MATHEMATICS AT THE SCHOOL - UNIVERSITY INTERFACE,

WITH SPECIAL REFERENCE TO THE NEEDS

OF ENGINEERING STUDENTS

by

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SUMMARY

There is at present a national concern about basic mathematics, particularly at school leavers' level. In this connection, this thesis is an attempt to cope with the problem of the mathematical competence of engineering students at the interface between school and university. Engineering students were given individual diagnosis by means of a test and tutorial discussion and by means of an approach through thinking aloud combined with observation and retrospective interview.

A common characteristic of the two approaches to individual diagnosis is that they require the presence of a tutor for the diagnosis. In the belief that students can diagnose themselves, if they are provided with the appropriate tools, a battery of mathematics diagnostic tests was developed. The items in this battery were based on the topics and level of difficulty mentioned by lecturers in a survey carried out with the purpose of finding out the mathematics needed in first year engineering courses at university, particularly during the first term.

As the battery of tests was being developed, the mathematical needs of sixth form school leavers and of newly enrolled engineering students were identified. It was found that both groups of students had difficulty with those topics which involved the use of formulae, and that the knowledge of concepts and numerical values tended to be
forgotten more rapidly than other knowledge.

Following this diagnosis, materials in algebra and trigonometry were developed for revision purposes, based on students' actual needs. These employed the principles of programmed learning, which were found to be useful and to have certain advantages over traditional textbooks.
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SUMMARY

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CHAPTER ONE

PRESENTATION OF THE PROBLEM AND METHODOLOGY
1.1 THE IDENTIFICATION OF THE PROBLEM

There is at present a national concern with standards in basic mathematics, particularly at school leavers' level. One of the problems many students encounter when they come from school to university is that they do not have some of the basic mathematical knowledge and skills which are needed to facilitate the effective learning of their first year courses. This is particularly so in science and engineering courses where mathematics is an essential tool to the understanding of the basic techniques of the discipline involved. Among the reasons for this situation are:

1. Ability of students,
2. Omission of topics at secondary level,
3. Forgetfulness of what has been learned.

In addition to these, there may be students who are familiar with the basic mathematics they have studied at secondary school, but who do not possess the comprehension to the extent commonly assumed for their first year courses. The fact that a lecturer attempts to teach something to his class carries with it the assumption that the students already have the necessary prerequisite knowledge and skills to accomplish the learning task. This would appear to be in line with the theory of hierarchical learning of Gagné (1962) which states that lower level needs must be mastered prior to higher level needs. If we compared this situation with the traditional scheme of the process of curriculum development (see fig. 1.1), it would appear that the starting point for this lecturer is the selection of objectives. But if there happen to be students who do not have the prerequisite knowledge and skills, the result may be that they will be delayed in understanding the new material whilst they revise what is assumed to be known. This was found by Kilty (1975) in his study of the problems involved in the change of
This situation had already been predicted by Gagné (1970) when he pointed out:

"The basic functional unit of a learning hierarchy consists of a pair of intellectual skills, one subordinate to the other ... the theoretically predicted consequence of a subordinate skill that has been previously mastered is that it will facilitate the learning of the higher-level skill to which it is related. In contrast, if the subordinate skill has not been previously mastered, there will be no facilitation of the learning of the higher-level".

(p.239).

The hierarchial theory of Gagné has given rise to a theory of diagnosis and remedial instruction which mainly involves the use of pretests designed to test students' knowledge of the various items in the hierarchy, and remedial instruction for those who do not have this knowledge, thus they overcome their difficulties (Orlosky and Smith 1978). This would appear to suggest that a new stage should be included in the process of curriculum. The revised scheme might then be seen as in Fig. 1.2.
As far as the lecturer is concerned, two questions emerge from this revised scheme: (a) What kind of mathematics performance is required to deal with a new particular course? (b) Will students have this mathematics performance at the level required by the course before they actually start it?

These questions are usually answered by listing those topics which are needed to deal with the new course and assuming that students have mastery of these topics, but when one speaks of mathematics in terms of topics, without a clear definition of the performance to be observed in each one, there is always the possibility that different people interpret the same topics in different ways. Psychologists working on mathematics learning point out that:

"To find out what kinds of mathematics performances are needed, find out what people do with these performances - that is, what more general aims they serve. Does the individual use the mathematics performances he has in order to learn certain specified kinds of advanced mathematics? If so, the particular performances required for such advanced learning are the ones he needs."
the person use the mathematics performances to solve problems
concerning the normality of chemical solutions? If so,
the particular performances required for the chemical problems
are the ones he needs. Does the individual use his performances
in mathematics to compute the resistances in electric circuits?
If so, that particular set of performances is the one he needs."

(Gagné 1972, p.169).

These views have originated the need for studying the process of
Diagnosis and Remedial instruction in mathematics for engineering students.

1.2 PURPOSE AND SCOPE OF THE RESEARCH

Based on the theory of diagnosis and remedial instruction originated as
a consequence of the theory of hierarchichal learning, I have suggested
in the previous section a revised scheme of the process of curriculum
which includes a stage of Diagnosis and Remedial instruction. This
stage is made up of (a) prerequisites for a new course, which involve
the level of performance required, (b) the means to ascertain whether
the students have these prerequisites, and (c) the remedial work suggested
accordingly. This stage is diagrammatically represented in Fig. 1.3.
The students' actual needs, besides being the basis for the suggestion
of appropriate remedial work, are also the basis for the construction of
new materials for subsequent groups of students. This scheme originated
the aims of this research. These were to:

1. Identify those topics in mathematics which are needed by first
year engineering students in their first year courses, particularly
during their first term at university, and the level at which these
topics are needed to facilitate the effective learning of the
material in the courses.

2. Develop approaches to give individual diagnosis in mathematics.
3. Find out to what extent engineering university entrants and secondary school leavers have mastery of the topics mentioned in aim 1.

4. Study schemes of revision work in mathematics for students entering engineering courses.

5. Develop revision materials in mathematics for the scheme mentioned in aim 4.

Fig. 1.4 shows a diagram of the different activities undertaken for the attainment of these aims and the interconnections among them.

The people who participated in the experiences described in this thesis were of three kinds, namely engineering undergraduates who were in their first term at university, sixth formers who were doing their second year of A-level mathematics, and staff lecturing first year engineering courses at universities and polytechnics. I should note here that rather than follow the engineering students involved throughout their course,
The key to this diagram is the following:

**ID**: Individual diagnosis  
**SRW**: Specific Remedial Work  
**CMRP**: Construction of Materials for Revision Purposes  
**PSSU**: Pilot Survey at Surrey University  
**PSDU**: Pilot Survey at Durham University  
**TFD**: Test for Diagnosis  
**TT**: Try out of Test  
**COR**: Construction of the test of Open-ended short Responses  
**MS**: Main Survey  
**CBDT**: Construction of Battery of Diagnostic Tests  
**RBDT**: Revision of the Battery of Diagnostic Tests  
**CST**: Construction of a Single Test
I investigated the situation at the point of entry (including their first term at university) on the basis of any information that could be obtained then.

1.3 OUTLINE OF THE THESIS

A review of the literature on general aspects of the mathematical education of non-specialists is presented in Chapter 2. This chapter also presents specific aspects of the mathematical education at the transition between school and university.

Due to the characteristics and interconnections of the different activities of this research, the main body of the thesis is based on the diagram in fig. 1.3, that is, Chapter 3 is concerned with the survey carried out to find out the mathematical performances prerequisite for engineering courses. Chapters 4 and 5 are concerned with the means employed to ascertain whether the students involved had these prerequisites - Chapter 4 deals with a battery of diagnostic tests developed during the research, and Chapter 5 deals with individual diagnosis in mathematics by means of a test and tutorial discussion (the T.T. method) and by means of the thinking-aloud approach in combination with observation and retrospective interview (the T.A.O.I. method). Chapter 5 is also concerned with the study of schemes of revision work carried out by the students involved in the T.T. and T.A.O.I. methods.

Chapter 6 presents the mathematical needs of students at the interface school-university. These needs were identified by means of the different versions of the mathematics test developed during the three years of the research, that is, each group of students was given a different version of the test being developed. In this chapter (Chapter 6) I treat each version of the test independently from the others. The way I constructed one version from the previous one is treated in depth in Chapter 4.
Chapter 7 deals with the development of the materials for revision purposes. The format of these materials were different from the format of traditional textbooks.

Chapter 8, the last, is concerned with the general conclusions of the research and with recommendations for further work. The rest of the present chapter deals with the methodology employed in the conduct of this research.

1.4 METHODOLOGY

One of the characteristics of educational research that includes developmental activities is that there may exist an overlap between method and aim of the research in some area (Johnson 1977). This is so for some of the aims and methods of the present research. The data gathered by means of a mathematics diagnostic test fulfilled two aims, namely it enabled me to get information about the students' mathematical needs at the same time that it enabled me to develop the diagnostic test. Since one of the aims of this research was to develop a battery of mathematics diagnostic tests for students entering engineering courses, the description of the process involved in its development represents part of the research and not of the methodology employed in it; therefore, this process is treated in the chapter concerned with the development of the battery of tests (i.e. Chapter 4). The same phenomenon occurs in the development of the T.A.O.I. method for individual diagnosis (i.e. the thinking aloud approach combined with observation and interview) where observation and interview are parts of the method. The corresponding process of this method is treated in depth in Chapter 5.
In order to achieve the aims of this research, I used different methods: Questionnaires, Interviews, Observation and Documentary Sources. What follows is a description of each of them.

1.4.1 Questionnaires

I used questionnaires in this research for the following purposes:

(a) To gather lecturers' opinions about the mathematics prerequisite for engineering courses (the SV questionnaire).

(b) To gather lecturers' opinions about the contents of the booklet on Algebraic Manipulation that I should construct (the AM questionnaire).

(c) To gather students' opinions about the scheme of diagnosis and revision work in which they took part (the DRW questionnaire).

(d) To gather students' comments about the booklets specially written for revision purposes (the T questionnaire and the A questionnaire).

1.4.1.1 The SV-questionnaire

The knowledge of what mathematics a lecturer needs for his course can be obtained by simply asking him about the mathematics he thinks he needs for his course. This is what is usually done in practice. In this research, I was concerned with what mathematics a lecturer actually needed at the beginning of the engineering course that he himself was giving to first year engineering students in his department and the level at which this mathematics would be used rather than with the mathematics
the lecturer thought he would use.

With this idea in mind, I designed a questionnaire in which I asked the lecturer to state precisely the topics of his course in which he needed mathematics and to give an example, taken from his own course, which showed how this mathematics was used.

The questionnaire (see Appendix A) consisted of two columns - column A: 'Engineering Topics', and column B: 'Example of maths. needed'. In column A there was a list of nineteen mathematical topics which were derived from a preknowledge test given to students at the University of Southampton (see section 3.2. ) The respondent was asked to list in the space provided under each topic those engineering topics in which the mathematical topic was needed, if it was needed, and then give in column B an example of a specific situation showing how the mathematical topic was used in his own engineering course. At the end of the questionnaire the respondents were also invited to add topics if they wished to do so. Two main issues emerged in the pilot study of the questionnaire, namely (1) some lecturers found filling in the questionnaire hard and time-consuming because they had to go through all topics of their own courses in order to look for any bit of mathematics, and (2) the questionnaire enabled me to detect the level of mathematics performance needed in each topic (i.e. column B) (see section 3.2.3.)

Since any busy lecturer could develop a negative attitude towards the questionnaire because it was 'time-consuming', I decided to change the format in such a manner that I could gather the same information I was interested in, and that the respondent spent less time filling it in. With this idea in mind, I decided not to ask the respondent to list
the engineering topics in which he needed each of the mathematical topics but to draw his attention to the level of difficulty of the mathematics he needed. The final version of the questionnaire (see Appendix B) consisted of four columns - column A: 'Mathematical topics', column B: 'Example of the level of difficulty', column C: 'Term in which the topic is needed', and column D: 'Example of the level of difficulty expected'.

Column A was very similar to the one in the previous version of the questionnaire and it included the additional topics mentioned in the pilot survey (a list of these topics is given in section 3.2.3.). Column B consisted of examples of the level of difficulty of each topic listed in column A (these examples were given by respondents in the pilot survey and are summarised in Tables 3.1 and 3.2). Column C was introduced in order to determine the term in which the respondents needed each topic, i.e. Autumn term, Spring term or Summer term. Finally in column D the respondents were asked to give examples of the level of difficulty they expected in their courses. The fact of having given in column B an example of each topic in column A facilitated the respondents to answer the questionnaire because he knew what was expected of him. The answers in column C enabled me to know exactly what topics they particularly needed in the first term of their courses, and column D gave me information about the level of difficulty of each topic.

The main feature of this questionnaire is column D because it allows one to collect information which is rarely sought in studies concerned with prerequisite knowledge and skills. The details of how the questionnaire was actually developed are given in Chapter 3.
1.4.1.2 The AM-questionnaire

In order to identify the topics I should include in the booklet on routine algebraic manipulation (see section 7.1), I designed a questionnaire to collect this information from first year engineering lecturers.

The questionnaire (see Appendix C) consisted of a list of basic routine algebraic manipulation covering eight topics that students normally study at secondary school (see section 7.3.1.2). Most of these topics were illustrated with examples in order to give the respondent an idea of what they meant. Respondents were asked to tick the appropriate boxes of those topics they thought students should know for their courses, but sometimes do not, and to add topics to the list if they wished to do so. The fact that most of the respondents were kind enough to add new topics to the list led me to design a complementary questionnaire with those topics added by one or two respondents. This questionnaire only had four topics and should be completed as the former questionnaire.

1.4.1.3 The DRW-questionnaire

This questionnaire was given to first year engineering students at the University of Southampton in 1976 and 1977 after they had completed the revision work in mathematics they carried out during their first week at University.

(a) The 1976 version

Basically the questionnaire (see Appendix D) consisted of open-ended questions which aimed to gather students' opinions about the following
aspects of the testing and revision work scheme:

1. Difficulty of the preknowledge test.

2. Material forgotten or new to students.

3. The testing-discussion session.

4. The revision material.

The questionnaire had also an open-ended question which asked the students for suggestions and recommendations for the improvement of the revision material they used. Questions regarding aspects 1 and 2 above had a list of mathematical topics derived from the mathematics pre-knowledge test. I derived this list because I was interested in knowing what specific topics students had found difficult in the pre-knowledge test, what specific topics they had forgotten and what specific topics they had not met at school.

To fill in the questionnaire, the students had only to tick the boxes appropriate to their answers; however, in those questions related to the testing-discussion session space was provided for students to give comments if they wished to do so.

(b) The 1977 version

This questionnaire had the same questions as in the 1976 version and also questions which aimed to gather students' opinions about the tests they were given at the end of the revision work, and about the whole scheme of diagnosis and revision work (see Appendix E).
1.4.1.4 The T-questionnaire and the A-questionnaire

These questionnaires were given to students who used the booklets on Trigonometry and Algebraic Manipulation during the week of revision work in mathematics at the University of Southampton in 1977. Basically, the questionnaires (see Appendices F and G) consisted of closed-ended questions which aimed to get feedback from students in order to improve the booklets. Each questionnaire mainly focused the following aspects:

(a) Usefulness of the booklet.
(b) Understanding of the instructions.
(c) Items failed in the pre-test included in the booklet.
(d) Comments about wording, presentation of the material, advantages, disadvantages.

Aspect (c) was included in the questionnaire because it would enable me to know what specific parts of the booklet each student had consulted.

1.4.2 Interviews

I used unstructured interviews in four facets of this research for different purposes. These are described below.

1.4.2.1 The survey

I interviewed respondents in the pilot survey of the questionnaire in order to (1) discuss with them my own interpretation of the data collected in the pilot survey, and (2) gather their reactions to the format of the questionnaire. This set of interviews enabled me to validate the data gathered in the pilot survey of the questionnaire and
1.4.2.2 The revision work

The main purpose of this set of interviews was to gather from students information related to the scheme of testing and revision work in order to design a questionnaire. Such interviews were held in three opportunities, namely (1) immediately after students had completed the pre-knowledge test, (2) immediately after the discussion of the test, and (3) when students were doing their revision work in the Reading Room. Tutors were also interviewed after the discussion of the test in order to gather their impressions about the testing scheme.

1.4.2.3 Development of materials

During the process of development of the booklets on Trigonometry and Algebraic Manipulation (see Section 7.1), I interviewed lecturers and students for the following purposes:

Lecturers: To discuss with them: (1) the outline of the contents and the objectives of the booklets, and (2) their reactions to the first draft of the two booklets.

Students: To gather their comments about the booklet on Trigonometry during the trial stage of the first draft.

1.4.2.4 Development of the test

I held individual interviews with six of the students who had taken part in the exercise of individual diagnosis in mathematics described in Section 5.3. The purpose of these interviews was to explore:
(a) what mathematics these students had actually used in their engineering courses during their first term at university, and (b) what changes they would like to see in the test used in the exercise.

1.4.3 Observation

In this study observation was used in the scheme of testing and revision work carried out by students at the University of Southampton. It was used to observe: (a) the discussion session of the test and (b) how students did their revision work.

1.4.4 Documentary Sources

In this study documentary sources refer specifically to: (a) A-level mathematics syllabuses of the Joint Matriculation Board (1976), The Associated Examining Board (1976), The University of London Examining Board (1978), and the Oxford and Cambridge School Examination Board (1978); and (b) Lecture notes of first year electrical engineering courses at the University of Surrey.

The former are kept by the corresponding Examining Board and the latter by the Surrey University Library.

1.5 Conclusions

I should like to conclude this chapter by adding some comments about research which involves development, like the one presented in this thesis.

The term research and development is encountered frequently in studies
concerned with educational innovation in curriculum (Harris et al 1975). The term is rather vague and can be defined in various ways. According to Havelock (1971) in Research, Development and Diffusion Model of Curriculum Development:

"R & D can be a process whereby ideas and tentative models of innovations are evaluated and systematically reshaped and packaged in a form that ensures benefit to users ..." (p.85).

In general, research and development involves the researcher in studying the problem over a period of time. It involves both research and developmental activities and attempts to link research findings with educational practice (Johnson 1977). Developmental activities may be directed to preparation of tests, programmes, etc., used in instruction and in the research activities. These research activities lead to further improvements of the tests and programmes which in turn could lead to more research. In the research presented in this thesis the development of a mathematics diagnostic test was used for determining students' needs at the transition between school and university. This induced to improvement of the test and to further research. Studies of this kind could involve a variety of methods for data collection but these have the characteristic of building the process, that is, an experience is based on research of previous experience. The order in which activities are carried out is important and therefore limits the study in time. If activity X must precede activity Y, or activity A is going to be used in activity B, or different activities should be carried out in parallel, the researcher has to look well ahead. In my own case, since I had to develop revision material in mathematics for engineering students before the beginning of the academic year 1977-78, the first version of the material was tried out with sixth form students and
Usually, the research that includes development involves different students and teachers during its different stages. This implies that the information has to be collected under different sets of conditions and that any experience could contribute valid information. In this research, sixth form students and first year engineering students were involved with their particular sets of conditions.
CHAPTER TWO

LITERATURE REVIEW
2.1 INTRODUCTION

There has in recent years been a growing concern about the problems of teaching mathematics to engineering students. In 1965 the Organization for Economic Co-operation and Development (OECD) published a report of a seminar on "The Mathematical Education of Engineers" held in Paris with the participation of engineers and mathematicians from Europe, North America and Japan (Organization for Economic Co-operation and Development 1965). This report included two alternative curricula for core courses in mathematics for engineers - a long curriculum and a short curriculum, which in view of the participants, represented the essential mathematical studies which are to be followed by all engineering students during the first two or three years of their university courses.

One of the most important points which emerged from the OECD study in this country was that some of the university courses in the United Kingdom were paying less than adequate attention to the mathematical education of engineers (Kerr 1970). For this reason, in 1968 the Council of Engineering Institutions and the Joint Mathematical Council of the United Kingdom set up a Committee on Mathematics in Engineering which was to investigate the relevance of mathematics in solving engineering problems. This Committee produced a questionnaire designed to obtain reactions to the recommendations of the OECD report. This questionnaire was then circulated to institutions in which engineering was taught as a degree subject throughout the United Kingdom. The report produced by the Committee (Bajpai and Francis 1970) gives recommendations which are basically based on syllabus content.

Since then, much of what has been written about the Mathematical Education of Engineers consists of a body of opinions about 'what
should be taught', 'who should teach it', 'how it should be taught' and 'why it should be taught'. Regarding this matter, Scott (1972) points out that:

"...it is not what is taught but how it is taught what really matters. But before answering the question "How?", an even more fundamental question "Why?" has to be considered. In other words, before any progress in the development of a course can be made we must determine the objective of the course" (p. 239).

Similarly, McLone (1971) points out that:

"...we need first to decide what are our aims and objectives in teaching and what qualities in a student's (mathematical) ability we are seeking to develop" (p. 344).

This would appear to indicate that both Scott and McLone suggest that the teaching of mathematics to non-specialists should be focussed on the aims and objectives rather than on anything else. This certainly is the viewpoint expressed by the OECD (1965), Elton (1971) and Bajpai et al (1975) who have stated aims and objectives of teaching mathematics to engineering students.

The kind of competence in mathematical skills with which I am concerned here is a necessary but not a sufficient condition for the achievement of these aims and objectives, that is, the competence in mathematical skills that newly enrolled engineering students bring to their first year courses.
"The main difficulties in the first year arise from the variety of experiences, knowledge and skills the new students bring to the course. The fact that nearly all have passed 'A'-level examinations ensures some uniformity but this does not extend to an acquaintance with any particular sections of the sixth-form syllabuses" (p. 37).

On the other hand, he also points out that:

"All types of engineering courses face certain problems. The first year of a course has to be constructed as an interface between school and university ..." (p. 36).

Although Allanson restricts his suggestion to engineering courses, this seems to be applicable to any university course in which the student needs to use and apply some of the knowledge and skills he has learned at secondary school. Two studies that give evidence of this are the studies carried out by Kilty (1975) and the Research Project in Accelerated Teaching of Higher Level Mathematics (Tagg 1970).

Kilty in his study of the problems involved in the change of students from Arts to Science concluded:

"...they found difficulty at an early stage in their Mathematics course in using and applying some of the knowledge and skills they had learned at school, in understanding the new concepts of the course and solving
The other study (Tagg 1970) tried to meet the difficulties experienced by students with O-level mathematics who intended to read subjects such as Biology and Economics. It was found that among the difficulties experienced by students in the new work were: (a) slowness and lack of certainty in the use of algebraic symbols; and (b) lack of knowledge due to forgetting and not having met various topics at school. One of the conclusions of the Committee of Mathematics in Engineering report (Bajpai and Francis 1970) which relates to the transition between school and university is that many institutions had provided remedial courses in mathematics for students with ONC, OND or poor A-level grades. This finding highlights the need for a matching of the end of the secondary education to the beginning of the first year at university; that is, the transition from school to university is also an important phase which needs attention. For this reason, I carried out a review of the literature with the purpose of finding out factors that have influence on the transition and different approaches that have been used to bring weak students in mathematics up to the mathematical level of their university courses. The remainder of this chapter is the result of this review, starting with the factors.

2.2 FACTORS THAT INFLUENCE THE TRANSITION FROM SCHOOL TO UNIVERSITY

2.2.1 Ability of Students

Sneed (1971) surveyed sixth form science students in 40 grammar schools to find out their attitude towards engineering. He found that there
existed a widespread prejudice among the ablest boys against engineering, as a career. According to the result of the survey, no engineers appeared in the first six of either prestige, pay, or intelligence ranking. A similar finding was obtained more recently at Loughborough Technical College (1977) where it was found that no engineers appeared in the first ten of either salary, status or intelligence ranking.

If at present, students who intend to follow university courses are also prejudiced against engineering as a career, this will probably reduce the number of applicants to courses and hence give little choice to engineering departments which will have to admit students who may be academically weak in mathematics. This had been previously hypothesized by Byrne (1975) in his investigation of the aspects of the transition between school and university when he pointed out that:

"The present decline in the popularity of science courses in recent years, will probably, in my opinion, have led to the admission of more students who are academically weak as judged by their 'A'-level grades. Therefore, if the standards of the course they enter has not altered, one might expect that the failure rate will not have been reduced since that time" (p. 7).

All these facts and opinions would appear to indicate that a factor which influences the transition from school to university mathematics is that engineering departments are now admitting more students who might not belong to the group of the ablest students.

It may be conjectured that such students are particularly likely to have been through courses which deliberately omitted parts of a syllabus and
will now be taken up.

2.2.2 Omission of Topics at Secondary Level

In a study conducted at the Universities of Durham, Newcastle and Nottingham (Cornelius 1972), a total of 224 students who were taking a mathematics degree, completed a questionnaire, at the beginning of their careers in October 1971, whose purpose was to identify problems encountered by students during the transition from school to university mathematics. One of the conclusions that emerged from analysis of replies was that there were some students who had not met several pieces of mathematics at school. A quote from a student which is given in the paper by Cornelius (1972) was:

"The worst problem is not having done some topics which are considered to be done. To improve this I think the individual examining boards should be abolished and replaced by a national one. This would mean that universities would have a much clearer idea about the topics candidates had covered" (p. 210).

The problem that I try to highlight here is not the discrepancy that there may exist among the wide variety of mathematics A-level syllabuses, but the fact that some students did not cover some parts of mathematics at school. This fact may obviously result in the students not understanding the material of their courses in which this part of mathematics is involved.


O'Connor et al. (1979) constructed a mathematics preknowledge survey that was designed to meet the requirements of three courses - mechanics of particles, structure of atoms and mathematics, and gave it to about 135 students of science and engineering in the first week of their first term at university. In this survey, the students had to associate certain symbols, formulae and concepts in a column with a matching one in a second column. The survey showed that the basic rules of calculus were effectively known by all students whereas about a half of the students were unfamiliar with certain topics - exponentials, dot notation, and distinction between a definite and indefinite integral, and even more, only a few students had met partial differentiation at school. Sutton (1977a) also found that students entering physics degree courses did not cover at school parts of mathematics which are prerequisite for their university courses.

If students knew what parts of mathematics are required by their courses and the level at which this mathematics is needed, it would be of benefit to them because it would give them an idea of what their university lectures may assume is known at the start of their first year courses.

Budden and Gilbart-Smith (1973), as a consequence of a survey of Oxford teachers and science and engineering students in the spring term in 1972, pointed out that there was a confusion among the respondents to the survey over what should be classed as 'pre-requisite', and came to the conclusion that:

"On the whole, undergraduates are of the opinion that any topic included in their course should already have been encountered at school ..... whereas the university staff tend to differ on this point" (p. 74).
... these facts clearly show that one of the factors that influences the transition from school to university mathematics is that not all students have met at school certain pieces of mathematics that are regarded as pre-requisites for their first year university courses.

In the past, some mathematics departments have been able to assume that nearly all their students will have studied mathematics as a double subject (Mathematical Education Committee 1976), but engineering departments normally demand a single A-level in mathematics as an entrance requirement, and even more, engineering departments are now recruiting students with ONC and HNC qualifications (Bajpai 1970).

Here again, there may be some students who either have not covered at school pieces of mathematics regarded as pre-requisites for their first year university courses, or may be weak in certain topics.

Allanson (1973) regarding National Certificates and Diplomas points out that:

"They are designed for the education of technicians but some of the successful students use them to gain admission to degree courses, mainly in engineering" (p. 35).

Due to the fact that National Certificates and Diplomas are not specifically designed for students who intend to enter university courses, it would appear that those students who get these qualifications and do enter university might encounter difficulties in coping with the mathematics needed in their courses. A reason for this may be that certain mathematical topics have not been treated at school at the level required by the courses the students are entering. Rees in her study of the difficulties experienced by craft and technician students (Rees 1973)
and first year engineering students who had a background of ONC, OND, HNC and HND (Rees 1974) found that there was a group of mathematical topics with which the students had difficulties, and that the main reasons for these difficulties were: lack of understanding of concepts, and lack of knowledge and understanding of formulae.

