THE PAULI EXCLUSION PRINCIPLE
AND THE SPIN-DEPENDENCE OF
THE DEUTERON-NUCLEUS INTERACTION

A Thesis submitted to the Faculty of Mathematical
and Physical Sciences of the University of Surrey
for the degree of Doctor of Philosophy

by

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Abstract

A study is made of the propagation of a deuteron through a nuclear medium. We consider the effect of the Pauli exclusion principle on the internal motion of the deuteron, for the idealized case of a deuteron propagating through infinite nuclear matter, assumed to be adequately described by the Fermi-gas model. This greatly simplifies the calculations and enables us to include explicitly the D-state component of the deuteron wavefunction in our model. It is shown that the action of the Pauli exclusion principle, on the D-state part of the deuteron, endows various properties of the deuteron with strong dependence on the relative orientation of the deuteron spin and its linear momentum. The dependence of the binding energy of the deuteron on this relative orientation is particularly important, because, it can be associated with a new tensor term, of the $T_p$ type, in the deuteron optical potential. The strength of this tensor potential is found to be non-negligible at high incident deuteron energies, for the realistic case of a deuteron propagating through a finite nucleus. Rough estimates of this strength at lower energies, indicate that this new tensor interaction is large for these energies too. The important implications of this, for the interpretation of experiment with polarized deuterons, are discussed.
ACKNOWLEDGEMENTS

I wish to express my gratitude to Dr. R.C. Johnson, my supervisor, for suggesting this problem and for providing invaluable guidance throughout the first two years of this work. His constant encouraging, helpful criticism and his devotion to his work were the driving force of this work.

I also wish to thank Professor D.F. Jackson for continuing so efficiently the supervision of this work in its final year.

My gratitude to these two persons extends beyond the academic side of my stay at Surrey. I will always remember with gratitude their humane attitude and kind understanding to me and my wife during the tragic circumstances we found ourselves in the Spring and Summer of 1974.

My stay at Surrey was a very happy one. I would like to thank for this all the members of the Physics Department. In particular I want to express my appreciation to my teachers and friends Drs. S. Cornbleet and M.C. Jones and my good friend from childhood K. Kouris.

I also wish to thank Drs. W.S. Pong and K. Ahmad for valuable discussions, and Messrs. G. Wales and M.R. Brown for lending me a number of items during the preparation of this thesis.

I would like to thank all the members of the Computing Unit of the Physics Department and in particular Mrs. J. Hilton for computing assistance. My thanks are also due to Mrs. D. Ioannides for preparing all the graphs in this thesis. I express my appreciation to Mrs. M. Fortuna for excellent typing of a difficult manuscript.
I also wish to thank the University of Surrey for the receipt of a University of Surrey studentship.

I also wish to thank my wife Kiki, for standing by me through the good and bad times of the last three years.
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INTRODUCTION

a) Historical background: The deuteron is the only bound state of the two nucleon system; it is therefore the simplest composite nucleus and at the same time it is the simplest, and may be the most useful, composite projectile. The study of the properties of the deuteron is one of the most important sources of information about nuclear forces. Nuclear reactions involving deuterons have been and continue to be an important source of detailed information on nuclear structure.

It is numerically and otherwise advantageous to analyse the data of scattering experiments in terms of an effective interaction between the projectile and target, which we call the optical potential. Over the last twenty years many authors have directed a lot of effort to the derivation of the deuteron nucleus optical potential starting from the nucleon-nucleus optical potential. Practically all these works revolve around Watanabe's original idea. Watanabe proposed that deuteron elastic scattering data should be analysed in terms of a deuteron-nucleus optical potential, $U_d$, which is the sum of the neutron and proton optical potentials $U_n$ and $U_p$, for the same nucleus evaluated at half the deuteron energy and folded over the internal wavefunction of the deuteron.

$$U_d(R) = \int d\mathbf{r} \phi^*(\mathbf{r}) [V_n(\frac{1}{2}|\mathbf{R} + \mathbf{r}|) + V_p(\frac{1}{2}|\mathbf{R} - \mathbf{r}|)]\phi(\mathbf{r})$$

where $\phi(\mathbf{r})$ is the deuteron wavefunction

$\mathbf{R}$ is the separation vector of the deuteron and nucleus centre of masses

and $\mathbf{r}$ the relative separation of the proton and neutron in the deuteron.
The Watanabe model was examined extensively over a wide range of nuclei and energies \(^2\) and was found to be essentially correct although the need for some refinements was obvious. Watanabe's work was followed by many authors \(^3\) who criticized the model and proposed a number of corrections. The corrections to the Watanabe model due to the D-state of the deuteron and the corrections due to the Pauli exclusion principle are of special interest to us.

SATCHLER first considered theoretically the spin-dependence of deuteron-nucleus elastic scattering. He showed, on general principles, and provided terms at most quadratic in the momentum operator are allowed in the optical potential, that the optical potential describing the elastic scattering of a spin one particle by a spin zero target can contain, apart from central and the usual spin orbit terms, three possible types of tensor terms of rank 2, which are usually denoted by \(T_R\), \(T_L\) and \(T_P\). The form of these potentials is as follows:

\[
V_R = U_R(R)T_R, \quad T_R = \frac{(S.R)^2}{R^2} - \frac{2}{3}
\]

\[
V_L = U_L(R)T_L, \quad T_L = (L.S)^2 + \frac{1}{3}(L.S) - \frac{2}{3}L^2
\]

\[
V_P = U_P(R)T_P + T_F U_P(R), \quad T_P = (S.P)^2 - \frac{2}{3}P^2
\]

In Watanabe's original model the D-state of the deuteron was neglected. SATCHLER showed that a \(T_R\) term can be generated by the Watanabe model if the D-state of the deuteron is included.

The spin-dependence of deuteron-nucleus elastic scattering was also examined in a series of papers by LYOVSHIN \(^9\). He proposed mechanisms for producing all three tensor potentials. However his \(T_L\) and \(T_P\) tensor potentials are much smaller than the \(T_R\) potential.
His $T_p$ tensor potential, which is relevant to the present work, is proportional to the energy derivative of the nucleon optical potential. A small tensor potential of the $T_L$ type was also proposed by Stamp $^{21}$.

During the last few years the spin-dependence of the deuteron-nucleus interaction has become the subject of considerable interest as the result of the development of elastic scattering experiments in which the five observables, $\sigma(0)$ (the cross-section for unpolarized deuterons), $I_{11}$, $I_{20}$, $I_{21}$, $I_{22}$ (the analysing power for polarized deuteron) are all measured over a considerable angular range and for a variety of targets and energies $^{22,23,24,25,26}$. The analysis of these data in terms of an optical potential usually assumes that in addition to a spin-orbit force of $L$-$S$ type, tensor forces of the $T_R$ and $T_L$ type are also present. It seems at the present time that the $T_R$ tensor alone is sufficient to explain the tensor analysing powers at sub-Coulomb energies $^{22,23,24}$. At higher energies however theory and experiment disagree considerably even when both the $T_R$ and $T_L$ tensor terms are included in the deuteron-nucleus optical potential $^{23,28}$.

It has been realised during the last few years that the Pauli exclusion principle plays a very important role in nuclear reactions involving multi-nucleon bound projectiles and targets. Deuteron-nucleus scattering is the simplest example. The effect of antisymmetrization on deuteron nucleus scattering was studied by many authors $^{13,14,19}$. Even in this case however a formal treatment of the problem is very complicated and the effect of antisymmetrization is interlocked with other effects in such a way that it is difficult to be recognised. Although formal treatments of deuteron nucleus scattering are very important, theoretically, they are very difficult to reduce to a calculable form. The need for simplifying assumptions
is obvious. The simplest and most successful theories for deuteron-nucleus elastic scattering are based on the Watanabe model.

The work of Gambhir and Griffith is most relevant to the present work. They investigate the variation of the deuteron binding energy when a deuteron is propagating through a nuclear medium because of the reduction of the neutron-proton interaction due to antisymmetrization. The binding energy of a deuteron in nuclear matter is obtained by solving a Bethe-Goldstone type of equation.

b) **Scope of the thesis:** A proper account of either the Pauli exclusion principle or of the D-state of the deuteron in the investigation of deuteron nucleus scattering makes the problem very difficult. Effects due to the Pauli exclusion principle and the D-state of the deuteron are usually presented as corrections to the theory. Practically all works investigating either Pauli effects or D-state effects are concerned with one of these two effects and neglect the other. The reason for this situation, apart from the complexity involved when both effects are included, is the idea that these effects are small, especially the D-state effect, because the D-state component in the deuteron wavefunction is small. In a number of cases however inclusion of the D-state of the deuteron, although small, has proven necessary.

The purpose of the present work is to investigate the importance of the combined effect of the Pauli exclusion principle and the inclusion of the D-state of the deuteron, when a deuteron is propagating through a nuclear medium. We primarily want to determine whether or not any experimentally observable effects can be produced in this way. In the absence of any previous similar study we could not foresee
from the beginning which experiments might be relevant. We did however realize that a formal treatment of the problem would result in expressions where various observable quantities would depend on a multitude of sources mixed together in an inseparable form. We are not therefore attempting to derive an exact theory, in order to avoid complex numerical calculations where the physical meaning of the results is obscure. Our aim at this stage is simplicity of ideas. We try as much as possible to present a picture of how the Pauli exclusion principle operates when a deuteron is propagating through a nuclear medium. The binding energy of the deuteron in nuclear matter, as a function of the deuteron incident energy and nuclear matter density, is obtained by solving a Bethe-Goldstone equation of the type used by Gambhir and Griffith generalised to include the tensor n-p interaction given by Yamaguchi and Yamaguchi 30. We found that the binding energy of the deuteron in nuclear matter depends strongly on the relative orientation of the total spin of the deuteron and the direction of its centre of mass motion. This effect can be accounted for by the inclusion of a $T_p$ type of tensor potential in the deuteron nucleus optical potential. The strength of this tensor potential may be of the same order as the strength of the $T_R$ tensor potential predicted by the Watanabe model. This new tensor potential may be relevant to suggestions that the Watanabe model does not give the correct spin-dependence at high deuteron incident energies $^{23,28}$. Soon after we obtained this result $^{31,32,33}$ Austern 29 proposed an alternative derivation of a $T_p$ tensor force from the combined effect of the Pauli exclusion principle and the D-state of the deuteron. At the end of the thesis an attempt is made to compare the two derivations. This part of the work was done in collaboration with Dr. W.S. Pong.
c) Outline of presentation: In chapter 1 we outline the action of the Pauli exclusion principle on a deuteron propagating through infinite nuclear matter, under the assumption that infinite nuclear matter is adequately described by the Fermi gas model. In the first chapter the Yamaguchi potential for the neutron-proton interaction is introduced.

In chapter 2 the Schrödinger equation for the relative motion, for a deuteron propagating through nuclear matter is established in the momentum representation and reduced to an eigenvalue determinant for the binding energy of the deuteron. The methods of solution of the eigenvalue determinant are then outlined. As a particular example, we examine at the end of chapter 2 the limiting case of a deuteron at rest in infinite nuclear matter.

In chapter 3 we present the variation of the binding energy of the deuteron as a function of nuclear matter density and centre of mass momentum vector. The dependence of these results on the D-state of the deuteron is also presented. The rest of chapter 3 is concerned with the distribution of orbital angular momenta in the deuteron wavefunction when the deuteron is propagating through nuclear matter. This distribution is compared with the corresponding distribution when the deuteron is in free space.

The results of chapter 3 are explained and justified by Chapter 4. A picture for the Pauli mechanism is developed for this purpose.
The relative momenta in the deuteron wavefunction are divided into three regions, two of which have the property of preserving, in nuclear matter, the symmetry of the deuteron wavefunction in free space, and one which does not possess this property. This division of the relative momenta enables us to explain simply the orbital angular momentum distribution of the deuteron wavefunction in nuclear matter.

In chapter 5 we connect the spin dependence of the binding energy of the deuteron in nuclear matter with a tensor potential of the $T_p$ type which must be added in the deuteron optical potential.

In chapter 6 we attempt to relate the nuclear matter calculation to the realistic case of a deuteron propagating through a finite nucleus. A very simple model is set up for this purpose and the validity of its basic assumption is examined. We find that this model is valid for incident deuteron energies greater than 150 MeV. Austern's derivation of the $T_p$ force is also examined and compared with our approach. Estimates of the shape and strength of the $T_p$ force are also obtained for low energies in this chapter. These estimates however are not expected to be accurate. They are however presented, in the absence of any previous estimates for this new force, as plausible forms for the $T_p$ force, for people wishing to include this force in their analysis of experiments with polarized deuterons.

In chapter 7 a simple calculation, in plane wave Born approximation is performed in which we calculate the tensor analyzing powers for elastic scattering of deuterons from a spinless target at high incident deuteron energies. In the absence of any experimental data, in this energy region, the usefulness of this calculation is limited to a comparison of the tensor analyzing powers produced when, in addition
to the central interaction, a $T_R$ tensor potential is present in the
deuteron nucleus interaction with the corresponding quantities when
both $T_R$ and $T_P$ tensor forces are present in the deuteron-nucleus
interaction.

Finally in the epilogue we suggest possible applications of
the $T_P$ tensor force we propose here to some fashionable problems
of nuclear physics.
CHAPTER 1
THE FERMI-GAS MODEL FOR NUCLEAR MATTER, THE PAULI EXCLUSION PRINCIPLE AND THE YAMAGUCHI POTENTIAL

1.A. Nuclear matter as a Fermi-gas and the Pauli exclusion principle

Infinite nuclear matter can be thought of as a collection of non interacting fermions confined in a region of space \( \mathcal{V} \). In this approximation the momentum distribution per unit volume in momentum space is a step function with constant value \( k < k_F \) and zero for \( k > k_F \). The momentum, \( k_F \), of the highest occupied state depends only on the density, \( \rho \), through

\[
\rho = \frac{2}{3\pi^2} k_F^3
\]

The central density of real nuclei with \( A \geq 12 \), \( \rho_c \), is practically a constant \(^{34}\)

\[ \rho_c = 0.172 \text{ particles/fm}^3 \]

\[ k_F = 1.36 \text{ fm}^{-1} \]

Let us now consider a deuteron propagating through free space with a momentum \( k_0 \). Its complete wavefunction may be expanded in plane waves

\[
\phi_0(x) e^{i k_0 \cdot R} = \int d^3k \phi_0(k) e^{ik_p \cdot r_p} e^{-ik_n \cdot r_n}
\]

Where \( R \) is the position vector of the centre of mass of the deuteron. \( k_p \) and \( r_p \) are the proton momentum wavenumber and position vector respectively. Similarly \( k_n \) and \( r_n \) are the neutron momentum...
wavenumber and position vector. The relative momentum wavenumber is $k$ and the relative position vector is $r$.

These quantities are related to each other through the usual relationships.

\[
k_p = a + k, \quad k_n = a - k \quad 1.5
\]

\[
r_p = R + \frac{r}{2}, \quad r_n = R - \frac{r}{2} \quad 1.6
\]

\[
a = \frac{K_0}{2}
\]

When the same deuteron is propagating through nuclear matter some values of the relative momentum $k$ will not be allowed because of the Pauli exclusion principle. If the Fermi gas model is assumed for nuclear matter the plane waves in the deuteron wavefunction corresponding to either $k_p$ or $k_n$ less than $k_F$ will be forbidden for the deuteron.

Let us now define the projection operator, $\hat{\Theta}$, which projects out of the Fermi-sea,

\[
\hat{\Theta} f(k) = f(k) \quad \text{if} \quad |a + k| > k_F \quad 1.7
\]

\[
\hat{\Theta} f(k) = 0 \quad \text{if} \quad |a + k| < k_F
\]

In terms of this projection operator, the deuteron wavefunction in nuclear matter becomes

\[
\hat{\Theta} \phi_0(r) e^{i K_0 \cdot R} = \int d^3k \hat{\Theta} \phi_0(k) e^{-ik \cdot p} e^{-i \frac{r \cdot n}{2}} \quad 1.8
\]
1.B. The action of the operator $\hat{\Theta}$

We summarize here some results, we will be using extensively in latter chapters. The integrals we will be using are of the form

$$I = \int d^3k \hat{\Theta} h(\theta) f(k)$$  \hspace{1cm} 1.9

where, $\theta$, is the angle between $k$ and $K_0$

$$\cos \theta = \frac{k \cdot K_0}{k K_0}$$  \hspace{1cm} 1.10.

For a non-zero contribution to the integral we require that both $|k_p|$ and $|k_n|$ are greater than $k_F$:

$$|a + k| > k_F < |a - k_F|$$  \hspace{1cm} 1.11

where $a$ is the same as before: $a = \frac{K_0}{2}$.

It is convenient to consider two distinct cases:

Case 1  $k_F > a$

Case 2  $k_F < a$

Case 1, $k_F > a$: In this case, there are no allowed values for $k$ for $k < \sqrt{k_F^2 - a^2}$ since either $k + a$ or $k - a$ have magnitude less than $k_F$.

In the region $\sqrt{k_F^2 - a^2} < k < k_F + a$ the allowed values of $k$ fall in the angular region

$$-\sigma \leq \cos \theta \leq +\sigma$$

where

$$\sigma = \frac{a^2 + k^2 - k_F^2}{2ak}$$  \hspace{1cm} 1.12
For higher values of k, k > k_F + a, there is no restriction on the value of k since both a + k_F - a have magnitude greater than k_F.

Hence the integral I becomes:

\[
I = 2\pi \left\{ \int_{-\sigma}^{+\sigma} d(\cos \theta) h(\theta) \int_{k_F+a}^{\infty} dk k^2 f(k) + \int_{-\sigma}^{+\sigma} d(\cos \theta) \right\}
\]

Case 2, a > k_F. In this case there is no restriction on the values of k if the magnitude of k obeys the inequalities 

a + k_F < k < a - k_F since then a ± k is always greater than k_F.

In the intermediate range a + k_F > k > a - k_F the allowed values for k fall in the angular region

\[-\sigma \leq \cos \theta \leq \sigma\]

with \(\sigma\) the same as for case 1.

Hence the integral I becomes:

\[
I = 2\pi \left\{ \int_{-\sigma}^{+\sigma} d(\cos \theta) h(\theta) \int_{a-k_F}^{a+k_F} dk k^2 f(k) + \int_{-\sigma}^{+\sigma} d(\cos \theta) \right\}
\]

\[
+ \int_{-\sigma}^{+\sigma} d(\cos \theta) h(\theta) \int_{k_F+a}^{\infty} dk k^2 f(k) \right\}
\]

1.13.A

1.13.B
1.C. The Yamaguchi potential

For the purpose of the present work we choose the Yamaguchi potential \(30\) for the description of the neutron-proton interaction. This simple non-local but separable potential accounts fairly well for most properties of the two nucleon system at low energies, including the properties of the deuteron. Its simple form enables us to retain analytic expressions in much of our work and keeps the discussion transparent. The Yamaguchi potential has the following form in the momentum representation:

\[
V_{np}(k,k) = -\frac{\lambda}{m} g(k) \hat{g}(k')
\]

\[
g(k) = C(k) + \frac{1}{\sqrt{8}} S(k) T(k)
\]

\[
S(k) = \frac{3}{k^2} (\sigma^p \cdot k)(\sigma^n \cdot k) - (\sigma^p \cdot \sigma^n)
\]

Where \(m\) is the nucleon mass and \(\lambda\) is the overall strength of the interaction. \(\sigma^p\) and \(\sigma^n\) are the usual spin matrices for the proton and the neutron respectively. \(C(k)\) and \(T(k)\) are functions of \(k = |k|\).

The simplest choice for the functions \(C(k)\) and \(T(k)\) corresponding to a potential without a long tail in the \(r\) representation \(30\) is the following:

\[
C(k) = \frac{1}{\beta^2 + k^2}
\]

\[
T(k) = \frac{-t k^2}{(\gamma^2 + k^2)^2}
\]

where \(t\), \(\beta\) and \(\gamma\) are constants. It should be noted that the tensor force is attractive for parallel nucleon spins and gives a positive
value for the quadrupole moment of the deuteron if $t$, the relative D-state strength, is positive.

The parameters of the Yamaguchi potential are chosen to be such as to reproduce correctly the observed properties of the deuteron and the low energy data for the nucleon-nucleon scattering. Neutron-proton scattering can be described fairly well with this potential for energies up to 100 MeV. In fact for energies below 20 MeV the central part of the interaction alone ($t = 0$) is adequate. The contribution of the D-state is more important at higher energies $^{30,35}$. We therefore expect the Yamaguchi potential to be useful in deuteron-nucleus scattering for deuteron incident energies no higher than 200 MeV. We have also used this potential for higher energies, because we do not expect the essential features of the effects under study here to alter qualitatively by errors introduced in this way.

If the tensor part of the interaction is neglected, ($t = 0$), a number of values for $\lambda$ and $\beta$ can be found which reproduce correctly the very low energy scattering data for the nucleon-nucleon system $^{17}$.

This freedom of choice for the parameters of the Yamaguchi potential disappears when the tensor part of the interaction is retained. There is only one set of parameters for a given D-state probability which accounts correctly for the observed deuteron properties and the low energy nucleon-nucleon scattering data. However the D-state probability is one of the least known properties of the deuteron with a value that can lie anywhere between 3% and 10% $^{36}$. 
CHAPTER 2

FORMULATION OF THE PROBLEM

2.A. Derivation of the eigenvalue equation

Let us consider a deuteron propagating through free space with centre of mass momentum $K_0$. Its wavefunction in the momentum representation is given by

$$\psi(k, k, \text{spin}) = \delta(k - K_0) \phi(k) \chi_1^m$$ \hspace{1cm} 2.1

where $\chi_1^m$ is the triplet spin function. The wavefunction for the relative motion, $\phi(k)$, satisfies the Schrödinger equation

$$\left( -\frac{\Delta^2_{fs}}{M} - \frac{k^2}{M} \right) \phi(k) \chi_1^m = \int d^3k' V_{np}(k, k') \phi(k') \chi_1^m$$ \hspace{1cm} 2.2

We use units such that $\hbar = M = 1$, energy has dimensions of length$^{-2}$, and the conversion factor is 1 fm$^{-2} = 41.5$ MeV.

Where $M$ is the nucleon mass and $\frac{\Delta^2_{fs}}{M}$ is the binding energy of the deuteron in free space. If the Yamaguchi potential is assumed for the neutron-proton interaction, $V_{np}$, the Schrödinger equation becomes

$$\phi(k) \chi_1^m = \frac{\lambda g(k)}{\Delta^2_{fs} + k^2} \int d^3k' g(k') \phi(k') \chi_1^m$$ \hspace{1cm} 2.3

The above equation is a $3 \times 3$ matrix equation because of the form of $g(k)$ (equation 1.15)

$$\left( \phi(k) I - \frac{\lambda g(k)}{\Delta^2_{fs} + k^2} \int d^3k' g(k') \phi(k') \right) \chi_1^m = 0$$ \hspace{1cm} 2.4

where $I$ is the unit $3 \times 3$ matrix.
Let us now consider the same deuteron propagating through nuclear matter. Assuming that nuclear matter is adequately described by the Fermi-gas model we can account for the effect of the Pauli exclusion principle by projecting out the forbidden momenta in the deuteron wavefunction with the aid of the operator $\hat{0}$.

We write for the Schrödinger equation

$$\hat{0}(\Delta^2 + k^2) \phi(k)\chi^m = \lambda \int d^3k' \hat{0} g(k) g(k') \phi(k')\chi^m$$

where

$$\hat{0} \phi(k)\chi^m = \hat{0} \frac{\lambda g(k)}{\Delta^2 + k^2} C_m$$

Substituting 2.6 in 2.7 we get

$$C_m = \lambda \int d^3k \hat{0} g(k) g(k) \frac{\Delta^2 + k^2}{\Delta^2 + k^2} C_m$$

Hence for a solution we require

$$\det \left[ I - \lambda \int d^3k \hat{0} g(k)g(k) \frac{\Delta^2 + k^2}{\Delta^2 + k^2} \right] = 0$$

this is our eigenvalue determinant.
As we mentioned earlier the simplest physically sound form for $g(k)$ is

$$g(k) = C(k) + \frac{1}{\sqrt{8}} T(k) S(k)$$

with

$$S(k) = \frac{3}{k^2} (\sigma^n \cdot k)(\sigma^n \cdot k) - (\sigma^p \cdot \sigma^n)$$

With this choice for $g(k)$ we have:

$$g(k)g(k) = C^2(k) I + \frac{1}{\sqrt{2}} C(k) T(k) S(k) + \frac{1}{8} T^2(k) S(k) S(k) \quad 2.11$$

$$S(k) S(k) = \frac{8}{k^4} (\sigma^p \cdot k)^2 (\sigma^n \cdot k)^2 + (\sigma^p \cdot \sigma^n)^2 - \frac{3}{p^2} (\sigma^p \cdot k)(\sigma^n \cdot k)(\sigma^p \cdot \sigma^n)$$

$$- \frac{3}{p^2} (\sigma^p \cdot \sigma^n)(\sigma^p \cdot k)(\sigma^n \cdot k) \quad 2.12$$

Using the identity for the Pauli spin matrices:

$$(\sigma \cdot \Lambda)(\sigma \cdot \beta) = \Lambda \cdot \beta \pm \sigma \cdot (\Lambda \times \beta) \quad 2.13$$

$$(\sigma \cdot k)(\sigma \cdot k) = k^2 \quad 2.14$$

$$\therefore \quad \frac{8}{k^4} (\sigma^p \cdot k)^2 (\sigma^n \cdot k)^2 = g \quad 2.15$$

also

$$[(\sigma^p \cdot \sigma^n)(\sigma^p \cdot k)](\sigma^n \cdot k) = [(\sigma^n \cdot k) + i \sigma^p \cdot (\sigma^n \times k)](\sigma^n \cdot k) \quad 2.16$$

Using the fact that $\sigma^p$, $\sigma^n$ and $k$ are commuting variables and the identity:

$$\Lambda \cdot (B \times C) = B \cdot (C \times \Lambda) \quad 2.17$$

$$\sigma^p \cdot (\sigma^n \times k) = \sigma^n \cdot (k \times \sigma^p) \quad 2.18$$
\[
(\sigma^p \cdot \sigma^n)(\sigma^p \cdot k)(\sigma^n \cdot k) = (\sigma^n \cdot k)^2 + i\sigma^n \cdot (k \times \sigma^p) (\sigma^n \cdot k)
\]

\[
= k^2 + i(k \times \sigma^p) \cdot k - \sigma^n \cdot (k \times \sigma^p) \times k \quad 2.19
\]

The second term of the last part of the above equation is identically zero. We use for the last term the identity:

\[
(A \times B) \times C = (A \cdot C) B - (B \cdot C) A \quad 2.20
\]

\[
\therefore \quad \sigma^n \cdot (k \times \sigma^p) \times k = (\sigma^n \cdot \sigma^p) k^2 + (\sigma^n \cdot k)(\sigma^p \cdot k) \quad 2.21
\]

\[
\therefore \quad - \frac{3}{k^2} (\sigma^p \cdot \sigma^n)(\sigma^p \cdot k)(\sigma^n \cdot k) = - 3 + 3(\sigma^n \cdot \sigma^p) - \frac{3}{k^2} (\sigma^n \cdot k)(\sigma^p \cdot k) \quad 2.22
\]

Similarly:

\[
- \frac{3}{k^2} (\sigma^p \cdot k)(\sigma^n \cdot k)(\sigma^n \cdot \sigma^p) = - 3 + 3(\sigma^p \cdot \sigma^n) - \frac{3}{k^2} (\sigma^p \cdot k)(\sigma^n \cdot k) \quad 2.23
\]

Now:

\[
(\sigma^p \cdot \sigma^n)(\sigma^p \cdot \sigma^n) = \sigma^n \cdot \sigma^n + i \sigma^p \cdot (\sigma^n \times \sigma^n) = 3 - 2(\sigma^p \cdot \sigma^n) \quad 2.24
\]

Hence:

\[
S(k) S(k) = g - 3 - 3(\sigma^n \cdot \sigma^p) - \frac{3}{k^2} (\sigma^n \cdot k)(\sigma^p \cdot k) - 3 + 3(\sigma^p \cdot \sigma^n)
\]

\[
= \frac{3}{k^2} (\sigma^p \cdot k)(\sigma^n \cdot k) + 3 - 2(\sigma^p \cdot \sigma^n) \quad 2.25
\]

\[
\therefore \quad S(k) S(k) = 2[3 - S(k) + \sigma^p \cdot \sigma^n] \quad 2.26
\]

for a triplet state \( \sigma^p \cdot \sigma^n = 1 \)

for a singlet state \( \sigma^p \cdot \sigma^n = - 3 \)
The tensor operator $S(k)$ can also be written as

$$S(k) = 2[3(S_{\hat{k}}^2 - S^2)]$$  \hspace{1cm} 2.28

where

$$S = \frac{1}{2} (\sigma^p + \sigma^n)$$  \hspace{1cm} 2.29

Now

$$6(S_{\hat{k}}^2) = 6(S_x \sin \theta \cos \phi + S_y \sin \phi \sin \theta + S_z \cos \theta)^2$$  \hspace{1cm} 2.30

Squaring and collecting terms with the same dependance on $\theta$ and $\phi$ we get

$$6(S_{\hat{k}}^2) = \frac{3}{2} \sin^2 \theta e^{2i\phi} [S_x^2 - S_y^2 - i(S_x S_y + S_y S_x)]$$

$$+ 3 \sin \theta \cos \theta e^{i\phi} [(S_x - i S_y)S_z + S_z(S_x - i S_y)]$$

$$+ 3 S_x^2 \cos^2 \theta + 3(S_x^2 + S_y^2) \sin^2 \theta$$

$$+ 3 \sin \theta \cos \theta e^{-i\phi} [(S_x + i S_y)S_z + S_z(S_x + i S_y)]$$

$$+ \frac{3}{2} \sin^2 \theta e^{-2i\phi} [S_x^2 - S_y^2 + i(S_x S_y + S_y S_x)]$$  \hspace{1cm} 2.31

Identifying the Spherical Harmonics of rank 2, $Y_{2q}$ and the vector and Tensor operators of point 2, $T_{2q}$ constructed out of the spin components

$$Y_{2\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$  \hspace{1cm} 2.32

$$Y_{2\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$  \hspace{1cm} 2.33
\[ Y_{20} = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \]  \hspace{1cm} 2.34

\[ S_{11} = S_+ = -\frac{1}{\sqrt{2}} (S_x + i S_y) \]  \hspace{1cm} 2.35.A

\[ S_{1-1} = S_- = \frac{1}{\sqrt{2}} (S_x - i S_y) \]  \hspace{1cm} 2.35.B

\[ S_{10} = S_0 = S_z \]  \hspace{1cm} 2.35.C

\[ T_{2\pm 2} = \sqrt{3} S_z^2 \]  \hspace{1cm} 2.36.A

\[ T_{2\pm 1} = \sqrt{\frac{3}{2}} (S + S_z + S_z S_\pm) \]  \hspace{1cm} 2.36.B

\[ T_{20} = \frac{1}{\sqrt{2}} (3 S_z^2 - S^2) \]  \hspace{1cm} 2.36.C

With the above and the definition for the tensor scalar product

\[ (L^{(k)} M^{(k)}) = \sum_{q=-k}^{+k} (-)^q L_{kq} M_{k-q} \]  \hspace{1cm} 2.37

we have

\[ S^{(k)} = \sqrt{\frac{32\pi}{5}} (T^{(2)} Y^{(2)}) \]  \hspace{1cm} 2.38

With \( T^{(2)} \) the irreducible tensor operator constructed out of the spin one operators and \( Y^{(2)} \) the spherical Harmonic of rank 2 constructed out of the directions of \( \hat{k} \).

