainly valid but at wide angles (\( 120^\circ \)-\( 180^\circ \)) which constitute the domain of interest, it is not completely satisfied. The maximum angle for which the assumption is valid is of course smaller for lower incident pi-meson energies and larger for higher incident energies. However the magnitude of \( K \) is still large enough for the approximation to be a reasonable improvement on plane-waves.

The eikonal method determines the modulating functions as
\[
\begin{align*}
\Delta^{+}_{\pi}(k, r_{\pi}) &= \exp[-i/(\hbar v_{\pi}) \int_{0}^{\infty} V_{\pi}(|r_{\pi} - k_s|)ds] \\
\Delta^{-}_{DC}(K, R) &= \Delta^{+}_{DC}(-K, R) \\
&= \exp[-i/(\hbar v_{DC}) \int_{0}^{\infty} V_{DC}(|R + k_s|)ds]
\end{align*}
\]
using time-reversal invariance. Here $v_{\pi} = \frac{\hbar k_{\pi} c}{E_{\pi}}$ is the relativistic velocity of the pi-meson and $v_{DC} = \frac{\hbar K}{2m}$ is the dinucleon velocity. We note that these equations imply an attenuation of the probability densities $|\chi|^2$ in a potential whose imaginary part is negative, as we pointed out earlier. The "attenuation lengths" $\lambda_{\pi}$ and $\lambda_{D}$ are given in tables VI and VIII at various energies.

We now rewrite the wavefunctions in the momentum transform $g_{LM}$ (in Equation 4) in terms of distorted waves. Since the functions $\Phi(R, r)$ and $\phi(R, r)$, which we expressed in terms of plane waves in section 3.2, are actually linear combinations of the wavefunctions at the points $r_1$ and $r_2$ it is necessary to introduce a further approximation here. We assume that the appropriate way to introduce the distorted waves is to replace $e^{-iQ \cdot R}$ (in equations 4 and 7a) by

$$e^{-iQ \cdot R} \Delta_{DC}(K, R)^* \Delta_{\pi}^{+}(k_{\pi}, \alpha R) = X_{DC}(K, R)^* X_{\pi}^{+}(k_{\pi}, \alpha R),$$

writing $\alpha$ for $(A-2)/A$. This is of course not exact since we are still bound to write $\Delta_{\pi}^{+}(k_{\pi}, \alpha R)$ for

$$\Delta_{\pi}^{+}(k_{\pi}, r_1) \text{ and } \Delta_{\pi}^{+}(k_{\pi}, r_2),$$

In addition we ought to replace the momenta $k_{\pi}$ and $K$ in $\delta^+$ (equation (4) of Section 3.2) by $-i\hbar \nabla$ with the appropriate gradient operators so that the new momentum components introduced by the distortion are fully taken into account. We have neglected doing this at the present time.
In order to evaluate the integrals in the matrix element we expand the scattering functions in partial waves.

We have, for the pi-meson wavefunction,

\[
\Delta^+_{\pi}(k, \alpha_R) = \frac{1}{2} \sum_{t} (2t+1) \mathcal{T}_t(k, \alpha_R) P_t(k, \alpha_R)
\]

and

\[
e^{-\nu} = \sum_{u} i^{u}(2u+1) j_u(\alpha_k R) P_u(k, \alpha_R)
\]

where \( P_t \) denotes the Legendre polynomial of degree \( t \), \( j_u \) is the spherical Bessel function of degree \( u \) which is regular at the origin, and

\[
\mathcal{T}_t(k, \alpha_R) = \int_{-1}^{1} d\mu \ P_t(\mu) \Delta^+_{\pi}(k, \alpha_R) \quad (\mu = k \cdot R)
\]

Using the result (see Appendix A) that

\[
(2u+1) P_t(\mu) P_u(\mu) = 4\pi \ \sum_{f f} (t0f0|u0) Y_f^\phi(k, \alpha_R) Y_f^\phi(R)^* \]

we find that

\[
\chi^+_{\pi}(k, \alpha_R) = 2\pi \ \sum_{f f} F_f(k, \alpha_R) Y_f^\phi(k, \alpha_R) Y_f^\phi(R)^*
\]

with

\[
F_f(k, \alpha_R) = \sum_{tu} i^{u}(2t+1)(t0f0|u0) T_t(k, \alpha_R) j_u(\alpha_k R). \quad (14)
\]

