STUDIES OF THE OPTICAL MODEL FOR ELASTIC SCATTERING

A thesis submitted to the Faculty of Mathematical and Physical Sciences of the University of Surrey for the degree of Doctor of Philosophy

by

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Abstract

Optical model studies of medium energy alpha particles and high energy pions using the microscopic model, are carried out to investigate the nuclear matter distribution. For medium energy alpha particles, the optical potentials are obtained by folding an effective alpha-nucleon interaction into the nuclear matter distributions which are constructed from single particle wave functions, and elastic scattering of alpha particles at 42 MeV is calculated. It is shown that the nuclear matter distributions obtained as above can explain reasonably well experimental data, if a suitable form for the effective interaction is chosen. Further, the ability of the microscopic model to reproduce the observed behaviour of the strong absorption radii, is examined and it is found that the strong absorption radii obtained, show systematic A-dependence. But it is noted that the small uncertainties in the range parameter of the effective interaction become significant when one is concerned with the precise behaviour of these radii. The same uncertainties also limit the accurate determination of the nuclear matter distribution.

For high energy pions, the optical potentials are obtained by folding the free pion-nucleon interaction in momentum space into the nucleon form factors which are calculated using neutron and proton distributions obtained as before. Using these pion optical potentials, the Klein-Gordon equation is solved numerically and the absorption and differential cross-sections at energies in the region 0.585-1.057 GeV are calculated. These calculations are carried out using the complete expression for the pion-nucleon interaction as well as the large A approximation. It is shown that the absorption and differential cross-sections for light and medium mass nuclei are
quite sensitive to the variations in the parameters of the pion optical potential and in particular, at the minima in the differential cross-sections, significant changes are produced. From a comparison of our results for the absorption cross-sections for such nuclei with some available experimental data, it is indicated that the use of the large A approximation may significantly alter the conclusions reached before about the nuclear matter distributions. It also appears that even for heavy nuclei, the use of this approximation is probably suspect, especially when one is concerned with the analysis of accurate experimental data.
Acknowledgements

I wish to thank Professor D.F. Jackson who has kindly supervised the work presented here. She has provided invaluable guidance and support during the various stages of its development. I would also like to thank Dr. R.C. Johnson and Dr. R.C. Barrett for valuable suggestions and discussions.

I am grateful to Professor L.R.B. Elton for his hospitality during my stay at the University of Surrey. I am indebted to him for the discussions during the development of the latter part of this thesis. It has been a great pleasure for me to work with the Nuclear Physics Group at the University, and I wish to thank all its members for their warmth and kind assistance.

I am indebted to Mr. K.R. Knight for programming some of the calculations. I would like to thank the staff of the Surrey University for providing me with computing facilities. Also, I gratefully acknowledge the grant of a Research Studentship by the University of Surrey during the period of my stay at the Physics Department.

I would like to thank Miss S.R. Lord for typing this thesis.

I wish to give warm thanks to my wife for her patience and encouragement during the various stages of preparation of this work.
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1. Introduction

The microscopic description of elastic scattering of projectiles, such as alpha particles and pions, by nuclei has been increasingly recognised as a valuable tool for the investigation of nuclear structure. In such a description, the many-body interaction between target nucleons in the nucleus and the incident projectile is reduced to a one-body potential, and elastic scattering is calculated from this one-body potential. The one-body or optical potential so obtained depends on the projectile-target nucleon interaction as well as on the structure of the target nucleus. This approach is significant in that the analysis of elastic scattering provides information on the structure of the target nucleus and further, on the projectile-target nucleon interaction. However, the projectile-target nucleon interaction is in general nonlocal, energy-dependent and complex. Further, it includes the effects of the exclusion principle and nuclear binding. In the case of alpha particles\(^{(23)(24)(63)(65)}\) at energies in the range 22-42 MeV and also, in the case of nucleons\(^{(2)}\) at energies in the range 14-40 MeV, this difficulty has been overcome with reasonable success by the use of an effective interaction. In this approximation, the relevant projectile-target nucleon interaction is replaced by a simple local energy-independent effective interaction which is treated as a phenomenological interaction. This effective interaction is assumed to simulate the effects of the exclusion principle and nuclear binding. But in the case of alpha particles a different point of view has been taken by other authors\(^{(47)(66)(67)}\) who have derived the alpha-target nucleon effective interaction by folding the free nucleon
nucleon interaction into the ground state distribution of the alpha
particle. In the latter method, the two-body (alpha-target nucleon)
strength and range parameters are determined in the process whereas
in the former method these two-body parameters are treated as variable
parameters.

At high energies, the situation is rather different, since at
these energies, the effects of the exclusion principle and nuclear
binding become less important and thus we may reasonably use the
impulse approximation which allows us to replace the projectile-target
nucleon interaction by the free projectile-nucleon interaction. The
impulse approximation has been successfully used, particularly in the
case of pions\(^{(34)(41)(48)(54)}\) and nucleons\(^{(28)}\) at energies in the
region of 1 GeV, to analyse elastic scattering of these projectiles
by nuclei. Although the use of the impulse approximation appears to
be reasonably valid at these energies, it has been noted for high
energy pions that the effects\(^{(48)}\) of the exclusion principle and the
direct absorption by pairs of nucleons (see Chapter 4) and further,
the effect of the Fermi motion\(^{(22)}\) of the target nucleons in the
nucleus may yet be significant. However, the analyses\(^{(48)(22)}\) of
high energy pion-nucleus elastic scattering have used the large A
approximation where it is assumed that the free pion-nucleon inter­
action in momentum space varies slowly compared with the nucleon form
factors (see Chapter 4). Although the large A approximation is
expected to be reasonably valid for heavy nuclei, its validity for
light and medium-heavy nuclei may be rather suspect. Since the
analyses\(^{(48)(22)}\) have been done for medium-heavy nuclei, the
conclusions reached\(^{(48)(22)}\) about the significance of the above
effects may need to be re-examined. Further, the information on
nuclear structure obtained\(^{(48)(22)}\) may be open to re-interpretation.
The calculation of the optical potential derived from the microscopic description involves, in addition to a knowledge of the projectile-target nucleon interaction, that of the proton and neutron distributions \( \rho_p(r), \rho_n(r) \) defined as:

\[
\rho_p(r) = \langle 0 | \sum_{i=1}^{Z} \delta(r-r_i) | 0 \rangle
\]

\[
\rho_n(r) = \langle 0 | \sum_{i=1}^{N} \delta(r-r_i) | 0 \rangle
\]

where \( |0\rangle \) is the ground state of the target nucleus, and \( Z, N \) are the numbers of protons and neutrons in the target nucleus. The proton and neutron distributions \( \rho_p(r), \rho_n(r) \) can be obtained using a suitable model for the target nucleus. In some analyses of elastic electron scattering, the proton distribution has been treated as a phenomenological distribution (spherically symmetric) which is chosen so as to simulate the behaviour in the interior of the target nucleus as well as in the surface region. A phenomenological distribution often used in such analyses is the Fermi distribution which has the required behaviour and is given by:

\[
\rho_p(r) = \rho_0 \left[ 1 + \exp \left( \frac{r-R}{a} \right) \right]^{-1}
\]

where \( R, a \) are the half-way radius and diffuseness parameters and \( \rho_0 \) is a parameter fixed by the normalisation condition. This Fermi distribution for protons has been found satisfactory in explaining the elastic electron scattering data. Further, the Fermi distribution has also been used successfully for neutrons in the analyses of nucleon-nucleus scattering\(^{(2)}\) and pion-nucleus scattering\(^{(7)(22)(34)(41)}\).

But in the analysis of elastic electron scattering, Elton and Swift\(^{(1)}\) have used the single particle description of the nucleus to obtain the proton distribution. In the single particle description of the nucleus, it is assumed that the interaction between a nucleon and the remaining nucleons in the target nucleus can be represented by a
(effective) single particle potential. The single particle wave functions $\psi_i$ (i denotes the state of the proton) are generated from these single particle potentials by fitting eigenenergies to mean proton separation energies (see Chapter 2) and the (spherically symmetric) proton distribution $\rho_p(r)$ is then constructed from these single particle wave functions $\psi_i$ as

$$\rho_p(r) = \sum_{i=1}^{Z} |\psi_i(r)|^2$$

The proton distribution so constructed by Elton and Swift\(^{(1)}\) has been found satisfactory in explaining elastic electron scattering data. From the success of the method of Elton and Swift\(^{(1)}\), it is indicated that the same procedure may be adopted for constructing the neutron distribution. However, if the neutron distribution is so obtained, it can be regarded as reliable only after it has been shown to explain satisfactorily and consistently a reasonable body of experimental data.

In this thesis, we have followed the method of Elton and Swift\(^{(1)}\) to obtain the proton distribution as well as the neutron distribution, and have been concerned with elastic scattering of medium energy alpha particles and high energy pions, using optical potentials as obtained from the microscopic description. We have discussed further some features of medium energy particles and high energy pions, in the following sections.

1.1 Medium energy alpha particles

For optical model studies using the microscopic model, medium energy alpha particles are more suitable than low energy alpha particles since at medium energies, unlike at low energies, many-body effects are expected to be less important. Further, because alpha particles are spin-zero particles, the spin-orbit term in the optical potential for alpha particles averages to zero for a spin-zero target nucleus, although
the alpha-target nucleon interaction involves such a term. In view of this, only the central part of the alpha-target nucleon interaction is involved in the calculation of the optical potential, and when the effective interaction approximation is used, this is replaced by an effective interaction. The effective interaction allows one to calculate the optical potential obtained from the microscopic description and subsequently, elastic scattering of alpha particles. But there remains still the problem of choosing a suitable form for the effective interaction. Various forms for the effective interaction have been used successfully in the analyses of elastic and inelastic scattering of medium energy alpha particles. In the analysis of inelastic scattering, it is assumed that the effective alpha-nucleon interaction should be very much similar to the free alpha-nucleon interaction. This is based on the argument that the phenomenon of strong absorption, in the case of a strongly absorbed projectile such as the alpha particle, has the effect that its interaction with the target nucleus is confined to the surface region of the nucleus, and hence the effects of the exclusion principle and nuclear binding are expected to be very much reduced compared with the situation for nucleons. However, in some analyses the effective interaction has been treated as a phenomenological interaction which is assumed to simulate the effects, if any, of the exclusion principle and nuclear binding. We have taken the latter point of view in the present work but later, have compared the effective interaction with the free interaction in order to see to what extent the former point of view can be justified and further, whether treatment of elastic and inelastic scattering of medium energy alpha particles is self-consistent as suggested.
As mentioned earlier, the alpha particle is a strongly absorbed particle and for such a particle, it has been well-known from analyses of elastic scattering (16)(17)(18)(19)(20) using the phenomenological optical potential, that discrete ambiguities in the optical potential exist and as a result, the mean square radius of the potential is a function of the depth. These ambiguities in the optical potential for alpha particles prevent the possible interpretation of the parameters of the optical potential in terms of nuclear size parameters. In view of this, the phenomenological analysis of elastic scattering (10) has examined, in some detail, whether there is any criterion for the equivalence of optical potentials for strongly absorbed particles and further, whether the size parameter as obtained from the phase analyses (9)(15) may be regarded as significant. This size parameter, called the strong absorption radius $R_{1/2}$ is defined (13) as:

$$kR_{1/2} = \gamma + \sqrt{\gamma^2 + L(L+1)}$$

where $\gamma$ is the Coulomb parameter, $k$ is the wave number of the incident alpha particle and $L$ is the angular momentum for which the real part of the reflection coefficient (scattering matrix element) is $\text{Re} \eta_L = 1$ (see Chapter 2). It has emerged from this analysis that the strong absorption radius $R_{1/2}$ is a very significant size parameter and that for the equivalence of the optical potentials, it is required that the real parts of the potentials should be equal at the strong absorption radius $R_{1/2}$. These observations have been confirmed by the recent phenomenological analysis (12). Further, the phenomenological analysis (12) has shown that the strong absorption radii exhibit systematic $A$-dependence for a wide range of nuclei, although some individual nuclei show deviations from this trend. In view of these results for the strong absorption radii, it has been pointed out (12) that the isotropic dependence of the strong absorption radii reflects,
presumably, the intrinsic structure of the nuclei. However, at present, the direct connection between the strong absorption radius and the root mean square radius of the nuclear matter distribution

\[ \rho_m(r) = \rho_p(r) + \rho_n(r) \]

is not obvious and because of this, the information obtained from phenomenological analyses of elastic scattering of alpha particles does not admit of easy comparison with that obtained from other analyses such as those of elastic nucleon-nucleus scattering and pion-nucleus scattering \(^{7)(22)(34)}\) where the nuclear matter distribution, and its root mean square radius have been the object of investigation. In view of this, the analysis of elastic scattering of alpha particles using the optical potential obtained from the microscopic description, is of much interest in that from the analysis of data, it appears possible to reach conclusions \(^{2}\) about the nuclear matter (or neutron) distributions and their root mean square radii which can be compared directly with the results for them obtained from analyses of elastic scattering of other projectiles and other analyses. However, since it is known, as mentioned before, from the phenomenological analyses \(^{10)(12}\) that for strongly absorbed particles, the strong absorption radius is a very significant size parameter, and these radii show systematic A-dependence, the microscopic description of elastic scattering of alpha particles should also be able to explain the behaviour of the strong absorption radii. It is indicated from \(^{24)(63}\) the work that the behaviour of these radii can be reproduced and this point has been examined later in the present work. Further, the use of the microscopic description of elastic scattering of alpha particles allows one to obtain information on the two-body (alpha-nucleon interaction) which might help to remove possible uncertainties in the parameters of the two-body interaction used in the inelastic scattering of alpha particles.
In the present work we have used two phenomenological forms for the effective alpha-nucleon interaction and have obtained the optical potential by folding the effective interaction into the nuclear matter distribution. Using this optical potential, we have analysed elastic scattering of medium energy alpha particles and examined the sensitivity of the elastic scattering cross-sections to the nuclear matter distribution. Further, we have investigated whether the microscopic description of elastic scattering of medium energy alpha particles can be used to interpret the behaviour of the strong absorption radii.

1.2 High energy pions

Optical model studies of pions have been successfully carried out to investigate nuclear structure, mainly at high energies. The reason for this is that at low energies, when the p-wave character of pion-nucleon scattering is taken into account, the pion optical potential turns out to be nonlocal in configuration space and thus, the calculations become somewhat involved. Besides, low energy optical model analyses have to restrict themselves to pion kinetic energies well below the (3,3) resonance energy (near 200 MeV) for pion-nucleon scattering. But at high energies, particularly in the GeV region, the situation is much better. At these energies, the impulse approximation is expected to be reasonably valid and further, in view of the fact that the free pion-nucleon matrix scattering matrix, in the GeV region, depends only on the momentum transfer as indicated by the analyses of pion-nucleon scattering data, it is possible to describe reasonably high energy pion-nucleus elastic scattering using a local pion optical potential. However, it is known that pion-nucleon total cross-sections at high energies show resonance behaviour but it so happens that the Fermi motion...
of target nucleons in the nucleus smooths out this resonance behaviour (see Chapter 4). Thus, the use of the optical model for high energy pions appears to be reasonable.

In the high energy optical model analysis, the pion optical potential has been obtained by Auerbach et al\(^{(34)}\) using the formalism of Watson\(^{(45)}\) but it has been assumed that the elastic scattering, because the pion-nucleus system is relativistic, is described by the Klein-Gordon equation. We have also made this assumption in the present work. Other analyses\(^{(22)(41)(48)}\) of high energy pion-nucleus scattering have also used pion optical potentials but in the semi-classical approximation. These analyses of high energy pion-nucleus elastic scattering have shown that the nuclear matter distribution can be investigated using high energy pions. However, in all these analyses, the large A approximation has been used and, as pointed out earlier, the validity of the large A approximation, particularly for light and medium-heavy nuclei, may be rather suspect. In view of this, the large A approximation needs to be further examined in order that one may be able to reach firm conclusions about the nuclear matter (or, neutron) distribution, from analysis of data. Further, this may also help to obtain more reasonably the corrections for the pion-nucleon total cross-section due to the effects\(^{(48)}\) of the exclusion principle and direct absorption by pairs of nucleons and the effect\(^{(22)}\) of the Fermi motion of target nucleons in the nucleus. We have examined in some detail the large A approximation in the present work.

Various studies of high energy pions, mentioned above, have been concerned with the information obtainable on the nuclear matter distribution from the analyses of data, mainly on absorption and elastic (diffraction) cross-sections. However, the studies of the differential elastic scattering cross-sections for pions at energies
in the GeV region, would be equally of much interest and the examination
of the sensitivity of these cross-sections to the nuclear matter
distribution and the pion-nucleon parameters will, no doubt, be valuable.
Further, it is in this situation that one can test further the use of
the large A approximation, possibly in a conclusive way. In view of
this, it appears therefore that analyses of differential cross-sections
might provide more accurate information on the nuclear matter distribution
as well as on the pion-nucleon parameters. It has been noted in the
analyses\(^{(22)}\)\(^{(54)}\) that the absorption and elastic diffraction cross-
sections for high energy pions are rather insensitive to the real part
of the pion-nucleon scattering amplitude. But one may be able to
determine this parameter from the analyses of differential cross-sections
for high energy pions. There is some support for this optimistic view
from the recent theoretical study\(^{(6)}\) of \(\pi-\text{He}^4\) elastic scattering at
high energies.

However, experimental data on differential cross-sections for
pions at energies in the GeV are not available at the present time,
but it appears possible that such measurements may become available
before long. In view of this, we have carried out this theoretical
study of high energy pions in order to see what degree of experimental
accuracy is required for such measurements so that when these become
available one may be able to reach some firm conclusions about the
nuclear matter (or neutron) distribution. In the present work, the
pion optical potential at energies in the GeV region is obtained using
the method of Kerman, McManus and Thaler\(^{(28)}\), originally developed for
nucleon-nucleus scattering, and as mentioned before, we have assumed
that elastic scattering is described by the Klein-Gordon equation. In
the calculation of the optical potential, we have used the form for the
pion-nucleon scattering matrix as taken in the high energy analyses of
pion-nucleon scattering data \(^{(42)}(43)\). Further, we have also used the large A approximation in the calculation of the pion optical potential. This is done in order to test the use of this approximation and to examine the possible consequences for the nuclear matter distribution.

1.3 Plan of thesis

Chapters 2 and 3 deal with the elastic scattering of medium energy alpha particles. In Chapter 2, the microscopic description of elastic scattering of alpha particles is discussed and the optical potential for alpha particles derived from the microscopic description is calculated using the effective interaction approximation, and the proton and neutron distributions obtained from single particle wave functions. With the optical potential, the Schrödinger equation is solved numerically and the phase shifts are obtained by the use of appropriate boundary conditions. Then, we deal with the strong absorption radii. In Chapter 3, we have first examined the sensitivity of the differential cross-sections at 42 MeV to the parameters of the effective interaction and then the theoretical results for \(^{40}\text{Ca}, {^{42}\text{Ca}}\)

\(^{44}\text{Ca}, {^{48}\text{Ca}}\) at 42 MeV are compared with the experimental data. The results for \(^{40}\text{Ca}, {^{42}\text{Ca}}\), \(^{44}\text{Ca}, {^{48}\text{Ca}}\) are further discussed.

Chapters 4 and 5 are concerned with the elastic scattering of pions at energies in the GeV region. In Chapter 4, the pion optical potential is derived using the method of Kerman, McManus and Thaler and this potential is calculated using the impulse approximation, and the proton and neutron distributions obtained as before. Using the pion optical potential so calculated, the Klein-Gordon equation is solved numerically and the phase shifts are determined again from boundary conditions. In Chapter 5, we deal with the calculations of absorption, elastic (diffraction) cross-sections and differential elastic cross-sections at energies in the range 0.585-1.057 GeV, and
in particular, the sensitivity of the cross-sections for $^{12}_{\text{C}}$, $^{40}_{\text{Ca}}$, $^{208}_{\text{Pb}}$ to the two-body (pion-nucleon) parameters is examined in some detail. Later, we discuss the theoretical results for $^{12}_{\text{C}}$, $^{40}_{\text{Ca}}$ and $^{208}_{\text{Pb}}$. 
2.1 Microscopic description

The microscopic description of elastic alpha particle scattering, as mentioned earlier, can be used to investigate nuclear structure. The incident alpha particle interacts with all the nucleons in the target nucleus and therefore, the interaction between the alpha particle and the target nucleus is essentially a many-body interaction. For simplicity, we assume that this many-body interaction can be taken to be the sum of two-body interactions,

\[ V = \sum_i^A v_i(R, r_i) \]  \hspace{1cm} (2.1)

where \( R \) is the distance between the centre of mass of the alpha particle and the centre of mass of the target nucleus, and \( r_i \) is the distance between a point nucleon and the centre of mass of the target nucleus. \( A \) is the number of nucleons in the target nucleus. The Hamiltonian for the system which consists of the incident alpha particle and the target nucleus is,

\[ H = H_N + H_\alpha + K_\alpha + V \]  \hspace{1cm} (2.2)

where \( H_N \) is the nuclear Hamiltonian, and \( H_\alpha, K_\alpha \) are the internal Hamiltonian and the kinetic energy operator for the alpha particle. The Schrödinger equation for the above system is,

\[ H|\psi> = E|\psi> \]  \hspace{1cm} (2.3)

where \( |\psi> \) is the state describing both the incident alpha particle and the target nucleus and \( E \) is the energy of the system. The initial state \( |\psi_o> \) before scattering is

\[ |\psi_o> = |\phi> |0> \]  \hspace{1cm} (2.4)
where $|\phi\rangle$ is the initial state of the alpha particle. The formal solutions of the equation (2.3) are given by Lippmann-Schwinger equations

$$|\psi\rangle = |\psi_0\rangle + \frac{1}{E - H_0 + i\varepsilon} V |\psi\rangle$$

(2.5)

where $H_0 = H_N + H_{\alpha} + K_{\alpha}$

Introducing the Møller wave matrix $\Omega$ we have

$$|\psi\rangle = \Omega |\psi_0\rangle$$

(2.6)

where the wave matrix $\Omega$ is given by

$$\Omega = 1 + \frac{1}{E - H_0 + i\varepsilon} V \Omega$$

(2.7)

and is related to the many-body scattering operator $T$ as

$$T = V \Omega$$

(2.8)

Because we are dealing with a many-body system, the solutions given by the equation (2.5) or (2.6) are many-body wave functions. Kerman, McManus and Thaler (28) have shown that a many-body problem can be reduced to a one-body problem and elastic scattering can be calculated from an optical potential. Following their method, the alpha particle optical potential can be derived as shown below. The many-body scattering operator $T$ is solution of the equation,

$$T = V + V \frac{1}{\alpha} T$$

(2.9)

where

$$\frac{1}{\alpha} = \frac{\mathcal{Q}}{E - H_0 + i\varepsilon}$$

(2.10)

and $\mathcal{Q}$ is the anti-symmetrisation operator for the nuclear states. We take the two-body (alpha particle-target nucleon) scattering operator $\tau$ as a solution of the equation,

$$\tau = v + v \frac{1}{\alpha} \tau$$

(2.11)

where $v$ is the two-body interaction (see equation (2.1)). [We have dropped the subscript $i$.] The choice for the propagator $\frac{1}{\alpha}$ in the
equation (2.11) is rather arbitrary but it is convenient to take the two-body scattering operator $\tau$ as above. In fact, one may choose any other propagator $1/\beta$ but the only restriction on this propagator is

$$<0|\frac{1}{\beta}|0> = 0|\frac{1}{\alpha}|0>$$  \hspace{1cm} (2.12)

The equation (2.12) ensures that we can describe elastic scattering from a single particle potential. Assuming that alpha particle-target neutron and alpha particle-target proton interactions are the same, we have

$$V = AV$$  \hspace{1cm} (2.13)

Using equations (2.9), (2.11) and (2.13) it can be shown that the one-body scattering operator $T'_{oo} = <0|T'|0>$ $(T' = \frac{A-1}{A} T)$ is given by

$$T'_{oo} = U_{oo} + \frac{1}{\alpha} T'_{oo}$$  \hspace{1cm} (2.14)

where

$$\frac{1}{\alpha} = \frac{1}{E-H-K+i\epsilon}$$

$$U_{oo} = <0|U|0>$$

$$= <0|U^0|0> + <0|AU|0>$$  \hspace{1cm} (2.15)

$$= U_{oo} + AU_{oo}$$

$$U^0 = (A-1)\tau$$

$$AU = U^0 \frac{1}{\alpha} QU$$  \hspace{1cm} (2.16)

and $Q$ is the projection operator which projects off the nuclear ground state. We have assumed above that the ground state energy of the nucleus is zero. We can write the equation (2.14) as

$$T'_{oo} = U_{oo} \Omega'_{oo}$$  \hspace{1cm} (2.17)

where

$$\Omega'_{oo} = \frac{1}{\alpha} T'_{oo}$$

and is related to

$$\Omega_{oo} = <0|\Omega|0>$$ as

$$\Omega_{oo} = \frac{A}{A-1} \Omega'_{oo} - \frac{1}{A-1}$$
It can be shown\textsuperscript{(28)} that the scattering state $\Omega^{t}\phi$ is the solution of the one-body Schrödinger equation,
\[ (E - H - K - V)\Omega^{t}\phi = 0 \tag{2.18} \]
The state $\Omega^{t}\phi$ still contains the internal co-ordinates of the alpha particle and if we assume that the internal energy of the alpha particle remains unchanged, we may write,
\[ \Omega^{t}\phi = \chi\xi_{\alpha} \tag{2.19} \]
where $\chi\xi_{\alpha}$ is the internal state of the alpha particle and $\chi\phi$ is the scattering state for the elastic channel. Using equation (2.19) the one-body Schrödinger equation reduces to the form,
\[ (E - K - V_{\alpha}^{0})\chi\phi = 0 \tag{2.20} \]
where
\[ V_{\alpha}^{0} = <\xi_{\alpha}|U_{oo}|\xi_{\alpha}> \tag{2.21} \]
and we have neglected the term $<\xi_{\alpha}|U_{oo}|\xi_{\alpha}>$ which contains multiple scatterings involving the intermediate excited states\textsuperscript{(28)} of the target nucleus. The lowest order potential $V_{oo}^{0}$ is, in general, nonlocal and energy-dependent. In the case of alpha particles, although the alpha particle-target nucleon interaction contains a spin-orbit term, the spin-orbit term in the optical potential averages to zero for spin-zero nuclei.

In the next section we have discussed the effective interaction approximation which we have used to calculate the lowest order potential $V_{oo}^{0}$.

