SYNTHESIS OF LINEAR SYSTEMS FOR PRESCRIBED TIME DOMAIN PERFORMANCE USING FREQUENCY RESPONSE DATA

by

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The first part of this work describes a method for calculating the impulse and step response of a linear system from the real part of its frequency response. The real part is approximated by confluent straight line segments up to a truncation frequency, and by an even inverse power of frequency beyond that. Criteria for choosing a truncation frequency and the required inverse power are discussed. Using the amplitudes of the real part at the junctions of the straight line segments, the truncation frequency, and the inverse power, the impulse and step response can be calculated using the tables and curves given.

The second part describes a synthesis procedure. The central idea in this is the description of the real part of the frequency response of systems by the harmonic content of the section up to a truncation frequency, and by an even inverse power of frequency beyond that. In this way real parts corresponding approximately to a wide range of systems with a specified pole-zero excess can be formed. Performance indices relating the transient and frequency performances corresponding to each such real part have been found, and are presented in the form of performance curves.

By means of these curves target systems meeting a range of performance indices may be chosen. From the frequency response of the target system and the open loop constraint, the frequency response of the required cascade compensation is found. This is then approximated by a rational transfer function from which the necessary compensating network can be synthesised.
ACKNOWLEDGMENT

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LIST OF PRINCIPAL SYMBOLS

\[ \begin{align*}
  \text{P}(s) & : \text{transfer function of the plant (open loop constraint)} \\
  \text{M}(s) & : \text{transfer function of the cascade compensation.} \\
  \text{H}(s) & : \text{transfer function of the parallel compensation.} \\
  \text{R}(s) & : \text{Laplace transform of the input signal.} \\
  \text{C}(s) & : \text{Laplace transform of the output signal.} \\
  \text{E}(s) = R(s) - C(s) & : \text{Laplace transform of the error.} \\
  \text{G}(s) = M(s)P(s) & : \text{open loop forward path transfer function.} \\
  \text{T}(s) = \frac{G(s)}{1 + G(s)L + H(s)} & : \text{closed loop transfer function.} \\
  \text{T}(j\omega) = \text{Re}T(j\omega) + j\text{Im}T(j\omega) & : \text{closed loop frequency response.} \\
  \text{h}(t) & : \text{impulse response.} \\
  \text{u}(t) & : \text{step response.} \\
  \text{uvel}(t) & : \text{velocity step response.} \\
  \text{uacc}(t) & : \text{acceleration step response.} \\
  \text{ha}(t)[\text{ua}(t)], \text{hb}(t)[\text{ub}(t)], \text{hc}(t)[\text{uc}(t)], \text{hd}(t)[\text{ue}(t)], & : \text{contributions to the impulse [step] response of sections of the approximate real part of the frequency response (relevant to ch. 2 and 3).} \\
  \text{ha}_k(t)[\text{ua}_k(t)] & : \text{contribution to the impulse [step] response of the harmonics comprising the real part of the frequency response up to } \omega_c. \\
  \omega_c & : \text{truncation frequency dividing the intermediate region from the asymptotic region of } \text{Re}T(j\omega). 
\end{align*} \]
N : the inverse power of frequency describing the asymptotic behaviour of \( \text{Re}T(j\omega) \).

\( A_0, A_1, \ldots, A_c \) : values of \( \text{Re}T(j\omega) \) at points where the true and approximate real part plots coincide.

\( x = \omega_1 t \) : normalised time variable used in ch. 2 and 3.

\( T = \omega_c t \) : normalised time variable used in ch. 4 - 8.

\( a_0, a_1, \ldots, a_n \) : Fourier coefficients specifying the harmonic content of \( \text{Re}T(j\omega) \) up to \( \omega_c \).

\( h_{ml}, t_{hl}, t_{sh} \) : indices relating to the impulse response (ch. 1).

\( u_{ml}, t_{ul}, t_{su} \) : indices relating to the step response (ch. 3).

\( M_{p}, \omega_p, \omega_s, \text{BW} \) : indices relating to the frequency response (ch. 1).

\( S_G \) : sensitivity of \( T(s) \) with respect to \( G(s) \) for unity negative feedback.

\( K_v \) : velocity constant.

\( K_a \) : acceleration constant.
GLOSSARY OF TERMS

Target system: system chosen to meet a given specification.

Truncation frequency $\omega_c$: frequency dividing the intermediate and asymptotic regions of the real part $[\text{Re}(j\omega)]$ of the frequency response $[T(j\omega)]$.

Intermediate region of the real part: section of $\text{Re}(j\omega)$ up to $\omega_c$ approximated either by confluent straight lines (ch. 2 and 3), or by a Fourier cosine series (ch. 4-8).

Asymptotic region of the real part: section of $\text{Re}(j\omega)$ beyond $\omega_c$ approximated by an even inverse power of frequency.

Class of system: the number of the highest harmonic used to describe the intermediate region of $\text{Re}(j\omega)$.

Formodd, formeven: parameters taking the values 0 and 1 only, used to specify the signs of the harmonics comprising the intermediate region of $\text{Re}(j\omega)$ (ch. 6).
INTRODUCTION

The overall problem of system design or synthesis contains within it several more or less well defined subsidiary problems. These may be listed under the following headings:

1) The preparation of a suitable specification.
2) The choice of a system to meet the specification.
3) The identification of the open loop constraints.
4) The identification of the necessary compensation.

The present work is concerned mainly with problems 2 and 4, but in order to put it into context within the overall design problem, each of the four subsidiary problems will be discussed a little more fully.

Problem one is the most elusive as far as the development of a logical and precise method is concerned. This is because practical systems may be required to perform a wide variety of tasks, and yet in each case in order to prepare a useable specification, their suitability or otherwise for these tasks must be judged by a few rather artificial performance indices, such as their response to a step input or their bandwidth. The present work makes no direct contribution to resolving this difficulty, the starting point in the synthesis method presented in the succeeding chapters being always a specification in terms of a number of performance indices. It may help to alleviate the difficulty somewhat, however, by allowing more rapid correlation of a larger number of performance indices than is possible by any other method.

Problem two marks the distinction between synthesis and design. The design approach, which omits problem two, might be termed a guided trial and error approach, in
which the performance of components chosen on the basis of power output requirements, economic and availability considerations (forming the open loop constraint), is modified by the addition of compensation networks. In general only a few types of networks are used, the qualitative effect of which on both the transient and frequency response of linear systems is known. By the successive incorporation of such compensation networks into the system, and successive re-calculation of their cumulative effect on the specified performance indices, the gradual fulfillment of the performance indices is attempted. The synthesis approach seeks to eliminate the successive re-calculation, by stating a clear target system, meeting the specified performance indices, right at the outset.

It is here that the present method is believed to make a useful contribution. The essential difficulty here is in choosing a target system which not only meets the specified performance indices but also is compatible with the open loop constraint i.e. the pole-zero excess of the target system must not be less than the pole-zero excess of the open loop constraint. It is believed that the method developed in the succeeding chapters provides correlation between the performance indices of systems with specified pole-zero excess for a much greater range of systems than is provided by any other method.

Problem three concerns the link between the physical reality of the hardware, and the mathematical manipulations used in the synthesis. The form of identification required by the present method is the frequency response. Inevitably, the measured frequency response of any physical
equipment will not be absolutely accurate, and further inaccuracies will result from the fact that the predicted performance is based on the assumption of linearity. It is primarily as a counter to this inaccuracy that the feedback configuration is resorted to in many practical systems, and the present method is principally concerned with the synthesis of feedback systems.

Problem four, like problem one, is difficult to pin down. It arises because, using the feedback configuration, there is not a unique solution to any particular problem. The division of the total required compensation partly into cascade compensation and partly into parallel compensation is governed by economic considerations (e.g. cost of a tachometer generator against cost of a network), space, weight, and availability considerations and sensitivity and reliability considerations. Since the present work aims at evaluating the feasibility of a completely new approach, and in order to concentrate on the essential advantages and disadvantages of the approach, problem four was simplified by restricting the configuration of the target system to unity feedback. With this restriction, the necessary cascade compensation becomes unique for any particular problem.

Throughout, it was attempted to make the method described in the succeeding chapters meet the following objectives:

To provide a more flexible means of choosing the target system than offered by the 'dominant pole pattern' approach.

To use the simplest form of system identification, which was considered to be the frequency response.
To make the method suitable for implementation on a digital computer.

Layout

In chapter I the commonly used performance indices are discussed and a set considered adequate for an accurate description of system performance arrived at. Existing design and synthesis methods are reviewed and an attempt is made to analyse their shortcomings.

In chapter 2 a method of obtaining the transient performance from the frequency response of a linear system is presented. The principle involved is not new (refs. 63, 65, 66) but it is believed that the method represents both an extension and an improvement on existing methods.

Chapter 3 contains numerical evaluation of the theoretical material of chapter 2.

It is believed that chapters 4-8 contain the main original contribution of the present author.

Chapters 4 and 5 contain the theoretical basis for a characterisation of linear systems in terms of the harmonic content of the real part of the frequency response up to a truncation frequency, and in terms of an even inverse power of frequency beyond the truncation frequency.

Chapter 6 contains the performance curves calculated on the above basis and used in the synthesis.

In chapter 7 the validity of the curves is investigated, and chapter 8 presents some examples of the use of the synthesis method.
SURVEY OF SOME DESIGN METHODS AND PERFORMANCE INDICES

1.1 Introduction.

It was noted in the introduction that one of the subsidiary problems of synthesis is the preparation of a useable specification, and this in itself involves two steps.

Firstly, an estimate is made of such things as, what the frequency response of the system should be like and what its transient response to impulse, step and ramp inputs should be like in order for it to carry out whatever it is intended to do. In other words, a set of performance indices are specified.

Secondly, a target system is chosen to meet the performance indices.

It is the purpose of this chapter to discuss some of the commonly used performance indices, in order to arrive at a set which can be considered sufficient to specify reasonably accurately both the transient and frequency responses of systems.

Once this has been done, a brief survey of some existing design and synthesis methods is carried out in order to investigate how close a control they afford over the chosen set of performance indices.

1.2 Choice of performance indices.

The method which will be developed in the succeeding chapters aims at establishing a close quantitative correlation between the frequency response and the transient response of systems, in order to allow a close
control over both these aspects of system performance.
As a first step, the indices commonly used to specify these
aspects of system performance are discussed.

1.2.1 Transient performance indices.

Conventionally, the transient performance of a
system is described in terms of its impulse and step
response. Because the two responses are related (the
impulse response being the derivative of the step
response), the great majority of the design methods
discussed in section 1.3 use the step response only. The
exceptions are the time-domain methods (63 ch.15, 31 ch.12,
1), but these specify the complete impulse response not
just some aspects of it.

The indices used in the various design methods are
shown in fig.1.1.a. Thus $u_{ml}$ and $t_{ul}$ are the values of
the overshoot and the time to reach it. The time delay
$t_d$ is generally taken as the time for the response to
reach half of its final value.

The rise time $t_r$ is variously defined as the time
interval between 10% and 90%, or 5% and 95% of the final
value, or as the reciprocal of the slope of the response
when the response has reached half of its final value
i.e. at $t_d$ (27).

The settling time $t_s$ is usually taken as the time
for the response to settle to within 5% or 2% of its
final value, but since for oscillatory responses this is
a discontinuous function of the system parameters,
approximations to this have been used (15).

The time for the response to reach the final value
for the first time, $t_{ol}$, has been used by Zemanian(49-53),
Fig. 1.1.a Indices commonly used to specify the step response

Fig. 1.1.b Indices which will be used to specify the frequency response

Fig. 1.2.a Indices which will be used to specify the step response.

Fig. 1.2.b Indices which will be used to specify the impulse response.
but does not appear in any of the other methods discussed in section 1.3.

For the purposes of the method developed in this work, it was decided to use indices relating both to the step and impulse responses.

Although they are mutually derivable, it was felt somewhat preferable to have both in evidence instead of inferring one from the other. After all, approximations to impulse inputs occur in practice, as for example wind gusts on aircraft or road irregularities on cars. Furthermore, by dividing the indices between the step and impulse, nothing is lost in the accuracy of the description. In fact it is enhanced by the specification of the approximate settling time both for the step and impulse responses, since these two settling times can be very different.

The transient response indices which will be used throughout the present work are shown in fig. 1.2a and b. The indices $t_{ul}$ and $u_{ml}$ are the same as in the other methods discussed in section 1.3. In place of $t_d$ and $t_r$, $h_{ml}$ and $t_{hl}$ convey similar information. The settling time $t_{su}$ has been taken to be the time of the extremum beyond which the step response lies wholly within 2% of the final value, and $t_{sh}$ of the extremum beyond which the unit impulse response is wholly within $\pm 0.02$. The reason for using these values rather than the true settling times was that they were quicker to compute, and in any case settling time is a very approximate quantity, being very dependent on stiction.
1.2.2 **Frequency performance indices.**

The frequency performance indices which will be used are shown in fig. 1.1.b. They consist of the generally accepted resonance peak $M_\omega$, resonant frequency $\omega_r$ and the 3db bandwidth $BW$. The last frequency response index is the frequency at which the sensitivity, assuming unity negative feedback, is one (c.f. section 6.4.2). This is termed $\omega_s$.

1.2.3 **Other performance indices.**

One other index which must be included relates to the accuracy of a system. This is the velocity constant $K_v$ or the acceleration constant $K_a$, whichever is applicable (c.f. section 6.5).

1.3. **Brief survey of some existing design methods.**

Existing methods fall broadly into five categories:

1) Frequency reponse methods.
2) Root locus methods.
3) Time domain methods.
4) Algebraic methods.
5) Special purpose methods.

By special purpose methods are meant those methods relying on the use of computers to permit investigation of a range of possible compensation schemes before a choice is made. It is of course true that by the use of a hybrid analogue computer, (or a digital computer), suitable compensation for any particular system may be found, and may even be optimised for some criterion (3). This approach however, useful though it is in practice, treats each problem as a special case. Since the purpose of the present work is to develop a general method
applicable to any system, special purpose methods will not be further considered.

Algebraic methods are those based on the manipulation of the coefficients of the system differential equation (5, 6 ch.10, 7,8,9). Their great disadvantage from an engineering point of view is the loss of insight into the physical system which their use entails. Since the aim throughout the present work was to develop a method which retained the scope for intuitive qualitative reasoning afforded by both the frequency response and root locus methods, but at the same time permitted a closer quantitative control over the transient performance than existing methods afforded, it was felt reasonable to omit algebraic methods from further consideration.

Time domain methods themselves are of two types; those which use the time domain specification of the required system response to obtain the necessary system transfer function, and those which carry out all design in the time domain without recourse either to the s-plane or the frequency response.

Examples of the first kind of time domain methods are Guillemin's method (63 ch.15) and the method of moments (31 ch.12). It was felt, however, that both approaches are difficult from an engineering point of view. This is because the starting point in each case is an impulse response (or step response) completely specified in the time domain, from which the necessary system transfer function is obtained. Now it would be a rare system indeed which required such detailed specification in the time domain; usually only a few performance indices are necessary (as discussed in section 1.2).
With only a few indices specified it would be very difficult with these methods to decide what price, in terms of compensation, one was paying by choosing one sort of shape for the complete transient response and not another.

An example of the second kind is a method developed by Tustin (1). The calculations involved, however, (similar to polynomial multiplication and division) are more suitable for analysis than synthesis. If used for synthesis, it suffers from the disadvantage that the desired transient response again must be completely specified, not merely described by a few indices.

Finally the time domain methods mentioned do not afford any direct link with the frequency response. Because of their essential difference, time domain methods likewise will not be further considered.

1.3.1 Frequency response methods

Design of systems by shaping their frequency response is historically the earliest method. Due to Nyquist's stability criterion (10) it is certainly possible by this method to design compensation to achieve a stable system. Developments introduced by Bode (11) have made this approach highly practicable, especially since the frequency response of practical systems may be obtained experimentally while a theoretical derivation of their differential equation may well be impracticable. The sole disadvantage of the frequency response approach lies in the difficulty of correlating the frequency response with the transient response.

The general principles of shaping the frequency
response by the addition of compensation are covered in virtually all text books dealing with control systems (12,13). The qualitative relationship between the frequency and transient responses is known and is expressed in terms of the gain and phase margins of the open loop frequency response and the height of the resonance peak of the closed loop frequency response. So long as the specification only requires a stable system satisfying frequency performance indices but with very flexible transient requirements (e.g. only setting a limit to the overshoot in the step response), the frequency response method is extremely satisfactory. When, however, as often happens in practice, a closer control over the transient response is required, difficulties arise.

A number of methods have been published seeking to solve these difficulties. Jaworski (56) presents empirically derived relationships between the rise time, overshoot, and settling time of the system step response and a 'form parameter' obtained from the system frequency response. As the author himself points out, however, the relationships are more useful in indicating general trends than in providing quantitative correlations. Even then they are further more restricted to frequency responses which are monotonically decreasing or have a single resonance peak. A number of similar relationships are likewise given by Horowitz, who also points out the lack of an easy yet accurate frequency-time correlation (31 p.190).

Westcott (57) describes a method which, in its essentials, consists of expressing the system impulse
response in a standard form as
\[ h(t) = (at + \beta t^2 + \lambda t^3) \exp(-\mu t) \]  
and the system frequency response by the first four terms of the series
\[
H(j\omega) = \int_0^\infty h(t) \exp(-j\omega t) \, dt \\
= a_0 - a_1(j\omega) + \frac{a_2(j\omega)^2}{2!} - \frac{a_3(j\omega)^3}{3!} \ldots
\]
where \( a_0 = \int_0^\infty h(t) \, dt \), \( a_1 = \int_0^\infty h(t) \, dt \).

i.e. the \( a \)'s are moments of area of the impulse response.

The correlation between the parameters describing the impulse and frequency responses is then established. A difficulty remains however, in that practical specification of the impulse response could normally be satisfied with a range of the \( a, \beta, \lambda \) and \( \mu \) parameters and this range would be difficult to find in practice. The same difficulty arises with the similar correlation the author derives between the step response and the first four terms of expression 1.3.2.

West and Potts (58) describe a method by means of which an extension of the gain and phase margins is used to determine approximately the principle mode (or dominant complex pole-pair) of the closed loop system. The approach essentially is of assuming that the system is approximately second order and its accuracy depends on the accuracy of the approximation.

More recently Meadows (14) analysed in detail the possible frequency responses of a third order type 1 system. Taken in conjunction with a paper by the present author (15) which summarised the relationship between the normalised pole-zero pattern of this class of systems
with the transient behaviour, it would provide a close frequency-time correlation, but regretfully only for the limited class of systems.

1.3.2 Root locus methods.

The root locus method was developed by Evans (16,17) in order to provide a better control over the transient response. While this approach does indeed by its very nature keep the general form of the transient response constantly in evidence, the quantitative correlation between the pole-zero pattern and the transient response for systems of any complexity is still difficult (27 p.285).

Root locus methods pose the additional difficulty that, before they can be attempted, the system being compensated (i.e. the open loop constraint) must be approximated by a transfer function. The transfer function parameters in practice are obtained from the measured frequency response (18-23), from transient measurements (24,70,71) or by the use of cross-correlating filters (25,26).

Once the transfer function of the open loop constraint has been obtained, various compensation schemes may be tried out. As with the frequency response methods, if the object is to produce a stable system having only a generally specified transient response, and now in addition, having only a generally specified frequency response, the method is satisfactory.

The effect on the root locus of introducing the commonly used forms of compensation are covered in standard texts (27,28) and a great deal of information about root locus shapes has been compiled by Williamson(29).
and Lovering and Williamson (41). If a close control over the transient and frequency response is required, however, difficulties arise.

For a synthesis method, the first step is to choose a closed loop transfer function which satisfies the specification. It is immediately in this first step that the difficult correlation between a pole-zero pattern and the corresponding frequency and transient performance becomes apparent. The most common answer to this difficulty is the adoption of the dominant pole pair approximation, i.e. approximation to a second order system (36). Extensions to third order systems have been described by Higgins and Levinthal (29), Hausenbauer and Lago (30) and the present author (15). Truxal (27, ch. 5) discusses a method originated by Guillemin in which the closed loop transfer function is assumed to be dominated by a complex pole pair and one zero. Any other poles are assumed to be sufficiently far in the l.h.p. or sufficiently close to a zero not to affect the performance significantly. Horowitz (31) presents information about the frequency and transient behaviour of a more complicated pole-zero pattern, given by

\[ T(s) = \frac{(s + a)}{(s + b \pm j\omega)(s + d)(s + e)(s + g)} \] 1.3.3

The number of variable parameters means, however, that even a number of families of curves can summarise only a few possible performances. Elgerd and Stephens (48) present a number of families of curves giving the step response of some selected pole-zero patterns.

Once the required closed loop transfer function \( T(s) \) is chosen, a number of methods are available for
calculating the required compensation. If a unity feedback configuration is assumed, then, once the closed loop transfer function $T(s)$ is chosen, the corresponding open loop transfer function $G(s)$ is uniquely specified. Since in practice $G(s)$ must incorporate the plant (open loop constraint) cancellation compensation is often called for (27 p. 309). This does not represent a serious difficulty since compensating zeroes can readily be located anywhere in the left half complex plane by means of passive RC networks (72) and, if necessary, compensating complex poles may be obtained by active RC networks.

A number of authors have specifically tried to choose the closed loop transfer function not only to meet the specification, but also to result in as simple a compensation network as possible given the open loop constraint (4, 32-35, 37). As pointed out by one of the authors however (33) there is no guarantee of convergence in this approach.

1.3.3 Standard form approach.

The frequency response and root locus methods considered in the last two sections sought to satisfy the commonly used performance indices. None of the methods allows an easy, simultaneous control of both frequency and transient response indices, and because of this a totally different approach also exists.

It consists of choosing some overall criterion of system performance instead of the individual performance indices. A number of possible performance criteria have been investigated (38, 39), the minimum ITAE
criterion being shown to have some advantages. Based on this criterion, standard forms of transfer function have been prepared (38).

This approach, however, is not a flexible one, in that a standard form gives only one combination of the frequency and transient performance indices, and if another combination is called for by the specification, the standard form approach offers no guidance.

1.4 Conclusions.

In the first part of this chapter, a set of performance indices was chosen, by means of which the transient and frequency performance of systems can be specified with reasonable precision.

In the second part, some existing systems and design methods are briefly considered. The conclusions reached are that the frequency response methods discussed do allow frequency performance indices to be satisfied but do not allow transient performance indices to be satisfied, other than by trial and error.

The conclusions relating to the s-plane methods considered are that while they do permit the design of compensation to provide any desired closed loop pole-zero pattern, they do not permit an easy correlation of such a pattern with the corresponding frequency and transient performance.

It is hoped to demonstrate in the succeeding chapters that the present method represents an improvement on these methods, in that it permits any easy quantitative correlation between the frequency and transient performance indices of systems with a specified pole-zero excess. This means that the pole-zero excess of the plant to be compensated can be taken into account right at the outset.
2.1 Introduction

The transient response of a system is often an important aspect of its performance, and even when it is not of primary importance, a design study would normally include its evaluation. Customarily, the impulse and step response is considered to give adequate information about the transient behaviour.

If the design is carried out in the s-plane in terms of the system transfer function, and if the transfer function is known in factored form, the transient response may straightforwardly be obtained by inverting the transform of the step and impulse response. Although the inversion is straightforward, it can be a laborious process and some methods have been developed to simplify it. Hazony and Riley (42) and also Billingsley and Rekoff (43) describe a simplified method for the evaluation of the residues of multiple poles. If the system transfer function is not known in factored form, Corrington (44) and also Boxer and Thaler (45) describe methods of obtaining the transient response from the transform expressed as a ratio of polynomials. Finally, if information about the overall form of the transient response is all that is required (e.g. the presence or otherwise of any overshoot), Mulligan (46, 47) Elgerd and Stephens (48), Zemanian (49, 50, 51, 52, 53) and also Matkowsky and Zemanian (54) have compiled a great deal of information about the correlation
between the general properties of the transient response and various pole-zero patterns in the s-plane.

If the design is carried out in terms of the frequency response, and if again only overall information about the transient response is required, then appropriate results and procedures are presented by Chu (55).

Approximate qualitative relationships between the frequency response and aspects of the transient response are presented by Jaworski (56). West and Potts (58) describe a procedure for evaluating the dominant complex pair of closed loop poles (assuming the system has one), from the open loop frequency response. Westcott (57) describes a procedure for correlating the impulse response with the frequency response. (All three references are discussed in more detail in section I.3.I.).

For the detailed evaluation of the transient response from the frequency response, methods are presented by Guillemin (63), Floyd (65), Stallard (59), Dawson (61), Hamos Jansson and Persson (64) Bedford and Fredandall (60), and also Solodovnikov, Topcheev and Krutikova (66).

The method presented in chapters 2 and 3 of the present work also falls into this category and, once it has been discussed, will be compared with the methods in references (59, 61, 63-66).

2.2 Mathematical background of frequency-time correlation.

The mathematical theory involved is treated in standard works (27, 63) but is briefly summarised here for the sake of completeness.

Let \( T(s) \) be the closed loop transfer function of the system whose transient response is required. The impulse of the system is then given by:
where $Br$ stands for the Bromwitch contour which must pass to the right of all singularities of $T(s)$.

It will be assumed that:

a) $T(s)$ is a rational function.

b) $T(s)$ has no singularities in the right half of the $s$-plane.

c) As $s$ tends to infinity, $T(s)$ tends to zero.

Bearing in mind that $T(s)$ is the closed loop transfer function, the first condition restricts the systems under consideration to be linear with lumped parameters.

The second condition excludes unstable systems and an ideal integrator. The latter in fact can be treated by the same method that is used in this chapter to relate the step response to $T(s)$.

The final condition implies the excess of poles over zeroes of $T(s)$ to be at least one. If this condition is not fulfilled, the system impulse response itself contains impulsive components. In such cases a transfer function fulfilling the stated conditions can be derived from the original transfer function by dividing the numerator polynomial by the denominator polynomial. Virtually all practical systems, however, would fulfill this condition.