Similarly we find, for the dinucleon wavefunction, that
\[
\chi_{DC}\overline{\langle K,R \rangle}^* = 2\pi \sum_{e\varepsilon} E_e\langle K,R \rangle Y_e^\varepsilon(K)Y_e^\varepsilon(R)^*
\]

with

\[
E_e\langle K,R \rangle = \sum_{t,u} i^u(2t+1)(t0e0|u0)^2 S_t\langle K,R \rangle j_u(KR)
\]  \hspace{1cm} (15)

\[
S_t\langle K,R \rangle = \int_{-1}^{1} d\mu P_t(\mu)A^+_t(K,R)
\]

\hspace{1cm} (\mu = K,R)

We now obtain for the distorted momentum transform

\[
g_{N\varepsilon}(K,R) = \int d^3\mathbf{R} \chi_{DC}(K,R)\chi_{\varepsilon\pi}\mathbf{R}^{N\varepsilon}(\mathbf{R}) Y_{\varepsilon}(\mathbf{R})
\]

\[
= \int_0^\infty R^2 e_{N\varepsilon}(R)dR
\]

where

\[
e_{N\varepsilon}(R) = \pi R^{N\varepsilon}(R) \frac{\Sigma(-1)^{\varepsilon} \sqrt{2\varepsilon+1} (2f+1)(e0f0|\varepsilon0) (e\pi0f0|\varepsilon0) ^{\varepsilon}}{\sqrt{2f+1}}
\]

\[
 \times E_e\langle K,R \rangle F_f(k_\pi,\alpha R).
\]

We have performed the integration over the angular coordinates \(\hat{R}\) and have chosen the axis of quantisation along the incident beam direction \(\hat{k}_\pi\) so that \(Y_{\varepsilon}(k_\pi) = \delta_{\varepsilon0} \sqrt{(2f+1)/4\pi}\).

If we only take the s-wave term, \(f = 0\) and the summation reduces to a single term,

\[
e_{N\varepsilon}(R) = \pi(-1)^{\varepsilon} R^{N\varepsilon}(R) Y_{\varepsilon}(K) E_{\varepsilon}(K,R) F_0(k_\pi,\alpha R).
\]

It has already been explained in section 3.2 that this term contributes to p-wave absorption.
The distorted momentum transform contains the essential nuclear structure information extracted from the reduced overlap integral. The plane wave momentum transform can be recovered from it by setting

\[ E_e(K, R) = 2i^e j_e(KR) \]

\[ F_f(k_f, \alpha R) = 2i^f j_f(\alpha k_f R). \]

We evaluated the integrals for \( g_{N,F}^\pi \) using Weddell's seven point rule (Abramowitz and Stegun 1965) with a step length of 0.125 fm, taking the integration up to 9.75 fm or 13.5 fm. In many cases the improvement using the longer range of integration was of the order of 1% or less.

An indication of the localisation of the \((\pi^+; 2p)\) reaction on Li\(^6\) is given by Figure 6 which shows

\[ R^2 \chi_{DC}^{-} (K, R)^* \chi_{\pi}^{+} (k_r, \alpha R) \]

with \( K = 0.45 \text{ fm}^{-1} \) and \( k_r = 0.81 \text{ fm}^{-1} \) (corresponding to pi-mesons incident with 78 MeV kinetic energy) for \( f = 0, 1 \) and 2 using plane waves. We have set \( e = f \) since we are taking \( J = 0 \) in the intercluster wavefunction for Li\(^6\). It is seen that the reaction probes deep into the nucleus. The extent to which the surface contributes depends on which intercluster function is folded in.

Figure 7 illustrates the effect of introducing the correct asymptotic form into the overlap integral. For single proton energies of 50 and 90 MeV there is
Fig. 6 Localisation of the $f = 0, 1$ and 2 contributions to the $(\pi^+, 2p)$ reaction.
Fig. 7 $|g_{NLM}|$ in the plane wave approximation as a function of the opening angle for fixed single proton energies $T_\pi = 50, 70, 90$ MeV with $T'_\pi = 78$ MeV. The full curve was calculated using $G$ for the intercluster wavefunction $\Phi^\text{IC}$, the dashed curve using $G_\text{F}$. 
Fig. 8 $|g|$ as a function of the opening angle for fixed single proton energies $T_1$ with $T_1' = 78$ MeV, normalised to unity at $180^\circ$. The full curve is the plane wave approximation, the short-dashed curve includes pi-meson distortion and the long-dashed curve is a full distorted wave approximation.
Fig. 9 $|g(K, k_x)|$ as a function of the recoil momentum, for $f = 0$. The curves are as described in Figure 8.
Fig. 10 \[ |g_{N\lambda m}| \] as a function of the opening angle showing the effect of including \( f = 1, 2 \) in \( \chi_{\nu} \).