2.2 Effective interaction approximation

We derived in section (2.1) the alpha particle optical potential and further it was shown there that the lowest order potential $V_{oo}^{0}$ is given by
\[ V_{oo}^0 = \langle \xi_\alpha | U_{oo}^0 | \xi_\alpha \rangle \]
\[ = (A-1)\langle 0 | \tau_{aa} | 0 \rangle \]
\[ \tau_{aa} = \langle \xi_\alpha | \tau | \xi_\alpha \rangle \quad (2.23) \]

The potential \( V_{oo}^0 \) given by the equation (2.22) can, in principle, be calculated if the two-body scattering operator \( \tau \), the alpha particle (matter) distribution and the nuclear matter distribution are known. However, the scattering operator \( \tau \) which describes the interaction between a target nucleon and the alpha particle is nonlocal and energy-dependent. Further, it involves the anti-symmetrisation operator \( Q \) and the propagator \( \frac{1}{\alpha} \) (see section (2.1)). The anti-symmetrisation operator \( Q \) takes into account the effect of the exclusion principle and the propagator \( \frac{1}{\alpha} \) which contains the nuclear Hamiltonian \( H_N \), allows for the effect of nuclear binding. Because of this, the scattering operator \( \tau \) differs from the free alpha particle-nucleon scattering operator as determined from the analysis of \( (p,a) \) elastic scattering data \( (30)(31)(64) \). So we cannot replace alpha particle-target nucleon scattering operator \( \tau \) by the free alpha particle-nucleon scattering operator in the energy region of interest in this work, which is around 42 MeV. In view of this difficulty, we follow the procedure adopted in the microscopic description of inelastic scattering of alpha particles \( (23)(24) \), and replace the complex interaction between the alpha particle and a target nucleon by an effective interaction. In particular, we replace the real part of \( \tau_{aa} \) as

\[ R \tau_{aa} = v_{\text{eff}}(r-R) \]
\[ \tau_{aa} = v_{\text{eff}}(r-R) \quad (2.24) \]

where \( v_{\text{eff}} \) is an effective interaction and \( r, R \) are the position vectors of the target nucleon and the centre of mass of the alpha particle from the centre of mass of the target nucleus (see Fig.1).
Fig. 1 Position vector diagram for the alpha-nucleus system.
The effective interaction \( v_{\text{eff}}^{(r-R)} \) takes into account the finite size of the alpha particle but the structure and the polarizibility of the alpha particle are neglected. Further, as remarked earlier, the interaction between the alpha particle and a target nucleon contains the spin-orbit term but this term averages to zero since the alpha particle has spin-zero. Hence, the effective interaction \( v_{\text{eff}}^{(r-R)} \) term is essentially a central interaction. Using the effective interaction approximation (2.24), the real part of the lowest order optical potential \( V_{oo}^{\sigma} \) is given by

\[
\Re V_{oo}^{\sigma} = (A-1)<0|v_{\text{eff}}|0> \tag{2.25}
\]

Since we know from analyses of elastic scattering of medium energy alpha particles using phenomenological optical potentials that the radial behaviour of the imaginary part of the optical potential may be taken to be the same as the real part, we take for \( \Im V_{oo}^{\sigma} \) as

\[
\Im V_{oo}^{\sigma} = \xi(A-1)<0|v_{\text{eff}}|0> \tag{2.26}
\]

where \( \xi \) is a parameter to be determined from fits to data.

We have used the equations (2.25) and (2.26) to calculate the real and imaginary parts of the lowest order optical potential.

However, these calculations also involve a knowledge of the nuclear matter distribution and the effective interaction \( v_{\text{eff}} \). We have discussed in the next section the forms for the effective interaction \( v_{\text{eff}} \) and in section (2.4) the model used for obtaining the nuclear matter distribution.

2.3 Forms for the effective alpha particle-nucleon interaction

Various forms for the effective interaction \( v_{\text{eff}} \) have been used in the analyses of elastic and inelastic scattering of medium energy alpha particles. In the analysis of inelastic scattering of alpha particles at 43 MeV, Madsen and Tobocman\(^{(23)}\) have used single
Yukawa and two Yukawa forms. The single Yukawa interaction has the form

\[ v_{\text{eff}}(s) = v_0 \frac{e^{-\lambda s}}{\lambda s} \]  

(2.27)

where \( s = r-R \), \( r, R \) are the position vectors as explained before (see Fig. 1), and \( v_0 \) and \( \lambda \) are the strength and the range parameters of the effective interaction. Madsen and Tobocman (23) have used various values for the range parameter \( \lambda \) and have found that if the range parameter is small enough the structure of the observed inelastic scattering cross-sections can be obtained using the above form. But if the range parameter is allowed to be greater than unity, a lower radial cut off of 6.45 F is needed in order to obtain the diffraction pattern which otherwise cannot be reproduced. The two Yukawa interaction taken by Madsen and Tobocman (23) has the form

\[ v_{\text{eff}}(s) = v_0 \left( e^{-\lambda_1 s} - e^{-\lambda_2 s} \right) \beta \left( \frac{\lambda_1}{\lambda_2} \right) \]  

(2.28)

where \( \lambda_1, \lambda_2 \) are the range parameters, \( \beta = \lambda_2 / \lambda_1 \), and \( v_0 \) is the strength of the interaction. \( s \) is the same vector as before. They have used the values of \( \lambda_1 = 1.09 \text{ F}^{-1} \), \( \lambda_2 = 1.4 \text{ F}^{-1} \) and \( v_0 = 949.4 \text{ MeV} \) and have found that the experimental data on inelastic scattering of alpha particles at 43 MeV can be explained satisfactorily. The above two Yukawa interaction has the same shape as that of the potential used by Gammel and Thaler (30) beyond about 2F but they differ in magnitude. Morgan and Jackson (24) and Peterson (32) have taken Gaussian form for the effective interaction \( v_{\text{eff}} \). The Gaussian form used by Morgan and Jackson (24) in the coupled-channels calculations of elastic and inelastic scattering of medium energy alpha particles has the form

\[ v_{\text{eff}}(s) = v_0 e^{-K^2 s^2} \]  

(2.29)

where \( v_0 \) and \( K \) are the strength and range parameters, and the vector \( s \)
is the same as before.

The value for the range parameter $K$ used\textsuperscript{(24)} is $0.5\ \text{F}^{-1}$ and the value for $v_o$ is in the region of $-36\ \text{MeV}$. In all the above analyses including that of Madsen and Tobocman\textsuperscript{(23)}, the form for the effective interaction $v_{\text{eff}}$ has been assumed and the two-body parameters are treated as variable parameters. But recently Bernstein\textsuperscript{(47)} has derived alpha particle-nucleon interaction by folding a Gaussian nucleon-nucleon interaction into the alpha particle (matter) distribution. The alpha particle-nucleon interaction so obtained by Bernstein\textsuperscript{(47)} has the Gaussian form whose parameters are very close to those given by Morgan and Jackson\textsuperscript{(24)}.

In the present work, we have used only single Yukawa and Gaussian forms. We attempted to determine range parameters and strengths of these two interactions from the data on phenomenological potentials in the literature relevant to medium energy alpha particles, as explained below. First we note that the volume integral $V_{\text{vol}}$ and the mean square radius $<R^2>_{\text{opt}}$ of the real part of the potential (2.25) are given by (33):

\begin{equation}
V_{\text{vol}} = \int \text{Re}V^O (R) dR \tag{2.30}
\end{equation}

\begin{equation}
= v_{\text{vol}}^{(A-1)}
\end{equation}

\begin{equation}
<R^2>_{\text{opt}} = \frac{\int \text{Re}V^O (R) R^2 dR}{\int \text{Re}V^O (R) dR} \tag{2.31}
\end{equation}

where $v_{\text{vol}}$ is the two-body volume integral

\begin{equation}
v_{\text{vol}} = \int v_{\text{eff}} (R) dR \tag{2.32}
\end{equation}

$<R^2>_{\text{TB}}$ is the two-body mean square radius,
\[ \langle R^2 \rangle_{TB} = \int \frac{V_{\text{eff}}(R)}{V_{\text{eff}}(R)} R^2 dR \]  

(2.33)

and \( \langle R^2 \rangle_{ND} \) is the mean square radius of the nuclear matter distribution \( \rho_m(R) \) (see section (2.4)),

\[ \langle R^2 \rangle_{ND} = \int \frac{\rho_m(R)}{\rho_m(R)} R^2 dR \]  

(2.34)

Introducing the equivalent radius \( R_{\text{eq}} \) of the real part of the potential (2.25) defined as

\[ R_{\text{eq}}^2 = \frac{5}{3} \langle R^2 \rangle_{\text{opt}} \]  

(2.35)

we obtain

\[ R_{\text{eq}}^2 = \frac{5}{3} \langle R^2 \rangle_{ND} + \frac{5}{3} \langle R^2 \rangle_{TB} \]  

(2.36)

But it is well known (1) that the equivalent radii \( \left( \frac{5}{3} \langle R^2 \rangle_{ND} \right)^{1/4} \) of nuclear matter distributions increase approximately as \( A^{1/3} \) and therefore, we can take

\[ \left( \frac{5}{3} \langle R^2 \rangle_{ND} \right)^{1/4} = r_o A^{1/3} \]  

(2.37)

where \( r_o \) is a radius parameter and has the value of 1.25\( \hat{A} \) for calcium isotopes (1).

Hence we have,

\[ R_{\text{eq}}^2 = r_o^2 A^{2/3} + \frac{5}{3} \langle R^2 \rangle_{TB} \]  

(2.38)

It is clear from the equation (2.30) that if we plot the volume \( V_{\text{vol}} \) against \( A^{-1} \), the gradient of the straight line graph gives the two-body volume integral \( V_{\text{vol}} \). Similarly from the equation (2.38), we can obtain the two-body mean square radius \( \langle R^2 \rangle_{TB} \) from the plot of \( \langle R^2 \rangle_{\text{opt}} \) against \( A^{2/3} \). The intercept at \( A=0 \) yields \( \langle R^2 \rangle_{TB} \). This procedure allows us to determine subsequently the range parameter and the strength of the effective interaction \( V_{\text{eff}} \). However, in order to do this, we need to know the volume integrals \( V_{\text{vol}} \) and the equivalent
radii \( R_{eq} \) of the real parts of the optical potentials for several target nuclei. We obtained \( V_{vol} \) and \( R_{eq} \) from the data on phenomenological potentials in the literature. The phenomenological potentials used in the analyses of elastic scattering of medium energy alpha particles, are assumed to be of the Saxon-Woods form,

\[
V(R) = -(V_o + iW_o) \left[ 1 + \exp\left(-\frac{R-R_o}{a}\right) \right]^{-1}
\]  

(2.39)

where \( V_o \) and \( W_o \) are the strengths of the real and the imaginary parts of the phenomenological potential, and \( R_o \) and \( a \) are the half-way radius and the diffuseness parameter. The volume integral \( V_{vol}^p \) and the equivalent radius \( R_{eq}^p \) of the real part of the phenomenological potential (2.39) are given by

\[
V_{vol}^p = \frac{4\pi}{3} V_o R_o^3 \left( 1 + \pi \frac{a^2}{R_o^2} \right)
\]

(2.40)

\[
(R_{eq}^p)^2 = R_o^2 + \frac{7}{3} \pi a^2
\]

(2.41)

We have used the values for \( V_o, R_o, \) and \( a \) listed in the literature for various target nuclei from the analyses of elastic scattering of 42 MeV alpha particles and have obtained \( V_{vol}^p \) and \( R_{eq}^p \) as given by the above equations. We may regard the values of \( V_{vol}^p \) and \( R_{eq}^p \) so obtained as 'experimental' values and use this information to test the A-dependence of the volume integral \( V_{vol}^p \) and the equivalent radius \( R_{eq}^p \) (or the mean square radius) predicted by the microscopic model. Fig.2 shows the mean square radii \( \langle R_p^2 \rangle \) of the real part of the phenomenological potential as a function of \( A^{2/3} \) and in Fig.3, the volume integrals \( V_{vol}^p \) are plotted against \( A^{-1} \). From these plots, we have obtained the following average values for \( V_{vol} \) and \( \langle R^2 \rangle_{TB} \),

\[
V_{vol} = -822.2 \text{ MeV F}^3
\]

\[
\langle R^2 \rangle_{TB} = 7.0 \text{ F}^2
\]
\[ <R_p^2> \]
\[ (F^2) \]

Fig. 2 Plot of the mean square radii of the phenomenological optical potentials against \( A^{1/3} \).
Fig. 3  Plot of the volume integrals of the phenomenological optical potentials against $A-1$. 

$$V_{\text{Vol}}^p \times 10^3 \quad (\text{MeV} \cdot \text{fm}^3)$$ 

$A-1$
We have attempted to determine the range parameter and the strength of the effective interaction from the above values for $\nu_{\text{vol}}$ and $<R^2>_{\text{TB}}$. However, the values for the range parameter and the strength depend on the form taken for $\nu_{\text{eff}}$ and we have discussed below single Yukawa and Gaussian forms which we have taken in the present work.

### 2.3.1 Single Yukawa and Gaussian forms

It can be easily shown that the single Yukawa form (see equation (2.27)) has the volume integral,

$$\nu_{\text{vol}} = \frac{4\pi \nu_0}{\lambda^3}$$

and the mean square radius,

$$<R^2>_{\text{TB}} = \frac{6}{\lambda^2}$$

Using the average values for $\nu_{\text{vol}} = -822.2$ MeV$^3$ and $<R^2>_{\text{TB}} = 7.0$ F$^2$ determined above, we obtain,

$$\lambda = 0.96 \text{ F}^{-1}$$

$$\nu_0 = -58 \text{ MeV}$$

Similarly for the Gaussian form (see equation (2.29)), we get

$$\nu_{\text{vol}} = \frac{\pi^{3/2} \nu_0}{K^3}$$

and

$$<R^2>_{\text{TB}} = \frac{1.5}{K^2}$$

As before, the two-body parameters are obtained as

$$K = 0.475 \text{ F}^{-1}$$

$$\nu_0 = -16 \text{ MeV}.$$ 

The above values for the two-body parameters for the two cases are those predicted by the relations (2.30) and (2.38) which follow from the microscopic description. In the case of the single Yukawa form, we first used the values of $\lambda = 0.96 \text{ F}^{-1}$ and $\nu_0 = -58 \text{ MeV}$ for
the range parameter and the strength respectively in order to test whether these predicted values would reproduce the structure of the observed elastic scattering cross-sections for 42 MeV alpha particles\(^{(32)}\). But it was found that the values of \(\lambda=0.96F^{-1}\) and \(\nu_o=-58\) MeV produced diffraction patterns which were different from those observed for calcium isotopes. The shape of the elastic scattering cross-section was found to be very sensitive to the value of the range parameter \(\lambda\) used. This meant that a much more accurate determination of the range parameter was required but it was not possible to do so because of the ambiguities in the optical potential for alpha particles which cause the mean square radius of the potential to be a function of the strength. In view of this difficulty, we attempted to fit the observed elastic scattering cross-sections by varying the range parameter \(\lambda\) and the strength \(\nu_o\) around the average values of \(\lambda=0.96F^{-1}\) and \(\nu_o=-58\) MeV. In this way, we obtained values of \(\lambda=0.8F^{-1}\) and \(\nu_o=-50\) MeV from fits to data on calcium isotopes. However, it was found that the deep minima in the observed elastic scattering cross-sections could not be produced by the single Yukawa interaction. This is further discussed in Chapter 3. We then used the Gaussian form for the effective interaction in order to see whether elastic scattering cross-sections can be explained by this choice of the effective interaction. However, we did not follow the above procedure and use the average values of \(K=0.5F^{-1}\) and \(\nu_o=-16\) MeV for the two-body parameters, since it was obvious from the case of the single Yukawa interaction that such a procedure was inadequate because of the ambiguities in the optical potential for alpha particles. This is confirmed by the recent work\(^{(65)}\) where it has been noted that there is no requirement for the constancy of the volume integral \(v_{vol} = \frac{3}{2} \lambda^2 \rho K^{-3}\) for the Gaussian form for the
effective alpha-nucleon interaction. Therefore, for the Gaussian interaction we used the values of $K = 0.5 \text{ F}^{-1}$ and $v_o = -35.6 \text{ MeV}$ given by Morgan and Jackson. It was found that the observed cross-sections for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$, $^{48}\text{Ca}$ could be explained satisfactorily by the Gaussian interaction but the value for the strength $v_o$ had to be adjusted slightly in order to obtain the correct magnitudes of the cross-sections except in the case of $^{42}\text{Ca}$. A further discussion of this is given in Chapter 3.

The single Yukawa and the Gaussian forms for effective interaction which we have used are compared below with the forms taken in the previous analyses of elastic and inelastic scattering of alpha particles, and we have also included in the comparison the free interaction of Sack, Biedenharn and Breit to see how the effective interactions compare with the free interaction.

2.3.1 Comparison of forms for the effective interaction

As mentioned in section (2.3), different forms for the effective interaction have been used in the analyses of elastic and inelastic scattering of alpha particles. Analyses of inelastic scattering of alpha particles have used the distorted-wave Born approximation where the interaction between a target nucleon and the alpha particle is replaced by an effective interaction. The wave functions for elastic scattering of the alpha particle which are required in these analyses are taken to be those obtained from best fits to elastic scattering of data using phenomenological optical potentials. The two-body parameters of the effective interaction are adjusted so as to obtain fits to inelastic scattering data. It has been pointed out by Jackson that the two-body parameters of the effective interaction obtained from fits to inelastic scattering data on alpha particles
should be consistent with those determined from the microscopic analysis of elastic scattering of alpha particles. Therefore, it is of interest to compare the different forms for the effective interaction used in the above analyses and to examine whether there is a reasonable consistency in the treatment of elastic and inelastic scattering of alpha particles as suggested by Jackson (21). Fig. 4 shows the different forms for the effective interaction used in the analyses of elastic and inelastic scattering of medium energy alpha particles and in the present work. Since in the effective interaction approximation, an involved two-body operator (see section (2.2)) is replaced by the effective interaction \( v_{\text{eff}} \), we get from Fig. 4 some indication of the behaviour of this two-body operator for which we have assumed simple forms (single Yukawa and Gaussian) in the present work. But this should not be taken seriously because multiple scatterings from the off-diagonal terms have been neglected. The two Yukawa interaction of Madsen and Tobocmann (23) and the Gaussian interaction used by Morgan and Jackson (24) and also, in the present work peak (see Fig. 4) at about 2\( R \). These two interactions are similar in shape but differ in magnitude. The single Yukawa interaction used in the present work differs substantially both from the two Yukawa and the Gaussian interactions. The single Yukawa interaction has a long tail and gives, when folded into nuclear matter distribution, optical potentials with rather large diffuseness. The range parameter \( \lambda \) and the strength \( v_0 \) of the single Yukawa interaction used in the calculation of optical potentials for \(^{40}\text{Ca}^+\), \(^{42}\text{Ca}^+\), \(^{44}\text{Ca}^+\) and \(^{48}\text{Ca}^+\) have the average values of 0.8\( R \) and ~50 MeV and we have used these values in Fig. 4.

As can be seen from Fig. 4, all these three interactions, (single Yukawa, two Yukawa and Gaussian) differ in magnitude and since the volume integrals are given by the areas under the curves in Fig. 4,
Fig. 4 Comparison for the forms for effective interactions. We have also shown the free interaction of Sack et al (see section 2.3.1)
they have all different values for $v_{\text{vol}}$. We have calculated the mean square radius $<R^2>_{TB}$ for each case and it is found that the Gaussian interaction of Morgan and Jackson (24) and used in the present work has a mean square radius smaller than that of either the single Yukawa or the two Yukawa interaction.

We have also shown in Fig.4 the free alpha-nucleon potential of Sack et al (31) which fits the alpha-nucleon elastic scattering at low energies. Sack et al (31) have used a Gaussian form with values of $k=0.435F^{-1}$ and $\nu_0=-50\,\text{MeV}$ for the range parameter and the strength respectively. Since the effective interaction is an approximation to the two-body (target nucleon-alpha) scattering operator, we should really compare it with the free alpha-nucleon scattering operator. However, it has been pointed out by Madsen and Tobocman (23) that the Born matrix element of the free alpha-nucleon potential is not a bad approximation to the alpha-nucleon elastic scattering and therefore, a comparison of the Gaussian potential of Sack et al (31) with the effective interactions is not altogether unrealistic. We see from Fig.4 that the effective interactions differ substantially from the Gaussian potential of Sack et al (31). This is not surprising in view of the fact that the alpha particle can scatter a target nucleon bound inside the target nucleus only according to the exclusion principle and hence, the effective interaction which simulates the effects of the exclusion principle, is expected to be different from the free interaction.

In Table I, we have presented two-body volume integrals $v_{\text{vol}}$ for the single Yukawa, two Yukawa and the Gaussian forms for the effective interaction. Further, we have also included in Table I the values of $v_{\text{vol}}$ for the free interactions of Sack et al (31) and Satchler (64) et al. Since the strength of the optical potential depends on the two-body volume integral, it is of interest to compare these volume integrals.
## TABLE I

Two-body volume integrals

<table>
<thead>
<tr>
<th>Reference</th>
<th>Alpha-nucleon interaction</th>
<th>$v_{vol}$ (MeV(\cdot)F(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Madsen &amp; Tobocman</td>
<td>Two Yukawa</td>
<td>-3628</td>
</tr>
<tr>
<td>Morgan &amp; Jackson</td>
<td>Gaussian</td>
<td>-1586</td>
</tr>
<tr>
<td>(also present work)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present work</td>
<td>Single Yukawa*</td>
<td>-1227</td>
</tr>
<tr>
<td>Satchler et al</td>
<td>Free</td>
<td>-2612–2885†</td>
</tr>
<tr>
<td>Sack et al</td>
<td>Free</td>
<td>-3200</td>
</tr>
</tbody>
</table>

* $v_o = -50$ MeV $\lambda = 0.8 F^{-1}$

† volume integral of the central part
in order to see what value for the strength of the optical potential is expected from each of the above forms for the effective interaction. The strength of the optical potential is very close to the value given by the product $v_{\text{vol}}\rho_o$ where $\rho_o$ is the central nuclear density. It is clear from Table I that the volume integral for the Gaussian interaction has larger value than that for the single Yukawa interaction. Thus, the Gaussian interaction produces deep optical potentials whose strengths are in the region of 300 MeV for calcium isotopes whereas the single Yukawa interaction gives less deep optical potentials having strengths less than 200 MeV. However, as can be seen from Table I, these two volume integrals are considerably smaller than those obtained for the free interactions of Sack et al $(31)$ and Satchler et al $(64)$ (at low energies), and the two Yukawa interactions of Madsen and Tobocman $(23)$. The two Yukawa interaction $(23)$, because of the largest value for $v_{\text{vol}}$ (see Table I), is expected to produce very deep potentials having strengths in the region of 600-700 MeV for calcium isotopes, but recent analysis of elastic scattering of alpha particles at 42 MeV (Jackson, private communication, 1970) indicates that these strengths are in the region of 150 MeV. However, we have not used this interaction in the present work.

2.4 Nuclear matter distribution

The calculation of the lowest order optical potential given by the equations $(2.25)$ and $(2.26)$ requires, in addition to knowing the forms for the effective interaction, the knowledge of the nuclear matter distribution. As mentioned in Chapter 1, the nuclear matter distribution $\rho_m(r)$ is defined as

$$\rho_m(r) = \rho_p(r) + \rho_n(r)$$

$(2.41)$
where, as before, $\rho_p(x) , \rho_n(x) $ are the proton and neutron distributions.

$$\rho_p(x) = \langle 0 | \sum_{i=1}^{Z} \delta(x-x_i) | 0 \rangle$$

$$\rho_n(x) = \langle 0 | \sum_{i=1}^{N} \delta(x-x_i) | 0 \rangle$$

$|0\rangle$ denotes the ground state of the target nucleus and we have used the normalisation,

$$\int \rho_m(x) \, dx = 1$$

In electron scattering, the proton distribution $\rho_p(x)$ is taken to be spherically symmetric and further, is usually assumed to be a phenomenological Fermi distribution given by

$$\rho_p(x) = \rho_0 [1 + \exp \left( \frac{x-R_0}{a} \right)]^{-1}$$

where $R_0$ and $a$ are the halfway radius and the diffuseness parameter. $\rho_0$ is obtained from the normalisation condition. Parameters $R_0$ and $a$ are determined from best fits to data on elastic electron scattering.

A Fermi distribution has also been taken in the work on muonic X-rays and in the analysis of nucleon-nucleus scattering by Greenless et al. Elton and Swift have recently analysed elastic electron scattering using single particle wave functions. In the single particle description of the nucleus, it is assumed that the interaction between a single nucleon and the remaining nucleons in the target nucleus can be represented by an effective one-body potential. Elton and Swift have taken the total ground state wave function $\psi_A$ of the target nucleus to be given by

$$\psi_A = \det \psi_i(x)$$

where $\psi_i(x)$ are single particle wave functions.

The proton distribution $\rho_p(x)$ is then given by

$$\rho_p(x) = \sum_{i=1}^{Z} |\psi_i(x)|^2$$

(2.42)
The single particle wave functions $\psi_i$ satisfy the Schrödinger equation

\[(T+V)\psi_i = \epsilon_i \psi_i \quad (2.43)\]

where the kinetic energy operator $T$ and the single particle potential $V$ refer to co-ordinates relative to the centre of the (single particle) potential. In the expression for $T$, the reduced nucleon mass $m(A-1)/A$ should be used.

It may be noted that functions $\psi_i$ contain in general the centre of mass of motion but for nuclei as heavy as calcium isotopes, the centre of mass motion may be ignored. It has been argued by Elton and Swift\(^{(1)}\) that the single particle wave functions $\psi_i$ obtained by fitting the eigenenergies $\epsilon_i$ to the mean separation energies as measured in poor resolution ($p,2p$) or ($e,e',p$) experiments are a good approximation to the single particle wave functions as given by the Hartree-Fock theory. Elton and Swift\(^{(1)}\) have taken for the single particle potential $V$ in the equation (2.43), the following form,

\[V(r) = -V_c f(r) + V_{\gamma\varepsilon} \left(\frac{\hbar}{Mmc}\right)^2 \frac{1}{r} \frac{df}{dr} \ell.\sigma + V_c \quad (2.44)\]

where $f(r)$ is given by

\[f(r) = \left[1 + \exp \left(\frac{r-R}{a}\right)\right]^{-1}\]

and

\[R = r_0 (A-1)^{1/3}\]

The first term on the right hand side is the central potential and the second term is the spin-orbit potential usually identified as the Thomas term. The third term $V_c$ is the Coulomb potential which is taken to be due to a uniformly charged sphere having root mean square radius as that of the target nucleus. The strengths of the central and spin-orbit parts, $V_c$ and $V_{\gamma\varepsilon}$, are taken to be different for the different shells by Elton and Swift\(^{(1)}\) in view of the fact that the single particle
potential (2.44) is, in general, expected to be nonlocal or, equivalently, 
energy-dependent (1). By adjusting the parameters of the potential (2.44) 
so as to fit eigenenergies ε 1 to proton separation energies obtained 
from poor resolution (p,2p) and similar experiments, Elton and Swift (1) 
have obtained the single particle wave functions ψ 1 for various nuclei 
in which the last shell is either closed or else consists of one 
particle, and they have shown that elastic electron scattering can be 
successfully explained using these single particle wave functions.