Bajpai (1970) regarding students with National Certificates and Diplomas suggests that:

"It might be advisable to provide such students with a separate course in mathematics in the first year to bring them up to the required standard" (p. 400).

In fact this kind of course was provided for students with ONC and HND qualifications by the Department of Engineering Mathematics at the University of Technology, Loughborough, but had to be discontinued due to lack of resources in terms of staff time (Bajpai 1979). A course with a similar purpose was also taught at King's College, London, in the autumn term of 1971 to overcome the deficiencies in mathematics of first year science and engineering undergraduates (Baker et al 1973). In the work presented in this thesis, staff who teach first year engineering students were surveyed with the purpose of finding out what topics in mathematics they considered as pre-requisite for the courses given and the level at which these topics are needed. (This matter is treated in detail in Chapter 3.) Apart from providing students with separate courses, other alternatives have also been used to overcome the deficiencies in mathematics of students entering new courses. These are described further in this chapter (Section 2.3).
2.2.3 Forgetting What has been Learned

According to the School Council (1973), some students spend a year on activities quite different from their studies between school and university, and some others go to further education institutions to attend a two or three year course after leaving school at 16. Allanson (1973) regarding the break between school and university points out:

"...some of the students will have interpolated a year in industry, or perhaps in voluntary service overseas, between school and university and will thus have become rather out of practice in handling mathematical problems" (p. 37)

This was confirmed by Cornelius and Marsh (1977) who found that the students who had done mathematics as a single subject at secondary school and had taken a break between school and university, experienced more difficulty with their mathematics course than the direct entrants.

On the other hand, Elton (1979) gave to two small groups of newly enrolled physics students a fifty-item multiple-choice test which covered the main mathematical needs of entrants to physics degree courses. He found that one of the main reasons why students may have failed to answer the test items correctly was that they had forgotten the relevant material. This was so even for students who had taken A-level earlier in the same year.

The fact that students had forgotten much of their mathematical knowledge and skills from their leaving of school to their entry to university may have some connection with the view expressed by the
"Students tend to forget topics fairly rapidly. While there is not statistical evidence to support the theory it has been found useful to assume that the sequence of 'remembering time' is roughly in geometric progression" (p. 164).

Cornelius and Marsh (1977) go on to suggest that:

"...any student who intends to spend a substantial period of time in industry between school and university should be advised to take a further course in mathematics in parallel with his industrial training" (p. 11).

If such students actually took such a course as Cornelius and Marsh suggest, one would expect that they will not become out of practice in handling mathematical problems and, therefore, will not encounter much difficulty in dealing with the mathematics of their university courses when they enter university.

To summarise, the break that for many students does exist between school and university is another factor that influences on the transition from school to university mathematics.

2.2.4 Discussion

Much of what has been written about the mathematical education of engineers consists of approaches to teaching particular topics of the
syllabus at university. As a consequence of this not much literature is devoted to the problem of the transition between school and university.

Three factors that do have influence on the mathematical education (competence of students) in the transition between school and university were identified. These factors are: firstly, engineering departments are now recruiting students with National Certificates and Diplomas - who might not have studied certain mathematical topics at the level required by their university courses - and students who have done A-level in mathematics but who might not belong to the group of the ablest students; secondly, not all students have met at school all the mathematics that is regarded as pre-requisite for their university courses; and thirdly, students may become rather out of practice in handling mathematical problems due to the fact that many of them spend a substantial period of time on activities quite different from their studies, between school and university.

In the present study, the mathematical knowledge and skills pre-requisite for engineering students were identified by means of a survey of staff lecturing first year engineering courses at institutions of higher education. These pre-requisites were used to construct and develop a battery of mathematics diagnostic tests whose purpose was to ascertain what the mathematical needs of a student entering engineering courses are. The details of the survey and the development of the tests are discussed in Chapters 3 and 4 respectively.

In the next section, I discuss different approaches which have been used to diagnose students' difficulties in mathematics and the means used to bring weak students up to the level required by their university courses.
The review carried out so far included only English studies because it was concerned with matters specific to the situation in this country but from now onwards, comparisons will be made also with relevant pieces of work related to the matter of revision and remedial work in mathematics which has been carried out elsewhere.

2.3.1 Remedial Courses

An approach which has been used to give students revision work in mathematics is by means of lectures. For example, at the Papua and New Guinea Institute of Technology, students entering engineering courses were given a 'two week crash revision course' at the beginning of their courses (Deakin 1972). This revision course was common for all students and was based on the students' performance on a test given to them at the beginning of the academic year.

In spite of the efforts of the teaching staff, the revision work was found not to have been very effective, as Deakin (1972) points out:

"...the effectiveness of the course is severely restricted by the fact that the students believe themselves to know the material already. Despite the best efforts of a high calibre teaching staff, they are bored. They do not realize what they do not know - even when it is demonstrated to them that there are areas of weakness in their mathematical background" (p. 230).

Another scheme which has been used with students entering engineering
courses is that used at the University of Technology, Loughborough, with students who are on sandwich courses (i.e. students entering mechanical engineering, transport technology, and production engineering courses), and whose academic courses start in January instead of October. This scheme consists of giving some remedial work in parallel with the standard mathematics course. This remedial work is provided by means of a lecture of one hour's duration each week during the first term at university (Bajpai et al 1975). This remedial work would apparently be profitable for the students only if they revise the topics involved before they actually encounter them in their university courses.

A third scheme which has also been used to give revision work in mathematics to students entering engineering courses is that used at the University of Durham. This scheme consists of giving students a short optional revision course at the beginning of their courses (Cornelius and Marsh (1977). In addition to this optional course, students are also provided with a booklet which contains a set of problems for the students to solve. The topics in this booklet are based on a note prepared by the Professors of Mechanical Engineering in the Northern Universities (1976). This note provided a list of topics in sixth form mathematics which were considered essential for students intending to enter a three year course leading to an honours degree course in engineering (see Section 3.1).

To summarise, according to these schemes all students were given the same instruction, that is, all of them had to attend common lectures and therefore no one could go either faster or slower in any particular topic. Regarding the matter of the same instruction for all students, Bloom et al (1971) points out that:

"If instruction for all students starts at the same point
the results can be that those who already mastered objectives well beyond this point fast become bored and disinterested while those who do not yet possess the pre-requisite fast become discouraged and frustrated" (p. 93).

This was what might have happened to students in the scheme used by Deakin (1972), though no conclusion can be drawn from the information given in his paper.

2.3.2 Diagnosis and Specific Remedial Work

Another approach which has been used to give students revision work in mathematics is that in which they are assigned revision work on the basis of their performance on a pre-test. The schemes under which this approach has been used are very similar in several respects, but differ in others. Among these schemes are the following:

At King's College, London, science and engineering students were given a 'crash course in calculus' when they arrived at university in October 1971. According to Baker et al (1973), the course was given for one whole day in each week of the first six weeks of the term and the students involved were chosen as needing intensive revision in calculus on the basis of their performance on a mathematics pre-test of forty multiple-choice items. Since the morning sessions of the course were devoted to individual revision work and the afternoon sessions were devoted to stimulate the students' interest, I shall concentrate on the former.

In the morning sessions, the students were expected to: (a) study a
certain number of frames in a programmed text which had been previously selected for the revision work, (b) work on a set of supplementary problems they were given, and (c) consult any of the three tutors available in the lecture room if they encountered difficulties (these tutors were postgraduate mathematics students). In general, students said, in a questionnaire given to them, that due to the characteristics of the scheme they could work at their own speed and could also go back over the text if they did not understand it.

Kilty (1975) in his study of the problems involved in the change of students from Arts to Science developed a scheme of revision work which besides involving a pre-test, also had the following features; (a) the test was accompanied with references to parts of a textbook appropriate to each item, (b) the test was marked and discussed with students (individually or in groups) immediately after they had completed it, (c) each individual student was then advised to do revision work according to his performance on the test, and (d) after the students had done the revision work, they were given a post-test whose items were similar to those they failed in the pre-test, and a procedure similar to that described in (b) was adopted.

According to Kilty, the scheme 'was found to be a very valuable introduction to the course by the majority of the students who were identified as needing the "remedial work"...' (p. 81). As a result of this experience, the scheme was integrated into the mathematics course formally.

At the University of Texas at Austin, a 60-item multiple-choice test on very basic algebra, geometry and trigonometry was devised to gather information about the mathematical competence of first year engineering
students (Fowler et al 1976). This test was given to a group of students in a normal class period and they were asked to retake it at home, using any reference material they wished, but without help from anyone else. It was found that there was a considerable difference between the scores on the test in the two opportunities and that almost all students improved their scores when they were allowed to use reference material to complete the test. The authors concluded that no attempt had been made to develop formal individualised remedial work programmes. Although the authors only intended to gather information about the mathematical competence of the students, it would appear that the fact of having asked the students to retake the test at home might have required them to revise part of the topics. This revision work has the characteristic that the students had to find out by themselves textbooks covering the topics involved.

At the Middlesex Polytechnic, students entering Social Science and Business study degree courses have been given since 1968, a remedial course in mathematics pre-requisite for their courses. According to Romiszowski et al (1976), the scheme employed involved the following:

(a) Two diagnostic tests. Each one covering eleven moduli of the remedial course material.

(b) Individualised self-teaching materials. These consisted of branching programmes on teaching machines, linear programmes and non-programmed booklets.

(c) Individual tutorials. These were available by telephone appointment in the tutors' offices at appointed hours.
(d) One post-test for each of the modulus of the course (i.e. 22 post-tests).

The procedure adopted was basically as follows:

(1) Students took one of the diagnostic tests on their first day on the course, and the second whenever they had completed the necessary study of the first eleven moduli.

(2) Each of the diagnostic tests was marked by tutors, if possible in the presence of the student, in order to discuss any problems immediately.

(3) As a result of this discussion the student was then advised to study, in the remedial maths room and within the month of October, the necessary moduli of the course.

Students had to study in the remedial maths room on a "learning by appointment" system at any convenient time of the day. Although they were expected to complete the necessary remedial work during the month of October, some students continued working on the course during November and early December.

A scheme which apparently intends to cope with the problem of the mathematical competence at the transition between school and university is that developed by the Physics Interface Project (PIP). According to Sutton (1977a), the students entering Physics degree courses at the six PIP universities are given one or more tests from a battery of five covering mathematics (two alternative tests), electricity, mechanics and Physics.
It seems that the purpose of the PIP is not exactly to give revision work to students but to know what they can do and what they cannot do, as Sutton points out the purpose of the test is:

"...to enable their teachers to discover how many students are unable to do what..." (p. 93).

Then, regarding what the students should do, he points out that:

"They are asked to make two copies of their answers, one in the question booklet and one on an answer grid to be handed in for analysis. At the end of the time allotted for a test, depending on local preferences, the correct answers are either issued in immediate exchange for the students' answer sheets, or they are made available at the next tutorial. In any event the students are encouraged to consult their tutors about the results, and different departments make their own arrangements for handling this part of the exercise" (p. 93).

It would appear that the PIP gave more importance to the analysis of the students' performance on the test(s) rather than to individual diagnosis of students. In spite of the PIP having devised a number of self-teaching materials for the students, these would appear to have been used only by those students who actually consulted their tutors about their results in the test. No numerical account was given of the number of students who actually consulted either their tutors after the completion of the test, or the self-teaching material (Sutton 1972, 1977a, 1977b; Taylor 1973).
At the University of Surrey, one of the PIP universities, the procedure adopted with the students entering the Physical Science degree in 1974 and 1975 laid emphasis on the individual diagnosis of students. According to Elton (1979), the students were given the PIP mathematics test and as soon as each student had finished, he went to an adjoining room where he met a tutor, who first marked the test against a key and then discussed it with him. As a result of this, each student was then advised to use, during the following two or three days, certain programmed materials which were available in a third room.

Elton concluded that the test and the tutors were important for the successful carrying out of the exercise, the former because it had a diagnostic motivational role, and the latter because without the tutors there is a risk that the data gathered in the test may be misinterpreted either by the students or by their tutors.

This procedure overcomes the inconvenience that may arise in the case described by Sutton (1977a), that is, the student does not have to wait for a future tutorial period to meet his tutor and discuss the results of the test with him.

To investigate the mathematical competence of students entering university to take Physics at the University of Waikato, New Zealand, a mathematics test was constructed and given to newly enrolled Physics students in the first week of their courses in 1977 (Osborne 1979a). As a consequence of the students' performance on this test, a self-teaching booklet on basic mathematics for Physics students was produced and given, in 1978, to all students entering Physics and who were considered to be weak in mathematics. This group of students was made up of students who scored less than 50% in a mathematics test they had been given before they
actually entered university (Osborne 1979b). The paper by Osborne concludes saying that:

"A detailed analysis of subsequent test results showed however that this had no obvious effect on the problem and hence that this measure alone is inadequate. Perhaps another alternative is to try to persuade the mathematicians to place greater emphasis on these skills. At least to be aware of the problem is a first step" (p. 21).

In my opinion, there might have been several reasons why this approach had no obvious effect on the problem. One which would appear to be obvious is that the students were advised to study the booklet but not asked to show that they had done so. Another reason might have been that there were some students who did not really need to study the whole booklet and, because of their not knowing exactly what they should revise, they simply did not bother with the revision. There might have been several other reasons but no further investigation was done in this sense.

2.3.3. Counselling

In 1971, the Department of Freshman Engineering at Purdue, U.S.A., started a project called "The Counselling-Tutorial Program" ('C-T program') whose main aim was to support and encourage students who were relatively poorly prepared to pursue engineering, and to help students who had been out of school for some time or whose mathematics and science backgrounds were inadequate. According to Deputy (1978), the newly enrolled students take, during their first term at university, a daily one-hour tutorial course plus any college preparatory course needed to strengthen their academic skills. In this tutorial course the students receive a course
"...introduces no new material, but shows the student how to solve problems that relate to material learned in other courses. Sometimes the material presented in a chemistry or math class is presented again but from a different point of view. Other times the hour is spent revising material and exchanging questions and answers or working problems students might have" (p. 314).

In addition to this course, the students also receive individual counselling sessions which are designed to help them to develop self-confidence, especially in solving tough engineering problems, and to decide whether they should remain in the 'C-T program' through the spring, to use only the individual tutoring and counselling sessions, or not to participate at all.

This 'C-T program' is of a complex nature in the sense that, on the one hand it requires the student to spend an extra semester at university to receive the engineering degree, and on the other hand it requires time from the members of staff involved.

2.3.4 Discussion

The schemes of diagnosis and revision work in the literature reviewed here vary from simple ones like those in which students are given remedial lectures to more complex ones like that in which each student takes daily tutorials in mathematics during his first semester at university.
If our aim is to bring weak students in mathematics up to the level required by the courses they intend to enter, one has to bear in mind that different students may be weak in different topic areas and therefore, need to do different revision work. If this is the case, it would appear that it is difficult to achieve this aim by means of lectures because those students who already know the material might become bored or uninterested and those who do not know it might like to spend longer time on it. An example of this situation would appear to be what happened in the scheme described by Deakin (1972). The approach in which students were just given a booklet with the mathematics that they should know (Osborne 1979a) was found to have been inadequate and the reason for this would appear to have been that the students did not get motivated enough as to study the booklet without any guidance, except that they should study all the material in the booklet.

Most of the schemes described here attempt to give diagnosis by means of a pre-test and then suggest revision work according to the students' performance on the test. In the scheme described by Baker et al (1973) although students were given a pre-test, the revision course was based on the overall students' performance on the test. The students could cover the material corresponding to any particular session as they pleased because they worked individually during each session. However, all of them had to do the same revision work because this was a course for all students rather than a programme to remedy individual deficiencies.

In the scheme described by Sutton (1977a), it would appear that what was important was that the teacher knew the areas of weaknesses of the students rather than that each individual knew his own areas of weakness. The answer grid the students handed in were used for the analysis of the test and not for the analysis of each student's performance on the test.
Since the students were left to themselves as regards their decisions about attendance at their next tutorial for the discussion of the results of the test, the result could have been that some students did not attend the tutorial and therefore, did not know the areas in which they needed to revise. Some schemes overcame this problem by carrying out individual discussions of the test immediately after each student had completed it (Kilty 1975, Romiszowski et al 1976, Elton 1979), and then as a result of this advice each student had to do the pertinent revision work. Since in these schemes each individual student was advised to revise the topics in which he needed revision, one would expect them to have been of more benefit to the students than those in which all students had to do the same revision work.

Although students had to do the same revision work in the scheme described by Deputy (1978), this case differs from the others in the sense that in the daily individual tutorials the emphasis could be put on those areas in which the students needed more work, and therefore, the individual needs of each student could be attended.

In the study reported here, a scheme similar to that used by Elton (1979) was used to give individual diagnosis and suggest revision work to a large number of students entering engineering courses at the University of Southampton. This is treated in detail in Chapter 5.

2.4 SUMMARY

I carried out a review of the literature with two purposes in mind. Firstly, to find out the main factors that have influence on education in mathematical competence of engineering students in the transition between school and university (mainly at the point of entry), and
secondly, to find out what approaches have been used to bring weak students in mathematics up to the level required by their university courses.

According to the literature reviewed here, there are three factors that have influence on the mathematical competence at the interface between school and university - the ability of students, omission of topics at secondary level, and forgetting what has been learned.

The approaches that have been used to bring weak students in mathematics up to the level required by the courses they intend to enter have been classified into three groups - remedial courses (by means of lectures); diagnosis and specific remedial work (usually based on a pre-test); and counselling (i.e. diagnosis and revision work based on tutorials).

In the present work, the factors mentioned above have been taken into consideration for the study of the mathematical knowledge and skills pre-requisite for engineering courses. A scheme of individual diagnosis and revision work similar to some in the second group was used. In addition to this, a battery of diagnostic tests based on knowledge and skills pre-requisite for engineering courses, and a new approach which intends to overcome the difficulties encountered in the schemes of diagnosis mentioned above was developed.
CHAPTER THREE

PREREQUISITE MATHEMATICAL KNOWLEDGE AND SKILLS: THE SURVEY
3.1 INTRODUCTION

Students entering engineering courses are required to have mastery of certain specific topics in mathematics. A list of such topics is given in the note prepared by the Professors of Mechanical Engineering in the Northern Universities (1976). This note provides a list of topics in sixth form mathematics which 'should have been covered by students before starting a three year honours course in mechanical engineering or engineering science'.

A test, which aims to diagnose the mathematics performances which students entering engineering courses should have for dealing with their first year courses, could not be based on the list of topics mentioned above because of the following two reasons: (a) the list does not specify 'when' during the three year course students will need to use the topics, and (b) it does not say the specific performance expected from students on each topic. As I have said earlier, the latter factor is of great importance for diagnosis because different people may interpret different topics in different ways and therefore, the interpretation of the topics could be misleading (see Section 1.1).

This situation led me to survey staff lecturing to first year engineering students, whom from now onwards I shall call 'lecturers', in order to have detailed information on the use of mathematics by first year engineering students. The information gathered by means of a questionnaire developed for this purpose (see Section 1.4.1.1) enabled me to know what mathematics students entering engineering courses should know at the point of entry, i.e. what mathematics was used throughout each respondent's (lecturer's) course, and the level at which he would use it in his first year course.
This chapter deals with the details of the survey and the results that came out of it.

3.2 THE PILOT STAGE OF THE SURVEY

A pilot study was carried out in two universities by means of a questionnaire. This exercise had the intention of finding out the validity and reliability of the questionnaire. The contents of this questionnaire were based on the mathematics prerequisite for the first year mathematics course for engineering students at the University of Southampton with which I had worked in a scheme of diagnosis and revision work in 1976 and 1977 (see Chapter 5). The topics in the questionnaire were the following:

1. Algebraic Manipulation
2. Solution of quadratic equations
3. Solution of inequalities
4. Binomial Theorem
   4.1 For positive integral exponent
   4.2 For fractional and negative exponent
5. Trigonometry
   5.1 Use of formulae for \( \sin(x \pm y); \cos(x \pm y); \tan(x \pm y) \)
   5.2 Use of formulae for \( \sin x \pm \sin y; \cos x \pm \cos y; \tan x \pm \tan y \)
   5.3 Sine and Cosine rules
   5.4 Sine and Cosine of angles expressed as a fraction or multiple of \( \pi \)
   5.5 General solution of equations such as \( \sin \theta = \frac{1}{2} \).
6. Evaluation of limits
7. Differentiation of trigonometric functions
8. Integral Calculus
Copies of this questionnaire were given, in the first instance, to three lecturers (who taught Hydraulics and Measurement in the Department of Civil and Electrical engineering at the University of Surrey) in order to get feedback on the format of the questionnaire and of the wording. These three lecturers were asked to state whether the topics in the questionnaire were needed in their own courses (by listing the engineering topics in which the mathematical topics were used), and to give examples taken from their own courses that showed the level of difficulty expected in each topic. The questionnaire also invited the lecturers to add new topics to the original list. After the three copies of the questionnaire were returned, I held individual interviews with the respondents with the purpose of discussing my interpretation of the data collected. As a result of these discussions, a revised outline of the topics for the questionnaire was agreed and no criticism was made to the format of the questionnaire. The changes introduced in the questionnaire can be summarised as follows:

(a) The term 'Algebraic Manipulation' was changed for Polynomial factorisation and simplification of rational functions because the meaning of the former (i.e. Algebraic Manipulation) was 'too wide'.

(b) The topic 'Solution of linear equations' was included.

Copies of the revised version of the questionnaire (see Appendix A) were
then sent to 15 lecturers of the Departments of Civil and Electrical Engineering at the University of Surrey, and to a member of the Department of Engineering Science at the University of Durham (6 copies) to distribute them among lecturers in his Department who were willing to complete the questionnaire. In the meantime, I carried out a review of lecture notes of first year engineering courses of the Department of Electrical Engineering at the University of Surrey in order to find out those engineering topics, in the corresponding courses, in which mathematics was needed either for understanding the new material of the course or for solving the coursework exercises, and the level at which this mathematics was used.

The notes reviewed were of the courses of Electrical Machines and Power (EMP), Electronics (E), Energy Sources (ES) and Measurements (M). Since these notes were not very intensive in content, I was unable to ascertain whether the mathematics involved was needed for understanding of new material or for problem-solving. The mathematical topics in question and the engineering topics associated with them can be summarised as follows:

**Trigonometry**

(a) Definition of sine and cosine of an angle:
Reactive Voltampers (EMP)

(b) Sine and cosine graphs:
Development of circuits: representation of sine waves (EMP)
Production of rotating magnetic fields (EMP)
Three phase waveform (EMP)
Waves (ES)
(c) Cosine rule.

Development of circuits (EMP)

(d) Use of formulae to expand $\sin(x \pm y)$ and $\cos(x \pm y)$:

Developments of circuits: Analysis of readings - the two Wattmeter method (EMP)

Production of rotating magnetic fields (EMP)

Harmonics in three phase transformers (EMP)

Waves (ES)

(e) Use of formulae for $\sin x \pm \sin y$ and $\cos x \pm \cos y$:

Harmonics in three phase transformers (EMP)

Superposition of waves (ES)

Dynamometer instruments (M)

Algebra

(a) Transposition of formulae

Direct current motors (EMP)

Direct current measurement (M)

(b) Solution of linear equations:

The Carnot efficiency (ES)

(c) Solution of simultaneous linear equations:

Permanent Magnet Movil Coil meter (M)

Exponentials

Thermionic Emission (E)
Properties of logarithms

Voltage and Current gain of Amplifiers (E)

Differential Calculus

(a) General rules of differentiation:
Triode valves: Calculation of gain for small signal (E)

(b) Differential of sine and cosine:
Production of rotating magnetic field (EMP)

Evaluation of Simple Integrals

Diodes (E)
Plasma oscillations (ES)
Debye Length in a plasma (ES)

3.2.1 The Surrey University Group

Of the 15 copies of the questionnaire distributed at the University of Surrey only 5 were returned by lecturers of the Electrical Engineering Department (in April 1978). There would appear that the main reason for this poor return was that no lecturer of the Civil Engineering Department returned the questionnaire (i.e. 7 lecturers) apparently because their courses (Architectural Forms, Computing, Geology,
Structural Design, Surveying) do not require mathematics above the G.C.E. O-level.

The courses taught by the 5 respondents from the Electrical Engineering Department were the following: Circuit Theory, Electrical Machines and Power, Electronics, Energy Source and Semiconductor Electronics. This would appear to indicate that although there were few respondents, the data collected came from lecturers who actually needed A-level mathematics for their courses and this was the kind of lecturer to whom the questionnaire was addressed.

After I had gone through the returned copies of the questionnaire, I interviewed four of the respondents individually. In these interviews, I discussed with them my own interpretation of the data gathered (by means of the questionnaire and the review of the lecture notes) with the purpose of clarifying any misunderstanding of the level of difficulty of the topics involved. Furthermore, some feedback on the format of the questionnaire was given.

As a result of the discussions, the topics needed and the level of difficulty expected in each of them was agreed. These are summarised in Table 3.1.

The analysis of the data gathered showed that there were:

(a) Names changed. Although the topic 'solution of linear equations' was mentioned by some lecturers, when they gave examples to illustrate the level of difficulty expected, they gave examples of 'Transposition of formulae' rather than of linear equations. This fact confirms what I said earlier in Section 1.1, that is, when one speaks of mathematics in terms of
<table>
<thead>
<tr>
<th>Topic</th>
<th>Example of the level of difficulty expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Operations with fractions</td>
<td>Addition, subtraction, multiplication and division of simple fractions</td>
</tr>
<tr>
<td>Algebraic expressions</td>
<td>Expansion of $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$</td>
</tr>
<tr>
<td>Solution of linear equations</td>
<td>Solve for $T$ the equation: $w = \frac{V - \frac{TR}{K^2}}{R}$, $R \neq 0$</td>
</tr>
<tr>
<td></td>
<td>Solve for $Rs$ the equation: $\frac{I_s}{X_m} = \frac{Rm+Rs}{Rs}$</td>
</tr>
<tr>
<td></td>
<td>Solve for $T_i$ the equation: $\eta = \frac{T_i - T}{T_i}$</td>
</tr>
<tr>
<td></td>
<td>Solve for $V_s$ the equation: $V_s = V_a + IR$</td>
</tr>
<tr>
<td>Solution of quadratic equations</td>
<td>Application of the general formula</td>
</tr>
<tr>
<td>Summation symbol</td>
<td>Meaning of the symbol $\Sigma$</td>
</tr>
<tr>
<td>Solution of inequalities</td>
<td>Solve for $x$: $</td>
</tr>
<tr>
<td>Logarithms</td>
<td>Application of the general properties</td>
</tr>
<tr>
<td>Complex numbers</td>
<td></td>
</tr>
<tr>
<td>Modulus</td>
<td>Find the modulus of a complex number $a+bi$</td>
</tr>
<tr>
<td>Basic operations</td>
<td>Addition, subtraction, multiplication and division of two complex numbers of the form $a+bi$</td>
</tr>
</tbody>
</table>
# TABLE 3.1 cont.
Level of Difficulty of the Mathematical Topics
Mentioned in the Pilot Survey (Surrey University).