The full line is for \( f = 0 \) only, \(-x-\) includes \( f = 1 \), and \(-o-\) includes \( f = 2 \) as well.
Fig. 11 $|g_{N^2M}|$ as a function of the energy of a single proton for symmetric emission at 140° and 180° with $T_\pi = 78$ MeV. The continuous curves include only the $f = 0$ component; the dash-cross curves include $f = 0, 1$ and 2 components.
Fig. 12 \( K^2 \mid g(K, \mathbf{k}_\pi) \mid^2 \) in the distorted wave calculation as a function of recoil momentum for several incident pi-meson energies (\( f = 0 \) component only).
quite a discrepancy near the minimum of the "momentum transfer" $Q$ (in the vicinity of $160^\circ$ opening angle) although this minimum is not small ($\approx 1 \text{ fm}^{-1}$).

The remaining graphs (Figures 8 - 12), have been calculated using the wavefunction $1T$. Comparing Figures 7 and 8 shows an immediate discrepancy. This is not surprising since the $f = 0$ component which is shown in these two diagrams is localised in the middle region of the nucleus in which the different wavefunctions have a completely different character. We see that the main effect of the pi-meson-nucleus optical potential is to reduce the magnitude of the form factor $g(K,k)$ for values of the recoil momentum $K$ between 0.5 and 2.0 fm$^{-1}$. The shape of the distribution is substantially unaffected. The neglect of the nucleon-nucleus optical potential, however, gives rise to major discrepancies. We have not completed the graphs for low $K$ values where the distortion effect is not approximated well by the eikonal method; the form factor remains finite, of course.

Figures 10 and 11 show that it is necessary to include the $f = 1$ and $f = 2$ terms in the calculations. Figure 10 shows $|g|$ as a function of opening angle (for symmetric coplanar geometry) for fixed single proton energies; Figure 11 shows $|g|$ as a function of the energy of a single proton for fixed opening angles. Both figures show plane-wave calculations.

In Fig. 12 we have plotted $K^2 |g(K,k)|^2$, which contains the $K$-dependence of the cross-section, as a function of $K$ for several pi-meson energies, in the distorted wave calculation. The increasing magnitude of the imaginary part of the optical potential as $T^\prime_\infty$ increases has the expected effect of dramatically reducing the form factor. The sums in Equations 14 and 15 were taken up to $t=6$ (for $\chi^\prime_\infty$) and $t=8$ (for $\chi_{DC}$) after calculations showed that higher partial waves do not contribute significantly.
3.5 The Relative Dinucleon Wave Function

Having disposed of the momentum transform \( g(\mathbf{q}) \) of the dinucleon relative to the core \( C \), we now turn to the transform \( a(\mathbf{q}) \) of the wavefunction which describes the internal relative motion of the nucleon pair in the nucleus \( A \). We introduced \( a(\mathbf{q}) \) in Section 3.3 (equation 7a) by

\[
a_{n\ell m}(\mathbf{q}) = \int d^3\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} d^n\mathbf{r} Y_{\ell}^m(\mathbf{r})
\]

for \( \mathbf{q} = k - \frac{1}{2}k_\pi \). It may be interpreted as the probability amplitude for finding in the nucleus \( A \) a pair of nucleons in a relative \((n\ell m)\) state with conjugate momentum (= half their relative momentum) equal to \( \mathbf{q} \).

The above definition of \( a(\mathbf{q}) \) arises from a description of the \((\pi,NN)\) process using plane waves. In a more accurate description we should improve \( a(\mathbf{q}) \) to take into account (i) distortion of the incoming pi-meson wave, since it contributes a term \( e^{i\mathbf{k}\cdot\mathbf{r}} \), and (ii) mutual scattering of the outgoing pair of nucleons which distorts the plane wave \( e^{i\mathbf{k}\cdot\mathbf{r}} \).

It is extremely difficult to take (i) into account; in our description we modified the incoming pi-meson wavefunction by the inclusion of a modulating term \( \Delta^+_\pi(k,\mathbf{r}) \). This term does not split easily.
in \( r, R \) coordinates (as is possible for plane waves).

We have therefore approximated by

\[
\Delta^+_{q_1}(k_{q_1}, r_1) = \Delta^+_{q_2}(k_{q_2}, r_2) = \Delta^+_R(k_{q_1}, \sim R)
\]

i.e. we assume that the distortion only affects the interaction in the \( R \)-channel, leaving a plane-wave term in the \( r \)-channel.

