THE SHOCK RESPONSE OF SUBMERGED MASTS

by

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Summary

A study of the response of submerged mast-like structures to shock loads has been carried out. An analytical model has been developed which uses a lumped parameter system whose equations of motion are solved by the use of finite difference time integration techniques. The effects of the stiffness and damping inherent in the supporting restraints are successfully included into the analysis by assuming that they provide linear rotational stiffness and a viscous type of damping.

An analysis of hydrodynamic interaction effects has been developed in order to investigate how the frequency response of the structure is affected by submergence. Changes in both the frequency and the damping characteristics are predicted as a function of the depth of submergence. Methods of incorporation of these hydrodynamic forces into the time domain solution of the lumped mass system are investigated and are found to be most accurate for slender structures.

In order to develop a fuller understanding of the hydrodynamic response, two experimental investigations have been carried out. The first is laboratory based and uses carefully controlled free vibration tests to excite a number of modal frequencies of a rigidly clamped vertical mast in both submerged and partially submerged conditions. Results show that the frequency changes are closely predicted by the hydrodynamic analysis developed in this thesis and also show that hydrodynamic damping is a linear function of amplitude of deflection for the first natural response mode over the mast's elastic range of response.

To investigate the shock response of submerged structures, a second test rig is used to subject a slender mast to a shock load radiating from an underwater explosion. The structural significance of the higher modal frequencies is evaluated through the use of Fourier Analysis and digital filtering techniques. The instrumentation performance including both damped and undamped transducers is assessed in order to make recommendations for future shock trials. The experimental investigation has shown that both the direct pressure loading and the movement of the supports are important factors in the mast response and that their combination produces important high frequency response modes.
Acknowledgements

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**Principal Notation**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>a</td>
<td>Structural dimension</td>
</tr>
<tr>
<td>A</td>
<td>Projected area of structure</td>
</tr>
<tr>
<td>$A_y$</td>
<td>Amplitude of vibration</td>
</tr>
<tr>
<td>C</td>
<td>Velocity of sound</td>
</tr>
<tr>
<td>$C_d$</td>
<td>Drag coefficient</td>
</tr>
<tr>
<td>$C_m$</td>
<td>Inertia coefficient</td>
</tr>
<tr>
<td>D</td>
<td>Diameter of mast</td>
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<tr>
<td>E</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>$E_d$</td>
<td>Damping energy</td>
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<tr>
<td>$E_f$</td>
<td>Energy of charge</td>
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<tr>
<td>$E_T$</td>
<td>Total energy</td>
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<tr>
<td>$f_{ij}$</td>
<td>Flexibility coefficient</td>
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<tr>
<td>$F_d$</td>
<td>Drag Force</td>
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<tr>
<td>$F_{el}$</td>
<td>Elastic Restoring Force</td>
</tr>
<tr>
<td>g</td>
<td>Acceleration due to gravity</td>
</tr>
<tr>
<td>G</td>
<td>Modulus of Rigidity</td>
</tr>
<tr>
<td>H</td>
<td>Height of water</td>
</tr>
<tr>
<td>$H_s$</td>
<td>Height of structure</td>
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<tr>
<td>I</td>
<td>Second moment of area</td>
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<tr>
<td>K</td>
<td>Stiffness coefficient</td>
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<tr>
<td>L</td>
<td>Structural length</td>
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<tr>
<td>m</td>
<td>Distributed mass</td>
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<tr>
<td>$m_a$</td>
<td>Added mass</td>
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<tr>
<td>M</td>
<td>Nodal mass</td>
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<tr>
<td>$M_1$, $M_2$</td>
<td>Moments</td>
</tr>
<tr>
<td>$M_1^<em>$, $M_2^</em>$</td>
<td>Support Moments</td>
</tr>
<tr>
<td>N</td>
<td>Number of nodes in a structure</td>
</tr>
<tr>
<td>$P_m$</td>
<td>Peak Shock pressure</td>
</tr>
<tr>
<td>$P_c$</td>
<td>Incident pressure</td>
</tr>
<tr>
<td>$P_r$</td>
<td>Reflected pressure</td>
</tr>
<tr>
<td>r</td>
<td>Radius of mast</td>
</tr>
<tr>
<td>R</td>
<td>Charge stand-off distance</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>$\delta t$</td>
<td>Time increment</td>
</tr>
</tbody>
</table>
T – Time constant of charge
u – Deflection of the structure
U – Velocity of the structure
x – Structural dimension
z – Structural dimension

α
β
γ
}
Explosive charge constants

Δ – Logarithmic decrement
ε – Strain
ζ – Damping coefficient
θ – Angular rotation
ν – Kinematic viscosity
ρ – Density
σ – Stress
φ(z) – Mode shape function
ω_N – Natural frequency of structure
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1. Introduction

Structures which are submerged in an offshore environment are often subjected to extreme loading conditions during the course of service. In some cases, the loading can be considered to be time invariant, thus allowing the structural design to be determined by static stress analysis. An example of such a case is the hydrostatic pressure loading associated with large depths of submergence. However, such simple loading conditions rarely occur alone and the loads on submerged structures are more often time dependent, making the design procedure far more complex and costly. From an analytical standpoint, it is convenient to divide these time dependent loads into two basic categories; periodic and random. In the offshore environment periodic loading may either arise as a result of fluctuations in the flowfield such as occur during oscillatory wave motion or may be generated by the movement of a structure within the surrounding medium creating fluid damping forces. A partially submerged structure can also receive random loadings such as wave slamming in stormy conditions, impact loads due to contact with other submerged structures and possibly shock loads from nearby explosions. The analysis of a structure under these types of loading requires the application of the principles of dynamic structural analysis. A structural dynamics problem of this form differs from its static-loading counterpart in two important respects. Firstly, because the input to the system is constantly changing, there is no single solution to the equation of motion and a succession of solutions is required during the response history. Secondly, the structural inertia force terms must be included in the force equilibrium equation that forms the basis of the dynamic analysis.

This present research has arisen from the need to gain a better understanding of the dynamic characteristics of slender mast structures under extreme explosive shock conditions. Such masts are a small but crucial structural component in sub-sea vehicles since they carry vital communication equipment. The mast can have a major role in the operation of a submarine vehicle, since, if damage occurs it may jeopardise the operational capability of the vessel. Consequently, it is necessary to develop a logical approach to the method of design of such a component. Furthermore, it is possible, due to improvements in the materials
technology such as the development of fibre reinforced composites, that future masts may no longer be produced in materials such as stainless steel. Composite materials are far more efficient in terms of their higher strength/weight ratio and can, if properly designed, give good shock resistance as well as beneficial radar silence. However, before such potential improvements can be implemented, a more comprehensive understanding of the dynamic characteristics of such a structural system is required. In the past, analyses of this type of structure have been carried out without adequate consideration of the implications of submergence. It is now apparent that such an analysis is not sufficiently accurate because both the loading characteristics and the structural response characteristics are dramatically changed by submergence. The shock pressure pulse radiating from an exploding charge tends to be far more effective than that of an equivalent charge in air. The shock front tends to be far steeper and the subsequent decay is more prolonged because of the effects of the secondary pressure pulse which is related to the oscillation of the gas bubble created at the charge centre. The response frequencies of the structure are also changed by submergence. The effect of the surrounding fluid imposes pressure forces on the moving structure which are in phase with the acceleration of the body. This force can be characterised in the equation of motion as an increased structural mass by including an 'added mass' distribution. Therefore, the effect of submergence is to decrease the natural frequency of the structure. Submergence has a second effect of changing the damping characteristics of the structure. The fluid forces acting on the body also have a component which is in phase with the structural velocity and this tends to damp the response of the structure. This damping phenomena is highly complex and can usually only be modelled by use of empirical relationships. However, the major problem in the design of submerged mast-like structures lies in understanding how the energy is transferred between an imposed shock wave and the structural surface. The presence of the structure changes the characteristics of the pulse due to the effect of surface reflections and must be included in the forcing function. Slender masts, because of their flexibility, move under the action of the pulse and this results in a reduction of the effectiveness of the shock load. Flow of the shock front around the structure also reduces the effective pressure acting on the projected area. Modelling of these phenomena on an analytical basis is
difficult and therefore requires further information from practical studies. It is also known that the frequencies associated with the pressure pulse occur in a range far wider than the normal mast response modes and the resulting deformation profiles are not easily predicted.

As a result of the excitation of higher modes, the way in which the structure is supported can, in some cases, become very important. The type of restraint/support provided for a slender mast is highly dependent on both its size and the type of sub-sea vehicle to which it will be fixed. Small masts tend to be rigidly fixed so that their mounting can be modelled as a cantilever support. However, taller masts may, because of their size, need to be made of a telescopic construction so that they are only fully extended when required. In such cases, the support conditions are very difficult to model correctly as they generally consist of two supporting collars along which the mast can slide before being locked into position. As a first approximation, these supports can be considered to be a pin joint type of restraint. However, in practice, this is not a particularly accurate model because a joint of this type must necessarily provide an element of local stiffness. The effect of this stiffness is to increase the modal frequencies of the beam to a value somewhere between the pin joint and the rigid cantilever approximations. The distance between the support also influences the frequency response of the mast since it generates additional frequencies whose wavelengths are fractional values of this distance. The frequencies associated with these modes are far higher than the normal modes of the unsupported length of the mast but can become significant in loading cases involving explosive shocks with broad band energy input. In addition to the frequency changes provided by the increase in support stiffness, it is also possible that the level of damping due to the energy losses at the supporting interfaces may highly influence the structure's dynamic response.

This research sets out to develop an analytical model which can be used to predict the shock response of both partially and fully submerged slender masts more accurately. Investigations into the importance of structural modifications such as changes in the support stiffness and changes in the level of damping due to the support restraints are outlined in the following chapters. The need to model hydrodynamic interaction effects more accurately has required the development of further analytical studies. The concept of added mass and fluid damping are re-evaluated by
means of a mode-superposition technique and the results are then used to improve the structural modelling technique used for the shock loaded mast. It is common for Engineers to use simplifying methods to model hydrodynamic effects. The accuracy of using an added mass distribution equal to the mass of water displaced by the structure and also the use of equivalent damping coefficients are fully examined.

It is also important that improvements in the analytical techniques are validated by experimental tests results. Because of the expense of carrying out full scale prototype shock trials, it is desirable to use scale model tests to investigate shock response. Fortunately, similarity principles do hold for scale models and hence the results from these tests can be used to predict full scale responses. To validate the improved analytical techniques, two experimental studies have been undertaken. A test rig has been designed specifically for the purpose of investigating the effects of hydrodynamic interaction on a mast which vibrates in a controlled manner for both fully submerged and partially submerged conditions. Several series of tests have been performed to measure the frequency and damping characteristics of the masts for several modes of response. The results are compared with those predicted from the analytical models developed in this thesis. A second test rig has been designed to investigate the response of a similar mast like structure to a shock load from an explosion. Results from two of the severest shock tests carried out are described in some detail. The importance of the high modes of response is analysed to gain a greater understanding of the transfer of energy between the shock pressure pulse and the surface of a flexible structure.

As a result of the shock trial, it has become necessary to re-evaluate the selection of transducers for the measurement of shock response. The broad energy spectrum associated with the shock pressure pulse creates major operational difficulties for transducers such as accelerometers. These transducers rely on internal moving components for their measurement and therefore have natural frequencies which when excited, add extra information to the signal which is not representative of structural motion. Methods which employ digital filtering techniques can be used to reject transducer resonance information. However, this obviously increases the time taken to analyse the structural motion and the suitability of using damped transducers in place of their undamped versions.
is therefore evaluated. Using the comparison of transducer performance, recommendations are made for instrumentation selection in further shock response trials.
2. Modelling of Structural Forms Under Dynamic Loads

2.1 Selection of the Analytical Technique

The specific structural forms of major interest to the present research are those of tall slender submerged cantilevered structures. The types of dynamic loading imposed on such structures may be wide ranging and may include forced excitation due to wave loading or earthquake response. Special attention is given to the case of shock loading which may be considered to be transmitted to the structure either via its structural support configuration or alternatively as a forcing function applied directly to its surface area. It is also envisaged that responses may be required where more than one type of loading is considered to act simultaneously. An example of a typical situation where this might occur is that of a structure subjected to explosive shock. In such a case it is common for the structure to see shock loads appearing firstly as a base displacement due to the action of the shock on the supporting structure, followed, after a short time delay, by a direct pressure load to the structure's surface.

Selection of the most suitable type of analysis to model the structural response depends on many factors. In particular, the type of structural system, the required accuracy of prediction, the time of computation and the size of available computer are all important considerations. The major constraint on the present analysis is that it should be carried out on a Hewlett Packard 9836 'desktop' mini-computer. It is also felt that the final analysis should be used as a design tool with which the practising Engineer can analyse several specific structural modifications during a working day. Hence, computation time was seen to be of greater importance than a high precision of accuracy. The need for a high precision of accuracy can only be justified by an exact knowledge of both structural properties and input loading. However, as is often the case in structural design, the final performance of the structure is improved by the inclusion of design factors which are considered to maintain safety in the event of unforeseen circumstances. On this basis it is therefore unnecessary to provide a high precision of accuracy.
Further study of the problem suggested that analyses such as large finite element packages were more suited to mainframe machines and that although some packages were readily available, they were of a too generalised nature to be of use to the present investigation. Therefore, it was considered necessary to develop special structural analysis software which would suit the foregoing requirements and could also model the two specific problem areas tackled by this research - namely the transient response of shock loaded structures and the hydrodynamic interaction phenomena present in submerged structures. At this stage there were no constraints that these two analytical studies had to be compatible; however, it was envisaged that a final analysis should be carried out by one software package.

Solutions to the equations of motion governing structural response can be obtained using one of four basic methods; direct step by step integration, analysis in the frequency domain, direct mode superposition analysis and response spectrum analysis. The suitability of each method is dependent upon the type of dynamic system under analysis. Both the mode superposition analysis and the spectral analysis involve the determination of the undamped mode shapes and their associated frequencies. The method of step by step integration reduces the dynamic equation by means of integration formulae such as the finite difference approximations. Finally, the frequency domain method solves the equation of motion in the frequency domain after reduction into a series of linear sets of complex equations.

Reviewing literature on both types of structural system, it became obvious that the most suitable methods for each were not compatible. The structural response to shock loads is most successfully modelled using finite difference time integration methods whereas hydrodynamic interaction is usually modelled using modal superposition techniques. In both cases, the available analysis would need development to suit the present requirements. It was decided that the final analysis of the complete structure would be most usefully carried out using the finite difference approach. The improvement in knowledge of hydrodynamic interaction from the modal technique would be subsequently used to develop hydrodynamic interaction in a finite difference scheme. The details of the use of analytical techniques to model the fluid interaction are presented later in chapter 3.
2.2 Development of the Analysis

In all real structures, dynamic loads can be broken down into component forces acting in three mutually orthogonal planes and the response of the structure can be separated in the same way. In the particular case under investigation the structure is assumed to have no vertically imposed loads and hence its response to transverse loads can be considered to be due only to flexural motions of the structure. Considering the displacement of the structure to be small, the induced vertical motion of the tower can be neglected. In most normal cases the tower-like structure is considered to be axisymmetric with the loads acting normal to the longitudinal axis of symmetry. It is therefore possible to model the structure in one dimension, considering it to be a beam with transverse loads. This method of dynamic response has been carried out by Taylor\(^{(1)}\) who was interested in the dynamic response of slender structures in air. It is the intention of this work to further develop Taylor's finite difference model to incorporate structural support stiffness. In the particular case of submerged slender structures the support conditions tend to be held in stiff flexible mountings which greatly affect the dynamic model.

The motion of a uniform beam with transverse loads is governed by the differential equation.

\[
\frac{EI}{kG} \frac{\partial^4 u}{\partial x^4} - \frac{EI}{G} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 u}{\partial t^2} = 0 \tag{2.1}
\]

where

- \(EI\) is the bending rigidity of the beam
- \(A\) is the cross sectional area
- \(G\) is the modulus of rigidity
- \(\rho\) is the density of the beam material
- \(u\) is the lateral deflection of the beam

and \(k\) is a constant dependent on the cross-sectional shape of the beam which allows for errors in the assumption that plane sections remain plane in simple bending theory.
For thin beams, as is the case for slender structures the effects of shear deformations can be neglected. The deflections due to flexure alone are then given by the simpler differential equation:

\[
\frac{EI}{\alpha^4} \frac{\partial^4 u}{\partial x^4} + \rho A \frac{\partial^2 u}{\partial t^2} = 0
\]  

(2.2)

However, most real structures which may consist of beams of irregular shape with varying cross-section or hollow beams with concentrated masses cannot be adequately represented by the relatively simple systems outlined above. One technique used to simplify the analysis of such structures is to split the complex structure into a subsystem with simple elastic and dynamic properties. This is achieved by creating a model which consists of a series of point 'lumped' masses which are connected up by a series of weightless beams/springs having the same characteristic stiffness as the structure. There is no limit to the number of masses that may be used to represent the structure. If the masses, like the beam, are considered to have motion only in the transverse direction, then the number of degrees of freedom is equal to the number of masses. The beam can therefore be represented by the model shown.

For each degree of freedom, and hence motion of each mass, a differential equation can be written

\[
m_j \ddot{u}_j = P_j
\]  

(2.3)

where, for free vibration, \( P_j \) is the elastic restoring force acting on the mass \( m_j \).

The spring stiffness of the structure with \( N \) degrees of freedom can be defined by a set of \( N^2 \) stiffness coefficients. The stiffness coefficient \( K_{jk} \) is defined as the change in spring force acting on the \( j^{th} \) mass when
the \( k \)th mass is displaced by a unit amount. Alternatively, for beam analysis, the concept of flexibility coefficients could be used, where the flexibility is equal to the reciprocal of the stiffness of the structure.

The total elastic force acting on the \( j \)th mass is the sum of the effects of the displacements of all the masses.

\[
F_{el} = - \sum_{k=1}^{N} K_{jk} u_k \tag{2.4}
\]

Combining equations (2.3) and (2.4) and adding the external forcing functions \( F_j \) at each mass, the dynamic equation becomes

\[
m_j \ddot{u}_j + \sum_{k=1}^{N} K_{jk} u_k = F_j \tag{2.5}
\]

Equation (2.5) represents \( N \) equations which can be replaced by a matrix equation of the form,

\[
[M] \{\ddot{u}\} + [K] \{u\} = \{F(t)\} \tag{2.6}
\]

where \([M]\) is a diagonal matrix containing the 'lumped' masses, \([K]\) is the stiffness matrix and \( \{F(t)\} \) is the vector of the externally applied forcing functions. Vectors \( \{\ddot{u}\} \) and \( \{u\} \) represent the nodal accelerations and displacements respectively. The order of the matrices is equal to the number of degrees of freedom of the structural system.

A solution to the matrix equation (2.6) can be obtained by replacing the differential by a corresponding finite difference equation obtained by truncation of the Taylor's series expansion. This then reduces the problem to a set of \( N \) simultaneous algebraic equations which can be solved by the computer.

If the relationship between the displacement and time is known for a series of time steps \( \delta t \) apart, then the velocity and acceleration at a point \( t \) can be obtained from the displacement at points \( \delta t \) before and after time \( t \).
The velocity,
\[
\frac{du}{dt} = \frac{1}{\delta t} \left( u_{t+\delta t} - u_t \right) \quad (2.7)
\]

and the acceleration
\[
\frac{d^2u}{dt^2} = \frac{1}{\delta t^2} \left( u_{t+\delta t} - 2u_t + u_{t-\delta t} \right) \quad (2.8)
\]

Substitution of equation (2.8) into (2.6) gives the solvable equation
\[
\{u\}_{t+\delta t} = 2\{u\}_t - \{u\}_{t-\delta t} + \delta t^2 [M]^{-1} \left( \{F(t)\}_t - [K] \{u\}_t \right) \quad (2.9)
\]

The form of equation (2.9) is particularly suited to the study of structural responses where the time dependant forcing function \(\{F(t)\}\) is complex. Knowledge of the structural characteristics in terms of the mass and stiffness matrix allows a value of the nodal displacement to be found in a step wise manner. At each time step, the forcing function must be re-evaluated to allow the prediction of the displacement at the end of the following time step to be calculated. However, the stability of the analysis is dependent on the size of time step. In a similar way to sampling criteria the Nyquist criteria must be satisfied by the time step. Hence, the time step frequency should be at least twice the highest frequency of any of the components of equation (2.9). In essence the requirement is that the time step must be at least twice the highest natural frequency of the structure or the highest frequency associated with the forcing function. The highest structural frequency can be obtained from knowledge that this is associated with the number of degrees of freedom and hence the number of masses in the model. An approximate value of frequency can then be obtained by making simplifying assumptions about the structural shape. In order to determine the highest frequency associated with the forcing function, the frequency characteristics may be obtained by use of a Fourier transform on the time history signal. However, in practice the effect of selecting a time step below the critical value results in a response which rapidly becomes unstable and can be easily recognised. In shock work, the types of force inputs tend to have a very broad band of frequency spread across the energy spectrum and this can have a limiting effect on the time step and hence the computation time.
2.3 Determination of the Stiffness Matrix

In Taylor's work (1), the stiffness matrix was obtained by assembling the flexibility coefficients into a flexibility matrix and carrying out a matrix inversion. The technique used to determine the coefficients was the strain energy analysis of a beam under multiple transverse loading. It is intended that this present analysis should adopt the same technique and should have the additional complication of including a support stiffness term.

The flexibility coefficient $f_{ij}$ is defined as the coefficient which represents the displacement at a nodal position $i$ due to the application of a unit force at a nodal position $j$. The general representation of the loaded beam is shown below.

![Diagram of a beam showing forces and reactions](image)

where a series of forces $F_n$ act on nodes $n$ and create opposing reaction forces $R_1$ and $R_2$ and two reactive moments $M_1$ and $M_2$ at the supports $A$ and $B$.

The strain energy due to flexural deformations of the beam, neglecting transverse shear effects, is given by

$$U = \int_0^L \frac{M^2}{2EI} \, dx$$

(2.10)

The equation can be broken down to represent the strain energy for each section of the beam

$$U = \int_{x_{n+1}}^{x_{n+2}} \frac{M_{n+1,n+2}^2}{2EI_{n+1}} \, dx + \sum_{q=1}^{n} \int_{x_q}^{x_{q+1}} \frac{M_{q,q+1}^2}{2EI_q} \, dx$$

(2.11)

where $M_{i,i+1}$ is the moment acting between nodes $i$ and $i+1$. The
deflection $\Delta_i$ of node $i$ under the action of a force $F_i$ is given by Castigliano's theorem,

$$\Delta_i = \frac{\partial U}{\partial F_i}$$  \hspace{1cm} (2.12)

The flexibility coefficient $f_{i,j}$ can be found from the application of equations (2.11) and (2.12). $f_{i} = \Delta_i$ when $F_j$ is unity and all other forces are set to zero. The resulting expression, valid for $j \leq i$ is

$$f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{n+1}I_{n+1}} \left\{ 2 \left( x_{q+1}^3 - x_q^3 \right) - 3 \left( x_{q+1}^2 - x_q^2 \right) + 6x_{q+1}x_q \right\}$$

$$+ \frac{1}{3E_{n+1}I_{n+1}} \left( x_{i,n+1} - x_{i,j} \right) \left( x_{j,n+1} - x_{j} \right) \left( x_{n+2,n+1} - x_{n+1} \right)$$

$$- \frac{1}{6E_{n+1}I_{n+1}} \left( x_{i,n+1} - x_{i} \right) \left( x_{n+2,n+1} - x_{n+1} \right) \left( 2M_{1*} - M_{2*} \right)$$  \hspace{1cm} (2.13)

where $M_{1*}$ and $M_{2*}$ are the support moments created by the application of the unit load at the point $j$ and as such have units only of length. $M_{1*}$ and $M_{2*}$ are also found by use of strain energy methods together with Castigliano's theorem applied to the rotation of the beam and are given as

$$M_{1*} = C_1 (x_{n+1} - x_j) \quad M_{2*} = C_2 (x_{n+1} - x_j)$$  \hspace{1cm} (2.14)

and coefficients $C_1$ and $C_2$ depend on the physical properties of the beam,

$$C_1 = \frac{- \left[ 1 - \frac{4E_{n+1}I_{n+1}}{(x_{n+2,n+1}) K} \right]}{1 - \frac{2E_{n+1}I_{n+1}}{(x_{n+2,n+1}) K}} \left[ 1 - \frac{6E_{n+1}I_{n+1}}{(x_{n+2,n+1}) K} \right]$$  \hspace{1cm} (2.15)
The full exposition of these calculations are given in Appendix A. The additional factor now required by this analysis is the effective support stiffness, \( K \), for which it has been assumed that the \( M/\theta \) relation is linear. In cases where the support is a simple diaphragm the stiffness can be calculated using small deflection plate theory. However, it is envisaged that empirical values may be required for complex supporting conditions.

From equations (2.14), (2.15) and (2.16) a relationship between the two supporting moments \( M_1^* \) and \( M_2^* \) can be obtained

\[
\frac{M_1^*}{M_2^*} = \left[ \frac{(x_{n+2} - x_{n+1}) + K}{2E_{n+1}I_n+1} \right] - 2
\]  

(2.17)

Two extreme cases, that of a cantilever beam and that of a beam held in two pin joints can be considered. In the case of the cantilever beam, the support stiffness becomes infinite and hence

\[
M_1^* = -(x_{n+1} - x_j)
\]

\[
M_2^* = 0
\]

hence the flexibility coefficients become

\[
f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{q}I_{q}} \left\{ 2(x_{q+1}^3 - x_q^3) - 3(x_{q+1}x_q) - 2x_{q+1}x_q + 6x_{q}x_{q+1} - x_{q+1} \right\}
\]  

(2.18)

In the case of the pin joint support, the support stiffness is zero and hence

\[
M_1^* = 0
\]

\[
M_2^* = 0
\]
hence the flexibility coefficients become

\[ f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{q} I_{q}} \left\{ 2(x_{q+1}^{3} - x_{q}^{3}) - 3(x_{i} + x_{j})(x_{q+1}^{2} - x_{q}^{2}) + 6x_{i}x_{j}(x_{q+1} - x_{q}) \right\} \]

\[ \frac{1}{3E_{n+1} I_{n+1}} (x_{i} - x_{n+1})(x_{j} - x_{n+1})(x_{n+2} - x_{n+1}) \]

(2.19)

which is exactly the same equation as determined by Taylor for his pin joint supported structure.
2.4 Determination of the Mass Matrix

Since the concept of lumped mass models was first introduced by Myklestad (2) there has been some dispute as to the most accurate method of subdividing the structure. It appears that the best methods used may be dependent upon the type of structure under analysis. However, due to the nature of the lumped mass technique, the calculated frequencies will always be subject to a small error. Two pioneers in the field of structural modelling were Rayleigh (3) and Duncan (4) and each have suggested different discretisation methods. Taylor (1) has discussed both methods of creating lumped mass models and found that for his purposes, a refined model called a symmetric model produced more accurate results. These three types of model are shown in figure 2.1.

In order to compare the three methods, a 10 node representation for each was made of a simple structure of constant section whose exact natural frequencies could be determined by calculation. This simple structural model will be used as an example throughout this report and hence will be referred to as 'Structure A' whose dimensions and material properties are defined in figure 2.2. The results for 'Structure A' showed that the frequency predictions in all cases were much lower than the theoretical solution using D'Alembert's principle (Ref 5) and that the closest prediction was made by the Rayleigh method. For this reason alone, the Rayleigh method was selected and is now used throughout the subsequent analysis.

It is known that as the number of 'lumped' masses is increased the accuracy of prediction is improved and that the predicted frequency approaches the theoretical value asymptotically. In order to obtain an idea of how many nodes are required to accurately determine the true frequency, a number of predictions have been made with an increasing number of nodes. The first natural frequency has been determined from an average of the time period over four cycles of the response. To obtain a greater understanding of how the determined frequency approaches the theoretical value, a graph of frequency versus the reciprocal of the number of nodes has been plotted. The theoretical value of frequency should be reached at an infinite number of nodes (represented as zero on the horizontal scale). The two extreme support cases described earlier, that of a cantilever and a pin joint mounting, have been investigated in this study. The results are shown in figure 2.3. The frequency results
Note: to improve the accuracy of the model and to allow for 'fixed bearing' modes, lumped masses must be included in the region between the supports.

Figure 2.1 Methods of Lumped Mass Structural Idealisation
Young's Modulus = 210 MN/m²
Density = 8000 Kg/m³
Outside Diameter = 30 mm
Inside Diameter = 26 mm

Figure 2.2 Geometric and Material Properties for 'Structure A'
Figure 2.3 Influence of Increasing the Number of Masses in the Structural Idealisation on the First Natural Frequency of Response

Note: translational stiffness of the support is assumed to be infinite.
appear to vary linearly with the reciprocal of the number of nodes and this is confirmed by a least squares fit through the data which has a correlation coefficient better than 0.999 for both graphs. This result agrees with the work of Duncan\(^4\). The predicted frequencies at the infinite condition are shown in table 2.1 to be within 0.8% of the D'Alembert exact beam analysis. It can be concluded from this study that the number of nodes required for an acceptably accurate response is 10 which produces a frequency error in the region of -6%.

Using this same technique, the effect of support stiffness can be analysed and the results can be used later for comparison in the practical investigation. The effect of support stiffness is also shown in figure 2.3.
Table 2.1 Values of the First Natural Frequency Predictions for 'Structure A' for Varying Levels of Support Stiffness

<table>
<thead>
<tr>
<th>Number of Nodes</th>
<th>FREQUENCY (Hz)</th>
<th>Rotational Support Stiffness (KNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>3600</td>
</tr>
</tbody>
</table>


Exact freq. | 10.675 | —— | —— | —— | —— | 12.125

% Error | +0.43 | —— | —— | —— | —— | +0.73
2.5 **Accuracy of the Model**

Some aspects of the accuracy of the 'lumped' mass modelling technique have already been discussed in detail in the previous section. However, this analysis has been used to model a uniform structure which is 'worst case' from the point of view of accuracy. It has been the intention throughout this study to make the model versatile thus enabling it to represent more complex structures which may have concentrated masses or variations in cross-sectional properties. In these cases, the lumped masses would be placed in more strategic positions and would improve the accuracy of the model. Taylor (1) has shown this to be true in his modelling of experimental structural forms with externally mounted concentrated masses.

However, a major consideration when choosing the number of nodes in the representation is the time requirement of the analysis. The number of equations requiring solution at each time step is equal to the number of nodes and hence the time taken for analysis increases dramatically with the number of nodes. An additional effect of increasing the number of nodes is to increase the possible frequencies of response and reduce the required stable time step. The resulting analysis time will be proportional to \((\text{number of nodes})^n\) where \(n\) is greater than 1. To some extent, there has to be a compromise between the time of processing and the accuracy of the predicted response.
2.6 Damping Phenomena for Structures in Air

The analytical model considered in the previous sections has been subjected only to forces which are due either to an external source such as pressure loads from an explosive charge or internal restoring forces due to the elasticity of the structure. In such cases, once the external load has been removed the motion of the structure would continue to oscillate in a repetitive manner without decay. However, in all real structures this will never occur because some of the kinetic energy associated with the structural motion will be 'lost' during the oscillation. This loss of energy is termed damping and can be related to a number of factors.

A considerable amount of research effort has been directed towards understanding how damping operates and a number of theories have been published. In many cases, the damping relationship is extremely complex and researchers have been forced to use empirical results to model structural response. The book by Lazan (6) is one of the most comprehensive studies of damping phenomena and he has shown that in the case of structures in air, there are two important damping mechanisms. The first is that of material damping which, as the name suggests, occurs internal to the structure on a molecular level and is common to all structures as it is a function of the structure's material properties. This type of damping is largely considered to be a function of stress amplitude. Although damping energy-stress relationships are normally complex, for most materials an approximate relationship can be expressed as

\[ D = J \sigma_a^n \]

where \( \sigma_a \) is the amplitude of alternating stress and \( J \) and \( n \) are material constants.

- \( D \) = energy loss/cycle per unit cube of material
- \( J \) = damping constant
- \( n \) = damping exponent

The damping exponent \( n \) is largely dependent on the stress amplitude and in the case of steel has three noticeable relationships. This three segment approximation is widely accepted and its significance is illustrated in figure 2.4.

In the low region (\( \sigma_a < 300 \) psi) the relationship between damping energy and stress amplitude is virtually linear i.e. the damping exponent \( n \)
Figure 2.4 The Three Segment Damping Relationship for Mild Steel Due to Material Damping
is constant with a value of 2 and hence the damping energy increases quadratically with stress.

In the central region \((300 \text{ psi} < \sigma_a < 29000 \text{ psi})\) the slope increases and hence \(n\) is greater than 2.

In the high region \((\sigma_a < 29000 \text{ psi})\) there is an abrupt change in the damping characteristic and \(n\) is now greatly dependent on the number of cycles of fatigue.

Conversion of the central region from a stress amplitude to a strain amplitude reveals the range to be

\[10.7 \mu\text{strain} < \varepsilon_a < 1036 \mu\text{strain}\]

and hence it can be concluded that for most normal cases of structural response the central range is of greatest importance.

The second and probably the most important damping mechanism for structures in air is energy loss at supporting interfaces. However, due to its nature, the relationship for this type of damping is highly dependent on the type of connection used. In the case of a supporting structure consisting of a bolted joint, the damping relationship can also be shown to be highly dependent on the joint interface pressure (Ref. 1). The only viable way of including damping of this nature into an analytical technique is to find the relationship between the energy loss and the structural response parameters from experiment. An analytical model can then be formulated, although the coefficients can only be confirmed by further experiment. In the work by Taylor (1), the structural supporting conditions basically consisted of a pair of self aligning roller bearings set into rigid support housings. It was found that the energy loss was a function of the rotational velocity of the supports. A model was derived by use of strain energy methods which placed viscous dashpots at the support positions so that they would produce resistive moments proportional to the angular velocity. This type of model has no dependence on amplitude of strain or displacement and hence the effective damping coefficient can be considered to be constant. In practice the damping coefficient was shown to have a value which had little dependence upon amplitude of displacement.

In this research, the type of support configuration has been changed to a diaphragm type support (see Chapter 6) which is considered to be more typical of practical supports. It is expected that this type of structure
will also be subjected to base dissipation moments which can be related to the rotation of the support. In order to include these effects, the strain energy method used by Taylor must be developed further to include rotational stiffness of the supports because this strongly affects the beam's angular rotation in the support area.

It has been shown in Appendix A that the rotation of the beam in the two support positions are:

\[
\theta_1 = -\frac{(x_{n+2} - x_{n+1})}{3E_{n+1}I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - \frac{M_2}{2} \right]
\]

\[
\theta_2 = -\frac{(x_{n+2} - x_{n+1})}{3E_{n+1}I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - 2M_2 \right]
\]

Using the results from the strain energy analysis, the displacement of the beam due to the moment at the supports can be determined from equation (2.13). It is actually the difference between the cases for the pin jointed structure and the cantilever beam, hence

\[
v_i = \frac{-3}{6E_{n+1}I_{n+1}} (x_i - x_{i+1})(x_{n+2} - x_{n+1})(2M_1 - M_2)
\]  

(2.20)

It is proposed to place viscous dampers opposing the rotation of the supports at these positions and hence

\[
M_1 = -c_1 \dot{\theta}_{n+1} \quad \text{and} \quad M_2 = -c_2 \dot{\theta}_{n+2}
\]

(2.21)

where \(c_1\) and \(c_2\) are the viscous damping coefficients.

It has been shown that the relationship between the rotations at the supports is given as (equation 2.17).

\[
\frac{\theta_{n+1}}{\theta_{n+2}} = \left[ \frac{(x_{n+2} - x_{n+1}) K}{2E_{n+1}I_{n+1}} - 2 \right]
\]

(2.22)

Combining equations (2.20), (2.21) and (2.22) gives

\[
v_i = -\left( \frac{2c_1}{c_1} - \frac{c_2}{c_2} \right) \frac{(x_{n+2} - x_{n+1})^2(x_{n+1} - x_i)^2}{18E_{n+1}^2I_{n+1}^2} \left[ M_{n+1} + \frac{M_{n+2}}{2} \right]
\]

(2.23)
where $M_{n+1}$ and $M_{n+2}$ are the moments acting on the beam at points $n+1$ and $n+2$.

These coefficients $V_i$ can now be assembled into a "base dissipation" vector $\{b\}$ which can then be added to the dynamic equation (2.6) to give

$$[M]\{\ddot{u}\} + [K]\{u\} - [K]\{b\} (\dot{M}) = \{F(t)\} \quad (2.24)$$

where $\dot{M}$ represents the change in base moments $(\dot{M}_{n+1} - \dot{M}_{n+2}/2)$

Using the finite difference equations then,

$$\{u\}_{t+\delta t} = 2\{u\}_t - \{u\}_{t-\delta t} + \delta t^2[M]^{-1} \left\{ \{F(t)\}_t - [K]\{b\} \left( \{M_{n+1} + \frac{1}{2} M_{n+2}\}_{t+\delta t} - \{M_{n+1} + \frac{1}{2} M_{n+2}\}_{t-\delta t} \right) \right\} \quad (2.25)$$

The effect of damping due to viscous forces proportional to the rotation of the base restraints with the inclusion of the base stiffness can now be incorporated into the dynamic analysis. The basic flow chart which shows the order of the calculation steps is given in figure 2.5.

The validity of equation (2.23) is shown by considering the extreme cases of base stiffness. When the stiffness is zero, i.e., when the beam is held in a spherical bearing type support, the equation reduces to the one previously used by Taylor.

$$V_i = \frac{- (4C_1 + C_2) \left( x_{n+2} - x_{n+1} \right)^2 \left( x_{n+1} - x_i \right) \dot{M}_{n+1}}{36E^2 I^2_{n+1}} \quad (2.26)$$

As the stiffness of the supports are increased the effect of damping is controlled more by the damping of the support at position $n+1$. At the extreme condition of infinite stiffness the equation would reduce to

$$V_i = \frac{- C_1 \left( x_{n+2} - x_{n+1} \right)^2 \left( x_{n+1} - x_i \right) \dot{M}_{n+1}}{9E^2 I^2_{n+1}} \quad (2.27)$$

This equation allows the inclusion of a damping force proportional to the base moment in the analysis of a cantilever structure.

Using equation (2.25) it is possible to show how the support stiffness affects the damped structural response. In order to understand the true
Figure 2.5 Basic Flow Chart for the Finite Difference Structural Analysis Program
effect of support stiffness, it is far clearer to specify the log decrement of the decay envelope for the mast tip. The response has been computed for a number of support stiffness values using a 5 node lumped mass model of 'Structure A' as defined in figure 2.2.

Figure 2.6 shows how the log decrement changes with the damping coefficients $C_{c_1}$ and $C_{c_2}$ which are assumed equal. It can be seen that for the case of a pin jointed beam the relationship is linear. Inclusion of rotational support stiffness has the effect of reducing the effective log decrement of the output. However, the relationship is no longer linear because the additional stiffness affects both the damping coefficient and the moments acting at the supporting positions. Using figure 2.6, it is now possible to obtain the correct values of the damping coefficients from an experimentally determined log decrement value with a pre-knowledge of the base stiffness value.
Figure 2.6 Variation in the Resulting Damped Response of 'Structure A' Due to Changes in the Support Stiffness
3. **Hydrodynamic Interaction Effects**

3.1 **Background**

Investigation of fluid-structure interaction problems can be traced back as far as Lamb\(^{(7)}\) whose original work in the early 1900's is still referenced by many researchers. Present knowledge of the fluid interaction phenomena, however, can be categorised into four distinct areas (ref 8).