With these restrictions, the Bromwich contour can be made coincident with the imaginary axis, so that $T(s)$ becomes $T(j\omega)$, the closed loop frequency response.

The impulse response is then given by:

$$h(t) = \frac{1}{2\pi j} \int_{Br} T(s)\exp(st) \, ds \quad 2.2.1.$$
which is the Fourier transform relationship for the system.

Writing:

\[ T(j\omega) = \text{Re}T(j\omega) + j\text{Im}T(j\omega) \] . . . 2.2.3.

expression 2.2.2. becomes

\[
 h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\text{Re}T(j\omega) + j\text{Im}T(j\omega)][\cos\omega t + j\sin\omega t] d\omega
\]

\[
 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\text{Re}T(j\omega)\cos\omega t - \text{Im}T(j\omega)\sin\omega t]
\]

\[
 + \frac{j}{2\pi} \int_{-\infty}^{+\infty} [\text{Re}T(j\omega)\sin\omega t + \text{Im}T(j\omega)\cos\omega t] \ldots 2.2.4.
\]

Both integrals are principal values, hence the second integral in expression 2.2.4 is zero, since the integrand is an odd function of \( \omega \). The first integral may be taken over the positive frequencies only, since the integrand is an even function of \( \omega \), giving

\[
 h(t) = \frac{1}{\pi} \int_{0}^{\infty} [\text{Re}T(j\omega)\cos\omega t - \text{Im}T(j\omega)\sin\omega t] d\omega \quad 2.2.5
\]

The first term is an even function of \( t \) and the second an odd function of \( t \). But with the stated restrictions on \( T(s) \), \( h(t)=0 \) for \( t<0 \) hence

\[
 h(t) = \frac{2}{\pi} \int_{0}^{\infty} \text{Re}T(j\omega)\cos\omega t \, d\omega \quad 2.2.6.
\]

and also

\[
 h(t) = -\frac{2}{\pi} \int_{0}^{\infty} \text{Im}T(j\omega)\sin\omega t \, d\omega \quad 2.2.7.
\]

The step response of the system is given by:

\[
 u(t) = \frac{1}{2\pi j} \int_{\text{Br}}^{\infty} \frac{T(s)}{s} \exp(st) \, ds \quad 2.2.8.
\]

Although there is now a singularity at the origin, the integral can still be evaluated along the imaginary axis because it is a principal value. Again the odd function in \( \omega \) gives a zero nett contribution, hence

\[
 u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\text{Re}T(j\omega)\sin\omega t + \text{Im}T(j\omega)\cos\omega t] \, d\omega \quad 2.2.9.
\]
Again it can be seen that the first term is an odd function of \( t \), while the second is an even function of \( t \). Since \( u(t) = 0 \) for \( t < 0 \) (as a consequence of the stated restrictions),

\[
    u(t) = \frac{\inf}{\pi} \int_{0}^{\infty} \frac{\text{Re}T(j\omega) \sin \omega t}{\omega} \, d\omega
\]

2.2.10.

and also

\[
    u(t) = \frac{\inf}{\pi} \int_{0}^{\infty} \frac{\text{Im}T(j\omega) \cos \omega t}{\omega} \, d\omega
\]

2.2.11.

Expressions 2.2.6 and 2.2.10 are the basis for the analysis and synthesis methods to be described. The choice between them and expressions 2.2.7 and 2.2.11 is arbitrary as far as practical system analysis is concerned. Theoretically the two sets of expressions differ in specifying the behaviour of the time function for \( t = 0 \), since \( \text{Re}T(j\omega) \) is an even function while \( \text{Im}T(j\omega) \) is an odd function of \( \omega \). In practical terms this difference is immaterial, since neither analysis nor synthesis of practical systems is carried out with absolute accuracy.

2.3 Approximation of the real part plot.

Expressions 2.2.6 and 2.2.10 link the impulse and step responses of a system with the real part of its frequency response. In order to evaluate the integrals, the real part plot is first approximated by simpler curves. The first step in this process is to divide the real part plot into two regions: A high frequency region, referred to as the asymptotic region, for frequencies higher than a chosen truncation frequency, and an intermediate region for frequencies lower than the truncation frequency. The choice of the truncation frequency and the details of the approximation are presented in the next section.
2.3.1 Approximation of the asymptotic region of the real part plot

To establish the behaviour of the real part of $T(j\omega)$ at high frequencies, let

$$T(j\omega) = \frac{b_0 + b_1(j\omega)}{a_0 + a_1(j\omega)} = \frac{b_n(j\omega)^n}{a_m(j\omega)^m} = \frac{B_r + jB_i}{A_r + jA_i}$$

where it is assumed that $m > n$. Hence

$$\text{Re}T(j\omega) = \frac{A_rB_r + A_1B_1}{A_r^2 + A_1^2}$$

The highest order term in the denominator of expression 2.3.2 is $\omega^{2m}$, whether the denominator of expression 2.3.1 is of even or odd order. If $m$ and $n$ are both even or both odd, the highest order term in the numerator of expression 2.3.2 is $\omega^{(m+n)}$, while if one is even and one odd, the highest order term is $\omega^{(m+n-1)}$.

Since the asymptotic behaviour is governed by the highest order terms in the numerator and denominator, it can be seen that:

For $m$ and $n$ both even or both odd,

$$\text{Re}T(j\omega) \bigg|_{\omega \to \text{inf.}} \rightarrow \frac{\omega^{(m+n)}}{\omega^{2m}} \rightarrow \frac{1}{\omega^{m-n}}$$

While for one even and one odd,

$$\text{Re}T(j\omega) \bigg|_{\omega \to \text{inf.}} \rightarrow \frac{\omega^{(m+n-1)}}{\omega^{2m}} \rightarrow \frac{1}{\omega^{m-n+1}}$$

Thus in both cases,

$$\text{Re}T(j\omega) \bigg|_{\omega \to \text{inf.}} \rightarrow \frac{1}{N} \quad \text{where } N \text{ is an even integer.}$$

$N$ is equal to the excess of poles over zeroes of $T(s)$ if the excess is even, or it is one greater than the excess if this is odd.
The approximation of the asymptotic region of the real part plot, therefore, consists of locating a truncation frequency, $\omega_c$, beyond which the true real part plot approaches the asymptotic behaviour within a desired tolerance, and replacing the true real part plot by

$$\text{Re}T(j\omega) = A_c \left(\frac{\omega}{\omega_c}\right)^N \quad \text{for } \omega > \omega_c$$

2.3.1

where $A_c$ is the value of the true real part plot at $\omega_c$.

In practice, the choice of a suitable $\omega_c$ and $N$ is most readily done with the aid of a plot of $\log \text{Re}T(j\omega)$ against $\log\omega$. The process involved is illustrated in fig.2.1.

Fig.2.1.a shows the frequency response (in polar form), for

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 1.4\omega_n(j\omega) + \omega_n^2}$$

2.3.2

Fig.2.1.b shows the corresponding $\text{Re}T(j\omega)$, and fig.2.1.c. shows a plot of $\log \text{Re}T(j\omega)$ against $\log\omega$. Shown also in fig.2.1.c is the -12 dB/octave asymptote. Bearing in mind that in practice the frequency response would only be known within the tolerance of the measurement technique used, it is clear that taking $\omega_c$ as say 10 should give good accuracy.

The effect of the choice of $\omega_c$ upon the accuracy is considered in more detail in the next chapter, where it is shown that a much lower value for $\omega_c$ still gives good results.

2.3.2 Approximation of the intermediate region of the real part plot

The intermediate region of the real part plot (i.e. for $\omega < \omega_c$), is approximated by equally spaced confluent straight line segments. In the application of the present
Fig. 2.1.a. Polar plot for expression 2.3.2.
   a) Real part of the polar plot.

Fig. 2.1.b. Magnitude of the real part on a logarithmic scale.

Fig. 2.2.a Approximation of the intermediate region of the real part
   of the frequency response by straight line segments.

Fig. 2.2.b Subdivision of the approximating trapezoids into triangles.
of the segments are required. For the development of the method, however, the trapezoids formed by the straight lines are further subdivided into triangles, as illustrated in fig.2.2.

Fig.2.2.a again shows the plot of ReT(jω) (as in fig.2.1.b), up to ωc and the approximating straight line segments. A0, A1-Ac are the values of the true real part at the junctions of the segments.

Fig.2.2.b shows the trapezoids formed by the segments subdivided into triangles. It will be seen that the area under the straight line segment approximation is thus the sum of the areas of a number of isosceles triangles plus two end triangles. The effect of the number of segments used upon the accuracy is considered in more detail in the next chapter.

2.3.3 Qualitative comments on the accuracy of the approximation.

The errors involved in the method are introduced in the approximation of the true real part plot by simpler curves. All subsequent calculations are exact.

Qualitatively the degree of approximation involved is readily assessable. This is because sections of the real part plot give components of the impulse and step response which are additive, as a consequence of the linearity of the integral relationships 2.2.6 and 2.2.10. Thus the degree of approximation in the time domain is linearly related to the degree of approximation in the frequency domain (63 p.662).

In practical terms, therefore, bearing in mind that the true real part plot is only known within the accuracy
of the measurements, a qualitative appraisal of the degree of approximation, guided by the quantitative results presented in chapter 3, should prove sufficient.

2.4 Contribution of sections of the approximate real part plot to the impulse and step response.

The contributions to the impulse response due to the intermediate region of the approximate real part plot area of three types:

Due to the isosceles triangles, designated $h_a(t)$.

Due to the end triangle at zero frequency, designated $h_b(t)$.

Due to the end triangle at $\omega_c$, designated $h_c(t)$.

The contribution to the impulse response due to the asymptotic region of the approximate real part plot is designated $h_e(t)$.

The corresponding contributions to the step response are designated $u_a(t)$, $u_b(t)$, $u_c(t)$, and $u_e(t)$ respectively.

2.4.1. Contributions $h_a(t)$ and $u_a(t)$.

Each isosceles triangle is of the form shown in fig. 2.3.a. The triangle, which forms a part of the intermediate region of $\Re T(j\omega)$ is described by

$$\Re T(j\omega) = \frac{A_k\omega}{\lambda} - \frac{A_k(\omega_k - \lambda)}{\lambda} \text{ for } \omega<\omega_k$$

2.4.1.

and

$$\Re T(j\omega) = -\frac{A_k\omega}{\lambda} + \frac{A_k(\omega_k + \lambda)}{\lambda} \text{ for } \omega>\omega_k$$

2.4.2.

Substitution of these expressions for $\Re T(j\omega)$ into expression 2.2.6 gives the contribution of the isosceles triangle to the impulse response ($h_a(t)$), and substitution into expression 2.2.10 gives the contribution to the step response ($u_a(t)$).
Fig. 2.4. Graphical presentation of expression 2.4.3.

Fig. 4.3. Triangles used in approximating the intermediate region of the real part.

Fig. 5.5. Illustrating the notation used in section 2.4.1.
Hence, the impulse contribution is
\[ h_a(t) = \frac{2}{\pi} \int \left[ -\frac{A_k}{\omega_k} \cos \omega_k t - \frac{A_k}{\lambda} (\omega_k - \lambda) \right] \cos \omega t \, d\omega \]
\[ + \frac{2}{\pi} \int \left[ -\frac{A_k}{\omega_k} \cos \omega_k t + \frac{A_k}{\lambda} (\omega_k + \lambda) \right] \cos \omega t \, d\omega \] 2.4.3

on evaluation this gives
\[ h_a(t) = \frac{4}{\pi} A_k \frac{\cos \omega_k t}{\lambda t^2} (1 - \cos \lambda t) \] 2.4.4

from which the contribution of any particular isosceles triangle, characterised by \( \omega_k, \lambda, A_k \), to the impulse response at time \( t \) could be found.

It was initially envisaged that expression 2.4.4 would be presented as a family of curves. It was re-arranged as
\[ h_a(\omega_k t) = \frac{4}{\pi} A_k \frac{\omega_k}{(\omega_k t)^2} \frac{1 - \cos \omega_k t}{k} \] 2.4.5

where \( k = \frac{\omega_k}{\lambda} \) and takes integer values only.

Such a family, for \( k=0 - 10 \) is shown in fig. 2.4. Although it is possible to use these curves, it is difficult to distinguish the individual lines in places. The corresponding family for the step response can be expected to be even more unsatisfactory in this respect, since it is formed by integrating the impulse response family.

It was therefore decided that the results would be presented in a tabulated form. Although this means that the impulse and step can only be found at discrete intervals, this is not likely to be a serious drawback in practice, provided the tabulated interval is fairly small. For tabulation, expression 2.4.4 was re-arranged yet again as:
\[ h_a(x) = A_k \frac{4}{\pi} \cos k \left( \frac{1 - \cos x}{x^2} \right) \] 2.4.6
where \( x = \omega_1 t \); and \( \omega_1 \) is the frequency of the first break in the intermediate region of the approximate real part plot. To clarify the notation figs. 2.2.b and 2.3.a are repeated as fig. 2.5 with the notation, marked.

It will be seen that once the approximation to the real part plot is made, \( \omega_1 \) is fixed so that for any time \( t \), \( x \) is fixed. The contribution of the \( k \)th triangle is then characterised by \( k \) and \( A_k \).

Similarly, substituting expressions 2.4.1 and 2.4.2 into expression 2.2.10 and evaluating the integral gives the step contribution as

\[
u_a(x) = A_k \left[ -\frac{4}{\pi} \cos kx \left( \frac{1}{x} - \frac{\cos x}{x^2} \right) + \frac{2}{\pi} (k-1) \text{Si}(x(k-1)) \right.
\]

\[
- \frac{4}{\pi} k \text{Si}(kx) + \frac{2}{\pi} (k+1) \text{Si}(x(k+1)) \] \quad 2.4.7.

where

\[
\text{Si}(x) = \int_0^x \frac{\sin z}{z} \, dz
\]

2.4.2 Contributions \( h_b(t) \) and \( u_b(t) \).

The end triangle at zero frequency is of the form shown in fig. 2.3.b. Following the same pattern of calculations as in the previous section, contributions to the impulse and step responses are obtained as

\[
h_b(x) = \omega_1 A_o \left[ \frac{2}{\pi} \left( \frac{1 - \cos x}{x^2} \right) \right] \quad 2.4.8
\]

and

\[
u_b(x) = A_o \left[ -\frac{2}{\pi} \left( \frac{1}{x} - \frac{\cos x}{x^2} \right) + \frac{2}{\pi} \text{Si}(x) \right] \quad 2.4.9
\]

2.4.3 Contributions \( h_c(t) \) and \( u_c(t) \).

The end triangle at \( \omega_c \) is shown in fig. 2.3.c, and its contributions are

\[
h_c(x) = \omega_1 A_c \left[ \frac{2}{\pi} \sin k_c x(x - \frac{\sin x}{x^2}) + \frac{2}{\pi} \cos k_c x \left( \frac{1 - \cos x}{x^2} \right) \right] \quad 2.4.10
\]
and

\[ uc(x) = A_c \left[ \frac{2}{\pi} \left( \cos(k_c-1)x - \cos k_c x \right) \right. \]

\[ \left. - \frac{2}{\pi} (k_c-1)(\text{Si}(k_c x) - \text{Si}((k_c-1)x)) \right] \quad 2.4.11 \]

2.4.4. **Contributions he(t) and ue(t)**

The contribution of the asymptotic region to the impulse response is given by:

\[ he(t) = \frac{2}{\pi} \int \omega_c A_c(\omega) N \cos \omega t \, d\omega \quad 2.4.12 \]

Evaluating the integral yields

\[ he(t) = \frac{2}{\pi} A_c \omega_c N \left[ \sum_{r=0}^{N/2-1} \sum_{q=1}^{\infty} \frac{(-1)^r t^{2r} \cos \omega_c t}{\omega_c^{N-(2r+1)} 2r+1(N-q)} \right] \]

\[ + \sum_{r=0}^{N/2-2} \sum_{q=1}^{\infty} \frac{(-1)^{r+1} t^{2r+1} \sin \omega_c t}{\omega_c^{(N-2r+2)} 2r+2(N-q)} \]

\[ + \sum_{q=1}^{2N-1} \frac{t^{N-1}}{(N-q)} \int_{\omega_c}^{\infty} \sin \omega t \, d\omega \quad 2.4.13 \]

Where \( r \) is valid only for positive integer values.

The final integral in the above expression can be converted into a proper integral by noting that

\[ \int_{\omega_c}^{\infty} \sin \omega t \, d\omega = \frac{\pi}{2} - \int_{0}^{\omega_c} \sin \omega t \, d\omega \quad 2.4.14 \]

It again is the intention to express 2.4.13 in terms of \( k \) and the normalised time variable \( x \). In this connection, it can be seen that each term of the cosine series in expression 2.4.13 contains the quotient

\[ \frac{t^{2r}}{\omega_c^{N-(2r+1)}} \]

Which can be re-arranged as \( \frac{(t\omega_c)^{2r}}{\omega_c^{N-1}} \) 2.4.15
The similar quotient in the sine series may be re-arranged as

\[
\frac{(tw_c)^{2r+1}}{\omega_c^{N-1}} \quad 2.4.16
\]

Finally in the last term of expression 2.4.13, \(t^{N-1}\) is changed to

\[
\frac{(tw_c)^{N-1}}{\omega_c^{N-1}} \quad 2.4.17
\]

Then, extracting the \(\omega_c^{N-1}\) term from the denominator and writing \(tw_c = k_c x\) expression 2.4.13 becomes

\[
he(x) = A_c \omega_c \left[ \frac{2}{\pi} \sum_{r=0}^{N/2-1} \frac{(-1)^r (k_c x)^{2r+1} \cos k_c x}{2r+1(N-q)} \right.
\]

\[
+ \frac{2}{\pi} \sum_{r=0}^{N/2-2} \frac{(-1)^r (k_c x)^{2r+1} \sin k_c x}{2r+2(N-q)}
\]

\[
+ \frac{2}{\pi} \sum_{q=1}^{N-1} \frac{(k_c x)^{N-1}}{N-1(N-q)} \left( \frac{\pi}{2} - \text{Si}(k_c x) \right) \quad 2.4.18
\]

For the purpose of actually evaluating \(he(x)\), it is simpler to use the following recursive formula, which is derived from expression 2.4.18.

\[
he(x)_N = A_c \omega_c \left[ \frac{2}{\pi} \cos k_c x - \frac{2}{\pi} \frac{(k_c x) \sin k_c x}{(N-1)} \right.
\]

\[
- \frac{2}{\pi} \frac{(k_c x)^2}{(N-1)(N-2)} \left( \frac{he(x)_{N-2}}{A_c \omega_c} \right) \quad 2.4.19
\]

for \(N=4, 6, 8---\)

and

\[
he(x)_{N=2} = A_c \omega_c \left[ \frac{2}{\pi} \cos k_c x - (k_c x) \left( \frac{\pi}{2} - \text{Si}(k_c x) \right) \right] \quad 2.4.20
\]
The step contribution is given by
\[ u_e(t) = \frac{2}{\pi} \int_{\omega_c}^{\infty} A_0 \left( \frac{\omega}{\omega_c} \right)^N \sin \omega t \, d\omega \] 2.4.21

Which on evaluation gives
\[ u_e(x) = A_0 \left[ \frac{2}{\pi} \frac{\sin k_c x}{N} + \frac{k_c x}{N} \left( \frac{h_c(x)}{A_0 \omega_c} \right) \right] \] 2.4.22

2.5. Numerical evaluation of the contributions

The expressions for the various contributions obtained in the preceding sections are summarised for convenient reference in table 2.1. They are given in per unit form and the reference of the original expression is quoted in each case.

2.5.1 Tabulated results for contributions \( h_a, h_b, h_c, \) and \( u_a, u_b, u_c. \)

As discussed in section 2.4.1, it was decided to present these contributions in tabulated form, because of the impracticability of a graphical presentation. The contributions have been evaluated for a range of values of \( k \) and \( x \) and the results are given in table 2.2.

It will be seen that values of \( k \) from 1 – 20 have been used. As will become apparent later, systems whose transient response is reasonably damped have a fairly smooth real part plot, so that normally less than twenty straight line segments are sufficient to approximate the plot with good accuracy up to the truncation frequency.

Values of \( x \) from 0 – 2 in steps of 0.1 have been used. This range should cover quite a large range of practical systems, because useful values of \( x \) tend to be similar for different systems, since those with small bandwidths have response times which are longer than those with large bandwidths. Thus just as the gain-
## Table 2.1

### Summary of Expressions Giving Per Unit Contributions to Step and Impulse Response

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{ha(x)}{A_k \omega_k} = \frac{4}{\pi} \cos k x \left[ \frac{1 - \cos x}{x^2} \right] ]</td>
<td>2.4.6</td>
</tr>
<tr>
<td>[ \frac{hb(x)}{A_0 \omega_0} = \frac{2}{\pi} \left[ \frac{1 - \cos x}{x^2} \right] ]</td>
<td>2.4.8</td>
</tr>
<tr>
<td>[ \frac{hc(x)}{A_c \omega_c} = \frac{2}{\pi} \left[ \sin k x \left( \frac{x - \sin x}{x^2} \right) + \cos k x \left( \frac{1 - \cos x}{x^2} \right) \right] ]</td>
<td>2.4.10</td>
</tr>
<tr>
<td>[ \frac{he(x)}{A_c \omega_c} = \frac{2}{\pi} \left[ \cos k x \left( \frac{1 - \cos x}{x^2} \right) - \left( \frac{k_c x}{N-1} \right) \right] - \left( \frac{k_c x}{N-1} \right)^2 \left( \frac{he(x)}{A_c \omega_c} \right) ]</td>
<td>2.4.19</td>
</tr>
<tr>
<td>and [ \frac{he(x)}{A_c \omega_c} = \frac{2}{\pi} \left[ \cos k x - \left( \frac{k_c x}{N-1} \right) \left( \frac{\pi}{2} - \text{Si}(k_c x) \right) \right] ]</td>
<td>2.4.20</td>
</tr>
<tr>
<td>[ \frac{ua(x)}{A_k} = \frac{2}{\pi} \left[ -2 \cos k x \left( \frac{1 - \cos x}{x} \right) + (k-1) \text{Si}(x(k-1)) - 2k \text{Si}(kx) + (k+1) \text{Si}(x(k+1)) \right] ]</td>
<td>2.4.7</td>
</tr>
<tr>
<td>[ \frac{ub(x)}{A_0} = \frac{2}{\pi} \left[ - \left( \frac{1 - \cos x}{x} \right) + \text{Si}(x) \right] ]</td>
<td>2.4.9</td>
</tr>
<tr>
<td>[ \frac{uc(x)}{A_c} = \frac{2}{\pi} \left[ \cos(k_c-1)x - \cos k_c x \left( \frac{k_c -1}{x} \right) - (k_c -1) \text{Si}(k_c x) \right] - \text{Si}(k_c x -1) ]</td>
<td>2.4.11</td>
</tr>
<tr>
<td>[ \frac{ue(x)}{A_c} = \frac{2}{\pi} \sin k_c x \left( \frac{k_c x}{N} \right) + \left( \frac{k_c x}{N} \right)^2 \left( \frac{ue(x)}{A_c \omega_c} \right) ]</td>
<td>2.4.22</td>
</tr>
<tr>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>0.60</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.61</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.62</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.63</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.10</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.11</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.12</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.13</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.19</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.20</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.21</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.22</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.23</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.24</td>
<td>0.63862</td>
</tr>
<tr>
<td>0.25</td>
<td>0.63862</td>
</tr>
</tbody>
</table>

**TABLE 2.2A IMPULSE RESPONSE CONTRIBUTIONS**

**UPPER VALUES**

- 0.44583
- 0.39483
- 0.29483
- 0.24483
- 0.19483
- 0.14483
- 0.09483
- 0.04483
- 0.01483

**LOWER VALUES**

- 0.35483
- 0.26483
- 0.15483
- 0.06483
- 0.00483
Fig. 2.6. Contributions $b_n(x)$, as defined in expression 2.4.19, for $N = 2, 4, 6$. 
Fig. 2.7: Contributions $u_{N}(x)$ as defined in expression 2.4.22, for $N=2, 4, 6$. 
bandwidth product of amplifiers tends to lie in a
definite range, so does the frequency-time product for
systems.

Table 2.2.A gives contributions to the impulse
response, and table 2.2.B to the step response. The
layout of parts A and B of the table is identical.

Thus, the first column contains values of x, and
the last column contains contributions \( h_b \) (or \( u_b \)).
The middle ten columns contain contributions \( h_a \) and \( h_c \)
(or \( u_a \) and \( u_c \)), arranged as two pairs of lines. In
the first pair of lines, the upper values are
contributions \( h_a \) (or \( u_a \)) for \( k=1 - 10 \), while the lower
values are contributions \( h_c \) (or \( u_c \)) for \( k_c=1 - 10 \). The
second pair of lines has the same information for \( k \) and
\( k_c=11 - 20 \).

The use of the table is illustrated in chapter 3.

2.5.2. Graphical results for contributions \( h_e \) and \( u_e \).

These contributions are presented in graphical
form in figs. 2.6 and 2.7. It was felt preferable to
present them in this form, rather than incorporate them
into table 2.2, since it would have meant a third and
sixth line of data for each value of x. (The extra
two lines would be necessary because \( h_e \) and \( u_e \) are
expressed in terms of \( k_c x \), so that for each value of
\( x \) they would have to be evaluated for \( k_c=1 - 20 \) in a
similar manner to contribution \( h_c \) and \( u_c \)).

2.6 Conclusions

A method has been presented by means of which
the impulse and step response of linear systems can be
calculated directly from its frequency response. By
tabulating or presenting in graphical form the
expressions involved, the remaining calculations are reduced to multiplication and addition. The method is also very suitable for computer implementation, with the pre-calculated contributions stored on tapes.

Practical applications of this method are presented in the next chapter.
3.1 Introduction

In the preceding chapter a method of finding the impulse and step response of a linear system from its frequency response was presented. It remains to investigate quantitatively the accuracy of the method, and to compare it with existing methods.