There are a number of approaches to (ii). Eckstein (1963) argues, in the case of the \( ^4\text{He} + (n, 2n) d \) reaction, that the relative momentum of the two nucleons should be sufficiently high that final state scattering can be safely ignored. (In addition, the phenomenological approach includes some of the scattering effect in the coupling constants.) In the absence of any other correlations this would give us

\[
a_n j_m(q) = 4\pi(-i)\ell Y^\ell_m(q) \int_0^\infty dr r^2 J_\ell(qr) d n_\ell(r).
\]

Some authors include the final state interaction by means of the asymptotic approximation. This replaces \( e^{-ik_{\pi} r} \) by an asymptotic form with phase shifts derived from free nucleon-nucleon scattering. Whilst this approach does not attempt to describe correctly the wavefunction at close nucleon-nucleon separations, it should be no worse there than a plane wave. Guy, Eisenberg and Letourneux (1966) have used this method, considering the nucleon-nucleon interaction to be more important than distortions due to nucleon-
nucleus scattering, and their results indicate considerable enhancement of the transition rate for the \((\pi,\text{NN})\) process as a consequence of including this interaction.

Kopaleishvili (1967) has replaced the plane wave by a function calculated by solving an appropriate set of coupled equations in a suitable nucleon-nucleon potential. He compares his results with those obtained using the asymptotic approximation, finding in general a further enhancement of the transition rates. Elsaesser and Eisenberg (1970) have also used a wavefunction obtained in a nucleon-nucleon potential and find a significant change in the shapes of the angular and energy distributions. These results correspond to those of Weise et al (1970) who find the asymptotic approximation unsatisfactory for the \((\gamma,\text{pn})\) reaction.

We have not included explicitly the effect of final-state interactions between the two outgoing nucleons in any of our calculations.

Parallel to these considerations is the question of nuclear pair correlations in the initial state. As we explained in Section 1.2, these correlations are a modification of the independent particle shell model wavefunction for the nucleus.
The usual method of introducing correlations is to multiply the shell model wavefunction by a correlation function \( f(|r_i - r_j|) \) for each correlated pair of nucleons \((i,j)\) in the nucleus. In a full treatment each pair of nucleons in the nucleus is correlated and the full correlation functions is a product \( \prod_{i<j} f(|r_i - r_j|) \) (Jastrow 1955). For a discussion of the \((\pi,\text{NN})\) reaction for which the interaction operator involves only the nucleons 1 and 2, the single term \( f(|r_1 - r_2|) \) is taken into consideration (Jibuti and Kopaleishvili 1964, Eisenberg and Letourneux 1967, and others).

The usual form chosen for the correlation function is that given by Dabrowski (1958):

\[
f(r) = \begin{cases} 
0 & r < r_c \\
1 - \exp\left[-\beta \left(\frac{r}{r_c} \right)^2 - 1\right] & r > r_c
\end{cases}
\]

This form corresponds to a hard core in the nucleon-nucleon potential of radius \( r_c \) (which is usually taken to be 0.4 fm.). The parameter \( \beta \) is a measure of the rate at which the correlated wavefunction "heals" to the uncorrelated form.

Another type of correlation function has been used by Weise, Huber and Danos (1970) in their discussion of the nuclear photo-effect. They use the correlation function
\[ f(r) = 1 - j_0(pr) \]

which corresponds to a nucleon-nucleon potential with a soft core. This form simulates the exchange of momentum \(\hbar p\) between the two nucleons.

Eckstein (1963) has proposed a phenomenological approach which involves inserting a term \(\delta(r)\) in the interaction Hamiltonian for the \((\pi,NN)\) process and replacing the coupling constant \(f\) by a pair of coupling constants \(g_0\) and \(g_1\) which depend on the transition \(S^T \rightarrow S'T'\). The new constants \(g\) may also differ for \(s\)- and \(p\)-wave \(\pi\)-mesons and could possibly vary with the momentum \(\hbar k_\pi\) of the incident beam. However Eckstein and most of the other authors who adopt the phenomenological approach consider absorption of \(\pi\)-mesons bound in an \(s\)-orbit and the published values for the constants \(g_0, g_1\) have been calculated for threshold reactions \((k_\pi = 0)\) and \(s\)-wave \(\pi\)-mesons only. The phenomenological approach has also been used by Figureau and Ericson (1969) for energetic \(\pi\)-mesons \((\pi^+)\) but they assume that \(g_0, g_1\) do not vary with momentum \(\hbar k_\pi\) and their detailed calculations only extend to \(s\)-wave \(\pi\)-mesons.
In our calculations the gradient operators in the interaction Hamiltonian are included in terms of \( \mathbf{k} \) and \( \mathbf{k}' \), as a result of which the s- and p-wave parts of the pi-meson wavefunction do not appear independently. This makes it difficult to introduce into our calculations the phenomenological approach to correlations without recalculating the constants, especially since they have been calculated for absorption of s-wave pi-mesons only (Eckstein 1963).