1. Static deformations achieved by steady fluid forces such as drag forces or static fluid pressure.

2. Static instability such as the buckling of fluid carrying tubes.

3. Forced vibration with periodicity due to vortex shedding or pulsating flows.

4. Self-excited vibration such as occurs with tubes carrying fluids with high velocity flows.

The relationship between these four areas of response can be represented by the flow diagram given in figure 3.1. It is obvious that some responses involve more than one of the four loops and that the relative importance of each may be controlled by factors such as flow velocity (as in the case of the lock-on region of vortex induced vibration). In the present research both free and forced vibration response of slender structures in an initially undisturbed fluid field will be considered. Hence, area (3) in figure 3.1 will be of major concern. Hydrodynamic interaction for such structures can be fully accounted for by use of an 'added mass' and a fluid damping force term. Before using such ideas, a full understanding of their origins must be obtained.

An early investigation into the response of a submerged slender mast to surface wave forces was made by Morison et al\(^{(9)}\) who identified the forcing function as having two components. One component was considered to be in phase with the acceleration of the body and the second was
Figure 3.1 Fluidelastic Responses of Circular Cylindrical Structures
(taken from reference 8)
considered to be in phase with the velocity. The magnitude of these two components is controlled by an 'inertia coefficient', $C_m$, and a 'drag coefficient', $C_d$, and the governing equation is of the form

$$F \cdot \dot{x} = \frac{C_m}{4} \pi D^2 \ddot{u} + \frac{C_d}{2} \rho D |\dot{u}| \dot{u}$$

(3.1)

where $D$ is the cylinder diameter, $\rho$ the fluid density, $U$ and $\dot{U}$ are the particle velocities and accelerations respectively (in the absence of the body). In more recent years, this equation has become known as the 'Morison equation'. The inertia coefficient representing the ratio of entrained water to structural mass was measured experimentally and was shown to vary between 1.3 and 1.7. The drag coefficient was shown to vary between 1.2 and 2.0.

Since the work of Morison, a large amount of research has been undertaken to determine the values of the inertia and drag coefficients by experiment for a wide range of submerged conditions. Sarpkaya\(^{10,11}\) has measured the in-line forces on a cylinder forced to vibrate in a uniform flow. Keulegan & Carpenter\(^{12}\) and Garisson et al\(^{13}\) have investigated forces imposed on a stationary cylinder due to wave motion. More recently McConnel and Jiao\(^{14}\) have considered the effect of structural support flexibility. Further discussion of experimentally determined drag coefficients will be made in section 3.3.3.

In a more recent study of structural vibrations in a fluid, Blevins\(^{15}\) has confirmed that the fluid force acting on a structure can be included in the governing equation of motion as a two component relationship. The dynamic equation for free vibration then being written as

$$(M + A_{xx}) \ddot{x} + (B_{xx} + C) \dot{x} + K x = 0$$

(3.2)

The inertia and drag terms of the Morison equation are now included as $A_{xx}$ and $B_{xx}$ which are more commonly known as the 'added mass' and 'added damping' coefficients respectively. The 'added mass' is considered to take into account the inertia of the entrained fluid which moves with the structure. The 'added mass' increases the effective mass of the structure and hence decreases its natural frequency.
An early investigation into how submergence affects the frequency response of a structure was made by Clough\(^{(16)}\). Using a simple energy analysis, he was able to show the significance of the kinetic energy imparted to the fluid and derived a 'virtual' or 'added' mass coefficient which could be used to analyse the complete system. An experimental investigation of cantilevers with various cross-sectional shapes subjected to both free vibration and shock input was undertaken. Both methods produced comparable added mass ratios for the first mode of response although information obtained on higher order modes was inconclusive. Some consideration was given to the effect of damping of the fluid and it was concluded that the damping effects were negligible. It can only be presumed that this statement meant that the fluid damping effect was negligible in comparison to the structural damping which was likely to be high due to the nature of the models used.

Experiments generally show\(^{(15)}\) that the added mass of a vibrating structure in a still fluid is a function of:

1. the geometry of the surface of the structure
2. the amplitude of vibration
3. a parameter similar to Reynolds number.

hence

\[
A_{\text{still fluid}} = \rho \cdot F\left( \text{geometry}, A_y, \frac{fD^2}{\nu} \right)
\]  (3.3)

where
\[
\rho \text{ is the density of the fluid} \\
A_y \text{ is the amplitude of vibration} \\
D \text{ is the characteristic diameter} \\
f \text{ is the frequency of vibration} \\
\nu \text{ is the kinematic viscosity of the fluid}
\]

It can be shown that added mass, unlike fluid drag or fluid damping, can exist in an incompressible, inviscid, irrotational fluid (i.e. \(\rho = \text{constant}, \nu = 0\)). This presents the possibility of evaluating added mass through the mathematical application of potential flow theory. The added mass determined using this method depends only on the structural surface.
\[ A_{\text{potential}} = \rho F(\text{geometry}) \]  

Early investigations into the validity of potential flow solutions for both the added mass and damping coefficients were made by Ackermann & Arbhabhirama\(^{(17)}\) and McConnel & Young\(^{(18)}\). Good agreement of potential flow solutions was found for experiments performed using fluids of low viscosity and hence high Stokes number. It was found that for Stokes number greater than \(10^5\) the potential flow solutions were no longer accurate.

Solutions to the hydrodynamic added mass have also been obtained through investigation of the pressure field created around a moving body. The velocity potentials in the vicinity of the body are defined by the Laplace equation

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]  

which applies to the flow of an inviscid incompressible fluid. Using boundary conditions applicable to the fluid-structure system, an expression for the hydrodynamic pressures and hence hydrodynamic mass can be derived. Jacobsen\(^{(19)}\) has used such an analysis with the additional assumptions:

(a) The loading of the structure appeared as a base excitation which only had horizontal components

(b) The body was not flexible

(c) Gravity wave effects were negligible.

Agreement between theoretical and experimental results was shown to be satisfactory. However, for vibrational response an improvement in the method of analysis may be obtained by use of the Poisson or wave equation.

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} \]  

An analysis of a vibrating slender bridge pier has been derived by
Goto & Toki (20) for circular cross-sections and by Kotsubu (21) for elliptic cross-sections. A solution to the wave equation is obtained by use of Bessel functions with the addition of relevant boundary conditions. The structure was assumed to be rigid and the results show that the added mass approaches the value of the mass of the displaced volume of water as the depth of water approaches infinity.

The effects of the compressibility of water and the flexibility of the structure were considered by Chandrasekaran et al. (22). The flexibility of the structure was included by defining a mode shape deformation function and the analysis was limited to the fundamental modal response. It was shown that the compressibility of water was important at certain frequencies which were dependent on the natural frequency of the structure. It was concluded that the flexibility of the structure has a considerable effect on predicted results and should be taken into account for an accurate analysis.

Landweber (23) has also investigated the response of submerged structures using a velocity potential function similar to those described above. The structural response was obtained using Lagrange's equation which is based on the principle of conservation of energy. It was also found that added mass could be considered equal to the displaced volume of fluid.

In 1974, Liaw and Chopra (24), (25) re-examined the previous ideas of assuming rigid structure and incompressible fluid. An analysis similar to that developed by Goto & Toki (20) was used with the added complication of allowing for structural flexibility. It was assumed that the cylindrical tower was subjected to small displacements in an undisturbed fluid field. The final equation in terms of Bessel functions produced a pressure distribution which could be determined for each mode of response of the structure. The analysis concentrated on large structures such as gravity dams and examined the accuracy of the much simpler constant added mass approach proposed by earlier researchers. It was concluded that the simple added mass representation produced negligible errors for the first mode of response and that these errors would increase significantly with higher modes. Two further points of interest raised by this paper were

(1) The compressibility of water is not important when considering the
effects on slender towers in low mode response.

(2) The effect of surface waves is negligible at high frequencies although it is possible that they could be significant at relatively low frequencies.

The ideas of 'added mass' and 'fluid damping' have also proved to be important in vibrations of a cylinder enclosed in a narrow annulus of fluid. Chen\textsuperscript{(26)} has used a fourth order equation to represent the velocity potential field in the annulus of fluid. Assuming that the amplitude of vibration is small, a solution can be obtained which consists of two components. The first component acts in phase with the acceleration of the body representing the added mass term, and the second is out of phase relating to the damping mechanism. These two components are shown both in experiment and theory to be dependent on a non dimensional factor \((\omega_0^2/v)\) as previously discussed by Blevins\textsuperscript{(15)}.

Approximate solutions to the Laplace equation can be obtained using the finite difference equations. Akkas\textsuperscript{(27)} has developed such a solution in the investigation of the transient response of a hemispherical shell. Unfortunately, no experimental results are available to verify the accuracy of this approach.

The possibility of negative added mass associated with bodies moving close to the water surface has been examined more recently by McIver & Evans\textsuperscript{(28)}. Using an analysis previously derived by Young\textsuperscript{(29)}, the added mass distribution for an upright circular cylinder has been determined. The method used and the final equations derived compare closely to those previously given by Jacobsen\textsuperscript{(19)}. The possibility of the occurrence of a negative value for the added mass can be verified using energy principles although its effect is relatively small for most normal response frequencies. In the case of slender bodies of the type presently under investigation, the occurrence of negative added mass would have negligible effect.

As previously discussed, the use of modal techniques to determine the dynamic response of submerged structures has naturally led to the use of finite element analyses. It is possible to use finite elements to represent both structure and fluid and to produce coupled equations in
terms of structural and fluid displacements. However, the resulting
equations are highly complex and a solution is required at every frequency
of interest. A far simpler method of solution is to evaluate 'added mass'
and 'added damping' matrices for 'dry modes' using Laplace's equation in a
similar way to the methods described earlier. The fluid model can then be
included into the finite element analysis and solved using standard
techniques. A solution for the dynamic response using such a procedure
has been reported by Eatock-Taylor & Young(30) and results obtained have
correlated well with simple experimental models vibrating in low modal
responses.

Finite element techniques have also been employed by Moan et al(31)
who have included the effect of submergence by using Morison's equation
together with empirically-based mass and drag coefficients. Similar
methods have been used by Thompson & French (32) who were interested in the
dynamic response of models of offshore platform components. Experimental
results have verified their approach to the vibration problem and further
tests are proposed for comparison of predicted results with full scale
tests.

In summarising previous hydrodynamic studies it has been shown that
submergence has the effect of creating hydrodynamic forces which decrease
the natural frequencies of the structure and increase the damping forces on
the structure. In the case where only the low modes of response of a
slender structure are excited, an adequate approximation to the decrease in
natural frequency can be obtained by inclusion of an added mass term equal
to the mass of the fluid displaced by the structure. The effect of
damping can be included by use of experimentally obtained relationships of
damping for oscillatory flows around a stationary cylinder. However, the
assumption of low modes of structural response is not always applicable to
slender structures which are excited by complex loadings such as those
associated with shock response. It is therefore necessary to re-examine
the added mass and damping methods to evaluate the significance of higher
modes of response.
3.2 Determination of the Inertia Coefficient using the Wave Equation.

3.2.1 Determination of Pressure Forces on a Submerged Structure.

The investigation by Chopra and Liaw\(^{(24),(25)}\) discussed in section 3.1 has been shown to be the most complete method used to date for the investigation of the 'added mass' phenomena. In order to improve the understanding of interaction effects it was proposed that the present research should investigate:

(1) the significance of higher mode response from a hydrodynamic viewpoint.

(2) the influence of hydrodynamic effects for a semi-submerged condition.

In order to carry out the above studies it is necessary to develop an analysis that describes the pressure field in the vicinity of the structure under consideration.

A cantilever mast of radius, \(r_0\), height, \(H_s\), submerged to a depth of \(H\) is shown in figure 3.2. Assuming water to be linearly compressible and neglecting the effects of internal viscosity, the pressure field around the tower is governed by the wave equation for small amplitudes of displacement.

\[
\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} + \frac{1}{r^2} \frac{\partial^2 P}{\partial \theta^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{C^2} \frac{\partial^2 P}{\partial t^2} \tag{3.7}
\]

where \(P(r,\theta,z,t)\) is the hydrodynamic pressure and \(C\) is the speed of sound in water.

The solution to the wave equation can be found by using Bessel functions (ref 33 and 34) together with boundary conditions applicable to the structural system.
Figure 3.2 Coordinate System for the Submerged Mast Undergoing Vibratory Motion
For the structure shown in figure 3.2 the boundary conditions are:

1. No vertical motion at the boundary \( z = 0 \)

\[
\frac{\partial P(r,0,z,t)}{\partial z} = 0 \quad (3.8)
\]

2. Symmetrical pressure distribution at the \( \theta = 0 \) plane

\[
\frac{\partial P(r,0,z,t)}{\partial \theta} = 0 \quad (3.9)
\]

\[
\frac{\partial P(r,\pi,0,z,t)}{\partial \theta} = 0 \quad (3.10)
\]

3. Wave condition at the free surface

\[
\frac{\partial^2 P(r,0,H,t)}{\partial t^2} = -g \frac{\partial P(r,0,H,t)}{\partial z} \quad (3.11)
\]

4. Radial component of motion of the boundary \( r=r_0 \) is the same as the radial motion of the tower

for any particle of water

\[
\frac{\partial P}{\partial r} = -\rho \frac{\partial^2 u^r}{\partial t^2} \]

where \( u^r \) is the radial component of velocity, \( \rho \) the density of water

now \( u^r = u \cos \theta \)

and \( \frac{\partial^2 u^r}{\partial t^2} = \ddot{u} \cos \theta = \phi_N(z) \cos \theta e^{i\omega t} \)

for a response where \( N \) is the mode number and \( \phi_N(z) \) is the shape function for that mode, hence

\[
\frac{\partial P(r_0,0,z,t)}{\partial r} = -\rho \phi_N(z) \cos \theta e^{i\omega t} \quad (3.12)
\]
Using the five boundary conditions, the description of the pressure distribution is complete and a solution can be found in terms of a modal contribution. The full exposition of the solution is given in Appendix B.

The resulting equation is

\[
P_N(r_0, \theta, z) = \delta \left\{ \begin{array}{l}
-\alpha_0 \frac{I_{k_0}}{\lambda_0} \frac{\cosh(\alpha_0 z)}{2\alpha_0 H + \sinh(2\alpha_0 H)} D_0(\lambda_0 r_0) \cos(\alpha_0 z) e^{i\varepsilon_0 r_0} \\
- \sum_{m=1}^{m_1} \alpha_m \frac{I_{km}}{\lambda_m} \frac{\cosh(\alpha_m z)}{2\alpha_m H + \sin(2\alpha_m H)} E_m(\lambda_m r_0) \cos(\alpha_m z) e^{i\varepsilon_0 r_0} \\
+ \sum_{m=m_1}^{\infty} \alpha_m \frac{I_{km}}{\lambda_m} \frac{\cosh(\alpha_m z)}{2\alpha_m H + \sin(2\alpha_m H)} D_m(\lambda_m r_0) \cos(\alpha_m z) e^{i\varepsilon_0 r_0}
\end{array} \right\} \cos \theta \quad (3.13)
\]

where \( m_1 \) is the smallest integer value of \( m \) such that \( \alpha_m > \omega/C \)

where

\[
\begin{align*}
\alpha_0 \tanh(\alpha_0 H) &= \frac{\omega^2}{g} \\
\alpha_m \tan(\alpha_m H) &= \frac{-\omega^2}{g}
\end{align*}
\]

\[
\begin{align*}
\lambda_0 &= \sqrt{\left(\frac{\omega^2}{c^2} + \alpha_0^2\right)} \\
\lambda_m &= \sqrt{\left(\frac{\omega^2}{c^2} - \alpha_m^2\right)} \\
\lambda_m' &= i \lambda_m
\end{align*}
\]

\[
\begin{align*}
I_{N\theta} &= \int_0^H \phi^N(z) \cosh(\alpha_0 z) \, dz \\
I_{Nm} &= \int_0^H \phi^N(z) \cos(\alpha_m z) \, dz
\end{align*}
\]
and \( D_m(\lambda_m r_0) = \sqrt{\frac{J_1(\lambda_m r_0)^2 + Y_1(\lambda_m r_0)^2}{[J_0(\lambda_m r_0) - J_2(\lambda_m r_0)]^2 + [Y_0(\lambda_m r_0) - Y_2(\lambda_m r_0)]^2}} \) \( (3.17) \)

\[
E_m(\lambda_m r_0) = \frac{K_1(\lambda_m r_0)}{K_0(\lambda_m r_0) + K_2(\lambda_m r_0)} \tag{3.18}
\]

and \( \tan(\varepsilon_m r_0) = \frac{[Y_0(\lambda_m r_0) - Y_2(\lambda_m r_0)]J_1(\lambda_m r_0) - [J_0(\lambda_m r_0) - J_2(\lambda_m r_0)]Y_1(\lambda_m r_0)}{[J_0(\lambda_m r_0) - J_2(\lambda_m r_0)]J_1(\lambda_m r_0) - [Y_0(\lambda_m r_0) - Y_2(\lambda_m r_0)]Y_1(\lambda_m r_0)} \) \( (3.19) \)

where \( J_n \) and \( Y_n \) are Bessel functions of the first and second kind of order \( n \) and \( K_n \) is the modified Bessel function of the second kind of order \( n \).

The contribution of dynamic pressure \( P_k(\theta, z, \omega) \) from each mode of response, \( k \), is available from this equation by using the applicable shape function. The use of shape functions is widespread in engineering mathematics and in many cases the functions are determined by the static deflection conditions of the beam. It will be shown later than the use of these functions can cause serious errors in the determination of pressures and consequent dynamic properties.
3.2.2 The Importance of Compressibility Effects and Surface Wave Effects

The equation derived in the previous section takes into account both the effects of compressibility and surface waves. It is of interest to investigate the importance of these parameters for slender structures.

If the effects of surface waves are ignored, the boundary condition (3.11) becomes $P(r, \theta, H, t) = 0$. This results in the disappearance of the first term in equation (3.13) and equation (3.14) becomes

$$\tan(\alpha_m H) = \infty$$

which is solved by $\alpha_m = (2m-1)\pi/2H$

hence,

$$P_N(r_0, \theta, z, \omega) = 8 \rho \left\{ - \sum_{m=1}^{m_1-1} \alpha_m \lambda_m \frac{I_{Nm}}{2\alpha_m H + \sin(2\alpha_m H)} D_m (\lambda_m r_0) \cos(\alpha_m z) e^{-\rho r_0} \right. \left. + \sum_{m=m_1}^{\infty} \alpha_m \lambda_m \frac{I_{Nm}}{2\alpha_m H + \sin(2\alpha_m H)} E_m (\lambda_m r_0) \cos(\alpha_m z) \right\} \cos\theta$$

(3.20)

To further simplify the analysis, the effect of compressibility can be neglected by letting $C \rightarrow \infty$ in equation (3.7) and results in the wave equation being reduced to the Laplace equation. The solution to the equation becomes

$$P_N(r_0, \theta, z, \omega) = 8 \rho \left\{ \sum_{m=1}^{\infty} \frac{I_{Nm}}{(2m-1)\pi} E_m (\lambda'_m r_0) \cos(\alpha_m z) \right\} \cos\theta$$

(3.21)

where $\lambda'_m = \alpha_m = (2m-1)\pi/2H$

The result of neglecting these two effects can be investigated by predicting the pressures acting on a structure for a range of values of $\omega$ using each of the equations (3.13), (3.20) and (3.21). In fact, this analysis can be carried out for all modes of the cantilever by substituting the relevant mode shape into the equation. Firstly, the influence of a zero mode 'step movement' of the structure can be investigated by
replacement of the structural mode shape as $\phi_0(z) = 1$. The results are shown in figure (3.3) as a function of the non-dimensionalised frequency term $\frac{\omega}{r_0}$ for a range of radius to height ($r_0/H$) ratios and for varying heights. These results can be shown to agree with those presented by Liaw & Chopra. It can be seen that the effect of compressibility is restricted to high frequencies and the effect of surface waves is restricted to low frequencies. It is also apparent that their effects diminish with both the ratio ($r_0/H$) and the height of the tower. In this present research the slenderness ratio ($H/r_0$) is likely to be of the order of 100.

The true significance of these two effects is fully appreciated when the total pressure force is given for the case of $H/r_0 = 100$ and $H = 1.5$ in higher order modal responses. The equations used for the mode shape functions will be discussed later, but the effects for modes 1, 2 and 3 are shown in figure 3.4 with the same frequency scale as previously used. The real effect, as far as the structure is concerned occurs at its natural frequencies. The approximate position for these 'dry' frequency values for a stainless steel tower with a wall thickness ratio $t/r_0 = 2/15$ is also shown in figure 3.4. The conclusion to be drawn from this analysis is that in normal response, the effects of compressibility and waves are negligible in comparison to the overall effect of submergence and can therefore be justifiably neglected.
Figure 3.3 Frequency Response of the Hydrodynamic Force Acting on a Fully Submerged Mast Vibrating in Mode 0
Amplitude of Total Hydrodynamic Force \( /w_r H \times 10^5 \)

--- Surface waves & compressibility included

--- Compressibility included

--- Solution to Laplace's equation

H=1.5m

\( r/H=0.01 \)

Mode 3
Mode 2
Mode 1

4 Natural Frequencies of a structure with \( t/r=2/15 \)

Excitation Frequency \( w/(\Pi C/2H) \)

Figure 3.4 Frequency Response of the Hydrodynamic Force Acting on a Slender Structure Vibrating in Various Modes of Response
3.2.3. Pressure Distribution Associated with Each Mode of Response

It must be strongly emphasised that the most important factor in the determination of the true pressure distribution acting on the structure is the use of the correct mode shape function. The previous determination of the pressure equation considered the structure to be a cantilever beam. However, the boundary conditions for the pressure field apply equally to a structure mounted in a flexible restraint. It is important to this research that both the case of a flexible restraint and the case of a rigid boundary are examined since most real structures will have boundary conditions that lie somewhere between these two extremes.

A full exposition of the determination of the corresponding modal shapes is given in Appendix C in which the basic dynamic equation for a vibrating beam is used together with the relevant boundary conditions. Note that it must be assumed, subsequently, that the effect of submergence has no effect on the mode shape equation. The results can be summarised as follows:

For a cantilever beam, the mode shape function is given by

\[
X(x) = 0.5 \left[ \frac{\cos(kx) - \cosh(kx) - (\cos(kL) + \cosh(kL)) (\sin(kx) - \sinh(kx))}{(\sin(kL) + \sinh(kL))} \right]
\]

(3.22)

where the values of \( k \) control the mode of response. The form of the equation and the values of \( k \) are given in figure 3.5.

For a beam supported by a pair of pin joints, the mode shape function is given by two equations (refer to figure 3.6)

Part 12 \( 0 < x_1 < a \)

\[
X(x) = \left( \frac{1 + \cos(kL) \cosh(kL)}{\cosh(kL) + \cos(kL)} \right) \left( \frac{\sin(ka) \sinh(kx_1) - \sinh(ka) \sin(kx_1)}{\cos(ka) \sinh(ka) - \sin(ka) \cosh(ka)} \right)
\]

(3.23)

Part 23 \( 0 < x_2 < L \)

\[
X(x) = 0.5 \left( \frac{(\sin(kx_2) + \sinh(kx_2)) (\cos(kL) + \cosh(kL)) - \cos(kx_2) - \cosh(kx_2)}{(\sin(kL) + \sinh(kL))} \right)
\]

(3.24)
Figure 3.5 Natural Mode Shapes for a Cantilever Beam
Figure 3.6 Natural Mode Shapes for a Pin Joint Supported Beam
where the values of \( k \) control the mode of response. The form of the equation and typical values of \( k \) are given in figure 3.6.

Using these determined mode shape functions, the coefficient \( I_{Nm} \) defined in equation 3.16 can be evaluated as a function of \( k \). For a cantilever beam

\[
I_{Nm} = \int_0^H \phi_N(z) \cos(\alpha_m z) \, dz
\]

\[
= \frac{1}{2\alpha_m} \left[ \frac{1}{C_1} \left[ \cosh(kH)\sin(\alpha_m H) + k \frac{\sinh(kH)\cos(\alpha_m H)}{\alpha_m} \right] \right.
\]

\[- \left. \frac{1}{C_2} \left[ \cos(kH)\sin(\alpha_m H) - k \frac{\sin(kH)\cos(\alpha_m H)}{\alpha_m} \right] \right]
\]

\[- \frac{C}{C_1} \left[ \sinh(kH)\sin(\alpha_m H) + k \left( \cosh(kH)\cos(\alpha_m H) - 1 \right) \right] \]

\[- \frac{C}{C_2} \left[ \sin(kH)\sin(\alpha_m H) + k \left( \cos(kH)\cos(\alpha_m H) - 1 \right) \right] \]

(3.25)

where \( C_1 = 1 + k^2 / \alpha_m^2 \); \( C_2 = 1 - k^2 / \alpha_m^2 \)

and \( C = \frac{\cos(kH) + \cosh(kH)}{\sin(kH) + \sinh(kH)} \)

The pressure distributions over the surface of the mast can now be evaluated for the case where the mast is submerged with the water surface in line with the tip of the mast. Equation (3.21) shows that the distribution of the pressure as a function of angle \( \theta \) is a cosine distribution as shown in figure 3.7. The peak pressure value of the cosine distribution along the height of the tower is shown for mode 0 through to mode 5 in figures 3.8 to 3.13. In the case of mode 0 the shape function is \( \phi_0(z) = 1 \) and \( I_{0m} = (-1)^{m-1} / \alpha_m \). Each graph of pressure distribution shows the effect of decreasing the slenderness ratio \( H/r_0 \) from 100 down to 10.

The pressure distribution for mode 0 shows that the boundary condition of having zero pressure at the surface has a much more dominant effect for lower slenderness ratios. The pressure for the most slender structure can
Figure 3.7 Hydrodynamic Pressure Distribution around the Circumference of the Mast

Figure 3.8 Mode 0
Figure 3.9 Mode 1

Peak Hydrodynamic Pressure Acting along the Height of the Mast in Various Response Modes
Figure 3.10 Mode 2

Figure 3.11 Mode 3

Figure 3.12 Mode 4

Figure 3.13 Mode 5

Peak Hydrodynamic Pressure Acting along the Height of the Mast in Various Response Modes
be seen to be constant over more than 90% of its height. The general shape of the pressure distribution for all modes can be seen to be of a similar form to the modal shape although, as for the case of zero mode, for lower slenderness ratios the pressure is reduced nearer the water surface.

3.2.4 The Frequency Response of Submerged Structures

Having determined the pressure variation acting on the structure for each mode of response, it is important to understand exactly how this affects the frequency response of the structure. Because of the way the pressure term has been derived, it is most convenient to investigate the frequency response characteristics by use of a modal technique. Such methods essentially split up the response into harmonic components. The equation of motion for free vibration of the tower shown in figure 3.2 can be shown to be (Ref 35)

\[
M_N \ddot{x}_N(t) + C_N \dot{x}_N(t) + K_N x_N(t) = P_N(t)
\]  

(3.26)

Where the generalised properties in this equation are defined as

\[
M_N = \int_0^{H_S} m(z) \phi_N(z)^2 \, dz
\]  

(3.27)

\[
K_N = \omega_N^2 M_N
\]  

(3.28)

\[
C_N = 2 \zeta_N \omega_N M_N
\]  

(3.29)

and \(m(z)\) is the mass of the tower per unit height, \(\phi_N(z)\) is the mode shape of deformation, \(\omega_N\) is the \(N^{th}\) natural modal frequency and \(\zeta_N\) is the damping ratio associated with the \(N^{th}\) mode of response.

The function \(P_N(t)\) is the generalised hydrodynamic loading due to the pressure variation \(P(\theta,z,t)\) acting over the surface of the structure. It is known from equation (3.21) that the pressure variation around the structure is of a cosine distribution hence

\[
P(\theta,z,t) = P(z,t) \cos \theta
\]  

(3.30)
The value of the total pressure per unit height can be obtained by integrating the function around the circumference of the tower

\[ P_{\text{tot}}(z,t) = \int_0^{2\pi} P(\theta,z,t) r_0 \cos \theta \, d\theta \]

\[ = \int_0^{2\pi} P(z,t) r_0 \cos^2 \theta \, d\theta \]

\[ = \pi r_0 P(z,t) \]  

(3.31)

The function of \( P_N(t) \) is therefore given as

\[ P_N(t) = \int_0^H P_{\text{tot}}(z,t) \phi_N(z) \, dz \]

\[ = \pi r_0 \int_0^H P(z,t) \phi_N(z) \, dz \]  

(3.32)

Assuming that the response is harmonic, the complex frequency response of the structure will be

Displacement, \( X_N(t) = \overline{X}_N(\omega) e^{i\omega t} \)  

(3.33)

Acceleration, \( \ddot{X}_N(t) = \overline{\ddot{X}}_N(\omega) e^{i\omega t} = -\omega^2 \overline{X}_N(\omega) e^{i\omega t} \)  

(3.34)

Hydrodynamic pressure, \( P(\theta,z,t) = \overline{P}(\theta,z,\omega) e^{i\omega t} \)  

(3.35)

Because the equations governing the pressure are linear, as are the boundary conditions, the principle of superposition of the pressure term can be applied. The complex pressure function can be expressed as

\[ \overline{P}(\theta,z,\omega) = \overline{P}_0(\theta,z,\omega) + \overline{X}_1(\omega) \overline{P}_1(\theta,z,\omega) + \overline{X}_2(\omega) \overline{P}_2(\theta,z,\omega) + \ldots \]

0 Mode 1st Mode 2nd Mode

\[ = \overline{X}_1(\omega) \overline{P}_1(\theta,z,\omega) + \overline{X}_2(\omega) \overline{P}_2(\theta,z,\omega) + \ldots \]  

(3.36)

in the case of no zero mode.
The complex frequency response can be obtained from equations (3.26), (3.27), (3.28), (3.29), and (3.36), giving

\[ [M_N + P_N] \ddot{X}_N(t) + 2 M_N \zeta_N \omega_N \dot{X}_N(t) + \omega_N^2 M_N X_N(t) = 0 \]  

(3.37)

where

\[ P_N(z) = \int_0^H P_N(z) \phi_N(z) \, dz \]  

(3.38)

and from equations (3.21) and (3.32)

\[ P_N(z) = 8 \rho r_0 \sum_{m=1}^{\infty} \frac{I_{Nm}}{(2m-1)} \cos(\alpha_m z) \]  

(3.39)

The changes in the natural period of vibration for the submerged and non-submerged case can be obtained from equation (3.37).

The ratio of natural frequency of a submerged structure \( \bar{\omega}_N \) to an equivalent one in an unsubmerged state \( \omega_N \) is given by

\[ \frac{\bar{\omega}_N}{\omega_N} = \frac{T_N}{\bar{T}_N} = \frac{1}{\sqrt{1 + \frac{P_N}{M_N}}} \]  

(3.40)

The form of this relationship can now be investigated by using the pressure distributions associated with each mode of response which were derived in section 3.2.3. The variation in the modal frequencies with depth of submergence can be obtained by computing the hydrodynamic pressure distribution for semi-submergence. Figures 3.14 to 3.18 show the change in natural frequency for two structures with different slenderness ratios. The first structure has a slenderness ratio of 100 which is typical of mast-like forms and the second has a slenderness ratio of 10.

In both cases the base support has been considered to be completely rigid and the mode shapes used are that of the cantilever which was given in section 3.2.3. In both cases the thickness/radius ratio \( t/r_0 \) was 2/15.
Figure 3.14 Change in Frequency Characteristics with Depth of Submergence for a Mast Vibrating in its 1st Natural Mode

Figure 3.15 Change in Frequency Characteristics with Depth of Submergence for a Mast Vibrating in its 2nd Natural Mode
Figure 3.16 Change in Frequency Characteristics with Depth of Submergence for a Mast Vibrating in its 3\textsuperscript{rd} Natural Mode.

Figure 3.17 Change in Frequency Characteristics with Depth of Submergence for a Mast Vibrating in its 4\textsuperscript{th} Natural Mode.
Figure 3.18 Change in Frequency Characteristics with Depth of Submergence for a Mast Vibrating in its 5th Natural Mode
The important features of the graphs are:

1. There is a certain depth to which a structure can be submerged without having significant influence on the response. This depth decreases with mode number and is given approximately as:

<table>
<thead>
<tr>
<th>Mode</th>
<th>Depth of Water/Height of Mast</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.07</td>
</tr>
</tbody>
</table>

2. Slight changes in the depth of submergence around a nodal position make very little difference to the frequency of response.

3. The effect of submergence is greater for more slender structures.

In order to investigate the extent to which the boundary support conditions influence the final frequency shift in the response, the analysis was carried out using both types of structure previously discussed in section 3.2.3, namely the cantilever and pin joint supported structure. The responses for modes 1 to 4 for the case of a slenderness ratio of 100 and $t/r_0$ of 2/15 are presented in figures 3.19 to 3.22 respectively. The differences in the response of the two structures can be seen to be greater at increased depths of submergence with the hydrodynamic forces having a greater effect on the cantilever structure. The explanation of this can be more easily understood by comparing the mode shapes and pressure distributions associated with each response. Concentrating on the first mode of response, the mode shape and pressure distributions are superimposed in figures 3.23 and 3.24. The greater flexibility of the pin joint supports allow the structure to deflect further in the region of the base and therefore increases the hydrodynamic pressure in this region. However, because the decrease in natural frequency is effectively proportional to a pressure and the reciprocal of deflection, the resulting change in natural frequency is not directly obvious. The percentage change in deflection can be shown to be higher
Figure 3.19 The Effect of Varying the Restraint Conditions on the Frequency Response of a Semi-submerged Mast Vibrating in its 1st Natural Mode

Figure 3.20 The Effect of Varying the Restraint Conditions on the Frequency Response of a Semi-submerged Mast Vibrating in its 2nd Natural Mode
Figure 3.21 The Effect of Varying the Restraint Conditions on the Frequency Response of a Semi-submerged Mast Vibrating in its 3rd Natural Mode

Figure 3.22 The Effect of Varying the Restraint Conditions on the Frequency Response of a Semi-submerged Mast Vibrating in its 4th Natural Mode
Figure 3.23 Comparison Between the 1st Mode Deflection Profiles for a Cantilever and Pin Joint Supported Mast

Figure 3.24 Comparison Between the 1st Mode Pressure Distributions for a Cantilever and Pin Joint Supported Mast
than the change in hydrodynamic pressure for the pin joint supported structure and this results in a lower prediction for the decrease in natural frequencies except for submergence levels close to the support. This can be shown to hold true for all modes of response of the pin joint supported structure. The analysis of these two extreme boundary conditions is advantageous since they provide the 'limits of response' for any similar structure which has an unknown support stiffness.

Having stated the importance of selecting the correct mode shape function to represent the response of the structure, it is interesting to show the errors incurred by selecting the wrong function. A common approach often used in the selection of a mode shape is to analyse the statically deformed structure and to model the shape by a polynomial expression which produces the correct boundary conditions. In the case of the cantilever, the important boundary conditions would be no displacement or rotation at the fixed end. These conditions are true for all modes of response, but extra boundary conditions are required to fully describe higher order responses. There are alternative methods available to obtain further boundary conditions. Firstly, for the $n^{\text{th}}$ mode shape, there are $n-1$ positions other than $z=0$ where the horizontal displacement of the beam is zero. Alternatively, for every $n^{\text{th}}$ mode shape there are $n-1$ positions other than at $z=0$ where the beam rotation is zero. However, neither of these two conditions completely specify the polynomial curve because they introduce an additional unknown position along the beam where they occur. One approach to overcome this problem is to concentrate on one mode in particular and investigate the effect of varying one of the unknown factors.

As an example, the second mode response of a cantilever beam using the exact mode shape function (section 3.2.3) was compared to the response using a cubic polynomial approximation of the shape. A family of cubic curves were obtained by controlling the ratio of the tip deflection to the deflection at the point of zero rotation (figure 3.25). The resulting frequency response predictions together with the response for the exact mode shape function is shown in figure 3.26. It can be seen that it is impossible to obtain a response which accurately represents second mode response and large errors will be incurred if the analysis is used for structures in a semi-submerged condition. In a fully submerged state, the errors may be small typically yielding a 2% increase in the time period of
Figure 3.25 Cubic Approximation to a 2\textsuperscript{nd} Mode Deformation Profile for a Cantilever Beam

Figure 3.26 Change in the Second Mode Frequency Characteristics for a Cantilever Beam using the Cubic Mode Shape Approximations
response.

It is clear that a cubic mode shape function can only be used for modes 1 and 2 since there is insufficient "flexibility" in the function. Higher order polynomials could be used but further complications in terms of numbers of available boundary conditions would limit their use. A possible solution would be to use a number of cubic splines dependent upon the mode of response. This method has been investigated but results indicate no improvement of accuracy over the continuous cubic representation for the second mode of response.

In conclusion to the investigation into the selection of mode shape functions it is seen that for hydrodynamic interaction problems, the choice of mode shape is critical and can only be determined from the overall dynamic equation. In cases where the structure is non-uniform this may prove to be a difficult requirement and may only be achieved by physical measurement of a prototype structure.

3.2.5 The Equivalence of Using an Added Mass Term

In equation 3.33 the effect of the hydrodynamic pressure is shown to be equivalent to an 'added mass', \( P_N \), which is included with the structural mass. This general 'added mass' term is therefore defined as

\[
P_N = \int_0^H P_N(z) \phi_N(z) \, dz
\]

where,

\[
P_N(z) = 8 \rho r_0 \sum_{m=1}^{\infty} \frac{I_{Nm}}{(2m-1)} E_m(\lambda r_0) \cos(\alpha_m z)
\]

and the value of \( I_{Nm} \) is defined in equation (3.15).

If the 'added mass' is considered to be a mass distribution \( m_a(z) \), then

\[
m_a(z) = \frac{P_N(z)}{\phi_N(z)}
\]

(3.41)

The form of the added mass distribution can therefore be determined from the pressure distribution and can therefore be determined from the
pressure distributions evaluated in section 3.2.4. The added mass profiles determined for the cases where the slenderness ratio are 10 and 100 are shown in figures 3.27 and 3.28 for modes 1 to 5. Care must be taken in using equation (3.41) because at the nodal positions the deflection must necessarily be zero thus producing a possible infinite value of mass if the pressure is not exactly in phase with the deflection. In practice, this is only a problem with non slender structures and is due to the finite numerical accuracy of the computer.

Figures 3.27 and 3.28 show that the added mass distribution tends to a constant value over more of the structure's length as the slenderness ratio is decreased. In the case considered here, where the depth of water is equal to the height of the tower, the added mass must always decay to zero at the top of the structure because of the surface pressure boundary condition. At the base, the opposite situation occurs. The displacement at the base must be zero but only the rate of change of pressure at the base must be zero. This has the effect of allowing the added mass at the base to approach infinity. These characteristics do not present problems in the modal analysis because it is the pressure distribution which is used for calculation. However, to include added mass into a time domain method such as the finite difference technique described in chapter 2, the infinite values present problems which will be discussed later in section 3.2.7.