We now define the basis vectors for the spin operators. We define as our basis vectors the vectors \( X_+ \), \( X_0 \) and \( X_- \) such that
$$
\begin{align*}
S_z \chi_+ &= \chi_+ \\
S_z \chi_0 &= 0 \chi_0 \\
S_z \chi_- &= -\chi_-
\end{align*}
\implies <\chi_\mu | S_z | \chi_\nu> = \mu \delta_{\mu\nu},  \hspace{1cm} 2.39
$$

Hence the operator $S_z$ in this representation is given by the matrix

$$
S_z = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}  \hspace{1cm} 2.40
$$

$$
S^2 = 2I \hspace{1cm} 2.41
$$

We will be concerned later on with the operator $T_{20}$. This operator is represented in the above representation by the matrix

$$
T_{20} = \frac{1}{\sqrt{2}} (3S_z^2 - S^2) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{pmatrix}  \hspace{1cm} 2.42
$$

Finally we can write

$$
g(k)g(k) = \left[ C(k) + T^2(k) \right] I + \sqrt{\frac{32\pi}{5}} \left[ \frac{C(k)T(k)}{\sqrt{2}} - \frac{T^2(k)}{4} \right] (T(2), y(2))  \hspace{1cm} 2.43
$$

Substituting the last result into the eigenvalue determinant

2.10 we get

$$
\det \left[ (1 - \lambda \ I_0) I + \lambda \ I_1 \right] = 0 \hspace{1cm} 2.44
$$

where

$$
I_0 = \int d^3k \hat{\phi} \left( \frac{C(k) + T^2(k)}{\Delta^2 + k^2} \right) \hspace{1cm} 2.45
$$
Choosing the direction of propagation of the deuteron, \( \hat{k}_0 \), as the z-axis the operator \( \hat{\theta} \) becomes independent of the angle \( \phi \). Hence the only term in the tensor scalar product \( (T^{(2)}, Y^{(2)}) \) that contributes to the integral \( I_1 \) is the \( T_{20} Y_{20} \) term.

Hence we can let

\[ T^{(2)} Y^{(2)} \rightarrow T_{20} Y_{20} \]  \( \tag{2.47} \)

With this choice for the z-axis \( I_1 \) reduces to

\[ I_1 = T_{20} \left[ I_2 - \frac{\sqrt{2}}{4} I_3 \right] \]  \( \tag{2.48} \)

where

\[ I_2 = \int d^3k \hat{\theta} \frac{C(k)T(k)}{\Delta^2 + k^2} (3 \cos^2 \theta - 1) \]  \( \tag{2.49} \)

\[ I_3 = \int d^3k \hat{\theta} \frac{T^2(k)}{\Delta^2 + k^2} (3 \cos^2 \theta - 1) \]  \( \tag{2.50} \)

Thus the eigenvalue determinant becomes

\[
\begin{bmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & A
\end{bmatrix}
\]

\[ \text{det} = 0 \]  \( \tag{2.51} \)

where

\[ A = 1 - \lambda \left[ I_0 + \frac{I_3}{4} - \frac{I_2}{\sqrt{2}} \right] = 1 - \lambda (I_0 + I_D) \]  \( \tag{2.52} \)

\[ B = 1 - \lambda \left[ I_0 - \frac{I_3}{2} + \sqrt{2} I_2 \right] = 1 - \lambda (I_0 - 2I_D) \]  \( \tag{2.53} \)

with

\[ I_D = \frac{I_3}{4} - \frac{I_2}{\sqrt{2}} \]  \( \tag{2.54} \)
Hence the solutions of the eigenvalue determinant are

\[ \lambda = \frac{1}{I_0 + I_D} \]

or

\[ \lambda = \frac{1}{I_0 - 2I_D} \]

Specification of the deuteron spin makes one of the solutions redundant. From equations 2.7, 2.8 and 2.10 we see that if the deuteron spin is parallel or antiparallel to the direction of motion \( \hat{k}_0 \), \( (\chi^m_1 = \chi_+ \text{ or } \chi_-) \), the solution is

\[ \lambda = \frac{1}{I_0 + I_D} \]

if the deuteron spin is perpendicular to \( \hat{k}_0 \) \( (\chi^m_1 = \chi_0) \) the solution is

\[ \lambda = \frac{1}{I_0 - 2I_D} \]

We concentrate now on the angular dependence of the integrals \( I_0 \) and \( I_D \). These integrals have the same form as the integral 1.9

\[ I = \int d^3k \, \hat{\theta} \, h(\theta) f(k) \]

The angular dependence of \( I_0 \) is particularly simple. The function \( h(\theta) \) is equal to one.

\[ I_0 = 2\pi \int \hat{\theta} \, d(\cos \theta) \int k^2 d \, k \, f(k) \]

The angular dependence of \( I_D \) comes through the function

\[ h(\theta) = (3 \cos^2 \theta - 1) \]
We observe here that

\[ \int_{-1}^{+1} d(\cos \theta)(3 \cos^2 \theta - 1) = 0 \]  \hspace{1cm} 2.57


\[ I_{D} = 2\pi \int_{-\sigma}^{+\sigma} dx(3x^2 - 1) \int_{\mu}^{\nu} dk k^2 F(k) \] \hspace{1cm} 2.58

\[ I_{D} = 4\pi \int_{\mu}^{\nu} dk (\sigma^3 - \sigma) k^2 F(k) \] \hspace{1cm} 2.59

where

\[ \nu = k_F + a \] \hspace{1cm} 2.60

\[ \mu = \sqrt{k_F^2 - a^2} \] if \( a < k_F \) \hspace{1cm} 2.61.A

\[ \mu = a - k_F \] if \( a > k_F \) \hspace{1cm} 2.61.B

\( \sigma \) is defined through equation 1.12

\[ \sigma = \frac{a^2 + k_F^2 - k_F^2}{2ak} \]

and

\[ F(k) = \frac{T(k)}{2(k_F^2 - 4k^2 - \Delta^2k^2)} \left[ T(k) - \frac{C(k)}{\sqrt{2}} \right] \] \hspace{1cm} 2.62

The eigenvalue determinant or alternatively equation 2.55.A and 2.55.B specifies the binding energy of the deuteron. We can easily see from these equations that the deuteron binding energy is not unique. It depends on the relative orientation of the spin and centre of mass momentum. This effect disappears whenever the integral \( I_D \) vanishes. There are two such cases.
The first case occurs when the angular part of the integral $I_D$ is spherically symmetric; this follows from equation 2.58. One such case is the deuteron in free space, as one might expect. In that case the operator $\hat{\theta}$ is the unit operator.

The second case occurs when the $D$-state of the deuteron is neglected. In this case the integral $I_D$ is identically zero. The effect we examine is therefore a direct consequence of a $D$-state part in the deuteron wavefunction.

2.B. Evaluation of the parameters of the Yamaguchi potential

The procedure for the evaluation of the parameters of the Yamaguchi potential we describe below was proposed by Yamaguchi 30.

The eigenvalue determinant for a deuteron propagating through free space has a unique solution:

$$
\frac{1}{\lambda} = I_0 = 4\pi \int_0^\infty \frac{dk}{k^2} \left( C^2(k) + T^2(k) \right)
$$

2.63

The functions $C(k)$ and $T(k)$ are defined by equations 1.17 and 1.18. With these functions 2.63 becomes

$$
\frac{1}{\lambda} = \pi^2 \left[ \frac{1}{\beta(\Delta+\beta)} + \frac{t^2(5\Delta^2+4\Delta\gamma+\gamma^2)}{8\gamma(\Delta+\gamma)^4} \right]
$$

2.64

we have dropped the subscript $fs$ for $\Delta$.

The constant $\Delta$ is found from the experimental value of the binding energy of the deuteron in free-space to be

$$
\Delta = 0.2316 \text{ fm}^4
$$

2.65
The deuteron D-state probability, $P_D$, is given by \(^{30}\)

$$P_D = \frac{t^2(5\Delta+\gamma)}{8\gamma(\Delta+\gamma)^5} \left[ \frac{1}{\Delta(\Delta+\beta)^3} + \frac{t^2(5\Delta+\gamma)}{8\gamma(\Delta+\gamma)^5} \right] \quad 2.66$$

From the last equation and the equation for the scattering length, $\alpha$,

$$\frac{1}{\alpha} = \frac{\Delta(\Delta+2\beta)}{2(\Delta+\beta)^2} + \frac{t^2\Delta^2\beta^4(\Delta^2+4\Delta\gamma+\gamma^2)}{16\gamma^3(\Delta+\gamma)^4} \quad 2.67$$

we get eliminating $t$

$$\frac{2(1-P_D)}{P_D} \left(\frac{\Delta+\beta}{\beta}\right)^3 \left[ \frac{1}{\alpha} - \frac{\beta(\Delta+2\beta)}{2(\Delta+\beta)^2} \right] = \frac{\Delta+\gamma}{\gamma(\Delta+\gamma)^4} \left[ 2\Delta + \frac{(\Delta+\gamma)^2}{\gamma} \right] \quad 2.68$$

This is a third order equation in $\gamma$. For a given value for $P_D$ and $\beta$ we get one real value for $\gamma$. With these values for $\beta$ and $\gamma$ t can be evaluated from either 2.66 or 2.67. Finally from 2.64 we obtain $\lambda$.

The quadrupole moment of the deuteron, $Q$, is given by \(^{30}\)

$$Q = \sqrt{2} \frac{\pi^2 N^2 t^2}{10(\Delta+\beta)^4} \left[ \Delta^{-1} \left\{ \frac{\Delta^2(5\Delta^2+4\Delta\beta+\beta^2)+8\gamma(10\Delta^3+33\Delta^2+22\Delta\beta^2+5\beta^3)}{(\beta+\gamma)^2(\Delta+\gamma)^5} \right. \right. \nonumber$$

$$\left. + \gamma^2(5\Delta^3+22\Delta^2\beta+33\Delta\beta^2+10\beta^3) + \gamma^3(\Delta^2+4\Delta\beta+5\beta^2) \left. \right\} \right.$$ 

$$+ \frac{2}{(\beta+\gamma)^3(\Delta+\gamma)^4} \left\{ \Delta(\beta+\gamma)^3+4(\Delta^2+\beta\gamma)(\beta^2+3\beta\gamma+\gamma^2) + 16\Delta\beta\gamma(\beta+\gamma) \right\}$$

$$+ \frac{2}{(\beta+\gamma)^4(\Delta+\gamma)^3} \left\{ \beta(\Delta+\gamma)^3+4(\beta^2+\Delta\gamma)(\Delta^2+3\Delta\gamma+\gamma^2) + 16\Delta\beta\gamma(\Delta+\gamma) \right\} \right]$$

$$- \frac{\pi^2 N^2 t^2 (7\Delta^3+49\Delta^2\gamma+91\Delta^2\gamma^2+33\gamma^3)}{160 \gamma^3(\Delta+\gamma)^7} \quad 2.69$$
where

\[ N^{-2} = \pi^2 \left\{ \frac{1}{\Delta \beta (\Delta + \beta)^3} + \frac{L^2 (5 \Delta + \gamma)}{8 \gamma (\Delta + \gamma)^5} \right\} \]

The desired parameters are then obtained by varying \( \beta \) until the parameters \( \beta, \gamma, \tau \) and \( \lambda \) give for \( Q \) the experimental value of the quadrupole moment, \( Q_{\text{exp}} \). The expression for \( Q \) in Yamaguchi's paper, ref. 30, is wrong, because the factor \( \Delta^{-1} \) to the right of our first left-hand bracket is missing.

For very low values of \( \beta \) \( Q \) is negative. It increases as \( \beta \) increases passing through zero and \( Q_{\text{exp}} \) reaches a maximum and decreases passing again through \( Q_{\text{exp}} \) and finally becoming negative again. For a given D-state probability we get two sets of values for the parameters satisfying the low energy scattering data and the observed properties of the deuteron. The second set however is rejected because its value for \( \gamma \) corresponds to a range for the D-state part of the interaction of about 10 fm. This is an unreasonable number because the generator of the long-range part of the nuclear interaction, the pion exchange process, has a maximum range of about 1.5 fm.

We therefore have a unique set of parameters for the Yamaguchi potential for a given D-state probability.

Table 1 lists the various quantities we used for the determination of the parameters of the Yamaguchi potential.

Table 2 is a list of the parameters for a number of D-state probabilities.
Table 1

\[ A = 0.2316 \text{ fm}^{-1} \]
\[ \alpha = 5.378 \text{ fm} \]
\[ Q = 0.27394 \text{ fm}^2 \]

Table 2

<table>
<thead>
<tr>
<th>D-state probability</th>
<th>Effective range ( r_0 ) (fm)</th>
<th>( \beta \text{(fm}^{-1}) )</th>
<th>( \gamma \text{(fm}^{-1}) )</th>
<th>( \lambda )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.704206</td>
<td>1.4151</td>
<td>0.6773</td>
<td>0.382</td>
<td>0.1124</td>
</tr>
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<td>0.3422</td>
<td>0.4701</td>
</tr>
<tr>
<td>3</td>
<td>1.703723</td>
<td>1.3607</td>
<td>1.3369</td>
<td>0.2979</td>
<td>0.9946</td>
</tr>
<tr>
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<td>1.5365</td>
<td>0.254</td>
<td>1.6634</td>
</tr>
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<td>1.7007</td>
<td>0.2138</td>
<td>2.4673</td>
</tr>
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<td>1.2842</td>
<td>1.8408</td>
<td>0.1786</td>
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</tr>
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<td>0.1487</td>
<td>4.4625</td>
</tr>
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<td>1.2366</td>
<td>2.0721</td>
<td>0.1238</td>
<td>5.6522</td>
</tr>
<tr>
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<td>1.2137</td>
<td>2.1701</td>
<td>0.1032</td>
<td>6.9706</td>
</tr>
<tr>
<td>10</td>
<td>1.690916</td>
<td>1.1913</td>
<td>2.2593</td>
<td>0.09622</td>
<td>8.4194</td>
</tr>
</tbody>
</table>
2.C. Details of the calculation

With the functions $C(k)$ and $T(k)$ as defined by equations 1.17 and 1.18 the integrals of the eigenvalue determinant become

$$I_0 = \hat{I}_0 - S + 4\pi \int_0^\nu \frac{(\sigma-1)}{\Delta^2 + k^2} \left[ \frac{k^2}{(\beta^2 + k^2)^2} + \frac{\tau^2 \kappa^6}{(\gamma^2 + k^2)^4} \right] dk$$

where

$$\hat{I}_0 = \int \frac{d^3k}{\Delta^2 + k^2} \left[ \frac{1}{(\beta^2 + k^2)^2} + \frac{k^4}{(\gamma^2 + k^2)^4} \right] = \pi^2 \left[ \frac{1}{\beta(\Delta + \beta)} + \frac{\tau^2(5\Delta^2 + 4\Delta \gamma + \gamma^2)}{\gamma(\Delta + \gamma)^4} \right]$$

$$S = 0 \quad \text{if} \quad a > k_F$$

$$S = 4\pi \int_0^\nu dk \frac{d^3k}{\Delta^2 + k^2} \left[ \frac{k^2}{(\beta^2 + k^2)^2} + \frac{\tau^2 \kappa^6}{(\gamma^2 + k^2)^4} \right] \quad \text{if} \quad k_F > a$$

$$\mu = \sqrt{k_F^2 - a^2} \quad \text{if} \quad k_F > a$$

$$\mu = a - k_F \quad \text{if} \quad a > k_F$$

$$\nu = a + k_F$$

Hence $I_0$ can be written as a sum

$$I_0 = \hat{I}_0 - 4\pi \left\{ \sum_{i=1}^\nu f_{i0} (k_F, a) \right\}$$

where

$$f_{01} = 1 \quad \text{if} \quad k_F > a$$

$$f_{01} = 0 \quad \text{if} \quad a > k_F$$

$$f_{02} = \frac{1}{2a} (k_F^2 - a^2) \quad , \quad f_{03} = 1 \quad , \quad f_{04} = -\frac{1}{2a}$$
\[ f_{05} = t^2 \quad \text{if} \quad k_F > a \]
\[ f_{05} = 0 \quad \text{if} \quad a > k_F \]
\[ f_{06} = t^2 f_{02} \]
\[ f_{07} = f_{03} \]
\[ f_{08} = t^2 f_{04} \]

The integrals \( I_{01} \) are

\[ I_{01} = \int_0^\mu \frac{k \, dk}{(\Delta^2 + k^2)(\beta^2 + k^2)^2} \]
\[ I_{02} = \int_0^\nu \frac{k \, dk}{(\Delta^2 + k^2)(\beta^2 + k^2)^2} \]
\[ I_{03} = \int_0^\nu \frac{k^2 \, dk}{(\Delta^2 + k^2)(\beta^2 + k^2)^2} \]
\[ I_{04} = \int_0^\nu \frac{k^3 \, dk}{(\Delta^2 + k^2)(\beta^2 + k^2)^4} \]
\[ I_{05} = \int_0^\nu \frac{k^6 \, dk}{(\Delta^2 + k^2)(\gamma^2 + k^2)^4} \]
\[ I_{06} = \int_0^\nu \frac{k^5 \, dk}{(\Delta^2 + k^2)(\gamma^2 + k^2)^4} \]
\[ I_{07} = \int_0^\nu \frac{k^6 \, dk}{(\Delta^2 + k^2)(\gamma^2 + k^2)^4} \]
\[ I_{08} = \int_0^\nu \frac{k^7 \, dk}{(\Delta^2 + k^2)(\gamma^2 + k^2)^4} \]

The integrals \( I_2 \) and \( I_3 \) are

\[ I_2 = -4\pi t \int_0^\nu \frac{dk \, k^4(\sigma^3 - \sigma)}{(\Delta^2 + k^2)(\beta^2 + k^2)(\gamma^2 + k^2)^2} = 2.75 \]
\[ I_0 = 4\pi t^2 \int_0^\infty \frac{dk}{\mu} \frac{k^6 (\sigma^3 - \sigma)}{(\Delta^2 + k^2)(\gamma^2 + k^2)^4} \]

Using 1.12 for \( \sigma \)

\[ \sigma^3 - \sigma = \frac{1}{8a^3} \sum_{n=1}^{\frac{4}{7}} F_n k^{2n-5} \]

where

\[ F_1 = (k_F^2 - a^2)^3 \]
\[ F_2 = (a^4 + 2a^2 k_F^2 - 3k_F^4) \]
\[ F_3 = (a^2 + 3k_F^2) \]
\[ F_4 = -1 \]

Hence the integrals \( I_2 \) and \( I_3 \) can also be expanded

\[ I_2 = -\frac{\pi t}{2a^3} \left\{ \sum_{n=1}^{\frac{4}{7}} f_{2n}(k_F, a) I_{2n} \right\} \]

where \( f_{2n} = F_n \)

\[ I_{2n} = \int_0^\infty \frac{dk}{\mu} \frac{k^{2n-1}}{(\Delta^2 + k^2)(\beta^2 + k^2)(\gamma^2 + k^2)^2} \]

\[ I_3 = \frac{\pi t^2}{2a^3} \left\{ \sum_{n=1}^{\frac{4}{7}} f_{3n}(k_F, a) I_{3n} \right\} \]

where \( f_{3n} = F_n \)

\[ I_{3n} = \int_0^\infty \frac{dk}{\mu} \frac{k^{2n+1}}{(\Delta^2 + k^2)(\gamma^2 + k^2)^n} \]

All the integrals of the eigenvalue determinant can therefore be expressed in terms of simple integrals of the form
\[ I_{ij} = \int_a^b \frac{dk \, k^\lambda}{(A^2 + k^2)(B^2 + k^2)^m(C^2 + k^2)^n} \]

with

\[ m = 0, 1 \text{ or } 2 \quad n = 0, 1, 2, 3 \text{ or } 4 \]

and \( \lambda = 1, 2, \ldots \text{ or } (2m + 2n + 1) \)

Each of these integrals can be expanded by partial fractions into integrals of the form

\[ I_{ijh} = \int_a^b \frac{dk \, k^\lambda}{(c^2 + k^2)^q} \]

with \( \lambda = 0 \text{ or } 1 \quad c = \text{ constant} \)

and \( q = 1, 2 \text{ or } 3 \)

These integrals can then be evaluated analytically.

Our basic assumption is that the neutron-proton interaction in a nuclear medium is of the same form as the corresponding interaction in free space. The only modifications to the interaction come about through the Pauli exclusion principle. Effects due to target excitation are not considered in the present work.

For a given D-state probability the parameters of the Yamaguchi potential are fixed. These parameters are used for given values of \( k_F \) and \( a \) to obtain the solution of the eigenvalue determinant. For this purpose \( \Delta \) is varied until equation 2.55.A or 2.55.B gives the correct value for the overall strength of the neutron-proton interaction \( \lambda \).

The integrals are solved analytically, as described earlier, and numerically. One method is used as a check for the other.
2.D. The deuteron at rest in nuclear matter

We now consider the limiting case of a deuteron propagating through nuclear matter with a very small centre of mass momentum

$$|a| \ll \Delta, \beta, \gamma$$

In this case terms of the form

$$(A^2 + k^2)^n, \quad (B^2 + k^2)^n, \quad (Y^2 + k^2)^n$$

can be assumed constant between $k = \sqrt{\frac{k_F^2}{k_F^2} - a^2}$ and $k = k_F + a$. With this assumption the integrals $I_2$ and $I_3$ become

$$I_2 = -\frac{\pi t}{2a^3(\Delta^2 + k_F^2)(B^2 + k_F^2)(Y^2 + k_F^2)} \int_{\frac{k_F}{\sqrt{k_F^2 - a^2}}}^{k_F + a} \frac{dk}{\sqrt{\frac{k_F^2}{k_F^2} - a^2}} \left[ \frac{\left( k_F^2 - a^2 \right)k + \left( a^4 + 2a^2k_F^2 - 3k_F^4 \right)k^3}{(a^2 + 3k_F^2)k^3 - k^7} \right]$$

$$= -\frac{\pi t(a + k_F)^3(3k_F^2 - 2a)a}{3(\Delta^2 + k_F^2)(B^2 + k_F^2)(Y^2 + k_F^2)^2}$$

Therefore, $I_2 = 2.83$

Similarly

$$I_3 = \frac{\pi t^2(a + k_F)^4(3k_F^2 - 0.8k_F^2 - 0.2a) + 0.2a^2k_F^2)k^2}{a + 0}$$

$$= \frac{3(\Delta^2 + k_F^2)(B^2 + k_F^2)(Y^2 + k_F^2)^2}{a + 0}$$

The integrals $I_2$ and $I_3$ therefore vanish when the deuteron is at rest in nuclear matter. The binding energy of the deuteron is then degenerate. This is to be expected because when $a$ goes to zero we cannot define the relative angle between $a$ and the direction of the spin of the deuteron.
The integral $I_0$ in this limiting case is

$$I_0 = \hat{I}_0 - 4\pi \int_{0}^{\sqrt{k_F^2 - a^2}} \frac{dk}{k^2 + \lambda^2} \left[ \frac{1}{(\beta^2 + k^2)^2} + \frac{\gamma^2 k^4}{(\gamma^2 + k^2)^4} \right]$$

with $\hat{I}_0$ defined by equation 2.72.

The eigenvalue determinant reduces for this particular case to

$$\frac{1}{\lambda} = I_0$$

The limiting case we examine here can be realized for nuclear matter. It is however an unphysical situation when one considers deuteron-nucleus scattering, because the deuteron needs some initial momentum to get inside a real nucleus.
CHAPTER 3

THE DEUTERON IN NUCLEAR MATTER

3.A. Some useful definitions

We have seen in chapter 2 that the solution of the eigenvalue determinant, equation 2.55, depends on the relative angle between the deuteron centre of mass momentum and deuteron spin. We now want to examine how strongly various properties of the deuteron, like its binding energy, depend on this relative angle. It is convenient to devise two "states" for the deuteron when it propagates through nuclear matter.

We name the "state" of the deuteron para-state if the deuteron centre of mass momentum is parallel or antiparallel to the deuteron spin. Quantities associated with this state will bare the subscript \( \pm \), or \( \mp \), e.g. the binding energy of the para-state will be symbolized by \( \text{BE}_\pm \) or \( \text{BE}_\mp \).

When the deuteron centre of mass momentum is perpendicular to the deuteron spin the state of the deuteron is named ortho-state. Quantities associated with this state will bare the subscript \( 0 \), e.g. the binding energy of the ortho-state will be symbolized by \( \text{BE}_0 \).

We will often want to compare various properties of the deuteron in nuclear matter with the corresponding properties of the deuteron in free space. Ratios of quantities in nuclear matter to those in free space will carry a circumflex over the appropriate symbol e.g.

\[
\frac{\text{BE}_\pm}{\text{BE}_{fs}} = \text{BE}_\pm
\]
3.B. **The binding energy of the deuteron in nuclear matter**

The dependence of the deuteron binding energy in nuclear matter on $K_0$, $k_F$ and the D-state probability is depicted on figures 1 to 9. For each case the binding energy is drawn for three different D-state probabilities; 2%, 4% and 6%. In figures 1 to 7 the total deuteron energy is fixed and the binding energy of the ortho-state and para-state in ratio to the binding energy of the deuteron in free space is drawn as a function of $k_F$. The difference of the binding energies of the two states is also drawn. Figures 8 and 9 show the binding energy variations as a function of $K_0$ (energy) for fixed nuclear matter density.

The physical reasons behind these results will be presented in the next chapter. Here we list their most important features.

1) The binding energy of the deuteron in nuclear matter is always less than the binding energy in free space.

2) The binding energy of the para-state is always greater than the binding energy of the ortho-state.

3) The binding energy difference as a function of either $k_F$ (fixed energy) or $K_0$ (fixed nuclear matter density) has a maximum.

4) When, in what follows, we speak of the D-state probability we imply that the parameters of the neutron-proton interaction are such that the deuteron we examine has a certain D-state probability in free space. For convenience we label the D-state probability thus defined as $D_f$. The actual angular momentum distribution of the deuteron in nuclear matter will be examined later on in this chapter.
4A) For deuteron energies up to about 120 MeV the binding energy of
the para-state is increased as \( D_f \) is increased.

For higher deuteron energies and large values of \( k_F \), \( \text{BE}_\pm \) is
increased as the D-state is increased up to about \( D_f = 4\% \) and decreases
thereafter as \( D_f \) is increased.

4B) The binding energy of the orthostate is increased as the D-state
probability is decreased for realistic values of \( D_f \).

4C) The binding energy difference is increased as the D-state
probability is increased for realistic values of \( D_f \).

3.C. The deuteron wavefunction

The deuteron wavefunction for relative motion must be of the
form

\[
\phi_m = \phi(k)x_1^m = \hat{\theta} \frac{N g(k)}{\sqrt{2+k^2}} x_1^m
\]

\[
\phi_m = \phi(k)x_1^m = N \hat{\theta}[u_S(k) + u_D(k) S(k)]x_1^m
\]

where

\[
u_S(k) = \frac{1}{\sqrt{2+k^2} (\beta^2 + k^2)}
\]

\[
u_D(k) = \frac{-k^2 t}{\sqrt{8(2+k^2)(\gamma^2 + k^2)}}
\]

and \( N \) is an overall normalization constant.

Choosing the z-axis along \( \hat{k}_0 \) and demanding

\[
<\phi(k)x_1^m | \phi(k)x_1^m> = 1
\]
we get for $N$:

\[ N^{-2} = \int d k \left\{ u_S^2(k) + 8u_D^2(k) - 2\sqrt{\frac{32\pi}{5}} \gamma_2(k) \langle x_1^m | \tau_2 | x_1^m \rangle (u_D^2 - u_S^2 u_D) \right\} \]

\[ N^{-2} = N_0^{-2} - 4\pi [I_{N1} + I_{N2} - I_{N3} - I_{N4}] \]

where

\[ N_0^{-2} = 4\pi \int_0^\infty d k k^2 (u_S^2(k) + 8u_D^2(k)) \]

\[ \cdots = \pi^2 \left[ \frac{1}{\Delta\beta(\Delta+\beta)^3} + \frac{\tau^2(5\Delta+\gamma)}{8\gamma(\Delta+\gamma)^5} \right] \]

\[ I_{N1} = 0 \quad \text{if} \quad a > k_F \]

\[ I_{N1} = \int_0^\mu d k k^2 (u_S^2(k) + 8u_D^2(k)) \quad \text{if} \quad a < k_F \]

\[ I_{N2} = \int_\mu^\nu d k k^2 (u_S^2(k) + 8u_D^2(k)) \]

\[ I_{N3} = \int_\mu^\nu d k k^2 \sigma(u_S^2(k) + 8u_D^2(k)) \]

\[ I_{N4} = \sqrt{8} \langle x_1^m | T_{20} | x_1^m \rangle \int_\mu^\nu d k k^2 (\sigma^3 - \sigma)(u_D^2(k) - u_S(k)u_D(k)) \]

and $\sigma$ as defined by 1.12.

