CALCULATIONS OF NUCLEAR SCATTERING

WITH

APPROXIMATE DISTORTED WAVES.

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by

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ABSTRACT

An attempt has been made to find realistic analytic wavefunctions suitable for use in \((p,2p)\) reactions. The success of McCarthy and others using modified plane waves in inelastic scattering is exploited. In order to estimate the validity of the use of these wavefunctions we have looked at the elastic scattering of alpha particles from \(^{88}\text{Sr}\) and \(^{42}\text{Ca}\) in an approach similar to that of Dar (D1), although the Gell-Mann, Goldberger two potential formula is not used explicitly. The results obtained using these modified plane waves are compared to those obtained using the Optical Model.

The limitations of the wavefunctions in this approach are discussed.
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CHAPTER 1

1.1 Introduction

The aim of this work was to make an attempt to construct the T-matrix element in a (p,2p) reaction such as $^{12}$C (p,2p) $^{11}$B or $^{16}$O (p,2p) $^{15}$N at medium incident proton energies (<100 MeV). The work most applicable to our line of approach is that of Wright, Storer and McCarthy (Wl).

1.2 The (p,2p) reaction

We consider here the method of describing the A (p,2p) A-1 reaction as in (Wl) and to consider the current major difficulties encountered when trying to construct the exact T-matrix element. This matrix element is given by the expression, see for example (J2)

$$T_{if} = \int \chi_k^* (r_0) \chi_{k2}^* (r_1) \psi_{j_l m_i} (r_2 ... r_A) \chi^{(+)} (r_0) \psi_{j_l m_i} (r_1 ... r_A) dr_0 dr_1 ... dr_A \quad (1.1)$$

Where $\psi_{j_l m_i} (r_1 ... r_A)$ is the initial nuclear wavefunction and $\psi_{j_l m_i} (r_2 ... r_A)$ is the final nuclear wavefunction. The $\chi_k^+ (r)$ is the incoming proton wavefunction of momentum $k$ and position $r$ while $\chi_k^- (r)$ is the time reversed outgoing wavefunction, and $V(0,1)$ is the two body interaction between the incoming and bound protons.

Now define the overlap integral

$$\psi_p (r_1) = \int \psi^*_{j_l m_i} (r_2 ... r_A) \psi_{j_l m_i} (r_1 ... r_A) dr_2 ... dr_A \quad , (1.2)$$
so that

$$T_{if} = \int \frac{\hat{x}_{k_1}(-) (r_0) \hat{x}_{k_2}(-) (r_1) V(0,1) \hat{x}_{k_0}(r_0) \psi_p(r_1) \, dr_0 \, dr_1}{r_1}$$  (1.3)

which is in fact the finite range DWBA matrix element assuming an infinitely heavy target.

It is not possible to evaluate this integral with exact distorted waves and a finite $V(0,1)$. A simplification can be made by using plane wave Born approximation in which the matrix element separates into the product of two integrals (J2) or alternatively a delta function potential may be used for $V(0,1)$. However, although PWBA is fairly easy to evaluate it does not give good results except at high energies. The idea in this work is to use McCarthy wavefunctions which have a plane wave structure and hence retain the properties of simplification that these give. These wavefunctions are discussed later.

Let us look at the wavefunctions $\chi^+(r)$ in equation (1.3). These are the distorted waves which are solutions of the Schrodinger equation

$$\left( T + V_{0c} \right) \chi_{k_0}^{(+)} = \epsilon_0 \chi_{k_0}^{(+)}$$  (1.4)

Where $V_{0c}$ is the interaction between the incident proton and the core of the target nucleus and

$$\epsilon_0 = \frac{\hbar^2 k_0^2}{2\mu}$$  (1.5)

where $\mu$ is the reduced mass of the proton target system. Thus the $\chi^{(+)}$ are solutions of the equation describing elastic scattering of protons and if we take $V_{0c}$ to be the phenomenological optical potential then the $\chi^{(+)}$ are the corresponding wavefunctions. It is still the case that these would have to be considered numerically. The object here is to try to use an analytic form for the distorted waves thereby possibly making the $T$-matrix element an analytic integral or at least one that could be computed in a simple way. These wavefunctions must be chosen such that they reproduce correctly the elastic scattering
cross section for if they did not, their use in the \((p,2p)\) reaction matrix element would not be valid.

The wavefunctions that we consider here are those used by McCarthy and others in several works (K1, J1, M4) mostly in consideration of inelastic scattering of alpha particles from several targets. The angular distribution of \((a,a')\) inelastic scattering from \(^{62}\text{Ni}, ^{88}\text{Sr}, ^{32}\text{S}, ^{48}\text{Ti}\) and \(^{24}\text{Mg}\) have been well reproduced over an energy range from 33 MeV to 100 Mev by the use of these so called modified plane waves provided these waves are restricted to the surface of the nucleus.

1.3 The importance of the nuclear surface

From the references above it appears that there has been a certain amount of success in applying the modified plane waves to inelastic scattering but there appears to be a lack of work with these on elastic scattering of alpha particles, although \((p,p)\) scattering has received some attention. (W1).

In this section we consider the work on inelastic scattering and its relationship to the nuclear surface. The work (J1) of Janus and McCarthy indicates the way in which the modified plane waves or preferably, model distorted waves, are used in a collective inelastic scattering model. The T-matrix element in the DWBA approximation for inelastic scattering can be written as

\[
T_{fi}(k',k) = \int \chi_f^{-}(k',\xi) \Phi^*_f(\xi) V(\tau,\xi) \Phi_i(\xi) \chi_i^{+}(k,\tau) \, d\tau \, d\xi ,
\]

where \(\Phi(\xi)\) are the core wavefunctions of the target and where \(\chi_{i,f}^{(+)}\) are the distorted waves of the projectile in the initial and final states, \(\xi\) represents the combined co-ordinates of the target and incident projectile.
The potential $V(r, \zeta)$ has a multiple expansion of the form

$$\sum_{\lambda} V_{\lambda}^{\zeta}(r, \zeta) Y_{\lambda}^{\zeta}(\zeta) i^{-\lambda}$$

and it is possible to obtain an expression for the $V_{\lambda}^{\zeta}(r, \zeta)$.

The idea of the collective model is to treat the nuclear excitation as an alteration in the vibrational mode of a nucleus that is spherical or in the alteration of the rotational properties of a nucleus that is an oblate or prolate spheriod, that is a deformed nucleus and is a completely macroscopic view. Generally it is assumed that the function $V(r, \zeta)$ follows the deformation of the nuclear surface. Under this assumption we write

$$V(r, \zeta) = V(r-R(\theta', \phi')),$$  \hspace{1cm} (1.7)

where the co-ordinates $(r, \theta', \phi')$ are the collective co-ordinates measured with respect to the axis of the collective nucleus. We can then expand $V(r-R)$ about a point $R_0$ in the form

$$V(r-R) = V(r-R_0) - \delta R \frac{d}{dr} V(r-R_0) + \ldots,$$ \hspace{1cm} (1.8)

The potential $V(r-R_0)$ in (1.8), that is, the first term in the Taylor series of $V(r, \zeta)$ is usually considered to correspond to the optical model potential for elastic scattering which we know is usually taken to be a spherically symmetric potential. The higher terms are then associated with inelastic scattering. To continue farther we assume the nucleus has a permanent deformation

$$R(\theta', \phi') = R_0 \left\{ 1 + \sum_{kq} \alpha_{kq} Y_{kq}(\theta', \phi') \right\},$$ \hspace{1cm} (1.9)

with respect to the fixed axis. In a similar way we may say that with respect to an arbitary fixed axis in space the surface of the nucleus is

$$R(\theta, \phi) = R_0 \left\{ 1 + \sum_{kv} \alpha_{kv} Y_{kv}(\theta, \phi) \right\},$$ \hspace{1cm} (1.10)
We do not go into full details here of the derivation of the inelastic amplitude as it may be found in several texts e.g. (J2) but it is worthwhile to explain the steps towards the result. Using the transformation properties of the spherical harmonics

\[ Y_{k\nu}(\theta,\phi) = \sum_{q} Y_{k\nu}(\theta',\phi') \mathcal{D}^{k}_{\nu q}(\alpha,\beta,\gamma), \]  