<table>
<thead>
<tr>
<th>TRIGONOMETRY</th>
<th>TOPIC</th>
<th>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Definitions of the trigonometric functions in a right angled triangle</td>
<td>Application of the definitions in problem-solving. eg. [ \frac{b}{a} = \sin \alpha ; ; ; c = b \cos \alpha ]</td>
</tr>
<tr>
<td></td>
<td>Numerical values for expressions involving the sine/cosine of angles expressed as a fraction or a multiple of ( \pi )</td>
<td>Evaluation of [ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \cos 2\pi t) , dt ] Values of the sine/cosine of ( \frac{\pi}{6} ; , ; \frac{\pi}{4} ; , ; \frac{\pi}{3} ; , ; \text{etc.} )</td>
</tr>
<tr>
<td></td>
<td>Use of formulae to expand ( \sin (x \pm y) ; ; ; \cos (x \pm y) )</td>
<td>Expansion of ( \sin (\omega t - \frac{2\pi}{3}) ) Expansion of ( \sin (30^\circ - \alpha) ) Expansion of ( y = r \sin \left[ 2\pi \left( \frac{4}{5} - \frac{\alpha}{3} \right) \right] )</td>
</tr>
<tr>
<td></td>
<td>Use of formulae for ( \sin x \pm \sin y ; ; ; \cos x \pm \cos y )</td>
<td>Expansion of ( \sin (3\omega t + \frac{2\pi}{3}) \sin (2\omega t + \frac{\pi}{3}) ) Express as a sum or difference of sines/cosines</td>
</tr>
<tr>
<td></td>
<td>Sine and Cosine rules</td>
<td>Application of the rules in problem-solving</td>
</tr>
<tr>
<td></td>
<td>Sine and Cosine graphs</td>
<td>Identification and sketching of the graphs</td>
</tr>
<tr>
<td>DIFFERENTIAL CALCULUS</td>
<td>Binomial Theorem for fractional exponents</td>
<td>Find a few terms of the expansion of ( (1 + x)^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>Differentiation of trigonometric functions</td>
<td>Differential of the sine and cosine</td>
</tr>
<tr>
<td></td>
<td>Differentiation of other types of functions</td>
<td>Application of the general rules of differentiation</td>
</tr>
<tr>
<td>TOPIC</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</td>
<td></td>
</tr>
<tr>
<td>-------------</td>
<td>---------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>INTEGRALS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Evaluation of integrals</td>
<td>Application of general techniques for evaluating integrals</td>
<td>Evaluation of integrals involving trigonometric functions, e.g., ( \int \sin(\omega t) , dt ), ( \int \sin^2(\omega t) , dt )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
topics, without a clear definition of the performance to be observed in each one, there is always the possibility that different people interpret the same topics in different ways. If I had only asked the lecturers to say whether they needed the topic 'solution of linear equations', they would have said 'yes' and then the information would not have been valid because what they really meant was 'transposition of formulae'. This fact reinforced the idea of asking people to give examples showing how they actually used the topics concerned. The variety of examples of this topic listed in Table 3.1 would appear to have more or less the same level of difficulty because all of them require the application of the basic rules to transpose terms and factors.

(b) Topics not mentioned. There were certain topics which were not mentioned by the respondents despite their being considered prerequisite for the first year mathematics course at the University of Southampton, from which the initial list of topics was derived (see Section 3.2). This is not surprising because these lecturers may expect students to use different mathematical topics than a lecturer who teaches mathematics. The topics in question were the following:

1.0 Algebra
   1.1 Polynomial factorisation
   1.2 Simplification of rational functions
2.0 Trigonometry
   2.1 General solution of equations such as \( \sin \theta = \frac{1}{2} \)
3.0 Binomial Theorem
   3.1 For positive and integral exponents
4.0 Limits
5.0 Exponentials

This topic was mentioned but only in connection with integral calculus.
to the original list were mainly topics which are covered by 0-level mathematics courses. However, these topics seem to be necessary for dealing with the lecturers' courses involved. These topics were:

1.0 Algebra
  1.1 Operations with fractions
  1.2 Operations involving algebraic expressions
  1.3 The summation symbol
2.0 Complex Numbers
  2.1 Modulus of a complex number
  2.2 Basic operations with complex numbers
3.0 Trigonometry
  3.1 Definition of the trigonometric functions in a right angled triangle
  3.2 Sine and cosine graphs
4.0 Differential Calculus
  4.1 General rules of differentiation

During the interviews with the lecturers some feedback about the format of the questionnaire was gathered. The lecturers said that the questionnaire was time-consuming to fill in because they had to list the different topics of their courses where they used each of the mathematical topics presented to them. According to them this exercise required a lot of time because they knew what mathematics they used but not exactly where in their courses.

3.2.2 The Durham University Group

The copies of the questionnaire distributed at the University of Durham
were returned in July 1978. These were filled in by six lecturers who taught Applied Electronics, Electrical Circuit Theory, Electronics, Engineering Thermodynamics, Fluids and Mechanics. My interpretation of the data collected was discussed with three of the respondents. Table 3.2 shows the resulting topics and the level of difficulty expected in each of them.

The analysis of the data collected showed that there were:

(a) **Omission of examples.** There were certain topics in which no examples showing the level of difficulty expected were given. However, additional information provided by respondents implied this level of difficulty.

(b) **Topics not mentioned or added to the list.** As it happened with the Surrey University Group, there were certain topics which were not mentioned by respondents in the questionnaire or were added to the original list. The reasons for this would appear to have been the same as for the Surrey University Group (see (b) and (c) in Section 3.2.1). The topics in question were:

(i) **Not mentioned:**

1.0 **Algebra**

1.1 Solution of linear equations
1.2 Polynomial factorisation
1.3 Simplification of rational functions

(ii) **Added to the list:**

2.0 **Complex numbers**

2.1 Basic operations
<table>
<thead>
<tr>
<th>Level of Difficulty</th>
<th>Example of the Level of Difficulty Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><strong>Solution of quadratic equations</strong></td>
</tr>
<tr>
<td>B</td>
<td><strong>Solution of simultaneous linear equations</strong></td>
</tr>
<tr>
<td>C</td>
<td><strong>The summation symbol</strong></td>
</tr>
<tr>
<td>D</td>
<td><strong>Solution of inequalities</strong></td>
</tr>
<tr>
<td>E</td>
<td><strong>Logarithms</strong></td>
</tr>
<tr>
<td>F</td>
<td><strong>Operations</strong></td>
</tr>
<tr>
<td>G</td>
<td><strong>Exponential form</strong></td>
</tr>
<tr>
<td>H</td>
<td><strong>For positive and integral exponents</strong></td>
</tr>
<tr>
<td>I</td>
<td><strong>For fractional exponents</strong></td>
</tr>
</tbody>
</table>

Example of the Level of Difficulty Expected:
- Use and interpretation: Evaluate $\sum_{i=1}^{5} (x^2 + 1)$
- Solve for $x$: $|x| - 7 < 9$
- Application of the general properties in problem solving
- Basic operations with complex numbers
- Use of the notation $e^x$
- Expansion of $(1 - x^2)^{1/2}$
<table>
<thead>
<tr>
<th>Topic</th>
<th>Example of the level of difficulty expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trigonometry</td>
<td>Numerical values for expressions involving the sine/cosine of angles expressed as a fraction or multiple of π Sine and cosine of angles such as π/6; π/4; etc. (whether in radians or in degrees)</td>
</tr>
<tr>
<td></td>
<td>Use of formulae to expand sin(x ± y); cos(x ± y) Expansion of sin(ωt + θ)</td>
</tr>
<tr>
<td></td>
<td>Use of formulae for sin x ± sin y; cos x ± cos y General use</td>
</tr>
<tr>
<td></td>
<td>Sine and Cosine rules Application of rules to problem-solving</td>
</tr>
<tr>
<td></td>
<td>Solution of equations such as sinθ = ½ Solution of the equations sinθ = p; cosθ = p (-1 ≤ p ≤ 1)</td>
</tr>
<tr>
<td>Differential Calculus</td>
<td>Differentiation of trigonometric functions Differential of sin x and cos x</td>
</tr>
<tr>
<td></td>
<td>General rules for differentiation Application in problem-solving</td>
</tr>
<tr>
<td>Vectors</td>
<td>Addition and subtraction of two vectors</td>
</tr>
<tr>
<td></td>
<td>Scalar product of two vectors</td>
</tr>
<tr>
<td>Topic</td>
<td>Example of the level of difficulty expected</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------------------------------</td>
</tr>
<tr>
<td>Integration of algebraic functions</td>
<td>Application of general rules Evaluate $\int x^2 , dx$</td>
</tr>
</tbody>
</table>
3.0 Differential Calculus

3.1 General rules of differentiation

3.2 Double integral

4.0 Vectors

4.1 Addition and subtraction

4.2 Scalar product of two vectors

3.2.3 Conclusions of the Pilot Survey

The pilot survey was carried out with the purpose of finding out how valid and reliable the information gathered by means of the questionnaire would be. On the other hand, as little was known about the possible effects of the format of the questionnaire on the respondent, some feedback of this kind was also expected.

The validity of this questionnaire is judged in terms of the kind of examples given in Column B by the respondents, and the reliability is judged by the interpretation given to the mathematical topics presented, that is, the interpretation given to each mathematical topic should be the same for all the respondents. Only one topic was found not to have been reliable (solution of linear equations) because lecturers gave an interpretation different from the one expected. However, the degree of similarity among the given examples showed that the interpretation of the topic was uniform. This seems to indicate that in this type of questionnaire, where examples are asked for, the validity of an item implies its reliability. This seems to be in line with what Oppenheim (1966) says about validity and reliability of an item in a
"...if we find that a measure has excellent validity, then it must be reliable." (Oppenheim 1966, p. 70)

Through the interviews held with some lecturers and the reviewed lecture notes (see Section 3.2), it was possible to ascertain whether the examples given in the questionnaire matched the right level of difficulty expected in the lecturers' courses. It was found that these actually matched. The mathematics topics mentioned as being used in first year engineering courses either for understanding of new material of the course or for solving coursework exercises are the following:

1.0 ALGEBRA
1.1 Operations with fractions
1.2 Multiplication of brackets
1.3 Laws of exponents
1.4 Transposition of formulae
1.5 Solution of linear equations
1.6 Solution of sets of simultaneous linear equations
1.7 Solution of quadratic equations
1.8 The summation symbol

2.0 INEQUALITIES
2.1 Solution of inequalities in one variable

3.0 LOGARITHMS
3.1 Properties of logarithms
4.0 COMPLEX NUMBERS
   4.1 Basic operations
   4.2 Use of notation $e^{i\phi}$
   4.3 Modulus of a complex number

5.0 TRIGONOMETRY
   5.1 Definition of the trigonometric functions in a right angled triangle
   5.2 Values of sine and cosine of $\pi/6$, $\pi/4$, $\pi/3$ etc.
   5.3 General solution of the equations $\sin\theta = n$ and $\cos\theta = n$, where $-1 \leq n \leq 1$
   5.4 Use of formulae to expand $\sin(x \pm y); \cos(x \pm y)$
   5.5 Use of formulae for $\sin x \pm \sin y; \cos x \pm \cos y$
   5.6 Sine and cosine rules
   5.7 Sine and cosine graphs

6.0 BINOMIAL THEOREM
   6.1 For positive and integral exponent
   6.2 For fractional exponents

7.0 DIFFERENTIAL CALCULUS
   7.1 Differentiation of trigonometric functions
   7.2 Differentiation of algebraic functions
   7.3 Differentiation of functions involving exponentials and logarithms
   7.4 General rules for differentiation
      7.4.1 Sum of two functions
      7.4.2 Product of two functions
      7.4.3 Quotient of two functions
      7.4.4 Function of a function
8.0 INTEGRAL CALCULUS

8.1 Integration of algebraic functions

8.1.1 Functions involving trigonometric functions
8.1.2 Functions involving e^{ax}
8.1.3 Functions involving ln(x)

8.2 Techniques for evaluating integrals

8.2.1 Integration by substitution
8.2.2 Integration by parts

The degree of difficulty of each of these topics is illustrated by the examples listed in Tables 3.1 and 3.2. I found that the examples listed for each topic by different respondents either were of the same kind, or they involved the same complexity.

As far as the format of the questionnaire is concerned, it was found that the lecturer needed to spend quite a lot of time on filling in the questionnaire. In order to avoid that this happened to future lecturers, the format of the questionnaire was changed, the examples given by lecturers were used in the new version to facilitate the process of filling in the questionnaire, and lecturers were not asked to list the engineering topics in which they used the mathematical topics presented but to draw their attention to the illustrative examples (see Section 1.4.1.1 and Appendix B).

3.3 THE MAIN SURVEY

The new version of the questionnaire (see Appendix B) was ready by the Autumn term of the academic year 1978-79. The main survey was limited to institutions of higher education in which students entering engineering courses were likely to have only a single A-level in mathematics, or an
equivalent. The reason for this criterion of selection was that students who have done a single A-level in mathematics are likely to have more weaknesses in their mathematical knowledge than those who have done double A-level in mathematics (González-León 1979).

3.3.1 Departments and Lecturers Involved

Ten institutions (see Appendix H) were approached by means of a letter sent to 10 lecturers at the corresponding Engineering Faculty of each institution. This letter contained:

(a) A paper which explained the purpose of the survey and asked the lecturer's collaboration in the distribution of copies of the questionnaire among colleagues in his department.

(b) A copy of the final version of the questionnaire.

(c) A form which the lecturer should return indicating whether he was willing to complete the questionnaire, and the number of copies of the questionnaire that he might distribute to colleagues in his department.

Three institutions did not request additional copies of the questionnaire and only one did not participate in the survey (the lecturer did not reply to the letter).

A total of 50 copies of the questionnaire were requested and after two months 25 copies had been returned. The breakdown of the numbers of copies requested and returned is given in Table 3.3.
TABLE 3.3
Numbers of Copies of the Questionnaire Requested and Returned in the Main Survey.

<table>
<thead>
<tr>
<th>INSTITUTION</th>
<th>No. Copies of the Questionnaire Requested</th>
<th>Returned</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>G</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>TOTAL</td>
<td>50</td>
<td>25</td>
</tr>
</tbody>
</table>

The courses taught by the 25 respondents in their Departments were the following:

1. Department of Aero/Mechanical Engineering
   (a) Engineering Thermodynamics
   (b) Fluid Mechanics
   (c) Mathematics
   (d) Numerical Methods
   (e) Statistics
   (f) Strength and Materials

2. Department of Civil Engineering
   (a) Applied Mechanics
   (b) Basic Mathematics
   (c) Statistics
   (d) Strength and Materials
Since the copies of the questionnaire were sent upon request to the lecturers approached in each institution, it is not possible to know the size of the sample that actually received the questionnaire. This means
that one cannot be sure of whether the number of copies returned represents the whole group who received the questionnaire. Nevertheless, my main concern in the survey was with the level of difficulty that lecturers expected in the topics they mentioned as needed in their courses (i.e. column D in the questionnaire) rather than with the tabulation of the responses given in the questionnaire. For this reason, I shall simply present the essence of the responses (i.e. the contents of column D in the questionnaire). This allows one to judge the meaning of the topics mentioned as used.

The resulting list of topics and their corresponding level of difficulty are summarised in Table 3.4. An analysis of this Table showed that:

(a) The fact that one example was provided in column B for each topic in column A made the respondents have uniformity in their interpretation of the topics. All examples given by them in column D corresponded to the mathematical topics in column A.

(b) In all topics the level of difficulty required could be identified. There were cases in which the examples listed for a particular topic apparently had different levels of difficulty. This was so for the example in the topic 'Solution of Inequalities' where the examples given by different respondents were: Solve for $x : 3 - 2x > 4$ and $\frac{3x - 1}{2x - 3} < 4$. This would appear to indicate that the level of difficulty expected in the topic depends on the course the lecturer teaches.

(c) There were examples in some of the cells in column D which could have been given to illustrate the level of difficulty of other topics in column A. For example the problem : Find $\frac{dy}{dt}$ if $y = te^{-4t}$, was given to
<table>
<thead>
<tr>
<th>TOPIC</th>
<th>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication of brackets</td>
<td>Expand and simplify ((a+x)(a-x)^2)</td>
</tr>
<tr>
<td>Laws of exponents</td>
<td>Express as a single power of (x: \frac{(x^3)^{1/2}}{(x^2)^3})</td>
</tr>
<tr>
<td>Operations with fractions</td>
<td>(\frac{3}{7} + \frac{5}{9})</td>
</tr>
<tr>
<td>Solution of quadratic equations</td>
<td>Solve the equation (2x^2 - 5x - 1 = 0)</td>
</tr>
<tr>
<td>Solution of linear equations</td>
<td>Solve the equation (10x + 20 - x - 10 = 40)</td>
</tr>
<tr>
<td>Transposition of formulae</td>
<td>Solve for (T:) (W = \frac{V}{K} - \frac{TR^2}{K^2}, R \neq 0)</td>
</tr>
</tbody>
</table>
| Solution of a set of simultaneous linear equations with two or three unknowns | Solve the sets of equations \[
\begin{align*}
3x + 4y &= 18 \\
4x - 2y &= 2
\end{align*}
\]  | Solve a set of three homogeneous equations with three unknowns |
<p>| The summation symbol                       | Evaluate (\sum_{r=1}^{5} (r^3 + 1))       |
| Solution of inequalities with one variable | Solve for (x:) (3 - 2x &gt; 4)          | Find the range of (x) for which (\frac{3x - 1}{2x - 3} &lt; 14)  |</p>
<table>
<thead>
<tr>
<th><strong>LOGARITHMS</strong></th>
<th><strong>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Theory</strong></td>
<td>Given $N = \alpha^x$, the student should be able to write $x = \log_\alpha N$.</td>
</tr>
<tr>
<td><strong>Properties</strong></td>
<td>Expand $\log \left( \frac{\alpha^n \beta^m}{\gamma^p} \right)$</td>
</tr>
<tr>
<td><strong>Complex Numbers</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Modulus</strong></td>
<td>Find the modulus of a complex number $a + bi$.</td>
</tr>
<tr>
<td><strong>Conversion of a complex number from binomial to trigonometric form</strong></td>
<td>Express a complex number $z = a + bi$ in the form $z = r (\cos \theta + i \sin \theta)$.</td>
</tr>
<tr>
<td><strong>Addition and Subtraction</strong></td>
<td>Given $z_1 = 9 + 5i$ and $z_2 = 12 + 4i$, find $z_1 + z_2$ and $z_1 - z_2$.</td>
</tr>
<tr>
<td><strong>Multiplication and division</strong></td>
<td>Given $z_1 = 2 + 3i$ and $z_2 = 4 - 2i$, find $z_1 z_2$ and $\frac{z_1}{z_2}$.</td>
</tr>
<tr>
<td><strong>Trigonometry</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Definition of the trigonometric functions in a right-angled triangle</strong></td>
<td>Application of the definitions in problem-solving.</td>
</tr>
<tr>
<td><strong>Numerical values of the sine and cosine of $\pi/6$, $\pi/4$, etc.</strong></td>
<td>Evaluate $\sin^2 \frac{\pi}{3} - \cos \pi$.</td>
</tr>
<tr>
<td><strong>Trigonometric equations</strong></td>
<td>General solution of $\sin \theta = \frac{1}{2}$. Find the values of $\theta$ which are solutions of $\cos \theta = \frac{\alpha}{\beta}$ $\left[-1 \leq \frac{\alpha}{\beta} \leq 1\right]$.</td>
</tr>
<tr>
<td>TOPIC</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</td>
</tr>
<tr>
<td>-------------------------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td><strong>TRIGONOMETRY</strong></td>
<td></td>
</tr>
<tr>
<td>Addition and subtraction formulae</td>
<td>Expand ( \sin (\omega t - \frac{2\pi}{3}) )</td>
</tr>
<tr>
<td></td>
<td>Expand ( \cos (\alpha + 12\theta) )</td>
</tr>
<tr>
<td></td>
<td>Expand and simplify ( 4\sin \omega t + 5\sin (\omega t + \frac{\pi}{3}) )</td>
</tr>
<tr>
<td>Sum and difference of sines and cosines</td>
<td>Simplify ( \sin (3\omega t + \frac{2\pi}{3}) + \sin (2\omega t + \frac{\pi}{4}) )</td>
</tr>
<tr>
<td>Sine and cosine rules</td>
<td>Application of the rules in problem-solving</td>
</tr>
<tr>
<td>Sine and Cosine graphs</td>
<td>Identification and sketching</td>
</tr>
<tr>
<td><strong>BI NOMIAL THEOREM</strong></td>
<td></td>
</tr>
<tr>
<td>For positive integral exponent</td>
<td>Expand ((1+x)^n ) and ((1-x)^n)</td>
</tr>
<tr>
<td>For fractional exponent</td>
<td>Find the first three or four terms of ((1+x)^{\frac{1}{4}}) and ((1+x)^{\frac{1}{5}})</td>
</tr>
<tr>
<td><strong>DIFFERENTIAL CALCULUS</strong></td>
<td></td>
</tr>
<tr>
<td>Differentiation of general functions</td>
<td>If ( y = 4x^2 - 17x + 2 ), find ( \frac{dy}{dx} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = \ln x ), find ( \frac{dy}{dx} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = t e^{-nt} ), find ( \frac{dy}{dt} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = \ln (a+bt) ), find ( \frac{dy}{dt} )</td>
</tr>
<tr>
<td>Differentiation of trigonometric functions</td>
<td>If ( y = 4\pi r^2 \cos \theta ), find ( \frac{dy}{d\theta} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = \sin^2 x - \cos x ), find ( \frac{dy}{dx} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = \sin^2 \theta ), find ( \frac{dy}{d\theta} )</td>
</tr>
<tr>
<td>Differentiation of the product of two functions</td>
<td>If ( y = x \sin x ), find ( \frac{dy}{dx} )</td>
</tr>
<tr>
<td></td>
<td>If ( y = (a+x)^2 (b+x)^3 ), find ( \frac{dy}{dx} )</td>
</tr>
<tr>
<td>Topic</td>
<td>Example of the Level of Difficulty Expected</td>
</tr>
<tr>
<td>-------------------------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td>Differentiation of the quotient of two functions</td>
<td>If $y = \frac{x-2}{x^2+2}$, find $\frac{dy}{dx}$</td>
</tr>
<tr>
<td>Differentiation of a function of a function</td>
<td>If $y = \sqrt{\frac{x}{a+x}}$, find $\frac{dy}{dx}$</td>
</tr>
<tr>
<td>Integration of algebraic functions</td>
<td>Evaluate $\int (7x^2 + 2x + 3) dx$</td>
</tr>
<tr>
<td>Integration of functions involving trigonometric functions</td>
<td>Evaluate $\int \sin^2 \omega t dt$</td>
</tr>
<tr>
<td>Integration of functions involving $e^{ax}$</td>
<td>Evaluate $\int e^{ax} dx$</td>
</tr>
<tr>
<td>Integration of functions involving logarithms</td>
<td>Evaluate $\int x \ln x dx$</td>
</tr>
<tr>
<td>Integration by substitution</td>
<td>Evaluate $\int \frac{dx}{x^2 + 2}$</td>
</tr>
<tr>
<td>Integration by parts</td>
<td>Evaluate $\int x^2 e^{2x} dx$</td>
</tr>
<tr>
<td>Integration by partial fractions</td>
<td>Evaluate $\int \frac{dx}{x + \sqrt{(x+2)}}$</td>
</tr>
</tbody>
</table>
General Functions', however, it also illustrates the level of difficulty expected in the topics 'Differentiation of the Product of two Functions' and 'Differentiation of the Function of a function'.

(d) The topic 'Integration by Partial Fractions' was added to the list by some respondents.

3.4 CONCLUSIONS

The main purpose of this survey was to ascertain those topics in mathematics which are actually used in first year engineering courses at institutions of higher education, particularly during the first term, and what is more important, to determine the level at which such topics are used.

The list of topics and the level of difficulty expected are summarised in Table 3.4. By comparing these topics with the A-level mathematics syllabuses analysed in the survey carried out by the Mathematical Education Committee (1976), I found that there were certain topics required by engineering lecturers which do not appear in the mathematics syllabuses of some Examining Boards (see Tables 3.5 and 3.6). The critical topics seem to be Solution of Inequalities, Binomial Theorem for Fractional Exponent, and Basic Operations with Complex Numbers. A conclusion that can be drawn from the above fact is that Engineering Departments might expect some of their newly enrolled students to know some mathematical topics, needed to understand the material in the course or for problem solving, which the students have not studied at school, because of the particular examination syllabus for which they had been prepared.
# TABLE 3-5

## MATHEMATICS SYLLABUSES AND EXAMINING BOARDS

<table>
<thead>
<tr>
<th>BOARD</th>
<th>SYLLABUS TITLE</th>
<th>SYLLABUS ABBREVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Associated Examining Board</td>
<td>Mathematics (pure and applied)</td>
<td>AEB PA</td>
</tr>
<tr>
<td></td>
<td>Mathematics (modern)</td>
<td>AEB M</td>
</tr>
<tr>
<td>University of Cambridge Local Examination</td>
<td>Mathematics Syllabus A</td>
<td>CL A</td>
</tr>
<tr>
<td>Syndicate</td>
<td>Mathematics Syllabus D</td>
<td>CL B</td>
</tr>
<tr>
<td>Joint Matriculation Board</td>
<td>Mathematics Syllabus A</td>
<td>JMB A</td>
</tr>
<tr>
<td></td>
<td>Mathematics Syllabus D</td>
<td>JMB B</td>
</tr>
<tr>
<td>School Examination Department, University of</td>
<td>Pure Mathematics with Statistics</td>
<td>JMB S</td>
</tr>
<tr>
<td>London</td>
<td>Mathematics Syllabus C</td>
<td>LC</td>
</tr>
<tr>
<td></td>
<td>Pure Mathematics with Statistics</td>
<td>LS</td>
</tr>
<tr>
<td>Northern Ireland GCE Examination Board</td>
<td>Mathematics Syllabus D</td>
<td>NIB</td>
</tr>
<tr>
<td>Oxford and Cambridge Schools Examination Board</td>
<td>Mathematics</td>
<td>O &amp; C</td>
</tr>
<tr>
<td>Welsh Joint Education Committee</td>
<td>Mathematics</td>
<td>W</td>
</tr>
</tbody>
</table>

# TABLE 3-6

## TOPICS MISSING IN SOME SYLLABUSES

<table>
<thead>
<tr>
<th>TOPIC</th>
<th>Syllabuses which do not contain the given topic.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution of inequalities</td>
<td>AEB PA AEB M AEBM CLA CLD JMB A JMB B JMS O &amp; C LS NIB OAC W</td>
</tr>
<tr>
<td>Basic operations with complex numbers</td>
<td>x x x x x x</td>
</tr>
<tr>
<td>Trigonometric equations</td>
<td>x x x x</td>
</tr>
<tr>
<td>Use of formulae for sin x sin y and cos x cos y</td>
<td>x x x x x x</td>
</tr>
<tr>
<td>Binomial Theorem for fractional exponent</td>
<td>x</td>
</tr>
</tbody>
</table>

(x) optional

(1) This Table is part of a Table that appears in the paper by The Mathematical Education Committee (1976).

(2) This Table is based on a similar Table that appears in the paper by The Mathematical Education Committee (1976).
By contrasting the results of the survey summarised in Table 3.4 with the list in the note prepared by the Professors of Mechanical Engineering in the Northern Universities (see Section 3.1), the following issues emerged:

(a) Some topics mentioned in the list in the note do not appear in the Table.

(b) Some topics mentioned in the survey do not appear in the list in the note.

Issue (a) seems to be logical because the list in the note covers those topics in sixth form mathematics which will be used by students sometime during their three year course whereas Table 3.4 only covers the mathematical topics used particularly during the first term of the first year courses (I should point out that respondents to the survey also said that all the topics in Table 3.4 were also used during the second and third terms of their courses).

Issue (b) seems to be justified by the following reasons:

(i) The list in the note includes only those topics which are needed by students entering courses in mechanical engineering and engineering science and therefore does not include those topics which are needed by students entering courses in other engineering departments.

(ii) Some of the topics in Table 3.4, such as operations with fractions and laws of exponents, do not appear in sixth form mathematics syllabuses but they are very elementary and are very likely to be in 0-level mathematics courses.
Some of the topics in Table 3,4 are implicit in other topics mentioned in the list in the note (e.g. modulus of a complex number is not mentioned in the note but is implicit in the topic 'Conversion of Complex Numbers from Rectangular to Polar Form' which actually is in the note).