The free space case is particularly simple;

\[ N_{fs} = N_0 \]
3.D. Expansion of the wavefunction in series of spherical harmonics

Let us expand the wavefunction \( \phi_m \) in a series of spherical harmonics

\[
\phi_m = \sum_{l=0}^{\infty} \sum_{\lambda=-l}^{l} \beta_{l}^{\lambda} y_{l}^{\lambda}(k) x_{1}^{\lambda}
\]

The coefficients \( \beta_{l}^{\lambda} x_{1}^{\lambda} \) are given by

\[
\beta_{l}^{\lambda} x_{1}^{\lambda} = \langle y_{l}^{\lambda}(k) | \phi_m \rangle
\]

The wavefunction \( \phi_m \) is

\[
\phi_m = N \delta | u_S(k) + \sqrt{\frac{32\pi}{5}} \sum_{l=2}^{\infty} C_{L}^{\lambda} x_{1}^{\lambda} \chi^m \]

The operator \( \hat{\delta} \) can also be expanded in a series of spherical harmonics

\[
\hat{\delta} = \sum_{L} \sum_{\lambda} y_{L}^{\lambda}(k) c_{L}^{\lambda}
\]

with

\[
c_{L}^{\lambda} = \int d k \hat{\delta} y_{L}^{\lambda}(k)
\]

Hence the coefficients \( \beta_{l}^{\lambda} x_{1}^{\lambda} \) become

\[
\beta_{l}^{\lambda} x_{1}^{\lambda} = N \langle y_{l}^{\lambda} | \sum_{L,\lambda'} C_{L}^{\lambda'} y_{L}^{\lambda'} u_S(k) + \sqrt{\frac{32\pi}{5}} \sum_{L,\lambda'} C_{L}^{\lambda'} y_{L}^{\lambda'} u_{D}(k) \rangle x_{1}^{\lambda}
\]

\[
\sum_{\mu} c_{\mu}^{\lambda} x_{1}^{\lambda}
\]

now

\[
\sum_{\mu} c_{\mu}^{\lambda} x_{1}^{\lambda}
\]
From the Wigner-Eckart theorem\textsuperscript{37}

\[
\langle x_1^m' | T_2^{-\lambda_2} | x_1^m \rangle = C_2 \langle 12m - \lambda | 1 \rangle = \sum_{\mu} C_{\mu} \frac{\lambda_2}{\mu} \delta_{m't\mu} = C_{m'} \lambda_2.
\]

\[
\therefore C_{\mu} = C_2 \langle 12m - \lambda | 1 \rangle >
\]

\[
e.g. \quad \langle x_1^0 | T_2^0 | x_1^0 \rangle = -\sqrt{2} = C_2 \langle 1200 | 10 \rangle
\]

\[
\therefore C_2 = \sqrt{2}
\]

\[
\therefore T_2^{-\lambda_2} x_1^m = \sqrt{15} \sum_{\mu} (-)^{\mu+1} x_1^m \begin{pmatrix} 1 & 2 \\ -\lambda_2 & -\mu \end{pmatrix}
\]

Here the quantities \(\langle j_1 j_2 m_1 m_2 | JM \rangle\) and \(\binom{j_1 j_2 J}{m_1 m_2 M}\) are the Glebsch-Gordon coefficients and 3j symbols as defined in Messiah\textsuperscript{37}.

The integral over all angles of the product of three spherical harmonics is

\[
\int d\Omega \ Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega) Y_{l_3}^{m_3}(\Omega) = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}
\]

\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 3.19
\]

Using 3.18 and 3.19 in 3.15 we get

\[
\frac{\delta_\lambda x_1^m}{N} = \sum_{\lambda'} C_{\lambda'} u_\lambda'(k) x_1^m + \sum_{L \lambda_2 \lambda_2'} \sum_{\mu} (-)^{\lambda+\lambda_2} u_D(k) C_{\lambda_2}^{\lambda_2'} \sqrt{120(2\lambda+1)(2L+1)}
\]

\[
\begin{pmatrix} l & 2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 2 & L \\ -\lambda & \lambda_2 & \lambda_2' \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ m & \lambda_2 & -\mu \end{pmatrix} 3.20
\]
Choosing the z-axis along $k_0$, we find that $\hat{\theta}$ becomes independent of $\phi$, hence

$$C_L^{\lambda'} = \int \hat{\theta} Y_{L}^{\lambda'}(\Omega)d\Omega = \delta_{\lambda'0} C_L^{\lambda'}$$  \hspace{1cm} 3.21

With this choice for the z-axis we have

$$\frac{1}{N^2} \beta_L^m x_1^m = \delta_{\lambda0} C_L^0 u_s(k)x_1^m$$

$$+ \sum_L (-)^{m-\lambda+1} u_d(k) C_L^0 \sqrt{\frac{120(2\lambda+1)(2L+1)}{12}} \left[ \begin{array}{ccc} \lambda & 2 & L \\ \lambda & -L & 0 \end{array} \right] \left[ \begin{array}{ccc} 1 & 2 & 1 \\ -m & \lambda & (m-\lambda) \end{array} \right] x_1^{m-\lambda}$$  \hspace{1cm} 3.22

Since $C_L^{\lambda'} = \delta_{\lambda'0} C_L^{\lambda'}$ we can write

$$C_L^0 = C_L$$

$$C_L = \int_{-\sigma}^{+\sigma} Y_L^0(x)dx$$

$$\therefore C_L = 0 \quad \text{if } L \text{ is odd}$$  \hspace{1cm} 3.23
3.E. Selection Rules for Orbital Angular Momentum

Equation 3.10 corresponds to the expansion of the wavefunction into states of definite angular momentum \( \ell \) with projection \( \lambda \) along the z-axis. The coefficients \( \beta_{\ell}^{\lambda} \chi_{1}^{m} \), which are functions of \( K_0, k_0, k_0 \cdot k \) and \( k_0 \cdot p \), define the probability of finding the deuteron with total spin projection \( m \) along its centre of mass direction of motion (chosen as the z-axis) and orbital angular momentum \( \ell \) with projection \( \lambda \) along the z-axis, \( p_{\ell}^{\lambda m} \), through

\[
p_{\ell}^{\lambda m} = \int d k_{z} \beta_{\ell}^{\lambda} \chi_{1}^{m} \beta_{\ell}^{\lambda} \chi_{1}^{m}
\]

3.24

The properties of the 3j symbols, which appear in the expression for \( \beta_{\ell}^{\lambda} \chi_{1}^{m} \), equation 3.22, define the selection rules for the orbital angular momentum.

From the "triangular inequalities" of the 3j symbols we have

\[
|\ell - 2| \leq L \leq \ell + 2
\]

3.25

The 3j symbol

\[
\begin{pmatrix}
\ell & 2 & L \\
0 & 0 & 0
\end{pmatrix}
\]

is zero if \( \ell + 2 + L \) is odd. Since \( L \) must be even (eq. 3.23) we have a non zero coefficient when \( \ell \) is even.

The 3j symbol

\[
\begin{pmatrix}
\ell & 2 & 1 \\
-m & \lambda & (m-\lambda)
\end{pmatrix}
\]

is non zero if \( |m - \lambda| \leq 1 \).

Hence for a non zero probability \( p_{\ell}^{\lambda m} \) we must have an even orbital angular momentum \( \ell \) with projection \( \lambda \) satisfying

\[
|m - \lambda| < 1
\]

3.26
Before we investigate the angular momentum distribution in nuclear matter we examine what this distribution is in free space. In free space we can not talk any more about ortho-state and para-state because there is no unique centre of mass direction of motion \( \hat{k}_0 \), since \( \hat{k}_0 \) can be changed at will by a change of the reference frame. Nevertheless we can choose two convenient reference frames. Frame 1 is chosen such that the deuteron total spin is along the z-axis and frame 2 is chosen such that the deuteron total spin is perpendicular to the z-axis. In either case we expect the total probability for an angular momentum \( \lambda \) to be the same in both frames. The projection of the angular momentum along the z-axis should however be different.

In free space the operator \( \hat{0} \) is the unit operator. Hence using equations 3.11, 3.12 and 3.18 with \( \hat{0} = 1 \) we get

\[
\frac{\beta \lambda x_1^m}{N} = \langle Y \lambda | \sqrt{4\pi} Y_0^0 u_S(k) \rangle + \sqrt{\frac{32\pi}{5}} \sum_{\lambda_2} (-)^\lambda u_D(k) \langle Y \lambda | Y_2 \lambda_2 \rangle T_{2,-\lambda_2} x_1^m.
\]

\[
\frac{\beta \lambda x_1^m}{N} = \delta_0 \delta_0 \sqrt{4\pi} u_S(k) + \sqrt{\frac{32\pi}{5}} u_D(k) \delta_{\lambda 2} (-)^\lambda \sum_{\mu} \frac{(-)^{\mu+1}}{\Gamma(\mu+1)}
\]

\[
\left[ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \\ -\lambda \\ -\mu \\ m \end{array} \right] x_1^\mu. 3.27
\]

The only non-zero coefficients have orbital angular momentum \( \lambda \)
equal to zero or two. This gives

\[
\beta_0 \lambda x_1^m = N \sqrt{4\pi} u_S(k) 3.28
\]

\[
\beta_2 \lambda x_1^m = N \sqrt{8\pi} (-)^{m+\lambda} u_D(k) \left[ \begin{array}{c} 1 \\ 2 \\ 1 \\ m \\ -\lambda \\ \lambda+\mu \\ m-\lambda \end{array} \right] x_1^\mu. 3.29
\]
We want also to evaluate the projection along the z-axis of the D-state ($\ell = 2$).

1) **Frame 1**: $\chi_1^m = \chi_1^1 \quad \lambda = 0, 1, 2$

\[ \beta_2^0 \chi_1^1 = N \sqrt{96} \ u_D(k) \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \chi_1^1 \]

\[ \therefore \beta_2^0 \chi_1^1 = \frac{4}{\sqrt{5}} N u_D(k) \chi_1^1 \quad 3.30 \]

\[ \beta_2^1 \chi_1^1 = N \sqrt{96} \ u_D(k) \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \chi_1^0 \]

\[ \therefore \beta_2^1 \chi_1^1 = 4 \sqrt{3} N u_D(k) \chi_1^0 \quad 3.31 \]

\[ \beta_2^2 \chi_1^1 = N \sqrt{96} u_D(k) \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{pmatrix} \chi_1^{-1} \]

\[ \therefore \beta_2^2 \chi_1^1 = 4 \sqrt{6} N u_D(k) \chi_1^{-1} \quad 3.32 \]

Hence if the total spin of the deuteron lies along the z-axis then the D-state of the deuteron is divided into components of relative probability 60% along the z-axis, 30% with projection 1 along the z-axis, and 10% projection perpendicular to the z-axis.

2) **Frame 2**: $\chi_1^m = \chi_1^0 \quad \lambda = -1, 0, +1$

\[ \beta_2^0 \chi_1^0 = (-) N \sqrt{96} u_D(k) \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \chi_1^0 \]

\[ \therefore \beta_2^0 \chi_1^0 = - \frac{8}{\sqrt{5}} N u_D(k) \chi_1^0 \quad 3.33 \]

\[ \beta_2^{-1} \chi_1^0 = (-) N \sqrt{96} \ u_D(k) \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \chi_1^1 \]

\[ \therefore \beta_2^{-1} \chi_1^0 = 4 \sqrt{3} N u_D(k) \chi_1^1 \quad 3.34 \]
\[ \beta_2^{-1} x_1^0 = \frac{(-) N \sqrt{96}}{u_D(k)} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \chi_1^{-1} \]

\[ \therefore \beta_2^{-1} x_1^0 = 4 \sqrt{\frac{3}{5}} u_D(k) \chi_1^{-1} \]

Hence, if the total spin of the deuteron lies along an axis perpendicular to the z-axis, the D-state is divided into components of relative probability 30\% projection + 1 along the z-axis, 30\% for projection - 1, and 40\% for projection zero along the z-axis.

3.F. Evaluation of the Amplitudes

The operator \( \hat{\theta} \) restricts the angular integration over \( \cos \theta \) in the interval \( \mu < k < \nu \) from \( -\sigma \) to \( +\sigma \) and reduces the \( \cos \theta \) integration to zero in the interval \( 0 < k < \mu \) when \( k_F > a \). In all other intervals \( \hat{\theta} \) is identical to the unit operator. For \( L \) greater than zero the coefficients \( C_L \) are non-zero only in the interval \( \mu < k < \nu \).

\[
\begin{align*}
C_0 &= 0 & 0 < k < \mu \quad \text{and} \quad k_F > a \\
C_0 &= 2 \sqrt{\pi} & 0 < k < \mu \quad \text{and} \quad k_F < a \\
C_0 &= 2 \sqrt{\pi} \sigma & \mu < k < \nu \\
C_0 &= 2 \sqrt{\pi} & \nu < k < \infty
\end{align*}
\]

\( C_L = 0 \) for \( L > 0 \) and \( k \) not in the interval \( \mu < k < \nu \). In the interval \( \mu < k < \nu \) we have

\[
C_2 = \sqrt{5\pi}(\sigma^3 - \sigma) \quad 3.37
\]

\[
C_4 = \frac{3}{4} \sqrt{\pi}(7\sigma^5 - 10\sigma^3 + 3\sigma) \quad 3.38
\]
Hence

\[
\begin{align*}
C_0^2 &= 4\pi \quad [v < k < \infty \text{ or } (a > k_F \text{ and } 0 < k < \mu)] \\
C_0^2 &= 0 \quad 0 < k < \mu \text{ and } k_F > a \\
C_0^2 &= 4\pi\sigma^2 \quad \mu < k < v
\end{align*}
\]

The following coefficients are zero except in the interval \(\mu < k < v\) where they take the values

\[
\begin{align*}
C_2 &= 5\pi(\sigma^6 - 2\sigma^4 + \sigma^2) \\
C_4 &= \frac{9\pi}{16}(49\sigma^8 - 140\sigma^6 + 142\sigma^4 - 60\sigma^2) \\
C_0^2 C_2 &= 2\pi \sqrt{5(\sigma^4 - \sigma^2)} \\
C_0^2 C_4 &= \frac{3}{2} \pi(7\sigma^6 - 10\sigma^4 + 3\sigma^2) \\
C_2^2 C_4 &= \frac{3\pi}{4} \sqrt{5(7\sigma^8 - 17\sigma^6 + 13\sigma^4 - 3\sigma^2)}
\end{align*}
\]

Using the coefficients \(C_L\) as just defined the individual amplitudes are as follows:

\[L = 0 \quad (L = 2, \lambda = 0)\]

\[
\begin{align*}
\beta_0^0 \chi_1^1 &= N[C_0 u_S(k) + \frac{2}{\sqrt{5}} C_2 u_D(k)]\chi_1^1 \\
\beta_0^0 \chi_1^0 &= N[C_0 u_S(k) - \frac{\mu}{\sqrt{5}} C_2 u_D(k)]\chi_1^0 \\
\beta_0^0 \chi_1^{-1} &= N[C_0 u_S(k) + \frac{2}{\sqrt{5}} C_2 u_D(k)]\chi_1^{-1}
\end{align*}
\]

\[L = 2 \quad m = + 1 \text{ (para-state) } ; \quad L = 0, 2, 4 \quad \lambda = 0, 1, 2\]

\[
\begin{align*}
\beta_2^0 \chi_1^1 &= N[C_2 u_S(k) + \frac{4}{\sqrt{5}} u_D(k)\frac{C_0}{2} + \frac{\sqrt{5}}{7} C_2 + C_4]\chi_1^1
\end{align*}
\]
\[ \beta_2 \chi_1 = 4 \sqrt{\frac{3}{5}} N u_D(k) \left[ \frac{C_0}{2} - \frac{\sqrt{5}}{14} C_2 + \frac{2}{7} C_4 \right] \chi_1 \] 3.50

\[ \beta_2 \chi_1 = 4 \sqrt{\frac{3}{10}} N u_D(k) \left[ C_0 - \frac{2}{7} \sqrt{5} C_2 + \frac{1}{7} C_4 \right] \chi_1^{-1} \] 3.51

\[ \lambda = 2 \quad m = 0 \text{ (ortho-state)} \quad L = 0, 2, 4 \quad \lambda = -1, 0, +1 \]

\[ \beta_2 \chi_1 = 4 \sqrt{\frac{3}{5}} N u_D(k) \left[ \frac{C_0}{2} + \frac{\sqrt{5}}{14} C_2 - \frac{2}{7} C_4 \right] \chi_1 \] 3.52

\[ \beta_2 \chi_1 = N \{ C_2 u_D(k) - \frac{\mu}{\sqrt{5}} u_D(k) \left[ C_0 + \frac{2}{7} \sqrt{5} C_2 + \frac{6}{7} C_4 \right] \} \chi_2 \] 3.53

\[ \beta_2 \chi_1 = 4 \sqrt{\frac{3}{5}} N u_D(k) \left[ \frac{C_0}{2} + \frac{\sqrt{5}}{14} C_2 - \frac{2}{7} C_4 \right] \chi_1^{-1} \] 3.54

The probability, \( P_\lambda^m \), of finding the deuteron with total spin projection \( m \) along its centre of mass direction of motion (\( \hat{k}_0 \)) and orbital angular momentum \( \lambda \) with projection \( \lambda \) along \( \hat{k}_0 \), can be evaluated from equation 3.24.

The probability, \( P_\lambda^m \), of finding the deuteron with total spin projection \( m \) along \( \hat{k}_0 \) and orbital angular momentum \( \lambda \), can be evaluated by summing over the orbital angular momentum projections.

\[ P_\lambda^m = \sum_\lambda P_\lambda^\lambda \] 3.55

The integrals involved in this calculation were evaluated numerically using Simpson's rule for numerical integration.
3.6. The Deuteron Orbital Angular Momentum Distribution in Nuclear Matter

The orbital angular momentum distribution of the deuteron in nuclear matter is different to the corresponding distribution in free space. In order to avoid confusion between the free-space and nuclear matter orbital angular momentum probabilities we use the following notation.

The S-state and D-state probabilities are denoted by the letters S and D respectively. The total probability of all higher orbital angular momentum states is denoted by HS. Each probability carries either the subscript f if it refers to deuteron in free space or the subscript n if it refers to a deuteron propagating through a nuclear medium. When the ratio of a probability in nuclear matter to that of free space is used, then the appropriate letter carries a circumflex and the subscript n. For example the D-state probability in nuclear matter is denoted by $D_n$. The ratio $D_n$ to $D_f$ is denoted by $\hat{D}_n$.

The orbital angular momentum distribution of the deuteron, as a function of $K_0$ (energy) for fixed $k_p$, in nuclear matter is shown in figures 10 to 18. The S-state and D-state probability are represented as a ratio to the corresponding probability in free space. We have drawn the results for $D_f = 4\%$. Different values of $D_f$ will produce similar graphs. These figures show the following features.

1) States of non zero orbital angular momentum have always greater probabilities, $D_n$, HS $n$ in the parastate configuration than in the ortho-state. $S_n$ for the parastate is always less than $S_n$ for the ortho-state.
2) The binding energy of either the parastate or the ortho-state approaches zero when the D-state probability, $D_n$, of the corresponding state approaches a minimum value.

3) The detailed shape of the D-state probability, $D_n$, follows one of three possible shapes.

Shape a: $\hat{D}_n$ increases sharply as $K_0$ increases. As $K_0$ increases further $\hat{D}_n$ reaches a maximum, $\hat{D}_n^{\text{max}}$, and soon after a minimum $\hat{D}_n^{\text{min}}$ followed by another maximum $\hat{D}_n^{\text{max}2}$. Thereafter $\hat{D}_n$ decreases steadily approaching 1 as $K_0$ increases. This shape occurs when $k_F$ is very small ($k_F < 0.4$ fm$^{-1}$). The maxima and the minimum have values greater than one, e.g. Fig. 10.

Shape b: $\hat{D}_n$ increases sharply as $K_0$ increases, it reaches a maximum, $\hat{D}_n^{\text{max}}$, and soon after a minimum, $\hat{D}_n^{\text{min}}$, which has a value less than one. Thereafter $\hat{D}_n$ increases steadily approaching 1 as $K_0$ increases further e.g. Fig. 13.

Shape c: $\hat{D}_n$ increases uniformly from near zero approaching 1 as $K_0$ increases. This shape occurs when $k_F$ is large e.g. Fig. 17.

3.A.) In the parastate configuration $\hat{D}_n$ follows shape a when $k_F < 0.4$ fm$^{-1}$. In the range $0.4$ fm$^{-1} < k_F < 1.4$ fm$^{-1}$, $\hat{D}_n$ follows shape b. For $k < 1$ fm$^{-1}$ $\hat{D}_n^{\text{max}}$ is greater than one and for $k_F > 1$ fm$^{-1}$ $\hat{D}_n^{\text{max}}$ is less than one. For higher densities, $k_F > 1.9$ fm$^{-1}$ $\hat{D}_n$ follows shape c.
3.B.) The D-state probability $\hat{D}_n$ of the ortho-state follows
shape a when $k_F < 0.3 \text{ fm}^{-1}$. In the range
$0.3 \text{ fm}^{-1} < k_F < 0.8 \text{ fm}^{-1}$ $\hat{D}_n$ for the ortho-state follows
shape b. For $k_F < 0.6 \text{ fm}^{-1}$ $\hat{D}_n^{\text{max}}$ has a value greater
than one and for $k_F > 0.6 \text{ fm}^{-1}$ $\hat{D}_n$ is always less than one.
When $k_F > 0.8 \text{ fm}^{-1}$ the ortho-state D-state probability
$\hat{D}_n$ follows shape c.

4) The probabilities $\hat{H}_n$ increase from zero as $K_0$ increases reaching a
maximum, near the maximum for the D-state probability $\hat{D}_n$. As
$K_0$ increases further $\hat{H}_n$ decreases approaching zero. Invariably
$\hat{H}_n$ are larger when $k_F$ is small and $K_0$ is of order $2 k_F$. 
Figure 1. Binding energy for para-state and ortho-state in nuclear matter for deuteron energy 50 MeV.
Figure 2. Binding energy for para-state and ortho-state in nuclear matter for deuteron energy 100 MeV.
Figure 3. Binding energy for para-state and ortho-state in nuclear matter for deuteron energy 200 MeV.
Figure 4. Binding energy for para-state and ortho-state in nuclear matter for deuteron energy 300 MeV.
Figure 5. Binding energy difference between para-state and ortho-state configurations in nuclear matter for deuteron energies 50 MeV and 100 MeV.
Figure 6. Binding energy difference between para-state and ortho-state configurations in nuclear matter for deuteron energy 200 MeV.
Figure 7. Binding energy difference between para-state and ortho-state configurations in nuclear matter for deuteron energy 300 MeV.
Figure 8. Binding energy for para-state and ortho-state in nuclear matter of Fermi momentum $1.36 \text{ fm}^{-1}$. 

$K_F = 1.36 \text{ fm}^{-1}$

- $D$-state 2%
- $D$-state 4%
- $D$-state 6%
Figure 9. Binding energy difference between para-state and ortho-state in nuclear matter of Fermi momentum $1.36 \text{ fm}^{-1}$. 

$K_F = 1.36 \text{ fm}^{-1}$

- D-state $2\%$
- D-state $4\%$
- D-state $6\%$
Figure 10. Orbital angular momentum distribution in nuclear matter of Fermi momentum 0.3 fm\(^{-1}\) for the para-state configuration.
Figure 11. Orbital angular momentum distribution in nuclear matter of Fermi momentum 0.3 fm$^{-1}$ for the ortho-state configuration.
Figure 12. High orbital angular momentum distribution in nuclear matter of Fermi momentum 0.3 fm$^{-1}$.
Figure 13. Orbital angular momentum distribution in nuclear matter of Fermi momentum 0.7 fm$^{-1}$ for the para-state configuration.
Figure 14. Orbital angular momentum distribution in nuclear matter of Fermi momentum 0.7 fm\(^{-1}\) for the ortho-state configuration.
Figure 15. High orbital angular momentum distribution in nuclear matter of Fermi momentum 0.7 fm$^{-1}$. 

$K_F = 0.7$ fm$^{-1}$

-- HS n $^\%$ Para-state
-- HS n $^\%$ Ortho-state
-- $\Delta\varepsilon$ (MeV)
Figure 16. Orbital angular momentum distribution in nuclear matter of Fermi momentum 1.3 fm⁻¹ for the para-state configuration.
Figure 17. Orbital angular momentum distribution in nuclear matter of Fermi momentum 1.36 fm$^{-1}$ for the ortho-state configuration.
Figure 18. High orbital angular momentum distribution in nuclear matter of Fermi momentum 1.36 fm\(^{-1}\).
CHAPTER 4

A PICTURE FOR THE PAULI EXCLUSION PRINCIPLE MECHANISM

4.A. Some comments on the action of the Pauli exclusion principle.

The main features of the results of the previous chapter can be understood in terms of a simple picture.

We neglect for the moment the neutron and proton intrinsic spin and we suppose that the total spin of the deuteron arises entirely through orbital motion of the neutron and proton. Figures 19 A and B describe the relative orientation of the momentum vector, \( \mathbf{k}_0 \), of the deuteron centre of mass and the plane of orbit of the neutron and proton (spin of the deuteron). The total momenta of the neutron and proton are the vector difference and sum respectively of half the centre of mass momentum, \( \mathbf{a} \), and the relative momentum \( \mathbf{k} \), equations 1.5

\[
\begin{align*}
\mathbf{k}_p &= \mathbf{a} + \mathbf{k} \\
\mathbf{k}_n &= \mathbf{a} - \mathbf{k} \\
\mathbf{a} &= \frac{1}{2} \mathbf{k}_0
\end{align*}
\]

The magnitudes of these momenta, \( k_p \) and \( k_n \), are

\[
\begin{align*}
k_p &= \left( a^2 + k^2 + 2ak \cos \theta \right)^{\frac{1}{2}} \\
k_n &= \left( a^2 + k^2 - 2ak \cos \theta \right)^{\frac{1}{2}}
\end{align*}
\]

where \( \theta \) is the angle between the centre of mass momentum, \( k_0 \), and the relative momentum \( k \).

These magnitudes, \( k_p \) and \( k_n \), determine the effect of the Pauli exclusion principle on the deuteron, equations 1.7. The Pauli
principle restricts or allows states in the deuteron wavefunction according to whether or not both \( k_p \) and \( k_n \) are smaller than \( k_F \).

In the para-state situation, figure 19.A, the plane of the n-p orbit is perpendicular to the centre of mass momentum, the angle \( \theta \) is therefore equal to \( \frac{\pi}{2} \). The magnitudes of the neutron and proton momenta are therefore

\[
k_p = k_n = (a^2 + k^2)^{\frac{3}{2}}
\]

In the ortho-state situation however, figure 19.B, the plane of the n-p orbit contains the centre of mass momentum vector \( K \) and the angle \( \theta \) is arbitrary. In that case either \( k_n \) or \( k_p \) will have magnitude less than the corresponding magnitude in the para-state, \( (a^2 + k^2)^{\frac{1}{2}} \). Some of the n-p relative momenta \( k \) are therefore such that the Pauli exclusion principle, equations 1.7, is satisfied by the para-state but not for the ortho-state. For example, when the Fermi-momentum, \( k_F \), is

\[
k_F = \sqrt{a^2 + c^2}
\]

with \( c \) a real number and 

\[
d = c + \delta
\]

with \( \delta \) a very small real number, practically all relative momenta \( k \) such that

\[
|k| = d
\]

will be forbidden for the ortho-state. These same relative momenta will be allowed for the para-state.
In the real deuteron a coupling of this type between the total spin of the deuteron and the plane of the n-p orbit is produced by the tensor force. The tensor force operator does not commute with the orbital angular momentum operator. The orbital angular momentum of the deuteron and its component are not observables. The deuteron wavefunction is a superposition of states with a definite orbital angular momentum vector. We can nevertheless discuss the effect of the Pauli exclusion principle on each such state. We make the usual choice for the z-axis.

\[ \hat{z} = \hat{\mathbf{k}}_0 \]

We represent the different components in the deuteron wavefunction by the Ket $|2\lambda\rangle$. \( \lambda \) refers to the magnitude of the orbital angular momentum and \( \lambda \) to its projection along the z-axis.

A state of zero orbital angular momentum, like the S-state of the deuteron ($|0\ 0\rangle$), has no special symmetry axis; it is a spherically symmetric state. The relative momentum vector, \( \mathbf{k} \), for that state is uniformly distributed over all angles for both the para-state and the ortho-state. The effect of the Pauli exclusion principle on the S-state part of the deuteron wavefunction is therefore identical for both the para-state and ortho-state.

A state with non-zero orbital angular momentum does not possess spherical symmetry. The D-state of the deuteron, $|2\lambda\rangle$, is such a state. Its main axis of symmetry is the direction of its orbital angular momentum.

The component of the D-state with \( \lambda = \pm 2 \), $|2 \pm \rangle$, has its orbital angular momentum axis along the direction of motion of the
deuteron centre of mass. The plane of the n-p orbit is therefore perpendicular to this axis. This component is therefore the component least affected by the Pauli exclusion principle.

The plane of the n-p orbit in the D-state component with \( \lambda = 0 \), \( |2 \, 0> \), contains the centre of mass momentum vector, \( K_0 \). This component is therefore the component most affected by the Pauli exclusion principle.

The effect of the Pauli exclusion principle on the D-state component with \( \lambda = \pm 1 \), \( |2 \, \pm 1> \), is stronger than the corresponding effect on the component \( |2 \, \pm 2> \) and weaker than the corresponding effect on the component \( |2 \, 0> \).

As a summary of the above discussion we list the components of the D-state part of the deuteron wavefunction with definite orbital angular momentum according to how strongly they are affected by the Pauli exclusion principle. We start with the most strongly affected component.

The D-state with projection \( \lambda = 0 \) along \( \hat{K}_0 \), \( |2 \, 0> \).

The D-state with \( \lambda = \pm 1 \), \( |2 \, \pm 1> \).

The D-state with \( \lambda = \pm 2 \), \( |2 \, \pm 2> \).