(1.11)

see (Bl), it is possible to obtain an expression for the \( \alpha'_{kq} \) in terms of the \( \alpha_{k\nu} \) and the rotation matrix element \( \mathcal{D}^{k}_{\nu q}(\alpha,\beta,\gamma) \) where \( (\alpha,\beta,\gamma) \) are the set of Euler angles taking the space fixed system \( \{ \theta, \phi \} \) to the body fixed system. We make a further simplification by assuming axial symmetry of the nucleus such that \( R(\theta,\phi) \) depends only on \( \theta \) and that

\[ \alpha'_{kq} + \alpha'_{k0} = \beta_{k} \]

The quantity \( \delta R \) may then be evaluated from its definition

\[ \delta R = R - R_{0} = R_{0} \sum_{k\nu} Y^{*}_{k\nu}(\theta,\phi) \beta_{k} \mathcal{D}^{k}_{\nu 0}(\alpha,\beta,\gamma), \]

(1.12)

This then gives the inelastic part of \( V(\xi,\zeta) \) as

\[ V_{\lambda\lambda}(r,\xi) = -i^{\ell} \beta_{k} R_{0} \frac{d}{dr} V(r-R_{0}) \mathcal{D}^{\ell}_{\lambda 0}(\alpha,\beta,\gamma), \]

(1.13)

Finally we compute the matrix element

\[ \int d\xi \phi^{*}_{f}(\xi) V(\xi,\zeta) \phi_{i}(\zeta), \]

(1.14)

\[ = \int d\xi \langle j_{f}^{m_{f}}|\xi> V(\xi,\zeta) \langle \zeta | j_{i}^{m_{i}}>, \]

(1.15)

\[ = \sum_{\lambda} \langle j_{i}^{m_{i}}| j_{f}^{m_{f}}| \xi> i^{-\lambda} Y^{*}_{\lambda \ell}(\hat{r}) \langle \lambda \xi | v_{\lambda \lambda} || j_{i}>, \]

(1.16)

by the Wigner Eckart theorem,

between \( K = 0 \) rotational bands and with \( j_{i} = 0, j_{f} = \lambda \), giving for the \( T \)-matrix element for inelastic scattering in the collective model as
What does this form tell us about the way the result depends on the nuclear surface? If we consider the case when we have the exact distorted waves clearly the radial part of the integral in 1.17) will depend strongly on the form of $dV/dr$, the derivative of the potential chosen. If $V(r)$ is the usual Woods-Saxon shape for the optical potential in elastic scattering then $dV/dr$ is sharply peaked at $r = R$ where

$$R = \frac{1}{3} r_0 A \text{ fm}$$

with $r_0$ having a value typically in the range 1.2-1.4 fm.

If the optical potential parameters are such that $dV/dr$ has a fairly sharp peak the contribution to the integral will arise mainly from those values of $r$ over which the value of the peak is significant, that is, in the region of the nuclear surface. This way of thinking is considered in the reference 11) in which the exact collective model expression for the inelastic $T$-matrix element is used, equation 1.17) but instead of using the exact distorted waves they use the model distorted waves, characterised by certain parameters that constrain or make these wavefunctions a good approximation to the exact distorted waves, that is, optical model wavefunctions at values of $r$ close to the nuclear radius. It is then hoped that any inaccuracies in these model distorted waves will not be significant due to the small value of $dV/dr$ away from the surface. The question that now presents itself is what form do we take for the model distorted waves?
CHAPTER 2

2.1 Background to the choice of wavefunctions.

The choice of the wavefunctions was influenced by the work of McCarthy and others that has appeared in several works \((M1,M2,M3,M4)\). A series of papers \((E1,M2,M3)\) considered the flux of particles inside and on the surface of the nucleus in typical elastic scattering of alpha particles. These lead to the observed feature in the flux that predicts a high probability of finding a particle in a particular region of space. This is called, by McCarthy, the focus region, its size and shape depending on the energy of the incoming projectile and the type of projectile considered. This focus is shown to be due to a converging effect on many different paths producing a region where they coincide and, consequently, a region having a high probability of finding a particle there. An extension to this was made by McCarthy and Pursey \((M4)\) which looked at the phase of the optical model wavefunctions. They conclude that this phase tends to be identical with the phase of the incident wave with an amplitude that tends to damp smoothly as one proceeds from the bright side of the nucleus, that is the side facing the incoming beam, to the dark side. The focus is superimposed on this general form. The features described above for the characteristics of the wavefunction are purely observed features coming from calculations using a good set of optical model parameters, that is a set describing well the elastic scattering.

2.2 Construction of the wavefunctions.

The form of the wavefunction used in this work has been constructed in such a way as to attempt to represent the exact optical model wavefunction at the surface of the nucleus as closely as possible. By the "exact optical model wavefunction" we mean that wavefunction constructed
using a set of potential parameters that reproduce the elastic scattering cross section.

It is shown in E1, M2, M3, that at high energies the difference between the quantal flux given by

\[ j(x) = \frac{\hbar}{2im} \left\{ \psi^*(x)\nabla\psi(x) - \psi(x)\nabla\psi^*(x) \right\}, \quad (2.1) \]

and the classical flux is very small in the low partial waves. This is the behaviour we would expect in high energy cases. For the type of energy range we are considering here we cannot use the results of classical calculations but it would be to our advantage when it comes to the calculation of the T-matrix element if we could use some sort of semi-classical argument for the form of the wavefunction. The results of McCarthy's work M2 for the \( \alpha \) particle flux, the absolute flux and the divergence of the flux are reproduced in the figures (Fig 1-3).

From these we can see the following features that have been used as a guide to the construction of the model wavefunctions.

a) At any given radial distance the value of the flux increases as the angle between \( \hat{k} \) and \( \hat{r} \) increases.

b) For protons, at any radius, the absolute flux is larger than that for alpha particles at the same angle. We would expect this because alpha particles are more strongly absorbed than protons.

c) The divergence of the flux is largest at large angles for any given radius.

d) For protons, the maxima in the divergence of the flux occurs nearer to the centre of the nucleus and is smaller in magnitude than the divergence of the flux for alpha particles at their maximum. (see b above).
FIGURES (1, 2, 3).

Elastic scattering of 18 MeV alpha particles and 72 MeV protons from 18 Ar. Reproduced from M2).

Figure 1) This shows the alpha particle flux in the potential $\alpha_1$. The arrows represent the flux vector and the dashed lines represent the divergence of the flux. The values are arbitrary and the 90% and 10% radii of the potential are also indicated.

Figure 2) Shows the flux from two sets of potentials that produce agreement with elastic scattering experiments ($\alpha_1, \alpha_2$) and one set for proton scattering (p) plotted against r for different values of $\theta$.

Figure 3) Here the divergence of the flux is shown calculated from the flux as in figure 2).
--- DIVERGENCE CONTOURS FIG(1)
$f = 0.9 - f = 0.1$

$\alpha_1$

$\alpha_2$

$p$

$|j|$

$179^\circ$

$139^\circ$

$81^\circ$

$0$

$5$

$10$

$r(f_m)$

FIG (2)
\( f = 0.9 - f = 0.1 \)

[Diagram of curves with angles labeled 179°, 139°, and 81°, with axes labeled \( \alpha_1 \), \( \alpha_2 \), and \( \gamma \).]
The region of large flux at small angles is associated with the region we have called the focus. From ref M3 it is also observed that as the energy of the projectile increases so the position of the focus is shifted outwards, away from the centre of the nucleus and shows a decrease in magnitude. This is due to the fact that the larger the energy of the particle, the more difference there is between any two particular paths acted on by the potential. The overall effect is a less distinct focus. We may compare this to spherical aberration of light.

We now know what general features we must look for in our 'model' wavefunction, that is the general features of the optical model wavefunctions discussed above. This has been looked at by McCarthy and Pursey (M4). They have found that in the case of strongly absorbed particles such as alpha particles the focus region is attenuated to a large extent. This is due to the imaginary part of the optical potential, the absorptive part. The attenuation is so strong that McCarthy and Pursey say that the focus may be neglected. This is rather a useful approximation to make because it reduces the number of unknown parameters that we will have to incorporate and eliminates the need to know the position of the focus which we have seen varies with the energy of the incident particle.