Furthermore, the calculations of the phenomenological constants \( g_0, g_1 \) were performed assuming that the wavefunction for the pi-meson is uniform throughout the nucleus i.e. writing \( \chi_{\pi}(\mathbf{k}, 0) \) for \( \chi_{\pi}(\mathbf{k}, r) \). It is not clear, therefore, whether these constants are appropriate in a calculation which treats the pi-meson wavefunction more carefully.

The function \( d^{nt}(r) \) which appears in the definition of \( a_{n'm}^2(g) \) arises from equation (7), which derives from a cluster expansion of the target wavefunction. In the cluster model this expansion is curtailed to a single term. For example, the \( \text{Li}^6 \) wavefunction has been described by a cluster model wavefunction whose relative dinucleon part is of the form \( \exp[-\alpha(\rho_1^2 + \rho_2^2)/2] \) (Tang, Wildermuth and Pearlstein 1962). In this expression \( \rho_1 \) and \( \rho_2 \)
are the coordinate vectors of the two p-shell nucleons in Li$^6$ relative to their centre of mass, i.e.

$$\rho_1 = \frac{1}{2} r, \quad \rho_2 = -\frac{1}{2} r,$$

so that

$$\mathbf{d}(r) = N \exp(-\alpha r^2/4)$$

with the normalisation constant given by

$$N^2 = (\alpha/2\pi)^{\frac{1}{2}} 2\alpha.$$

Values for the parameter $\alpha$ are given in Table IX.

As we explained in Section 3.3 we take the internal motion of the D cluster in the Li$^6$ ground state to be $1s$ in the cluster model. Since the above form has no nodes and does not vanish for $r = 0$ it has the properties of a $1s$ wavefunction. The angular wavefunction $Y^m_l(r)$ with $l = m = 0$ is $1/\sqrt{4\pi}$.

Table IX

Values for the parameter in the deuteron cluster wavefunction for Li$^6$

- $\alpha = 0.66$ (Tang et al 1962)
- $\alpha = 0.27$ (Lodhi 1967)
We have given a description of the $(\pi,\text{NN})$ reaction and have discussed the matrix element which appears in the expressions for the reaction cross-section. We have developed this matrix element in terms of a cluster model description of the target nucleus. Only a slight modification is required to accommodate a more general cluster series description.

We pointed out in Chapter II that whilst much theoretical effort has been directed at an investigation of the short-range effects in pi-meson absorption studies, virtually no consideration has been given to an adequate description of the long-range components.

One long-range effect which we notice is (Figure 7) that using the wrong tail for the overlap integral in the matrix element can cause a large discrepancy in the angular distributions but a comparison with Figure 8 shows that the asymptotic behaviour does not dominate to the exclusion of any dependence on the behaviour of the wavefunction in the surface region of the nucleus. In our calculations the surface dependence may be accentuated by the restriction to s-waves in a plane wave approximation. Since, as we see from Figures 10 and 11, it is necessary to include other partial waves in the calculations the momentum space form factor $g$ should be more dependent on the asymptotic
region than our results indicate (cf. Figure 6). We should also mention that our calculations do not relate to very low values of the momentum transfer $Q$, for which there would undoubtedly be little sensitivity to the 3-6 fm. region.

Another factor which must be expected to reduce the probability of the pi-meson penetrating the nuclear surface is inclusion of pi-meson-nucleus distortion.

Our results indicate that this effect is most noticeable for values of $K$, the recoil momentum of the nucleus, in the region of $1\text{ fm}^{-1}$. It seems unlikely that inclusion of the extra "gradient" term in the pi-meson - nucleus optical potential should make much difference to this conclusion.

The other long-range effect which we have considered is the distortion of the final state by the nucleon-nucleus interaction. Here we have found the effect to be considerable (Figure 9), both as regards shape and magnitude.

The relative importance of $f = 0$, 1 and 2 contributions to $g$ must be interpreted in light of the way which $\chi_\pi$ appears in the matrix element. For the energies which
we are considering $\zeta$ (see Section 2.1) and the main contribution is through the term $-iV_\pi^2$, i.e., $\frac{1}{2m}\chi_\pi^2$ in the transition operator. Thus the $f = 0$ and $f = 2$ contributions come (mainly) from absorption of p-wave pi-mesons, whereas s-wave pi-mesons contribute to the $f = 1$ term. It is not unreasonable therefore to find a significant contribution from the $f = 2$ term. For large opening angles the $f = 1$ contribution is minimal since odd $f$-values do not affect the back-to-back amplitude (because $\hat{\kappa}_k \cdot \hat{\kappa} = 0$).