We will now outline the conclusions stemming out of this picture for the Pauli mechanism and compare them with the results of the previous chapter.
4.B. The binding energy of the deuteron in nuclear matter

The Yamaguchi potential we use in this work for the neutron-proton interaction is attractive for both the S-state and the D-state. The exclusion of certain relative momenta due to the Pauli principle leads naturally to a reduction of binding energy; hence the binding energy of the deuteron in nuclear matter is expected to be always less than the binding energy of the deuteron in free space.

We neglect for the moment the Pauli exclusion principle and we consider a deuteron propagating through nuclear matter with centre of mass momentum parallel to the $z$-axis. The para-state corresponds to frame 1 and the ortho-state to frame 2. Frames 1 and 2 are defined in chapter 3. The D-state part of the deuteron wavefunction is then composed of the following components with definite angular momentum projection along the $z$-axis.

**Para-state:**

$|2\ 0\rangle\ 10\%$

$|2\ \pm\ 1\rangle\ 30\%$

$|2\ \pm\ 2\rangle\ 60\%$

**Ortho-state:**

$|2\ 0\rangle\ 40\%$

$|2\ +\ 1\rangle\ 30\%$

$|2\ -\ 1\rangle\ 30\%$

We now allow the Pauli exclusion principle to operate. Its effect will be stronger on the ortho-state than on the para-state because the ortho-state contains, in its D-state, larger components affected strongly by the Pauli principle, than the para-state. Since the neutron-proton interaction we use is always attractive we expect the binding energy of the para-state to be always greater than the binding energy of the ortho-state in nuclear matter.
We denote the difference in binding energy between the para-state and ortho-state by $\Delta \varepsilon$ such that

$$\Delta \varepsilon = BE_\perp - BE_0$$

We have examined, earlier on, the variation of $\Delta \varepsilon$ as a function of either $K_0$ (fixed $k_F$) or as a function of $k_F$ (fixed $K_0$). We examine here the limits of these variations.

When $k_F$ approaches zero the para-state and ortho-state are degenerate, $\Delta \varepsilon$ therefore also approaches zero.

For very large values of $k_F$ both the para-state and ortho-state binding energies are reduced to zero, hence $\Delta \varepsilon$ is zero.

When the deuteron energy is very large all relative momenta in the deuteron wavefunction are allowed. The binding energies of both the ortho-state and para-state approach the binding energy of free space. $\Delta \varepsilon$ therefor approaches zero.

The limiting case when the deuteron is at rest in nuclear matter ($K_0 \to 0$) was examined in chapter 2 section D. $\Delta \varepsilon$ for this case was found to be zero.

Since $\Delta \varepsilon$ must always be positive the graphs of $\Delta \varepsilon$ as a function of either $k_F$ or $K_0$ must show at least one maximum.

In order to investigate the dependence of the binding energy of the deuteron in nuclear matter on the D-state probability, $D_f$, we make some comments about the neutron-proton interaction arising from table 2.
As $D_f$ increases, the overall strength, $\lambda$, of the neutron proton-interaction decreases while the relative D-state strength, $t$, increases. As $D_f$ increases, the parameter $\beta$ for the S-state is slowly decreased while the parameter $\gamma$ for the D-state is rapidly increased. Hence as $D_f$ increases, the neutron-proton interaction depends more on the D-state, the S-state contains smaller relative momenta $k$ and the D-state is peaked at larger values for the relative momenta. We therefore expect the following things to happen to the deuteron binding energy as $D_f$ increases:

1) The binding energy has more contributions from the D-state.

2) The S-state contributions to the binding energy come through smaller values of $k$.

3) The D-state contributions to the binding energy come through higher values of $k$.

We therefore expect for low deuteron energies, where the Pauli exclusion principle depresses strongly all deuteron components except the D-state part with $\lambda = 2$, the binding energy of the para-state to be increased as $D_f$ is increased. In the same range we expect the binding energy of the ortho-state to be decreased as $D_f$ is increased.

For high deuteron energies, where the Pauli exclusion principle affects strongly only the D-state component with $\lambda = 0$ and specially its high $k$ values, we expect the binding energy of the para-state to decrease as $D_f$ is increased. This decrease however should be small because the component affected is only 10% of the D-state. In the same range we expect the binding energy of the ortho-state to decrease.
rather rapidly as $D_f$ is increased, since the component affected is 40% of the D-state.

In all ranges we expect the binding energy difference, $\Delta \varepsilon$, to increase as $D_f$ is increased.

All these conclusions for the behaviour of the binding energy of the deuteron in nuclear matter stemming out of the simple picture for the Pauli mechanism we propose here are in full agreement with the results of the previous chapter.

4.C. The deuteron orbital angular momentum distribution in nuclear matter

We now want to apply the Pauli picture we propose in this chapter to the orbital angular momentum distribution of the deuteron in nuclear matter. In order to accomplish that we first examine the distribution of relative momenta $k$ in the deuteron wavefunction when the deuteron propagates through free space. The probability of finding the neutron-proton system in the deuteron with relative momentum lying between $k$ and $k + dk$ is $P(k)$

$$ P(k) = 4\pi k^2 \phi^*(k) \phi(k) dk $$

We can divide this probability into two parts according to whether the n-p system with relative momentum between $k$ and $k + dk$ forms an S-state or a D-state. Using equations 3.2 we can define the S-state and D-state distributions of relative momentum by $P_s$ and $P_d$.

$$ P_s = N_s k^2 u_s^2(k) $$

$$ P_d = N_d k^2 u_d^2(k) $$
Where the normalization constants $N_s$ and $N_D$ are chosen for convenience to be such that the maximum values of $P_s$ and $P_D$ are equal to one. These distributions are plotted in Fig. 20. We see from the figure that the S-state is concentrated at small relative momenta with the maximum just above $k = 0.2 \text{ fm}^{-1}$. The D-state is concentrated at larger relative momenta with its maximum around $k = 0.95 \text{ fm}^{-1}$.

When the deuteron is propagating through a nuclear medium, of Fermi momentum $k_F$, with centre of mass momentum $K_0$, the range of values for the magnitude of the relative momentum $k$ is naturally divided into three regions. These regions are defined as follows:

Region 1: $0 < k < \mu$

Region 2: $\mu < k < \nu$

Region 3: $\nu < k < \infty$

where: $\mu = a - k_F$ if $a > k_F$

$\mu = \sqrt{k_F^2 - a^2}$ if $a < k_F$

$\nu = a + k_F$, $a = \frac{K_0}{2}$

The effect of the Pauli exclusion principle on each region is described by one of the three adjectives, allowed, forbidden and semi-allowed. An allowed region is one where all angular orientations of the vector $k$ are allowed. A forbidden region is one where all angular orientations of the vector $k$ are forbidden. A semi-allowed region is one where some angular orientations of the vector $k$ are allowed where others are forbidden.
Region 1 is an allowed region if $a > k_F$. If $a < k_F$, region 1 is forbidden. Region 2 is always a semi-allowed region. Region 3 is always an allowed region. Fig. 21 depicts the three regions and their allowed and forbidden parts. The forbidden space lies within the two spheres of radius $k_F$.

It is useful to compare the deuteron wavefunction in free space to the corresponding wavefunction in nuclear matter. The latter can be thought of as the former modified by the Pauli exclusion principle. For a given nuclear matter density and deuteron centre of mass momentum we can define the three regions as specified in this section. We then divide the deuteron wavefunction in nuclear matter, $\phi_n$, and the free space wavefunction, $\phi_f$, into three parts.

$$\phi_n = \phi_{n1} + \phi_{n2} + \phi_{n3}$$

$$\phi_f = \phi_{f1} + \phi_{f2} + \phi_{f3}$$

The second subscript refers to which region the relative momenta in the wavefunction belong. Each wavefunction $\phi_{ij}$ can be expanded into a sum of wavefunctions of definite orbital angular momentum. The free space wavefunction $\phi_f$ will of course contain S-state and D-state parts only.

In regions 1 and 3 the allowed space is spherically symmetric, in the case $k_F > a$ the allowed space of region 1 is spherically symmetric in the trivial way that it does not exist. The symmetry of the free space wavefunction is therefore preserved in the nuclear matter wavefunction for regions 1 and 3. The wavefunctions $\phi_{n1}$ and $\phi_{n3}$ can therefore be described by an S-state and a D-state part only.
Region 2, allowed + forbidden parts, is also a spherically symmetric region. The allowed part of region 2 alone however is not spherically symmetric. This non-symmetry destroys the symmetry which the deuteron wavefunction has in free space, when the deuteron is propagating through nuclear matter. For example, the allowed S-state part of $\phi_{f2}$ in $\phi_{n2}$ cannot any longer be described by an S-state wavefunction alone. It must be described by an infinite sum of states of definite orbital angular momentum (with even $\ell$ to conserve parity). Similarly the allowed D-state part of $\phi_{f2}$ which forms the remaining part of $\phi_{n2}$ must also be described by an infinite sum of states with definite orbital angular momentum with even $\ell$. The presence of high orbital angular momentum states in $\phi_n$, which are not present in the deuteron wavefunction in free space, must not create the impression that the Pauli exclusion principle "creates" new states for the n-p system. The high even $\ell$ states arise from the states already present in the free space deuteron wavefunction. The probability of finding in the deuteron wavefunction orbital angular momenta with $\ell$ greater than 2 depends on the position and extent of region 2.

In the limiting case when $K_0$ approaches zero the semi-allowed region vanishes. Therefore for extremely small $K_0$ values a deuteron propagating through nuclear matter is composed almost entirely by an S-state and a D-state. The relative proportion of S-state to D-state will however be different to the corresponding proportion in free space.

We can now "predict" the orbital angular momentum distribution of the deuteron in nuclear matter according to the ideas introduced so far in this chapter.
We first consider the case when the nuclear matter density is very small, \( k_F < 0.3 \text{ fm}^{-1} \). When \( K_0 \) is extremely small, \( K_0 << k_F \), there is practically no semi-allowed region. Region 1, which contains low magnitude relative momenta, is a forbidden region. A major part of the S-state, which is peaked at low relative momenta, is therefore destroyed in this region. A small part of the D-state is also destroyed in region 1. As \( K_0 \) increases a semi-allowed region is formed around region 1 which is still a forbidden region. Both regions 1 and 2 affect strongly the S-state. The effect of regions 1 and 2 on the D-state is much weaker than the corresponding effect on the S-state. In the semi-allowed region the dominant effect is the transformation of a large part of the S-state into D-state and higher even \( \ell \) states. We therefore expect the D-state probability \( \hat{D}_n \) to increase rapidly, from a value greater than 1 at \( K_0 = 0 \), as \( \frac{K_0}{2} \) increases from zero to \( k_F \). The probability \( HS_n \) of the higher states is also expected to increase, from near zero when \( K_0 \) tends to zero, as \( \frac{K_0}{2} \) increases from zero to \( k_F \). As \( K_0 \) increases further, region 1 becomes allowed as soon as \( \frac{K_0}{2} \) becomes greater than \( k_F \). Most of the S-state in \( \phi_f \) becomes allowed as an S-state in \( \phi_n \). We therefore expect \( D_n \) and \( HS_n \) to reach a maximum around \( \frac{K_0}{2} \approx k_F \) and decrease as \( K_0 \) increases further. As \( K_0 \) increases further region 1 will expand sufficiently to contain practically all the S-state. Further expansion of region 1 (increase in \( K_0 \)) will result in an increase of \( \hat{D}_n \) and a further decrease of \( HS_n \), since, the net effect there will be that more D-state in \( \phi_f \) will lie within region 1 and hence remain as a D-state in \( \phi_n \). Eventually region 1 will contain all of the S-state and D-state when \( K_0 \) is large enough. At this point the distribution of relative momenta in nuclear matter will be identical to that of free space. We therefore expect \( \hat{D}_n \) to reach a maximum after it
increases for the second time as $K_0$ increases. As $K_0$ increases further we expect $\hat{D}_n$ to decrease approaching one for large values of $K_0$. We also expect the distance in $K_0$ between the two maxima in $\hat{D}_n$ to be small because the spread of relative momenta in the $8$-state part of the deuteron wavefunction is small.

When $k_F$ is very small the forbidden space is also small. The major modification of the free space wavefunction comes through modifications of its $S$-state component. The major part of the $D$-state wavefunction is unaffected by the Pauli exclusion principle. We therefore expect the effect of the Pauli exclusion principle to be similar for both the para-state and ortho-state configurations. At these small nuclear matter densities the deuteron is always bound. This is a direct consequence of the small extent of the forbidden region. In other words the allowed region contains sufficient $n$-$p$ states to form a bound deuteron.

For larger values for the Fermi momentum, $k_F$, regions 1 and 2 contain together the major part of all possible states of the $n$-$p$ system. The $S$-state is contained mostly in region 1 and the $D$-state is contained mostly in region 2. The $D$-state component with orbital angular momentum projection along the $z$-axis ($\hat{k}_0^z$), $|2 \pm 2\rangle$, is concentrated around a plane perpendicular to $\hat{k}_0^z$; that is the plane through the middle of the line joining the centres of the two Fermi spheres and at right angle to this line. This $D$-state component has therefore the least overlap with the forbidden region. For a given centre of mass momentum $K_0$ and $k_F \geq 0.3$ fm$^{-1}$ the para-state configuration will contain a lot more allowed $n$-$p$ states belonging to the $D$-state than the ortho-state configuration. The $D$-state alone however does not produce a bound state. Since most of the $S$-state lies in region 1, the
The deuteron should be unbound when region 1 is forbidden, that is when \( k_F > \frac{K_0}{2} \). As \( K_0 \) increases from zero the para-state will first form a bound state soon after \( \frac{K_0}{2} \) becomes greater than \( k_F \). The D-state probability, \( \hat{D}_n \), of the para-state will increase rather sharply as \( K_0 \) increases, after a bound state is formed. This increase will come partly through contributions from the S-state of \( \phi_f \) lying in the semi-allowed region and partly because the component \( |2 \pm 2\rangle \) will become allowed. As \( K_0 \) increases further region 1 will contain practically all the S-state in \( \phi_f \). A further increase in \( K_0 \) will make region 1 large enough to contain more and more of the strongly affected D-state components, \( |2 \pm 1\rangle \) and \( |2 0\rangle \). We therefore expect \( \hat{D}_n \) to reach a maximum as \( K_0 \) increases. As \( K_0 \) increases further we expect \( \hat{D}_n \) to reach a minimum of value less than 1. Thereafter \( \hat{D}_n \) is expected to increase as \( K_0 \) increases further approaching 1 when \( K_0 \) is very large.

The shape of \( \hat{D}_n \) as a function of \( K_0 \), soon after the deuteron becomes bound, depends on how soon the deuteron becomes bound after \( \frac{K_0}{2} \) becomes greater than \( k_F \). This in turn depends on how many allowed D-state components there are in \( \phi_f \) when \( \frac{K_0}{2} \) becomes greater than \( k_F \). As \( k_F \) increases these allowed D-state components reduce. When \( k_F \) is large enough even the D-state component \( |2 \pm 2\rangle \) has its major part in the forbidden space. For the deuteron to be bound in this case the allowed region 1 should be large enough to contain within it practically all the S-state of \( \phi_f \). The S-state part of \( \phi_f \) is never transformed into other states in \( \phi_n \) for this case. For large enough \( k_F \), \( k_F > 1.4 \text{ fm}^{-1} \), we expect \( \hat{D}_n \) to increase uniformly, for the para-state configuration, from a small value, as \( K_0 \) increases, approaching one for very large values of \( K_0 \).
The argument for the ortho-state configuration is similar to the above. The D-state component \( |2 \pm 2> \) however is not present in the ortho-state. For a given \( K_0 \) and \( k_F \), \( (k_F > 0.3 \text{ fm}^{-1}) \), there are a lot less allowed D-state components in the ortho-state than in the para-state. For \( k_F > 0.3 \text{ fm}^{-1} \) we therefore expect the ortho-state configuration to produce a bound state for larger \( K_0 \) value than the para-state. The initial increase in \( \hat{D}_n \) as \( K_0 \) increases, after a bound state is formed, is expected to be less sharp than the corresponding increase in \( \hat{D}_n \) for the para-state. The value of \( k_F \) when the maximum in \( \hat{D}_n \) graph disappears for the ortho-state is expected to be smaller than the corresponding value of \( k_F \) for the para-state.

The higher even \( \ell \) states in the deuteron wavefunction \( \phi_{n2} \) arise from the semi-exclusion of the S-state and D-state of \( \phi_{f2} \); since the S-state is much more populated than the D-state in \( \phi_f \), we expect HS\(_n\) to be appreciably greater than zero when region 2 contains a large portion of the S-state and the deuteron forms a bound state. These conditions are fulfilled when both \( k_F \) and \( K_0 \) are very small and about equal to each other.

Comment:

The Pauli picture we have developed in this chapter explains practically all the features of a deuteron propagating through nuclear matter as derived from the use of the Yamaguchi potential.

This picture however is quite general, it can be applied to any potential. The division of the values of the relative momenta in the deuteron wavefunction into three regions (two of which have the property of preserving the initial symmetry of the wavefunction and one not possessing this property) is not particular to the Yamaguchi potential.
Figure 19. Relative orientation of the centre of mass momentum, $\mathbf{K}_0$, and the plane of orbit of the neutron and proton, (spin of the deuteron $S_d$) When the intrinsic spins of the neutron and proton are neglected.

In figure A $\hat{S}_d \cdot \hat{\mathbf{K}}_0 = 1$

In figure B $S_d \cdot \mathbf{K}_0 = 0$
Figure 20. Distribution of relative momenta in the S-state, $P_s$, and the D-state $P_D$. $P_s$ and $P_D$ are normalized in such a way that their respective maxima are equal to one.
Figure 21. Division of the relative momenta space, $k$-space, into forbidden, allowed and semi-allowed regions.
CHAPTER 5

A new tensor potential of the $T_p$ type in the deuteron optical potential

We have examined in the last two chapters the effect of the Pauli exclusion principle on the internal motion of the deuteron when the deuteron is propagating through nuclear matter. We have shown that the ortho-state and para-state configurations are non degenerate in nuclear matter. This is a spin dependent effect. The para-state and ortho-state configurations correspond to different eigenvalues of the operator $\hat{M}$,

$$\hat{M} = S \cdot \hat{K}_0$$

5.1

The para-state corresponds to $\hat{M} = \pm 1$ and the ortho-state corresponds to $\hat{M} = 0$. In the presence of nuclear matter therefore, there are in general two $1^+$ spin triplet states of the $n-p$ system. The degeneracy of the states with $\hat{M} = +1$ and $\hat{M} = -1$ suggests that the effect we examine here has a quadratic dependence on $\hat{M}$. As we will shortly show the binding energy difference, $\Delta \varepsilon$, between the states with $\hat{M} = \pm 1$ and $\hat{M} = 0$ can be attributed to a tensor force of the $T_p$ type in the deuteron optical potential, with the strength of this tensor force simply related to $\Delta \varepsilon$.

5.A. Need for a tensor force of the $T_p$ type

The binding energy difference, $\Delta \varepsilon$, between the states with $\hat{M} = \pm 1$ and $\hat{M} = 0$ increases as the nuclear matter density increases. For nuclear matter densities similar to the central density of real nuclei $\Delta \varepsilon$ has a maximum value slightly below 2 MeV at deuteron kinetic
energies around 200 MeV. Such an effect would have been very difficult
to observe experimentally a few years ago. However the spin depen­
dence of this effect and the recent developments of elastic scattering
experiments, in which the five observables, $\sigma(0)$ (the cross-section
for unpolarized deuterons), $i T_{11}, T_{20}, T_{21}, T_{22}$ (the analysing powers
for polarized deuterons) are all measured over a considerable angular
range and for a variety of targets and energies $^{22-26}$, make the
experimental detection of the effect under study here possible.
Furthermore, the results of the above mentioned experiments show that
for incident deuteron energies above the Coulomb barrier present day
theories are inadequate to explain the data $^{23,28}$.

The first theoretical study of the spin-dependence of the
deuteron-nucleus elastic scattering was done by Satchler $^{4,20}$.
Starting from general principles, based on the present day knowledge of
the symmetry properties of nuclear forces, and assuming that terms
at most quadratic in the momentum operator are present in the optical
potential, he showed that the optical potential describing the
elastic scattering of a spin one particle by a spin zero target can
contain apart from the usual spin orbit term three possible types
of tensor terms of rank 2, which are usually denoted by $T_R$, $T_L$ and
$T_P$. The form of these potentials is as follows.

\[ V_R = U_R(R)T_R \]  
\[ T_R = (S \cdot R)^2 - \frac{2}{3} \]  \hspace{1cm} 5.2

\[ V_L = U_L(R)T_L \]  
\[ T_L = (L \cdot S)^2 + \frac{1}{2} L \cdot S - \frac{2}{3} L^2 \]  \hspace{1cm} 5.3

\[ V_P = U_P(R)T_P + T_P U_P(R) \]  
\[ T_P = (S \cdot P)^2 - \frac{2}{3} P^2 \]  \hspace{1cm} 5.4
Satchler also showed how the $T_R$ type of tensor potential can be generated from the Watanabe model when the deuteron D-state is included.

Elastic scattering experiments with polarized deuterons are mainly performed at low deuteron incident energies. It seems at the present time that for subcoulomb energies the only tensor potential needed in the deuteron nucleus optical potential is of the $T_R$ type. For higher incident deuteron energies, theory and experiment disagree considerably even when both $T_R$ and $T_L$ types of tensor interactions are included in the optical potential. The inclusion of a $T_p$ type of tensor potential might prove very useful in bringing the theoretical predictions closer to the experimental data.

Lyovshin first proposed a mechanism for producing such a potential. However, this contribution is proportional to the energy derivative of the nucleon optical potential and is therefore much smaller than the $T_R$ potential predicted by the folding model. A mechanism for producing a $T_p$ type tensor potential of non-negligible strength has not so far been proposed.

5.B. The $T_p$ tensor term in the deuteron optical potential

The Schrödinger equation for a deuteron propagating through nuclear matter is

$$(E - T_R - H_{np} - V)\psi_n = 0$$

where $T_R$ is the kinetic energy operator, $H_{np}$ is the internal Hamiltonian for the deuteron and $V$ the optical potential of the deuteron in nuclear matter.
The exact solution is of the form

\[ \psi_m = \phi_m(r_0 K_0^2) e^{iK_0 \cdot R} \]  \hspace{1cm} 5.6

where \( \phi_m(r_0 K_0^2) \) is the deuteron wavefunction for relative motion, and it is the solution of the Schrödinger equation in nuclear matter, equation 2.10.

The subscript \( m \) refers to the eigenvalue of the operator

\[ \hat{\mathcal{H}} = \sum \hat{K}_0, \] and

\[ H_{np} \phi_m = (BE)_m \phi_m \]  \hspace{1cm} 5.7

with \( (BE)_m \) the binding energy of the deuteron.

Substituting 5.6 in 5.5 and using 5.7 we see that \( K_0 \) must satisfy

\[ \left( E - \frac{\hbar^2 K_0^2}{4m} + (BE)_m - V \right) = 0 \]  \hspace{1cm} 5.8

where \( m \) is the nucleon mass, or

\[ \frac{\hbar^2 K_0^2}{4m} = E - V_m \]  \hspace{1cm} 5.9

where

\[ V_m = V - (BE)_m \]  \hspace{1cm} 5.10

Hence

\[ V_\pm - V_0 = [(BE)_\pm - (BE)_0] = \Delta \epsilon \]  \hspace{1cm} 5.11
The binding energy of the deuteron in nuclear matter depends on the square of \((S \cdot \hat{K}_0)\). The only physically reasonable term in the deuteron optical potential, having such a dependence on \(M^{4,20}\) is the tensor term of the \(T_p\) type. We therefore write the deuteron optical potential as a sum of a central term \(V_c\) and a tensor term of the \(T_p\) type.

\[
v_m = V_c + \frac{1}{2} [V_p S(\hat{K}_0) + S(\hat{K}_0) V_p]
\]

where

\[
S(\hat{p}) = 2[3(S \cdot \hat{K}_0) - S^2]
\]

Hence

\[
V_\pm - V_0 = \frac{1}{2} \left\{ V_p [2(3 - 2) - 2(-2)] + [2(3 - 2) - 2(-2)] V_p \right\}
\]

\[
\therefore \quad V_\pm - V_0 = 6V_p
\]

\[
\therefore \quad \Delta \epsilon = -6V_p
\]
CHAPTER 6

THE PROPAGATION OF A DEUTERON THROUGH A FINITE NUCLEUS

The work presented so far was concerned with the propagation of a deuteron through nuclear matter. When a deuteron is propagating through a finite nucleus effects similar to the ones discussed for nuclear matter will arise. Their evaluation however is much more difficult than the corresponding nuclear matter calculations. We know from the work presented this far that a new tensor term of the $T_p$ type is present in the deuteron-nucleus optical potential. An exact theory accounting for this term is beyond the scope of the present work. Our objective is limited to a rough estimate of the strength and shape of this term.

In this chapter we propose a simple model for the propagation of a deuteron through a finite nucleus. This model enables us to obtain the para-state and ortho-state binding energy of the deuteron, and their difference, which is related to the strength of the new tensor force, as a function of the deuteron nucleus separation. The assumptions underlying the model are examined and their range of validity established.

6.A. A simple model for the propagation of a deuteron through a finite nucleus and the underlying assumptions

The model we propose here, for the propagation of a deuteron through a finite nucleus, is based on the following five assumptions.

a) Validity of the local density approximation (L.D.A.). We assume that every point in the target nucleus is surrounded by a
neighbourhood which can be approximated by nuclear matter of
practically constant density.

b) The spatial separation of the neutron and proton in the deuteron
can be neglected. In effect we assume that the deuteron is a
point particle!

c) The energies of the neutron and proton in the deuteron are both
equal to half the deuteron total energy.

d) The centre of mass motion of the deuteron is adiabatic in
comparison to its relevant internal motion.

e) Inelastic processes have no significant contribution to the
elastic channel.

Some of these assumptions look unrealistic at first. We will
however see in the next section, where the validity of these assumptions
is examined, that the model we propose here is physically meaningful
for certain energy ranges of the incident deuteron.

The propagation of a deuteron through a finite nucleus is
divided, in this model, into steps where the deuteron is propagating
through neighbourhoods of constant density, \( \rho \), with constant energy,
\( E \). The numerical part of the problem is therefore conveniently reduced
to a simple nuclear matter calculation. The energy of the deuteron,
in a neighbourhood located a distance \( R \) from the centre of the target,
is

\[
E = E_d - V_p(R) - V_n(R)
\]

where \( E_d \) is the incident deuteron energy and \( V_p(R) \) and \( V_n(R) \) are the
nucleon optical potentials, including the Coulomb field due to a
uniformly charged spherical nucleus for the proton. The potential
$V_n$ and the strong interaction part of $V_p$ are taken from equation 13 of reference 42 evaluated at $\frac{1}{2} E_d$\textsuperscript{43}. The spin orbit and imaginary parts of $V_n$ and $V_p$ are neglected. The potential $V_p$ and $V_n$ have the form\textsuperscript{42}

$$V_p(R) = V_c(R) - V_{op} \left[ 1 + e^{-\frac{R-C}{a}} \right]^{-1}$$

$$V_n(R) = -V_{on} \left[ 1 + e^{-\frac{R-C}{a}} \right]^{-1}$$

where \textsuperscript{42}

$$C_v = 1.17 \text{ fm} \quad , \quad a_v = 0.75 \text{ fm}$$

and the Coulomb potential $V_c$ is

$$V_c(R) = \frac{Zze}{2R_c} \left[ 3 \right. - \left. \frac{R^2}{R_c^2} \right] \quad R \leq R_c$$

$$V_c(R) = \frac{Zze}{R} \quad R \geq R_c$$

The radius $R_c$ is given by Elton\textsuperscript{44}

$$R_c = (1.123 A^{1/3} + 2.352 A^{-1/3} - 2.07 A^{-1}) \text{ fm}$$

$A$ is the mass number of the target, $Z$ is the atomic number of the target and $ze$ is the charge of the deuteron, $z = 1$.

The strengths of the proton and neutron optical potentials $V_{op}$ and $V_{on}$ are given by\textsuperscript{42}

$$V_{op} = [55.2 + 24(N - Z)/A + 0.27 Z/A^{1/3} - 0.32 \frac{E_d}{2}] \text{ MeV}$$

$$V_{on} = [55.2 - 24(N - Z)/A - 0.32 \frac{E_d}{2}] \text{ MeV}$$
Hence
\[ E(R) = E_d - V_c(R) + \frac{110.4 + 0.27Z/A^3 - 0.32E_d}{R - C_v} \text{ MeV} \]

The optical potentials \( V_n(R) \) and \( V_p(R) \) are valid for neutron and proton energies less than 50 MeV and for targets with mass number \( A > 40 \). The expression for the deuteron total energy, \( E(R) \), is expected to be accurate for \( E_d \leq 100 \) MeV. We have however used this expression for higher energies as well, because errors introduced in this way do not affect drastically the effect of the Pauli exclusion principle.

The Fermi momentum, \( k_F \), prevailing at a neighbourhood located a distance \( R \) from the centre of the target is determined from the local density, \( \rho(R) \), at \( R \) through
\[ k_F(R) = \left[ \frac{3\pi^2 \rho(R)}{2} \right]^{\frac{1}{3}} \]

We have assumed that the neutron and proton densities are the same and used the Fermi charge distribution in table 3 of reference 41 for the density distribution. We have not unfolded the electromagnetic size of the proton; errors due to this are completely negligible for the purposes of this work. The Fermi charge distribution in reference 41 is obtained from X-ray transition energies in muonic atoms; it has the form
\[ \rho(R) = \rho_0 \left[ 1 + e^{-R_C_d/a_d} \right]^{-1} \]
where
\[ \rho_0 = \frac{3}{4\pi C_d^2} \left[ 1 + \left( \frac{R_C_d}{a_d} \right)^2 + \frac{6}{k^3} \left( e^{-k} - \frac{1}{2} e^{-2k} \right) \right]^{-1} \]
The values of the parameters of the Fermi distribution \( ^4 \)1 for a number of nuclei are given in table 3.