In the present work, we have followed the method of Elton and 
Swift (1) and calculated numerically both proton and neutron distributions 
for ⁴₀Ca, ⁴⁴Ca and ⁴⁸Ca from single particle potentials given by them. 
In the case of neutrons, the Coulomb potential in the equation (2.44) 
is switched off and the single particle wave functions are obtained 
from fitting eigenenergies ε 1 to neutron separation energies obtained 
from (p,d) and other experiments. We have also obtained proton and 
neutron distributions for ⁴²Ca from single particle potentials. However, 
for ⁴²Ca, single particle potential well parameters were not available 
and because of this, we have taken potential well parameters for ⁴²Ca 
to be the same as those for ⁴⁴Ca but have adjusted the depth of the 
single particle potential for the 1f ⁷/₂ neutrons to give the correct 
separation energy binding energy of the last neutron. In all the 
above calculations of proton and neutron distributions for ⁴₀Ca, ⁴²Ca, 
⁴⁴Ca and ⁴⁸Ca, we have neglected the residual two-body interactions 
as in the work of Elton and Swift (1), and have taken the point of view that the single particle description of the nucleus as given 
by the equation (2.43) is reasonably valid.

We have given in Table II single particle potential well parameters 
for ⁴₀Ca, ⁴²Ca, ⁴⁴Ca and ⁴⁸Ca used in the present calculations, and have
### TABLE IX

**Single particle potential well parameters for calcium isotopes**

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Level</th>
<th>For protons</th>
<th>For neutrons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$V_c$ (MeV)</td>
<td>$V_{sc}$ (MeV)</td>
</tr>
<tr>
<td>$^{40}\text{Ca}$</td>
<td>$1s^{1/2}$</td>
<td>85</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>$1p^{3/2}$</td>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$1p^{1/2}$</td>
<td>53</td>
<td>12</td>
</tr>
<tr>
<td>$^{42}\text{Ca}$</td>
<td>$1s^{1/2}$</td>
<td>85</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>$1p^{3/2}$</td>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$1p^{1/2}$</td>
<td>55</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>$1d^{5/2}$</td>
<td>46</td>
<td>7.6</td>
</tr>
<tr>
<td>$^{44}\text{Ca}$</td>
<td>$1s^{1/2}$</td>
<td>85</td>
<td>1.26</td>
</tr>
<tr>
<td></td>
<td>$1p^{3/2}$</td>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$1p^{1/2}$</td>
<td>57</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>$1d^{5/2}$</td>
<td>44</td>
<td>8.0</td>
</tr>
</tbody>
</table>

* These parameters are due to Elton and Swift, except for $^{42}\text{Ca}$. 
Fig. 5  Nuclear matter distribution for $^{40}_{\text{Ca}}$, calculated from single particle wave functions.
Fig. 6  Nuclear matter distribution for $^{42}_{\text{Ca}}$, as Figure 5.
Fig. 7  Nuclear matter distribution for \(^{44}\text{Ca}\), as Figure 5.
Fig. 8 Nuclear matter distributions for $^{48}\text{Ca}$, calculated from single particle wave functions, as explained in section 2.4
shown in Figs. 5-8 nuclear matter distributions for these nuclei as obtained within the framework of the single particle description of the nucleus. For $^{48}\text{Ca}$, we have used two neutron distributions. In the first case, neutron distribution is obtained as described above by fitting eigenenergies $\varepsilon_i$ to neutron separation energies and in the second case, neutron distribution is generated using proton well parameters and switching off the Coulomb potential. Although the latter neutron distribution does not correspond to the case where eigenenergies $\varepsilon_i$ fit neutron separation energies obtained from experiment, this is done in order to see whether elastic alpha particle scattering is sensitive to the change in the neutron distribution. The corresponding nuclear matter distribution is also shown in Fig. 8 and it can be seen from Fig. 8 that the two nuclear matter distributions (1 and 2 refer to the first and second cases) differ substantially. A further discussion of the sensitivity of elastic alpha particle cross-sections to changes in nuclear matter distributions is given in Chapter 3.

In section (2.5) we have presented optical potentials for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$, $^{48}\text{Ca}$ which are calculated using nuclear matter distributions as obtained within the framework of the single particle description of the nucleus.

2.5 Optical potentials for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$, $^{48}\text{Ca}$

The nuclear matter distribution constructed from the single particle wave functions as explained in section (2.4), and the forms for effective two-body (alpha-target nucleon) interactions described in section (2.3) allow us to calculate optical potential $V^0_{oo}$ in lowest order given by the equations (2.25) and (2.26) (see section (2.2)). Since we have assumed that the imaginary part of the optical potential $V^0_{oo}$ has the same radial behaviour as the real part, we need only
evaluate the real part $\text{Re}V^O_{\infty}$. The equation (2.25) (see section (2.2))
gives for the real part $\text{Re}V_{\infty}$,

$$\text{Re}V^O_{\infty} = \langle 0 | v_{\text{eff}} | 0 \rangle$$

$$= (A-1) \int \rho_m(r) v_{\text{eff}}(r-R) \, dr$$

where $\rho_m(r)$ is the nuclear matter distribution as defined before in
section (2.4) and $v_{\text{eff}}(r-R)$ is the effective two-body interaction
discussed in section (2.3).

We have used the Gaussian and single Yukawa forms for the
effective interaction $v_{\text{eff}}(r-R)$ and numerically calculated $\text{Re}V^O_{\infty}$.
For the Gaussian form for the effective two-body interaction $v_{\text{eff}}$, we
have used a value of $K=0.5 F^{-1}$ for the range parameter. The strengths
$v_o$ of the effective two-body interaction required to fit the elastic
scattering data on $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$ for 42 MeV alpha particles
varied slightly from the value of $v_o=-35.6$ MeV given by Morgan and
Jackson(24), except in the case of $^{42}\text{Ca}$. We have given in Table III
the two-body parameters $K$ and $v_o$ used in the present work and also
the strengths of $\text{Re}V^O_{\infty}$ and $\text{Im}V^O_{\infty}$ for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$,
obtained from the best fits to the data. For the single Yukawa
interaction, the average values of $\lambda=0.8 F^{-1}$ for the range parameter
and $v_o=-50$ MeV for the strength have been used. However, for $^{40}\text{Ca}$
it was found necessary to vary very slightly the range parameter $\lambda$ from
the average value of $0.8 F^{-1}$ in order to obtain the correct spacing of
the diffraction minima in the cross-sections (see Chapter 3). The
two-body parameters $\lambda$ and $v_o$ and the strengths $\text{Re}V^O_{\infty}$ and $\text{Im}V^O_{\infty}$ obtained
in the present work are listed in Table IV.

The real parts of the optical potential $V^O_{\infty}$ calculated from the
equation (2.45) using the two-body parameters mentioned above for the
two cases are shown in Figs. 9-16. It can be seen from Figs. 9-12 and
### TABLE III

The strengths of potentials $V_{oo}^0$ obtained using the Gaussian interaction

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Range parameter $K(F^{-1})$</th>
<th>$V_o$ (MeV)</th>
<th>$ReV_{oo}^0$ (MeV)</th>
<th>$ImV_{oo}^0$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{40}\text{Ca}$</td>
<td>0.5</td>
<td>-40.7</td>
<td>-274.6</td>
<td>-84.6</td>
</tr>
<tr>
<td>$^{42}\text{Ca}$</td>
<td>0.5</td>
<td>-35.6</td>
<td>-235.4</td>
<td>-134.2</td>
</tr>
<tr>
<td>$^{44}\text{Ca}$</td>
<td>0.5</td>
<td>-50.0</td>
<td>-337.6</td>
<td>-132.6</td>
</tr>
<tr>
<td>$^{48}\text{Ca}$</td>
<td>0.5</td>
<td>-38.4</td>
<td>-255.0</td>
<td>-108.9</td>
</tr>
</tbody>
</table>

### TABLE IV

The strengths of potentials $V_{oo}^0$ obtained using the single Yukawa interaction

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Range parameter $\lambda(F^{-1})$</th>
<th>$V_o$ (MeV)</th>
<th>$ReV_{oo}^0$ (MeV)</th>
<th>$ImV_{oo}^0$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{40}\text{Ca}$</td>
<td>0.8</td>
<td>-50.0</td>
<td>-171.0</td>
<td>-34.2</td>
</tr>
<tr>
<td></td>
<td>0.76</td>
<td>-50.0</td>
<td>-192.9</td>
<td>-38.6</td>
</tr>
<tr>
<td>$^{42}\text{Ca}$</td>
<td>0.8</td>
<td>-50.0</td>
<td>-167.4</td>
<td>-33.5</td>
</tr>
<tr>
<td>$^{44}\text{Ca}$</td>
<td>0.8</td>
<td>-50.0</td>
<td>-165.8</td>
<td>-33.2</td>
</tr>
<tr>
<td>$^{48}\text{Ca}$</td>
<td>0.8</td>
<td>-50.0</td>
<td>-166.9</td>
<td>-33.4</td>
</tr>
<tr>
<td></td>
<td>0.84</td>
<td>-50.0</td>
<td>-148.0</td>
<td>-29.6</td>
</tr>
</tbody>
</table>
Fig. 9 Folded potential for $^{40}\text{Ca}$, obtained using the single Yukawa interaction.
Fig. 10 Folded potential for $^{42}\text{Ca}$, as Figure 9.
Fig. 11 Folded potential for $^{44}$Ca, as Figure 9.
Folded potential for $^{48}\text{Ca}$, as Figure 9 and using distribution 1 shown in Fig. 8.
Table IV that the single Yukawa interaction produces potentials whose strengths are in the region of 200 MeV for $^{40}$Ca, $^{42}$Ca, $^{44}$Ca and $^{48}$Ca. Blair and Fernandez\(^{(12)}\) have recently analysed the elastic scattering data on the calcium isotopes for 42 MeV alpha particles, using the phenomenological optical potential and in this analysis, they have used deep potentials in the region of 200 MeV. Although, the potentials produced by the single Yukawa interaction have strengths which are close to those used by Blair and Fernandez\(^{(12)}\), the single Yukawa interaction gives too diffuse potentials (see Figs. 9-12). Because of this, the single Yukawa interaction does not produce the pronounced diffraction minima in the elastic scattering data. This is further discussed in Chapter 3. The Gaussian interaction used here produces potentials (see Figs. 13-16) which have strengths in the region of 300 MeV for $^{40}$Ca, $^{42}$Ca, $^{44}$Ca, $^{48}$Ca and these potentials have the required behaviour in the surface region.

We find that the potentials produced by the Gaussian interaction can satisfactorily explain the diffraction pattern of the observed elastic scattering cross-sections (see Chapter 3). It has also been observed by Morgan and Jackson\(^{(24)}\) who have analysed elastic data on $^{42}$Ca using the coupled-channels method, the Gaussian interaction produces theoretical cross-sections which are in satisfactory agreement with experimental data.

In the following section, we have compared the folded potentials discussed here with the phenomenological potentials used by Jackson and Morgan\(^{(10)}\).

2.5.1 Comparison between folded and phenomenological potentials

Phenomenological optical potentials have been successfully used by Blair and Fernandez\(^{(12)}\), Jackson and Morgan\(^{(10)}\) and many other authors in the analyses of elastic scattering of alpha particles. The phenomenological optical potential is usually taken to be of Saxon-Woods form mentioned...
Fig. 13 Folded potential for $^{40}\text{Ca}$ obtained using the Gaussian interaction

$V_0 = -4.7 \text{ MeV}$

$K = 0.5 \text{ F}^{-1}$
Fig. 14 Folded potential for $^{42}\text{Ca}$, as Figure 13.
Fig. 15  Folded potential for $^{44}\text{Ca}$, as Figure 13
Fig. 16  Folded potential for $^{48}$Ca, as Figure 13.
earlier in section (2.3) and the real part of the potential is given by

\[ V(R) = -V_0 \left[ 1 + \exp\left(\frac{R-R_0}{a}\right) \right]^{-1} \]  

The imaginary part of the phenomenological potential is assumed to have the same radial behaviour as the real part. The well parameters \( V_0, R_0 \), and \( a \) are obtained from best fit to elastic scattering data. Since the theoretical cross-sections calculated from the phenomenological potential \( V(R) \) are in good agreement with the experimental data, we may take the point of view that the average potential which the incident alpha particle sees inside the target nucleus is reasonably well described by the phenomenological Saxon-Woods potential given by equation (2.46).

It is of interest therefore to compare the folded potentials which we have calculated in the present work using the microscopic description with the phenomenological potentials used in the analyses of elastic scattering. However, as is well known, the ambiguities which exist in the phenomenological potentials for strongly absorbed particles, such as the alpha particle, do not permit easy comparison. In view of this, we have restricted ourselves to phenomenological potentials which have strengths comparable to those of the folded potentials obtained here and we have shown in Fig. 17 the phenomenological potentials of Jackson and Morgan \(^{(10)}\) and the folded potentials for \(^{42}\)Ca. The folded potential for \(^{42}\)Ca produced by the single Yukawa interaction is rather too diffuse and differs substantially in shape from the phenomenological potentials of Jackson and Morgan \(^{(10)}\) which fit elastic scattering data or \(^{42}\)Ca at \(42\) MeV. But the folded potential for \(^{42}\)Ca produced by the Gaussian interaction has the required shape in the surface region and is in reasonable agreement, in the surface region, with the phenomenological potentials of Jackson and Morgan \(^{(10)}\). It appears, therefore, that the Gaussian form for the effective two-body interaction
Fig. 17 Comparison of folded potentials and phenomenological potentials.
is satisfactory whereas the single Yukawa form is far from adequate. Since the effective two-body interaction is an approximation to the two-body scattering \( \tau \) (see section (2.2)), it may be remarked that the behaviour of the two-body scattering \( \tau \) is reasonably well described rather by the sharply peaked Gaussian interaction than the single Yukawa interaction which has a long tail (see Fig.4).

### 2.6 The Schrödinger equation for alpha particles

Within the framework of the microscopic description of elastic scattering, we obtain, as shown in section (2.1), the one-body Schrödinger equation given by the equation (2.20). The lowest order potential \( V_{oo}^0 \) obtained in the effective interaction approximation (see section (2.2)) and given by the equation (2.45) (section (2.5)), is local in configuration space and using this potential \( V_{oo}^0 \), the one-body Schrödinger equation becomes,

\[
\left( \frac{\hbar^2}{2\mu} v^2 + E - V_{oo}^0 - V_c \right) <R|\chi> = 0
\]

where we have included the Coulomb potential \( V_c \) to take into account the fact that the alpha particle is charged and \( \mu \) is the reduced mass of the alpha particle given by

\[
\mu = m \left( \frac{A_\alpha A}{A_{\alpha} + A} \right)
\]

\( A_\alpha \) is the mass number of the alpha particle and \( m \) is the nucleon mass. The equation (2.47) describes the elastic scattering of alpha particles from the optical potential \( V_{oo}^0 \) in the centre of mass of the alpha particle-nucleus system. The scattering wave function \( <R|\chi> \) can be expanded in terms of partial waves and it is easy to show that the equation (2.47) reduces to the following radial equation,

\[
\left( \frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} + E - V_{oo}^0 - V_c - \frac{\ell(\ell+1)}{R^2} \right) u_\ell(R) = 0
\]
where \( u_{l}(R) \) is a radial wave function.

The optical potential \( V_{oo}^{0} \) is given by

\[
V_{oo}^{0}(R) = (1+i\xi) \int \rho_{m}(r) v_{\text{eff}}(r-R)dr
\]  

(2.49)

\( \xi \) is the strength parameter of the imaginary part of the optical potential to be determined from best fit to experimental data. \( V_{c} \) the Coulomb potential is taken to have the usual form due to a uniformly charged sphere of radius \( R_{c} \) is given by

\[
V_{c}(R) = \frac{Z_{c}Z_{\alpha}}{2R_{c}} \left[ 3 - \left( \frac{R}{R_{c}} \right)^{2} \right] , \quad R \leq R_{c}
\]

\[
= \frac{Z_{c}Z_{\alpha}}{R} , \quad R > R_{c}
\]  

(2.49.1)

where \( R_{c} = r_{0} A^{1/3} \), and \( Z_{c} \) and \( Z_{\alpha} \) are the charges of the alpha particle and the target nucleus. We have numerically calculated the radial wave functions \( u_{l}(R) \) from equation (2.48) using the optical potential \( V_{oo}^{0} \) given by equation (2.49), and we have obtained the phase shifts by matching the radial wave functions \( u_{l}(R) \) to those given by

\[
\left( \text{for large } R \right)
\]

\[
\left[ \frac{\hbar^{2}}{2\mu} \frac{d^{2}}{dR^{2}} + E - V_{c} - \frac{\ell(\ell+1)}{R^{2}} \right] u_{l}(R) = 0
\]  

(2.50)

The equation (2.50) is the usual non-relativistic Coulomb equation and two independent solutions are \( F_{\ell}(R) \) and \( G_{\ell}(R) \) which have the asymptotic forms,

\[
F_{\ell}(R) \sim \sin(kR - \frac{\ell\pi}{2} + \sigma_{\ell} - \gamma \log(2kR))
\]

\[
G_{\ell}(R) \sim \cos(kR - \frac{\ell\pi}{2} + \sigma_{\ell} - \gamma \log(2kR))
\]

\( \sigma_{\ell} \) is the Coulomb phase given by

\[
\sigma_{\ell} = \arg \Gamma(\ell+1+i\gamma)
\]

and \( \gamma \) is the Coulomb parameter,
$k$ is the wave number corresponding to the centre of mass energy $E$ of the incident alpha particle. The two independent Coulomb functions $F_L^R(R)$ and $G_L^R(R)$ are obtained numerically from the equation (2.50). Once, the functions $F_L^R(R)$, $G_L^R(R)$ and $\psi_L^R(R)$ are calculated numerically, the phase shifts $\delta_L^R$ can be obtained from the boundary conditions (for large $R$)

$$u_L^R(R) = A_L^R F_L^R(R) + B_L^R G_L^R(R)$$  (2.51)

and

$$u_L'(R) = A_L^R F_L'(R) + B_L^R G_L'(R)$$

where the primes denote derivatives with respect to $R$ and $u_L^R(R)$ and $u_L'(R)$ are calculated from equation (2.48). Since the radial wave functions $u_L^R(R)$ have the asymptotic behaviour,

$$u_L^R(R) \sim \sin(\theta_L^R + \delta_L^R)$$

where

$$\theta_L^R = kR - \frac{\pi}{2} + \sigma_L^R - \gamma \log(2kR)$$

the equations (2.51) take the form,

$$u_L^R(R) = A_L^R (F_L^R(R) + \tan\delta_L^R G_L^R(R))$$

$$u_L'(R) = A_L^R (F_L'(R) + \tan\delta_L^R G_L'(R))$$  (2.52)

It is easy to show from the equations (2.52) that the phase shifts are given by

$$\tan\delta_L^R = -\frac{(F_L' - Y_L^R F_L)}{(G_L' - Y_L^R G_L)}$$  (2.53)

where $Y_L^R = u_L'(R)/u_L^R(R)$.

The phase shifts $\delta_L^R$ given by the equation (2.53) are complex because of the presence of the imaginary part of the optical potential $V_0^o$ in the radial equation (2.48). We have calculated the elastic scattering cross-sections for 42 MeV alpha particles using the phase shifts $\delta_L^R$ given by equation (2.53) and the results for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$.
and $^{48}$Ca are presented in Chapter 3.

In the next section, we have discussed strong absorption radii for alpha particles which can be determined from the analysis of elastic scattering of alpha particles.

2.7 Strong absorption radii

From the various analyses (10)(16)(18)(19) of elastic scattering of strongly absorbed particles, such as the alpha particle, it is known that ambiguities in the optical potential exist and because of the ambiguities which cause the equivalent radius to be a function of the strength of the optical potential, the well parameters such as the half-way radius and the diffuseness of the phenomenological optical potential, usually taken to be of the Saxon-Woods form, cannot be regarded as significant nuclear size parameters, in contrast to the case of nucleon-nucleus scattering, where nuclear size parameters can be extracted from the proton (neutron) optical potentials (2). In view of this difficulty, we follow the method of Blair (13) and define the strong absorption radius as given by the classical turning point where $d^2u_R/dR^2$ vanishes. From this definition of the strong absorption radius, we have from the equation (2.50)

$$E - V_c - \frac{\ell(\ell+1)}{R^2} = 0 \quad (2.54)$$

Since $E=\hbar^2k^2/2\mu$ and $V_c = \frac{ZqZe^2}{R}$ (at the classical turning point) it follows that

$$kr = \gamma + \left[\gamma^2 + L(L+1)\right]^{1/2}$$

where $\gamma$ is the Coulomb parameter.

The strong absorption radius $R_{1/2}^s$ is then taken to be given by

$$KR_{1/2}^s = \gamma + \left[\gamma^2 + L(L+1)\right]^{1/2} \quad (2.55)$$

where the (critical) angular momentum $L$ is defined as the angular
momentum $\xi$ for which the real part of the reflection coefficient

$$\eta_\xi = e^{2i\delta_\xi}$$

is

$$\text{Re} \eta_\xi = \frac{1}{2}$$

The strong absorption radii as given by the equation (2.55) have been obtained by Frahn et al. and Faivre et al. over a range of incident energies using the method of direct parametrisation of phase shifts, and it is found by them that the strong absorption radii can be reasonably well represented by the formula,

$$R = r_0 A^{1/3} + c$$

where $r_0$ is the radius parameter with the value of 1.446-1.52F and the constant $c$ has the value of 2.14-2.29F.

The above formula describes the general trend for a wide range of nuclei but individual nuclei are known to show deviations from this trend. Recently, Blair and Fernandez have carefully analysed the elastic scattering data on the calcium isotopes for alpha particles at 42 MeV, using the phenomenological optical potential, and from the reflection coefficients determined from best fits to data, they have obtained strong absorption radii for the calcium isotopes. The values for strong absorption radii obtained by Blair and Fernandez differ by less than 1% from those obtained from direct parametrisation of phase shifts. Jackson and Morgan have carried out a detailed analysis of elastic scattering of alpha particles at 42 MeV and 30.5 MeV, with a view to understanding the significance of the strong absorption radius. They have used the phenomenological optical potential and varied the strengths of the potential from very shallow to very deep values in order to see whether the strong absorption radius may be regarded as a significant size parameter. They have shown that the different potentials used for $^{42}\text{Ca}$ produce almost the same strong absorption radius and the moduli of the scattering wave functions of
the alpha particle given by these potentials exhibit very similar behaviour on the illuminated side of the nucleus. In particular, Jackson and Morgan\textsuperscript{(10)} have pointed out that the absorption effect on the wave function begins in the vicinity of the strong absorption radius and the strong absorption radius may be significantly interpreted as the distance from the origin (the centre of the potential) at which the process of absorption begins to take place.

We have calculated the strong absorption radii according to the definition given above and corresponding to the folded potentials given in section (2.5). The results are presented in section (3.3), Chapter 3.
CHAPTER 3

Elastic scattering of medium energy alpha particles

3.1 Theoretical elastic scattering cross-sections

As mentioned in section (2.1), Chapter 2, the scattering wave function $\langle R|\chi\rangle$, for large $R$, behaves as

$$ (2\pi)^{3/2} \langle R|\chi\rangle = \chi_o(R) + f'(k',k) \frac{e^{ikR}}{R} $$

(3.1)

where $\chi_o(R)$ and $f'(k',k)e^{ikR}$ are the incident and the scattered waves, and $k, k'$ are the wave vectors in the initial and the final scattering states of the alpha particle. We are interested in the scattering amplitude $f'(k',k)$ and it can be shown (46) that

$$ f'(k',k) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)(\eta_\ell-1) P_\ell(\cos\theta) $$

(3.2)

where $\theta$ is the angle of scattering and is given by the angle between the wave vectors $k$ and $k'$, and $\eta_\ell$ are the reflection coefficients (scattering matrix elements) given by

$$ \eta_\ell = e^{-i\delta_\ell} $$

(3.3)

where $\delta_\ell$ are the phase shifts from the potential $V^{\alpha\alpha}_0$ (see section (2.6) Chapter 2). Further it can be shown (46) that when the Coulomb potential $V_c$ is present, elastic scattering amplitude $f(k',k)$ due to both the potential $V^{\alpha\alpha}_0$ and $V_c$ is given by

$$ f(k',k) = f_c(k',k) + \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) \frac{2i\xi_\ell}{\eta_\ell-1} P_\ell(\cos\theta) $$

(3.4)

where $\xi_\ell = \sigma_\ell + \delta_\ell$. $\delta_\ell$ is now the phase shift due to the nuclear and
modified Coulomb potential in the presence of the Coulomb potential for a point charge for which the Coulomb phase shift is \( \sigma_\ell \) defined by

\[
\sigma_\ell = \arg \Gamma(\ell+1+i\gamma)
\]

is the Coulomb parameter \( \gamma = \frac{i2Ze^2}{\hbar^2k} \).

\( f_c(k',k) \) is the Coulomb scattering amplitude \(^{(60)}\),

\[
f_c(k',k) = e^{-\gamma \log(\sin^2 \theta) + 2i\sigma_0}
\]

Then, the elastic differential scattering cross-sections \( \frac{d\sigma}{d\Omega} \) are given by

\[
\frac{d\sigma}{d\Omega} = |F(k',k)|^2
\]

We have calculated the differential cross-sections for \(^{40}\text{Ca}, ^{42}\text{Ca}, ^{44}\text{Ca}, ^{48}\text{Ca} \) at 42 MeV as given by the expression (3.6) but in the calculations corrections of order \( A^{-1} \) have been ignored. It may be noted that because we have taken the Coulomb potential as that due to uniformly charged sphere of radius \( R_c \), for \( R<R_c \) (inside the nucleus) our calculated phase shifts \( \delta_\ell \) include the contribution from this potential. The Coulomb phase shifts \( \sigma_\ell \) corresponding to the case of the point nucleus are obtained from the recurrence relations \(^{(60)}\)

\[
\sigma_{\ell-1} = \sigma_\ell - \tan^{-1} \frac{\gamma}{\ell}
\]

We have first calculated \( \sigma_\ell \) for \( \ell=60 \) from the Stirling's formula (see reference 60) and then generated the rest from the relation (3.7).

