THE USE OF PSEUDO RANDOM DATA IN THE ANALYSIS
OF VEHICLE DYNAMICS

by

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ABSTRACT

A method of quantifying the response of a vehicle to steering wheel inputs is examined. The technique will aid designers to adapt a prototype vehicle to give a desired response or 'feel'. Also it may be used to investigate the steering performance of current vehicles to make comparison between vehicles, or to examine the effect of changes in a single vehicle such as tyre types, pressures and load distribution.

A model driver/vehicle simulation is set up and it is postulated that a vehicle's steering qualities may be rated by assessing the contribution the model driver needs to make to place the closed loop response in an acceptable area on the pole/zero diagram.

For a single vehicle it is shown that a better steering response is achieved with the car fitted with radial ply tyres than with cross-ply tyres.
SUMMARY

The equations which relate a vehicle's steering wheel angle to the position of its centre of gravity, relative to a ground plane, are established. This is achieved by spectral analysis of the recorded transverse and angular acceleration response from random inputs to the steering wheel. A synopsis of random data techniques is given and the advantages in using a pseudo random sequence input to the steering are outlined.

The measured equations are presented as transfer functions (pole/zero patterns), and are shown to represent closely the true vehicle dynamics by good correlation of model output with the vehicle data.

The technique is sufficiently sensitive to show migrations in the pole/zero patterns due to changes in vehicle speed and side force.

Sets of transfer functions are given which cover a vehicle speed range from 16 km/h to 96 km/h, and from zero to 0.4 g equivalent side force.

The effects on the transfer functions are examined, of changes in vehicle parameters such as tyre types (cross-ply to radial), tyre pressure and vehicle loading. An attempt is made to determine the dynamic characteristics that a driver prefers by making changes to the vehicle which predictably degrade the vehicle's 'feel' (eg low tyre pressures and excessive rear loading) and observing the effect upon the pole/zero pattern. The acceptable areas on the pole/zero diagram are outlined and are supported by work carried out in aircraft studies.

Driver difficulty in controlling a particular 'set' of measured transfer functions is quantified by root locus examination of a multiloop vehicle/driver feedback model. The driver is represented by a model
which contains terms such as gain, reaction time, neuromuscular lag and anticipation. Driver difficulty is assessed by determining the contribution in terms of the amount of gain and anticipation the model driver needs to supply to place the closed loop dynamics (the poles) in an acceptable area on the pole/zero diagram. The technique allows comparisons to be made between sets of transfer function before and after a change in a vehicle parameter. The results show that the model driver requires less gain and anticipation to control a particular vehicle when fitted with radial-ply tyres compared with cross-ply tyres.

The results indicate that there is a small difference between the vehicle's natural frequency in the horizontal transverse plane and the horizontal rotational plane.
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GLOSSARY OF SYMBOLS

a \hspace{1cm} \text{distance of front wheel from C of G}
a_y \hspace{1cm} \text{transverse acceleration}
a_x \hspace{1cm} \text{forward acceleration}
b \hspace{1cm} \text{distance of rear wheel from C of G}
B(t) \hspace{1cm} \text{box car function}
B_e \hspace{1cm} \text{bandwidth resolution}
C(f) \hspace{1cm} \text{coincident spectral density function}
C_F \hspace{1cm} \text{front tyre cornering stiffness}
C_R \hspace{1cm} \text{rear tyre cornering stiffness}
D(S) \hspace{1cm} \text{denominator polynomial}
f \hspace{1cm} \text{frequency (Hz)}
f_{\text{max}} \hspace{1cm} \text{maximum frequency}
f_{\text{min}} \hspace{1cm} \text{minimum frequency}
F(S) \hspace{1cm} \text{transfer function}
G_{XX}(f) \hspace{1cm} \text{input auto PSD}
G_{YY}(f) \hspace{1cm} \text{output auto PSD}
G_{XY}(f) \hspace{1cm} \text{cross PSD}
h(t) \hspace{1cm} \text{impulse response}
H(f) \hspace{1cm} \text{transfer function}
I \hspace{1cm} \text{moment of inertia}
I.F. \hspace{1cm} \text{impact factors}
L_{ST} \hspace{1cm} \text{static load}
M \hspace{1cm} \text{mass}
M \hspace{1cm} \text{maximum lag number}
n \hspace{1cm} \text{number of bits in a shift register}
n(t) \hspace{1cm} \text{noise contamination}
N(S) \hspace{1cm} \text{numerator polynomial}
\( P \)  
maximum number of bits in a binary sequence

\( P(S) \)  
pole

\( Q(f) \)  
quadrature PSD

\( r \)  
lag number, correlation coefficient

\( r(t) \)  
yaw rate

\( R_{xx}(\tau) \)  
input autocorrelation function

\( R_{yy}(\tau) \)  
output autocorrelation function

\( R_{xy}(\tau) \)  
cross correlation function

\( R_{yx}(\tau) \)  
reverse cross correlation function

\( S \)  
Laplace operator

\( S_F \)  
side force at front wheel

\( S_R \)  
side force at rear wheel

\( T \)  
period of sample data

\( u(t) \)  
forward speed

\( v(t) \)  
side slip velocity

\( V_I(t) \)  
inertial velocity

\( \omega_d \)  
damped natural frequency

\( \omega_n \)  
natural frequency

\( x(t) \)  
input time function

\( y(t) \)  
output time function

\( Y_I(t) \)  
inertial displacement

\( Z(S) \)  
pole

\( \alpha_F \)  
front tyre slip angle

\( \alpha_R \)  
rear tyre slip angle

\( \beta(t) \)  
side slip angle

\( \gamma^2_{xy}(f) \)  
coherece function

\( \Delta F \)  
discrete frequency interval

\( \Delta \)  
discrete time interval

\( \epsilon \)  
phase angle

\( \zeta \)  
damping factor
\( \theta \)  
steering wheel angle

\( \lambda \)  
front wheel steer angle

\( \mu_x \)  
mean value of \( x(t) \)

\( \sigma_x \)  
standard deviation of \( x(t) \)

\( \tau \)  
time delay

\( \psi_x^2 \)  
mean square value of \( x(t) \)

\( \omega \)  
frequency (radians)
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1. Introduction

The performance of a dynamic system, whether it be mechanical, electrical, biological, or a nation's economy, depends on the contributory factors within that system and their interconnection. If it is required to adjust the system to achieve a desired result, it is necessary to recognise and understand the separate factors within the system. A dynamic system is referred to here as simply an assembly of connected parts which produce an output in response to an input stimulus. The examination of the input stimuli and the resulting output signals may lead to the description or identification of the dynamics which caused the output.

The subject of this thesis is to describe and demonstrate a method of identifying the dynamic steering response of a motor vehicle using random data methods, and to attempt to interpret the meaning of the measured response.

1.1 The need for the measurement of vehicle dynamic response

A quantitative method of measuring vehicle dynamic response would provide vehicle manufacturers with a design tool to aid the development of prototype vehicles to a predetermined performance of steering response, ride and stability. Also it would give the safety engineer the means to study the relationship between vehicle dynamic response and parameters
such as tyre characteristics, road profile, vehicle speed, load distribution, etc.

1.2 An introduction to the statistical identification approach

A considerable quantity of information has been published on the analytic examination of vehicle response by considering mathematical models in the form of differential equations of motion, and establishing the validity of the models by comparing model outputs with practical experiments (Ref.1).

The present investigation is somewhat a reverse process in that data is taken from a practical experiment and reduced by a statistical process to give a mathematical description of the vehicle dynamics in the form of a frequency response or transfer function written in the form of a ratio of polynomials (Ref.2).

A control systems engineering approach has been adopted such that the vehicle is treated as a 'black box' which receives an input and transmits an output, for example, these might be a vehicle's steering wheel angle and angular acceleration. The contents of the 'black box', that is, the transfer function operates upon the input signal to form the output; thus the output data contains the input data together with the imprint of the transfer function. It follows that, given a suitable data processing technique, there is sufficient information in the input and output signals to recover the transfer function.

A part of the essential requirements for any measuring device is repeatability of results. Initially the task was to determine that
recorded vehicle signals could be treated as stationary random data which could be analysed by established statistical correlation procedures. The degree of repeatability achieved is described under Section 5.2.

Unfortunately the relationships between steering wheel angle and vehicle motions contain non-linearities which are attributable to the tenuous contact between the tyre and road surface. It is a general feature of non-linear systems that the output signal is amplitude dependent, however a linear approach was made possible by careful selection of the input signal. The chosen input was statistically well defined and restrained in amplitude to follow small deviations about the mean input signal level. As a further precaution, to support a linear approach, the coherence function was generated in the data reduction process, this quantity gives a fractional measure of the dependence of the output upon the input; a value of 1.0 implies complete dependence. Non-linearities in the system have the effect of deprecating the value of the coherence function. Thus the coherence function serves the dual purpose of monitoring the dependence of the output upon the input and gives an indication of the degree of non-linearity. The chosen input was a pseudo-random binary sequence (PRBS) generated by an electrical logic network. PRBS has two distinct properties which suggest its use in the identification of vehicle dynamics. Firstly it can be superimposed upon the normal input whilst the vehicle is undergoing normal manoeuvres. Secondly, over a selected bandwidth it has an equal energy spectrum which ensures that the dynamic system is excited over all modes of interest.
1.3 A review of the methods of system identification

It is constructive at this stage to examine the possible methods of describing and exciting dynamic systems and to show some of the advantages and disadvantages of the various techniques and to justify the chosen method.

1.3.1 The impulse and step response

An impulse or step input is applied to the system and a record of the response is taken. Information is available in the time domain and quantities such as rise time, damping factor, settling time and natural frequency are available. The method is somewhat impracticable to use on line, particularly on a vehicle at high speed. The magnitude of the step or impulse may have to be large to generate a measurable output, so driving the system considerably into the non-linear region.

In practice it is possible only to approximate an impulse or step input to a mechanical system without the risk of structural damage, it follows that using this technique in the vehicle application the input is ill defined.

1.3.2 Harmonic analysis

The system is excited by sinusoidal inputs at discrete frequencies, and a single measurement is made at each frequency. The magnitude and phase are extracted from the recorded information and are plotted as functions of frequency, a curve fitting procedure may be employed to produce the transfer function in the form of a polynomial. The choice
of amplitude may cause difficulty from the consideration that large values may introduce vehicle handling problems and a small signal amplitude may produce a poor signal to noise ratio. The process can be very time consuming which introduces the possibility of system variations during experimentation.

1.3.3 Random data analysis

Data representing a system can be classified over a range from deterministic to completely random. Deterministic data are those that can be described to reasonable accuracy by an explicit mathematical relationship. Phenomena that can be described by nearly random data (white noise) occur frequently in nature. At what point a chosen set of data is placed on the deterministic-random scale is debatable; it may be argued that an unforeseen event may upset the phenomena producing the deterministic data or on the other hand random data may be predictable given sufficient mathematical knowledge of the mechanism producing the data.

Data have been classified into groups throughout the deterministic to random range (Ref.3). The techniques which are described below apply to data which are both stationary and ergodic, that is for any chosen sample function the sample mean value and the sample autocorrelation functions do not vary from sample to sample, or in other words they equate to the ensemble mean and autocorrelation function.

The identification of a system transfer function calls for the calculation of auto and cross correlation coefficients in the time domain and power spectral density and coherence functions in the frequency
domain. Phase information is present in the cross spectral and cross correlation functions.

Data may be sampled during normal operation of the vehicle but to ensure a sufficient frequency bandwidth and energy content in the input signal a small perturbation can be superimposed upon the normal input; it is chosen to have a spectral bandwidth sufficient to excite all modes of vibration of interest in the vehicle. The result is a description of the system over the spectral bandwidth, made from an averaged ensemble of sampled data. Uncorrelated errors which may arise from spurious changes in experimental conditions are minimised by the averaging process and any long term drifts can be extracted from the data by regression analysis.

A perturbation which takes only two discrete levels and occurs in a random fashion constitutes a pseudo random binary sequence (PRBS) (Fig.1.1). Its correlation function approaches a delta function which is indicative of its random property (Fig.1.2), and its power spectral density function takes the form of a function (Fig.1.3). By careful selection of the PRBS clocking rate and sequence length it can be arranged to have an equal energy spectrum over the bandwidth of interest. The signal bandwidth and the frequency content are under the control of the experimenter. The PRBS may be superimposed on the normal input of the system and the perturbations may be made sufficiently small to allow continued operation of the system without the signal amplitude causing undesirable disturbance of the vehicle control. Improvement in the signal to noise ratio can be made by averaging over several sample sequences, the effects due to uncorrelated noise tend to cancel with increasing sets of samples.
Fig. 1.1  EXAMPLE OF 15 BIT PRBS

Fig. 1.2  AUTOCORRELATION FUNCTION OF 15 BIT PRBS
\[ G_{xx}(f) = \text{func.} \left( \frac{K_{II}}{K_p} \right) \]

\[ f_k = \frac{K}{p\Delta t} \]

where \( K = 1, 2, 3, \ldots, \ldots \).

-3db at 0.45 f_{\text{clock}}

\[ f_{\text{clock}} = \frac{1}{\Delta t} \]

\[ f_{\text{clock}} = \frac{1}{4} \frac{3}{p\Delta t} \]

\[ f_{\text{min}} = \frac{1}{p\Delta t} \]

Fig. 13 POWER SPECTRUM OF PRBS

Relative power \( G_{xx}(F) \) (db)
2. A synopsis of the mathematics and computer programs for vehicle identification

The mathematical relationships required for the identification of a transfer function are briefly stated in the following paragraphs, a comprehensive treatment is given under Reference 3. An outline of the computer programs is given in this chapter, an algorithm for computing the frequency response from the time data is developed in Appendix I. Reference 4 gives details of a complex curve fitting routine which generates the pole/zero pattern from the frequency response: Reference 5 describes the method of generating the time function from the transfer functions.

Calculations can be made either in the time domain or in the frequency domain, one domain does not supply more information than the other but gives a particular insight into a system's performance; it is the Fourier Transform which provides the link between the two domains. Some concepts are better visualised and manipulated as time functions and others as functions of frequency. For example the time domain provides information about the frequency response of a system and the Fourier Transform of the impulse response is the transfer function in the frequency domain.

2.1 Time domain relationships

Correlation is a time domain procedure which provides a coefficient of similarity between two waveforms (Fig.2.1). It is generated by
Correlation coefficient $R_{x_1x_2}(\tau)$ is computed by multiplying waveforms ordinate by ordinate and averaging the product. For $\tau = 0$ correlation between waveforms $x_1(t)$ and $x_2(t)$ is unity. Waveform $x_3(t)$ is identical to $x_2(t)$ but shifted Lag $\tau = k$, so correlation is less than unity.

Fig. 2.1 REPRESENTATION OF CORRELATION PROCEDURE
multiplying waveforms ordinate by ordinate and computing the average product.

Thus the autocorrelation function \( R_{xx}(\tau) \) of a waveform \( x(t) \) is defined as

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) x(t + \tau) \, dt
\]  

(2.1)

and the cross-correlation function of two waveforms \( x(t) \) and \( y(t) \) (where \( x(t) \) is the input and \( y(t) \) is the output) of a system is

\[
R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} y(t) x(t + \tau) \, dt
\]  

(2.2)

If the autocorrelation is impulsive in shape as in the case of the autocorrelation of PRBS then by means of the convolution integral

\[
y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) \, d\tau
\]  

(2.3)

where \( h(\tau) \) is the impulse response it can be shown (Appendix I) that the cross-correlation function is proportional to the impulse response. That is

\[
R_{xy}(\tau) = Kh(\tau)
\]  

(2.4)

where \( K \) is a constant of proportionality.
2.2 Power spectral density and transfer functions

The power spectral density of time series data describes its composition as a function of frequency in terms of the mean square value $\psi^2$.

That is

$$\psi^2_x(f, \Delta f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t, f, \Delta f) \, dt$$

where $x(t, f, \Delta f)$ is that portion of $x(t)$ in the frequency range from $f$ to $\Delta f$. Thus for small $\Delta f$ a power spectral density function $G_x(f)$ can be defined as

$$\psi^2_x(f, \Delta f) = G_x(f) \Delta f$$

(2.5)

In terms of the power spectral density function the mean value of $x(t)$ is given by

$$\mu_x = \left[ \int_{o^-}^{o^+} G_x(f) \, df \right]^\frac{1}{2}$$

(2.6)

where $o^-$ means the lower limit of integration is approached from below and the upper limit $o^+$ is approached from above. This indicates that the mean value of $x(t)$ will appear in the spectral distribution $G_x(f)$ as a Dirac delta function at zero frequency. The mean value is the square root of the area under the delta function. It is usual for the raw data to be zero biased by subtracting the mean value of the time data from each discrete data point to remove the DC bias which appears at zero frequency.
The mean square value of $x(t)$ given by

$$\psi_x^2 = \int_0^\infty G_x(f) \, df$$

shows that the total area under a power density spectral plot is itself the mean square value, this provides a useful check on the calculation procedures of the power spectral density functions such that the mean square values calculated from time domain data can be crosschecked with the algebraic sum of the mean square values calculated in the frequency domain.

An important property of the power spectral density is its relationship by the Fourier Transform with the correlation functions. That is for the autocorrelation function $R_{xx}(t)$ the auto power spectral density $G_{xx}(f)$ is

$$G_{xx}(f) = 2 \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-2\pi j f \tau} \, d\tau \quad (2.8)$$

Similarly for the cross correlation function $R_{xy}(f)$ the cross power spectral density $G_{xy}(f)$ is

$$G_{xy}(f) = 2 \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-2\pi j f \tau} \, d\tau \quad (2.9)$$

The principal application for a power density measurement is to establish the frequency composition of the data, which in turn bears important relationships to the basic characteristics of the system involved. The relationships which are relevant to this work are given below. The Gain Factor, $H(f)$ which relates the magnitude of the input to the output of a system, is given by
There is no phase content present in the auto spectra $G_{yy}(f)$ and $G_{xx}(f)$, consequently there is no phase content present in the gain factor $H(f)$, generation of phase content requires cross spectral analysis.

The Transfer Function $F(f)$ contains both magnitude and phase information and can be determined from

$$F(f) = \frac{G_{xy}(f)}{G_{xx}(f)}$$  \hfill (2.11)

$G_{xy}(f)$ is complex and can be written as

$$G_{xy}(f) = C_{xy}(f) - j Q_{xy}(f)$$  \hfill (2.12)

where $C_{xy}(f)$ is the coincident spectral density function and the imaginary part $Q_{xy}(f)$ is called the quadrature spectral density function. The magnitude $|G_{xy}(f)|$ and the phase angle $\theta_{xy}(f)$ are related to $C_{xy}(f)$ and $Q_{xy}(f)$ by

$$|G_{xy}(f)| = \left[C_{xy}^2(f) + Q_{xy}^2(f)\right]^{\frac{1}{2}}$$  \hfill (2.13)

and

$$\theta_{xy}(f) = \tan^{-1} \frac{Q_{xy}(f)}{C_{xy}(f)}$$  \hfill (2.14)

When applying cross spectral density methods to physical problems it is desirable to have a fractional measure of the dependence of the output upon the input, this is given by the value of the coherence function $\gamma_{xy}^2(f)$ defined as
\[ \gamma_{xy}^2(f) = \frac{|G_{xy}(f)|^2}{G_x(f) G_y(f)} \]  

(2.15)

When at a particular frequency \( \gamma_{xy}^2 = 0 \) then \( x(t) \) and \( y(t) \) are said to be incoherent, if \( \gamma_{xy}^2(f) = 1 \) for all frequencies of interest then \( y(t) \) is completely causal to \( x(t) \).

If an estimate of \( y(t) \) is desired when extraneous noise is present and the noise is uncorrelated with \( x(t) \) then the minimum mean square error is given by

\[
R_e(0) = \int_0^\infty G_y(f) \left[ 1 - \gamma_{xy}^2(f) \right] df \tag{2.16}
\]

and indicates how closely an actual system approximates the optimum system.

2.3 Pole/zero patterns

The magnitude and phase characteristic of a system plotted as a function of frequency provide a complete description of the system transfer function, but a graphical presentation can be unwieldy when sets of transfer functions need to be visually assessed or implemented in a computer simulation.

Control system theory makes frequent use of the pole zero pattern for representing system characteristics. Consider the Laplace transfer function

\[
F(S) = \frac{K(S + Z_1)(S + Z_2)(S + Z_3) \ldots (S + Z_i)}{(S + P_1)(S + P_2)(S + P_3) \ldots (S + P_r)} \tag{2.17}
\]

where \( r \geq i \).
There are values of $S$ which put $F(S)$ equal to zero, and there are values of $S$ which cause $F(S)$ to approach infinity. That is

$$F(S) = 0 \text{ when } S = -Z_1, -Z_2, -Z_3 \ldots -Z_i$$

and

$$F(S) \to \infty \text{ when } S = -P_1, -P_2, -P_3 \ldots -P_r$$

thus the $-Z_i$ are the zeros and the $-P_r$ are the poles of the transfer function $F(S)$. The poles and the zeros occur in either real or conjugate complex pairs. It is common practice to plot the poles and zeros of $G(S)$ in the complex plane (argand diagram), this plot represents the pole zero pattern (Ref.6).