3.2.6 The Validity of Assuming a Constant Added Mass Distribution

It is common in many hydrodynamic interaction analyses to use an added mass which is assumed to be constant over the structure's height in order to take into account the mass of the entrained fluid. This concept has largely arisen from the work of researchers such as Morison et al(9) who have measured the fluid force acting on a structure, finding it to be proportional to the structural acceleration. The acceleration coefficient was defined as being equal to the mass of water displaced by the structure multiplied by an 'inertia coefficient' $C_m$ which was found to be approximately 2.0 for many applications. It has been shown subsequently that the added mass coefficient is equal to $(C_m - 1)$ from potential flow theory. It should be realised that this work has concentrated on non-flexible structures which were considered to move predominantly in the zero
Figure 3.27  Added Mass Distribution for a Slender Mast of $r_0/H = 0.01$
Figure 3.28 Added Mass Distribution for a Dumpy Mast of $r_0/H = 0.1$
mode (ie step movement). It has been shown in earlier sections that for zero mode, the pressure distribution is approximately constant over most of the structure's height for slender forms. This would produce a near constant added mass equal to the mass of fluid displaced. However, for non slender structures this constant added mass distribution would not be a good approximation and the errors would increase with mode number.

The use of constant added mass distributions should not be completely discounted because it greatly simplifies the dynamic equations. However, because the added mass term is dependent only on the external dimensions of the tube, the frequency changes due to submergence will be a function of the tube's cross-sectional properties and not its slenderness ratio. This confirms that the error associated with such an approximation would increase with decreasing slenderness ratio. In order to fix limits for which the errors in the predicted response are acceptable, results are plotted in figure 3.29 and 3.30 for 1st and 2nd modes of response for the following cases.

1. True added mass for a structure of slenderness ratio 100
2. True added mass for a structure of slenderness ratio 10
3. Added mass equal to the displaced volume of water.

The results show that for slenderness ratios greater than 100, the simple approximation is acceptable for both modes of response.
Figure 3.29 Validity of the Constant Added Mass Approximation in the Prediction of the 1st Mode Response of a Slender Mast

Figure 3.30 Validity of the Constant Added Mass Approximation in the Prediction of the 2nd Mode Response of a Slender Mast
3.2.7 Inclusion of Hydrodynamic Effects in a Time Domain Finite Difference Analysis

It was an aim of this research that the final structural analysis should be carried out using a finite difference method which computes the response of a structural system as a function of time. Therefore, the feasibility of including the hydrodynamic interaction effects into such a scheme must be investigated. It has been shown that the effect of submergence on the frequency of response can be fully accounted for by increasing the mass of the structure through the inclusion of a fluid 'added mass' distribution. In a finite difference method, this added mass distribution could be divided up in a similar way to the structural mass and could then be added directly into the mass matrix. It has been shown that this added mass distribution is dependent upon both the mode of response and the slenderness ratio of the structure. However, inclusion of an added mass distribution for one mode would have an effect on all natural response frequencies and it is therefore impossible to satisfy the frequency requirement for all modes by the use of a single mass distribution.

In using a lumped mass model, the number of natural response frequencies is limited by the number of nodal masses. It is therefore possible to determine a range of added mass distributions which are likely to be of importance to the response. A logical approach to the problem of selection of appropriate added mass profiles for inclusion in the numerical model is to first analyse the structure in air under an identical shock loading using the finite difference method. The modal content of the response can be determined by Fourier analysis of the output to obtain a frequency representation of the response. Identification of the dominant frequencies allows an order of importance to be ascribed to the significant modes. If it is found that the response is dominated by one particular mode, then the 'added mass' distribution of only this mode should be included in the mass matrix. The shock analysis can then be repeated. A check should then be made to ensure that the change in modal content is not significant enough to warrant the use of a different added mass distribution. The magnitude of the added mass distribution tends to decrease with mode number (see figures 3.27 and 3.28) and hence modal frequencies below the dominant one will be in error having frequencies
higher than expected and conversely frequencies above the dominant one will be lower than expected. In the case where there are two or more dominant frequencies, then a compromise must be sought by averaging the added mass distributions so that the errors in the frequencies are of the same order in the dominant modes. The frequency errors are highly dependent on the slenderness ratio of the structure. In the present research the structures of prime concern are mast-like having a slenderness ratio of the order of 100. For such cases the added mass distributions are very similar and hence the frequency errors are very small.

For the specific case in which a structure vibrates predominantly in one mode, a different approach can be taken for the inclusion of hydrodynamic effects in a finite difference method. Instead of determining the added mass distribution from the knowledge of the modal shape, the deformed shape of the structure during the response can be used. The nodal coordinates at any time can be used to calculate an approximate value of $I_{Nm}$ defined in equation (3.16) and hence to obtain a pressure distribution from equation (3.39). A typical deformation profile for a structure in its fundamental response mode is shown in figure 3.31. A piecewise approximation of the mode shape is given by

$$\phi_{qN} = \frac{1}{y_{\text{max}}} \left( y_{q+1} + \frac{(z - z_{q+1})}{(z_{q} - z_{q+1})} \right)$$

which is valid for $z_{q-1} < z < z_q$ where $1 < q < n$ and $n$ is the number of nodes and $N$ represents the mode of response.

The value of $I_{Nm}$ given in equation (3.16) can be divided up as

$$I_{Nm} = \int_{z_2}^{z_1} \phi_{1N}(z) \cos(\alpha_m z) \, dz + \int_{z_3}^{z_2} \phi_{2N}(z) \cos(\alpha_m z) \, dz + \cdots + \int_{z_n}^{z_{n-1}} \phi_{n-1,N}(z) \cos(\alpha_m z) \, dz + \int_{z_{n+1}}^{z_n} \phi_{nN}(z) \cos(\alpha_m z) \, dz$$

$$= \sum_{q=1}^{n} \int_{z_{q+1}}^{z_q} \left( A_q + B_q z \right) \cos(\alpha_m z) \, dz$$

$$= \sum_{q=1}^{n} \left( \left[ \frac{A_q + B_q z}{\alpha_m} \right]_{z_{q+1}}^{z_q} + \left[ \frac{B_q \cos(\alpha_m z)}{\alpha_m^2} \right]_{z_{q+1}}^{z_q} \right)$$

(3.43)
Figure 3.31 Coordinate System of the Lumped Mass Idealisation
where 
\[ A_q = \frac{(y_{q+1} - B_q z_{q+1})}{y_{\text{max}}} \]

and 
\[ B_q = \left( \frac{y_q - y_{q+1}}{z_q - z_{q+1}} \right) \]

Expansion and simplification of equation (3.43) gives

\[
I_{Nm} = \frac{1}{\alpha_m y_{\text{max}}} \left[ y_1 (-1)^{m-1} + \frac{1}{\alpha_m} \left( (B_2 - B_1)\cos(\alpha_m z_2) + (B_3 - B_2)\cos(\alpha_m z_3) + \ldots \right) \right]
\]

In most normal mode shapes for a cantilever, the maximum deflection occurs at the tip of the structure and hence \( y_{\text{max}} = y_1 \). \( I_{Nm} \) becomes

\[
I_{Nm} = \frac{1}{\alpha_m} \left[ (-1)^{m-1} + \frac{1}{\alpha_m y_1} \left( \sum_{q=1}^{n-1} (B_{q+1} - B_q)\cos(\alpha_m z_{q+1}) - B_n \right) \right]
\]

The equation can now be substituted into equation (3.39) to evaluate the pressure distribution on the cantilever and hence the added mass distribution.

This hydrodynamic analysis has been included as a subroutine in the finite difference analysis described in chapter 2 to investigate the response of a fully submerged slender structure. A cantilever beam was given an initial 1st mode deflection profile calculated from the expressions derived in section 3.2.3 and was then released. The added mass distribution was calculated at each time step of the response and was found to be similar in form to the first mode mass distribution shown earlier in figure 3.27. The added mass distribution was included into the mass matrix in exactly the same manner as structural mass. In this way, the problem of having a large value of added mass close to the support position is ignored because the Rayleigh method absorbs part of the mass distribution into the supporting nodes. The model used was a five node approximation, and the changes in the added mass term with time are shown
Figure 3.32 Variation of the Lumped Added Mass Term with Time of Response
in figure 3.32. The occasional spurious result is associated with the cantilever tip passing through the zero position whilst there are small 'residual' displacements at the other nodes. The graph shows that the initial modal deflection is a good approximation and that the added mass remains constant throughout the response. The results also confirm the fact that the added mass is a function of mode number and would therefore need evaluation only once.

The effect of semi-submergence has also been investigated using this method. The structural properties of the mast were the same as that previously given for 'Structure A' in figure 2.2 except that the supports were now considered to have a rotational stiffness of 5200Nm. The number of masses used for this hydrodynamic model was varied between 5 and 15 and the determined first mode frequencies, averaged over four cycles of response, are shown in table 3.1. The final prediction of frequency for each height of water as the number of nodes approaches infinity was obtained using the method discussed in chapter 2. The inconsistencies evident at low levels of submergence are due to the errors associated with using only a four cycle average to determine the frequency. The predicted results are also shown in figure 3.33 to compare with the modal predictions from section 3.2.4. The results show that the support stiffness of 5200 Nm creates a beam which reacts, from a hydrodynamic stand-point, very much as though it were in a cantilever support.

The results have generally shown that the finite difference method can be used, quite successfully, for structural analysis of fluid-structure interaction problems in a few limited cases. If the structure is slender, so that its modal 'added mass' distributions are similar, an acceptable response can be obtained by using the distribution associated with the dominant mode. Similarly, if the structure moves only in a single mode, it can be accurately modelled. This second case may prove extremely useful where structures are of a complex nature, making it difficult to obtain the modal shape functions.
<table>
<thead>
<tr>
<th>HEIGHT OF WATER /m</th>
<th>FREQUENCY FOR NUMBER OF NODES (Hz)</th>
<th>%DECREASE IN FREQU.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>0.00</td>
<td>9.588</td>
<td>9.828</td>
</tr>
<tr>
<td>0.45</td>
<td>9.560</td>
<td>9.794</td>
</tr>
<tr>
<td>0.60</td>
<td>9.515</td>
<td>9.775</td>
</tr>
<tr>
<td>1.50</td>
<td>8.120</td>
<td>8.275</td>
</tr>
</tbody>
</table>

*Table 3.1 Frequency Values Predicted by the Finite Difference Structural Analysis Program*
Figure 3.33 Frequency Characteristics Predicted by the Finite Difference Program
3.3 Effect of Damping Due to Submergence

In using the Laplace equation to investigate the surrounding pressure field it has been assumed that the flow is irrotational and that water is an inviscid incompressible fluid (ref 36). In a real fluid, the action of viscosity, particularly at the structural surfaces allows shearing forces to build up in the flow which have the effect of placing a drag force on the structure. A wake forms behind the structure and this can lead to vorticity.

The occurrence of drag forces has been observed since the early work of Morison and since this time many researchers have worked towards the determination of drag coefficients by experiment. The drag force in Morison's equation is proportional to the square of the velocity since the flow is separated and the drag is primarily due to pressure rather than the skin friction.

3.3.1 The Use of an Equivalent Damping Term

Because of the difficulty in quantifying the exact mechanism of damping, researchers have developed the idea of an 'equivalent damping model' which dissipates the same amount of energy per cycle as the real system. Blevins (37) has used the ideal viscous damper as his equivalent damping model. This produces a resistive force proportional to the velocity.

\[ F_d = C \dot{y}(z,t) \]

The damping for one mode \( \phi(z) \) can be given as

\[ F_d = 2 m \zeta \omega_N \dot{y}(t) \phi(z) \]  \hspace{1cm} (3.45)

where
- \( m \) - mass per unit length of structure including added mass
- \( \zeta \) - equivalent viscous damping ratio
- \( \omega_N \) - natural frequency of the structure
The ratio of expended energy per cycle per unit length, $E_d$, to the total energy of the elastic structure, $E_T$, for the ideal viscous damper can be shown to be given as

$$\left( \frac{E_d}{E_T} \right)_{\text{ideal}} = 4 \pi \zeta$$ (3.46)

For the system under investigation in the present research the ratio is given as

$$\left( \frac{E_d}{E_T} \right)_{\text{real}} = \frac{\int_{0}^{T} \int_{0}^{H} F_d \dot{y}(t) \phi(z) \, dt \, dz}{\frac{1}{2} \int_{0}^{H} m(z) [y(t)]_{\text{max}}^2 \phi^2(z) \, dz}$$ (3.47)

By equating the real and ideal systems, an equivalent damping coefficient $\zeta$ can be given as

$$\zeta = \frac{\int_{0}^{H} \int_{0}^{T} F_d \dot{y}(t)(z) \, dt \, dz}{2 \pi \int_{0}^{H} m(z) [y(t)]_{\text{max}}^2 \phi^2(z) \, dz}$$ (3.48)

The main attraction of this method is that $\zeta$ can now be used as the equivalent damping coefficient and can be included into the structural dynamics equation as an ideal viscous damper acting at every node on the structure.

From Morison's equation, the drag force due to damping can be given by

$$F_d = \frac{1}{2} \rho D C_d |U_{\text{rel}}| U_{\text{rel}}$$ (3.49)

Substitution into the damping equation (3.48) gives

$$\zeta = \frac{\int_{0}^{H} \int_{0}^{T} \rho |\dot{y}|^3 D C_d \phi^3(z) \, dz}{4 \pi y_{\text{max}}^2 \int_{0}^{H} m(z) \phi^2(z) \, dz}$$ (3.50)

This equation is applicable only to the case of full submergence and can be simplified by the use of an approximation for both the value of drag coefficient $C_d$ and the added mass $m_a(z)$. 
The drag coefficient $C_d$ is dependent on both Reynold's number and a frequency coefficient dependent on the structure's dynamic characteristics.

$$C_d = C_d(\text{Re}, \frac{U_m D}{\nu})$$  \hspace{1cm} (3.51)

and Reynold's number

$$\text{Re} = \frac{U_{rel} D}{\nu} = A_y \omega_N \phi(z) \left| \cos(\omega_N t) \right| \frac{D}{\nu}$$  \hspace{1cm} (3.52)

where $A_y$ is the amplitude of displacement

$\omega_N$ is the natural frequency of the structure

$\nu$ is the kinematic viscosity of water

The drag coefficient of the circular cross section in a steady flow is a complex function as shown in fig 3.34. Empirical results are given by Schlichting(38) and can be approximated by a polynomial function of Reynolds number, of the form

$$C_d = b_1 + \frac{b_2}{\text{Re}}$$  \hspace{1cm} (3.53)

which is valid for $1 < \text{Re} < 10^4$ with values of constants as

$$b_1 = 1.3$$

$$b_2 = 10.0$$

The equivalent viscous damping is now given as

$$\zeta = \frac{\rho D^2}{4 m \pi} \left( \frac{8 a b_1 (A_y)}{3 \omega_D} + \pi b_2 \left( \frac{\nu}{\omega_N D^2} \right) \right)$$  \hspace{1cm} (3.54)

where $a = \int_0^{H_S} \phi^3(z) \, dz / \int_0^{H_S} \phi^2(z) \, dz$

The mass, $m$, includes both the mass of the structure, $m_s$, and the mass of entrained water, $m_a$, which can also be considered as part of the kinetic energy of the system. The added mass, $m_a$, can be considered to be $\pi D^2/4$ for these purposes, this being the approximate value determined for slender structures.

The form of equation (3.54) is linear as shown in figure 3.35. If the maximum value of parameter $\omega_N D^2/\nu$ is greater than 10,000 then the drag
Figure 3.34 Drag Coefficient versus Reynold's Number for a Smooth Circular Cylinder in a Steady Flow

Figure 3.35 Damping in a Still Fluid for a Fully Submerged Circular Cylinder
force is proportional to the square of the velocity over most of the cycle and the equivalent damping becomes directly proportional to the amplitude of displacement.

### 3.3.2 Variation of Damping Ratio with Depth of Submergence

Considering a semi-submerged system, equation (3.50) is no longer valid because the added mass on the system is only acting over part of its height. The corrected equation is now

\[
\zeta = \frac{\int_0^H \int_0^T \rho \ y^3 \ D \ C_d \ \phi^3(z) \ dz}{4 \ \pi \ y_{\text{max}}^2 \ \int_0^H (m_s(z) + m_a(z)) \ \phi^2(z) \ dz + \int_0^H m_s(z) \ \phi^2(z) \ dz}
\]  

(3.55)

which, using the \( C_d \) approximation given in equation (3.53), simplifies to

\[
\zeta = \frac{\rho \ D^2}{4 \ m_s \ \pi} \left( \frac{8}{3} \ a_1 \ A_y \ \left( \frac{A_y}{D} \right) + \ \pi \ a_2 \ \left( \frac{v}{\omega_N \ D^2} \right) \right)
\]  

(3.56)

where

\[
a_1 = \frac{\int_0^H \phi^3(z) \ dz}{m_a/m_s \ \int_0^H \phi^2(z) \ dz + \int_0^H \phi^2(z) \ dz}
\]  

(3.56a)

\[
a_2 = \frac{\int_0^H \phi^2(z) \ dz}{m_a/m_s \ \int_0^H \phi^2(z) \ dz + \int_0^H \phi^2(z) \ dz}
\]  

(3.56b)

The values of \( a_1 \) and \( a_2 \) are constant for a particular level of submergence and hence the form of equation (3.56) is also linear. The variation in the damping relationship can now be determined as a function of depth in submergence. Equation (3.56) is only strictly correct in cases where the mode shape is positive along the whole length of the structure, however, higher modal responses can be found by forcing the energy imparted to the fluid to be positive, this being achieved by summing only the magnitude of the energy.
As an example, the damping characteristics for the slender mast considered earlier (see figure 2.2) have been investigated. For this case it can be shown that the constant damping term, $\zeta_0$, where

$$
\zeta_0 = \left( \frac{\rho \, D^2}{4 \, \frac{m}{s} \, \pi} \right)^{\frac{1}{2}} \, b_2 \, a_2 \left( \frac{\nu}{\omega_N \, D^2} \right)
$$

has very little influence on the damping characteristics because the value of $\omega_N D^2/\nu$ is approximately 51,000 for first mode (see figure 3.35). For higher modes of response this value increases and the form of equation (3.56) is even closer to a direct proportionality between $\zeta$ and $A_y/D$. However, it is of interest to understand how the depth of submergence changes the gradient of this relationship. Equation (3.56) has been evaluated over a series of depths ranging from the non submerged to the fully submerged cases for both 1st and 2nd modes of response and the results are shown in figure 3.36 and 3.37. The important characteristics shown by the graphs can be summarised as follows.

1. The fluid damping has no effect on the response of the mast until the depth of submergence exceed 40% of its height for mode 1 and 20% of its height for mode 2.

2. The second mode damping coefficient is higher than the first mode for submergences up to 90% of its height after which the relationship is reversed. (However, the true effect of damping on the motion of the structure is governed by equation 3.45 which also includes the natural frequency of the structure and hence the second mode damping forces are always far greater than those of the first mode.)

3.3.3 Validity of Blevins' 'Equivalent Damping' Term

The main source of error in the approach developed by Blevins (31), and discussed in the previous section is the approximation used for the drag coefficient. The graphical representation of the change in drag coefficient with Reynolds number has been obtained from Schlichting's (38) presentation of experimental data determined from Wieselberger's original
Variation in the Damping Coefficient with the Depth of Submergence (using Blevins' $C_d$-Re approximation)
work. The data is a compilation of results for a number of tubes of varying dimensions thus confirming that the principle of similarity holds for the drag coefficient. However, the drag coefficient has been determined for a cylindrical tube in a uniform flow field and questions must be raised as to how this relationship is altered for a tube whose displacement varies with both time and distance along its length.

A large amount of literature has been produced which shows the variation in $C_d$ for a structure oscillating in a still fluid and alternatively for a stationary structure in an oscillating flow field. Using the principles of dynamic similarity, the drag coefficient can be shown to be dependent on four factors.

$$C_d = f(K, Re, k/D, t/T)$$  \hspace{1cm} (3.57)

where \[ K = \frac{U_m T}{D} \hspace{1cm} Re = \frac{U_m D}{\nu} \]

and \[ T \] is the period of vibration \[ U_m \] is the peak velocity \[ t \] is the time of response \[ k \] is the roughness characteristic dimension

The difficulty presented by the time dependency is normally overcome by using time invariant averages. Hence,

$$C_{d_{av}} = f(K, Re, k/D)$$  \hspace{1cm} (3.58)

Both $K$ and $Re$ are dependent upon fluid velocity and hence make it difficult to interpret experimental results. Replacing $Re$ by $Re/K = D^2/\nu T$ gives

$$C_{d_{av}} = f(K, \beta, k/D)$$  \hspace{1cm} (3.59)

where $\beta = D^2/\nu T$ and has been termed the 'frequency parameter' by Sarpkaya.

The relationship between the drag coefficient and the value $K$ has been
found empirically by both Keulegan & Carpenter (12) and Sarpkaya (10), (11), (39), (40) for varying values. Using a rigid stationary cylinder subjected to forces from an oscillating flow, the results have shown that the relationship is extremely complex and is best represented graphically as shown in figure 3.38. It can be seen that the drag coefficient has a maxima at a K value which decreases with increasing value of $\beta$. The values of Reynold's number shown in figure 3.38 are time averages over one cycle of the flow. These results can also be represented on a graph of $C_d$ versus Re and this allows a comparison with Schlichting's (38) steady flow results as shown in figure 3.39. It can be concluded that the drag coefficient for a cylinder in a harmonically oscillating flow is not closely approximated by the values obtained for steady flow at the same Reynolds numbers. It is also apparent that a relationship between $C_d$ and Re cannot be represented by one equation. However, for a particular structure, the coefficient $\beta$ is constant which opens up the possibility of determining the relationship for each value of $\beta$. Considering 'Structure A' (refer to figure 2.2), which has been used throughout this text, the values of the coefficients $Re$, $\beta$ and $K$ can be determined assuming the structure to vibrate at its first natural frequency (approximately 10 Hz).

The maximum Reynold's number $= \frac{U m D}{v} = \frac{2 \pi f A y D}{v} = 84700$

The value of $\beta = \frac{D^2}{v T} = \frac{f D^2}{v} = 8980$ (constant)

The value of $K = \frac{2 \pi A}{D}$ ; $K_{max} = \frac{2 \pi A_y}{D} = 12.6$

with the additional values:

Kinematic viscosity of water $= 1.002 \times 10^{-6}$ m$^2$/sec

Maximum amplitude of vibration $A_y = 2.0$ D

During one cycle of vibration the value of $K$ and $Re$ are considered constant. $K$ is determined by the amplitude of vibration at the start of the cycle and $Re$ is the time average over the cycle. Hence, during the response of the structure, the damping coefficient varies along a line.
Figure 3.38 $C_d$ versus $K$ for Various Values of the Reynolds Number and the Frequency Parameter (taken from reference 39)

Figure 3.39 $C_d$ versus Reynolds Number for Various Values of $K$ (taken from reference 39)
where $\beta$ is constant and for this structure the response can be approximated in figure 3.38 by the line where $\beta = 8370$. Using Sarpkaya's results from reference 3.33 and considering $K$ not to exceed 20 then an approximate relationship for $C_d$ can be defined by

$$C_d = C_1 + \frac{C_2}{Re}$$  \hspace{1cm} (3.60)

where $C_1 = 0.427$

$C_2 = 32600$

Now, for a structure oscillating at its first natural frequency, the value of $Re$ will vary along the structure and can be given by

$$Re = 2 \pi f A_y D \phi(z)$$

where $A_y$ is now the maximum amplitude during one cycle of the structure's response.

Equation (34.6) would now be simplified to

$$\zeta = \frac{\rho D^2}{\pi m_s} \left[ \frac{2 C_1}{3} \left( \frac{A_y}{D} \right) a_1 + \frac{C_2 a_2}{3 \pi \beta} \right]$$  \hspace{1cm} (3.61)

where $a_1$ and $a_2$ take the same values as given in equations (3.56a) and (3.56b). It should be remembered that the coefficients $C_1$ and $C_2$ are now dependent on the value of $\beta$ and are therefore only applicable to one particular structure. The equation remains a linear function of amplitude of deflection and the gradient $\zeta/(A_y/D)$ can be represented as a function of the depth of submergence in figure 3.40 and 3.41 for modes 1 and 2 respectively. The second term in equation (3.61) is no longer a direct function of Reynold's number and can be shown to be relatively dominant in comparison to the first term at low amplitudes of response. The value of the second term $\zeta_0$ is plotted in figures 3.42 and 3.43 for modes 1 and 2 respectively. The relationship given by equation (3.61) should be more accurate for a structure oscillating in a still fluid than those previously developed using Blevins' $C_d$ relationship, although some experimental verification is now required.
Figure 3.40 1st Mode Response of the Mast

Figure 3.41 2nd Mode Response of the Mast

Change in the Amplitude Dependence of the Equivalent Damping Term with Depth of Submergence (using Sarpkaya's $C_d$-Re approximation)
Figure 3.42 1st Mode Response of the Mast

Change in the Constant Fluid Damping Term with the Depth of Submergence (using Sarpkaya's $C_d$-Re approximation)
3.3.4 Inclusion of Hydrodynamic Damping into the Finite Difference Scheme

The use of the equivalent damping term in the previous section has shown that the effective damping coefficient is mode dependent and varies as a function of the depth of submergence and the amplitude of vibration. As with the case of hydrodynamic mass, it is impossible to satisfy the damping characteristics for each mode in a finite difference model and hence a compromise must be found. One such compromise would be to use the damping coefficient associated with the most dominant mode.

Unlike the structural damping term, hydrodynamic damping forces act along the submerged section of the structure. Using a lumped mass model would therefore require the summation of damping forces in the region of the node so that a lumped mass parameter could be placed at the node. However, the use of the equivalent damping requires only that a value of the damping coefficient $\zeta$ needs to be specified at each node. The damping force would then be given as

\[ \text{Damping force} = 2 \zeta \omega_n [M] \{u\} = [M^*] \{\dot{u}\} \]

where $\omega_n$ is the natural frequency of the active mode. The components in the matrix $[M^*]$ are equal to the components in the matrix $[M]$ multiplied by the scalar value $(2\zeta \omega_n)$. The equation of motion would then be given as

\[ [M] \{\ddot{u}\} + [K] \{u\} + [M^*] \{\dot{u}\} = \{F(t)\} \quad (3.62) \]

In the case of semi-submergence, the matrix $[M^*]$ would only include the masses of the submerged nodes, having all other coefficients set to zero. In the solution to the above equation, matrix $[M^*]$ must be updated during the response because the damping coefficient, $\zeta$, changes with the maximum amplitude of vibration. However, the use of a peak amplitude during each cycle of response is only meaningful in the case where the structure is undergoing a simple pure mode response. In complex transient responses this method would become inaccurate and difficult to implement. One possible solution to some of these difficulties would be to identify the most dominant mode of response and to satisfy its damping requirements.
However, the inclusion of hydrodynamic damping in the equation of motion affects each response mode in different ways because the effective damping applied to each mode is also dependent on the modal frequency. Therefore, it is possible that the most dominant mode of response may vary during the time of structural analysis. It can be seen that such a method of including the effects of damping would be difficult to incorporate into a dynamic analysis which uses a time domain solution and would be limited in its ability to model complex structural response.
4. Shock Loads Due to Underwater Explosions

4.1 Literature Survey

Having developed a hydrodynamic model for free vibration of a submerged structure, it is possible to include the effect of typical external loadings. The shock load is of particular interest to the present study and may take several forms, such as a pressure pulse from a detonated explosive charge or alternatively a base movement due to an earthquake. In such cases, the forcing function can be considered to be a shock loading because of the high rates of change of input which gives rise to a broad energy band in the frequency spectrum. The effectiveness of such shock loads essentially lies in the amount of energy from the input excitation that corresponds to the structure's natural frequencies. The shock load due to an underwater explosion can often produce a severe form of dynamic load and will now be considered in greater detail.

Since the inception of this study, it has become clear that a large amount of research effort has been directed towards understanding the form of the pressure pulses radiating from a detonated charge. However, it seems that there has been little progress towards predicting the way in which structural forms react to this type of loading. Part of the reason for this has been that the majority of the research was undertaken during and directly after the Second World War. Over this period, the performance of available measuring and recording equipment precluded the accurate determination of structural response. It is only since the late 1960's that the use of digital computers has enhanced the ability of the Engineer to investigate the dynamic response of shock loaded structures. It is also clear that many of the studies carried out by Defence Departments since this time have had restricted availability due to security implications. The present survey is an attempt to compile and also improve the knowledge available on the types of shock load and responses produced by underwater explosives.

It should be noted that a mixed system of units has been used in conjunction with this discussion of explosive loadings because of the continued use of traditional unit systems by present researchers.
4.1.1 The Form of the Shock Wave

The sequence of events after initiation of an underwater explosion is well established (Ref 4) and can be summarised as follows:

(1) Upon detonation of a spherical charge, a pressure wave will travel radially outwards at a velocity dependent on the type of explosive used. The explosive is converted into an incandescent gas at high pressure in a finite time while the material in front of the pressure wave remains unaffected.

(2) The high pressure gas expands and quickly compresses a spherical layer of water around the charge. Further layers are also compressed as the pressure wave is propagated away from the charge centre. The velocity of propagation decreases until it reaches a value of 1500m/s (the speed of sound in sea water), after which it remains relatively time-invariant.

(3) The remaining gas bubble expands and contracts, radiating further pressure peaks at full contraction. The peak pressure of these secondary peaks is much weaker than the original pulse and in practice only the first oscillation is visible on the overall pressure trace. This oscillation is usually termed 'the secondary pressure pulse'.

(4) The gas globe rises with time under the influence of buoyancy until it eventually breaks the surface.

As a first approximation, the pressure pulses can be considered to behave as intense sound pulses. The pressure at a point distance \( r \) from the charge centre is given by

\[
P = \frac{\rho}{r} f \left( t - \frac{r}{c} \right)
\]

(4.1)

where
- \( P \) = pressure in pulse (additional to hydrostatic)
- \( \rho \) = density of water
- \( c \) = velocity of sound in water
- \( t \) = time after initiation of the charge
The pressure $P_1$ at a distance $r_1$ can be related to a second pressure $P_2$ at a distance $r_2$ at a later time by the equation

$$P_1r_1 = P_2r_2 \quad (4.2)$$

This equation has been proved by experiment to hold for all explosive pulses except in the close vicinity of the detonation.

A typical experimental pressure/time record, obtained by a tourmaline gauge for a 0.23Kg charge of pentolite at a stand off distance of 3m is shown in figure 4.1 (Taken from reference 42). The important feature of the pulse is a vertical shock front, corresponding to an instantaneous rise in pressure to a maximum value followed by a more gradual decrease in pressure. Empirical analysis of typical results has indicated that the initial decreasing portion following the front is exponential in shape. At a later stage the rate of change in pressure reduces and the decay curve has a higher pressure than the exponential approximation. The potential damaging power, however, is related to the impulse from the shock rather than the direct pressure and hence the final tail of the graph is relatively unimportant. For this reason the exponential approximation is generally accepted as a reliable and useful forcing function. The pressure pulse approximation is given by

$$P = P_m e^{-nt'} \quad (4.3)$$

where $P_m$ is the maximum pressure in the pulse, $n$ determines the rate of decay of pressure, and $t'$ is the time measured from the arrival of the pulse. Since the pulse has an indefinite tail, there is neither a definite duration of the pulse at a given point nor a definite length of the pulse in space at a given time. However, for the theoretical analysis, the value of $n$ in the above equation determines the rate at which the pressure changes. The reciprocal of $n$ gives a time constant which is a measure of the order of time for which the pressure is of a significant magnitude.

Some empirical relationships for the peak pressure, impulse and energy in terms of the weight of the charge and the scaled distance from the
Figure 4.1 Pressure-Time Characteristics for a 0.23Kg Pentolite Charge

<table>
<thead>
<tr>
<th>Explosive</th>
<th>Peak Pressure $P_m$ ($K \times 10^{-4}$)</th>
<th>Impulse $I$ ($\alpha$)</th>
<th>Energy Density $E$ ($m \times 10^{-3}$)</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TNT (Density 1.52)</td>
<td>2.16 (2.60)</td>
<td>1.13 (1.21)</td>
<td>2.41 (3.78)</td>
<td>2.05</td>
</tr>
<tr>
<td>Loose Tetryl (Density 0.93)</td>
<td>2.14 (2.50)</td>
<td>1.15 (1.22)</td>
<td>3.00 (3.20)</td>
<td>2.10</td>
</tr>
<tr>
<td>Pentolite (Density 1.60)</td>
<td>2.25 (2.85)</td>
<td>1.13 (1.23)</td>
<td>3.27 (4.23)</td>
<td>2.12</td>
</tr>
</tbody>
</table>

Table 4.1 Values of the Coefficients Controlling the Characteristics of the Pulse from an Explosive Charge
charge have been presented by Cole (42)

\[
P_m = k \left( \frac{W^{1/3}}{R} \right)^\alpha
\]  

(4.4)

\[
I \left( \frac{t}{W^{1/3}} \right) = 1 \left( \frac{W^{1/3}}{R} \right)^\beta
\]  

(4.5)

\[
E_k \left( \frac{t}{W^{1/3}} \right) = m \left( \frac{W^{1/3}}{R} \right)^\gamma
\]  

(4.6)

where \( W \) is the weight of the charge in lbs.

\( R \) is the radial distance from the charge centre in ft.

\( k, \alpha, \beta, \gamma \) are dependent on the type of explosive

(Typical values are given in table 4.1)

The above relationships allow comparisons to be made between different explosive materials and allow an equivalent charge to be found which has the same destructive power. Experimental results have verified these relationships over the range \( 0.03 < \left( \frac{W^{1/3}}{R} \right) < 1.0 \) (lb \( 1/3 \)/ft) the latter giving \( P_m = 20,000 \) lb/in \(^2\). Many researchers have carried out further experimental investigations to expand the range of charge weight. In one specific test programme, carried out by Arons (43), it was shown that the relationships (4.4), (4.5) and (4.6) continued to operate down to values of \( \left( \frac{W^{1/3}}{R} \right) = 0.0005 \) (lb \( 1/3 \)/ft). This result is significant because it is this range of charge which is more likely to be of importance in scale model testing.

An investigation into the way in which the depth of the charge affects the pressure characteristic of the emitted pulse has been studied by Christian and Blaik (44). Rather an extreme depth range of 900 to 6,700m was used for charges whose weight ranged between 0.02 and 4.5Kg. The conclusion was that the shock wave peak pressure exhibited no charge depth effects and continued to follow the same power function that described Aron's (43) horizontal long-range measurements. Further investigation (45) of depth effects have concentrated on the frequency spectra of the pressure pulses. Peak energy levels have been shown to occur at the bubble pulse
natural frequency which is dependent on both charge weight and depth. The 'tail off' of energy at the high frequency end of the spectrum depends also on the charge depth. It has been shown that this varies from a $\frac{1}{(frequency)^2}$ relationship near the surface to a $\frac{1}{(frequency)^3}$ at extreme depths. This factor would have little significance on model testing as it seems unlikely that the charge would be submerged to such large depths. However, care must be taken in attempting to use model tests to predict the response of prototype structures which may be designed to work at far greater depths.

Another difficulty in attempting to scale up model tests is that real pressure pulses do not behave as small amplitude pulses and suffer finite amplitude effects. Simple theory is only valid for pressure pulses at distances for which the pressure is less than 30 MN/m$^2$. The passage of finite amplitude pulses involves an irreversible heating of the water and a consequent dissipation of energy by conduction through water. These effects are more pronounced in the steep front of the pulse where the most rapid changes occur and tend to decrease the pressure in the front. Studies by Marsh et al$^{(46)}$, Pocchi$^{(47)}$ and Sternberg$^{(48)}$ have confirmed the problem of finite amplitude effects although their results show some disagreement as to the amount of high frequency absorption.

To enable the pressure pulse to travel through the fluid, each fluid particle must move and thus transmit kinetic energy to neighbouring particles. Cole$^{(41)}$ has shown that the particle velocity can be found from basic shock theory and can be defined as

$$u_r(t) - u_r(t_0) = \frac{P - P_0}{\rho_0 c_0} + \frac{1}{\rho_0 c_0} \int_{t_0}^{t} \left[ \frac{P(r,t') - P_0}{\rho_0 r} \right] dt'$$

where $u_r$ - radial particle velocity  
$t_0$ - initial time  
$P_0$ - hydrostatic pressure  
$P$ - pressure at time $t$.  
$\rho_0$ - density of water  
$c_0$ - velocity of sound in water  
r - radial distance from the charge centre.

An indication of the form of this equation is given in figure 4.2 for the case of a 0.5lb charge in sea water. As expected, the particle
Figure 4.2 Particle Velocity - Time Relationship for a 0.23 Kg Pentolite Charge
velocity reduces as the stand off distance is increased. However, it should be noted that the particle velocity cannot stay positive indefinitely as suggested by figure 4.2 as this would require the gas globe to expand continually. The exponential approximation is not accurate enough to predict particle velocities for times much greater than the time constant for the charge which in this case is 0.4 ms at a stand off distance of 10 ft. (see figure 4.1).

The inevitable presence of the sea surface can lead to important modifications of the pressure profile from an underwater explosion. The pressure pulse emitted by the charge arrives at the surface with a pressure profile which can be approximated by a steep fronted shock with an exponential decay. At the surface, the pressure pulse is divided into two components, a reflected pulse and a transmitted pulse. However, due to the large differences between both the density and compressibility of air and water, the transmitted pulse tends to be much smaller than the reflected pulse and can be ignored in most cases. The pressure in the reflected pulse must be of opposite sign to the pressure at the surface due to the original pulse. The amount of tension that water can withstand is relatively small and this can lead to cavitation at the reflective surface. Wentzell et al (49) have studied surface cavitation and have produced a mathematical model which predicts the occurrence at far lower pressures than achieved experimentally. This result is not particularly surprising since the ability of water to resist tensile loads is highly dependent on factors such as the amount of air dissolved in the water and the amount of impurities. Further work on cavitation has been carried out by Driels(50),(51),(52) who has used a pneumatic gun and shock tube to produce pressure pulses of a similar form to explosive pulses. This 'more carefully' controlled experiment has produced much better correlation with cavitation theory.

In the case of an explosion close to the sea-bed, the reflected pulses may be seen as pressure pulses behind the main pulse. The transmitted pulse is much larger than in surface reflections and in the case of a rocky sea-bed the reflected pulse may be of the order of 30-50% of the main pulse. In the case of a muddy sea-bed the reflected pulse may be negligible.

An interesting report concerning the design of model test rigs has been produced by Bjorno and Levin(53). Realising that full scale tests can be very expensive, they have attempted to reduce the cost by validating
the use of model tests. The results of experiments performed using small amounts of explosive, ranging from 0.2 to 0.6g, show that small scale tests are valid for modelling much larger practical applications. However, in using scaling rules, there are erroneous effects which must be understood to obtain realistic results. For example, the finite size of the transducer may lead to distortions in the recorded leading edge of the shock wave. This is particularly troublesome in the case of short waves arising from the detonation of small amounts of explosive; however, empirical expressions can be used to correct these errors.