This procedure enabled us to obtain also the Coulomb amplitude \( f_c(k',k) \) given by expression (3.5). For 42 MeV alpha particles, it is found that the reflection coefficient \( \eta_\ell \) approach the values of 1, when \( \ell \) is >30 and therefore only about 30 partial waves contribute to the elastic scattering of 42 MeV alpha particles.
We have presented in the next section the theoretical fits to experiments and the results for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$, and $^{48}\text{Ca}$ obtained from the microscopic description are discussed.

3.2 Theoretical fits to experiments

We have first investigated the sensitivity of the differential cross-sections to variations in the two-body parameters and the parameter $\xi$ which gives the strength of the imaginary part of the optical potential, but we have confined ourselves to the single Yukawa form. It is found that variations in the strength $v_0$ of the single Yukawa interaction and $\xi$ essentially changed only the magnitude but the shape of the cross-section is very sensitive to the range parameter $\lambda$. First, we adopted the procedure described in section (2.3), Chapter 2 to determine the range parameter $\lambda$ and obtained the average value of $\lambda = 0.96F^{-1}$. But because of the ambiguities in the optical potential for alpha particles, it was not possible to obtain accurately the range parameter in this way and therefore we varied the range parameter around this average value of $0.96F^{-1}$. The effect of variations in is shown in Fig. 18 and it can be seen from there that a sufficient change in $\lambda$ can cause a maximum in the cross-section to change into a minimum. As the value of $\lambda$ is increased from $0.8F^{-1}$ to $1.6F^{-1}$, deeper diffraction minima appear in the calculated cross-section and, in addition, the spacing of the diffraction pattern becomes wider. This may be easily explained by examining the shapes of the optical potentials obtained with these values of $\lambda$. It is found that for values of $\lambda$ in the region of $1.1-1.6F^{-1}$, the potentials obtained are less diffuse and hence, one would expect deeper diffraction patterns in the cross-sections. However, as can be seen from Fig. 18, the calculated cross-sections for $^{44}\text{Ca}$ for larger values of $\lambda$, although deeper diffraction patterns can be obtained, do not show the correct spacing of the observed
Fig. 18 Cross-sections for elastic alpha particle scattering from $^{44}$Ca at 42 MeV, calculated for different values of the range parameter of the single Yukawa interaction.
diffraction pattern (32) (see Fig. 23). For smaller values of (near 0.8F⁻¹) the correct spacing of the diffraction pattern can reasonably be obtained but not the deep diffraction pattern. In the case of the Gaussian interaction, we have not attempted to vary the range parameter, since the value of K=0.5F⁻¹ for the range parameter used by Morgan and Jackson (24) was found to be satisfactory for all the four calcium isotopes but we have adjusted the strength of the two-body interaction in cases where it was found necessary in order to fit the magnitudes of the cross-sections.

In sections (3.2.1-3.2.4) we have presented the results for ⁴⁰Ca, ⁴²Ca, ⁴⁴Ca, ⁴⁸Ca as obtained from the microscopic description of elastic scattering. In section (3.3), we have presented the strong absorption radii for the calcium isotopes, and the results obtained from the present analysis are discussed in section (3.4).

3.2.1 Results for ⁴⁰Ca

We have shown in Fig. 19 the best fit to the data, obtained using the single Yukawa interaction. The experimental data used is that of Peterson (32). The accuracy of the measurements of the absolute cross-sections by Peterson (32) is about 17% and the accuracy of the locations of the minima in the cross-sections is ±0.5°. The value of ν₀ = -50 MeV for the strength of the single Yukawa interaction and the value of ξ=0.2 have been used. For the range parameter, we have obtained the value of λ=0.76F⁻¹ but repeated the calculations using the value of λ=0.8F⁻¹ in order to see whether the data on all the calcium isotopes can be explained with a fixed set of two-body parameters (ν₀ = -50 MeV, λ=0.8F⁻¹). If the two-body parameters ν₀, λ remain fairly constant, as is reasonable to expect, through the calcium isotopes, then on this basis we may regard the nuclear matter...
Fig. 19. Cross-sections for elastic alpha particle scattering from $^{40}\text{Ca}$ at 42 MeV obtained using the single Yukawa interaction, and compared with the data of Peterson.
Fig. 20. Cross-sections for elastic alpha particle scattering from $^{40}\text{Ca}$ at 42 MeV obtained using the Gaussian interaction, and compared with the data of Blair and Fernandez.
distributions used for these isotopes as adequate. However, as can be seen from Fig.19, the value of $\lambda=0.8F^{-1}$ does not yield the correct spacing of the diffraction pattern. But it is clear from Fig.19 that the calculated cross-sections are very sensitive to variations in the range parameter $\lambda$. Although this is so, the potential for $^{40}\text{Ca}$ produced by the single Yukawa interaction does not yield both the deep diffraction and the correct spacing of it for any value of $\lambda$ in the range 0.8-1.6$F^{-1}$. The correct spacing of the diffraction pattern can be reproduced up to about 35°-40° for the value of $\lambda=0.76F^{-1}$ but beyond this the calculated cross-sections do not show much structure.

We have presented in Fig.20 the best fit to the data on $^{40}\text{Ca}$ at 42 MeV, for the case of the Gaussian interaction. We have used the data of Blair and Fernandez (12), which have been obtained with an accuracy of 10-15.1%. In particular, Blair and Fernandez (12) have determined the locations of the minima near $\theta_{\text{cm}}=35^\circ$ to within $^{0.1^\circ}$. In this respect, these data are much more accurate than those of Peterson (32) and at present, are the best available. As can be seen from Fig.20, the potential produced by the Gaussian interaction yields much more improved fit to the data and the calculated cross-sections show the required deep diffraction pattern. The values of $v_o=-40.7$ MeV and $K=0.5F^{-1}$ for the two-body parameters, and the value of $\xi=0.31$ have been used in this case. The value of $K=0.5F^{-1}$ for the range parameter is the same as that used by Morgan and Jackson (24). Further, from Fig.20, it is clear that the correct spacing of the diffraction pattern is also reproduced in this case. Hence, it follows that the Gaussian form for the two-body interaction which is sharply peaked (see Fig.4), can explain reasonably well the experimental data. But there is a slight discrepancy between theory and experiment at about $\theta_{\text{cm}}=53^\circ$ and the reason for this is not clear at present. There has also been some difficulty in obtaining accurate fits to elastic nucleon-scattering...
data on this isotope. However, it may be pointed out that only three parameters ($v_0, K, \xi$) are available in the present analysis and in view of this, the quality of the fit obtained (see Fig.20) is indeed remarkable.

3.2.2 Results for $^{42}$Ca

The best fit to the data on $^{42}$Ca is shown in Fig.21, for the single Yukawa interaction. The data is again that of Peterson. We have used the values of $v_0 = 50$ MeV and $\lambda = 0.8 F^{-1}$ for the two-body parameters, and the value of 0.2 for the parameter $\xi$. The single particle potential well parameters for $^{42}$Ca were not available and because of this we adopted the procedure described in section (2.4), Chapter 2, to generate the nuclear matter distribution for this isotope. Using the nuclear matter distribution so obtained and the above-mentioned two-body parameters, we have obtained the potential for $^{42}$Ca (see Fig.10) and calculated from this potential the cross-sections shown in Fig.21. It can be seen from Fig.21, the calculated cross-sections agree qualitatively with the data up to about $30^\circ-35^\circ$ but beyond this the single Yukawa interaction does not reproduce as in the case of $^{40}$Ca, any of the features shown by the data. The potential (see Fig.10) produced by the single Yukawa interaction is rather too diffuse and hence, does not have the required behaviour in the surface region shown by the phenomenological potentials which yield cross-sections in agreement with data.

We have shown in Fig.22 the best fit to the data on $^{42}$Ca for the Gaussian interaction. We have used the data of Blair and Fernandez in the comparison between the theory and the experiment shown in Fig.22. The values for the two-body parameters used are $v_0 = 35.6$ MeV, $K = 0.5 F^{-1}$ and the parameter $\xi$ is found to have the value of 0.586. The values for the two-body parameters are the same as used by Morgan and
Cross-sections for elastic alpha particle scattering from $^{42}\text{Ca}$ at 42 MeV as Figure 19.
Fig. 22  Cross-sections for elastic alpha particle scattering from $^{42}\text{Ca}$ at 42 MeV as Figure 20. Two-body parameters used are $v_0 = -35.6$ MeV.
Jackson (24). The nuclear matter distribution used is the same as before. It is clear from Fig.22 that the potential (see Fig.14) produced by the Gaussian interaction yields cross-sections in much better agreement with the data than those obtained in the case of the single Yukawa interaction. Both the deep diffraction pattern and the correct spacing of it are produced by the Gaussian interaction. However, at the diffraction minima, there is a small discrepancy between the calculated and the measured cross-sections. The diffraction minima in the calculated cross-sections at $\theta_{\text{cm}} \approx 24^\circ$ and $\theta_{\text{cm}} \approx 35^\circ$ are less deep than those in the data (12). Although at these angles, the Coulomb scattering is not expected to contribute significantly to the elastic scattering, it may yet influence the filling of the diffraction minima at these angles. It may be possible to obtain slightly more improved fit to the data at these angles by varying the radius parameter $r_c$ (see section (2.6), Chapter 2) of the Coulomb potential seen inside the target nucleus by the alpha particle. But we have taken the radius $R_c$ of the uniformly charged sphere to be the same as that of the nucleus and have not attempted to vary $r_c$ (or $R_c$).

At $\theta_{\text{cm}} \approx 46^\circ$ the diffraction minimum in the calculated cross-section is deeper than that given by the data and this may be due to the fact that there is a slight uncertainty in the measurement of the cross-section at this angle. However, the optical model code used here does not incorporate the facility, as do some codes, of folding experimental angular resolution into theoretically predicted cross-sections in order to take account of the experimental uncertainties and as such, a slight discrepancy between theory and experiment is probably inevitable. Apart from these slight discrepancies, the Gaussian interaction reproduces all the features present in the data and the fit obtained in Fig.22 is comparable, in terms of quality,
3.2.3 Results for $^{44}\text{Ca}$

Fig. 23 shows the best fit to the data on $^{44}\text{Ca}$ at 42 MeV for the single Yukawa interaction. The measurements on $^{44}\text{Ca}$ at 42 MeV are those of Peterson (32). The values for the two-body parameters are $\nu_0 = -50$ MeV, $\lambda = 0.8 F^{-1}$, and the value of $\xi = 0.2$ has been used. We see from Fig. 23 that the correct spacing of the diffraction minima is qualitatively reproduced by the single Yukawa interaction up to about $\theta_{cm} = 35^0$ but the diffraction minima, as in the cases of $^{40}\text{Ca}$, $^{42}\text{Ca}$ are not as deep as shown by the measured cross-sections (32). At larger angles, the calculated cross-sections again do not show any diffraction pattern at all whereas the measured cross-sections, even at these angles, exhibit considerable structure. It may be seen from Fig. 18 that although the calculated cross-sections are very sensitive to variations in the range parameter $\lambda$, no value of $\lambda$ in the range $0.8-1.6 F^{-1}$ produces the deep diffraction pattern and the correct spacing of it, as shown by the measured cross-sections (32) for $^{44}\text{Ca}$.

We repeated the calculations using the Gaussian interaction, in place of the single Yukawa interaction, and the best fit to data is shown in Fig. 24. The experimental data is that of Blair and Fernandez (12). We have used the value of $\xi = 0.5 F^{-1}$ for the range parameter and the values of $\nu_0 = -50$ MeV, $\xi = 0.395$ have been obtained. The potential for $^{44}\text{Ca}$ (see Fig. 15) produced by the Gaussian interaction yields theoretical cross-sections which are in satisfactory agreement with the measured cross-sections (12) and in particular, the theoretical cross-sections near the diffraction minima except the first one agree very well with the experiment. The second diffraction minimum at $\theta_{cm} = 35^0$ has been of interest and Blair and Fernandez (12) have closely examined the cross-sections near this diffraction minimum in order to obtain the strong absorption radii for the calcium isotopes.
Fig. 23  Cross-sections for elastic alpha particle scattering from $^{44}$Ca at 42 MeV as Figure 19.

$v_0 = -50$ MeV

$\lambda = 0.8$ F$^{-1}$

$\zeta = 0.2$
Fig. 24 Cross-sections for elastic alpha-particle scattering from $^{44}$Ca at 42 MeV as Figure 20.
and to test the $A$-dependence of these strong absorption radii.

### 3.2.4 Results for $^{48}\text{Ca}$

The best fit to the data\(^{(32)}\) on $^{48}\text{Ca}$ at 42 MeV is shown in Fig. 25, for the single Yukawa interaction. We have used the same values for $v_0$, $\lambda$, and $\xi$ as in the cases of $^{42}\text{Ca}$, $^{44}\text{Ca}$ except that we have also examined the calculated cross-sections using $\lambda = 0.84 F^{-1}$. For $^{48}\text{Ca}$, we have used both the nuclear matter distributions 1 and 2 as discussed in section (2.4), Chapter 2, and this is done in order to test the sensitivity of the cross-sections to the nuclear (or neutron) distribution. The proton distribution is identical in both the cases and fits the elastic electron scattering\(^{(1)}\). It can be seen from Fig. 25 that the calculated cross-sections in each case agree qualitatively with the data\(^{(32)}\) and the first two diffraction minima in the calculated cross-sections are reproduced at the correct positions. But as in the cases of $^{40}\text{Ca}$, $^{42}\text{Ca}$ and $^{44}\text{Ca}$ the single Yukawa interaction does not yield the deep diffraction pattern and at larger angles, it fails to reproduce the features shown by the data\(^{(32)}\). The two sets of calculated cross-sections are almost similar and the effect of the substantial change in the nuclear (or neutron) distribution is not very well reflected in the cross-sections. This is further discussed in section (3.4).

Fig. 26 shows the best fit to the data of Blair and Fernandez\(^{(12)}\) for the Gaussian interaction. The value for the range parameter used is the same as for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ but the values for the two-body strength $v_0$ and $\xi$ obtained are $-38.4$ MeV and $0.427$ respectively. In this case, we have used only the nuclear matter distribution 1. The calculated cross-sections are in satisfactory agreement with the experimental data\(^{(12)}\) (see Fig. 26). The second and third diffraction minima in the calculated cross-sections, in particular, agree very well with the corresponding minima in the measured cross-sections\(^{(12)}\).
Cross-sections for elastic alpha particle scattering from $^{48}\text{Ca}$ at 42 MeV, calculated using the two distributions shown in Fig. 8 and various values for the range parameter of the single Yukawa interaction. The data is that of Peterson (Ref. 32)
Fig. 26 Cross-sections for elastic alpha particle scattering from $^48\text{Ca}$ at 42 MeV as Figure 20.
quality of the fit obtained here is again comparable, as in the cases of \(^{40}\text{Ca}, {^{42}\text{Ca}}\) and \(^{44}\text{Ca}\) to those obtained from the phenomenological potentials \(^{(10)(12)}\).

We have given in section (3.3) the strong absorption radii for \(^{40}\text{Ca}, {^{42}\text{Ca}}, {^{44}\text{Ca}}\) and \(^{48}\text{Ca}\) as obtained from the microscopic description and in section (3.4), we have presented the discussion of the results for \(^{40}\text{Ca}, {^{42}\text{Ca}}, {^{44}\text{Ca}}\) and \(^{48}\text{Ca}\).

### 3.3 Results for strong absorption radii

We have calculated the strong absorption radii for \(^{40}\text{Ca}, {^{42}\text{Ca}}, {^{44}\text{Ca}}\) and \(^{48}\text{Ca}\) and the results are presented in Table V for both the single Yukawa and the Gaussian forms for the effective two-body interaction. Fig.27 shows the plot of the strong absorption radius against \(\frac{1}{\sqrt[3]{A}}\), and it is clear from Table V and Fig.27 that the strong absorption radii for the calcium isotopes show a systematic trend in both the cases (single Yukawa and Gaussian forms). The change in the nuclear matter distribution through the calcium isotopes (see Figs. 5-8) is, in general, reflected in the strong absorption radius and the values of the strong absorption radii can be reproduced by taking a reasonable nuclear matter distribution as also noted recently by Jackson \(^{(63)}\). In the case of the single Yukawa interaction, the strong absorption radii for the calcium isotopes obtained from the present work increase with \(\frac{1}{\sqrt[3]{A}}\) in reasonable agreement with the general trend obtained from phase analyses \(^{(9)}\) whereas in the case of the Gaussian interaction, they increase with \(\frac{1}{\sqrt[3]{A}}\) in accordance with the trend of Blair and Fernandez \(^{(12)}\) obtained using phenomenological optical potential. However, in view of much more accurate data \(^{(12)}\) used in the latter case and further, improved fits to these data which we have been able to obtain, these results for the strong absorption radii are therefore more accurate than those in the former case.

In Table V, we have also given the strong absorption radius
# TABLE V

Results for strong absorption radii obtained using Gaussian and single Yukawa interactions

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Alpha-nucleon interaction</th>
<th>Range parameter (F⁻¹)</th>
<th>R&lt;sub&gt;1&lt;/sub&gt; (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>⁴⁰Ca</td>
<td>Gaussian</td>
<td>0.5</td>
<td>7.30</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.8</td>
<td>7.43</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.76</td>
<td>7.66</td>
</tr>
<tr>
<td>⁴²Ca</td>
<td>Gaussian</td>
<td>0.5</td>
<td>7.30</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.8</td>
<td>7.55</td>
</tr>
<tr>
<td>⁴⁴Ca</td>
<td>Gaussian</td>
<td>0.5</td>
<td>7.38</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.8</td>
<td>7.59</td>
</tr>
<tr>
<td>⁴⁸Ca</td>
<td>Gaussian</td>
<td>0.5</td>
<td>7.44</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.8</td>
<td>7.77</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.84</td>
<td>7.43</td>
</tr>
<tr>
<td></td>
<td>Single Yukawa</td>
<td>0.8</td>
<td>7.62&lt;sup&gt;a)&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

<sup>a)</sup> Using density 2 (see section 3.3)
Fig. 27 Strong absorption radii obtained using the Gaussian and single Yukawa interactions plotted versus $A^{1/3}$. Strong absorption radii used here for the case of the single Yukawa interaction are those produced by the average values, $v_0 = -50$ MeV, $\lambda = 0.8 F^{-1}$. 
for $^{48}$Ca obtained using the nuclear matter distribution 2 (see section (2.4), Chapter 3) and this is done in order to test whether the strong absorption radius is sensitive enough to a substantial change in the nuclear matter (or equivalently, neutron) distribution. As can be seen from Table V (nuclear matter distribution 2, $\lambda=0.8F^{-1}$ for $^{48}$Ca), a 3% change in the equivalent radius of the nuclear matter distribution for $^{48}$Ca causes a 2% change in the strong absorption radius for the same isotope. Further, we have also investigated the effect of variations in the range parameter on the strong absorption radius. We have varied the range parameter $\lambda$ for the single Yukawa interaction by 5% and it can be seen again from the Table V (nuclear matter distribution 1, $\lambda=0.84F^{-1}$ for $^{48}$Ca), this produces a change of the same order of magnitude in the strong absorption radius for $^{48}$Ca. It follows from the above results for $^{48}$Ca that in the elastic scattering of alpha particles, the strong absorption radius can be regarded as a sensitive size parameter.

3.4 The discussion of the results for $^{40}$Ca, $^{42}$Ca, $^{44}$Ca, $^{48}$Ca

It can be seen from the best fits to the data on $^{40}$Ca, $^{42}$Ca, $^{44}$Ca and $^{48}$Ca at 42 MeV shown in Figs. 19-26 that the results obtained using the Gaussian interaction are in much better agreement with the data than those obtained using the single Yukawa interaction. In the case of the Gaussian interaction, both the qualitative and quantitative agreement with the data (12) can be obtained with reasonable success and the value of $K=0.5F^{-1}$ for the range parameter used by Morgan and Jackson (24) from the coupled-channels analysis is found to be satisfactory. But the strengths of the Gaussian interaction required to fit the magnitudes of the measured cross-sections for the calcium isotopes varied from the value of $-35.6$ MeV given by Morgan and Jackson (24) except in the
case of $^{42}$Ca (see Table III). In the case of the single Yukawa interaction, although the shape of the cross-sections is very sensitive to the range parameter $\lambda$, it was not possible to obtain both the required deep diffraction pattern and the correct spacing of it, even when the range parameter $\lambda$ was varied over a sufficiently wide range of values. However, for values of $\lambda$ near $0.8F^{-1}$, the correction spacing of the diffraction pattern can be reasonably reproduced and the qualitative features of the data can be explained up to about $\theta_{\text{cm}} = 35^\circ - 40^\circ$ (see Figs. 19, 21, 23, 25). But the potentials produced by the single Yukawa interaction for values of $\lambda$ near $0.8F^{-1}$ are rather too diffuse and hence, the calculated cross-sections do not show the deep diffraction pattern observed in the data (32). However, for values of $\lambda$ near $0.8F^{-1}$, at least qualitative agreement with the experiment can be obtained in the case of elastic scattering of alpha particles at 42 MeV whereas it has been found by Madsen and Tobocman (23) that the single Yukawa form for the effective two-body interaction did not reproduce any relevant features of the data on the inelastic scattering of alpha particles at 43 MeV unless a radial cut off of $6.45F$ was used.

From the results obtained for the calcium isotopes using the Gaussian and the single Yukawa forms for the effective two-body interaction, it is clear that the sharply peaked Gaussian form (see Fig. 4) is more satisfactory in explaining the observed deep diffraction pattern and the correct spacing of it than the single Yukawa form which has a long tail. This shows that the microscopic description of elastic scattering, as presented in Chapter 2, is indeed feasible but the choice of the form for the effective two-body interaction is of crucial importance. It follows that if the form for the effective two-body interaction is chosen carefully, improved fits to data can be obtained and the fits so obtained from the microscopic description, are comparable in terms of quality, to those obtained using the phenomenological
The importance of the form chosen for the effective two-body interaction can be understood if we examine the effective interaction approximation which we have used in the present work. In the effective interaction approximation, we have replaced a complicated two-body (target nucleon-alpha particle) scattering operator $\tau$ by an effective interaction, $v_{\text{eff}}$. As remarked in section (2.2), the effects of the exclusion principle and the nuclear binding are included in the scattering operator $\tau$ and hence the effective interaction $v_{\text{eff}}$ which is used to parametrise the complicated two-body scattering operator $\tau$, should simulate these effects. In view of this, the form for the effective two-body interaction must be chosen with care and if this is done the effective interaction approximation can be successfully used as can be seen from the case of the Gaussian form taken in the present work.

The same conclusion has been reached by Greenlees et al.\(^{(2)}\) in the microscopic description of elastic nucleon-nucleus scattering but in the work of Greenlees et al.\(^{(2)}\) the imaginary part of the proton optical potential has been taken to have the phenomenological Saxon-Woods form and thus, many more parameters were available for the fitting procedure, whereas in the present work, there are only three parameters $\nu_0$, $\lambda(\xi)$, and $\xi$.

The results for \(^{48}\text{Ca}\) shown in Fig.25 are obtained using the two nuclear matter distributions as mentioned earlier. We have explored the sensitivity of the cross-sections to the neutron distribution but have confined ourselves to the single Yukawa form. As can be seen from Fig.25 a substantial change in the neutron distribution produces only a small change in the cross-sections for \(^{48}\text{Ca}\). If a comparable change had been made in the proton distribution, the effect would have been easily detected in the analysis of elastic electron scattering or muonic X-rays. In order to detect the small changes in the cross-sections for
alpha particles produced by the changes in the neutron distribution and consequently, to determine more accurately the neutron distribution, it is further necessary to remove the uncertainty in the two-body parameters. This can be easily seen again from Fig. 25 where we have also shown the calculated cross-sections using the nuclear matter distribution 1 and the value of $\lambda = 0.84F^{-1}$. We have varied the range parameter $\lambda$ by 5% and it is found that the change produced in the cross-section for $^{48}\text{Ca}$, although small, is nearly of the same order of magnitude as in the case discussed above using the nuclear matter distribution 2 and $\lambda = 0.8F^{-1}$.

Greenlees et al.\(^{(2)}\) have also found that the uncertainty in the range parameter (or the mean square radius) of the two-body (in their case, nucleon-nucleon) interaction prevents accurate determination of the differences between the proton and neutron distributions, although nucleons are not strongly absorbed and are more sensitive to the interior of the nucleus. In view of this difficulty, Greenlees et al.\(^{(2)}\) have reanalysed the elastic nucleon-nucleus scattering data using a carefully chosen value for the range parameter and have been able to obtain the differences between the root mean square radii of the proton and neutron distributions with accuracies ranging from 14-63% for various nuclei. However, the results of Greenlees et al.\(^{(2)}\) may yet be open to reinterpretation in view of the fact that what they essentially determine from their analysis is the root mean square radius of the nucleon optical potential. From the present analysis we have obtained the value of $0.24 \pm 0.095F$ for the difference between the root mean square radii of the proton and the neutron distributions for $^{48}\text{Ca}$. Although the root mean square radius of the nuclear matter distribution for $^{48}\text{Ca}$ is determined within about 2%, the error of about 2% in the estimate of that quantity introduces a large error of about 40% in the estimate
of the difference between the root square radii of the proton and the neutron distributions. Therefore, a small error in the estimate of the root mean square radius of the nuclear matter distribution becomes significant when the differences between the proton and the neutron distributions are examined and the same conclusion has also been reached by Greenlees et al.\(^2\). In view of the slight uncertainty in the range parameter \( \lambda \) involved and, in addition, the qualitative nature results obtained in the case of the single Yukawa interaction, it is not possible to obtain much more accurate estimate of the root mean square radius of the nuclear matter distribution. The Gaussian form for the effective two-body interaction used in the present work produces much more improved fits to the data and hence, one may argue that the differences between the proton and the neutron distributions can be determined with much better accuracy from the elastic scattering of alpha particles using the Gaussian form. Although this appears plausible, a small uncertainty in the value of the range parameter \( K \) of the Gaussian interaction will prevent again from reaching any firm conclusion about the differences between the proton and the neutron distributions. As mentioned earlier, we have used the values of \( K=0.5F^{-1} \) as given by Morgan and Jackson\(^ {24} \) and do not know at the present time the error limits on this range parameter. But it appears that there is some ambiguity in the values of \( v_0 \) and \( K \) (Jackson, 1970, private communication). In view of this, we have not pursued further the investigation of the differences between the proton and the neutron distributions from the elastic scattering of alpha particles. But it is encouraging that the elastic scattering can be reasonably well described using the nuclear matter distribution obtained from the single particle description of the nucleus, although more precise information on the nuclear matter distribution cannot be extracted because of the uncertainty
in the parameters of the two-body interaction. The present analysis is promising in that if the theoretical basis of the microscopic description is made more sophisticated so that the two-body interaction is determined more precisely, the microscopic description of elastic scattering of alpha particles can be used successfully to investigate the nuclear matter distribution and, in particular, the differences between the proton and the neutron distributions.