For example, consider the transfer function

$$F(S) = \frac{K(S + 3)(S + 2 \pm j5)}{(S + 4)(S + 1 \pm j3)}$$

The pole zero pattern is, (Fig.2.2),

![Pole/Zero Pattern](image-url)
The poles may be physically interpreted as the principal modes of vibration of a system. A pole in the left hand quadrants represents a decaying time function which is a stable condition, and a pole in the right hand quadrants represents a time function that grows without limit, that is an unstable condition.

2.3.1 Dominant second order poles

Many physical systems can be represented by a second order transfer function of the form, (Fig.2.3).

\[ G(S) = \frac{K}{(S + \alpha \pm jb)} \]  \hspace{1cm} (2.18)

Its pole zero pattern is

![Diagram](image)

Fig.2.3 INTERPRETATION OF SECOND ORDER POLES
It can be shown that

\[
\left[ \alpha^2 + \beta^2 \right]^\frac{1}{2} = \omega_n \text{ (natural frequency)} \tag{2.19}
\]

\[
\beta = \omega_d \text{ (damped natural frequency)} \tag{2.20}
\]

\[
\cos \eta = \zeta \text{ (damping factor)} \tag{2.21}
\]

2.3.2 Pole/zero patterns from complex curve fitting

In order to simplify the analysis and understanding of a dynamic system under investigation it is advantageous for analytic purposes to form the equation which represents the pole zero pattern. The form of the equation that is usually quoted in the literature is a ratio of polynomials which may be factorised to generate the poles and the zeros. The polynomial is formed by applying a curve fitting procedure to the frequency response data, which may be in the form of real and imaginary parts or magnitude and phase.

That is

\[
F(S) = \frac{A_0 + A_1S + A_2S^2 + A_3S^3 + \ldots + A_iS^i}{B_0 + B_1S + B_2S^2 + B_3S^3 + \ldots + B_rS^r}
\tag{2.22}
\]

where the Laplace 'S' has been substituted for jw.

In factored or pole/zero form this becomes

\[
F(S) = \frac{K(S + Z_1)(S + Z_2)(S + Z_3) \ldots (S + Z_i)}{(S + P_1)(S + P_2)(S + P_3) \ldots (S + P_r)}
\]

\[
= \frac{N}{D}(S)
\]

18
If the function \( P(S) \) is used to represent one which is ideal in that it provides a perfect fit to the data, then the error \( \epsilon(S) \) between the two functions \( F(S) \) and \( P(S) \) is

\[
\epsilon(S) = P(S) - F(S)
\]

or

\[
\epsilon(\omega) = P(\omega) - \frac{N}{D}(\omega)
\]

(2.23)

Multiplying both sides by \( D(\omega) \)

\[
D(\omega) \epsilon(\omega) = D(\omega) P(\omega) - N(\omega)
\]

which may be written in terms of complex numbers as

\[
D(\omega) \epsilon(\omega) = a(\omega) + jb(\omega)
\]

(2.24)

where \( a \) and \( b \) are functions of frequency and the unknown coefficients \( A_i \) and \( B_i \).

If \( E \) is defined as the above function summed over all frequencies of interest then

\[
E = \sum_{n=0}^{N} \left[ a^2(\omega_n) + b^2(\omega_n) \right]
\]

(2.25)

The unknown coefficients may be evaluated (Ref. 7) by minimizing the function \( E \) by differentiating with respect to each of the unknown coefficients and setting the result to zero.
This produces a matrix of equations

\[
[T][U] = [V]
\]

which is solved to produce the values of the polynomial coefficients.

A further weighting function may be used to provide a better fit at the lower frequencies. Some restriction is placed on the algorithm in that it cannot be used to fit a polynomial to data which describes a transfer function with a pole at the origin, that is the magnitude of the function approaches infinity as the frequency approaches zero. If this condition is suspected then a solution can be made by premultiplying the data by \(j\omega\).

2.3.3 Root Locus analysis

Fig.2.4 A SIMPLE CLOSED LOOP UNITY FEEDBACK SYSTEM

Root Locus analysis provides a means for examining the location and stability of the closed loop poles of a feedback system as the static loop gain sensitivity \(K\) varies, similar methods may be used to examine the effect of changes in any other transfer function coefficients (Ref.8).
The application in the context of vehicle dynamics is to examine the closed loop behaviour of a driver/vehicle simulation when the feedback loop is closed by a driver model. For example the transfer functions derived from a vehicle when fitted with cross-ply and radial ply tyres may be compared (see Section 7.3).

Consider the closed loop system of Figure 2.4. The closed loop transfer function is

\[
\frac{C}{R}(S) = \frac{\text{KF}(S)}{1 + \text{KF}(S)} \tag{2.27}
\]

We are interested in the behaviour of the roots or closed loop poles of the characteristic equation

\[
1 + \text{KF}(S) = 0 \tag{2.28}
\]

writing

\[
\text{F}(S) = \frac{\text{N}(S)}{\text{D}(S)}
\]

gives

\[
\text{KN}(S) + \text{D}(S) = 0 \tag{2.29}
\]

The roots of equation (2.29) as K varies describe the root locus, or in other words, the equation defines the track of the closed loop poles from the open loop poles to the open loop zeros as a function of K. When K takes a zero value the roots are those of D(S) which are the open loop poles, when K is large the roots are those of N(S) which are the open loop zeros.
The method used in the computer algorithm to determine the roots is due to Baristow (Ref.9), this produces complex roots of a polynomial equation by applying Newton's method of approximation.

That is

\[ x_n = x_{n-1} - \frac{f(x_{n-1})}{F'(x_{n-1})} \]  \hspace{1cm} (2.30)

for a root of \( f(x) = 0 \).

The root locus method is equally applicable to non-unity multiloop feedback systems.

2.4: Computer programs for the identification of vehicle dynamics

A suite of four computer programs has been implemented to aid the investigation of vehicle dynamics. The first program named SPECTR was developed particularly for use in the identification of vehicle dynamics, the second, third and fourth called BODFIT, TRYSYM and ROOTL, which, although written for use in the analysis of other dynamic systems were general enough to apply to motor vehicle investigation.

2.4.1 Program SPECTR - Frequency response from time series data

Program SPECTR (Appendix I) takes raw input and output time data logged from a practical system and generates the frequency response as the real and imaginary parts of a complex number and as values of magnitude and phase (Ref.10). In the process of calculation the auto and cross correlation coefficients are produced. If the autocorrelation function is impulsive in nature then the cross correlation function
represents the impulse response of the system. The auto and cross power spectral density and coherence functions are generated by digital transformation of the correlation coefficients. The computer algorithm uses a straight line approximation between the data points. The raw data are accepted as a number of blocks from which an ensemble average is taken thus reducing errors arising from uncorrelated noise.

A regression line is drawn through the data and from this a correction is made for any long term drift; the data are scaled and zero meaned to remove any DC content which would appear in the frequency response at zero frequency.

The correlation and spectral response calculations are performed upon this prepared data.

2.4.2 Program BODFIT - Pole/zero patterns from the frequency response

Program BODFIT (Ref.4) takes the real and imaginary parts of a set of complex numbers which are a function of frequency and applies a weighted least squares complex curve fitting procedure. The result is presented as a ratio of frequency dependant polynomials, the pole/zero locations are available from factorisation of the polynomials. A percentage fit is requested for each computer run, the program stops when the fit has been reached otherwise fitting will continue until a 14th order polynomial has been reached.

2.4.3 Program TRYSYM - Transfer function simulation

Program TRYSYM (Ref.5) is a digital simulation language written in the form of block structures in which procedures represent the blocks and simulate the following functions.
\[ F(S) = \frac{K}{S + a} \quad (2.31) \]

\[ F(S) = \frac{K}{S^2 + aS + b} \quad (2.32) \]

\[ F(S) = (S + a) \quad (2.33) \]

Each procedure is implemented using Euler integration. Simulation of a complex system is achieved by interconnecting the appropriate transfer functions by means of simple programming instructions, an example is given in Figure 2.5.

2.4.4 Program ROOTL - Determination of the closed loop poles

The closed loop transfer function of a system is first written in canonical form, (see Fig.2.4), that is

\[ \frac{C}{R}(S) = \frac{KF(S)}{1 + KF(S)} \]

The program ROOTL calculates and plots the closed loop poles or roots of the equation

\[ 1 + KF(S) = 0 \]

as \( K \) is varied.

The open loop transfer function \( G(S) \) may be entered in the program in the form of polynomial coefficients or polynomial factors.
Procedure for a single pole (simple lag)

\[ \frac{K}{S + a} \]

Procedure for double or complex pole

\[ \frac{K}{S^2 + aS + b} \]

Procedure for a single zero (simple lead)

\[ S + a \]

Example

\[ G(s) = \frac{3.9(S^2 + 3S + 12)}{(S + 4)(S^2 + 5S + 8)} = \frac{3.9S((S+3) + \frac{12}{S})}{(S+4)(S^2+5S+8)} \]

Computer program listing

\[ Y2: = \text{Grate} \ 2(Y1, 3.9, 5, 8, 1) \]
\[ Y3: = \text{Grate} \ 1(Y2, 1, 4, .1) \]
\[ Y4: = \text{Deriv} \ (Y3, 3, 1) \]
\[ Y5: = \text{Grate} \ 1(Y3, 0, 2) \]
\[ Y6: = Y4 + Y5 \]
\[ Y7: = \text{Deriv} \ (Y6, 0, 2) \]

Fig. 2.5 DIGITAL SIMULATION OF DYNAMIC SYSTEMS
An upper and lower limit is set for the value of $K$ and the program is arranged to give a numerical output for the poles together with a graphical plot of the roots on a line printer.

2.5 Validation of computer programs

Assuming that good time data are logged on each occasion the repeatability of a measurement depends upon the efficiency of SPECTR to compute the power spectral densities and upon the accuracy of the curve fitting program BODFIT to generate the pole/zero pattern.

The validity of the computer programs was substantiated by the following procedures, (Fig.2.6). A transfer function was written into TRYSYM and driven by computer generated PRBS to produce a set of computer perfect time data, that is the computed data contained no noise. The data was reduced by SPECTR to form the frequency from which the pole/zero pattern was formed by program BODFIT. A completely successful result would produce a pole/zero pattern identical to the function originally written into TRYSYM. The following paragraphs describe the practical checks made to examine the accuracy of reconstruction, also the effects on the computation of random noise in the data, and variations in the lag number of the correlation procedure are examined.

2.5.1 The accuracy of computation of a specimen transfer function from computer perfect data

A second order transfer function

$$F(S) = \frac{16}{S^2 + 35 + 16}$$
The computed transfer function will be identical to the inscribed transfer function if computer programs TRYSYM, SPECTR and BODFIT are efficient.
was written into TRYSYM and driven by PRBS to produce perfect output data. The transfer function chosen describes a system with a natural frequency of 4 rads/sec and a damping factor of 0.375. The frequency response was computed by SPECTR from the perfect data and submitted to BODFIT to generate the pole/zero pattern.

A cross check had previously been made on the accuracy of the pole/zero pattern produced by BODFIT by comparing fitted data with the frequency response calculated from a fourth order fitted model. The results are shown plotted in Figure 2.7. The maximum difference in magnitude occurred at a frequency of 4 rads/sec, the fit for both magnitude and phase was better than 5 per cent over the bandwidth of interest.

If programs TRYSYM, SPECTR, and BODFIT were perfect the transfer function resulting from the analysis of the computer perfect data would be identical to the specimen transfer function. It was expected that some difference would arise due to quantization errors introduced by the program algorithms. The difference was measured by comparing the frequency response produced from both the specimen transfer function and from the computed transfer function and also by comparing the time function quantities $\omega_n$ (natural frequency) and $\zeta$ (damping factor).

Figure 2.8 shows a plot of the specimen compared with the computed response over a bandwidth from 0.1 to 20 rads/sec. Table 2.1 lists the results and shows that the error in magnitude was less than 5 per cent out to a frequency of 16 rads/sec, (30 dBs down), the maximum error of 7.5 per cent occurred at 20 rads/sec, (40 dBs down). The pole/zero locations computed by BODFIT over the bandwidth 0.1 to 20 rads/sec gave a transfer function of
Fig 2.7 CHECK ON CURVE FITTING PROGRAM BODFIT

F(s) = \[0.1448 \frac{(s+8.81+j14.17)(s-0.28+j11.98)}{(s+3.27+j8.95)(s+2.52+j6.4)}\]

Fitted equation

Phase

Magnitude

Data points

Fitted curve

Rads/s

Magnitude (db)
Fig. 2.8 COMPARISON OF SPECIMEN TRANSFER FUNCTION WITH FUNCTION CALCULATED BY SPECTR FROM TIME DATA OUTPUT OF SPECIMEN
<table>
<thead>
<tr>
<th>FREQUENCY</th>
<th>CALCULATED RESPONSE</th>
<th>SPECIMEN RESPONSE</th>
<th>% ERROR</th>
<th>% ERROR</th>
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<td>-171.1</td>
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</table>

Table 2.1. Comparison of specimen transfer function with computed response

**SPECIMEN FUNCTION (SP)**

\[ F(S)_{SP} = \frac{16}{S^2 + 3S + 16} \]

**COMPUTED FUNCTION (CP)**

\[ F(S)_{CP} = \frac{15.95}{S^2 + 3.07S + 15.92} \]
\[ F(S)_{\text{calc}} = \frac{15.95}{S^2 + 3.07S + 15.92} \]

The percentage differences in the natural frequencies \(\omega_n\) and the damping factor \(\zeta\) between the fitted and the specimen transfer function were 2.5 per cent and 0.25 per cent respectively.

2.5.2 The effect of noise on the computation of the frequency response

Program SPECTR provides an averaging process in order to reduce the effect of noise present in the recorded data (Appendix I). The effectiveness of the averaging was checked by injecting noise \(n(t)\) into the sample output data as shown in Figure 2.9.

Fig 2.9 CONTAMINATION OF OUTPUT DATA \(y(t)\) BY NOISE \(n(t)\)

32.
Early analysis of results taken from a vehicle at constant speed had shown that a typical transfer function relating steering wheel angle $\theta$ to transverse acceleration of the vehicle centre of gravity $a_y$ could be described by a fourth order equation of the form:

$$\frac{\theta}{a_y} = \frac{0.1734 (S + 1.13 \pm j9.15)(S + 3.07 \pm j11.95)}{(S + 5.08 \pm j4.96)(S + 2.77 \pm j13.62)}$$

The frequency response of this function calculated from noise free data $x(t)$ and $y(t)$ is shown plotted in Figure 2.10.

Random noise ($n(t)$) was added to contaminate $y(t)$ to form $z(t)$. The noise was generated by calling a computer package procedure RANDREAL which produces random numbers of uniform distribution. The probability density function of steering wheel angles was assumed to have a normal distribution about the mean straight ahead position. To avoid biasing the data it was necessary that the noise contamination data also was normally distributed.

The normal distribution was produced from the uniform distribution by a method due to Box and Muller (Ref.11) who show that if $P_1$ and $P_2$ are defined as:

$$P_1 = (-2 \ln R_1)^{\frac{1}{2}} \sin 2\pi R_2$$
$$P_2 = (-2 \ln R_1)^{\frac{1}{2}} \cos 2\pi R_2$$

then $P_1$ and $P_2$ are independent variables each having Gaussian distribution where $R$ is a sample from a uniform random distribution.

The set of random numbers may be formed from the algorithm
Fig. 2.10 EFFECT OF ADDING 10% NOISE CONTAMINATION TO TIME DATA

\[ F(s) = 0.173 \left( \frac{S + 11.3 \pm 9.15}{S + 11.95} \right) \left( \frac{S + 3.07 \pm 11.95}{S + 5.08 \pm 4.96} \right) \]
The implemented computer program allowed the random distribution to be called with chosen RMS and mean values.

As a check on the effect of processing contaminated data, noise of 10 per cent RMS value, with zero mean, was added to y(t) to form z(t) (Fig.2.9). The result is plotted in Figure 2.10, and shows that SPECTR provides an adequate rejection of noise, indicated by only a small shift in the frequency response due to the added noise. The largest effect was noticeable in the value of the coherence function, at a frequency about 10 rads/sec, where the value fell from about 0.91 (0 per cent noise) to about 0.74 (10 per cent noise). Figure 2.10 shows that the loss in coherence was not reflected in the magnitude and phase plots. In practise it seems that a loss in coherence as large as 25 per cent may be tolerated, for the particular system under investigation, and yet produce dependable estimates of transfer functions.

2.5.3 The effect of correlation lag numbers on the frequency response

The autocorrelation function $R_{xx}(\tau)$ for zero biased data is

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) x(t + \tau) \, dt \quad (2.34)$$

In discrete form the autocorrelation coefficient for a sample with N data points is

$$\phi_{xr}(r, \Delta t) = \frac{1}{N-r} \sum_{n=1}^{N-r} x_n x_{n+r} \quad (2.35)$$

for $r = 0, 1, 2 \ldots M$
where \( r \) is the lag number, \( M \) is the maximum lag number and \( \phi_{xr} (r, \Delta t) \) is the value of the correlation coefficient at lag \( r \), and \( \Delta t \) is the time period of the lag.

The time displacement \( \tau \) is related to the maximum lag number by

\[
\tau = r \cdot \Delta t \tag{2.36}
\]

so that the maximum displacement is

\[
\tau_{\text{max}} = M \cdot \Delta t \tag{2.37}
\]

The power spectral density \( G_x \) has been stated as the Fourier Transform of the correlation functions (the Wiener-Khinchine relations, Ref.12).

That is

\[
G_{xx} (f, \tau_{\text{max}}) = \int_{-\tau_{\text{max}}}^{+\tau_{\text{max}}} \phi_{xx} (\tau) e^{-2\pi jf\tau} d\tau \tag{2.38}
\]

in discrete form

\[
\phi_{xx} (\tau) = \phi_{xr} (r, \Delta t) \tag{2.39}
\]

where

\[
\tau = r \cdot \Delta t
\]

\( r = 0, 1, 2, 3, \ldots, M \)
Hence the discrete form of

\[ G_{xx}(f, \tau_{\text{max}}) = G_{xr}(r, M\Delta t) \]  (2.40)

is

\[ G_{xr}(f, M\Delta t) = \Delta \tau \sum_{r=0}^{M} \phi_{rx}(r, \Delta \tau) e^{-2\pi j r \Delta t}. \]  (2.41)

The computation of \( G_{xr}(f, M\Delta t) \) is made at each discrete frequency so that

\[ f_k = k \Delta f = \frac{k}{\tau_{\text{max}}} = \frac{k}{M \cdot \Delta t} \]  (2.42)

\( k = 1, 2, 3, \ldots, N \)

Hence

\[ G_{xr}(f_k, M\Delta \tau) = \Delta \tau \sum_{r=0}^{M} \phi_{xr}(r, \Delta \tau) e^{-2\pi j \frac{kr}{M}} \]  (2.43)

\( k = 1, 2, 3, \ldots, M \)

The value of \( G_{xr}(f_k, M\Delta \tau) \) is unique only out to \( M/2 \), the Nyquist cut off frequency, where at this point the calculations recur in a reversed sequence.

The quantity \( \phi_{xr}(r, \Delta \tau) \) can be normalised by dividing by the sample variance \( \phi_{xr}(0) \), then the ratio of the autocorrelation coefficient at lag \( r \) to the variance should lie between plus and minus unity.
That is

\[ 1 \leq \frac{\phi_{Xr}(r, \Delta t)}{\phi_{Xr}(0)} \leq 1 \]  

(2.44)

It is convenient to calculate the power spectral density from normalised data because the correlation coefficients are then of predictable numerical range suitable for visual appraisal and comprehension, and convenient for handling by a digital computer. Allowance must be made if the absolute value of the power spectral density is subsequently required. Similar reasoning to the above formulation applies to the cross-correlation and cross spectral density coefficients.

A consequence of taking the discrete Fourier Transform of the cross-correlation function in order to provide the power spectral density is that the resolution bandwidth \( B_e \), of the resulting spectral estimate, is related to the lag number \( M \) by the relationship

\[ B_e = \frac{1}{\tau_{\text{max}}} = \frac{1}{M \Delta t} \]

(2.45)

This shows that the choice of lag number determines the degree of resolution of the calculation, or in other words, it provides a filtering or averaging process. Clearly the degree of filtering chosen depends on the spectral content of the system under investigation and must be arranged not to discard any valuable information as a result of the averaging process.