The form of the wavefunction chosen by McCarthy and Pursey is of the form M4),

$$\psi(r) = Ne^{iD \cdot \mathbf{r}}$$  \hspace{1cm} (2.2)

where $N$ is a real normalisation factor and $D$ is a complex number to be written in the form

$$D = \beta + i\alpha$$  \hspace{1cm} (2.3)

where $\alpha$, $\beta$ are real.
We can write the wavefunction as

\[ \psi(r) = N e^{-\alpha k \cdot r} e^{i \beta k \cdot r} \quad (2.4) \]

so the phase of \( \psi(r) \) is

\[ \phi = \beta k \cdot r \quad (2.5) \]

This indicates that if we plot a graph of \( \phi \) against \( \cos \theta \) or \( \theta \) for a fixed value of \( r \) we should obtain a straight line. This linear relationship of the phase with angle obtained by this simple model is shown to exist approximately in one case discussed in (M4) for 40 MeV alpha particles on \( ^7 \text{Al} \), as calculated from the optical model. In (M4) it is pointed out that the variation in phase may be expressed roughly in the form

\[ \phi = A r \hat{k} \cdot \hat{r} + B \quad (2.6) \]

\[ = A' \hat{k} \cdot \hat{r} + B \quad (2.7) \]

which is of the form in (2.5). The constant \( B \) may be taken as some multiple of \( 2 \pi \) to make the phase positive for all angles. The main points to make note of from McCarthy's work may then be summarised as follows:

a) If the interaction is localised on the surface of the nucleus only the angular dependence of the wavefunction is of importance. The use of the wavefunction (2.2) must be restricted to some value \( R_0 \) which describes the position of the localised interaction. It cannot therefore be used in an overlap integral for example where it would have to be a good approximation to the optical model wavefunction at all values of \( r \).

b) The dependence of \( |\psi| \) on angle is similar at all radii near to the surface radius.

c) The intensity is greatest on the front side of the nucleus, facing the beam and falls off towards the back except at the focus.
d) For strongly absorbed particles we can neglect the focus term.

Whether or not point d) above is valid will be discussed later.

We are then left with a simple picture of the wavefunction containing a parameter that will control the ratio of maxima to minima in the cross section and a parameter that will control the rate at which the cross section decreases.

2.3 Reactions chosen for investigation

The reactions chosen for investigation of $(\alpha,\alpha)$ scattering were

$^{88}\text{Sr} (\alpha,\alpha) ^{88}\text{Sr}$ at 42.0 MeV Reaction a)

$^{42}\text{Ca} (\alpha,\alpha) ^{42}\text{Ca}$ at 30.5 MeV Reaction b)

both being selected for different reasons. Reaction a) is the elastic case of the reaction $^{88}\text{Sr} (\alpha,\alpha) ^{88}\text{Sr}$ studied by Janus and McCarthy in J1) using a collective model. The parameters used, quite successfully, are tabulated in J1) and we use them here directly to test the consistency in the parameters. That is if they can reproduce both the elastic and inelastic scattering cross sections. However there seems to be no data or calculations concerning the optical model wavefunctions for this reaction available to study the relationship between the set of parameters for the 'model' wavefunction and the exact optical model wavefunction. We therefore choose reaction b) as an example of a reaction where a calculation of the modulus of the optical model wavefunction has been made. This is from Morgan M5), Fig 4. Here we see presented the plot of $|\psi|$ against $r$. We clearly see the focus effect described by McCarthy. This time however we have no indication as to the values of the parameters we should use in the 'model' wavefunction for this reaction. This will give us then an indication whether it is a simple matter to estimate these knowing the optical model parameters that reproduce elastic scattering.
**TABLE 1**

<table>
<thead>
<tr>
<th>Reaction</th>
<th>E (MeV)</th>
<th>V (MeV)</th>
<th>W (MeV)</th>
<th>r (fm)</th>
<th>a (fm)</th>
<th>r' (fm)</th>
<th>a' (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ref 1) a</td>
<td>42.0</td>
<td>-50.0</td>
<td>-20.0</td>
<td>1.582</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ref 2) b</td>
<td>30.5</td>
<td>-54.6</td>
<td>-16.3</td>
<td>1.622</td>
<td>0.599</td>
<td>1.703</td>
<td>0.604</td>
</tr>
</tbody>
</table>

ref 1) Alster, Shreve and Peterson  
ref 2) Morgan

**FIGURE 4)**

Reproduced from M5)

Graph showing the variation of $|\psi|$ with $\theta$ along the axis for the elastic scattering $^{42}\text{Ca} (\alpha, \alpha) ^{42}\text{Ca}$ at 30.5 MeV.
We are at this point effectively testing our 'model' wavefunction against the exact optical model wavefunction, it is therefore advantageous to have the results of elastic scattering for these two reactions available. These calculations were made using an existing optical model code using the optical model parameters listed in table 1. See Figures 5) and 6).

2.4 Relationship of this approximation with other approximations.

In the W.K.B approximation the Schrödinger equation

\[
\frac{i \hbar \delta \psi}{\delta t} = \frac{\hbar^2}{2\mu} \nabla^2 \psi + V(r) \psi
\]

is solved by assuming the wavefunction is of the form

\[
\psi(r,t) = A e^{iW(r,t)/\hbar}
\]

which leads to the following equation for \( W(r,t) \)

\[
\frac{\delta W}{\delta t} + \frac{1}{2\mu} (\nabla W)^2 + V - \frac{i\hbar}{2\mu} \nabla^2 W = 0
\]

If we now set

\[
W(r,t) = S(r) - Et
\]

and

\[
\psi(r,t) = U(r) e^{-iEt/\hbar}
\]

we have

\[
U(r) = A e^{iS(r)/\hbar}
\]

So that \( S(r) \) satisfies the equation

\[
\frac{1}{2\mu} (\nabla S)^2 - (E-V) - \frac{i\hbar}{2\mu} \nabla^2 S = 0
\]

If now \( s(r) \) is slowly varying with \( r \) we may set

\[
\nabla s(r) = \hbar k(r) = \left\{ \frac{2\mu}{E} (E-V(r)) \right\}^{\frac{1}{2}}
\]

or

\[
k(r) = k \left\{ \frac{1-V(r)}{E} \right\}^{\frac{1}{2}}
\]
where \[ k^2 = \frac{2\mu E}{\hbar^2} \]

or \[ k(r) = k\beta(r) \]

If \( V(r) \) is sufficiently slowly varying at the region of the nuclear surface we may take \( \beta(r) \) as a constant value in this region so

\[ \beta \approx \left( 1 - \frac{V}{E} \right)^{\frac{1}{2}} \]

This is the form chosen for the value of \( \beta \) in our model wavefunction in the first instance. We must bear in mind that we are not approximating \( k(r) \) for all values of \( r \) as a constant but only for those values of \( r \) close to the nuclear surface where \( r = R_0 \).
3.1 The Scattering amplitude

We here consider the scattering amplitude for scattering of alpha particles from a heavy nucleus. The general form may be written exactly as \((S1,J2)\)

\[ f(\theta) = f_c(\theta) + f_N(\theta), \quad (3.1) \]

where \(f_c(\theta)\) is the scattering amplitude due to the long range coulomb interaction \(V_c(r)\) and is given by

\[ f_c(\theta) = \frac{n}{k(1 - \cos\theta)} e^{-i\eta n \sin^2\theta} + 2i\sigma_0 \quad (3.2) \]

Here \(\eta\) is the coulomb parameter

\[ \eta = \frac{\mu Z_1 Z_2 e^2}{\pi^2 k}, \quad (3.3) \]

\[ k^2 = \frac{2\mu E}{\pi^2}, \]

\(k\) is the momentum of the alpha particle with \(\mu\) the reduced mass of the system, \(Z_1\) and \(Z_2\) are the charges on the alpha particle and target nucleus respectively, \(\theta\) is the scattering angle and \(\sigma_0\) is the coulomb phase shift in the \(l = 0\) partial wave given by

\[ e^{2i\sigma_0} = \frac{\Gamma(l + in)}{\Gamma(l - in)}, \quad (3.4) \]

\(f_N(\theta)\) is the scattering amplitude due to the short range nuclear potential \(V_N(r)\) in the presence of \(V_c(r)\) and is given in terms of a sum over partial amplitudes as

\[ f_N(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)e^{2i\sigma_0} e^{i\delta_2} \sin^{2l} P_2(\cos\theta), \quad (3.5) \]
In this equation $\delta_\ell$ is the phase shift due to $V_N(r)$ in the presence of $V_c(r)$ in the $\ell$th partial wave. We shall refer to $f_c(\theta)$ as the coulomb scattering amplitude and $f_N(\theta)$ as the nuclear scattering amplitude.