In the present work, we have confined ourselves to the lowest order potential $V_{\infty}^0$ and neglected the term $\langle \xi_\alpha | U_{\infty} | \xi_\alpha \rangle$ (see section (2.1), Chapter 2) which contains the multiple scatterings involving the intermediate excited states of the target nucleus. It is not known at present how these multiple scatterings may influence the elastic scattering of alpha particles and to what extent the elastic scattering cross-sections for alpha particles are modified when the effects of these multiple scatterings are included, although these effects can be investigated using coupled-channels method. In the case of elastic electron scattering, it has been shown that these corrections due to these multiple scatterings are small. However, it may be pointed out again that all the intermediate states in which the target nucleus remains in its ground state are properly taken into account by the present analysis.

We have also obtained the strong absorption radii for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$ from the present analysis (see Section (3.3)) and it is found that they show systematic $A$-dependence. In the case of the single Yukawa interaction, the strong absorption radii for the calcium isotopes obtained are in satisfactory agreement with the results of the direct phase analyses whereas in the case of the Gaussian interaction, they are in much better agreement with those of Blair and Fernandez and the results obtained for $^{42}\text{Ca}$ by Jackson.
and Morgan. The results of the present work differ from those of Blair and Fernandez by about 2-3% in the case of the single Yukawa interaction and by about 0.5-1% in the case of the Gaussian interaction. From the present analysis, we note the remarkable result that the microscopic model can reproduce the isotopic behaviour of the strong absorption radius. This indicates clearly that it is properly described by conventional scattering theory and no special mechanism need be invoked to explain this behaviour of the strong absorption radius. The same conclusion has also been reached in the recent work of Jackson. Further, we have investigated, as is mentioned in section (3.3), the sensitivity of the strong absorption radii to the changes in the nuclear matter distribution and the range parameter. But we have confined ourselves to the isotope $^{48}$Ca. It is found that the changes in the nuclear matter distribution and the range parameter of the two-body interaction produce changes in the strong absorption radius of the same order of magnitude and this provides yet more evidence that in the elastic scattering of alpha particles, the strong absorption radius may well be treated as a sensitive size parameter whose magnitude can in principle be determined from a microscopic calculation.
4.1 Microscopic description

High energy pions have been used recently in the investigation of nuclear structure and from this point, one of the topics of much interest has been the elastic scattering of high energy pions by nuclei. In view of the success of the optical model in the case of other high energy projectiles such as nucleons, the same model has also been used to describe pion-nucleus elastic scattering at high energies. However, it is known that at high energies, as at low energies, the pion-nucleon total cross-sections exhibit resonance behaviour (35)(36)(37) and the resonances persist up to about 1.5 GeV. Because of this, the use of the optical model in the case of pions near resonance energies might appear to be rather difficult to justify, although it is reasonable to use at energies away from the resonance energies. But, fortunately, when one takes into account the effect of the motion of the target nucleons in the nucleus (22), the resonance behaviour at high energies is smoothed out and therefore, one is probably justified in extending, as done in various analyses of high energy pions, the use of the optical model to energies near resonances. We have briefly examined later the validity of this argument by performing an actual calculation near a resonance energy (see Chapter 5) but apart from this, we have confined ourselves in the present work to energy regions of overlapping resonances. We have given below the derivation of the pion optical potential using the microscopic description.
Auerbach et al\(^{(7),(34)}\) have used the formalism of Watson to derive the pion optical potential but in the present work, we follow the method of Kerman, McManus and Thaler\(^{(28)}\) originally developed for nucleon-nucleus scattering. The Hamiltonian \(H\) for the pion-nucleus system is given by

\[
H = H_N + K + V
\]

(4.1)

where \(H_N\) is the nuclear Hamiltonian and \(K\) is the pion kinetic energy operator. \(V\) is the many-body interaction between the incident pion and the target nucleons and is taken to be the sum of the two-body (pion-target nucleon) interactions \(v_i\),

\[
V = \sum_i v_i (r_i - r_M)
\]

(4.2)

where \(r_i\) is the distance between the pion and the centre of mass of the target nucleus and \(r_i\) is the distance between a target nucleon and the centre of mass of the target nucleus. The many-body scattering operator \(T\) is given by

\[
T = V + V \frac{1}{\alpha} T
\]

(4.3)

where the propagator \(1/\alpha\) is

\[
\frac{1}{\alpha} = \frac{\mathcal{A}}{E - H_N - K + i\epsilon}
\]

\(\mathcal{A}\) is the antisymmetrisation operator for the nuclear states and \(E\) is the total energy of the pion-nucleus system. The pion-target nucleon scattering operator \(\tau_i\) is defined as

\[
\tau_i = v_i + v_i \frac{1}{\alpha} \tau_i
\]

(4.4)

We can express the many-body scattering operator \(T\) in terms of two-body scattering operator \(\tau_i\) in the same way as before (see section 2.1, Chapter 2) and we obtain,

\[
T'_{oo} = U_{oo} + U_{oo} \frac{1}{\alpha} T'_{oo}
\]

(4.5)

where
The pion optical potential is given by

\[ U = \langle 0 | U | 0 \rangle \]

where

\[ U = \langle 0 | U° | 0 \rangle + \langle 0 | U° - QU | 0 \rangle \]

The second term in the equation (4.6) can be obtained as a multiple scattering expansion in which intermediate excited states of the target nucleus are involved. But in the present work, we have neglected this term and concerned ourselves only with the lowest order approximation for the pion optical potential \( U° \). In the lowest order, the pion optical potential is given by

\[ U° = \sum_{i} \tau_i \]

and \( Q \) is the projection operator which projects off the nuclear ground state. The second term in the equation (4.6) can be obtained as a multiple scattering expansion in which intermediate excited states of the target nucleus are involved. But in the present work, we have neglected this term and concerned ourselves only with the lowest order approximation for the pion optical potential \( U° \). In the lowest order, the pion optical potential is given by

\[ U° = \sum_{i} \tau_i \]

where \( \tau_i \) are the pion-target proton (neutron) scattering operators and \( Z, N \) are the numbers of the protons and neutrons in the target nucleus. Using the equations (4.8) and (4.9), the pion optical potential in lowest order can be rewritten as
We have used the equation (4.10) in the calculation of high energy pion-nucleus elastic scattering discussed in the next section.

4.1.1 High energy pion-nucleus elastic scattering

Because the pion is spinless and further, the system that we are dealing with is relativistic, we assume that pion-nucleus elastic scattering is described by the Klein-Gordon equation,

\[
\left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi(r) = (E - V - U^o) \psi(r) \quad (4.11)
\]

where \(m\) is the rest mass of the pion, \(E\) is the pion total energy in the centre of mass frame for the pion-nucleus system and we have added the Coulomb potential \(V_c\) in order to take account of the fact that the pion is charged. The lowest order pion optical potential \(U^o\) is as given by the equation (4.10), \(\psi(r) = (2\pi)^{3/2}. \psi(r)\) is the pion (elastic) scattering wave function. The equation (4.11) involves, on the right hand side, the calculation of matrix elements such as \(\langle r|U^o|k\rangle\) and \(\langle r|V_c U^o|k\rangle\). The matrix element \(\langle r|U^o|\psi\rangle\) can be calculated as follows:

\[
\langle r|U^o|\psi\rangle = \int \langle r|U^o|k\rangle \langle k|\psi\rangle dk \quad \text{d}r'
\]

To evaluate \(\langle r|U^o|k\rangle\), we note that

\[
\langle r|U^o|k\rangle = \int \langle k|U^o|k'\rangle \langle k'|r\rangle dk \quad \text{d}k'
\]

\[
= \frac{1}{(2\pi)^3} \int e^{i(k\cdot r - k'\cdot r')} \langle k|U^o|k'\rangle \quad \text{d}k \quad \text{d}k'
\]

where we have used

\[
\langle r|k\rangle = \frac{1}{(2\pi)^{3/2}} e^{i k \cdot r}
\]

\[
\langle k'|r\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i k' \cdot r'}
\]

From the equation (4.10), we have
where \( F_p(k-k'), F_n(k-k') \) are the proton and the neutron form factors (see section (4.6.1)). We assume that the scattering matrices \( \langle k | \pi \rangle | k' \rangle, \langle k | \eta \rangle | k' \rangle \) depend only on the momentum transfer \( q = |k' - k| \), for high energy scattering. This is suggested by the fact that the high energy free pion-nucleon scattering can be described reasonably well making such an assumption \((42)(43)\). Hence, we may write the matrix element \( \langle k | U^0 | k' \rangle \) as

\[
\langle k | U^0 | k' \rangle = Z \pi_p(q) F_p(q) + N \pi_n(q) F_n(q)
\]

Using this equation, it is easy to show

\[
\langle r | U^0 | r' \rangle = \frac{1}{(2\pi)^3} \int e^{i k \cdot (r-r')} e^{i q \cdot r'} [Z \pi_p(q) F_p(q) + N \pi_n(q) F_n(q)] dk dq
\]

\[
= \delta(r-r') U^0_{oo}(r')
\]

where

\[
U^0_{oo}(r') = \int e^{i q \cdot r'} [Z \pi_p(q) F_p(q) + N \pi_n(q) F_n(q)] dq
\]

It follows then from the equation (4.14),

\[
\langle r | U^0 | \psi \rangle = U^0_{oo}(r) \psi(r)
\]

Similarly, it is possible to show,

\[
\langle r | V_c U^0 | \psi \rangle = V_c(r) U^0_{oo}(r) \psi(r)
\]

where we have assumed that the Coulomb potential \( V_c \) is local as usually done.

Using the equations (4.16) and (4.17), the Klein-Gordon equation (4.11) takes the form

\[
\left[ \hbar^2 c^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial r^2} + V_c^2 + (U^0_{oo})^2 - 2E V_c - 2E U^0_{oo} + 2V U^0_{oo} \right] \psi(r) = 0
\]
where the pion optical potential $U^{0}_{oo}(r)$ is given by equation (4.15) and $k$ is the pion wave number in the centre of mass frame for the pion-nucleus system. We have used the radial form of equation (4.18) in the present calculations of pion-nucleus elastic scattering and these radial equations are discussed in section (4.8).

In the next section, we have discussed the impulse approximation which has been used in the calculations of the pion optical potentials.

4.2 Impulse approximation

The calculation of the pion optical $U^{0}_{oo}(r)$ (see equation (4.15)) involves the knowledge of the pion-target nucleon scattering matrices $\tau_{\pi N}(q) (N=p,n)$. The pion-target nucleon scattering matrices can be, in principle, evaluated but the actual calculations are involved. Therefore, the impulse approximation has often been used in the analyses of high energy scattering of pions and nucleons by nuclei. Auerbach et al. have used the impulse approximation both at low and high energies in the optical model analyses of pion-nucleus scattering and have been reasonably successful in explaining the experimental data. Further, Cronin et al. have also used the impulse approximation in the calculations of pion-optical potentials and have been able to obtain reasonable agreement with their data.

The pion-target nucleon scattering operator $\tau_{\pi N}$ as given by the equation (4.4) (see section (4.1)) is:

$$\tau_{\pi N} = v + \frac{1}{\alpha} \tau_{\pi N}' , \quad N=p,n$$

where as before,

$$\frac{1}{\alpha} = \frac{\alpha}{E-H_{N}-K+i\epsilon}$$

In the impulse approximation, the propagator $\frac{1}{\alpha}$ which contains, as can be seen from the equation (4.19), the nuclear Hamiltonian $H_{N}$ and the anti-symmetrisation operator $\alpha$ for the nuclear states, is replaced by
the free propagator \( \frac{1}{E - K_\pi - i\epsilon} \) where \( K_\pi \) is the kinetic energy operator for the free nucleon. This is, in fact, equivalent to taking,

\[
\tau_{\pi N} \simeq t_{\pi N}
\]

\[
t_{\pi N} = v + \sqrt{\frac{1}{E - K_\pi - i\epsilon} E - K_\pi + i\epsilon}
\]

(4.20)

where \( t_{\pi N} \) is the free pion-nucleon scattering operator.

In the present work, we are interested in the calculations of pion-nucleus elastic scattering at high energies (0.5-1 GeV) and hence, it is reasonable to use the above impulse approximation. However, the use of the impulse approximation implies that the second order corrections to the pion optical potential are not important. We have further examined this in section (4.5). But now, using the impulse approximation, we obtain

\[
\tau_{\pi N}(q) \simeq t_{\pi N}(q)
\]

\[
\tau_{\pi N}(q) = \frac{W_1 W_2}{E_1 E_2} t_{\pi N}(q)
\]

(4.21)

where we have expressed the free two-body scattering matrix \( t_{\pi N}(q) \) which is in the centre of mass frame for the pion-nucleus system, in terms of the free two-body scattering matrix \( t_{\pi N}(q) \) in the centre of mass frame for the pion-nucleon system (see Appendix A) and \( W_1 W_2(E_1 E_2) \) are the total energies of the incident pion and a target nucleon in the centre of mass frame for the pion-nucleon (pion-nucleus) system. Using the relation

\[
t_{\pi N}(q) = -\frac{2\epsilon}{(2\pi)^2} \left( \frac{W_1 W_2}{W_1 W_2} \right)^2 \tau_{\pi N}(q)
\]

(4.22)

where \( f_{\pi N}(q) \) is the free two-body scattering amplitude in the centre of mass frame for the pion-nucleon system, it follows

\[
\tau_{\pi N}(q) = -\frac{2\epsilon}{(2\pi)^2} \left( \frac{W_1 W_2}{E_1 E_2} \right)^2 f_{\pi N}(q)
\]

In the analyses of pion-nucleon elastic scattering at high energies...
(≥1 GeV), the form usually taken for \( f^{(2)}_{\pi N}(q) \) is

\[
f^{(2)}_{\pi N}(q) = \frac{\sigma_{\pi N} k}{4\pi} (i + \alpha_{\pi N}) e^{-\frac{1}{2} \beta_{\pi N}^2 q^2}
\]

(4.23)

where \( k \) is the wave number in the centre of mass frame for the pion-nucleon system, \( \sigma_{\pi N} \) is the pion-nucleon total cross-section and

\[
\alpha_{\pi N} = \frac{\text{Re} f^{(2)}_{\pi N}(0)}{\text{Im} f^{(2)}_{\pi N}(0)}
\]

(4.23.1)

\( \beta_{\pi N}^2 \) is a two-body parameter and all the three \( \sigma_{\pi N}, \alpha_{\pi N}, \beta_{\pi N}^2 \) are obtained from the two-body data available at high energies (see section 4.4). The parameter \( \alpha_{\pi N} \) which determines the phase of \( f^{(2)}_{\pi N}(q) \) depends, in general, on the square of the four momentum transfer \( q \) but we have assumed that \( \alpha_{\pi N} \) is independent of \( q \). We shall hereafter drop the superscript \( 2 \) on the two-body amplitude and simply write as \( f_{\pi N}(q) \). Using the form (4.23) for the amplitude \( f_{\pi N}(q) \) and the equation (4.23.1), the pion-optical potential \( U_{oo}^\circ(\tau) \) is given by

\[
U_{oo}^\circ(\tau) = -\frac{\hbar^2 c^2}{(2\pi)^2} \left( \frac{W + W}{E_1 E_2} \right) \int_{0}^{\infty} e^{iq\cdot\tau} \left[ f_{\pi P}(0) F_{p}(q) + N f_{\pi p}(0) F_{n}(q) \right] \text{exp}\left(-\frac{1}{2} \beta_{\pi N}^2 q^2 \right) dq
\]

(4.24)

where

\[
\begin{align*}
\sigma_{\pi P}(o) &= \frac{\sigma_{\pi N} k}{4\pi} (i + \alpha_{\pi N}) \\
\sigma_{\pi N}(o) &= \frac{\sigma_{\pi N} k}{4\pi} (i + \alpha_{\pi N})
\end{align*}
\]

(4.24.1)

and we have taken

\[
\beta_{\pi N}^2 = \beta_{\pi P}^2 = \beta^2
\]

(4.24.2)

For the nuclei, \( A=2z \), the proton and the neutron distributions may be assumed to be identical (apart from normalisation). Therefore, for these nuclei, we have

\[
U_{oo}^\circ(\tau) = -A b \frac{\hbar^2 c^2}{(2\pi)^2} \left( \frac{W + W}{E_1 E_2} \right) \int_{0}^{\infty} e^{iq\cdot\tau} F_{p}(q) e^{-\frac{1}{2} \beta_{\pi N}^2 q^2} dq
\]

(4.25)
where \( b = \frac{1}{2}(f_{\pi p}(o) + f_{\pi n}(o)) \).

In the next section, we have discussed the large \( A \) approximation which is often used in the analysis of high energy scattering.

4.3 Large \( A \) approximation

In the limit of large \( A \) (\( A \) is mass number), it is often assumed that the two-body scattering amplitude \( f_{\pi N}(q) \) varies slowly with the momentum transfer \( q \) compared with the nucleon form factors \((F_p(q) \) and \( F_n(q))\), and this enables one to obtain the pion optical potential approximately as

\[
\omega^0_{\pi n}(r) = -2\hbar^2 c^2 \left( \frac{W_1 + W_2}{E_1 E_2} \right) \left[ 2 f_{\pi p}(o) \rho_p(r) + N f_{\pi n}(o) \rho_n(r) \right] 
\]

(4.26)

where the proton and the neutron distributions \( \rho_p(r) \), \( \rho_n(r) \) are normalised as

\[
\int \rho_p(r) \, dr = 1 \quad \text{ and } \quad \int \rho_n(r) \, dr = 1 \]

(4.27)

The expression (4.26) for the pion optical potential has been used by Cronin et al. (41), Crozon et al. (22), Auerbach et al. (34) and in various other analyses (11)(48)(54). Further, the large \( A \) approximation has also been used in the analyses of nucleon-nucleus scattering at high energies. However, the large \( A \) approximation is expected to be reasonably valid only for heavy nuclei but it is not clear whether this approximation can be equally used for analysing high energy scattering data on medium-heavy and light nuclei as done, for instance, in the analyses (22)(41). Recently, Jackson (11) has analysed high energy reactions initiated by pions and nucleons in order to study the sensitivity of the effective nucleon number \( N(A) \) to the two-body parameters. It appears from this work that a deviation in \( N(A) \) of approximately 1\% arises from the use of the large \( A \) approximation in the case of \(^{40}\)Ca.
Although this approximation may be reasonable to use as shown by the work of Jackson\cite{11}, it is yet difficult to say whether it can be used with confidence in the analyses such as those of high energy pion-nucleus scattering from which one hopes to draw subsequently conclusions about the nuclear matter distribution. In view of this, we have attempted to examine closely this approximation and have used both the expressions (4.24) and (4.26) (see section 4.2) in the calculations of the pion optical potentials for $^{12}$C, $^{40}$Ca and $^{208}$Pb. It is found that the pion optical potentials calculated in the two cases differ substantially in the case of $^{12}$C, $^{40}$Ca and reasonably in the case of $^{208}$Pb, (see section 4.7).

4.4 Pion-nucleon parameters

The calculation of the pion optical potential obtained in the impulse approximation (see equation (4.24)) involves a knowledge of the two-body parameters $\sigma_{\pi N}$, $\alpha_{\pi N}$, $\beta$. But in the large A approximation we need to know only the parameters $\sigma_{\pi N}$ and $\alpha_{\pi N}$. Since in both these approximations, the two-body parameters involved are the free parameters, these can be obtained from the two-body data available at high energies. However, it has been shown by Beg\cite{48} that the effects of the exclusion principle and the direct absorption by pairs of nucleons, are important and, in view of this, these effects need to be properly taken into account. We have discussed in section 4.5 the corrections for the two-body parameters due to these effects. But first we need to know the free two-body parameters and therefore we have discussed in the following sections (4.4.1-4.4.3) the information available on these two-body parameters in the GeV region.
4.4.1 Pion-nucleon total cross-sections

There have been several measurements of pion-nucleon total
cross-sections at energies in the GeV region\((35)(36)(37)\) but the
recent measurements of Carter et al\((37)\) are much more accurate and
at the present time, they are the best available. Carter et al\((37)\)
have measured both the \(\pi^+ p\) and \(\pi^- p\) total cross-sections up to about
2 GeV/C pion momentum (laboratory). We have used the two-body data
of Carter et al\((37)\) in the calculations of pion optical potentials
at energies around 1 GeV. The total cross-sections \(\sigma \pi^- n\), \(\sigma \pi^+ n\)
are obtained from charge symmetry,
\[
\begin{align*}
\sigma \pi^- n &= \sigma \pi^+ p \\
\sigma \pi^+ n &= \sigma \pi^- p.
\end{align*}
\]
(4.28)
The values for the \(\pi^+ p\) total cross-sections in the energy region
0.585-1.057 GeV used are given in Table VI. But in addition, we
have also used the values for two-body total cross-sections
obtained by Crozon et al\((22)\) (see Table VI). It is easy to see
from the equation (4.24) the strength of the imaginary part of the
pion optical potential depends on the values of the total cross-
sections \(\sigma \pi^+ p\). For the nuclei, \(A=2Z\), only the average value of
\(\sigma \pi^- p\) and \(\sigma \pi^+ p\) (see equations (4.25) and (4.27) appears in the
expression for the pion optical potential and because of this, the
optical potentials for positive and negative pions are identical in
the case of these nuclei.

4.4.2 Real part of pion-nucleon forward scattering amplitude

Some measurements of the real parts \(Re f_{\pi N}(o)\) of forward
scattering amplitude at high energies are available but it is not
clear how accurate these measurements are. Actually, the (elastic)
differential cross-sections \(\frac{d\sigma}{d\Omega}(o)\) are measured and the real parts
Re \( f_{\pi N}(o) \) are obtained from the relation,

\[
\frac{d\sigma}{d\Omega}(o) = (\text{Re} \ f_{\pi N}(o))^2 + (\text{Im} \ f_{\pi N}(o))^2
\]

\[
= (\text{Re} \ f_{\pi N}(o))^2 + \frac{k^2\sigma_{\pi N}}{(4\pi)^2} \tag{4.29}
\]

where we have used the optical theorem

\[
\sigma_{\pi N} = \frac{4\pi}{k} \text{Im} \ f_{\pi N}(o) \tag{4.30}
\]

and \( k \) is the photon wave number in the centre of mass frame for the two-body system.

Although the magnitudes of \( \text{Re} \ f_{\pi N}(o) \) may be obtained from equation (4.29), its sign cannot be determined from a knowledge of \( \frac{d\sigma}{d\Omega}(o) \) and total cross-section \( \sigma_{\pi N} \). There have been some attempts to obtain the real parts \( \text{Re} \ f_{\pi N}(o) \) of the forward scattering amplitude using the dispersion relations (27)(38)(39)(40). This approach provides a method of determining the magnitude of \( \text{Re} \ f_{\pi N}(o) \) as well as its sign and the calculations (38)(40) which have been made for pion kinetic energies in the GeV region, indicate that the sign of \( \text{Re} \ f_{\pi N}(o) \) is negative in this energy region. The negative sign for \( \text{Re} \ f_{\pi N}(o) \) means the real part of the pion optical potential is 'repulsive'. In the present work, we have mainly used the values for \( \text{Re} \ f_{\pi N}(o) \) which are obtained from the dispersion relations (38)(40) and these values (in the laboratory frame for the two-body system) are given in Table VII for the energy region 0.585-1.057 GeV.

But we varied the values for \( \text{Re} \ f_{\pi N}(o) \) at 1.057 GeV over a wide range in order to test the sensitivity of the cross-sections to the changes in \( \text{Re} \ f_{\pi N}(o) \) and in particular, to the sign of \( \text{Re} \ f_{\pi N}(o) \).

It is found that the differential cross-sections (see section (5.2-5.3, Chapter 5) are very sensitive to the magnitude as well as the sign of \( \text{Re} \ f_{\pi N}(o) \). The same result has also been noted in the theoretical studies of \( \pi-\text{He}^4 \) elastic scattering at 6 GeV(6) and 8 GeV(25) using the Glauber theory(3). But although it has been
TABLE VI

Pion-nucleon total cross-sections

<table>
<thead>
<tr>
<th>Pion kinetic energy (GeV)</th>
<th>$\sigma^+_{\pi^-p}$ (mb)</th>
<th>$\sigma^-_{\pi^-p}$ (mb)</th>
<th>$\overline{\sigma}$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.585</td>
<td>15.16$^a$</td>
<td>46.31$^a$</td>
<td></td>
</tr>
<tr>
<td>0.700</td>
<td>16.89$^a$</td>
<td>35.80$^a$</td>
<td></td>
</tr>
<tr>
<td>1.057</td>
<td></td>
<td></td>
<td>33.85$^b$</td>
</tr>
</tbody>
</table>

*$\overline{\sigma} = \frac{1}{2}(\sigma^+_{\pi^-p} + \sigma^-_{\pi^-p})$

1. Carter et al (Ref.37)
2. Ref.35.
3. Crozon et al (Ref.22).

---

TABLE VII

Real parts of pion-nucleon forward scattering amplitudes

<table>
<thead>
<tr>
<th>Pion kinetic energy (GeV)</th>
<th>Ref$^-_{\pi^-p}$ (F)</th>
<th>Ref$^+_{\pi^+p}$ (F)</th>
<th>Ref(o)$^-_{\pi^-p}$ (F)</th>
<th>Ref(o)$^+_{\pi^+p}$ (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.585</td>
<td>0.17$^a$</td>
<td>-0.78$^a$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.700</td>
<td>-0.14$^a$</td>
<td>-0.42$^a$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.057</td>
<td></td>
<td></td>
<td>-0.30$^b$</td>
<td></td>
</tr>
</tbody>
</table>

*$\overline{\text{Ref}(o)} = \frac{1}{2}(\text{Ref}$$_{\pi^-p}(o)$+Ref$^+_{\pi^+p}(o))$

1. Cronin (Ref.38)
2. Guisan (Ref.40)

---

TABLE VIII

Single particle potential well parameters* for $^{12}$C

<table>
<thead>
<tr>
<th>Proton level</th>
<th>$V_{\pi}$ (MeV)</th>
<th>$V_{s}\pi$ (MeV)</th>
<th>$r_o$ (F)</th>
<th>a (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1s$^1$</td>
<td>60</td>
<td>1.36</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>1p$^3/2$</td>
<td>55</td>
<td>9</td>
<td>1.36</td>
<td>0.55</td>
</tr>
</tbody>
</table>

* See equation (2.44) (Section 2.4, Chapter 2)
shown by Formanek et al.\(^{(6)}\) that the calculated differential cross-section near the minimum for pions at 6 GeV is sensitive to the real part \(\text{Re } f_{\pi N}^{N}(o)\) (or the phase) of the two-body scattering amplitude, it is not clear from their work whether the cross-section near the minimum is also sensitive to its sign.