3.2. Evaluation of the accuracy of the method.

This section is intended to illustrate in detail the effect of the choice of the truncation frequency and the number of straight line segments used on the accuracy of the method.

A second order system was used in the calculations because the expressions for its real part and for its transient response are simple, thus avoiding long computation times. As far as the application of this method is concerned, the system order is immaterial since the mode of approximation and calculation is always the same.

The second order system used has the transfer function

\[ T(s) = \frac{1}{s^2 + 1.4s + 1} \]

3.2.1 Effect of the choice of the truncation frequency

Fig. 3.1 shows the plot of \( \log |\text{Re}(j\omega)| \) against \( \log \omega \), and also marked is the asymptote at -12db/octave.

As mentioned in section 2.3.3, a qualitative
Fig. 3.1. Logarithmic plot of the real part of expression 3.2.1.
appraisal of the situation indicates that by \( \omega = 10 \) the true real part plot is for all practical purposes coincident with the asymptote. A much lower truncation frequency can be used, however, without introducing serious errors.

To investigate quantitatively the effect of the choice of \( \omega_c \), the true contribution of what is being called the asymptotic region of the real part plot (i.e. for \( \omega > \omega_c \)) was found from:

\[
\text{true contribution} = h(t) - \frac{2}{\pi} \sqrt{\text{Re}\{T(j\omega)\}} \cos \omega t \, \omega_c \quad 3.2.2
\]

where \( h(t) \) is obtained from the expression for the true impulse response of the second order system considered, and \( \text{Re}\{T(j\omega)\} \) is the expression for the true real part of \( T(j\omega) \). The integration was done on a digital computer with an accuracy of five decimal places. The approximate contribution, \( h_e(t) \), was obtained from

\[
h_e(t) = \inf_{\omega_c} \frac{2}{\pi} \int_{-\infty}^{\infty} \text{Re}\{A_c(\omega/\omega_c)^N\} \cos \omega t \, \omega_c \quad 3.2.3
\]

where, for this example, \( N = 2 \). Expression 3.2.3 was not evaluated as such, instead expression 2.4.18 which is derived from 3.2.3 (identical with 2.4.12) was used.

Fig. 3.2.a, b, c shows the true contribution of the asymptotic region (plotted as a continuous line), and spot values of the approximate contribution for \( \omega_c = 4, 3, 2 \) respectively.

The greatest error in each case occurs at \( t = 0 \), and this gets progressively smaller as \( \omega_c \) is increased. For \( t > 0 \), fig. 3.2.a shows that with \( \omega_c = 4 \), the spot values of \( h_e(t) \) are virtually indistinguishable from the true values. From Fig. 3.2.b it can be seen that for \( \omega_c = 3 \), the accuracy is still good, while fig. 3.2.c shows that
FIG. 3.2. THE CONTRIBUTION OF THE ASYMPTOTIC REGION OF THE REAL PART TO THE
IMPULSE RESPONSE AND SPATIAL VARIATION OF THE APPROXIMATE CONTRIBUTION
FOR $a_1$ EQUAL T1:

4. a.
3. b.
2. c.
for $\omega_c=2$, quite large differences between the true and $h_e(t)$ values arise.

While fig. 3.2 supports the validity of making the asymptotic region approximation described in section 2.3.1, the acid test as far as accuracy is concerned is the effect upon the predicted impulse response. To isolate the effect of the asymptotic region approximation on the overall accuracy, the true impulse response was compared with spot values calculated from

$$\text{spot values} = \frac{2}{\pi} \int \text{Re}\{T(j\omega) \cos \omega t \} \, d\omega + A_c \omega_c [h_e(x)]$$

The first integral again was evaluated numerically to five decimal places, while the approximate asymptotic region contribution was found from the expression (shown in table 2.1) used to obtain the graphs of $h_e(x)$ given in figs. 2.6 and 2.7.

The results, showing the true impulse response as a solid line with the spot values superimposed, are shown in fig. 3.3.a, b, c for $\omega_c=4,3,2$ respectively. It can again be seen that, apart from the value at $t=0$, the true and spot values for $\omega_c=4$ are virtually indistinguishable. For $\omega_c=3$, the difference is still very small, while for $\omega_c=2$ noticeable differences occur.

Identical calculations were done for the step response. Fig. 3.4.a, b, c shows the true and approximate contributions of the asymptotic region for $\omega_c=4,3,2$ respectively. Fig. 3.5.a, b, c again isolates the effect of the asymptotic region approximation on the step response for $\omega_c=4,3,2$ respectively.

From the results it is concluded that if a high accuracy was required, $\omega_c=3$ would be a suitable truncation frequency, representing a reasonable compromise between
FIG. 3.3. EFFECT OF THE EXPONENTIAL REJECTION APPROXIMATION ON THE IMPULSE RESPONSE FOR $H_1$, SIGNAL: D.

4. A.
5. B.
6. C.
FIG. 3.4. THE CONTRIBUTION OF THE ASYMPTOTIC REGION OF THE REAL PART TO THE STEP RESPONSE AND SPATIALLY LOCALIZED APPROXIMATE CONTRIBUTION FOR N_eq = 10^4:

- 1.
- 2.
- 3.
- 4.
FIG. 35. EFFECT OF THE ASYMPOTIC REGION APPRECIATION ON THE STEP RESPONSE FOR $n_k = n_k = T_n$

4. a.
5. b.
6. c.
accuracy and the desirability of having as low an $\omega_c$ as possible, in order to have as few segments in the intermediate region approximation as possible. For most engineering purposes, however, $\omega_c=2$ would be adequate.

It has not been found practicable to establish any simple general rule linking $\omega_c$ with accuracy. This is because the way in which the true real part plot approaches the asymptote is different for every system. The examples of this section, however, give a guide which should prove adequate for most practical applications.

3.2.2. Effect of the number of straight line segments on accuracy.

To investigate the effect of the number of segments used to approximate the intermediate region of the real part plot, $\omega_c$ was taken as 2. Since the error involved in using the approximate contribution of the asymptotic region for $\omega_c=2$ was explored in the last section, it was felt unnecessary to isolate the effect of the approximation of the intermediate region. Instead, the true impulse and step responses were compared with those calculated from $he(x)$ and $ue(x)$ for $\omega_c=2$ plus the contributions of the triangle forming the intermediate region.

Since the object of this section, as also of the preceding one, is to give a feel for the kind of error occurring in the time domain for a given approximation in the frequency domain, the true real part plot up to $\omega_c=2$, and its straight line approximation using 10, 7 and 5 segments is shown in fig.3.6.a, b and c. The corresponding predicted impulse responses are shown in
FIG. 5.6. APPROXIMATION OF THE INTERRELATION BETWEEN THE REAL PART ($\omega = 2$) USING:

10 SEGMENTS, A.
7 = B.
5 = C.
FIG. 3.7. THE IMPULS response and spot values of the predicted impulse response for \( \omega_0 = 2 \) and the intermediate region approximated by

10 segments, A.
7 = B.
5 = C.
FIG. 3.8. TIME STEP RESPONSE AND SPOT VALUES OF THE PRELIMINARY STEP RESPONSE
FOR \( \gamma = 2 \) AND THE INTERMEDIATE REGION APPROXIMATED BY:

10 RESONTS, 4.
7 = 5.
3 = 6.
fig. 3.7 a, b and c, while the step responses are shown in fig. 3.8 a, b and c.

From the results it is concluded that a close approximation of the true real part plot in the intermediate region is required in order to get good accuracy in the time domain. 10 segments for this example seems a suitable value.

3.3 Example: Transient response of a fourth order system

By way of an example, the impulse and step response will be evaluated for a system whose transfer function is:

$$T(s) = \frac{18.72}{(s+1)^2(s+0.6+\jmath)}$$ 3.3.1

The logarithmic real part plot is shown in fig. 3.9, from which the truncation frequency was chosen as 4 rad/sec.

3.3.1 Computer calculation of the transient response.

It was decided to use 10 segments for the intermediate region, and fig. 3.10 shows the degree of approximation this entails.

The true and spot values of the approximate impulse and step response are shown in fig. 3.11. To indicate the sort of computing times involved, fig. 3.11 was produced in about 1 minute. To obtain the approximate impulse and step values without plotting them out takes about 10 seconds.

3.3.2 Example of hand calculation.

The calculation necessary to obtain one value of the impulse response will be made to show the feasibility of applying the method by hand calculation and to illustrate the use of table 2.2.
Fig. 3.9. Illustrating the choice of the truncation frequency for the example of section 3.3.
Fig. 3.10. Approximation of the Intermediate Region of the Real Part for the Example in Section 3.3.

Fig. 3.11. Time Step and Impulse Response and Spot Values of the Approximate Step and Impulse Response for the Example in Section 3.3.
Suppose it is required to find the impulse response at $t=1$ sec. In this example, $\omega_c=4$ and since 10 segments were used, $\omega_1=0.4$. Hence $x=\omega_1 t=0.4$.

A convenient way of doing the calculation is to arrange the numbers in table form as shown in table 3.1. Thus the first column identifies the number of the triangle, the second column gives its height. It is as well to distinguish the first and the last number in the second column by putting in $A_0$ and $A_c$ respectively. The third column contains the contributions for $x=0.4$.

The contribution due to $A_0$ is $h_b$ and is taken from the last column in table 2.2. The contributions due to $A_1$ to $A_9$ are $h_a$ and are taken from the top line of the numbers corresponding to $x=0.4$. The contribution due to $A_c$ is taken from the second line in column 10.

Since these contributions were calculated for unit height of the triangles, the true contributions are obtained by multiplying them by the heights, taking into account the signs. It is convenient to put positive products in one column, negative products in the next column. Once this has been done, the numbers are added as shown.

Now it is recalled that the contributions were calculated for unit height of the triangles and for unit $\omega_1$ (as indicated in table 2.1.). Hence to get the nett contribution due to the intermediate region, the positive and negative contributions are added and the result multiplied by $\omega_1=0.4$.

It now remains to add the contribution due to the asymptotic region. This can be read off fig.2.6 for $N=4$ and $k_0 x=4$ giving 0.02. This value must be multiplied
Table 3.1.  
Hand calculation of the impulse response. (Relevant to section 3.3.2).

<table>
<thead>
<tr>
<th>k</th>
<th>$A_k$</th>
<th>$\frac{h_a(x)}{A_k\omega_1}$</th>
<th>$\frac{h_a(x)}{\omega_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$A_0 = 1$</td>
<td>0.314</td>
<td>0.314</td>
</tr>
<tr>
<td>1</td>
<td>0.907</td>
<td>0.578</td>
<td>0.525</td>
</tr>
<tr>
<td>2</td>
<td>0.570</td>
<td>0.438</td>
<td>0.250</td>
</tr>
<tr>
<td>3</td>
<td>0.048</td>
<td>0.228</td>
<td>0.011</td>
</tr>
<tr>
<td>4</td>
<td>-0.353</td>
<td>-0.018</td>
<td>0.064</td>
</tr>
<tr>
<td>5</td>
<td>-0.552</td>
<td>-0.261</td>
<td>0.143</td>
</tr>
<tr>
<td>6</td>
<td>-0.652</td>
<td>-0.463</td>
<td>0.302</td>
</tr>
<tr>
<td>7</td>
<td>-0.588</td>
<td>-0.592</td>
<td>0.348</td>
</tr>
<tr>
<td>8</td>
<td>-0.193</td>
<td>-0.627</td>
<td>0.121</td>
</tr>
<tr>
<td>9</td>
<td>0.029</td>
<td>-0.563</td>
<td>-0.016</td>
</tr>
<tr>
<td>10</td>
<td>$A_c = -0.237$</td>
<td>2.078</td>
<td>-0.030</td>
</tr>
</tbody>
</table>

\[ h_e(k_c x) = (0.059)(4)(0.02) = 0.0047 \quad \text{and} \quad 0.4(2.048) = 0.8192 \]

\[ h(t) = 0.8192 + 0.0047 = 0.8239 \]
by $A_c \omega_c$ and the result added to the total.

The answer obtained by this slide-rule calculation is 0.824 and the true value is 0.811. Percentage errors in a quantity which goes through zero are of course misleading, but the error expressed as a percentage of the maximum value of the impulse response is less than 1.5%.

3.4. Comparison of the present method with some existing ones.

As mentioned in the introduction to chapter 2, the present method is comparable to a number of existing methods and these are considered in turn.

3.4.1. Comparison with Floyd's method

This method (65 ch. 11) is considered first because it is the most similar. In fact the work described in chapter 2 and 3 arises as a result of trying to improve Floyd's method.

The starting point in both cases is the real part plot. The main difference lies in the fact that in Floyd's method the entire real part plot, including the asymptotic region, is approximated by straight line segments. There is no guide as to how best to choose the final segment which makes the approximate real part amplitude zero. It is in this that it is felt the present method has greater accuracy.

The second difference lies in the treatment of the approximate real part formed by the straight line segments. In Floyd's method the trapezoids formed are further subdivided into other trapezoids, a process which is quite time consuming (see ref. 65 p. 345). It is felt that the greater simplicity of the present method is an
Finally Floyd's method provides the impulse response only, while the present method provides both the impulse and step response by identical calculations.

3.4.2. Comparison with Guillemin's method.

An example of the application of Guillemin's method can be found in ref. 27 p. 386. The method relies on approximating a derivative of the real part plot by straight lines which, by successive differentiation, are reduced to an impulse train. The impulse response may then be written by inspection from this impulse train.

The main difficulty in implementing Guillemin's method lies in getting the derivatives of the real part plot, which in practical cases is available only as a measured plot. The amount of work involved seems greater than in the present method, while the accuracy does not seem better.

3.4.3. Comparison with other methods.

The method of Stallard (59) and also of Hamos et al. (64) gives the step response only by considering the response of the system to a square wave of suitably long period. The square wave is analysed into harmonics and the response due to each harmonic is obtained from the system frequency response. The calculations involved are longer than in the present method and the choice of a suitable period for the square wave not as simple as the choice of \( \omega_c \) in the present method.

Dawson's (61) method uses the imaginary part plot as a starting point. The contribution of the asymptotic section of this plot is neglected, on the argument that if \( \omega_c \) is chosen large enough, the contribution of the
asymptotic region is small. The remainder of the imaginary part plot is analysed into a Fourier sine series by 18 point graphical integration. This calculation greatly exceeds in complexity the present method.

Solodovnikov et al (66) present a method of calculating the step response from the real part. Their approach is identical with Floyd's, with trapezoids being used and the asymptotic region approximated by a straight line, so that the same comments apply as were made in section 3.4.1.

3.5. Conclusions

It is felt that the method presented in chapter 2 and 3 offers the following advantages over the methods mentioned in section 3.4.

a) Improved accuracy due to consideration of the contribution of the asymptotic region of the real part.

b) Greater simplicity in the actual calculation, since the amplitude of the real part at the junctions of the straight line segments only are used i.e. the subdivision into component trapezoids is avoided.

c) An identical calculation for both impulse and step responses.

While this method is suitable for obtaining the transient response of any particular system from its measured frequency response, it is not suitable for correlating the frequency and transient responses of systems generally, as an aid to their synthesis. The difficulty is the same as with the correlation of the
transient response and pole-zero patterns, namely the number of variable coefficients, which makes any kind of detailed presentation of the results impractical.

To overcome this difficulty, an alternate way of approximating the intermediate region of the real part is presented in the next chapter.
4.1. Introduction

As discussed Chapter I, one of the principal difficulties in system synthesis lies in choosing a target system which

a) meets the specification

b) is capable of being realised by the addition of compensation to the open loop constraint.

For the target system to be practically realisable, it must have at least as great an excess of poles over zeroes as the open loop constraint, otherwise the required compensation will include pure differentiators which are normally ruled out on noise and saturation grounds. Yet as pointed out by Truxal (27 p.285), a designer can intelligently handle only a fairly simple pole-zero pattern, with the result that the 'dominant pole' concept is invoked. However, as pointed out by Horowitz (31 p.221), this generally leads to a design which is either prone to saturation or does not fully exploit the capabilities of the plant.

This is the price which is paid for the fact that the pole-zero representation gives exact information about the transient performance.

The method presented in the succeeding chapters uses an approximate characterisation of a system in terms of its real part plot. It is based on the premise
that for engineering purposes, the exact information of the pole-zero approach can with advantage be traded for approximate information, in return for a greatly increased capacity to correlate systems of specified pole-zero excess with their transient behaviour.

The approximate characterisation consists of specifying the asymptotic region in the manner described in section 2.3.1 (i.e. as an inverse power of the frequency), and the intermediate region by a Fourier cosine series.

Before considering the implications of such a form of approximate characterisation, the general behaviour of the real part of the frequency response of lumped linear systems is investigated.

4.2 Behaviour of the real part of the frequency response of lumped linear systems.

The object of this section is to investigate in broad terms the possible behaviour of the real part plot. The complete real part plot will again be considered as consisting of an asymptotic region (for \( \omega > \omega_c \)) and an intermediate region (for \( \omega < \omega_c \)).

4.2.1 Asymptotic behaviour of the real part plot.

It was shown in section 2.3.1 that the asymptotic region behaviour of the real part of \( T(j\omega) \) is given by:

\[
\lim_{\omega \to \infty} \frac{\text{Re}T(j\omega)}{\omega^N} = \text{constant}
\]

4.2.1

where \( N \) is an even integer equal to the excess of poles over zeroes of \( T(s) \) if the excess is even, or one greater than the excess if it is odd.

Expression 4.2.1 is a consequence of the fact that for a lumped linear system, \( T(j\omega) \) is a rational
function. By this same token:

\[ T(j\omega) \bigg|_{\omega \to \infty} = \text{constant} \quad (j\omega)^{m-n} \]

where \( m-n \) is the excess of poles over zeroes of \( T(s) \)

(It is assumed that \( m > n \) and that \( T(s) \) is minimum-phase.

The case of non minimum-phase \( T(s) \) is considered in section 7.3.2).

The possible asymptotic behaviour of \( T(j\omega) \), in polar form, is shown in fig.4.1 for different values of \( N \). The angle of \( T(j\omega) \) approaches \( \pi/2 \) times the pole-zero excess of \( T(s) \), so that to each value of \( N \) correspond two possible final angles.

4.2.2 Intermediate region behaviour of the real part plot.

The zero frequency value of \( T(j\omega) \) depends upon its type. Since \( T(j\omega) \) is used here to represent a closed loop transfer function, it will be assumed of type 0. Furthermore, it will be assumed that the corresponding open loop transfer function, \( G(j\omega) \) is of type I or higher, as is true of a large proportion of practical control systems. Under those circumstances

\[ T(j\omega) \big|_{\omega=0} = 1 \]

Knowing the behaviour of \( T(j\omega) \) at the two extremes of the frequency range, the possible behaviour in the intermediate region can readily be deduced. Fig.4.2.a shows some possible shapes of \( T(j\omega) \) for a pole-zero excess of 2 (i.e.\( N=2 \)), and fig.4.2.b shows the corresponding real part plots.

It will be seen that shapes A, B and C of \( T(j\omega) \), shown as continuous lines, produce the same sort of shape of \( \text{Re}T(j\omega) \), in the sense that the real part plot changes from positive to negative only once. Shapes
Fig. 4.1. Polar diagrams illustrating the possible asymptotic behaviour of a minimum-phase \( T(s) \) for different values of \( N \).

Fig. 4.2. Illustrating possible frequency responses and the corresponding real parts.

Excess 3

Excess 5

Excess 4

Excess 6

3a. Illustrating some possible frequency responses and the corresponding real parts for \( N = 4 \).

Fig. 4.3.b. Illustrating some possible frequency responses and the corresponding real parts for \( N = 6 \).
D and E, shown dashed, produce three changes in sign.
In fact, since for $N=2$ $T(j\omega)$ approaches the negative
real axis as $\omega$ approaches infinity, it follows that the
real part plot can only have an odd number of changes
of sign.

Now, while shapes D and E are possible (as are
even more convoluted ones), only shapes A, B and C
 correspond to likely practical systems. The reason is
that shapes D and E would correspond to systems whose
transient response would have a very long settling time.
This statement is further clarified in section 4.4.3.

Thus for a pole-zero excess of two, the shape of
the real part plot of practical systems is most likely
to be that shown as a solid line in fig.4.2.b.

Fig.4.3* shows some possible shapes of $T(j\omega)$ and
Re$T(j\omega)$ for systems having a pole-zero excess of 3 and 4
(i.e. $N=4$) and 5, 6 (i.e. $N=6$). Again solid lines show
those shapes which correspond to likely practical systems,
while the dashed lines represent systems which, while
feasible, would have long settling times, and hence are
not considered practicable.

4.3. Approximate description of the real part plot.

The object, as in chapter 2, is to approximate the
real part plot by simpler curves in order to evaluate
the integrals linking the real part with the impulse
and step response (expressions 2.2.6. and 2.2.10).

4.3.1 Approximate description of the asymptotic region
of the real part plot.

The approximation is exactly the same as described
in section 2.3.1 i.e.

$$\text{Re}T(j\omega) = A_c \left(\frac{\omega}{\omega_c}\right)^N \text{ for } \omega > \omega_c$$

4.3.1
where $\omega_c$ divides the asymptotic region from the intermediate region, and $A_c$ is the value of $\text{Re}T(j\omega)$ at $\omega_c$.

### 4.3.2 Approximate description of the intermediate region of the real part plot

The intermediate region of the real part plot will be described by a Fourier cosine series, i.e.

$$\text{Re}T(j\omega) = \sum_{k} a_k \cos \frac{\pi k \omega}{\omega_c} \text{ for } \omega < \omega_c \quad \text{(4.3.2)}$$

where $a_0, a_1, \ldots$ etc. are the Fourier coefficients.

### 4.3.3 Qualitative comments on the form of the approximation.

The asymptotic region approximation has already been discussed in chapter 2 and 3 and will not be further considered. However, some comments seem appropriate at this stage about the new form of approximation of the intermediate region of the real part plot.

Firstly, as is well known, any continuous periodic waveform can be approximated to any required degree of accuracy by a Fourier series. The series defined by expression 4.3.2 has a period of $2\omega_c$, i.e. it describes the real part up to $\omega_c$ plus its mirror image up to $2\omega_c$. Thus it is describing a continuous periodic waveform, though it will only be used in the range $0-\omega_c$.

Secondly, the greatest error in the approximation occurs at $\omega_c$, i.e. at the junction of the intermediate and asymptotic regions. This is because at this point the cosine series has zero slope, while the real part it is approximating may not have a zero slope. Nevertheless the difference in the areas can be made as small as desired. By way of illustration, fig.4.4 shows the true real part plot of a third order system, and the
Fig. 4.4. True real part plot of a third order system and its approximation by a Fourier cosine series containing 11 terms.

Harmonic amplitudes used

\[ a_0 = 0.09935 \quad a_1 = 1.02464 \quad a_2 = -0.07966 \]
\[ a_3 = -0.19748 \quad a_4 = 0.26134 \quad a_5 = -0.14551 \]
\[ a_6 = 0.02656 \quad a_7 = 0.05386 \quad a_8 = -0.03967 \]
\[ a_9 = 0.01435 \quad a_{10} = 0.00478 \]
approximation using 10 coefficients.

Finally, and of greatest importance, while the preceding comments have been made to justify the use of the Fourier series approximation and to indicate where the greatest discrepancy between this approximation and a true real part plot lies, it must be stressed that it is not intended to take a specified real part plot and approximate it in this way. To get the transient response for any specified real part, the method described in chapter 2 and 3 is much more suitable.

The advantage of the Fourier series approximation is that dealing with only a few coefficients, a large number of approximate real part plots and their corresponding transient performances can be considered.

4.4 Correlation between the transient performance and the approximate real part.

Since the object is to develop a synthesis method giving close control over the transient performance, the correlation between the impulse and step response and the approximate real part description given in the last section is now developed.

4.4.1 Contribution of the asymptotic region to the impulse and step response.

This has been fully covered in section 2.4.4, the contributions being given by expressions 2.4.19 and 2.4.22

4.4.2 Contribution of the intermediate region to the impulse and step response.

The contribution of the intermediate region to the impulse response is obtained by substituting
expression 4.3.2 into expression 2.2.6. Thus,

\[
\text{contribution of intermediate region} = \frac{2}{\pi} \sum_{\omega} a_k \cos k \omega [\cos \omega t] \cos \omega t \, dw
\]  

4.4.1

For convenience, the contribution due to any one term of the series is designated as \( h_k(t) \). Evaluating expression 4.4.1 gives

\[
h_k(t) = -\frac{2}{\pi} a_k \omega c \left[ \frac{\omega c t \sin \omega c t}{(k\pi)^2 - (\omega_c t)^2} \right] \cos k \pi
\]  

4.4.2

It is convenient at this stage to define a normalised time variable as

\[
T = \omega_c t
\]  

4.4.3

In terms of which all future expressions will be written. Thus

\[
h_k(T) = -\frac{2}{\pi} a_k \omega c \left[ \frac{T \sin T}{(k\pi)^2 - T^2} \right] \cos k \pi
\]  

4.4.4

The shapes of such contributions for two values of \( k \) are shown in fig. 4.5.a.

The corresponding contributions to the step response, designated as \( u_k(t) \) are obtained by substituting expression 4.3.2 into expression 2.2.10. Thus,

\[
\sum_{\omega} u_k(t) = \frac{2}{\pi} \int \sum_{\omega} a_k \cos \frac{k\pi \omega}{\omega c} \frac{\sin \omega t}{\omega} \, dw
\]  

4.4.5

Evaluating the integral yields for the contribution of the \( k \)th harmonic of the series, in terms of the normalised time variable \( T \),

\[
u_k(T) = \frac{a_k}{\pi} \left[ \text{Si}(T-k\pi) + \text{Si}(T+k\pi) \right]
\]  

4.4.6

The shapes of such contributions for two values of \( k \) are shown in fig. 4.5.b.

4.4.3 Relationship between the harmonic content of the intermediate region and the settling time.

The complete impulse response, as defined by the
Fig. 4.5a Contributions of the harmonics to \( h(t) \).

Fig. 4.5b Contributions of the harmonics to \( u(t) \).