The lag number was chosen for the system under present investigation, by making a number of exploratory calculations on the same set of typical data over a range of correlation lag numbers. The frequency response produced is shown in the form of a Nyquist plot in Figure 2.11 and a
Bode plot in Figure 2.12. For a data sample of \( N = 252 \) points there is little difference in the magnitude or phase for the correlation lag number \( r = 100 \) or \( r = 200 \), although the 100 lag calculation tended to give a smoother result as might be expected from an averaging process. There is a marked loss in magnitude for lag numbers of \( r = 20 \) and \( r = 50 \) (Fig. 2.11); this is consistent with the tail of the cross-correlation function being truncated as confirmed by Figure 2.13 which shows that the information contained in the cross-correlation function is negligible after a lag number of about \( r = 60 \).

The frequency response data generated from the 100 and 200 lag calculations was processed by BODFIT to give the dominant second order pole/zero patterns which were

\[
F(S) = \frac{0.0395 (S^2 + 2.65S + 121.65)}{(S^2 + 16.78S + 78.13)} \quad \text{for 100 lags}
\]

\[
F(S) = \frac{0.0383 (S^2 + 2.65S + 121.88)}{(S^2 + 16.38S + 75.52)} \quad \text{for 200 lags}
\]

The difference in damping factor and natural frequencies were 2.3 per cent and 3.3 per cent respectively.

On the basis of these exploratory calculations the lag number was chosen to be 100 for subsequent calculations of frequency response.

It is worth noting that the effort made in establishing the lowest acceptable lag number paid not only in giving improved smoothed results for easier curve fitting, but also gave a saving of over 50 per cent in computer time.
Fig. 2.11 EFFECT OF CORRELATION LAG NUMBER ON FREQUENCY RESPONSE
Fig. 2.12. EFFECT OF LAG NUMBER ON FREQUENCY RESPONSE
Fig. 2.13 CROSS CORRELATION COEFFICIENTS

Δt = 0.104 seconds
Summary (Chapter 2)

The relationships which in the time domain (correlation) and in the frequency domain (power spectra) are used in the identification of system dynamics, have been reviewed.

The concept of pole/zero patterns has been introduced and their interpretation (second order system) in terms of physical quantities such as, damping factor and damped and natural frequencies has been demonstrated.

The method of fitting the poles and zeros to a complex frequency function has been outlined together with a brief introduction to root locus analysis.

The computer programmes which implement the above concepts are:-

SPECTR = frequency response from time series data
BODFIT = pole/zero patterns from frequency response
TRYSYM = digital simulation of transfer functions
ROOTL = root locus analysis

The programs have been validated and shown to adequately reproduce a fourth order system subjected to 10 per cent noise contamination in the output signal.

It has been shown that the method of obtaining the PSD's by transformation of the correlation coefficients allows the experimenter to smooth the spectral estimate to his requirement by selection of the appropriate lag number in the correlation procedure; an added advantage in the particular transfer function investigated was a 50 per cent saving in computer time.
3. Instrumentation

The instrumentation developed for the recording of vehicle motions consists of the following three main units. Firstly the PRBS GENERATOR (Plates I and II) for producing the perturbation signal to be superimposed on the normal steering input, secondly the SERVO-MECHANISM (Plates III and IV) for converting the PRBS signals into steering wheel angle movement, and thirdly, the SIGNAL CONDITIONING and RECORDING UNIT (Plates I and V) for filtering and recording the vehicle's angular and transverse motions and the steering wheel angles. These units are described in the following paragraphs.

3.1 Random signals and their generation

An adequate identification of a system by analysis of recorded data depends upon the system being excited over all the important modes of vibration. If the normal input to the system contains sufficient frequency content to do this then the on-line data is sufficient. However, in general the frequency content of an input signal is unknown at the time of the experiment and so it is prudent to add to the input a random time signal of known frequency content which will excite the system over all its vibrational modes.

White noise, which has an equal energy spectrum is clearly a choice for the added signal. The generation of white noise from noisy diodes, thyatrons, photocells, etc, suffers from difficulty in controlling the statistical properties, that is if on-line data or 'pink' noise
PLATE I  Front panel of PRBS generator and signal conditioning unit
PLATE II  PRBS generator and signal conditioning unit
PLATE III  Steering servo mechanism
PLATE IV  Steering linkage
PLATE V Transducers, signal conditioning and record units
generated as described are used as the excitation signal, it is necessary to average an ensemble set of data for the following reasons.

An estimate of the spectral power $G_X(f)$ in a sample record $x(t)$ of length $T$ is

$$\hat{G}_X(f) = \frac{2}{T} |X(f, T)|^2$$

(3.1)

where $|X(f, T)|$ is the Fourier Transform of $x(t)$.

Now, $|X(f, T)|^2$ has coincident and quadrature components $C^2(f, T)$ and $Q^2(f, T)$ respectively,

$$|X(f, T)|^2 = C^2(f, T) + Q^2(f, T)$$

(3.2)

Equation (3.2) can be interpreted as the sums of squares of two independent Gaussian variables which define a Chi-squared distribution with two degrees of freedom ($n$). The mean and variance of a Chi-squared distribution are $n$ and $2n$ respectively. The normalised standard error $\epsilon$ is

$$\epsilon = \frac{\sqrt{2n}}{n} = \sqrt{\frac{2}{n}}$$

For $n = 2$, $\epsilon = 1$, this means that the standard deviation of the estimate is as great as the quantity being measured which indicates no confidence at all. Increasing the record length does not alter the distribution function defining the random error of the estimate. The solution is to take a number of sets of data and average the sum, thereby increasing the number of degrees of freedom of the Chi-squared distribution. This
argument applies to a sampled data noise input where the considerations of Gaussian statistics applies. In the case of PRBS the signal is periodic and of known frequency content hence one sample $x(t)$ of a single sequence will be the same as the next, that is by choosing PRBS as the exciting function we have eliminated the statistical need for ensemble averaging, however there remains the need to sample sets of data in order to reduce unwanted noise in the output signal (Appendix I).

The analog generation and use of white noise was abandoned, for the reasons stated above, in favour of a digitally generated binary noise with output levels $\pm 1$ (Fig.1.2).

Binary noise waveforms permit relatively simple control of DC levels and RMS output through accurate clamping of the binary output levels, but the stationarity and randomness properties of the noise generation mechanism still require careful attention. A digital shift register sequence (Refs.12,13) complies with this requirement and can produce a pseudo-random noise of frequency content up to about $4 \, \text{MHz}$. For example, Figure 3.1 shows a 3-bit shift register whose output is fed through an EX-OR gate to produce a maximum length sequence (M-seq), its reverse and a non-maximum length sequence.

The maximum sequence length ($P$) obtainable with an EX-OR (modulo 2 adder) feedback is $2^n - 1$ bits; for an $n$-bit shift register we can have $2^n$ states but the first state will contain all zeros which when fed back will produce only zeros, for this reason a shift register random sequence generator must be primed with an initial non-zero count in any one of the register locations to commence the generation of a sequence.
Fig. 3.1 EXAMPLES OF (a) M-SEQUENCE, (b) REVERSE M-SEQUENCE, (c) NON M-SEQUENCE

Fig. 3.2 EXCLUSIVE OR-GATE TRUTH TABLE
Figure 3.2 shows a truth table defining the EX-OR logic and Table 3.1 lists the maximum length sequences generated for shift registers of n-bits using feedback through an EX-OR gate.

<table>
<thead>
<tr>
<th>M</th>
<th>Sequence length</th>
<th>Feedback other than last stage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>1 or 2</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>1 or 3</td>
</tr>
<tr>
<td>5</td>
<td>31</td>
<td>2 or 3</td>
</tr>
<tr>
<td>6</td>
<td>63</td>
<td>1 or 5</td>
</tr>
<tr>
<td>7</td>
<td>127</td>
<td>1 or 3 or 4 or 6</td>
</tr>
<tr>
<td>8</td>
<td>255</td>
<td>4 and 5 and 6</td>
</tr>
<tr>
<td>9</td>
<td>511</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>1023</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td>2047</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 3.1 Feedback loops for some maximum length sequences

It is a feature of every PRBS, which supports the substitution for true random sequences, that they have the following properties.

(a) In each maximum length sequence every state is repeated except the all zero state, so that for each maximum length of number P, bits

\[ P = 2^n - 1 \]

and the number of states is \( \frac{P + 1}{2} \) at one level and \( \frac{P - 2}{2} \) at the other.
(b) In every maximum length sequence one half of the number of the number of bits (1 and 0's) are of length one clock period; 1/4 are of length 2 clock periods, (00 or 11) etc, and for each run of 0's there are an equal number of 1's. This property is approximate for short sequence lengths.

(c) If successive bits are \( a_1, a_2, a_3 \ldots a_n \) then the autocorrelation function \( \phi(k) \) is

\[
\phi(k) = \frac{1}{2^n - 1} \sum_{i=1}^{2^{n-1}} a_i a_{i+k} \tag{3.4}
\]

where

\[
\phi(0) = a^2 \tag{3.5}
\]

and

\[
\phi(pk) = -\frac{a^2}{2^n - 1} \quad p > 1 \tag{3.6}
\]

Different initial priming of the shift registers produces identical sequences but shifted in time, on the other hand different EX-OR feedback conditions produces different length sequences. An n-bit shift register can produce \( \psi(P) \) different maximum length sequences where \( \psi(P) \) is the number of positive integers less than n and relatively prime to n.

The power density spectrum of PRBS is a line spectrum expressed as

\[
G_{xx}(\omega) = \frac{P + 1}{p} a^2 \Delta t \sum_{\tau=0}^{P-1} \left[ \sin \frac{\pi \frac{\tau}{P}}{\frac{\pi}{P}} \right]^2 \tag{3.7}
\]

where \( P = 2^n - 1 \).
Spectral lines are separated by the reciprocal of the period $P\Delta t$, that is

\[ \Delta f = \frac{1}{P\Delta t} \quad (3.8) \]

and the power content is proportional to

\[ \left( \sin \frac{\pi}{P} \right)^2 \left( \sin \frac{\pi}{P} \right)^2 \quad (3.9) \]

which gives an energy spectrum with 0.1 dB fall off out to

\[ f = \frac{1}{2\Delta t} \quad (3.10) \]

The above considerations were paramount in deciding to use PRBS as the exciting signal for the vehicle steering in contrast to the normal steering wheel motions. In summary they were:

(1) By suitable choice of PRBS a virtually equal energy spectrum may be generated to excite all modes of the vehicle's dynamics.

(2) The sequence is statistically well defined and controllable.

(3) It can be superimposed upon the normal steering wheel movements, a small signal amplitude has a linearizing effect on the transfer function estimate.

(4) It is easily generated and repeatable.
(5) The bandwidth is simply proportional to the clock rate and sequence length.

(6) The sequence can be reset to repeat a sequence of events at any time.

3.2 Electrical circuits for the generation of PRBS

A simple circuit for the generation of PRBS is shown in Figure 3.1, the principle of generation by digital shift register was enlarged to produce the PRBS unit shown in Plate II. The PRBS unit consists of two basic circuit configurations, firstly the clock pulse generator circuit and secondly the EX-OR and shift register circuits.

The clock pulse generator circuit is shown in Figure 3.3. The signal pulse is derived from an XR-220 frequency generator and the pulse is sharpened by a 'one-shot' integrated circuit SN 7412N to give a pulse width of 100 microsecs. The circuit design allows for selection of the range of clock pulses 1, 5, 10, 20, 50, 100 pulses per second by selection at front panel switch Sw.1. The individual ranges are formed by dividing down the basic frequency of 1000 pulses/sec produced by the XR-220 generator, by decade counters. For example, if Sw.1 is set to position Sw.1a then Pin 13 of AND-gate IC-I is set to logic 1 and all other comparable AND-gate pins (eg I2, I5, I10) are at logic 0. The 1000 p/s from the XR-220 is divided down by the divide 10 counter D, hence Pin 1 on AND-gate I receives 100 p/s which is reflected at gate output pin 8. The clock pulse is directed to clock output terminal T2 via OR-gate B and NAND-gate G. A facility is provided for manually inching the pulse generator by front panel push button PB.3 driving 'one-shot' C which outputs to NAND-gate G.
The clock pulse is fed from terminal F2 (Fig.3.3) to clock input terminal T2 (Fig.3.4) and so to the two series 5-bit shift registers N and M making a shift register of total length 10-bits. Selection of a PRBS length is made by front panel switch Sw.3, the selectable sequence lengths are 15, 31, 63, 127, 255, 511, 1023 bits.

For example, to generate a 15-bit PRBS length we see from Table 3.1 that a shift register of at least 4 bits is necessary with feedback to an EX-OR gate from the third and fourth bit positions, ie terminals C and D on shift register N Figure 3.4. This is realised in Figure 3.4 by closing Sw.2a which brings Pins 2 and 4 of AND-gate R to the logic 1 level leaving all other comparable inputs (eg R10, R12: Q2, Q4:) at logic 0. Thus, when register cells C or D are at logic 1 there is a logic 1 output at pin 3 of EX-OR gate X. The clocking rate of the shift registers determines the speed of the PRBS generated. Light emitting diodes (LED's) are included in the circuit to indicate the state of each of the 10 bits of the shift registers and the state of the PRBS signal.

We see from Table 3.1 that four feedback paths are required to generate a 255 bit sequence. This was implemented by using 3 EX-OR gates as shown in Figure 3.5, the Boolean algebra and the truth table supporting the circuit are also shown in the figure.

The PRBS drive to the steering wheel servo-mechanism is made through a constant current amplifier W to the base of transistor AA (BC214K, Fig.3.4). A logic 1 at AAb will raise the base above cut-off and cause the transistor to conduct, thus providing a constant current drive via jack-plug JPB to the servo amplifier. The magnitude of the current may be set by potentiometer RV.1, this in turn sets the amplitude of perturbation of the vehicle's steering wheel.
Therefore, 

\[ F = (\overline{AB} + \overline{BA})(\overline{CD} + \overline{DC})(\overline{CD} + \overline{DC})(\overline{AB} + \overline{BA}) \]

\[ F = \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} + \overline{ABCD} \]

Truth table for 4-input EX-OR

<table>
<thead>
<tr>
<th>AB</th>
<th>00</th>
<th>01</th>
<th>11</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>00</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>01</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3.5 CIRCUIT AND TRUTH TABLE FOR A 4-INPUT EX-OR
The voltage and current supply to both the clock circuits and the shift register circuits are controlled by a chopper voltage stabilizer and an LM309K current stabilizer.

The accuracy of the clock pulse generator was checked against a crystal controlled timer, the results are given in Table 3.2. The time periods were close to that expected by appropriate selection of resistors and capacitors required for the operation of the XR-220.

<table>
<thead>
<tr>
<th>PRBS unit switch position (pulses/sec)</th>
<th>Measured period (secs)</th>
<th>Measured frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a (100)</td>
<td>0.0104</td>
<td>96.15</td>
</tr>
<tr>
<td>b (50)</td>
<td>0.0208</td>
<td>48.08</td>
</tr>
<tr>
<td>c (20)</td>
<td>0.0520</td>
<td>19.23</td>
</tr>
<tr>
<td>d (10)</td>
<td>0.1040</td>
<td>9.62</td>
</tr>
<tr>
<td>e (5)</td>
<td>0.2080</td>
<td>4.81</td>
</tr>
<tr>
<td>f (1)</td>
<td>0.1040</td>
<td>0.962</td>
</tr>
</tbody>
</table>

Table 3.2 Clock pulse generator calibration (by crystal controlled oscillator)

3.3 The vehicle steering servo mechanism

The steering wheel servo mechanism is shown in Plates III, IV, V and VI. Its original purpose was for use in the remote radio control of vehicles involved in high speed test impacts with road safety barriers and street furniture (Ref.14). It was adapted for use in its present mode by replacing the radio receiver by the PRBS generator and making connection to the servo amplifier through a constant current source.
The servo mechanism is a velocity feedback position control system, the position control signal is taken from a ten turn potentiometer geared to the steering wheel. The servo takes the form of an on-off controller in which a 1/4 hp DC motor is coupled to the steering wheel via two electro-magnetic clutches and a worm and wheel drive. A command signal to turn right engages the appropriate clutch and so makes the connection between the DC motor and the steering wheel, the wheel will continue to turn until a null position is registered by the feedback potentiometer; a similar sequence takes place for a left steer command. Clearly the left and right steer commands originate from the PRBS generator. In addition a hand wheel is available which provides a voltage at the summing junction of the servo amplifier, this enables an over ride signal to be input to the steering, thus setting the vehicle on a chosen mean path.

The frequency response of the steering wheel servo mechanism was measured and is shown in Figure 3.6. A 40 dB/decade roll off occurred at a break point of about 2 Hz, this is slightly lower than might be of importance in steering response measurements. However, the upper limit of a human operator is about 2.5 Hz to 3 Hz so a description of the vehicle's dynamics within this bandwidth is sufficient.

Later discussion with workers in vehicle handling suggested that oscillatory motions above about 3 Hz would result mainly from compliance in such components as steering bush rubber linkages and spring couplings, these vibrations would not play an important part in the overall response of the vehicle.
PLATE VI General view of vehicle instrumentation
Fig. 3.6 FREQUENCY RESPONSE OF STEERING SERVO MECHANISM
3.4 Signal conditioning circuits

Initially the instrumentation was set up to record five variables, these were, the steering wheel angle, the transverse and angular accelerations of the vehicle and the pitch and roll angles. The roll angle measurement was required both for producing a roll/steer angle transfer function and to make gravitational corrections to the transverse accelerometer output. Subsequently measurement of pitch and roll were abandoned on the basis that the 1/4 degree resolution of the available displacement gyroscope was too large to measure accurately the vehicular angular displacements produced from the small perturbations generated by the PRBS, but it follows that the errors introduced in rejecting the measurements were also small.

The recorded variables were conditioned by operational amplifiers and logged on an ultra-violet galvanometer recorder. Figure 3.7 shows that the clock signal pulse is input to a high impedance buffer amplifier A1 and connected to a simple variable gain op. amp. multiplier. Resistor R5 acts as a current limiter for galvanometer G whose sensitivity is 5 μA/cm with a bandwidth of 100 Hz. The same type of galvanometer was used in all circuits.

The circuit used for the PRBS signal is shown in Figure 3.8.

The PRBS signal is taken from the emitter of the transistor drive to the steering servo amplifier, the PRBS amplitude level may be set by potentiometer RV1 (Fig.3.4), thus the amplitude of the PRBS signal is reflected in the recorded trace.

The steering wheel angle is picked up from potentiometer R13 (Fig.3.9) geared to the steering column through anti-backlash
gears. A voltage proportional to the steering angle is fed from R13 to buffer amplifier Ala and then to multiplier A2. A summing junction at the input to A2 is supplied with a bias voltage from amplifier A16 via Sw.1 for initial setting of the galvanometer spot position. The output from A2 is filtered by a two pole filter made up of two first order lags. The break point of 4 Hz is determined by 1/CR and is followed by a roll off of 40 dB./decade. The transfer function relating the input voltage $V_{in}$ to the output voltage $V_{out}$ is given by

$$\frac{V_{out}}{V_{in}} = \left( \frac{R_2}{R_1} \right)^2 \left( \frac{1}{SCR + 1} \right)^2$$

(3.11)

The measured frequency response is shown in Figure 3.10. The signal conditioning circuits for the transverse and angular accelerometer outputs were similar and are shown in Figures 3.11 and 3.12. Amplifier Ala provides a buffered input to a multiplier which has selectable switched ranges, the output from the multiplier is filtered in a similar manner to the steering angle circuits. It was arranged that the cut-off point would be the same on all filters to avoid the inherent phase shift in the filter circuits introducing erroneous errors in phase difference between vehicle input and output signals. Frequency response plots for the transverse and angular amplifier channels were similar to the steering channel amplifier.

Summary (Chapter 3)

Methods of producing random signals have been reviewed and in particular the electronic circuits for the generation of Pseudo Random Binary Sequences (PRBS) by digital feedback to shift register have been detailed.
The method of superimposition of the random signal upon the steering wheel motions through a steering servo-mechanism has been outlined.