For the method to be used in this work it is not convenient to express $f_N(\theta)$ in the form (3.5) but rather in integral form. We can do this by writing the total potential in the form

$$V(r) = V_c(r) + V_N(r), \quad (3.6)$$

assuming both $V_c$ and $V_N$ to be spherically symmetric.

Now let $v_\ell(r)$ be the solution of the $\ell$th radial Schrodinger equation in the presence of $V_c(r)$ only, that is $v_\ell(r)$ satisfies the equation

$$v''_\ell(r) + \left\{ k^2 - \frac{\ell(\ell+1)}{r^2} - U_c(r) \right\} v_\ell(r) = 0, \quad (3.7)$$

where $U_c(r) = \frac{2\mu}{\pi^2} V_c(r)$,

and with $v_\ell(r)$ normalised such that

$$v_\ell(r) \longrightarrow e^{i\sigma_\ell} \sin(kr - \frac{\ell\pi}{2} + \sigma_\ell - \eta\ell n2kr), \quad (3.8)$$

Now let $w_\ell(r)$ be the solution of the $\ell$th radial Schrodinger equation in the presence of both $V_c(r)$ and $V_N(r)$, so that $w_\ell(r)$ satisfies the equation

$$w''_\ell(r) + \left\{ k^2 - \frac{\ell(\ell+1)}{r^2} - U_c(r) - U_N(r) \right\} w_\ell(r) = 0, \quad (3.9)$$

normalised such that

$$w_\ell(r) \longrightarrow e^{i\sigma_\ell} \sin(kr - \frac{\ell\pi}{2} + \delta_\ell - \eta\ell n2kr), \quad (3.10)$$

where

$$\sigma_\ell^T = \sigma_\ell + \delta_\ell$$
Now multiplying equation (3.9) by \( v_\ell(r) \) and equation (3.7) by \( w_\ell(r) \) and subtracting leaves us with

\[
v_\ell(r) w_\ell''(r) - w_\ell(r) v_\ell''(r) = v_\ell(r) U_N(r) w_\ell(r),
\]

(3.11)
or

\[
\frac{d}{dr} \left\{ v_\ell(r) w_\ell'(r) - w_\ell(r) v_\ell'(r) \right\} = v_\ell(r) U_N(r) w_\ell(r),
\]

(3.12)

Now we integrate both sides from \( r = 0 \) to \( r = R \) where \( R \) is some arbitrary radial distance

\[
\int_0^R dr \left[ v_\ell(r) w_\ell'(r) - w_\ell(r) v_\ell'(r) \right] = \int_0^R dr v_\ell(r) U_N(r) w_\ell(r),
\]

(3.13)

We now take the limit as \( R \to \infty \) and use the asymptotic forms (3.8) and (3.10) for the radial wavefunctions

\[
\int_0^\infty dr v_\ell(r) U_N(r) w_\ell(r) = ke^{i\sigma_\ell} e^{i\delta_S} \sin(\sigma_\ell - \delta_\ell),
\]

Using \( \delta_\ell = \delta_S^T - \sigma_\ell \) we can write

\[
e^{2i\delta_\ell} e^{i\delta_\ell} \sin\delta_\ell = -\frac{1}{k} \int_0^\infty dr v_\ell(r) U_N(r) w_\ell(r),
\]

(3.14)

We can now insert the integral on the right in equation (3.14) into the expression for the nuclear scattering amplitude, (3.5)

\[
f_N(\theta) = \frac{1}{k^2} \sum_{\ell} (2\ell + 1) P_\ell(\cos\theta) \int_0^\infty dr v_\ell(r) U_N(r) w_\ell(r),
\]
Using the addition theorem of spherical harmonics (Bl) this can be written as

\[ f_N(\theta) = -\frac{1}{k^2} \sum_{\ell, m} \int_0^\infty dr \, v_{\ell m}(r) u_N(r) \omega_{\ell m}(r) 4\pi Y_{\ell m}^* \left( \hat{k}_1 \right) Y_{\ell m} \left( \hat{k}_2 \right), \]

\[ = -\frac{4\pi}{k^2} \sum_{\ell, m} \int_0^\infty dr \, v_{\ell m}(r) u_N(r) Y_{\ell m} \left( \hat{k}_2 \right) Y_{\ell m} \left( \hat{k}_1 \right) \delta_{\ell \ell'} \delta_{m m'}, \]

By the orthogonality of the spherical harmonics over the solid angle this may be written as

\[ f_N(\theta) = -\frac{4\pi}{k^2} \int_0^\infty dr \int d\Omega_{\hat{k}} \sum_{\ell, m} i^{-\ell'} v_{\ell m}(r) Y_{\ell m} \left( \hat{k}_1 \right) U_N(r) i^\ell Y_{\ell m} \left( \hat{k}_2 \right), \]

\[ (r) Y_{\ell m}^* \left( \hat{k}_1 \right) Y_{\ell m}^* \left( \hat{k}_2 \right), \]

\[ = -\frac{4\pi}{k^2} \frac{2\mu}{\hbar^2} \int_0^\infty dr \int d\Omega_{\hat{k}} \left\{ \sum_{\ell, m} i^{-\ell'} v_{\ell m}(r) Y_{\ell m} \left( \hat{k}_1 \right) U_N(r) i^\ell Y_{\ell m} \left( \hat{k}_2 \right) \right\} \]

\[ V_N(r) \left\{ \sum_{\ell, m} i^\ell Y_{\ell m} \left( \hat{k}_1 \right) Y_{\ell m} \left( \hat{k}_2 \right) \right\}, \quad (3.15) \]

Bearing in mind the definition of the \( w_{\ell}(r) \) and the \( v_{\ell}(r) \) we can make the identification that the first term in the integral is the partial wave expansion of the incoming distorted wave while the second term in the integrand is the partial wave expansion of the outgoing wave, the eigenfunction of the full Schrödinger equation. That is,

\[ \chi\left(-\right)^* \left( r \right) = \frac{4\pi}{k r} \sum_{\ell, m'} i^{-\ell'} v_{\ell m'}(r) Y_{\ell m'}^* \left( \hat{k}_1 \right) Y_{\ell m} \left( \hat{k}_2 \right), \]

and

\[ \psi\left(+\right) \left( r \right) = \frac{4\pi}{k r} \sum_{\ell, m} i^\ell v_{\ell m}(r) Y_{\ell m} \left( \hat{k}_1 \right) Y_{\ell m} \left( \hat{k}_2 \right), \]
where

\[(T + V_c + V_N)\psi^{(+)} = E\psi^{(+)}, \tag{3.16}\]
\[(T + V)\chi^{(+)} = E\chi^{(+)}, \tag{3.17}\]

we can therefore write equation (3.15) as

\[f_N(\theta) = \frac{-\hbar}{2\pi\hbar^2} \int d\hat{r} \chi^{(-)*}(\hat{r}) V_N(\hat{r}) \psi^{(+)}(\hat{r}), \tag{3.18}\]

So that knowing the wavefunctions this expression may be evaluated and is the form we choose to work with.

3.2 Conversion of the scattering amplitude to a surface integral

As has been stressed earlier the validity of the type of wavefunction, equation 2.2), that we are to use is only good at one particular radius which we are identifying as the nuclear surface. We cannot therefore use equation (3.18) as it stands. In order to proceed we make use of Greens Theorem, transforming the volume integral (3.18) to an integral over a surface. We do this as follows.