Table 3

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( \rho_0 )</th>
<th>( C_d ) (fm)</th>
<th>( a_d ) (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{16}\text{O})</td>
<td>0.1806</td>
<td>2.4421</td>
<td>0.52339</td>
</tr>
<tr>
<td>(^{56}\text{Fe})</td>
<td>0.1652</td>
<td>4.1177</td>
<td>&quot;</td>
</tr>
<tr>
<td>(^{92}\text{Zr})</td>
<td>0.1667</td>
<td>4.9117</td>
<td>&quot;</td>
</tr>
<tr>
<td>(^{120}\text{Sn})</td>
<td>0.1614</td>
<td>5.4591</td>
<td>&quot;</td>
</tr>
<tr>
<td>(^{208}\text{Pb})</td>
<td>0.1593</td>
<td>6.6475</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

6.B. **Validity of basic assumptions**

A treatment of the Pauli exclusion principle for finite nuclei, where the density is not uniform throughout the nucleus, is very much more complicated than the corresponding nuclear matter calculation. The apparent successes of nuclear matter calculations \(^3\),\(^4\) and the short range character of nuclear forces make the local density approximation (L.D.A.) a logical simplification of the problem. The basic assumption for this approximation is that the reaction matrix for an interacting pair of nucleons located at a position \( R \) inside a finite nucleus can be replaced by the reaction matrix for nuclear matter evaluated for the local density at \( R \), or some average value.
of it in a small region surrounding $R$. The justification for this is that the density of real nuclei, especially heavy ones, does not change very much over a distance corresponding to the range of the nucleon-nucleon force. L.D.A. was proven very successful for heavy nuclei $^{45-49}$.

The foundations of assumption b look at first glance very shaky. The separation of the neutron and proton in the deuteron is for long times much larger than the nucleon-nucleon force range $^{35}$, unlike the corresponding separation of two bound interacting nucleons in a finite nucleus. The variation of density over a distance corresponding to the size of the deuteron is certainly not negligible throughout the volume of real nuclei. This variation can be considered small only in the central region of medium and heavy nuclei. At the surface region however the density variation over a distance corresponding to the size of the deuteron is large even for heavy nuclei. The large size of the deuteron however is associated with the small relative momenta prevailing in its $S$-state component. The $D$-state part of the deuteron wavefunction contains large relative momenta and it is therefore much more localised in space. The tensor force, of the $T_p$ type, which we want to investigate here arises from the action of the Pauli exclusion principle on the $D$-state part of the deuteron wavefunction. L.D.A. might therefore be adequate for a rough estimate of the strength of this force even at the surface region, at least for medium and heavy nuclei. At low and high incident deuteron energies L.D.A. can be justified further. For very low incident deuteron energies, subcoulomb region, the effect of the Pauli exclusion principle on the internal motion of the deuteron, at the surface region of the target (low density region), is the destruction of a large part of the low relative momentum states of the $S$-state, without much effect on the $D$-state.
The binding energy difference, and hence the strength of the $T_p$ tensor force, is small in this case. The destruction of the low relative momenta implies that the size of the deuteron is reduced. L.D.A. might not therefore be a bad approximation in this case.

We have seen in chapter 4 that for high incident deuteron energies, $E_d > 100$ MeV the S-state is practically unaffected by the Pauli exclusion principle. The excluded relative momenta belong to the D-state and they therefore correspond to small neutron-proton separations. Furthermore at the surface region (low density region) the binding energy difference is small. The most important property of the target affecting the strength of the $T_p$ tensor force is most likely the local density. L.D.A. must therefore be a reasonable approximation at high incident deuteron energies.

For medium incident deuteron energies however, Coulomb barrier energy $< E_d < 100$ MeV, neither of the above two arguments is valid. We do not expect L.D.A. to be a good approximation for this range of incident deuteron energies.

Assumption C was incorporated in Watanabe's original model and it was examined by Johnson and Soper, who found that possible corrections related to this assumption would not amount to more than a 4% correction to the optical potential.

We do not expect the adiabatic assumption, assumption d, to be valid for all incident deuteron energies. We now examine in a semi-quantitative fashion the range of validity for this assumption. We define an external time, $t_{ext}$, as the time taken by the centre of mass of the deuteron to travel from an area of the target where the effect of the Pauli principle is negligible to an area where the same
effect is important. A reasonable time is therefore the time taken by the deuteron to cross the nuclear surface. The surface thickness of medium size nuclei is of the order of 2-2.5 fm. An appropriate external time is therefore

\[ t_{\text{ext}} = \frac{2}{V_d} = \frac{4}{K_0} \]  

where \( K_0 \) is some average value of the energy wavenumber \( K_0 \) over the target surface thickness.

Assumption d asserts that the external time, \( t_{\text{ext}} \), is long compared to the internal time, \( t_{\text{int}} \), characterising the relevant internal motion of the deuteron. The relevant internal motion of the deuteron is the internal motion of the deuteron affected strongly by the Pauli exclusion principle. We define the ratio of \( t_{\text{int}} \) to \( t_{\text{ext}} \) as

\[ \zeta = \frac{t_{\text{int}}}{t_{\text{ext}}} = \frac{K_0}{4} t_{\text{int}} \]  

When \( \zeta \) is small the deuteron is able to modify its internal motion according to the requirements of the Pauli exclusion principle as it travels through different regions of the target; in this case assumption d is expected to be valid.

Equation 6.13 might give the impression that the adiabatic approximation used here is a low energy approximation because \( t_{\text{ext}} \) becomes large as the incident deuteron energy decreases. This is not the case however because, as we will shortly see, the internal time, characterising the relevant internal motion of the deuteron, depends strongly on the incident deuteron energy.
As the deuteron crosses into the nuclear interior some of the relative momenta, \( k \), in its internal wavefunction are forbidden by the Pauli exclusion principle. These relative momenta are associated with an energy \( \varepsilon_k \)

\[
\varepsilon_k = k^2 \tag{6.15}
\]

The internal times, \( t_{\text{int}} \), characteristic of the rate of change of these momenta, is related to the spread of the energies \( \varepsilon_k \), through the time-energy uncertainty relation

\[
t_{\text{int}} \times \Delta \varepsilon_k > 1 \tag{6.16}
\]

Taking the value \( \varepsilon_k \) for \( \Delta \varepsilon_k \) we have

\[
t_{\text{int}} \approx \frac{1}{k^2} \tag{6.17}
\]

The longer internal times are therefore associated with the shorter excluded relative momenta, \( k \),

\[
t_{\text{int}}^g = \frac{1}{k_s^2} \tag{6.18}
\]

Hence the maximum value of \( \zeta \) is of order

\[
\zeta_{\text{max}} = \frac{k_0}{4k_s^2} \tag{6.19}
\]

The shorter internal times, \( t_{\text{int}}^g \), are associated with the longest excluded relative momenta \( k_G \)

\[
t_{\text{int}}^g = \frac{1}{k_G^2} \tag{6.20}
\]

The minimum value for \( \zeta \) is therefore

\[
\zeta_{\text{min}} = \frac{k_0}{4k_G^2}
\]
For very low incident deuteron energies i.e. energies of the order of the Coulomb barrier energy, the deuteron local energy wave-number, $K_0$, is less than the local Fermi momentum $k_F$ at the nuclear interior. Since the shortest excluded relative momenta are practically zero the adiabatic assumption, $d$, is not valid at these energies. For these energies

$$\zeta_{\text{max}} \gg 1 \quad \text{6.22}$$

It is interesting to see whether or not the opposite adiabatic assumption, $d'$, is applicable in this case ($d'$). The deuteron internal motion is adiabatic in comparison to the centre of mass motion.

For this to be true $\zeta$ must be large. The longer excluded relative momenta, $k_{\xi}$, are of order

$$k_{\xi} = \frac{\hat{K}_0}{2} + k_F \quad \text{6.23}$$

$$\therefore \quad \zeta_{\text{min}} = \frac{\hat{K}_0}{(\hat{K}_0 + 2k_F)^2} \quad \text{6.24}$$

Since $k_F \approx 1.3 \text{ fm}^{-1}$ and $k_F$ is larger than $\frac{\hat{K}_0}{2}$

$$\zeta_{\text{min}} \ll 1 \quad \text{6.25}$$

For very low incident deuteron energies therefore no adiabatic assumption is valid because the Pauli exclusion principle affects strongly a large range of relative momenta.

The adiabatic assumption $d'$ is also invalid at higher energies, because the minimum value of $\zeta$ is again given by 6.24. In fact $d'$ becomes more difficult to satisfy as the energy increases.
Assumption d however, which is the one used in our work, might be valid at high energies. When the incident deuteron energy is high enough for \( K_0 \) to be always greater than \( k_F \), the shorter excluded momenta are of order

\[
k_s = \frac{K_0}{2} - k_F
\]

Hence the maximum value of \( \xi \) is

\[
\xi_{\text{max}} = \frac{K_0}{(K_0 - 2k_F)^2}
\]

We expect the adiabatic approximation used here to be applicable if the external time, \( t_{\text{ext}} \), is at least comparable to the longer internal time, \( t_{\text{int}} \). Roughly, we expect the adiabatic approximation, \( d \), to give a rough estimate of the shape and strength of the \( T_p \) type of tensor force when \( \xi_{\text{max}} \approx 1.5 \). When \( \xi_{\text{max}} < 1 \) we expect the results of this approximation to be accurate. For a given \( \xi_{\text{max}} \), \( K_0 \) is given by

\[
\xi_{\text{max}} \frac{\hat{K}}{K_0} = -(4\xi_{\text{max}} k_F + 1)\hat{K}_0 + 4\xi_{\text{max}} k_F^2 = 0
\]

\[
\hat{K}_0 = \frac{4\xi_{\text{max}} k_F + 1 + \sqrt{8\xi_{\text{max}} k_F^2 + 1}}{2\xi_{\text{max}}}
\]

We list in the following table 4, the minimum values of \( \hat{K}_0 \) and the corresponding values for the incident deuteron energy, \( E_d \), for a range of \( \xi_{\text{max}} \). We also give for comparison purposes the corresponding value of \( \xi_{\text{min}} \). We have taken the value of 1.3 fm\(^{-1}\) for \( k_F \).
Table 4

<table>
<thead>
<tr>
<th>$\xi_{\text{max}}$</th>
<th>$\hat{K}_0(\text{fm}^{-1})$</th>
<th>$E_d(\text{MeV})$</th>
<th>$\xi_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>7.65</td>
<td>750</td>
<td>0.072</td>
</tr>
<tr>
<td>0.6</td>
<td>5.67</td>
<td>345</td>
<td>0.083</td>
</tr>
<tr>
<td>0.8</td>
<td>5.13</td>
<td>255</td>
<td>0.085</td>
</tr>
<tr>
<td>1</td>
<td>4.788</td>
<td>205</td>
<td>0.0877</td>
</tr>
<tr>
<td>1.2</td>
<td>4.546</td>
<td>170</td>
<td>0.089</td>
</tr>
<tr>
<td>1.5</td>
<td>4.29</td>
<td>140</td>
<td>0.09</td>
</tr>
<tr>
<td>2</td>
<td>4.02</td>
<td>115</td>
<td>0.091</td>
</tr>
</tbody>
</table>

Assumption d is therefore expected to be valid for the purposes of this work for incident deuteron energies greater than $E_d = 140$ MeV.

The deuteron is a loosely bound particle and it can therefore be broken up quite easily. The validity of assumption e, which asserts that only the ground state of the deuteron should be included in the calculation, is doubtful. A formalism, developed by W.S. Pong, based on the perturbed stationary state method is presented in Appendix 1. In this formalism the non-local nature of the Pauli exclusion principle operator, $\hat{\theta}$, in $\mathbb{R}$, the separation of the centre of masses of the deuteron and target, gives rise to additional terms, $C_{mn}$, in the deuteron optical potential. These terms have the form

$$C_{mn}(R) = 2 \left[ \int dR_2 \phi_m^*(R_1R) V_R \phi_n(R_1R) \right] \cdot V_R + \int dR_2 \phi_m^*(R_1R) V_R \phi_n(R_1R)$$

(6.30)
where \( \phi_m(R, r) \) represents the eigenstate of the n-p system, and it satisfies a Schrodinger equation
\[
\{ t_R + Q(R)V_{np}(r) \} \phi_m(R, r) = E_m \phi_m(R, r)
\]

\[6.31\]

The diagonal terms, \( C_{00} \), gives the reaction of the deuteron ground state to a change in \( R \). The off diagonal terms \( C_{m0} \) give the coupling of the excited states \( m \) with the ground state. Assumption e is valid if the amplitudes of all excited states are small compared to the ground state amplitude. It is shown in the appendix that the quantity, \( \delta \),
\[
\delta = \left| \frac{C_{00}(R)}{B(R)} \right| 
\]

\[6.32\]
gives an upper estimate of the sum of all amplitudes of the excited states, where \( B(R) \) is the binding energy of the deuteron at \( R \).

Assumption e is therefore valid if
\[
\delta \ll 1
\]

\[6.33\]

Taking only the first term in 6.30 and replacing the last \( V_R \) by \( K_0(R) \) we have
\[
C_{00}(R) = \lim_{\Delta R \to 0} \frac{2}{\Delta R} \int dR \phi^*_{00}(R, r) [\phi(R + \Delta R, r) - \phi(R, r)] \cdot K_0(R)
\]

\[6.34\]

and transforming into the momentum representation and labelling the wavefunction at \( R \) by \( 1\phi \) and the wavefunction at \( R + \Delta R \) by \( 2\phi \) gives
\[
C_{00}(R) = \lim_{\Delta R \to 0} \frac{2}{\Delta R} K_0(R) \int dk \frac{d\phi^*_0(k)}{dk} [2\phi(k) - 1\phi(k)]
\]

\[6.35\]
At every point $R$ inside the target both the binding energy of the deuteron and the quantity $C_{00}(R)$ will depend on the relative orientation of the spin of the deuteron and the centre of mass momentum. The two extreme values for $\delta$ are the value corresponding to the para-state, $\delta_\perp$, and the value corresponding to the orthostate, $\delta_0$.

\[ \delta_m = \frac{2 K_0(R)}{A R B_m(R)} I_m \]  

where

\[ I_m = \int dk \frac{1}{k} \left[ \phi_m^*(k) \left( 2 \phi_m(k) - 1 \phi(k) \right) \right] = I_{1m} - I_{2m} \]  

\[ I_{1m} = \int dk \phi_m^*(k) 2 \phi_m(k) \]  

\[ I_{2m} = \int dk \phi_m^*(k) \phi_m(k) \]  

We can use for the deuteron wavefunction the normalised wavefunction of chapter 3 section C.

\[ n_{\phi_m} = \phi(k) X_1^m = n_0 N \frac{g(k)}{\Delta^2 + k^2} X_1^m \]  

The Pauli operators at $R$, $1\hat{\Theta}$, and at $(R + \Delta R)$, $2\hat{\Theta}$, are different in general; they depend on the local momentum, $K_0(R)$ and at the local Fermi momentum $k_F(R)$. With this choice for $\phi_m(k)$, $6.39$ becomes

\[ I_{2m} = \int dk \phi_m^*(k) \phi_m(k) = 1 \]  

The wavefunction $\phi_m(k)$ can also be expanded in series of spherical harmonics, as in chapter 3, section 4.
\[ \phi_m(k) = \sum_{\ell, \lambda} \lambda \mu \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda 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\lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \la
The integral $I_{1m}$ becomes with these definitions

$$I_{1m} = \int dk \phi^*_m(k) \phi_m(k)$$

$$= \int k^2dk \int d\vec{k} \sum_{l',\lambda} \gamma_{l',\lambda}(\vec{k})\gamma^*_{l',\lambda}(\vec{k}) \lambda_1^{\mu_1} \lambda_2^{\mu_2} \xi_1^{\mu_1} \xi_1^{\mu_2} 6.54$$

$$I_{1m} = \sum_{l',\lambda} k^2dk \mathcal{D}_{l'\lambda}^{\lambda m} \quad 6.55$$

where

$$\mathcal{D}_{l'\lambda}^{\lambda m} = 1_{l'\lambda}^{\lambda m} \lambda_2^{\mu_2}$$

The integrals $I_{1m}$ was evaluated numerically using Simpson's rule for numerical integration. The results are shown on Fig. 22 and 23. The ratio $\delta_1$ is always much smaller than $\delta_0$. This is to be expected, because the binding energy of the para-state is always larger than the binding energy of the ortho-state. The ortho-state is therefore expected to lead to break up much more easily than the para-state since the former is always much nearer to the break-up states of the deuteron, energy wise, than the latter. A reasonable limit for the validity of assumption e is set by requiring that $\delta_0(R)$ is always less than 1. Adopting this criterion we can consider assumption e valid for incident deuteron energies, $E_d$, greater than 150 MeV.

The discussion presented in this section suggests that the model used here for the propagation of a deuteron through a finite nucleus is valid for high incident deuteron energies, $E_d \geq 150$ MeV. If we take an optimistic view we can hope that the model is useful for even lower energies, in the region of 100 MeV. For $E_d < 100$ MeV we can
however not trust the model at all. This is unfortunate because deuteron-nucleus elastic scattering experiments with polarized deuterons are performed at much lower incident deuteron energies. Our results can not therefore be compared with experiment, at this stage.

The deuteron and proton optical potentials we use in this model are accurate for incident deuteron energies not greater than 100 MeV. However this is a superficial problem because inaccuracies introduced by the use of these optical potentials at higher energies are not expected to be large. Any conclusions we might draw about the strength and range of the $T_p$ type of tensor force should not be affected fundamentally by the errors in the optical potentials.

### 6.C The binding energy of the deuteron in a finite nucleus

The results of the calculation are presented in figures 24 to 29. Figs. 24 to 26 depict the variation of the binding energy of the deuteron, as a ratio to the free space binding energy $\frac{BE}{BE_{fs}}$, and as a function of $R$, for D-state probabilities, $D_f$, 2%, 4%, 6% and 8%; and incident deuteron energies, $E_d$, 50 MeV, 150 MeV and 250 MeV. Figures 27 to 29 depict the binding energy difference, $BE_\pm - BE_0$, as a function of $R$, for the same values of $D_f$ and $E_d$. The target nucleus is $^{92}$Zr. These results are typical for all targets; similar graphs were obtained for other nuclei exhibiting the same features as the ones presented here.

For incident deuteron energies of the order of 50 MeV the binding energy of the ortho-state reduces rapidly to zero, just inside the nuclear surface. The binding energy of the para-state is also reduced as the deuteron penetrates the nuclear surface but remains finite and constant at the nuclear interior. The binding energy
difference, $\Delta \varepsilon$, is calculated when both the para-state and ortho-state configurations form a bound state. At these energies $\Delta \varepsilon$ exhibits a maximum at the nuclear surface, just before $BE_0$ is reduced to zero. $\Delta \varepsilon$ goes to zero very fast as the deuteron is moving outwards from the nuclear surface.

At higher incident deuteron energies, $E_d > 150$ MeV, fig. 28 and 29 both the para-state and ortho-state configurations are bound throughout the target nucleus. Their binding energies are reduced to constant values in the nuclear interior as the deuteron crosses the nuclear surface; the reduction for $BE_0$ is much bigger than the corresponding reduction for $BE_\pm$. $\Delta \varepsilon$ is almost constant at the nuclear interior and falls rapidly to zero just outside the nuclear surface. The shape of $\Delta E(R)$ is very similar to the shape of the density for the target nucleus.

The maximum value of the binding energy difference as a function of energy increases as the energy increases up to about 180 MeV and decreases for higher energies. The magnitude of $\Delta \varepsilon$, and hence the strength of the $T_p$ potential, increases significantly as the D-state probability increases. The shape of these curves however changes very little with variations of the D-state.

The shape of $BE_\pm$, $BE_0$ and $\Delta \varepsilon$ follows directly from the model we use here for the propagation of a deuteron through a finite nucleus and the nuclear matter calculation of chapter 3. The physical reasons for these shapes can be understood from the picture for the Pauli mechanism presented in chapter 4.
Soon after we published the results of the last few sections of this chapter, Austern proposed an alternative derivation for the spin dependent effects discussed earlier on. His derivation is based on the orthogonalised deuteron-nucleus scattering analysis of Pong and Austern. Austern's approach uses the adiabatic assumption discussed in section B of this chapter. We have seen that this assumption is wrong for all energy ranges of the incident deuteron. However it is logically possible that assumption is less bad than the adiabatic assumption we use for our binding energy approach, at lower energies. We restate these two assumptions for the convenience of the reader.

\( d \) The centre of mass motion of the deuteron is adiabatic in comparison to its relevant internal motion.

\( d' \) The relevant internal motion of the deuteron is adiabatic in comparison to its centre of mass motion.

The relevant internal motion of the deuteron is the internal motion of the deuteron affected by the Pauli exclusion principle.

The correction term to the optical potential resulting from Austern's approach, \( V_A \), is of the form

\[
V_A = \langle \phi_d | \hat{P} v_{np} | \phi_d \rangle
\]

where \( \phi_d \) is the deuteron internal wave-function, \( v_{np} \) the neutron proton interaction and the projection operator, \( \hat{P} \), projects into the occupied states of the target.
\[ \hat{P} + \hat{\theta} = 1 \]  \hspace{1cm} 6.57

The adiabatic assumption \( d' \) was also used by Johnson \(^5\) in conjunction with a variational \(^5\) approach. We outline this approach now.

After separation of plane-wave centre of mass motion in nuclear matter (or in the model we use in this chapter for the propagation of a deuteron through a finite nucleus with assumption \( d \) replaced by assumption \( d' \)) the equation for the relative motion is

\[ [\hat{T} + \hat{\theta} V_{np}] |\psi\rangle = \varepsilon |\psi\rangle \]  \hspace{1cm} 6.58

where \( \hat{T} \) is the relative kinetic energy operator.

From 6.57 we get, using the fact that \( \hat{P} \) and \( \hat{\theta} \) commute with \( \hat{T} \),

\[ \hat{T} (|\psi\rangle) = \varepsilon (|\psi\rangle) \]  \hspace{1cm} 6.59

if \( \hat{T} \) has no bound state

\[ \hat{P} |\psi\rangle = 0 \quad \text{or} \quad \hat{\theta} |\psi\rangle = |\psi\rangle \]  \hspace{1cm} 6.60

In this case 6.57 becomes

\[ [\hat{T} + \hat{\theta} V_{np} |\psi\rangle = \varepsilon |\psi\rangle \]  \hspace{1cm} 6.61

In order to get an expression for the energy, \( \varepsilon \), we can use a variational approach. The average value of the energy, \( E[\psi] \), of a quantum system in a state described by the state vector \( |\psi\rangle \) is given by \(^5\)

\[ E[\psi] = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \]  \hspace{1cm} 6.62
where $H$ is the Hamiltonian of the system. If the state vector $|\phi> \text{ is not known we can still use equation 6.62 to obtain an estimate for the energy.}$ In that case we replace $|\phi>$ by a trial wavefunction $|\psi>$ in 6.62. In a variational approach we seek for a trial wavefunction $|\psi>$ by varying the parameters contained in $|\psi>$ until expression 6.62 is stationary. For this approach to be useful we need a simple trial wavefunction in order to get a simple expression for 6.62. In the problem in hand we can not obtain simple expressions for 6.62 for the Hamiltonian in 6.58. We therefore, instead of varying the parameters in the trial wavefunction, $|\psi>$, we seek a trial wavefunction as close to the exact wavefunction as possible. When the deuteron propagates through free space the exact wavefunction is known; it is given by equation 6.40 with $\hat{\theta} = 1$. We take the part of this wavefunction that lies in the allowed space and normalized to one as our final wavefunction

$$|\psi> = \frac{\hat{\theta}|\phi_m>}{<\phi_m|\phi_m>^{\frac{1}{2}}} \quad 6.63$$

where $m$ refers to the projection of the deuteron spin along the $z$-axis, chosen such that $\hat{z} = \hat{K}_0$. The wavefunction, $|\phi>$, satisfies the free-space Schrödinger equation

$$[\hat{T} + V_{np}]|\phi_m> = \epsilon_0 |\phi_m> \quad 6.64$$

We can then use equation 6.62 with $|\phi>$ replaced by $|\psi>$ to give us an estimate for the energy $\epsilon$. 

-112 -
\[
\varepsilon = \frac{\langle \psi | T + \Theta V_{\text{np}} | \psi \rangle}{\langle \psi | \psi \rangle} + O(|\phi\rangle - |\psi\rangle)^2
\]

\[
= \frac{\langle \phi_m | \hat{T} + \hat{\Theta} V_{\text{np}} | \phi_m \rangle}{\langle \phi_m | \hat{\Theta} | \phi_m \rangle} + O(|\phi\rangle - |\psi\rangle)^2
\]

\[
= \frac{\langle \phi_m | \hat{\Theta} V_{\text{np}} - \hat{\Theta} V_{\text{np}} + \hat{\Theta} V_{\text{np}} | \phi_m \rangle}{\langle \phi_m | \hat{\Theta} | \phi_m \rangle} + O(\delta\phi)^2
\]

\[
\therefore \quad \varepsilon = \varepsilon_0 + \frac{\langle \phi_m | \hat{\Theta} V_{\text{np}} - \hat{\Theta} V_{\text{np}} | \phi_m \rangle}{\langle \phi_m | \hat{\Theta} | \phi_m \rangle} + O(\delta\phi)^2
\]

\[
6.65
\]

\[
\varepsilon = \varepsilon_0 - V_J + O(\delta\phi)^2
\]

This method will be accurate if the neglected term of order \((\delta\phi)^2 = (|\phi\rangle - |\psi\rangle)^2\) is small. The energy shift \(V_J\) is the correction to the optical potential. Rearranging this correction term and comparing with Austern's correction term 6.56 and assuming a normalized wavefunction \(|\phi_m\rangle\) we get

\[
V_J = \frac{\langle \phi_m | \hat{\Theta} V_{\text{np}} - \hat{\Theta} V_{\text{np}} | \phi_m \rangle}{1 - \langle \phi_m | \hat{P} | \phi_m \rangle}
\]

\[
= \frac{\langle \phi_m | (1 - \hat{\Theta}) V_{\text{np}} - (1 - \hat{\Theta}) V_{\text{np}} (1 - \hat{\Theta}) | \phi_m \rangle}{1 - \langle \phi_m | \hat{\Theta} | \phi_m \rangle}
\]

\[
= \frac{\langle \phi_m | \hat{\Theta} V_{\text{np}} | \phi_m \rangle - \langle \phi_m | \hat{\Theta} V_{\text{np}} \hat{\Theta} | \phi_m \rangle}{1 - \langle \phi_m | \hat{\Theta} | \phi_m \rangle}
\]

\[
6.66
\]

Hence

\[
V_J = \frac{V_A - V_C}{1 - N_J}
\]

6.67
The variational approach agrees with Austern's approach when the correction term $V_C$ and the normalization correction term $N_J$ are negligibly small.

$$V_C = \langle \phi_m | \hat{P}_n p | \phi_m \rangle$$  \hspace{1cm} 6.68

$$N_J = \langle \phi_m | \hat{P} | \phi_m \rangle$$  \hspace{1cm} 6.69

Using the Yamaguchi potential for $V_{np}$ we have

$$V_{np} = -\frac{\lambda}{M} |g\rangle <g| , \quad \pi = M = 1$$  \hspace{1cm} 6.70

$$|\phi_m\rangle = \frac{N}{\Delta^2 + k^2} |g\rangle |x_1^m\rangle$$  \hspace{1cm} 6.71

The overall strength of the interaction, $\lambda$, is given by

$$\frac{N}{\lambda} |x_1^m\rangle = <g| \phi_m\rangle$$  \hspace{1cm} 6.72

Hence

$$V_A = -\frac{\lambda}{\pi} \langle \phi_m | \hat{P} | g\rangle <g| \phi_m \rangle$$

$$= -\lambda \langle \phi_m | p | g\rangle \frac{|x_1^m\rangle}{\lambda}$$

$$= -N^2 <x_1^m| <g| \frac{\hat{p}}{\Delta^2 + k^2} |g\rangle |x_1^m\rangle$$  \hspace{1cm} 6.73

$$V_C = -\lambda \langle \phi_m | \hat{P} | g\rangle <g| \phi_m \rangle$$

$$= -\lambda N^2<x_1^m|<g| \frac{\hat{p}}{\Delta^2 + k^2} |g\rangle <g| \frac{\hat{p}}{\Delta^2 + k^2} |g\rangle |x_1^m\rangle$$  \hspace{1cm} 6.74

Similarly

$$N_J = N^2<x_1^m|<g| \frac{\hat{p}}{\Delta^2 + k^2} |g\rangle |x_1^m\rangle$$  \hspace{1cm} 6.75
The potentials $V_J$ and $V_A$ and the correction terms $V_C$ and $N_J$ depend on the relative orientation of the total spin of the deuteron and the direction of the centre of mass motion of the deuteron. The strength of the $T_P$ type of tensor force, $V_p$, resulting from each of these approaches is as follows:

Variational approach:

$$V_P^J = \frac{1}{6} \left[ [V_J]_{m=0} - [V_J]_{m=1} \right]$$  

Austern's approach

$$V_P^A = \frac{1}{6} \left[ [V_A]_{m=0} - [V_A]_{m=1} \right]$$

The form of the correction terms, $V_C$ and $N_J$ suggests that $V_P^J$ and $V_P^A$ should agree when the excluded space is small, that is when

$$\hat{P} |\phi> \propto 0$$

In the opposite limit however

$$\hat{P} |\phi> \propto |\uparrow>$$

the two methods should differ considerably. We expect $V_P^J$ to be more accurate than $V_P^A$ because the Pauli exclusion principle is treated more carefully. To see this more clearly we try the free space wavefunction $|\phi_m>$ as our trial wavefunction in 6.62. Then we get the following estimate for $\epsilon$
The energy shift in this case is identical to the energy shift derived by Austern's approach. Equation 6.62 is expected to give an accurate estimate for the energy shift if the trial wavefunction resembles the exact wavefunction. The wavefunction in 6.63 must bear a closer resemblance to the exact wavefunction than the free space wavefunction because the Pauli exclusion principle is satisfied for the former and it is not satisfied for the latter. We therefore expect $V_p^J$ to be a better estimate of the strength of the $T_p$ tensor force than $V_A^J$.