Circuits are given for the signal conditioning of the transducer outputs which reflect the steering wheel angles, the angular and linear accelerations and the gyroscope outputs.
FIRST ANGLE PROJECTION

RESISTORS
R1-R12  R13  R14  R15  R16  R17  R18  R19  R20  R21  R22  R23  R24
C1  C2
CAPACITORS
SEMICONDUCTORS
SWI  PLI  ME1
MISCELLANEOUS

COMP REF  ITEM TECHNICAL DATA
R1  RESISTOR 4K7
R2  RESISTOR 240R
R3  RESISTOR 200R
R4  RESISTOR 16R
R5  RESISTOR 16R
R6  RESISTOR 15R
R7  RESISTOR 16R
R8  RESISTOR 16R
R9  RESISTOR 16R
R10 RESISTOR 200R
R11 RESISTOR 240R
R12 RESISTOR 4K7
R13 POTentiOMETER 10K
R14 RESISTOR 10K
R15 RESISTOR 10K
R16 POTentiOMETER 100K
R17 RESISTOR 47K
R18 RESISTOR 10K
R19 RESISTOR 47K
R20 RESISTOR 47K
R21 RESISTOR 10K
R22 RESISTOR 47K
R23 RESISTOR 25R
R24 RESISTOR 3K5
IC-A1 INTEGRATED CIRCUIT SN7556
IC-A2 INTEGRATED CIRCUIT SN7556
IC-A3 INTEGRATED CIRCUIT SN7556
C1 CAPACITOR 0.15µF
C2 CAPACITOR 0.15µF
ME1 GALVOMETER
SKTI SOCKET 3 PIN
PLI PLUG 3 PIN
SWI SWITCH

Dimensions in millimetres
Materials
TOLERANCES
No place of decimals ± 1mm
One place of decimals ± 0.5mm
Two places of decimals ± 0.1mm
(Unless otherwise stated)

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Department of the Environment
Crowthorne Berkshire

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Fig. 3.10 FREQUENCY RESPONSE OF SIGNAL CONDITIONING AMPLIFIERS
4. The choice of PRBS

Three main factors are involved in the setting up of the PRBS perturbation signal, they are the amplitude, the sequence length and the clock rate. In general these factors decide the energy content and the bandwidth of the system frequency spectrum over which the spectral estimate is valid.

The lowest frequency \( f_{\text{min}} \) and also the resolution of the system spectrum is

\[
f_{\text{min}} = \frac{1}{P \cdot \Delta t}
\]  

(4.0)

\( P \) = number of bits in PRBS
\( \Delta t \) = clock period

so that the total time period of data collection must be at least the same as the period of the lowest frequency, a rule of thumb is to log data over a period about five times the settling time.

The highest frequency \( f_{\text{max}} \) of the system spectrum is determined from the sampling theorem which states for band limited frequencies the signal is completely determined by samples taken at intervals of \( \Delta t \) where

\[
\Delta t \leq \frac{1}{2f_{\text{max}}}
\]
Consequently some assessment must be made of the vehicle's frequency spectrum by exploratory experiments in order to set up suitable values for the clock frequency and sequence length.

From the above equations, we see that the ratio of the maximum to the minimum frequency present in the PRBS spectrum for a chosen clock interval is

\[
\frac{f_{\text{max}}}{f_{\text{min}}} = \frac{P}{Z}
\]

Figure 4.1 shows a nomograph relating PRBS bandwidth to sequence length and clock interval, equation 4.2 and Figure 4.1 may be used as aids to the choice of the PRBS signal.

The amplitude of the PRBS is also a practical consideration, a first order choice is an amplitude which represents levels about those of error corrections made to the system under normal use. Experiments were made to select a perturbation amplitude which gave minimal interference to normal operation of the vehicle, yet provided a coherent relationship between the input and output signals as estimated by the coherence function.

In practice a limitation was placed on the measured bandwidth by a 40 dB/decade roll off at about 11 rads/sec in the frequency response of the servo-mechanism; however sufficient energy was present in the steering wheel input angle spectrum for coherent measurements to be made up to a frequency of about 15 rads/sec. Data was collected by adjusting the vehicle direction with the 'electric' steering wheel to a steady condition and then switching on the PRBS signal. The effect of noise in the output signals was reduced by collecting data over at least 5 sequence lengths. At the higher vehicle speeds (90 km/h) track...
length was insufficient to record continuously 5 sets of data (Plate VII) so that sets needed to be assembled from several test runs, the facility of starting the PRBS from the beginning of its sequence proved advantageous.

Since the vehicle is in a steady state before switching on the PRBS it is necessary to allow a 'settling' time, after switch on, before records are taken.

4.1 Transducers and calibration

The task in the identification of vehicle dynamics is to determine the relationship between the steering wheel angle and the vehicle dynamic response, where the response is the transverse motion of the vehicle which locates it with reference to inertial space or a ground plane such as the roadway. The following paragraphs describe the variables which need to be measured to achieve this.

We see from Figure 4.2 that, with reference to a vehicle borne axis, an accelerometer located with its sensitive axis in a transverse plane will measure $a_y(t)$ where

$$a_y(t) = v(t) + ur(t)$$  \hspace{1cm} (4.3)

$v = $ side slip velocity
$u = $ forward speed
$r = $ heading or yaw rate

Translational motions of the vehicle relative to the roadway differ from motions relative to body axis but are derivable there from
Fig. 4.1: NOMOGRAPH FOR SELECTION OF PRBS SEQUENCE LENGTH, BANDWIDTH AND CLOCK INTERVAL.
PLATE VII Vehicle on test track
\(\Delta v_{P_1X_1} = (u + \delta u) \sin \delta \psi + (v + \delta v) \cos \delta \psi - v\)

Vehicle acceleration \(a\) is change in \(\Delta v\) in time \(\delta t\)

\[
a_{P_1X_1}(t) = \frac{\delta v_{P_1X_1}}{\delta t} = \frac{u \sin \delta \psi}{\delta t} + \frac{\delta u \sin \delta \psi}{\delta t} + \frac{v \cos \delta \psi}{\delta t} + \frac{\delta v \cos \delta \psi}{\delta t} - \frac{v}{\delta t}
\]

Limit
\[
\delta \psi \rightarrow 0, \quad \frac{d}{dt} a_{P_1X_1}(t) = u \frac{d}{dt} \psi + \frac{dv}{dt} = u \dot{\psi} + \ddot{v}
\]

\[
\delta t \rightarrow 0
\]

Change in velocity parallel to \(P_1Y_1\) axis

\(\Delta u_{P_1Y_1} = (u + \delta u) \cos \delta \psi - (v + \delta v) \sin \delta \psi - u\)

\[
a_{P_1Y_1}(t) = \frac{u \cos \delta \psi}{\delta t} - \frac{\delta u \cos \delta \psi}{\delta t} - \frac{v \sin \delta \psi}{\delta t} - \frac{\delta v \sin \delta \psi}{\delta t} - \frac{u}{\delta t}
\]

Limit
\[
\delta \psi \rightarrow 0, \quad \frac{d}{dt} a_{P_1Y_1}(t) = \frac{du}{dt} - v \frac{d}{dt} \psi = \dot{u} - v \dot{\psi}
\]

\[
\delta t \rightarrow 0
\]

Fig. 4.2 VEHICLE ACCELERATION RELATIVE TO BODY AXIS
as shown in Figure 4.3. For angular motions, quantities related to spatial and body co-ordinates are the same. The transverse velocity $V_I(t)$ relative to an inertial co-ordinate system initially coincident with the unperturbed body axis is given by the integral of the transverse acceleration $a_y$ of the centre of gravity of the vehicle.

That is

$$Y_I(t) = \int a_y(t) \, dt \quad (4.4)$$

$$= \int v(t) + ur(t) \, dt \quad (4.5)$$

In Laplace notation this becomes

$$Y_I(s) = \frac{v(s) + ur(s)}{s} \quad (4.6)$$

$$Y_I(s) = \frac{v(s) + u_0 r(s)}{s^2} \quad (4.7)$$

Hence for small intervals of time, the double integration of the accelerometer output gives the value of the transverse inertial displacement $Y_I(t)$ relative to a pre-fixed set of co-ordinates, e.g., the roadway, and double integration of the angular acceleration $\dot{r}(t)$ provides values of the heading or yaw angle $\psi(t)$.

The path angle may be determined from the slip angle $\beta(t)$ plus the heading angle $\psi(t)$.

Where

$$\beta(t) = \frac{v}{u}(t) \quad (4.8)$$
\[ y_I(t) = \int_{0}^{t} \left[ (v + u \dot{\psi}) \cos \theta + (\dot{u} - v \dot{\psi}) \sin \theta \right] \, dt \] 

Therefore

\[ y_I(t) = \int_{0}^{t} [v \cos \theta + u \sin \theta] \, dt \]

Similarly

\[ x_I(t) = \int_{0}^{t} (u \cos \theta - v \sin \theta) \, dt \]

Note:-

When the vehicle axis is coincident with the inertial axis then for small angles

\[ y_I(t) \approx \Delta t \int_{0}^{\Delta t} (u + v \dot{\psi}) \, dt \]

\[ x_I(t) \approx \Delta t \int_{0}^{\Delta t} (u - v \dot{\psi}) \, dt \]

Fig. 4.3 PREDICTION OF POSITION RELATIVE TO INERTIAL AXIS
therefore, the path angle $\gamma(t)$ becomes,

$$
\gamma(t) = \frac{V}{u}(t) + \psi(t)
$$

(4.9)

It was assumed, for the small angles superimposed by the PRBS that the path angle would approximate closely to the heading angle and consequently a measurement of $v(t)$, the side slip velocity was not made.

In summary, the quantities measured to determine the transfer function which would identify the vehicle's steering response relative to an inertial axis were the steering wheel angle, the transverse acceleration and the angular acceleration.

Erroneous results could arise from the output of the transverse accelerometer due to gravitational errors generated by vehicle roll angle displacement and roll acceleration. The measurement of vehicle roll angle as a function of time was attempted with a displacement gyroscope, the resolution was about $\frac{1}{4}$ degree. Roll angle signal levels were found to be too small to give consistent results. Figure 4.4 shows one of the better results plotted as a frequency response transfer function relating steering wheel angle to roll angle, measurement was made at a vehicle speed of 64 km/h.

The effect of roll angle on the measurement of transverse acceleration was estimated by plotting the frequency response of the transverse acceleration transfer function with and without roll corrections (Fig.4.5).

The fitted transfer functions gave the following equations
The RMS value of the transverse acceleration time series data was increased by about 14 per cent due to the addition of roll correction, and Figure 4.5 shows a small shift in the magnitude plot of the frequency response. The coherence value was degraded by non roll correction and occurred about a frequency bandwidth of 1.2 Hz to 2.0 Hz although the fitted transfer functions (equations 4.10 and 4.11) showed little change in the dominant pole positions.

As a consequence of these checks and the difficulty of repeating results due to the poor stability and low resolution of the gyroscopes, roll measurement for the correction of transverse acceleration was abandoned with the knowledge that fairly small errors would be present in the transverse acceleration transfer function, and were likely to occur only in the higher order poles.

The accelerometer transducers were rigidly mounted at the centre of gravity of the vehicle. Figure 4.6 shows the method of locating the height of the C of G by tilting the vehicle and measuring the angle of tilt and the front axle load, the measurements were repeated for the rear axle loads and an overall mean taken from both results. During the measurements the springs were blocked to prevent the C of G moving under
Fig. 6.4  ROLL ANGLE TRANSFER FUNCTION
Fig. 4.5 TRANSVERSE ACCELERATION TRANSFER FUNCTION WITH AND WITHOUT ROLL ANGLE CORRECTION
By taking moments about P we may write the distance of the C of G from the front axle as,

\[ b = \frac{W_R B}{(W_R + W_F)} \]

Taking moments about Q and rearranging we get,

\[ W_F = \frac{Wh}{B} \tan \theta + W(1 - \frac{b}{B}) \]

where \( W = W_F + W_R \)

Which is in the form of a straight line equation \( Y = mX + c \)

The height (H) of the C of G above ground is, \( H = r + h \)

Fig. 4.6 METHOD OF LOCATING THE VERTICAL HEIGHT OF THE C OF G
load transference. The mean result gave the measured height of the C of G as 0.56 metres.

Static calibration of the linear accelerometer was made by comparison with component values of the earth's gravitational field using an accurately scaled protractor table. Results are given in Figure 4.7.

The angular accelerometer was calibrated on an oscillating platform using the linear accelerometer as a reference, results are shown in Figure 4.8.

4.2 Sampling and aliasing

An analogue signal is usually sampled at equal time intervals $\Delta t$, the problem is to decide the length of $\Delta t$. If the record is sampled too frequently a mass of redundant data wastes computer time, and if the sampling is too infrequent contamination of the low frequencies by the higher frequencies will result. This phenomenon is known as aliasing, and has the effect of folding back the higher frequencies to erroneously enhance the lower frequencies. An illustration of this in cine photography is the apparent reversal of a revolving wheel caused by a frame speed too slow to record progressive movements of the spokes in the wheel.

Figure 4.9 shows two continuous waveforms, one twice the frequency of the other; at least two samples per cycle are necessary to define a particular frequency component. The sampling procedure indicated in Figure 4.9 has confused the higher with the lower frequencies in such a manner that the higher waveform could not be recovered from the sampled
Fig. 4.7: CALIBRATION OF LINEAR ACCELEROMETER
Angular accelerometer placed at any position on rigid beam

For small angles $\theta$, $\ddot{x} = r \ddot{\theta}$

Fig. 4.8 CALIBRATION OF ANGULAR ACCELEROMETER
data, $t_1$, $t_2$, and $t_3$. This confusion of waveforms or folding back can best be illustrated in the frequency domain.

![Waveform Confusion Caused by Aliasing](image)

**Fig. 4.9 WAVEFORM CONFUSION CAUSED BY ALIASING**

The sampling process may be interpreted as the multiplication of the original waveform by a train of equally spaced, constant height Dirac pulses. Since multiplication in the time domain is equivalent to convolution in the frequency domain, the resulting spectrum may be found by convolving the individual spectra of the continuous waveform with the pulse train (Fig. 4.10).
Consider the frequency spectrum of Figure 4.10b, its characteristics are such that it is repetitive, real and symmetrical about $\Delta t = 0$ and therefore has a line spectrum containing only cosine terms with fundamental frequency $\frac{1}{\Delta t}$ Hz $\left(\frac{2\pi}{\Delta t}$ rads/sec$\right)$. The harmonics may be simply determined from the Fourier Series by observing that the magnitude ($a_r$) of any harmonic term in a periodic waveform may be found by multiplying the waveform by a cosine of the chosen frequency and integrating over a complete period.

$$a_r = \int_{0}^{2\pi} P(t) \cos 2\pi ft \, dt \quad (4.12)$$

We see pictorially from Figure 4.11 that the Dirac pulse will pick out only that value of the cosine waveform at the time at which the pulse exists, that is at $\Delta t = 0$ and at every complete period $\Delta t$. This observation is supported by the sifting property of the Dirac function $\delta$, which states that,
The conclusion is that the spectrum of the pulse train of Figure 4.11 is also a pulse train as shown in Figure 4.12 of interval $\frac{2\pi}{\Delta t}$ having equal amplitude and an infinite number of harmonics.

Hence convolving a continuous spectrum $X(\omega)$ with the pulse train $P(\omega)$ gives $Y(\omega)$. 

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That is,

\[ Y(\omega) = P(\omega) * X(\omega) \quad (4.14) \]

where * is the convolving process.

Writing equation 3.26 in the form of the convolution integral we have,

\[ Y(\omega) = \int_{-\infty}^{+\infty} P(\omega) \cdot X(\omega - u) \, du \quad (4.15) \]

where \( u \) is a dummy variable.

We have already concluded that,

\[ P(\omega) = \ldots \delta\left(\omega + \frac{2\pi}{\Delta t}\right) + \delta(\omega) + \delta\left(\omega - \frac{2\pi}{\Delta t}\right) \ldots \quad (4.16) \]

and substitution in equation 4.15 gives,

\[ Y(\omega) = \int_{-\infty}^{+\infty} X(\omega - u) \left[ \ldots \delta\left(\omega + \frac{2\pi}{\Delta t}\right) + \delta(\omega) + \delta\left(\omega - \frac{2\pi}{\Delta t}\right) \ldots \right] \, du \quad (4.17) \]

The sifting property of the Dirac function \( \delta \) will produce from \( \delta\left(\omega + \frac{2\pi}{\Delta t}\right) \) a pulse at \( u = -\frac{2\pi}{\Delta t} \). So, for a single term \( Y_1(\omega) \) we have,

\[ Y_1(\omega) = \int_{-\infty}^{+\infty} X_1(\omega - u) - \frac{2\pi}{\Delta t} \, du \]

\[ = X_1\left(\omega + \frac{2\pi}{\Delta t}\right) \quad (4.18) \]
Thus for a train of pulses we get,

\[ Y(\omega) = \ldots X(\omega + \frac{2\pi}{\Delta t}) + X(\omega) + X(\omega - \frac{2\pi}{\Delta t}) \ldots \]  

(4.19)

Equation 4.19 is represented in Figure 4.13 from which we see \( X(\omega) \) is repeated indefinitely at intervals of \( \frac{2\pi}{\Delta t} \) rads/sec. In general the convolution of a continuous function with a Dirac pulse train will produce the frequency spectrum of the function at intervals of the reciprocal of the pulse spacing.

![Fig.4.13 REPETITIVE EFFECT OF DIGITAL SAMPLING](image)

Thus for the bandwidth, \(-\omega \) to \(+\omega \), of the signal spectrum shown in Figure 4.13 a sampling time of greater than \( \Delta t \) will cause overlap or folding back of the higher frequencies with the lower frequencies (Fig.4.13).

In summary, the minimum sampling rate \( \Delta t \) which will faithfully reproduce a continuous time series signal of bandwidth \( \pm \omega \) is,

\[ \Delta t < \frac{1}{2f} \]  

(4.20)
Summary (Chapter 4)

The method of choosing the PRBS has been outlined and is aided by reference to a nomograph.

A synopsis is given of the mathematics of vehicle response to a steering wheel input with reference to both a vehicle borne axis and a ground inertial axis; this information defined the variables which need to be measured in order to locate a vehicle on a ground plane and consequently indicates the type and location of the transducers within the vehicle.

A method of finding the height of the centre of gravity of the test vehicle is developed and also a technique for calibrating the acceleration transducers is given.

The need for compliance with Shannon's sampling theorem is established both in the time and frequency domains, in brief it is necessary to sample time series data at least twice per cycle of the highest frequency present.
Experimental verification of the analytical technique

5. Validity of the measured transfer functions

The results of the identification of a vehicle's steering dynamics are presented under two headings, firstly open loop analysis (Chapter 6) and secondly, closed loop analysis (Chapter 7). The open loop results consider the migration of the poles and zeros as a result of intentional changes made in vehicle parameters, in general the results are presented as pole/zero patterns. The closed loop analysis attempts to give preference to sets of measured vehicle dynamics from a model driver's viewpoint, the results are presented in terms of root locus analysis.

Initially the task was to establish whether linear transfer functions generated from random time data were sufficiently accurate and repeatable to serve as an acceptable engineering measuring tool when applied to road vehicles which are known to have non-linear characteristics. A successful result would provide engineers and research workers with a means of quantifying the effect on vehicle response of a change in the vehicle's mechanics. The degree of accuracy and repeatability is given under Section 5.1 and Section 5.2.

Clearly it is a consideration of prime importance to know whether a particular movement in the pole/zero pattern is an improvement or not. The first intention was to measure a number of different vehicles which subjectively had been classified, by popular opinion, to have predominantly good or bad steering characteristics, and so in this way to establish groups of desirable pole/zero patterns. The cost and time required to
achieve this was prohibitive, consequently the work needed to be constrained to a single vehicle. The procedure was to make selective changes to the vehicle and observe the effect as migrations in the pole/zero patterns. Conclusions could then be drawn that a change which could be accepted generally as an improvement (or degradation) in the vehicle's performance revealed a particular trend in the pole/zero pattern. For instance it was postulated that tyre pressure substantially below the manufacturers quoted value decreased the desirable response of the vehicle, this provided the opportunity to observe the effect of such a change in the pole/zero pattern (Section 6.3).

It was important to have an understanding of the required magnitude of change in vehicle parameters which would reflect movement in the poles and zeros. Here the vehicle speed could be changed by discrete amounts and the resulting pattern migrations observed (Section 6.1).

The stability and linearity of the steering response could be indicated by observation of the pole zero migrations resulting from varying degrees of side force; placing the vehicle in a circular path provided the desired levels of side force (Section 6.2).

The tyre plays an important role in the make up of a vehicle's response, a simple and topical change that could be made was from cross-ply to radial ply tyres and assess the resulting response (Section 6.1).

Finally, a gross attempt was made to degrade the steering response by altering the vehicle's moment of inertia about a vertical axis and make observations on the result (Section 6.4).
5.1 The accuracy of a measured transfer function

An overall check was made in the time domain on the authenticity of one of the computed transfer functions by driving the computed model in program TRYSYM, with the original recorded steering wheel data, and comparing the time series output, from the model, with the data recorded in the vehicle.

The outputs from a fourth order transfer function and the recorded vehicle data, given in Figure 5.1, are of a transfer function relating the steering wheel angles to the transverse acceleration of the centre of gravity of the vehicle at a vehicle speed of 32 km/h. That is,

\[
F(S) = \frac{a_y}{\theta} = \frac{0.145 (S + 9.81 \pm j14.17)(S - 0.28 \pm j11.98)}{(S + 3.27 \pm j8.95)(S + 2.53 \pm j16.4)}
\]  
(5.1)

\(a_y\) = transverse acceleration  
\(\theta\) = steering wheel angle

Checks on several other derived transfer functions at different vehicle speeds gave similar results to the following.