We know that \(\psi^{(+)}\) is a solution of equation 3.16) which can be written as

\[\left\{ -\frac{\hbar^2}{2\mu} \frac{\nabla^2}{r} + V_c(r) + V_N(r) \right\} \psi^{(+)}(r) = E\psi^{(+)}(r), \tag{3.19}\]

so adding and subtracting \((V_c(r)-E)\) we have

\[f_N(\theta) = \frac{-\hbar}{2\pi\hbar^2} \left\{ \int d\hat{r} \chi^{(-)*}(\hat{r}) (V_N(\hat{r})+V_c(\hat{r})-E) \psi^{(+)}(\hat{r}) \right. \]
\[\left. - \int d\hat{r} \chi^{(-)*}(\hat{r}) (V_c(\hat{r})-E) \psi^{(+)}(\hat{r}) \right\}, \]

Now using equation (3.19) we have
\[ f_N(\theta) = \frac{-\mu}{2\pi \hbar^2} \left\{ - \int \frac{dr}{r} \chi^{(-)*}(r) \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} \psi^{(+)}(r) \right. \\
\left. - \int \frac{dr}{r} \left[ \psi^{(+)*}(r) \left( V_C(r) - E \right) \chi^{(-)}(r) \right]^* \right\}, \quad (3.20) \]

But \( \chi^{(-)}(r) \) is a solution of
\[ \left( \frac{-\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) \right) \chi^{(-)}(r) = E \chi^{(-)}(r), \]
so that equation (3.20) becomes
\[ f_N(\theta) = -\frac{1}{4\pi} \left\{ \int \frac{dr}{r} \chi^{(-)*}(r) \frac{d^2}{dr^2} \psi^{(+)}(r) - \int \frac{dr}{r} \psi^{(+)}(r) \frac{d^2}{dr^2} \chi^{(-)*}(r) \right\}, \]
\[ = -\frac{1}{4\pi} \int d\mathbf{r} \cdot \left\{ \chi^{(-)*}(r) \frac{d}{dr} \psi^{(+)}(r) - \psi^{(+)}(r) \frac{d}{dr} \chi^{(-)*}(r) \right\}. \]

Now using Green's Theorem this may be rewritten as
\[ f_N(\theta) = -\frac{1}{4\pi} \int_s ds \cdot \left\{ \chi^{(-)*}(r) \frac{d}{dr} \psi^{(+)}(r) - \psi^{(+)}(r) \frac{d}{dr} \chi^{(-)*}(r) \right\}, \quad (3.21) \]

where \( s \) is the bounding surface of the volume throughout which the integral in equation (3.18) is evaluated. This then implies that the surface should be at infinity to incorporate the total contribution to the integral. To overcome this difficulty consider the radial part of the integral of equation (3.18) which may be written
\[ \int_0^\infty dr \ r^2 \chi^{(-)*}(r) V_N(r) \psi^{(+)}(r) \int_0^{R_0} dr \ r^2 \chi^{(-)*}(r) V_N(r) \psi^{(+)}(r) \]
\[ = \int_0^\infty dr \ r^2 \chi^{(-)*}(r) V_N(r) \psi^{(+)}(r) + \int_0^{R_0} dr \ r^2 \chi^{(-)*}(r) V_N(r) \psi^{(+)}(r), \quad (3.22) \]
so that if we now assume that the potential \( V_N(r) \) is of a form such that for values of \( r \) greater than \( R_0 \), the nuclear surface radius, the second term in the last equation may be neglected we may say that it is reasonable to evaluate equation 3.21) at the point \( r = R_0 \). We are assuming therefore that

\[
V_N(r) = V_{ws}(r) \quad r \leq R_0
\]

\[
V_N(r) = 0 \quad r > R_0
\]

where \( V_{ws}(r) \) is the usual Woods-Saxon form for the optical model potential. In this sense we are considering \( V_N(r) \) to have a sharp boundary.

3.3 Calculation of the nuclear scattering amplitude

We are now in a position to evaluate the expression for the nuclear part of the scattering amplitude but before we proceed it is worth considering the normalisation we place on \( \psi(+) (r) \), that is \( N \) in equation 2.2). It is found to be useful if we choose to normalise \( \psi(+) (r) \) to a plane wave at \( r = R_0 \), that is

\[
\psi(+) (r) = e^{-\alpha k_1 R_0} e^{iD k_1 \cdot r}, \quad (3.23)
\]

It is possible that we could have selected some other point but our choice reduces the number of parameters involved.

Further to this we need to know the wavefunction \( \chi(-)^* \). Strictly speaking this should be a coulomb distorted wave but use of this functional form would not lend itself to a simple analytic treatment of \( f_N(\theta) \). Instead we choose in the first instance that \( \chi(-) (r) \) shall be a plane wave

\[
\chi(-) (r) = e^{i k_2 \cdot r}, \quad (3.24)
\]

with \( |k_2| \) real.
If we now substitute our expression for \( \psi^+(x) \) and \( \chi^-(x) \) into equation 3.21) we find

\[
f_N(\theta) = -\frac{NR_0}{4\pi} \int d\Omega \frac{e^{i\mathbf{q} \cdot \mathbf{r}}}{r} \cdot \frac{\mathbf{K} \cdot \mathbf{r}}{r}, \tag{3.25}
\]

where we have defined

\[
\mathbf{K} = Dk_1 + k_2,
\]

\[
\mathbf{q} = Dk_1 - k_2,
\]

so that \( \mathbf{q} \) may be thought of as a complex momentum transfer and \( \mathbf{K} \) as a complex total momentum. We now write

\[
f_N(\theta) = -\frac{NR_0}{4\pi} \mathbf{K} \cdot \mathbf{q} \int d\Omega \frac{e^{i\mathbf{q} \cdot \mathbf{r}}}{r},
\]

\[
= -\frac{NR_0}{4\pi} \mathbf{K} \cdot \mathbf{q} \int d\Omega \sum_{\ell, m} \sqrt{4\pi j_0(qR_0)} Y_{\ell m}^*(\hat{\mathbf{q}}) Y_{\ell m}^*(\hat{\mathbf{r}}),
\]

Now \( Y_{00}^*(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}} \),

so this may be written as

\[
f_N(\theta) = -\frac{NR_0}{4\pi} \mathbf{K} \cdot \mathbf{q} \int d\Omega \sum_{\ell, m} \sqrt{4\pi j_0(qR_0)} \sqrt{4\pi} Y_{\ell m}^*(\hat{\mathbf{q}}) Y_{\ell m}^*(\hat{\mathbf{r}}),
\]

The integral here gives \( \delta_{\ell 0} \delta_{m0} \) so on summing over \( \ell, m \) we have

\[
f_N(\theta) = -NR_0 \mathbf{K} \cdot \mathbf{q} j_0(qR_0) Y_{00}^*(\hat{\mathbf{q}}),
\]

\[
= -NR_0 \mathbf{K} \cdot \mathbf{q} j_0(qR_0),
\]
Using $j_0(qR_0) = \frac{\sin qR_0}{qR_0}$ this can be written as

$$f_N(\theta) = -\frac{NR^2_0}{q^2} Kq \left\{ \frac{\cos qR_0}{qR_0} - \frac{\sin qR_0}{(qR_0)^2} \right\}, \quad (3.26)$$

with $Kq = k^2_2 \left( \frac{k_1^2D^2}{k_2^2} - 1 \right)$.

Equation (3.26) may then be rewritten as

$$f_N(\theta) = -\frac{NR^2_0 k^2_2(f^2 - 1)}{q^2} \left\{ \frac{\cos qR_0}{qR_0} - \frac{\sin qR_0}{qR_0} \right\}, \quad (3.27)$$

where we have defined

$$q^2 = k^2_2(1+f^2 - 2f \cos \theta),$$

$$f = \frac{k_1}{k_2}$$

So that the complete scattering amplitude is given by the equation (3.1)

$$f(\theta) = f_c(\theta) + f_N(\theta) \quad (3.28)$$

with $f_c(\theta)$ given by equation (3.2) and $f_N(\theta)$ given by the equation (3.27).

The form of $f(\theta)$ is thus obtained as an analytic function of the quantities concerned. The cross section may then be found by application of

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \times 10 \text{ mbsr}^{-1}.$$
4.1 Reaction parameters

What we now must do is to choose what we think will be suitable values for the reaction parameters $\alpha$, $\beta$ and $R_0$. We have said we are following McCarthy's values for these in reaction a) however we must consider the factors leading to the values chosen. Already a rough estimate of $\beta$ may be obtained by using the local W.K.B. prediction

$$\beta = \left\{ 1 - \frac{(V_N(R_0) + V_C(R_0))}{E} \right\}^\frac{1}{2} \tag{4.1}$$

which we find useful as a guide to the value of $\beta$ but do not stick firmly to it.

The parameter $\alpha$ is rather harder to estimate and it was decided to take the value as given in J1) for reaction a) as a starting value, again varying it if need be. What we may say is that $\alpha$ will give a measure of the absorption of the beam and as such will be relatively large if $W$, the imaginary part of the optical potential is large and vice versa. We can refer to $\alpha$ as an absorption parameter.