The three estimates for the strength of the $T_p$ tensor force, $V_p^A$, $V_p^J$ and $V_p^B$, derived from our binding energy approach, are plotted in figures 30 to 33, as a function of $R$ for incident deuteron energies 10 MeV, 50 MeV, 150 MeV and 250 MeV. The target nucleus is $^{92}$Zr and the D-state of the deuteron is 4%. $V_p^A$ is always greater than $V_p^J$ which in turn is always greater than $V_p^B$. The shape of these curves at high incident deuteron energies is of a Wood-Saxon type which roughly follows the density distribution of the target nucleus. At lower energies, $E_d < 10$ MeV for $V_p^A$, $E_d < 20$ MeV for $V_p^J$ and $E_d < 130$ MeV for $V_p^B$ the corresponding curves show an additional small surface peak. In both the extreme surface
region \((R > 6 \text{ fm})\) and for incident deuteron energies greater than 250 MeV all three estimates for \(V_p\) are very similar in both magnitude and shape.

The maximum value for \(V_p^A\) occurs for incident deuteron energy around 10 MeV; for \(V_p^J\) around \(E_d = 55\) MeV and for \(V_p^B\) around \(E_d = 150\) MeV.

As we mentioned earlier, deuteron nucleus elastic scattering experiments with polarized deuterons, where the effect of the tensor term we propose here might show up, are performed so far for low incident deuteron energies, \(E_d < 30\) MeV. In the absence of any other theoretical estimate for the shape and strength of the \(T_p\) type of tensor force, we suggest here some plausible limits for the strength and shape of this force. These comments arise from the results of this section. We stress again that because neither \(V_p^A\) nor \(V_p^J\) are expected to be accurate estimates, nor is \(V_p^B\) expected to be a good estimate at low energies, the following comments are nothing more than plausible guesses.

The strength of the \(T_p\) type of tensor force, at low incident deuteron energies, \(E_d < 30\) MeV should lie within the limits 0.1 to 0.7 MeV. The shape of this tensor potential should predominantly be of a Wood-Saxon type with possibly an additional surface peak.
Figure 22. The quantity $\delta$, which is a measure of the amplitude of the excited states of the deuteron generated from the ground state, as a function of $R$ for the ortho-state.
Figure 23. The quantity $\delta$, which is a measure of the amplitude of the excited states of the deuteron generated from the ground state, as a function of $R$ for the para-state.
Figure 24. The ratios of the binding energy of the deuteron in a finite nucleus to the free-space binding energy, for the ortho-state and para-state, for various D-state probabilities at $E_d = 50$ MeV.
Figure 25. The ratios of the binding energy of the deuteron in a finite nucleus to the free-space binding energy, for the ortho-state and para-state, for various D-state probabilities at $E_d = 150$ MeV.
Figure 26. The ratios of the binding energy of the deuteron in a finite nucleus to the free-space binding energy, for the ortho-state and para-state, for various D-state probabilities at $E_d = 250$ MeV.
Figure 27. Binding energy difference between para-state and ortho-state configuration in a finite nucleus for $E_d = 50$ MeV.
Figure 28. Binding energy difference between parastate and ortho-state configuration in a finite nucleus for $E_d = 150$ MeV.
Figure 29. Binding energy difference between para-state and ortho-state configuration in a finite nucleus for $E_d = 250$ MeV.
Figure 30. The strength of the tensor interaction of the $T_p$ type in the deuteron optical potential as a function of the deuteron nucleus separation, as predicted by Johnson, adiabatic approach, Austern approach and our binding energy approach, for $E_d = 10$ MeV.
Figure 31. The strength of the tensor interaction of the $T_p$ type in the deuteron optical potential as a function of the deuteron nucleus separation, as predicted by Johnson adiabatic approach, Austern approach and our binding energy approach, for $E_d = 50$ MeV.
Figure 32. The strength of the tensor interaction of the $T_0$ type in the deuteron optical potential as a function of the deuteron nucleus separation, as predicted by Johnson adiabatic approach, Austern approach and our binding energy approach, for $E_d = 150$ MeV. 

$E_d = 150$ MeV

D-state 4%

Target $^{92}$Zr

- $V^A_{\rho}$
- $V^B_{\rho}$
- $V^A_{\rho}$
- $V^B_{\rho}$

$R$(fm)
Figure 33. The strength of the tensor interaction of the $T_p$ type in the deuteron optical potential as a function of the deuteron nucleus separation, as predicted by Johnson adiabatic approach, Austern approach and our binding energy approach, for $E_d = 250$ MeV.
CHAPTER 7

THE TENSOR ANALYZING POWERS FOR DEUTERON-NUCLEUS ELASTIC SCATTERING

IN PLANE WAVE BORN APPROXIMATION (P.W.B.A.).

We have seen in earlier chapters that, when a deuteron is propagating through a nuclear medium, the restrictions imposed by the Pauli exclusion principle on the internal motion of the deuteron give rise to a tensor potential of the $T_p$ type in the deuteron optical potential. The realistic case of the propagation of a deuteron through a finite nucleus was examined in the previous chapter. The strength and shape of the $T_p$ potential in this case can be calculated accurately by our binding energy approach for high incident deuteron energies. There are however no experimental data available at these energies. The usefulness of the calculations of this chapter is therefore limited to a comparison of polarization quantities produced when both $T_p$ and $T_R$ type of tensor potentials are present in the deuteron nucleus optical potential with the corresponding quantities when $T_p$ is not present. The polarization quantities of interest to us here are the tensor analyzing powers for deuteron nucleus elastic scattering. These quantities were calculated in the P.W.B.A. We chose P.W.B.A. because this approximation is valid for high incident energies and it leads to simple expressions for the polarization quantities of interest to us here.

7.A. The Born Approximation

We consider the elastic scattering of a deuteron from a spinless target. For present purposes the deuteron nucleus optical potential contains a central term $V_0(R)$, and tensor terms of the $T_p$ and $T_R$ types.
We have neglected the spin-orbit interaction because this interaction is of rank one and it is not therefore expected to alter significantly the comparison of polarization quantities (of rank 2) when both \( T_p \) and \( T_R \) tensor terms are present with the corresponding quantities when only the \( T_R \) term is present.

We denote the total potential by \( V \)

\[
V = V_0(R) + R_2(R)S_{12}(\hat{S},\hat{R}) + \frac{1}{2}[P_2(R)S_{12}(\hat{S},\hat{R}) + S_{12}(\hat{S},\hat{R})P_2(R)]
\]

where

\[
S_{12}(\hat{S},\hat{R}) = A \sum_q (-)^{2q} T_{2q}(\hat{S})Y_{2q}(-\hat{R})
\]

and

\[
A = \sqrt{\frac{32\pi}{5}}
\]

Some of the quantities in this chapter are defined differently than in previous chapters. In this chapter we follow the notation prescribed in the Madison Convention. Polarization quantities are defined as follows.

The irreducible tensor operators, \( \tau_{kq} \), constructed out of the spin 1 operators are normalized so that

\[
\text{Tr}(\tau_{kq}^\dagger \tau_{k'q'}) = 3 \delta_{kk'} \delta_{qq'}
\]

The tensor analyzing powers, or efficiency tensors describing the effect of initial polarization of a beam or target on the differential cross-section are denoted by \( T_{kq} \) and they are referred to a right handed co-ordinate system which the positive z-axis is along the beam direction of the incident particles, \( k \), and the y-axis is along \( k \times k' \) for the reaction in question, where \( k' \) is the direction of the outgoing beam, fig. 34.
The scattering amplitude, \( f_{m'm_s}^{m_m}(\theta,\phi) \), relating the initial beam of polarized deuterons, with spin projection \( m_s \), to the outgoing deuteron beam, with spin projection \( m'_s \), is given by

\[
f_{m'm_s}^{m_m}(\theta,\phi) = -\frac{2m}{4\pi\hbar^2} <k'_{m'm_s}^m|V|k_{m_s}^{m'}> 7.4
\]

In Born approximation the scattering ket \( |\psi^{(+)}_{k_{m_s}^{m'}}> \) becomes

\[
|\psi^{(+)}_{k_{m_s}^{m'}}> = |k_{m_s}^{m'}>
\]

Hence the scattering amplitude in Born approximation,

\[
f_{m'm_s}^{m_m}(\theta,\phi) = -\frac{2m}{4\pi\hbar^2} \left[ \delta_{m_s}^{m_m} \delta^F_{m'm_s} A^{2m} + A^{2m} \right]
\]

where

\[
\delta^F_{m_s}^{m'_m} = <k'_{m'_m}^m|V_0(R)|k_{m_s}^m> = <k_{m'_m}^m|k_{m_s}^m>|v_0(Q)
\]

with

\[
<v_0(Q)> = \int dR e^{i\mathbf{Q}\cdot\mathbf{R}} V_0(R)
\]

Also

\[
2m^{m_m}_{m'm_s} = \sum_q (-)^q <k_{m'_m}^m|T_{2q}^m|k_{m_s}^m>V_{2q}(Q)
\]
where

\[ W_{2q} = \int dR \bar{Y}_{2q}(\theta) e^{iQ \cdot R} W_{2q}(R) \]  \hspace{1cm} (7.13)

and using Wigner-Eckart's theorem for \( <S_{m'q}|T_{2q}|S_{m}> \) we get

\[ <S_{m'q}|T_{2q}|S_{m}> = (-)^{k-S-m} \sqrt{2S+1} <S_{m'-m}|k_q> \]

\[ = \sqrt{2S+1} \langle (-)^{s-m} <S_{m'-m}|k_q> \]  \hspace{1cm} (7.14)

\[ 2m_{m'} = \sum_{q} (-)^{q-S-m} \sqrt{2S+1} <S_{m'-m}|k_q> W_{2q} \]

\[ = \sum_{q} (-)^{q+S-m} \sqrt{2S+1} <S_{m'-m}|k_q> W_{2q} \]  \hspace{1cm} (7.15)

The tensor analyzing powers, \( T_{2q} \), are given, in relation to the scattering amplitude by

\[ T_{kq} = \frac{\text{Tr}(f_{T_{kq} f}^+)}{\text{Tr}(f f^+)} \]  \hspace{1cm} (7.16)

We have dropped the superscript Born, but all quantities calculated in this chapter are in P.W.B.A.

We need to evaluate the traces of \( (f f^+) \) and \( (f f_{T_{kq} f}^+) \).

\[ f f^+ = \sum_{m} f_{m'm} f_{m'm}^+ = \sum_{m} f_{m'm} f_{m'm}^+ \]  \hspace{1cm} (7.17)

\[ T_{f}(f f^+) = \sum_{m} f_{m'm} f_{m'm}^+ \]

\[ = \frac{m^2}{4\pi^2 \hbar^4} \sum_{m} \left[ \delta_{m'm} \cdot \delta_{m'm} + A_{m'm}^2 \right] \]  \hspace{1cm} (7.18)
\[ \text{Tr}(\mathbf{f f}^\dagger) = \frac{m^2}{4\pi^2\hbar^4} (\phi_1 + \phi_2 + \phi_3 + \phi_4) \quad 7.19 \]

\[ \phi_1 = \sum_{m_s} \delta_{m_s} \phi_1 \quad 7.20 \]

\[ \phi_2 = \sum_{m_s} \delta_{m_s} \phi_2 \quad 7.21 \]

Similarly

\[ \phi_3 = 0 \quad 7.22 \]

\[ \phi_4 = \lambda^2 \sum_{m_s} \phi_4 \quad 7.23 \]

\[ = \lambda^2 (2S+1) \sum_{m_s} \phi_4 \quad 7.24 \]

but

\[ \sum_{m_s} \phi_4 = \lambda^2 \sum_{m_s} \phi_4 \quad 7.25 \]
hence
\[
\text{Tr}(f f^+) = \frac{3m^2}{4\pi^2\hbar^2} \left[ v_{00}^Q v_{0}^Q + A^2 \sum_q \mathbf{M}_q v_{2-q}^Q \mathbf{M}_q^* v_{2-q}^Q \right] 7.25
\]

We examine now the term \( f \tau_{kq} f^+ \)

\[
f \tau_{kq} f^+ = \sum_{m_s, m'_s} f_{m', m_s}^{m''} \langle S m_s | \tau_{kq} | S m''_s \rangle f^+_{m''_s}\]

\[
= \sum_{m_s, m'_s} f_{m', m_s}^{m''} \langle S m_s | \tau_{kq} | S m''_s \rangle f^+_{m''_s} 7.26
\]

\[
\text{Tr}(f \tau_{kq} f^+) = \sum_{m_s, m'_s} f_{m', m_s}^{m''} \langle S m_s | \tau_{kq} | S m''_s \rangle f^+_{m''_s} 7.27
\]

with

\[
F_1 = \sum_{m_s, m'_s} \delta_{m', m_s}^{m''} \langle S m_s | \tau_{kq} | S m''_s \rangle \delta_{m'', m_s}^{m'''} f^+_{m''_s}
\]

\[
= \sum_{m_s} \delta_{m, m_s}^{m''} \langle S m_s | \tau_{kq} | S m_s \rangle
\]

\[
= \delta_{k0} \delta_{q0} F_1 = \delta_{k0} \delta_{q0} 7.28
\]
\[
F_2 = \sum_{m_s m''_s} A^o F \delta_{m''_s m''} <S m_s \leftrightarrow \tau_{kq} S m''_s> \sum_{m'_s} (-)^q <S m'_s T_{2q'_s} | S m'_s>
\]
\[
\times M^*_{V_{2-q'_s}}
\]

\[
= A^o F (2S+1) \sum_{q'_s} (-)^q \sum_{m''_s} ^{m''_s} <S m''_s m''_s-k-q> (-)^{m'_s} <S m'_s m'_s|2-q'> M^*_{V_{2-q'}}
\]

\[
= (2S+1) A^o F \sum_{q'_s} (-)^q M^*_{V_{2-q'}} \sum_{m''_s} <S m''_s-m''_s-k-q> <S m''_s-m''_s|2-q'> M^*_{V_{2-q'}}
\]

\[
= (2S+1) A^o F \sum_{q'_s} (-)^q M^*_{V_{2-q'}} \delta_{k2} \delta_{qq'} 7.29
\]

\[
F_2 = (2S+1) A^o F (-)^q V_{00}^*_{2-q} = 3A V_{00}(-)^q V_{00}^*_{2-q} 7.30
\]

With a similar analysis we get for \( F_3 \)

\[
F_3 = 3 A V_{00}^* V_{2q}^* 7.31
\]

\[
F_4 = A^2 \sum_{m_s m''_s} F_{m''_s m''_s} <S m_s \leftrightarrow \tau_{kq} S m''_s> F_{m''_s m''_s}^* =
\]

\[
= A^2 \sum_{m_s m'_s} (-)^{q'_s+q''} <S m''_s T_{2q'_s} | S m''_s> <S m''_s \leftrightarrow \tau_{kq} S m'_s> <S m'_s | T_{2q'_s} | S m'_s>
\]

\[
\times M^*_{V_{2-q'_s}} M^*_{V_{2-q''}}
\]

\[
= A^2 (2S+1)^2 \sum_{q'_s q''} (-)^q M^*_{V_{2-q'_s}} M^*_{V_{2-q''}} A^q_{2-q} 7.32
\]
where

\[ \Lambda^{q}_{q', q''} = \sum_{m_s, m'_s, m''_s} (-1)^{3S_m - m'^m''_m} <SS \ m_s - m'_s | 2-q' > <SS \ m''_s - m'_s | 2-q'' > <SS m''_s - m'_s | 2-q'' > \]

\[ = (-1)^{3S+q''+q'+q''} \sqrt{k} \sum_{m_s, m'_s, m''_s} (-1)^{m_s - m'^m''_m} \]

\[ = (-1)^{q'+q''} \sqrt{k} \sum_{m_s, m'_s, m''_s} 3S+m_s + m'^m''_s \]

\[ = (-1)^{q'+q''} \sqrt{k} \left\{ \begin{array}{ccc} 2 & k & 2 \\ 2 & 2 & 2 \\ -q'' & q & q' \end{array} \right\} \]

Transforming the Clebsch-Gordon coefficients into 3j symbols, using their symmetry properties and the relationship between 3j symbols and 6j symbols in reference (equation C.33) we get

\[ \Lambda^{q}_{q', q''} = \sum_{m_s, m'_s, m''_s} (-1)^{3S_m - m'^m''_m} <SS \ m_s - m'_s | 2-q' > <SS \ m''_s - m'_s | k-q'' > <SS m''_s - m'_s | 2-q'' > \]

\[ = (-1)^{3S+q''+q'+q''} \sqrt{k} \sum_{m_s, m'_s, m''_s} (-1)^{m_s - m'^m''_m} \]

\[ = (-1)^{q'+q''} \sqrt{k} \sum_{m_s, m'_s, m''_s} 3S+m_s + m'^m''_s \]

\[ = (-1)^{q'+q''} \sqrt{k} \left\{ \begin{array}{ccc} 2 & k & 2 \\ 2 & 2 & 2 \\ -q'' & q & q' \end{array} \right\} \]

\[ = (-1)^{q'+q''} \sqrt{k} \left\{ \begin{array}{ccc} 2 & k & 2 \\ 2 & 2 & 2 \\ -q'' & q & q' \end{array} \right\} \]

7.34
Hence

\[ F_4 = A^2 (2S+1)^2 \frac{3}{4\sqrt{k}} (-)^q \sum_{q',q''} \left\{ \begin{array}{ccc} 2 & k & 2 \\ s & s & s \end{array} \right\} \left(\frac{-}{q'} \right)^{q'} \left(\frac{-}{q''} \right)^{q''} \]

\[ \left. \begin{array}{ccc} 2 & 2 & k \\ -q'' & q' & q \end{array} \right\} M_{V_2-q',2-q''} \]

\[ \ldots \text{Letting } s = 1 \]

\[ F_4 = 15\sqrt{3} (-)^q A^2 \left\{ \begin{array}{ccc} 2 & k & 2 \\ 1 & 1 & 1 \end{array} \right\} \sum_{q',q''} (-)^{q''} q''<2 2 q''-q' \left| 2q> M_{V_2-q',2-q''} \right. \]

\[ \ldots \]

\[ F_4 = 15\sqrt{3} A^2 \left\{ \begin{array}{ccc} 2 & k & 2 \\ 1 & 1 & 1 \end{array} \right\} \sum_{q',q''} (-)^{q''} q''<2 2 q''-q' \left| 2q> M_{V_2-q',2-q''} \right. \]

Hence

\[ \text{Tr}(f f^+) = \frac{m}{\mu \hbar^2} \left[ \delta_{k0} \delta_{q0} F_{00} F + 3A(V_{00} M_{V_2-q'} + (-)^q V_{00} M_{V_2-q''}) \right. \]

\[ + 15\sqrt{3} A^2 \left\{ \begin{array}{ccc} 2 & k & 2 \\ 1 & 1 & 1 \end{array} \right\} \sum_{q',q''} (-)^{q''} q''<2 2 q''-q' \left| 2q> M_{V_2-q',2-q''} \right. \]

\[ \ldots \]

The tensor analyzing powers are therefore given by

\[ T_{2q} = A \frac{V_{00} M_{V_2-q'} + (-)^q V_{00} M_{V_2-q''} + 5\sqrt{3} A \left\{ \begin{array}{ccc} 2 & 2 & 2 \\ 1 & 1 & 1 \end{array} \right\} \sum_{q',q''} (-)^{q''} q''<2 2 q''-q' \left| 2q> M_{V_2-q',2-q''} \right. \]

\[ \ldots \]
7.B. The deuteron nucleus optical potential

The tensor term of $T^\text{type}_R$ and the central part of the deuteron-nucleus optical potential were taken from Keaton and Armstrong folding model without deuteron break-up. The spin orbit interaction was neglected and the imaginary parts of these potentials were zero at the energies considered here.

The central part of the potential, $V_0(R)$, contains a strong nuclear part and a Coulomb part $U_q$.

$$V_0(R) = U_q(R) - U_r f_r(R)$$

where

$$U_q(R) = \frac{Z}{R} e^2 \quad R > R_c$$

$$= \frac{Z}{2 R} \left[ 3 - \frac{R^2}{R_c^2} \right] \quad R \leq R_c$$

The Coulomb radius $R_c$ is given by Elton.

The strength of the strong nuclear part of the central potential is

$$U_r = 0.917(110.4 + 0.272/A - 0.32 E_d) \text{MeV}$$

The radial function $f_r(R)$ has Wood-Saxon shape

$$f_r(R) = \left[ 1 + e^{-\frac{R-r}{a_r}} \right]^{-1}$$

with

$$r_r = 1.14 \text{ fm} \quad a_r = 0.97 \text{ fm}$$
The radial part of the $T_R$ tensor term, $V_{2R}(R)$ is \(^{23,42}\)

\[ V_{2R}(R) = - U_{Tr} m_{Tr}(R) \] \(7.44\)

where the strength $U_{Tr}$ is

\[ U_{Tr} = 0.0771 U_r \] \(7.45\)

and the radial function $m_{Tr}(R)$ is

\[ m_{Tr}(R) = \lambda^2 \frac{d}{dR} \left[ 1 - \left( \frac{R}{r_{Tr}} \right)^{1+e} \right] \] \(7.46\)

with $r_{Tr} = r_r = 1.14$ fm, $a_{Tr} = 1.02$ fm \(7.47\)

\[ \lambda^2 = 2.0 \text{ fm}^2 \]

The radial part of the $T_P$ tensor term, obtained in the last chapter, was fitted to a Wood-Saxon shape

\[ V_{2P}(R) = \frac{U_{T_P}}{6} g_{T_P}(R) \] \(7.48\)

\[ g_{T_P}(R) = \left[ \frac{1}{\left( \frac{R-r_{T_P}}{a_{T_P}} \right)^{1+e}} \right]^{-1} \] \(7.49\)

The parameters $U_{T_P}$, $r_{T_P}$ and $a_{T_P}$ were found to be strongly dependent on the incident deuteron energy. Some of these values are presented in table 5, for the target \(^{92}\text{Zr}\) and deuteron D-state
probability of 4%.

The approximation of the shape of the $T_p$ term to a Wood-Saxon shape is very good at high energies, $E_d > 150$ MeV. At lower energies this approximation amounts to the neglect of the small peak at the target's surface.

Table 5

<table>
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<tr>
<th>$E_d$ (MeV)</th>
<th>$U_{Tp}$ (MeV)</th>
<th>$r_{Tp}$ (fm)</th>
<th>$a_{Tp}$ (fm)</th>
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<td>1.126</td>
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</tr>
<tr>
<td>330</td>
<td>0.94812</td>
<td>1.101</td>
<td>0.51974</td>
</tr>
</tbody>
</table>

7.C. The angular part of the integrals

The central term $V_{00}$ can be split up into a Coulomb and a strong nuclear part.

$$V_{00}(Q) = \int dR e^{iQ\cdot R} V_0(R)$$

$$= V_{00}^d(Q) + V_{00}^n(Q)$$

7.50
where
\[ V_{00}^q = \int dR e^{\frac{iQ \cdot R}{R}} U_q(R) \]  
7.51

and
\[ V_{00}^n = \int dR e^{\frac{iQ \cdot R}{R}} V_n(R) \]  
7.52

The tensor terms \( M_{V_{2q}} \) have the form
\[ M_{V_{2q}}(Q) = \int dR e^{\frac{iQ \cdot R}{R}} Y_{2q}^{*}(R) M_{V_{2}}(R) \]  
7.53

The angular part of these integrals can be performed straight-away. We use the formula for the expansion of the plane wave into a series of spherical harmonics 60

\[ e^{\frac{iQ \cdot R}{R}} = 4\pi \sum_{L=0}^{\infty} \sum_{q'=\pm 2}^{2L} i^{L} \bar{j}_L(QR) Y_{q'}^{*}(\hat{\Omega}) Y_{q'}(\hat{\Phi}) \]  
7.54

Substituting this formula into 7.51

\[ V_{00}^q = \frac{4\pi Ze^2}{Q} (I_1^q + I_2^q) \]  
7.55

where
\[ I_1^q = \int_0^{\infty} \sin QR e^{-\frac{R}{a_q}} \]  
7.56

\[ I_2^q = \frac{1}{2R_c} \int_0^{R_c} R \sin QR \left[ 3 - \frac{R^2}{R_c} \right] - \frac{R}{a_q} \]  
7.57

The term \( e^{-\frac{R}{a_q}} \) comes from screening. The parameter \( a_q \) is let to become infinite at the end of the calculation.
Performing the integrals and letting $a_q$ become infinite

$$v_{00}^0(Q) = \frac{12\pi e^2}{Q^3R_c^3} \left[ - R_c Q \cos QR_c + \sin QR_c \right]$$ 7.58

The strong nuclear term becomes

$$v_{00}^n(Q) = 4\pi I_{00}^n$$ 7.59

$$v_{00} = 4\pi I_0$$ 7.60

where

$$I_0^0 = I_0^n + \frac{3Ze}{Q^3R_c^3} \left[ - R_c Q \cos QR_c + \sin QR_c \right]$$ 7.61

$$I_0^0 = \frac{1}{Q} \int_0^\infty R dR \sin(QR) v_n(R)$$ 7.62

The tensor terms take the following form

a) $T_R$ type of tensor term: $M = \hat{R}$

$$M_{Y_{2q}} = \int dR Y_{2q}(\hat{R}) e^{iQ \cdot R} R v_2(R) =$$

$$= 4\pi \sum_{l,q'} i^l Y_{lq'}(\hat{Q}) \int_0^\infty dR R^2 j_l(QR) R v_2(R) \int d\Omega Y_{lq'}(\hat{R}) Y_{2q}(\hat{R})$$ 7.63

Using the orthogonality of the spherical harmonics

$$R v_{2q} = - 4\pi Y_{2q}(\hat{Q}) I_R$$ 7.64
where
\[ I_R = \int_0^\infty dR R^2 j_2(QR) V_2(R) \] 7.65

b) \( T_p \) type of tensor term: \( \hat{\mathbf{M}} = \hat{\mathbf{p}} \)

The total potential in this case is:
\[ V = V_0(R) + \frac{1}{2} \left[ P V_2(R) S_{12}^P + S_{12}^P V_2(R) \right] \] 7.66

The Born approximation to the scattering amplitude is
\[ f_{m'm's's} = -\frac{2m}{4\pi \varepsilon^2} \left[ T_{m's'm's} + 2P_{m's'm's} \right] \]

with
\[ T_{m's'm's} = \int dR V_0(R) e^{i\mathbf{Q} \cdot \mathbf{R}} \]

and
\[ 2P_{m's'm's} = \frac{1}{2} \left[ \langle k' s'm' | P V_2(R) S_{12}^P + S_{12}^P V_2(R) | k s m \rangle \right] \]

\[ = \frac{A}{2} \sum_q (-)^q (Y_{2-q}(\hat{k}) + Y_{2-q}(\hat{k}')) \langle s'm' | T_{2q} | s'm \rangle \times \int dR e^{i\mathbf{Q} \cdot \mathbf{R}} P V_2(R) \] 7.67

The quantities \( M_{V_2q} \) are defined by equation 7.12. Comparing 7.67 with 7.12 we see that

\[ P_{V_2q}(Q) = \frac{1}{2} [Y_{2q}(\hat{k}) + Y_{2q}(\hat{k}')] \int_0^\infty dR e^{i\mathbf{Q} \cdot \mathbf{R}} P V_2(R) \]

Using 7.54 we get
\[ P_{V_2q}(Q) = 2\pi [Y_{2q}(\hat{k}) + Y_{2q}(\hat{k}')] I_p \] 7.68
where

\[ I_p = \frac{1}{Q} \int_0^\infty R \, dR \sin (QR) P_2 (R) \quad 7.69 \]

7.D. Analytic expressions for integrals involving Wood-Saxon integrands

The integrands we want to evaluate are of the form

\[ \int_0^\infty dR R^n CS(QR) \frac{d^n f(R)}{dR^n} \quad 7.70 \]

where

\[ CS(QR) \text{ is either } \cos(QR) \text{ or } \sin(QR) \text{ and} \]

\[ f(r) = \begin{cases} \left( \frac{R-C}{a} \right)^{-1} & R > C \\ \frac{1}{1 + e^{\frac{R-C}{a}}} & R < C \end{cases} \quad 7.71 \]

We now consider the integral \( I' \) of the complex function \( F(z) \) around the close loop \( OABO \), fig. 35.

\[ I' = \int_{OABO} F(z) \, dz \quad 7.72 \]

\[ I' = \sum_n 2\pi i R_n \quad 7.73 \]

where \( R_n \) are the residues of \( F(z) \), also

\[ I' = \int_0^A F(R) \, dR + \int_{AB} F(z) \, dz + i \int_B^0 F(iy) \, dy \quad 7.24 \]

\[ I' = I_1 + I_2 + I_3 \quad 7.75 \]
where

\[
I_1 = \int_0^A F(R)dR
\]

\[
I_2 = \int_{AB} F(z)dz
\]

\[
I_3 = i \int_B^0 F(iy)dy
\]

For \( F(z) \) we use the function

\[
F(z) = \frac{e^{iQz}}{1 + e^{z/c}}
\]

\( F(z) \) thus defined has poles at \( z = a_n \) where

\[
a_n = (2n + 1)\pi a i + c
\]

The residues are

\[
R_n = \lim_{\delta \to 0} \frac{i Qc e^{-Qn/Q} (e^{-2\pi Qa})^n}{\delta e^{z/c} - 1 - e^a}
\]

\[
\therefore R_n = -\frac{a e^{iQc - Qn/Q} (e^{-2\pi Qa})^n}{1 - e^a}
\]

The residues of \( f(z) \) form a geometric series. Summing up this series for all \( R_n \) we have

\[
\sum_{n=0}^{\infty} R_n = -\frac{a e^{iQc}}{e^{Qnd} - e^{-Qwd}} = -\frac{a e^{iQc}}{2\sinh Qwa}
\]

Hence when the points A and B are removed to infinity the integral \( I' \) is
\[ I' = \frac{-ima e^{iQc}}{\sinh Q \pi a} \]

We now consider the integral \( I_2 \). We let \(|OA| = |OB|\) and \( z = Re^{i\theta} \), then \( dz = iR e^{i\theta} d\theta \) hence

\[
I_2 = \int_0^\frac{\pi}{2} \frac{iR e^{i\theta} e^{iQe^{i\theta}}}{1 + e^{i\theta}} d\theta R e^{i\theta} f(R, \theta) \left| \int_0^\frac{\pi}{2} d\theta |f(R, \theta)| \right|
\]

where

\[
|f(R, \theta)|^2 = R^2 \left[ \frac{e^{iQR \cos \theta} - QR \sin \theta - iQR \cos \theta - QR \sin \theta}{1 + e^{-\theta}} \right] \left[ \frac{e^{iQR \cos \theta} - QR \sin \theta - iQR \cos \theta - QR \sin \theta}{1 + e^{-\theta}} \right]
\]

\[
|f(R, \theta)| = \frac{Re^{-QR \sin \theta}}{(1 + e^{-\theta})^2 + 2e^{-\theta} \cos \theta + e^{-\theta} \cos \theta^2}
\]

Hence as \( R \) becomes infinite

\[
|f(R, \theta)| \rightarrow 0 \quad \text{for all } \theta
\]

\[
\therefore \quad I_2 \rightarrow 0 \quad R \rightarrow \infty
\]

The integral \( I_3 \), for \( R \rightarrow \infty \) is

\[
I_3 = i \int_\infty^0 e^{-Qy} dy \left( -\frac{c}{a} \frac{iY}{e^{aY}} \right)
\]

For the purposes we want this integral \( c \) is always greater than \( a \). Hence

\[
|e^{aY} e^{-\frac{c}{a} iY}| < 1 \text{ for all values of } y. \quad \text{Hence we can expand the denominator}
\]
in a Taylor series

\[
\frac{1}{1+e^{-\frac{e^y}{a}}} = 1 - \frac{c}{a} e^{-\frac{y}{a}} + e^{-\frac{2c}{a} e^{-\frac{y}{a}}}
\]

\[
= \sum_{n=0}^{\infty} \left( -\frac{c}{a} e^{-\frac{y}{a}} \right)^n = \sum_{n=0}^{\infty} \left( -\frac{nc}{a} e^{-\frac{y}{a}} \right)^n
\]

\[I_3 = i \sum_{n=0}^{\infty} \left( -\frac{nc}{a} e^{-\frac{y}{a}} \right)^n \int_0^\infty dy e^{-\frac{y}{a} (\frac{n}{a} - Q)} \]

\[I_3 = \sum_{n=0}^{\infty} (-\frac{nc}{a}) \frac{n a e^{-\frac{y}{a}}}{n^2 + Q^2 a^2} = i Q a^2 \sum_{n=0}^{\infty} \left( -\frac{nc}{a} \right) \frac{e^{-\frac{y}{a}}}{n^2 + Q^2 a^2}
\]

Hence \( I = I' - I_3 \)

\[I = \frac{\pi a \sin Q}{\sinh Q \pi a} - \sum_{n=0}^{\infty} \left( -\frac{nc}{a} \right) \frac{n a e^{-\frac{y}{a}}}{n^2 + Q^2 a^2}
\]

\[+ i a \left[ -\frac{\cos QC}{\sinh Q \pi a} + Q a \sum_{n=0}^{\infty} \left( -\frac{nc}{a} \right) \frac{e^{-\frac{y}{a}}}{n^2 + Q^2 a^2} \right]
\]

The series in 7.86 are very rapidly convergent series due to the exponential \( e^{-\frac{y}{a}} \); the first few terms in these series suffice to give a very accurate answer.