We therefore conclude that:

(a) it is necessary to treat the asymptotic behaviour of the overlap integral correctly for the $(\pi^+, 2p)$ reaction on light nuclei; this corresponds with a similar result found by Jain, Sarma and Banerjee (1970) for the (p,pd) reaction;

(b) within our approximations the $(\pi^+, 2p)$ reaction is also sensitive to the form of the overlap integral in the surface region at momentum transfer $Q \gtrsim 1 \text{ fm}^{-1}$ from the pi-meson to the two-nucleon centre of mass;

(c) nucleon-nucleus distortion in the final state plays a significant role; and

(d) in a partial wave analysis of the incoming pi-meson beam both s- and p-waves are important for two-nucleon emission.
We have not shown any results for the differential cross-section and it would now be instructive to calculate these for the purpose of comparing with experimental results. There has been a preliminary report of a recent experiment (Amato et al 1971) on the Li$^6(\pi^+, 2p)$ reaction with asymmetric geometry, for which theoretical calculations can be made using our formalism to compare with the detailed experimental results when they become available.

There are three improvements to our formalism which should be made in future calculations:

(a) making the pi-meson-nucleus optical potential somewhat more realistic by including the "gradient" term;

(b) using a more realistic wavefunction for the target wavefunction which reproduces the known properties of Li$^6$, possibly with another term in the cluster series; and

(c) replacing the eikonal approximation by a Schrodinger equation solution, at least for low values of the dinucleon momentum $K$. 
A.1 Spherical Harmonics

The spherical harmonic function $Y^m_\ell(\hat{r})$ is defined as a normalised simultaneous eigenfunction of the angular momentum operators

$$L^2 = \frac{1}{\sin^2 \theta} \left\{ \left[ \sin \theta \frac{\partial}{\partial \theta} \right]^2 + \left[ \frac{\partial}{\partial \phi} \right]^2 \right\}$$

and

$$L_0 = -i \frac{\partial}{\partial \phi}$$

with eigenvalues $\ell(\ell+1)$ and $m$ respectively (Landau and Lifshitz 1965). In the above expressions $\theta$ is the polar angle of $\hat{r}$ and $\phi$ is the azimuthal angle. We give, in this section, formulas for the spherical harmonics which have been used in our calculations and we derive the result which we used in Section 3.4.

The definition of the spherical harmonics is undetermined to the extent of a phase factor. In our calculations we have used the phase convention implied by

$$Y^m_\ell(\hat{r}) = (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi (\ell+m)!}} P^m_\ell(\cos \theta)e^{im\phi}, \quad m \geq 0$$

where the associated Legendre function $P^m_\ell$ is real. (The phase convention of Landau and Lifshitz differs from this by a factor $i^\ell$.) The spherical harmonics for $m < 0$ are obtained using the relation

$$Y^m_\ell(\hat{r})^* = (-1)^m Y^{-m}_\ell(\hat{r}).$$
Since the calculations in this thesis all relate to cases in which $\phi = 0$ or $\phi = \pi$ the spherical harmonics are real with our phase convention.

The functions $P^m_l(cos \theta)$ have been evaluated using the recurrence relations (Abramowitz and Stegun 1965):

\begin{align*}
P^0_0 &= 1 \\
P^0_l &= (2l-1) \sin \theta P^0_{l-1} \\
P^1_l &= (2l-1) \cos \theta P^0_{l-1} \\
P^m_l &= ((2l-1) \cos \theta P^m_{l-1} - (l+m-1) P^m_{l-2}) / (l-m)
\end{align*}

The integral of three spherical harmonics

$$\int_{\text{unit sphere}} Y^\gamma_c(\hat{r})^* Y^\alpha_a(\hat{r}) Y^\beta_b(\hat{r}) \, d^2r$$

is found by the Wigner-Eckart theorem to be

$$\sqrt{(2a+1)(2b+1)} \frac{4\pi}{(2c+1)} (a a b | c y) (a 0 b 0 | c 0).$$

It follows, by the orthonormality and completeness of the spherical harmonics, that

$$Y^\gamma_a(\hat{r}) Y^\beta_b(\hat{r}) = \sum_{c} \sqrt{\frac{(2a+1)(2b+1)}{4\pi (2c+1)}} (a a b | c y)(a 0 b 0 | c 0) Y^\gamma_c(\hat{r}).$$

Thus

$$\sum_{a\beta} Y^\alpha_{a} (\hat{r}_1)^* Y^\alpha_{a} (\hat{r}_2) Y^\beta_{b} (\hat{r}_1)^* Y^\beta_{b} (\hat{r}_2) =$$
\[
\sum_{cY} \left( \frac{(2a+1)(2b+1)}{4\pi (2c+1)} \right) (a0b0|c0)^2 \gamma^c(r_1) \gamma^c(r_2)
\]

where we have used the closure relation for the Clebsch-Gordan coefficients

\[
\sum_{\alpha\beta}(a\alpha\beta|cY)(a\alpha\beta|cY) = \delta_{cc} \delta_{\gamma\gamma}.
\]