Fig. 4.6a Harmonic amplitudes obtained from eqn. 4.4.8 for damping ratios less than 1.

Fig. 4.6b Harmonic amplitudes obtained from eqn. 4.4.8 for damping ratios greater than 1.
approximate intermediate and asymptotic regions is given by

\[ h(t) = \sum_{k} h_{k}(t) + h(t) \quad 4.4.7 \]

Inspection of expression 4.4.2 shows that at \( \omega_{c}t = k\pi \), only the kth term contributes, all others are zero. Inspection of the graph of \( h_{k}(t) \) in fig. 2.6 shows that \( h_{k}(t) \) is small in the vicinity of \( \omega_{c}t = k\pi \). Hence it can be said that the amplitude of \( h(t) \) at \( \omega_{c}t = k\pi \) is largely determined by the amplitude of the kth harmonic in the series description of the intermediate region of its real part. In other words the higher the significant harmonic in the series description, the longer is the settling time of the impulse response.

At this stage the statement made in section 4.2.3, namely that the real part shapes D and E of fig. 4.2.6 correspond to systems with a long settling time, can be amplified.

It is seen that the real part shapes D and E of fig. 4.2.6 change sign three times. The third harmonic in the series description of the intermediate region of shapes D and E would be much larger than that corresponding to shape A, B, C. It follows therefore that the settling time corresponding to shapes D and E would be much longer than that corresponding to shapes A, B, C.

By way of illustration, fig. 4.6 shows the harmonic content of the intermediate region of a second order system for various values of damping ratio. The harmonic amplitudes were obtained by the numerical solution on a computer of the expression

\[ a_{k} = \frac{2}{\omega_{c}} \int_{0}^{\frac{\omega_{c}}{2}} \frac{1 - \left(\frac{\omega}{\omega_{n}}\right)^{2}}{(1 - \left(\frac{\omega}{\omega_{c}}\right)^{2} + 4\delta^{2}(\frac{\omega}{\omega_{n}})^{2}} \cos k\omega \, \omega \quad 4.4.8 \]
4.5. **Conclusions.**

The method of approximating the real part plot described in this chapter would permit the approximation of any given real part to any desired degree of accuracy.

The real aim of using this form of approximation, however, is not to approximate specified real part plots (for this chapters 2 and 3 are relevant), but to permit the consideration of the transient response corresponding to a large number of approximate real part plots which can be formed by manipulating only a few constants of the Fourier series.

The next chapter considers the problem of finding the rational transform of a system from a specification of its real part. This step is clearly necessary for the development of a synthesis procedure.
5.1 Introduction

In chapter 4 a method of approximating the real part plot was presented. It consisted of approximating the asymptotic region by an inverse power of frequency, and the intermediate region by a Fourier cosine series.

With the aid of this approximation the frequency response–transient response correlation of a large number of systems will be investigated and summarised in performance curves. Before such results can usefully be applied in synthesis, however, it is necessary to get a rational transform description of the necessary compensation.

Two approaches seem possible: One may either seek to obtain a rational transfer function $T(s)$ directly from the description of $\text{Re}T(j\omega)$ discussed in the last chapter. Then, provided one can obtain a rational transfer function for the open loop constraint, the rational transfer function of the compensation can be found. Alternatively, one may first find the complete $T(j\omega)$ from the description of $\text{Re}T(j\omega)$. An attempt at obtaining a rational transfer function $T(s)$ may be made at this stage. As a modification of this approach, $T(j\omega)$, together with the frequency response of the open loop constraint may be used to get the frequency response of the necessary compensation, and only at this stage an attempt made to get a rational transfer function for the compensation. This approach in fact works best in practice, as will be shown in the
succeeding chapters (c.f. sections 7.2.4 and 7.3).

5.2. 'Pole-zero' approach along the lines of Guillemin's method.

Guillemin's method (63 ch.15) has been mentioned in section 1.3. One feature of the method is that the impulses forming the impulse train characterisation of a desired transient response are approximated by what Guillemin calls 'quasi impulses', whose rational transforms are known, and the linear combination of their transforms gives the transform of the desired transient response.

The 'quasi impulses' are obtained by truncating the true real part of an impulse (which is coswt) at some truncation frequency \( \omega_c \).

The present method may be viewed in a similar manner; thus, using the correlation between the transient response and the harmonic and asymptotic description of the real part developed in chapter 6, a desired real part is chosen. This step corresponds to the representation of the desired transient response (or one of its derivatives) by an impulse train in Guillemin's method.

The desired real part is specified as a sum of asymptotically terminated cosines.

Following the lines of Guillemin's method, the rational transform corresponding to each such terminated cosine could be found. Fig.5.1 shows the constant term \( a_0 \) and the first and second harmonics with the asymptotic termination corresponding to \( N=2 \). Beside each is the appropriate rational transform. The real part calculated from the transform is also shown (dashed), to indicate the degree of correspondence. The rational transform of a real part defined in terms of the constant and the
Fig. 5.1a Real part due to $a_0$ and its rational transform.

Fig. 5.1b Real part due to $a_1$ and its rational transform.

Fig. 5.1c Real part due to $a_2$ and its rational transform.
first and second harmonics (with asymptotic section corresponding to N=2) is the linear combination of the individual transforms.

5.2.1 Illustration of this approach.

A number of comments on this approach seem appropriate. In Guillemin's method the cosines are terminated abruptly, in the present method they are terminated by an asymptotic section. Guillemin has considered the possibility of non-abrupt termination (63 p.693) with reference to the amount of ripple present in the corresponding 'quasi-impulses' which affects the accuracy of the time domain approximation of the true impulse by the 'quasi impulses'. He concluded that the reduction of ripple by the introduction of a non-abrupt truncation was not sufficient to compensate for the loss of simplicity.

The reason for introducing the asymptotic termination in the present method, however, was not in order to get a less ripply time domain contribution corresponding to each harmonic, but to introduce at the outset the constraint imposed by a pole-zero excess which is at least as great as that of the plant. In Guillemin's method this would correspond to using different 'quasi impulses' to build up the transient response of systems having different pole-zero excesses.

Looked at in another way, Guillemin's method represents complete freedom; any transient response can be specified and its (approximate) rational transform found. In the present method the designer considers only those transient responses relevant to the pole-zero excess indicated by his plant.
The disadvantage of this approach, however, lies in the complexity of the pole-zero pattern of specified real parts. As fig. 5.1 shows, the transforms corresponding to the individual harmonics all have different poles, hence their linear combination contains all the poles of the individual transform. For example, consider the case of a real part defined as:

$$\text{Re}(T(j\omega)) = 0.5 + 0.75 \cos\frac{\pi}{\omega_c} - 0.25 \cos\frac{2\pi}{\omega_c} \text{ for } \omega<\omega_c \quad 5.2.1$$

and

$$\text{Re}(T(j\omega)) = -0.5\left(\frac{\omega_c^2}{\omega^2}\right) \text{ for } \omega>\omega_c \quad 5.2.2$$

This approximates the real part of some system with a pole-zero excess of 2. (The choice of the coefficients $a_0$, $a_1$ and $a_2$ at present seems arbitrary. Rules governing their possible values are developed in chapter 6).

Using the rational transforms of the individual components the transform of $T(s)$ corresponding to 5.2.1 is:

$$T(s) = 0.5 \left[ \frac{1.278(s+0.5309\pm j0.4657)(s+0.1536\pm j0.9153)}{(s+0.8178)(s+0.5224\pm j0.6798)(s+0.1471\pm j0.9395)} \right]$$

$$+ 0.75 \left[ \frac{-0.636(s+0.7645)(s+0.5735\pm j0.4946)}{(s+0.4058\pm j0.6902)(s+0.5987\pm j0.2622)} \right]$$

$$\times \left[ (s+0.1666\pm j0.9317) \right]$$

$$- 0.25 \left[ \frac{0.643(s-0.3662\pm j0.3917)}{(s+0.3637)(s+0.2876\pm j0.6295)} \right]$$

$$\times \left[ (s+0.1898\pm j0.8841) \right]$$

$$\left( s+0.1401\pm j0.9433 \right)$$

The above expression is hardly an encouraging start to practical system design. Furthermore, a much simpler expression for $T(s)$ corresponding to 5.2.1 can be obtained and is:

$$T(s) = \frac{0.4801(s+0.1773\pm j0.8874)}{(s+0.3310\pm j0.6086)(s+0.1367\pm j0.8946)} \quad 5.2.4$$
To indicate the accuracy of the above expression, fig. 5.2.2 compares the real part defined in 5.2.1 (solid line) with that calculated from 5.2.4 (dashed line) and fig. 5.2.2.b compares the step and impulse obtained directly from the intermediate and asymptotic regions according to their definitions in expressions 5.2.1 and 5.2.2 (solid line) (cf. section 4.4.1, 4.4.2) with the step impulse corresponding to the $T(s)$ given in expression 5.2.4 (dashed line). It is readily apparent that for engineering purposes the accuracy is adequate.

5.2.2 Comments on this approach.

Firstly, it will be noticed that the pole-zero excess of expression 5.2.3 is one, and of expression 5.2.4 is two. This discrepancy stems in part from the ambiguity in the description of the asymptotic region of the real part i.e. $N=2$ corresponds to an excess of one or two, and in part from the difficulty of ensuring the correct asymptotic behaviour while specifying the frequency response over a finite frequency range only.

Expression 5.2.4 was fitted to the complete real part as specified by 5.2.1 and 5.2.2, and the best fit had a pole-zero excess of two. Expression 5.2.3 is composed of the individual fits to the asymptotically terminated harmonics, each of which is best fitted by a transfer function having a pole-zero excess of one. Apart from the ambiguity mentioned, there is also the difficulty of ensuring that the asymptotic region defined over only a finite frequency range does in fact result in the correct pole-zero excess, and this difficulty is in fact part of the larger problem of fitting a rational transform to a specified frequency response. The above
1.2

Fig. 5.2.c Comparison of the real and imaginary parts obtained from exp. 5.2.1 and from exp. 5.2.4.

Fig. 5.2.d Comparison of the step and impulse responses obtained from exp. 5.2.1 and from exp. 5.2.4.
ambiguity and difficulty will be considered further in the succeeding chapters. (c.f. sections 7.2.4 and 7.3.2).

Secondly, the complexity of expression 5.2.3 is in direct contrast with the comparative simplicity of expression 5.2.4.

On both these counts, it is felt that this approach is not fruitful, and an alternative approach will be developed.

5.3 **Approach based on a frequency response characterisation.**

The approach of the previous section sought to pass directly from the specification of \( \text{Re} T(j\omega) \) to the corresponding \( T(s) \), by a linear combination of the transforms of the asymptotically terminated harmonics making up \( \text{Re} T(j\omega) \).

The alternative approach developed in this section aims at first specifying the imaginary part corresponding to each asymptotically terminated harmonic making up \( \text{Re} T(j\omega) \), then, by combining the real and imaginary parts obtaining \( T(j\omega) \), and only then attempting to obtain a rational \( T(s) \).

This may seem at first sight merely to postpone the difficulty of getting a rational \( T(s) \). In practice it has been found to result in considerable advantages, which will be brought out in the succeeding chapters.

5.3.1 **Obtaining the imaginary part corresponding to a particular real part via the Hilbert transform.**

It is well known that the real and imaginary parts of a function of the complex variable are uniquely related by the Hilbert transform, provided the function is analytic in the right half of the s-plane.
For the purposes of the present work, the of interest is \( T(s) \), the closed loop rational transfer function. For stable systems, \( T(s) \) fulfills the requirement of r.h.p analyticity. Thus if

\[
T(j\omega) = \text{Re}T(j\omega) + j\text{Im}T(j\omega)
\]

then the Hilbert transform states that

\[
\text{Im}T(j\omega) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Re}T(\mu)}{\omega - \mu} \, d\mu
\]

For the purposes of the present synthesis method, \( \text{Re}T(j\omega) \) will always be specified as

\[
\text{Re}T(j\omega) = \sum_{k=0}^{\infty} a_k \cos \frac{k\pi}{\omega_c} \omega, \quad k=0, 1, 2, \ldots \text{for } \omega < \omega_c
\]

(c.f. 4.3.2)

and

\[
\text{Re}T(j\omega) = A_c \left( \frac{\omega}{\omega_c} \right)^N \quad \text{for } \omega > \omega_c
\]

(c.f. 4.3.1)

As has been commented on in section 4.3.3, the real part of any rational \( T(s) \) may be approximated to any degree of accuracy in this way. Because of the linearity of the integral relationship 5.3.2, it follows that each asymptotically terminated harmonic used in the description of \( \text{Re}T(j\omega) \) by expressions 5.3.3 and 5.3.4 will have a corresponding imaginary part, the sum of which is \( \text{Im}T(j\omega) \).

Each such imaginary part can be obtained by evaluating expression 5.3.2 with \( \text{Re}T(\mu) \) replaced by the relevant asymptotically terminated harmonic. The method of so doing is discussed in the following section.

It is worthwhile to point out at this stage the advantage of using the real and imaginary parts of \( T(j\omega) \), rather than the magnitude and phase. It lies in the fact that, for the real and imaginary parts to be related by the Hilbert transform, the only restriction is that \( T(j\omega) \) must correspond to a stable system, and this would
in any case be required on practical grounds. For the magnitude and phase to be so related, on the other hand, requires \( T(j\omega) \) to correspond to minimum phase systems only.

5.3.2 Computer algorithm for the evaluation of the Hilbert transform.

For the evaluation of the transform, an alternative form of expression 5.3.2 was used, namely

\[
\text{Inf} \quad \text{Im}T(j\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re}T(\mu)}{\mu - \omega} \, d\mu \quad 5.3.5
\]

Substituting a general term \( a_k \cos \frac{k\pi}{\omega_c} \) from expression 5.3.3 into 5.3.5 gives

\[
\text{Imaginary part} = - \frac{k}{\omega_c} a_k \int_{-\infty}^{\infty} \frac{\sin \frac{k\pi}{\omega_c}}{\int_{-\infty}^{\infty} \text{log} \left| \frac{\mu + \omega}{\mu - \omega} \right| d\mu} \quad \text{for } \omega < \omega_c \quad 5.3.6
\]

and substituting expression 5.3.4. into 5.3.5 gives

\[
\text{Imaginary part} = - \frac{A_c N \omega_c N}{\pi} \int_{-\infty}^{\infty} \frac{\text{log} \left| \frac{\mu + \omega}{\mu - \omega} \right| d\mu} \quad \text{for } \omega > \omega_c \quad 5.3.7
\]

At this stage it is convenient to introduce a normalised frequency variable

\[
w = \frac{\omega}{\omega_c} \quad 5.3.8
\]

(and a normalised dummy variable \( \theta = \mu/\omega_c \)) in terms of which expressions 5.3.6 and 5.3.7 become

\[
\text{Imaginary part} = - k \cdot a_k \int \frac{\sin k\omega \omega_c}{\omega_c} \cdot \text{log} \left| \frac{\theta + w}{\theta - w} \right| d\theta \quad 5.3.9
\]

and

\[
\text{Imaginary part} = - \frac{A_c N}{\pi} \int \frac{\text{log} \left| \frac{\theta + w}{\theta - w} \right| d\theta}{\theta^{N+1}} \quad 5.3.10
\]

Thus expression 5.3.5, in terms of the normalised frequency \( w \), is the sum of expressions 5.3.9 (for each harmonic) and 5.3.10. As they stand, however, the above
expressions are still not suitable for numerical evaluation, because the integrand becomes infinite when \( \theta = w \) for \( w \) in the range 0 - 1.

To get over this difficulty, both expressions are further expanded. Expression 5.3.9 can be written as

\[
\text{imaginary part} = -\frac{a_k}{\pi} \int_0^1 \sin \pi \theta \log(\theta + w) \, d\theta
\]

\[-k^2 \int_0^1 \cos \pi \theta \, (\theta - w) \log |\theta - w| \, d\theta
\]

\[-\frac{a_k}{\pi} (1 - \cos k\pi) 5.3.11
\]

The second integral in expression 5.3.11 still contains a troublesome term, namely \( (\theta - w) \log |\theta - w| \). The difficulty is that while the whole term becomes zero at \( \theta = w \), the logarithm by itself becomes infinite at this point. The easiest way out of this difficulty seemed to be to omit this one point, provided the inaccuracy involved could be found. For \( w \) in the range 0 - 1 therefore, expression 5.3.11 was evaluated as

\[
\text{imaginary part} = -\frac{a_k}{\pi} \int_0^1 \sin \pi \theta \log(\theta + w) \, d\theta
\]

\[-k^2 \int_0^1 \cos \pi \theta \, (\theta - w) \log(w - \theta) \, d\theta
\]

\[+ \int_{\theta - w}^{\theta + w} \cos k\pi \theta \, (\theta - w) \log(\theta - w) \, d\theta
\]

\[-\frac{a_k}{\pi} (1 - \cos k\pi) 5.3.12
\]

The relationship between the inaccuracy in the imaginary part and the size of \( \epsilon \) is developed in appendix A at the end of this chapter.

Expression 5.3.10 can be written as:

\[
\text{imaginary part} = -\frac{A_c}{\pi} \log \left| \frac{1+w}{1-w} \right| - \frac{A_c}{\pi w N} \left[ \int_{1}^{(\beta-1)N-1} \frac{\inf \{ \beta \} \log(\beta-1)N-1}{\beta} \, d\beta
\]

\[-\int_{1}^{(\alpha-1)N-1} \frac{\inf \{ \alpha \} \log(\alpha-1)N-1}{\alpha} \, d\alpha \right] 5.3.13
\]

where \( \alpha = \frac{\theta + w}{\theta} \) and \( \beta = \frac{\theta - w}{\theta} 5.3.14 \)
For $N=2$, expression 5.3.13 becomes

\[
\text{imaginary part} = -\frac{A_c}{\pi} [(1-\frac{1}{w^2}) \log \left| \frac{1+w}{1-w} \right| + \frac{2}{w}] \quad 5.3.15
\]

When $w=1$, expression 5.3.15 reduces to

\[
\text{imaginary part} = -\frac{2}{\pi} A_c \quad 5.3.16
\]

For $N=4$, it becomes

\[
\text{imaginary part} = -\frac{A_c}{\pi} [(1-\frac{1}{w^4}) \log \left| \frac{1+w}{1-w} \right| + \frac{2}{w^3} + \frac{2}{3} \frac{1}{w}] \quad 5.3.17
\]

When $w=1$, expression 5.3.17 reduces to

\[
\text{imaginary part} = -\frac{8}{3\pi} A_c \quad 5.3.18
\]

Finally, for $N=6$, it becomes

\[
\text{imaginary part} = -\frac{A_c}{\pi} [(1-\frac{1}{w^6}) \log \left| \frac{1+w}{1-w} \right| + \frac{2}{w^5} + \frac{2}{3} \frac{1}{w^3} + \frac{2}{5} \frac{1}{w}] \quad 5.3.19
\]

When $w=1$, this reduces to

\[
\text{imaginary part} = -\frac{46}{15\pi} A_c \quad 5.3.20
\]

Now it had been decided to evaluate $\text{Im}(T(jw))$ up to $w=3$ (reasons for this are discussed in section 7.3), and this was done in three steps:

For $1<w>0$

\[
\text{Im}(T(jw)) = \sum_{k=1}^{\text{class no}} \left[ -a_k \left[ \frac{1}{w} \int \sin k\pi \log (\theta + w) \, d\theta - \frac{w-\epsilon}{\pi} \cos k\pi \log (w-\theta) \cos k\pi \log (\theta-w) \, d\theta + \frac{1}{\pi} (1 - \cos k\pi) \right] + \text{expression 5.3.15}, 5.3.17 \text{ or } 5.3.19 \right]
\]

depending on $N$ \quad 5.3.21

For $w=1$

\[
\text{Im}(T(jw)) = \sum_{k=1}^{\text{class no}} \left[ -a_k \left[ \frac{1}{w} \int \sin k\pi \log (\theta + 1) \, d\theta \right] \right]
\]
\[
+ \pi \kappa^2 \int_0^{1-\epsilon} \cos k\pi \theta (\theta-1) \log (1-\theta) \, d\theta + \frac{1}{\kappa^2} (1-\cos k\pi)
\]

expression 5.3.16, 5.3.18 or 5.3.20 depending on \( N \). 

Finally for \( w > 1 \)

\[
\text{class no } 1
\]

\[
\text{ImT}(jw) = - k_{a_k} \int_0^{\infty} \sin k\pi \theta \log \left( \frac{w+\theta}{w-\theta} \right) \, d\theta
\]

expression 5.3.15, 5.3.17 or 5.3.19 depending on \( N \).

The numerical evaluation of expressions 5.3.21 - 23 required a comparatively long computing time. It was decided therefore, that the only practical approach was to tabulate the imaginary parts of individual harmonics once, and to use this stored data in all further calculations. The details of the calculations are discussed in section 6.4.2.

5.4 Obtaining a rational transform for a specified real part.

The evaluation of a rational transform to fit an experimentally measured frequency response is a problem of long standing in system design. It is of course an essential first step if design is carried out in terms of the root locus. In the present method it is the last step, but it is still essential for a true synthesis method. The present work makes no new contribution in this field, but relies on existing methods.

Methods of varying complexity and accuracy exist, ranging from highly mathematical approaches (69) to comparatively rough approximations based on fitting asymptotes to the experimental Bode plots of magnitude. Although this last approach is comparatively rough, both
Horowitz (31 p. 684) and Truxal (27 p. 346) note that for many engineering applications it is sufficient, bearing in mind the inaccuracies anyway introduced by the experimental evaluation of the frequency response and by the assumption of linearity. Furthermore, the accuracy of this approach can be improved by Linvill's procedure (68).

Since the present work in any case required a great deal of digital computation, and in order to eliminate as much as possible time consuming manual curve fitting, a method due to Levy (67) was used throughout. This was programmed and found to give excellent accuracy in most cases, as will be apparent from the examples presented in chapters 7 and 8.

5.5 Conclusions

In order that the present method may be regarded as a true synthesis method, the rational transform of the required compensation must be obtained, because only then can the compensating network be synthesised without trial and error.

The possibility of obtaining a rational transform \( T(s) \) for the closed loop target system directly from the specification of \( \text{Re}T(j\omega) \) was investigated and rejected in favour of an alternate approach. This is to calculate \( \text{Im}T(j\omega) \) corresponding to the specified \( \text{Re}T(j\omega) \), thus forming the closed loop target system frequency response \( T(j\omega) \). This, together with the measured frequency response of the open loop constraint, specifies the frequency response of the required cascade compensation. As the final step, this frequency response is used to obtain the rational transform of the cascade compensation.
Appendix A

The error involved in expression 5.3.12 is

\[ \text{error} = \int_{w-\xi}^{w+\xi} \cos k\pi \theta \cdot (\omega - w) \log |w-\theta| d\theta \]  \hspace{1cm} A.1.

Since \( \xi \) will be a very small number \( (10^{-6}) \), over the interval \( 2\xi \)

\[ \cos k\pi \theta \div \cos k\pi w \div \text{constant} \]  \hspace{1cm} A.2

Since \( (\omega - w) \log |w-\theta| \) is anti-symmetrical about \( \theta = w \) the integral of this function (multiplied by \( \cos k\pi w \), assumed constant) over an interval equally spaced about \( w \) must be zero. The value of the error, therefore, will only differ from zero as a result of the variation in the magnitude of \( \cos k\pi \theta \) over a variation of \( \theta \) equal to \( 2 \times 10^{-6} \). Clearly for engineering applications such an error is negligible.

There is one condition where the argument used above needs modification, and that is when \( w \) is an odd multiple of \( \pi/2 \), since then \( \cos k\pi \theta \) changes sign at \( \theta = w \). This condition, however, does not arise in the present calculation, since expression 5.3.12 is used only for \( w \) in the range 0-1.
6.1 Introduction.

As mentioned in the conclusions to chapter 4, the idea behind the approximation of $\text{Re}T(j\omega)$ by asymptotically terminated harmonics is that by manipulating only a few harmonics, a large number of real parts can be formed and the corresponding transient performance investigated.

The present chapter is concerned with the details of the manipulation of the harmonics; in particular with the question of what combinations of amplitudes of the harmonics are useful or even permissible.

6.2 Constraints upon the harmonic content of the intermediate region of the real part plot.

As has been discussed in chapter 4, it is clear that the real part of the frequency response of any rational transfer function can be approximated to any degree of accuracy by approximating the intermediate region by a Fourier cosine series, and the asymptotic region by an inverse power of frequency.

The inverse of the above statement is not so clear, i.e. if the intermediate and asymptotic regions are specified in the above terms first, how closely does the resulting real part approximate to the real part of some rational transfer function?

This question is not answered explicitly in this work, the object of which was to investigate the feasibility of what is a completely new approach. Under the circumstances, it was felt not justifiable to spend
too much time on that aspect of the approximation, before the overall effectiveness of the method was tested. In the event, the results presented in chapters 7 and 8 suggest that the accuracy is quite adequate for engineering purposes. In fact, in practice the main inaccuracy might well be that in the measured frequency response of the plant.

The choice, therefore, of the harmonic content of the intermediate region is not guided by considerations of the accuracy with which the resulting real part matches up to the real part of some rational transfer function, but by more general requirements on the resulting real part and on the corresponding transient response. These matters are now considered in detail.

6.2.1 Constraints due to the transient behaviour of lumped linear systems.

The intention is to consider real parts corresponding to systems with a pole-zero excess of 2-5 by using the appropriate asymptotic region description i.e. N=2, 4 or 6.

It is known that for systems with a pole-zero excess > 1

$$h(t) \bigg|_{t=0} = 0$$

6.2.1

Now the contribution of the kth harmonic of the intermediate region to the impulse response was given in expression 4.4.2, and putting t=0 gives

$$h_{a_k}(t) \bigg|_{t=0} = -\frac{2}{\pi} a_0 \omega_c$$

6.2.2

In other words, only the constant term contributes at t=0, the contributions of the other harmonics
being zero.