Those topics mentioned in the survey and which do not appear in the note prepared by the Professors of Mechanical Engineering of the Northern Universities are the following:

**Algebra**

(a) Multiplication of brackets  
(b) Laws of exponents  
(c) Operation with fractions  
(d) Solution of linear equations  
(e) Transposition of formulae  
(f) The summation symbol  
(g) Solution of inequalities with one variable

**Trigonometry**

(a) Numerical values of the sine and cosine of $\pi/6$, $\pi/4$, $\pi/3$ etc  
(b) Trigonometric equations

**Logarithms**

Properties of logarithms

**Complex Numbers**

(a) Modulus of a complex number  
(b) Addition and subtraction of complex numbers
Binomial Theorem

Binomial Theorem for fractional exponent

Integral Calculus

(a) Integration of algebraic functions
(b) Integration of functions involving $e^{ax}$
(c) Integration of functions involving logarithms

I should note here that to know what is expected in each of the topics listed above it is necessary to refer to Tables 3.1, 3.2 and 3.4, and to the list of objectives given in Section 4.7.1.

Two different levels of difficulty were found in the examples given by lecturers in the questionnaire. In one case, the level of difficulty expected in a topic depended on the mastery of the knowledge and/or skill regarded by the topic; for example, the solution for $x$ of the inequalities $3 - 2x > 4$ and $\frac{3x - 1}{2x - 3} < 4$ basically depends on the skill in solving inequalities. In the other case, the level of difficulty expected in a topic involved a prior mastery of a prerequisite knowledge and/or skill; for example, the problem: Find $\frac{dy}{dx}$ if $y = te^{-4t}$ was given as an example to illustrate the level of difficulty expected in the topic 'Differentiation of General Functions'. However, it requires the prior mastery of the knowledge of the differentiation of the product of two functions and of the differentiation of the function of a function.

The data gathered by means of this survey was used in the development of a battery of mathematics diagnostic tests for students entering engineering courses. This is the subject of the next chapter.
CHAPTER FOUR

A BATTERY OF MATHEMATICS
DIAGNOSTIC TESTS
One of the aims of the present work was to design a mathematics 'diagnostic test' based on the mathematical knowledge and skills that students entering engineering courses are expected to have by the time they actually start their university courses, i.e. based on the prerequisite mathematical knowledge and skills. The purpose of this chapter is to discuss how such a test was developed over the past three years. Before going into the details of how this test was developed, I shall present some general aspects of tests which are relevant to the design of the 'diagnostic test'.

4.2 DEFINITION OF THE TEST

The test required should have the following three characteristics:

It should:

(1) Cover those aspects in mathematics which are prerequisites for the attainment of the objectives of the first year engineering courses.

(2) Pinpoint the causes of deficiencies in each of the prerequisites mentioned in (1).

(3) Enable us to give a description of what a student can and cannot do without reference to the performance of others.

According to the literature on tests, the test to be developed had the functions of a placement test and of a diagnostic test, and the interpretation of the results should be criterion-referenced due to the following reasons:

(a) It had the function of a placement test in the sense that a
placement test either attempts to ascertain the extent to which students possess the skills and ability that are needed to succeed in a particular task - Readiness test, or it attempts to ascertain the extent to which students have already achieved the intended objectives of the planned instruction - Advanced Placement Test (Brown 1976, Gronlund 1977, Chase 1978). In this case, aspect (1) above is in line with the purpose of a Readiness Test.

(b) It has the function of a diagnostic test in the sense that the purpose of a diagnostic test is to determine the underlying causes of a student's learning difficulties in a particular area of the subject-matter of a course (Thorndike and Hagen 1969, Bloom et al 1971, Collins et al 1976, Mehrens and Lehmann 1978) and this is what is stated in aspect (2) above.

(c) The interpretation of the test results should be criterion-referenced because the function of this type of interpretation is to ascertain a student's status with respect to some explicit criterion (Popham and Husek 1969, Ebel 1972, Gage and Berliner 1975, Gronlund 1977). The tests that are interpreted using this method are called Criterion-referenced tests (CRTs) in contrast with Norm-referenced tests (NRTs).

Since the primary function of the test to be developed is to diagnose, I shall refer to it by simply saying a 'diagnostic test', but it should be borne in mind that this test is extensive in coverage of content and may therefore be called extensive diagnostic test in contrast with the common diagnostic test which may well be called intensive diagnostic test due to its coverage of content being more restricted and intensive (i.e. the test has a large number of items on each topic area).
With these observations in mind, I gathered from the literature on placement, diagnostic and criterion-referenced tests those aspects that seemed to be particularly relevant to the design and development of the mathematics diagnostic test. These aspects are discussed in the following section. Since the bulk of the literature on CRTs usually examines the characteristics on CRTs by contrasting them with the characteristics of NRTs, I shall discuss in the same manner the aspects mentioned above.

4.3 THE CONSTRUCTION OF THE TEST: PRELIMINARY CONSIDERATIONS

4.3.1 Variability

The variability of a set of scores refers to the extent to which the scores are spread out or dispersed from the average score of the group. With NRTs the more variability in the scores the better, but with CRTs, variability is irrelevant because in CRTs the meaning of the score is not dependent on comparisons with other scores (Popham and Husek 1969, Smith 1974, Chase 1978). If the whole group of students meets the criterion (as it is hoped in a CRT), the variability will be very small and this does not mean that the test is poor.

In our own case, it is expected that students have a grasp of the material in the test because it tests very basic prerequisite knowledge, that is, the variability of the scores is expected to be very small.

4.3.2 Validity

In a very general sense, the validity of a test is concerned with the extent to which the test measures what it is supposed to measure. Of the
great varieties of validity (see Ebel 1972) most authors agree that
'content validity' is of prime importance for CRTs because a CRT requires
items which are constructed from objectives expressed in behavioural terms
and the content validity is concerned with the degree to which the test
adequately assesses the behaviour as specified in the objective.

The content validity of a test is defined as:

'a way of estimating how adequately the content of
the test actually samples the behaviour or content
domain about which inferences are to be made.'

(Popham 1978, p. 34)

Two principal techniques for determining the content validity of a CRT
have been suggested: One of these techniques suggests the use of the
test designer as a judge (Popham and Husek 1969, Klein 1974, Gronlund
1977, Chase 1978). According to this technique, the designer of the test
should: (a) identify the subject-matter topics and behavioural outcomes
to be measured, (b) judge relative importance of each topic, (c) build a
table of specifications which specifies the sample of items to be used,
and (d) construct a test that closely fits the table of specifications.

The other technique suggests the use of judges other than the designer
of the test, i.e. people who are familiar with the subject-matter topics
of the test (Bloom et al 1971, Klein 1974, Wedman 1974). These judges
should be given the table of specifications and as full explanations as
possible of the meaning of the rows and columns in the table (i.e. the
subject-matter and the behavioural outcomes). Given this information the
judges should be asked to match the items in the test to cells in the
table. If the agreement is less than 50 percent, the items should then be
re-examined.
The reliability of a test 'refers to the consistency of test-scores - that is, to how consistent they are from one measurement to another' (Gronlund 1977, p. 138). With NRTs there are five types of reliability indices typically used, namely, (1) stability (Test-retest method), (2) equivalence (Equivalent forms method), (3) equivalence and stability (Test-retest method with equivalent forms), and two indices of internal consistency - one is estimated by using the Spearman-Brown formula (split-half method), and the other is estimated by using the Kuder-Richardson formulae (Method of Rational Equivalence). Unfortunately, these indices are inappropriate for CRTs because of their dependence on score variability (Popham and Husek 1969, Smith 1974, Gronlund 1977; Chase 1978).

Since it is hoped that all students meet the criterion in a CRT (see Section 4.3.1), all students could get perfect or near perfect score on the test. This means that the test will have little or no variability at all and therefore, the traditional procedures for estimating the reliability of a test would produce a zero reliability coefficient, even though the test may be very adequate for our purpose.

Popham and Husek (1969), who popularized the use of CRTs, suggest the use of indices that reflect the ability of a test to produce variation from pre-instruction to post-instruction testing, but they do not give any concrete procedure for estimating the reliability.

A number of attempts have been made to modify or create procedures for estimating the reliability of CRTs (see Wedman 1974) but a satisfactory procedure has not yet been agreed upon (Gronlund 1976). To increase the
Reliability of a CRT, Gronlund (1977) recommends including five items for each specific objective in which a decision concerning mastery-nonmastery should be taken. Klein (1974) found in an informal survey of CRTs that the usual practice is to construct approximately three to five items per objective.

In the early version of the mathematics diagnostic test only one item each specific objective was included for most objectives because the test should be kept as short as possible (see Section 4.5.2). However, students' performance on an open-ended version of this test (see Appendix I) revealed that there were some items (i.e. problems) in which it was not possible to ascertain whether the reason why students answered them incorrectly was because of deficiency in their knowledge or because of carelessness in their working (see Section 4.6.1). For this reason, it was decided to increase to three the number of items per objective. This is discussed in Section 4.7.3.

4.3.4 Usability

This is another attribute which has to be considered in the design of the mathematics diagnostic test. Lyman (1978), regarding usability of a test points out:

"We include here all the many practical factors that go into our decision to use a particular test ... Under usability, we deal with all sorts of practical considerations. A longer test may be more reliable, even more valid; however, if we have only a limited time for testing, we may have to compromise with that idea." 

(Lyman 1978, p. 39)
In our own case, the time to be spent in completing the test was limited to about sixty minutes because this was the length of time of a normal class period. Since the subject-matter content of the test was quite extensive, it was not possible to include a large number of items for each specific objective. At this point we should also bear in mind that attribute usability is also important for our test. This matter is treated in more detail in section 4.5.4.3.

I shall now turn to the details of the design of the mathematics diagnostic test.

4.4 EARLY EXPERIENCES AT THE UNIVERSITY OF SOUTHAMPTON

Students entering engineering courses at the University of Southampton in 1976 and 1977 were given a mathematics pre-knowledge test during their first week at the University. This test was used in conjunction with discussions for the purpose of individual diagnosis (see the T.T. method in Section 5.2) but in itself it did not have a diagnostic function as defined in Section 4.2. However, this test was the starting point of the battery of mathematics diagnostic tests that I have developed.

The following sub-sections describe the test used at the University of Southampton on both occasions.

4.4.1 The 1976 Experience

The test used in 1976 was a modified version of the test constructed by the Physics Interface Project (see Appendix J). The Physics Interface Project mathematics test consisted of fifty multiple-choice items which had been constructed with the help of a number of university lecturers in
considered to be basic to the study of first year physics by the
lecturers involved (Sutton 1972).

The reason why this test was modified before it was used at the University
of Southampton was because, as I said above, it had been designed for
students entering physics degree courses and it would be used at the
University of Southampton with students entering engineering courses. The
modifications were carried out bearing in mind the mathematics considered
prerequisite for first year mathematics course for engineers. I should
point out here that I did not participate in the construction of this
version because it had already been done when I actually got involved in
the present research work. The test in question dealt with the following
topics:

<table>
<thead>
<tr>
<th>TOPIC</th>
<th>NO. ITEMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td>4</td>
</tr>
<tr>
<td>Inequalities</td>
<td>1</td>
</tr>
<tr>
<td>Binomial Theorem</td>
<td>1</td>
</tr>
<tr>
<td>Limits</td>
<td>2</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>4</td>
</tr>
<tr>
<td>Logarithms and Exponentials</td>
<td>1</td>
</tr>
<tr>
<td>Co-ordinate Geometry</td>
<td>1</td>
</tr>
<tr>
<td>Differential Calculus</td>
<td>2</td>
</tr>
<tr>
<td>Integral Calculus</td>
<td>4</td>
</tr>
</tbody>
</table>

Since the main purpose of the test was to test mathematics which was
considered prerequisite for the first year mathematics course, the
material in the test was compared with the content outline of the
syllabus of the mathematics course in order to see whether the test
content was valid, i.e., to see whether the material in the test would actually be needed in the mathematics course. From this comparison I found that there was a topic - Co-ordinate Geometry, which did not appear to be connected with the content outline of the mathematics syllabus. Discussing this matter with the mathematics course lecturers, it was agreed that what was really needed was certain knowledge of rectangular co-ordinates in order to work with geometrical representation of complex numbers in a late unit of the main mathematics course (see Section 5.5.1). The overall students' performance on the test is discussed in Section 6.2.1.

4.4.2 The 1977 Experience

In the light of the 1976 experience, the test was revised.* The main aspects taken into account for the revision were the following:

(a) The item on Co-ordinate Geometry was excluded from the test due to the reason given above.

(b) Since a high percentage of students had difficulties with Limits because they had forgotten the material or found the material new for them (see Section 6.2.1), Limits was included in the main mathematics course for the following year and those items regarding Limits were also excluded from the test.

This test (see Appendix K) was given to all students entering engineering courses at the University of Southampton in October 1977. Students'
performance on this test (described in Section 6.2.2) revealed that students had difficulty with two items in particular - one of these was on the Binomial Theorem and the other on Trigonometry. Students said (in interviews held with them) that the wording of the stem of the item on the Binomial Theorem was unclear. The stem of this item was:

The first three terms of the expansion \( \frac{1}{(1 - x)^{\frac{3}{2}}} \),

where \( 0 < x < 1 \), are:

(alternatives)

A proposed solution to this problem was to re-write the stem using the following wording:

If \( \frac{1}{(1 - x)^{\frac{3}{2}}} \) is expanded as a series in powers of \( x \), for \( 0 < x < 1 \), then the first three terms of the series are:

(alternatives)

The stem of the item on Trigonometry was:

\( \cos(\theta + \frac{\pi}{2}) \) is the same as:

(alternatives)

and it seems that students interpreted \( \cos(\theta + \frac{\pi}{2}) \) as \( \cos(\frac{\theta + \pi}{2}) \). This misunderstanding could be avoided by changing \( \cos(\theta + \frac{\pi}{2}) \) to either \( \cos(\theta + \frac{\pi}{2}) \) or \( \cos(\theta + \frac{3\pi}{2}) \).
The mathematics test used in 1976 and 1977 at the University of Southampton was a revised version of the Physics Interface Project mathematics test. The latter had to be revised because it had been designed for students entering Physics degree courses and it would be used with students engineering courses. The revised test was used in conjunction with discussions for the purpose of individual diagnosis but it itself did not have a diagnostic function as defined in Section 4.2.

At this point the necessity of constructing a mathematics diagnostic test as defined in Section 4.2 emerged. The following sections deal with the steps towards the construction of this mathematics diagnostic test.

4.5 TOWARDS A MATHEMATICS DIAGNOSTIC TEST

After the pre-knowledge test mentioned in the previous section had been given to students entering engineering courses at the University of Southampton in 1977, it was felt necessary to construct a mathematics diagnostic test as it is defined in Section 4.2, that is, a test which has the functions of a placement test and of a diagnostic test, and whose results are interpreted as for criterion-referenced tests. In the following sub-sections, I discuss how such a test was developed.

4.5.1 The Placement Function

I have said earlier that a placement test attempts to ascertain the extent to which students possess the skills and ability that are needed to succeed in a particular task (see Section 4.2), for this reason the first step towards the construction of the mathematics diagnostic test was to
find out those topics in mathematics which were actually needed in the study of first year engineering courses other than mathematics. To do this I surveyed first year engineering lecturers in two universities with the purpose of gathering their opinions about the mathematics required by their courses and which was assumed to be known by students when they entered university (see Section 3.2).

The information gathered in this survey, which is summarized in Table 3.1, led me to the formulation of the following list of objectives:

The student entering engineering courses should be able to:

1. Apply the correct rules for addition, subtraction, multiplication and division of two fractions.

2. Apply the correct laws of exponents in expressions involving a single variable.

3. Multiply and simplify expressions involving \((a + b)(c + d)\) or \((a \pm b)^2\).

4. Transpose formulae.

5. Solve quadratic equations using the general formula.

6. Solve inequalities with one unknown.

7. Apply the properties of logarithms.

8. Find the modulus of a complex number \(z = a + bi\).

9. Find the numerical values of expressions involving the sine and
cosine of angles as a fraction or multiple of π.

10. Apply the corresponding formula to expand sin(a ± b) and cos(a ± b).

11. Apply the corresponding formula to evaluate sin a ± sin b and cos a ± cos b (in both senses).

12. Expand as a series of powers of x the binomial \((1 + x)^n\), where \(n\) is a fraction.

13. Apply the general rules for the differentiation of algebraic functions.

14. Differentiate functions that involve sine and cosine.

15. Apply different techniques for evaluating integrals.

This list of objectives does not include some of the topics in Table 3.1 because the objectives were written before the pilot survey ended.

After the writing of these objectives the next task was to prepare the test specifications. This is treated in the following sub-section.

4.5.2 Test Specifications and Test Length

The relationship between the subject-matter content of the test and the objectives listed in the previous sub-section are shown in Table 4.1. This Table also shows how these objectives are related to Bloom's taxonomy (Bloom et al 1971) and the number of items for each objective. In constructing the Table, the length of the test was also taken into
TABLE 4-1

TABLE OF SPECIFICATIONS: A MATHEMATICS DIAGNOSTIC TEST

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Objective</th>
<th>Level</th>
<th>No. Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 ALGEBRA</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1 Operations with fractions</td>
<td>1</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Laws of exponents</td>
<td>2</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Multiplication of brackets</td>
<td>3</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>1.4 Transposition of formulae</td>
<td>4</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>1.5 Quadratic equations</td>
<td>5</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>2.0 INEQUALITIES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.1 Solution of inequalities</td>
<td>6</td>
<td>✗ ✗ ✗</td>
<td>2</td>
</tr>
<tr>
<td>3.0 LOGARITHMS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.1 Properties of logarithms</td>
<td>7</td>
<td>✗ ✗</td>
<td>2</td>
</tr>
<tr>
<td>4.0 COMPLEX NUMBERS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.1 Modulus of a complex number</td>
<td>8</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>5.0 TRIGONOMETRY</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.1 Trigonometric function values of 3ε, 7/4, etc</td>
<td>9</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>5.2 Addition and subtraction formulae</td>
<td>10</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>5.3 Sum and difference of sines and cosines</td>
<td>11</td>
<td>✗ ✗</td>
<td>2</td>
</tr>
<tr>
<td>6.0 BINOMIAL THEOREM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.1 For fractional exponent</td>
<td>12</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>7.0 DIFFERENTIAL CALCULUS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.1 Differentiation of algebraic functions</td>
<td>13</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>7.2 Differentiation of trigonometric functions</td>
<td>14</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>8.0 INTEGRAL CALCULUS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.1 Evaluation of standard integrals</td>
<td>15</td>
<td>✗</td>
<td>1</td>
</tr>
<tr>
<td>8.2 Techniques of integration</td>
<td>15</td>
<td>✗ ✗ ✗</td>
<td>2</td>
</tr>
<tr>
<td>TOTALS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

A1: Knowledge of specific facts (Knowledge)
A3: Ability to carry out algorithms (Knowledge)
C1: Ability to solve routine problems (Application)
D2: Ability to discover relationships (Analysis)
consideration. There were three factors that influenced the length of the test, namely (1) the purpose of the test, (2) the type of test items, and (3) the time available for testing.

Ideally, the student should be given as much time as he needs to complete the test because the purpose of the test is to ascertain whether the student has a grasp of the mathematics regarded as prerequisite for his engineering courses, but in practice, a normal class period is of about fifty minutes. This in turn led me to include only one item per specific objective and to use items of the multiple-choice type. This particular type of item was selected because of the following reasons:

(a) Multiple-choice items require less time to be answered than open-ended items.

(b) Multiple-choice items can be used to measure both knowledge and intellectual skills.

(c) Multiple-choice items can be constructed in such a way that they have a diagnostic function.

(d) No skill is required to mark the test and the marking can be done in a very short time. Due to this, the student himself can mark the test if he is provided with the answers.

I shall now turn to the actual construction of the items.

4.5.3 Construction of the Items

Once the specifications of the test had been delineated, the next step was to construct the test items to measure the objectives. I took 12 items
because they were suitable according to the specifications in Table 4.1, and I constructed new items (i.e. items 4, 9, 10, 11, 12, 13, 17 and 19 in Appendix L) according to Table 4.1 and the examples given in Table 3.1. Two of the 12 items taken from the Southampton test were slightly modified because of the following reasons:

(a) In item no. 14, alternative (c) was originally $x = \frac{e^a}{e^b}$ and it was changed to $x = e^{a-b}$ to avoid the involvement of algebraic manipulation within the item. This change also made the item more homogeneous (i.e. this alternative had been the only one that included e twice).

(b) In item no. 15 (Binomial Theorem) a new wording was given in the stem because it had been found unclear by students (see Section 4.4.2). The other 8 items were constructed using the standard rules for construction of multiple-choice items (see for example Ebel 1972, Gronlund 1976, Mehrens and Lemann 1978, Popham 1978). In order to make the distractors plausible responses to the items, they were based on common misconceptions or errors that students have (these were based on my own experience as a mathematics teacher). The details of how these eight items were actually constructed are given in Appendix M.

Before the test was ready for the trial stage, some additional details about its construction were considered. These are discussed in the following sub-section.

4.5.4 Some Additional Considerations

4.5.4.1 Item Difficulty

As was noted earlier, the variability of the scores in our test is expected to be very small because the students are expected to have
mastery of the material in the test (see Section 4.3.1). Due to this, the level of difficulty of most of the items in our test should be very low.

Gronlund (1977) regarding this matter of item difficulty points out that:

"If the test is to be criterion referenced, item difficulty is determined by the difficulty of the learning task described in the specific learning outcome...item difficulty should match the difficulty of the task. If the task is easy, the test items should be easy. If the task is difficult, the test items should be difficult. No item should be eliminated simply because most students might be expected to answer it correctly, or because it might be answered incorrectly by most students."

(Gronlund 1977, p. 31)

In conclusion, the level of difficulty of the items in our test depends on the difficulty of the task for which the items are designed.

4.5.4.2 The Test Results Interpretation: The Criteria

Gronlund (1977) regarding the matter of CRTs interpretation points out that:

"In both the construction and the interpretation of a criterion-referenced test, the focus is on the specific behavioural objectives the test is intended to measure. Each set of items is designed to measure
a particular objective as directly as possible,  
and success on the items is interpreted with reference  
to the objectives being measured. Thus, the results  
from a criterion-referenced test are typically  
organized in terms of the measured objectives."

(Gronlund 1977, P. 117-118)

According to Gronlund, an individual's success on the items of a CRT  
depends on the level of proficiency obtained with respect to the  
achievement of the objectives on which the test is based.

In a CRT, the level of proficiency for the success with the items  
designed to measure a particular objective (i.e. the criterion) might  
be set at 90% and then use an 'on-off' approach, that is, either 90%  
minimum has been achieved or it has not (Popham and Husek 1969). Although  
Popham and Husek (1969) regard 90% as the level of proficiency for the  
success on the items for a particular objective, 80% or more is usually  
adopted, that is, the student has to answer 80% or more of the items  
correctly (Smith 1974). An aspect that is important to note here is  
that the criteria of the test should be set prior to the administration  
of the test (Smith 1974, Gage and Berliner 1975) because then the  
criteria cannot be influenced by how well students do on the test.

Since in our own case there is only one item per objective, the criterion  
should be 'to answer the item correctly'. This means that there are  
only two courses of action available to the student (i.e. success or  
failure with respect to the criterion), and therefore, we need to report  
this performance as success or failure (Popham and Husek 1969).
To establish the content validity of the test I used the first of the two procedures described in Section 4.3.2, that is, basing the construction of the test on the test specifications. Then I gave a colleague the test specifications and the items and asked him to match the items to cells in the table (i.e. a procedure similar to the second described in Section 4.3.2). This led to the clarification of some of the levels given to the items according to Bloom's taxonomy (see Table 4.1). It was noted earlier that a method of increasing the reliability of a CRT is by constructing three to five items for each objective (see Section 4.3.3). However, in the present case, only one item for each objective could be included in the test due to the constraints of time for its administration. This means that, in this particular case, attention had to be drawn to the usability of the test rather than to its reliability.

4.5.5 Trying out the Test

When the test was ready for its administration, it was given to 67 sixth form students at three schools* a few weeks before they sat their A-level mathematics examination in 1978. At this point I shall only make reference to the distractors of the items. An analysis of students' performance on each of the items in the test is given in Section 6.3.1.

The analysis of the items showed that there were certain distractors in nine items which were not selected by students. These were the following:

<table>
<thead>
<tr>
<th>ITEM NO.</th>
<th>NO SELECTED DISTRACTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B, C, D</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
</tr>
</tbody>
</table>

* The author is indebted to Mr Howlett from Charterhouse School, to Mr May from Godalming College, and to Mr Weston from Park Barn County Secondary School, the courses mathematics teachers, for their co-operation in the administration of the test.
Since nothing could be said about why students did not select these distractors, the test was lacking of the diagnostic function established in Section 4.2. This fact led me to think of how to improve the distractors in the test. This matter is treated in the section following the summary (i.e. Section 4.6).

4.5.6 Summary

In this section I have discussed the first steps taken towards the construction of a mathematics diagnostic test as it is defined in Section 4.2. The characteristics of placement test and of the criterion-referenced interpretation have been discussed here. Within the context of the definition of our mathematics diagnostic test, the placement function refers to the coverage of the test whereas the criterion-referenced interpretation refers to the method of interpreting the results of the test.

The coverage is given by the objectives derived from the data gathered in the pilot survey of first year engineering lecturers (see Section 4.5.1). These objectives led to the construction of the items in the test (see Sections 4.5.2 and 4.5.3).
I have noted earlier that to increase the reliability of a CRT it was recommendable to construct three to five items per objective (see Section 4.3.3). However, in this test, the attention has to be drawn to the attribute usability rather than to the attribute reliability due to the constraints of time for the administration of the test; that is, the test should be designed so that it could be completed within sixty minutes (see Section 4.5.2). This led me to include only one item per objective and to set the criteria at 100%, that is, the student has already achieved an objective or he has not.

The test was made up of 20 multiple-choice items, 12 of which were taken from the mathematics test used at the University of Southampton in 1977 and 8 items were specially constructed (see Appendix M). This test was tried out with a group of sixth form students a few weeks before they sat their A-level mathematics examination. An analysis of the test items showed that the test was lacking in the diagnostic function established in Section 4.2. The subject of the following sections is what was done to overcome this problem.

4.6 THE DIAGNOSTIC FUNCTION OF THE TEST

It has been noted earlier that the mathematics test should have a diagnostic function (see Section 4.2), i.e. it should pinpoint the causes of deficiencies in each of the topics covered by the test. This means that the distractors of the items in the test play a very important role, namely, they should be able to give a hint as to why a student who lacks the knowledge called for the item selects one of them in preference to the correct answer. If this happened to be so, it would be possible to provide the student with information relevant to the form of remedial work that he needs.
We found in the test mentioned in the previous section that there were nine items whose distractors were not all selected by students (see Section 4.5.5). These distractors had been based on common errors and misconceptions students have according to the teaching experience of their constructors (i.e. the people involved in the construction of the PIP mathematics test on the one hand, and my own experience on the other hand).

In order to select distractors that really played a diagnostic role, I converted the items of this test into open-ended short items, that is, the stems of the items were worded as problems for the student to solve (see Appendix I). This open-ended version was then given to groups of students at the University of Durham\(^1\), and at Queen Mary College, London\(^2\).

The overall students' performance on this test is treated in Section 6.2.3.

The scripts of the test were analysed by going through the solutions for each of the items, particularly through those which led to incorrect answers, trying to find out how students came to particular answers, and pinpointing the causes of errors in the cases in which there were errors. Several problems in the test required the use of a formula in their solutions. It was assumed that the students knew these formulae but it was found on the scripts of the test that some students used wrong formulae and therefore got answers to the problems incorrect. Other items in the test required the student to have some basic knowledge (i.e. a rule, a definition or certain trigonometric function values). Some students had misconceptions of this knowledge or did not have the knowledge at all. To sum up, there were several items in the test which tested knowledge at a higher level than the level some students actually had. A detailed

\(^1\) The author is indebted to Professor H Marsh and Mr H Neill, the course lecturers, for their co-operation in the administration of the test.