Now, the expression

\[
I = \sum_{\alpha} Y^\alpha_1(\hat{r}_1) Y^\alpha_2(\hat{r}_2)
\]

is invariant under rotations of the coordinate system (Rose 1957). By choosing the polar axis along \( \hat{r}_1 \), we easily obtain the result

\[
I = \frac{2a+1}{4\pi} P_a(\hat{r}_1, \hat{r}_2)
\]

where \( P_a = P^0_a \) is the Legendre polynomial of order \( a \).

Therefore

\[
P_a(\hat{r}_1, \hat{r}_2)P_b(\hat{r}_1, \hat{r}_2) = \sum_{cY} \left( \frac{4\pi}{2\pi c+1} \right) (a0b0|c0)^2 \gamma^c(\hat{r}_1) \gamma^c(\hat{r}_2),
\]

which is the result we used in Chapter III.

A.2 Spherical Bessel Functions

The Rayleigh expansion of a plane wave

\[
e^{ik \cdot r} = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)j_{\ell}(kr)P_{\ell}(\hat{k}, \hat{r})
\]

involves the spherical Bessel function \( j_{\ell} \) which is regular.
at the origin. The Legendre polynomial $P_\ell$ has been defined in Section A.1.

To compute the functions $j_\ell$ we have used the recurrence relation (Abramowitz and Stegun 1965)

$$x j_\ell(x) = (2\ell-1) j_{\ell-1}(x) - x j_{\ell-2}(x).$$

We obtain regularity at the origin by starting off with

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$
The set of equations which arises from a cluster expansion in which the non-central potential $V_{\text{coup}}$ is retained is (cf. Section 3.3)

$$T'_D + V_D - S_{\alpha \beta}^{AD} R^\gamma + \sum_\delta U^\delta R^\delta = 0,$$

or in a simpler notation

$$Q^\gamma [\lambda^\gamma] R^\gamma + \sum_\delta U^\delta R^\delta = 0$$

where $Q^\gamma$ is the operator

$$-\frac{d^2}{dR^2} + \frac{\ell^\gamma (\ell^\gamma + 1)}{R^2} + V_D(R) - S_{\alpha \beta}^{AD}$$

and $\ell^\gamma$ is the angular momentum quantum number corresponding to $R^\gamma$. $S_{\alpha \beta}^{AD}$ may be different for differing choices of $\gamma$ and $\lambda^\gamma$ is the eigenvalue of $Q^\gamma$ for the uncoupled problem

$$Q^\gamma [\lambda^\gamma] R^\gamma = 0.$$

Let us denote the solutions of the uncoupled problem by $\eta^\gamma$ and those of the coupled problem by $\psi^\gamma$. Each of the $\eta^\gamma$ has an undetermined normalisation factor but the $\psi^\gamma$ are unique except for an overall normalisation factor. In
each case the boundary conditions are given by

\[ \eta \gamma(0) = \psi \gamma(0) = 0 \quad (1) \]

\[ \lim_{R \to \infty} \eta \gamma = \lim_{R \to \infty} \psi \gamma = 0 \quad (2) \]

The uncoupled equations are easy to solve (for example, using the method of Buck 1960). We may obtain an approximate solution to the coupled equations as follows. For each \( \gamma \) we solve the ordinary inhomogeneous equation

\[
\left( Q^\gamma \left[ \lambda_\gamma \right] + U^\gamma \right) \psi (1) + \sum_{\delta \neq \gamma} U^{\gamma \delta} \eta \delta = 0,
\]

whose solution is given by

\[ \psi_j = \xi + \alpha \gamma_1 + \beta \gamma_2 \]

where \( \xi \) is any solution of the inhomogeneous equation, \( \gamma_1 \) and \( \gamma_2 \) satisfy

\[
\left( Q^\gamma \left[ \lambda_\gamma \right] + U^\gamma \right) \gamma = 0, \quad (3)
\]

and \( \alpha \) and \( \beta \) are constants chosen to make \( \psi(1) \) satisfy the boundary conditions. Clearly \( \lambda_\gamma \) will not be an eigenvalue for this problem and there will be no solution \( \gamma \) which satisfies conditions (1) and (2) simultaneously. Therefore let \( \gamma_1 \) and \( \gamma_2 \)
be, respectively, the solutions of Equation 3 satisfying conditions (1) and (2). We now divide the domain $[0, \infty)$ into two regions divided by the point $R_m$, which is the largest value of $R$ such that \( \frac{d^2 y}{dR^2} = 0 \). This ensures the maximum stability in the numerical integration to follow.