The contribution due to the asymptotic region was given in expression 2.4.19 and putting \( t=0 \) gives (recalling that in expression 2.4.19 \( x=\omega_1 t \)),

\[
he(\omega_1 t)_N \bigg|_{t=0} = \frac{2 A_c \omega_c}{\pi (N-1)}
\]

Combining expressions 6.2.2 and 6.2.3, therefore, the condition for the impulse response to be zero at \( t=0 \) is

\[
a_0 = -\frac{A_c}{N-1}
\]

6.2.2 Constraints due to the real part behaviour of lumped linear systems.

As mentioned in section 4.2.3, it is intended that the real part resulting from specified harmonics should correspond to an open loop system of type I or higher, i.e.

\[
\text{Re}T(j\omega) \bigg|_{\omega=0} = T(j\omega) \bigg|_{\omega=0} = 1
\]

Thus, since the intermediate region will be described by

\[
\text{Re}T(j\omega) = \sum a_k \cos \frac{k\pi}{\omega c} \omega \text{ for } \omega < \omega_c
\]

it follows that

\[
\sum a_k = 1
\]

The asymptotic region will be described by

\[
\text{Re}T(j\omega) = A_c \left(\frac{\omega_c}{\omega}\right)^N \text{ for } \omega > \omega_c
\]

At the junction of the intermediate and asymptotic regions, i.e. at \( \omega_c \) the amplitude of the real part plot must be \( A_c \), therefore

\[
\sum a_k \text{ (k even)} - \sum a_k \text{ (k odd)} = A_c
\]
The last requirement is that \( \text{Re}(j\omega) \) should change sign the correct number of times. This was discussed in some detail in section 4.2.2, and it was concluded that in order to restrict the possible shapes of \( \text{Re}(j\omega) \) to those likely to have practical usefulness (in particular to avoid long settling times), the number of changes of sign of \( \text{Re}(j\omega) \) to correspond to systems with a pole-zero excess of 2 and 3 should be one, and for systems with an excess of 4 and 5 should be two.

This requirement is difficult to express analytically. To indicate what is involved, consider a real part whose intermediate region is defined as

\[
\text{Re}(j\omega) = a_0 + a_1 \cos \frac{\pi}{\omega_c} \omega + a_2 \cos \frac{2\pi}{\omega_c} \omega \quad (6.2.10)
\]

Suppose it is intended that this corresponds to a system with a pole-zero excess of 2. In that case the asymptotic termination will be for \( N=2 \), so that applying expression 6.2.4 gives

\[
a_0 = -A_c \quad (6.2.11)
\]

Applying conditions expressed in 6.2.7 and 6.2.9 and 6.2.10 gives

\[
a_0 + a_1 + a_2 = 1 \quad (6.2.12)
\]

\[
a_0 - a_1 + a_2 = A_c \quad (6.2.13)
\]

Hence combining with expression 6.2.11

\[
a_2 = \frac{1 + 3A_c}{2} \quad \text{and} \quad a_1 = \frac{1 - A_c}{2} \quad (6.2.14)
\]

It may be noted in passing that for a \( \text{Re}(j\omega) \) description containing at most the second harmonic, the harmonic amplitudes are uniquely specified for any value of \( A_c \). Such is not the case if higher harmonics
are present, necessitating the discussion presented in section 6.3.

Expression 6.2.10 can therefore be written in terms of \( A_c \)

\[
\text{Re}\{T(j \omega)\} = - A_c + \left(\frac{1-A_c}{2}\right) \cos \frac{\pi}{\omega_c} \omega + \left(\frac{1+3A_c}{2}\right) \cos \frac{2\pi}{\omega_c} \omega \tag{6.2.15}
\]

Calling \( \cos \frac{\pi}{\omega_c} \omega = \beta \) and expanding the double angle results in

\[
\text{Re}\{T(j \omega)\} = - \left(\frac{1+5A_c}{2}\right) + \left(\frac{1-A_c}{2}\right) \beta + (1+3A_c) \beta^2 \tag{6.2.16}
\]

Now for \( \text{Re}\{T(j \omega)\} \) to be zero once for a value of \( \omega \) between 0 and \( \omega_c \) requires the above quadratic to have one root between +1 and -1, and the other root outside these limits. (since for \(\omega_c > \omega > 0\), \(1 > \cos \frac{\pi}{\omega_c} \omega > -1\)).

The roots of this quadratic for varying \( A_c \) can readily be found and their locus is sketched in fig. 6.1.a. Inspection of this shows that the requirement of a single root between +1 and -1 is met for any \( A_c \).

An analogous development for \(N=4\) gives

\[
\text{Re}\{T(j \omega)\} = - (3+7A_c) + 3(1-A_c) \beta + (6+10A_c) \beta^2 \tag{6.2.17}
\]

The locus of the roots is shown in fig. 6.1.b. This shows that for a negative \( A_c \), corresponding to a system with a pole-zero excess of 3, there is only one root between +1 and -1, while for a positive \( A_c \), corresponding to an excess of 4, there are two.

Thus for a \( \text{Re}\{T(j \omega)\} \) description containing at most the second harmonic, the situation with regard to sign changes can be investigated, and in fact no restrictions need to be placed on the possible values of \( A_c \) on this score. As the harmonic content in the description of \( \text{Re}\{T(j \omega)\} \) rises, it is clear that the difficulty in
Fig. 6.2. Variation of the logarithm of the harmonic amplitude.
specifying combinations of harmonic amplitudes to ensure only one root between +1 and -1 in the resulting polynomial increases. The polynomial can certainly be formed in any specific case. Applying the conditions expressed in 6.2.7, 6.2.9 and 6.2.4 to the general case gives.

\[ a_1 + a_3 + a_5 \ldots = \frac{1-A_c}{2} \quad \text{and} \quad a_2 + a_4 + a_6 \ldots = \frac{1+A_c}{2} \frac{N+1}{(N-1)} \]

6.2.18

but now \( A_c \) no longer uniquely specifies the harmonic content.

Rather than considering the number of sign changes of \( \text{Re}T(j\omega) \) as a separate problem, and trying to establish limits on the permissible harmonic combinations for a specified number of changes, it was decided to lump this problem with the general question of how to specify the harmonic content (treated in the next section).

Thus in practice, the required number of sign changes was not expressed as an analytical condition, but was checked by factoring the polynomial for each combination of harmonics for which the transient performance was evaluated.

The constraints upon the harmonic content, therefore, are expressed by 6.2.4, 6.2.7 and 6.2.9. It is interesting to note that these constraints also ensure that

\[ u(t) \bigg|_{t \rightarrow \text{inf}} \rightarrow 1 \] 6.2.19

This can be seen from expression 4.4.6, giving the contribution of the kth harmonic to the step response. Allowing the time to go to infinity in expression 4.4.6 gives

\[ u_{a_k}(t) \rightarrow a_k \] 6.2.20
since \( Si(x) \) tends to \( \pi/2 \) as \( x \) tends to infinity. The contribution of the asymptotic region to the step response tends to zero as time tends to infinity, so that

\[
\sum_{t \to \inf} u_{ak}(t) \to \sum_{t \to \inf} a_k \to 1 \quad 6.2.21
\]

by virtue of condition 6.2.7.

Finally it can be seen that the conditions are equivalent to the requirement that the nett area under the real part plot be zero.

6.3. Specification of the harmonic content of the intermediate region of the real part plot.

The previous section established general constraints upon the permissible harmonic content of the intermediate region of the real part plot. Even with these constraints however, there are still infinitely many possible combinations of harmonic amplitudes. For the purposes of carrying out a systematic correlation of harmonic content (asymptotically terminated), with the resulting transient performance, some precise means of specification of the harmonic content was required. In particular, since it was intended to present the correlation in the form of a family of performance curves, it was desirable to specify the harmonic content by two main parameters.

6.3.1 Investigation of the harmonic content of the real part of some rational transfer functions.

Since it is clearly desirable that the harmonic content specified should correspond to that of rational transfer functions, it was decided to evaluate the
harmonic content of some rational transfer functions.

The transfer function used has already been given in expression 4.4.8 and the harmonic amplitudes obtained in fig 4.6. In the present discussion, interest centres on how the harmonic amplitudes vary, and to this end fig.6.2 shows the variation of the logarithm of the harmonic amplitude against harmonic number, for a damping ratio of 0.1, 1 and 1.5.

Inspection of fig.6.2 and fig.4.6 shows two patterns:

a) Odd and even harmonics decreasing approximately logarithmically (For damping ratios 1 and 1.5).

b) Even harmonics small, odd harmonics alternating in sign with amplitudes decreasing approximately logarithmically.

6.3.2 Means of specification adopted.

For any value of $A_c$ (the amplitude of the real part at $\omega_c$), the harmonics are given as

$$\sum a_k = \frac{1-A_c}{2} \quad k = 1, 3, 5 \ldots \quad 6.3.1$$

$$\sum a_k = \frac{1+A_c(N+1)}{2} \quad k = 2, 4, 6 \ldots \quad 6.3.2$$

The above expressions are consequences of the conditions given in expressions 6.2.4, 6.2.7 and 6.2.9.

In view of the results discussed in the last section, it was decided to specify the harmonic amplitudes as decreasing exponentially, the rate of decrease being governed by a parameter $m$. Considering the odd harmonics first, they will be specified as

$$a_3 = a_1 e^{-2\pi m}, \quad a_5 = a_1 e^{-4\pi m}, \quad a_7 = a_1 e^{-6\pi m} \text{ etc.}$$
so that expression 6.3.1 becomes
\[ a_1 + a_1 e^{-2\pi m} + a_1 e^{-4\pi m} + a_1 e^{-6\pi m} \ldots = \frac{1-A_c}{2} \] 6.3.3

The signs of the harmonics are specified as follows: \( a_1 \) always takes the sign of the right hand side of expression 6.3.3, the higher odd harmonics are then either of the same sign as \( a_1 \), or alternately positive and negative. This information is carried by a single parameter, which becomes 0 if the signs are all the same, and 1 if the signs alternate. This parameter had to be given a name for programming purposes, and was called formodd. To illustrate, consider a negative \( A_c \) so that the r.h.s. of expression 6.3.3 is positive, and hence \( a_1 \) is positive. Then for formodd=0, the harmonics are specified as
\[ a_1 + a_1 e^{-2\pi m} + a_1 e^{-4\pi m} + a_1 e^{-6\pi m} \ldots = \frac{1-A_c}{2} \] 6.3.4
while if formodd=1, this changes to
\[ a_1 - a_1 e^{-2\pi m} + a_1 e^{-4\pi m} - a_1 e^{-6\pi m} \ldots = \frac{1-A_c}{2} \] 6.3.5

Similarly, the even harmonics are specified as
\[ a_4 = a_2 e^{-2\pi m}, \ a_6 = a_2 e^{-4\pi m}, \ a_8 = a_2 e^{-6\pi m} \text{ etc} \]

so that expression 6.3.2 becomes
\[ a_2 + a_2 e^{-2\pi m} + a_2 e^{-4\pi m} + a_2 e^{-6\pi m} \ldots = \frac{1+A_c N(N-1)}{2(N-1)^2} \] 6.3.6

\( a_2 \) always takes the sign of the r.h.s. of expression 6.3.6, and the signs of the higher even harmonics are either all the same as of \( a_2 \) (indicated by a parameter called formeven being zero), or alternating (formeven=1).

Thus, for a particular asymptotic behaviour i.e. value of \( N \), the harmonic content is specified by two continuous variables, \( A_c \) and \( m \), and two parameters,
formodd and formeven, which are either 0 or 1. The final piece of information required is the highest harmonic present in the description of ReT(jω). This has been called the class of the system.

For convenience, this information is always presented in the same order, namely:
Value of N, class of system, value of formodd, value of formeven, value of m, value of A_0.

Thus the harmonic content of a system specified by:

N=4, class 4, 1, 0, 0.225, -1.1,

can be found as follows. For m=0.225, e^{-2πm}=0.2435.
Considering the odd harmonics first, since formodd=1 the signs will be alternating. Expression 6.3.1 therefore becomes

\[ a_1 - 0.2435 a_1 = 1.05 \] 6.3.7

Hence

\[ a_1 = 1.3875 \text{ and } a_3 = -a_1 e^{-2πm} = -0.3375 \]

The signs of the even harmonics will all be the same, since formeven=0. \( a_0 \) is fixed by expression 6.2.4, which gives \( a_0 = 0.3667 \). Expression 6.3.2 therefore becomes

\[ a_2 + 0.2435 a_2 = -0.4167 \] 6.3.8

Hence \( a_2 = -0.3351 \), and \( a_4 = a_2 e^{-2πm} = -0.0815 \). The complete description of the intermediate region of the real part is therefore

\[
\text{ReT}(jω) = 0.3667 + 1.3875 \cos\frac{π}{ω_c}ω - 0.3351 \cos\frac{2π}{ω_c}ω
\]

\[ - 0.3375 \cos\frac{3π}{ω_c}ω - 0.0815 \cos\frac{4π}{ω_c}ω \text{ for } ω<ω_c \] 6.3.9

Since N=4, the asymptotic region is described by

\[
\text{ReT}(jω) = -1.1\left(\frac{ω_0}{ω}\right)^4 \text{ for } ω>ω_c \] 6.3.10
6.4 Evaluation of the performance indices.

The performance indices which were chosen as a set which gave a reasonably good definition of the performance, were discussed in section 1.2.

Their approximate values were evaluated by digital computation. Approximate values only were evaluated, for two reasons. Firstly, it reduced computation times by an order. This is because, while it is perfectly feasible to find, say, the time for the step response to reach its maximum (by finding the time for $h(t)=0$), the computation involved is long, due to the comparative complexity of the expression given in table 2.1. Secondly, in the course of the synthesis procedure, a rational transfer function is fitted approximately to the system as defined by the asymptotically terminated harmonic content of its real part. Thus the performance of the end product, which is the system defined by the rational transfer function, in any case only approximates to the performance predicted from the harmonic content.

On the closeness with which the resulting rational system performance matches the performance predicted from the harmonic content, hinges the whole usefulness of this method. This will be investigated in chapters 7 and 8.

The way the approximate values were computed, and the degree of approximation involved, is now considered.

6.4.1 Transient performance indices.

The contributions of the harmonics (without the asymptotic terminations), to the impulse and step response are given in expressions 4.4.4 and 4.4.6. These were evaluated, for unit harmonic amplitudes, for the
normalised time variable $T$ varying from 0 to 20 in steps of 0.2.

The contributions of the asymptotic region are given in expressions 2.4.19 and 2.4.22. These also were evaluated over the same range of $T$, for $N=2, 4, 6$. These results were stored on tapes.

The programmes for evaluating the transient response, therefore, used the above data as input, together with a specification of the system to be considered, given in the terms discussed in section 6.3.2 (i.e. $N$, class, formodd, formeven, $m, A_c$).

The time indices (i.e. $T_{ul}$ etc.), therefore, may have an error of up to 0.2. Since the minimum value for $T_{ul}$ found was 3.6, the maximum error could be about 6%. The maximum error in $T_{hl}$ can be about twice as much. The error in the magnitude indices (i.e. $H_{ml}$, $H_{m1}$) ..., due to them being calculated at the wrong times, cannot easily be established, but for practical responses the curvature near an extremum is never great, so that the error should be quite small.

While it is appreciated that the above discussion of the errors is very imprecise, it is again stressed that the purpose of the present work was to check the overall feasibility of the method, and unlimited computing time was not available. Improvements in the accuracy may constitute a later refinement. Furthermore, it will be shown in chapters 7 and 8 that in practice the method gives an accuracy entirely acceptable for most engineering applications.
6.4.2 Frequency performance indices.

The calculation of the imaginary part corresponding to an asymptotically terminated real part, was discussed in sections 5.3.1 and 5.3.2. The imaginary parts corresponding to unit amplitude real part harmonics, for N=2, 4 and 6 were calculated for the normalised frequency variable, \( w \), varying from 0.05 to 3 in steps of 0.05.

Again the programmes for evaluating the frequency response used this, together with the system specification as data. The resonance peak, resonant frequency and bandwidth, were obtained by the linear combination of the real and corresponding imaginary parts, in the specified proportions.

The smallest value of the resonant frequency found was 0.1 so that the maximum error in this index could be 50%. This, however, occurs for a very small resonance peak (0.25 dB). For a resonance peak of 1 dB, the smallest value of the resonant frequency was found to be 0.35, giving a maximum error of 14%. For larger values of the resonance peak, \( V_r \) is larger so that the error is smaller. The smallest value of the bandwidth found was 0.44 so that the maximum error could be 11.5%. Again it is not possible to be certain what the maximum error in the closed loop response resonance peak \( M_{\nu\nu} \). Again reliance is placed on the fact that the curvature of practical characteristics is not large, and the results of chapter 7 and 8 bear this out.

The sensitivity, assuming unity feedback, is defined as

\[
S_T^G = \frac{\delta T}{G} \frac{G}{G}
\]
since
\[ T(s) = \frac{G(s)}{1 + G(s)} \quad 6.4.2 \]
it follows that
\[ \frac{dT(s)}{dG(s)} = \frac{1}{[1 + G(s)]^2} \quad 6.4.3 \]
Hence
\[ S_G^T = \frac{1}{1 + G(s)} = 1 - T(s) \quad 6.4.4 \]
and
\[ |S_G^T(\omega)| = \sqrt{[ (1 - \text{Re}T(j\omega))^2 + (\text{Im}T(j\omega))^2] \quad 6.4.5} \]
Thus again the sensitivity can be calculated from the real and imaginary parts.

Expression 6.4.4 shows that for unity feedback, the sensitivity is a function of \( T(s) \) only, so that once \( T(s) \) is chosen (to give a required transient performance), the sensitivity is fixed.

For independent control of both the transient performance determined by \( T(s) \), and the sensitivity, parallel compensation must be used. If feedback is through a transfer function \( 1 + H(s) \), expression 6.4.4 changes to
\[ S_G^T = \frac{1}{1 + G(s) [1 + H(s)]} = 1 - T(s)[1 + H(s)] \quad 6.4.6 \]

In the present work unity feedback only has been considered, on the grounds that this gave the simplest means of evaluating the usefulness of the method developed. Extension to non-unity feedback is clearly a desirable development which remains to be done.

6.5 Relationship between the harmonic content and the error coefficients.

In addition to the performance indices considered in section 6.4, the remaining performance index of the
set chosen in section 1.2 is the velocity constant \( K_v \) or acceleration \( K_a \) (whichever is applicable). The relationship between this and the harmonic content will now be developed.

6.5.1 **Contribution of the intermediate and asymptotic regions to the velocity step response.**

The contribution of the kth harmonic to the step response is given by expression 4.4.6. Integration of this gives the corresponding contribution to the velocity step as:

\[
 u_{vel,k}(T) = \frac{a_k}{\omega_0 \pi} [(T-k\pi) \text{Si}(T-k\pi) + (T+k\pi) \text{Si}(T+k\pi) + 2 \cos k\pi (\cos T - 1) - 2k\pi \text{Si}(T)]
\]

6.5.1

The shapes of such contributions (for unity harmonic amplitude) for a few values of \( k \) are shown in fig. 6.3.

The contribution of the asymptotic region to the velocity step response could be obtained by integrating expression 2.4.22. The object however, is not to get a general expression for the velocity step response but only to find the final error in that response. For this purpose the contribution of the asymptotic region may be disregarded since it tends to zero as time tends to infinity.

The fact that the contribution of the asymptotic region to the velocity step response (and also to the step and impulse response) tends to zero as time tends to infinity, may readily be seen from expression 2.4.12 which gives the asymptotic region contribution to the
Fig. 6.3. Contributions of the harmonics to a velocity step input.

Fig. 6.4. Relevant to the discussion in section 6.5.5.
impulse response, rewritten as

\[ h_e(t) = \frac{2}{\pi} \int_{t_0}^{t} \frac{\cos \omega t}{t} \, dt \]  

6.5.2

As \( t \) tends to infinity the lower integration limit approaches the upper limit and the integrand tends to zero so that \( h_e(t) \) tends to zero.

An expression similar to 6.5.2 obtains for the contribution of the asymptotic region to the step, velocity step and acceleration step responses. In each case as \( t \) tends to infinity the lower limit of integration approaches the upper limit so that the integral approaches zero.

Hence the final value of the velocity step response is given by

\[
\lim_{T \to \infty} \left[ \sum_{k=1}^{\text{class no}} uvel_k(T) \right] = \left[ \lim_{T \to \infty} uvel(T) \right]  
\]

6.5.3

6.5.2 Relationship between the harmonic amplitudes and \( K_v \).

The velocity constant \( K_v \), is of course the reciprocal of the final value of the lag (in seconds) in the velocity step response, provided this is finite.

Now expression 6.5.1 (illustrated in fig. 6.3) shows that the contribution of the kth harmonic tends to a velocity step having a rate of rise equal to \( a_k \) and delayed by a certain amount.

Thus the total response to a unit velocity step input tends to a delayed unit velocity step output (since \( \sum a_k = 1 \) c.f. expression 6.2.7). The delay is equal to the sum of the delays in the contributions of the individual harmonics. To find these delays reasonably accurately the following approach was adopted:
Assuming in each case that \( a_k = 1 \), the contributions of harmonics 0-6 was found for

\[ T = n\pi \text{ using } n=1, 2, 3 \ldots 18, \]

from expression 6.5.1. Once the value of the contribution \( u_{vel_k}(n\pi) \) was known, the time at which a unit ramp would reach the same value was found, the difference between \( n\pi \) and this time being the delay. The delay in each case oscillated about some value, the amplitudes of the oscillations decreasing as \( n \) increased. By taking \( n \) up to 18, the oscillations were reduced sufficiently to permit a reasonably accurate estimate of the final value of the delay. The values of the delay for \( n = 12 - 18 \) together with the final estimate made by taking the mean of the extremes of oscillations, are shown in table 6.1.

On the basis of these results, the delay in the velocity step response of any particular system was found from:

\[
\text{delay} = 0.6363a_o + 3.0672a_1 + 6.3093a_2 + 9.4119a_3 \\
+ 12.574a_4 + 15.7030a_5 + 18.8531a_6 \\
\]

It is apparent that apart from the constant term, the delay is very nearly \( \pi \) times the harmonic number.

It must be noted that this delay is in terms of the normalised time variable \( T = \omega_c t \), so must be converted into seconds by dividing by \( \omega_c \). \( K_v \) is then the reciprocal of this delay.

6.5.3 Qualitative discussion of the relationship between the harmonic amplitudes and \( K_a \).

The relationship between the harmonic amplitudes and \( K_a \) cannot be established by finding the contributions
<table>
<thead>
<tr>
<th>k</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<td>1.00</td>
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<td>9714</td>
<td>8892</td>
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<td>6228</td>
<td>8721</td>
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<td>4425</td>
<td>3093</td>
<td>12.57</td>
</tr>
<tr>
<td>18</td>
<td>Final estimate</td>
<td>0.6363</td>
<td>3.0672</td>
<td>6.3093</td>
<td>9.4119</td>
<td>12.5740</td>
<td>18.8521</td>
</tr>
</tbody>
</table>

Values of the delay in the contributions of the harmonics to a unit velocity step input. (Relevant to sect.6.5.2)
of the harmonics to the acceleration step response
i.e. integrating expression 6.5.5.

The reason is that, due to the non-linearity of
the acceleration step response a steady static error
corresponds to a continuously (but non-linearly)
decreasing delay.

It could be established by considering the
difference between the velocity step contributions of
the harmonics and corresponding ideal delayed velocity
steps. To clarify the situation consider fig. 6.4.a
which shows a unit velocity step and the reponse to
this of the second harmonic. The difference between
the actual response of the second harmonic and a delayed
unit velocity step is the shaded area in fig. 6.4.a. and
similar conditions obtain for the other harmonics.

Now \( K_a \) can have a finite value only if \( K_v \) is in-
finite i.e. if there is zero final following error on
velocity step input. If the harmonic contributions to
the velocity step were ideal delayed velocity steps, the
condition for zero final following error on velocity
step input would ensure that the output velocity step
was identical with the input velocity step. (This
situation arises because for negative \( a_k \)'s the delay
also is negative of fig. 6.4.c) Thus the acceleration
step input and response would also be identical so that
\( K_a \) would also be infinite. But, since the harmonic
contributions to the velocity step are not ideal
delayed velocity steps, the condition for infinite \( K_v \)
results in the output velocity step being of the form
shown in fig. 6.4.d and the shaded area would equal
the inverse of $K_a$. Since the shaded area of fig. 6.4.d could be expressed in terms of the shaded areas of the type shown in fig. 6.4.a, a link between $K_a$ and the harmonic content could theoretically at least be established.

In fact no attempt has been made to do so because it has been found that correlation between $K_y$ predicted from the harmonic content via expression 6.5.4, and the $K_y$ obtained from the rational transform of the final system do not correspond closely. Because of this only the infinite $K_y$ locus is marked on the performance curves. It is useful in that it is a boundary, to the right of which lie system specifications resulting in unstable open loop transfer functions (assuming unity feedback). No attempt has been made to indicate $K_a$ values on the infinite $K_y$ locus.

The reasons why the correlation between the $K_y$ predicted from the harmonic content, and that obtained from the final rational transform, is poor (in direct contrast to the very good transient performance correlation) are discussed in section 7.4.

6.6 Performance curves.

The performances indices, chosen in section 1.2 and calculated as discussed in section 6.4 and 6.5, are presented in the form of families of curves for $N = 2, 4$ and 6.

The first set of curves specifies the indices $T_{ul}$ and $u_{ml}$. These are the main curves which determine the range of $A_c$ and $m$ values used on all the other curves. $T_{ul}$ is the normalised time variable $\omega_c t_{ul}$ (of section
The index $u_{ml}$ is unaltered by $\omega_c$.

The first four families correspond to negative $A_c$ values. These show the infinite $K_V$ locus as a dashed line, to the right of which lies the region corresponding to unstable open loop systems for the unity feedback configuration (cf. sections 6.5.2 and 6.5.3). It was found that for the curves with positive $A_c$ values, the infinite $K_V$ locus lay outside the range of values shown.