\(^2\) The author is indebted to Dr B Chirgwin and Mr D R J Mudge, the course lecturers, for their co-operation in the administration of the test.
4.6.1 Items Which Required the Use of a Formula

Seven items required the use of a formula in their solutions (items no 5, 9, 10, 11, 12, 15 and 20 in Appendix I) and there were several reasons why students got wrong answers to these particular items. These can be categorized in the following three groups:

(a) The use of a wrong formula

Although some students did not actually write down the formula they used to solve the problems in these items, it was obvious from their working on the scripts that they did not use the correct formulae. The following conclusions could be drawn from the analysis of their working on these seven items:

Item No 5

Students used one of the following expressions as the correct formula to find the solutions to the a quadratic equation of the form \( ax^2 + bx + c = 0 \):

\[
(1) \quad x = \frac{-b \pm \sqrt{b^2 - 2ac}}{2a}
\]

\[
(2) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}
\]

\[
(3) \quad x = \frac{-b \pm \sqrt{4ac}}{2a}
\]

These expressions seem to indicate that the students were aware of the existence of \(-b\) in the numerator, of the product \(ac\) under the radical sign,
and of the $z$ in the denominator, however, it seems that they were, for some reason, unable to use the correct formula.

**Items No 9 and 10**
The purpose of these two items was to test whether the students knew how to use the formulae $\cos (A - B) = \cos A \cos B + \sin A \sin B$ (item No 9) and $\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$ (item No 10). The solutions on the scripts of the test revealed that some students used the formulae to expand $\cos(A + B)$, $\sin(A + B)$ or $\sin(A - B)$ in item No 9, and the formulae for $\cos A - \cos B$, $\sin A + \sin B$ or $\sin A - \sin B$ in item No 10. This would appear to have happened because the students were used to work with formula books and these were not allowed for the completion of the test. The fact that they used in these two items the formulae mentioned above is not surprising because for each item, the formulae are apparently similar.

**Item No 11**
This item tested the use of the formula $\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$ but reading it from right to left. Although this fact apparently made the item more difficult than item No 10, the difficulties experienced by students on it were, to some extent, the same as in item No 10, i.e. they were able to identify the kind of formula they needed but were unable to use the correct one.

**Item No 12**
Students used one of the following expressions as the correct formula to find the modulus of a complex number $a + bi$:

1. $|Z| = \frac{b}{a}$
2. $|Z| = \frac{a}{b}$
3. $|Z| = \sqrt{a^2 - b^2}$
4. $|Z| = a^2 - b^2$
Students who used the expression (1) or (2) would appear to have mixed up the definition of 'modulus of $Z$' and the definition of 'argument of $Z$'
\[ \theta = \tan^{-1} \frac{b}{a} \]. Those who used the expression (3) and (4) would appear to have had a misunderstanding of the definition of modulus. This seems to be so because for a complex number $Z = a + bi$, the modulus $|Z|$ is given by $|Z| = \sqrt{a^2 + b^2}$ or $|Z|^2 = a^2 + b^2$ and the students would appear to have taken $|Z| = \sqrt{a^2 + (bi)^2} = \sqrt{a^2 - b^2}$ or $|Z| = a^2 + (bi)^2 = a^2 - b^2$.
Another possibility might have been that they were used to working with formula books and they could only recall what the formula looked like.

**Item No 15**
From the analysis of students' working on this item it was very difficult to draw patterns of errors because the students did not write down the formula they used. They obviously did not use the correct one. This was apparent because not even the coefficients of the variable in the series were right.

The fact that the given expression in this item was $\frac{1}{(1 - x)^2}$ and a large number of students re-wrote it as $(1 - x)^{-\frac{3}{2}}$ makes one think that they knew that they had to use a formula to expand a binomial with a negative exponent but that they might not have remembered the correct expression of the formula.

**Item No 20**
In this item, there were some students who did not seem to know what to do to integrate by parts. They should have used either the expression
\[ \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \] or \[ \int udv = uv - \int vdu \], but since they did not write down the formula they used in their working, it is not possible to say what exactly they did.
It may be convenient to include an item that tests this on a future occasion.

(b) Carelessness in the Application of the Formula

Although some students seemed to have used the correct formula in Item No 5 (they did not actually write it down), their working would appear to indicate that they made careless mistakes when substituting the numerical values in it. This carelessness happened to occur under the radical sign in all cases. The student involved wrote down $25 - 4 \times 2 \times 1$ instead of $25 - 4 \times 2 \times (-1)$ or $25 + 4 \times 2 \times 1$.

(c) Cases in Which it was not Possible to Tell What Went Wrong

There were two cases in which students did not write down the formula they used in Item No 5 and it was not possible to tell from the working what went wrong. In spite of this, the errors committed appeared under the radical sign and the kind of error might be put under either of the two previous categories.

4.6.2 Lack of Basic Knowledge

There were four items which required the students to know either a rule, a definition or certain trigonometric function values which the student did not know. These items were the following:

Item No 6

This item assumes that the student knows the rules for transposition of factors in an inequality but the reality was that some students did not
Know these rules. The item asked the student to solve for $x$ the inequality:

$$3 - 2x > 4$$

I shall solve this inequality using the following four steps in order to make reference to them in the discussion:

(i) $-2x > 4 - 3$

(ii) $-2x > 1$

(iii) $2x < -1$

(iv) $x < -\frac{1}{2}$

Some students failed to proceed correctly in (iii), as they wrote $2x < 1$, and some others went from (ii) directly to the expression $x > -\frac{1}{2}$.

As we can appreciate from these observations, the students did not seem to know the rules for transposition and therefore got the answer to the problem wrong. This would appear to indicate that it may be convenient to include in the test some items that test whether the student knows the rules for the transposition of factors in an inequality.

**Item No 7**

In this case the item in question required to solve the inequality:

$$|x - 3| < 2$$

In spite of having answered Item No 6 correctly, some students seemed to have got confused or did not know the meaning of the bars, as some did not even attempt to solve the problem. Some others solved the inequality as if it had been proposed without the bars (i.e. $x - 3 < 2$). There were also some students who wrote:

$$|x - 2| < 5 \Rightarrow |x| < 5$$

This seems to indicate that they had no idea of the meaning of the bars.
To sum up, the item assumes that the students know the meaning of modulus (i.e. the bars) but their working on the item revealed that some of them did not have this knowledge.

**Item No 8**

This item asked the student to evaluate $\sin^2 \frac{\pi}{2} - \cos \pi$. Students mixed up the values of sine and cosine of $\pi/2$ and $\pi$, as they took $\cos \pi = 1$, $\cos \pi = 0$, $\sin \frac{\pi}{2} = \frac{1}{2}$ or $\sin \frac{\pi}{2} = 0$.

Apparently some students either did not know or remember the values of $\sin \frac{\pi}{2}$ and $\cos \pi$, or had been taught these values in terms of degrees (i.e. $\sin 90^\circ$ and $\cos 180^\circ$). If the latter case happens to be true, then the results may indicate that the students involved cannot convert from radians to degrees and therefore, are unable to evaluate expressions of the type involved in this item. It might be convenient to include in the test an item that tests conversion from radians to degrees.

**Item No 17**

This item asked to find $\frac{dy}{dx}$ when $y = \sin^2 x - \cos x$. The types of errors committed by students in this item can be categorized in the following three groups:

(a) **No Knowledge of the Differential of $\sin x$ and $\cos x$**

Some students apparently took $\frac{d}{dx}(\cos x) = \sin x$. They did not actually write down this expression but this fact can be concluded from their working on the scripts.
To find \( \frac{dy}{dx} \) the students have to be aware of the fact that to differentiate \( \sin^2{x} \) they have to apply the 'chain rule', however, some of them did not seem to have realised this situation and gave one of the following expressions for \( \frac{d}{dx} (\sin^2{x}) \):

(1) \( 2\sin^2{x} \)  
(2) \( 2\cos^2{x} \)  
(3) \( 2\sin{x} \)

In order to ascertain the kind of difficulty that any particular student might have, it may be convenient to test whether the student can apply the chain rule to functions not involving trigonometric functions.

(c) Lack of Skill in Applying the Chain Rule

One student was aware of the need to use the chain rule to find \( \frac{d}{dx}(\sin^2{x}) \) however, he wrote that:

\[
\frac{d}{dx}(\sin^2{x}) = 2\sin{x}\cos^2{x}
\]

This would appear to indicate that he either had a lack of skill in applying the chain rule or he did not actually know how to apply the chain rule.

Item No 19

In this item, students gave one of the following substitutions for \( x \):

\[
x = \sin^2{u}; \quad x = \cos^2{u}; \quad x = \tan{u}
\]

but they did not actually check whether the substitution would allow them
4.6.3. Conclusions

In order to select distractors that really played a diagnostic role, the items in the test mentioned in Section 4.5.5 were converted into open-ended items and the test was given to groups of students at two universities. The students' performance on this test led me to draw the following conclusions:

(a) There were four reasons why students failed items in the test, namely (1) Lack of basic knowledge, (2) Use of wrong formulae, (3) Carelessness in the application of formulae, and (4) Lack of skill in applying rules.

(b) Students' performance on certain items would appear to indicate that they used to work with formula books and therefore had no need to memorize the formulae. If the students are not used to working with formula books and they are expected to know the formulae involved in the test items, the knowledge of these formulae should also be tested. This would also give the test items a diagnostic function.

(c) Certain items in the test should be broken into several items each (i.e. Items No 5, 6, 8, 12, 17 and 20) in order to make them have a diagnostic function. This would enable us to include items that test basic prerequisite mathematical knowledge (i) at the simplest level (i.e. recall of specific information), and (ii) at more complex levels (i.e. solution of problems requiring knowledge of facts, a formula or a rule). This will enable us to pinpoint more precisely where students have difficulty, and suggest the appropriate form of remedial treatment.
(d) Since students may fail items due to carelessness and not to lack of knowledge or skill (see (b) in Section 4.6.1), it may be convenient to increase (to at least three) the number of items per objective involving calculation (e.g. Items No 5, 6, 8 and 17). This would increase the reliability of the test.

4.7 A BATTERY OF DIAGNOSTIC TESTS

4.7.1 The Placement Function of the Battery

In order to increase the placement function of the test the survey mentioned in Section 4.5.1 was extended to other institutions of higher education (this is treated in detail in Chapter 3). The information gathered by means of this survey (summarised in Table 3.4) and the conclusions that came out of the analysis of items given in Section 4.6 led me to the formulation of a list of objectives which students entering engineering courses should have already achieved by the time they enter university. These objectives, which also include the objectives listed in Section 4.5.1 are the following:

The student entering engineering courses should be able to:

1. Apply the correct rules for addition, subtraction, multiplication and division of two fractions.

2. Apply the correct laws of exponents in expressions involving a single variable.

3. Multiply and simplify expressions involving \((a \pm b)(c \pm d)\) and \((a \pm b)^2\).
4. Transpose formulae.

5. Solve quadratic equations using the general formula.

6. Solve sets of simultaneous linear equations with two and three unknowns.

7. Evaluate expressions involving the summation symbol $\Sigma$.

8. Solve simple inequalities with one unknown.

9. Apply the properties of logarithms.

10. Find the modulus of a complex number $Z = a + bi$.

11. Convert a complex number from rectangular to trigonometric form.

12. Find the product of two complex numbers.

13. Find the quotient of two complex numbers.

14. Identify the expressions defining the sine and cosine of an angle in a right angled triangle.

15. Convert angles from degrees to radians.

16. Know the values of sine and cosine of $\frac{\pi}{6}$, $\frac{\pi}{4}$, etc.

17. Solve the equations $\sin \theta = p$ and $\cos \theta = p$ where $-1 \leq p \leq 1$. 
18. Apply the rules for expanding $\sin(A \pm B)$ and $\cos(A \pm B)$.

19. Apply the rules for $\sin A \pm \sin B$ and $\cos A \pm \cos B$.

20. Apply the cosine rule.

21. Recognize the sketch of $\sin nx$ and $\cos nx$, where $n$ is an integer.

22. Expand as a series of powers of $x$ the binomial $(a + b)^n$, where $n$ is a positive integer.

23. Find up to the first three terms of the expansion of powers of $x$ of the binomial $(1 + x)^n$, where $n$ is a fraction.

24. Know the derivative of the sine and cosine.

25. Apply the chain rule to simple functions.

26. Apply the chain rule to functions involving sine and cosine.

27. Differentiate the product of two functions.

28. Differentiate the quotient of two functions.

29. Integrate functions involving standard integrals.

30. Solve simple integrals using the technique of substitution.

31. Solve simple integrals using the technique of partial fractions.
10. Solve simple integrals using the technique of integration by parts.

Table 4.2 shows how the subject-matter content of the test relates to the objectives listed in the previous sub-section. It also shows how these objectives are related to Bloom's taxonomy (Bloom et al. 1971), and the number of items for each objective. These numbers were determined according to the reliability and the diagnostic function of the test (see Sections 4.7.2 and 4.7.3).

Since the number of items in the test should be 105 according to the table of specifications (i.e. Table 4.2), I felt that it was convenient to construct a battery of four short tests which had approximately the same number of items, instead of one long test. Thus the student concerned could take these four tests at his own convenience in one, two, three or four sittings without getting bored or fatigued.

In order to have approximately the same number of items in each test, the tests were designed as follows:

TEST ONE : Algebra (28 items)

TEST TWO : Inequalities, Logarithms, Complex Numbers, Binomial Theorem (24 items)

TEST THREE : Trigonometry (25 items)

TEST FOUR : Differential Calculus and Integral Calculus (28 items).
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A1: Knowledge of specific facts
A2: Knowledge of terminology
A3: Ability to carry out algorithms
B1: Knowledge of concepts
Cl: Ability to solve problems
C2: Ability to make comparisons
D2: Ability to discover relationships
4.7.2 The Diagnostic Function

A conclusion that emerged from an early analysis of a test which had one item for each objective (for most objectives) was that certain items should be broken into several items each in order to make them have a diagnostic function. The new items for each objective should be of different level in the sense that some of them should test knowledge of specific information and the others should test more complex knowledge. Thus enabling one to pinpoint more precisely where a student has difficulty, and suggest the appropriate treatment for each objective (see Section 4.6.3). A similar conclusion came out of the analysis of the survey where two different levels of difficulty were given by lecturers in the examples (see Section 3.4).

The diagnostic function was introduced in the battery described here as follows: when the objective had implied the knowledge of a definition, the meaning of a symbol, a formula or a rule, I included an item which tested this knowledge. In addition to this, I included three more items for those objectives which required algebraic manipulation or calculation (the reasons why three items were included are given further in Section 4.7.3).

The test items in the battery are of two types: (i) multiple-choice items, and (ii) matching items. The reasons for using the former are given in Section 4.5.2, the latter type was introduced because it allows the sampling of larger content and therefore enables one to ask many questions in the limited amount of testing time, and also because in matching items the same set of possible responses can be used for several questions. The test items were constructed according to the specifications in Table 4.2, using the examples given in Tables 3.1, 3.2.
and 3.4, and based on the analyses of items given in this chapter and in Chapter 6. (The battery of tests and their corresponding criteria are given in Appendices P and Q respectively.)

Since each matching exercise should consist of homogeneous items (i.e., items that deal with only a single concept, classification or area), and every response in one column should be a plausible answer to every premise in the other column (Gronlund 1976, Mehrens and Lehmann 1978), the matching exercises were divided into sections by horizontal lines so that each section could be answered independently of the others. Each of these sections corresponded to a single objective.

4.7.3. Reliability, Validity and Usability

Students' performances on the previous version of the mathematics tests revealed that they sometimes got items wrong due to carelessness and not to lack of knowledge or skills (see Sections 6.2.1. and 4.6.3). Since there was only one item per objective in these versions, students were diagnosed as not having mastery of the subject-matter tested. If only one item is included per objective, there is always the possibility that students who get an item wrong due to carelessness in their working, get it right if they are given the opportunity to attempt the item again, and vice versa. To avoid this and to increase the reliability of the test, I included three items per objective where algebraic manipulation or calculation was involved. The reasons for taking three items per objective were the following:

(a) If two items had been included, no conclusion could be drawn if a student got one item right and one wrong.
(b) With three items, the criterion could be set at 66%, that is, to getting two items right.

(c) No more than three items per objective could be included because the attribute usability, that is, the tests of the battery would have been too lengthy due to the fact that they should cover 32 objectives (see Section 4.7.1).

It has been noted earlier in this chapter that the type of validity that is of prime importance for CRTs is the content validity and that a technique for establishing it uses the test designer as a judge (see Section 4.3.2). Since the tests of the battery were CRTs, it was necessary to establish their content validity, indeed this was done using the technique mentioned above.

Since the battery had a diagnostic function, it should also have certain validity for diagnosis, that is, the items in the battery should really play a diagnostic role. That is why I have been concerned with this type of validity throughout the latter two sections of this chapter (i.e. Sections 4.5 and 4.6). One of the main conclusions that emerged from the analysis of items from the point of view of diagnosis was that certain items should be broken into several items each and that these new items should test prerequisite mathematical knowledge at two levels according to Bloom's taxonomy (Bloom et al 1971) - recall of specific information, and solution of problems requiring knowledge of facts, a formula or a rule (see Section 4.6.3).

Two other activities which served to validate items were: (i) the discussions I held with students in the individual diagnosis described in Section 5.3, and (ii) a single test, derived from Table 4.2, tried out
with some sixth form students. The former activity enabled me to see the kind of errors students made when working on some of the problems in the items and to pinpoint the causes of their difficulties. This in turn enabled me to construct appropriate types of items for diagnosis.

For activity (ii) I sampled items which were under categories A3 (Ability to carry out algorithms) and C1 (Ability to solve problems) from the table of specifications of the battery (i.e. Table 4.2) and assembled a test with them. This test was made up of 47 items (see Appendix N) and was given to 85 sixth form students who were getting prepared for their A-level mathematics exam.* Feedback about the performance on the test was given to each student and the results of the diagnosis of the test, which are summarized in Section 6.3.2, were discussed with the Head of the Mathematics Department of the Sixth Form College. It was agreed that the items would appear to have actually diagnosed the students' difficulties, that is, that the items in the test played a diagnostic role.

* The author is indebted to Mr T Roberts, Head of the Science Faculty, and to Mr J Connibear, Head of the Mathematics Department at Farnborough (Sixth Form) College, for their co-operation in the administration of the test.
CHAPTER 5

INDIVIDUAL DIAGNOSIS AND
REVISION WORK IN MATHEMATICS
5.1 INTRODUCTION

I have noted earlier that several approaches have been used to give students individual diagnosis by means of diagnostic tests and tutorial discussions, and that specific remedial work was assigned to each student according to his performance on the tests (see Section 2.3.2).

This chapter deals with two approaches to give individual diagnosis in mathematics in which I was involved and the procedure employed by students to revise those topic areas in which they needed revision. These two approaches are: (a) diagnosis by means of a pre-knowledge test and tutorial discussions, and (b) diagnosis by means of the 'thinking-aloud' approach combined with observation and retrospective interviews. The former was used with a large number of students entering engineering courses at Southampton University in 1976 and 1977, and the latter was used with a small group of students entering engineering courses at the University of Surrey in 1978.

For simplicity I shall call these two approaches for diagnosing mathematical difficulties the T.T. method (i.e. diagnosis by means of tests and tutorial discussions) and the T.A.O.I. method (i.e. diagnosis by means of the thinking-aloud approach combined with observation and retrospective interview).

5.2 DIAGNOSIS BY MEANS OF A TEST AND TUTORIAL DISCUSSION : THE T.T. METHOD

5.2.1 The Procedure

In 1976, and again in 1977, a mathematics pre-knowledge test was given to students entering engineering courses at the University of Southampton
(a description of the test used on each occasion is given in Sections 4.4.1 and 4.4.2). In 1976 the test was given to 61 students entering the Civil Engineering Department and in 1977 the test was given to all 333 students entering engineering courses.

The procedure adopted in both years was very similar to that used by Elton (1979) with students entering the Physical Science degree course at the University of Surrey (see Section 2.3.2). This procedure was as follows:

(a) During the first mathematics class period, the students were given a brief explanation about the purpose of the pre-knowledge test and then the test was administered.

(b) As each student had completed the test, he went to an adjoining room where he met a tutor who immediately marked the test and discussed it with him for some 10 minutes.

(c) As a result of this discussion the tutor gave the student a card with the number of the items he had got wrong and advised him to revise the subject matter concerned with those items (the actual revision work is fully treated in Section 5.5).

5.2.2 General Findings

(a) Although the test was to be answered within 30 minutes, no student was stopped after the 30 minutes were over. This made it possible for them to finish the whole test.

(b) In both years, the students' performance on the test was quite
inadequate in most of the topic areas covered by the test (this is fully discussed in Sections 6.2.1 and 6.2.2).

(c) By means of the interviews I held with students before and after they discussed the test and with tutors when the marking/discussing session was over, it was possible to determine that the main reason why a large number of students had difficulties with the topics in the test was because they made careless mistakes or had forgotten the material. The latter fact was confirmed by the replies given by students in a questionnaire that I gave to them (60 in 1976 and 103 in 1977) during the second and third week of the Autumn term.

By means of the questionnaire it was also found that:

(d) The majority of students found the discussion of the pre-knowledge test useful. Typical comments they gave in this sense were:

"It's a good idea, it helps you to see there and then ...... you usually take the test and you have something wrong but you don't know what." (October 1976)

"You know what you got wrong and why, and also what you have to do." (October 1976)

"It's a good idea..... It's more personal and you can follow exactly what you had wrong. He (the tutor) can tell you, and help you and advise you." (October 1977)

"He (the tutor) advised me to study inequalities and particularly pointed out the basic necessary points. I was incited to look it up." (October 1977)
A few students found the discussion of the pre-knowledge test not very useful, and there were two main reasons for this which clearly refer to quite different kinds of students. Either they found the pre-knowledge test easy or very easy because they got only one or two items wrong in the test and had no discussion at all with their tutors, or they got several items wrong and therefore considered that the time allowed for such discussion was too short. Typical comments given by both groups of students were:

"It was not really useful at all since there were only a couple of things I had forgotten." (October 1977)

"The time allowed to find my mathematical background was too short." (October 1977).

5.3 DIAGNOSIS BY MEANS OF THE THINKING-ALOUD APPROACH IN COMBINATION WITH OBSERVATION AND RETROSPECTIVE INTERVIEW: THE T.A.O.I. METHOD

5.3.1 The Thinking-Aloud Approach

The thinking-aloud approach is a method which has been used in Psychology of Thinking 'in order to get protocols from which typical sequences of steps towards the solution in a problem solving situation may be reconstructed' (Lüer 1973, p 301). This method has been widely used in problem solving situations for a variety of reasons. For example, it has been used to formulatetheories of human problem solving (Newell et al 1958), to examine the role of mediating response in anagram problem solving (Mayzner et al 1964), to seek stages in the process of problem solving (Green 1966), to relate methods of problem solving to factors of intelligence test performance (Groot 1966), to develop strategies of
proDiem solving (Vorner 1973, Luer 1973), to seek 'bases' from which students and graduate structural engineers attack problems concerned with the behaviour of structures (Cowan 1977).

As we can appreciate from the above examples, the thinking-aloud approach has basically been used with the idea of finding out the process of thinking of human problem solving. I used this method in combination with observation and interviews for a rather different purpose, namely to diagnose mathematical difficulties students have in solving mathematical problems based on mathematics they are supposed to know. All students who took part in the exercise described in the following sections were volunteers. I had explained to them the object of the exercise and that the problems they would be asked to solve were based on mathematics they were expected to know by the time they arrive at University.

5.3.2 The Procedure

5.3.2.1 A first group of students

A group of 8 students entering engineering courses at the University of Surrey in 1978 were given a mathematics test of twenty open-ended short items (see Appendix I) and were asked to 'think aloud' as they worked towards the solution of each of the problems in the test. In the meantime, I observed them and took notes of 'when' and 'where' they had difficulty in their working.

After they had finished with the test, they were interviewed retrospectively about the problems in which they appeared to have had difficulty, as revealed by the observation. The reason why this criterion was adopted as the basis for the interview was because the overall object of
the exercise was to diagnose the mathematical difficulties that each
student encountered when solving the problems in the test. I should point
out here that this exercise was carried out with each student
individually.

There were some cases in which I asked students about what there was in
their minds when they hesitated or stopped working on a problem for a few
seconds. Their reply was that they did not remember. It would appear
that one reason why this happened was that the students took between half
an hour and 45 minutes to complete the test and some of the questions I
asked them in the interviews required them to remember what had been in
their minds perhaps 30 minutes before the interview situation. In spite
of this inconvenience, it was possible to say what mathematical
difficulties students had in their solution to most of the problems and
then suggest appropriate revision work (the actual revision work is fully
treated in Section 5.5.4).

The following two examples and diagnoses of what came out of the use of
the T.A.O.I. method give some indication, it is believed, of the value of
the method (see also examples Nrs 1 and 2 in Appendix R).

The problems in these examples are:

1. Rewrite as a single power of x : \( \frac{(x^3)^{\frac{1}{2}}}{(x^2)^3} \)
2. Solve for x the inequality \( 2 - 3x < 7 \)

Keys:
1. Remarks in round brackets ( ) are remarks made by the interviewer.
2. Remarks in square brackets [ ] are explanatory notes.
(a) Solving the Problem

**STUDENT'S PROBLEM**

He writes the question and the fraction that follows but then stops writing.

"I am not sure whether I should add or multiply the powers of x at the moment. This is holding me up."

[Long pause]

He crosses out the 5, writes 6 instead and completes the solution.

\[
\begin{align*}
\left(\frac{x^3}{2}\right)^{\frac{1}{3}} &= \frac{x^{3/2}}{(x^2)^3} \\
&= x^{3/2} \times x^{-6} \\
&= x^{-2}
\end{align*}
\]

(b) Interview (Transcribed audiotape)

(You wrote this bit \(\left(\frac{x^3}{2}\right)^{\frac{1}{3}} = \frac{x^{3/2}}{x^5}\) but then you decided to change the 5 for 6...)

Yes, what I was trying to work out was that .... you have got \(x^2\) in brackets ... cubed ... um... now, in that case you either are going to multiply the two powers of x together or add them together, and when you have got \(x^2\) times \(x^2\), if you multiply before you can add at that time, and vice versa

[what he meant was that when you have \(x^2 \times x^2\), you can either multiply the two powers together or add them together]
I was trying to work it out. I did that [i.e. he added] and I thought a bit harder that it was really rather silly, I was going the other way round, so I changed it [i.e. he changed from 'add' to 'multiply' the two powers in \((x^2)^3\)]

(c) **Diagnosis**

i) In his working on the problem the student was not sure of whether he should add or multiply the two powers in \((x^2)^3\). He first added the two powers together but then he changed his mind and multiplied them together.

ii) It would appear that he got confused by the rules \(x^m \times x^n = x^{m+n}\) and \((x^m)^n = x^{m \times n}\), as both rules give the same result in the cases \(x^2 \times x^2\) and \((x^2)^2\), and this was the example that he had in his mind when he got to the point in which he had to decide which of the two rules he should apply.

iii) Since the student stopped talking when he got stuck and he said no word during the rest of the time he spent on the solution of the problem, I would not have been able to detect his difficulty except through the interview that followed his completion of the test (and the observation which enabled me to note that he seemed confused).