### 4.4.3 Parameter \(\beta^2\)

When the complete expression (4.23) for the two-body scattering amplitude is used, the calculation of the pion optical involves the knowledge of the parameter \(\beta^2\). As mentioned earlier, the form (4.23) for the two-body amplitude has been used in the analyses of free pion-nucleon elastic scattering data\(^{(42),(43)}\) at pion momenta (laboratory) \(\geq 3\) GeV/c and the value of \(7.64\text{ (GeV/c)}^{-2}\) for \(\beta^2\) has been obtained from the analyses of pion-proton data\(^{(42)}\) at about 3 GeV/c. But the values for \(\beta^2\) in the energy region 0.585-1.057 GeV are not available and therefore, we have used the above value of \(7.64\text{ (GeV/c)}^{-2}\). But later we allowed the parameter \(\beta^2\) to vary by about 50\% in order to examine the sensitivity of the cross-sections to the variations in this parameter.

### 4.5 Effects of the exclusion principle, the direct absorption by pairs of nucleons and the Fermi motion

We assumed in section 4.2 that at high energies, the pion-target nucleon scattering matrix could be replaced by the free pion-nucleon scattering matrix. The use of the impulse approximation has been justified by Cronin et al.\(^{(41)}\) on the ground that the (second order) effects of the exclusion principle and the direct absorption by pairs of nucleons, being approximately equal in magnitude but opposite in sign, cancel out each other. The exclusion principle forbids elastic collisions in which the momentum transferred to the
struck nucleon (bound inside the target nucleus) is less than the
Fermi momentum and Cronin et al. have argued that this effect
reduces the pion-target nucleon total cross-section by about 10% for
negative pions at about 1 GeV whereas the direct absorption of
negative pions through the reactions such as \( \pi^- + p \rightarrow n + n \) and \( \pi^- + p \rightarrow p + n \)
which can take place in nuclei, enhance the pion-target nucleon
total cross-section by the same amount. However, it has been
pointed out by Beg that in the high energy region (~1 GeV) where
the imaginary parts of the free two-body forward scattering amplitude
are greater than the real parts, the exclusion principle enhances,
contrary to the argument of Cronin et al., the pion-target nucleon
total cross-sections. Using the Fermi gas model of the nucleus,
Beg has shown that the 'effective' \( \pi^- - p \) and \( \pi^- - n \) forward scattering
amplitudes \( f^{\text{eff}}_{\pi^- - p} (o) \) and \( f^{\text{eff}}_{\pi^- - n} (o) \) are given by

\[
f^{\text{eff}}_{\pi^- - p} (o) = f_{\pi^- - p} (o) - i \frac{k_F}{5k} \left[ (f_{\pi^- - p} (o))^2 + (f_{\text{CE}} (o))^2 \right] + i \frac{\Gamma \sigma_a}{4\pi}
\]

\[
f^{\text{eff}}_{\pi^- - n} (o) = f_{\pi^- - n} (o) - i \frac{k_F}{5k} (f_{\pi^- - n} (o))^2
\]

where \( f_{\pi^- - p} (o) \), \( f_{\pi^- - n} (o) \) are the free two-body forward scattering
amplitudes, \( f_{\text{CE}} (o) \) is the forward amplitude for the process (charge
exchange scattering) \( \pi^- + p \rightarrow \pi^0 + n \), and \( \sigma_a \) is the absorption cross-
section in deuterium. \( \Gamma \) is a proportionality factor which is given
by the ratio of the probability of finding two nucleons close together
in a nucleus to that in the loosely bound deuteron and the value of
10 for \( \Gamma \) has been taken by Cronin et al. and Beg. \( \hbar k_F \) is the
Fermi momentum and the value of \( \hbar k_F \approx 200 \text{ MeV/c} \) has been used. Defining the effective \( \pi^- - p \) and \( \pi^- - n \) total cross-sections as

\[
\sigma^{\text{eff}}_{\pi^- - N} = (4\pi/k) \cdot \text{Im} f^{\text{eff}}_{\pi^- - N} (o)
\]
and taking the value of 0.5 mb for the absorption cross-section \( \sigma_a \) in deuterium over the energy region 0.6-1.2 GeV, Beg\(^{18}\) has reanalysed the data of Cronin et al and the corrections due to the effects of the exclusion principle and the direct absorption by pairs of nucleons are, as estimated by Beg\(^{18}\), about 17% for the pion-nucleon total cross-section (averaged over isospin) at 0.97 GeV. But Beg\(^{18}\) has ignored, as in the work of Cronin et al\(^{41}\), the motion of target nucleons. Crozon et al\(^{22}\) have investigated the effect of the motion of target nucleons, in addition to that of the exclusion principle which has been taken into account in the same way as in the work of Watson et al\(^{52}\). Using the Fermi gas model of the nucleus, they have calculated the effect of the Fermi motion of target nucleons on the pion-nucleon total cross-sections in the energy region 0.5-1.3 GeV and have noted that the effect of Fermi motion is, in contrast to the result at low energies\(^{50}\), significant. In the energy region 0.5-1.3 GeV, the measured two-body total cross-sections\(^{35}\)(37) show resonance behaviour and the notable feature of the effect of the Fermi motion, as shown by Crozon et al\(^{22}\), is to smooth out the behaviour of the total cross-sections in this energy region. The correction due to this effect, as calculated by Crozon et al\(^{22}\), is approximately 5-9% at 0.9-1.3 GeV. It appears therefore that the total correction due to all the effects mentioned above is about 22-26% for the pion-nucleon total cross-section (averaged isospin).

However, the effects of the charge exchange scattering and the "off-the-energy-shell terms" have not been taken into account by the analyses\(^{18}\)(22)(41). It has been noted by Amblard et al\(^{51}\) that in the high energy region (1-1.9 GeV), the real parts of the charge exchange scattering amplitude \( f_{CE}(0) \) are large compared to the imaginary parts and in view of this, it appears that the effect of the
charge exchange scattering would be to reduce the pion-nucleon cross-section at these energies. But the corrections due to this effect are probably within the experimental errors. It is difficult to estimate the correction due to the effect of the off-the-energy shell terms and at present we do not know much about them. However, it may be possible in future to obtain some estimate of these corrections using the method suggested recently by Feshbach et al\(^{(53)}\) to calculate high energy elastic scattering.

Further, the corrections due to all the above effects for the real part of the forward amplitude cannot be estimated with any reasonable accuracy by analysing the data\(^{(22),(41)}\) because of the fact that the absorption and elastic cross-sections are rather insensitive, as noted by Batty\(^{(54)}\) to the real part. But it is indicated from the present work that some estimate of these corrections may possibly be obtained from the examination of differential cross-sections for high energy pions.

In the present work, we have not attempted to investigate these effects (mentioned in this section) as in the earlier analyses\(^{(18),(22)}\) but have varied the pion-nucleon total cross-section by 10-20\% and the real part of the forward amplitude over a wide range of values in order to examine the sensitivity of the calculated cross-sections to the possible uncertainties in these parameters arising from the neglect of the second order corrections to the pion optical potential. This is further discussed in sections (5.2-5.4), Chapter 5.

In the next section we have first discussed the nucleon distributions for \(^{12}\text{C}, \, ^{40}\text{Ca}, \, ^{208}\text{Pb}\), and later, the nucleon form factors for \(^{12}\text{C}, \, ^{40}\text{Ca}, \, ^{208}\text{Pb}\).
4.6 Proton and neutron distributions for $^{12}$C, $^{40}$Ca, $^{208}$Pb

We have constructed the proton and the neutron distributions (spherically symmetric) from single particle wave functions, as discussed before (see Chapter 2) but in the case of $^{208}$Pb, in addition, we have used the Fermi distribution. Fig. 28 shows the proton distribution for $^{12}$C obtained using the single particle wave functions. This proton distribution for $^{12}$C has been found satisfactory in explaining elastic electron scattering\(^{(1)}\). The single particle potential well parameters for $^{12}$C have been taken from the work of Elton and Swift\(^{(1)}\) and are listed in Table VIII. For $^{40}$Ca the proton distribution used is the same as before (see Chapter 2). Figs. 29a and 29b show the neutron and the proton distributions for $^{208}$Pb obtained using the single particle wave functions. The proton distribution for $^{208}$Pb shown in Fig. 29b fits elastic electron scattering\(^{(56)}\). The single particle potentials for neutrons have been taken from the work of Rost\(^{(55)}\) (see Table IXa) and we have used the single particle potentials for protons as given by Elton and Webb\(^{(56)}\) (see Table IXb). Further, we have also shown in Fig. 29b the proton distribution for $^{208}$Pb (Fermi distribution) as taken by Auerbach et al\(^{(34)}\) (see Table X). But we have assumed in this case that the neutron distribution is identical to the proton distribution. We have used this Fermi distribution in order that later the results for $^{208}$Pb from the present work can be compared with those obtained by Auerbach et al\(^{(34)}\).

4.6.1 Proton and neutron form factors for $^{12}$C, $^{40}$Ca, $^{208}$Pb

One needs to know the proton and the neutron form factors $F_p(q)$, $F_n(q)$ when the expression (4.15) (see section 4.1.1) for the pion optical potential is used. These form factors are defined as
### TABLE IXa

**Single particle neutron potential well parameters** for $^{208}\text{Pb}$

(For all levels)

<table>
<thead>
<tr>
<th>Reference</th>
<th>$v_\epsilon$ (MeV)</th>
<th>$v_{s\epsilon}$ (MeV)</th>
<th>$r_o$ (F)</th>
<th>$a$ (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rost</td>
<td>40.5</td>
<td>8.3</td>
<td>1.349</td>
<td>0.7</td>
</tr>
</tbody>
</table>

### TABLE IXb

**Single particle proton potential well parameters** for $^{208}\text{Pb}$

(For all levels)

<table>
<thead>
<tr>
<th>Reference</th>
<th>$v_\epsilon$ (MeV)</th>
<th>$v_{s\epsilon}$ (MeV)</th>
<th>$r_o$ (F)</th>
<th>$a$ (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elton and Webb</td>
<td>59.6</td>
<td>9.5</td>
<td>1.262</td>
<td>0.7</td>
</tr>
</tbody>
</table>

* See equation 2.44 (section 2.4, Chapter 2)
Fig. 28 Proton distribution for $^{12}_C$, calculated from single particle wave functions.
Fig. 29a Neutron distribution for $^{208}\text{Pb}$, calculated using single particle potential well parameters of Röst.
Proton distribution for $^{208}$Pb calculated from single particle wave functions, together with the Fermi distribution as taken by Auerbach et al.
Fig. 30  Proton form factor for $^{12}_C$, obtained using the distribution shown in Figure 28.
Fig. 31 Proton form factor for $^{40}\text{Ca}$, obtained using single-particle wave functions.
Proton form factor for $^{208}$Pb, obtained using the Fermi distribution as taken by Auerbach et al.
\[ F_p(q) = \int e^{iq\cdot r} \rho_p(r) \, dr \quad (4.34) \]
\[ F_n(q) = \int e^{iq\cdot r} \rho_n(r) \, dr \quad (4.35) \]

where \( F_p(o) = F_n(o) = 1 \).

We have obtained numerically the form factors for \( ^{12}_C \), \( ^{40}_Ca \), \( ^{208}_Pb \) as given by the above equations. Figs. 30 and 31 show the proton form factors for \( ^{12}_C \), \( ^{40}_Ca \). For these nuclei the neutron and the proton form factors are identical. It can be seen from Figs. 30-31 that relative more high momentum components are present in the proton (or equivalently, nuclear) form factors for \( ^{12}_C \), \( ^{40}_Ca \) than expected from the argument of Glauber (3) (see also section 4.3). In view of this, the use of the large \( A \) approximation in the case of these nuclei may be suspect. For \( ^{208}_Pb \) the pion optical potential as given by the expression (4.15) has been calculated using only the Fermi distribution and hence, we have shown in Fig.32 the proton form factor for \( ^{208}_Pb \) obtained using just this distribution. Again, because of the assumption made earlier, the neutron and the proton form factors are identical in this case. As can be seen from Fig.30, the proton form factor for \( ^{208}_Pb \) shows the expected behaviour (3)(28) and hence, one is probably justified in using as done by Auerbach et al.,(34) the large \( A \) approximation. However, it is found from the present work (see Chapter 5) that the errors resulting from the use of large \( A \) approximation are not insignificant.

4.7 Pion optical potentials for \( ^{12}_C, ^{40}_Ca, ^{208}_Pb \)

We have calculated the pion optical potentials for \( ^{12}_C, ^{40}_Ca, ^{208}_Pb \) using the large \( A \) approximation as well as the complete expression (see section 4.2) for the two-body scattering amplitude. In the calculations for the latter case, we have used the nucleon
Fig. 33  Pion optical potentials for $^{12}_C$ at 1.057 GeV.
Fig. 34. Pion optical potentials for $^{40}$Ca at 1.057 GeV.
TABLE X

Parameters of the Fermi distribution for protons

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Half-way radius R(F)</th>
<th>Diffuseness parameter a(F)</th>
<th>$&lt;r^2&gt;^{1/2}$ (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{208}_{\text{Pb}}$</td>
<td>6.628</td>
<td>0.5348</td>
<td>5.5</td>
</tr>
</tbody>
</table>

* As taken by Auerbach et al (Ref. 34)
† Root mean square radius.

TABLE XI

Pion-nucleon parameters used in the calculations of pion optical potentials for $^{12}_{\text{C}}$ and $^{40}_{\text{Ca}}$ shown in Figs. 33-34

<table>
<thead>
<tr>
<th>Pion kinetic energy (GeV)</th>
<th>Pion-nucleon* total cross-section $\tilde{\sigma}$ (mb)</th>
<th>Real part* of forward amplitude $\text{Re}(\sigma)(F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.057</td>
<td>33.85</td>
<td>-0.3</td>
</tr>
</tbody>
</table>

* Averaged over isospin.

TABLE XII

The strengths of pion optical potentials $U_{oo}^{0}$ for $^{12}_{\text{C}}$ and $^{40}_{\text{Ca}}$ at 1.057 GeV

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$\beta^2$ (GeV/c)$^{-2}$</th>
<th>$\text{Im}U_{oo}^{0}$ (MeV)</th>
<th>$\text{Re}U_{oo}^{0}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{12}_{\text{C}}$</td>
<td>7.64</td>
<td>-62.0</td>
<td>11.5</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-62.8</td>
<td>11.6</td>
</tr>
<tr>
<td>$^{40}_{\text{Ca}}$</td>
<td>7.64</td>
<td>68.1</td>
<td>12.6</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-78.5</td>
<td>14.5</td>
</tr>
</tbody>
</table>
form factors discussed in the previous section. The imaginary parts of the pion optical potentials for $^{12}$C and $^{40}$Ca at 1.057 GeV obtained using the complete expression for the two-body amplitude, are shown in Figs. 33 and 34. The value of $7.64 \text{(GeV/c)}^{-2}$ for $\beta^2$ has been used and the values used for the other two-body parameters are given in Table XI. Further, in Figs. 33 and 34 we have also shown the imaginary parts of the pion optical potential for $^{12}$C and $^{40}$Ca at 1.057 GeV obtained using the large A approximation. The real parts of the potentials in both the cases have the same shapes as the imaginary parts but differ in magnitudes. In Table XII, we have given the strengths of the potentials for $^{12}$C and $^{40}$Ca obtained for the two cases. It can be seen from Fig.33 that the pion optical potential for $^{12}$C obtained using the complete expression for the two-body amplitude is more diffuse than that obtained using the large A approximation, and there is a substantial difference between them in the transition region. Further, from Fig.34 it is easy to see that the two pion optical potentials for $^{40}$Ca differ similarly in the transition region. It has been noted by Jackson (11) that pion-nucleus elastic scattering in the GeV region is sensitive to this transition region and in view of this, it is interesting to see whether these differences between the potentials in the transition region can be detected in the analyses of high energy pion-nucleus elastic scattering and consequently, whether the nuclear matter distributions, at least for these nuclei with $A=2Z$, may be obtained accurately in the transition region which is also the region most accurately determined from elastic electron scattering. Using the pion optical potentials for $^{12}$C and $^{40}$Ca shown in Fig.33 and 34, we have calculated the cross-sections and it is found that the differential cross-sections for $^{12}$C and $^{40}$Ca are very sensitive to the differences between the potentials whereas the absorption cross-sections are reasonably
Fig. 35a Pion optical potentials for $^{208}$Pb at 0.7 GeV, obtained using the neutron distribution of Rost. We have used here the large $A$ approximation.
Fig. 35b Pion optical potential for $^{208}$Pb at 0.7 GeV, calculated using the complete expression for the two-body amplitude; together with the potential using the large A approximation. The Fermi distribution as taken by Auerbach et al has been used.
sensitive to these differences. This shows clearly that it is possible to determine more accurately the nuclear matter distributions for the nuclei with $A=2Z$, from the analyses of the differential cross-sections for high energy pions. This is further discussed in Chapter 5.

We have shown in Fig. 35a the imaginary parts of the pion (both $\pi^-$ and $\pi^+$) optical potentials for $^{208}$Pb at 0.7 GeV obtained using the large $A$ approximation. We have used the neutron distribution constructed from single particle wave functions by Rost\textsuperscript{(55)} and the proton distribution similarly constructed (see section 4.6). The strengths of the pion optical potentials obtained and the two-body parameters used are given in Tables XIIIa and XIIIb. The neutron distribution of Rost\textsuperscript{(55)} extends beyond the proton distribution\textsuperscript{(56)} (see Figs. 29a and 29b) and because of this, the imaginary parts of the pion optical potentials for positive and negative pions shown in Fig. 35a differ. Further, we have also calculated the pion optical potential for $^{208}$Pb at 0.7 GeV assuming that the neutron distribution is identical to the proton distribution\textsuperscript{(56)} (apart from normalisation) but these calculations have been made again using the large $A$ approximation. The imaginary part of the potential for $^{208}$Pb obtained in this case is also shown in Fig. 35a. The two-body parameters used are the same as before and the strengths of the imaginary and the real parts of the potential obtained are listed in Table XIIIa. In this case, the optical potentials for positive and negative pions are identical (apart from magnitude) and moreover, the real parts of the potential have the same shape as the imaginary part. The results for $^{208}$Pb obtained using the pion optical potentials shown in Fig. 35a are discussed in section 5.4, Chapter 5.

We have shown in Fig. 35b the pion optical potential for $^{208}$Pb
### TABLE XIIIa

The strengths of pion optical potentials $U_\infty^0$ for $^{208}_{\text{Pb}}$ at 0.7 GeV

<table>
<thead>
<tr>
<th>Neutron distribution</th>
<th>$\text{Im}U_\infty^0$ (MeV)</th>
<th>$\text{Re}U_\infty^0$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rost</td>
<td>-38.3$^a$</td>
<td>11.4$^a$</td>
</tr>
<tr>
<td></td>
<td>-36.3$^b$</td>
<td>12.3$^b$</td>
</tr>
<tr>
<td>Identical to $\rho_p$</td>
<td>-46.9$^a$</td>
<td>17.9$^a$</td>
</tr>
<tr>
<td></td>
<td>-54.7$^b$</td>
<td>14.6$^b$</td>
</tr>
</tbody>
</table>

a) Calculated for negative pions  
b) Calculated for positive pions.

### TABLE XIIIb

Pion-nucleon total cross-sections and real parts of forward amplitudes used in the calculations of potentials for $^{208}_{\text{Pb}}$ (at 0.7 GeV) shown in Figs. 35a-35b

<table>
<thead>
<tr>
<th>$\sigma_{\pi^-p}$ (mb)</th>
<th>$\sigma_{\pi^+p}$ (mb)</th>
<th>$\text{Ref}_{\pi^-p}(o)$ (F)</th>
<th>$\text{Ref}_{\pi^+p}(o)$ (F)</th>
<th>$\alpha_{\pi^-p}$</th>
<th>$\alpha_{\pi^+p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.8</td>
<td>16.89</td>
<td>-0.14</td>
<td>-0.42</td>
<td>-0.117</td>
<td>-0.745</td>
</tr>
</tbody>
</table>

* $\alpha_{\pi p} = \text{Ref}_{\pi p}(o)/\text{Imf}_{\pi p}(o)$
at 0.7 GeV obtained using the complete expression for the two-body amplitude. But we have used the nucleon form factors obtained using the Fermi distribution (see section 4.6.1). The value used for the parameter $\beta^2$ is 7.64 (GeV/c)$^{-2}$ and the values taken for the other two-body parameters are the same as before (see Table XIIIb). In Fig.35b we have also shown, for purposes of comparison, the pion optical potential obtained from the same Fermi distribution but using the large A approximation. It can be seen from Fig.35b that the two potentials do not differ substantially but, however, the differences between these potentials may yet become significant in the analysis of data much more accurate than those of Abashian et al (57) (see section 5.4, Chapter 5).

Further, in order to examine the use of the optical model near resonance energies, we have also calculated the pion ($\pi^+$ and $\pi^-$) optical potentials for $^{208}$Pb at 0.585 GeV which is near a resonance energy. We have used the neutron distribution of Rost (55) and the proton distribution (56). Later, assuming that the neutron and the proton distributions are identical, we repeated the calculations. But in all these calculations, we have used the large A approximation. The results obtained in this case are briefly discussed in sections 5.4 and 5.5, Chapter 5.

4.8 Klein-Gordon equation for high energy pions

We assumed in section 4.1.1 that pion-nucleus scattering at high energies can be described by the Klein-Gordon equation (4.18). Using the partial wave expansion for the scattering wave function $\psi(r)$, it can be shown that the equation (4.18) can be reduced to the radial form

$$\left[ \frac{d^2}{dp^2} + \frac{(E' + E)^2}{c^2} - 2E'U' + 2V'U' + U'' - \frac{\lambda(\lambda+1)}{\rho^2} \right] u_{\lambda} = 0 \quad (4.36)$$
where $u_\ell$ is a radial wave function, $\rho = kr$, $E' = E' / \hbar c$, $V'_c = V_c(\rho) / \hbar c$, and $U'_o(\rho) / \hbar c$. The Coulomb potential $V'_c$ is taken to be as that due to a uniformly charged sphere. The radial equation (4.36) involves terms such as $(U'_o)^2, V'_cU'_o$ and in the analysis of high energy pion-nucleus scattering by Auerbach et al. these terms have been ignored. But we have included all these terms in the present analysis and have calculated the radial wave functions $u_\ell$ from the equation (4.36) using numerical method (see Appendix B). For large $\rho$, the equation (4.36) becomes,

$$\left[ \frac{d^2}{d\rho^2} + 1 - \frac{2\alpha}{\rho} - \frac{\xi(\ell+1)-\xi^2}{\rho^2} \right] u_\ell = 0$$

where $\xi = \frac{Ze^2}{\hbar c}$, $\alpha = (\frac{\pi}{\hbar c})^2 Z e^2$.

$Z$ and $v$ are the charge of the pion and the pion velocity in the centre of mass frame for the pion-nucleus system.

Two independent solutions of equation (4.37) are $F_\ell(\rho)$ and $G_\ell(\rho)$ which have the asymptotic forms

$$F_\ell(\rho) \sim \sin \left( \rho - \frac{\xi \pi}{2} + \alpha \log(2\rho) \right)$$

$$G_\ell(\rho) \sim \cos \left( \rho - \frac{\xi \pi}{2} + \alpha \log(2\rho) \right)$$

where $\sigma_\ell$ is the Coulomb phase shift given by (59)

$$\sigma_\ell = \arg \Gamma(p+i\alpha) - \frac{\pi}{2} (\rho-z-1)$$

and

$$p = \frac{1}{2} + \left[ (\ell+\frac{1}{2})^2 - \xi^2 \right]^{\frac{1}{2}}$$

It may be noted that the term $\xi^2 / \rho^2$ in equation (4.37) arises from the inclusion of $V'_c$. In the absence of this term, Coulomb functions $F_\ell(\rho)$ and $G_\ell(\rho)$ reduce to the usual non-relativistic Coulomb functions. But including this term, we have calculated the Coulomb functions $F_\ell(\rho)$ and $G_\ell(\rho)$ from equation (4.37) starting with series solutions for small $\rho$ (see Appendix B). The Coulomb phase shifts $\sigma_\ell$. 


are obtained from equation (4.38) for each partial wave as in reference 59 by making use of Stirling's formula since, in this case, recurrence relations are not available.

By matching the radial wave functions calculated from equation (4.36) to those obtained from equation (4.37), we have obtained the phase shifts in the same way as described in section 2.6, Chapter 2. The results obtained are presented in sections 5.2-5.4, Chapter 5.
CHAPTER 5

Pion-nucleus elastic scattering at high energies

5.1 Theoretical elastic scattering cross-sections

We derived in Chapter 4 the pion optical potential at high energies and also showed there that the phase shifts $\delta_{\lambda}$ from the pion optical potential may be obtained using boundary conditions on the radial wave functions $u_{\lambda}$. Using these phase shifts $\delta_{\lambda}$, the pion-nucleus elastic scattering amplitude $F_{\pi N}(\theta)$ (in the centre of mass of the pion-nucleus system) can be calculated from the partial wave expansion,

$$F_{\pi N}(\theta) = \frac{1}{2ik} \sum_{\lambda=0}^{\infty} (2\lambda+1)(e^{2i\delta_{\lambda}}-1)P_{\lambda}(\cos\theta) + \frac{1}{2ik} \sum_{\lambda=0}^{\infty} (2\lambda+1)(e^{2i\delta_{\lambda}}-1)P_{\lambda}(\cos\theta)$$

where $\sigma_{\lambda}$ are the Coulomb phase shifts as obtained before (see equation (4.38), section (4.8), Chapter 4), $\theta$ is the angle of scattering and $k$ is the pion wave number in the centre of mass of the pion-nucleus system. The above expansion for $F_{\pi N}(\theta)$ can be written in the following form so that it converges more rapidly (59),

$$F_{\pi N}(\theta) = f_c(\theta) + \frac{1}{2ik} \sum_{\lambda=0}^{\infty} (2\lambda+1)e^{2i\sigma_{\lambda}^0}(e^{2i\Delta\sigma_{\lambda}^0} - 1)P_{\lambda}(\cos\theta)$$  (5.1)

where $\sigma_{\lambda}^0$ are the Coulomb phase shifts calculated from equation (4.37) (section (4.8), Chapter 4) without the term $\epsilon^2/\rho^2$, $f_c(\theta)$ is the corresponding Coulomb amplitude,

$$f_c(\theta) = \frac{-\alpha}{2k\sin^2 \frac{\theta}{2}} e^{-i\alpha \log(\sin^2 \theta) + 2i\sigma^0}$$  (5.2)

and $\Delta\sigma_{\lambda}^0 = \sigma_{\lambda} - \sigma_{\lambda}^0$

Further, $\eta_{\lambda}$ are the reflection coefficients (scattering matrix elements)

$$\eta_{\lambda} = e^{2i\delta_{\lambda}}$$

and the Coulomb parameter $\alpha$ which appears in equation (5.2) is the same
as before (see section (4.8), Chapter 4). Ignoring corrections of order $A^{-1}$, we obtain the differential elastic cross-sections (in the centre of mass frame for the pion-nucleus system)

$$\frac{d\sigma}{d\theta} = |F_{\pi A}(\theta)|^2$$

and the absorption cross-sections are given by

$$\sigma_a = \frac{4\pi}{k^2} \sum_{k=0}^{\infty} (1 - |n_k|^2)$$

The elastic (diffraction) cross-sections $\sigma_{el}$, because of the Coulomb interaction between the target nucleons and the incident pion, is not finite but if the Coulomb interaction is ignored, as in the analyses of the analyses $^{(22)}$, $^{(41)}$, $^{(48)}$, it can be shown

$$\sigma_{el} = \frac{4\pi}{k^2} \sum_{k=0}^{\infty} (1 - |n_k|^2)$$

We have calculated the absorption and differential cross-sections for $^{12}$C, $^{40}$Ca using the expressions (5.4) and (5.3) but later, we obtained the differential cross-sections in the laboratory frame for the pion-nucleus system using the fact that $|F_{\pi A}(\theta)|^2 d\Omega$ is an invariant. The results for $^{12}$C, $^{40}$Ca are presented in sections (5.2) and (5.3).