In conclusion to the study of previous research into the form of the explosive pressure pulse, it has been shown that the transmission of shock waves through a fluid medium is a complex process. Reflection effects, depth effects and finite amplitude effects have negligible influence except in positions of close proximity to boundary surfaces or the charge centre or at extreme depths. Therefore, it is possible to accurately predict the free field pulse resulting from an explosion under normal conditions.

4.1.2 Interaction of Shock Waves and Submerged Structures

The presence, form and behaviour of a structure in the path of a radiating shock wave modifies the resulting pressure, and it is not correct to assume that the target is subjected to the free field pressure. As an example, the pressure produced on an air backed plate when there is assumed to be no cavitation in the fluid would be as shown in figure 4.3. The period of negative pressure would have the effect of placing tension on the water and because water is unable to withstand even low tensile forces, cavitation may occur.

A large number of the investigations into structural response due to transient pressure pulses has concentrated on plate like structures. The boundary conditions have been varied and both elastic and plastic responses have been studied. However, the majority of published results are for the plastic response of both water backed and air backed plates, and have concentrated upon the analysis of the final deformed shape rather than transient response characteristics.

One of the few transient analyses of plates has been developed by Driels\(^{54}\) from the theory first outlined by Fox\(^{41}\). Treating the explosive pulse as a plane wave which strikes the plate at normal incidence, the resulting pressure pulse is shown to be dependent on the
Figure 4.3 Pressure Load Acting on the Surface of a Plate Resulting from a Nearby Explosion
motion of the plate. The motion of the plate is governed by an equation of the form (rigid body motion)

\[
x = \frac{p_0}{m n^2 (\varepsilon-1) \varepsilon} \left\{ (\varepsilon-1) + e^{-n\varepsilon t} - \varepsilon e^{-nt} \right\}
\] (4.7)

and the pressure acting on the plate is given as

\[
p = \frac{p_0}{(\varepsilon-1)} \left\{ \varepsilon e^{-n\varepsilon t} - e^{-nt} \right\}
\] (4.8)

where \(p_0\) is the maximum pressure in the incident pulse.

\(n\) is the exponential parameter in the decay of pressure

\(t\) is the time measured from first arrival of pulse.

\(m\) is the mass of plate per unit area of surface

\(\rho\) is the density of water

\(c\) is the speed of sound in water

and

\[
\varepsilon = \frac{\rho c}{m n}
\]

Driels\(^\text{(54)}\) has examined the effect of assuming non-zero cavitation tension and has shown that the predicted plate damage would increase as a consequence. However, due to the motion of the plate being restricted by its stiffness, the motion of the fluid on the opposite side of the cavitation void will catch up and 'reload' the structure. This phenomena is known as 'spray reloading' and cancels out most of the dynamic effects of cavitation.

Another important transient plate analysis by Lindh\(^\text{(55)}\) has studied the transmission of the shock through an immersed plate. This project was concerned with the pressure reducing ability of a plate protecting a secondary structure. It was shown that the most important parameter in the attenuation effect was dependent on the factor \(d/c\cdot t_c\) where \(d\) is the thickness of the plate, \(t_c\) is the time constant of the incident pulse and \(c\) is the propagation velocity of the shock.
The way in which a pressure field is affected by the presence of a structure has been widely studied both numerically and experimentally. Bjorno\textsuperscript{(56)} has shown that when a shock wave passes around the edge of a plate which has been placed at right angles to the direction of shock propagation, a distinct 'shadow and light' region appears behind the plate. That is to say that the line trailing from the top edge of the plate marks the boundary where the pressure gradient is maximum. The pressure behind the area of the plate is at its lowest value and rises suddenly to a maximum in the free stream where the pressure field is unaffected by the plate.

The two dimensional pressure field created around a cylindrical object due to the passage of a pressure transient can also be shown to be a complex distribution. Collins and Chen\textsuperscript{(57)} have solved the shock relationships numerically and have shown that their predicted results correlate well with the experimental pressure field measured by Heilig\textsuperscript{(58)} using shadowgrams. The passage of the shock front is shown in figure 4.4 as a function of time, and the surface pressure in figure 4.5 is shown to be a complex function. The peak pressure occurs at a position around the circumference of the cylinder dependent on the occurrence of irregular reflection. A line trailing from this point defines the region where the shock front is affected by the cylinder. Eventually, the shock front reforms to the form of a plane wave front. If the resulting pressure field is considered to be acting over the projected area in that region, then the impulse imparted to the front face of a rigid structure can be considered to be approximately the same as that seen by a flat plate placed at right angles to the shock front. However, it is also clear from figure 4.5 that the pressure field acting behind the cylinder is much weaker than that on the front face. Collins and Chen have expressed concern that the movement of the cylinder wall would have a dramatic effect on the pressure field. However, in the present research the problem of deformation of the circular cross-section is not considered significant in comparison to the full body motion of the structure. The frequencies of structural response are very much lower than those associated with the shock pulse and hence movement of the structure during the loading phase can be considered to be a function of the acting pressure alone.

The most applicable report obtained from the literature survey has described a shock trial carried out by the Swedish Ministry of Defence. Hellquist\textsuperscript{(59),(60),(61)} has reported on a trial which made an experimental
**Figure 4.4** Diffracted Shock Shapes on a Cylinder with Initial Shock of Mach Number 1.92 in Air

**Figure 4.5** Surface Pressure on a Cylinder Due to the Passage of a Shock of Mach Number 1.92 in Air
study of the dynamic response of a submarine hull to shock loading from an
explosive charge. The size of the submarine section was 11m in length
with an outside diameter of 5.7m and an overall weight of 283 tons. The
size of the explosive charge was 137Kg and a total of 21 shots were carried
out for stand-off distances varying from 5.7 metres to 23 metres in all
directions around the submerged structure. In order to record the
response, 600 strain gauges, 300 accelerometers and 50 pressure transducers
were used giving 2700 recorded signals. However, very little information
about the results is given, possibly for security reasons. Although a
large amount of instrumentation was provided, it seems that the
investigators were more interested in the ability of the structure to
survive the shock rather than being able to predict the response.
Attempts to obtain more information about the shock trial have proved
fruitless.

In looking at the structural effect of shock loads due to underwater
explosions, it is important not to ignore work carried out in related
fields such as explosions in air. Both chemical and nuclear explosions
produce pressure forms similar to figure 4.1, although the analysis is
often carried out assuming the shock wave to be of a linear form. The
similarity between chemical and nuclear explosions is confirmed by the use
of an equivalent weight of chemical explosive to quantify the power of the
explosion. Bulson(62) has discussed this similarity and has summarised
studies carried out over the last 30 years in an attempt to produce a
reference for designers of blast resistant structures.

More detailed information about the pressure signatures from
explosions in air has been given by Stoner & Bleakney(63) and
Gubkin(64). Relationships between peak pressures, charge weight and
stand-off distance were produced in the same way as that used by Cole(42)
and Arons(43) for the submerged case. The peak pressure is shown to vary
with shape of charge and also with type of explosive. For a spherical
charge of pentolite, the peak pressure is given as:

\[
P_m = \frac{8.63}{Z} + \frac{295.1}{Z^2} + \frac{7823}{Z^3}
\]

(4.9)

and \( Z = \frac{R}{(\rho T)^{1/3}} \)
where \( P_m \) is the excess peak pressure in atmospheres
\( R \) is the distance from the charge
\( \rho \) is the specific gravity of the explosive
\( \tau \) is the volume of charge.

This relationship is valid for \( 18 < Z < 110 \).

The work of Newmark\(^{(65),(66)}\) in the late 1950's has shown that for blast design the form of the free field pressures has both a negative and a positive phase. In most cases this can be approximated by a linear form which neglects the period of the negative pressure (figure 4.6). The pressure effect due to the air mass movement is known as the dynamic pressure and can be approximated as

\[
P_d = P_{d_0} \left(1 - \frac{t}{t_{d_0}}\right)^2 e^{-2t/t_{d_0}}
\]

(4.10)

If the pressure is not reflected by the surface then it is called the side-on overpressure and can be approximated as

\[
P_s = P_{s_0} \left(1 - \frac{t}{t_0}\right) e^{-t/t_0}
\]

(4.11)

where \( P_{d_0} \) is the peak dynamic pressure
\( P_{s_0} \) is the peak side-on overpressure
\( t_0 \) is the duration of the positive phase of the over pressure
\( t_{d_0} \) is the duration of the outwardly going directed phase of the dynamic pressure
\( t \) is the time after initiation of the blast pressure.

These relationships are applicable for overpressures less than 180 KN/m\(^2\).

Similar to the submerged case, placing a structure in the pressure field greatly changes the distribution. The effective load on the front face of a structure subjected to an air blast would be a superposition of overpressure, dynamic pressure and reflected pressure. The maximum reflected overpressure is given as

\[
P_{r_0} = P_{s_0} \left(2 + \frac{6P_{s_0}}{7P_{r_0} + P_{s_0}}\right)
\]

(4.12)
Figure 4.6 Free Field Pressure Pulse from an Air Blast

Figure 4.7 Effective Loading Pressure on a Rectangular Structure

Figure 4.8 Force - Time Loading from an Air Blast for a High Frequency Structure
The form of the effective pressure on a rectangular structure is shown in figure 4.7.

Scale model investigations into the effects of blast load on small cantilever beams were implemented by Baker(67). Charges of 0.06Kg pentolite with stand off distances of 2 to 2.5m were used to shock load cantilevers of lengths ranging from 0.15 to 0.6m in length. Strain gauges were used to measure the response and both elastic and plastic responses were investigated. It was shown that the same similarity principles which operate for air blast pressures, impulses, etc., as discussed earlier, operate also in the response of structures to these blast effects. Some uncertainty, however, has been expressed for structures whose behaviour varies markedly with strain rates. Further tests carried out at the Ballistics Research Laboratory, USA by Sperrazza(68) has confirmed that scaling laws such as those presented by Baker and the similarity laws proposed by Sachs are correct. Using Buckingham's \( \pi \) theorem, similarity equations were derived which showed that the peak over-pressure and scaled impulse \( \frac{I}{W^{1/3}} \) were both functions of the scaled distance, \( \frac{R}{W^{1/3}} \).

It has, however, been shown by Kornhauser(69), that the over-pressure time history of a shock wave has little correspondence with the load experienced by a structure enveloped by the shock wave. If the structure has a very high natural frequency, the effect of the diffraction phase of the blast may be severe, whereas its effect will be negligible for cases involving low natural frequencies. During the diffraction phase, the pressure sustained by the structure, as previously reported by Newmark(65), will be much greater than that of the static side-on overpressure. The much longer drag phase which comes after the relatively short diffraction phase subjects the structure to a far weaker loading which is a function of the dynamic pressure behind the shock front. A typical force time relationship is shown in figure 4.8.

The determination of the response of structures in air to shock loads from explosive charges has been attempted, both experimentally and theoretically, by many researchers. The response of cylindrical cantilevers has been investigated by Bothel et al (70) and Kim et al(71). The response of flat plates has been investigated by Mazumdar(72),(73), Ross(74) and Cost(75). In most cases finite difference techniques have proved to be the best method to compute the shock conditions particularly because of the ease of controlling the excitation using such a method. In the case of cantilever structures, the use of a
varying drag coefficient has been shown to give greater improvements in correlation of theory and results than previously achieved by Baker\(^{(67)}\).

In conclusion, it can be seen that although a considerable research effort has already been spent, further work is required for a full understanding of the phenomena involved. In the case of underwater explosions it is known that frequencies associated with the high propagation velocity of the shock and the bubble pulse are far higher than the normal modes of structural response. Factors not completely understood are the relative importance of direct loads imposed by the pressure wave and those due to the shock movement of supporting structures. In order to obtain basic information about the importance of such factors, some experimental trials must be undertaken.
4.2 Damage Criteria for Shock Loaded Structures

It has been shown that the effectiveness of an explosive charge is highly dependent on the type of structure being loaded. In the case of a completely rigid structure, the peak pressure and hence damage capability can be shown, by using Cole's (41) similarity expressions, to be dependent on the factor $W^{1/3}/R$ (where $W$ is the weight of charge and $R$ is the stand off distance). If the structure is flexible, it is able to move under the influence of the pressure loading and this has been shown by Driels (54) to reduce the effective pressure loading on its surface. In such cases Cracknell (76) has shown that the damage capability of a charge becomes dependent on the factor of $W^{1/2}/R$.

Using Cole's (42) expression for the pressure-time relationship and Fox's (41) expression for the pressure-distance relationship, the pressure field resulting from the detonation of an explosive charge can be represented by a three dimensional graphical plot as shown in figure 4.9. It can be seen that the ability of a charge to deliver a certain damage level can be controlled by varying either the weight of the charge or the stand-off distance.

The damage law,

$$\left(\frac{W^{1/2}}{R}\right) = \text{constant} \quad (4.13)$$

can be represented by figure 4.10. The value of the constant is usually termed the 'shock factor' and can vary between 0, for a shock whose damage is considered to be minimal, and 1.0 ($\text{lb}^{1/3}/\text{ft}$) for severe damage.
Figure 4.9 Relationship of Pressure with Both Time and Distance from the Charge

Figure 4.10 Dependence of the Shock Factor on the Stand-off Distance from the Explosion
4.3 Inclusion of Explosive Loads into the Structural Analysis

The shock loading imposed on a submerged slender mast due to the detonation of an explosive charge may appear in two dominant forms:

(1) A direct loading of the mast due to interaction between the pressure pulse and the structure's surface area.

(2) A dynamic input to the support positions due to the response of the supporting structure to the same pressure transient.

The relative importance of these loading conditions is dependent on many factors such as the shape, size and natural frequency of the structure and its support.

4.3.1 Direct Pressure Loading

It has been shown that the free field pressure pulse can be approximated by the exponential form

\[ P = P_m e^{-t/T} \]

where \( t \) is the time after arrival of the peak pressure

\( T \) is the time constant of the charge

and \( P_m \) is the initial pressure in the pulse and is given as

\[ P_m = k \left( \frac{W^{1/2}}{R} \right)^\alpha \]

where, for the case of pentolite \( K = 2.25 \times 10^4 \)

\[ \alpha = 1.13 \]

The effect of this pressure pulse on a structure is highly dependent on the shape of the structure. In the case of a cylindrical mast, the major form of loading is due to the pressure pulse acting on the projected area of the mast. For far more slender bodies, the effect of increased particle velocity associated with the transmission of the pulse would create 'skin friction' type forces on the body which may be of the same order of magnitude as the direct loads.

The effect of the direct load acting on a cylindrical structure would not be as severe as the case of a plate with the same projected area because the flow around the surface would allow a reduction in incident
pressure. However, an analysis can be formulated using the theory developed for a plate by including an area coefficient whose value is less than 1.0 to take the reduction of pressure into account. The analysis can be simplified further by considering the pressure pulse to arrive at the curved surface of the cylindrical mast instantaneously.

If $u_i$ and $u_r$ are the particle velocities due to the incident and reflected pulses respectively, then the continuity of the structure gives

$$\dot{x} = u_i - u_r \quad (4.14)$$

where $\dot{x}$ is the velocity of the structure

and if

$$P_i = \rho c u_i$$
$$P_r = \rho c u_r \quad (4.15)$$

then,

$$\rho c \dot{x} = P_r - P_i \quad (4.16)$$

where $\rho$ = density of water
$c$ = speed of sound in water
$P_i$ = pressure in the incident pulse
$P_r$ = pressure in the reflected pulse

The total pressure acting on the structure is the sum of both the incident and the reflected pulse, hence

$$P = P_i + P_r$$
$$= 2 P_i - \rho c \dot{x}$$

giving,

$$P(t) = 2 P_m e^{-t/T} - \rho c \dot{x}(t) \quad (4.17)$$

It can be seen that for very stiff structures, the second term in the above equation will be small in comparison to the incident pressure term and hence the pressure will be twice the incident loading as previously discussed by Fox(41). In the case of flexible structures, the movement of the structure will result in a reduction of the pressure loading.

The form of equation (4.17) needs only minor modification for incorporation as a loading function in a finite difference structural analysis program. Due to the nature of the lumped mass model, the forcing
function can only be placed at the nodal positions. The loading term must therefore be lumped at these positions by multiplying the acting pressure by the 'effective' projected area modelled at each node. This 'effective' area takes into account the effect of pressure reduction due to flow around the cylinder. The projected area at each node must be determined by the same procedure used for the lumped masses. Hence, a load vector \( \{ F(t) \} \) can be created which requires an updated time \( t \) and structural velocity \( \{ x \} \) at each time step of analysis. The pressure loading can be considered to have a negligible effect after a time approximately equal to twice the charge's time constant after arrival of the pulse. The general loading term is:

\[
\{ F(t) \} = \{ P(t) \} \{ A^* \}
\]

where

\[
P(t) = 2 P_m e^{-t/T} - \rho C A \dot{x}_{t-\delta t}
\]

and

\[
A^* = C_a A
\]

\( A \) = the projected area of the structure

\( C_a \) = coefficient to account for flow around the structure

As a result of the flow around the cylindrical structure, the back of the structure will also see a rise in pressure due to the passage of the pulse. However, it has been shown by Collins and Chen that the pressure is far weaker than the incident pulse and can be neglected. It has also been shown that an approximate value of \( C_a \) would be in the region of 0.5.

### 4.3.2. The Dynamic Response of Support Positions

In shock tests carried out on submerged masts, the supporting structure is not necessarily fully rigid and may also respond to the shock input. In the trials undertaken as part of this research and described later in section 5.1, it will be seen that bodily motion of the supporting structure is an important factor in understanding the response of model masts. A sketch showing the layout of the test rig is shown in figure 4.11. In this case, the support vessel used was a large cylindrical tube with tapering end caps which held the model masts in a transverse orientation. The vessel was oriented so that its axial centre line aligned with the charge centre.
Figure 4.11 Experimental Layout of the Shock Test Rig

Test rig body supporting transverse masts

 Explosive Charge
The analysis of this supporting vessel can be carried out in exactly the same way as the model masts. However, the importance of skin friction effects needs re-evaluation. The ends of the cylindrical vessel are covered by a conical cap which has a tapering angle of 45° and makes the structure relatively blunt in comparison to an aerofoil shape. Skin friction effects were again considered to be of negligible importance in comparison to drag effects on the body. The loading function could therefore be considered to be due only to direct pressure loading and because of the bluntness of the structure, the coefficient \( C_a \), defined earlier, could be considered to have a value close to 1.0. The pressure acting on the front face of the body is of the form

\[
P(t) = 2 P_m e^{-t/T} - \rho c \dot{x}_{t-\delta t}
\]

where \( \dot{x}_{t-\delta t} \) is now the velocity of the body. However, the size of the body is far more massive than that of the masts and can be considered to be relatively rigid. In such a case, the pressure loading reduces to

\[
P(t) = 2 P_m e^{-t/T}
\]  

(4.19)

Unlike the shock wave travelling over the cylindrical masts, as the wave travels along the side of the body, its shock front is likely to reform and when it reaches the end of the body it will place a restoring pressure force on the back face. The pressure equation is now of the form

\[
P(t) = 2 P_{m1} e^{-t/T} \quad \text{for } t < \delta t'
\]

\[
= 2 \left( P_{m1} e^{-t/T} - P_{m2} e^{-(t+\delta t')/T} \right) \quad \text{for } t \geq \delta t'
\]

(4.20)

where \( t' \) is the time taken for the shock wave to travel over the length of the body. The peak pressures \( P_{m1} \) and \( P_{m2} \) are not identical as their position of occurrence is considered to be different. It can be seen that the excursion of the body is now also dependent on the length of the body.

The forcing function can be written as

\[
F(t) = 2 A^* P_{m1} e^{-t/T} \quad \text{for } t < \delta t'
\]

\[
= 2 A^* \left( P_{m1} e^{-t/T} - P_{m2} e^{-(t+\delta t')/T} \right) \quad \text{for } t \geq \delta t'
\]

(4.21)
and \[ A^* = C_a A \]

\( A \) = the projected area of the structure  
\( C_a \) = coefficient to account for flow around the structure \( (\approx 1.0) \)

By way of example, the function described by equation (4.21) is shown in figure 4.12 for the case of a 0.23Kg charge of pentolite at a stand-off distance of 3m for a body 0.3m in diameter and having three different body lengths. However, the response of this structure to these forcing functions is highly dependent on the natural frequencies of the body.

The one dimensional rigid body motion can be computed using a one degree of freedom finite difference model. The system can be represented by the free body diagram

\[
\begin{align*}
\ddot{x} = &\pm C_x \dot{x}^2 \\
M = &\text{elastic restoring forces on the structure} \\
K_x = &\text{fluid damping forces}
\end{align*}
\]

Resolving the forces in the direction of motion produces the general dynamic equation

\[ M \ddot{x} + C \dot{x}^2 + K x = F(t) \]  \hspace{1cm} (4.22)

where \( K_x \) represents the elastic restoring forces on the structure and \( C \dot{x} \) represent the fluid damping forces. Using the same finite difference equations defined in chapter 2 (equations 2.7 and 2.8) for the velocity and acceleration, the motion of the body to the forcing function \( F(t) \) can be obtained.
Figure 4.12 Force - Time Relationship Acting on a Cylindrical Body
4.3.3 The Dynamic Response of Submerged Masts

The effects of both the direct loads and those on the supporting structure can now be included into the dynamic analysis program for the mast. The movement of the support structure can be included into response as a horizontal displacement of both support nodes and the direct pressure as a forcing function at the free nodal positions. A time delay between the occurrence of the two phenomena must be included to account for the time taken for the pulse to travel between the front of the body and the surface of the masts. A flow chart representation of the load calculation phase is shown in figure 4.13.
INPUT VALUES
Explosive Material
Charge Size
Charge Time Constant
Stand-off Distance
— Body front face
— Body back face
— Mast surface

PRESSURE-TIME CALCULATION
— Mast
— Body

TIME COUNTER

Has shock wave reached the mast?

Calculation of Body Response
Calculation of Nodal Forces
Calculation of Mast Response

Has the shock wave left the region of interest?

Calculation of Mast Response

Time Condition Satisfied?

STOP

Figure 4.13 Flow Chart for the Explosive Loading Routine
5. **Offshore Test Rig**

5.1 **Test Rig Requirements**

It has become apparent that there is a lack of understanding of shock transmission from a free field pressure pulse into a dynamically responding structure. In the case of a structure which has large external fixtures, such as masts, it is not known whether it is the shock transmitted through its supports or the direct transient loading that creates the 'worst case' condition. There is also uncertainty as to how much the flow around the structure changes the effectiveness of the pressure pulse. To obtain a greater understanding of such phenomena, it is necessary to carry out experimental trials on a scale model structure.

The requirements of the model structure were that a mast with a slenderness ratio of approximately 100 should be held in a submerged condition by a large structure whose motion could be determined from rigid body analysis. This supporting structure was to be of a shape large enough to support the mast firmly, yet slender enough not to cause large disturbances to the shock passage. These contradicting requirements were satisfied by using a tubular body with tapering ends, which held the mast at right angles to the length of the body. In order to preserve symmetry of the structural loading, two masts were used in a back-to-back configuration and were joined by a bolted flange. The mast fixing supports were provided by angular contact bearings secured by taper-locking sleeves - these essentially created a pin joint support. The layout of the model is sketched in figure 5.1.

The offshore site for the tests was chosen to provide the structure with a free expanse of water of at least 4.5m on all sides. Allowing for changes in the tide, the available depth of water was 14m.

The main body of the rig was supported in a pendulum arrangement by a crane extending out from the jetty. Restraints on body motion were imposed only in the vertical direction. Symmetrical loading of the structure was assured by the alignment of the charge centre with the axis of symmetry of the main body.

It should be noted that this work was carried out in collaboration with the Admiralty Research Establishment who assumed responsibility for
the control of the experiments and provided the data collection facilities. However, aspects of the test rig design, instrumentation selection and trials procedure were handled as part of this research programme and full responsibility for the subsequent data analysis was assumed.
5.2 Instrumenting an Underwater Shock Rig

5.2.1 Transducer Resonance

The problems of measuring the response of an underwater structure which is shock loaded by a pressure pulse are formidable. The high energy, high frequency shock has an ability to excite the most unexpected modes in a structure. Secondly, because some transducers rely on movement for their measurement they necessarily have a natural frequency and in extreme shock conditions can themselves be excited by the pulse. This tends to have the effect of swamping the true response signal with the unwanted response of the transducer. In some extreme cases the amplitude of the resonant signal may be higher than that of the true response signal. This problem of transducer resonance can be eliminated by use of damped transducers which are designed so that their natural frequency characteristics are controlled. However, because their outputs are damped, the response suffers from small phase distortions whereas the output from the undamped transducer is far more accurate at frequency levels below the transducer's natural frequency. Analytical techniques are available which use digital signal processing methods to filter out mechanical noise due to transducer resonance. Essentially, these methods produce a time domain weighting sequence which has the characteristics of a band filter in the frequency domain. This weighting function is then multiplied onto the noisy signal to produce a final signal which has the characteristic frequencies only in the frequency band of the filter. However, great care must be taken when using these techniques. To correctly determine the response signal, the complete signal including the noise must be fully measured. That is to say, if the signal exceeds the range of measurement because of the high amplitude resonance, then no information can be obtained from that area of the signal. If this 'clipping' has been due to overloading of the measuring device, then once the signal returns to within the measurement 'window', the subsequent information can be regarded as true response data. However, if the limiting of the signal was due either to the amplifier reaching full output or the transducer hitting its mechanical 'stops', then it is very unlikely that any of the data is of any value. Once saturation levels are reached, it takes a certain time for the electrical systems to recover. The
indeterminancy of this time makes it difficult to interpret the data in such cases. A more detailed description of filtering techniques is given in Appendix H.

When using filtering techniques care must also be taken to check whether the true data obtained after filtering is significant in comparison to the noise levels. As is often the case with high 'g' accelerometers, the signal-to-noise ratio is low and the true signal can sometimes be strongly influenced by the bit level of machine recording the signal.

5.2.2. Types of transducer available

The requirement of response measurement for the shock rig is to measure the movement of both the supporting structure and mast to obtain a fuller understanding of the way shock is input to the system. This also requires the measurement of the pressure pulse input to the system.

To measure the response there are many options available:

(i) Accelerometers

Accelerometers are either one of two specific types. The first are piezoelectric devices which have a seismic mass supported by the transducer body through quartz elements. The quartz elements produce a signal proportional to the force applied which, for the constant seismic mass, is proportional to acceleration. The second type are resistive devices which consist of a seismic mass supported by a cantilever beam which has semiconductor strain gauges bonded to the surface. In this case the movement of the mass produces changes in the gauge resistance proportional to the acting acceleration levels. Both types of accelerometer have the option of being damped or undamped.* The damped accelerometers have basically the same low frequency response as the non damped version but have highly suppressed natural frequencies. The damping is commonly provided by using a damping medium in which the seismic mass vibrates. However, another form of damping may be provided externally in the form of a rubber mounting device which filters out the high frequency at the signal input stage. This method will be referred to as mechanical isolation.

* Note: "undamped accelerometer" is the manufacturer's terminology and should actually be defined as a lightly damped transducer.
The selection of accelerometers for the shock trials was based predominantly on experience of shock tests in air\(^{(1)}\). This work had been carried out solely with the use of Entran piezoresistive damped accelerometers. In the main, this type of accelerometer was used; however, as an extension to the shock investigation, the opportunity was taken to compare other types of measuring device.

The positioning of the devices were as follows.

(1) A bi-axial undamped piezo resistive accelerometer at the tip of the mast

(2) A bi-axial undamped piezo resistive accelerometer at the axis of symmetry of the supporting structure.

(3) A single axis accelerometer at the support position of the mast

(4) Several alternative accelerometers placed at the same location as (2).
   
   (a) Piezoelectric high 'g' accelerometer
   (b) Damped piezoresistive high 'g' accelerometer
   (c) Mechanical damped piezoresistive low 'g' accelerometer.

(ii) **Strain Gauges**

The obvious advantage of strain gauges for measuring shock response is that because of their nature, they do not have moving parts and hence do not have a true resonant frequency. The output from such devices will contain only real response data with no requirement of time consuming filtering operations. The other advantage of not having resonant frequencies is that the sample rate of the measurement needs only to be fast enough to eliminate aliasing of the highest response frequency. In the case of accelerometers it is strictly necessary to measure at twice the natural frequency of the transducer.

The major drawback of using strain gauges is that it is difficult to interpret the data from the device in the form of a displacement...
profile. To obtain a reasonable idea of the deflected shape would require the use of many strain gauges. However, the strain gauge is very good at producing modal details such as the natural frequencies of interest.

In the shock trial only one full bridge circuit of gauges was used. It was placed inside the mast in-line with the support position, where it was considered that it would see maximum strain due to the deflection of the mast alone.

(iii) **Displacement Transducers**

There is a vast range of types of displacement transducer available but in most cases they are not recommended for dynamic measurement because of size and natural frequency problems. The solution taken in this shock trial was to use a non contacting device working on an inductance principle to measure the proximity of a structural surface. This had the advantage that it could be rigidly fixed to the main body of the rig without imposing any restraint on the mast. The transducer was positioned at the centre of the body so that it could measure movement of the flange joining the two masts together.

(iv) **Pressure Transducers**

Because of the severe conditions under which the test rig operates, the pressure transducer selected was a steel diaphragm, piezoelectric type. These transducers are very sensitive to pressure variation, operate over a large range of pressures and have a fast response time; all of which makes them seemingly suitable for shock application. It will be shown later that their use in the field has suggested that they are highly influenced by acceleration effects and also by shock waves travelling through their mountings, all of which mask the true pressure signal.

In the shock trials, the pressure transducer was positioned in either one of two locations. The first was at the tip of the body, closest to the explosive charge - this would enable a measurement of the pressure loading as seen by the body. The second position was on the surface of the body in line with the mast position - this would measure the pressure in the pulse as it passed over the transducer.
5.2.3. **Transducer Requirements for Submergence**

Some of the transducers described in the previous section are not suitable for use in direct contact with water. Accelerometers, however, can be hermetically sealed but the extra expense has prevented their use in the present series of tests. All measuring devices were therefore mounted within the sealed structure. In positions such as the mast tip, internal mounting of the accelerometer is made particularly difficult because of the limited space available. In order to fix the transducer firmly in position, a special taper-locking mounting device has been designed. However, before using such a device, it must be proved that the performance of the accelerometer is unaffected. It is possible that a device of this nature may shake loose after repeated shock tests and this may lead to erroneous signals being output by the transducer. Preliminary tests were carried out on a similar tubular mast held in an impact shock rig. Comparison of the output with an external accelerometer which had been mounted in a conventional way showed that the device created no ill effects. This mounting device was subsequently adopted for both internally mounted accelerometers used for measuring the mast response.

The pressure transducers were fitted so that their faces were flush with the surrounding structure and were unaffected by submergence because the diaphragm inside the transducer which reacts to the pressure transient was seam welded in position thus protecting the sensing elements from the water. The remaining transducers were all fitted internally and would not come into contact with sea water as long as the tubular body remained sealed.

5.2.4 **Amplification and Recording Devices**

The transducers discussed in section 5.2.2 are only capable of generating either small voltage outputs in the case of strain gauge bridges or small current outputs in the case of piezoelectric gauges. In order to measure and store these signals accurately, a certain amount of amplification must be applied to the signal. The amplifiers used for the accelerometer outputs were Entran EM series amplifiers and for the strain gauge bridge output a Fylde FE359 transducer amplifier was used. In the case of the pressure transducer a Fylde FE128 charge amplifier was used.
For all of these amplifiers, the sensitivity could be varied and a maximum output of 10v was possible.

In order to reduce electrical noise pick-up in the connection leads running between the transducer and amplifier it is necessary to minimise their length. It was decided that this would be achieved most effectively by placing the amplification units inside the main body of the rig. All of the amplifiers were fixed to one place and shock absorbing support, using compressed foam rubber, was provided to reduce shock loads to the instrumentation. Some basic experiments were carried out to find the foam rubber's damping characteristics and an analytical simulation was carried out to check that the foam rubber provided enough support to prevent damage to the expensive amplifiers.

The output from the amplifiers was taken to shore via co-axial screened cables which were protected from the sea water by an outer flexible covering. On site recording was carried out using a 'Kontron' transient recorder with a Teac FM tape recorder. The Kontron transient recorder is a 12 bit machine which can sample the signals at frequencies up to 10 MHz and has 32 Kbytes of memory for each of its 10 channels. The collected data can be dumped down onto flexible discs for permanent storage. Full details of the instrumentation specifications are given in Appendix I.

The selection of sampling frequencies used for measuring the output signals is dependent on three factors. Firstly, the levels of frequency expected from the response of the structure - this usually requires a maximum frequency of 2KHz. Secondly, the requirement of sampling at frequencies at least twice as fast as the natural frequency of the transducer gives a sample rate of 8KHz. Finally, to obtain good resolution on the pressure transducer output a sample frequency of up to 20 KHz is required. It can be seen that the requirement of pressure measurement creates the greatest restraint on the required sampling rate. In this case, the 20 KHz sample rate was adopted at the expense of losing valuable low frequency information for the structural response.

In the actual shock trials, the data was recorded at 2 msec intervals over a period of 65 msec. Triggering to initiate the recording was taken from the pressure transducer and 16.5 msec of data was recorded before the initiation of the response. In hindsight, the recording period was far
too short and too much high frequency data was obtained. It was found that only one point in eight was required for the highest frequency of interest and hence 88% of the storage capacity was wasted. In frequency transforming the signals it was found that the definition of the measured spectrum was limited to 20 Hz. It was therefore impossible to measure frequencies under 20 Hz and all frequencies of interest were subject to an error of at least ±10 Hz. In further tests, it would be advisable to decrease the sample rate to the range of 20–50 KHz and continue to record over a period of at least 600 msec. This would allow for much more accurate frequency determination in the structural frequency range. However, the test site used for the shock trial was only available for a limited period and the tests were not repeated with the above recommendations. The results obtained from the shock trial have provided some interesting information even with the limited frequency definition. The points of major interest will be discussed in the subsequent sections of this chapter.
5.3 **Natural Frequency of Mast Vibrations**

To correctly interpret the results of the shock trials it is useful to have a knowledge of the mast's deflected mode shapes and their associated natural frequencies. Firstly, the structure must be analysed in air to obtain the mode shape characteristics. Each mast can be assumed to be a slender beam of constant section which is held at its centre by two bearings which act as pin joints (see figure 5.1). An analysis similar to that carried out in chapter 3 on the beam with the double support configuration can now be developed. In this case, both masts must be considered in order to take into account the symmetric and anti-symmetric modes which may occur. A full exposition of the analysis is given in the latter part of Appendix C. The values of the natural frequencies for the mast are given in tables 5.1 and 5.2 and the form of the mode shape is shown in figure 5.2 and 5.3 for the symmetric and anti-symmetric modes respectively.

To include the effect of submergence on the model masts, the pressure and hence the added mass distribution for each mode of response can be evaluated as previously described in chapter 3. The values of the submerged frequencies can then be obtained and the results are also given in tables 5.1 and 5.2.
Table 5.1 Symmetric Modal Frequencies of the Mast

<table>
<thead>
<tr>
<th>MODE</th>
<th>Kn</th>
<th>Non Submerged Frequency/Hz</th>
<th>Submerged Frequency/Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.254583</td>
<td>15.39</td>
<td>14.15</td>
</tr>
<tr>
<td>2</td>
<td>3.245759</td>
<td>103.0</td>
<td>93.38</td>
</tr>
<tr>
<td>3</td>
<td>5.508587</td>
<td>296.7</td>
<td>268.1</td>
</tr>
<tr>
<td>4</td>
<td>7.719808</td>
<td>582.7</td>
<td>526.9</td>
</tr>
<tr>
<td>5</td>
<td>9.625967</td>
<td>906.0</td>
<td>823.5</td>
</tr>
<tr>
<td>6</td>
<td>11.007829</td>
<td>1185</td>
<td>1078</td>
</tr>
<tr>
<td>F.B.M</td>
<td>12.898960</td>
<td>1627</td>
<td>1481</td>
</tr>
</tbody>
</table>

Table 5.2 Anti-symmetric Modal Frequencies of the Mast

<table>
<thead>
<tr>
<th>MODE</th>
<th>Kn</th>
<th>Non Submerged Frequency/Hz</th>
<th>Submerged Frequency/Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.345910</td>
<td>17.71</td>
<td>15.96</td>
</tr>
<tr>
<td>2</td>
<td>3.389048</td>
<td>112.3</td>
<td>100.8</td>
</tr>
<tr>
<td>3</td>
<td>5.694412</td>
<td>317.1</td>
<td>284.7</td>
</tr>
<tr>
<td>4</td>
<td>7.996662</td>
<td>625.3</td>
<td>561.5</td>
</tr>
<tr>
<td>5</td>
<td>10.302805</td>
<td>1038</td>
<td>933.4</td>
</tr>
<tr>
<td>6</td>
<td>12.601796</td>
<td>1553</td>
<td>1410</td>
</tr>
<tr>
<td>7</td>
<td>14.868525</td>
<td>2162</td>
<td>1968</td>
</tr>
<tr>
<td>F.B.M</td>
<td>20.113099</td>
<td>3955</td>
<td>3597</td>
</tr>
</tbody>
</table>

F.B.M – Fixed bearing mode
Figure 5.2 Symmetric Mode Shapes for the Model Mast
Figure 5.3 Anti-symmetric Mode Shapes for the Model Mast
5.4 Results of the Trials

5.4.1 The Pressure Pulse

The pressure traces from two particular shots during the shock trial are shown in figures 5.4 and 5.5. Both have been measured whilst the pressure transducer is in the 'side on' position. The traces show that the pressure, as it passes over the body, is exponential in form with a short initial rise time estimated to be of the order of 2 microseconds. Superimposed on the trace, are high frequency pressure fluctuations of unknown origin at a frequency of 5KHz. It is possible that these signals could be related to the resonance of the transducer as they have a frequency much higher than those associated with the structural response. This high frequency response appears to decay after 4msec which would be consistent with the damped response of the transducer elements. The pressure signals also have a lower residual frequency in the region of 200 Hz which may be related to the transducer responding to acceleration effects. It is unlikely that they are true pressure fluctuations as they require the water to withstand a tensile force of 138 KN/m².

The charge size used in shot B was twice that of shot A using the same stand-off distance. It is interesting to note that the pressure peaks are not scaled as expected from theory by use of equation (4.4). However, the explosive used for the test is not one of the standard types of charge and it may not be valid to assume the same characteristic $\alpha$ as a pentolite charge. (Using the pentolite constants, the peak pressure is 88% of that calculated for shot B and only 64% in the case of shot A. This would seem to indicate that for a plastic explosive the constants given in equation (4.4) would be $\alpha = 2.44$ and $K = 2.515 \times 10^6$. However, it is unlikely that these free field relationships hold in the vicinity of the submerged structure).