**EXAMPLE No. 2**

(a) **Solving the Problem**

<table>
<thead>
<tr>
<th>STUDENT'S COMMENT</th>
<th>STUDENT'S WRITTEN ANSWER</th>
</tr>
</thead>
<tbody>
<tr>
<td>[He writes the problem ]</td>
<td>2 - 3x &lt; 7</td>
</tr>
<tr>
<td>[He made no comments when working</td>
<td></td>
</tr>
</tbody>
</table>
through this problem and he solved it quite quickly)

\[ -3x < 9 \]
\[ x < \frac{9}{3} \]

(b) Interview

(Can you tell me what you did from here \([2 - 3x < 7]\) to here \([-3x < 9]\)?)
Well, I had to get rid of ... I had \(2 - 3x < 7\) so I wanted to get the 2 in the same side of 7
(How did you do it?)
So, I wanted to take if from both sides ... subtract it from both sides, what leaves me with minus 3x in this side \([\text{left hand side of the inequality}]\) but to put it across the inequality, I changed the sign on this side \([\text{right hand side of the inequality}]\), so I had to take that to the 7... to take 2 to the 7 to give me minus 3x is less than 9.
(If I asked you to transpose this 2 \([i.e. 2 - 3x < 7]\) to this side \([\text{right hand side of the inequality}]\), how would you do it?)
Ah... if it had just been an ordinary set of equal signs in the middle, I had just simply subtracted 2 from each side ... ah ... leaving me with \(-3x = 5\), but because of the inequality sign, I changed the sing of the 2 on the other side of ... I took it away from one \([i.e. \text{the left hand side of the inequality}]\) but when I went round to take it away from the other one \([\text{right hand side of the inequality}]\), since I sort of got through the inequality sign, I changed the sign of the 2.

(c) Diagnosis

The mathematical difficulty that this student had was a misunderstanding of the rules of transposition of terms and factors in an inequality, as he
Here again, the interview situation enabled me to detect what mathematical difficulty the student had and which would not have been possible to detect by other means.

5.3.2.2 A second group of students

With a second group of 5 students the procedure was slightly changed to avoid that the students forget what was in their minds at the time they hesitated or stopped working on a problem. The procedure adopted then was as follows:

(a) The problems were written on index cards.

(b) The students were given one problem at a time and were asked to think aloud as they worked towards the solution of the problem. In the meantime, I observed what they were actually writing on the paper.

(c) Every time that students stopped working or hesitated, before or after they wrote something down, they tended to stop talking too. At this moment, I asked them to tell me what difficulty they had. This enabled me to pinpoint the causes of the difficulty they were having with that particular problem at that particular point in their solutions.

I should note here that I did not make students stop working when their working was not leading them to the correct answer of the problem. If this was the case, I asked them to explain, after they had finished the
the problem, whatever solution they had without telling them about the correctness of their solutions. As the explanation went along, I asked the students to tell me why they had proceeded as they did. Since they had just worked on the problem they had no difficulty in explaining how they had solved it. With this procedure it was easier to diagnose students' difficulties in each of the problems they tackled.

The following example and diagnosis of what came out of the use of the T.A.O.I. method gives an indication of the value of the method when the student is given a problem at the time and the interview take place when he is solving the problem or he has just solved is (see also Examples Nos 3 and 4 in Appendix R).

**EXAMPLE No. 3**

**PROCESS (Transcribed audiotape)**

[He writes down the problem]

I'd make $2x^2 - 5x$ equal to 1

(pause)

(Could you say what you intend to do?)

Probably ... take the $x$s out of the ...

I don't really know

(pause)

I don't really think I will get anywhere because of the different powers

(pause)

I will leave it

(In an exam situation would you leave it?)

Yes, because whatever you do, you will never separate the $x$ on its own

**STUDENT'S WRITTEN ANSWER**

\[2x^2 - 5x - 1 = 0\]

\[2x^2 - 5x = 1\]
Have you solved quadratic equations before?
Oh yes! That is true, with a formula
... actually, I will start again
(If the question had said 'solve this quadratic equation'... you would have been able to do it)
[He crosses out the equation above]
So, it either factorises or it does not.
I could use an equation

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

I do not think it factorises into rational factors, even if it does, it would not come out using the... straight on. So
x equals [he writes]
I cannot remember it ... um ... I cannot remember the equation, sorry
(You have had the equation before?)
Yes, in the OND mathematics we were given formula sheets with all the relevant formula in it for the exam, so I never bothered to learn them.
(Why did you say you could not factorise it?)
Well, I just could not see any way ... and I do not think there is any other way apart from just guessing...
(O.K., let us say that we have a book of tables which says \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \] [the formula is written by the interviewer on the paper], will that help you or not?)
Yes, I will try it. [He writes] 
so that is 25 - 8 is 17 [He writes] 
(What is a, b and c for you?) 
a is 2, b is -5 and c is -1 [He writes] 
Oh, that is -1 so, it is a plus [He changes in the substitution above, He writes] 
so it is ... 33 instead [he writes] 
(Do you have trouble with the signs?) 
Yes, I suppose so if I am not careful.

Diagnosis

(a) This student did not recognise that the given equation was a quadratic equation.

(b) Having been told that it was a quadratics, he knew that he had to use a formula but had forgotten it.

(c) Having been given the formula he was able to substitute correctly but still made a silly mistake.

5.3.3 General Findings

(a) The T.A.O.I. method enabled me to ascertain the difficulties in mathematics that students had when working on the problems they were asked to solve. These difficulties resulted in being due to the following reasons:

   (i) Lack of knowledge of the relevant material

   (ii) Forgetfulness
(iii) Misunderstanding of concepts or rules

(iv) Silly mistakes

(b) With a first group of students, each student was given all the problems at the same time, as is usually done in an exam situation, and they were interviewed about their solution processes after they had finished with the test. The procedure would appear not to have been very profitable due to the fact that the test took 30 - 45 minutes to complete and some students had forgotten by then what they had in their minds when they were solving some of the problems. This difficulty was overcome when a second group of students were given one problem at a time and interviewed as they went along in their working on each problem.

(c) The observation-interview played an important role in the approach because it enabled me to ask students about their difficulties every time they were not sure about what they had just done or were about to do. This observation-interview would appear to be particularly useful when the student stops talking when working towards the solution of a problem.

5.4 CONCLUSIONS

In the previous sections I described how I used the T.T. and the T.A.O.I. methods for individual diagnosis in mathematics. The former would appear to be particularly useful in the case in which a large number of students is involved. However, a disadvantage of the method is that it requires the involvement of a large number of tutors. The experience at the University of Southampton showed that students had to queue up (up to 30 minutes) in order to discuss the test, and then only had about 10 minutes to actually discuss it. An average of 10 minutes for the discussion of
the test might perhaps be enough for students who only get one or two items wrong but not enough for those students who get a larger number of items wrong.

The other method - the T.A.O.I. method, would appear to be particularly useful in those cases in which a small number of students is involved because it requires that the student solve the problem(s) in the presence of the person who is to diagnose his mathematical difficulties.

The experience with two groups of students at the University of Surrey revealed that it is better to give the student one problem at a time and to interview him either whilst he is working toward the solution of the problem or immediately after he has solved it.

5.5 THE REVISION WORK

I have noted earlier that students entering engineering courses at the University of Southampton in 1976 and 1977, and at the University of Surrey in 1978 were advised to revise certain topics in mathematics according to their performance on a mathematics test (see Sections 5.2.1 and 5.3.2.1). This section aims to discuss the schemes under which such revision work was carried out. Since the main mathematics course at the University of Southampton was different from the traditional type in both years and it is related to some extent to the schemes of revision work, I shall first give some details of the mathematics course before going into the details of the scheme.

5.5.1 The Main Mathematics Course

The main mathematics course was a self-paced course, divided into a
a number of well-defined subject areas called blocks. These blocks are divided into units; Each unit is designed to cover approximately one week's work. Each student works at his own pace and in his own way until he feels that he has mastered the material of the unit. He then attempts a short test, taking about 25 minutes, which is marked by a course tutor in his presence and discussed with him immediately. The student has to exhibit a satisfactory level of competence to be allowed to proceed with the next unit. If he does not pass the test he is told to study the unit again for a subsequent test. A flow chart of the self-paced mathematics course is shown in Figure 5.1.

5.5.2 The 1976 Scheme

5.5.2.1 The procedure

After each student discussed his pre-knowledge test with a tutor, he was given a card with numbers of the items he got wrong (see Section 5.2.1) and also the Unit 1 of the main mathematics course.

The revision work was carried out in the Departmental Reading Room where selected books and programmed materials were available, some of which had been specially written for revision purposes (e.g. Flexer and Flexer 1967, Nuttall 1973, Potter 1977). In the Reading Room, students had available two folders: one called 'Initial Procedure' which contained a list of all the materials associated with each item in the pre-knowledge test and which students were advised to consult in the first instance; the other folder, which was called 'Resource Material', had a list of all the books and materials concerned with the pre-knowledge test and which were available in the Reading Room. Students could use the resource material whenever they liked because it would be available for them to
FIGURE 5-1: FLOW CHART OF THE MAIN MATHEMATICS COURSE
AT SOUTHAMPTON UNIVERSITY.

START

Get the first unit

Study the unit and the textbook

Any difficulty with the material?

YES
Consult your tutor during any of your tutorial/testing sessions
Revise the unit and the textbook

NO

Do you feel prepared for taking a test of the unit?

NO

Any difficulty with the material?

YES
Consult your tutor during any of your tutorial/testing sessions

NO

Do you feel prepared for taking a test of the unit?

YES
Take a test during any of your tutorial/testing sessions
Discuss your test with your tutor

NO

Have you passed it?

NO

Was it the last unit?

YES
You have completed the self-paced course

NO

Get next unit

END

S

R
consult through the whole term, but it was clearly best to use it within
the one or two days following the day of the test because of two reasons,
namely (1) because one tutor would be available in the Reading Room for
students to consult if it was necessary, and (2) because students could
already take the test on Unit 1 of the mathematics course on the third
day (in fact 13 students took the test on the third day and 47 took it on
the fourth day).

5.5.2.2 General findings

Students' opinions about the scheme of revision work were gathered by
means of interviews held with them when consulting the material in the
Reading Room and by means of the questionnaire given out in the second
week of the term (see Section 5.2.2). This questionnaire was
distributed at the beginning of a lecture on another subject, thus it
was possible to ensure a high number of respondents.

The most important findings that emerged from the analysis of the
information collected by these means can be summarized as follows:

(a) Students found the format of the programmed learning materials
useful for their revision. Typical comments given by them were:

"[The] booklets explained in fairly simple terms the maths
in it."

"The booklets tend to be more useful than books."

(b) There were too many materials on the same topic in the Reading Room
and students sometimes wondered which ones would be the best to consult.
(c) New materials on Algebra, inequalities and Trigonometry based on students' needs were necessary.

(d) A few students did not have time to revise all the topics they were advised to revise because they were already expected to take the test on Unit 1 of the mathematics course two days after the pre-knowledge test.

5.5.3 The 1977 Scheme

5.5.3.1 The procedure

The procedure in 1977 was similar, except that in the light of the 1976 experience the following changes were introduced:

(a) The pre-knowledge test was revised (see Section 4.4.1).

(b) The test was given to all students entering engineering courses. The breakdown of these students was as follows:

<table>
<thead>
<tr>
<th>Course</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acoustical</td>
<td>9</td>
</tr>
<tr>
<td>Aeronautics</td>
<td>44</td>
</tr>
<tr>
<td>Civil</td>
<td>72</td>
</tr>
<tr>
<td>Electrical</td>
<td>41</td>
</tr>
<tr>
<td>Electronics</td>
<td>85</td>
</tr>
<tr>
<td>Mechanical</td>
<td>56</td>
</tr>
<tr>
<td>Nautical Science/Ship Science</td>
<td>26</td>
</tr>
</tbody>
</table>

(c) The period of revision work was extended to one week.

(d) New materials on Algebra, Inequalities and Trigonometry were specially written for revision (Cohen and D'Inverno 1977, González-León 1977a, 1977b).
The following criteria to recommend revision work was adopted:

(1) Students who failed items on those parts of the test associated with Algebra, Inequalities and Trigonometry were advised to consult the new material mentioned in (d). Those who were strongly advised to consult these materials were required to take, within the revision week, a corresponding post-test* on the topics they had revised (see Appendix S). Table 5.1 shows the percentages of students who were advised to consult material and of those who actually took the post-test.

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>% of Students Advised to Consult the Material</th>
<th>% of Students who Took the Post-Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALGEBRA</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>INEQUALITIES</td>
<td>31</td>
<td>19</td>
</tr>
<tr>
<td>TRIGONOMETRY</td>
<td>29</td>
<td>14</td>
</tr>
</tbody>
</table>

(2) Since most of the topics tested by the other items in the test would be covered in the main mathematics course, those students who failed any of these items were advised to consult a formula sheet given out after the discussion of the pre-knowledge test. Those who were very weak in any of the topics tested by these items were also advised to consult any textbook which dealt with the topics, but no post-test was required to be taken.

* designed by the course lecturers
5.5.3.2 General findings

As it was done with the 1976 group, students' opinions about the scheme of revision work were gathered by means of interviews when consulting the material in the Reading Room and by means of the questionnaire given to 103 of the 169 students from the Civil, Electrical and Mechanical Engineering departments in the tutorial/testing sessions of the mathematics course during the third week of the term (see Section 5.2.2).

These three particular groups of students were chosen because they make up what might be called the 'classical engineering students', that is, their specialities are the ones most frequently found in Engineering Faculties. The breakdown of the 103 students who filled in the questionnaire was as follows:

<table>
<thead>
<tr>
<th>DEPARTMENT</th>
<th>NO. OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Civil</td>
<td>57</td>
</tr>
<tr>
<td>Electrical</td>
<td>19</td>
</tr>
<tr>
<td>Mechanical</td>
<td>27</td>
</tr>
</tbody>
</table>

The most important findings that emerged from the activity of revision work were the following:

(a) On the whole students found the scheme of revision work quite useful. Typical comments given by them in this sense were:

"The revision work was very helpful. It acted as a light which helped me to see the work I did in my A-level but had forgotten."
I found the revision work very useful, as I had not done mathematics for some time. The knowledge gained has been of assistance in the units of the mathematics course since then.

"The revision work was very useful in helping me to remember my A-level mathematics. It also covers points that my teacher forgot to verify when I was doing my A-level."

(b) Some of the weakest students would have liked to have longer time for revision or to take home the revision materials because they did not have enough time to revise all the topics they should revise.

Critical comments included:

"(I) could have taken longer time over it - for people who have forgotten a great deal - but obviously more than a week was not available." (This particular student got 8 marks in the pre-knowledge test and took the post-tests on Inequalities and Trigonometry.)

"I wish I had had more time to look at the other revision booklets, or perhaps they should be made available to take out on a loan basis so that I could have more time to read them."

(c) Students found the programmed material useful for their revision work due to the characteristics of the format of the material. Some of the advantages they found were: (i) the material is presented 'bit-by-
bit', that is, the step-by-step format of the programmed texts; (ii) the student only needs to go through those frames related to the area in which he needs revision. One student pointed out:

"I found it most useful and far better than slogging through a text book trying to find the relevant pieces of information."

Similar comments were also made about the booklet on Algebraic Manipulation. One student pointed out:

"Many examples which are given are useful and very simple steps are shown."

(d) I went through the scripts of the post-tests to see how students' performance on them compared to their performance on the pre-knowledge test and I found that some students had difficulties with Item No 10 in the post-test on Trigonometry and with Item No 5 in the post-test on Inequalities (see Appendix S). The reasons why this happened would appear to have been that: (i) the revision material did not cover the topic area involved in this item despite the topic area being also tested in the pre-knowledge test, and (ii) the revision material on Inequalities did not have enough worked examples of the type involved in Item No 5 in the post-test.

(e) The post-tests on Algebraic Manipulation and Trigonometry besides testing the material tested in the pre-knowledge test also included items on the following topic areas:

Algebraic Manipulation: Polynomial factorisation
- Solution of sets of simultaneous linear equations
- Rationalisation
- Division of polynomials

Trigonometry:
- Application of Sine and Cosine rules
- Application of formulae to evaluate $\sin(A + B)$ and $\sin A + \sin B$

It would appear that students did not have difficulties with these new topic areas because they were covered by the corresponding materials. However, if these new topic areas are really necessary, they should also be tested in the pre-knowledge test, otherwise, the two post-tests should be revised to match exactly the pre-knowledge test.

5.5.3.3 Follow-up activities

I followed-up the Civil, Electrical and Mechanical engineering students (169 students) by looking at their performance on the items in the early tests of the mathematics course which were directly concerned with the revision work. The findings that came out of this follow-up can be summarized as follows:

(a) Algebraic Manipulation: The students had to put into practice their skills on algebraic manipulation in the evaluation of limits in the test on the first unit of the main course. Their performance was fairly good, as nobody failed in evaluating these limits due to algebraic manipulations; however, the kind of problems they had to deal with, in this case, were less hard than the kind of problems they had to deal with in the revision work.
(b) Trigonometry: In the test on the second unit of the self-paced course there was the following question:

"Given that \( \theta = \cos^{-1} (-\frac{1}{2}) \) find the values of:

i) \( \theta \) ii) \( \sin \theta \).

(This question is similar to one which appeared in the post-test on trigonometry.)

50 of the 169 students of Civil, Electrical and Mechanical engineering failed in this question (36%). One of the reasons for this is that in some cases the students sketched the graph of the sine function instead of the cosine function to find the value of \( \theta \). According to the answers the students gave on the test, a large number of them were confused by the minus sign in the problem as some of them gave a solution of \( \theta = \cos^{-1} (\frac{1}{2}) \) and some chose the wrong quadrants.

(c) Inequalities: In the tests on the first and second units of the main mathematics course students had to deal with the symbols < and >, and also with the meaning of absolute value. They did not have any problem with this. No test dealt with solving inequalities.

5.5.4 The 1978 Scheme

5.5.4.1 The procedure

After each student had completed the session of diagnosis with interview (see the T.A.O.I. method in Section 5.3.2.1) he was given a card with the number of the items he had got wrong and a list of materials (books and booklets) associated to each item in the test. The list made reference to
those of each material which were particularly relevant to each item in the test, that is, the references were 'item-material-pages'.

Students were advised to use these materials, which were available in the Resources Centre of the Institute, whenever they liked within the following week because they should take a post-test on items similar to those in which they had encountered difficulty in the pre-test after they had finished the revision work. Students' performance on the post-test was generally satisfactory with the exception of one student who changed his department after three weeks because he realised that he was very weak in mathematics. (This student had Grade E in his A-level mathematics and had a break of one year between school and university.)

5.5.4.2 General findings

Students' opinions were gathered by means of an interview held with each of them immediately after they took the post-test. The main findings that came out of these interviews were:

(a) Students liked the idea of having a list 'item-material-pages' because it enabled them to find very rapidly the material they should revise.

(b) Students found the post-test as a motivational factor for the revision work.

The follow-up meetings were held with the students with the purpose of gathering their feedback about the usefulness of the revision work activity. Indeed, they said that they had used all the topics covered by the test during their first term at university. They also said that it
was an advantage to have participated in the activity of revision work because they could do the actual revision at the beginning of the term and not when they were already immersed in their courses.

5.6 CONCLUSIONS

On the whole, the three groups of students involved in the schemes discussed in this chapter found the activity of revision work useful and a good start for their university courses, as they could revise those topic areas in mathematics which they would need in their first year courses and they had forgotten or had not covered at school.

Four elements have characterized the schemes of revision work used in this study. These elements are the following:

Materials Materials specially written for revision purposes were used. Students found that this type of material, most of which was programmed, was quite appropriate for their revision work. In this respect, a word of caution should be added here. The material for revision purposes may not be suitable for those students who did not cover the subject matter at school, as they assume that the student has studied the topic before and therefore do not cover the theory in detail.

Management The process by means of which the students selected and used the materials played an important role in the carrying out of the exercise of revision work. Two folders - Initial Procedure and Resource Material - served as a guide for the students of the 1976 and 1977 groups because they indicated which materials they should use according to their individual needs. The 1978 group used a list similar to the one included in the folder Initial Procedure but the references in
it were 'item-material-pages' instead of 'item-material'. This enabled the students to know more precisely what they should look for in each material. Students said in interviews that this type of reference (item-material-pages) was an advantage because they could find what they needed to revise more quickly.

**Time**  The time and facilities allotted for the revision work seem to be of great importance for the weakest students. In spite of having extended the period of revision work from two days (in 1976) to one week (in 1977), some of the weakest students did not have time to revise all the topic areas they were advised to revise. This problem was overcome in 1978 when the students were allowed to take home, on an overnight loan basis, some of the revision materials, and then to return them early in the morning of the following day.

**Motivation**  The post-tests seem to have acted as a motivational factor for the students to do the recommended revision work. It should be made clear here that the post-tests should only cover the same knowledge tested in the pre-test.
CHAPTER 6

MATHEMATICAL NEEDS OF STUDENTS
AT THE INTERFACE SCHOOL-UNIVERSITY
INTRODUCTION

We have seen in the chapter on the literature review (i.e. Chapter 2) that there are three factors which have influence on the mathematical competence at the transition between school and university, namely, (1) topics not covered at school, (2) admission of students who might have not studied at the level required by their university courses some topics in mathematics, and (3) admission of students who may have become out of practice in handling mathematical problems. Due to this, students entering engineering courses in particular might not have the mathematical knowledge and skills needed to cope with their first year engineering courses. This fact led the Department of Mathematics at the University of Southampton to give students entering engineering courses revision work in mathematics based on their performance on a mathematics pre-knowledge test. I found that in 1976 and 1977, the students' performance on the pre-knowledge test was inadequate and that one of the reasons why this happened was because they had forgotten, due to the break between school and university, part of the material tested, i.e. the students' mathematical needs were affected by the break. To see whether these needs were different at the point of leaving the sixth form, a mathematics test was administered to sixth form school leavers in 1978 and 1979, a few weeks before they sat their mathematics A-level examination.

This chapter deals with the mathematical needs of students entering engineering courses (in 1976, 1977 and 1978) and of sixth form students (in 1978 and 1979) according to their performance on the test mentioned above. I should point out here that since one of the purposes of testing students was to describe which learning tasks they could and could not perform, rather than to rank students in order of achievement, the
analyses discussed here are criterion-referenced (considering the whole group of students) rather than norm-referenced (see sections 4.2 and 4.5.4.2).

6.2 UNIVERSITY ENTRANTS NEEDS

In October 1976, 1977 and 1978, groups of students entering engineering courses were given a mathematics test which had been developed over the period 1976-1978. I shall present the analyses of students' performance on these tests in a chronological order and discuss the mathematical needs in each case according to their performance on each item.

The test was administered to the 1976 and 1977 groups in a scheme of revision work in which each student should (a) take the test, (b) discuss it with a tutor, and (c) revise those topics in which his performance on the test was inadequate.

6.2.1 The 1976 group

A group of 61 students entering engineering courses at the University of Southampton was given a test covering mathematics prerequisite for their first year mathematics course (see section 5.5.1). This test consisted of twenty multiple-choice items (see Appendix J) and covered the following topics:

<table>
<thead>
<tr>
<th>TOPIC</th>
<th>ITEM NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebra</td>
<td>1-4</td>
</tr>
<tr>
<td>2. Inequalities</td>
<td>5</td>
</tr>
<tr>
<td>3. Binomial Theorem</td>
<td>6</td>
</tr>
<tr>
<td>4. Limits</td>
<td>7-8</td>
</tr>
<tr>
<td>5. Trigonometry</td>
<td>9-12</td>
</tr>
<tr>
<td>6. Logarithms and Exponentials</td>
<td>13</td>
</tr>
<tr>
<td>7. Co-ordinate Geometry</td>
<td>14</td>
</tr>
<tr>
<td>8. Differential Calculus</td>
<td>15-16</td>
</tr>
<tr>
<td>9. Integral Calculus</td>
<td>17-20</td>
</tr>
</tbody>
</table>

In the diagram, the horizontal axis represents different mathematical topics, and the vertical axis represents the percentages of wrong answers. The topics include Algebra, Inequalities, Binomial Theorem, Limits, Trigonometry, Logarithms, Exponentials, Co-ordinate Geometry, Differential Calculus, Integral Calculus, and Calculus. The diagram shows the distribution of wrong answers across these topics.
Fig. 6.1 shows a histogram of percentages of all students' wrong answers in the test. From it we can appreciate that the overall students' performance on the test was quite inadequate in all the topics in the test except Algebra and Differential Calculus. The other topics tested by the remainder items (13 items) were answered wrongly by more than 25% of students.

I found through interviews and a questionnaire (see section 5.2.2) that the main reasons why students failed these 13 items were either because they had forgotten the material tested, because they found the material new to them, or because they made careless mistakes. Table 6.1 is derived from the results of the questionnaire filled in by students. It shows

**Table 6.1**

**Mathematical Needs in 1976:**

*Topics Forgotten by Students or New to Them (N = 61)*

<table>
<thead>
<tr>
<th>Topic</th>
<th>Students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% Forgotten</td>
</tr>
<tr>
<td>Inequalities</td>
<td>15</td>
</tr>
<tr>
<td>Binomial Theorem</td>
<td>37</td>
</tr>
<tr>
<td>Limits</td>
<td>42</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>40</td>
</tr>
<tr>
<td>Integral Calculus</td>
<td>28</td>
</tr>
</tbody>
</table>

the percentages of students who had forgotten the material in the test or found the material new to them.

According to the results presented in Fig. 6.1, it can be concluded that the students entering Civil engineering courses at the University of Southampton in 1976 presented a very low performance on the different topics
tested, with the exception of Algebra and Differential Calculus, that is, the mathematical needs of these students were in Inequalities, Binomial Theorem, Limits, Trigonometry, Logarithms and Exponentials, Co-ordinate Geometry and Integral Calculus.

6.2.2 The 1977 group

In 1977, all 333 students entering engineering courses at the University of Southampton were given a revised version of the test given to the 1976 group (see Appendix K). This test covered the following topics:

<table>
<thead>
<tr>
<th>TOPIC</th>
<th>ITEM NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebra</td>
<td>1-4</td>
</tr>
<tr>
<td>2. Inequalities</td>
<td>5-6</td>
</tr>
<tr>
<td>3. Binomial Theorem</td>
<td>7</td>
</tr>
<tr>
<td>4. Trigonometry</td>
<td>8-12</td>
</tr>
<tr>
<td>5. Logarithms and Exponentials</td>
<td>13-14</td>
</tr>
<tr>
<td>6. Differential Calculus</td>
<td>15-16</td>
</tr>
<tr>
<td>7. Integral Calculus</td>
<td>17-20</td>
</tr>
</tbody>
</table>

Fig. 6.2 shows a histogram of percentages of all students wrong answers in the test. From it we can appreciate that as it happened with the 1976 group, the overall students' performance on the test was quite inadequate in most topics. It shows that there were 11 items which were answered wrong by more than 25% of students, that is, again all topics covered by the test except Algebra and Differential Calculus.

Here again there were students who found part of the material tested new to them, although the main reason for failing items in the test seems to have been that they had forgotten the material (see Table 6.2).
FIGURE 6.2: MATHEMATICAL NEEDS IN 1977: UNIVERSITY ENTRANTS (N = 333)
The mathematical needs of this group of students were similar to those of the 1976 group. It should be noticed that Limits and Co-ordinate Geometry were not included in the test of this group (see section 4.4.2).

6.2.3 The 1978 group

A new mathematics test was given to two groups of students entering engineering courses at the University of Durham, and at Queen Mary College, London, during their first two weeks at their universities (see section 4.6). One of the groups (51 students) took the test in a regular class period, whereas the other group (42 students) was given the test to take home. The latter group was asked not to use any reference material or help from anyone else, that is, to take the test as if it had been taken in a regular class period. The only reason why this group was given the test to take home was because a regular class period for the administration of the test could not be made available.
The test consisted of twenty open-ended short items (see Appendix I) and covered the following topics:

<table>
<thead>
<tr>
<th>TOPIC</th>
<th>ITEM NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebra</td>
<td>1-5</td>
</tr>
<tr>
<td>2. Inequalities</td>
<td>6-7</td>
</tr>
<tr>
<td>3. Trigonometry</td>
<td>8-11</td>
</tr>
<tr>
<td>4. Complex Numbers</td>
<td>12</td>
</tr>
<tr>
<td>5. Logarithms</td>
<td>13-14</td>
</tr>
<tr>
<td>6. Binomial Theorem</td>
<td>15</td>
</tr>
<tr>
<td>7. Differential Calculus</td>
<td>16-17</td>
</tr>
<tr>
<td>8. Integral Calculus</td>
<td>18-20</td>
</tr>
</tbody>
</table>

The main reason why this test was made up of open-ended short items was because the analysis of each of the items in the test could then be used to construct distractors for multiple-choice items. However, since in this chapter I am only concerned with students' mathematical needs, here I treat the test in exactly the same way as I did with the ones in the two previous sub-sections, that is, in order to diagnose deficiencies (A detailed analysis of the students' performance on the items in the test is treated in section 4.6).