Visually the two sets of time data (Fig.5.1) gave good comparison, but to put a quantitative value to the result a regression and correlation analysis was made on the two sets of data, that is between the recorded vehicle acceleration data and the generated transfer function output data.
The computation assumes that a linear equation of the form

\[ Y = a_0 + a_1 X \]

where \( Y \) = model data \( (y(t)) \)
and \( X \) = vehicle data \( (x(t)) \)
relates the two sets of variables, and the coefficients \( a_0 \) and \( a_1 \)
are determined from a least square fit procedure (Ref.15).

If \( Y_{(est)} \) represents the value of \( Y \) for given values of \( X \) then
a measure of the scatter of the data about the regression line of \( Y \) on \( X \)
is given by

\[ \sigma_{yx} = \sqrt{\frac{\sum (Y - Y_{(est)})^2}{N}} \]  \hfill (5.2)

\( \sigma_{yx} \) = standard deviation
\( N \) = number of samples in time \( T \).

The total variation of \( Y \) is \( \sum (Y - \bar{Y})^2 \), this may be written,

\[ \sum (Y - \bar{Y})^2 = \sum (Y - Y_{(est)})^2 + \sum (Y_{(est)} - \bar{Y})^2 \]  \hfill (5.3)

\( \sum (Y - Y_{(est)}) \) = unexplained variation (unpredictable)
\( \sum (Y_{(est)} - \bar{Y}) \) = explained variation (predictable)

The correlation coefficient \( (r) \) or product moment is given by
\[ r = \pm \frac{\text{Explained variation}}{\text{Total variation}} \]
\[ = \sqrt{\frac{\sum (Y_{\text{est}} - \bar{Y})^2}{\sum (Y - \bar{Y})^2}} \quad (5.4) \]

For two variables this reduces to,

\[ r = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{(\sum (X - \bar{X})^2)(\sum (Y - \bar{Y})^2)} \quad (5.5) \]

The value of the product moment lies between ±1 and gives a measure of the symmetry between waveforms \( X \) and \( Y \). It remains to determine the slope of the regression line \( a_1 \) to establish the value of the constant term relating \( X \) to \( Y \), assuming that \( a_0 \), the point where the regression line cuts the Y-axis, is zero.

The slope is given by

\[ a_1 = \frac{N \sum XY - \sum X \sum Y}{N \sum X^2 - (\sum X)^2} \quad (5.6) \]

In summary, the value of the product moment \( r \), determines the similarity in the oscillatory nature of the two sampled waveforms, and the slope \( a_1 \) determines the constant linear relationship; that is, if \( r = 1 \), and \( a_1 = 0.5 \), the sampled waveforms are identical in shape, but one has half the amplitude of the other.

This method of analysis was used to compare the similarity between the sampled waveforms computed from the derived transfer functions and that recorded in the vehicle.

The product moment of the waveforms in Figure 5.1 was 0.976, this value of \( \sigma \) suggests that there was about a 2.5 per cent difference between
**Fig. 5.1** A SAMPLE COMPARISON OF A TRANSFER FUNCTION OUTPUT WITH THE ORIGINAL RECORDED DATA
the oscillatory similarity of the waveforms. The slope of the regression line was calculated to be 0.931, which gives the error, for the estimate of the transfer function gain term, to be about 7 per cent.

5.2 Repeatability of measurements

In a practical test in which an attempt is made to produce two identical sets of time data, the situation arises where the vehicle may not be at exactly the same speed for each test, or the noise content of the data may change because of wind forces, or the vehicle may pass over a different track in the road surface, etc.

In order that the transfer function method could be established as an acceptable method of identifying vehicle dynamics, it was essential to know whether these extraneous variables would reflect in the results, and if so, whether they were large compared with those differences arising due to an intended change in a vehicle parameter. To make this assessment, time data were recorded of the steering wheel angles and the transverse and angular accelerations, under nominally similar conditions, on two separate occasions.

The erroneous contribution to horizontal vehicle motions, which could arise due to inputs from the road wheels, was assessed by setting the vehicle to run over various paths and maintain a steady steering wheel position. The resulting signals in the horizontal plane were found to be minimal compared with those caused by the PRBS input to the steering wheel. This exploratory measurement excluded the particular road surface profile as a cause of extraneous inputs, and also eliminated the need for treating the vehicle as a multi input system and so simplified the analysis accordingly.
Two nominally similar sets of data taken on the same day were processed by SPECTR and the computed frequency response functions gave almost identical results (Fig. 5.2).

As a check on the long term repeatability, data taken nearly six months apart relating steer angle to transverse and angular accelerations were examined and are shown as frequency transfer functions in Figure 5.3 and Figure 5.4.

The curves give good visual comparison between the spectral estimates. Curve fitting program BODFIT computed the dominant second order transfer functions to be, for the transverse acceleration transfer functions:

\[
F(S)_{17.9.74} = \frac{a_y}{\theta} (S) = \frac{0.44 \left( S + 0.03 \pm j10.86 \right)}{S + 4.69 \pm j6.68} \frac{g}{\text{deg} \cdot \text{100}} (5.7)
\]

\[
F(S)_{4.2.75} = \frac{a_y}{\theta} (S) = \frac{0.49 \left( S + 0.19 \pm j11.11 \right)}{S + 4.87 \pm j7.19} \frac{g}{\text{deg} \cdot \text{100}} (5.8)
\]

and for the angular transfer functions:

\[
F(S)_{17.9.74} = \frac{r}{\theta} (S) = \frac{1.05 \left( S - 0.46 \right) \left( S + 32 \right)}{S + 5.38 \pm j7.66} \left[ \frac{\text{rads}}{S^2 \text{deg}} \right] \cdot \frac{1}{\text{100}} (5.9)
\]

\[
F(S)_{4.2.75} = \frac{r}{\theta} (S) = \frac{0.29 \left( S - 0.96 \right) \left( S + 147 \right)}{S + 5.57 \pm j8.08} \left[ \frac{\text{rads}}{S^2 \text{deg}} \right] \cdot \frac{1}{\text{100}} (5.10)
\]

The percentage differences in the dominant pole real and imaginary parts for the transverse transfer function were 3.5 per cent and 7 per cent.
Speed 32 km/h

Date: 29.11.74

\[ F(s)_1 = 0.1561 \left( \frac{S + 7.03 + j16.13}{S + 3.06 + j8.72} \right) \]

\[ F(s)_2 = 0.1008 \left( \frac{S + 7.90 + j18.0}{S + 3.11 + j8.94} \right) \]

Fig. 5.2 REPEATABILITY OF MEASUREMENT ON SAME DAY (TRANSVERSE ACC. TRANSFER FUNCTION)
Fig. 5.3 REPEATABILITY OF MEASUREMENT (TRANSVERSE ACC. TRANSFER FUNCTION)
Fig. 5.4 | REPEATABILITY OF MEASUREMENT (ANGULAR ACC. TRANSFER FUNCTION)
respectively and for the angular transfer functions, 3 per cent and 5 per cent.

Summary (Chapter 5)

The accuracy with which a measured transfer function describes a vehicle's response has been established and found to be within limits which permit the technique to be used as an engineering tool. The method was found to be consistent, results taken after an interval of six months were found to give good comparison.
6. **The effect on the frequency response of a change in vehicle speed**

So far we have considered the accuracy with which a measured transfer function represents the dynamic response of a vehicle; and the degree to which consistent results may be produced. Both the accuracy and consistency were found to be within acceptable engineering limits.

The next step was to consider whether a transfer function measurement could reflect a deliberate adjustment to a vehicle parameter. To achieve this, sets of time data of vehicle motions were recorded with car speed increased from 32 km/h to 64 km/h. The data were processed by SPECTR and the spectral frequency plots are shown in Figure 6.0 and Figure 6.1 for the transverse and angular transfer functions respectively.

For both transfer functions a clear shift of the frequency response magnitude was noticeable, toward the lower frequency end of the spectrum as a result of the increase in vehicle speed. Also, the increase in vehicle speed produced an increase in phase lag for both transfer functions, although the effect was more prominent for the transverse acceleration transfer function.

The fitted transfer functions for the two sets of measurements were computed to be,
Equations 6.0 to 6.3 show that the dominant pole natural frequencies tended to reduce with vehicle speed, (-2 per cent transverse T.F. and -13 per cent angular T.F.) and the damping coefficient tended to increase with speed (45 per cent transverse T.F. and 9 per cent angular T.F.)

6.1 The effect of speed and tyre characteristics on the pole/zero pattern

The previous paragraphs have established that a change in vehicle speed is reflected as a shift in the frequency response curves with a corresponding movement in the pole/zero pattern. The next step was to examine in finer detail the migration of the poles and zeros over a large speed range, and to observe the extent to which speed affects the effective dynamics of the vehicle. It is worth noting here that the pole/zero method considerably simplifies this type of analysis, for it would be a formidable task to examine and comprehend, the large number of frequency response curves involved, when written as functions of both magnitude and phase.
Date: 17.9.74

\[
\begin{align*}
F(s) &= 0.44 \frac{(S + 0.03 + j 10.86)}{S + 4.70 + j 6.68} \\
F(s) &= 0.15 \frac{(S + 0.04 + j 11.46)}{S + 3.32 + j 7.64}
\end{align*}
\]

Fig. 6.0 THE EFFECT ON THE TRANSFER FUNCTION OF A CHANGE IN VEHICLE SPEED FROM 32 km/h TO 64 km/h (TRANVERSE ACC.)
Fig. 6.1: THE EFFECT ON THE TRANSFER FUNCTION OF A CHANGE IN VEHICLE SPEED FROM 32km/h TO 64km/h
(ANGULAR ACC.)
It was expected, through the work of other people, who have described vehicle dynamics by analytic equations derived from Newton's laws of motion etc, that the poles and zeros of a transfer function were related to vehicle speed. However, the laws of motion method depends upon estimates of vehicle constants such as moments of inertia, centre of gravity, roll height, tyre stiffness, etc, which are often difficult to determine, and in themselves may be functions of speed. Authors have attributed discrepancies between model outputs and practical results to such quantities as, compliance in the suspension, non-linearities or dynamic factors in the tyre stiffness characteristics, and in general, difficulties arising in obtaining values of vehicle constants. The vehicle is essentially a number of masses, supported by springs, containing both linear and non-linear characteristics.

It was with these difficulties in mind that the power spectral method appeared attractive, for it might offer new insight into the description of vehicle dynamics. Intrinsically it includes all the unknown constants and non-linearities and gives the result as a linear approximation in the form of a transfer function.

Vehicle speed was one of the important variables to be investigated, and together with this, the opportunity was taken to examine the effect of a change from cross-ply to radial ply tyres by repeating the measurements throughout the speed range after the change. Initially the vehicle was fitted with cross-ply tyres and time data of vehicle motions were gathered at discrete vehicle speeds over a range from 16 km/h to 80 km/h, with the car set on a mean-straight path; a second set of records was taken for radial ply tyres.
The data were processed by computer programs SPECTR and BODFIT and the results are presented in the form of transfer functions relating transverse acceleration/steer angle, and angular acceleration/steer angle, in Figure 6.2 and Figure 6.3 respectively.

The plots show three main features; firstly there is a continuous migration of the poles and zeros, for increasing vehicle speed, this indicates that spectral methods are sufficiently sensitive to allow investigation of say a prototype vehicle's response throughout its speed range.

Secondly, the plots show that a vehicle fitted with cross-ply tyres gives a distinctly different pattern from one fitted with radial ply tyres; this provides the possibility of selection of tyre types to achieve a desired vehicle response. Figures 6.2 and 6.3 show that the dominant poles are grouped closer for the radially ply rather than the cross-ply tyres.

In broad terms, the driver compensation required to control a vehicle may be related to the driver effort or concentration. Therefore, the vehicle configuration which provided the closer grouping of poles would need less driver compensation, than the wider, cross-ply grouping, and so probably the vehicle fitted with the radial ply tyres would be easier to control over the speed range.

Thirdly, the pole/zero plot shows that the pole patterns of the transverse motion transfer function are different from those of the angular transfer functions. This result is contrary to the conventional mathematical analytic characteristic equations which are usually shown to be identical, for both the angular and transverse motion
Fig. 6.2 THE DOMINANT POLE / ZERO PATTERN OVER A RANGE OF SPEED FOR A VEHICLE FITTED WITH CROSS-PLY AND RADIAL-PLY TYRES.

Transverse acceleration T.F.
Fig. 6.3 THE DOMINANT POLE/ZERO PATTERN OVER A RANGE OF SPEED FOR A VEHICLE FITTED WITH CROSS-PLY AND RADIAL-PLY TYRES.
transfer functions. One explanation could be that the spectral method includes all vehicle parameters in the recorded time data, including linearization of non-linear quantities, whereas the analytic laws of motion method necessarily needs to make some simplifying assumptions or exclusions in the formulation of the equations.

6.2 The effect of side force on the pole/zero pattern

The power spectral density method of system identification through pole/zero patterns is a linear technique, if the system under test is non-linear then the method gives the best linear approximation from a least squares criterion. Earlier paragraphs have mentioned that the PRBS method provides a means of linearization through the application of small perturbations which are sufficiently small to be considered to give a linear result. Typical RMS values for steering wheel angular inputs were about 20 to 30 degrees. Other workers have shown that a vehicle behaves in a linear fashion, that is side force in relation to steer angles, up to steering wheel angles of 90 degrees or side force equivalent to 0.3g.

The opportunity was taken to examine the linearity of a vehicle using PSD methods by observing the migration of the poles resulting from increased levels of vehicular side force, that is, a linear method was applied to a possibly non-linear system by moving the operating point through the non-linear range and applying small perturbations about that point.

The level of side force was set by observing the output of a transverse accelerometer reflected in a meter reading, and adjusting the steer angle until the particular value of transverse acceleration (side force) was reached for the experiment under test. The PRBS was
switched on and the vehicle executed random motions about a mean circular path. Tests were made for a number of speeds from 32 km/h to 64 km/h over a range of side forces equivalent to 0.1g to 0.4g. Limitations on speed and side forces were imposed by the size of the track area and the power in the steering servo to sustain high values of side force.

Some practical difficulties were experienced in keeping the vehicle on a continuous course over the time period for each test (about 60 secs). Some scatter and inconsistency in the results may be attributed to this difficulty.

Figures 6.4 and 6.5 show the dominant pole/zero plots for steering wheel angle/transverse acceleration transfer functions. A complete set of pole/zero patterns for each speed and side force were not analysed; considerable effort is required for the generation of each pole/zero plot, since each is based on over 2500 data points read from an analogue trace and digitized for computer processing, however a sufficient number of pole/zero patterns were completed to enable general conclusions to be drawn. Both figures give clear groupings of low side force dominant poles for each vehicle speed of 32, 40, 48 and 64 km/h.

To assist presentation of the results an envelope of diameter equivalent to a maximum damping coefficient tolerance of ±0.07 has been drawn and positioned to include the maximum number of low value side force poles at the particular speed of interest. The envelopes show a continuous migration of poles with speed, similar in nature to that described in Section 6.2. But within each envelope there is some scatter. Had the results produced identical dominant pole positions for each value of side force then this would have indicated that the vehicle dynamics were independent of side force, however in practise this was not the case,
nevertheless we may say that within the scope of each envelope the vehicle describes a linear system. There is some indication particularly in the case of the angular motion transfer function that at the lower speeds, side force produces non-linearities, the pole position describe a system of greater natural frequency and greater damping.

On those occasions where zero side force measurements have been repeated, the values give good agreement with previous results (see Section 6.1) also the earlier results fall within the appropriate speed envelopes. It is worth noting that there was a gap of about 5 to 6 months between the original and repeated zero side force measurements, this observation reinforces the degree of repeatability of the PSD method.

Summary

The results have shown that within a boundary (wherein the scatter is likely to be due to experimental variations) a linear transfer function measured by PSD techniques is valid up to a side force equivalent to about 0.2g to 0.3g for vehicle speed above 48 km/h, below this speed non-linearities may be present in both angular and transverse acceleration transfer functions.

Improvement could be made to the experimental technique used by arranging say, a preset RMS value of side force to act as a feedback signal to the steering servo to maintain a steady mean side force.

6.3 The effect of tyre pressure on the pole/zero pattern

The vehicle chosen for these exploratory tests is in common usage and so far as popular opinion may be determined, it is accepted as having
○ Original zero side force measurement

△ Vehicle speed 32 km/h

□ " " 40 km/h

▽ " " 48 km/h

◊ " " 56 km/h

Note:
(a) Circle diameters are equivalent to a damping coefficient tolerance of ± 0.07
(b) Numerals in symbols are values of side force quoted as 1/10 th of g

All zeros fell in this area

Fig. 6.4 EFFECT OF SIDE FORCE ON ANGULAR ACCELERATION TRANSVERSE FUNCTION
○ Original zero side force measurement
△ Vehicle speed 32 km/h
□ " " 40 km/h
▽ " " 48 km/h
◊ " " 56 km/h

All zeros fell in this area

Note:-
(a) Circle diameters are equivalent to a damping coefficient tolerance of ± 0.07
(b) Numerals in symbols are values of side force quoted as $1/10$ th of g

---

**Fig. 6.5 EFFECT OF SIDE FORCE ON TRANSVERSE ACCELERATION TRANSVERSE FUNCTION**
characteristic and acceptable handling qualities for its type, that is a medium sized family estate car. Its characteristic dynamics having been identified as described in previous section, it was of interest to alter the response of a vehicle, which is considered to have acceptable dynamic qualities, by making drastic changes to a vehicle parameter, and so produce a vehicle with degenerated response, and to observe the result in the pole/zero pattern.

The tyre pressure is a simple parameter to change and most drivers would agree that a vehicle with severely reduced tyre pressures behaves badly. A limited number of measurements were made at speed of 32 and 48 km/h with all round tyre pressures reduced in steps down to 1.05 kg/cm$^2$ (15 lb/in$^2$).

The results are shown in Figures 6.6 and 6.7 as pole/zero plots for the angular and transverse motion transfer function. For both planes of motion little change in the dominant pole locations is noticeable from reduction in tyre inflation pressures as large as 0.7 kg/cm$^2$, but significant movement in the pole is evident from a further reduction of 0.35 kg/cm$^2$. This result suggests that the PSD method can detect gross changes in tyre pressures in the case of the particular vehicle and speeds; the movement of the dominant pole shows that the response becomes more damped with reduction in inflation pressure and a lower damped natural frequency.

The number of measurements made was insufficient to draw more detailed conclusions, but further work of this nature, that is of making known reductions or improvements in vehicle parameters, particularly under high speed and side force conditions, may lead to an understanding of the
preferred area in the pole/zero plot which will achieve a desired vehicle response over a range of conditions.

6.4 The effect of vehicle loading on the pole/zero pattern

As a continuation of Section 6.3 (tyre inflation pressure reductions), a predictable degradation in the vehicle handling quality was made by placing an excess load of 1400 N on the rear of the vehicle. Measurement of the vehicle motions in response to a steering wheel PRBS input was made at the single speed of 48 km/h. Driver opinion was quite firm in declaring that the vehicle handled badly.

The change in the dynamics due to the extra load was reflected in the pole/zero patterns (Figs.6.8 and 6.9), by producing a large movement in the dominant poles of both the angular and the transverse transfer functions to a position which indicates a vehicle with lower damped natural frequencies and considerably higher damping.

Further measurements are required in order to draw more detailed conclusions, but it is of interest to note that the PSD method of transfer function identification may be used to observe the effect on the vehicle dynamics of load distribution.

Summary (Chapter 6)

The selectivity of the technique allowed description of distinct transfer functions from response measurements taken for vehicle speed intervals as little as 16 km/h.
Fig. 6.6  EFFECT ON THE ANGULAR ACCELERATION TRANSFER FUNCTION OF TYRE INFLATION PRESSURE

Fig. 6.7  EFFECT ON THE TRANSVERSE ACCELERATION TRANSFER FUNCTION OF THE TYRE INFLATION PRESSURE

Note: - Suffixes indicate reduction in pressure (lbs/sq in)
Vehicle speed 48 km/h

Fig. 6.8 EFFECT ON THE ANGULAR ACCELERATION POLE/ZERO PATTERN OF 140 kg LOAD ON REAR OF VEHICLE

Fig. 6.9 EFFECT ON THE TRANSVERSE ACCELERATION POLE/ZERO PATTERN OF 140 kg LOAD ON REAR OF VEHICLE
Examination of a vehicle fitted with cross-ply and then radial ply tyres showed closer grouping of the first order poles throughout the speed range when the vehicle was fitted with radial ply tyres. Since the vehicle response changes less with speed in the case of the radial ply tyres a driver would find this vehicle easier to control than the 'cross-ply vehicle' for he would need to provide less compensation himself.

Transfer functions derived whilst the vehicle was undergoing various levels of side force indicated that, within a boundary of speed and side force, the vehicle behaved as a linear system. The results suggest that non-linearities are introduced at low speed (40 km/h and below) and at equivalent side forces in excess of 0.2g to 0.3g. The maximum speed in this series of tests was 56 km/h.