The second set of curves specifies the indices $T_{hl}$ and $h_{ml}$. The values of $h_{ml}$ are for $\omega_c = 1$, and so must be multiplied by the $\omega_c$ used in any particular problem (cf. expression 4.4.4).

The third and fourth sets specify the indices $M_{pw}$, $W_r$, $BW$, and $w_s$. They are expressed in terms of the normalised frequency variable $w = \frac{\omega}{\omega_c}$ (cf. expression 5.3.8). The magnitude of the resonance peak is not altered by $\omega_c$, i.e. $M_{pw} = M_{pw}^*$. The above four sets of families of curves can be read easily and form the bulk of the information needed to choose target systems having a specified performance. The remaining set specifies the indices $T_{su}$ and $T_{sh}$, the approximate settling times (cf. section 1.2.1). Those proved difficult to present clearly, due to their discontinuous nature, and are tedious to interpret. They are seen as a rough guide only, particularly in view of their very approximate nature (cf. section 1.2.1).

6.7 Conclusions.

The constraints imposed on the harmonic content of the intermediate region by both the transient and
frequency behaviour of lumped linear systems was investigated.

Within those constraints, a form of specification of the harmonic content using few parameters was devised, based on the evaluated harmonic content of some rational transforms.

Using this form of specification, a large scale correlation of the transient and frequency performance with the specified real part was undertaken, and the results are presented in the form of a set of performance curves.

The next chapter illustrates the use of the performance curves.
CHAPTER 7

SOME GENERAL ILLUSTRATIONS OF THE SIGNIFICANCE OF THE PERFORMANCE CURVES.

7.1 Introduction

The scope and validity of the performance curves presented in chapter 6 will be investigated in the present chapter, prior to their application to specific examples.

While by no means exhaustive, it is felt that the number of examples presented in this chapter provides a reasonable check on the curves and indicates the potential of the method.

7.2 Some examples of target system specification using the performance curves.

The purpose of the curves is to provide the designer with information about the performance of a large number of systems of varying complexity, as characterised by the excess of poles over zeroes in the system transfer function.

On the basis of this information a target system meeting the required specifications and having the correct pole-zero excess can be chosen, and the necessary compensation calculated.

At present, the performance curves cover systems with a pole-zero excess of 2-5. An excess of 1 was felt not worthwhile, while an excess of 6 or more would require a modification in the method of system specification used. The reason is that the method of specification used at present (discussed in section 6.3.2) is
not suitable for specifying real parts having three changes of sign, for which the third harmonic must predominate. Since the present work aims at the initial exploration of the feasibility of this method, it was felt reasonable to stop at an excess of 5, which would, in any case, cover a large number of practical systems.

The real interest in the evaluation of the curves centres on two things.

a) How closely does the predicted performance read from the curves (which are approximate, c.f. section 6.4) correspond to the true predicted performance.

b) How closely does the performance obtained from the rational transfer function fitted to the specified real and imaginary parts correspond to the predicted performance.

To investigate this a number of examples are now presented. For each example the predicted real and imaginary parts and the predicted transient performance (shown by solid lines), are compared with the corresponding quantities obtained from the rational transforms (shown by dashed lines).

In addition, after the examples, the performance indices read from the curves are compared with the values obtained from a plot of the predicted transient performance and with the transient performance obtained from the rational transform (shown), and with the values obtained from the predicted frequency performance and with the frequency performance obtained from the rational transform(not shown).

7.2.1 Examples for \( N = 2 \).

Example a. It seems reasonable to start with the very
simplest system description possible. This is for a class 1 system with $N = 2$. Thus the full specification is:

$$N = 2, \text{ class 1, } -, - , - , - , 1/3, \omega_c = 1$$  \hspace{1cm} 7.2.1

(formodd, formeven and $m$ are relevant to systems of class 3 or greater. In class 2 systems, the harmonic amplitudes are uniquely specified by $A_\psi$, as shown in section 6.2.1 expression 6.2.14. A class 1 system is a special case of class 2 with $a_2 = 0$, and occurs for one value of $A_c$ only).

Hence, the real part is

$$\text{Re}T(j\omega) = 1/3 + 2/3 \cos \omega/\omega_c$$  \hspace{1cm} 7.2.2

This real part together with the corresponding imaginary part is shown in fig. 7.1a which also shows the real and imaginary parts calculated from the rational transfer function fitted to expression 7.2.2 (by the method described in section 5.4.). This rational transfer function was found as

$$T(s) = \frac{0.343(s+0.5646 \pm j0.6149)}{(s+0.5145 \pm j0.3111)(s+0.3706 \pm j0.7227)}$$  \hspace{1cm} 7.2.3

The transient responses, directly from the real part and from the rational transfer function, are compared in fig. 7.1b.

From fig. 7.1 it will be seen that the correspondence between the system performance as specified by the harmonic content of the intermediate region and the asymptotic termination, and that obtained from the rational transform fit is good.

It is also remarkable how such a simple form of specification (expression 7.2.2) in terms wholly unrelated to the physical behaviour of systems yields a
Fig. 7.1.a Real and imaginary parts of system specified by exp. 7.2.1.

Fig. 7.1.b Transient response of system specified by exp. 7.2.1.

Fig. 7.2.a Real and imaginary parts of system specified by exp. 7.2.4.

Fig. 7.2.b Transient response of system specified by exp. 7.2.4.
Transient response which corresponds so closely to that of a physical system (admittedly idealized by a transfer function description).

Example b: The system specification is:

\[ N = 2, \text{ class } 2, - , - , - , - 0.2, \omega_c = 1 \]  

Thus the real part is given by

\[ \text{Re} T(j\omega) = 0.2 + 0.6 \cos \frac{\pi}{\omega_c} \omega + 0.2 \cos \frac{2\pi}{\omega_c} \omega \]  

The rational transfer function fitted to the above real part and corresponding imaginary part is

\[ T(s) = \frac{0.208(s+0.6200 \pm j0.8785)}{(s+0.4242 \pm j0.2974)(s+0.4863 \pm j0.8108)} \]

Fig. 7.2a compares the real and imaginary parts specified (solid lines) with those obtained from the rational transform fit, (dashed lines) and fig. 7.2b compares the specified transient response (solid lines) with that obtained from the rational transform (dashed lines).

Example c: The system specification is:

\[ N = 2, \text{ class } 2, - , - , - , - 0.5, \omega_c = 1 \]

The real part is

\[ \text{Re} T(j\omega) = 0.5 + 0.75 \cos \frac{\pi}{\omega_c} \omega - 0.25 \cos \frac{2\pi}{\omega_c} \omega \]

and the corresponding rational transform is

\[ T(s) = \frac{0.4801(s+0.1773 \pm j0.8874)}{(s+0.1367 \pm j0.8949)(s+0.3310 \pm j0.6086)} \]

Again the performance from the specification and from the rational transform are compared in figs 7.3a and 7.3b. The performance indices for the examples used, obtained from the performance curves and from plots of the predicted performance, are compared in table 7.1.
Fig. 7.3.a Real and imaginary parts of system specified by exp. 7.2.7.

Fig. 7.3.b Transient response of system specified by exp. 7.2.7.

Fig. 7.4.a Real and imaginary parts of system specified by exp. 7.2.10.

Fig. 7.4.b Transient response of system specified by exp. 7.2.10.
Further examples of more complicated specification for \( N = 2 \) will be presented later on in this chapter.

7.2.2 Examples for \( N = 4 \).

Example a: The simplest form of specification is again considered first:

\[
N = 4, \text{ class 1, } -, -, -, - - 0.6, \omega_c = 1
\]

The real part is

\[
\text{Re}(T(j\omega)) = 0.2 + 0.8 \cos\frac{\pi}{\omega_c} \omega
\]

The corresponding rational transform is

\[
T(s) = \frac{0.4345(s+0.1691 + j0.9541)}{(s+0.4032 \pm j0.7504)(s+0.1258 \pm j0.9654)} \times \frac{1}{(s+0.5892)}
\]

The specified performance and that obtained from the rational transform is compared in fig. 7.4.

Example b: The specification is:

\[
N = 4, \text{ class 2, } -, -, -, - 1.2, \omega_c = 1
\]

The real part is

\[
\text{Re}(T(j\omega)) = 0.4 + 1.1 \cos\frac{\pi}{\omega_c} \omega - 0.5 \cos\frac{2\pi}{\omega_c} \omega
\]

The corresponding rational transform is

\[
T(s) = \frac{0.801(s+0.1455 \pm j0.9666)}{(s+0.1095 \pm j0.9713)(s+0.3397 \pm j0.7768)} \times \frac{1}{(s+0.5202 \pm j0.2533)}
\]

Fig. 7.5 compares the specified performances with that obtained from the rational transform.

Example c: The specification is:

\[
N = 4, \text{ class 3, } 1, 0, 0.192, 0.1, \omega_c = 1
\]
Fig. 7.5. a Real and imaginary parts of system specified by exp. 7.2.13.
Fig. 7.5. b Transient response of system specified by exp. 7.2.13.

Fig. 7.6. a Real and imaginary parts of system specified by exp. 7.4.11.
Fig. 7.6. b Transient response of system specified by exp. 7.4.16.
The real part is

$$\text{Re}\,T(j\omega) = -0.0333 + 0.6422 \cos\frac{\pi}{\omega_c} + 0.5833 \cos\frac{2\pi}{\omega_c} - 0.1922 \cos\frac{3\pi}{\omega_c}$$ 7.2.17

The corresponding rational transform is

$$T(s) = \frac{0.106(s+0.2023 \pm j0.9058)}{(s+0.3354 \pm j0.3216)(s+0.2194 \pm j0.6845)} \times \frac{1}{(s+0.1174 \pm j0.9011)}$$ 7.2.18

The specified performance is compared with that obtained from the rational transform in fig. 7.6.

The performance indices from the performance curves, plots of predicted performance and performance obtained from the rational transforms are compared in table 7.1.

7.2.3 Examples for \(N = 6\).

Example a: A class 1 description is not possible with an asymptotic termination of \(N = 6\). This is because such a termination corresponds to systems with a pole-zero excess of 5 or 6, the real part of which must have at least two sign changes up to \(\omega_c\) (c.f. section 4.2.2), while a class 1 description produces only one sign change. Thus a class 2 description is the simplest applicable to the asymptotic termination of \(N = 6\).

The specification is.

$$N = 6, \text{ clas 2}, - , - , - , 0.2, \omega_c = 1$$ 7.2.19

The real part is

$$\text{Re}\,T(j\omega) = -0.04 + 0.4 \cos\frac{\pi}{\omega_c} + 0.64 \cos\frac{2\pi}{\omega_c} \omega$$ 7.2.20

The corresponding rational transform is

$$T(s) = \frac{0.102}{(s+0.4198)(s+0.2418 \pm j0.8189)} \times \frac{1}{(s+0.3791 \pm j0.4363)}$$ 7.2.21
Fig. 7.7.a Real and imaginary parts of system specified by exp. 7.2.19.
Fig. 7.7.b Transient response of system specified by exp. 7.2.19.

Fig. 7.8.a Real and imaginary parts of system specified by exp. 7.2.22.
Fig. 7.8.b Transient response of system specified by exp. 7.2.22.
The performance is illustrated in fig. 7.7.

Example b: The specification is:

\[ N = 6, \text{ class } 3, 1, 0, 0.082, 0.1, \omega_c = 1 \]

The real part is

\[ \text{Re} T(j\omega) = -0.02 + 1.1176 \cos\frac{\pi}{\omega_c} \omega \]

\[ + 0.57 \cos\frac{2\pi}{\omega_c} - 0.6676 \cos\frac{3\pi}{\omega_c} \omega \]

The corresponding rational transform is

\[ T(s) = \frac{0.0921(s+0.7339)}{(s+0.1370 \pm j0.8957)(s+0.1967 \pm j0.6748)} \]

\[ \times \frac{(s+0.3391 \pm j0.9780)}{(s+0.1946 \pm j0.3637)} \]

The performance is illustrated in fig. 7.8.

The performance indices read from the performance curves, from plots of predicted performance and finally obtained from the rational transform are compared in table 7.1. The upper values in table 7.1 are those read off the performance curves (section 6.6). The second set are those obtained from a plot of the performance obtained from the rational transform. The last set are those obtained from a plot of the predicted performance. The object of including both the first and the last set was to show that despite the fact that the performance curves are only approximate, the performance indices read off them are very nearly the same as the exact predicted indices (c.f. section 7.2 and 6.4.).

7.2.4 Interim conclusions.

The first observation about the examples is that the correlation between the performance predicted from the performance curves, the true predicted performance,
Comparison of the performance indices predicted from the performance curves and obtained from the rational transforms for the examples of section 7.2.

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<th>Target system specification</th>
<th>$u_{m_l}$</th>
<th>$T_{ul}$</th>
<th>$h_{m_l}$</th>
<th>$T_{hl}$</th>
<th>$M_{pW}$</th>
<th>$w_r$</th>
<th>BW</th>
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<td>2.50</td>
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<td>0.218</td>
<td>2.40</td>
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<td>0.470</td>
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<td>1.030</td>
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<td>0.90</td>
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<td>1.80</td>
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<td>8.02</td>
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<td>4.65</td>
<td>0.42</td>
<td>0.37</td>
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<td>8.20</td>
<td>0.278</td>
<td>4.80</td>
<td>0.56</td>
<td>0.30</td>
<td>0.750</td>
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<td>15.8</td>
<td>8.20</td>
<td>0.278</td>
<td>4.60</td>
<td>0.46</td>
<td>0.35</td>
<td>0.750</td>
</tr>
<tr>
<td>$N = 6, \text{ class } 2$</td>
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<td>-</td>
<td>0.603</td>
</tr>
<tr>
<td>$A_c = 0.2$</td>
<td>2.63</td>
<td>9.6</td>
<td>0.228</td>
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<td>0</td>
<td>-</td>
<td>0.580</td>
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<tr>
<td></td>
<td>2.90</td>
<td>9.4</td>
<td>0.239</td>
<td>5.40</td>
<td>0</td>
<td>-</td>
<td>0.580</td>
</tr>
<tr>
<td>$N = 6, \text{ class } 3.1,0$</td>
<td>60.0</td>
<td>7.13</td>
<td>0.408</td>
<td>4.22</td>
<td>4.91</td>
<td>0.472</td>
<td>0.950</td>
</tr>
<tr>
<td>$m = 0.082, A_c = 0.1$</td>
<td>67.5</td>
<td>7.20</td>
<td>0.437</td>
<td>4.20</td>
<td>4.55</td>
<td>0.50</td>
<td>0.910</td>
</tr>
<tr>
<td></td>
<td>62.5</td>
<td>7.0</td>
<td>0.422</td>
<td>4.20</td>
<td>5.23</td>
<td>0.450</td>
<td>0.910</td>
</tr>
</tbody>
</table>
and the performance obtained from the rational transform is good. This is despite the fact that the specified real part always contains a discontinuity at \( w = 1 \), due to the different descriptions used in the intermediate and asymptotic regions. The rational transform does not contain such a discontinuity, so that there is always some discrepancy between the specified real part and that obtained from the rational transform. This discrepancy is small, however and its effect on the transient response is likewise small. Many more examples will be presented both in this chapter and the next and it will be found that the good correlation is consistently maintained.

The second observation about the examples is that in some cases (Examples 7.2.3b, see also expression 7.4.4 and 7.4.6) the rational transforms do not have the pole-zero excess implied by the asymptotic termination of the real part. The fact that nevertheless the specified real and imaginary parts and those obtained from the rational transform are in good agreement, serves to highlight the difficulty of ensuring the correct asymptotic behaviour while specifying the frequency response over a finite frequency range only (c.f. section 5.2.2). This and related matters are further discussed in the next section.

7.3 Ambiguity in the form of system specification used and difficulties associated with the asymptotic behaviour.

The purpose of this section is to discuss further the difficulties associated with obtaining the correct
asymptotic behaviour and in particular to distinguish the difficulty arising from the fact that the frequency response is specified over a finite frequency range only, from difficulties stemming from a basic ambiguity which would arise even if the frequency response was specified over a range approaching infinity.

7.3.1 Difficulties arising from the specification of the frequency response over a finite frequency range only.

In the present method, the frequency response is specified up to $3\omega_c$ i.e. up to 1.5 octaves beyond the frequency at which the real part is made to start its asymptotic behaviour.

As noted in section 7.2, in some cases the rational transform fitted gives good agreement with the specified real and imaginary parts yet has a pole-zero excess other than that implied by the value of $N$ used to define the asymptotic region.

Now it may be that specifying the frequency response up to say $5\omega_c$ might improve matters in this respect, but this has not been done for two reasons:

Firstly, by the time $\omega = 3\omega_c$, the magnitude of the frequency response is small (generally $\sim 1\%$ of the zero frequency value which is 1). Now in the practical application of the present method, frequency responses measured from physical systems would be used, and it would not be feasible to obtain meaningful values of the frequency response at such low levels of output.

Secondly, it has been found in practice that this difficulty can be avoided by not attempting to fit
a rational transform to the frequency response of the target system, but to the frequency response of the required compensation (c.f. section 5.1). The frequency response of the required cascade compensation (assuming unity feedback) is uniquely specified, given the frequency response of the target system and of the open loop constraint. Since the target system would be chosen to have the same pole-zero excess as the open loop constraint, the compensation would always have equal number of poles and zeroes. The many examples presented in chapter 8 bear out the practicability of this approach.

7.3.2 Ambiguity in the form of system specification used for minimum-phase and non minimum-phase systems.

Apart from the difficulties discussed in the preceding section, there is a basic ambiguity inherent in the method of specification used which has not been resolved in the present initial evaluation of this approach.

It has already been touched on in section 6.2 where it was pointed out that having specified a real part in terms of the harmonic content of its intermediate region, and the behaviour of its asymptotic region, it was not possible at present to say to what degree of accuracy this real part corresponded to the real part of a rational transform. The particular difficulty being considered is the asymptotic behaviour; now at present, specifying the asymptotic region of the real part as say \( N = 4 \), \( A_c \) +ve can in fact correspond to a minimum phase system with pole-zero excess of 3 or 4. This is illustrated in fig. 7.9b by curves A and B. The same specification can also correspond to a non-minimum phase
Fig. 7.9 Illustrating the possible frequency responses corresponding to the specified real part asymptotic behaviour.

Fig. 7.10a - Real and imaginary parts of system specified by exp. 7.3.1

Fig. 7.10b - Transient response of system specified by exp. 7.3.1
system with a pole-zero excess of three and one right half plane zero (curve C).

(It must be noted that an odd number of right half plane zeroes would produce a negative final displacement. Since only systems having a positive final displacement are of interest here, it is assumed that a system with an odd number of right half plane zeroes has a negative value of gain to produce a positive final displacement).

As an example of such a system consider the following specification:

\[ N = 4, \text{ class } 3, 1, 0, 0.082, 0.6, \omega_c = 1 \quad 7.3.1 \]

The real part is

\[
\text{ReT}(j\omega) = -0.2 + 0.4967 \cos \frac{\pi}{\omega_c} \omega \\
+ \cos^2 \frac{\pi}{\omega_c} - 0.2967 \cos \frac{3\pi}{\omega_c} \omega \quad 7.3.2
\]

and the corresponding transform is

\[
T(s) = \frac{-0.3056(s-0.6315)}{(s+0.3938 \pm j0.3957)(s+0.2791 \pm j0.7493)} \\
\times \frac{(s+0.1858 \pm j0.9571)}{(s+0.1282 \pm j0.9447)} \quad 7.3.3
\]

The specified real and imaginary parts are compared with those obtained from the transfer function in Fig. 7.10a, and the corresponding comparison for the transient response is shown in fig. 7.10b.

To resolve this ambiguity, two approaches seem possible: It might be possible to establish what combination of harmonics specifying the intermediate region can permissibly be coupled to an asymptotic region specified by a given \( N \) in order to produce a frequency response which closely approximates a rational transform having a particular asymptotic behaviour (i.e. final
Such an investigation, it is felt, would best be done by a mathematician rather than an engineer and might well produce a quite different method of specifying the harmonic content of the intermediate region than used at present (discussed in section 6.3).

Alternately, keeping the present method of specification, it should be possible to mark on the performance curves zones indicating the final angle and minimum or non-minimum phase character of the systems within them.

This has not been done yet, because in the present exploratory use of the performance curves, the full predicted response of any target system is always obtained in any case, at which stage its final angle is evident. This may well however constitute a future refinement.

7.4 Control over the velocity constant $K_v$

The relationship between the harmonic content and $K_v$ was given in expression 6.5.4 (recalling that $K_v$ is the inverse of the delay) and it was noted in section 6.5.3 that the correlation between the $K_v$ predicted from the harmonics and that obtained from the rational transform is poor.

The reason for this is that $K_v$ is determined by the behaviour of $T(j\omega)$ for zero frequency, or in general, the steady state error is given by:

$$ e(t) \bigg|_{t \to \inf} = sR(s) \left[ \frac{1+G(s)H(s)}{1+G(s)(1+H(s))} \right]_{s \to 0} = sR(s)[1-T(s)]_{s \to 0} \quad 7.4.1 $$
for velocity step response, therefore, the steady state error is given by:

\[
e(t) \bigg|_{t \to \infty} = \frac{1 - T(s)}{s} \bigg|_{s \to 0}
\]

Clearly therefore \( K_v \) (which is the reciprocal of the above error) is very sensitive to the behaviour of \( T(j\omega) \) near zero frequency.

Now the difficulty arises because the rational transform fitted is never an exact fit, and even a small difference near zero frequency between the behaviour of \( T(j\omega) \) specified from the harmonics and that resulting from the rational transform, results in large changes in \( K_v \). To illustrate this, the next section presents the extreme case of infinite \( K_v \) specification.

7.4.1 Infinite \( K_v \) specification

Using expression 6.5.4 the harmonic contents yielding infinite \( K_v \) (or zero lag) can be found, and the following specifications are examples of this type:

- \( N = 2, \) class 2, \( A_0 = -0.64272005 \)
- \( N = 4, \) class 3, \( A_0 = 0.00218368 \)

The values of \( A_0 \) are quoted to eight decimal places because it has been found that, due to the sensitivity of \( K_v \) to the values of \( T(s) \), a difference in even the sixth decimal place produces significant changes in \( K_v \).

Fig. 7.11 shows the low frequency parts of the open loop polar plots (assuming unity negative feedback) for the above systems (curves A, B respectively), predicted from the harmonics (solid line) and obtained from the rational transforms (dashed line) which are,
Fig. 7.11.a Low frequency part of the polar plot for system specified by exp.7.4.3.

Fig. 7.11.b Low frequency part of the polar plot for system specified by exp.7.4.4.
respectively

\[ T(s) = \frac{0.6640(s+2.2535)(s+0.5472 \pm j1.2278)}{(s+0.3740)(s+0.5433 \pm j1.1685)(s+0.1661 \pm j0.9244)} \times \frac{(s+2.140 \pm j0.9453)}{(s+3.706 \pm j0.6141)} \]

\[ T(s) = \frac{0.1341(s+0.2860)(s+0.4631 \pm j0.7503)}{(s+0.2136 \pm j0.2780)(s+0.2276 \pm j0.5719)} \times \frac{(s+0.1231 \pm j0.9702)}{(s+0.1991 \pm j0.7957)(s+0.1015 \pm j0.9358)} \]

The \( K_V \) values obtained from the transforms are respectively

\[ K_V = 1.002 \]
\[ K_V = 2.630 \]

It can be seen from Fig. 7.11 that while the shape of the polar plots is what would be expected from high \( K_V \) systems, and while the corresponding rational transform plots are again in good agreement, nevertheless the \( K_V \) values obtained from the rational transforms are far from infinite.

Although the extreme case of infinite \( K_V \) was deliberately chosen for the examples, and the correlation for finite \( K_V \) values is better, it is nevertheless not reliable. It will be shown by means of an example in chapter 8 that it is more satisfactory to control \( K_V \) in the conventional manner by inserting an appropriate lag network (section 8.3.1, specification 8.3.25). In this way a required \( K_V \) value can be obtained at the same time as a required transient performance, the latter being produced by suitably shaping the intermediate region of the real part.

Although in practice control of \( K_V \) will be obtained
conventionally by means of lag networks, and this will be illustrated in the next chapter, two further items associated with $K_v$ will now be considered before this topic is left.

7.4.2 **Control of $K_v$ by high harmonics.**

Inspection of expression 6.5.4 shows that the higher the harmonic, the greater is its effect on $K_v$. It is tempting to speculate therefore, that having chosen a low class number target system yielding a desired transient response, the addition of small amounts of the higher harmonics would permit control over $K_v$ without significantly affecting the transient response.

To illustrate this approach, the following system specification was chosen:

$$N = 2, \text{ class } 2, -, -, -, \ldots, -0.5, \omega_c = 1 \quad 7.4.9$$

The real part being given by

$$\text{ReT}(j\omega) = 0.5 + 0.75 \cos \frac{\pi}{\omega_c} \omega - 0.25 \cos \frac{2\pi}{\omega_c} \omega \quad 7.4.10$$

The predicted $K_v$ is 0.965

The rational transform fit for this system is

$$T(s) = \frac{0.4801(s+0.1773 \pm j0.8874)}{(s+0.1367 \pm j0.8949)(s+0.3310 \pm j0.6086)} \quad 7.4.11$$

and $K_v$ obtained from this is 0.794.

Now, in order to raise the predicted $K_v$ value without significantly affecting the predicted transient response, the values of the harmonics given in expression 7.4.10 are left unaltered, but harmonics up to the sixth are added to give:

$$\text{ReT}(j\omega) = 0.5 + 0.75 \cos \frac{\pi}{\omega_c} \omega - 0.25 \cos \frac{2\pi}{\omega_c} \omega$$

$$+ 0.0713 \cos \frac{3\pi}{\omega_c} \omega + 0.0713 \cos \frac{4\pi}{\omega_c} \omega$$

$$- 0.0713 \cos \frac{5\pi}{\omega_c} \omega - 0.0713 \cos \frac{6\pi}{\omega_c} \omega \quad 7.4.12$$
With these values, the predicted $K_v$ is 6.89.