The shape of the pressure pulses, ignoring the superimposed additional frequencies, can be approximated to exponential curves given by equation (4.3). The exponential constant can be found by plotting the pressure trace on logarithmic scales and determining the gradient. The resulting expressions are:

- Shot A: $P = 164 \ e^{-4820t} \ \text{lbf/in}^2$
- Shot B: $P = 288 \ e^{-5940t} \ \text{lbf/in}^2$
Figure 5.4 Resulting Pressure Pulse for Shot A

Figure 5.5 Resulting Pressure Pulse for Shot B
5.4.2 **Accelerometer Performance Under Shock Loading**

In addition to the information obtained regarding the response of the masts, several recording channels were dedicated to assessing accelerometer performance. Accelerometers with varying ranges, sensitivities and types of damping were compared by analysing their ability to measure the movement of the main body of the test rig in a particular shock test. The comparison was carried out for shot B, in which the instrumentation was subjected to the most severe shock.

Of particular interest is the comparison of transducers with different methods of damping. Three transducers were mounted on a vertical transverse beam spanning the diameter of the main body as shown in figure 5.6. The specifications of the transducers were as follows:

- **Channel 3** - Undamped Entran EGA-C accelerometer
  Dynamic range ± 250g Natural frequency 4.8 KHz.

- **Channel 9** - Damped Entran EGAX accelerometer
  Dynamic range ± 250g.

- **Channel 12** - Kistler 8642A-5 accelerometer mounted on a mechanical damping unit. Dynamic range ± 5000g.

The response of the undamped accelerometer, on channel 3, is shown in figure 5.7 together with its associated frequency spectrum in figure 5.8 which indicates a large amount of energy at the transducer natural frequency. The signal can be seen to be outside of its nominal range although this is not a problem unless it exceeds this range by more than 100%. Expansion of the frequency spectrum in figure 5.9 indicates dominant frequencies at 1450 and 1680 Hz. To eliminate the effect of the transducer resonance, a digital filter with a 2KHz cut-off frequency has been applied to the signal. The reconstructed signal is shown in figure 5.10 to have maximum and minimum acceleration levels of +160g and -160g.

In a similar way, the data has been given for the damped accelerometer on channel 9 in figures 5.11 to 5.14. Dominant frequencies are seen to lie at 1400, 1680, 2300 and 4000 Hz. The value of 4000 Hz represents the damped natural frequency of the transducer. Filtering the signal at 2KHz
Figure 5.6 Location of Accelerometers on the Transverse Beam
Figure 5.7 Acceleration Signal from Channel 3 Shot B

Figure 5.8 Frequency Spectrum for the Signal from Channel 3 Shot B
Figure 5.9 Expanded Frequency Spectrum for the Signal from Channel 3 Shot B

Figure 5.10 Reconstructed Acceleration Signal from Channel 3 Shot B after Applying a Low-pass Digital Filter with a 2 KHz Cut-off Frequency
Figure 5.11 Acceleration Signal from Channel 9 Shot B

Figure 5.12 Frequency Spectrum for the Signal from Channel 9 Shot B
Figure 5.13 Expanded Frequency Spectrum for the Signal from Channel 9 Shot B

Figure 5.14 Reconstructed Acceleration Signal from Channel 9 Shot B after Applying a Low-pass Digital Filter with a 2 KHz Cut-off Frequency
appears to attenuate the signal from ±225g to ±175g.

Data from the third accelerometer on channel 12 which has the mechanical damper is shown in figures 5.15 to 5.18. Dominant frequencies are confirmed to lie at 1400 and 1680 Hz. Filtering the signal at 2KHz appears to attenuate the signal from ±275g to ±175g.

The results suggest that damped accelerometers are far better at measuring shock loads. After filtering all three signals at 2KHz, there are remarkable similarities, especially considering the level of resonance on channel 3. This shows that resonating transducers are still able to measure much lower frequency responses simultaneously and that digital filtering techniques are very powerful in coping with such signals. However, in using undamped accelerometers, the need to measure the full range detracts from the need to make accurate measurements at lower frequencies. This is especially true when the bit level of the recording device becomes significant. It is therefore recommended that in any future shock trials a form of damped accelerometer is used although the choice between internal damping and mechanical damping is somewhat arbitrary.

As a further comparison, two high range undamped accelerometers were also attached to the body of the structure. The results showed that both accelerometers suffered badly because the high dynamic range implies a poor signal to noise ratio. As a result, it was only just possible to identify the dominant frequencies at 1400 and 1680 Hz with the prior-knowledge obtained from the previous channels. Results therefore indicate that increasing the dynamic range does not improve an undamped transducer's ability to measure shock response.
Figure 5.15 Acceleration Signal from Channel 12 Shot B

Figure 5.16 Frequency Spectrum for the Signal from Channel 12 Shot B
Figure 5.17 Expanded Frequency Spectrum for the Signal from Channel 12 Shot B

Figure 5.18 Reconstructed Acceleration Signal from Channel 12 Shot B after Applying a Low-pass Digital Filter with a 2 KHz Cut-off Frequency
5.4.3 **Response of the Masts**

The response of the masts has been measured by a variety of transducers in a number of strategic positions. A number of shock tests have been carried out on the test rig. Early shots were used to prove both the recording the measuring techniques and did not use the full complement of instrumentation at any one time. The final test, shot B, produced the most complete set of results and, for this reason, will be examined in detail.

(a) **At the Mast Supports**

A single axis accelerometer mounted internally has measured the acceleration levels of the fore and aft motion at the support position of the mast. The natural frequency of this accelerometer is given as 4200 Hz and its nominal range is ±500g. Figure 5.19 shows that the raw acceleration trace exceeds the manufacturers range. Frequency analysis of the signal produces the spectrum shown in figure 5.20. It can be seen that the majority of the energy associated with the signal lies at the transducer's natural frequency. Expansion of the lower range of frequencies in figure 5.21 shows that there are significant frequencies at 1500, 1850, 1950, 2300 and 2850 Hz. However, it is believed that the frequencies of interest, as far as mast deformation is concerned, occurs at frequencies not exceeding 2KHz. To obtain an indication of the significance of such low frequencies within the signal, a digital filtering technique, as described earlier, was used to suppress the higher frequencies. The result of applying a 2KHz cut-off filter to the signal is shown in figure 5.22. This then suggests that maximum acceleration levels of +190g and -170g are actually induced at the mast support.

(b) **At the Mast Tip**

The accelerometer mounted internally at the tip of the mast measured both the transverse movement and the fore and aft movement. The bi-axial transducer used had a natural frequency of 4800Hz and an operating range of ±500g. The results of the fore and aft motion are shown in figure 5.23. It can be seen that the signal has been corrupted at both extremes
Figure 5.19 Acceleration Signal from Channel 5 Shot B

Figure 5.20 Frequency Spectrum for the Signal from Channel 5 Shot B
Figure 5.21 Expanded Frequency Spectrum for the Signal from Channel 5 Shot B

Figure 5.22 Reconstructed Acceleration Signal from Channel 5 Shot B after Applying a Low-pass Digital Filter with a 2 KHz Cut-off Frequency
Figure 5.23 Acceleration Signal from Channel 1 Shot B

Figure 5.24 Frequency Spectrum for the Signal from Channel 1 Shot B
of acceleration. At high accelerations the signal has been limited to 1000g by the transient recorder, whereas, at negative acceleration levels the signal has been limited to -500g by the amplification units. Fortunately, because of the large amount of recording time used, the signal can still be analysed at a time long after the limited part of the signal. The frequency characteristic, like the previous acceleration signal, is dominated by the transducer resonant frequency. Expanding the frequency range below 2KHz as given in figure 5.24 identifies dominant frequencies to be 230,450,750,1650 and 1860 Hz.

The transverse motion also measured by this accelerometer is shown in figure 5.25 together with the frequency characteristics over the range 0 to 2KHz in figure 5.26. Dominant frequencies are seen to occur at 240,450, 750,1150,1650 and 1860 Hz.

(c) At the Connecting Flange

The displacement transducer positioned to measure fore and aft movement of the flange connection has provided some useful confirmatory data. This transducer has none of the ill effects due to resonance, although the time history shown in figure 5.27 has been corrupted by the occasional spike. These spurious data points are mostly likely related to the input voltage levels. Frequency analysis of the signal produces the spectrum given in figure 5.28 which confirms the previously obtained dominant frequencies of 300,450 and 1670 Hz. The effect of applying a 2KHz cut off filter to the signal is shown in figure 5.29. This 'cleaned up' signal shows that the displacements associated with the above frequencies are very small with a maximum value of $8 \times 10^{-4}$ mm.

(d) At the mid section between the mast supports

The output from the strain gauge bridge is shown in figure 5.30 to be completely free of high frequency resonance effects. The frequency spectrum given in figure 5.31 shows dominant frequencies at 250,450,760,1100,1470,2350,2680,2870,3100 Hz. A maximum of 80 µstrain occurs 4 msec after the arrival of the pulse. The maximum moment associated with this value is 544 lbf.in (61.4Nm) which confirms the low value of deflection measured by the displacement transducer.
Figure 5.25 Acceleration Signal from Channel 2 Shot B

Figure 5.26 Frequency Spectrum for the Signal from Channel 2 Shot B
Figure 5.27  Displacement Signal from Channel 6 Shot B

Figure 5.28  Frequency Spectrum for the Signal from Channel 6 Shot B
Figure 5.29 Reconstructed Displacement Signal from Channel 6 Shot B after Applying a Low-pass Digital Filter with a 2KHz Cut-off Frequency
Figure 5.30 Strain Gauge Signal from Channel 10 Shot B

Figure 5.31 Frequency Spectrum for the Signal from Channel 10 Shot B
A summary of the important response frequencies and amplitudes is given in tables 5.3 and 5.4 for both shot B and shot A where half the weight of explosive was used. It can be seen from the results that use of strain gauges is by far the best method of obtaining frequency information relating to shock loaded structures. The frequencies detected by the strain gauge in shots A and B compare very closely except for the case of the first dominant frequency. In comparing the frequencies it should be noted that they are subject to an error band of at least ±10 Hz which is due to the unfortunate selection of sample rate combined with the limited storage capacity of the recorder. Comparing the frequencies with the theoretical values determined in section 5.3, there appears to be significant correlation. The first dominant frequency measured by the instrumentation at 250 Hz is most closely represented by 3rd symmetric mode of response which has a frequency of 268 Hz. The more dominant frequency of 450 Hz seen by all of the transducers measuring the mast movement, is most closely represented by the 4th symmetric mode which has a calculated frequency of 527 Hz. The symmetric modes 5, 6 and 7 can be seen clearly occurring at frequencies of 760, 1100 and 1470 Hz. It can also be seen that the calculated and measured values are closer for higher modes. Mode 7 is particularly relevant because it represents the mode where the between bearing distance is highly dominant (refer to figure 5.2). This mode can be considered to occur at the same frequency as if the central span was fixed at its bearings.

The values of the frequency for both symmetric and anti-symmetric modes are relatively unaffected by the distance between bearings for low modes of response and hence their frequencies are relatively close. For this reason, it is difficult to establish whether anti-symmetric modes do have any relevance in the overall response although it is unlikely in the case of a perfectly symmetric loading. At frequencies much closer to the fixed bearing modes, the distance between bearings obviously takes on a greater significance. However, the first anti-symmetric fixed bearing mode occurs at a frequency in the region of 4 KHz and is therefore difficult to measure in this study.

Frequencies higher than 2 KHz identified as components of the strain signal are difficult to relate to higher deformation modes. It is possible that they are related to modes higher than mode 7, although it is equally possible that they may be related to deformations of the body such as ovalling of the cylindrical section.
<table>
<thead>
<tr>
<th>Number</th>
<th>Measured Movement</th>
<th>Dominant Frequencies /Hz</th>
<th>Peak Filtered Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mast Tip F/A Acceleration</td>
<td>270,460,1720,1840</td>
<td>Over-ranged</td>
</tr>
<tr>
<td>2</td>
<td>Mast Tip</td>
<td>240,450,750,1150,1650,1860</td>
<td>+40g, -15g</td>
</tr>
<tr>
<td>5</td>
<td>Mast Support F/A Acceleration</td>
<td>1500,1850,1950,2300,2850</td>
<td>+180g, -170g</td>
</tr>
<tr>
<td>6</td>
<td>Mast Mid Span F/A Deflection</td>
<td>300,450,1670</td>
<td>+9x10mm, -8x10mm</td>
</tr>
<tr>
<td>10</td>
<td>Mast Mid Span F/A Bending Strain</td>
<td>250,450,760,1100,1470,2350,2630,2870,3100</td>
<td>+60μst, -80μst</td>
</tr>
</tbody>
</table>

Table 5.3 Compiled Results from Shot B

<table>
<thead>
<tr>
<th>Number</th>
<th>Measured Movement</th>
<th>Dominant Frequencies /Hz</th>
<th>Peak Filtered Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mast Tip F/A Acceleration</td>
<td>210,1720,1870</td>
<td>Over-ranged</td>
</tr>
<tr>
<td>2</td>
<td>Mast Tip Transverse Accn.</td>
<td>240,480,730,1140,1710,1850</td>
<td>+10g, -15g</td>
</tr>
<tr>
<td>5</td>
<td>Mast Support F/A Acceleration</td>
<td>450,700,1160,1310,1440,1710,1850</td>
<td>+90g, -100g</td>
</tr>
<tr>
<td>6</td>
<td>Mast Mid Span F/A Deflection</td>
<td>350,470,1420,1660</td>
<td>+3x10mm, -3x10mm</td>
</tr>
<tr>
<td>10</td>
<td>Mast Mid Span F/A Bending Strain</td>
<td>350,490,730,1100,1420,1710,2320,2640,3090</td>
<td>+30μst, -54μst</td>
</tr>
</tbody>
</table>

Table 5.4 Compiled Results from Shot A
Discussion

The results from the shock trial have provided a very large data base of information regarding shock wave/structure interaction. Analysis of the data has shown that the response of the structure to a high frequency, high energy input has many modal forms. Response modes up to and including mode 7 has been shown to be important. The most important response frequency, as far as the movement of the mast is concerned, is mode 4. However, the strain gauge data has shown that the "fixed bearing" mode (mode 7) is also important. Both strain gauge and displacement transducer data have shown that the critical phase of response occurs at some time after the first cycle which confirms the importance of more than two modes. The fact that the fixed bearing mode is also important suggests that both the load transmitted to the mast by the movement of the supporting structure and the direct pressure load to the area of the mast are significant. It would seem necessary, therefore, to include both forcing functions discussed in chapter 4 into the structural model in order to obtain a realistic response.
6. Experimental Investigation of Hydrodynamic Effects

6.1 Background

In the previous chapter it was indicated that hydrodynamic interaction does indeed vary both the natural frequency and the damping characteristics of the shock response of a fully submerged slender mast. However, it has also become apparent that a far more rigorous test programme is required to fully verify the theories presented in chapter 3. Despite the documented studies that have to be made into hydrodynamic interaction effects, there is still a lack of readily useable practical information.

It was decided that a test rig was required that would allow conditions to be controlled more accurately than is possible in offshore trials. The test rig was to be laboratory-based with both the supporting structure and the initial condition of the fluid being stationary. The parameters to be measured were, firstly the change in the natural frequency with depth of submergence and secondly, the damping characteristics and their dependence on depth of submergence.

Several types of test are used for the investigation of structural response and their suitability is very much dependent on the parameter to be determined. Those commonly used are:

(1) Free vibration response
(2) Forced vibration response using a shaker.
(3) Shock/impact transient response

The free response method is most suited to the present study because the structure is isolated, and once it had been released from an initial deformed position, it can only respond in a combination of its own natural frequencies. Determination of the frequencies can be obtained by use of the Fourier Analysis of any measure of the time dependent motion of the structure such as acceleration, velocity, displacement, strain etc. The damping coefficients can be found from the decay envelope of free response traces and this also allows for an assessment of whether the damping coefficients are dependent on factors such as the amplitude of the
response. However, there are limitations to this method of testing. The main difficulty is to obtain higher mode information as most normal structures tend to have free responses that are highly dominated by their fundamental mode. To some extent, these limitations can be overcome by releasing the mast in ways other than the use of a simple tip deflection.

In the case of forced response, the structure is subjected to cyclic loading and the response is measured over a range of frequencies to find positions of 'resonance'. The determination of damping coefficients can be found in one of two ways. Either the difference between the input and output response can be measured to determine the out of phase damping forces (phase lag method) or the width of the frequency spectrum at the half power point can be measured (band width method). The second method, however, is suited to the measurement of damping where the coefficient is considered to be independent of amplitude which is not strictly correct for all types of damping. If the damping is strongly influenced by amplitude then many tests would need to be performed with the shaker providing different input amplitudes. This is seen to be a major drawback in the use of such a method, as is the cost of a shaker. It is also inevitable that the presence of such a shaker would influence the frequency characteristics of the structure either by virtue of its size or by changing the local stiffness in the vicinity of the connection between the structure and shaker.

The method of structural impact using transient shock impulses has the advantage of being able to excite high mode responses as shown in earlier shock trials described in chapter 5. However, due to the nature of the shock impulses, it is difficult to predict or control the modal content of the response.
6.2 Design of the Test Rig

6.2.1 Description

The basic requirement for the test rig was to place a tubular mast underwater in the largest expanse of water available with it fixed in a rigid support so that free vibration tests could be carried out to investigate hydrodynamic interaction effects. An additional system was required which could hold the beam in a relatively complex deformation profile and release it instantaneously. It was also necessary to provide an instrumentation system which could measure the response of the structure over a wide range of frequencies.

The final design of the test rig is shown in figure 6.1 and consists of a 1.5m long slender mast 30mm in diameter, held firmly in a rigid supporting base below a large tank of water. The mast supports consist of diaphragm plates which are firmly attached by use of taper locking bushes (figure 6.2 and plates 1 and 2). The width of the webs in the diaphragms were minimised to keep the stiffness and material damping coefficient low. The diaphragm plates are supported in bearing-type housings which are subsequently bolted rigidly to a large machine bed. This sub-assembly is completely detached from the tank assembly except for a rubber gaiter which forms a water tight seal between the two units.

The tank is raised above the floor so as not to interfere with the machine bed. It is supported by four cross member assemblies made of 100mm x 50mm channel section which are joined by cross bracing to produce a sturdy framework. The tank measures 2.4m long x 1.2m wide x 1.8m deep with an open top and is made of 4mm thick mild steel plate. The tank size was limited by the available laboratory space and its size allows 1.5msec of response time before reflection waves from the tank walls reach the structure. Under normal conditions the response of the mast would be monitored over a period of seconds and hence complete avoidance of reflection effects is not possible. However, in such a test rig, the reflected waves and their associated impulses are considered to have a very minor influence, unlike the case of reflected pressure waves from explosive sources.
Figure 6.1 Laboratory Test Rig Layout
Figure 6.2 Mast Support Assembly
The release mechanism consists of a pulley/wire system which is held under tension using an electromagnet. Release of the tension is achieved by a switch which cuts the power supply to the magnet. The ability to apply a lateral load at any position along the mast has been provided by a rigid vertical support at one end of the tank on which a pulley can be fixed (figure 6.3). This support consists of a channel-section fixed by flanges to the tank support framework at its bottom end and to the tank reinforcement at the top end. The pulley can then be fixed firmly to this support so that the tension wire runs horizontally from the mast. To reduce any noise inducing mechanical interference between the release mechanism and the mast, a strip of rubber has been placed around the tip of the mast before wrapping the wire around and fixing. The wire is passed around the pulley and connected to a 'bottle screw' arrangement which is fixed to a rectangular plate. The plate is held by the energised electromagnet which is fixed at the top of the vertical support with its magnetic faces directed towards the bottom of the tank. The tension applied to the mast can be varied by tightening the bottle screw. This adjustment can be used as a method of ensuring the repeatability of the initial deformation conditions for each test. On release, the plate and bottle screw are 'caught' by a length of wire connected between the magnet and plate. This reduces the likelihood of the wire being snagged around the pulley and prevents the falling mass from imposing any secondary forcing function on the structure.

In order to obtain higher order response modes, it was considered necessary to release the beam from an initial complex deformation shape provided by imposing two displacements on the beam at different axial locations in opposing directions. The use of two electromagnets to release the beam was seen to be undesirable since the timing of the release would be highly dependent on the force applied to the plates by the wire. Instead, a more complex pulley/wire arrangement was devised which releases both wires from the same electromagnet as show in figure 6.4. The ability to impose different values of load by each wire was provided by the use of an adjustable connector between the wires. The overall magnitude of the loads was controlled by the 'bottle screw' adjuster. The repeatability of the tests using this method was shown to be good as was the ability to improve the amount of higher mode response.
Figure 6.3 Pulley/Wire System for a Simple Tip Deflection

Figure 6.4 Complex Pulley/Wire System for the Dual Loading Condition
6.2.2. Selection of Instrumentation

The requirement for the instrumentation was that it should provide an accurate response signal which could be used to obtain information about both the frequency and the damping characteristics of the mast. The resulting signals had to be accurate in both time of response and amplitude of response. The possible parameters that could be measured were:

1. Strain
2. Displacement
3. Velocity
4. Acceleration

Both displacement and velocity were rejected mainly due to their lack of suitability and the availability of reliable dynamic sensors. Strain gauges and accelerometers had been widely used in past investigations and had been found to be suitable for response measurement. The strain gauges selected were foil resistance type 90° rosette with a gauge length of 5mm. The great advantage of strain gauges is that, due to their size, they do not alter the structure's mass or stiffness and hence do not affect the response characteristics. The output from a strain gauge is highly linear and has no limit to its ability to measure high frequency. In all cases, a full bridge circuit with four active gauges was used. Two 90° rosettes were fixed in diametrical opposition and by virtue of using the "Poisson gauges", the output from the bridge was increased to 2.6 times the true strain reading.

The accelerometers were of the piezo-resistive type, consisting essentially of an undamped cantilever beam which has semi-conductor strain gauges bonded to it. Due to the type of construction of the accelerometer, the measured frequency response has a limited bandwidth which is dependent upon the natural frequency of the transducer. This creates some limitations on the use of this type of transducer especially in the shock environment as discussed previously.

To measure the response of the structure, it was decided to use five transducers - two accelerometer and three strain gauge bridge positions. The tip and mid span response would be measured using accelerometers and the strain values at the mid length position and the base of the mast would
also be measured. Finally, to obtain information about the stiffness of the support configuration, another strain gauge bridge was required on the mast in a central position between the supports. The positions are shown in figure 6.5. The problem of isolating the accelerometers from the water was resolved by fixing them on mounting plates inside the mast (figure 6.6). The upper two strain gauge bridges were fixed on the outside of the mast and coated in quick setting epoxy resin to seal them from water. The 'between support' strain gauge had to be fixed inside the mast because the supports were a sliding fit and were pushed onto the lower part of the mast during assembly. To fix the strain gauge 150mm down the inside of the mast, a rather novel approach was used. The more usual method of placing a gauge in a deep hole firstly fixes the gauge to a flexible celluloid strip which is then placed down the hole. The gauge is then bonded to the internal surface under pressure from an inflated piece of surgical tubing as discussed in reference (78). However, such methods are prone to alignment errors especially with such remote access. Instead, a collar was machined from a block of Araldite CT200 to fit accurately inside the internal diameter of the tube. The shape and dimensions of the collar are given in figure 6.7. Two 90° rosettes were accurately aligned and bonded face down onto the external surface of the collar. The gauge wires were passed directly through the wall of the collar and fixed to terminals on the inside. Connections for a full bridge were then made. In order to glue the collar in place, two small glue holes were machined in the steel tube. Positioning of the collar was achieved by accurately marking a cross on the collar which was subsequently aligned with the centre of one of the glue holes. To check the influence of the collar on the behaviour of the mast, the bending stiffness of each component was calculated and the collar was shown to have only 1% of the beam's bending stiffness and hence its influence is negligible.

The instrumentation system is completed by connecting the transducers to a rack of Fylde transducer amplifiers which can supply the transducers with up to 15V stabilised energisation. The output sensitivity can be varied from 1mV to 1V full scale output making the signal conditioning very suitable for both strain gauges and accelerometers. Analogue filters with a range of cut-off frequencies are provided at the input to the amplifiers to limit the high frequency signals associated with the resonance of the transducer. Throughout the tests, a 2 stage low pass filter with a 1KHz
Figure 6.6 Internal Accelerometer Mounting

Figure 6.7 Araldite Collar for Mounting the Internal Strain Gauge
cut-off was used which can be considered not to affect the structural response frequencies. Measurement and capture of the signals output from the amplifiers is achieved by means of a Datalab 2800 transient recorder. The time base of the signal recording is controlled by a crystal oscillator and the sample rate can be varied from 5Hz to 2MHz. The analogue signal from the amplifier is digitally converted to a 10 bit word and each channel is stored in 4 Kbyte memory modules. Instantaneous display of the captured signals is given on an inter-connected oscilloscope. Final permanent storage is achieved by transferring the information via a parallel interface to a Hewlett Packard computer and dumping down onto flexible disc storage. A diagram showing the instrumentation system is given in figure 6.8. Further details of the transducer and instrumentation are given in Appendix I.
Figure 6.8 Instrumentation System

Accelerometer #1

Accelerometer #2

Strain Gauge #1

Strain Gauge #2

Strain Gauge #3
6.3 Calibration Tests

In order to be confident of the experimentally determined results, it is necessary to check the performance of the transducer system under some simple 'calibration' tests. Firstly, the manufacturer's values of the material constants such as the Youngs modulus and the Poissons ratio for the mast must be verified. The most convenient method of measuring these properties is to carry out either a tensile or compressive test. Because of the difficulties raised in holding a tubular test specimen, the compression test method was adopted. In practice, the difference between the tensile and compressive value of Youngs modulus is small at low loading. The ASTM specification for compression testing (ref 79) specifies a length/diameter ratio of 8:1 and hence a 210mm long piece of tube was used with accurately machined ends. Two 90° strain gauge rosettes were fixed at the mid length position and each gauge was connected separately as a quarter bridge to a Solartron strain gauge data logging system. The full calibration results are given in Appendix D - the determined values being Young's Modulus = 208 MN/m² and Poisson's ratio = 0.288.

In order to check the accuracy of the positioning of the strain gauges on the model mast, a test was carried out with the mast in a simple support configuration. The accuracy of the output signal from a strain gauge may be in error for a number of reasons. The quoted resistance for the gauge has an associated tolerance and the accuracy of the output will depend largely on the quality of the gauge. Other errors could appear because of glue layer thickness, alignment of the gauges, the curved surface of the tube, etc. The most convenient way of checking the gauge performance is to support the mast in simple support configuration and compare the output to a theoretically calculated value. A full description of the test is given in Appendix D. The test was repeated four times and the average strain outputs are given overleaf.
<table>
<thead>
<tr>
<th>Transducer</th>
<th>GRADIENT (µstrain/Load (N))</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measured</td>
<td>Predicted</td>
</tr>
<tr>
<td>#3</td>
<td>0.3815</td>
<td>0.4080</td>
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<td>#4</td>
<td>0.6976</td>
<td>0.7394</td>
</tr>
<tr>
<td>#5</td>
<td>0.3129</td>
<td>0.3325</td>
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</table>

(Refer to figure 6.5 for the strain gauge locations)

The results show that all three strain gauges have approximately the same error which suggests that the discrepancies are more likely due to the curved surface of the beam rather than to alignment errors. This calibration data has validated the procedure for installing the internally mounted gauges. To adjust for these errors, compensation factors were used to bring the strain gauge signals to their calculated values throughout the experimental test programme.

A check on the accelerometer performance was also considered necessary as the available devices had been used for shock work for a number of years. Calibration was carried out by using a 'back to back' test with an accurate Kistler piezoelectric accelerometer which had not been used since manufacturer's calibration. The test used a small cantilevered strip of mild steel with the accelerometers mounted at the free tip on opposing faces. The output signals were recorded for four cycles whilst the beam vibrated predominantly in the first mode and maximum acceleration levels exceeding 70% of the manufacturers specified range were obtained. The performance check was carried out on both piezo-resistive accelerometers and the maximum variation was found to be 1.5% of the peak amplitude. This discrepancy was most likely related to variations in the voltage supply to the accelerometers.

Once the accelerometers had been calibrated, they were mounted on small back plates and fitted into the mast using a specially made alignment tool. The mast was then fitted into the taper locking supports in an
upright position. Alignment of the transducers with the tanks' axes of symmetry was achieved by loading the mast at its tip and rotating it, whilst monitoring the output from strain gauges mounted at the base, until a maximum value was obtained. The mast was then sealed by fixing a rubber insert at the tip using glue to prevent water from damaging the instrumentation system.

Further tests were required to investigate the characteristics of the support configuration. An approximate stiffness for the plate had been calculated using Roark's (80) theory which predicted a value of 4660 Nm (see Appendix E). However, it was felt that this value may not be entirely accurate as the analysis ignores any influence of blend radii at the intersection of the web and the flange of the support diaphragms. To determine the stiffness of the support, an experiment was carried out in which the beam was subjected to a transverse tip loading of up to 130N. The influence of the support stiffness was measured by the strain gauge output from the transducer mounted internally at the mid point between supports. The results are given in Appendix E and show that the strain output remains linear at the highest load of 130N which produces an output of 200 μstrain. It was not possible to continue to load above this value because the maximum strain measured at a position just above the upper support was approaching 50% of the material yield value. Four separate tests were performed and the resulting strain/load relationships were linear with correlation coefficients of better than 0.99. The average gradient of the output was shown to be 2.21 μstrain/N and this value was subsequently used to evaluate the true stiffness of the supports. For the system shown,
using a strain energy analysis as discussed previously in Chapter 2, the value of the rotational stiffness, $K$, is given by the equation,

$$K = \left[ \frac{W L y}{\epsilon a} - \frac{2 E I}{a} \right]$$

where $y = d/2$ \hspace{1cm} (6.1)

Substitution of the experimentally determined value ($W/\epsilon$) gives a stiffness of 5200 Nm. This indicates that a large error would have been incurred had the "Roark value" of 4620 Nm been used. A further experiment using only one supporting diaphragm was also conducted to investigate whether it was the combination of the two supports or the effect of axial loads created by the taper locking devices that was responsible for the differences in the theoretical and experimental values of rotational stiffness. However, these tests confirmed that the effective rotational stiffness was 5200 Nm.

Having determined the support stiffness, it was possible to predict the natural frequency of the mast using a finite difference model as discussed in chapter 2. By increasing the number of nodes, it was possible to determine by extrapolation the 'infinite node' condition which gave a natural frequency of 10.921 Hz. This was later confirmed by tests on the mast in air which measured the beam's natural frequency as being between 10.74 and 10.84 Hz. The error in the predicted frequency is in the range 0.7 and 1.7% which is of the same order as the error incurred in the use of the finite difference method to predict the exact frequency of the cantilever beam as discussed in chapter 2.
6.4 Test Programme for the Measurement of the Frequency Characteristics of the Model Mast

The aim of this test programme was to investigate how the mast's natural frequencies changed with depth of submergence. Having developed a numerical model of the experimental structure, it was possible to simulate the effect of providing different initial deformation profiles. Because it was desirable to keep the release mechanism simple so as to avoid repeatability problems, the effect on the modal content of the response of placing a single transverse load at varying positions along the length of the mast was investigated. After several attempts it was found that the highest percentage of second mode information could be obtained by placing the transverse loading at the mid-height of the structure. However, in using such a method, the response spectrum contained little energy for modes higher than the second natural frequency. Furthermore, by using the numerical model, it was found that by providing two transverse loads, one at the tip and one at the mid-height position, higher modes of response were possible depending on the relative transverse displacements at each position.

Having determined the natural frequencies of the beam in its non-submerged condition, it was possible to evaluate the corresponding frequencies for the fully submerged condition from the information provided in chapter 3. It has been shown that the increase in the natural frequency is between 17 and 18% for the first four modes of response. It was therefore possible to decide on the optimum sample frequency for the transient recording system to enable an accurate frequency measurement to be made. For the tests in which the first and second modes were of particular interest, a sampling rate of 200 Hz was used which provided 20.5 seconds of response on each channel. This allowed frequencies of up to 100 Hz to be measured with an accuracy of ± 0.05 Hz this being suitable for the measurement of the first and second modes which have a range of 8.8 to 10.8 Hz and 56.4 to 68.0 Hz respectively. To measure the third mode of response the sampling frequency was increased to 500 Hz thus allowing frequencies of up to 250 Hz to be determined with an accuracy of ± 0.12 Hz. This was suitable for the third mode which varies between 160 and 200 Hz. To measure fourth mode the sampling frequency was raised to 1000 Hz.
When using accelerometers, the sample rates used must be high enough to enable measurement of all frequencies up to transducers resonance. This requires the sample rate to be increased to at least 8KHz using the Nyquist sampling criterion. At this level, the accuracy of the determined frequency values precludes assessment of the normal structural frequencies because of the limited memory of the recorder. In the early test series, two sample rates were used, one to enable accurate structural frequency measurement and a second to determine the transducer characteristics. It was found that transducer resonance was not a problem at the levels of excitation used and that the energy levels associated with the structural frequencies were far higher than those of the transducer. However, in further tests in which the sample rate rendered it impossible to measure the transducer frequencies, a 1 KHz low-pass analogue filter was introduced into the instrumentation system to remove these 'unwanted' signals before they were amplified.

In preliminary tests carried out on the mast, it was important to analyse the output data closely to determine whether the frequency of the mast changed during the response, i.e. whether the frequency has any dependence on the amplitude of vibration or other parameters. In order to check this, a test was carried out with the mast in its fully submerged condition. The mast was subjected to an initial tip deflection and the response was monitored via the base strain gauge for tip displacements decaying from ±20mm to ±0.5mm. It was found that the variation in the time period was less than ±2%, this being the bit level of the recording device and it was concluded that the frequency showed no amplitude dependence over the range of output strains considered. This result is important since it allows a fast Fourier transform (FFT) to be applied to the experimental data so as to accurately determine the frequency characteristics of the mast response.

In the first series of tests the mast was subjected to an initial tip loading which was monitored by the output from the base strain gauge which represented a tip deflection of approximately 17mm. Both accelerometers and strain gauges #3 and #4 were monitored throughout the response. The designation of the instrumentation and the positions of the transducers on the mast are given in table 6.1 overleaf.
Table 6.1 - Transducer Notation and Location

<table>
<thead>
<tr>
<th>Channel Number</th>
<th>Transducer</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Accelerometer Entran EGC 240</td>
<td>Mast Tip</td>
</tr>
<tr>
<td>2</td>
<td>Accelerometer Entran EGC 240</td>
<td>Mast Mid-position</td>
</tr>
<tr>
<td>3</td>
<td>Strain Gauge</td>
<td>Mast Mid-position</td>
</tr>
<tr>
<td>4</td>
<td>Strain Gauge</td>
<td>Mast base</td>
</tr>
<tr>
<td>5</td>
<td>Strain Gauge</td>
<td>Between supports</td>
</tr>
</tbody>
</table>

The tests were repeated at sample rates of 10 KHz and 200Hz for the mast with the same initial starting condition. A total of 23 tests were performed for depths of submergence from 0 to 1650mm in 150mm steps. The results from the tests carried out with the 10KHz sampling rate (T104 to T124 in steps of 2) served only to show the negligible influence of the transducer resonance. The data from the tests with the 200Hz sampling rate (T103 - T125 in steps of 2) was analysed using a 4096 point FFT to convert the signals into the frequency domain. Using a sorting routine, the peak amplitudes of the spectrum and their associated frequency values were determined over the available response. A typical time dependent acceleration signal with its frequency spectrum is shown in figures 6.9 and 6.10. These figures show the base strain gauge output for the case of full submergence. It can be seen that the response is dominated by a frequency of 8.89Hz which corresponds to the first modal frequency. A more attenuated second natural frequency is seen at 56.4Hz. A summary of the measured first and second modal frequencies for this test series is given in table Fl of Appendix F. The errors associated with the values can be considered to be due only to the definition of the frequency spectrum which is dependent upon the sample frequency of the recorder and has a value of 0.05Hz at a sampling rate of 200Hz. Timing errors in the sampling frequency are negligibly small and errors in the amplitude of measurement have very little effect on the measured frequency. A summary of how the frequencies change with depth of submergence are shown by a graph of percentage change in frequency as shown in figures 6.11 and 6.12.
Figure 6.9 Typical Damped Acceleration Signal from the Test Series 100

Figure 6.10 Frequency Spectrum for Shot 123 Channel 1
for first and second mode respectively. The errors associated with this data are shown in figure 6.11 to be significant for first mode - they are not shown in figure 6.12 as the error bars would not be visible.

It was concluded from the foregoing experiments that a satisfactory amount of information had been obtained for first mode response. However, further information was required about the second mode frequencies. A second series of experiments was carried out with the mast loaded at the mid-span position, this having been shown, by analysis, to improve the second mode content. It was also thought that it may be possible to induce some third mode response using this technique. To make sure that the third mode data was correctly captured without signal 'aliasing' the sampling rate was increased to 500 Hz. In the previous tests, it had become obvious that only limited information could be determined from the signals obtained using the 10 KHz sampling frequency and therefore this sampling frequency was not used in the subsequent test programmes. In total, 34 tests were carried out for the mast with its depth of submergence varied between 0 and 1600mm in steps of 100mm. The use of different submergence depths served to provide further information to that previously obtained from the first test series. The repeatability of the tests was again controlled by maintaining the initial output from the base strain gauge at a constant level. The results from the tests carried out with a sampling frequency of 200 Hz (T203-T235 in steps of 2) were analysed as before and a summary of the frequency contents is given in Table F2 of appendix F. A typical time dependent tip acceleration signal for a submergence depth of 1500mm measured using a sampling rate of 500Hz is shown in figure 6.13 together with its frequency spectrum in figure 6.14. It can be seen that far more second mode information is available and even some third mode is visible. A summary of the values of first and second mode frequencies for the test series using a 500Hz sampling rate (T204-T236 in steps of 2) is given in table F3 of Appendix F. The limited information about third mode data available from the tip accelerometer is given in Table F4 of Appendix F. The error associated with these measured frequencies is ± 0.122 Hz. The values of the decrease in frequency from table F3 are also given in figure 6.11 and 6.12. The third mode information from table F4 is shown in figure 6.15.

Finally, to improve third mode information and possibly induce some fourth mode response, a third test program was carried out using the double
Figure 6.11 Experimentally Determined Changes in the Mast's First Natural Frequency of Response
% Decrease in Frequency

* exp. values T100 series
* exp. values T200 series
* exp. values T250 series

Depth of Water/Height of Mast

Figure 6.12 Experimentally Determined Changes in the Mast's Second Natural Frequency of Response
Figure 6.13 Typical Damped Acceleration Signal from the Test Series 200

Figure 6.14 Frequency Spectrum for Shot 234 Channel 1
Figure 6.15 Experimentally Determined Changes in the Mast's Third Natural Frequency of Response
wire system discussed earlier. In order to measure fourth mode data the sampling frequency was increased to 1KHz. In total 36 tests were performed with depths of submergence varying from 150mm to 1650mm in 100mm steps and also including a test in the non-submerged and fully submerged conditions. The repeatability of the experiment was again controlled by monitoring the strain gauge outputs for the initial deflection profile. In this case both the base and the mid-height strain gauges were monitored and initial outputs of 250 and 200 µstrain respectively were used as initial starting conditions. A sampling rate of 200Hz was also used to provide further information about the first and second mode responses. A summary of the results with the 200Hz sample rate (T255-T289 in steps of 2) is given in table F5 in Appendix F. The values of the associated decrease in frequency is also given in figures 6.11 and 6.12. A summary of the first and second mode results with the 1KHz sample rate (T254-288 in steps of 2) is given in table F6 in appendix F. A typical time dependent base acceleration signal is shown in figure 6.17 together with its frequency spectrum in figure 6.18. It can be seen that both first an second modes are dominant, however, some third and fourth mode information can be found by expanding the scale in the corresponding frequency range. A summary of the third and fourth mode frequencies obtained from the test series with a sample rate of 1KHz is given in table F7. Information from channel 3 has been omitted from the results because the amplitudes of the 3rd and 4th mode frequencies on this signal were not significantly greater than the background levels. The dependence of 3rd and 4th mode frequencies upon the depth of submergence is shown in figures 6.15 and 6.16 respectively.