A reduction in tyre pressure of 15 lb/sq in. had a large effect on the dominant pole positions, the vehicle became considerably higher damped with a lower damped natural frequency. Under these conditions it would be difficult to control and would respond in a very sluggish manner.

Similarly, an excess load on the rear of the vehicle caused significant movement in the dominant poles towards a more highly damped system: measurements of this nature could provide assistance in finding the optimum location and distribution of vehicle mass at vehicle development stage or when revising a loading schedule.

In general exploratory investigation of vehicle response in this way, allows area of 'good' vehicle dynamics to be located on the pole/zero diagrams.
7. **A driver/vehicle model**

So far the results have considered the response of the vehicle without driver control, this has allowed examination of the vehicle's dynamic characteristics independent of the driver contribution. Consequently it has been possible to study the vehicle as a simple mechanical system, without introducing the difficulties which arise due to day to day variation in driver behaviour. However the merit of a vehicle rests in the manner in which it behaves from the driver viewpoint, or in other words, the closed loop response, with the driver closing the feedback loop between the vehicle position and the road input (Fig.7.0).

![Diagram of a simple vehicle/driver closed loop system](image)

**Fig.7.0 A SIMPLE VEHICLE/DRIVER CLOSED LOOP SYSTEM**

To follow an analytic approach of this nature demands the formulation of an acceptable driver/vehicle control system model, and a description of the driver in the form of an input/out transfer function. The intention was to measure the driver transfer function by
practical field experiments, unfortunately concentrated effort was needed in the work on vehicle suspensions and so information on human transfer functions had to be drawn from published works (Ref.16). The adoption of a characteristic description of the human operator restricted the analysis to be of a comparative rather than an absolute nature.

The technique used was to postulate a driver/vehicle closed loop system and, by root locus techniques, compare sets of measured vehicle transfer functions using the same driver model for both sets. Also, an attempt was made to assess the driver compensation needed to match the dynamics of one closed loop system with another. The implication is that the amount of compensation required provides a measure of driver concentration or difficulty.

7.1 A survey of some single loop closures

To assist the understanding of a vehicle driver model it was informative to examine measured transfer functions in a single closed loop unity feedback configuration, with the driver represented by a simple gain term, K. The simulation may be interpreted as a system controlled by a robot driver, who can make no anticipatory or smoothing compensation, but acts simply on the observed output signal in a manner which tends to reduce the error signal, and in addition, he can alter the gain factor K to place the closed loop poles at some optimum working point. The aim of the analysis was to seek those loop closures which provided possible control of the vehicle on the assumption that a driver would select the simplest and most profitable closures. For example, if adjustment of gain is synonymous with a degree of driver concentration, then his task will be simpler with a gain insensitive system; a profitable closure would be one that required a low gain value but will
maintain the closed loop poles in an acceptable area on the pole/zero diagram from a control engineering viewpoint, i.e. 0.6 to 0.8 damping factor.

The likely possible loops examined were vehicle angular and transverse acceleration and their integrals, velocity and displacement. Figures 7.1 and 7.2 show the root loci of a vehicle's transfer functions, measured at 64 km/h, relating steering wheel angle to angular and transverse acceleration, velocity and displacement. The task is to examine the loci and attempt to determine those which might give good single loop closure.

Referring to Figures 7.1a and 7.2a it is unlikely that a driver would use acceleration as a feedback signal, his primary stimulus must be visual and acceleration is a difficult quantity to assess visually. Figure 7.1a shows that the root locus for the angular acceleration transfer function moves to a highly damped position in the pole/zero diagram for small increase in the gain value K. The transverse acceleration dominant closed loop poles remain within an acceptable area (Fig.7.2a) although the secondary closed loop pole moves towards a zero close to the $jw$-axis, that is towards an unstable area.

It must be emphasized that the gain values (K) for our robot driver are not absolute, but may be used for comparative purposes.

Angular velocity (Fig.7.1b) seems a better choice for the position of the closed loop pole is not so sensitive to $K$ as the angular acceleration locus, although the locus moves towards a higher damped system with increasing $K$. The transition towards higher damping may be
interpreted as an undesirable effect for our model driver, for he would find his vehicle responding slower as he increased the steering wheel angle in his attempts to reduce the error signal. The transverse velocity loop (Fig.7.2b) does not seem such a good possibility as the transverse acceleration loop from a control viewpoint, for it moves to a lightly damped and higher oscillatory area although it may be better visually observed than transverse acceleration.

Angular displacement seems a good candidate for a single loop closure (Fig.7.1c), the closed loop poles move toward a less damped system yet remain within the set damping criterion for a comparatively large increase in K; also the non oscillatory pole at the origin moves towards the stable more responsive region. The transverse displacement loop (Fig.7.2c) is inherently unstable, the locus from the pole at the origin moves to the unstable right hand plane and returns to a stable area only for large values of K, but it is easily observable as the distance between the vehicle and the roadside, and so could make a secondary loop in a multiloop model.

A practical interpretation of the unstable nature of the transverse displacement loop is that at speed, it is not possible to steer a vehicle by looking out of the side window. This agrees with practical tests; the author with the help of a colleague to warn for approaching obstacles, found a vehicle to be uncontrollable at speeds above about 32 km/h when using only the kerb line as a visual steering queue. This indicates that drivers are well advised to travel at a speed within adequate sight distance in foggy conditions not only from fear of colliding with vehicles ahead but, on entering a fog patch at speed where forward sight distance is impeded, they would experience difficulty in controlling the vehicle from any kerb side queues which might be available.
In summary it appears from these simplified observations that angular velocity and angular displacement are good loops, but either singly would be insufficient to track a vehicle along a road, a measure of the transverse location is essential, hence a secondary loop must be closed. Transverse velocity seems a good choice from a control engineering viewpoint, but transverse displacement although inherently unstable is easily observed visually. So it seems that the simplest driver vehicle model which could follow a tortuous road input must be multiloop. However further work is necessary to substantiate this prognosis, other workers (Ref.17) have verified a multiloop model in an overtaking manoeuvre.

7.2 Simulation of a driver/vehicle model

The simulated driver/vehicle closed loop system adopted in this work for assessment of driver difficulty is one due to Weir and McRuer who showed their model to be acceptable by comparing the simulated output with practical measurements (Ref.17).

The driver model $H(S)$ has been formulated and referenced in many publications (Ref.16,18). The one chosen for this analysis is due to Weir et al (Ref.23).

$$H(S) = \frac{K_p}{s} \frac{a_T}{\sigma_T} \frac{e^{-S\tau}}{(1 + S\tau_L)(1 + S\tau_K)} \frac{1}{(1 + S\tau_N)\left[\left(\frac{s}{\omega_n}\right)^2 + \frac{2\xi}{\omega_n}\frac{s}{\omega_n} + 1\right]}$$

where

- $K_p =$ gain
- $K_1\left[\frac{a_T}{\sigma_T}\right] =$ indifference threshold describing function
- $e^{-S\tau} =$ pure time delay
\[ G(s) = \frac{0.0241(s - 0.03)(s + 6.89)}{(s + 4.88 \pm j4.68)} \]

(a) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Ang. accel}}{\text{Steer angle}} \)

(b) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Ang. velocity}}{\text{Steer angle}} \)

(c) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Ang. displacement}}{\text{Steer angle}} \)

Fig. 7.1 ROOT LOCUS OF CLOSED LOOP VEHICLE ANGULAR TRANSFER FUNCTION (UNITY FEEDBACK) (64 km/h)
\[ F(s) = 0.40 \left( \frac{S + 4.44 \pm j 0.64}{S + 3.99 \pm j 5.5} \right) \left( S + 0.93 \pm j 11.17 \right) \]

(a) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Transverse accel.}}{\text{Steer angle}} \)

(b) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Transverse velocity}}{\text{Steer angle}} \)

(c) \( \frac{\text{Out}}{\text{In}} = \frac{\text{Transverse displ.}}{\text{Steer angle}} \)

**Fig. 7.2** ROOT LOCUS OF CLOSED LOOP VEHICLE TRANSVERSE TRANSFER FUNCTION (UNITY FEEDBACK) (64 km/h)
\[
\frac{(1 + ST_L)}{(1 + ST_I)} = \text{equalization characteristics}
\]

\[
\frac{(1 + ST_K)}{(1 + ST_{KI})} = \text{low frequency equalization}
\]

\[
\frac{1}{(ST_N + 1)\left[\left(\frac{S}{\omega_n}\right)^2 + \frac{2\zeta \omega_n S}{\omega_n} + 1\right]} = \text{high frequency neuromuscular lag}
\]

The indifference threshold, from studies by Goodyear (Ref.19) approximates to unity for large forcing functions. The low frequency equalization is important for conditionally stable systems but for a directionally stable vehicle this value may also be set to unity (Ref.17). Published data supports a first order lag term to give,

\[
\frac{1}{(1 + ST_N)}
\]

This simplification reduces \( H(S) \) to,

\[
H(S) = \frac{K_p e^{-\tau} (1 + ST_L)}{(1 + ST_I)(1 + ST_N)} \quad (7.1)
\]

\[
= \frac{K_p e^{-S(\tau+T_N)} (1 + ST_L)}{(1 + ST_I)} \quad (7.2)
\]

The values used to replace the constant terms were as follows,

\( \tau = 0.2 \) secs.

\( T_L = 0.5 \) (and 0) secs.

\( T_I = 0 \)

\( T_N = 0.1 \) (secs).

\( K_P \) was varied in the root locus analysis.
The chosen simulation was a multiloop system with two feedback loops to the driver, they were the vehicle's transverse position ($Y(t)$) and heading angle ($\psi(t)$) (Fig.7.3).

![Fig.7.3 MULTILOOP SIMULATION OF VEHICLE/DRIVER SYSTEM](image)

The driver is assumed to respond to the vehicle heading and transverse position as separate stimuli. The heading angle would be observed as the angle between an object in the driver's field of view, say the centre line of the road, and a point on the car such as the bonnet mascot; the transverse position would be seen as a distance from the kerb. The driving task is assumed to be that of following a desired path as described by $Y$, the horizontal road curvature.

If Figure 7.3 is redrawn as a signal flow diagram (Fig.7.4),
then the closed loop response can be written down directly by using Mason’s rules (Ref. 8). That is,

\[
\frac{Y_I}{Y}(S) = \frac{K_{HT} H_T(S) K_T F_T(S)}{1 + K_{HT} H_T(S) K_T F_T(S) + K_{HA} H_A(S) K_A F_A(S)}
\]  \quad (7.3)

The root locus equation is

\[
1 + K_{HT} H_T(S) K_T F_T(S) + K_{HA} H_A(S) K_A F_A(S) = 0
\]  \quad (7.4)

Analysis of equation 7.4 was made by program ROOTL which takes as its input the loop transfer function \( L(S) \):-

\[
L(S) = K_A K_{HA} H_A(S) F_A(S) + K_T K_{HT} H_T(S) F_T(S)
\]  \quad (7.5)

and tracks as a function of \( K \) (the driver gain constant) the closed loop poles from the open loop poles to the open loop zeros. The
computed output is presented as a digital print out of the roots and a root locus plot on a line printer.

Several approximations were tried for the pure time delay (Fig.7.6) the one chosen for use in the analysis was a second order Padé approximation:-

$$e^{-ST} = \left(\frac{S - \frac{4}{\tau}}{S + \frac{4}{\tau}}\right)^2$$

(7.7)

Figure 7.5 shows that the approximation is exact for the modulus, the phase plot shows a deviation from the true curve above a frequency of about 3 Hz.

The analytic task consisted of substituting sets of measured vehicle dynamics $K_T F_T(S)$ and $K_A F_A(S)$ for different vehicle conditions (ie cross-ply and radial ply tyres) and comparing the results by means of the root locus plots.

7.3 A comparison of driver difficulty in controlling a vehicle when fitted with cross-ply compared with radial ply tyres

It has been shown that the closed loop response of the driver/vehicle multiloop model of Figure 3.4 is:

$$\frac{Y_I}{Y}(S) = \frac{-K_{HT} H_T(S) K_T F_T(S)}{1 + K_{HT} H_T(S) K_T F_T(S) + K_{HA} H_A(S) K_A F_A(S)}$$

(7.8)

$K_{HT} =$ driver gain constant for transverse position response

$K_{HA} =$ driver gain constant for angular response

$H_T =$ driver transfer function transverse response

$H_A =$ driver transfer function angular response

$K_T F_T(S) =$ vehicle gain and transfer function transverse response

$K_A F_A(S) =$ vehicle gain and transfer function angular response
Fig. 7.5: FREQUENCY PLOT OF PURE TIME DELAY ($\tau = 0.3\,\text{s}$) AND APPROXIMATIONS

- Pure time delay and Padé gain ($1.0$)
- Binomial gain $\frac{1}{(1 + 0.3)^2} \approx e^{-st}$
- 2nd order Padé and binomial
- 4th order Padé
- Pure time delay $-\omega \, 0.3$
The intention of the analysis is to procure an insight into the driver contribution required to control a vehicle fitted with cross-ply tyres compared with radial ply tyres, or indeed to compare the effect of any other change in vehicle geometry or mechanics. The assumption is that the greater the gain or anticipation term the driver is required to produce to achieve control then the more arduous is his task. The comparison of the 'cross-ply' and 'radial ply' dynamics is made by observing the position of the closed loop poles in the S-plane and noting the amount of driver gain or anticipation needed to attract the closed loop poles to the selected position. Since the method is comparative it is necessary to choose only an approximate yet acceptable area for the closed loop poles. Work in the aircraft industry (Ref.9) has produced a correlation between pilot opinion and the position of the pitch mode poles of the aircraft (Fig.7.6). The yaw response equations of a vehicle are similar to those of the pitching mode of an aircraft, so it is likely that a similar rating could be applied for vehicle driver opinion. Furthermore, it is likely that a driver will attempt to adjust his own dynamics to place the closed loop poles within the acceptable area on the pole/zero plane. It is the effort on the part of the driver to achieve this, measured in terms of gain and anticipation, that is proposed as a basis for comparing sets of vehicle dynamics.

The calculation of the closed loop pole positions is as follows:

In canonical form equation 7.8 may be written as

\[
\frac{Y_I}{Y}(S) = \frac{KF(S)}{1 + KF(S)} \quad (7.9)
\]
The root locus equation is,

\[ 1 + K F(S) = 0 \]  

(7.10)

where \( K \) is a function of the root locus.

Comparing equation 7.8 with equation 7.10 we see that,

\[ K F(S) = K_H A (K_A F_A(S) H_A(S)) + K_H T (K_T F_T(S) H_T(S)) \]  

(7.11)

The terms \( K_A F_A(S) \) and \( K_T F_T(S) \) have been determined previously by power spectral means; the transfer functions chosen for the analysis were those of the vehicle when at speed 48 km/h.

These were,

**CROSS-PLY TYRES**

\[ K_A F_A(S) = \frac{\psi}{\theta} (S) = \frac{0.0418 (S + 78.97)(S - 0.84)}{S^2 (S + 11.78 \pm j5.66)} \]  

(7.12)

\[ K_T G_T(S) = \frac{Y_I}{\theta} (S) = \frac{0.1352 (S + 0.744 \pm j11.22)}{S^2 (S + 5.89 \pm j10.32)} \]  

(7.13)

**RADIAL PLY TYRES**

\[ K_A F_A(S) = \frac{\psi}{\theta} (S) = \frac{0.3982 (S + 57)(S + 0.056)}{S^2 (S + 6.45 \pm j9.09)} \]  

(7.14)

\[ K_T F_T(S) = \frac{Y_I}{\theta} (S) = \frac{0.1183 (S - 0.055 \pm j11.43)}{S^2 (S + 4.93 \pm j8.38)} \]  

(7.15)
Fig. 7.6 PILOT OPINION OF PITCH MODE AIRCRAFT DYNAMICS
This particular set of transfer functions was chosen because a request for a 5 per cent fit in BODFIT produced simple second order equations for all functions, this considerably simplified the algebraic manipulations involved.

The terms for the driver model dynamics $H_A(S)$ and $H_T(S)$ in equation 7.11 were equated.

That is

$$H_A(S) = H_T(S) = H(S) = \frac{e^{-St} (1 + ST_A)}{(1 + ST_N)(1 + ST_I)} \quad (7.16)$$

The neuromuscular lag term $(1 + ST_N)$ was included in the time delay term to give:

$$H(S) = \frac{e^{-S(t+T_N)} (1 + ST_A)}{(1 + ST_I)} \quad (7.17)$$

where

$$t = 0.2 \text{ secs}$$

and $T_N = 0.1 \text{ secs}$

Using the Pade approximation

$$e^{-S(0.3)} = \frac{(S - 13.3)^2}{(S + 13.3)^2}$$

this becomes

$$H(S) = \frac{(S - 13.3)^2 (1 + ST_A)}{(S + 13.3)^2 (1 + ST_I)} \quad (7.18)$$
The remaining terms in equation 7.11, \( K_{HA} \) and \( K_{HT} \) are the driver gain factors and are the function of the root locus.

Several root loci were plotted with \( K_{HA} \) and \( K_{HT} \) of different selected ratios.

That is

\[
K_{HA} = \beta \times K_{HT}
\]

where \( \beta = \frac{1}{3}, 1, 3, 6. \)

Equation 7.11 may now be written as,

\[
K F(S) = K_{HF} (K_T F_T(S) H(S) + \beta (K_A F_A(S) H(S))) \tag{7.19}
\]

Thus the root locus equation is:

\[
1 + K_{HF} (K_T F_T(S) H_T(S) + \beta (K_A F_A(S) H(S))) \tag{7.20}
\]

Program ROOTL requires the data to be entered in the form,

\[
K F(S) = K \frac{N}{D} (S) \tag{7.21}
\]

as coefficients of the polynomial \( \frac{N}{D}(S) \); writing equation 7.20 as numerator (N) and denominator (D) terms we have:

\[
K_{HF} \cdot \frac{N_H}{D_H} \left\{ \frac{K_T D_A N_T + \beta K_A D_T N_A}{D_A D_T} \right\} \tag{7.22}
\]
The roots of the above equation were found for,

$$T_A = 0 \text{ and } T_A = 0.5$$

for each value of $\beta$, the smoothing lag term constant $T_I$ was equated to zero. For this condition ($T_A = 0$), the driver equalization characteristics are nil, the best aircraft pilot ratings occur under these conditions and a human transfer function of this form could become the basis for predicting 'good' closed loop dynamics (Refs.17,22).

Figure 7.7 shows a typical plot of the root loci arising from equation 7.22 on substitution of values for the radial ply tyre transfer functions (with $K_{HA} = K_{HT}$). The loci start at the open loop poles and head towards the open loop zeros or infinity.

We see from Figure 7.7 that the angular motion poles are more sensitive to driver gain than the transverse motion poles. Also, the two poles at the origin, arising from the double integration of the acceleration transfer functions, will have the greatest influence over the closed loop dynamic response; and for this reason they will be studied more closely in order to compare radial and cross-ply tyre vehicle response.

Figure 7.8 to Figure 7.9 shows the loci near the origin in greater detail for both cross-ply and radial ply tyres over a range of driver gains and for a range of driver ratios from $K_{HA} = 6K_{HT}$ to $K_{HT} = 3K_{HA}$ with the anticipation time constant $T_A = 0$. The loci are repeated in Figure 7.10 to Figure 7.11 with the anticipation time constant equal to 0.5 secs.
Fig. 7.7 ROOT LOCUS OF MULTILOOP VEHICLE/DRIVER MODEL
RADIAL-Ply TYRES $K_{HA} = K_{HT} = K$
Fig. 7.8 ROOT LOCUS OF MULTILOOP VEHICLE/DRIVER MODEL NEAR ORIGIN
CROSS-PLY TYRES $T_A = 0$
Fig. 7.9  ROOT LOCUS OF MULTILOOP VEHICLE/DRIVER MODEL NEAR ORIGIN
RADIAL-PLY TYRES  $T_A = 0$
Fig. 7.10 ROOT LOCUS OF MULTILOOP VEHICLE/DRIVER MODEL NEAR ORIGIN
CROSS-PLY TYRES $T_A = 0.5$ secs
**Fig. 7.11** ROOT LOCUS OF MULTILOOP VEHICLE/DRIVER MODEL NEAR ORIGIN
RADIAL-PLY TYRES  $T_A = 0.5$ secs
With the anticipation time constant at $T_A = 0$, we see that for the cross-ply tyre closed loop system the loci are predominantly in the unstable right hand quadrants of the S-plane, except when the driver angular gain $K_{HA}$ is about six times greater than the transverse driver gain (Fig. 7.8). For radial ply tyres a somewhat similar locus to the latter is achieved for equal angular and transverse human gain factors (Fig. 7.9). The implication is that the driver controlling the 'radial ply' system is required to provide less compensation in terms of gain than for the 'cross-ply' system. This is not to say that either system is basically unstable, practise has borne out that this is not so, but for only the simple driver model chosen, is the system unstable, nevertheless a comparative measure or preferred choice can be made between the 'cross-ply' and 'radial ply' closed loop system.