In connection with expression 7.4.12, it should be noted that since the sum of the harmonic amplitudes must be one (c.f. expression 6.2.7), then for the constant and first two harmonics of expression 7.4.12 to be the same as those of expression 7.4.10 it is necessary to go straight from a two harmonic description to a six harmonic description, with $a_4 = -a_6$ and $a_3 = -a_5$.

The rational transform corresponding to expression 7.4.12 is

$$T(s) = \frac{0.4915(s+0.3763)(s+0.1162 \pm j0.3100)}{(s+0.6147)(s+0.1118 \pm j0.2522)(s+0.226 \pm j0.6178)}.$$

Fig. 7.12b compares the predicted transient performances of the closed loop systems specified by expressions 7.4.10 (solid line) and 7.4.12 (dashed line) with the transient performance obtained from expression 7.4.13 (chain dotted). It is readily apparent that the increased harmonic content affects the settling time without significantly changing the main shape of the predicted transient performance, and that the correlation with the performance obtained from the rational transform is good.

Fig. 7.12a compares the predicted open loop frequency response of the systems specified by expression 7.4.10 (solid line) and 7.4.12 (dashed line) with the open loop frequency response obtained from expression 7.4.13 assuming unity negative feedback (chain dotted). It will again be seen that the change in the predicted response due to the addition of the high harmonics does indeed represent a change which would be expected to yield an increased $K_v$, and that the correlation with the rational
Fig. 7.13.a  Frequency response of system specified by exp.7.4.14.

Fig. 7.13.b  Transient response of system specified by exp.7.4.14.
transform is quite good, though showing increasing divergence as the frequency decreases.

Regrettably however, the value of $K_v$ obtained from expression 7.4.12 is only 1.26, which is only a small increase on the original value and nowhere near the predicted value. This example, therefore, serves to re-emphasise that the correlation between the predicted and final $K_v$ values is poor, and to indicate that because of this, control over $K_v$ by the specification of the harmonic content of the real part is not feasible.

7.4.3 Negative $K_v$ values.

A brief mention of negative $K_v$ systems is also worthwhile at this stage. Negative $K_v$ values can readily occur in systems utilising feed-forward or parallel compensation, and nothing special can be said about those cases. In the special case of unity negative feedback, however, a negative value of $K_v$ must correspond to open loop unstable systems with an odd number of right half plane poles. Non-minimum phase systems would not produce this effect provided that, as discussed in section 7.3.2 the sign of the gain is such as always to produce a positive final displacement (on closed loop) due to a step input.

An example of such a system is afforded by the specification:

$$N = 2, \text{ class 2, } -, -, -, -0.8, \omega_c = 1 \quad 7.4.14$$

for which

$$\text{Re}T(j\omega) = 0.8 + 0.9 \cos\frac{\pi}{\omega_c} \omega - 0.7 \cos\frac{2\pi}{\omega_c} \omega \quad 7.4.15$$
which gives the predicted $K_y$ as -0.877

The corresponding rational transform is

$$T(s) = \frac{0.8250(s+0.1830)}{(s+0.3167)(s+0.3200 \pm 0.6122)}$$

Assuming unity negative feedback, the open loop is

$$G(s) = \frac{0.8250(s+0.1830)}{s(s-0.1508)(s+1.0548)}$$

Fig. 7.13b compares the predicted and final closed loop transient responses, and fig. 7.13a compares the open loop frequency responses.

7.5. Conclusions.

A number of target systems have been specified and the corresponding rational transforms found. The correlation between the transient and frequency performances predicted from the specification and those obtained from the rational transform has been found to be good. Furthermore, the performance predicted from the curves (contained in section 6.6) is seen to agree well with that obtained from a plot of the predicted performance. It is concluded therefore, that the preparation of the curves from an approximate evaluation of the predicted performance is justified on the grounds of greatly reduced computing time (cf section 7.2.4 and 6.4)

The difficulty of ensuring the correct pole-zero excess for target systems has been discussed, and has been shown to stem both from the fact that the target systems are specified over a finite frequency range only, and from the basic ambiguity that any one asymptotic behaviour of the real part of the frequency response corresponds to a number of possible asymptotic
behaviours of the complete frequency response. A means of circumventing the first part of this difficulty has been found, but the second part of the difficulty remains.

Finally, the correlation between the predicted and final $K_v$ values has been investigated and found to be poor.

The next chapter considers the use of the performance curves for system synthesis.
8.1. Introduction

The purpose of this chapter is to present some specific examples of synthesis, by the method developed in the preceding chapters.

In all the examples, cascade compensation only has been considered. The reason is that the essential contribution of the present method lies in presenting the designer with detailed information about the performance of a much wider range of systems (with specified pole-zero excess) than is available by any other means. Once a target system is chosen from the performance curves, then, for any given plant, the necessary cascade compensation is uniquely specified. Since the object of the present work was to develop the method and investigate its feasibility, it was felt that cascade compensation provided the simplest test.

It might be felt that some of the examples have overshoots which are larger than would be likely to be acceptable in practice. The justification again is that in evaluating the method it was felt that a spread of transient performance should be checked.

8.2 General comments on the application of the method.

For the purpose of this chapter, the twin starting points are taken to be the transfer function of the plant (open loop constraint) and a chosen target system.

In practice the plant would be characterised by a measured frequency response while the choice of a target system would involve plotting boundaries of the maximum
and minimum specified performance indices on the performance curves before the choice was made. If necessary a re-appraisal of the specification would be made at this point.

Since the performance curves are plotted in terms of normalised time and frequency variables, a suitable \( \omega_c \) has to be chosen to satisfy the actual time and frequency specifications.

Once the target system is chosen from the performance curves, the full target system frequency and transient performance can readily be obtained with the aid of a computer. Although this step would be fairly laborious by hand calculation, the computing time is short, since the contributions of the harmonics and of the asymptotic section are tabulated and need only be added in the correct proportion for any particular problem.

The frequency response of the necessary cascade compensation is thus obtained directly. At this point it must be approximated by a rational transfer function. This may be done by any of the methods available, but in the present work the method discussed in chapter 5 was exclusively used.

As mentioned in section 7.3.1 the advantage of fitting a rational transfer function to the compensation frequency response rather than fitting it to the target system frequency response, lies in the fact that the compensation network asymptotic behaviour is always the same i.e. it has equal numbers of poles and zeroes. This is a consequence of choosing a target system with the same pole-zero excess as that of the open loop constraint.

Two precautions only are necessary in the application of the method. The first relates to the behaviour of the open loop transfer function \( G(j\omega) \) as \( \omega \) tends to
From the fact that the target system and hence also the frequency response of the compensation is controlled only down to $0.05\omega_c$ (it is of course not feasible to control them to zero since $G(j\omega)$ tends to infinity). Now provided the resulting frequency response results in a stable closed loop system, control of $G(j\omega)$ down to $0.05\omega_c$ is sufficient to ensure the desired transient response. But it has been found, and is illustrated in detail in section 8.3.3 that the uncontrolled variation of $G(j\omega)$ below $0.05\omega_c$ can result in an unstable system.

While this possible pitfall needs to be borne in mind, it does not represent a serious difficulty since the fact that the resulting system is unstable is readily deducible from the final shape of the $G(j\omega)$ plot, and the means of correcting this are straightforward. These matters will be brought out in section 8.3.3.

The second precaution consists of smoothing out the frequency response of the required compensation near $\omega=\omega_c$, if it has a marked discontinuity there (e.g. fig. 8.17.b). This always occurs due to the discontinuity in the real part specification used (c.f. section 7.2.4.), and in practice all that is necessary is to omit the point at $\omega=\omega_c$, and possibly one or two values on either side, from the curve defining the frequency response of the required compensation.

Apart from the above precautions, the application of the method is absolutely straightforward and requires no trial and error changes. On this is based the claim that it is a true synthesis method as opposed to a design method.
Further aspects of the method will be brought out in the course of the presentation of the examples which follow.

8.3 Examples of synthesis.

In the examples of synthesis which follow, comparison is made in each case between the open loop frequency response and the closed loop transient response obtained

a) directly from the specification of the real part in terms of its harmonic content and asymptotic termination

and b) from the corresponding rational transform.

The closed loop frequency response $T(j\omega)$ is not shown since the open loop frequency response $G(j\omega)$ provides a more sensitive check on the agreement between the predicted frequency response and that resulting from the rational transform. This is particularly true for small $\omega$, when $T(j\omega) \approx 1$. The predicted response is always shown by a solid line, while that obtained from rational transforms by dashed lines. (Also by chain dotted and dotted lines if more than one transform response is shown).

8.3.1 Example 1.

For this example the open loop constraint was taken as

$$P(s) = \frac{1}{s(s+1 \pm j\omega)}$$

To illustrate the facility with which a range of target systems may be specified, and the necessary compensation calculated, five target systems have been selected, each with a predicted step response overshoot of 15%.
Further, to illustrate the effect of the choice of $\omega_c$, two sets of $\omega_c$ values were used: in the first instance values of $\omega_c$ were selected for each of the target systems to give the same $t_{ul}$, and for this value to be similar to $t_{ul}$ values produced by the system without compensation.

Then a second set of $\omega_c$ values was used with the same target systems, again so chosen as to give the same $t_{ul}$ for all the systems, but now this $t_{ul}$ to be much shorter than that produced by the system without compensation.

Fig. 8.1 shows the predicted open loop frequency response and the closed loop transient response for a system specified as

$$N = 4, \text{class 4, 1, 0, 0.170, -0.06, } \omega_c = 2.1$$

by fitting a rational transform to the frequency response of the necessary cascade compensation, (c.f. section 5.4) the compensated open loop transform was obtained as:

$$M(s)P(s) = 1.133 \left[ \frac{(s+9685 \pm j1.3393)(s+9465 \pm j1.9352)}{(s+5907 \pm j1.5019)(s+1840 \pm j1.9072)} \right]$$

$$\times \left[ \frac{1}{s(s+1 \pm j1)} \right]$$

8.3.3

giving the closed loop transform as:

$$T(s) = \frac{1.133(s+9685 \pm j1.3392)}{(s+8496)(s+8055 \pm j1.1034)}$$

$$\times \frac{(s+2905 \pm j1.9352)}{(s+3652 \pm j1.4049)(s+1793 \pm j1.8714)}$$

8.3.4

The frequency response obtained from expression 8.3.3 and the transient response obtained from expression 8.3.4, are shown dashed in fig. 8.1. It will be seen that the predicted and final performances are almost identical.

Inspection of expression 8.3.3 shows that the compensation consists of two complex pole-zero pairs,
Fig. 8.1.a Transient response of system specified by exp.8.3.5.

Fig. 8.1.b Frequency response of system specified by exp.8.3.5.

Fig. 8.2.a Transient response of system specified by exp.8.3.7.

Fig. 8.2.b Frequency response of system specified by exp.8.3.7.
in each pair the poles and zeroes lying close together. This is a consequence of the required performance being similar to the uncompensated system performance. To show, however, that the compensation does materially affect the performance, the performance of the uncompensated system with gain set so as to give the same $K_v$ value has also been calculated. The uncompensated system is given by

$$P(s) = \frac{1.241}{s(s+1 \pm j1)}$$

resulting in

$$T(s) = \frac{1.241}{(s+1.1954)(s+0.4023 \pm j.9361)}$$

and its performance is shown chain dotted in fig.8.1.

Fig.8.2 relates to a system specified as

$$N = 4, \ \text{clas} \ 4, \ 1, \ 0, \ 0.27, -0.42, \ \omega_c = 1.9$$

The necessary compensation is given by:

$$M(s)P(s) = 2.358\left[\frac{(s+.7779 \pm j1.1983)(s+.2652 \pm j2.1816)}{(s+.9506 \pm j1.4849)(s+.2585 \pm j2.1748)}\right] \times \left[\frac{1}{s(s+1 \pm j1)}\right]$$

giving

$$T(s) = \frac{2.358(s+.7779 \pm j1.1983)}{(s+1.4881)(s+.5302 \pm j.9710)} \times \frac{(s+.2652 \pm j2.1816)}{(s+.6760 \pm j1.4878)(s+.2585 \pm j2.1714)}$$

Shown chain dotted in fig.8.2 is the performance of the uncompensated system having the same $K_v$ value and defined by:

$$P(s) = \frac{1.5581}{s(s+1 \pm j1)}$$

yielding the closed loop transfer function

$$T(s) = \frac{1.5581}{(s+1.3703)(s+.3149 \pm j1.0188)}$$

It will again be seen that the predicted and final performance are very close.
Fig. 8.3 relates to a system specified as:

\[ N = 4, \text{ class 2, } - , - , - , -0.765, \omega_c = 1.5 \] 8.3.12

The necessary compensation is defined by:

\[
M(s)P(s) = 1.934 \left( \frac{s + 0.6767 \pm j 0.9870}{s + 0.7962 \pm j 1.1580} \right) \left( \frac{s + 0.2570 \pm j 1.511}{s + 0.2321 \pm j 1.5053} \right)
\]

\[
x \left[ \frac{1}{s(s + 1 \pm j 1)} \right] \] 8.3.13

with the corresponding closed loop transfer function given by:

\[
T(s) = \frac{1.934(s + 0.6767 \pm j 0.9870)}{(s + 0.4334)(s + 0.5965 \pm j 0.8198)}
\]

\[
x \left( \frac{s + 0.2570 \pm j 1.511}{s + 0.5066 \pm j 1.3036} \right) \] 8.3.14

The high degree of correspondence between the predicted and final performance is again in evidence.

The uncompensated system with the same \(K_v\) value is given by:

\[ P(s) = \frac{1.425}{s(s + 1 \pm j 1)} \] 8.3.15

and

\[ T(s) = \frac{1.425}{(s + 1.3043)(s + 0.3479 \pm j 0.9857)} \] 8.3.16

and its performance is shown chain-dotted in fig. 8.3.

The last two target systems are included only as illustrations of the close correspondence between the performance predicted from the design curves contained in chapter 6 and the final performance obtained from a rational transform fitted to the specified frequency response, not as useful practical systems. The reason is that both predict a long settling time, with the second overshoot comparable or even larger than the first overshoot.

Thus fig. 8.4 relates to a system specified as:

\[ N = 4, \text{ class 4, } 0, 0, 0.195, -1.1, \omega_c = 1.3 \] 8.3.17
Fig. 8.3.a Transient response of system specified by exp. 8.3.12.

Fig. 8.3.b Frequency response of system specified by exp. 8.3.12.

Fig. 8.4.a Transient response of system specified by exp. 8.3.17.

Fig. 8.4.b Frequency response of system specified by exp. 8.3.17.
The compensated open loop transfer function is

\[ M(s)p(s) = 1.701 \left( \frac{s+0.2217 \pm j0.5358}{s+0.2627 \pm j0.5984} \right) \left[ \frac{1}{s(s+1 \pm j1)} \right] \] 8.3.18

giving

\[ T(s) = \frac{1.701(s+0.2217 \pm j0.5358)}{(s+1.4439)(s+0.2509 \pm j0.4854)} \]

\[ \times \frac{1}{(s+0.2898 \pm j1.1144)} \] 8.3.19

The uncompensated system with the same value of \( K \) is given by

\[ P(s) = \frac{1.341}{s(s+1 \pm j1)} \] 8.3.20

and

\[ T(s) = \frac{1.341}{(s+1.2575)(s+0.3712 \pm j0.9636)} \] 8.3.21

The final target system is specified as

\( N = 4, \ \text{class} \ 4, \ 0, \ 0, \ 0.066, \ -1.42, \ \omega_c = 1.1 \) 8.3.22

The compensated open loop transfer function is given by

\[ M(s)p(s) = 1.206 \left( \frac{s+.9831}{s+.7256} \right) \left( \frac{s+.0619 \pm j.5191}{s+.0537 \pm j.6282} \right) \]

\[ \times \left[ \frac{1}{s(s+1 \pm j1)} \right] \] 8.3.23

and

\[ T(s) = \frac{1.206(s+.9831)}{(s+.1688 \pm j.4786)(s+.1758 \pm j.9398)} \]

\[ \times \left( \frac{s+.0619 \pm j.5191}{s+.0718 \pm j.4743} \right) \] 8.3.24

The performance is illustrated in fig. 8.5.

The foregoing examples show a very high degree of correspondence between the predicted transient performance and that obtained from the rational transform, fitted to the frequency response of the compensation.

The same target systems are now presented with \( \omega_c \) values chosen so as to give the same \( t_{ul} \) for all the systems, but its value now is 1.5, which is about 2.5
Fig. 3.5.a Transient response of system specified by exp. 8.3.42.

Fig. 3.5.b Frequency response of system specified by exp. 8.3.42.

Fig. 3.6.a Transient response of system specified by exp. 8.3.45.

Fig. 3.6.b Frequency response of system specified by exp. 8.3.45.
times faster than for the uncompensated system.

Fig. 8.6 relates to a system specified as

\[ N = 4, \text{ class } 4, 1, 0, 0.17, -0.06, \omega_c = 4.8 \quad 8.3.25 \]

The compensated open loop transfer function is given by

\[ M(s)P(s) = 13.52 \left[ \frac{(s+0.9561 \pm j1.0785)(s+0.9524 \pm j4.2617)}{(s+0.5187 \pm j4.2450)(s+1.5598 \pm j2.7849)} \right] \]

\[ \times \left[ \frac{1}{s(s+1 \pm j1)} \right] \quad 8.3.26 \]

Hence

\[ T(s) = \frac{13.52(s+0.9561 \pm j1.0765)}{(s+1.8680)(s+0.8751 \pm j1.0340)} \]

\[ \times \left[ \frac{(s+0.9524 \pm j4.2617)}{(s+0.8065 \pm j2.9027)(s+0.4629 \pm j4.1175)} \right] \quad 8.3.27 \]

This example is also used to illustrate the conventional approach of increasing \( K_v \) by means of a lag network. Thus the predicted \( K_v \) for this system was \( 1.4 \), and the value finally obtained from expression 8.3.26 is \( 1.43 \)

To increase the \( K_v \) value by a factor of 5, lag compensation is added resulting in

\[ M(s)P(s) = 13.52 \left[ \frac{(s+0.9561 \pm j1.0785)(s+0.9524 \pm j4.2617)}{(s+0.5187 \pm j4.2450)(s+1.5598 \pm j2.7849)} \right] \]

\[ \times \left[ \frac{1}{(s+0.1)(s+0.02)} \right] \left[ \frac{1}{s(s+1 \pm j1)} \right] \quad 8.3.28 \]

which gives

\[ T(s) = \frac{13.52(s+0.9561 \pm j1.0785)}{(s+1.0620)(s+1.8051)(s+0.8673 \pm j1.0312)} \]

\[ \times \left[ \frac{(s+0.9524 \pm j4.2617)(s+0.01)}{(s+0.7992 \pm j2.8871)(s+0.4667 \pm j4.1170)} \right] \quad 8.3.29 \]

The step response obtained from expression 8.3.29 is shown chain dotted in fig. 8.6. It must be stressed at this point that this is meant to illustrate the approach to increasing \( K_v \) by means of lag compensation, and leaves out the question of realising the compensation in practice.
Thus the required time constant of 50 would be difficult to obtain with a conventional lag network. Nevertheless, if it is required not to affect the transient response appreciably, then the compensating dipole must be much smaller than the other system poles.

Fig. 8.7 relates to a system specified as

\[ N = 4, \text{ class } 4, 1, 0, 0.27, -0.42, \omega_c = 4.4 \]  

The compensated open loop transfer function is given by

\[ M(s)P(s) = 29.31 \frac{(s+1.0137 \pm j1.0850)(s+3.909 \pm j4.2667)}{(s+3.769 \pm j4.3001)(s+2.7209 \pm j3.1487)} \]

\[ \times \frac{1}{s(s+1 \pm j1)} \]

and

\[ T(s) = \frac{29.31(s+1.0137 \pm j1.0850)}{(s+2.9735)(s+3.9363 \pm j1.1381)(s+1.3010 \pm j2.8443)} \]

\[ \times \frac{(s+3.909 \pm j4.2667)}{(s+3.648 \pm j4.3142)} \]

Fig. 8.8 relates to a system specified as

\[ N = 4, \text{ Class } 2, - , - , - , 0.765, \omega_c = 3.5 \]  

The compensated open loop transfer function is given by

\[ M(s)P(s) = 23.61 \frac{(s+0.9890 \pm j1.0446)(s+5.356 \pm j3.3179)}{(s+4.715 \pm j3.3418)(s+2.4855 \pm j2.8909)} \]

\[ \times \frac{1}{s(s+1 \pm j1)} \]
Fig. 8.7.a Transient response of system specified by exp.8.3.30.

Fig. 8.7.b Frequency response of system specified by exp.8.3.30.

Fig. 8.8.a Transient response of system specified by exp.8.3.33.

Fig. 8.8.b Frequency response of system specified by exp.8.3.33.
and

\[ T(s) = \frac{23.61(s+0.9890 \pm j1.0446)}{(s+2.7917)(s+0.9401 \pm j1.0556)(s+1.2122 \pm j2.6629)} \]

\[ \times \frac{(s+5.356 \pm j3.3179)}{(s+4.088 \pm j3.3725)} \]

As before, the last two examples have pronounced second overshoots in the step response, since a change of \( \omega_c \) affects the time scale of the step response only, without changing its shape.

Thus fig. 8.9 relates to a system specified as

\[ N = 4, 4, 0, 0, 0.195, -1.1, \omega_c = 3.1 \]

with

\[ M(s)P(s) = 22.1 \frac{(s+0.7353 \pm j0.9904)(s+1.5154 \pm j3.3099)}{(s+1.6252 \pm j2.6848)(s+1.9359 \pm j3.1973)} \]

\[ \times \frac{1}{s(s+1 \pm j)} \]

and

\[ T(s) = 22.1(s+0.7353 \pm j0.9904)(s+1.5154 \pm j3.3099) \]

\[ \frac{(s+3.2101)(s+0.6812 \pm j0.8668)(s+0.5615 \pm j2.7509)}{(s+1.7134 \pm j3.4045)} \]

and fig. 8.10 relates to a system specified as

\[ N = 4, \text{class 4, } 0, 0, 0.006, -1.42, \omega_c = 2.8 \]
Fig. 8.9.a Transient response of system specified by exp. 8.3.36.

Fig. 8.9.b Frequency response of system specified by exp. 8.3.36.

Fig. 8.10.a Transient response of system specified by exp. 8.3.39.

Fig. 8.10.b Frequency response of system specified by exp. 8.3.39.
with
\[ M(s)P(s) = 19.9\left[\frac{(s+1.366 \pm j1.2740)(s+1.3496 \pm j1.0888)}{(s+1.709 \pm j1.5829)(s+2.5080 \pm j2.6588)}\right]\]
\[ \times \left[\frac{1}{s(s+1 \pm j1)}\right] \]

and
\[ T(s) = \frac{19.9(s+1.366 \pm j1.2740)}{(s+1.9419)(s+1.3141 \pm j1.1334)(s+1.4085 \pm j2.3816)} \]
\[ \times \left[\frac{(s+1.3496 \pm j1.0888)}{(s+1.9854 \pm j1.5482)}\right] \]

It will be seen from inspection of figs. 8.6-8.10 that the effect on the transient response of increasing \( \omega_c \) is to increase the value of \( h_{ml} \).

8.3.2 Example 2.

For this example the open loop constraint was still taken as
\[ P(s) = \frac{1}{s(s+1 \pm j1)} \]

but the set target systems chosen have a greater overshoot than in the previous example.

Fig. 8.11 relates to a system specified as

\[ N = 4, \text{ class 2, } - , - , - , -1.3349, \ \omega_c = 1.25 \]

The compensated open loop transfer function is
\[ M(s)P(s) = 1.86\left[\frac{(s+0.5980 \pm j0.6938)(s+0.3243)}{(s+0.7040 \pm j0.7712)(s+0.0320)}\right]\]
\[ \times \left[\frac{1}{s(s+1 \pm j1)}\right] \]

and
\[ T(s) = \frac{1.86(s+0.5980 \pm j0.6938)}{(s+1.3371)(s+0.5783)(s+0.3210 \pm j1.0559)} \]
\[ \times \left[\frac{(s+0.3243)}{(s+0.4413 \pm j0.5842)}\right] \]

The performance calculated from these rational transforms is shown dashed in fig. 8.11.

As noted in some of the examples of section 8.3.1, the
Fig. 8.11a: Transient response of system specified by exp. 8.3.45.

Fig. 8.11b: Frequency response of system specified by exp. 8.3.45.

Fig. 8.12a: Transient response of system specified by exp. 8.3.46.

Fig. 8.12b: Frequency response of system specified by exp. 8.3.46.
compensation again contains complex pole-zero pairs with small separation between the poles and zeroes (i.e. small residues in the poles). On practical grounds it would be preferable to avoid compensation requiring complex poles, and to see the effect of omitting that part of the compensation, the performance of the following modified systems (having the same $K_v$ values as for expression 8.3.44) was found:

$$M(s)P(s) = 1.431\left[\frac{s+0.3243}{s+0.0326}\right] \frac{1}{s(s+1 \pm j1)} \quad 8.3.46$$

and

$$T(s) = \frac{1.431(s+0.3243)}{(s+0.9980)(s+0.6023)(s+0.2159 \pm j0.8517)} \quad 8.3.47$$

The performance from these expressions is shown chain dotted in fig. 8.11, and shows that despite the small residues in the complex poles their compensating effect is considerable.

Fig. 8.12 relates to a system specified as

$$N = 4, \text{ class 2, } - , - , - , -1.24 \quad \omega_c = 1.26 \quad 8.3.48$$

with

$$M(s)P(s) = 1.791\left[\frac{(s+3.5158 \pm j1.7134)(s+3.363)}{(s+3.602 \pm j0.7691)(s+0.0945)}\right]$$

$$\times \left[\frac{1}{s(s+1 \pm j1)}\right] \quad 8.3.49$$

and

$$T(s) = \frac{1.791(s+3.5158 \pm j1.7134)}{(s+1.3588)(s+5.0777)(s+3.957 \pm j6.190)}$$

$$\times \frac{(s+3.363)}{(s+3.273 \pm j1.0707)} \quad 8.3.50$$

The performance with a simplified compensation and gain adjusted to give the same value of $K_v$ is again calculated from:

$$M(s)p(s) = 1.452\left[\frac{s+3.363}{s+0.0945}\right] \frac{1}{s(s+1 \pm j1)} \quad 8.3.51$$

and

$$T(s) = \frac{1.452(s+3.363)}{(s+1.0906)(s+5.311)(s+2.364 \pm j9.872)} \quad 8.3.52$$
and is shown chain dotted in Fig. 8.12.