Discussion of Results

Figures 6.11, 6.12, 6.15, 6.16 summarise the experimental results obtained from the three series of tests for modes 1,2,3 and 4 respectively. The data giving the change in frequency with depth of submergence for 1st mode of response shows extremely good correlation with the predicted results from chapter 3 which are also superimposed on this graph. It seems that, from a hydrodynamic viewpoint, the mast behaviour is represented better by the cantilever model. This agrees with the finite difference analysis of the first mode response taking into account the effect of the support's rotational stiffness. Unfortunately, due to
Figure 6.16 Experimentally Determined Changes in the Mast's Fourth Natural Frequency of Response
Figure 6.17 Typical Acceleration Signal from the Test Series 250

Figure 6.18 Frequency Spectrum for Shot 284 Channel 1
the need to measure the second mode response simultaneously, rather large
error bars are associated with these frequency measurements and this
somewhat limits the ability to conclude further as to the accuracy of the
cantilever model. The three series of results for the first mode are seen
to be consistent and differ only in terms of the associated errors.

The results of the second response mode are also shown to be
consistent for all three test series. The characteristic trends show good
correlation with the predicted results. For depths of submergence of up
to 50% of the mast's height, the decrease in natural frequency is
accurately predicted. However, once the depth of submergence approaches
the mast's node position (i.e. the position where the mast's deflection for
this pure mode remains zero throughout the oscillation), the differences
between the predicted and measured responses are much more pronounced. As
the depth of submergence increases further, these differences reduce in
magnitude until finally, at the fully submerged condition, the mast
response appears to lie between the predicted results for the cantilever
and pin joint supported systems. The frequency measurement errors
associated with the accuracy of the frequency transform are now very small
in comparison to the measured frequency and are not significant.

The results of the third response mode show similar characteristic
trends to those of the second mode except that the mast has two modal
positions and hence has two regions where the changes in the depth of
submergence show little influence on the frequency characteristics. The
frequency changes at both low levels of submergence and at full submergence
seem to be the most accurately predicted results and the largest variation
between experimental and predicted results occur at the position of the
nodes.

Finally the results of the fourth response mode follow logically from
the previous modes. The results represent only one test series and do
show some experimental scatter between the frequencies measured by the
strain gauges and the accelerometers. The predicted results show good
correlation at both low submergences and the fully submerged condition and
maximum variation again appears to occur at the nodal positions.

The trends of the four modes of response show that the analytical
model of the submerged system is most accurate at partial submergences
close to both the non-submerged and the fully submerged conditions. This
suggests that it is the accuracy of the mode shape rather than the hydrodynamic simplifications, such as the neglect of compressibility and wave effects, which is responsible for these errors. The assumption that the structure continues to oscillate in its 'dry' mode shape is obviously in question. At low levels of partial submergence the effect of the water on both the frequency and the determined mode shape is very small. A fully submerged slender structure has been shown to have a near constant added mass distribution and hence the total effective load distribution would again be constant and the 'dry' mode shape would be accurate, as would the determined frequency. However, at positions of partial submergence between these two extremes, the mode shape will be changed from its 'dry' profile because the effective total mass distribution of the structure would vary along its length. The effect of this variation depends on both the slenderness ratio and on the wall thickness/diameter ratio of the mast. Having a larger proportion of mass distributed closer to the base of the structure would reduce the deflection of the structure in this area with respect to the deflection at the tip. The effect of such a change in the deflection profile has been shown in section 3.2.4 where changes in the cubic approximation for the mode shape were considered. The ratio of the maximum deflections for a second mode response were varied and the effect on the frequency response was shown in figure 3.26 to reduce the level of the 'plateau' region associated with the modal position. Therefore, it can be concluded that the effect of partial submergence is to alter the mode shape by reducing the displacement in the region of submergence with respect to the tip deflection for that mode. This would reduce the changes in the predicted natural frequencies obtained from the hydrodynamic model, towards the experimentally measured values.
6.5 Test Programme for the Evaluation of Damping

As discussed in chapters 2 and 3, the damping of structures depends on many factors and in some cases can only be modelled by use of empirical data. Throughout the experiments described in the previous section, the influence of damping has been notable both in the non-submerged and semi-submerged conditions. The use of free vibration response tests gives the possibility of determining the damping coefficients in several ways. The damping coefficients used conventionally are either the logarithmic decrement $\Delta$ or the fraction of critical damping $\zeta$, and there is a simple relationship between the two. The dynamic equation for a single degree of freedom system is

$$M \ddot{x} + C \dot{x} + K x = F(t) \quad (6.2)$$

The form of the solution of this equation depends on the proximity of the damping coefficient $C$ to the critical damping coefficient $C_c$, where

$$C_c = 2\sqrt{K/M} = 2M\omega_n \quad (6.3)$$

The coefficient $\zeta = C/C_c$ and is defined as the fraction of critical damping. For the case where $\zeta<<1$ (which is more representative of the structure in the present study), the solution of the equation motion is

$$x = A e^{-\zeta\omega_n t} \sin(\omega_d t + \theta) \quad (6.4)$$

The damped natural frequency, $\omega_d$, is related to the undamped natural frequency, $\omega_n$, by the equation

$$\omega_d = \omega_n \left( 1 - \zeta^2 \right)^{\frac{1}{2}} \quad (6.5)$$

The equation of response (6.4) can be represented graphically as shown overleaf.
It can be seen from equation (6.5) that the coefficient $\zeta$ has some influence on the frequency of the response. However, in normal cases $\zeta$ is less than 0.14 and thus has little effect on the results (error $< 1\%$).

The degree of damping in a system where $\zeta < 1$ can be determined from the change in successive peak values from a free response displacement versus time plot as shown:

The logarithmic decrement $\Delta$ is the natural logarithm of the ratio of the amplitude of two successive cycles of the damped free vibration.

$$\Delta = \log_e \left( \frac{x_1}{x_2} \right)$$  \hspace{1cm} (6.6)

However, from equation (6.4), if the peaks occur at times $t_1$, and $t_2$,
then

\[ x_1 = A e^{-\zeta_1 \omega t_1} \sin(\omega_d t + \theta) \]

\[ x_2 = A e^{-\zeta_2 \omega t_2} \sin(\omega_d t + \theta) \]

and it is known that \((t_2 - t_1) = T\), the time period of response

\[ = \frac{2\pi}{\omega_d} \]

so that,

\[ \Delta = \log_e \left[ e^{-\zeta \omega(t_1 - t_2)} \right] \]

\[ = \frac{2\pi \omega}{\omega_d} = \frac{2\pi \zeta}{(1 - \zeta^2)^{\frac{1}{2}}} \]  

(6.7)

Assuming small values of \( \zeta \) (i.e. \( \zeta < 0.1 \) which is usually the case for the damping of structures in air and water) an approximate relationship between the fraction of critical damping and the log decrement is

\[ \Delta \approx 2\pi \zeta \]  

(6.8)

The choice of measuring the log decrement or the value of zeta is arbitrary and the suitability of each is largely dependent on the type of damping acting. In the case where the damping is considered to be constant throughout the response, a continuous decay envelope can be fitted to the peak values of oscillation. A curve fitting routine using a least squares method can then be used to fit the best exponential decay to the data and obtain the decay coefficient \( \zeta \). However, this technique is not suited to the measurement of damping characteristics which change during each cycle of response. Such a variable characteristic is the inherent material damping which is often dependent on strain amplitude. Methods of fitting decay curves over a shorter range of amplitudes, say 10 peak values, have been used in the past(1). The amplitude dependency can be found by determining the damping ratio which is considered to remain constant over, say, 10 cycles and relating it to the peak amplitude value of the first cycle. The curve fitting 'window' can then be stepped through the data.
set to obtain the relationship between the decay coefficient and maximum amplitude. However, in cases where the value of the decay coefficient is high, the assumption of a constant value over a ten cycle period can often lead to erroneous results and it is often more realistic to determine the coefficient by use of the log decrement term.

In order to investigate the damping characteristics of experimentally obtained acceleration and strain signals, software can be written to pick up the maximum and minimum points of the signal. The logarithmic decrement can be determined by use of equation (6.6) and the relationship between the logarithmic decrement and the amplitude of response can then be found for each signal. However, in using this method it must be assumed that only one response mode is excited. Inclusion of both higher or lower modes can create major errors in the log decrement values. In the experimental results reported in section 6.4, it has been shown that more than one mode was operative in all tests. To overcome these difficulties, digital filtering techniques can be employed, in much the same way as described in section 5.2.1, to extract only the modal information required. Care must be taken whilst using such techniques because the damping characteristics are controlled by a range of frequencies around the natural frequency value. As the damping coefficient increases, the number of neighbouring frequencies providing these characteristics is also increased. It is important not to corrupt any of these component frequencies whilst carrying out the filtering process and hence a relatively wide pass band should be placed around the modal frequency of interest. Using this method, the results from the frequency tests described in section 6.4 were analysed. However, the predicted results for each test series showed large discrepancies. Further tests were carried out using the mast subjected to a tip deflection and the analysis of these results indicated that although the mast's frequencies could be measured repeatedly, the damping characteristics were not repeatable. The deviations were shown by later experiments to be due mainly to the electromagnetic release system and also frictional effects from the cable/pulley system.

Several options for a new type of release method were subsequently considered and tested. Use of an electrical fusing link was considered but it was envisaged that the need for submergence of the system would create major technical difficulties. Even if the fusing took place above
the surface of the water, friction at the pulley/wire interface would still exist. A method of cutting a tensioned string was tried, but it became increasingly difficult to cut the string cleanly at high levels of load. Further tests showed that the most effective method of release was achieved by imposing the load by hand with no use of tensioning wires. Although relatively crude, this method could accommodate a large range of input amplitudes. Unfortunately this method of release greatly restricts the ability to excite higher modal responses. However, it is important to understand first mode behaviour before considering the higher modes. The major weakness in this method was the directional control on the input displacement. However, some improvement was provided by fixing a length of cord across the longitudinal axis of the tank slightly above the tip of the mast to facilitate alignment of the deflected tip.

In an initial set of damping tests, the difference between the decay coefficient/strain amplitude characteristic was investigated for the two extreme cases with the mast in a non-submerged and a fully submerged condition. For each condition, five initial mast deflections were used so that the initial voltage output from the amplifiers was at the full scale of the respective voltage ranges on the transient recorder. (It was realised that the errors in the analysis were controlled by the 'bit level' of the recorder and that its maximum range should be used to maintain the lowest errors). At each value of the initial deflection, the experiment was repeated four times, thus providing 20 experimental records for each channel. Only the information provided by the base strain gauge #4 was subsequently stored because this output could be related to tip deflection for comparison with the theoretical values given in chapter 3. For each extreme mast condition, the values of the log decrement and the maximum amplitude during the cycle of response have been compiled into one graph for the 20 experiments. The results of the non-submerged and fully submerged cases are given in figures 6.19 and 6.20 respectively. It can be seen that the results from the non-submerged case exhibit approximate linearity. This is verified by the least squares straight line fit having a correlation coefficient better than 0.97. In the case of a fully submerged mast, it can be seen that the straight line approximation is relatively accurate over the central 80% of the range investigated. At both high and low values of strain amplitude, the experimental results appear to differ markedly from the straight line. However, from the
\[ Y = Mx + C \]

- \( M = 3.38 \times 10^{-5} \)
- \( C = 0.00338 \)

Amplitude of Strain (\( u_{strain} \))

Figure 6.19 Experimentally Determined Damping Relationship with the Amplitude of Strain for the Non Submerged Mast

\[ Y = Mx + C \]

- \( M = 0.000263 \)
- \( C = 0.00956 \)

Amplitude of Strain (\( u_{strain} \))

Figure 6.20 Experimentally Determined Damping Relationship with the Amplitude of Strain for the Fully Submerged Mast
theory developed in chapter 3, there is no justification for fitting a polynomial of higher order.

A final series of tests was undertaken to investigate the mast's damping characteristics whilst in a semi-submerged condition. The experiments were carried out for submergence levels increasing in steps of 100mm using the same method as described above. The complete set of graphs for this series of tests is given in Appendix G. In all cases, a straight line was fitted using a least squares method and a summary of the values of the resulting damping coefficients is given in table 6.2. To enable a comparison to be made with the theoretical values of damping due to hydrodynamic interaction derived in chapter 3, the amplitude of strain must be converted into an amplitude of deflection. This presents no real problems because the mast is considered to be vibrating in a single mode. The relationship between the radius of curvature and the strain in the outer fibres of the beam in the region of the support can be obtained from simple bending theory. Hence,

$$
\varepsilon = y \frac{d^2X}{dx^2} \quad (6.9)
$$

The value of $\frac{d^2X}{dx^2}$ can be obtained from the mode shape of the response.

$$
\frac{d^2X}{dx^2} = \frac{A_y k^2}{2} [-\cos(kx) - \cosh(kx) + C (\sin(kx) + \sinh(kx))] \quad (6.10)
$$

and at the support position, $x=0$

$$
\frac{d^2X}{dx^2} = -A_y k^2 \quad (6.10)
$$

Using equations (6.9) the relationship becomes

$$
\left( \frac{\varepsilon}{A_y} \right) = -\frac{k^2 D}{2} \quad (6.11)
$$

Considering the beam in the experimental tests to vibrate only in first mode then substitution of the relevant coefficients gives

$$
\left( \frac{\varepsilon}{A_y} \right) = 23.4 \ (\mu\text{strain/mm})
$$
Table 6.2 Experimentally Determined Linear Damping Relationships

Considering the damping characteristic to be of the form

$$\Delta = \Delta_0 + K_1 \varepsilon$$

<table>
<thead>
<tr>
<th>Depth of Water /m</th>
<th>$K_1$ (μstrain) x10^5</th>
<th>$\Delta_0$ x10^3</th>
<th>Correlation Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>3.38 ± 0.08</td>
<td>3.38 ± 0.10</td>
<td>0.975</td>
</tr>
<tr>
<td>0.10</td>
<td>2.77 ± 0.06</td>
<td>6.06 ± 0.08</td>
<td>0.975</td>
</tr>
<tr>
<td>0.20</td>
<td>2.47 ± 0.07</td>
<td>7.22 ± 0.10</td>
<td>0.955</td>
</tr>
<tr>
<td>0.30</td>
<td>2.06 ± 0.07</td>
<td>8.58 ± 0.09</td>
<td>0.939</td>
</tr>
<tr>
<td>0.40</td>
<td>3.18 ± 0.09</td>
<td>5.72 ± 0.14</td>
<td>0.952</td>
</tr>
<tr>
<td>0.50</td>
<td>2.55 ± 0.06</td>
<td>7.88 ± 0.08</td>
<td>0.972</td>
</tr>
<tr>
<td>0.60</td>
<td>2.64 ± 0.06</td>
<td>8.00 ± 0.08</td>
<td>0.976</td>
</tr>
<tr>
<td>0.70</td>
<td>2.47 ± 0.07</td>
<td>9.85 ± 0.09</td>
<td>0.963</td>
</tr>
<tr>
<td>0.80</td>
<td>3.63 ± 0.08</td>
<td>8.64 ± 0.10</td>
<td>0.978</td>
</tr>
<tr>
<td>0.90</td>
<td>4.34 ± 0.09</td>
<td>10.3 ± 0.09</td>
<td>0.975</td>
</tr>
<tr>
<td>1.00</td>
<td>6.07 ± 0.15</td>
<td>9.51 ± 0.16</td>
<td>0.966</td>
</tr>
<tr>
<td>1.10</td>
<td>9.02 ± 0.34</td>
<td>7.85 ± 0.30</td>
<td>0.955</td>
</tr>
<tr>
<td>1.20</td>
<td>12.3 ± 0.27</td>
<td>9.54 ± 0.27</td>
<td>0.982</td>
</tr>
<tr>
<td>1.30</td>
<td>16.7 ± 0.57</td>
<td>8.80 ± 0.43</td>
<td>0.940</td>
</tr>
<tr>
<td>1.40</td>
<td>19.3 ± 0.54</td>
<td>15.5 ± 0.40</td>
<td>0.967</td>
</tr>
<tr>
<td>1.50</td>
<td>26.9 ± 0.70</td>
<td>9.56 ± 0.40</td>
<td>0.954</td>
</tr>
</tbody>
</table>
Thus, the output value of strain measured by the base strain gauge can be related to the dynamic tip deflection. The effect of damping due only to submergence can be found by subtracting the damping due to material and support effects. It has been shown in chapter 3 that the effect of submergence is not significant until the structure is submerged further than 1/3 of its height. The results for the total damping relationship up to this level show large variations and it is reasonable to average the results over this length. The effect of submergence can then be found by subtracting these averaged coefficients from the determined coefficients at each height. The results are shown in table 6.3 and are plotted in figure 6.21 for comparison with the theories developed in chapter 3.

So far in this damping analysis, only the characteristics of the first mode have been considered. The difficulty in obtaining reliable damping information for this mode has been due to the limitations of the test rig and it therefore seems unlikely that good second mode information can be obtained by the free response method. However, in analysing the first mode signals, a certain amount of higher mode information has been rejected by the filtering operation and it is worthwhile to investigate this rejected data. Care must be taken in analysing this data because the amplitude levels for the second mode signal are far lower than those of the dominant mode. The effect of errors in the filtering technique together with the significance of the 'bit level' of the machine place large uncertainties on the predicted results. For these reasons alone, only the characteristic trends of such signals are discussed. The free response signals can be filtered using a narrow band filter which only allows frequencies between 40 and 80 Hz to remain. The resulting second mode signals have maximum amplitudes of 20 μstrain which relates to 20 bits in the original signal and hence the accuracy of log decrement values is questionable. However, the trends show that the second mode damping is far more significant than first mode damping at the same amplitudes of strain. This result correlates with theory since the equation of motion has a frequency term attached to the damping force which will obviously increase in magnitude with mode of response.
Table 6.3 Linear Hydrodynamic Damping Relationship Derived from the Experimental Results

Converting the previously given experimental results into a form which is suitable for comparison with the theoretical values determined in chapter 3.

\[ \zeta = \zeta_0 + K_1 \left( \frac{A_y}{D} \right) \]

<table>
<thead>
<tr>
<th>Depth of Water /m</th>
<th>( K_1 \times 10^2 )</th>
<th>( \zeta \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-0.50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.011 ± 0.015</td>
<td>0.243 ± 0.029</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.030 ± 0.016</td>
<td>0.537 ± 0.030</td>
</tr>
<tr>
<td>0.80</td>
<td>0.100 ± 0.017</td>
<td>0.345 ± 0.032</td>
</tr>
<tr>
<td>0.90</td>
<td>0.179 ± 0.018</td>
<td>0.609 ± 0.030</td>
</tr>
<tr>
<td>1.00</td>
<td>0.373 ± 0.025</td>
<td>0.483 ± 0.041</td>
</tr>
<tr>
<td>1.10</td>
<td>0.702 ± 0.046</td>
<td>0.219 ± 0.064</td>
</tr>
<tr>
<td>1.20</td>
<td>1.069 ± 0.038</td>
<td>0.488 ± 0.058</td>
</tr>
<tr>
<td>1.30</td>
<td>1.560 ± 0.072</td>
<td>0.370 ± 0.084</td>
</tr>
<tr>
<td>1.40</td>
<td>1.851 ± 0.068</td>
<td>1.437 ± 0.080</td>
</tr>
<tr>
<td>1.50</td>
<td>2.700 ± 0.086</td>
<td>0.491 ± 0.080</td>
</tr>
</tbody>
</table>
Figure 6.21 Comparison Between the Experimentally Measured and Predicted Results of the Relationship of Hydrodynamic Damping
Confidence Limits of the Determined Slope

In quoting the determined slope from the experimental damping data, it is important to determine the reliability of the calculated slope. An indication of the 'goodness' of fit of a least squares straight line fit is given by the correlation coefficient, $R$, which is defined as:

$$ R = \frac{\sum (x_i - \bar{x}) (y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 (y_i - \bar{y})^2}} \quad (6.12) $$

where $x_i$ is the independent variable, $y_i$ is the dependent variable and $\bar{x}$ and $\bar{y}$ are the mean values of the respective variables. The values of the correlation coefficient determined from this equation has been given in table 6.2 for each of the data sets. However, it is very difficult to place a level of significance on the value of this coefficient. The coefficient is very useful for detecting totally unreasonable fits but cannot detect gentle curves in the relationship between the data values.

A more meaningful assessment of the data is to establish a range of values within which the true slope has 95% likelihood of falling. In this way, the error in the slope can be compared to the value of the slope, thus indicating the validity of the straight line fit. The true slope can then be given as,

$$ \text{Slope} \pm t \cdot \text{standard error of the slope} $$

where $t$ is the 95% student's $t$ value with $(n - 2)$ degrees of freedom. The standard error of the slope is given by (ref 81),

$$ \text{Standard error of slope} = \frac{\text{Standard deviation of the points from the line}}{\sqrt{\sum (x_i - \bar{x})^2}} \quad (6.13) $$

and the standard deviation is given by,

$$ \text{Standard deviation} = \frac{\sqrt{\sum (y_i - (mx_i + c))^2}}{n-2} \quad (6.14) $$
In addition to this slope error, there is also an uncertainty in the exact position of the cross-over point \( y_0 \). This is due to errors in the individual \( y^1 \) values. The value of the cross over point can be similarly quoted as

\[ y_0 \pm t \times \text{standard error of } y \]

Standard error of \( y = \frac{\text{standard deviation of the points from the line}}{\sqrt{n}} \)  \hspace{1cm} (6.15)

Using equations (6.13) and (6.15) the errors in both the gradient and intercept have been determined from the experimental results used earlier, and are given in tables 6.2 and 6.3.

In considering the scatter of experimental results it is also important to investigate the significance of the individual errors associated with the evaluation of the data points. The value of the logarithmic decrement has been calculated from the equation

\[ \Delta = \log_e \left( \frac{\varepsilon_1}{\varepsilon_2} \right) \]  \hspace{1cm} (6.16)

However, both \( \varepsilon_1 \) and \( \varepsilon_2 \) are subject to error because of the accuracy of the output signal and the accuracy of the recording device.

The accuracy of the output strain signal is dependent on many factors. However, by carrying out calibration tests, as has been described earlier, some of the inherent errors such as the effects of gauge misalignment or the effects of the curved surface of the mast, can be eliminated through use of a 'compensation factor'. The output signal from the gauge can now be considered to have no error as long as it is set up with exactly the same alignment and energisation voltage. It is therefore inevitable that the gauge will have some element of erroneous systematic signal. The mis-alignment of the tube, even with great care, may be of the order of \( \pm 1^\circ \) and the strain gauge bridge voltage applied may vary between experiments by 1%. The effect of misalignment is not great since the variation in the measured strain is governed approximately by the natural sine of the angular misalignment. A 1% error in angle would produce a 1.7% error in the strain reading and the total error on the output voltage would be of the order of 2.7%. However, these errors can
be considered to remain constant throughout the experiment and hence their effect on the value of the log decrement calculation would be negligible. Only the amplitude of strain at which the value of log decrement was calculated would be subjected to this small error.

The accuracy in the recording device may, in some cases, be the most significant source of error. The major error is associated with the bit level of the machine and errors in the amplification system can be considered to be small in comparison. The 'bit level' error is constant for a selected range of voltage and hence the percentage error on the signal increases with decreasing levels of amplitude. The error can be given as:

\[
\text{Error in Strain} = \pm \frac{0.5}{1024} \times \text{Max. range of strain amplitude} \quad (6.17)
\]

(This equation is valid for a 10 bit recording device)

The errors determined from a typical set of data consisting of 5 tests carried out at different release amplitudes using equation (6.16) and (6.17) are shown in figure 6.22. The graph indicates that the most accurate information is obtained in the first oscillations of the response. If the information was shown as percentage errors in the data points it would be seen that the most accurate points were those at large amplitudes. However, it is difficult to obtain further information about the way in which these individual errors actually affect the accuracy of the predicted straight line since it is impossible to ascertain the likelihood of the errors occurring at the same time. The predicted straight line fit for this data is also shown in this figure and it is interesting to note that the error bars do seem to account for the discrepancies between the data and the curve fit over a majority of the points. This suggests that the relationship is linear and that the error analysis has fully accounted for the major errors in the measurement of the damping characteristics.
Figure 6.22  Typical Error Bands Associated with the Determined Damping Coefficients
Discussion of Results

The results of the non submerged test show that the damping has a strong linear dependence upon the amplitude of strain. The results also show that the value of damping for very small amplitudes of strain approach a finite value of log decrement. This indicates that at least two mechanisms of damping are operating during the dry response of the mast. The first, which remains constant throughout the response is considered to be a support effect and can therefore be modelled using the base viscous dampers discussed in section 2.6 with the damping coefficients in equation (2.23) equal to $2.5 \times 10^{-4}$. The second damping term is directly proportional to the strain amplitude and is far more dominant than the offset value at high values of strain. This damping can be considered to be a material damping phenomena. It has been shown in chapter 2 that the central relationship in the 3 segment approximation of material damping is dependent on an amplitude of stress which has an index $(n-2)$ where $n$ varies between 2 and 3. The results of the non-submerged test suggest that $n$ is closer to three in the range $100 < \varepsilon < 900 \mu$strain. The results also indicate a value of $n$ lower than 3 in the region below 100 \mu strain. This agrees closely with the results presented by Lazan.(6) The value of log decrement occurring at an amplitude of 800 \mu strain due solely to material effects can be shown from the experimental results to have a value of around 0.032 whereas that predicted by Lazan is 0.037. It can therefore be concluded that the results from the non-submerged trial is exactly as expected and are completely accounted for by the inclusion of both material and structural support losses.

The damping results of the semi-submerged tests through to the fully submerged test shows good linearity with respect to the output strain and practically all of the correlation coefficients are better than 0.95. At low submergences, however, the coefficients of the straight line fit to the data vary quite erratically and no real upward trend in the gradient is visible until the depth of submergence exceeds half of the mast's height. After this depth has been reached, the change in gradient becomes quite significant and increases rapidly until full submergence when the gradient becomes eight times steeper than at non-submergence. The offset coefficient of the straight line does not show such a definite trend and behaves very erratically in the region of full submergence. This
behaviour is most likely related to the individual errors in the determined coefficients of the straight line fit which become much more significant at higher levels of damping and tend to affect the offset much more than the gradient.

The fact that the $\zeta$ - amplitude of strain relationships remain linear with increasing depths of submergence suggests that the method of using an equivalent viscous damping coefficient is suitable for modelling fluid damping. However, it has been shown that the accuracy of such a method is highly dependent on the type of $C_d$-Re relationship selected. The graphs in figure 6.21 show the predicted values of gradient assuming $C_d$-Re relationships which have been determined for either of two systems.

(1) The steady flow past a stationary cylinder (Blevins$^{(37)}$)

(2) The oscillatory flow past a stationary cylinder (Sarpkaya$^{(40)}$)

The results using the available data for system (1) show that the gradient of the $\zeta$ - amplitude of strain relationship is far more dominant than the offset which can be considered to be practically zero for all depths of submergence. Alternatively, use of approximation (2) gives a reduced dependence upon amplitude of strain and a far more dominant offset value. The results for the 15 partially submerged experiments show close correlation with the gradient predictions of the second approximation but predict offset values between the two extremes of the above approximations. However, both of the approximations used to predict the experimental results do not model the fluid-structure interaction entirely correctly. The true relationship required is one in which the structure is vibrating with varying amplitudes of deflection along the structure's length whilst the surrounding fluid remains relatively still. The way in which the damping information has been collected prevents the experimental results from being used to accurately predict the true $C_d$-Re relationship for the vibrating model mast. In order to obtain this type of information, the mast would need to be subjected to forced vibration where the maximum amplitude of oscillation was kept constant over repeated cycles. Integration of the forcing function would allow the accurate determination of the time averaged $C_d$ value over the period of oscillation. In using the method of equivalent damping, the output
damping value is a measure of the summation of the $C_d$ values for a range of Reynold's numbers and hence some of the finer detail of the $C_d$-Re relationship is obscured. However, it is possible, with the pre-knowledge of how the $C_d$ affects the equivalent damping, to predict the trends of the $C_d$-Re relationship for the model mast. Because the damping gradients of the experimental results match closely with the equivalent damping values obtained using Sarpkaya's experimental results, it would seem that the value of $C_d$ which is approached at high values of Reynold's number would be around 0.43. The results also suggest that the offset coefficient, which is controlled by a constant associated with the reciprocal of Reynold's number, is much closer to that predicted by Blevins. The approximate form of the equation for $C_d$ would be

$$C_d = C_1 + \frac{C_2}{Re}$$

where $C_1=0.43$ and $C_2=800$. The value of $C_2$ may be subject to large errors because of the nature of the measured offset data. However, these errors do not significantly affect the damping values because it is the coefficient $C_1$ which strongly influences the damping behaviour over the range of measured values.

The form of the above $C_d$ relationship is shown in figure 6.23 for comparison with the two relationships previously defined. The applicable range of the steady flow approximation shown is limited to Reynold's numbers below $10^4$ because such a simple relationship does not operate in the region of the transition to turbulent flow. The applicable range for the approximation to Sarpkaya's oscillatory flow data is governed by the range of values used to determine the relationship. Finally, the range over which the damping characteristics of the model mast have been measured can be shown to be $3700 < Re < 72000$. It should be noted in the case of predictions for the oscillatory flows, that the higher values of $C_d$ predicted outside these ranges might not be correct although other results(39) have suggested that the $C_d$ values obtained at these lower Reynold's numbers do increase to beyond the steady flow values.

The differences in the predicted results for the damping coefficients can be seen to be the direct result of the use of $C_d$ relationships which are not an accurate representation of the flow system. The Blevins approach predicts much higher changes in damping gradient because, at high
Figure 6.23 Approximate Relationships of $C_d$ versus Reynolds Number for Various Flow Regimes
Reynold's numbers, the steady state value of \( C_d \) predicted from Schlichting's work is much greater than that achieved in oscillatory flow. The lower value of \( C_d \) associated with high Reynold's number oscillatory flow is largely influenced by the formation of vortices and large scale turbulence on both sides of the cylinder. The occurrence of separation is very sensitive to factors such as the slenderness ratio, the natural frequency, the amplitude of vibration and the surface roughness of the cylinder. It has even been shown that the \( C_d \) relationship is influenced by the characteristics of the surrounding water container/tunnel. These factors also help to explain the discrepancies between Sarpkaya's results and the experimental results obtained in the present study. Although both experiments have values in the region of 8500, it seems that it is not correct to assume that same \( C_d \) relationship. In fact, the cylinder used by Sarpkaya has a diameter 5.5 times larger and a slenderness ratio 10 times smaller than that used in the present study. More importantly, the structure used in the present study relies on its flexibility to create the oscillatory flow and hence the drag coefficient must be lower than that for rigid body flow because the system allows vertical motion of the fluid which relieves the pressure on the cylinder's surface. Lastly, the \( C_d \) values have been obtained in the previous studies for a structure in a fully developed steady state flow condition. For the steady flow experiments, the \( C_d \) value was obtained after the flow was given time to settle to its steady state. The oscillatory flow experiments had the amplitude of oscillation at a constant value and also allowed time for a steady state to be achieved before the measurement of the \( C_d \) value. In the present tests, the steady state is never achieved because the amplitude of oscillation reduces with each cycle of response. In considering all of the dissimilarities between the past and present experimental fluid/structure systems, it would be anticipated that discrepancies would exist between the measured \( C_d \) relationships.
7. **Concluding Discussion**

7.1 **Discussion of Experimental and Analytical Results**

It has been shown that the most suitable model for numerical analysis of the shock response of a submerged mast is the lumped parameter system. A time domain solution to the equations of motion for the discretised structural system can be obtained through the use of finite difference techniques. The advantages of such an approach are:

1. The complexity of the model can be varied to suit the needs of the analysis. Increasing the number of nodal masses increases both the accuracy and the computation time. However, this also has the effect of increasing the numbers of degrees of freedom of the system and hence reduces the size of the stable time step which also has implications on the required computation time.

2. The inclusion of local changes in material properties, such as the effect of stiffness provided by the supporting positions, presents no computational difficulties. A mast supported in diaphragm-type restraints has been successfully modelled by the analysis, the associated errors in predicting the fundamental frequency being typically 5% for a 12 node idealisation.

3. Structural damping effects, such as those occurring at the support positions, can be modelled in the analysis by use of viscous damping terms.

4. Hydrodynamic interaction effects for mast-like structures can be accurately modelled by including an added mass distribution and an equivalent damping term at each submerged nodal position.

5. Complex forcing functions can be included in the analysis in a number of ways. Loads imposed by relative movement of the supporting structure can be modelled by step inputs, ramp inputs and complex recorded inputs can even be used such as the displacement trace measured from a seismic event. Loads imposed on the structural surface can be incorporated by lumping the
distributed load profile at the nodal positions. In this way, loads from the pressure pulses which radiate from detonated explosives can be included. It is also possible to impose initial deformation profiles before releasing the structure and such methods may be used to model the effect of failure of supporting guy lines, for example.

(6) Because of the nature of the analysis, more than one forcing function can be applied to the structure either simultaneously or with a specified time delay. This becomes very important for explosive shock loading where the loads on the mast may be both a pressure loading and a base movement with a delay between their times of arrival.

In order to validate this analytical approach, it is necessary to obtain further information from both experimental and other analytical investigations. To study the response of submerged structures from a practical standpoint, a laboratory based experimental test rig was built which supported a mast in a vertical position using two diaphragm restraints. The effect of partial and full submergence was investigated by filling a surrounding tank with water. The response was measured by the use of three strain gauge bridges and two undamped accelerometers. In order to analytically model the structure correctly, the stiffness of the supporting mounts was required. The stiffness has been determined by measurement of the strain at a position between the two mounts using a strain gauge bridge which was bonded to the internal surface of the mast. The experimentally measured stiffness showed some deviation from a theoretical calculation based on simple plate theory. However, the measured value has been confirmed as providing the correct frequency shift in the mast's measured frequency response. The damping of the non-submerged structure has been investigated by use of free vibration tests. It has been shown that the damping is controlled by two separate factors. The first is a material damping effect which is dependent on the amplitude of strain. However, over the range of strains used, only the central portion of the three segment approximation commonly used to represent material damping was visible. The second form of damping is that due to the losses at the structural supports. Assuming that the form
of structural damping is proportional to the angle of rotation of the support, it can be shown in theory to provide a damping force independent of strain amplitude. This component of damping has been shown, by experiment, to be small in comparison to material losses.

Hydrodynamic interaction has also been shown, by experiment, to influence the structural response in two ways. Firstly, the natural frequency of the structure is decreased and secondly the level of damping is increased. The analysis of the frequency changes in the response of submerged structures has been developed with the use of the wave equation. Using suitable boundary conditions which apply to the particular structural/fluid system, the pressure field in the region around the mast has been evaluated. The effects of compressibility and surface waves are subsequently shown to be negligible at the normal levels of structural frequency response. The boundary condition specifying the structure's displacement profile allows the pressure distribution to be computed as a function of mode number. The change in frequency can then be computed for each mode as a function of the depth of submergence. Comparisons with experimental data obtained from the laboratory test rig have confirmed that the mode shape associated with the response of the structure must be obtained from the dynamic response equation and that approximations to this mode shape could lead to large errors in the hydrodynamic analysis. To obtain these characteristic mode shapes it has been necessary to assume that they are unaffected by submergence. It is shown later that such an assumption provided acceptably accurate results although it is possible that improvements could be made by inclusion of the small changes in shape due to partial submergence.

The percentage change in the natural frequency has been shown to depend on both the geometry and the material properties of the system. The effect of either increasing the slenderness ratio or the diameter/wall thickness ratio of the structure produces a greater change in the structure's frequency during submergence. The way in which the structure is supported also influences the change in the frequency response and the two extreme cases of a cantilever support and a double pin joint support have been considered.

It has been shown that the hydrodynamic pressure effect is completely accounted for, by inclusion of an added mass distribution into the normal
mass distribution of the structure. This added mass distribution is mode dependent and varies significantly with mode number for non-slender structures. However, in the case where the diameter/wall thickness ratio is around 15, then for structures with slenderness ratios above 100, the added mass distribution can be considered to be constant over its full depth of submergence and independent of mode of response. The value of the added mass per unit height is approximately equal to the mass of fluid displaced by the structure per unit height which is given as $\frac{\pi \rho D^2}{4}$. In this form, the added mass is now suitable for inclusion into the finite difference analysis. However, the usefulness of this method, in the case of non slender structures, is somewhat limited. In such cases, the added mass is mode dependent and hence an accurate response can only be achieved at one particular modal frequency. The most dominant frequency must be used to determine the added mass profile and the errors incurred by the other modal frequencies must be accepted as part of the approximation. In some limited cases where the structure oscillates in one pure mode, the added mass distribution can be calculated using the structure's deformation profile predicted by the program. This method has been shown to be highly accurate for the modelling of the response of a tip loaded cantilever.

Experimental results of the frequency changes experienced by a structure as the level of submergence is increased until it is just submerged have shown good correlation with the predicted modal frequency changes. The test rig has been very successful in producing responses which provide information on the first four modes of normal response. The measured frequency values from all four monitored transducers have been consistent for all experiments except in the case of the fourth mode where the signal was obscured in the background 'noise' for one of the strain gauge bridges. The results have confirmed the accuracy of the predicted results for first mode response from both the modal technique and the finite difference approach. In the cases of 2nd, 3rd and 4th mode response, discrepancies of up to a 1.5% change in the response frequency predictions have been shown for submergence levels around the nodal points of the response mode. These errors are most likely to be the result of the use of 'dry' modes in the analysis. The effect of partial submergence is to increase the mass distribution over the submerged section which would locally reduce the deflection. The result of this reduction in deflection has been shown by using cubic approximations to lower the
predicted frequencies towards the measured values.