Finally the question of what value to take for $R_0$ must be asked. We again follow the work of J1) by choosing

$$R = (1.523 \ A^\frac{1}{3} + 2.14) \ \text{fm} \tag{4.2}$$

which is the general form for the strong absorption radius J2). This radius is given more fundamentally by

$$R_0 = \frac{1}{k} \left\{ \eta \pm (\eta^2 + L(L+1))^{\frac{1}{2}} \right\}$$

with the value of the angular momentum $L$ set equal to $L_{\frac{1}{2}}$ defined from the expression

$$\text{Re} \ \eta(L_{\frac{1}{2}}) = \frac{1}{2},$$

that is where the real part of the reflection coefficient has the value $\frac{1}{2}$. 
In the case of the three parameters we must use these values as a guide and assume we are free to vary each within bounds that keep them physically acceptable. Also note in (11) \( \beta \) is defined differently to our \( \beta \), they are related by

\[
\beta_{J1} + 1 = \beta \quad \beta_{J1 < 0}
\]

Using the ideas above we can estimate a value for all the parameters in both reactions except that of \( \alpha \) in reaction b). If we consider the work of Morgan (M5), specifically the plots of \(|\psi^+|\) for reaction b) it is possible to make an estimate of \( \alpha \). The plots are made by taking the exact solution of the radial Schrodinger equation for each \( \ell \) value and forming the sum

\[
\psi^{(+)}(r,\theta) = \sum_{\ell} i^\ell (2\ell + 1)f_\ell(kr)P_\ell(\cos \theta), \quad (4.3)
\]

and have been shown in Fig 4. From this it can be seen that at \( r = R_0 \) along the axis the value of \(|\psi^+|\) is roughly 0.38 on the 'dark side' of the nucleus. So assuming the form of the McCarthy 'model' wave-function we have

\[
|\psi^+| = e^{-2\alpha k R_0}, \quad (4.4)
\]

which provides a value for \( \alpha \) knowing \(|\psi^{(+)}|\) at \( r = R_0 \). Table 2) gives the sets of parameters used as initial estimates from which it is hoped better values may be obtained.

It is noticed from (M5) and is in fact well known that the optical model parameters that produce agreement with elastic scattering data are not unique. It is seen that for different potentials the graphs of \(|\psi^{(+)}|\) differ largely inside the nucleus but all tend to produce roughly the same value on the nuclear surface. It is the focus that appears to be effected the most. This implies that the value of
Table showing parameters used to calculate the elastic cross section.

<table>
<thead>
<tr>
<th>Target</th>
<th>$k(\text{fm}^{-1})$</th>
<th>$R (\text{fm})$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
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<tr>
<td>Sr</td>
<td>2.68</td>
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<td>2.21</td>
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</table>
a will not change significantly with a change in the optical model parameters. In addition to this β is also influenced by the optical model parameters but here again it is found that changes in β are not very large. This is due to the fact that if the depth of the potential is altered, in order to fit the elastic scattering a change in the radius of the potential is also needed, these two effects roughly cancel to produce an almost constant value of $V_N(r)$ at $r = R_0$.

4.2 Results

Initially calculations were made based on the set of parameters shown previously in table 2). However these were found to be very poor and considerations were made to improve on this. The idea was to treat $\chi^(-)(r)$ a little more realistically by assuming it to take the form of

$$\chi^(-)(r) = e^{i\beta_2 k_2 \cdot r}, \quad (4.5)$$

where $\beta_2$ is a real number playing a similar role to β, that is to allow for a change in momentum in the region of the nucleus. The wavefunction should be a coulomb distorted wave so that $\beta_2$ can be obtained by the use of an expression similar to 4.1),

$$\beta_2 = \left\{ 1 - \frac{V_C}{E} (R_0) \right\}^{\frac{1}{2}}, \quad (4.6)$$

We consider this to be fixed at a value determined by this equation.

Figures 5) and 6) show the results obtained using these parameters and the results produced by an optical model code using the sets of parameters given in table 1). We see that the results here are really very poor, there is a complete lack of structure to the graphs coupled
FIGURES 5 and 6

Figure 5  Elastic scattering $^{88}$Sr ($\alpha$,\,$\alpha$) $^{88}$Sr at 42.0 MeV.

   a) Optical Model calculation
   b) Calculation using equation 3.28 with parameters from table 2 and with $\beta_2 = 0.85$

Figure 6  Elastic scattering $^{42}$Ca ($\alpha$,\,$\alpha$) $^{42}$Ca at 30.5 MeV.

   a) Optical Model calculation
   b) Calculation using equation 3.28 with parameters from table 2 and with $\beta_2 = 0.83$
$^{88}\text{Sr}$ 42 MeV

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta^\circ$ (cm)

FIG(5)
$^{42}_{\text{Ca}}$ 30.5 MeV

$\frac{d\sigma}{d\Omega}$ (mb/sr)

$\theta^\circ$ (c.m.)

FIG(6)
with the fact that the decrease towards large angles is too slow consequently the overall magnitude is too large.

The question to be asked now is what is the reason for the poor fit obtained in the above calculation? To investigate this the optical model code was calculated in two different ways. Firstly by calculating the cross section obtained by setting the charge of the $\alpha$ - particle to zero. This will give the cross section without any coulomb effects, and secondly by simply not adding $f_c(\theta)$ to the nuclear amplitude $f_N(\theta)$.

So the first case is equivalent to

$$f(\theta) = f_N^{(1)}(\theta)$$

where

$$f_N^{(1)}(\theta) = -\frac{-\mu}{2\pi\hbar^2} \int d \tau e^{-ik_2 \cdot \tau} v_N(\tau) \psi^{(1)}(\tau),$$

in which $\psi^{(1)}(\tau)$ would be the solution of

$$\left\{ \frac{\hbar^2}{2\mu} \psi^2 + V_N(\tau) \right\} \psi^{(1)}(\tau) = E\psi^{(1)}(\tau),$$

while in the second case we take

$$f(\theta) = f_N(\theta),$$

with $f_N(\theta)$ equivalent to equation 3.18)

See figs 7) and 8)
FIGURES 7) and 8)

Figure 7)

a) Graph showing the differential cross section obtained from the optical model code for reaction a) when the charge on the alpha particle is assumed to be zero. That is the coulomb effects are absent

b) As above for reaction b)

Figure 7)

a) Graph showing the differential cross section obtained by omitting the coulomb amplitude only

b) As above for reaction b)
We see here that the difference between simply switching off \( V_c(r) \), that is, setting \( Z = 0 \) and just omitting \( f_c(\theta) \) from the calculation is quite apparent. We therefore must make an attempt to reproduce the results shown in Fig 7) and then add in to it the coulomb amplitude. This can be done by using the parameters of table 2) and calculating the cross section from

\[
\frac{d\sigma}{d\Omega} = |f_N(\theta)|^2 \times 10 \text{ mbsr}^{-1}
\]

with \( f_N(\theta) \) given by equation 3.27). The results of this calculation are shown in Fig 9) for the two reactions under consideration. It is clear from this that even though the parameters used to calculate these results were estimated on the assumption that \( V_c(r) \) was present the results are more characteristic of Fig 8) in which the coulomb potential has been switched off. We now ask what type of results do we get if we make our \( \beta \) and \( \beta' \) correspond to \( V_c(r) = 0 \)? These are shown in Fig 10) and 11).

There is obviously a poor fit here although the general shape of the curves produced from our calculation indicates that by searching on the parameters a better fit could be obtained. This idea was followed resulting in the best fits shown in Figs 12) and 13). A table of the best fit parameters is also displayed (Table 4). The parameter labelled \( q - \) factor has been introduced because it was found to be very difficult to obtain any sort of matching in the position of the maxima and minima. This is due to the fact that as \( \alpha, \beta, \beta' \) are altered in an attempt to get a good comparison in magnitude they also effect the terms \( \cos qR_0 \) and \( \sin qR_0 \) in equation 3.27). Now these terms are the over­riding influence on the positions of the maxima and minima, hence changing the above three parameters by any significant amount also changes these positions. The \( q - \) factor has been introduced by letting

\[ q \to (q \text{ factor}) q. \]
to try to compensate for this change. Even so it was found to be
impossible to fit the positions exactly.

Before considering any conclusions that can be drawn from these
sets of results we look at the following figures which indicate the
influence that the parameters $\alpha$ and $\beta$ have over the resulting cross
section. Figure 14) shows the effect of a change in $\beta$ while figure 15)
shows the effect of a change in $\alpha$. Finally figures 16) and 17) show
the resulting cross section obtained by using the set of best fit para-
ters and switching on the coulomb potential.