The last system chosen is specified as:

\[ N = 4, \text{class}4, 1, 0, 0.192, -0.6455, \omega_c = 1.6 \]

with:

\[ M(s)P(s) = 2.05\left[\frac{(s+0.9216 \pm j0.9438)(s+0.3095)}{(s+1.0773 \pm j1.2236)(s+0.0255)}\right] \]

\[ \times \left[\frac{1}{s(s+1 \pm j1)}\right] \]

and

\[ T(s) = \frac{2.05(s+0.9216 \pm j0.9438)}{(s+0.6529 \pm j3.533)(s+0.8961 \pm j7601)} \]

\[ \times \left[\frac{1}{s(s+0.3095)}\right] \]

\[ \left(s+0.5410 \pm j1.0761\right) \]

and again the simplified compensation is given by

\[ M(s)P(s) = 1.341\left[\frac{s+0.3095}{s+0.0255}\right]\left[\frac{1}{s(s+1 \pm j1)}\right] \]

and

\[ T(s) = \frac{1.341(s+0.3095)}{(s+0.8890)(s+0.6451)(s+0.2457 \pm j0.8144)} \]

The performance is shown in Fig. 8.13.

It can be seen that for the larger overshoots specified here, the correlation between predicted and final performance is still very close.

The foregoing examples illustrate the routine application of the present method. They pose no special problems and the necessary compensation is obtained in a straightforward manner.

The next example is chosen to show a case where the calculated compensation requires some interpretation before a satisfactory performance is obtained.

8.3.3 Example 3

For this example, the open loop constraint is taken as

\[ P(s) = \frac{1}{s^2} \]
Fig. 8.13a. Transient response of system specified by exp. 8.3.53.

Fig. 8.13b. Frequency response of system specified by exp. 8.3.53.

Fig. 8.14. Frequency response of the compensation for the system specified by exp. 8.3.59.

Fig. 8.15. Qualitative illustration of unstable behaviour with r.h.p. compensation zero, and stable behaviour with l.h.p. compensation zero.
and a target system specified by
\[ N = 2, \text{ class } 2, -, -, -, 1.2, \omega_c = 1 \]

Applying the method in the usual manner, the frequency response of the open loop constraint is divided into the specified open loop frequency response to give the cascade compensation frequency response. This is shown in fig. 8.14 (solid line).

Fitting a rational transfer function to this frequency response by the method discussed in section 5 gives the following answer.

\[ M(s) = 0.406 \left( \frac{(s-0.0029)(s+0.3329 \pm j0.7937)}{(s+0.7650)(s+0.2701 \pm j0.7978)} \right) \]

The frequency response of the above transfer function is shown dashed in fig. 8.14 and indicates that it is a good fit. Yet it is readily apparent from a root locus plot that a zero on the positive real axis in conjunction with a double pole at the origin will always lead to an unstable system (whether positive or negative feedback is used).

The explanation of this apparent failure of the method to give the correct compensation lies in the fact that the specified frequency response is in fact specified only down to 0.05\( \omega_c \). By then the specified open loop magnitude is large so that the closed loop frequency response (and therefore its real part) are insensitive to the exact behaviour of the open loop. In other words, to get a specified transient performance it is sufficient to control the behaviour of the open loop frequency down to 0.05\( \omega_c \), provided that its behaviour for \( \omega < 0.05\omega_c \) does not result in an unstable system.

The course of action which must be adopted in the present case therefore, is to modify the compensation so that
the frequency response down to $0.05\omega_c$ is not appreciably changed, while the behaviour for $\omega < 0.05\omega_c$ becomes that of a stable system. To do this it is only necessary to move the zero into the left half plane, leaving its magnitude the same. Since the zero is very small, this will affect only very low frequencies, and leave the bulk of the frequency response unchanged. This is illustrated qualitatively in fig. 8.15, which shows the specified open loop frequency response down to $0.05\omega_c$ as a solid line, its behaviour with the positive zero as a dashed line and with the negative zero chain dotted.

Thus finally the compensated open loop transfer function is taken as

$$M(s)P(s) = 0.406 \left[ \frac{(s+0.0029)(s+0.3329 \pm j0.7937)}{(s+0.7650)(s+0.2701 \pm j0.7978)} \right] \frac{1}{s^2}$$ 8.3.61

and

$$T(s) = 0.406 \frac{(s+0.0029)(s+0.3329 \pm j0.7937)}{(s+0.002915)(s+0.4268 \pm j4.747)}$$

$$\times \left[ \frac{(s+0.2243 \pm j0.7937)}{(s+0.1224 \pm j0.8271)} \right]$$ 8.3.62

The specified and final performances are compared in fig. 8.16, which also shows (chain dotted) the performance resulting from a simplified compensation, giving the same $K_v$, and defined as

$$M(s)P(s) = 0.4246 \left[ \frac{(s+0.0029)}{s+0.7650} \right] \frac{1}{s^2}$$ 8.3.63

and

$$T(s) = 0.4246 \frac{(s+0.0029)}{(s+0.002915)(s+0.3810 \pm j0.5265)}$$ 8.3.64

The correspondence between the predicted and final performance for the full compensation is again good, while the modified compensation shows the sort of difference which has been observed in the previous examples.
Fig. 8.16. a Transient response of system specified by exp. 8.3.59.

Fig. 8.16. b Frequency response of system specified by exp. 8.3.59.

Fig. 8.17. a Transient response of system specified by exp. 8.3.65.

Fig. 8.17. b Frequency response of system specified by exp. 8.3.65.
A second target system is now considered, for which the predicted and final performances do not agree so closely. The target system is specified by

\( N = 2, \text{ class } 4, 1, 0, 0.256, -0.18, \omega_c = 1 \)  

The compensated open loop transfer function is given by

\[
 M(s)P(s) = 0.1698\left[\frac{\left(s+0.0066\right)\left(s+0.2972 \pm j0.7645\right)}{\left(s+0.5584\right)\left(s+0.2079 \pm j0.6943\right)}\right]\frac{1}{s^2} 
\]

Hence

\[
 T(s) = \frac{0.1698\left(s+0.0066\right)}{\left(s+0.006714\right)\left(s+0.3274 \pm j0.3536\right)} \times \frac{\left(s+0.2972 \pm j0.7645\right)}{\left(s+0.1563 \pm j0.6761\right)} 
\]

The predicted performances (solid line) and that obtained from the above expressions (dashed line) are compared in fig. 8.17.

The discrepancy is greater than in the previous examples. The reason is that again the real zero of the compensation came out positive, and again its sign was changed. Since however, this zero is larger than the previous one, this caused a bigger difference between the predicted and final frequency responses than in the last example.

Shown chain dotted in fig. 8.17 is the performance using the simplified compensation given by

\[
 M(s)P(s) = 0.2173\left[\frac{s+0.0066}{s+0.5584}\right]\frac{1}{s^2} 
\]

resulting in

\[
 T(s) = \frac{0.2173\left(s+0.0066\right)}{\left(s+0.006714\right)\left(s+0.2758 \pm j0.3708\right)} 
\]

8.3.4 Example 4

For this example the open loop constraint was taken as

\[ P(s) = \frac{1}{s^3} \]
and a target system specified as

\[ N = 4, \text{ class 2, } - , - , - , -0.77, \omega_c = 1 \]

The rational approximation to the required compensation came out as

\[ M(s) = 0.547 \left[ \frac{(s+0.0341)(s-0.0231)}{s+0.7208 \pm j0.8279} \right] \]

changing the sign of the right half plane zero gave

\[ M(s)P(s) = 0.547 \left[ \frac{(s+0.0341)(s+0.0231)}{s+0.7208 \pm j0.8279} \right] \frac{1}{s^3} \]

and

\[ T(s) = \frac{0.547(s+0.0341)}{(s+0.0424)(s+0.0214)(s+0.7370)} \times \frac{(s+0.0231)}{(s+0.3204 \pm j0.7357)} \]

The predicted performance is compared with that obtained from the above expression (dashed) in fig. 8.18. It is seen that the correlation is poor and again this is attributed to the fact that the change in the sign of the zero produced an appreciable difference between the specified frequency response and that resulting from expression 8.3.73.

To illustrate how sensitive the performance is to the values of the zeroes, fig. 8.18 also shows (chain dotted) the performance obtained from

\[ M(s)P(s) = 0.371 \left[ \frac{(s+0.0341)^2}{s+0.7208 \pm j0.8279} \right] \frac{1}{s^3} \]

and

\[ T(s) = \frac{0.371(s+0.0341)^2}{(s+0.0265)(s+0.0591)(s+0.3873)} \times \frac{1}{(s+0.4844 \pm j0.6911)} \]

and also from

\[ M(s)P(s) = 0.808 \left[ \frac{(s+0.0231)^2}{s+0.7208 \pm j0.8279} \right] \frac{1}{s^3} \]
Fig. 8.10.a Transient response of system specified by exp.8.3.71.

Fig. 8.10.b Frequency response of system specified by exp.8.3.71.

Fig. 8.11.a Transient response of system specified by exp.8.3.80.

Fig. 8.11.b Frequency response of system specified by exp.8.3.80.
and
\[
T(s) = \frac{0.808(s+0.0231)^2}{(s+0.0197)(s+0.0291)(s+1.0059)} \times \frac{1}{(s+0.1934 \pm j0.8426)}
\]

which is shown dotted.

This example highlights a difficulty associated with the present method which will need further work. The difficulty lies in calculating the compensation required not only to control the frequency response down to 0.05\(\omega_c\) but to ensure that its behaviour for \(\omega < 0.05\omega_c\) results in a stable system. This will be further considered in the conclusions after this chapter.

8.3.5 Example 5.

For this example the open loop constraint was chosen as
\[
P(s) = \frac{1}{s^2(s+2)}
\]

The first target system was specified as
\[
N = 4, \text{ class } 4, 1, 0, 0.170, -0.06, \ \omega_c = 2.1
\]

the compensated open loop transfer function was
\[
M(s)P(s) = 1.139\left[\frac{(s+0.0148)(s+3.607 \pm j1.8854)}{(s+2.1490)(s+2.035 \pm j1.8700)}\right]
\times \left[\frac{1}{(s+0.6724 \pm j1.2827)}\right] \frac{1}{s^2(s+2)}
\]

giving
\[
T(s) = \frac{1.139(s+0.0148)(s+3.607 \pm j1.8854)}{(s+7.332)(s+0.01519)(s+2.2605)(s+1.8336)}
\times \frac{(s+2.0628 \pm j1.4658)}{(s+3.444 \pm j1.2928)(s+1.848 \pm j1.8241)}
\]

The predicted and final performance is compared in fig. 8.19. It will be seen that the correspondence is good, though some discrepancy is evident in the latter part of the
transient. This corresponds to the increased divergence between the specified and final frequency response at very low frequencies.

The performance obtained from a simplified compensation is given by

\[
M(s)P(s) = 1.181 \left[ \frac{(s+0.0148)(s+2.0628 \pm j1.4658)}{(s+2.1490)(s+.6724 \pm j1.2827)} \right]
\]

\[
x \left[ \frac{1}{s^2(s+2)} \right]
\]

and

\[
T(s) = \frac{1.181(s+0.0148)(s+2.0628 \pm j1.4658)}{(s+.7827)(s+.01519)(s+2.2699)(s+.8144)}
\]

\[
x \frac{1}{(s+.3058 \pm j1.2267)}
\]

is shown chain dotted in fig. 8.19.

a second target system was chosen as

\[ N = 4, \text{ class } 4, 1, 0, 0.270, -0.42, \omega_c = 1.9 \]

with

\[
M(s)P(s) = 2.313 \left[ \frac{(s+2.2489)(s+.01289)}{s+1.2745 \pm j1.3634} \right] \left[ \frac{1}{s^2(s+2)} \right]
\]

and

\[
T(s) = \frac{2.313(s+2.2489)(s+.01289)}{(s+.01311)(s+.5568 \pm j1.1781)}
\]

\[
x \frac{1}{(s+1.7111 \pm j2.2871)}
\]

The predicted and final performance is compared in Fig. 8.20.

The same target systems have also been considered

with increased \( \omega_c \) values. Thus fig. 8.21 relates to a

\[ N = 4, \text{ class } 4, 1, 0, 0.170, -0.06, \omega_c = 4.8 \]

with

\[
M(s)P(s) = 13.52 \left[ \frac{(s+1.8188)(s+.0591)(s+.8984 \pm j4.2313)}{(s+1.5230 \pm j2.7640)(s+.8481 \pm j4.2324)} \right]
\]

\[
x \left[ \frac{1}{s^2(s+2)} \right]
\]
Fig. 8.20.a Transient response of system specified by exp. 8.3.85.

Fig. 8.20.b Frequency response of system specified by exp. 8.3.85.

Fig. 8.21.a Transient response of system specified by exp. 8.3.86.

Fig. 8.21.b Frequency response of system specified by exp. 8.3.86.
and
\[ T(s) = \frac{13.52(s+1.8188)(s+0.0591)(s+.8984 \pm j4.2313)}{(s+.0621)(s+.3213)(s+.12353)(s+.7731 \pm j2.8901)} \]
\[ \times \frac{1}{(s+.4246 \pm j4.1025)} \quad 8.3.90 \]

and fig. 8.22 relates to a system specified as
\[ N = 4, \text{class 4, 1, 0, 0.270, -.42, } \omega_c = 4.4 \quad 8.3.91 \]

with
\[ M(s)P(s) = 28.61[(s+2.1097)(s+.03308)]\left(\frac{1}{s^2(s+2)}\right) \quad 8.3.92 \]

and
\[ T(s) = \frac{28.61(s+2.1097)(s+.03308)}{(s+.03373)(s+2.4827 \pm j3.3838)} \]
\[ \times \frac{1}{(s+.2506 \pm j2.7819)} \quad 8.3.93 \]

8.3.6 Example 6.

For this example the open loop constraint was taken as
\[ P(s) = \frac{(s+.6)}{(s+.2)(s+1 \pm j)} \quad 8.3.94 \]

and a target system specified as
\[ N = 4, \text{class 2, - - - - -1.3349, } \omega_c = 1.25 \quad 8.3.95 \]

The compensated open loop transfer function was obtained as
\[ M(s)P(s) = 1.819[(s+.1271)(s+.9658 \pm j4.032)]\left(\frac{(s+.6)}{s(s+.2)(s+1 \pm j)}\right) \]
\[ \times \left(\frac{(s+.6)}{(s+.2)(s+1 \pm j)}\right) \quad 8.3.96 \]

and
\[ T(s) = \frac{1.819(s+.6)(s+.1271)(s+.9658 \pm j4.032)}{(s+.1313)(s+.3976 \pm j4.778)(s+.3478 \pm j1.0430)} \]
\[ \times \frac{1}{(s+.4235 \pm j.6678)} \quad 8.3.97 \]

the performances are compared in fig. 8.23.

8.3.7 Example 7.

For the last example having a pole-zero excess of
Fig. 8.22.a Transient response of system specified by exp. 8.3.91.

Fig. 8.22.b Frequency response of system specified by exp. 8.3.91.

Fig. 8.23.a Transient response of system specified by exp. 8.3.95.

Fig. 8.23.b Frequency response of system specified by exp. 8.3.95.
three the following open loop constraint was chosen.

\[ P(s) = \frac{(s+0.3)}{s(s+0.1)(s+1 \pm j1)} \]  

8.3.98

Using the same target system as in the previous example resulted in

\[ M(s)P(s) = 1.8\left[ \frac{(s+.1076)(s+.2148 \pm j1.1131)(s+1 \pm j1.0197)}{(s+.02825)(s+.1984 \pm j1.1249)(s+.0594 \pm j1.1919)} \right] \]

\[ \times \left[ \frac{(s+1)}{s(s+.1)(s+1 \pm j1)} \right] \]  

8.3.99

and

\[ T(s) = \frac{1.8(s+0.3)(s+.1076)(s+.2148 \pm j1.1131)}{(s+.0224)(s+.6366)(s+.1072)(s+.1548 \pm j1.1145)} \]

\[ \times \left[ \frac{(s+1 \pm j1.0197)}{(s+.9956 \pm j1.0285)(s+.3885 \pm j0.9634)} \right] \]  

8.3.100

The predicted and final performances are compared in fig. 8.24.

8.3.8 Example 8.

The last example in this chapter was chosen with a pole-zero excess of four, the open loop constraint being

\[ P(s) = \frac{1}{s^2(s+1 \pm j1)} \]  

8.3.101

Two target systems were used. The first was specified as

\[ N = 4, \text{ class 3, 1, 0, 0.192, 0.1, } \omega_c = 5.5. \]  

8.3.102

The compensated open loop transfer function corresponding to this was found to be

\[ M(s)P(s) = 112\left[ \frac{(s+.0181)(s+4.4488)(s+1.2823)}{(s+.7831)(s+.7794 \pm j4.9462)} \right] \]

\[ \times \left[ \frac{1}{s^2(s+1 \pm j1)} \right] \]  

8.3.103

and

\[ T(s) = \frac{112(s+.0181)(s+4.4488)(s+1.2823)}{(s+.01833)(s+.7831)(s+.8481)(s+.6946 \pm j4.6404)} \]

\[ \times \left[ \frac{1}{s+.6899 \pm j1.8457} \right] \]  

8.3.104
Fig. 8.24.a Transient response of system specified by exp. 8.3, 95.
Fig. 8.24.b Frequency response of system specified by exp. 8.3, 95.

Fig. 8.25.a Transient response of system specified by exp. 8.3, 102.
Fig. 8.25.b Frequency response of system specified by exp. 8.3, 102.
The predicted and final performance is compared in fig. 8.25.

The second target system was specified as

\[ N = 4, \text{ class 3, 1, 0, 0.192, 0.15 } \omega_c = 2.35 \]

For this case the compensated open loop transfer function was

\[
M(s)P(s) = 3.573 \left[ \frac{(s+0.002908)(s+1.0841 \pm j1.1207)}{(s+1.7634)(s+4.647 \pm j2.008)} \right] x \left[ \frac{(s+0.2738 \pm j2.2101)}{(s+1.824 \pm j2.2349)} \right] \frac{1}{s^2(s+1 \pm j1)}
\]

and

\[
T(s) = \frac{3.673(s+0.002908)(s+1.0841 \pm j1.1207)}{(s+0.002922)(s+1.7487 \pm j9.672)(s+1.2068 \pm j5.462)} x \left[ \frac{(s+0.2738 \pm j2.2101)}{(s+3.998 \pm j1.8168)} \right]
\]

Fig. 8.26 compares the predicted and final performances. Also shown in fig. 8.26 (chain dotted) is the performance using a simplified compensation given by

\[
M(s)P(s) = 3.638\left[ \frac{(s+0.002908)(s+1.0841 \pm j1.1207)}{(s+1.7634)(s+4.647 \pm j2.008)} \right] x \left[ \frac{1}{s^2(s+1 \pm j1)} \right]
\]

and

\[
T(s) = \frac{3.638(s+0.002908)(s+1.0841 \pm j1.1207)}{(s+0.002922)(s+3.338 \pm j7.353)(s+4.106 \pm j1.9257)} x \left[ \frac{1}{(s+1.6205 \pm j9.307)} \right]
\]

It will be seen that the method used to fit a rational transform to the frequency response (c.f. section 5) has produced poorer results for this example than for the other examples, and this is reflected in the greater discrepancy between the specified transient performance and that obtained from the rational transform.
Fig. 8.26.a Transient response of system specified by exp.8.3.105.

Fig. 8.26.b Frequency response of system specified by exp.8.3.105.
8.4 Conclusions

On the evidence of the examples presented in this chapter, it is concluded that the method developed in the present work is feasible, and that in fact in many cases it gives very accurate results.

A possible difficulty associated with the very low frequency behaviour of the open loop frequency response has been considered (section 8.2. and example 3), and a way round it suggested. This point, however, needs some further work to ensure that both stability and the desired transient performance can be achieved simultaneously.

The calculated cascade compensation has, in many instances, complex poles and zeroes, which may well be undesirable in practice. It should be borne in mind, however, that cascade compensation only has been considered purely in order to evaluate the method, and that in many practical situations a combination of cascade and parallel compensation would be used, giving quite different compensation transfer functions.
CONCLUSIONS

A method of deriving the transient response from the measured frequency response, accurately and simply, has been developed. Evaluation of previous methods indicated that there was room for improvement on two counts: 1) increasing the accuracy by a more exact treatment of the asymptotic behaviour (chapter 2) 2) simplifying the mode of approximating the frequency response. It was felt particularly important to be able to use directly the values of the measured frequency response. This led to the use of triangles in the approximation of the frequency response (chapter 2).

The final method is a development of existing methods (as discussed in chapter 3) and it is felt, represents an improvement in accuracy and simplification, particularly as both step and impulse are obtainable by an identical calculation (chapter 3). The calculation is certainly feasible by hand (chapter 3), but of greater importance, it is in a particularly suitable form for programming on a digital computer since as input it only needs the measured frequency response.

The aim of the work, contained in chapters 4-8, was to develop a synthesis method giving close control over transient and frequency performance. The specification of the transient and frequency performance by performance indices is discussed in chapter 1. The essential difficulty here lies in choosing a target system which not only has the required transient and frequency performance, but also has a pole-zero excess
(i.e. asymptotic behaviour) corresponding to that of the plant (i.e. open loop constraint), as discussed in the Introduction.

It has been found possible, by using a particular form of description of the real part of the frequency response, to catalogue in advance a range of systems, of specified pole-zero excess, giving a spread of transient and frequency performances. (The link between the real part of the frequency response and the transient performance is discussed in chapter 2). The real part is divided into an asymptotic region and an intermediate region. The asymptotic region is described by an inverse power of frequency, the value of the power implying a particular pole-zero excess, and the intermediate region is described by a Fourier cosine series (since the real part is even). (chapter 4).

It is possible in this manner to approximate the real part of the frequency response of lumped linear systems to any degree of accuracy (chapter 4). Of much greater importance, it has been found that such a description, using only a few terms of the cosine series, corresponds to a transient performance very similar to that of physical systems (see section 7.2.1 example a). It is this which permits the large scale correlation of identifiable systems with their transient and frequency response. The details of the approximation and the means of evaluating the corresponding transient performances are treated in chapter 4.

Since for the practical realisation of a system either its complex frequency response or its rational transform is necessary, the means of obtaining these
quantities from a specification of the real part of the frequency response is discussed in chapter 5. After an exploration of the possibility of passing directly from the specification of the real part to the corresponding rational transform, it is concluded that a more fruitful approach is to obtain the complex frequency response first and then to approximate it by a rational transform. A computer algorithm for calculating the imaginary part from a specification of the real part via the Hilbert transform has been developed in chapter 5. The complex frequency response is then approximated by a rational transform, using one of the existing methods which is particularly suitable (chapter 5).

A set of performance curves has been produced from which a target system having a known transient and frequency performance, and a specified pole-zero excess, can be chosen. In order for such curves to be practicable, it must be possible to specify a system by as few parameters as possible. In chapter 6 the details of the specification of the intermediate region by the cosine series and the asymptotic region by the inverse power of frequency are discussed. Limitations imposed by both the transient and the frequency behaviour of lumped linear systems are explored, and a method of specifying the terms of the cosine series adopted. The details of the computation of the performance curves are considered, and the curves themselves are presented.

The possibility of specifying the velocity constant $K_v$, or (if applicable) the acceleration constant $K_a$, in terms of the harmonic content of the intermediate region was investigated. The conclusions reached are
that control over this aspect of the performance is not reliable and that better results are obtained by the conventional form of lag compensation.

A difficulty associated with the method is discussed in chapter 7. It lies in a basic ambiguity which exists in the specification of the asymptotic region of the real part of an inverse even power of frequency, since, in certain cases, any one value of this power can correspond to more than one pole-zero excess. In the practical application of the method this point is not troublesome, but from the theoretical point of view, this problem, of considerable mathematical difficulty, remains for further development.

Since, in the final analysis, the real test of the method lies in the results it produces, the remainder of chapter 7 and the whole of chapter 8 comprise of examples of its application. On the evidence of these examples it is concluded that the method provides an absolutely straightforward means of choosing target systems of specified pole-zero excess and having known transient and frequency performances, as characterised by the indices discussed in chapter 1. In handling systems with a pole-zero excess of 2-5, it removes the necessity of resorting to the 'dominant pole' approximation, and hence represents a considerable improvement in flexibility over existing methods.

The accuracy of the method was found to be consistently good, and in fact in many of the examples it was found to be exceptionally so.

Two final comments of interest can be made about
the method. The first is that since it uses the measured frequency response of the plant as one set of input data, it automatically guards against any attempt at achieving a performance beyond the physical capacity of the plant. For example, an attempt to make the plant respond very much faster than it inherently is capable of doing, immediately runs into the difficulty that in order to calculate the necessary compensation, information is required about the behaviour of the plant at frequencies at which the output is so small that it cannot be measured.

The second point concerns the correlation of the frequency response characterisation used and a pole-zero characterisation. In chapter 5, the pole zero patterns corresponding to a few asymptotically terminated harmonics are given, and are found to contain several poles and zeroes. In this lies the success of the present method; instead of manipulating individual poles and zeroes, and thereby getting bogged down in a mass of parameters, simplicity is achieved by manipulating whole sets of poles and zeroes, corresponding to the harmonics. But while reasoning via the frequency response shows that by the combination of the harmonics any lumped linear system may be approximated to any degree of accuracy, it would have been very difficult to arrive at the same conclusion regarding the combination of the pole-zero sets via s-plane reasoning.
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