The second effect of hydrodynamic interaction, that of increased damping on the output response, has been studied semi-analytically and experimentally. The hydrodynamic analysis used for the determination of pressure and hence effective added mass distributions neglects the viscosity of water and hence provides no damping information. The damping of structures is highly influenced by fluid effects such as vorticity and turbulence and with the present knowledge of this field, it is impossible to predict damping forces accurately by purely analytical methods. The approach generally used to model fluid damping effects is to employ empirically based damping coefficients which control the damping forces defined within the 'Morison equation'. In order to reduce the complexity of adding such force terms into the dynamic equation, a method of equivalent viscous damping is commonly used. This method produces an equivalent damping coefficient for the viscous damper which is then included at each submerged nodal position on the structure. To evaluate the damping coefficient, some assumption about the relationship between the drag coefficient, $C_d$, and the Reynolds number, $Re$, must be made. Empirical results are available for both the steady flow and the oscillatory flow around stationary cylinders. Over the range of amplitudes used in the experimental investigation, the empirical results discussed above can be approximated by an equation of the form:

$$C_d = C_1 + C_2/Re$$

Analysis using such an approximation yields linear dependence on the amplitude of deflection and an offset which is dependent on the reciprocal of a frequency parameter related to the structural geometry. In the case of steady flow, this offset has negligible influence whereas for oscillatory flow, it becomes quite significant. Experimental results from the test rig have shown that the change in the gradient of the damping coefficient is more closely predicted by the oscillatory flow relationship whereas the offset is closer to the one produced by the steady flow approximation. From the point of view of the $C_d$ relationship, it seems that at high Reynold's number the vibrating cylinder acts as though it was in an oscillatory fluid whereas at low Reynold's number it approaches the steady flow condition. It may be that the vibrating cylinder changes
between flow regimes, however, it is more likely that it is due to more fundamental differences between these two regimes and the true experimental flow conditions. Firstly, the $C_d$ values obtained in the previous experiments have been determined for a steady state whereas, in the present experimental tests, the peak amplitudes are changing at every cycle of response and 'steady' conditions are never achieved. Discrepancies between the oscillatory flow case and the present study may be due to the differences in both the diameter and the slenderness ratio. However, previous investigations have suggested that the $C_d$-$Re$ relationship is dependent only on a $\beta$ ratio, where

$$\beta = \frac{D^2}{\nu T}$$

The results obtained from this present study indicate that the $C_d$ relationship is also dependent on an additional factor which is a function of the slenderness ratio of the mast.

This research programme has also studied the response of a submerged structure to a shock load resulting from a nearby explosion. The equations governing the characteristics of a pressure pulse resulting from such a blast are well established. However, the way in which the energy is transferred between the pulse and the structure is far more uncertain. Shock factors are commonly used to define the severity of the explosion on the structure and vary from a $W^{1/3}/R$ to a $W^{1/2}/R$ for rigid and flexible structures respectively. A value of the shock factor approaching 1.0 (lb $^{2}$/ft) for a flexible structure represents severe damage. However, it is unlikely that such a simple factor holds true for all materials and geometries. To incorporate the loading resulting from an explosive pressure pulse into the finite difference structural analysis, the pressure loading must be determined by considering both the incident and reflective pressures. The loading function can then be obtained by considering the pressure to be acting over the projected area of the mast. However, the effectiveness of the pressure will be reduced because of flow around the structure and will be governed by a factor similar to the drag coefficient of the cylinder. The pressure will be reduced further by the induced motion of the structure. Structures such as slender masts must be supported by rigid bodies which will, themselves, be subjected to the shock loading. The movement of the supporting structure in response to the
shock loading can be modelled as a rigid body motion. This response may be critical because any movement will be transferred to the mast at the supporting positions and for an accurate model of the mast response both forcing functions must be included into the analysis. To gain further understanding of the transfer of energy from an explosive pressure pulse to the structural surface and the relative importance of both types of loading, an experimental study has been carried out on a submerged slender mast subjected to explosive shock.

Measurement of the shock response of such a dynamic system is very difficult because of the broad band of energy supplied by the charge. Several types of transducer have been used and comparisons have been made between their ability to measure shock movement. The output signals from strain gauges mounted on the mast are exceptionally clear and strain gauge installations have been shown to be good for obtaining reliable modal frequency information. However, it is very difficult to interpret such data in terms of displacements, especially in the case where only one strain gauge bridge is used. The shock trials have also shown that non-contacting proximity displacement gauges, which work on an inductance principle, can also provide clear information about the response. The use of accelerometers for shock measurement was far more troublesome because of the natural frequencies associated with these transducers. Both damped and undamped accelerometers have been tested and it has been shown that the damped transducers provide the most accurate information even though a slight phase distortion is present because of the effect of damping on the phase characteristics. The requirement to measure the resonance signal from the undamped accelerometer before applying digital filtering techniques somewhat limits the accuracy of the measured signal due to the limited resolution of the recorder. The performance of two types of damped transducer system has been investigated - an internally damped transducer and an undamped transducer connected to a mechanical damping adaptor. The output signals during a severe shock loading have been compared, showing close correlation. It is recommended that either type of damped unit could be successfully used although it is felt that the manufacturer's damped version is probably the most reliable. The information provided by the filtered signal from the undamped accelerometer and the signal from the damped accelerometer did confirm the frequencies previously measured by both the strain gauges and the displacement transducer.
The results from the shock trial on the submerged mast showed, firstly, that the pressure pulses measured by the pressure transducer as they travelled over the submerged body were not the same as those calculated for a pentolite charge. This would indicate that the type of explosive used was less powerful. Unfortunately, there does not appear to be any published information to enable the characteristics of such a charge to be determined from similarity expressions. The effect of the pressure pulses has been seen to excite mast modes up to and including the seventh mode. It has been shown that mode 4 was most dominant in terms of deflection although mode 7, (the fixed bearing mode), was also seen to be important. This suggests that it is necessary to include both types of forcing function into the analytical model. It is also important to note that the maximum level of strain measured in the shock trial was 80 μstrain as opposed to levels of 400 μstrain being achieved in the laboratory based test rig. The lower strain values are confirmed by the displacement transducer which indicates movements of only microns. Although the hydrodynamic information obtained from the laboratory based test rig still remains applicable at these low levels of response, it was hoped that the level of severity created by the shock load would be much higher. Further tests should now be conducted for higher shock factors with the structure being subjected to far more extreme loading conditions.
7.2 Suggestions for Further Work

There are several areas where the author feels that further effort should be expended in order to gain a greater understanding of the shock response of submerged masts:

(1) The controlled response of higher mast modes up to and including the 'fixed bearing modes', which are seen to be excited by shock loads, should be investigated in a laboratory environment. The limits of free response testing have been exhausted and these higher modes may only be available by forced response methods. Using such methods would allow mode shape characteristics to be obtained by use of photographic techniques and would allow a study of the possible changes in mode shape in partially submerged conditions.

(2) A parametric study of the relationship between the drag coefficient $C_d$ and either the Reynold's number or the amplitude/diameter ratio for a vibrating mast must be made. The importance of factors such as the diameter or the slenderness ratio must be evaluated before the analysis of the damping of model masts can be applied to prototype structures. At first, a series of steady state tests using a forced response method which is controlled by tip movement of the mast should be carried out on a test rig similar to the laboratory based rig described within this text. Keeping factors such as the $\beta$ value constant, various slenderness ratios can be obtained by using tubes with different wall thicknesses. In a similar way, the damping characteristics of the second and higher modes could also be investigated. Finally, the free vibration test results could be further examined to ascertain whether the steady state results can be applied to a transient analysis where the damping terms are updated at every cycle of response.

(3) In order to completely understand the instantaneous behaviour of the drag coefficient, advances must be made to analytically model the behaviour of vortices in terms of their formation, growth and subsequent motion. At present, only limited solutions to relatively simple flow conditions have been developed.
Further offshore shock trials should be conducted with either larger charges or closer stand-off distances to induce a far more severe shock loading. The instrumentation should be uprated by use of further strain gauges placed in strategic positions where high strain rates are due predominantly to one mode. The positioning of the strain gauges at places with remote access can be achieved by use of 'araldite inserts' as used in the laboratory test rig. Having far more information from strain gauges, would enable a more accurate assessment of the response mode shapes which could then be related to overall displacements. A greater number of pressure transducers should be used to obtain a further understanding of how the pressure reduces as it travels over the submerged body. The instrumentation should also be supplemented with damped accelerometers to measure the mast tip movement and the rigid body motion.

The damping characteristics of the supporting arrangement have been shown to be invariant with angular rotation and can be accurately modelled by use of viscous dampers placed at the support nodes of the structural representation. However, it is only possible to determine the coefficients governing the level of damping from empirical measurements. Therefore, it would be rewarding to undertake a parametric study of this type of diaphragm support in the absence of material damping of the mast so that the damping coefficients can be pre-determined by virtue of their geometric properties.

To improve the accuracy of the analysis, it may be considered necessary to extend the analysis to a two dimensional model. It is realised that not all mast forms can be adequately modelled in one dimension. The shock loads on the mast do not necessarily strike the structure in a transverse direction and it is possible that the shock may cause rotation of the base supports. In such cases, there may be significant deflections in the vertical direction because of asymmetry of the mast. The analysis of such a 2-dimensional model would be more complex and would require a much greater computation time. However, with the continuing improvements in computer technology and the lower cost of using much larger machines, it is becoming increasingly possible to carry out this type of analysis on 'desktop' computers.
Finally, the growing use of composite materials such as fibre reinforced composites for mast-like structures, makes it worthwhile developing the analysis to include such non-linear materials. The major advantage of using a time stepping analysis as developed in this study is the ease with which non-linear effects can be incorporated into the computational algorithm.
7.3 Conclusions

(1) An analytical method which uses a lumped parameter model and solves the equations of motion by means of finite difference approximations has been shown to be highly successful at predicting the shock response of a submerged slender mast.

(2) The effect of support stiffness and damping can be included into the model by assuming that they provide linear rotational stiffness and viscous damping.

(3) The effect of hydrodynamic interaction can be fully accounted for by including an added mass distribution and an equivalent damping constant into the equation of motion of the structure.

(4) It is necessary to include both the direct pressure and the shock movement transmitted through the structural support arrangement to correctly predict the mast's shock response to explosive loading.

(5) The added mass distribution can be obtained through the determination of the hydrodynamic pressure acting on a vibrating structure. The effects of compressibility and surface waves are negligible for slender structures at the normal response modes.

(6) The added mass distribution is mode dependent and for slender structures is most successfully included into the finite difference analysis as a constant term over the submerged length, having a magnitude equal to the mass of displaced fluid. The errors incurred using such a technique are small for low response modes although the errors do gradually increase with mode number.

(7) The 'constant' added mass distribution approximation produces greater errors as the structure becomes less slender.

(8) The change in natural frequency due to submergence as modelled by the inclusion of the added mass term is dependent upon both the slenderness ratio and the wall thickness/diameter ratio of the structure. It is
also dependent upon the way in which the structure is supported.

(9) An experimental investigation of the change in natural frequency with depth of submergence has successfully measured the first four modes of response of a slender mast using free vibration tests in a purpose built laboratory test rig. The results have confirmed the values predicted by the hydrodynamic analysis although some discrepancies are shown to lie in the region of submergence around the nodal position.

(10) The hydrodynamic damping of a submerged mast has been shown by experiment to be linearly dependent upon the amplitude of displacement of the tip of the structure for the first mode of response. The damping coefficient is controlled by the drag coefficient which in turn is a function of both the amplitude of deflection and a Reynolds number term. The relationship for the drag coefficient of a vibratory structure is not accurately modelled by either the steady flow coefficients or the oscillatory coefficients obtained by previous researchers.

(11) An offshore shock trial has been carried out on a simply supported fully submerged slender mast and the response has been successfully recorded in terms of accelerations, displacements and strain gauge data. The input shock was obtained from a detonated explosive charge for which a peak pressure of $65 \, \text{N/m}^2$ was measured.

(12) Examination of the output signals has shown that the broad energy spectrum of the input pressure signal excites the natural frequencies of the transducer as well as a broad range of mast frequencies. Digital signal processing techniques have been shown to be very effective in removing these unwanted signals and identifying the natural frequencies of the mast.

(13) The value of transducers which do not suffer from resonance problems is seen from the ability of the strain gauge bridge and the non-contacting displacement transducer to clearly display the genuine structural response information. It is therefore recommended that greater use of such transducers should be made in future trials.
The performance of undamped accelerometers under shock conditions has been compared with that of similar transducers which have been damped either internally or by a mechanical damping unit. It is concluded that damped accelerometers provide the more accurate responses although they may suffer phase distortions.

The experimental trial has demonstrated that the effect of explosive shock loads is to excite a wide range of frequencies. The most important frequency of response is the fourth mode and modes up to and including the seventh mode are also present. The seventh mode is particularly important because it represents the 'fixed bearing mode' which is related to the rigid body movement of the supporting structure. This confirms the need to use both the direct pressure loading and the support movement for a realistic model of the shock response of submerged masts to explosive loads.

The conclusions drawn from this work represent a useful contribution to the assessment and understanding of the shock loading response of submerged and semi-submerged masts. The analytical techniques that have been developed allow a more realistic computer simulation of the associated structural dynamics to be undertaken. The experimental programme has yielded much useful information regarding water borne shock conditions and has indicated the way forward for monitoring the structural response to such events. In conjunction with further work in the areas outlined in section 7.2, the information contained within this thesis will lead to the more rational and efficient design of this structural form for the sub-sea shock loading environment.
Plate 1  Mast and Support Assembly
Determination of the Flexibility Matrix

The displacement of a beam subjected to a series of transverse loads can be analysed as follows:

\[
M_1 + M_2 + R_1(x_{n+2} - x_{n+1}) = -F_1(x_{n+2} - x_1) - F_2(x_{n+2} - x_2) - \ldots \nonumber
\]
\[
\ldots - F_k(x_{n+2} - x_k) - \ldots - F_n(x_{n+2} - x_n) \nonumber
\]

\[
\Rightarrow M_1 + M_2 + R_1(x_{n+2} - x_{n+1}) = - \sum_{k=1}^{n} F_k(x_{n+2} - x_k) \nonumber
\]

Hence,

\[
R_1 = - \sum_{k=1}^{n} \frac{F_k(x_{n+2} - x_k)}{(x_{n+2} - x_{n+1})} (A.1) \nonumber
\]

Let \( M_{q,q+1} \) be the moment acting between nodes \( q \) and \( q+1 \), then

\[
M_{q,q+1} = \sum_{k=1}^{q} F_k(x - x_k) \quad \text{for} \ 1 \leq q \leq n \nonumber
\]
The total strain energy in the beam, neglecting energy due to transverse shear, is given by,

\[
U = \int_{x_1}^{x_2} \frac{M_1^2}{2E_1I_1} \, dx + \int_{x_2}^{x_3} \frac{M_2^2}{2E_2I_2} \, dx + \ldots + \int_{x_n}^{x_{n+1}} \frac{M_{n+1,n+1}^2}{2E_{n+1,n+1}I_{n+1}} \, dx + \ldots
\]

Hence,

\[
U = \sum_{q=1}^{n} \int_{x_q}^{x_{q+1}} \frac{M_q^2}{2E_qI_q} \, dx + \int_{x_n}^{x_{n+2}} \frac{M_{n+1,n+2}^2}{2E_{n+1,n+1}I_{n+1}} \, dx
\]

The displacement \( \delta_i \) of the point of application of the \( i \)th force, \( F_i \), can be obtained by Castigliano's theorem,

\[
\delta_i = \frac{\partial U}{\partial F_i} = \sum_{q=1}^{n} \frac{1}{E_qI_q} \int_{x_q}^{x_{q+1}} M_{q,q+1} \frac{\partial M_{q,q+1}}{\partial F_i} \, dx
\]

\[
+ \frac{1}{E_{n+1,n+1}I_{n+1}} \int_{x_{n+1}}^{x_{n+2}} M_{n+1,n+2} \frac{\partial M_{n+1,n+2}}{\partial F_i} \, dx
\]

(A.2)

where

\[
M_{n+1,n+2} = \sum_{k=1}^{n} F_k (x - x_k) + R_1 (x - x_{n+1}) + M_1
\]

(A.3)

Considering the partial differentials in equation (A.2)

\[
\frac{\partial M_{q,q+1}}{\partial F_i} = \frac{\partial}{\partial F_i} \sum_{k=1}^{n} F_k (x - x_k)
\]
and \( M_{1,2} = F_1 (x - x_1) \)
\( M_{2,3} = F_1 (x - x_1) + F_2 (x - x_2) \)
\( M_{3,4} = F_1 (x - x_1) + F_2 (x - x_2) + F_3 (x - x_3) \)

e.tc.

Hence, \( \frac{\partial M_{1,2}}{\partial F_1} = (x - x_1) \); \( \frac{\partial M_{2,3}}{\partial F_1} = (x - x_1) \); \( \frac{\partial M_{3,4}}{\partial F_1} = (x - x_1) \)

\( \frac{\partial M_{1,2}}{\partial F_2} = 0 \); \( \frac{\partial M_{2,3}}{\partial F_2} = (x - x_2) \); \( \frac{\partial M_{3,4}}{\partial F_2} = (x - x_2) \)

\( \frac{\partial M_{1,2}}{\partial F_3} = 0 \); \( \frac{\partial M_{2,3}}{\partial F_3} = 0 \); \( \frac{\partial M_{3,4}}{\partial F_3} = (x - x_3) \)

Hence,
\[
\frac{\partial M}{\partial F_i} = \begin{cases} 
(x - x_i) & \text{for } q < i \\
(x - x_i) & \text{for } q > i
\end{cases}
\]  \( \) (A.4)

Also,
\[
\frac{\partial M_{n+1,n+2}}{\partial F_i} = \frac{\partial}{\partial F_i} \sum_{k=1}^{n} F_k (x - x_k) + R_1 (x - x_{n+1}) + M_i
\]

\[
\frac{\partial M_{n+1,n+2}}{\partial F_i} = (x - x_i) + \frac{\partial R_1}{\partial F_i} (x - x_{n+1})
\]  \( \) (A.5)

From equation (A.1)
\[
\frac{\partial R_1}{\partial F_i} = \frac{(x_{n+2} - x_i)}{(x_{n+2} - x_{n+1})}
\]  \( \) (A.6)
Substitution of equations (A.4), (A.5) and (A.6) into the governing equation (A.2) gives,

\[ \delta_i = \sum_{q=1}^{n} \frac{1}{E_{q_{q}}} \int_{x_i}^{x_{q+1}} \sum_{k=1}^{q} F_k(x-x_k)(x-x_i) \, dx \]

\[ + \frac{1}{E_{n+1} n+1} \int_{x_{n+1}}^{x_{n+2}} \left\{ \sum_{q=1}^{n} F_k(x-x_k) + R_1(x-x_{n+1}) + M_1 \right\} \left\{ \frac{(x-x_i)}{(x_{n+2} - x_{n+1})} \right\} \, dx \]

Integration and simplification gives

\[ \delta_i = \sum_{q=1}^{n} \frac{1}{6E_{q_{q}}} \sum_{k=1}^{q} F_k \left( \frac{2}{x_{q+1} - x_q} - 3 \frac{1}{x_i - x_k} \frac{x_{q+1}^2 - x_q^2}{x_{q+1}^2 - x_q^2} + 6x_i x_k \frac{x_{q+1} - x_q}{x_{q+1} - x_q} \right) \]

\[ + \frac{1}{6E_{n+1} n+1} \left\{ \frac{2}{x_{n+2} - x_{n+1}} \left( \sum_{k=1}^{n} F_k + R_1 \right) \right\} \]

\[ - 3 \frac{2}{x_{n+2} - x_{n+1}} \left[ \sum_{k=1}^{n} F_k x_k + \frac{x_{n+2} - x_{n+1}}{x_{n+2} - x_{n+1}} \sum_{k=1}^{n} F_k x_k + R_1(x_{n+1} - x_1) \right] \]

\[ + \frac{2}{x_{n+1} \left( \frac{x_{n+2} - x_i}{x_{n+2} - x_{n+1}} \right)} - \left( \frac{x_i - x_{n+1}}{x_{n+2} - x_{n+1}} \right) M_1 \right\} \]

\[ + 6(x_i - x_{n+1}) x_{n+2} \sum_{k=1}^{n} F_k x_k + R_1 x_{n+1} - M_1 \right\} \quad (A.7) \]

where \( R_1 \) is given by equation (A.1)
The flexibility coefficient \( f_{i,j} \) can be found from equation (A.7) by setting \( F_j \) to unity and all the other forces to zero. For the present time the moments are given unity values of \( M_1^* \) and \( M_2^* \) and have the dimensions of length.

The flexibility coefficient is given as

\[
f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{i,j} I_{q,q}} \left\{ 2\left(\frac{x_{i+1}}{x_{q+1} - x_q} - 3x_i x_j (x_{q+1} - x_q) + 6x_i x_j (x_{q+1} - x_q)\right) \right\}
\]

\[
+ \frac{1}{6E_{n+1} I_{n+1}} \left[ 2\left(\frac{x_{i,n+1}}{x_{n+2} - x_{n+1}}\right) \left(1 - \frac{x_n - x_{j,n+1}}{x_{n+2} - x_{n+1}}\right) \right] (x_{n+2} - x_{n+1}) \]

\[
- 3 \left\{ \frac{(x_i - x_{n+1})x_j + x_{n+2}\left(x_i - x_{n+1}\right) - (x_{n+2} + x_{n+1})\left(x_i - x_{n+1}\right)}{\left(x_{n+2} - x_{n+1}\right)^2} \right\} \left(\frac{x_i - x_{n+1}}{x_{n+2} - x_{n+1}}\right)^m \]

\[
- \left(\frac{x_i - x_{n+1}}{x_{n+2} - x_{n+1}}\right)^m \left(x_{n+2} - x_{n+1}\right) \}
\]

\[
- 6x_{n+2} (x_i - x_{n+1}) \left\{ x_j - x_{n+1} \left(\frac{x_{n+2} - x_{j,n+1}}{x_{n+2} - x_{n+1}}\right) - M_1^* \right\} \right\}
\]  (A.8)
This expression can be simplified to give

\[
f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{q}I_{q}} \left\{ 2(x_{q+1}^{3} - x_{q}^{3}) - 3(x_{i}^{2} + x_{j}^{2})(x_{q+1}^{2} - x_{q}^{2}) + 6x_{i}x_{j}(x_{q+1}^{2} - x_{q}^{2}) \right\}
\]

\[
+ \frac{1}{3E_{n+1}I_{n+1}} (x_{i} - x_{n+1})(x_{j} - x_{n+1})(x_{n+2} - x_{n+1})
\]

\[
- \frac{1}{6E_{n+1}I_{n+1}} (x_{i} - x_{n+1})(x_{n+2} - x_{n+1})(2M_{1}^{*} - M_{2}^{*})
\]  \(\text{(A.9)}\)

To understand how the flexibility coefficients change with support stiffness, a further strain energy analysis needs to be performed to obtain the values of \(M_{1}^{*}\) and \(M_{2}^{*}\).

The rotation \(\theta_{1}\) of the beam at node \((n+1)\) is given by application of Castigliano's theorem for angular displacement.

\[
\theta_{1} = \frac{\partial U}{\partial M_{1}} = \sum_{q=1}^{n} \frac{1}{E_{q}I_{q}} \int_{x_{q}}^{x_{q+1}} M_{q,q+1} \frac{\partial M_{q,q+1}}{\partial M_{1}} \, dx
\]

\[
+ \frac{1}{E_{n+1}I_{n+1}} \int_{x_{n+1}}^{x_{n+2}} M_{n+1,n+2} \frac{\partial M_{n+1,n+2}}{\partial M_{1}} \, dx \quad \text{(A.10)}
\]

But

\[
\frac{\partial M_{q,q+1}}{\partial M_{1}} = 0
\]

\[
\frac{\partial M_{n+1,n+2}}{\partial M_{1}} = \frac{\partial R_{1}}{\partial M_{1}} (x-x_{n+1}) + 1
\]

and

\[
\frac{\partial R_{1}}{\partial M_{1}} = \frac{-1}{(x_{n+2} - x_{n+1})}
\]
Hence \[
\frac{\partial M_{n+1,n+2}}{\partial M_1} = \left( \frac{x_{n+2} - x}{x_{n+2} - x_{n+1}} \right)
\]

Using these results, equation (A.10) becomes

\[
\theta_1 = \frac{1}{E_{n+1} I_{n+1}} \int_{x_{n+1}}^{x_{n+2}} \left( \sum_{k=1}^{n} F_k (x - x_k) + R_1 (x - x_{n+1}) + M_1 \right) \frac{(x - x_{n+2})}{(x_{n+2} - x_{n+1})} \, dx
\]

Integration gives

\[
\theta_1 = \frac{(x_{n+2} - x_{n+1})}{3 E_{n+1} I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - M_2 / 2 \right] \quad (A.11)
\]

Similarly,

\[
\theta_2 = \frac{\partial U}{\partial M_2} = \sum_{k=1}^{n} \frac{1}{E_{q} I_{q}} \int_{x_q}^{x_{q+1}} M_{q,q+1} \frac{\partial M_{q,q+1}}{\partial M_2} \, dx
\]

\[
+ \frac{1}{E_{n+1} I_{n+1}} \int_{x_{n+1}}^{x_{n+2}} M_{n+1,n+2} \frac{\partial M_{n+1,n+2}}{\partial M_2} \, dx \quad (A.12)
\]

But \[
\frac{\partial M_{q,q+1}}{\partial M_2} = 0
\]

\[
\frac{\partial M_{n+1,n+2}}{\partial M_2} = \frac{\partial R_1}{\partial M_2} (x - x_{n+1})
\]

and \[
\frac{\partial R_1}{\partial M_2} = \frac{-1}{(x_{n+2} - x_{n+1})}
\]
Using these results, equation (A.12) becomes

$$\frac{\theta}{2} = \frac{1}{E_{n+1}I_{n+1}} \int_{x_{n+1}}^{x_{n+2}} \sum_{k=1}^{n} F_k (x - x_k) + R_1 (x - x_{n+1}) + M_1 \frac{x_{n+1} - x}{x_{n+2} - x_{n+1}} \, dx$$

which after substituting for $R_1$ and integrating gives

$$\frac{\theta}{2} = \frac{-(x_{n+2} - x_{n+1})}{6 E_{n+1} I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - 2M_2 \right] \quad (A.13)$$

If it is assumed that the stiffness of the support is constant and equal to the moment applied divided by the angle of deformation, then equations (A.11) and (A.13) become,

$$\frac{M_1}{K} = \frac{(x_{n+2} - x_{n+1})}{3 E_{n+1} I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - \frac{M_2}{2} \right] \quad (A.14)$$

$$\frac{M_2}{K} = \frac{-(x_{n+2} - x_{n+1})}{6 E_{n+1} I_{n+1}} \left[ \sum_{k=1}^{n} F_k (x_{n+1} - x_k) + M_1 - 2M_2 \right]$$

where $K$ is the stiffness of the support and is assumed to be the same for both supports.

Solving (A.14) gives

$$M_1 = C_1 \sum_{k=1}^{n} F_k (x_{n+1} - x_k) \quad (A.15)$$

$$M_2 = C_2 \sum_{k=1}^{n} F_k (x_{n+1} - x_k)$$
where

\[
C_1 = - \left[ 1 - \frac{4E_{n+1}I_{n+1}}{(x_{n+2} - x_{n+1})K} \right] \left[ 1 - \frac{6E_{n+1}I_{n+1}}{(x_{n+2} - x_{n+1})K} \right] \tag{A.16}
\]

\[
C_2 = -\frac{2E_{n+1}I_{n+1}}{(x_{n+2} - x_{n+1})K} \left[ 1 - \frac{6E_{n+1}I_{n+1}}{(x_{n+2} - x_{n+1})K} \right] \tag{A.17}
\]

To include the effect of support stiffness in the flexibility coefficients, then the moments \(M_1\) and \(M_2\) must be evaluated when \(F_k\) is unity and all other forces are zero. Hence,

\[
M_1^* = C_1 (x_{n+1} - x_j)
\]

and

\[
M_2^* = C_2 (x_{n+1} - x_j)
\]

Equation (A.9) becomes

\[
f_{i,j} = \sum_{q=1}^{n} \frac{1}{6E_{q}I_{q}} \left[ 2\left( x_{q+1}^3 - x_q^3 \right) - 3(x_{i} + x_{j})(x_{q+1}^2 - x_q^2) + 6x_{i}x_{j}(x_{q+1} - x_q) \right] + \frac{1}{3E_{n+1}I_{n+1}} (x_{i} - x_{n+1})(x_{j} - x_{n+1})(x_{n+2} - x_{n+1})(1 - C_1 + C_2) \tag{A.18}
\]
Solution to the Wave Equation

Assuming water to be linearly compressible and neglecting internal viscosity, the small amplitude irrotational motion is governed by the wave equation, which in cylindrical coordinates is given as,

\[
\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} + \frac{1}{r^2} \frac{\partial^2 P}{\partial \theta^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{C^2} \frac{\partial^2 P}{\partial t^2}
\]  

(B.1)

and the system yields several boundary conditions.

\[
\frac{\partial P}{\partial z} (r, \theta, 0, t) = 0 \quad (B.2)
\]

\[
\frac{\partial P}{\partial \theta} (r, 0, z, t) = 0 \quad (B.3)
\]

\[
\frac{\partial P}{\partial \theta} (r, \pi, z, t) = 0 \quad (B.4)
\]

\[
\frac{\partial^2 P}{\partial t^2} (r, \theta, H, t) = -g \frac{\partial}{\partial z} P (r, \theta, H, t) \quad (B.5)
\]

\[
\frac{\partial P}{\partial r} (r_0, \theta, z, t) = -\rho \phi_N(z) \cos(\theta) e^{i\omega t} \quad (B.6)
\]

the solution using boundary conditions (B.3) and (B.4) can be broken down to

\[
P(r, \theta, z, t) = P(r, z) \cos(\theta) e^{i\omega t}, \text{ where}
\]

\[
\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial z^2} + \left( \frac{\omega^2}{C^2} - \frac{1}{r^2} \right) P = 0
\]

or

\[
P(r, z) = R(r) Z(z), \text{ where}
\]

\[
Z \frac{d^2 R}{dr^2} + \frac{r}{Z} \frac{dR}{dr} + R \frac{d^2 Z}{dz^2} + \left( \frac{\omega^2}{C^2} - \frac{1}{r^2} \right) R Z = 0
\]

i.e.

\[
\frac{1}{R} \left\{ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right\} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + \left( \frac{\omega^2}{C^2} - \frac{1}{r^2} \right) = 0
\]
Putting, \[ \frac{1}{Z} \frac{d^2Z}{dz^2} = \alpha^2 \] (B.7)

then \[ \frac{1}{R} \left( \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \left( \frac{\omega^2}{C^2} - \frac{1}{r^2} \right) + \alpha^2 = 0 \] (B.8)

which is a form of 'Bessel's equation'.

Solutions are now given by the two differential equations:

\[ \frac{d^2Z}{dz^2} \pm \alpha^2 = 0 , \quad \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left\{ \left( \frac{\omega^2}{C^2} + \alpha^2 \right) - \frac{1}{r^2} \right\} R = 0 \] (B.9)

Solutions to the first equation may have two forms depending on the ± sign.

for \( \alpha^2 > 0 \)
\[ Z = \begin{cases} \cosh(\alpha z) & \lambda = \sqrt{\frac{\omega^2}{C^2} + \alpha^2} \\ \sinh(\alpha z) & \end{cases} \]

for \( \alpha^2 < 0 \)
\[ Z = \begin{cases} \cos(\beta z) & \lambda' = \sqrt{\frac{\omega^2}{C^2} - \beta^2} \\ \sin(\beta z) & \end{cases} \]

(B.10)

The solution to the second equation is a combination of Bessel functions

\[ R = f \left\{ J_1(\lambda r), Y_1(\lambda r) \right\} \]

From boundary condition (B.2) for \( r \geq r_0 \), \( \frac{dP}{dz} = 0 \) on \( z=0 \). This implies that only \( \cos(\beta z) \) and \( \cosh(\beta z) \) appear in the solutions.

Using boundary condition (B.5) then

for \( \alpha^2 > 0 \)
\[ -\omega^2 P = -g \frac{dP}{dz} \quad \text{on} \quad z = H \]

\[ \Rightarrow -\omega^2 \cosh(\alpha H) = -\alpha g \sinh(\alpha H) \]

i.e. \( \alpha \tanh(\alpha H) = \omega^2/g \) (B.11)

for which there is only one positive root \( \alpha_0 \) which gives the solution of the form

\[ \left\{ J_1(\lambda_0 r_0), Y_1(\lambda_0 r_0) \right\} \cosh(\alpha_0 z) \]
for \( \alpha^2 < 0 \)

\[-\omega^2 \cos(\beta H) = g \beta \sin(\beta H)\]

\[\beta \tan(\beta H) = -\frac{\omega^2}{g}\]  \hspace{1cm} (B.12)

which has an infinite number of positive roots \( \alpha_1, \alpha_2, \ldots \).

Suppose \( \alpha_1, \alpha_2, \ldots, \alpha_{m_1-1} \) have \( \frac{\omega^2}{C^2} - \alpha_i^2 > 0 \)

then \( \lambda_i = \sqrt{\frac{\omega^2}{C^2} - \alpha_i^2} \quad \alpha = \alpha_i, \ i = 1, 2, \ldots, m_1-1 \)

for \( m \geq m_1 \)

\[\frac{\omega^2}{C^2} - \alpha_i^2 < 0 \]

\[\lambda_i' = \sqrt{\alpha_i^2 - \frac{\omega^2}{C^2}}\]

hence,

For \( i = 1, \ldots, m_1-1 \), solutions are of the form \( J_1(\lambda_i, r), Y_1(\lambda_i, r) \cos(\alpha_i z) \)

\( i = m_1, \ldots, \infty \) solutions are of the form \( K_1(\lambda_i', r) \cos(\alpha_i z) \)

Hence the general solution for \( P(r, z) \) (eigenfunction expansion)

\[P(r, z) = A_0 \, H_1^{(2)}(\lambda_0 r) \cosh(\alpha_0 z) + \sum_{m=1}^{m_1-1} A_m \, H_1^{(2)}(\lambda_m r) \cos(\alpha_m z)\]

\[+ \sum_{m=m_1}^{\infty} C_m K_1(\lambda_m' r) \cos(\alpha_m z) \quad r \geq r_0\]  \hspace{1cm} (B.13)

Finally, using the boundary condition (B.6), the constants \( A_0, A_m \) and \( C_m \) can be evaluated.

\[\frac{\partial P}{\partial r} = -\rho \phi(z) \quad \text{on} \quad r = r_0\]

i.e.

\[\frac{\partial}{\partial r} \left\{ A_0 \, H_1^{(2)}(\lambda_0 r) \cosh(\alpha_0 z) + \sum_{m=1}^{m_1-1} A_m \, H_1^{(2)}(\lambda_m r) \cos(\alpha_m r)\right\} = -\rho \phi(z) \quad (B.14)\]

\[0 \leq z \leq H\]
Hence,

\[ A_0 \lambda_0 H^{(2)}_1(\lambda_0 r_0) \cosh(\alpha_0 z) + \sum_{m=1}^{m_1-1} \lambda_m A_m H^{(2)}_1(\lambda_m r) \cos(\alpha_m z) \]

\[ + \sum_{m=m_1}^{\infty} C_m \lambda'_m K'_1(\lambda'_m r) \cos(\alpha_m z) = -\rho \phi_N(z) \]  

(B.15)

\[ 0 \leq z \leq H \]

Using orthogonality i.e. multiplying through by \( \cosh(\alpha_0 z) \) and integrating between 0 and H.

Hence,

\[ A_0 \lambda_0 H^{(2)}_1(\lambda_0 r_0) \int_0^H \cosh^2(\alpha_0 z) \, dz = -\rho \int_0^H \phi_N(z) \cosh(\alpha_0 z) \, dz \]

simplifies to

\[ A_0 \lambda_0 H^{(2)}_1(\lambda_0 r_0) \left( \frac{2\alpha_0 H + \sinh(2\alpha_0 H)}{4\alpha_0} \right) = -\rho I_0 \]  

(B.16)

where \( I_0 = \int_0^H \phi_N(z) \cosh(\alpha_0 z) \, dz \)

Similarly, multiplying through by \( \cos(\alpha_m z) \), then for each value of \( m \) the equation produces values only when \( \alpha_m = \alpha_0 \) i.e. \( n=m \)

\[ A_m \lambda_m H^{(2)}_1(\lambda_m r_0) \int_0^H \cos(\alpha_m z) \cos(\alpha_0 z) \, dz = -\rho \int_0^H \phi_N(z) \cos(\alpha_0 z) \, dz \]

which gives,

\[ A_m \lambda_m H^{(2)}_1(\lambda_m r_0) \left( \frac{2\alpha_m H + \sin(2\alpha_0 H)}{4\alpha_m} \right) = -\rho I_m \]  

(B.17)

\[ I_m = \int_0^H \phi_N(z) \cos(\alpha_m z) \, dz \]

and

\[ C_m \lambda'_m K'_1(\lambda'_m r) \int_0^H \cos^2(\alpha_m z) \, dz = -\rho \int_0^H \phi_N(z) \cos(\alpha_m z) \, dz \]
Giving,
\[
C_m \lambda_m' K_1(\lambda_m' r_0) \left( \frac{2a_m H + \sin(2a_m H)}{4a_m} \right) = - \rho I_m
\]  (B.18)

\[
I_m = \int_0^H \phi_N(z) \cos(\alpha z) \, dz
\]

Hence the solution is now,
\[
P(z, r_0) = 4 \rho \left\{ -\frac{\alpha_0}{\lambda_0} \frac{I_0}{2a_0 H + \sin(2a_0 H)} \frac{H^{(2)}_1(\lambda_0 r_0)}{H^{(2)'}_1(\lambda_0 r_0)} \cosh(\alpha z) \\
- \sum_{m=1}^{m-1} \frac{\alpha_m}{\lambda_m} \frac{I_m}{2a_m H + \sin(2a_m H)} \frac{H^{(2)}_1(\lambda_m r_0)}{H^{(2)'}_1(\lambda_m r_0)} \cos(\alpha z) \\
+ \sum_{m=m+1}^{\infty} \frac{\alpha_m}{\lambda_m} \frac{I_m}{2a_m H + \sin(2a_m H)} \frac{K_1(\lambda_m r_0)}{K_1(\lambda_m r_0)} \cos(\alpha r_0) \right\}
\]  (B.19)

Now \( \frac{H^{(2)}_1}{H^{(2)'}_1} \) can be expressed in complex form i.e. as \( R e^{i\theta} \)

The magnitude of \( R \) is given by,
\[
\left| \frac{H^{(2)}_1(\lambda_0 r_0)}{H^{(2)'}_1(\lambda_0 r_0)} \right|^2 = \frac{(J_1 - i Y_1)}{(J_1' - i Y_1')} \frac{(J_1 + i Y_1)}{(J_1' + i Y_1')}
\]
\[
= \frac{J_1^2 + Y_1^2}{[J_1']^2 + [Y_1']^2}
\]

but \( J_1'(x) = \frac{1}{2} (J_0(x) - J_2(x)) \)

Hence,
\[
\left| \frac{H^{(2)}_1(\lambda_0 r_0)}{H^{(2)'}_1(\lambda_0 r_0)} \right|^2 = 2 \left[ D_0(\lambda_0 r_0) \right]^2 = \frac{2(J_1[\lambda_0 r_0]^2 + Y_1[\lambda_0 r_0]^2)}{(J_0[\lambda_0 r_0] - J_2[\lambda_0 r_0])^2 + (Y_0[\lambda_0 r_0] - Y_2[\lambda_0 r_0])^2}
\]
and similarly,

\[
\left| \frac{H_1^{(2)}(\lambda_m r_0)}{H_1^{(2)'}(\lambda_m r_0)} \right|^2 = \frac{2( J_1[\lambda_m r_0]^2 + Y_1[\lambda_m r_0]^2 )}{(J_0[\lambda_m r_0]-J_2[\lambda_m r_0])^2 + (Y_0[\lambda_m r_0]-Y_2[\lambda_m r_0])^2}
\]

Now,

\[
\frac{H_1^{(2)}}{H_1^{(2)'}} = \frac{(J_1 - iY_1)(J_1' + iY_1')}{(J_1' - iY_1')(J_1 + iY_1')} = \frac{(J_1 J_1' + Y_1 Y_1') - i(J_1 Y_1' - J_1' Y_1)}{J_1^2 + Y_1'^2}
\]

hence, \( \text{Arg} \left( \frac{H_1^{(2)}}{H_1^{(2)'}} \right) = \tan^{-1} \left[ \frac{Y_1 J_1 - J_1' Y_1'}{J_1 J_1' + Y_1 Y_1'} \right] = \tan^{-1} \left[ \frac{(Y_0 - Y_2) J_1 - (J_0 - J_2) Y_1}{(J_0 - J_2) J_1 + (Y_0 - Y_2) Y_1} \right]
\]

Finally \( \frac{K_1'(\lambda'_m r_0)}{K_1(\lambda_m r_0)} = \frac{2 K_0'(\lambda'_m r_0)}{K_0(\lambda_m r_0) + K_2(\lambda_m r_0)} = 2 \text{E}_m(\lambda'_m r_0)
\]

Hence the equation becomes

\[
P(\theta, z, \omega, r_0) = 8 \rho \left\{ \frac{-\alpha_0}{\lambda_0} \frac{I_{k0} \cosh(\alpha_0 z)}{2\alpha_0 H + \sinh(2\alpha_0 H)} D_0(\lambda_0 r_0) \cos(\alpha_0 z) e^{i\epsilon_0 r_0} \right. \\
- \sum_{m=1}^{m_1-1} \frac{\alpha_m}{\lambda_m} \frac{I_{km} \cosh(\alpha_m z)}{2\alpha_m H + \sin(2\alpha_m H)} D_m(\lambda_m r_0) \cos(\alpha_m z) e^{i\epsilon_m r_0} \\
+ \sum_{m_1}^{\infty} \frac{\alpha_m}{\lambda'_m} \frac{I_{km} \cosh(\alpha'_m z)}{2\alpha'_m H + \sin(2\alpha'_m H)} E_m(\lambda'_m r_0) \cos(\alpha'_m z) \left. \right\} \cos(\theta) (B.20)
\]

where \( \epsilon_m r_0 = \theta_m = \text{Arg} \left( \frac{H_1^{(2)}(\lambda_m r_0)}{H_1^{(2)'}(\lambda_m r_0)} \right) \)
Determination of Exact Mode Shapes

The deflection \( u(x) \) of the elastic line of a beam of bending stiffness \( EI(x) \) under a distributed load \( p(x) \) is

\[
\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 u(x)}{\partial x^2} \right] = p(x) \quad (C.1)
\]

Including the inertia of the beam, then

\[
\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right] = p(x,t) - \rho_s A(x) \frac{\partial^2 u(x,t)}{\partial t^2} \quad (C.2)
\]

where \( \rho_s \) is the density of the structure and \( A(x) \) is the cross-sectional area.