From equation 7.86 and the formula

\[e^{iQR} = \cos QR + i \sin QR \]

we get

\[SW = \int_0^{\pi} dR \sin QR = -\frac{\pi a \cos QC}{\sinh Q \pi a} + Q a^2 \sum_{n=0}^{\infty} \left( -\frac{nc}{a} \right) \frac{e^{-\frac{y}{a}}}{n^2 + Q^2 a^2}
\]
and
\[ CW = \int_0^{\infty} \frac{dR \cos QR}{R-c} = \frac{\pi a \sin Qc}{\sinh Q\alpha} - a \sum_{n=1}^{\infty} \frac{n e}{n^2+Q^2a^2} \] 7.89

Integrals of the form \[ \int_0^{\infty} R^{2n+1} \sin(QR)f(R)dR \] can be evaluated from 7.88 and 7.89 by the relationship
\[ \int_0^{\infty} R^{2n+1} \sin(QR)f(R)dR = (-)^{n+1} \frac{d^{2n+1}}{dR^{2n+1}} \left[ \int_0^{\infty} f(R)\sin(QR)dR \right] \] 7.90

Hence
\[ SRW = \int_0^{\infty} \frac{R \sin(Q\alpha)}{R-c} \frac{d}{dR} \left[ \int_0^{\infty} f(R)\sin(QR)dR \right] = \frac{\pi a^2}{\sinh^2 Q\alpha} \left[ \frac{c}{a} \cos(Qc)\sinh(Q\alpha) - \pi \cosh(Q\alpha) \right] \times \sin(Qc) - 2Q a^3 \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{n e}{n^2+Q^2a^2} \] 7.91

Similarly
\[ CRW = \int_0^{\infty} R \cos(QR)f(R)dR = \frac{d}{dQ} \left[ \int_0^{\infty} f(R)\sin(QR)dR \right] \]
\[ = \frac{\pi a^2}{\sinh^2 Q\alpha} \left[ \pi \cosh(Q\alpha)\cos(Qc) + \frac{c}{a} \sinh(Q\alpha)\sin(Qc) \right] \]
\[ + a^2 \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{e}{a} \left[ \frac{n^2-Q^2a^2}{(n^2+Q^2a^2)} \right] \] 7.92

Integrals involving derivatives of Wood-Saxon can be expressed in terms of the four integrals SW, CW, SRW and CRW. We first consider the integral, \( D_1 \)
\[ D_1 = \int_0^{\infty} R \sin(QR) \frac{d f(R)}{dR} dR \] 7.93
Integrating by parts we have

\[ D_1 = R \sin(QR)f(R) \bigg|_0^\infty - \int_0^\infty \sin(QR)f(R)dR - Q \int_0^\infty R \cos(QR)f(R)dR \]

The first term becomes zero at both limits if \( f(R) \) is of a Wood-Saxon shape, hence

\[ D_1 = - SW - QX \text{CNI} \] \hspace{1cm} 7.94

Integrating by parts the integral \( D_2 \) we get

\[ D_2 = \int_0^\infty dR R^3 J_2(QR) \frac{d}{dR} \left( \frac{1}{R} \frac{df}{dR} \right) = \]

\[ = R^2 J_2(QR) \frac{df}{dR} \bigg|_0^\infty - \int_0^\infty \frac{dR}{R} \frac{df}{dR} \frac{d}{dR} \left[ R^3 J_2(QR) \right] \] \hspace{1cm} 7.95

Using the following recursion relation for the Bessel function \( J_2(QR) \)

\[ \rho^{\ell+1} J_{\ell-1}(\rho) = \frac{d}{d\rho} \left[ \rho^{\ell+1} J_\ell(\rho) \right] \] \hspace{1cm} 7.96

we get

\[ D_2 = - \int_0^\infty Q R^2 J_1(QR) \frac{df}{dR} dR \] \hspace{1cm} 7.97

Integrating by parts again and using 7.96 we get

\[ D_2 = Q \int_0^\infty dR R \sin(QR)f(R) = Q \text{ (SNR)} \] \hspace{1cm} 7.98

All the radial integrals we need can now be evaluated. \( I_p \) and \( I_0 \) can be evaluated from 7.91 and \( I_R \) can be evaluated from 7.98.
7.E. The tensor analysing powers

The expression 7.38, for $T_{2\omega}$, can be written as

$$T_{2\omega} = A \frac{f_{ct} + C f_{\theta\theta}}{f_{cc} + \Delta^2 f_{\theta\theta}}$$  \hspace{1cm} 7.99

where

$$f_{ct} = (-)^q V^{(0)}_{q0} M^{(0)}_{q0} + V^{(0)}_{q0} M^{(0)}_{q0}$$  \hspace{1cm} 7.100

$$C = 5\sqrt{3} A \left\{ \begin{array}{ccc} 2 & 2 & 2 \\ 1 & 1 & 1 \end{array} \right\} = \sqrt{\frac{7}{4}} A$$  \hspace{1cm} 7.101

$$f_{\theta\theta} = \sum_{q',q''} (-)^{q''} <2q'2q''|2\omega> V^{(0)}_{q'0} M^{(0)}_{q''0}$$  \hspace{1cm} 7.102

$$f_{cc} = V^{(0)}_{q0} V^{(0)}_{q0} = 16\pi^2 I_{0} I_{0}$$  \hspace{1cm} 7.103

$$f_{\theta\theta} = \sum_{q'} M^{(0)}_{q'2-q} M^{(0)}_{q'2-q}$$  \hspace{1cm} 7.104

These expressions can be simplified further, with the aid of the results of section C. The $T_{2\omega}$ are referred to a right handed co-ordinate system in which the positive z-axis is along the direction of the incident beam, $\hat{k}$, and the y-axis is along $k \times k'$, where $\hat{k'}$ is the direction of the outgoing beam. With this choice of axes the polar angle $\phi$ is always zero for the vectors $k$, $k'$ and $Q = k - k'$. We shall express the $T_{2\omega}$ in terms of the scattering angle, $\theta$, between $k$ and $k'$. Other angles we will use are the angle between $k$ and $Q$, $\theta_{Q}$; and the angle between $k'$ and $Q$, $\theta'_{Q}$. We are concerned here with elastic scattering;
the magnitudes of \( k, k, \) and \( k', k' \), are therefore equal. The angles \( \theta_q \), \( \theta_q' \), and \( \theta_q'' \) are simply related in this case, as can be easily shown from fig. 34

\[
\theta_q = \frac{\theta}{2} - 90 \quad \theta_q' = 90 + \frac{\theta}{2}
\]

7.105

Since \( \hat{k} = \hat{z} \) and \( \phi = 0 \) for \( \hat{k} \)

\[
V_{kq}(\hat{k}) = \sqrt{\frac{2k+1}{4\pi}} \delta_{q0}
\]

7.106

Some other useful properties of the spherical harmonics are the following \( ^{58,60} \)

\[
\sum_{q} Y_{kq}(\hat{\rho}) Y_{k'q}(\hat{\rho}') = \frac{2k+1}{4\pi} P_{k}[\cos(\hat{\rho}.\hat{\rho}')] \]

7.107

\[
Y_{kq}(\hat{\rho}) Y_{k'q'}(\hat{\rho}) = \sum_{L,M} \left[ \frac{(2k+1)(2k'+1)}{4\pi(2L+1)} \right]^{\frac{1}{2}} <kk'00|L0><kk'qq'|LM>Y_{LM}(\hat{\rho})
\]

2.108

The product tensor of rank \( k \) constructed out of the directions \( \hat{\rho} \) and \( \hat{\rho}' \),

\[
B_{kq}^{k'k''}(\hat{\rho},\hat{\rho}')
\]

\[
B_{kq}^{k'k''}(\hat{\rho},\hat{\rho}') = \frac{4\pi}{\sqrt{(2k'+1)(2k''+1)}} \sum_{q',q''} <k'q'q''|k\rho>Y_{k'q'}(\hat{\rho})Y_{k''q''}(\hat{\rho}')
\]

7.109

is called bipolar spherical harmonic \( ^{58} \).

Some properties of the bipolar spherical harmonics of use to us here are the following

\[
B_{kq}^{k'k''}(\hat{\rho},\hat{\rho}') = B_{kq}^{k''k'}(\hat{\rho}',\hat{\rho})
\]

7.110
From equation 7.108 and the orthogonality relation for the Clebsch-Gordon coefficients we have

$$B_{kq}^{k''k''} (\hat{r}, \hat{r}'') = \frac{4\pi}{(2k'+1)(2k''+1)} \sum_{q''} < k''q'' | kq > \sqrt{\frac{2k'+1}{4\pi}} \delta_{q''0} Y_{k''q''}(\hat{r}'')$$

Where the direction $\hat{r}$ is along the z-axis, we have from equation 7.106

$$B_{kq}^{k''k''} (\hat{z}, \hat{r}'') = \sqrt{\frac{4\pi}{2k''+1}} < k''0q | kq > Y_{k''q''}(\hat{r}'')$$

We let the deuteron-nucleus optical potential to be of the type in equation 7.1

$$V = V_0(R) + R V_2(R) S_{12}(\hat{S}, \hat{R}) + \frac{1}{2} \left[ \frac{P V_2(R) S_{12}(\hat{S}, \hat{R}) + S_{12}(\hat{S}, \hat{R}) P V_2(R)}{1 - 2 I_{2q}} \right]$$

In this case

$$R_{2q} = 2\pi \left[ (Y_{2q}(\hat{k}) + Y_{2q}(\hat{k}')) I_{P} - 2 I_{R} Y_{2q}(\hat{Q}) \right]$$

From equations 7.100 and 7.106 we get

$$\xi_{2q} = \frac{2\pi}{\sqrt{5}} \left[ (I_{P} I_{P}^* + I_{D} I_{D}^*) \sqrt{\frac{5}{4\pi}} Y^{(\hat{R}'')} - 2 I_{R} Y_{2q}^{(\hat{Q})} \right]$$

From equations 7.102 and 7.106-7.112 we get
\[ f_{\text{tt}}^q = 4\pi^2 \sqrt{\frac{5}{4\pi}} \langle 2200 | 20 \rangle \left\{ I_{\text{P}}^* I_{\text{R}}^* \left[ \sqrt{\frac{5}{4\pi}} \delta_{\alpha 0} + 2 \frac{\langle 2200 | 2\alpha \rangle}{\langle 2200 | 20 \rangle} Y_{2\alpha}(\hat{k}') \right] + Y_{2\alpha}(\hat{k}') \right\} + 4I_{\text{R}}^* I_{\text{R}} Y_{2\alpha}(\hat{Q}) \\
- \frac{2}{\langle 2200 | 20 \rangle} \left( I_{\text{R}}^* I_{\text{P}}^* \langle 2\alpha | 20 \rangle Y_{2\alpha}(\hat{Q}) + \sqrt{\frac{5}{4\pi}} B_{2\alpha}^2(\hat{k}', \hat{Q}) \right) \right\} \]

\[ f_{\text{tt}}^q = 4\pi^2 \sqrt{\frac{5}{4\pi}} \left[ I_{\text{P}}^* I_{\text{R}}^* Y_{2\alpha}(\hat{Q}) + I_{\text{R}}^* I_{\text{P}}^* \left[ \sqrt{\frac{5}{4\pi}} \delta_{\alpha 0} + (1-\sqrt{14\langle 2200 | 2\alpha \rangle}) Y_{2\alpha}(\hat{k}') \right] + \sqrt{14}(I_{\text{R}}^* I_{\text{P}}^* + I_{\text{P}}^* I_{\text{R}}^*) \langle 2200 | 2\alpha \rangle Y_{2\alpha}(\hat{Q}) + \sqrt{\frac{5}{4\pi}} B_{2\alpha}^2(\hat{k}', \hat{Q}) \right] \] 7.115

From equations 7.104, 8.107 and 7.113 we get

\[ f_{\text{tt}}^q = 4\pi^2 \frac{5}{4\pi} \left[ I_{\text{P}}^* I_{\text{R}}^* (2P_2(1) + 2P_2(\cos 0)) \right] \\
- 2(I_{\text{P}}^* I_{\text{R}}^* + I_{\text{R}}^* I_{\text{P}}^*) \left[ 2P_2(\cos 0) + 2P_2(\cos 0') \right] + 4I_{\text{R}}^* I_{\text{P}}^* (1) \right\} \\
= 5\pi \left\{ I_{\text{P}}^* I_{\text{R}}^* (3 \cos^2 0 + 1) + 4I_{\text{R}}^* I_{\text{R}}^* - 2(I_{\text{P}}^* I_{\text{R}}^* + I_{\text{R}}^* I_{\text{P}}^*) \left[ 3(\cos^2 0 + \cos^3 0') - 2 \right] \right\} \\
7.116

In terms of the scattering angle \( \theta \) \( f_{\text{tt}}^q \) becomes

\[ f_{\text{tt}}^q = 5\pi \left[ I_{\text{P}}^* I_{\text{R}}^* (3 \cos^2 0 + 1) + 4I_{\text{R}}^* I_{\text{R}}^* - 4(I_{\text{R}}^* I_{\text{P}}^* + I_{\text{P}}^* I_{\text{R}}^*) (3 \sin^2 \frac{\theta}{2} - 1) \right] \\
7.117
Hence

\[ T_{2q} = A \left\{ 2\left( I_{0}^{*} I_{P}^{*} + I_{P}^{*} I_{0}^{*}\right) \left\{ \sqrt{\frac{5}{4\pi}} \delta_{q0} + Y_{2q}(\hat{k})\right\} - 2\left( I_{0}^{*} I_{R}^{*} + I_{R}^{*} I_{0}^{*}\right) Y_{2q}(\hat{\phi}) \right\} \]

\[ - A \sqrt{\frac{5}{8\pi}} \left\{ 4I_{R}^{*} Y_{2q}(\hat{\phi}) + I_{P}^{*} \left[ \sqrt{\frac{5}{4\pi}} \delta_{q0} + (1 - \sqrt{14} Y_{220q} |2q> \right) Y_{2q}(\hat{k}) \right\} \]

\[ + \sqrt{14}(I_{R}^{*} I_{P}^{*} + I_{P}^{*} I_{R}^{*}) \left\{ Y_{2q}(\hat{\phi}) + \sqrt{\frac{5}{4\pi}} B_{22q}(\hat{k}) \right\} \]

\[ \left[ 4I_{0}^{*} I_{0}^{*} + \frac{5h^{2}}{4} \left\{ I_{P}^{*} I_{P}^{*} - 4I_{R}^{*} - 4\left( I_{P}^{*} I_{R}^{*} + I_{R}^{*} I_{P}^{*}\right) \left( \sin^{2} \frac{\theta}{2} - 1 \right) \right\} \right] \]

7.118

If the central potential is neglected

\[ T_{2q} = - \sqrt{\frac{2\pi}{5}} \left\{ 4I_{R}^{*} I_{R}^{*} Y_{2q}(\hat{\phi}) + I_{P}^{*} \left[ \sqrt{\frac{5}{4\pi}} \delta_{q0} + (1 - \sqrt{14} Y_{220q} |2q> \right) Y_{2q}(\hat{k}) \right\} \]

\[ + \sqrt{14}(I_{R}^{*} I_{P}^{*} + I_{P}^{*} I_{R}^{*}) \left\{ Y_{2q}(\hat{\phi}) + \sqrt{\frac{5}{4\pi}} B_{22q}(\hat{k}) \right\} \]

\[ \left[ I_{P}^{*} I_{P}^{*} + \frac{5h^{2}}{4} \left\{ I_{P}^{*} I_{P}^{*} - 4I_{R}^{*} - 4\left( I_{P}^{*} I_{R}^{*} + I_{R}^{*} I_{P}^{*}\right) \left( \sin^{2} \frac{\theta}{2} - 1 \right) \right\} \right] \]

7.119

If both the central potential and the \( T_{P} \) term are neglected the \( T_{2q} \) become

\[ T_{2q} = - \sqrt{\frac{2\pi}{5}} Y_{2q}(\hat{\phi}) \]

7.120
In particular

\[ T_{20} = -\frac{1}{\sqrt{8}} (3 \sin^2 \frac{\theta}{2} - 1) \]

\[ T_{21} = -\sqrt{3} \sin \frac{\theta}{2} \cos \frac{\theta}{2} = -\frac{\sqrt{3}}{4} \sin \theta \]

\[ T_{22} = -\frac{\sqrt{3}}{4} \cos^2 \frac{\theta}{2} \]

If the central potential and the \( T_R \) term are neglected we have

\[ T_{2q} = -\sqrt{\frac{2\pi}{5}} \frac{\sqrt{\frac{5}{4\pi} \delta_{q0} + (1-\sqrt{14}\langle 220q | 2q \rangle)Y_{2q}(\hat{k}')}}{3 \cos^2 \theta + 1} \]

In particular

\[ T_{20} = -\frac{1}{\sqrt{8}} \frac{9 \cos^2 \theta - 1}{3 \cos^2 \theta + 1} \]

\[ T_{21} = \sqrt{3} \frac{\sin \theta \cos \theta}{3 \cos^2 \theta + 1} = \frac{\sqrt{3}}{2} \frac{\sin(2\theta)}{3 \cos^2 \theta + 1} \]

\[ T_{22} = +\frac{\sqrt{3}}{4} \frac{\sin^2 \theta}{3 \cos^2 \theta + 1} \]

Equations 7.119-7.123 have of course no physical significance because the central potential is much greater than the tensor potentials. Nevertheless these equations are quite an interesting result; the tensor analyzing powers in P.W.B.A. are independent of the shape of the potential and the bombarding energy when the optical potential consists of only one type of tensor interaction. This simple result provided us with an easy check for the computer program used for the evaluation of the full expression for \( T_{2q} \).
7.F. Validity of P.W.B.A.

P.W.B.A. is essentially a high energy approximation. This approximation is expected to be valid if the following inequality is satisfied

\[ \left( \frac{m}{2\pi \hbar} \right)^2 \left| \int_0^\infty V(r)(e^{2ikr} - 1)dr \right| \ll 1 \quad 7.124 \]

By far the strongest potential in our case is the strong nuclear central potential, \( V_0 \).

\[ V_0 = U_r f(R) \quad 7.125 \]

with \( U_r \) and \( f(R) \) defined in equations 7.41 and 7.42 respectively.

Hence an estimate for the validity of P.W.B.A. is given by

\[ \left( \frac{m}{2\pi \hbar} \right)^2 |I_1 - I_2|^2 \ll 1 \quad 7.126 \]

where

\[ I_1 = \int_0^\infty \frac{e^{2ikr}dr}{r-c \left( 1 + \frac{a}{e^a} \right)} \quad 7.127 \]

and

\[ I_2 = \int_0^\infty \frac{dr}{r-c \left( 1 + \frac{a}{e^a} \right)} \quad 7.128 \]

The integral \( I_1 \) can be evaluated from equation 7.86 and \( I_2 \) from 7.89, e.g.

\[ I_2 = \lim_{Q \to 0} \int_0^\infty \frac{dR \cos QR}{R-c \left( 1 + \frac{a}{e^a} \right)} \]
\[
\frac{\ln \frac{a}{n} - \frac{nc}{a}}{\pi a} - a \sum_{n} (-1)^{n} \frac{e^{n - \frac{c}{a}}}{n}
\]

\[
\ln \left(1 + \frac{c}{a}\right)
\]

but

\[
\frac{c}{a} \ll 1
\]

hence \( I_2 \) reduces to

\[
I_2 = c
\]

At high energies, where P.W.B.A. is expected to be valid, \( I_2 \) is much larger than \( I_1 \). The criterion for the validity of P.W.B.A. is then conveniently reduced to

\[
\left(\frac{m_d U c}{n^2 k}\right)^2 \ll 1
\]

With this criterion P.W.B.A. is expected to be a reasonable approximation for \( E_d \geq 250 \text{ MeV} \), at least for small scattering angles.

7.G. Results of the calculation

The results of the calculation are shown in figures 36 to 41, for incident deuteron energies 250 MeV and 310 MeV. The target is \( ^{92}\text{Zr} \) and the strength and shape of \( T_p \) is derived from our binding energy approach, with a deuteron D-state of 4%. The deuteron-nucleus optical potential contains, apart from the central potential, either a tensor potential of the \( T_R \) type only, or both \( T_R \) and \( T_p \) tensor potentials. The tensor analyzing powers corresponding to the former case, \( R_{T_2q} \), are the dashed curves, and the tensor analyzing powers corresponding to the latter case, \( R_{TP_{T_2q}} \), are the solid curves.
We are not very much interested in the actual shape of $T_{2q}$, because there are at present no experimental data to compare with our results, at these high energies, instead we are interested in the differences between $R_{T_{2q}}$ and $P_{T_{2q}}$, because these differences can provide a test for the existence of the $T_p$ tensor force we propose here when experimental data at these high energies become available.

We can easily see in the figures that $R_{T_{2q}}$ and $P_{T_{2q}}$ show important differences at forward angles for $T_{20}$ and $T_{21}$. At backward angles and for all angles for $T_{22}$, $R_{T_{2q}}$ and $P_{T_{2q}}$ are very similar. The result suggests that measurements of $T_{20}$ and $T_{21}$ at high energies and at forward angles are expected to be the most sensitive polarization quantities on the $T_p$ potential.

We can easily understand the reasons for this result, from equation 7.38. This equation expresses the tensor analyzing powers in terms of the Fourier transforms of the potentials, $M_{v_{kq}}$. The largest of these potentials is of course $v_{00}$. The $T_{2q}$ are therefore essentially determined from terms linear in $M_{v_{2q}}$, that is $M_{T_{2q}}$ essentially depends on $v_{00} M_{v_{2q}}$. Differences between $R_{T_{2q}}$ and $P_{T_{2q}}$ are therefore expected to be more pronounced when $P_{v_{2q}}$ is large compared with $R_{v_{2q}}$. Using equations 7.64, 7.65, 7.68, 7.69, 7.91 and 7.98 we get

\[
R_{v_{2q}} = -4\pi Q \left[ SEW(a_T, c_T, Q) \right] Y_{2q}(\hat{Q}) \tag{7.133}
\]

\[
P_{v_{2q}} = \frac{2\pi}{Q} \left[ SRW(a_T, c_T, Q) \right] \left[ Y_{2q}(\hat{k'}) + \sqrt{\frac{5}{4\pi}} \delta_{q0} \right] \tag{7.134}
\]
Hence the ratio of $P V_{2q}$ to $P V_{2q}$ is proportional to the quantity $D_{2q}$

$$D_{2q} = -\frac{\sqrt{\frac{5}{4\pi}} Y_{2q}(\hat{k}')}{2q^2 Y_{2q}(\hat{q})} \quad 7.135$$

The momentum transfer $Q$ expressed in terms of $k$ and the scattering angle $\theta$ is

$$Q = 2k \sin \frac{\theta}{2} \quad 7.136$$

Hence

$$D_{2q} = -\frac{\sqrt{\frac{5}{4\pi}} Y_{2q}(0)}{8k^2 \sin^2 \frac{\theta}{2} Y_{2q}(0)} \quad 7.137$$

In particular

$$D_{20} = -\frac{3 \cos^2 \theta + 1}{8k^2 \sin^2 \frac{\theta}{2} (3 \sin^2 \frac{\theta}{2} - 1)} \quad 7.138$$

$$D_{21} = \frac{\cos \theta}{4k^2 \sin^2 \frac{\theta}{2}} \quad 7.139$$

$$D_{22} = -\frac{1}{2k^2} \quad 7.140$$

Differences between $R T_{2q}$ and $R T_{2q}$ are expected to be important when $|D_{2q}|$ is large. $D_{20}$ and $D_{21}$ are largest for $\frac{\theta}{2}$ small. $D_{22}$ is always small at high energies. Differences between $R T_{2q}$ and $R T_{2q}$ are therefore expected at forward angles for $T_{20}$ and $T_{21}$. 
Figure 34. The choice of axes as prescribed by the Madison convention.
Figure 35. The integration path for the integral in section 7.
Figure 36. The tensor analyzing power $T_{20}$ in PWBA for $E_d = 250$ MeV.
Figure 37. The tensor analyzing power $T_{21}$ in PWBA for $E_d = 250$ MeV
Figure 38. The tensor analyzing power $T_{22}$ in PWBA for $E_d = 250$ MeV.
Figure 39. The tensor analyzing power $T_{20}$ in PWBA for $E_d = 310$ MeV.
Figure 40. The tensor analyzing power $T_{21}$ in PWBA for $E_d = 310$ MeV.

for $E_d = 310$ MeV.
Figure 41. The tensor analyzing power $T_{22}$ in PWBA

for $E_d = 310$ MeV.
EPILOGUE

In this work we have proved that a new tensor potential of the \( T_p \) type must exist in the deuteron-nucleus interaction. The fundamental source of this new potential is the tensor force in the nucleon-nucleon interaction and the Pauli exclusion principle. We have shown that this new tensor potential has a significant magnitude over a wide range of high incident deuteron energies. Although our model for the propagation of a deuteron through a finite nucleus is not valid at low energies, \( E_d \leq 100 \text{ MeV} \), we do expect the tensor potential of the \( T_p \) type to be significant for these energies also, because the effect of the Pauli exclusion principle is strong at these energies. The only energy region where we expect this new potential to be negligibly small is for incident deuteron energies below or of the order of the Coulomb barrier energy, because, there the action of the Pauli principle, although very strong, is very similar on all possible spin states of the deuteron.

These remarks are reinforced by the state of agreement between experiment and theory at present. Experimental data from deuteron nucleus elastic scattering, with polarised deuterons, are analysed in terms of an optical potential which contains in addition to the usual spin orbit term, tensor forces of the \( T_R \) and \( T_L \) types.

The results of such an analysis agree fairly well with experiment at subcoulomb energies. This agreement, between theory and experiment disappears, however, for deuteron energies a little higher than the Coulomb barrier energy. Sources for this discrepancy, other than the neglect of the tensor potential of the \( T_p \) type, are conceivably deuteron break-up and Coulomb effects. Both these effects become small at high energies, e.g. \( E_d \approx 100 \text{ MeV} \). Experiments with polarized deuterons are
unfortunately performed so far at much lower energies where both Coulomb
and break-up effects are expected to be important. Similar experiments
at high energies would be very useful in establishing whether or not
a strong tensor potential of the $T_p$ type exists in the deuteron nucleus
interaction. Hooton and Johnson\textsuperscript{8} showed that certain combinations
of the tensor analyzing powers are expected to be very sensitive on the
tensor interactions. These ideas were extended by Santos\textsuperscript{62,63} who
showed that, if the polarizations of both the ingoing and outgoing beams
are measured, combinations of observable quantities can be constructed
which are sensitive to particular types of tensor interactions.
Simonius\textsuperscript{64} has shown that these experiments are in principle the most
complex experiments required to completely determine the transition
matrix elements at a given energy and angle. Only a handful of such ex-
periments have been done or are in progress so far. These are
performed at very low energies and very light targets, and measurements
have been primarily obtained at 0° where it is simpler to obtain the
polarization transfer coefficients. Such experiments for heavier
targets and at a number of angles could be extremely useful in deter-
mining the correct tensor potentials, $T_R$, $T_L$, and $T_p$, in the deuteron
nucleus interaction.