Further, ignoring the Coulomb interaction, we have calculated the total elastic cross-sections for $^{12}$C, using the expression (5.5). We have also calculated the absorption cross-sections for $^{208}$Pb but in the calculations for $^{208}$Pb, we have ignored the term $\frac{e^2}{\rho^2}$ in the equation (4.37). The results for $^{208}$Pb are presented in section (5.4). Further, we have presented in section (5.5) the discussion of the results for $^{12}$C, $^{40}$Ca, $^{208}$Pb.

5.2 Results for $^{12}$C

Using the pion optical potentials for $^{12}$C discussed in section (4.7), Chapter 4, we have obtained the absorption and the differential cross-sections for negative pions at 1.057 GeV but later, we varied the two-body parameters in order to test the sensitivity of these cross-
sections. First, we have presented below the results for the absorption cross-sections and later, the results for the differential cross-sections.

Table XIVa shows the results for absorption cross-sections for $^{12}$C at 1.057 GeV. It can be seen from there that these cross-sections remain almost unchanged when the value of the real part,
\[
\text{Re } \overline{f(0)} = \frac{1}{2} \text{Re } f_{\pi p}(0) + \text{Re } f_{\pi p}(0),
\]
of the two-body forward scattering amplitude is varied from $-0.3F$ to $0.3F$. We have, of course, held fixed the values of the pion-nucleon total cross-section $\sigma = \frac{1}{2} (\sigma + \sigma_{\pi n})$ and the parameter $\beta^2$ ($\sigma = 33.85(35)$, $\beta^2 = 0$ (large $A$ approximation)). It is clear that the absorption cross-sections are rather insensitive to the real part $\text{Re } \overline{f(0)}$, as also noted before. But these cross-sections are found to be reasonably sensitive to the pion-nucleon total cross-section $\overline{\sigma}$. We have varied $\overline{\sigma}$ by 10% and 20% from the measured value (35) of 33.85 mb, but have held fixed this time the values of $\text{Re } \overline{f(0)}$ and $\beta^2$ at $-0.3F$ and 0 respectively. As can be seen from Table XIVa, the changes in the absorption cross-sections produced by variations in $\overline{\sigma}$ of 10% and 20% are respectively 4% and 9%. Further we have investigated the effect of variations in the parameter $\beta^2$, keeping fixed the values of $\overline{\sigma}$ and $\text{Re } \overline{f(0)}$ at 33.85 mb and $-0.3F$. We used first the value of $7.64 (\text{GeV/c})^{-2}$ for $\beta^2$ and then varied it by 50%. It can be seen again from the Table XIVa that the absorption cross-sections are reasonably sensitive to the parameter $\beta^2$. In particular, the absorption cross-sections obtained using the values of 3.82 and 7.64 $(\text{GeV/c})^{-2}$ for $\beta^2$ differ from that obtained using the value of $\beta^2 = 0$ and the same values for $\overline{\sigma}$ and $\text{Re } \overline{f(0)}$ by 4% and 9%.

It may be pointed out that these changes in the cross-sections are approximately of the same order of magnitude as these changes in the cross-sections produced by variations in $\overline{\sigma}$ of 10% and 20%.

Further, ignoring the Coulomb interaction between the target
### TABLE XIVa

Results for absorption cross-sections for $^{12}$C at 1.057 GeV

<table>
<thead>
<tr>
<th>$\frac{\beta^2}{(\text{GeV})^{-2}}$</th>
<th>$\sigma$ (mb)</th>
<th>$\text{Ref}(\sigma)$ (F)</th>
<th>Absorption cross-section $\sigma_a$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>33.85</td>
<td>-0.30</td>
<td>229.8</td>
</tr>
<tr>
<td>0</td>
<td>33.85</td>
<td>0.00</td>
<td>231.0</td>
</tr>
<tr>
<td>0</td>
<td>33.85</td>
<td>+0.30</td>
<td>230.0</td>
</tr>
<tr>
<td>0</td>
<td>37.235</td>
<td>-0.30</td>
<td>240.9</td>
</tr>
<tr>
<td>0</td>
<td>40.62</td>
<td>-0.30</td>
<td>251.0</td>
</tr>
<tr>
<td>7.64</td>
<td>33.85</td>
<td>-0.30</td>
<td>246.0</td>
</tr>
<tr>
<td>3.82</td>
<td>33.85</td>
<td>-0.30</td>
<td>238.0</td>
</tr>
</tbody>
</table>

### TABLE XIVb

Results for absorption and diffraction cross-sections for $^{12}$C at 1.057 GeV

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\bar{\sigma}$ (mb)</th>
<th>$\text{Ref}(\sigma)$ (F)</th>
<th>Absorption cross-section $\sigma_a$ (mb)</th>
<th>Diffraction cross-section $\sigma_{el}$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present work</td>
<td>37.5</td>
<td>0.8</td>
<td>238</td>
<td>99</td>
</tr>
<tr>
<td>Crozon et al</td>
<td>37.5</td>
<td>0.8</td>
<td>234</td>
<td>93</td>
</tr>
</tbody>
</table>

### TABLE XV

Results for absorption cross-sections for $^{40}$Ca at 1.057 GeV

<table>
<thead>
<tr>
<th>$\frac{\beta^2}{(\text{GeV}/c)^{-2}}$</th>
<th>Absorption cross-section $\sigma_a$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>544</td>
</tr>
<tr>
<td>3.82</td>
<td>553</td>
</tr>
<tr>
<td>7.64</td>
<td>568</td>
</tr>
</tbody>
</table>
Fig. 36  Cross-sections for elastic scattering of pions from $^{12}\text{C}$ at 1.057 GeV calculated for different values of pion-nucleon total cross-sections.

$\pi^{-}-^{12}\text{C}$  $1.057\text{ GeV}$

- $\beta^2=0$  $\overline{\sigma}=33.85\text{mb}$  $\text{Re}\phi=0.3F$
- $\beta^2=0$  $\overline{\sigma}=37.235\text{mb}$  $\text{Re}\phi=0.3F$
- $\beta^2=0$  $\overline{\sigma}=40.62\text{mb}$  $\text{Re}\phi=0.3F$
Fig. 37 Cross-sections for elastic scattering of pions from $^{12}\text{C}$ at 1.057 GeV, calculated for different values of the parameter $\beta^2$. 

- $\beta^2 = 0 \quad \bar{\sigma} = 33.85\text{mb}$
  $\Re f(\omega) = -0.3F$

- $\beta^2 = 3.82(\text{GeV/c})^2 \quad \bar{\sigma} = 33.85\text{mb}$
  $\Re f(\omega) = -0.3F$

- $\beta^2 = 7.64(\text{GeV/c})^2 \quad \bar{\sigma} = 33.85\text{mb}$
  $\Re f(\omega) = -0.3F$
nucleons and the incident pion, we have also calculated the absorption
and elastic scattering cross-sections for $^{12}\text{C}$ at 1.057 GeV but we have
used the large $A$ approximation in the calculations. We have done so
because this enables us to compare the results obtained using the micro-
scopical description as in the present work with the results obtained
by Crozon et al\textsuperscript{(22)} who have used the semi-classical approximation.
Of course, the two-body parameters used are the same as in that work\textsuperscript{(22)}
($\sigma=37.5\text{mb}$, Re $f(0)=0.8F$). The results for $^{12}\text{C}$ obtained from the
present work are presented in Table XIVb and for purposes of comparison,
we have also reproduced there the results obtained by Crozon et al\textsuperscript{(22)}.

It can be seen from Table XIVb that the results for the absorption and
elastic cross-sections at 1.057 GeV obtained from the present work
agree satisfactorily with those obtained by Crozon et al\textsuperscript{(22)} and the
differences between the two sets of results are within experimental
errors quoted\textsuperscript{(22)}. This is further discussed in section (5.5).

Fig.36 shows the differential cross-sections for $^{12}\text{C}$ at
1.057 GeV (negative pions) obtained using the large $A$ approximation.

We have used the value\textsuperscript{(40)} of Re $f(0)=-0.3F$ and the value\textsuperscript{(35)} of
$\sigma=33.85\text{mb}$. Later we varied the value of $\sigma=33.85\text{mb}$ by 10% and 20%.

It can be seen from Fig.36 that the differential cross-sections for
$^{12}\text{C}$ are sensitive to the pion-nucleon total cross-section $\sigma$ and in
particular, the second minimum in the cross-section at about $\theta_L=27^\circ$
is much more sensitive to the variations in $\sigma$. Fig.37 shows the
effect of variations in the parameter $\beta^2$. The values of $\sigma$ and Re $f(0)$
have been held fixed at 33.85mb and $-0.3F$. As can be seen from Fig.37
that both minima in the cross-sections ($\theta_L=15^\circ$, and $\theta_L=27^\circ$) are
very sensitive to the value of $\beta^2$ and as the value of $\beta^2$ is increased
from 0 to 7.64 (GeV/c)$^{-2}$, deeper minima in the cross-sections are
Fig. 38 Cross-sections for elastic scattering of pions from $^{12}$C at 1.057 GeV, calculated for different values of the real part of the two-body forward amplitude.
Fig. 39 Comparison of elastic scattering cross-sections for negative and positive pions at 1.057 GeV from $^{12}_C$.
produced, in contrast to the more shallow minima produced by larger values of \( \sigma \). In particular, the cross-sections at the second minimum calculated using the values of \( \beta^2=0 \) and \( \beta^2=-7.64 \text{ (GeV/c)}^{-2} \) differ as much as by nearly an order of magnitude. We have shown in Fig.38 the effect of variations of \( \text{Re} f(0) \) on the cross-sections. Here, the values of \( \sigma \) and \( \beta^2 \) have been held fixed at 33.85 mb and 0. As can be seen from Fig.38, the differential cross-sections for \(^{12}\text{C} \) at the minima are very sensitive to the magnitude of \( \text{Re} f(0) \) as well as its sign. Again, deeper minima in the cross-sections are produced as the value of \( \text{Re} f(0) \) is increased from -0.3F to 0.3F, and the cross-section at the second minimum changes approximately by an order of magnitude between these values. It is clear that the real part \( \text{Re} f(0) \) becomes a significant parameter in the analysis of the differential cross-sections, as also noted by Formanek et al\(^{(6)}\) at higher energies (>1 GeV). Finally, we have also obtained the differential cross-sections for positive pions at 1.057 GeV, and the results for positive and negative pions are compared in Fig.39. It can be seen from there that the cross-sections for positive and negative pions differ substantially at the minima and the examination of these differences may provide yet more accurate information on the nuclear distribution for \(^{12}\text{C} \) (see section (5.5)).

It is evident from the above theoretical results for \(^{12}\text{C} \) that the analysis of data on differential cross-sections for pions at about 1 GeV will provide valuable information on the nuclear matter distribution as well as the two-body parameters. However, such data is not available at the present time but we have discussed in section (5.5) the degree of accuracy of the experimental data needed in order to obtain more accurate information on the nuclear matter distribution.
5.3 Results for $^{40}\text{Ca}$

We have calculated the absorption and differential cross-sections for $^{40}\text{Ca}$ at 1.057 GeV and have examined the effect of variations in the parameter $\beta^2$ and consequently, the use of the large $A$ approximation which is often taken to be a good approximation for a nucleus such as $^{40}\text{Ca}$. We have presented in Table XV the absorption cross-sections for $^{40}\text{Ca}$ calculated using the values of $\beta^2=0$, 3.82, 7.64 (GeV/c)$^{-2}$. The values of the pion-nucleon total cross-section $\sigma$ and the real part $\text{Re} f(0)$ of the two-body scattering amplitude have been held fixed at 33.85 mb$^{(35)}$ and $-0.3F^{(40)}$. It can be seen from Table XV that as the value of the parameter $\beta^2$ is increased from 0 to 7.64 (GeV/c)$^{-2}$, larger absorption cross-sections for $^{40}\text{Ca}$ are produced and the absorption cross-sections calculated using the value of $\beta^2=7.64$ (GeV/c)$^{-2}$ and the large $A$ approximation differ by about 5%. However, the changes in the absorption cross-sections produced by the variations in $\beta^2$ of the order 50% cannot be easily detected (see Table XV).

We have shown in Fig.40 the differential cross-sections for $^{40}\text{Ca}$ at 1.057 GeV (negative-pions) calculated using the values of $\beta^2=0$, 3.82, 7.64 (GeV/c)$^{-2}$. It can be seen from Fig.40 that the differential cross-sections for $^{40}\text{Ca}$ are very sensitive to the parameter $\beta^2$ and in particular, the minima in the cross-sections, as in the case of $^{12}\text{C}$, become deeper as the value of $\beta^2$ increased from 0 to 7.64 (GeV/c)$^{-2}$. Substantial changes in the cross-sections at the minima are produced by the variations in the parameter $\beta^2$ and the cross-section at the fourth minima ($\theta_L=34^\circ$) changes by approximately an order of magnitude between the values of 0 and 7.64 (GeV/c)$^{-2}$ for $\beta^2$. It appears therefore that it is possible to determine the parameter $\beta^2$ and consequently, the nuclear matter distribution for $^{40}\text{Ca}$ from the analysis of these cross-sections at 1.057 GeV, particularly near the minima. However,
Fig. 40 Cross-sections for elastic scattering of pions from $^{40}\text{Ca}$ at 1.057 GeV calculated for different values of the parameter $\beta^2$.
Fig. 4.1 Cross-sections for elastic scattering of pions from \( \text{Ca} \) at 1.057 GeV calculated for different values of the real part of two-body forward amplitude.
Fig. 42 Comparison of elastic scattering cross-sections for negative and positive pions at 1.057 GeV from $^{40}$Ca.

$\beta^2 = 0 \ \bar{\sigma} = 33.85\text{mb}$

$\text{Re} F(\omega) = -0.3F$

$\pi^- - Ca^{A_0}$

$\pi^+ - Ca^{A_0}$
as can be seen from Fig. 41, the effect of variations in $\text{Re} \, f(q)$ is similar to the effect of variations in $\beta^2$ at the minima in the cross-sections and hence, unless one of the two parameters is determined beforehand, it may not be possible to obtain unambiguously these parameters. This is further discussed in section (5.5). We have compared in Fig. 42 the differential cross-sections for positive and negative pions at 1.057 GeV. As can be seen from Fig. 42 the differential cross-sections for positive and negative pions differ substantially at the minima and as remarked before, it may be possible to derive more accurate information on the nuclear matter distribution for $^{40}\text{Ca}$ by examining these differences (see section (5.5)).

We have not been able to compare the present results for differential cross-sections with the experimental results, since such experimental data on $^{40}\text{Ca}$ do not exist at present. However, such data will be no doubt, valuable, in the investigation of the nuclear structure for $^{40}\text{Ca}$ as well as the pion-nucleon interaction at high energies.

5.4 Results for $^{208}\text{Pb}$

We have obtained the $\pi^+$ and $\pi^-$ absorption cross-sections for $^{208}\text{Pb}$ at 0.7 GeV using the pion optical potentials discussed before. The results for $q = \frac{\sigma(\pi^-\text{Pb})}{\sigma(\pi^+\text{Pb})} - 1$ at 0.7 GeV are presented in Table XVI. As can be seen from Table XVI, the value for $q$ obtained using the neutron distribution of Rost (55) and the proton distribution (56), is $-0.024$ whereas the experimental value obtained by Abashian et al (57) is $+0.05 \pm 0.01$. But if we take the neutron distribution for $^{208}\text{Pb}$ to be the same as the proton distribution (56), the value of $+0.046$ for is obtained (see Table XVI) and this is in reasonable agreement with the experimental value. A similar result for $q$ ($=+0.042$) is obtained
TABLE XVI
Results for \(^{208}\text{Pb}\) at 0.7 GeV

<table>
<thead>
<tr>
<th>Proton distribution</th>
<th>Neutron distribution</th>
<th>(\frac{&lt;r^2&gt;^{1/2}_n}{&lt;r^2&gt;^{1/2}_p})</th>
<th>(\beta^2) (GeV/c)^{-2}</th>
<th>(q^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elton &amp; Webb</td>
<td>Rost</td>
<td>1.12</td>
<td>0</td>
<td>-0.024</td>
</tr>
<tr>
<td>Elton &amp; Webb</td>
<td>Identical to the proton distribution</td>
<td>1.00</td>
<td>0</td>
<td>+0.046</td>
</tr>
<tr>
<td>Fermi (Auerbach et al)</td>
<td>Fermi</td>
<td>1.00</td>
<td>0</td>
<td>+0.042</td>
</tr>
<tr>
<td>Fermi (Auerbach et al)</td>
<td>Fermi</td>
<td>1.00</td>
<td>7.64</td>
<td>+0.039</td>
</tr>
</tbody>
</table>

\(q^+ = \frac{\sigma(\pi^-\text{Pb})}{\sigma(\pi^+\text{Pb})}^{-1}\)

\(<r^2>^{1/2}_n\) and \(<r^2>^{1/2}_p\) are the root mean square radii of neutron and proton distributions.

TABLE XVII
Results for \(^{208}\text{Pb}\) at 0.585 GeV
(calculated using the large A approximation)

<table>
<thead>
<tr>
<th>Proton distribution</th>
<th>Neutron distribution</th>
<th>(\frac{&lt;r^2&gt;^{1/2}_n}{&lt;r^2&gt;^{1/2}_p})</th>
<th>(q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elton &amp; Webb</td>
<td>Rost</td>
<td>1.12</td>
<td>-0.065</td>
</tr>
<tr>
<td>Elton &amp; Webb</td>
<td>Identical to the proton distribution</td>
<td>1.00</td>
<td>+0.033</td>
</tr>
</tbody>
</table>
using the Fermi distribution for protons (as taken by Auerbach et al\textsuperscript{(34)}) and assuming that the neutron and proton distributions are identical. As mentioned before, we have used the values for the $\pi^+ p$ total cross-sections as measured more accurately by Carter et al\textsuperscript{(37)} and since these values differ by about 20\% from those used by Auerbach et al\textsuperscript{(34)} who fit the experimental value\textsuperscript{(57)} for $q$, it follows the value of $q$ changes by about 19\%. (see Table XVI) when the values of the $\pi^+ p$ total cross-sections are changed by about 20\% as also noted before\textsuperscript{(34)}. Further, it is found that the value of $q$ is rather insensitive to the real parts $\pi^+ p$ forward scattering amplitudes, in agreement with the conclusion reached before\textsuperscript{(34)} and in view of this, we have not reproduced here those results. But in Table XVI, we show the results for $q$ obtained using the value of $\beta^2=7.64 \text{ (GeV/c)}^{-2}$. Here, we have used the Fermi distribution for protons mentioned above and assumed again that the neutron and the proton distributions are identical. It can be seen from Table XVI that the value of $q$ changes by about 7\% between the values of $\beta^2=0$ (large $\Lambda$ approximation) and $\beta^2=7.64 \text{ (GeV/c)}^{-2}$. Hence, it is clear that the change in the value of $q$ is reasonable and such a change may become significant in the analyses of data much more accurate than those of Abashian et al\textsuperscript{(57)}.

As mentioned earlier, we have also calculated the $\pi^+$ and $\pi^-$ absorption cross-sections at the pion energy 0.585 GeV. We have used the neutron and proton distributions constructed from single particle wave functions as before. But the calculations have been made using only the large $\Lambda$ approximation in view of the exploratory nature of this study. We have used the values for the $\pi^+ p$ total cross-sections as given by Carter et al\textsuperscript{(37)} and the values for the real parts of the $\pi^+ p$ forward amplitudes as obtained by Cronin\textsuperscript{(38)}. The results for $^{208}\text{Pb}$ at 0.585 GeV are presented in Table XVII. As can be seen from
there, the value of -0.065 for q is obtained if the neutron distribution of Rost\(^{(55)}\) is used. However, if it is assumed that the neutron distribution is identical to the proton distribution\(^{(56)}\), the value of +0.033 for q is obtained. Although these theoretical results for \(^{208}\)Pb at 0.585 GeV indicate that it is possible to distinguish between the above neutron distributions, it is necessary to know first whether these results are reliable. At the energy of 0.585 GeV, which is very close to a resonance, the off-the-energy-shell effects may become very important and thus, the use of the optical model itself may be questionable. Some indication as to the magnitude of these effects may probably be obtained from the work of Crozon et al\(^{(22)}\), if we treat the values of the two-body parameters required to fit data there, as 'experimental' and further, regard the resulting differences between these values and the values for the free two-body parameters after having allowed for the corrections due to the effects of the exclusion principle, the direct absorption by pairs of nucleons and the Fermi motion, as qualitative estimates, possibly, of the corrections due to the off-the-energy-shell effects. As mentioned earlier, the corrections due to the effects of the exclusion principle and the Fermi motion have been already obtained in the work of Crozon et al\(^{(22)}\) but the effect of the direct absorption by pairs of nucleons have not been taken into account by that work. If we allow for this effect as in the work of Beg\(^{(48)}\) (see also section (4.5) Chapter 4), it appears that the corrections for the pion-nucleon total cross-section near 0.585 GeV due to the off-the-energy-shell effects are, at most, of the order of about 0.13%. In view of this, the use of the optical model is probably reasonable but if experimental data on q at 0.585 GeV were available, it would have been possible to compare the results obtained here with those data and to examine further the use of the optical model at this energy.
5.5 The discussion of the results for $^{12}\text{C}$, $^{40}\text{Ca}$ and $^{208}\text{Pb}$

As can be seen from the results for $^{12}\text{C}$ presented in section (5.2), the absorption cross-sections for negative pions at 1.057 GeV (negative pions) are reasonably sensitive to the pion-nucleon total cross-section $\sigma$ (averaged over isospin) and the parameter $\beta^2$, whereas these cross-sections are rather insensitive to the real part of the two-body forward scattering amplitude (averaged over isospin) as noted before (22)(41)(54). Further, it appears that if the accuracy of the data is of the order of 2-4%, as in the work of Crozon et al (22), the significance of the differences between the values for absorption cross-sections obtained using the microscopic description as in the present work and the semi-classical approximation, is not apparent and one may equally use the semi-classical approximation in the analyses of such data. However, the complete expression for the two-body scattering amplitude should be used and unless this is done, the result for the nuclear matter distribution for $^{12}\text{C}$ may be open to re-interpretation as can be seen from the identical results for the absorption cross-section at 1.057 GeV obtained using the set of values $\sigma=37.5$ mb and $\beta^2=0$ and the set of values $\sigma=33.85$ mb and $\beta^2=3.82$ (GeV/c)$^2$ (see Tables XIVa and XIVb). It may be noted further that as a consequence of using the complete expression for the two-body scattering amplitude, the value for the pion-nucleon total cross-section $\sigma$ obtained is now close to the measured value (37) and is almost the same as the measured value (35) near 1 GeV. It follows therefore that if the value of $\beta^2$ lies in the region 3.82-7.64 (GeV/c)$^2$, the corrections due to the effects of the exclusion principle, the direct absorption by pairs of nucleons and the Fermi motion may be over-estimated, as in the work of Crozon et al (22) and in the analysis of Beg (48). In view of this, accurate estimate of the parameter $\beta^2$ from the analysis of pion-nucleon elastic scattering data around 1 GeV is necessary in order to examine
the above corrections for $\sigma$ and consequently, to obtain more accurately
the nuclear matter distribution for $^{12}\text{C}$.

As shown in section (5.2), the differential cross-sections for
$^{12}\text{C}$ at 1.057 GeV (negative pions) are quite sensitive to the pion-
nucleon total cross-section $\sigma$ and the parameter $\beta^2$, as well as to the
real part Re $\vec{f}(0)$ of the two-body forward scattering amplitude.

Changes in the differential cross-sections produced by variations of
the order 10 to 20% in the pion-nucleon total cross-section could
be detected from the analysis of data on $^{12}\text{C}$ at 1.057 GeV, if such
data were available (see Fig.36). The effect of variations in $\sigma$
can be seen much more clearly at the second minimum in the cross-section
($\theta_L \approx 27^\circ$) and this minimum becomes more shallow as the value of $\sigma$ is
increased from 33.85 to 40.62 mb (see Fig.36). Since the corrections
due to the effects of the exclusion principle, the direct absorption
by pairs of nucleons and the Fermi motion of target nucleons, are of
the order 20-26%, as shown before, for the pion-nucleon total cross-
section at 1.057 GeV, it is clear that these corrections may become
significant in the analyses of differential cross-sections for $^{12}\text{C}$ at
1.057 GeV. Further, it is found that the differential cross-sections
for $^{12}\text{C}$ at 1.057 GeV calculated using the large A approximation and the
complete expression for the two-body scattering amplitude differ
substantially and at the minimum ($\theta_L \approx 27^\circ$) changes of nearly an order
of magnitude are produced [see Fig.37 $\beta^2=0$ (large A approximation) and
$\beta^2=7.64 \text{ (GeV/c)}^{-2}$]. In addition, the differential cross-sections for
$^{12}\text{C}$ at 1.057 GeV are also very sensitive to the magnitude, as well
as the sign, of the real part Re $\vec{f}(0)$ of two-body scattering amplitude
(see Fig.38), in sharp contrast to the insensitivity of the absorption
cross-sections $^{(22)(41)(54)}$ to this parameter. Again, deeper minima
in the cross-sections are produced by larger values for Re $\vec{f}(0)$ as
in the case of the parameter $\beta^2$. This indicates that in the analyses of differential cross-sections at about 1 GeV, the real part $\text{Re} f(0)$ of the two-body scattering amplitude may be regarded as a significant two-body parameter.