The addition of an anticipation term $(1 + 0.5S)$ to the human transfer function improves the 'cross-ply' dynamics by moving all the loci into the stable left hand quadrants of the S-plane (Fig. 7.10). Figure 7.11 shows that the addition of an anticipation term to the 'radial ply' dynamics is unnecessary, in fact for $K_{HA} = 3K_{HA}$ the closed loop response tends to be degraded after a human gain value of about 5.

The points which may be drawn are:-

(1) 'Cross-ply' dynamics are more difficult for the simple driver model to control than the 'radial ply' tyre dynamics.

(2) An anticipation term in the 'cross-ply' loop improves the closed loop response but at the cost of greater effort on the part of the driver model.
(3) The 'radial ply' dynamic system does not need an anticipation term, the closed loop system is 'better' than the 'cross-ply' system with this term excluded than it is with the anticipation term included.

Specific effort on the part of the driver model, as described above, has directed the loci toward the 'good' area of the S-plane (Fig.7.11) and the amount of model driver compensation required to achieve this may be used as a measure of the difficulty in controlling the vehicle.

It is emphasised that the interpretation given of the root loci trajectories is comparative not absolute, but may be used for comparing changes made in a vehicle driver characteristic, however if a typical description were available of a driver transfer function it is likely that absolute results could be derived.

Summary (Chapter 7)

The closed loop response has been examined by root locus analysis firstly in a unity feedback configuration and then by setting up a two loop simulation with a model driver closing the feedback loops.

Examination of single unity feedback loops showed angular displacement to be a good likely visual queue in the steering task, but transverse displacement, although easily observable by a driver, produced an unstable system. One consequence of this is that a driver would find difficulty in controlling his vehicle from visual queues of the road side through his side window. It follows that in foggy conditions drivers are well advised to travel at speeds appropriate to the forward sight distance not only from the possibility of colliding with other vehicles, but on striking a dense patch of fog they would have difficulty in
controlling their vehicle from any kerb side queues which might be available.

Although transverse displacement alone is an unstable loop it may, due to its easy observability, form a secondary loop in a multi-loop system. On this basis a multi-loop system containing angular and transverse displacement was set up and examined by root locus analysis. Two sets of dynamics were considered, those with the test vehicle fitted with cross-ply and then with radial ply tyres (vehicle speed 48 km/h). The feedback loops were closed by a human model which contained terms such as gain, pure time delay, neuromuscular lag and anticipation.

In general root locus examination showed that stable systems were produced when driver gain was greater in the angular plane than in the transverse plane. In addition the model driver needed to provide greater gain and anticipation to control the vehicle when fitted with cross-ply tyres.

On the basis of the above analysis it is postulated that changes made in vehicle parameters at the prototype or usage stage may be quantified by consideration of the amount of feedback required of a model driver to place the closed loop poles in an acceptable area on the pole/zero diagram.
Conclusions and discussion

8. Steering dynamics

The need for the work carried out was based on the assumption that a mathematical description of vehicle dynamics which related steering wheel angles to steering response would aid vehicle engineers in the development of prototype vehicles, and assist other workers in the vehicle field such as safety engineers, and tyre manufacturers, to assess quantitatively, the effect on the vehicle response of making a change in a vehicle parameter such as, load, distribution, steering geometry, tyre pressure, speed etc.

In the past mathematical modelling has been confined mostly to producing models derived from laws of motion and justifying the models by practical tests.

The current work presents a rather reverse approach in that the mathematical model was produced from practical data recorded from vehicle motions, thus producing a mathematical description of the overall system including all the parameters as a lumped system. This approach has avoided the difficulties which arise in the 'laws of motion' method of quantifying the vehicle constants such as tyre stiffness, moments of inertia, etc and attempting to formulate non-linearities where considered necessary.

The random data PSD method is essentially a linear technique, any non-linearities in the system will be approximated on at least squares fit basis. The use of a small perturbation upon the normal input
provides a linearising effect if non-linearities are present, the result appears as a linear function formed about an operating point which may be moved throughout the non-linear regions. Normal, operational steering wheel inputs may have been used in place of the small perturbations, the same mathematics is applicable but the linearisation would not have been so effective because of the lack of control over the steering angle inputs.

The particular perturbation was chosen to be a pseudo random binary sequence, this offers several distinct advantages, these are, its amplitude may be quite small compared with the normal inputs, its bandwidth may be predetermined and data may be recorded over a considerable period of time; this has the effect of rejecting unwanted noise and so improves the signal to noise ratio.

The initial task was to ascertain the accuracy and repeatability of the method, for these qualities are basic to any engineering measurement device or method. The accuracy of the identification was established by driving the transfer function, which had been derived by PSD methods, with recorded data of the original steering wheel motions, and comparing the output of the model with the recorded output motions of the vehicle. The accuracy and repeatability were found to be within limits which allowed investigation of the effects on vehicle dynamics, of changes such as speed and tyre characteristics. Consequently if the desired open loop pole/zero pattern was known then the experimental changes in the vehicle mechanics may be quantified and compared with some target result.
The initial intention of the work was to establish the desired open loop pole/zero locations by measurement of several vehicles which had been graded by driver opinion, unfortunately the work had to be curtailed and confined to a single vehicle. A secondary approach was adopted in that changes which would have a clearly predictable effect were made, such as gross reduction in tyre pressures, and excess loading at the rear of the vehicle, the result then was observed in the pole/zero pattern. From these measurements on one vehicle (which is rated by popular opinion to have typical and acceptable handling qualities for its type) and by making comparison with work done in the aircraft industry and also by assessment using traditional control theory, an area for good vehicle response was outlined.

It was established that a set of dominant poles measured when the vehicle was fitted with radial ply tyres, and plotted as a function of speed, produced a closer grouping and thus better overall dynamics than when the vehicle was fitted with cross-ply tyres. This conclusion is based on the assumption that a driver would need to provide less compensation to achieve a desired response throughout the speed range, for the closer grouped set of poles.

A feature noticed throughout the measurements was a difference between the characteristic equations of the angular and transverse transfer functions, this is at variance with the 'equations of motion' method of vehicle modelling and implies that the vehicle has a slightly different natural frequency and damping in one plane compared with the other.

An attempt was made to assess the driver difficulty in controlling the 'radial ply tyre' vehicle model compared with the cross-ply tyre model by setting up a multiloop driver/vehicle feedback model for each
set of transfer functions (cross-ply and radial) and making a root locus analysis. Driver difficulty was assessed by the amount of compensation he needed to contribute to place the closed loop poles in an acceptable area in the pole/zero plane. For this particular set of transfer functions, measured from a popular family estate car (at 48 km/h) it was found that the radial ply tyres produced a dynamic system which required less effort in the form of gain and anticipation on the part of the model driver than the cross-ply tyres.

Thus, if a set of standard driver models could be established which represent a range of drivers from say learner through to highly skilled, then it may be possible to use vehicle/driver closed loop models to examine vehicle/driver compatibility for measured sets of vehicle dynamics, from say prototype vehicles. That is, the vehicle may be tailored to optimised to suit the driver dynamics. Additionally, the contribution that a model driver needs to make to place the closed loop poles in an acceptable area on the pole/zero diagram can be used to 'rate' the measured open loop dynamics of the vehicle.

A single vehicle was examined for likely non-linear characteristics between steering amplitudes and vehicle response by applying increasing levels of side force and determining the transfer function at each discrete side force value. Some scatter was noticeable in the results, but areas could be drawn on the pole/zero diagram from which conclusions could be drawn which suggest that the vehicle response tended to be non-linear at low speeds but above 48 km/h a linear system was observed up to side forces equivalent to a transverse acceleration of about 0.2g to 0.3g.
8.1 Further work

Further work on vehicle steering dynamics could include:-

(a) The measurement of driver transfer functions to set up one or more 'standard' models for use in closed loop assessment of vehicle dynamics. Several models may be required to cover a range of driving conditions such as, town driving, high speed driving and different types of drive.

(b) The measurement of the transfer function of several vehicles which had been placed subjectively in a good/bad handling range.

(c) Improvement in instrumentation to control accurately the mean side force during examination of vehicle dynamics.

(d) On board digital logging and analysis to produce pole/zero patterns as an aid to field examination of vehicle dynamics.

(e) Adaptive techniques to sense possible abnormalities in the steering 'error' signal. This could be used to detect whether a driver is behaving in an abnormal manner through ill health or the influence of alcohol. The detection of such a condition could be used to immobilise the vehicle.

The present need to examine ways of conserving national resources has resulted in the work on vehicle steering dynamics, at TRRL, being replaced by investigation of vehicle suspension dynamics from the point of view of minimizing damage to road surfaces caused by dynamic wheel loads.
The force/damage relationship approximates to a fourth power law so that a small reduction in the dynamic loads would result in considerable reduction in road damage, with a commensurate saving in national expenditure on repair costs. Also it is likely that an improved suspension may provide a better environment for the transportation of delicate goods which again could reflect a financial saving.

PRBS techniques are being used in the form of a road profile (Plate VIII) to identify the suspension dynamics of current vehicles. It is proposed that measured transfer functions may be optimised to produce a desired output by:-

(a) modification of parameter coefficients
(b) active compensation

The next stage is to produce hardware to match the optimised suspension dynamics.

In addition suspension models driven by simulated road profiles produced from filtered PRBS are being examined by computer techniques to establish the magnitude of the problem and to determine the effect and importance of variables such as, damping, spring stiffness, body mass and axle configurations.

This work is being carried out as part of the TRRL research program.
PLATE VIII PRBS road surface
APPENDIX I

The calculation of transfer functions from time series data

1. The relevant equations for the calculation of the transfer functions via the autocorrelation functions may be stated as,

   **Autocorrelation Function**
   
   \[ R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) x(t + \tau) \, dt \]  
   \[ (I.0) \]

   **Cross-correlation Function**
   
   \[ R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) y(t + \tau) \, dt \]  
   \[ (I.1) \]

   **Auto Power Spectral Density Function (single sided)**
   
   \[ G_{xt}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j\omega \tau} \, d\tau \]  
   \[ (I.2) \]

   **Cross Power Spectral Density Function (single sided)**
   
   \[ G_{xy}(\omega) = \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-j\omega \tau} \, d\tau \]  
   \[ (I.3) \]

2. Impulse response and noise reduction

   For a linear system \( F(\omega) \) the output \( y(t) \) is given by the convolution integral,
   
   \[ y(t) = \int_{-\infty}^{+\infty} h(u) x(t - u) \, du \]  
   \[ (I.4) \]
The output including noise is,

\[ y_n(t) = y(t) + x(t) \]  \hspace{1cm} (I.5)

(see Fig.I.1).

Fig.I.1 LINEAR SYSTEM OUTPUT PLUS NOISE

Using equation I.1 in the form,

\[ R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} y(t) x(t - \tau) \, dt \]  \hspace{1cm} (I.6)

Substituting \( y(t) \)

\[ R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \left[ \int_{-\infty}^{+\infty} (h(u) x(t - u) \, du) + n(t) \right] x(t - \tau) \, dt \]  \hspace{1cm} (I.7)

rearranging

\[ R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \left[ \int_{-\infty}^{+\infty} h(u) x(t - u) \, du \right] x(t - \tau) \, dt \]

\[ + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} n(t) x(t - \tau) \, dt \]  \hspace{1cm} (I.8)
The last term defines the correlation between the noise \( n(t) \) and the input function \( x(t) \). If \( n(t) \) is random and uncorrelated with \( x(t) \) then the second term tends to zero for increasing number of data samples.

Changing the order of integration we get,

\[
R_{yx}(\tau) = \text{LIMIT}_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} h(u) \, du \int_{-\infty}^{+\infty} x(t - u) \, x(t - \tau) \, dt \quad (I.9)
\]

Now,

\[
\int_{-\infty}^{+\infty} x(t - u) \, x(t - \tau) \, dt = R_{xx}(\tau - u) \quad (I.10)
\]

by definition.

If \( x(t) \) is band limited white noise then the autocorrelation function \( R_{xx}(\tau - u) \) is an impulse of strength \( K \) at time \( \tau - u \).

That is,

\[
K\delta(\tau - u) = K \text{ when } \tau = u = 0 \text{ for all other } \tau. \quad (I.11)
\]

Thus the cross-correlation function is proportional to the impulse response if the auto correlation function is impulsive.

That is,

\[
R_{xy}(\tau) = kh(\tau) \quad (I.12)
\]
3. Derivation of the transfer function $F(S)$

From equations I.9 and I.10 we may write

$$R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} h(u) R_{xx}(u - \tau) \, du \quad (I.13)$$

Substituting I.13 in I.3

$$G_{xy}(\omega) = 2 \int_{-\infty}^{+\infty} h(u) e^{-j\omega u} \, du \int_{-\infty}^{+\infty} R_{xx}(t) e^{-j\omega t} \, dt$$

$$= F(\omega) G_{xx}(\omega)$$

or,

$$F(\omega) = \frac{G_{xy}(\omega)}{S_{xx}(\omega)} \quad (I.14)$$

$G_{xy}(\omega)$ is made up of the coincident spectral density function, (real part) $C_{xy}(\omega)$, and the quadrature spectral density function, (imaginary part) $Q_{xy}(\omega)$.

Thus,

$$G_{xy}(\omega) = C_{xy}(\omega) - Q_{xy}(\omega) \quad (I.15)$$

The phase angle $\varepsilon_{xy}(\omega)$, between the time signals $x(t)$ and $y(t)$ is given by,

$$\varepsilon_{xy}(\omega) = \tan^{-1} \frac{Q_{xy}(\omega)}{C_{xy}(\omega)} \quad (I.16)$$
4. Spectral windows

The theoretical justification for the identification of system transfer functions by Fourier Transform and Correlation techniques assumes data are collected over an infinite period of time.

In practice, data must be collected over a finite length of time, the effect of this is to multiply the continuous time data by a box car function (Fig. I.2).

So that,

\[ x_s(t) = x_c(t) \times B(t) \]  \hspace{1cm} (I.17)

Multiplication in the time domain is equivalent to convolution in the frequency domain, so the effect in the frequency domain of the box car function sampling window is,

\[ G_S(\omega) = \int_{-\infty}^{+\infty} G_C(v) B(\omega - v) \, dv \]  \hspace{1cm} (I.18)

The Fourier transform of the box car function B(t) is,
Which has the form of a $\frac{\sin x}{x}$ function as shown in Fig.I.3.

If $G_c(\omega)$ is a line spectrum equation I.18 shows that it is convolved with the continuous $\frac{\sin x}{x}$ function resulting in a weighted version of the $\frac{\sin x}{x}$ pattern about each of the spectral lines of interest, causing a smearing of the spectral estimate, as indicated in Figure I.4.
In simple terms we may say that the effect of observing continuous data through a time window and taking its transform to achieve an independent spectral estimate at a discrete frequency, causes smearing or leakage of the power spectrum by spreading the main lobe of the spectral density function, and generates an infinite number of lesser side lobes.

Figure I.4 shows that half of the side lobes are negative and under certain conditions this introduces the possibility of computing an erroneous negative power density, particularly when the power content of the spectrum is changing rapidly.

The solution is to introduce a filter which shapes the $\sin x/x$ function, many types of filter have been proposed, the main ones include,

Hanning $\omega(t) = (0.5 + 0.54 \cos (\pi t/T))$

Hamming $\omega(t) = (0.54 + 0.46 \cos (\pi t/T))$

Bartlett $\omega(t) = 1 - |t|/T$, for $|t| < T$
\[
\text{Parzen } \quad (t) = 1 - 6 \left( \frac{\pi t}{T} \right)^2 + 6 \left( \frac{\pi t}{T} \right)^3 \quad r = 0, 1, 2, \ldots, \frac{T}{2}
\]
\[
(t) = 2 \left( 1 - \frac{\pi t}{T} \right)^3 \quad r = \frac{T}{2} + 1, \ldots, T
\]
\[
= 0 \quad t > T
\]
\[
\text{Tukey } \quad (t) = 0.5 \left( 1 + \cos \frac{\pi t}{T} \right)
\]

The Tukey filter was used in the SPECTR algorithm and is shown superimposed on the Sin x/x function in Figure I.3.

The disadvantages of data windows have been examined, but in some circumstances the smearing of the spectral estimate can act to advantage as a smoothing operation, particularly if the unwanted fluctuations are of an uncorrelated random nature. The smoothing may best be done by transforming the autocorrelation functions to obtain the power spectra rather than the raw time data, for the truncation of the time data to produce a wider spectral lobe may omit wanted information, whereas truncation of the correlation function by choice of correlation lag numbers includes all the recorded time data in the calculation of each correlation coefficient, and allows examination of the smoothing effect for each chosen lag number (see section 2.5.3).

5. **The coherence function**

The cross spectrum \( G_{xy}(\omega) \) is a complex quantity composed of the cross spectral amplitude which may be considered to be a measure of the average power shared between the input and output signals as a function of frequency, and the phase angle \( \phi(\omega) \) which gives the average angle between the common frequency components.
If we consider the product of the input and output auto spectra (which have no phase components) in relation to the cross spectra, we have a measure of the correlation between the input and output records at each discrete frequency. This ratio is defined as the coherence $\gamma^2(\omega)$.

Where,

$$\gamma^2(\omega) = \frac{|G_{xy}(\omega)|^2}{G_{xx}(\omega) G_{yy}(\omega)}$$  (I.20)

In summary, the coherence is a fractional measure of the dependence of the output upon the input on a scale ranging from 0 to 1; a value of unity implies that the output signal is completely dependent upon the input, and a value of zero indicates that the output is completely uncorrelated with the input.

6. Regression analysis

The purpose of the analysis is to eliminate long term drifts in the recorded time data. The basis of the technique is to find a best straight line fit to the data by the method of least squares. The data is then normalised at each data point by having a value subtracted from it that is the value of the regression line at the co-ordinates of the point.

The method of least squares

Given a pair of data values $(x_r, y_r)$, we assume $x_r$ is correct and $y_r$ is subject to drift.
If the relationship is true then,

\[ y = ax + b \]  \hspace{1cm} (I.21)

the error is given by,

\[ y_r - y = y_r - ax_r - b \]  \hspace{1cm} (I.22)

If the sum of the errors is to be zero then,

\[ \sum_{r=1}^{n} (y_r - ax_r - b) \]  \hspace{1cm} (I.23)

where \( n \) is the number of samples.

Expanding we get,

\[ \sum_{r=1}^{n} y_r - a \sum_{r=1}^{n} x_r - nb = 0 \]  \hspace{1cm} (I.24)

and dividing by \( n \),

\[ \bar{y}_r - a\bar{x}_r - b = 0 \]  \hspace{1cm} (I.25)

Making the sum of the squares of the error equal zero,

\[ \sum_{r=1}^{n} (y_r - ax_r - b)^2 = 0 \]  \hspace{1cm} (I.26)

Differentiating equation I.26 with respect to \( a \) and \( b \) and equating to zero we get,

\[ \sum_{r=1}^{n} x_r y_r = a \sum_{r=1}^{n} x_r^2 + b \sum_{r=1}^{n} x_r \]  \hspace{1cm} (I.27)
\[ y = ax_r - b \quad (I.28) \]

From equations I.4 and I.7,
\[
\frac{\sum_{r=1}^{n} x_r y_r - nxy}{\sum_{r=1}^{n} x_r^2 - nx^2} \quad (I.29)
\]

The regression line is defined as,
\[ (y - \bar{y}) = a(x - \bar{x}) \quad (I.30) \]

where \( a \) is the regression slope.

The variance is
\[
\sigma_x^2 = \frac{\sum_{r=1}^{n} x_r^2 - nx^2}{n} \quad (I.31)
\]

therefore,
\[
a = \frac{P}{\sigma_x^2} \quad (I.32)
\]

where
\[
P = \frac{1}{n} \sum_{r=1}^{n} (x_r - \bar{x})(y_r - \bar{y}) \quad (I.33)
\]

To find a line passing through \( (x_r, y_r) \), we have
\[ (y_r - \bar{y}) = \frac{P}{\sigma_x^2} (x_r - \bar{x}) \quad (I.34) \]

where \( P \) is the covariance of the sample, and \( \sigma_x^2 \) is the variance.
Hence for a particular value for $x$ a value of $y$ (ie $y_1$) may be calculated from the regression slope and subtracted from the value of $y$ at the sample point $y_1$.

That is,

$$y_{(corrected)} = y_1 - y \quad (I.35)$$

7. The correlation algorithm

The algorithm for computer program SPECTR (transfer function from time data) requires the calculation of the forward correlation coefficients $R_{xy}(\tau)$ and the reverse correlation coefficients $R_{yx}(\tau)$, these are represented in the correlation algorithm by $R1[K]$ and $R2[K]$ respectively.