When the shape and strength of this new tensor force is estab-
lished the $T_p$ force can be used to obtain new information on a variety of
important problems currently fashionable in nuclear physics. Some of
these are the following.

The strength of the tensor potential we propose here depends
strongly on the D-state of the deuteron. Experiments with polarized
deuterons can in principle yield information about the parameter $D_2$\textsuperscript{65}
obtained from $(d,p)$ measurements at low energies\textsuperscript{66}. 
At high energies the tensor force we propose here comes about from the large relative momenta of the n-p systems it is therefore associated with the medium and short range parts of the nucleon-nucleon force which are not very well known at present. Experiments with polarized deuterons at high energies can therefore yield new information about the short and medium range parts of the nucleon-nucleon tensor force. At low energies, on the other hand, where the well known part of the nucleon-nucleon force contributes more to the generation of the $T_p$ tensor force, experiments with polarized deuterons can give information about the nuclear density at the nuclear surface.

It is presently believed that a mechanism, involving the nucleon-nucleon tensor force and the Pauli exclusion principle, similar to the one generating the $T_p$ tensor force, is responsible for binding nuclei at the observed densities. The spin structure of the $T_p$ tensor force can provide a way of separating out this mechanism, for a range of energies and densities, in experiments with polarized deuterons.
APPENDIX

The perturbed stationary state method (P.S.S.) and the Pauli correction term in d-nucleus scattering (by W.S. Pong).

The Hamiltonian describing the scattering between two composite particles is

\[ H = T(R) + H_A(\xi) + H_B(\eta) + V_{AB}(R, \xi, \eta) + U_c(R) \quad \text{(A.1.1)} \]

where \( H_A(\xi), H_B(\eta) \) are the internal Hamiltonians of particles A and B; \( \xi, \eta \) their internal coordinates respectively; \( U_c(R) \) the Coulomb interaction which is assumed to depend only on the relative coordinate \( R \); \( V_{AB}(R, \xi, \eta) \) their mutual interaction; and \( T(R) \) their relative kinetic energy.

The wavefunction, at each point \( R \), can be expanded in terms of the complete set

\[ \{ \phi_n(\xi, \eta; R') \delta(R - R') \}; \quad \text{\eta discrete or continuous} \quad \text{(A.1.2)} \]

with \( \phi_n \) satisfying the equation

\[ \{ H_A(\xi) + H_B(\eta) + V_{AB}(R', \xi, \eta) \} \phi_n(\xi, \eta; R') = E_n(R) \phi_n(\xi, \eta, R) \quad \text{(A.1.3)} \]

Here \( R' \) is treated as a parameter.

This expansion will fail if the interaction \( V_{AB}(R, \xi, \eta) \) is non-local.

\[ \psi(R, \xi, \eta) = \int \int dR' \phi_n(\xi, \eta; R') \delta(R - R') \phi_n(R') \]

\[ = \int \phi_n(\xi, \eta; R) \phi_n(R) \quad \text{(A.1.4)} \]
Substituting A.1.4 into the Schrödinger equation

\[(E - H) \psi = 0\]  \hspace{1cm} \text{A.1.5}

and projecting onto \(\phi_m(\xi_n;R')\delta(R - R')\) we get

\[
\left[ (E - E_m(R)) + \frac{\hbar^2}{2\mu} \nabla_R^2 - \left\{ (U_c)_{mn} - \frac{\hbar^2}{2\mu} C_{mm} \right\} \right] \phi_m(R) = \\
= \int \frac{\hbar^2}{2\mu} C_{mn} \phi_n(R) \hspace{1cm} \text{A.1.6}
\]

where

\[(U_c)_{mn} = U_c(R) = \langle \phi_m(\xi_n;R) | U_c | \phi_m(\xi_n;R) \rangle \text{ over } \xi_n, \text{A.1.7} \]

\[C_{mn}(R) = 2 \left[ \int d\xi_m \phi_m^*(\xi_n;R) \mathcal{V}_R \phi_n(\xi_n;R) \right] \nabla_R = \\
\hspace{1cm} + \int d\xi_m \phi_m^*(\xi_n;R) \nabla_R^2 \phi_n(\xi_n;R) \hspace{1cm} \text{A.1.8} \]

Defining \(\Delta E_m(R)\) by

\[\Delta E_m(R) = E_m(\infty) - E_m(-) \hspace{1cm} \text{A.1.9} \]

we can rewrite A.1.6 as

\[
\left\{ E - E_m(\infty) + \frac{\hbar^2}{2\mu} \nabla_R^2 - \left[ U_c - \frac{\hbar^2}{2\mu} C_{mn} - \Delta E_m(R) \right] \right\} \phi_m(R) = \\
= - \int \frac{\hbar^2}{2\mu} C_{mn} \phi_n(R) \hspace{1cm} \text{A.1.10} \]
The diagonal terms, $C_{mn}(R)$, describe the reaction of $\phi_n(r,n,R)$ to a change in $R$, and the off-diagonal terms, $C_{mn}(R)$ give the coupling of different states as $R$ changes.

In d-nucleus scattering, the hamiltonian can be written as

$$H_{d-N} = t_n + t_p + U_p\left[\frac{R - R'}{2}\right] + U_n\left[\frac{R + R'}{2}\right] + \hat{h}V_{np}(r) + U_c\left[\frac{R - R'}{2}\right]$$

A.1.11

where $\hat{h}$ projects out of occupied states.

The internal motion of the $n$-$p$ system is described by

$$\{t_p + \hat{h}V_{np}(r)\} \phi_n(r,R) = E_n(R)\phi_n(r,R)$$

A.1.12

It is impossible to find a solution to the above equation of the form $\phi_n(r,R)\delta(R - R')$ because $\theta$ is non-local in $R$.

$$\langle k, K_0 | \theta | k', K_0' \rangle = \theta\left(-k_F + \left[\frac{K_0}{2} - k\right]\right)\theta\left(-k_F + \left[\frac{K_0}{2} - k\right]\right)\delta(k - k')\delta(K_0 - K_0')$$

A.1.13

where $k_F = k_F(R)$ is the local Fermi momentum, and

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

A.1.14

The solution of equation A.1.12 must therefore depend on the centre of mass momentum, $K_0$, because $\theta$ depends strongly on $K_0$. No such momentum dependence is built into $\phi_n(r,R)\delta(R - R')$.

However a remedy can be found. We can solve A.1.12 using $iK_0'(R - R')$ as an ansatz, with $\phi_n(r,R',K_0)$ and $E_n(R',K_0)$ the deuteron wavefunction and eigenergy (binding energy) at the point $R'$,
obtained in chapter 6. Replacing $K_0$ by $-i\nabla_R$, we can regard
\[
\{\phi_n(r,R), -i\nabla_R\delta(R' - R)\}
\]
as the complete set at the point $R$. Substituting into A.1.4, we have
\[
\psi(R,r) = \sum_n \phi_n(r,R, -i\nabla_R)\phi_n(R)
\]
A.1.15

This makes sense only if
\[
-i\nabla_R\phi_n(R) = K_0(R)\phi_n(R)
\]
A.1.16

The function $K_0(R)$ can be identified as the local momentum. When
\[
\phi_n(r,R,K_0)
\]
is a singular function of $K_0$, or rapidly varying function of $K_0$ the expansion of the wavefunction in terms of \{\phi_n(r,R,K_0(R))\} is

useless, even if A.1.16 holds, because any conclusions drawn from such
an expansion are inaccurate. However this corresponds to a long-range
non-locality in $R$ and does not arise in the model under consideration here.

Substituting A.1.12, with $\phi_n(r,R)$ replaced by $\phi_n(r,R,K_0)$, into

the Schrödinger equation we get
\[
\left\{ (E - E_m(\infty)) + \frac{\hbar^2}{2\mu} \nabla^2_R - \left[ U_{mn}(R) + U_c(R) - \frac{\hbar^2}{2\mu} C_{mn} - \Delta E_m(R) \right] \right\}\phi_n(R)
\]
\[
= \sum_{n\neq m} \left[ U_{mn}(R) - \frac{\hbar^2}{2\mu} C_{mn}(R) \right] \phi_n(R)
\]
A.1.17

where
\[
U_{mn}(R) = \langle \phi_m(r,R,K_0) | U + U_p | \phi_n(r,R,K_0) \rangle_{\text{over } r}
\]
A.1.18

and $\phi_m(R)$ describes the centre of mass motion.
The coupling terms $C_{mn}(R)$ are

$$C_{mn}(R) = 2 \left[ \int dr \, \phi_m(R,r)^* \frac{\partial}{\partial r} \phi_n(R,r) \right] \phi_R + \int dr \, \phi_m(R,r)^* \frac{\partial}{\partial r} \phi_n(R,r)$$

A.1.19

and they arise from the dependence of $\phi$ on the centre of mass co-ordinates $R$. We restrain ourselves to excited states $m$ generated from the ground state only. The amplitudes of these states in first order perturbation are

$$\frac{\hbar^2}{2\mu} \frac{C_{m0}(R)}{E_m - E_0^0} \phi_0(R)$$

A.1.20

where $(E_m - E_0)$ is the energy difference between the two eigenstates. Taking only the first term in A.1.19 and applying the local W.K.B. approximation to $\phi_0(R)$, we get

$$\frac{\hbar^2}{2\mu} \frac{C_{m0}(R)}{E_m - E_0} \phi_0(R) \approx i \frac{\hbar^2}{2\mu} \left[ B_{m0} \cdot \chi_0(R) \right] \phi_0(R)$$

A.1.21

where

$$B_{m0} = 2 \int dr \, \phi_m(R,r)^* \frac{\partial}{\partial r} \phi_0(R,r)$$

A.1.22

We want to show that $\frac{\hbar^2}{2\mu} \frac{C_{00}(R)}{E_1 - E_0^0}^2$ gives an upper estimate of

$$\sum_{m \neq 0} \left( \frac{\hbar^2}{2\mu} \frac{C_{m0}}{E_m - E_0^0} \right)^2.$$  $E_0$ is the ground state energy and $E_1$, the energy of the first excited state. For the n-p system there is only one bound state, hence

$$E_1 - E_0 = 0 - E_0 = B$$

A.1.23
where $B$ is the binding energy of the $n$-$p$ system.

Now

$$ B_{m0} = \lim_{\Delta R \to 0} \left| \frac{d \phi_m^{\ast}(R, r)}{d \phi_0^{\ast}(R + \Delta R, r) - \phi_0^{\ast}(R, r)} \right| $$

where $\phi_0^{\ast}(R + \Delta R, r)$ and $\phi_0^{\ast}(R, r)$ satisfy the equations

$$ \{ t_r + Q(R + \Delta R) V_{np}(r) \} \phi_0^{\ast}(R + \Delta R, r) = E_0^{\ast}(R + \Delta R, r) \phi_0^{\ast}(R + \Delta R, r) \quad \text{A.1.24} $$

and

$$ \{ t_r + Q(R) V_{np}(r) \} \phi_0^{\ast}(R, r) = E_0^{\ast}(R) \phi_0^{\ast}(R, r) $$

Thus $\phi_0^{\ast}(R + \Delta R, r)$ can be obtained from \{ $\phi_0^{\ast}(R, r)$ $i = 0, -\infty$ $\} \}$ by perturbation, the perturbing potential being

$$ \Delta V = \{ \phi_0^{\ast}(R + \Delta R) - \phi_0^{\ast}(R) \} V_{np}(r) \quad \text{A.1.25} $$

To first order in perturbation

$$ 2 \int d R \phi_m^{\ast}(R, r) [\phi_0^{\ast}(R + \Delta R, r) - \phi_0^{\ast}(R, r)] = \frac{2(\Delta V)_{m0}}{E_0^{\ast}(R) - E_m^{\ast}(R)} \quad \text{A.1.26} $$

and to second order in perturbation,

$$ 2 \int d R \phi_0^{\ast}(R, r) [\phi_0^{\ast}(R + \Delta R, r) - \phi_0^{\ast}(R, r)] = - \sum_{m \neq 0} \left[ \frac{(\Delta V)_{m0}}{E_0^{\ast}(R) - E_m^{\ast}(R)} \right]^2 $$

where

$$ (\Delta V)_{m0} = \langle \phi_m^{\ast}(R, r) | \Delta V | \phi_0^{\ast}(R, r) \rangle \quad \text{over } r \quad \text{A.1.27} $$

$$ (\Delta V)_{m0} = \langle \phi_m^{\ast}(R, r) | \Delta V | \phi_0^{\ast}(R, r) \rangle \quad \text{A.1.28} $$
Thus
\[ B_{m^0} = \lim_{|\Delta R| \to 0} \frac{2(\Delta V)_{m^0}}{E_0(R) - E_m(R)} \quad A.1.29 \]

\[ B_{0^0} = \lim_{|\Delta R| \to 0} \left( \frac{(\Delta V)_{m^0}}{E_0(R) - E_m(R)} \right)^2 \quad A.1.30 \]

Thus
\[ B_{0^0} \cdot \Delta R = \frac{1}{4} \sum_{m \neq 0} |B_{m^0} \cdot \Delta R|^2 \quad A.1.31 \]

Now
\[ \frac{\hbar^2}{2\mu} C_{00}(R) \approx \frac{\hbar^2}{2\mu} B_{00}(R) \cdot k(R) = \frac{\hbar^2}{2} B_{00}(R) \cdot \nu d \quad A.1.32 \]

where \( \nu d \) is the velocity of the deuteron. Using A.1.31 and assuming that the deuteron travels a distance \( \Delta R \) in a time \( \Delta t \)

\[ \frac{\hbar^2}{2\mu} C_{00}(R) = -\frac{\hbar}{2} \sum_{m} |B_{m^0} \cdot R|^2 \cdot \frac{1}{\Delta t} \quad A.1.33 \]

Hence
\[ \frac{\hbar^2}{2\mu} C_{00}(R) = -\frac{\hbar}{2} \sum_{m} |B_{m^0} \cdot \nu d|^2 \cdot \Delta t \]

\[ = -\frac{\hbar^2}{8} \sum_{m} \left| C_{m^0}(R) \right|^2 \cdot \Delta t \]

\[ = -\frac{1}{2\hbar} \sum_{m} \frac{\hbar^2}{2\mu} C_{m^0}(R)^2 \cdot \Delta t \quad A.1.34 \]

A.1.34 is accurate the smaller \( \Delta t \) is. However \( \Delta t \) cannot be chosen to be less than

\[ \frac{\hbar}{E_1 - \nu} = \frac{\hbar}{E} \quad A.1.35 \]
otherwise one would not be sure that the deuteron is in its ground state. Thus

\[ \frac{\hbar^2}{2\mu} C_{00}^{(R)} > \frac{i}{2} \sum_m \left| \frac{\hbar^2}{2\mu} C_{m0}^{(R)} \right|^2 \cdot \frac{1}{B} \]

Hence

\[ \left| \frac{\hbar^2}{2\mu} C_{00}^{(R)} \right| > \frac{1}{2} \sum_m \left| \frac{\hbar^2}{2\mu} C_{m0}^{(R)} \right|^2 \frac{1}{B^2} \]

\[ > \frac{1}{2} \sum_m \left| \frac{\hbar}{2\mu} C_{m0}^{(R)} \right|^2 \frac{1}{(E_m - E_0)^2} \quad \text{A.1.36} \]
REFERENCES

1) S. Watanabe, Nucl. Phys. 8, (1958) 484.
   B.L. Gambhir and J.J. Griffin, in "Qualitative Features of Pauli Break-up from Configuration Space Viewpoint", University of Maryland. Reprint.
34) A. De Shalit and H. Feshbach, Theoretical Nuclear Physics (John Wiley & Sons, 1974) Vol.1, Nuclear Structure, Chapters II and III.
36) J.P. Swenne and S.F.J. Wilks, Proceedings of the 4th Int. Symp. on Polarization Phenomena in Nuclear Reactions, Zurich, Switzerland, August 1975; and references therein.
37) A. Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1962) Vol. II, Ch. XIII.
45) A. De Shalit and H. Feshbach, Theoretical Nuclear Physics (John Wiley & Sons, 1974), Vol. I, Nuclear Structure, Chapters IV to VII.


50) W.S. Pong, Private Communication.

51) T.Y. Wu and T. Ohmuna, (1963), The Quantum Theory of Scattering, pages 228-231.


53) R.C. Johnson, Private Communication.

54) A. Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1962), Vol. II Ch. XVII.


56) A clear discussion for the definitions of the irreducible tensor operators \( T_{kq} \) and a picturesque description for the efficiency tensors can be found in S.E. Darden's paper, in Pol. phen. in Nucl. Reac., ed. H.H. Barschall and W. Haeberli (Univ. of Wisconsin, Madison, 1971) p.39.


58) A. Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1962), Vol. II, Appendix C.

59) A. Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1962), Vol. II, Chapter XIX.


61) A. Messiah, Quantum Mechanics (North-Holland, Amsterdam, 1962), Vol. I, Appendix B.

63) F.D. Santos to be published.
    and Polarization Phenomena in Nuclear Reactions, ed. H.H.
    Barschall and W. Haeberli (Univ. of Wisconsin Press, Madison, 1971).
The Pauli Principle and the Spin-Dependence of the Deuteron-Nucleus Interaction. R. C. Johnson, U. of Surrey, U.K., and U. of Maryland; and A. A. Ioanides, U. of Surrey, U.K.—It is shown that the effect of the Fermi sea on the binding energy of a deuteron in nuclear matter depends strongly on the relative orientation of the momentum and spin of the deuteron. This effect can be interpreted as a contribution to the deuteron optical potential of the type

\[(S \cdot P)^2 - \frac{2}{3} P^2\]

where \(S\) and \(P\) are the deuteron spin and momentum operators respectively. Calculations of the strength of this interaction and its influence on polarization effects in elastic deuteron scattering will be presented.

*Supported by U.S. Energy Research and Development Adm.
Spin dependence of the deuteron optical potential, the deuteron D-state and the Pauli exclusion principle.

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The effect of the Pauli exclusion principle on the binding energy of a deuteron propagating in nuclear matter has been the subject of renewed interest recently \(^1,^2,^3\). In the work presented here these calculations have been extended to include the influence of a tensor force component of the neutron-proton interaction of the Yamaguchi type \(^4\). A consequence of the presence of D-state components in the deuteron is that for a given Fermi momentum and deuteron centre-of-mass momentum, \(k\), the deuteron binding energy is found to depend strongly on \((s.k)^2\), where \(s\) is the deuteron spin operator. The effect of the Fermi sea is to produce less binding on a deuteron with \(s.k = 0\) than with \(s.k = \pm 1\). This can be understood in a simple semi-classical picture in which account is taken of the orbital contribution to \(s\) from the deuteron D-state.

The quantitative results shown in Fig. 1 have been obtained by assuming that at each value, \(R\), of the radial distance of the deuteron centre-of-mass from the centre of the nucleus the deuteron has a definite centre-of-mass momentum, \(k\), determined by the local value of the Coulomb energy and the nucleon optical potentials \(^5\). The binding energies at the Fermi momentum corresponding to the local density (obtained from the point charge distribution \(^6\)) is then determined by solving a Bethe-Goldstone type of equation \(^2\). The binding energy difference \((\epsilon_{+1} - \epsilon_0)\), where the suffixes refer to the value of \((s.k)\), for a series of incident deuteron energies \(E_d\) are shown in Fig. 1 for the target \(^{56}\)Fe. For values of \(R\) at which no binding energy is shown, the state with \((s,k) = 0\) is unbound.

The difference \((\epsilon_{+1} - \epsilon_0)\) can be interpreted as a tensor term in the deuteron optical potential of the \((s.k)^2\) type \(^7\). It should be noted that the potentials of Fig. 1 have a similar magnitude to the tensor force of the \((s.R)^2\) type predicted by the Watanabe model and which gives an excellent account of all 3 tensor asymmetries observed in low energy deuteron elastic scattering \(^8\). The new tensor force discussed here may be relevant to suggestions that the Watanabe model does not give correct spin-dependence at higher energies \(^9,^10\).

This work was supported by the S.R.C.

References
4. Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. 95, 1635 (1954)
6. R. Engfer, et al., to be published
Figure 1. Spin dependence of deuteron binding energy $\varepsilon$ as a function of position for $^{56}$Fe for the indicated incident deuteron energies $E_d$. 
THE PAULI PRINCIPLE AND THE SPIN-DEPENDENCE OF THE DEUTERON-NUCLEUS INTERACTION

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Received 4 December 1975

It is shown that the effect of the Pauli principle on the propagation of a deuteron in nuclear matter depends strongly on the relative orientation of the momentum and spin of the deuteron. Possible implications for experiments with polarized deuterons are discussed.

The spin-dependence of the deuteron-nucleus interaction has become the subject of considerable interest as the result of the development of elastic scattering experiments in which the five observables, $\sigma(0)$ (the cross-section for unpolarized deuterons), $T_{11}$, $T_{20}$, $T_{21}$, $T_{22}$ (the analysing powers for polarized deuteron) are all measured over a considerable angular range and for a variety of targets and energies [1–4]. The analysis of these data in terms of an optical potential usually assumes that in addition to a spin-orbit force of the $L\cdot S$ type, tensor forces proportional to the operators $T_R$ and $T_L$ are also present. Here

$$T_R = (S\cdot R)^2 - \frac{3}{5},$$

$$T_L = (L\cdot S)^2 + \frac{1}{2} L\cdot S - \frac{2}{5} L^2,$$

where $R$ is the deuteron-nucleus separation and $S$ is the deuteron spin operator. Forces of this type were first proposed on general ground by Satchler [5] who also showed how a $T_R$ potential is generated by the D-state of the deuteron. A small $T_L$ potential has been predicted by Stamp [6].

The purpose of the present letter is to point out that a potential of the type

$$V_P(R) T_F,$$

where

$$T_F = (S\cdot P)^2 - \frac{3}{8} P^2,$$

and $P$ is the deuteron centre-of-mass momentum operator conjugate to $R$, is also expected when account is taken of the Pauli exclusion principle on the propagation of a non-spherical deuteron through nuclear matter. It is estimated here that the tensor potential generated in this way has a significant magnitude over a wide range of incident deuteron energies.

The physical basis for this effect can be understood from the consideration of a model in which a deuteron of centre-of-mass momentum $K_0$ is introduced into nuclear matter, assumed to be adequately described by a Fermi gas model of Fermi momentum $K_F$. It is well known that the presence of the Fermi sea has a strong effect on the deuteron binding energy (see, e.g. the recent calculations in [7]). In addition deuteron states corresponding to different eigenvalues of $(S\cdot \hat{K}_d)^2$ are no longer degenerate, i.e. in the presence of nuclear matter there are in general two $1^+$ spin triplet states of the n-p system corresponding to $S\cdot \hat{K}_d = \pm 1$ and $S\cdot \hat{K}_d = 0$. This energy splitting arises because of the tensor force in the n-p interaction, a consequence of which is that the deuteron spin $S$ has a contribution from orbital relative motion of the n-p system. A classical argument shows that a state with $S\cdot \hat{K}_d = \pm 1$ is expected to be more strongly bound than a state with $S\cdot \hat{K}_d = 0$ if $S$ arises entirely from orbital motion. In the former case the Pauli principle will be satisfied by n-p relative momenta such that $(\frac{1}{2} K_0^2 + k^2)^{1/2} > k_F$. The same values of $k$ may violate $|k| > k_F$ in the case $S\cdot \hat{K}_d = 0$ where possible momenta $k$ have a component parallel to $K_d$. In the real deuteron a coupling of the required type between $S$ and the plane of the n-p orbit is produced by the D-state component.

These simple considerations are borne out by a quantum calculation. The binding energy of a deuteron in nuclear matter as a function of $K_d$ and $K_F$ is obtained by solving a Bethe–Goldstone equation of the type used in [7] generalized to include the tensor n-p interaction given by Yamaguchi and Yamaguchi [8],

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Fig. 1. Deuteron binding energy splitting ($e_{\pm 1} - e_0$) as a function of position, $R$, and incident deuteron energy, $E_d$, for Zr, which generates the experimental deuteron quadrupole moment and free space binding energy and corresponds to a D-state probability of 4%.

The solution of the Bethe–Goldstone equation depends on $K_d, M(=S \cdot \vec{K}_d)$, and $k_F$, and when a bound state solution exists is given in momentum space by

$$\phi_M(K_d, k_F, p) = \frac{\theta(K_d, p) N}{(\Delta^2(M, K_d, k_F) + p^2)} \times \left( \frac{1}{(\beta^2 + p^2)} - \frac{rp^2}{(\beta^2 + p^2)^2} \right)^{1/2} \chi^M,$$

where the force parameters $\beta, \gamma,$ and $t$ are defined in [8], $\chi^M$ is a triplet spin wavefunction $S_{12}(p)$ is the usual tensor force operator, $N$ is a normalization constant, and $\theta$ is the Pauli operator given by

$$\theta(K_d, p) = 0, \quad |\frac{1}{2} K_d \pm p| < k_F,$$

$$= 1, \quad |\frac{1}{2} K_d \pm p| > k_F.$$

The deuteron binding energy is

$$\epsilon_M = \frac{\hbar^2}{2 \mu} \Delta^2(M, K_d, k_F),$$

where $\mu$ is the n-p reduced mass.

For present purposes it is most useful to display the dependence of $\epsilon_M$ on the variables $K_d, k_F$ in terms of a dependence on $R$, the distance of the deuteron from the centre of finite nucleus. For an incident deuteron energy $E_d$ this can be achieved by endowing $k_f$ and $K_d$ with an $R$ dependence through the relations

$$E_d = \frac{\hbar^2 K_d^2}{2M_d} + V_p(R) + V_n(R),$$

$$k_F(R) = \left( \frac{3\pi^2}{2} \rho(R) \right)^{1/3},$$

where $V_p$ and $V_n$ are nucleon optical potentials, including the Coulomb field due to a uniformly charged spherical nucleus for the proton, and $\rho(R)$ is the nuclear matter density at $R$. For the latter, published [9] point charge densities for the nucleus considered have been used. The potential $V_n$ and the strong interaction part of $V_p$ are taken from eq. (13) of ref. [10] evaluated at $\pm E_d$ [11]. The spin-orbit and imaginary parts of $V_n$ and $V_p$ are neglected. Some of the results quoted below correspond to nucleon energies outside the range considered in [10] but errors introduced in this way are not expected to alter qualitatively the nature of the Pauli principle effect of interest here.

Full details of the dependence of $\epsilon_M$ and the internal structure of the deuteron on $R$, mass number and energy will be published elsewhere. Here emphasis will be placed on the properties of the binding energy difference

$$e_{\pm 1} - e_0.$$  

(10)

Because $M$ is the eigenvalue of $(S \cdot \vec{K}_d)$, this difference can be interpreted as giving information about the strength and radial dependence of $V_f(R)/K_3(R)$ of eqs. (3) and (4). (Note that $T_p((S \cdot \vec{P})^2 = 1) - T_p((S \cdot \vec{P})^2 = 0) = p^2$).

Calculations of this difference have been confined to the case where both the $M = \pm 1$ and $M = 0$ states are bound. In agreement with the classical picture, binding in the $M = 0$ state always goes to zero faster than in the $M = \pm 1$ case as the density increases. Some features of the results are shown in fig. 1 for the nucleus Zr. At energies above about $E_d = 100$ MeV the radial dependence roughly follows the density. At lower energies there is some evidence for an additional surface peaking effect. The maximum value of $(e_{\pm 1} - e_0)$ increases with energy up to about $E_d = 150$ MeV and then decreases to zero as the binding energy approaches the free space value $\epsilon_f = 2.23$ MeV, for both values of $M^2$. 

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The more detailed energy dependence can be understood as follows. At large $R$ and small density, $\frac{1}{2}K_d \gg k_F$, and $\epsilon_{e1}$ is very close to $\epsilon_f$ as the classical picture suggests. However $\epsilon_0$ decreases rapidly with decreasing energy with the result that $d(\epsilon_{e1} - \epsilon_0)/dE_d < 0$. Inside the nuclear surface $\frac{1}{2}K_d \approx k_F$ for $E_d < 100$ MeV and $\epsilon_0$ has become very small compared with $\epsilon_{e1}$. The main energy dependence of $(\epsilon_{e1} - \epsilon_0)$ therefore comes from $\epsilon_{e1}$ i.e. $d(\epsilon_{e1} - \epsilon_0)/dE_d > 0$. Well above $E_d \approx 150$ MeV $\frac{1}{2}K_d \gg k_F$ for all $R$ and $(\epsilon_{e1} - \epsilon_0)$ becomes very small because both $\epsilon_{e1}$ and $\epsilon_0$ approach $\epsilon_f$.

These results are relevant to the description of the tensor analysing powers mentioned at the beginning. It is interesting to compare the potential energies shown in fig. 1 with the $T_R$ potential

$$V(R) \left[ (S \cdot \tilde{R})^2 - \frac{3}{2} \right],$$

(11)
given in [2]. According to [1] and [2], $V(R)$, which measures the potential energy difference between a deuteron with $(S \cdot \tilde{R})^2 = 1$ and $(S \cdot R)^2 = 0$, has the radial shape of the 2nd derivative of the real part of the nucleon optical potential and with a peak value similar to the energies shown in fig. 1. The $T_P$ potential introduced here will produce an angular and energy dependence for polarization quantities quite different from the $T_R$ potential but the simple comparison of strengths does suggest that the $T_P$ potential cannot be ignored. Detailed calculations of the polarization using a coupled channels formalism are planned.

Finally it should be noted that the effects shown here are not determined simply by the deuteron free space quadrupole moment, $Q$. When the parameters of the Yamaguchi potential are altered keeping the deuteron free space binding energy and $Q$ fixed the magnitude of the difference $(\epsilon_{e1} - \epsilon_0)$ changes in a non-negligible fashion [12]. Thus in principle the measurement of this effect will yield complementary information to the parameter $D_2$ [14] obtained from $(d, p)$ measurement at low energies [15].

An alternative derivation of the spin-dependent effect discussed here is given by Austern in the following article [17].

The authors wish to thank Mrs. J. Hilton for computing assistance and A.I. acknowledges receipt of a University of Surrey studentship.

References