It is clear that it is possible to determine the two-body parameters $\bar{\sigma}$, $\text{Re} f(0)$, $\beta^2$ from the analyses of data on these cross-sections in the energy region around 1 GeV, and consequently, to obtain the nuclear matter distribution for $^{12}\text{C}$. In particular, the examination of the structure of the differential cross-sections near the minima may provide information on the nuclear matter distribution as well as on the two-body parameters. However, if the accuracy of the experimental data is limited to about 25-30%, it appears that it may not be possible to determine the two-body parameters unambiguously since two sets of values such as $\bar{\sigma}=33.85$ mb, $\beta^2=3.82$ (GeV/c)$^{-2}$, $\text{Re} f(0)=-0.3F$ and $\bar{\sigma}=33.85$ mb, $\beta^2=0$ $\text{Re} f(0)=0$ may produce equally good fits to such data, within experimental errors (see Figs. 37 and 38). Hence, in the event that the accuracy of the experimental data is limited to 25-30% or so, an accurate estimate of the parameter $\beta^2$ from the analysis of pion-nucleon elastic scattering data in the energy region around 1 GeV may be necessary in order to remove these possible ambiguities. If the value of $\beta^2$ is determined independently as mentioned above, the other two parameters $\bar{\sigma}$, $\text{Re} f(0)$ can be obtained from the analysis of data on differential cross-sections but any remaining uncertainties in $\bar{\sigma}$ may be removed by fitting further consistently the data on absorption cross-sections. This procedure may determine accurately the parameter $\bar{\sigma}$ and hence the parameter $\text{Re} f(0)$ and eventually the nuclear matter distribution for $^{12}\text{C}$. It may be noted that it is possible to obtain in addition to the magnitude, the sign of the phase of the two-body forward scattering amplitude from
the analysis of data on differential cross-sections in the high energy region.

Further, the differential cross-sections for $^{40}$Ca at 1.057 GeV (negative pions) are also sensitive to the two-body parameters (see Figs. 40 and 41) and, in particular, it is found that contrary to what one may reasonably expect from the argument of Glauber and the analysis of high energy reactions by Jackson, there are substantial changes in the cross-sections calculated using the large $A$ approximation and the complete expression for the two-body scattering amplitude and changes of nearly an order of magnitude, as in the case of $^{12}$C, may be produced at the minima. (See Fig. 40, the first minimum at $\theta_L=10^\circ$). The absorption cross-sections for $^{40}$Ca at 1.057 GeV are also reasonably sensitive to the parameter $\beta^2$ and the changes in the absorption cross-sections produced by the values between 0 and $7.64 \text{ (GeV/c)}^{-2}$ for $\beta^2$ are of the order of 5%, and can be detected if the accuracy of the measurements is as good as in the work of Crozon et al. But the absorption cross-sections for $^{40}$Ca are rather insensitive, as in the case of $^{12}$C, to the real part $\text{Re} f(0)$ of the two-body scattering amplitude whereas the differential cross-sections are very sensitive to the magnitude as well as the sign of this parameter. It is clear that it is possible to obtain more accurately the nuclear matter distribution for $^{40}$Ca from the analysis of data on these cross-sections but again it may be preferable that the parameter $\beta^2$ is determined independently as mentioned before.

As can be seen from Figs. 39 and 42, the differential cross-sections for $^{40}$Ca and $^{12}$C calculated for positive and negative pions at 1.057 GeV, differ substantially only at the minima and since the minima in the cross-sections, as discussed before, are sensitive to the two-body parameters, it follows that the examination of the differences between the cross-sections for positive and negative pions
at 1.057 GeV would also provide information on the two-body parameters and consequently, the nuclear matter distributions for $^{40}$Ca and $^{12}$C. This is significant in view of the fact that it may be possible to measure the differences between the cross-sections for positive and negative pions in the energy region around 1 GeV, with good accuracy even though the absolute value of each cross-section cannot be measured to the same accuracy. If this can be done, it is clear that the nuclear matter distributions for $^{12}$C and $^{40}$Ca can be obtained much more accurately from the analyses of such data than from the analyses of individual data on either positive or negative pions.

From the results for $^{208}$Pb at 0.75 GeV presented in Table XVI, it is clear that the value for $q = \frac{\sigma(\pi^-Pb)}{\sigma(\pi^+Pb)} - 1$ obtained using the single particle neutron distribution as constructed by Rost and the proton distribution similarly constructed, is in disagreement with the experimental value for $q$ obtained by Abashian et al. However, if the neutron distribution is assumed to be identical to the single particle proton distribution or the phenomenological Fermi distribution for protons as taken by Auerbach et al, the value for $q$ obtained in this case is in reasonable agreement with the experimental value for $q$ and the discrepancies between theory and experiment are well within experimental errors quoted. These conclusions are consistent with those reached before. Further, it is found that the changes in the value of $q$ produced by the variations of the order 20% in the $\pi^+p$ total cross-sections are approximately of the same order of magnitude as also noted before but these changes in the value of $q$ cannot be detected if the accuracy of the data is such as that in the work of Abashian et al. The value of $q$ is also reasonably sensitive to the parameter $\beta^2$ and, in particular, the difference between the values of $q$ calculated using the largest approximation ($\beta^2=0$) and the value of $\beta^2=7.64$ (GeV/c)$^{-2}$ is about 7%
(see section (5.4)) but again, this difference cannot be detected from the analysis of data such as those of Abashian et al. It appears therefore that the difference between the root mean square radii of the neutron and the proton distributions for $^{208}$Pb, present cannot be determined better than $75\%$ from the analysis of data of Abashian et al. In order to do so, more accurate data on $q$ at $0.7$ GeV are required but, in addition, a precise estimate of the corrections due to the effects of the exclusion principle, the direct absorption by pairs of nucleons and the Fermi motion of target nucleons, for the $n^+p$ total cross-sections $^{22}/^{48}$ at $0.7$ GeV is necessary, since these corrections which are of the order of $20\%$ or less may now become significant. Further, if the accuracy of data is better than $7\%$, the differences between the values of $q$ calculated using the large $A$ approximation and the complete expression for the two-body scattering amplitude which are of the order of $7\%$ as mentioned above, also become significant. If the parameter $\beta^2$ is determined independently, then it is possible to obtain much more accurately the neutron distribution for $^{208}$Pb from the analysis of these data when they become available. However, it may be noted that even if the accuracy of the data is better than $7\%$, it may not yet be possible to reach any firm conclusion about the neutron distribution for $^{208}$Pb from the analysis of these data on $q$ before the small uncertainties in the proton distribution which might change the value of $q$ as much as by about $10\%$ are removed, as can be seen from the results for $q$ (see Table XV) obtained using the two proton distributions which differ very slightly (difference of $0.07\%$ in the root mean square radii) but both fit elastic electron scattering. If this is done, the difference between the root mean square radii of the neutron and the proton distributions for $^{208}$Pb can be obtained with a high degree of accuracy.
The theoretical results for $^{208}$Pb at 0.585 GeV indicate that it is possible to distinguish between the two neutron distributions such as the above. However, this is based on the argument that the use of the optical model at this energy, is probably reasonable. Although there appears to be some indirect support for this argument (see section (5.4)), it remains to be seen whether the use of the optical model at 0.585 GeV can be fully justified, when the experimental data on q at this energy become available. But if it proves to be so, it would, no doubt, be a significant outcome because it may be possible to obtain, later, consistently and hence, accurately the neutron distribution for $^{208}$Pb from the analyses of measurements with good accuracy, in the entire energy region of 0.5-1 GeV. It may be worth pointing out here that accurate measurements of q at higher energies (in the region of 2 GeV) although this is a more energy region for the investigation of the neutron distribution in view of the smooth behaviour of the $\pi^+\text{-p}$ total cross-sections in this energy region,\textsuperscript{(35)}\textsuperscript{(37)} are more difficult to obtain.
CHAPTER 6
Summary and Conclusions

This work has been concerned with elastic scattering of medium energy alpha particles and high energy pions, using optical potentials as obtained from the microscopic description. First, we considered elastic scattering of medium energy alpha particles and using the method of Kerman, McManus and Thaler(28), we derived the optical potential for alpha particles. We concerned ourselves only with the lowest order potential and assumed that we could ignore higher order terms involving intermediate excited states of the target nucleus. The lowest order potential involved a knowledge of the alpha-target nucleon interaction as well as that of the nuclear matter distribution. Although one could, in principle, evaluate this lowest order potential, it was found necessary in view of the complicated nature of the alpha-target nucleon, to use an effective interaction. This approximation allowed us to replace the alpha-target nucleon by an effective interaction which was treated as a phenomenological interaction. We first investigated whether we could determine the volume integral and the mean square radius of this interaction from the plot of the volume integral of the potential against $A-1$ and from the plot of the mean square radius of the potential against $A^{2/3}$ respectively, as predicted by the microscopic description. In doing so, we used the data on the phenomenological potentials for alpha particles, over a wide range of nuclei, available in the literature and treated the values obtained for the mean square radii and the volume integrals of these potentials, as 'experimental'. This procedure yielded the average values for the mean square radius and the volume integral of the two-body (alpha-nucleon) interaction and taking first the single Yukawa form for
the effective interaction, we examined whether this procedure was adequate. It was found that the value so obtained for the range parameter of the single Yukawa interaction did not correctly reproduce the diffraction pattern as shown by the experimental data. In view of this, we treated, subsequently, the parameters of the single Yukawa interaction as variable parameters and attempted to fit the data. Although the single Yukawa interaction could reproduce the qualitative features of the experimental data, it was found that this interaction generally failed to yield the required deep diffraction pattern of the data. The lack of ability of this interaction to explain satisfactorily the measured cross-sections led us to try the Gaussian form for the effective interaction. In this case, we did not attempt to vary the range parameter but used the value as given by Morgan and Jackson (24). However, the strengths of the interaction were adjusted in order to obtain the correct magnitudes of the cross-sections. In sharp contrast to the poor quality fits produced by the single Yukawa interaction, the Gaussian interaction yielded much improved fits to data and these fits obtained were comparable in terms of quality to those obtained using the phenomenological potentials. In view of this success with the Gaussian interaction, it was clear that optical potentials containing only three parameters, as in the present work, could reproduce reasonably the features of the experimental data. However, from the case of the Gaussian interaction, it also emerged that the choice of the form for the effective interaction was rather crucial. and if a suitable choice was made, one could explain successfully the experimental data on medium energy alpha particles.

The calculations of the optical potentials involved the nuclear matter distributions and we followed the method of Elton and Swift (1)
to obtain the proton distributions as well as the neutron distributions. It was found that the neutron (or nuclear matter) distributions so obtained for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$, $^{48}\text{Ca}$ could explain reasonably well the experimental data (12)(32) on those nuclei, in particular when the Gaussian interaction was used. However, it appeared that the cross-sections were not very sensitive to the neutron distribution, since a substantial change in the neutron distribution for $^{48}\text{Ca}$ produced only a small change in the cross-sections for this nucleus. Further, it was observed that these small changes in the cross-sections produced by the changes in the neutron distribution were nearly of the same order of magnitude as those produced by the changes of about 5% or so in the range parameter. It was therefore clear that the small uncertainties in the two-body parameters became significant, when one was concerned with more accurate determination of the neutron distribution. In view of this, it was not possible to examine further the reliability of the method of Elton and Swift (1) for neutrons, although the neutron distributions for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$ obtained using their method were found to reproduce reasonably well the observed features (12)(32) of the cross-sections.

We investigated further whether the microscopic description of elastic scattering of alpha particles could also explain the behaviour of the strong absorption radii for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$ as has been found from the phenomenological analysis of Blair and Fernandez (12). Defining the strong absorption radii as explained in Chapter 2, we calculated these radii for $^{40}\text{Ca}$, $^{42}\text{Ca}$, $^{44}\text{Ca}$ and $^{48}\text{Ca}$ using both the single Yukawa and Gaussian forms for the effective interaction and in each case it was found that the resulting strong absorption radii showed systematic A-dependence. However, it was noted that the Gaussian form reproduced better the behaviour of these radii as observed from the phenomenological analysis (12) than
the single Yukawa form. From this outcome, it was evident that
the behaviour of the strong absorption radii could be explained
using the microscopic description if, as also noted earlier, the
form for the effective interaction was chosen suitably. In view of
the ability of the microscopic description of elastic scattering of
alpha particles to reproduce, with reasonable success, the behaviour
of the strong absorption radii, we examined further the sensitivity
of the strong absorption radius to the changes in the neutron
distribution and the range parameter. It was found that the strong
absorption radius was sensitive to the changes in the neutron
distribution as well as to the changes in the range parameter and
that the changes in the neutron distribution and the range parameter
produced changes in the strong absorption radius of the same order
of magnitude. This provided yet more evidence that in the elastic
scattering of alpha particles the strong absorption radius may be
regarded as a sensitive size parameter.

Later, we examined the elastic scattering of pions at energies
in the range 0.585-1.057 GeV and here again we followed the method of
Kerman, McManus and Thaler, which allowed us to derive the pion
optical potential. But in the present work we concerned ourselves
only with the lowest order pion optical potential and, as before,
eglected the higher order terms involving the intermediate excited
states of the target nucleus. Using the lowest order pion optical
potential, we calculated elastic scattering of high energy pions from
the Klein-Gordon equation. However, the lowest order pion optical
potential involved the pion-target nucleon scattering matrix as well
as the neutron and proton form factors. We used again the method of
Elton and Swift to obtain the neutron and proton distributions and,
subsequently, the nucleon form factors and used the impulse approxi-
mation which has been found satisfactory at high energies. We
were then able to calculate the lowest order pion optical potentials for $^{12}$C, $^{40}$Ca, $^{208}$Pb as well as the (elastic) differential cross-sections for $^{12}$C, $^{40}$Ca. The calculations were carried out in each case using the complete expression for the two-body scattering amplitude and also the large A approximation. This was done in order to examine the use of the large A approximation in various high energy analyses of pion-nucleus scattering data on light and medium-heavy nuclei and in particular, to see how the use of this approximation influenced the determination of neutron (or nuclear matter distributions). From the theoretical results obtained for the absorption cross-sections for $^{12}$C at 1.057 GeV, it was found that these cross-sections obtained using the complete expression for the two-body amplitude and the large A approximation, differed by 4-9%, and these changes of 4-9% in the cross-sections were nearly of the same order of magnitude as those changes in the cross-sections produced by variations of 10-20% in the pion-nucleon total cross-section (see Chapter 5). It was indicated from this investigation and the later comparison of our results for $^{12}$C at 1.057 GeV with those of Crozon et al. that the corrections due to the effects of the exclusion principle, the direct absorption by pairs of nucleons and the Fermi motion might be over-estimated by those analyses using the large A approximation and further, the information obtained there on the nuclear matter distribution, particularly in the surface region, might need to be re-examined.

Further, the theoretical results for the differential cross-sections for $^{12}$C, $^{40}$Ca at 1.057 GeV showed that these cross-sections were quite sensitive to the variations in the parameter $\beta^2$ and the pion-nucleon total cross-section as well as to the variations in the real part of two-body forward amplitude. In particular, it was noted...
that at the minima in the cross-sections, changes of nearly an order of magnitude were produced, when the parameter $\beta^2$ was varied from 0 to $7.64 \text{ (GeV/c)}^{-2}$. Although this indicated that the examination of the structure of the cross-sections near the minima might provide information on the parameter $\beta^2$ and consequently, the nuclear matter distribution, it appeared that the parameter $\beta^2$ and the real part of the two-body scattering amplitude could not possibly be determined unambiguously. In view of this, it emerged that an independent estimate of the parameter $\beta^2$ from analysis of pion-nucleon scattering data at about 1 GeV was needed in order to be able to obtain more accurately the nuclear matter distribution. In addition, such an independent information on $\beta^2$ would also help to determine the magnitude, as well as it sign, of the real part of the two-body forward amplitude.

From the results obtained for $^{208}$Pb at 0.7 GeV, it was found that the neutron distribution of Rost$^{(55)}$ did not yield the experimental value$^{(57)}$ for $q = \frac{a(\pi^-\text{Pb})}{a(\pi^+\text{Pb})}$ but when the neutron distribution for $^{208}$Pb was taken to be identical to the proton distribution of Elton and Webb$^{(56)}$, it was possible to reproduce reasonably that experimental value for $q$. This outcome was consistent with the conclusion reached before$^{(34)(62)}$. However, it appeared that the difference between the root mean square radii of the neutron and proton distributions for $^{208}$Pb could not possibly be determined better than 75%, if the accuracy of the data on $q$ at 0.7 GeV was of the order of 20% as in the work of Abashian et al$^{(57)}$. Although this showed clearly more accurate data were needed, it was further revealed from the present work that in the analyses of such data, particularly when $q$ was measured better than about 7%, it was necessary to use the complete expression for the two-body scattering amplitude in order to obtain
accurately the neutron distribution for $^{208}$Pb. However, it was noted that in this situation, the small uncertainties in the proton distribution obtained from elastic electron scattering might also become probably significant and in view of this, it emerged that these small uncertainties in the proton distribution might have to be first removed before reaching firm conclusions about the neutron distribution for $^{208}$Pb.
Appendix A

It is convenient to express the free two-body scattering matrix $t_{\pi N}(q)$ in the centre of mass frame for the pion-nucleus system in terms of the free two-body scattering matrix $t^{(2)}_{\pi N}(q)$ in the centre of mass frame for the pion-nucleon system. In order to do this, we note that although the two-body scattering matrix $t_{\pi N}(q)$ is not invariant, the quantity $E_1 E_2 t_{\pi N}(q)$ is an invariant \(28\), where $E_1 E_2$ are the total energies of the incident pion and a target nucleon in the centre of mass frame for the pion-nucleon system. Therefore, we have

$$E_1 E_2 t_{\pi N}(q) = W_1 W_2 t_{\pi N}(q) \quad (A1)$$

Where $W_1, W_2$ are the total energies of the incident pion and a target nucleon in the centre of mass frame for the pion-nucleon system. It may be noted that the square of the four momentum transfer $q$ is also an invariant. The relation (A1) has been used in obtaining the equation (4.21), section 4.2 (Chapter 4).
Appendix B

Radial wave functions \( u_\ell \)

The Klein-Gordon equation is solved numerically, first noting that the radial wave function \( U_\ell (\rho) \), for very small \( \rho \), can be taken to be given by the power series,

\[
U_\ell (\rho) = \rho^{\ell + 1} \sum_{n=0}^{\infty} b_{n\ell} \rho^n
\]

(B1)

The series solution (B1), for very small \( \rho \), satisfies the Klein-Gordon equation (4.36) and thus we have

\[
\left[ \frac{d^2}{d\rho^2} + a_1 a_2 \rho^2 + a_3 \rho^4 - \frac{\ell (\ell + 1)}{\rho^2} \right] U_\ell (\rho) = 0
\]

(B2)

Where \( U_\ell (\rho) \) is given by the equation (B1),

\[
a_1 = 1 - 3 \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} + \frac{9}{4} \alpha_1^2 - \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} \frac{\alpha_1}{2} \right)
\]

\[
+ 3 \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} \right)
\]

\[
a_2 = \frac{1}{\rho c} \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} - \frac{3}{2} \alpha_1 \frac{2}{\rho c} - \frac{1}{2} \frac{2}{\rho c} \left( \frac{m}{c^2} \right) \frac{\alpha_1}{\lambda k} \right)
\]

\[
a_3 = \frac{4}{\rho c} \frac{2}{\rho c}
\]

\[
\lambda = \frac{\hbar}{m c} \quad (m \text{ is the rest mass of the pion},
\]

\[
\alpha_1 = \frac{2mZ}{137 \rho c}
\]

and we have used the appropriate expression for the Coulomb potential

\[
V_c(\rho) = \frac{Z^2 Ze^2}{2\rho_c} \left( 3 - \frac{\rho^2}{\rho_c^2} \right)
\]

where \( \rho_c = kR_c \), \( R_c = r_c A^{1/3} \).
Equating, in (B2), the coefficient of $\rho^n$ to zero, we get

$$\left[(n-1)+2(n+1)\right] b_{n,\ell+1} + a_1 b_{n-2,\ell} + a_2 b_{n-4,\ell} + a_3 b_{n-6,\ell} = 0$$  \hspace{1cm} (B3)

It follows from the recurrence relation (B3),

$$b_{1,\ell} = 0,$$

$$b_{2,\ell} = -\frac{a_1 b_{0,\ell}}{6+4\ell},$$

$$b_{3,\ell} = 0,$$

$$b_{4,\ell} = -\frac{a_1 b_{2,\ell} + a_2 b_{0,\ell}}{20+8\ell},$$

and similarly, other $b_{n,\ell}$'s can be obtained easily. We have obtained the series solution (B1) using $b_{n,\ell}$'s as determined above, and starting with this series solution, we have calculated, later, numerically the radial wave functions $U_\ell(\rho)$, using the Runge-Kutta method. It may be noted that it is not necessary here to determine the coefficients $b_{0,\ell}$, since they do not appear in the final calculations of phase shifts $\delta_\ell$.

**Coulomb Functions $F_\ell$**

The Coulomb function $F_\ell(\rho)$ which is regular at the origin, is given by the power series

$$F_\ell(\rho) = \rho^p \sum_{h=0}^{\infty} c_{n,\ell} \rho^n$$  \hspace{1cm} (B4)

where $p = 1 + \left[\left(\ell + \frac{1}{2}\right)^2 - \frac{1}{4}\right]^{\frac{1}{2}}$. Noting that the Coulomb function $F_\ell(\rho)$ as given by (B4) is a solution of the Klein-Gordon equation (4.37) and following the above procedure, it is easy to show (59),

$$\left[2p + n(n-1)\right] c_{n,\ell} + c_{n-2,\ell} - 2a c_{n-1,\ell} = 0$$  \hspace{1cm} (B5)

The above recurrence relation determines the coefficients $c_{n,\ell}$'s and the first four coefficients $c_{n,\ell}$, for example, are obtained as
\[ c_{1\ell} = \frac{\alpha}{p} c_{0\ell} \]
\[ c_{2\ell} = \frac{(2\alpha^2 - p)}{2p(2p+1)} c_{0\ell} \]
\[ c_{3\ell} = \frac{(2\alpha^2 - 3p - 1)c_{0\ell}}{6p(2p+1)(p+1)} \]
\[ c_{4\ell} = \frac{(2\alpha c_{3\ell} - c_{2\ell})}{(8p+12)} \]

where \( c_{0\ell} \) is given by \(^{(59)}\)
\[ c_{0\ell} = \frac{z^{p-1} - \frac{\pi}{2\alpha} \Gamma(p+i\alpha)}{\Gamma(2p)} \]

Using the above relations and starting with reasonable values of \( \rho \),
we have calculated numerically as before the Coulomb function \( F_{\ell}(\rho) \)
and repeated the calculations for each partial wave, since the recurrence
relations for \( F_{\ell}(\rho) \) are not available.

Coulomb functions \( G_{\ell} \)

We find it convenient to take a second independent solution \( G_{\ell}(\rho) \)
of the Klein-Gordon equation \((4.37)\) having the following asymptotic form,
\[ G_{\ell}(\rho) \sim \cos(\rho - \frac{\ell\pi}{2} + \sigma_{\ell} - \alpha \log(2\rho)) \]

Therefore, we choose the function \( G_{\ell}(\rho) \) as
\[ G_{\ell}(\rho) = \frac{(\tilde{G}_{\ell}(\rho) - \cos \phi_{\ell} F_{\ell}(\rho))}{\sin \phi_{\ell}} \]

where \( F_{\ell}(\rho) \) is the Coulomb function as before,
\( \tilde{G}_{\ell}(\rho) \) is the Coulomb function (not regular at the origin) as given
by Elton (Ref. 59, thesis), \( \phi_{\ell} \) is given by
\[ \sin \phi_{\ell} = \sin(\rho - p)/((1+2X \cos(p-p) + X^2), \]
\[ X = e^{2\pi\alpha}, \quad \overline{p} = \frac{1}{2} - \left[\left(\frac{\ell+1}{2}\right)^2 - e\right]^{\frac{1}{2}} \]
and \( p \) is the same as before.

Since the Coulomb function \( \tilde{G}_{\ell}(\rho) \) has the asymptotic form \(^{(59)}\)
\[ \tilde{G}_{\ell}(\rho) \sim \sin(\rho - \frac{\ell\pi}{2} + \sigma_{\ell} - \alpha \log(2\rho) + \phi_{\ell}) \]
it is easy to show that the Coulomb function $G_k(\rho)$ as given by (89) has the required asymptotic form (B8). We may obtain the Coulomb functions $G_k(\rho)$ from the relation (B9) but first we need to calculate the Coulomb functions $\tilde{G}_k(\rho)$. (The calculation of $F_k(\rho)$ has already been described.) $\tilde{G}_k(\rho)$ may be obtained as a power series (59)

$$\tilde{G}_k(\rho) = \frac{\rho}{\rho} \sum_{n=0}^{\infty} a_{nk} \rho^n$$

We note that $\tilde{G}_k(\rho)$ satisfies the Klein-Gordon equation (4.37). Thus, the coefficients $a_{nk}$ can be obtained as before and $\tilde{G}_k(\rho)$ can be calculated numerically. However, during the calculations, it was found that because of the factor $\rho^{1/2}$ appearing in (B10), numerical results obtained for $\tilde{G}_k(\rho)$ varied significantly when the starting value for $\rho$ was allowed to be slightly different. In view of this, we adopted a somewhat different method and calculated first the functions $\nu_k(\rho)$ where $\tilde{G}_k = \rho \nu_k$. The functions $\nu_k$ satisfy the equation,

$$\left[ \frac{d^2}{d\rho^2} + 2\rho \frac{d}{d\rho} - (1 - \frac{2\alpha}{\rho}) \right] \nu_k = 0$$

Writing $\nu_k(\rho)$ as a power series,

$$\nu_k(\rho) = \sum_{n=0}^{\infty} a_{nk} \rho^n$$

it is easy to show that $a_{nk}$ are given by,

$$n(n-1+2\rho) a_{nk} + a_{n-2,k} - 2n a_{n-1,k} = 0$$

The coefficients $a_{nk}$ as given by Elton (59) are

$$a_{nk} = \frac{\rho^{p-1} \rho^{\pi\alpha}}{\rho(2\rho)}$$

= $\frac{\sin k \pi}{c_{nk}(\rho-p)}$

Where $c_{nk}$ are the same as before (see equation (B7)).

This method of calculation produced consistent numerical results for $\tilde{G}_k(\rho)$ when reasonably different starting values for $\rho$ were used.
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