Since the purpose of the exercise was not to see which group of students performed better, no comparison was made between them and the analysis of their performance was done considering all students as making up a single group (i.e. 93 students).

Fig. 6.3 shows a histogram of percentages of all students' wrong answers in the test. From it we can appreciate that there were seven items in which students' performance was not satisfactory, i.e. more than 25% of the students had difficulty with them. The topics tested by these items were: Inequalities, Trigonometry, Complex Numbers, Binomial Theorem and Integral Calculus. The difficulties encountered by students in each of
FIGURE 6.3: MATHEMATICAL NEEDS IN 1978: UNIVERSITY ENTRANTS (N = 93)

Integral Calculus
Differential Calculus
Binomial Theorem
Logarithms
Complex Numbers
Trigonometry
Inequalities
Algebra
the items in the test are fully discussed in section 4.6. However, I shall summarise here the main mathematical needs these students had in each of the topics covered by the test.

Algebra

There were students who did not know the general formula to solve quadratic equations and students who knew it but who did not know how to apply it.

Inequalities

There were students who did not know the rules for transposition of factors in an inequality. There also were students who seemed to have worked with solution of inequalities before but who apparently had not worked with inequalities involving modulus. They got right Item Number 6 (i.e. Solve for $x$: $3-2x > 4$) but got wrong item Nr. 7 (i.e. Solve for $x$: $|x-3| < 2$).

Trigonometry

There were students who did not know the trigonometric formulae they had to use and there were students who did not know the values of $\sin \frac{\pi}{2}$ and $\cos \pi$.

Students' performance on this topic area was quite inadequate.

Complex Numbers

There were students who did not know how to find the modulus of the complex number $4-3i$
Logarithms

A small number of students did not know the elementary properties of logarithms.

Binomial Theorem

A large number of students did not know the formula to expand a binomial as a series. Another group of students committed errors in the substitution when using the correct formulae.

Differential Calculus

There were two aspects in which students had difficulties in this topic. The first one was that some students did not know the derivatives of the sine and cosine, and the second one was that there were students who apparently did not know that they should apply the chain rule.

Integral Calculus

There were students who did not seem to have known what they should do to integrate by parts.

6.2.4 Discussion

The mathematical needs of the three groups of students described above are summarised in Table 6.3. This table shows the specific knowledge and skills in which students encountered difficulties, and the range of percentages of students involved in each case. A study of the results presented in this table reveals some points of interest. These are:
<table>
<thead>
<tr>
<th>KNOWLEDGE AND SKILLS</th>
<th>RANGE OF PERCENTAGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 ALGEBRA</td>
<td></td>
</tr>
<tr>
<td>1.1 Application of the laws of exponents</td>
<td>14-20</td>
</tr>
<tr>
<td>1.2 Solution of quadratic equations</td>
<td>10-25</td>
</tr>
<tr>
<td>2.0 INEQUALITIES</td>
<td></td>
</tr>
<tr>
<td>2.1 Rules for transposition of factors in an inequality</td>
<td>17-32</td>
</tr>
<tr>
<td>** 2.2 Solution of inequalities involving modulus</td>
<td>30-48</td>
</tr>
<tr>
<td>3.0 TRIGONOMETRY</td>
<td></td>
</tr>
<tr>
<td>3.1 Knowledge of trigonometric function values of $\frac{\pi}{2}$ and $\pi$</td>
<td>37-49</td>
</tr>
<tr>
<td>** 3.2 Solution of equations such as $\cos \theta = \frac{1}{2}$</td>
<td>46</td>
</tr>
<tr>
<td>* 3.3 Knowledge (and perhaps application) of formulae to expand $\sin (x \pm y)$ and $\cos (x \pm y)$</td>
<td>32</td>
</tr>
<tr>
<td>* 3.4 Application of formulae for $\sin x \pm \sin y$ and $\cos x \pm \cos y$</td>
<td>77</td>
</tr>
<tr>
<td>4.0 COMPLEX NUMBERS</td>
<td></td>
</tr>
<tr>
<td>* 4.1 Find the modulus of a complex number $a + bi$</td>
<td>25</td>
</tr>
<tr>
<td>5.0 LOGARITHMS</td>
<td></td>
</tr>
<tr>
<td>5.1 Application of general properties</td>
<td>5-30</td>
</tr>
<tr>
<td>6.0 BINOMIAL THEOREM</td>
<td></td>
</tr>
<tr>
<td>6.1 Expansion of a binomial with negative and fractional exponent</td>
<td>47-59</td>
</tr>
<tr>
<td>7.0 DIFFERENTIAL CALCULUS</td>
<td></td>
</tr>
<tr>
<td>** 7.1 Application of the chain rule</td>
<td>9-18</td>
</tr>
<tr>
<td>8.0 INTEGRAL CALCULUS</td>
<td></td>
</tr>
<tr>
<td>8.1 Knowledge of standard integrals</td>
<td>15-20</td>
</tr>
<tr>
<td>8.2 Use of the technique of integration by parts</td>
<td>48-58</td>
</tr>
</tbody>
</table>

* DATA FROM THE 1978 GROUP ONLY

** DATA FROM TWO GROUPS
(1) Students' ability to solve inequalities seems to be very poor, especially in those inequalities involving modulus.

(2) Trigonometry is maybe one of the topic areas where the students showed more difficulties. The problem of forgetfulness and use of formulae seems to have its roots in the fact that students are used to working with formula books.

(3) The formula to expand a binomial as a series and the technique of integration by parts seem to be two topics that students forget most.

Although these topics cannot be generalised as being the mathematical needs of students entering engineering courses, they give an indication of the percentages of newly enrolled engineering students whose mathematical competence in the corresponding topic may be inadequate and where it is more likely that a high proportion of students have difficulties.

6.3 SCHOOL LEAVERS NEEDS

As I have said earlier, one of the reasons why students' performance on a mathematics pre-knowledge test was inadequate (in 1976 and 1977) was because they had forgotten part of the material due to the break between school and university (see section 6.1). Bearing this in mind, I gave tests covering mathematics pre-requisite for first year engineering courses to sixth form students before they sat their A-level mathematics examinations in June. This section deals with the analyses of the students' performance on these tests.
Three groups of sixth form students from three different schools were given the mathematics test (see section 4.5.5 and Appendix L). The breakdown of these three groups of students and their respective type of school are given in Table 6.4. Since the purpose of this subsection is not to see which group of students performed better, but to see what students knew and what they did not know according to their performance on the test, the analysis was done considering all students as making up a single group (i.e. 67 students). I should note at this point that although the students of the three schools studied syllabuses from different Examining Boards (i.e. London and Oxford and Cambridge Joint), the material in the test was covered by all these syllabuses.

### Table 6.4

**Breakdown of the Sixth Form Students Who Took the Mathematics Test in May 1978**

<table>
<thead>
<tr>
<th>SCHOOL</th>
<th>TYPE OF SCHOOL</th>
<th>NUMBER OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Comprehensive</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>Sixth Form College</td>
<td>34</td>
</tr>
<tr>
<td>C</td>
<td>Public School</td>
<td>30</td>
</tr>
</tbody>
</table>

A histogram of percentages of all students' wrong answers in the test is shown in Fig. 6.4. From it we can appreciate that students' performance on the items testing Inequalities, Trigonometry, Binomial Theorem and Integral Calculus was quite inadequate, as more than 25% of
students got them wrong. From a scrutiny of the alternatives of the items ticked by students on the test answer sheets, the following conclusions could be drawn:

**Inequalities**

(1) The fact that in Item Number 6 a large number of students did not change the direction of the inequality when they divided both sides of it by -2 (i.e. they selected distractor (A) of the item) would appear to indicate that they did not know the rules for transposition of factors in an inequality.

(2) The fact that the distractor most selected in Item Number 7 was (D) would appear to indicate that students made a slip when adding up -2 and 3 (i.e. they seem to have taken $-2 + 3 = -1$).

**Trigonometry**

(1) In Item Number 8 it would appear that some students either were not aware of the sign minus of the operation, or they took

$$\sin \frac{\pi}{2} = \cos \pi = 0,$$

as they mainly chose distractor (b) of this item.

(2) Students might have got items Nrs. 9,10 and 11 wrong due to the fact that they usually work with formula books to do their course-work exercises, and they were required to complete the test without the aid of such formula books.

**Logarithms**

The fact that a large number of students selected distractor
(A) \( x = e^a - e^b \) in Item Number 14, would appear to indicate that they confused this expression with the correct one (C) \( x = e^{a-b} \). This seems to indicate that these students have not a clear understanding of what \( \log_e x = a-b \) means.

**Binomial Theorem**

The distractor most selected in Item Number 15 was (B). This would appear to indicate that they only had difficulty with the signs in the expansion of the binomial.

**Differential Calculus**

Although some students apparently used the correct rule in Item Number 16 to differentiate \( y = x^3 - 2x \), the distractor most selected in Item Number 17 was (B). This would appear to indicate that these students got confused by the exponent of \( \sin x \), as they took \( \frac{d}{dx} (\sin^2 x) = \cos^2 x \) instead of \( \frac{d}{dx} (\sin^2 x) = 2 \sin x \cos^2 x \). This might have happened because no alternative in Item Number 17 began with 2 (in the correct answer \( \sin 2x \) was written instead of \( 2 \sin x \cos x \)). These, and some other students, did not realise that they should use the chain rule.

**Integral Calculus**

(1) The fact that the distractor most selected in Item Number 18 was (B) would appear to indicate that the students involved mixed up the rules for integration and differentiation, that is, they took

\[ \int e^{kx} \, dx = k e^{kx} \text{ instead of } \int e^{kx} \, dx = \frac{1}{k} e^{kx}. \]
(2) In Item Number 19, the students were asked to select the substitution that allowed them to evaluate most readily the integral \( \int \frac{dx}{x \sqrt{1-x}} \).

Since they were not asked to try their selection to see whether it worked, it would appear that they selected what they thought might work. This maybe was the reason why all the distractors for this item were more or less equally selected.

(3) The distractor most selected in Item Number 20 was (C). This would appear to indicate that the reason why a large number of students selected this distractor is the same as the reason given in (1). This seems to be so because distractor (C) is obtained when taking \( v = x^2 \) and \( du = e^{2x} \, dx \) and then taking \( dv = 2x \, dx \) and \( u = \frac{1}{2} e^{2x} \) instead of \( u = \frac{1}{2} e^{2x} \). I should point out here that the students who chose distractor (C) in Item Number 20 were not exactly the same as those who chose distractor (B) in Item Number 18; that is there were some students who chose distractor (B) in Item Number 18 and did not choose distractor (C) in Item Number 20. Therefore it may be concluded that some students encountered this difficulty when they were differentiating and some when they were integrating.

6.3.2 The 1979 group

In May 1979, a different version of the test on mathematics pre-requisite for engineering courses (see section 4.7.3) was given to 85 sixth form students at a Sixth Form College. This test was divided into two parts. Part I consisted of 17 matching items, and Part II consisted of 30 multiple-choice items (see Appendix N). The relation between the topic areas and the items in the test was as follows:
<table>
<thead>
<tr>
<th>TOPIC</th>
<th>ITEMS NUMBERS</th>
<th>TOTAL NUMBER OF ITEMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebra</td>
<td>1-13, 18-23, 26-27</td>
<td>21</td>
</tr>
<tr>
<td>2. Binomial Theorem</td>
<td>14-17</td>
<td>4</td>
</tr>
<tr>
<td>3. Inequalities</td>
<td>24-25</td>
<td>2</td>
</tr>
<tr>
<td>4. Logarithms</td>
<td>28-29</td>
<td>2</td>
</tr>
<tr>
<td>5. Complex Numbers</td>
<td>30-33</td>
<td>4</td>
</tr>
<tr>
<td>6. Trigonometry</td>
<td>34-41</td>
<td>8</td>
</tr>
<tr>
<td>7. Differential Calculus</td>
<td>42-45</td>
<td>4</td>
</tr>
<tr>
<td>8. Integral Calculus</td>
<td>46-47</td>
<td>2</td>
</tr>
</tbody>
</table>

Students were allowed to use their formula books to complete the test because they usually used them to solve the coursework exercises. For the administration of the test, the students had to be divided into two groups (one had 38 students and the other 47 students) because in the College there did not exist a lecture room in which all students could be accommodated at the same time. The only criterion for grouping the students was timetable availability. This means that within each of the two groups there were students who had different mathematics teachers (there were 12 mathematics teachers for all 85 students). Since the test had 47 items and the time available to complete it was 60 minutes, a half of each group of students was asked to start the test with Part I and the other half with Part II. They were also asked to answer as many items of the whole test as they could and to leave those items that they did not know.

Histograms of percentages of students’ wrong and omitted answers for the two groups of students are shown in figure 6.5(a) and 6.5(b). From these diagrams the following aspects can be seen:

**Algebra**

(1) Few students got most items on Algebra wrong. This fact would appear to indicate that most students did not have difficulty with this topic except for the material tested by Item Numbers 18, 21 & 23.
FIGURE 6-5a: MATHEMATICAL NEEDS IN 1979: SCHOOL LEAVERS (N = 85)
FIGURE 6.5 (b): MATHEMATICAL NEEDS IN 1979: SCHOOL LEAVERS (N = 85)
(2) The fact that in Item Number 18 (quadratic equations) the distractor most selected was (D) would appear to indicate that either students used the expression \( x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a} \) as the correct formula to solve quadratic equations, or made a slip in the substitution in the correct formula. These students took \( 49 - 4(3)(1) = 37 \) instead of \( 49 - 4(3)(-1) = 61 \) under the radical sign.

In this case no conclusion can be drawn but if the test had included another quadratic equation in which \( c > 0 \), it might have been possible to ascertain which of the two kinds of errors mentioned above a student might have had. I should point out here that of the two kinds of errors mentioned above, the latter was more likely to occur because students completed the test using their formula books.

(3) In Item Number 21 (transposition of formulae), the distractor most selected was (B). This seems to indicate that students omitted a sign in transposition, i.e. from \( WK^2 = VK - TR \) they implied that \( TR = WK^2 - VK \). This might have happened due to carelessness in the working on the item, but to ascertain this it would have been necessary to have had another item on transposition of formulae in which similar steps had to be taken in the transposition.

(4) In Item Number 23 (Solution of set of simultaneous homogeneous equations) the distractors were selected as follows:

<table>
<thead>
<tr>
<th>DISTRACTER</th>
<th>% STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>34</td>
</tr>
<tr>
<td>(B)</td>
<td>18</td>
</tr>
<tr>
<td>(D)</td>
<td>12</td>
</tr>
</tbody>
</table>
Discussing these figures with the Head of the Mathematics Department of the school involved, he said that what might have happened was that the material in the item had been studied by students long before they took the test and, therefore, they might have got confused when they dealt with the item.

**Binomial Theorem**

Very low percentages of students got wrong the items regarding binomials with positive integer. This would appear to indicate that they did not encounter any difficulty. However, in the items regarding the binomial theorem for fractional exponents, they seemed to have encountered certain difficulties.

(1) In Item Number 16 students were asked to find the first three terms of the series obtained in the expansion in powers of $x$ of the binomial $(1 + x)^{\frac{3}{2}}$, where $0 < x < 1$. These three terms are the following:

$$1 + \frac{3}{4}x + \frac{(\frac{3}{4})(\frac{3}{4}-1)}{2} x^2 = 1 + \frac{3}{4}x - \frac{3}{32} x^2$$

The responses most selected in this item were:

(D) $1 + \frac{3}{4}x + \frac{3}{32} x^2$ and (F) $1 + \frac{3}{4}x + \frac{5}{32} x^2$

This would appear to indicate that the students involved took $\frac{3}{4} - 1 = \frac{3}{4}$, or $\frac{3}{4} + 1$ instead of $\frac{3}{4} - 1$

(2) The fact that 18% of students did not answer Item Number 17 would appear to indicate that either they did not have enough time to answer it, or they did not know how to apply the Binomial Theorem to a binomial with a negative exponent.
I should note here that all students, except two, did apparently have time to answer Part I of the test. The two who did not have time did not answer any of the items in Part I.

**Inequalities**

(1) It seems that in Item Number 24 students either forgot to change the direction of the inequality when they transposed -2, or they did not know they should do it (the distractors most selected in this item was (B)).

(2) It would appear that students implied -3<x-2<3 from the inequality |x-2|<3 in Item Number 25, but then they added 2 to the number on the extreme right-hand side in the former inequality, and subtracted 2 to the number on the extreme left-hand side, that is, from -3<x-2<2 they implied that -3-2<x<3+2 and therefore -5<x<5.

**Logarithms**

(1) The fact that the distractor most selected in Item Number 28 was (A) $x = e^a - e^b$ would appear to indicate that the students involved did not have a clear meaning of what the logarithm of a number is.

(2) Some students did not know how to expand the logarithm of a product and of a quotient. The only distractor they selected in Item Number 29 was (A).
Complex Numbers

The overall students' performance on this topic area was quite good. However, there were students who (1) did not know how to find the modulus of a complex number \(a + bi\), and (2) did not know how to multiply and divide two complex numbers. In spite of having used the expressions:

\[
\text{(B) } \frac{(a + bi)(c + di)}{(c + di)(c - di)}; \quad \text{(C) } \frac{(a + bi)(c + di)}{(c + di)(c - di)} \quad \text{but taking} \quad (c + d)(c - di) = c^2 - d^2; \\
\text{(D) } \frac{a + b}{c + d}i
\]

to construct the distractor for Item Number 33, all the distractors were more or less equally selected.

Trigonometry

The main conclusion that came out of the analysis of the items on trigonometry was that a high proportion of students did not know how to read (or use) their formula books. As noted earlier in this section, all students taking the test were allowed to use their formula books (those who did not have one were given one from the Mathematics Departmental Library). In spite of working on the problems with these formula books, some students used wrong formulae to solve the problems in the items. This is clear from the following facts:

(1) All distractors in Items Numbers 35, 36, 37 and 39 were more or less equally selected.

(2) A minimum of 15% of students did not answer Items Numbers 37, 38 and 39. In Item Number 39 the reason might have been that the
(3) The distractor most selected in Item Number 38 involves the product of two cosines when the correct formula involves the product of sine and cosine.

(4) The distractors most selected in Item Number 40 were (A) and (C). This means that the students involved used either the expression \( m^2 = n^2 + p^2 + 2np \cos M \) or the expression \( m^2 = n^2 + p^2 - np \cos M \) as the cosine rule. (These two expressions were used to construct distractors (A) and (C), respectively.

In Item Number 41 the distractor most selected was (A) \( \sin x \). This would appear to indicate that some students realised that the given sketch began from 0 and that this occurs with the sine curve. However, they did not seem to have realised that the curve in the item reaches its maximum value when the angle is \( \frac{\pi}{4} \) and not \( \frac{\pi}{2} \).

**Differential Calculus**

(1) The main reason why students got Items Numbers 42 and 44 wrong was because (a) they did not know the derivative of \( \sin x \) and \( \cos x \), and (b) they did not know the rule to differentiate the product of two functions.

(2) Some students failed Item Number 43 mainly because they applied the chain rule once and they should have applied it twice (i.e. the distractors most selected were (B) and (C)).
(1) Several students mixed up standard rules of derivation and integration.

(2) Some students did not know how to integrate by parts (all the distractors in Item 47 were more or less equally selected).

6.3.3 Discussion

The school leavers' mathematical needs are summarised in Table 6.5. This table shows the range of percentages of students involved in each case, and also includes topics which were in the test of the 1979 group (though not in the test of the 1978 group) on which students' performance was inadequate. Looking in detail at the results presented in this table, some interesting points emerged. These are the following:

(1) It seems that students do not know the rules of transposition of factors in inequalities neither how to solve inequalities involving modulus.

(2) In spite of having given the students of the 1979 group the opportunity to answer the test with their formula books, they had the same kind of difficulties as those who did not use it (i.e. the students of the 1978 group). This seems to imply that the students either have not got a grasp of the trigonometric concepts and meanings of formulae, or they did not know how to use their formula books.
<table>
<thead>
<tr>
<th>KNOWLEDGE AND SKILLS</th>
<th>RANGE OF PERCENTAGES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.0 ALGEBRA</strong></td>
<td></td>
</tr>
<tr>
<td>1.1 Solution of quadratic equations</td>
<td>11-22</td>
</tr>
<tr>
<td>1.2 Transposition of formulae</td>
<td>11-20</td>
</tr>
<tr>
<td><strong>2.0 INEQUALITIES</strong></td>
<td></td>
</tr>
<tr>
<td>2.1 Rules for transposition of factors in an inequality</td>
<td>19-38</td>
</tr>
<tr>
<td>2.2 Solution of inequalities involving modulus</td>
<td>21-33</td>
</tr>
<tr>
<td><strong>3.0 TRIGONOMETRY</strong></td>
<td></td>
</tr>
<tr>
<td>3.1 Solution of equations such as ( \cos \theta = \frac{1}{2} )</td>
<td>35</td>
</tr>
<tr>
<td>3.2 Application of formulae to expand ( \sin (x \pm y) ) and ( \cos (x \pm y) )</td>
<td>31*-44</td>
</tr>
<tr>
<td>3.3 Application of formulae for ( \sin x \pm \sin y ) and ( \cos x \pm \cos y )</td>
<td>35*-66</td>
</tr>
<tr>
<td><strong>4.0 LOGARITHMS</strong></td>
<td></td>
</tr>
<tr>
<td>4.1 Knowledge of definition of logarithm</td>
<td>12-26</td>
</tr>
<tr>
<td><strong>5.0 BINOMIAL THEOREM</strong></td>
<td></td>
</tr>
<tr>
<td>5.1 Expansion of a binomial with negative and fractional exponent</td>
<td>18*-56</td>
</tr>
<tr>
<td><strong>6.0 DIFFERENTIAL CALCULUS</strong></td>
<td></td>
</tr>
<tr>
<td>6.1 Application of the chain rule</td>
<td>14-26</td>
</tr>
<tr>
<td><strong>7.0 INTEGRAL CALCULUS</strong></td>
<td></td>
</tr>
<tr>
<td>7.1 Knowledge of standard integrals</td>
<td>15-29</td>
</tr>
<tr>
<td>7.2 Use of technique of integration by parts</td>
<td>11-42</td>
</tr>
</tbody>
</table>

** DATA FROM ONE GROUP ONLY

* THIS FIGURE INCLUDES WRONG AND OMITTED ANSWERS
(3) Students seem to have difficulties with the expansion of a binomial as a series when this has a negative and fractional exponent.

(4) Students mix up rules of differentiation and integration.

To end this section I should point out that mathematics teachers say that students do not need to memorise formulae because they have got a formula book. However, the main finding that came out of the analysis of this test was that the fact of the students having a formula book is no guarantee that they know how to use it. It may be important that teachers make sure that students know how to use their formula books.

6.4 CONCLUSIONS

The mathematical needs of students entering engineering courses and of sixth form school leavers have been discussed in this chapter. These needs were identified by means of mathematics tests administered to students at the point of entry to university (engineering students) and at the point of leaving school (upper sixth students). The mathematics test used were based on mathematics regarded as prerequisite for dealing with engineering courses, and they were different versions of a test which was being developed.

Comparing the mathematical needs of the two groups of students, taking into consideration the kind of task involved in each of the items in the test, and relating the items to the topics they covered, it was found that students at university and at school level had difficulty with those topics which involved the use of formulae (e.g. solution of quadratic equations; application of formulae to expand \( \sin (x \pm y) \) and
cos (x ± y), application of formulae for sin x ± sin y and cos x ± cos y; expansion as a series of a binomial; and the use of the technique of integration by parts). This would appear to indicate that students do not actually know formulae because they are used to work with formula books at school. From this analysis it was also found that there were some topics which involved the knowledge of a concept or of a numeric value which seemed to have been forgotten by the students entering university (e.g. the concept of modulus, trigonometric function values, and standard integrals). This would appear to indicate that this kind of knowledge is forgotten more rapidly than others.

The evidence presented here leads to the conclusion that whatever the reasons for the lack of knowledge and skills are, a student entering engineering courses may need to revise some of the topics presented in Tables 6.3 and 6.5. To know exactly what he can and cannot do, it is necessary to diagnose which of the mathematical topics required as prerequisite for his first year engineering courses he is lacking.

Some approaches by means of which this could be done have been discussed in Chapter 5.
CHAPTER 7

DESIGN AND DEVELOPMENT OF MATERIALS FOR REVISION PURPOSES
Three schemes of revision work have been presented and discussed in Chapter 5. The evaluation of these schemes have showed that those materials where the content is presented step by step were found particularly useful for revision purposes (see section 5.6). This would appear to indicate that the type of materials needed for revision work should be written in a style different from that of normal textbooks.

When one thinks about development of materials designed for the revision of topics students have lost, for whatever reasons, the following aspects should be taken into consideration:

1. As it is expected that the student has previously studied, by whatever means, the subject matter presented in the material, this does not necessarily include details of the theory involved in it.

2. The material should emphasize the mastery of working techniques, rather than the understanding of the techniques in depth, otherwise the student may be learning new subject matter rather than revising it.

3. It is assumed that the student could only spend a limited period of time on the revision of the topic, therefore, the material should facilitate as much as possible the acquisition of the skills needed by students.

These three considerations were taken into account for the design of two revision materials - Trigonometry and Algebraic Manipulation, which
were needed for the revision work in mathematics at the University of Southampton (see section 5.5.2.2). This chapter deals with the design and development of such materials.

7.2 THE BOOKLET ON TRIGONOMETRY

When preparing the booklet on trigonometry, the first problem I faced was to find a suitable way of presenting the topics bearing in mind the three considerations stated in the previous section.

In the revision work in mathematics at the University of Southampton in 1976, the students used certain programmed texts (see section 5.5.2), some of which had been specially written for revision purposes (e.g. Flexer and Flexer 1967; Nuttall 1973). This experience revealed that the students preferred to work with programmed materials rather than with the books written in normal textbook style. They argued that in the programmed texts the material was presented 'bit by bit' whereas in normal textbooks it was not.

The literature on programmed texts shows that they have been used for:

(a) complementation - to replace the customary form of instruction for a particular objective of the curriculum (e.g. Bajpai and Calus 1970, Stroud 1969),

(b) enrichment - to provide intensified understanding of a particular topic (e.g. Hogg 1967),

(c) remediation - to give the student another opportunity to tackle the subject matter in which he has encountered difficulties (e.g. Unwin and Spencer 1967), and

(d) revision - to aid the student to revise portions of previous instruction (e.g. Kilty 1975).

It must be borne in mind that the circumstances surrounding the use of programmed texts for enrichment, remediation and revision purposes are certainly different from those surrounding the use
of programmed texts for the purpose of complementation because in the former case it is assumed that the student has previously studied, by whatever means, the subject matter in the programme whereas it is not so in the case of complementation. In addition to this, remediation is very similar indeed to revision, as in both cases it is assumed that the student needs to go through the material whereas it is not necessarily so in the cases of complementation and enrichment. Studies carried out in the U.S.A. revealed that programmed texts seemed to have major value for these four purposes, but not for taking over a major part of instruction in a given subject (Gage and Berliner 1975). It has also been found that learning time is less in those subjects where the objectives are clearly defined, and where graded exercises and repetitive practice are required (Henderson 1969).