For free vibration:

\[
\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right] = - \rho_s A(x) \frac{\partial^2 u(x,t)}{\partial t^2} \quad (C.3)
\]

and assuming \( u(x,t) = X(x) f(t) \), then separating the variables gives

\[
\frac{1}{\rho_s A(x) X(x)} \frac{d^2}{dx^2} \left[ EI(x) \frac{d^2 X(x)}{dx^2} \right] = \frac{1}{f(t)} \frac{d^2 f(t)}{dt^2} = \omega^2 \quad (C.4)
\]

where \( \omega \) is a constant.

Two ordinary differential equations satisfied by \( X(x) \) and \( f(t) \) follow:

\[
\frac{d^2}{dx^2} \left[ EI(x) \frac{d^2 X(x)}{dx^2} \right] - \rho_s \omega^2 A(x) X(x) = 0 \quad (C.5)
\]

\[
\frac{d^2 f(t)}{dt^2} + \omega^2 f(t) = 0
\]
For a beam of constant section, the first equation becomes

\[ \frac{d^4X}{dx^4} - k^4X = 0 \quad \text{where} \quad k^2 = \sqrt{\frac{\omega_n^2 \rho_s A}{E I}} \quad (C.6) \]

The solution to these equations are

\[ f(t) = A \cos(\omega_n t) + B \sin(\omega_n t) \quad (C.7) \]

and

\[ X(x) = C_1 \left[ \cos(kx) + \cosh(kx) \right] + C_2 \left[ \cos(kx) - \cosh(kx) \right] + C_3 \left[ \sin(kx) + \sinh(kx) \right] + C_4 \left[ \sin(kx) - \sinh(kx) \right] \quad (C.8) \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are determined by the boundary conditions of the beam. Three cases will now be considered; firstly, a cantilever beam, secondly a beam supported in a double pin joint configuration and finally two beams supported in pin joints in a back-to-back configuration.

(a) Considering the Beam to be a Simple Cantilever

The boundary conditions are:

\[ \begin{align*}
\text{at } x = 0 & \quad X = 0 \quad (C.9a) \\
\text{at } x = 0 & \quad X' = 0 \quad (C.9b) \\
\text{at } x = L & \quad X'' = 0 \quad (C.9c) \\
\text{at } x = L & \quad X''' = 0 \quad (C.9d)
\end{align*} \]
From b.c. (C.9a) \[ C_1 = 0 \]

From b.c. (C.9b) \[ C_3 = 0 \]

From b.c. (C.9c) \[ 0 = k^2 \left\{ C_2\left[-\cos(kL)-\cosh(kL)\right] - C_4\left[\sin(kL)+\sinh(kL)\right]\right\} \]

From b.c. (C.9d) \[ 0 = k^3 \left\{ C_2\left[\sin(kL)-\sinh(kL)\right] - C_4\left[\cos(kL)+\cosh(kL)\right]\right\} \]

Hence,

\[
\begin{align*}
C_2\left[\cos(kL)+\cosh(kL)\right] + C_4\left[\sin(kL)+\sinh(kL)\right] &= 0 \\
C_2\left[\sin(kL)-\sinh(kL)\right] - C_4\left[\cos(kL)+\cosh(kL)\right] &= 0
\end{align*}
\]  \hspace{1cm} (C.10)

This is a homogeneous solution in the two unknowns \(C_2\) and \(C_4\). For a non trivial solution, the determinant must equal zero. This leads to the frequency equation

\[ \cos(kL) \cosh(kL) = -1 \]  \hspace{1cm} (C.11)

The roots \(k_nL\) of this equation then determine the values of the natural frequencies and the corresponding modes of vibration.

\[ \omega_n = k_n^2 \sqrt{\frac{E \, I}{\rho \, A}} \]  \hspace{1cm} (C.12)

\[ X_n(x) = \frac{1}{2} \left[ \cos(kx) - \cosh(kx) - \frac{(\cos(kL)+\cosh(kL))(\sin(kx)+\sinh(kx))}{(\sin(kL)+\sinh(kL))} \right] \]  \hspace{1cm} (C.13)

The values of \(k_n\) and the resulting mode shapes are shown in Chapter 3, figure 3.5.
(b) Considering the Beam to be Held in a Double Pin Joint Configuration

To analyse such a structure, it must be considered to be two separate beams which have identical boundary conditions at the connecting position.

Considering the beam 12, the characteristic equation is:

\[ x^{(1)}(x_1) = C_1^{(1)}[\cos(kx_1) + \cosh(kx_1)] + C_2^{(1)}[\cos(kx_1) - \cosh(kx_1)] + C_3^{(1)}[\sin(kx_1) + \sinh(kx_1)] + C_4^{(1)}[\sin(kx_1) - \sinh(kx_1)] \]  

(C.14)

The boundary conditions are:

at \( x_1 = 0 \) \( x^{(1)} = 0 \) \( \Rightarrow C_1^{(1)} = 0 \)  

(C.15)

at \( x_1 = 0 \) \( x^{(1)''} = 0 \) \( \Rightarrow C_2^{(1)} = 0 \)  

(C.16)

at \( x_1 = a \) \( x^{(1)} = 0 \) \( \Rightarrow C_3^{(1)} = -\frac{C_4^{(1)}(\sin(ka) - \sinh(ka))}{(\sin(ka) + \sinh(ka))} \)  

(C.17)

Considering the beam 23, the characteristic equation is

\[ x^{(2)}(x_2) = C_1^{(2)}[\cos(kx_2) + \cosh(kx_2)] + C_2^{(2)}[\cos(kx_2) - \cosh(kx_2)] + C_3^{(2)}[\sin(kx_2) + \sinh(kx_2)] + C_4^{(2)}[\sin(kx_2) - \sinh(kx_2)] \]  

(C.18)
The boundary conditions are:

\[ \begin{align*}
& \text{at } x_2 = 0 \quad x^{(2)}'' = 0 \quad \Rightarrow \quad c^{(2)}_2 = 0 \\
& \text{at } x_2 = 0 \quad x^{(2)}'' = 0 \quad \Rightarrow \quad c^{(2)}_4 = 0 \\
& \text{at } x_2 = L \quad x^{(2)} = 0 \\
& \quad \Rightarrow \quad c^{(2)}_3 = -c^{(2)}_1 \frac{[\cos(kL) + \cosh(kL)]}{[\sin(kL) + \sinh(kL)]}
\end{align*} \]

Hence equations (C.14) and (C.18) become:

\[ \begin{align*}
X^{(1)}(x_1) &= C^{(1)}_3 \left[ \sin(kx_1) + \sinh(kx_1) - \frac{(\sin(ka) + \sinh(ka))(\sin(kx_1) - \sinh(kx_1))}{(\sin(ka) - \sinh(ka))} \right] \\
X^{(2)}(x_2) &= C^{(1)}_3 \left[ \sin(kx_2) + \sinh(kx_2) - \frac{(\sin(kL) + \sinh(kL))(\cos(kx_2) + \cosh(kx_2))}{(\cos(kL) + \cosh(kL))} \right]
\end{align*} \]

At position (2), for the beam to be continuous then two further boundary conditions must apply.

\[ \begin{align*}
\frac{dX^{(1)}(a)}{dx_1} &= -\frac{dX^{(2)}(L)}{dx_2} \quad \text{(C.23)} \\
\frac{d^2X^{(1)}(a)}{dx_1^2} &= \frac{d^2X^{(2)}(L)}{dx_2^2} \quad \text{(C.24)}
\end{align*} \]

using b.c. (C.23) gives

\[ \begin{align*}
- C^{(1)}_3 \frac{[\sin(ka) \cosh(ka) - \cos(ka) \sinh(ka)]}{\sin(ka) - \sinh(ka)} &= C^{(2)}_3 \frac{[1 + \cos(kL) \cosh(kL)]}{\cos(kL) + \cosh(kL)}
\end{align*} \]
Using b.c. (C.24) gives

\[ 2 C_3^{(1)} \frac{\sin(ka) \sinh(ka)}{\sin(ka) - \sinh(ka)} = C_3^{(2)} \frac{\sinh(kL) \cos(kL) - \sin(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} \]

Hence the resulting equations can be written as

\[
\begin{align*}
C_3^{(1)} & \left( \frac{\cos(ka) \sinh(ka) - \sin(ka) \cosh(ka)}{\sin(ka) - \sinh(ka)} \right) - C_3^{(2)} \left( \frac{1 + \cos(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} \right) = 0 \\
C_3^{(1)} & \left( \frac{2 \sin(ka) \sinh(ka)}{\sin(ka) - \sinh(ka)} \right) - C_3^{(2)} \frac{\sinh(kL) \cos(kL) - \sin(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} = 0
\end{align*}
\]

(C.25)

Again, this is a homogeneous solution in the two unknowns \( C_3^{(1)} \) and \( C_3^{(2)} \). The determinant must be equal to zero, which leads to the frequency equation:

\[ [\cos(ka) \sinh(ka) - \sin(ka) \cosh(ka)] [\sinh(kL) \cos(kL) - \sin(kL) \cosh(kL)] - 2 \sin(ka) \sinh(ka) [1 + \cos(kL) \cosh(kL)] = 0 \]  

(C.26)

The roots \( k_n \) of this equation then determine the values of the natural frequencies and their corresponding natural modes of vibration.

\[ \omega_n = k_n^2 \sqrt{\frac{E}{\rho A}} \]

The mode shape is represented by two equations:

**Part 1** \( 0 < x < a \)

\[ X(x) = \left( \frac{1 + \cos(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} \right) \left( \frac{\sin(ka) \sinh(kx) - \sinh(ka) \sin(kx)}{\cos(ka) \sinh(ka) - \sin(ka) \cosh(ka)} \right) \]

\[ \text{Part 2} \quad 0 < x < L \]

\[ X(x) = \frac{1}{2} \left( \frac{(\sin(kx) + \sinh(kx)) (\cos(kL) + \cosh(kL)) - \cos(kx) - \cosh(kx)}{(\sin(kL) + \sinh(kL))} \right) \]

\[ \text{(C.27)} \]

\[ \text{(C.28)} \]

The values of \( k_n \) and the resulting modal forms are given in Chapter 3, fig.3.6
(c) Considering a System where Two Beam are Supported in Pin Joints in a
Back-to-Back Configuration

(i) Symmetric Modes

To analyse the symmetric modes of response, only half of the beam needs to be considered. This can then be treated as two beams which must then have identical boundary conditions at the connecting positions.

Considering beam 12, the characteristic equation is (from equation C.22)

\[ X^{(1)}(x_1) = c_3^{(1)} \left[ \sin(kx_1) + \sinh(kx_1) - \frac{(\sin(kL) + \sinh(kL))(\cos(kx_1) + \cosh(kx_1))}{(\cos(kL) + \cosh(kL))} \right] \]  

(C.29)

Considering beam 23, the characteristic equation is

\[ X^{(2)}(x_2) = c_1^{(2)} \left[ \cos(kx_2) + \cosh(kx_2) \right] + c_2^{(2)} \left[ \cos(kx_2) - \cosh(kx_2) \right] + c_3^{(2)} \left[ \sin(kx_2) + \sinh(kx_2) \right] + c_4^{(2)} \left[ \sin(kx_2) - \sinh(kx_2) \right] \]  

(C.30)

and the boundary conditions are:

- at \( x_2 = 0 \), \( X^{(2)} = 0 \) \( \Rightarrow c_3^{(2)} = 0 \)
- at \( x_2 = 0 \), \( X^{(2)'} = 0 \) \( \Rightarrow c_4^{(2)} = 0 \)
- at \( x_2 = a \), \( X^{(2)} = 0 \)
  \[ \Rightarrow c_2^{(2)} = -c_1^{(2)} \left( \frac{\cos(ka) + \cosh(ka)}{\cos(ka) - \cosh(ka)} \right) \]
Hence,

\[
\chi^{(2)}(x_2) = c_1^{(2)} \left\{ \frac{\cos(kx_2) + \cosh(kx_2)}{\cos(kx_2) - \cosh(kx_2)} \right\}
\]

Further boundary conditions are provided by virtue that the beams are continuous, hence

\[
\frac{dX_1^{(1)}}{dx_1} = - \frac{dX_2^{(2)}}{dx_2} \tag{C.32}
\]

\[
\frac{d^2X_1^{(1)}}{dx_1^2} = \frac{d^2X_2^{(2)}}{dx_2^2} \tag{C.33}
\]

which leads to the equations:

\[
\begin{align*}
& c_3^{(1)} \left[ \frac{1 + \cos(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} \right] + c_1^{(2)} \left[ \frac{\sin(ka) \cosh(ka) + \sinh(ka) \cos(ka)}{\cos(ka) - \cosh(ka)} \right] = 0 \\
& c_3^{(1)} \left[ \frac{\sinh(kL) \cos(kL) - \sin(kL) \cosh(kL)}{\cos(kL) + \cosh(kL)} \right] - 2 c_1^{(2)} \frac{\cos(ka) \cosh(ka)}{\cos(ka) - \cosh(ka)} = 0
\end{align*}
\]

Again, this is a homogeneous solution in the two unknowns \( c_3^{(1)} \) and \( c_1^{(2)} \). The determinant must equal zero and hence,

\[
2 \left[ 1 + \cos(kL) \cosh(kL) \right] \left[ \cos(ka) \cosh(ka) \right] + \\
\left[ \sin(ka) \cosh(ka) - \sinh(ka) \cos(ka) \right] \left[ \sinh(kL) \cos(kL) - \sin(kL) \cosh(kL) \right] = 0 \tag{C.34}
\]

The roots \( k_n \) of this equation determine the values of the natural frequencies from equation (C.12) and their corresponding natural modes of vibration from equations (C.29) and (C.31). The values of \( k_n \) and the resulting mode shapes are given in table 5.1 and figure 5.2 in Chapter 5.
(ii) Anti-Symmetric Modes

To obtain the anti-symmetric modes, different boundary conditions must be applied at the axis of symmetry.

Considering the beam 12, the characteristic equation will again be given as

\[ X^{(1)}(x_1) = C_3^{(1)} \left[ \sin(kx_1) + \sinh(kx_1) - \sin(kL) - \sinh(kL) \right] \]
\[ \frac{\cos(kx_1) + \cosh(kx_1)}{\cos(kL) + \cosh(kL)} \]

(C.35)

Considering the beam 23, the characteristic equation is

\[ X^{(2)}(x_2) = C_2^{(2)} \left[ \cos(kx_2) + \cosh(kx_2) \right] + C_4^{(2)} \left[ \sin(kx_2) - \sinh(kx_2) \right] \]
\[ C_3^{(2)} \left[ \sin(kx_2) + \sinh(kx_2) \right] \]

(C.36)

and the boundary conditions are:

at \( x_2 = 0 \) \( x^{(2)} = 0 \) \( \Rightarrow C_1^{(2)} = 0 \)

at \( x_2 = 0 \) \( x^{(2)*} = 0 \) \( \Rightarrow C_2^{(2)} = 0 \)

at \( x_2 = a \) \( x^{(2)} = 0 \)

\( \Rightarrow C_3^{(2)} = -C_4^{(2)} \frac{\sin(ka) - \sinh(ka)}{\sin(ka) + \sinh(ka)} \)

Hence,

\[ X^{(2)}(x_2) = C_3^{(2)} \left\{ \frac{\sin(kx_2) + \sinh(kx_2)}{\sin(ka) + \sinh(ka)} \right\} \]

(C.37)

The other boundary conditions previously defined in equations (C.32) and (C.33) remain applicable, hence

\[ \frac{dX^{(1)}(L)}{dx_1} = \frac{-dX^{(2)}(a)}{dx_2} \]

and

\[ \frac{d^2X^{(1)}(L)}{dx_1^2} = \frac{d^2X^{(2)}(a)}{dx_2^2} \]
which leads to the equations:

\[
\begin{align*}
C_3^{(1)} \frac{[1+\cos(kL)\cosh(kL)]}{\cos(kL)+\cosh(kL)} + C_3^{(2)} \frac{[\cosh(ka)\sin(ka)-\cos(ka)\sinh(ka)]}{\sin(ka)-\sinh(ka)} &= 0 \\
C_3^{(1)} \frac{[\sinh(kL)\cos(kL)-\sin(kL)\cosh(kL)]}{\cos(kL)+\cosh(kL)} - 2C_3^{(2)} \frac{[\sin(ka)\sinh(ka)]}{\sin(ka)-\sinh(ka)} &= 0
\end{align*}
\]

The determinant must again be zero and hence,

\[2[1+\cos(kL)\cosh(kL)][\sin(ka)\sinh(ka)]\]
\ [+\left[\cosh(ka)\sin(ka)-\cos(ka)\sinh(ka)\right][\sinh(kL)\cos(kL)-\sin(kL)\cosh(kL)] = 0 \quad (C.38)\]

The roots $k_n$ of this equation determine the values of the natural frequencies from equation (C.12) and their corresponding modes of vibration from equations (C.35) and (C.37). The values of $k_n$ and the mode shapes for these masts are given in table 5.2 and figure 5.3 in chapter 5.
APPENDIX D

Compression Test on Mast Material

A compression test on a 210mm long piece of stainless steel mast material O/D 30mm and I/D 26mm was carried out over a range of loads with a maximum of 40 KN. The values of strain at the mid length position were obtained from two 90° strain gauge rosettes at diametrically opposite positions. Each gauge was monitored individually by connection as quarter bridges to a Solartron data logging system.

The positions of each strain gauge channel are shown in figure D1. A graphical representation of the results is given in figure D2 and using the method of least squares a linear fit has been made for each channel of data. The linear fit to the data is justified because the resulting correlation coefficient for the results is better than 0.998. The values of gradient for each channel are as follows:

<table>
<thead>
<tr>
<th>Channel</th>
<th>Gradient (μstrain/KN)</th>
<th>Correlation Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-25.93</td>
<td>0.999</td>
</tr>
<tr>
<td>2</td>
<td>-28.80</td>
<td>0.999</td>
</tr>
<tr>
<td>3</td>
<td>8.21</td>
<td>0.999</td>
</tr>
<tr>
<td>4</td>
<td>7.54</td>
<td>0.998</td>
</tr>
</tbody>
</table>

The variation between channel 1 & 2 and 3 & 4 is due to the test specimen having initial imperfections such as not being perfectly straight. The applied load would then be seen to have a component of bending moment in addition to the compressive load. The bending moment component can be removed from the signal by simply adding the two longitudinal and the two transverse signals together. The Youngs Modulus and Poissons ratio can then be obtained as follows.

Young's Modulus, \[ E = \frac{F L}{A \varepsilon} = 2.08 \times 10^{11} \text{ N/m}^2 \]

Poisson's Ratio = \[ \frac{\text{Average of Transverse Strain}}{\text{Average of Longitudinal Strain}} = 0.288 \]
Figure D1 Strain Gauged Compression Specimen of Mast Material
Figure D2 Experimentally Measured Strain Gauge Output
Comparison of Strain Gauge Performance

To check the accuracy of the reading from the three strain gauge full bridge circuits, a simple test can be performed on the model mast to allow the results to be compared with theoretically predicted values. A simple knife edge support system was used for the test as shown in figure D3. The output strain values were recorded on a Vishay Strain indicator box for loads of up to 89N. The experiment was repeated four times and the results from each are shown in figures D4 to D7. The results of straight line curve fits by the method of least squares are summarised in table D2.
Figure D3  Experimental Arrangement for the Strain Gauge Calibration Tests
TEST #1

Figure D4 Experimentally Measured Strain Gauge Output

TEST #2

Figure D5 Experimentally Measured Strain Gauge Output
Figure D6  Experimentally Measured Strain Gauge Output

Figure D7  Experimentally Measured Strain Gauge Output
<table>
<thead>
<tr>
<th>Transducer</th>
<th>Experimentally Determined Gradient (µst/N)</th>
<th>Predicted Gradient (µstrn/N)</th>
<th>%Error</th>
</tr>
</thead>
<tbody>
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<td>Test #3</td>
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Table D1  Experimentally Measured and Predicted Values of the Strain/Load Relationship
Experimental Evaluation of Support Stiffness

Four experiments were carried out with the mast clamped vertically by two diaphragm supports in order to determine their rotational stiffness. The mast was loaded at the tip and the strain output was measured from the gauges positioned midway between the bearings. The results are given in figures E1 to E4 and a summary of the determined gradients using straight line curve fits by the method of least squares are shown in table E1. The validity of using linear approximations was supported by the correlation coefficients for the curve fits having values better than 0.99. The average value of strain output per load was found to be 2.211 μstrain/N.

Using the equations relating the angular rotation to the applied loads derived in Appendix A (equations (A.11) and (A.13)), the rotation at the supporting positions 1 and 2 for a tip loaded uniform beam is given as

\[ \theta_1 = \frac{a}{6EI} \left[ 2WL + 2M_1 - M_2 \right] \]  
\[ \theta_2 = \frac{-a}{6EI} \left[ WL + M_1 - 2M_2 \right] \]
Figure E1  Experimentally Measured Strain/Load Relationship

Figure E2  Experimentally Measured Strain/Load Relationship
**Figure E3** Experimentally Measured Strain/Load Relationship

**Figure E4** Experimentally Measured Strain/Load Relationship
### Table E1  Experimentally Measured Values of the Strain/Load Relationship

<table>
<thead>
<tr>
<th>Test Number</th>
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<th>Correlation Coefficient</th>
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<td>E2</td>
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<td>E3</td>
<td>2.206</td>
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<td>E4</td>
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<tr>
<td><strong>Average</strong></td>
<td><strong>2.211</strong></td>
<td></td>
</tr>
</tbody>
</table>
Assuming that the rotation of the support is equal to the effective rotational stiffness, \( K \) of the diaphragm multiplied by the moment acting at that point, then

\[
K \theta_1 = -M_1 \\
K \theta_2 = -M_2
\]

(E.3)

The solution of equations (E.1), (E.2) and (E.3) gives

\[
M_1 = -\frac{W L [4K^* + 1]}{1 + 2K^* [1 + 6K^*]}
\]

(E.4)

\[
M_2 = \frac{2K^* W L}{1 + 2K^* [1 + 6K^*]}
\]

(E.5)

where \( K^* = \frac{E I}{K a} \)

The moment at a position \( x \), between the supports is given as

\[
M_x = W x + M_1 + R_1 (x - L)
\]

where

\[
R_1 = \frac{-M_1 - M_2 - W (L + a)}{a}
\]

At a position where \( x = L + a/2 \) then the moment \( M_x \) becomes

\[
M_x = \left( W L + M_1 - M_2 \right)/2
\]

(E.6)

Applying bending theory together with equations (E4) and (E5) then equation (E6) reduces to give

\[
K = \left[ \frac{W L y}{\varepsilon a} - \frac{2 E I}{a} \right]
\]

(E.7)

where \( (\varepsilon/W) \) is the strain per unit load measured at a position, distance \( y \) from the neutral axis.
The experimentally measured values of the coefficients of equation (E7) are as follows:

\[ \frac{\varepsilon}{W} \text{average} = 2.211 \, (\mu \text{strain}/N) \]

\[ y = 13 \, \text{mm} \]
\[ L = 1.51 \, \text{m} \]
\[ a = 0.324 \, \text{m} \]
\[ E = 2.08 \times 10^9 \, \text{N/m}^2 \]
\[ I = 1.733 \times 10^{-8} \, \text{m}^4 \]

Hence, the predicted value of rotational stiffness is 5200 Nm.

**Theoretical Value of Stiffness**

The angle of rotation, \( \theta \), produced by a central moment, \( M \), acting on a circular plate of constant thickness, whose edges are considered to be in a built-in condition is of the form

\[ \theta = \frac{\alpha M}{E \, t^3} \]  

(E.8)

where \( E \) is the Young's modulus of the plate material, \( t \) is the thickness of the plate and \( \alpha \) is dependent on the dimensions of the plate.

The rotational stiffness \( K \), is given by

\[ K = \frac{M}{\theta} = \frac{E \, t^3}{\alpha} \]  

(E.9)
The relationship between the value of $\alpha$ and the ratio $\frac{b_2}{b_1}$ is shown in figure E5. The values of these coefficients for the plate are as follows:

\[
\begin{align*}
    t &= 1.1 \times 10^{-3} \text{ m} \\
    b_1 &= 0.043 \text{ m} \\
    b_2 &= 0.023 \text{ m} \\
    E &= 210 \times 10^9 \text{ N/m}^2
\end{align*}
\]

$\frac{b_2}{b_1} = 0.535$

The value of $\alpha$ for the above values can be seen to be 0.06 which from equation E9 gives a rotational stiffness of 4660 Nm.
Figure E5 Relationship between the Coefficient $\alpha$ and the $b_2/b_1$ ratio
APPENDIX F

Summary of experimental results of the frequency characteristics of a semi-submerged mast in free response
<table>
<thead>
<tr>
<th>Datafile</th>
<th>Channel</th>
<th>Height of Water/m</th>
<th>Mode 1</th>
<th>Mode 2</th>
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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
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<th>Mode 2 Frequency /Hz</th>
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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
### TABLE F1 (continued)

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<th>Mode 2 Frequency /Hz</th>
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**Sampling Frequency 200Hz**  
**Accuracy of Frequency Values ±0.02Hz**
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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
TABLE F2 (continued)

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<th>Height of Water/m</th>
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<td>%Increase in Freq.</td>
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Sampling Frequency 500Hz
Accuracy of Frequency Values ±0.06Hz
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- **Sampling Frequency**: 500Hz
- **Accuracy of Frequency Values**: ±0.06Hz
### TABLE F4 Experimentally Determined Frequency Values for the Centrally Deflected Mast

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<th>%Increase in Freq.</th>
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Sampling Frequency 500Hz
Accuracy of Frequency Values ±0.06Hz
TABLE F5  Experimentally Determined Frequency Values for the Dual Loaded Mast

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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
TABLE F5 (continued)

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Sampling Frequency 200Hz
Accuracy of Frequency Values ±0.02Hz
TABLE F6  Experimentally Determined Frequency Values for the Dual Loaded Mast

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Accuracy of Frequency Values ±0.12Hz
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Sampling Frequency 1000Hz
Accuracy of Frequency Values ±0.12Hz
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Sampling Frequency 1000Hz
Accuracy of Frequency Values ±0.12Hz
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Sampling Frequency 1000Hz
Accuracy of Frequency Values ±0.12Hz
APPENDIX G

Experimental results of the change in damping relationship with amplitude of strain for an increasing depth of water.
Damping Relationship for Water Depth 0 m

\[ Y = Mx + C \]
\[ M = 3.38 \times 10^{-5} \]
\[ C = 0.00338 \]

Damping Relationship for Water Depth 0.1 m

\[ Y = Mx + C \]
\[ M = 2.77 \times 10^{-5} \]
\[ C = 0.00606 \]
Damping Relationship for Water Depth 0.2 m

\[ Y = Mx + C \]

\[ M = 2.47 \times 10^{-5} \]
\[ C = 0.00722 \]

Damping Relationship for Water Depth 0.3 m

\[ Y = Mx + C \]

\[ M = 2.06 \times 10^{-5} \]
\[ C = 0.00858 \]
Damping Relationship for Water Depth 0.6 m

\[ Y = Mx + C \]
M = \(2.64 \times 10^{-5}\)
C = 0.008

Damping Relationship for Water Depth 0.7 m

\[ Y = Mx + C \]
M = \(2.47 \times 10^{-5}\)
C = 0.00985
Damping Relationship for Water Depth 0.8 m

\[ Y = Mx + C \]
\[ M = 3.63 \times 10^{-5} \]
\[ C = 0.00864 \]

Damping Relationship for Water Depth 0.9 m

\[ Y = Mx + C \]
\[ M = 4.34 \times 10^{-5} \]
\[ C = 0.0103 \]
Damping Relationship for Water Depth 1.0 m

\[ Y = Mx + C \]

\[ M = 6.07 \times 10^{-5} \]
\[ C = 0.00951 \]

Amplitude of Strain (\( \mu \text{strain} \))

Damping Relationship for Water Depth 1.1 m

\[ Y = Mx + C \]

\[ M = 9.02 \times 10^{-5} \]
\[ C = 0.00785 \]

Amplitude of Strain (\( \mu \text{strain} \))
Damping Relationship for Water Depth 1.2 m

\[ Y = Mx + C \]

\[ M = 0.000123 \]

\[ C = 0.00954 \]

Damping Relationship for Water Depth 1.3 m

\[ Y = Mx + C \]

\[ M = 0.000167 \]

\[ C = 0.0088 \]
Damping Relationship for Water Depth 1.4 m

\[ Y = Mx + C \]

\[ M = 0.000193 \]
\[ C = 0.0155 \]

Damping Relationship for Water Depth 1.5 m

\[ Y = Mx + C \]

\[ M = 0.000269 \]
\[ C = 0.00956 \]
APPENDIX H

The Use of Digital Filtering Techniques

High frequency noise superimposed on transducer signals obtained from tests carried out in the study of the effect of shock loads on submerged structures can somewhat restrict the accurate determination of the essential characteristics of the response information. Fortunately, the noise created by the resonance of the transducer is usually at much higher frequencies than those of the normal response of the structure. This allows the use of filtering techniques to separate the two components of the signal. As a first step in the analysis of the signal, a Fourier transform can be used to obtain a frequency representation of the time dependent motion of the structure. This transform essentially splits the signal into a series of sine and cosine signals and the signal can be represented as:

\[ F(t) = A_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t) \]

where \( A_0, A_n, B_n \) represent the amplitudes of the harmonic components of the original signal. The Fourier transform can be computed more efficiently with the development of the Fast Fourier Transform (FFT). A typical acceleration signal and its associated frequency spectrums obtained using an FFT is shown in figures H1 and H2. It can be seen that the noise on the signal associated with the resonance of the transducer occurs at much higher frequencies than those of the structural behaviour. Filtering techniques can be used to suppress these unwanted frequencies.

Having the 'raw' transducer signals stored as digitised data leads naturally to the use of digital filtering techniques. Analogue filtering may also be used in the recording system, but has become unpopular as a principal filtering technique because of improvements in the ease, accuracy and cost effectiveness of employing digital filtering on a modern computer system. Digital filtering methods also have the significant advantage of being a 'non-destructive' method, i.e. the original signal is stored prior to the application of the technique and therefore remains uncorrupted.

There are two basic forms of digital filter - the recursive filter and the non-recursive filter. The recursive filter has the advantage of minimal
Figure H1 Acceleration Signal from Channel 3 Shot B

Figure H2 Frequency Spectrum for the Signal from Channel 3 Shot B
computation time but has non-linear phase characteristics whereas the non-recursive filter is inherently stable with linear phase characteristics. For the reasons of stability the latter was considered more useful in the filtering of accelerometer signals.

The design of the non-recursive filter is carried out by firstly deciding on the required frequency characteristics (82). The types of filter which can be employed by these methods are limitless and may include low pass filters, high pass filters, band pass filters and even band stop filters. Having decided on the cut-off frequency or frequencies, the transfer function of the filter can be obtained analytically. These characteristics can be represented in the time domain as a series of filter weights which make up the filter and are evaluated by use of an inverse Fourier transform. However, since this Fourier series is necessarily finite, it cannot produce a step change in the frequency characteristics of the filter. To overcome some of the difficulties, a time-limited weighting function can be multiplied on to the filter weights to minimise the effects of truncation. It was found that the Hamming window (83) smoothing function was satisfactory for use with low pass filters and was subsequently used throughout the digital filtering processes discussed in this study. The number of points in the weighting function determines both the roll-off characteristics and the stop band attenuation. The minimum number of points needed to represent the filter characteristics depends both on the sample frequency of the signal and the cut-off frequency of the filter. However, the accuracy of the filter may be improved by increasing the number of points in the filter but this incurs the expense of a longer processing time. The required accuracy of the filter needs only to be the same as that of the recording accuracy for the signal and hence the stop-band attenuation level was matched to the bit level of the transient recorder. It was found that 256 points in the weighting function produced an acceptably accurate filter for the data obtained from both the explosive shock rig and the laboratory based test rig. The weighting sequence obtained for a low pass filter with a cut off frequency of 2KHz which is suitable for use with a data set which has a sample rate of 12 sec is shown in figure H3. The true filter characteristics can be obtained by Fourier transforming the weighting sequence back into the frequency domain and the result is shown in figure H3.
256 Point Weighting Sequence with Hamming Window.

Sample rate 12 usec
Cut-off frequency 2 KHz

Figure H3  Digital Filter Characteristics
The filter consists of a set of weights \( h_k \) and the process of filtering determines the sum of the products \( h_k x_{i-k} \) form the set of filter weights at each sampled data value \( x_i \). The resulting filtered value \( y_i \) is given by

\[
y_i = \sum_{k=0}^{M-1} h_k x_{i-k} \quad (i = 1, 2, \ldots, N; k = 0, \ldots, M-1)
\]

where \( M \) is the number of weights in the filter and \( N \) is the number of points in the data set. At each point along the digital record the average of the product \( h_k x_i \) is determined so that a moving average is carried out as the filter moves along the time signal (see figure H4). In the case of a low pass filter, frequencies in the time signal below the cut-off point will remain unaffected whereas the frequencies above will be cancelled out during the summation process.

One disadvantage of the digital filtering technique is that a small amount of the signal's length must be lost during the processing. If the weighting sequence has \( N \) points, then the summation process distorts both the first or last \( N/2 \) points of the signal because there is not sufficient information over the full width of the filter. In the case of transient signals which have a certain amount of zero-level pre-trigger data, further 'zero' points can be added in front of the data set so that no 'true' information is lost at the start of the signal. The loss of information at the end of the signal can be shown to represent only 3% of the total signal and is therefore not considered to present major restrictions on the transient study.

As an example to show the ability of such a filtering technique, the accelerometer signal previously shown in figure H1 has been 'cleaned up' using the low-pass filter shown in figure H3. The frequency spectrum of the signal given in figure H2 has shown that the true response of the structure lies below 2KHz and that the dominant natural frequency of the transducer occurs at 5KHz. The result of applying this 2KHz low-pass filter is shown in figure H5 to provide far clearer information about the structural response. The peak values of acceleration are now seen to be \( \pm 100g \) whereas before filtering the 'noise' on the transducer gave a maximum acceleration of \( \pm 650g \).
Average product taken over this period

Figure H4  Non-recursive Filtering in the Time Domain
Figure H5  Reconstructed Acceleration Signal from Channel 3 Shot B after Applying a Low-pass Digital Filter with a 2 KHz Cut-off Frequency
APPENDIX I

Specification for the Transducers and Amplification System used in the Laboratory Based Test Rig.

Accelerometers (2 off)

Manufacturer: ENTRAN DEVICES, Inc.
Model: EGC - 240 - 500
Type: Piezoresistive high 'g' accelerometer
Range: ± 500g
Excitation: 15.0 Volts
Sensitivity: 0.385 mV/g at 25g, 500Hz, 75°F
Nominal Resonant Frequency: 4000 Hz

Strain Gauges (6 off forming 3 full bridge circuits)

Manufacturer: TML TOKYO SOKKI KENKOJO Co.Ltd.
Model: FCA - 5 - 11
Type: 90° stacked double element gauge
Gauge Length: 5mm
Gauge Width: 2mm
Resistance: 120Ω ± 0.5Ω
Gauge Factor: 2.12

Amplification and Signal Conditioning Units (4 off)

Manufacturer: FYLDE ELECTRONIC LABORATORIES Ltd.
Model: FE359 TA
Type: DC bridge/transducer amplifier
Bridge voltage: 2.5 - 15V
Linearity: ± 0.1%
Gain Accuracy: ± 0.25%
Gain Stability: ±0.1% 12 months
Bandwidth: DC - 50 KHz, -3dB
Internal Filter: Low pass 6dB/Octave filter, 2Hz to 10KHz.
Output: ± 10V ± 10mA anti-aliasing filters
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<td><strong>Model:</strong></td>
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<td><strong>Amplitude Resolution:</strong></td>
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<td><strong>Memory Size:</strong></td>
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<td>(64)</td>
<td>GUBKIN, K.E.</td>
<td>&quot;The Similarity of Explosions&quot;</td>
<td>Physics of the Solid Earth, 14, 10, May 1979, pp 714-721.</td>
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