4.3 **Summary**

We can say that the cross sections produced by this procedure are
very poor when compared to those produced by conventional optical model
calculations. They appear to be lacking in structure and detail over
the range and have a magnitude which at some angles is at least two
orders of magnitude too large. A limited amount of success has been
achieved by looking at the case when the coulomb interaction is assumed
to be zero but the parameter sets are inconsistent with those expected
from approximate methods of calculating their values, and with the
values used in J1) for the equivalent inelastic scattering.

The influence of the parameters $\alpha$, $\beta$ and $\beta_2$ on the shape of the
resulting cross section are consistent with that predicted by McCarthy.
It is clearly seen from figure 14) that an increase in $\beta$ increases the
overall magnitude of the cross section and has a slight influence on
the position of the maxima and minima especially at the larger angles.
It is noted that the effect of the influence on the magnitude is
roughly uniform over all the angular range, that is no one angle is
effected more than any other. Figure 15) shows that $\alpha$ indeed does have
the role of a damping parameter, the oscillations becoming smaller
towards larger angles. The influence exerted by these can be seen by
looking at the form of the expression for $f_N(\theta)$, see Appendix.
FIGURE 9)

a) Calculations made using equation 3.27) for reaction a)
   with the parameters as in table 2) and with $\beta_2 = 0.83$

b) Calculations made using equation 3.27) for reaction b)
   with the parameters as in table 2) and with $\beta_2 = 0.85$
FIGURES 10 and 11

Figure 10  Reaction a)

a) Optical model calculation with $V_c(r) = 0$

b) Calculation using equation 3.28 with $V_c(r) = 0$

Figure 11  Reaction b)

As above

TABLE 3

Parameters used in figures 10 and 11

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<th>Target</th>
<th>k(fm$^{-1}$)</th>
<th>$R_0$(fm)</th>
<th>$\beta$</th>
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TABLE 4

Best fit parameters (see Figures 12 and 13)

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</table>

Table showing those parameters needed to produce the best fit to the optical model calculation assuming $V_C(r) = 0$. 
FIGURES 12) and 13)

Figure 12) Reaction a)

Graph showing the best fit obtained to the elastic scattering cross section with the coulomb potential switched off. Dashed curve is the same as curve a) of figure 11)

Figure 13) As above for reaction b)

Dashed curve is the same as curve a) of figure 10)
FIGURE 14) and 15)

Figure 14) Graph showing the effect of changing $\beta$

Figure 15) Graph showing the effect of changing $\alpha$

In each case above only reaction a) is considered.
FIGURES 16 and 17

Figure 16    Reaction a)

   a) Optical model calculation

   b) Cross section using equation 3.28 with
       best fit parameters

Figure 17    Reaction b)

   As above
\[ \frac{d\sigma}{d\Omega} \, (\text{mb}/\text{sr}) \]

\[ \Theta^\circ \, (\text{c.m.}) \quad \text{FIG(16)} \]
CHAPTER 5

In this chapter we discuss the possible reasons for the poor results obtained and make an attempt to indicate ways they may be improved.

5.1 Further investigation of the 'model' wavefunctions

Calculations indicate, using simple Born approximation, that the contribution to the scattering amplitude $f_N(\theta)$, in a Woods-Saxon type potential, is well approximated by neglecting the contribution from values of $r$ greater than $R_0$. This is typically of the order of 0.1 to 0.01 of the scattering amplitude from $r$ less than $R_0$ at $90^\circ$. If we accept the assumption made to neglect this region of the integral, it seems reasonable to assume the lack of a good set of results is due to the choice of wavefunctions. For this reason we can now take a closer look at their representation of exact optical model wavefunctions from ref M5).

An interesting feature arises if we consider the form of the modulus of the wavefunction, equation 3.23). That is we have

$$|\psi_+ (r)| = e^{-\alpha kR_0} e^{-\alpha kZ}$$

for values of $r$ along the Z axis. That is the bright side of the nucleus has $Z<0$. The results are shown in figures 18) and 19) for both reaction a) and reaction b) and for the parameters given in J1) and the set of parameters listed in table 4). These show a clear exponential decrease. We remember that inside the nucleus we have no reason to expect agreement between the results of Fig 19) and the results of Morgan, Fig 4) but at $Z = \pm R_0$ the values are the same due to the choice of $\alpha$. This is by construction. Now consider in Fig 19) the gradient of $|\psi_+ |$, that is
FIGURES 18) and 19)

Figure 18) Reaction a)
   a) Graph showing $|\psi|$ against $\varepsilon$ for the parameter set of table 2)
   b) As above for parameter set of table 4)

Figure 19) As figure 18) for reaction b)
\[
\frac{d|\psi^+|}{dZ} = -\alpha k |\psi^+|
\]

So that when \( Z = -R_0 \), i.e. when \( \theta = \pi \) the bright side of the nucleus
\[
\frac{d|\psi^+|}{dZ} = -\alpha k = -0.099
\]

While when \( Z = R_0 \), i.e. when \( \theta = 0 \) the dark side of the nucleus
\[
\frac{d|\psi^+|}{dZ} = -\alpha k e^{-2\alpha k R_0} = -0.023
\]

It is possible from Fig 4) to read off, by construction, the corresponding gradients of \( |\psi^+(\theta)| \) they come out to be roughly
\[
\frac{d|\psi^+|}{dZ} = -0.065 \quad \text{for } \theta = \pi
\]
\[
\frac{d|\psi^+|}{dZ} = 0.038 \quad \text{for } \theta = 0
\]

We see then that there is poor agreement between the two sets, the most significant point being the sign change in the \( \theta = 0 \) case. The reason why this is of importance is seen by considering equation 3.21) where we need \( \nabla_r \psi^+(r) \)
\[
\nabla_r \psi^+(r) = e^{-\alpha k R_0} i De^{i\beta k.R} \frac{d}{dZ} e^{-\alpha k Z}
\]
\[
= i(\beta+i\alpha) e^{-\alpha k R_0} e^{i\beta k.R} e^{-\alpha k Z}
\]
\[
= -i\beta \frac{d|\psi^+(\theta)|}{dZ} e^{i\beta k.R}
\]

where \( \frac{d|\psi^+(\theta)|}{dZ} = -\alpha k e^{-\alpha k R_0} e^{-\alpha k Z} \)

with \( Z = r \cos \theta \).

Consequently because we have the wrong sign for the gradient of the modulus this will be reflected in the expression for the scattering amplitude through the expression for the gradient of the wavefunction itself. This could be a likely reason for the poor results of the method used for two reasons. Firstly, if we had wished to make a calculation of \( f_N(\theta) \) using equation 3.18) we would have to know exactly the form of \( \psi^+(\theta) \) for all values of \( r \) in order to carry out the
integration. In making the conversion to a surface integral the integration is made simpler because we now regard the wavefunction as just a function of angle over the surface so we only need to know its value at one radius, but in addition to this we now also need to know the value of the gradient at that radius. While the idea of McCarthy succeeds in modelling the wavefunction on the surface it says nothing about the gradient. Secondly we can understand why success has been achieved with inelastic scattering, because the gradient of the wavefunction is not involved, only the value of $\psi^{(+)}(r)$.

Another interesting feature is to investigate the predicted flux and its divergence for the model wavefunction and to compare it with calculations made in (M2, M3) and (E1), see Fig (1-3). These are calculated using equation 2.1) and then taking the divergence. We find

$$j(r) = \frac{n}{\mu} k e^{-2akR_0} e^{-2ak \cdot r} \beta(\cos 0 - \hat{\nabla} \sin 0),$$

so $|j(r)| = \frac{2E\beta}{k_n} e^{-2akR_0} e^{-2ak \cdot r}$,

and $V \cdot j = -2k \alpha |j|$. 