Fig. I.5 THE CROSS CORRELATION PROCEDURE

$Y(t)$ stepped in time through lag number $K$; we may write down the following directly from Figure I.5.
\[
R_1 [K] = \frac{1}{S_L - K} \sum_{r=1}^{S_L-K} \frac{(x[r] - \bar{x})(y[r + K] - \bar{y})}{\sigma_x \sigma_y}
\]  
(I.36)

\[
R_2 [K] = \frac{1}{S_L - K} \sum_{r=1}^{S_L-K} \frac{(y[r] - \bar{y})(x[r + K] - \bar{x})}{\sigma_x \sigma_y}
\]  
(I.37)

where,

\[K = 0, 1, 2, 3, 4, \ldots, CL.\]

\[CL = \text{number of correlation lags.}\]

\[S_L = \text{number of data points.}\]

and,

\[
\bar{x} = \frac{1}{S_L} \sum_{r=1}^{S_L} x[r]
\]  
(I.38)

\[
\bar{y} = \frac{1}{S_L} \sum_{r=1}^{S_L} y[r]
\]  
(I.39)

\[
\sigma_x^2 = \frac{1}{S_L} \sum_{r=1}^{S_L} (x - \bar{x})^2
\]  
(I.40)

\[
\sigma_y^2 = \frac{1}{S_L} \sum_{r=1}^{S_L} (y - \bar{y})^2
\]  
(I.41)

8. **The algorithm for the spectral transfer function program SPECTR**

The power spectral density functions \(G_{xx}(\omega), G_{yy}(\omega)\) and \(G_{xy}(\omega)\) may be calculated via the Fourier transform of the correlation functions \(R_{xx}(\omega), R_{yy}(\omega)\) and \(R_{xy}(\omega)\), as defined for the cross spectral density function \(G_{xy}(\omega)\) as follows.
\[ G_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(t) e^{-j\omega t} \, dt \quad \text{(I.42)} \]

we may write

\[ G_{xy}(\omega) = \int_{0}^{\infty} R_{xy}(t) e^{-j\omega t} \, dt + \int_{-\infty}^{0} R_{xy}(t) e^{-j\omega t} \, dt \]

\[ = \int_{0}^{\infty} R_{xy}(t) e^{-j\omega t} \, dt - \int_{0}^{\infty} R_{xy}(-t) e^{+j\omega t} \, dt \]

\[ = \int_{0}^{\infty} R_{xy}(t) e^{-j\omega t} \, dt + \int_{0}^{\infty} R_{xy}(-t) e^{+j\omega t} \, dt \]

\[ = \int_{0}^{\infty} R_{xy}(t) e^{-j\omega t} \, dt + \int_{0}^{\infty} R_{xy}(t) e^{+j\omega t} \, dt \quad \text{(I.43)} \]

Since,

\[ R_{xy}(-t) = R_{yx}(t) \]

In trigonometric form we have,

\[ G_{xy}(\omega) = \left[ \int_{0}^{\infty} R_{xy}(t) \cos \omega t \, dt + \int_{0}^{\infty} R_{yx}(t) \cos \omega t \, dt \right] \]

\[ - j \left[ \int_{0}^{\infty} R_{xy}(t) \sin \omega t \, dt - \int_{0}^{\infty} R_{yx}(t) \sin \omega t \, dt \right] \quad \text{(I.44)} \]
Rewriting the above equation in abbreviated form

\[ G_{xy}(\omega) = [\text{REAL1} + \text{REAL2}] - [\text{IMAG1} - \text{IMAG2}] \] (I.45)

Consider the term,

\[ \text{REAL1} + \text{REAL2} = \int_{0}^{\infty} R_{xy}(t) \cos \omega t \, dt + \int_{0}^{\infty} R_{yx}(t) \cos \omega t \, dt \]

Fig I.6 STRAIGHT LINE APPROXIMATION TO THE CORRELATION FUNCTION

Using the straight line approximation

\[ R1 = mt + Q \]

where,

\[ R1 \] = a point on the correlation function.
\[ m \] = slope at that point.
\[ Q \] = constant.
Then from Figure I.6,

\[ m = \frac{R_k^{[K+1]} - R_k^{[K]}}{\Delta t} \quad (I.46) \]

near the point \( R_k^{[K]} \).

And,

\[ R_k^{[K]} = m(K - 1) \Delta t + Q \]

or

\[ Q = R_k^{[K]} - m(K - 1) \Delta t \quad (I.47) \]

therefore,

\[
\text{REAL } 1 = \int_{(K-1)\Delta t}^{K\Delta t} R_k(t) \cos \omega t \, dt
\]

\[
= \int_{(K-1)\Delta t}^{K\Delta t} (mt + Q) \cos \omega t \, dt \quad (I.48)
\]

Integrating by parts we get,

\[
\text{REAL } 1 = \left[ \frac{Q}{\omega} \sin \omega t + \frac{mt}{\omega} \sin \omega t + \frac{m}{\omega^2} \cos \omega t \right]_{(K-1)\Delta t}^{K\Delta t}
\]

or,

\[ \text{REAL } 1 = \text{RE1} + \text{RF1} + \text{RG1} \quad (I.50) \]
where,

\[ R_E 1 = \frac{2}{\omega} \sin \left( \frac{\omega \Delta t}{2} \right) \sum_{K=1}^{CL} \cos \left( \frac{\omega \Delta t(2K - 1)}{2} \right) \{R1[K + 1]\} \]

\[ R_F 1 = \frac{2}{\omega^2 \Delta t} \sin \left( \frac{\omega \Delta t}{2} \right) \sum_{K=1}^{CL} \sin \left( \frac{\omega \Delta t(2K - 1)}{2} \right) \{R1[K] - R1[K + 1]\} \]

\[ R_G 1 = -\frac{2}{\omega} \sum_{K=1}^{CL} \sin (\omega \Delta t(K - 1)) \{R1[K] - R1[K + 1]\} \]

(I.51)

and

\[ \omega = \frac{2\pi J}{SL \Delta t} \]  

(I.52)

where J = 1, 2, 3, 4, ..... SL/2.

The same operation may be performed to obtain an expression for REAL2, then upon substituting in equation I.45 and including the Tukey filter,

\[ D = (1 + \cos (\pi \times (K - 1)/CL))/2 \]  

(I.53)

\[ E = (1 + \cos (\pi \times K/CL))/2 \]  

(I.54)

for R1[K] and R1[K - 1] we get,

\[ \text{REAL1} + \text{REAL2} = R_E 1 + R_E 2 + R_F 1 + R_F 2 + R_G 1 + R_G 2 \]  

(I.55)
where,

\[
RE_1 + RE_2 = \frac{SL \Delta t}{\pi} \sin \frac{\pi J}{SL} \sum_{k=1}^{CL} \cos \frac{\pi J}{SL} (2k - 1) E_{(K + 1)} \left\{ R_1_{(K + 1)} + R_2_{(K + 1)} \right\}
\]

\[
RF_1 + RF_2 = \frac{SL^2 \Delta t}{2\pi^2 J^2} \sin \frac{\pi J}{SL} \sum_{k=1}^{CL} \sin \frac{\pi J}{SL} (2k - 1) \left\{ D_{(K)}(R_1_{(K)} + R_2_{(K)})
- E_{(K + 1)} (R_1_{(K + 1)} + R_2_{(K + 1)}) \right\}
\]

\[
RG_1 + RG_2 = -\frac{SL \Delta t}{\pi J} \sum_{k=1}^{CL} \sin \frac{\pi J}{SL} (K + 1) \cos \frac{\pi J}{SL} (K + 1) \left\{ D_{(K)} (R_1_{(K)} + R_2_{(K)})
- E_{(K + 1)} (R_1_{(K + 1)} + R_2_{(K + 1)}) \right\}
\]

where \( J = 1, 2, 3, 4, \ldots \) SL/2

A similar calculation may be made for the imaginary spectral components \( IMAG_1 \) and \( IMAG_2 \).

Thus,

\[
IMAG_1 + IMAG_2 = IE_1 + IE_2 + IF_1 + IF_2 + IG_1 + IG_2 \quad (I.57)
\]

where,
\[ IE1 + IE2 = \frac{S}{K} \sum_{K=1}^{\pi J} \cos \frac{\pi J}{N} (2K - 1) \left\{ E[K + 1] R2[K + 1] - R1[K + 1] - D[K] (R2[K] - R1[K]) \right\} \]

\[ IF1 + IF2 = \frac{S}{K} \sum_{K=1}^{\pi J} \sin \frac{\pi J}{N} (2K - 1) \left\{ E[K + 1] R2[K + 1] - R1[K + 1] \right\} \]

\[ IG1 + IG2 = -\frac{S}{K} \sum_{K=1}^{\pi J} \cos \frac{\pi J}{N} (K - 1) \left\{ E[K + 1] R2[K + 1] - R1[K + 1] - D[K] (R2[K] - R1[K]) \right\} \]

where \( J = 1, 2, 3, 4, 5, \ldots \), \( S/2 \).

The auto power spectral density functions \( G_{xx}(\omega) \) and \( G_{yy}(\omega) \) involve only the terms REAL1 and REAL2.

To summarise

\[ G_{xy}(\omega) = (\text{REAL1} + \text{REAL2}) - (\text{IMAG1} + \text{IMAG2}) \]

and

\[ G_{xx}(\omega) \text{ or } G_{yy}(\omega) = (\text{REAL1} + \text{REAL2}) \text{ as appropriate.} \]

The following quantities may be calculated using the values derived from the above spectral estimates.
1.0 CROSS MODULUS

\[ |G_{xy}(\omega)| = \sqrt{(\text{REAL1} + \text{REAL2})^2 + (\text{IMAG1} + \text{IMAG2})^2} \quad (I.59) \]

2.0 FREQUENCY

\[ 2\pi J/SL \Delta t \text{ where } J = 1, 2, 3, 4, \ldots, SL/2 \quad (I.60) \]

3.0 TRANSFER FUNCTION

\[ F(\omega) = \frac{|G_{xy}(\omega)| \sqrt{\sigma_x^2 \sigma_y^2}}{G_{xx}(\omega) \sigma_x^2} \quad (I.61) \]

4.0 GAIN

\[ \text{GAIN} = \sqrt{\frac{G_{yy}(\omega) \sigma_y^2}{G_{xx}(\omega) \sigma_x^2}} \quad (I.62) \]

5.0 COHERENCE

\[ \gamma^2 = \frac{|G_{xy}(\omega)|^2(\omega)}{G_{xx}(\omega) G_{yy}(\omega)} \quad (I.63) \]

6.0 PHASE

\[ \epsilon = \tan^{-1} \left( \frac{\text{REAL1} + \text{REAL2}}{\text{IMAG2} - \text{IMAG1}} \right) \quad (I.64) \]

7.0 REAL PART OF TRANSFER FUNCTION

\[ \text{RE} \left( F(\omega) \right) = \frac{F(\omega)}{\sqrt{1 + \tan^2\epsilon}} \quad (I.65) \]
8.0 IMAGINARY PART OF TRANSFER FUNCTION

\[
F(\omega) = \frac{\sqrt{1 + \frac{1}{\tan^2 \epsilon}}}{1.66} \tag{I.66}
\]
The convolution integral is stated as,

\[ y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) \, d\tau \]  \hspace{1cm} (II.1)

The operation of the integral may be seen clearly in terms of a sample data plot as given in Figure II.1.

![Diagram of sampled impulse response and input signal](image)

**Fig. II.1** SAMPLED IMPULSE RESPONSE \( h(t) \) AND INPUT SIGNAL \( x(t) \)

Each input \( x_n(t) \) weights a delayed version of the impulse response as depicted in Table II.1. The summation of each column gives the value of the output at that instant in time.
\[ T = 0 \quad t \quad 2t \quad 3t \quad 4t \]
\[ x_0 h_0 + x_1 h_1 + x_2 h_2 + x_3 h_3 + x_4 h_4 \quad \text{from } x_0 \]
\[ + x_1 h_0 + x_1 h_1 + x_2 h_2 + x_3 h_3 \quad \text{from } x_1 \]
\[ + x_2 h_0 + x_2 h_1 + x_3 h_2 \quad \text{from } x_2 \]
\[ + x_3 h_0 + x_3 h_1 \quad \text{from } x_3 \]
\[ + x_4 h_0 \quad \text{from } x_4 \]

**Table II.1 Convolution of \( x(t) \) with \( h(t) \) up to \( T = 4t \)**

The output values up to time \( T = 4t \) are given in the table, that is,

\[ y(4t) = x_0 h_4 + x_1 h_3 + x_2 h_2 + x_3 h_1 + x_4 h_0 \]  \hspace{1cm} (II.2)

In words we see from equation II.2 that the output at time \( 4t \) is the sum of the inputs multiplied by a delayed version of the impulse response. The summation may be expressed by,

\[ y(n) = \sum_{n=0}^{n} x(m) h(m - n) \]  \hspace{1cm} (II.3)

\[ n = 1, 2, 3, \ldots \quad T \]
APPENDIX III

PSD results in relation to analytic equations

Analytic mathematic models of various levels of complexity, that describe the dynamic steering response of a vehicle, have been constructed (Refs.1,23,24). The equations of vehicle motions in response to a steering input are obtained by equating transverse tyre forces to transverse inertial vehicle forces and tyre angular moments to vehicle angular momentum. If the values of vehicle constants such as mass, moment of inertia, tyre stiffness, speed etc are known for a particular vehicle of interest, then a transfer function relating to that vehicle may be written down. In general the degree of validity is substantiated by comparison of model output with practical field measurements.

Whitcomb (Ref.1) has shown that a vehicle with high roll stiffness or undergoing manoeuvres in which roll angles are small may be approximated by a two degree of freedom model.
\( S_F = \) Side force at front wheel  
\( S_R = \) Side force at rear wheel  
\( a = \) Distance of front wheel from C of G  
\( b = \) Distance of rear wheel from C of G  
\( M = \) Mass of vehicle  
\( I = \) M of I of vehicle about vertical axis through C of G  
\( r = \) Yaw rate  
\( C_F = \) Front tyre cornering stiffness  
\( C_R = \) Rear tyre cornering stiffness  
\( \alpha_F = \) Front slip angle  
\( \alpha_R = \) Rear slip angle  
\( U = \) Forward velocity  
\( V = \) Sideslip velocity  
\( \lambda = \) Front wheel steer angle

**Fig. III.1 TWO DEGREE OF FREEDOM VEHICLE MODEL**

From Figure III.1

\[ S_F + S_R = \text{transverse inertial force} = M(V + U_r) \]  \hspace{1cm} (III.0)

\[ aS_R + bS_R = \text{angular torque} = Ir \]  \hspace{1cm} (III.1)
where

\[ S_F = C_F \alpha_F; \quad S_R = C_R \alpha_R \] (III.2)

and

\[ \alpha_F = \lambda - \frac{V}{U} - \frac{ar}{U}; \quad \alpha_R = \frac{br}{U} - \frac{V}{U} \] (III.3)

Equations III.1 to III.4 reduce to,

\[
\begin{align*}
\left[ M_U + \frac{aC_F}{U} - \frac{bC_R}{U} \right] r + \left[ \frac{C_F + C_R}{U} \right] V + M \dot{V} &= C_F \lambda \\
\dot{r} \left[ \frac{a^2C_F + b^2C_R}{U} \right] r + \left[ \frac{aC_F - bC_R}{U} \right] V &= aC_F \lambda
\end{align*}
\] (III.4, 5)

Now

\[ a_y(t) = [V(t) + Ur(t)] \] (III.6)

where \( a_y(t) \) = transverse acceleration.

So that equations III.4, III.5 and III.6 may be solved for \( \frac{a_y}{\lambda} \) and \( \frac{r}{\lambda} \) to give the normalised transverse and angular transfer functions

\[
\frac{a_y}{\lambda}(s) = \frac{K_1(s^2 + 2\zeta_a \omega_s S + \omega_s)}{(s^2 + 2\zeta_a \omega_s S + \omega_s)}
\] (III.7)

and

\[
\frac{r}{\lambda}(s) = \frac{K_2 S(S + \omega_r)}{S^2 + 2\zeta_{\lambda r} \omega_{\lambda r} S + \omega_{\lambda r}}
\] (III.8)

where \( \lambda a = \lambda r \).
In general the measured value of $\omega_{\lambda r}$ tended to be greater than $\omega_{\lambda a}$, (Table III.1), for both angular and cross-ply tyres, that is the vehicle has a marginally greater natural frequency in the rotational plane than in the transverse plane. This phenomenon is not apparent in the analytic equations; factors which may contribute to the difference between the measured and analytic result are:

(a) non-linearities in the vehicle,

(b) difference in tyre characteristics under rotational stress compared with transverse stress,

(c) flexibilities in the structure of the vehicle including body to axle fixings,

(d) differences in response between angular and transverse acceleration transducers.

(Precautions were taken against errors arising from (d) by calibrating the angular accelerometer against the linear accelerometer on an oscillating beam (Viz. Chapter 4.1)).

(e) roll acceleration and gravitational acceleration components affecting the transverse accelerometer output.

(Roll effects are assumed to be small (Viz. Chapter 4.1). No errors would arise from roll acceleration contaminating the estimates of transverse acceleration measurements if the accelerometer was mounted at the roll height of the vehicle - in these tests errors arising were likely to be small since the accelerometer was mounted close to the roll centre.)
If we assume that the dynamics between the steering wheel and the road wheels can be represented by a simple gain factor it is possible to compare the coefficients in equations III.7 and III.8, with those obtained by power spectral density methods, particularly where a second order equation has produced an acceptable fit.

Table III.1 shows measured values of $\omega_a$, $\zeta_a$, $\omega_{\lambda a}$, $\zeta_{\lambda a}$, $\omega_r$, $\omega_{\lambda r}$ and $\zeta_{\lambda r}$ over a range of speed for the same vehicle fitted with first cross-ply and then radial ply tyres.

<table>
<thead>
<tr>
<th>SPEED (km/h)</th>
<th>TRANSVERSE FUNCTION</th>
<th>ANGULAR FUNCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega_a$</td>
<td>$\zeta_a$</td>
</tr>
<tr>
<td>CROSS-PLY TYRES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>12.9</td>
<td>0.14</td>
</tr>
<tr>
<td>32</td>
<td>12.1</td>
<td>0.19</td>
</tr>
<tr>
<td>48</td>
<td>11.2</td>
<td>0.26</td>
</tr>
<tr>
<td>64</td>
<td>11.2</td>
<td>0.18</td>
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<tr>
<td>80</td>
<td>10.8</td>
<td>0.13</td>
</tr>
<tr>
<td>96</td>
<td>10.5</td>
<td>0.14</td>
</tr>
<tr>
<td>RADIAL PLY TYRES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>11.5</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>11.5</td>
<td>-</td>
</tr>
<tr>
<td>48</td>
<td>11.0</td>
<td>-</td>
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<tr>
<td>64</td>
<td>11.0</td>
<td>-</td>
</tr>
<tr>
<td>80</td>
<td>11.4</td>
<td>-</td>
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</tbody>
</table>

Table III.1 Coefficients of vehicle angular and transverse transfer functions with cross-ply and radial ply tyres
For the transverse acceleration transfer function coefficients the lead term $\omega_a$ changes little with speed or type of tyre, but the associated damping factor $\zeta_a$, tends to increase towards the middle speeds (48 km/h) in the case of cross-ply tyres, and was insignificantly small for radial ply tyres.

The difference in $\omega_{\lambda a}$ between cross-ply and radial ply tyres does not show a continuous trend with speed, although there is an indication that radial ply tyres, at speeds above 48 km/h, produce a vehicle of higher natural frequency and lower damping.

In the case of the angular transfer function, the lead term break point $\omega_r$, was found to be insignificant up to about 48 km/h for cross-ply tyres, and was not detectable over the whole speed range for radial ply tyres. The same tendency for the radial ply tyres to produce a higher natural frequency, $\omega_{\lambda r}$, at the higher speed is noticeable in the results; similarly the cross-ply damping factor, $\zeta_{\lambda r}$, tends to increase towards the middle speeds. The radial ply vehicle, in angular motion, produced a less damped system than the cross-ply over the whole speed range excluding 16 km/h.

Natural frequencies and damping factors, calculated by insertion of measured vehicle parameters, were not easily available for comparison with those values generated by power spectral means, however Weir et al (Ref.16) quote values for a typical American sedan at 96 km/h of $\omega_{\lambda} = 4.3$ rads/sec and $\zeta_{\lambda} = 0.87$, which as might be expected, describes a vehicle of lower natural frequency and higher damping than typical measured values from the Cortina (Table 3.1).
In order to arrive at an overall assessment of cross-ply and radial ply tyres from the driving viewpoint, a set of transfer functions at a single speed (48 km/h) have been compared by Root Locus method with a 'standard' model driver in the feedback loop (Viz. Chapter 7.3).
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