Let $\xi_1, \xi_2$ be arbitrary solutions of the inhomogeneous equation in the regions $[0, R_m]$ and $[R_m, \infty)$, respectively. Then in general

\[ \xi_1(R_m) \neq \xi_2(R_m) \]

\[ \xi'_1(R_m) \neq \xi'_2(R_m) \]

but we can now choose $\alpha, \beta$ such that

\[ \xi_1 + \alpha y_1 = \xi_2 + \beta y_2 \]

\[ \xi'_1 + \alpha y'_1 = \xi'_2 + \beta y'_2 \]

at $R_m$ so that we obtain a smooth solution given by

\[ \xi_j(R) = \xi_1(R) + \alpha y_1(R) \quad \text{if} \quad R < R_m \]

\[ = \xi_2(R) + \beta y_2(R) \quad \text{if} \quad R > R_m \]
Further approximations may be obtained by iterating the above procedure but it is necessary to perform a detailed analysis to determine whether the iteration converges.

The choice of functions \( \eta \) to start off the approximation is not well-determined by the problem. We can however impose the condition that the \( \eta \) each be normalised to unity to give them equal starting weight, i.e.

\[
\int_0^\infty R^2 |\eta^r(R)|^2 \, dR = 1.
\]

The functions \( \psi^r \) must be normalised by the condition (Jackson 1967b)

\[
\sum_{\gamma} \int_0^\infty R^2 |\psi^r(R)|^2 \, dR = 1.
\]

A different initial normalisation for the \( \eta \) will lead to a different set of solutions \( \psi^r \). In principle, however, the true set of solutions \( \{\psi^r\} \) which are smooth and satisfy the boundary conditions is unique, so that if the process is convergent it should not matter what starting value we take.
For the Li\textsuperscript{6} intercluster wavefunction we have performed some preliminary calculations coupling two 2s states with separation energies 1.47 MeV and 3.70 MeV (the latter corresponding to the break up Li\textsuperscript{6} \(\longrightarrow\) \(\alpha\) + p + n). Taking the coupling potentials to be of the form \(dV/dR\), where \(V\) is a Saxon-Woods form, and to have a depth of 2-4 MeV we obtain a result with the relative normalisation

\[
\psi_{1.47} : \psi_{3.70} \approx 55 : 45.
\]

Our calculations indicate that the two wavefunctions have essentially the same shape as in the uncoupled case, but the coupling introduces a relative minus sign in the asymptotic region.
References


J. AMATO et al, 1971, LASL preprint LA-DC 12389


N. AUSTERN, 1964, Phys. Rev. 136: 1743


K. A. BRUECKNER, R. SERBER and K. M. WATSON, 1951, Phys. Rev. 84: 258

B. BUCK, 1960, Report No. 15, Oxford Computing Laboratory,

Nucl. Phys. Computing Group


D. L. CHESHIRE and S. E. SOBOTTKA, 1969, Phys. Letters 30B: 244

P.P. DIVAKARAN, 1965, Phys. Rev. 139B: 387

C. ECKART, 1930, Rev. Mod. Phys. 2: 305
S. G. ECKSTEIN, 1963, Phys. Rev. 129: 413


1969, Selected Topics in Pion-Nucleus Interactions, Hercegnovi lectures.


1967b, Nuovo Cim. 51B: 49
1968, Adv. in Phys. 17: 481
1970a, Nuclear Reactions, Methuen, London


B. K. JAIN, 1972, preprint


L. S. KISSLINGER, 1955, Phys. Rev. 98: 761


(Soviet J. Nucl. Phys. 9: 51)


T. LAURITSEN and F. AJZENBERG-SELOVE, 1966, Nucl. Phys. 78: 1

O LOCK 1960, High Energy Nuclear Physics, Methuen, London.

J.M. McKINGLEY, 1963, Rev. Mod. Physics 35: 788


1965, Nuc. Phys. 61: 219


ROSENFELD et al, 1967 Rev. Mod. Phys. 39: 1


R.SHERR et al., 1965, Phys.Rev. 139: B1272


S.WATANABE, 1958, Nucl. Phys. 8: 484


W.WEISE, M.G.HUBER and M.DANOS, 1970 Z. Physik 236: 176

E.WIGNER, 1931, Groupentheorie, Vieweg

K.WILDERMUTH, 1962, Nucl. Phys. 31: 478

K.WILDERMUTH and Th.KANELLOPOULOS, 1959 CERN Report 59-23