Clearly these two quantities are multiples of each other and have the same basic form. They are shown in Figs 20) and 21) for reaction a) at various angles and for the parameters of J1 and those of table 4).
Figure 20) Reaction a)

Graph showing the comparison between $|i|$ for the parameters of table 2), from ref J1) and the parameters of table 4). These have been evaluated at $\theta = 0$, $\frac{\pi}{2}$ and $\pi$.

i) Table 4) parameters for $\theta = 0$

ii) Table 2) parameters for $\theta = 0$

iii) Table 4) parameters for $\theta = \frac{\pi}{2}$

iv) Table 2) parameters for $\theta = \frac{\pi}{2}$

v) Table 4) parameters for $\theta = \pi$

vi) Table 2) parameters for $\theta = \pi$

Figure 21) Reaction a)

Graph showing comparison between $-V_j$ for parameters of table 2) and table 4)

Labelling of graphs as in figure 20).
Figures 22) and 23) show the variation of $|\psi|$ with angle in both reactions and both sets of data while fig 24) shows the variation of the phase angle $\phi$ with $\theta$ for reaction a) only using the parameter set of table 4).

It is clear from figures 22) and 23) that $|\psi|$ has the shape predicted by McCarthy and Pursey (M4). However, calculations made by Austern (A2) indicate a completely different variation with angle, there is a largely oscillatory nature to the modulus rather than the smooth shape we have been considering. This oscillatory shape is clearly visible in calculations made by Amos (A3), (M1) and by Austern (A4). It is however neglected in the work of McCarthy in forming the expression for the wavefunction in his model, in fact very little consideration seems to have been given to the variation of $|\psi|$ with angle. The variation of the phase with angle is however largely the same in all cases, that is, at large angles, the side of the incoming beam, it is the phase of a plane wave that should depart from this character for smaller angles. Although our phase here does not do this the discrepancy is not large, even at these small angles, see (A1).

Conclusions

From the results obtained here it appears that the wavefunctions chosen are inadequate to describe the scattering of strongly absorbed particles, such as $\alpha$-particles, from heavy and intermediate target nuclei. The principal reasons for this are the inability of the model wavefunction to describe the gradient of the correct wavefunction and its variation with angle. From the simple form of the wavefunction used it is not possible to improve on these two points and to do any better than this one would certainly need a more complex model wavefunction that would have to incorporate into it the focus region in such a way that the change in gradient is accounted for.
FIGURES 22) and 23)

Figure 22) Reaction a)

Graph showing $|y|$ against $\theta$ for the parameters of table 2), graph i) and parameters of table 4), graph ii).

Figure 23) Reaction 6)

As above for figure 22).
Another interesting feature is the resemblance between the results of this calculation compared with optical model calculations and early calculations using sharp cut off models or square well potentials. The latter show marked oscillations in the cross section but produce results consistently too large. It would appear in our calculation that the oscillations have been damped or averaged out but are still consistently too large. See for example reference (G1).

The feeling generally is the 'model' wavefunctions tend to be an oversimplification of the exact wavefunctions. The lack of consistency in being able to use the same set of parameters for inelastic scattering and elastic scattering is disappointing and makes the extension to (p,2p) and other reactions unrealistic. Observing the features of $|\psi|$ as indicated by Morgan's work M4) the question as to whether neglecting the focus, even for strongly absorbed particles, is valid must be asked. It appears to be exactly this feature that changes the sign of the gradient and so doubt must be cast onto this approximation. Clearly if a focus term were to be included it would reflect in the shape of the flux and gradient of the flux which as they stand are poor representations of the same quantities as shown in fig 2,3).

We do however find that the parameters used in this calculation have the same influence over the resulting form of the cross section as predicted by McCarthy.
COMPUTATION OF THE CROSS SECTIONS

The computer code used for the calculation of the optical model cross sections was due to F. Pursey and named JIB IV a modified version of JIB III. This has been very widely used.

The code to calculate the cross sections of the approach used here was written on the IBM 360/195 at the Rutherford Laboratory using FORTRAN IV. The complex arithmetic was handled using the built-in routines for this purpose. As an additional check, above those carried out by considering the low angle behaviour, \( \cos \theta = 1 \) where the expression for \( f_N(\theta) \) simplifies and for \( \cos \theta = \pi/2 \) where again simplifications result, several hand calculations at various intermediate angles showed exact agreement with the computer calculated results.
Appendix

We can write the closed form expression for the nuclear scattering amplitude equation (3.27) in the form

\[
 f_N(\theta) = -\frac{NR k_2^2(f^2-1)}{k_2^2(1+f^2-2f\cos\theta)} \left\{ \frac{\cos qR_0}{q^2 R_0} \right\} \quad (A1)
\]

where \( f = \frac{k_1}{k_2} (\beta+i\alpha) \)

and hence

\[
 k_2^2(f^2-1) = k_1^2(\beta^2\alpha^2+2i\alpha\beta) - k_2^2 \quad (A2)
\]

For elastic scattering we would take \( k_1 = k_2 \) but to take into account the coulomb distortion we have introduced \( \beta_2 \) such that \( k_2 = \beta_2 k_1 \)

where

\[
 k_1^2 = \frac{2\mu E}{h^2}
\]

is the asymptotic value of \( k \).

Then

\[
 k_2^2(f^2-1) = k_1^2 \left\{ (\beta^2\beta_2^2) - \alpha^2 + 2i\alpha\beta \right\} \quad (A3)
\]

This shows us that under conditions when \( \alpha \) is small \( \alpha^2 < (\beta_2^2\beta_2^2) \) the magnitude of \( f_N(\theta) \) will depend to a large extent on the value of \( (\beta^2\beta_2^2) \). If \( \beta = \beta_2 \) then only the \( \alpha^2 \) term contributes to the real part of (A3). As the difference between \( \beta^2 \) and \( \beta_2^2 \) increases so an increase in the overall magnitude will be seen.
It was found quite difficult to observe the effect of changing $a$ due to the complexity of the expression $A_l$ when evaluated in real and imaginary parts as indicated in the following. We have

$$q^2 = k_2^2 (1 + f^2 - 2f \cos \theta)$$

$$= k_2^2 \left\{ (\beta_2^2 + \bar{\beta}_2^2) - \alpha^2 - 2\beta_2 \bar{\beta}_2 \cos \theta \right\} + ik_1^2 \{ 2\alpha \beta - 2\alpha \bar{\beta} \cos \theta \}$$

$$= q_r + iq_i$$

or we can say

$$q^2 = Q e^{i\phi}$$

where

$$Q = (q_r^2 + q_i^2)^{\frac{1}{2}}$$

$$\phi = \tan^{-1}(q_i/q_r)$$

Then

$$q = Q' e^{i\phi/2}$$

$$= q_r' + iq_i'$$

with

$$Q' = \sqrt{Q}$$

and

$$q_r' = \left( \frac{Q}{1 + \tan^2 \phi/2} \right)^{\frac{1}{2}}$$

$$q_i' = q_r' \tan \phi/2$$

The expression for $f_N(\theta)$, equation $A_l$ can now be written

$$f_N(\theta) = -\frac{NR_0 k_1^2}{Q e^{i\phi}} \{ (\beta_2^2 + \bar{\beta}_2^2) - \alpha^2 + 2i\alpha \beta \}$$

$$\left\{ \cos(q_r' R_0 + iq_i' R_0) - \frac{\sin(q_r' R_0 + iq_i' R_0)}{Q' R_0 e^{i\phi/2}} \right\}$$
After a lengthy expansion of the complex cos and sin terms the scattering amplitude may be written

\[ f_N(\theta) = \frac{NR_0k_1^2}{Q} \left\{ (A-iB) (C-iD) \right\} \]

where

\[ A = (\beta_2 - \beta^2 + \alpha^2) \cos \phi - 2\alpha \beta \sin \phi \]
\[ B = (\beta_2 - \beta^2 + \alpha^2) \sin \phi + 2\alpha \beta \cos \phi \]
\[ C = \cos q_1 R_0 \cosh q_1 R_0 - \frac{1}{R_0 Q} \left\{ \cos \frac{q_1}{2} \sin q_1 R_0 \cosh q_1 R_0 \right\} \]
\[ + \sin \frac{q_1}{2} \cos q_1 R_0 \sinh q_1 R_0 \]
\[ D = \sin q_1 R_0 \sinh q_1 R_0 + \frac{1}{R_0 Q} \left\{ \cos \frac{q_1}{2} \cos q_1 R_0 \sinh q_1 R_0 \right\} \]
\[ - \sin \frac{q_1}{2} \sin q_1 R_0 \cosh q_1 R_0 \]

Thus we can see that keeping track of \( \alpha \) in this is very difficult. What we can say though, knowing the influence of \( \alpha \) in damping the cross section is that it must have a strong influence in the exponential terms, that is \( \cosh \) & \( \sinh \).
REFERENCES