All this seems to indicate that programmed materials are useful for a student who has studied the topic area before and wishes to revise it in a short time.

Bearing in mind all these facts, I decided to write the booklet on trigonometry using programmed learning techniques. Before going into the details of the design and development of the booklet, I shall first present some general aspects of programmed learning.

7.2.1 Some aspects of programmed learning

The field of programmed learning or programmed instruction as it is also called, is based in part on certain principles of psychology derived from laboratory studies of human and animal learning.

Three psychologists, Sidney Pressey, Norman Crowder and specially B.F. Skinner, have been the pioneers in the development of self-teaching
Sidney Pressey is regarded as the originator of the devices. He issued the first published report of a teaching machine in 1926 (Pressey 1926). His original machine taught one student at a time by asking questions to which the student responded by pressing one of four buttons. If he was correct, the machine responded by going on to the next question; if he was wrong, the machine did not move on until the student selected the correct response. The present interest in programmed learning and teaching machines is specifically attributable to the writings of Professor B. F. Skinner in 1954 and 1958 (Skinner 1954, 1961). Skinner's thinking on learning theory and his experiments brought him to the concept of teaching machine programming.

Skinner's central convictions were:

(a) Any unit of subject matter that had to be learned could be broken up into a large number of very small steps, each one with an increment of successive approximation to the final mastery of the whole topic,

AND

(b) Frequent rewards after small units of work (i.e. immediate knowledge of results).

Norman Crowder is regarded as the originator of the type of programme called Branching or Intrinsic (Crowder 1959).

A typical definition of a programme is that given by Lysaught and Williams (1963):

"... is a carefully ordered and organized sequence of material to assure the best possible learning conditions for a student" (p.16).
I carried out a review of textbooks on Programmed Learning with the purpose of finding out the characteristics or principles of programming. Lysaught and Williams (1963), Meyer (1969), Rowtree (1966) and Skinner (1961), among others, have described a number of characteristics or principles which might take the following form:

(1) **Assumptions about the learners.** A programmer has to make a certain assumption about the students to whom his programme is directed.

(2) **Explicitly stated objectives.** The programme aims to get the student to perform in a very specific way. The objectives of the programme must be carefully expressed in clear behavioural terms.

(3) **Logical sequence of small steps.** A logical sequence of information is presented - one small unit at a time - to the student. These steps or units are usually called items or frames.

(4) **Active overt response.** The student responds actively to each frame.

(5) **Immediate knowledge of results.** The student gets immediate knowledge of whether his result is correct. The correct answer or "confirmation" of the frame appears immediately in the following frame.

(6) **Self-pacing.** Each student works at his own best pace.

A group of American psychologists put on paper - in a form of textbook - some programmes used by teaching machines (Crowder 1959, Homme and Glaser 1959, Meyer 1962). This was the beginning of the emphasis upon programmes as a separate entity in the form of
There are many types of programmes and they range from the entirely linear or extrinsic programme at the one extreme, to a full branching or intrinsic at the other. The former was developed by Professor B.F. Skinner and is based on the operant conditioning theory of learning (Skinner 1961). In a linear programme, each student goes through the same instructional sequence, along a single line, i.e. all students see the same sequence of information. A diagram of the process would look like that in fig. 7.1. In a linear programme the learner has to 'construct' his response to each frame. Typically the frames are short, although there is no general agreement on what is meant by 'short'.

FIGURE 7.1: Sequence in a linear programme

As human learning takes place in a variety of ways, Glaser et al (1962) points out:

"The differential effectiveness and characteristics of learning by means of programmed text procedures should be investigated for high - , how - , and exceptional - aptitude students to the degree that it would be feasible to construct differently programmed textbooks or different subsequences within a program to facilitate optimal learning".

(p.443)
Due to this, some modifications have been introduced to the basic linear programme. Most of them are in the same line as the linear programme and are usually called skip linear. This type of programme has, at certain key points, frames which test whether the student has mastered what he has studied in the previous frames. In each of these frames, if the student's response is correct, he is instructed to skip ahead to another part of the programme, if not, he goes through the normal sequence of the programme in order to revise the subject matter involved in the previous frames. A diagram of the process would look like in fig. 7.2. In this example, if the student's response to frame 10 is correct, he skips to frame 14, otherwise he goes on to frame 11.

![Diagram of skip linear programme](image)

The other type of programme - branching programme, was developed by Norman Crowder from his experiences in training armed forces personnel to understand and use electronic equipment (Lysaught and Williams 1963). According to Crowder (1962), the student is given the material to be learned in small logical units (i.e. frames). Each of these frames is followed by a multiple-choice question and the student's choice determines directly and automatically what frame he will see next. If the student chooses the correct answer to the frame, he is referred to the next frame in the main sequence of the programme, but if he chooses an incorrect answer, he is referred to frames written specifically to correct the particular error he has just made (remedial frames).
At the end of the remedial frame(s), the student will be either directed to return to the original frame to have a second try at the original question, or asked a question similar to that in the original frame. At this point there will be another branching. A diagram of this process would look like that in fig. 7.3.

**FIGURE 7.3 : Example of a sequence in a branching programme (Source: Appendix V)**

7.2.1.2 Programmed texts

Let us now relate the principles of programming mentioned at the beginning of section 7.2.1 to the specific cases of programmed texts. A general feature of a programmed text is that the material being taught is being broken down into small steps which are called frames. Each frame presents a certain amount of information and also requires the student to use what he has learned by asking him to give an active response - fill in a blank, solve a problem, derive a formula, select the correct answer from among several alternative answers, complete a diagram, etc., before he goes on to the next frame. After the student has given his response, he gets immediate knowledge of whether his result is correct. If his answer is right, he is asked to go to the next frame in the main sequence of the programme, otherwise he is asked either to revise the frame again or to go to another frame in
the programme in which he is given additional explanations and/or examples, or simply to go to the next frame in the main sequence.

Now I shall turn to the details of the design and development of the booklet 'Programmed Trigonometry'.

7.2.2 Design and development of the booklet 'Programmed Trigonometry'

7.2.2.1 Stages in its design and development

The characteristics of programming mentioned in section 7.2.1 were the basis for the different steps I followed in the design and development of the booklet. These steps are represented in the diagram in fig.7.4 and are described overleaf.

(1) Assumptions about the learners

The first thing in the design of the booklet was to ask myself the question: To whom is the booklet addressed? (i.e., what is the audience?). The answer to this question was: 'The booklet is addressed to students entering university courses in science and engineering who need revision in trigonometry'. Thus the booklet should be written bearing in mind that the students whom it is addressed to would have studied trigonometry at school and are therefore familiar with the subject matter.

(2) Outline of the contents

The outline of the contents of the booklet was initially derived from the contents of a number of A-level mathematics syllabuses (Joint Matriculation Board 1976, The Associated Examining Board 1976). This outline was then discussed with some mathematics lecturers at the Universities of Southampton and Surrey, and with some other
FIGURE 7.4: DIAGRAM OF THE STEPS FOLLOWED IN THE DESIGN AND DEVELOPMENT OF THE BOOKLET 'PROGRAMMED TRIGONOMETRY'.

1. Assumptions about the learners
2. Outline of the contents
3. Writing and ordering the objectives
4. Selection of programming model
5. Writing the frames
6. Writing the ancillary parts
7. Field testing
8. Analysis of the information gathered
9. Writing the second version
people familiar with the subject matter. As a result of these discussions, a revised outline of the contents of Trigonometry was agreed for the booklet, and it was also agreed that the booklet should have an introductory section with the basic definitions of an angle and some related topics in Cartesian Geometry because they were prerequisite for dealing with Trigonometry. As a consequence of the latter, I produced an outline of the topics in Geometry and Co-ordinate Geometry I thought were prerequisite for the mastery of the topics on Trigonometry, and discussed it later with the same people. As a result of these discussions, we agreed on the outline of the contents of this introductory section.

(3) Writing and ordering the objectives

After having decided on the contents of the booklet, my next task was to write down a list of the behavioural objectives for the booklet, i.e. what the student was expected to do upon completing the booklet, and to arrange them in a logical order. Each objective was based on the three characteristics of a behavioural objective according to Mager (1975), that is, (a) Performance, (b) Conditions, and (c) Criterion. All these objectives were also discussed with lecturers familiar with the subject matter and then revised accordingly.

(4) Selection of programming model

Once the objectives had been identified, the next step was the selection of programming model. Since the booklet should have, as it was noted earlier, an introductory section on Geometry and
Co-ordinate Geometry, I split the booklet into two sections - One: Angles and Cartesian Geometry, Two: Trigonometry - and selected programming models for each one (linear and branching, respectively). I chose the linear programme for the first section for two reasons: (i) because linear programmes require each student to go through the same instructional sequence (see section 7.2.1.1) and in our case all students were expected to revise all the section, and (2) because the student was expected to go fairly quickly through the material included in the section.

The branching programme was chosen for the second section because the contents of this section (i.e. Trigonometry) was the main concern of the booklet and the branching technique made it possible to give more remedial work for those students most in need (see section 7.2.1.1).

(5) Writing the frames

Before starting the actual writing of the frames of the two programmes, I carried out a review of textbooks on programmed learning with the purpose of finding out the characteristics of a frame.

A classical definition of a frame is:

"The segment of material which the student handles at one time .... In almost all programming methods, it will require at least one response .... and provide for knowledge of results before the student proceeds to the next frame".

(Meyer 1969, p.302.)
Following this definition, Lysaught and Williams (1963), Thomas et al (1963), Taber et al (1965), Leight (1966) and Meyer (1969), among others, classify the frames according to their design in the following three groups:

1. **Copying frames**: tell the student something and ask him to copy one or more words of what he is told.

2. **Prompted frames**: frames which have a prompt (i.e. a stimulus that makes the correct response more likely while the student is learning).

3. **Terminal frames**: have no prompts and test whether the student has mastery of what is contained in the previous sequence of frames of the programme.

A classification which does not follow the definition of frame given above is the one given by Gilbert (1958), based on the function of the frame. This classification, which may be called **Content classification**, can be summarised as follows:

1. **Lead-In Items**: do not give new information but prepare the student for new.

2. **Augmenting Items**: give new information but do not require the student to make any relevant response.

3. **Interlocking items**: require the student to revise established skills while new information is being presented.
(4) **Rote Review Items:** present a problem identical to one presented earlier. (This type of item is particularly suggested only for those cases in which something has to be learned by memory).

(5) **Restated-Review Items:** require the rehearsal of a skill in which the problem presented is restated in a different way.

(6) **Delayed Review Items:** allow for further practice in material which has been previously presented in the programme.

(7) **Fading Items:** give revision while prompts or cues are gradually withdrawn.

(8) **Generalizing Items:** point out a common characteristic of several specific problems previously presented in the programme.

(9) **Specific Items:** give examples following a rule.

(10) **Dovetailing Items:** require the student to discriminate between stimuli in cases in which the student may become confused.

As we can appreciate from the above classifications, the characteristics of a frame depend on the purpose of the frame within a programme.

To write the frames of the programmes of the two sections of the booklet, I used the aspects suggested by Whalley (1966) for the writing of frames as guidelines. These aspects are:

(a) Each sequence of frames should be developed until a terminal frame (or possibly several terminal frames) in which the student's assimilation of the topic is tested.
(b) The student should be given the feeling that he is always progressing. To refer the student back to something that he has already read should be avoided.

(c) Each frame should make the student think and make a response of some kind whenever possible.

(d) The student should always be encouraged by informing him immediately that his answer is correct, if it is correct.

(e) If a student makes an incorrect answer, it must be explained to him why the answer of his choice is incorrect, before informing him of his mistake.

I wrote the initial frames of the programmes on small index cards to facilitate any necessary correction and rearrangement of the sequence of frames. After the writing was completed, each frame was critically examined and then the two programmes were given to some lecturers familiar with the subject matter. With the feedback received from these people, second drafts of the programmes were written and the cycle analysis-feedback-revision was repeated, leading to third drafts of the programmes. These constituted the blue prints of the programmes for the field testing.

(6) Writing the ancillary parts

After the blue prints of the programmes were ready for the field testing, the ancillary parts of the booklet were written. These ancillary parts were the following:

(a) To the student. It introduced the booklet telling the student
some general characteristics of programmed texts, and indications
of how to work with the booklet.

(b) Complement to each of the two programmes of the booklet. It
included, for each of the two programmes, the following:

(b.1) A pre-test with answers. The pre-test consisted of
a criterion-referenced test of problem type questions
which were based on the objectives of the programme.

(b.2) Objectives and pre-test correspondence. This section
told the student the relationship between the objectives
of the programme and the items in the tests.

(b.3) Objectives and frames correspondence. This section told
the student the relationship between the objectives and
the frames of the programme.

(b.4) Plan of the programme. This consisted of an outline of
the contents of the programme (referred to the frames).

(b.5) Instructions. These told the student how to work
with the programme.

(c) Complement to the second section of the booklet. It included

(c.1) List of formulae. This comprised the relevant formulae
which appeared in the section.
(c.2) **Supplementary problems with answers.** This consisted of additional problems at the same level of difficulty as those included in the programme.

(c.3) **Appendices.** They comprised the derivation of the formulae used in the programme, which were not so important as to include them within the programme.

(7) **Field Testing**

After the whole booklet was ready, the next step was the field testing. This was carried out with 47 school leavers from three schools in the area of Guildford a few weeks before they sat their mathematics A-level examinations. The sample was distributed as follows:

<table>
<thead>
<tr>
<th>SCHOOL</th>
<th>TYPE</th>
<th>NUMBER OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Comprehensive School</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>Sixth Form College</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>Public School</td>
<td>12</td>
</tr>
</tbody>
</table>

All students in schools A and C worked individually with the booklets whereas those in school B worked in pairs (i.e. there were 12 pairs). The procedure followed in the field testing was as follows: when the students were given the booklets they were told what the activity was all about, and were also given a sheet of paper with the main information of the whole activity (see Appendix T). Students were orally told the purpose of the field testing, asked to go through the whole booklet and to write into it: (a) their responses to the frames, (b) any comments about wording, instructions and explanations
given in the frames, and (c) anything that they did not understand or found difficult.

By the end of the first week I interviewed all of them (individually or in pairs) and gathered their impressions and comments up to that moment, then by the end of the second week when they were expected to have gone through the whole booklet, I interviewed them again with the same purpose. A total of 30 booklets out of the 35 given out were returned.

(8) Analysis of the information gathered

After the collection of the booklets I went through each one to gather all the information written into them. Such information could be classified into four groups, namely:

(a) General comments
(b) Wording and figures
(c) Troublesome frames
(d) Comments given on the frames

I analysed this information and drew some conclusions which should be considered for the second version of the booklet. The most relevant conclusions were:

(a) The first section of the booklet (i.e. Geometry) could be left out because most of the students found it 'too elementary' and this made it 'tedious' for a student who has studied Trigonometry before. Since the basic contents of this section was the definition of an angle and some students may need to go through it before they actually revise the section on Trigonometry, this
(b) The number of items in the pre-test ought to be extended in order to be more specific when relating them to the frames and therefore have less frames per item in the pre-test.

(c) The frames in the new version should be re-arranged in such a way that the student does not have to turn back in the text (either to go on to the next frame in the sequence or to check his answer to a frame), as it was found that this may discourage the student.

(d) The frames dealing with tables should be re-written in such a way that if a student knew how to read tables, he would skip some frames otherwise he would go through several remedial frames.

(e) There were some frames which required the student to solve a problem and then check his answer in the following frame, where the problem was solved in full. Students who got the right answer to the problem in the first frame said that they should have skipped the second frame. Due to this fact a branching should be introduced in frames such as these, i.e. the student will be given the correct answer to the problem and asked to check his answer against it. If his answer is right, he will be asked to continue with the main sequence of the programme, otherwise he will be asked to go through a sequence of remedial frames.
Bearing in mind the results of the field testing, the second version of the booklet was written. It consisted of the following sections:

* (a) To the student
* (b) A list of behavioural objectives
* (c) A pre-test with answers
* (d) Objective-Pretest-Frame Correspondence
* (e) Plan of the programme

(f) Introduction: This presents to the student some terms and symbols which are concerned with the idea of an angle and which are used throughout the booklet.

(g) Further instructions: They explain to the student how he should proceed to work with the programme.

(h) The programme: This consisted of a branching programme of 137 frames.

(i) A post-test with answers: The purpose of this post-test is to test whether the student has grasped the material he has revised. The number of the items in it are directly related to those in the pre-test (i.e. question 1 in the post-test tests the same objective as question 1 in the pre-test), and the student has to answer only those he failed in the pre-test.

* (j) Supplementary problems with answers
* (k) A list of formulae

* This section had basically the same contents as in the first version
Appendices: There are four appendices, three of them are concerned with demonstrations or deductions of some of the formulae which appear in the programme, and the fourth consists of additional information about the development of the booklet.

To work with the booklet, the student has to proceed in the way shown in the flow chart in fig. 7.5

7.2.2.2 Format of the second version

Most of the existing branching programmed texts are presented in the form of a 'scrambled book'. A scrambled book is defined as:

"A book that presents an intrinsic program. The pages are not read consecutively. Following the information presentation, a multiple-choice question is given. The answer that the student selects refers him to a particular page for confirmation or correction. He may be sent either forward or backward in the text, the number of pages in either direction being randomized. Thus no clue as to which alternative is correct can be found in the page reference accompanying each alternative".

(Meyer 1969, p.304).

In the second version of the booklet on Trigonometry, the programme was 'unscrambled' by having one frame on each page and by giving the student the impression that he is always progressing - the student moves always forwards. Figure 7-6 is an example taken from the
FIGURE 7.5: FLOW CHART OF HOW A STUDENT SHOULD WORK WITH THE BOOKLET ON TRIGONOMETRY

1. Read general instructions
2. Answer the pre-test
3. Check your answers
4. All answers right? (Yes/No)
   - Yes: The end
   - No: Find out the numbers of the frames you should revise
   - Go through these frames
   - Answer the corresponding questions in the post-test
actual sequence of the programme on Trigonometry (see Appendix T) which shows how the student moves forwards if he fails the question asked in a frame.

If he fails the question in frame 84, he is asked to go to frame 85 (remedial frame) in which is explained how to solve the problem in frame 84, and he is then asked to solve a similar problem. If he fails the question in this frame, he is asked to go to frame 86 (remedial frame similar to frame 85). If he fails the question of this frame, he is then asked to repeat the frame. If he gives the correct answer either in frame 84, 85 or 86, he is asked to go to frame 87.

There are some occasions in which the student is asked to go back to the initial frame in a subsequence - if his answer to a further frame indicates that he has not grasped the theory involved. An example of this situation is given in fig. 7.7.
The actual size of the format of the booklet was A5. The frames and their corresponding questions appeared on the right hand side pages, and the answers to the questions and instructions appeared on the following pages (i.e. left hand side), as it is shown in fig. 7.8.
Since the second version of the booklet differed substantially from the first version, I interviewed 24 students who used it during their revision work in mathematics at the University of Southampton in 1977 and gave them a questionnaire (see Appendix F). The information collected showed that, apart from a few misprints, the students found the booklet between interesting and average, and between useful and very useful. General comments included:

'I found it most useful and far better than slogging through a textbook trying to find the relevant pieces of information'.

It would appear that the fact of having 'unscrambled' the programme was a very great advantage because no student made criticism either of the programme itself or of the manner they had to go through it.

The sections more consulted according to the students' performance on the pre-test were those concerned with: (a) Trigonometric functions for angles greater than $2\pi$ ($360^0$), (b) Trigonometric formulae, (c) Sum and difference of sines and cosines, and (d) Small angles.

7.3 THE BOOKLET ON ALGEBRAIC MANIPULATION

As it happened in the case of the booklet on trigonometry, the first problem I faced with the booklet on Algebraic Manipulation was to find a suitable way of presenting the topics bearing in mind the three considerations stated in section 7.1. In addition to these, I should also take into account that students' needs for Algebra are somehow different from their needs for Trigonometry, as they are more likely to know more Algebra than they know Trigonometry because they usually have to deal with Algebraic Manipulation more than they have to do with Trigonometry. Under these circumstances, I discarded the idea of using a programmed text format for this booklet due to the following reasons:
(1) Because the main purpose of the booklet was to give the student the opportunity to regain the ability to deal with routine algebraic manipulation.

(2) Since many students may only need to practise specific types of routine algebraic manipulation, the programmed learning format could be found boring (see (8) in section 7.2.2.1).

(3) Because the writing of programmed material is time consuming.

It should be pointed out here that although I discarded the idea of using a programmed text format, I did not discard the idea of using the principles of programmed learning in this booklet, that is, logical sequence of small steps, students are required to respond actively to each step, and checking the correctness of his response before going on to the following step.

7.3.1 Stages in its design

Based on the experience with the booklet on trigonometry, I followed the following steps in the design of the booklet on 'Algebraic Manipulation' (see fig. 7.9). :
(1) **Assumptions about the learners**

The booklet (see Appendix V) should be used by those students who need practice in routine algebraic manipulation that they have studied at school and which are frequently needed.

(2) **Outline of the contents**

In order to gather engineering lecturers' opinions about the topics that should be treated in the booklet, I produced a questionnaire (see Appendix C) which covered the following topic areas:

(a) Removal and Multiplication of brackets
(b) General laws of exponents
(c) General laws of radicals
Copies of the questionnaire were given to five engineering lecturers at the University of Southampton and to five engineering lecturers at the University of Surrey.

In the questionnaire the respondents had only to tick, according to their experience, the appropriate boxes concerning the algebraic manipulation that students should know, but sometimes do not, when they enter university.

After all the forms of the questionnaire were returned, I produced a short 'complementary questionnaire' regarding the new topics that had been mentioned by one or two respondents in the first questionnaire, and sent it to the same respondents to gather their opinions. Since no forms of this complementary questionnaire were returned, it would appear that the additional topics were not relevant for the courses the respondents taught.

Every topic in the first questionnaire was mentioned by at least 45% of the respondents. To validate this data, I discussed this matter with two mathematics lecturers at the University of Surrey. The outline of the contents agreed in such discussion was the following:

(d) Rationalization of the denominator of a fraction with radical in the denominator
(e) Operations with polynomials
(f) Polynomial factorisation
(g) Operations with rational functions
(h) Solution of linear equations
1. Removal of brackets
2. Multiplication of brackets
3. Exponents
4. Radicals
5. Polynomials
6. Polynomial factorisation
7. Radical functions
8. Solution of linear equations
9. Solution of simultaneous linear equations
10. Solution of quadratic equations

The latter two topics were mentioned by more than 45% of respondents in the first questionnaire.

(3) Writing the objectives

After it was decided what the contents of the booklet should be, the next step was to write the behavioural objectives of the booklet. These are listed below:

At the end of the booklet the student should be able to:

(i) Apply the rules for the removal of brackets preceded by either a positive sign, a negative sign, or no sign at all.

(ii) Expand and simplify products of the form $(ax + by)^2$ and $(ax + by)(cx + dy)$.

(iii) Evaluate powers such as $5^{-2}$ and $(0.17)^{2.8}$ using logarithms.
(iv) Apply the rules for the multiplication and division of powers which have equal basis.

(v) Simplify expressions which involve the application of the property $e^{\ln x} = x$

(vi) Apply the laws for radicals

(vii) Rationalise the denominator of fractions which have radical sign in the denominator

(viii) Add, subtract, multiply and divide polynomials

(ix) Factorise polynomials

(x) Add, subtract, multiple and divide rational fractions

(xi) Solve linear equations

(xii) Solve sets of two simultaneous linear equations with two unknowns

(xiii) Solve quadratic equations

(4) Selection of a paradigm

For the selection of a paradigm, the following aspects were taken into account:
(a) The main purpose of the booklet was to offer the student routine algebraic manipulation in which he may need practice.

(b) The booklet should try to attain the objectives mentioned above.

(c) Each student may only need to work through some of the sections of the booklets and different students may need to work through different sections.

(d) The exercises in each section should be graded.

Based on the principles of programmed learning (but not on the format of programmed learning) and on the aspects mentioned above, I decided to split the booklet on Algebraic Manipulation into ten sections independent of each other, and each one covering one of the topics mentioned earlier (see (2) Outline of the contents). Each section should consist of (i) revision notes, (ii) some worked examples, and (iii) a set of graded exercises for the student to solve. The answers to each of these exercises should be provided at the back of the booklet. To work with any of the sections (or sub-sections) of the booklet, the student should proceed as it is indicated in the diagram in fig. 7.10.

(5) Writing the sections

After the paradigm for the booklet had been selected, the next step was to write the different sections. I carried out the actual writing of each of the sections in the following order:
FIGURE 7.10: Flow chart of how the student should work with each section of the booklet on Algebraic Manipulation

1. **Revise Notes**
2. **Study Worked Examples**
3. **Solve Exercise 1**
   - **Check answer with the one provided**
   - **Is it right?**
   - **Yes**: **WAS IT THE LAST EXERCISE?**
     - **Yes**: **End**
     - **No**: **Solve the next exercise**
   - **No**: **Revise relevant worked example**
   - **Attempt to solve the exercise again**
1. Revision notes, where necessary.
2. Design of the set of graded exercises.
3. Worked examples of the section.

To design the sets of graded exercises, I initially based on my own experience the decision about the level of difficulty of the exercises within each set, then I gave the ten sets of graded exercises to two University mathematics lecturers and to other persons familiar with the subject matter and asked them to grade the exercises within each of the sets.

After this validation of the order of the exercises, the necessary changes were introduced and the resulting sets made up the final sets of exercises for the booklet.

(6) Writing the ancillary parts

After all the material of the ten sections were ready, the next step was to write the ancillary parts of the booklet. These ancillary parts were the following:

(a) To the student: This introduces the booklet to the student and tells him how to work with it.

(b) A pre-test with answers: The pre-test consists of a Criterion-referenced test of problems based on the objectives of the booklet.

(c) Pre-test Topic Correspondence: This consists of the relationship between the items in the test and the sections in the booklet.
(d) Instructions: These tell the student how to work with the sections of the booklet. To work with the booklet, the student has to proceed in the same way as for the booklet on trigonometry (see fig. 7.5).

7.3.2. Students' impressions about the booklet

In order to gather students' comments about the booklet, I interviewed 8 students who used it during their revision work in mathematics at the University of Southampton and gave them a questionnaire (see Appendix G). The information collected revealed that all students found the booklet useful and that the sections more consulted were: Exponents, laws of radicals and Simplification of rational functions, particularly the latter. No comments regarding the design of the booklet were given, except that it was easy to understand and that some of the answers given at the back in the booklet needed to be revised. A design similar to the one of the booklet on Algebraic Manipulation is the one used by the Schaums' Outline Series (e.g. Spiegel 1959, Ayres 1964). Although each of the books of this series is designed 'to be used either as a textbook for a formal course, or as a very useful supplement to all current standard texts', each chapter begins with the statements of pertinent definitions, principles and theorems, and is then followed by a set of worked examples and a set of graded problems for the student to solve.

The success of this series would appear to indicate that the way of presenting the material in the books has been successful. The main difference between the books of this series and the booklet on Algebraic Manipulation is in the audience, that is, in my case the booklet assumes that the student has studied the material some time
before (i.e. the booklet is designed for revision purposes) whereas in the case of the Shaum's series the books are basically addressed to beginning students (i.e. students who will be studying the subject matter for the first time). For this reason, the books of the series include theorems and derivations of formulae among the solved problems whereas the booklet on Algebraic Manipulation does not include any theorem or derivation of formulae at all.

7.4 CONCLUSIONS

As was noted earlier (see section 7.1), the design of the two booklets discussed in this chapter was based on three aspects, namely (1) the student is expected to have studied the subject matter presented in the material, (2) the material should emphasize the mastery of working techniques, and (3) the material should require the student to spend little time on it to regain the skills he has lost. From my experience in the development of the two booklets, the following general conclusions and/or suggestions could be given/drawn:

1. In any material designed for revision purposes there may be certain definitions, rules, etc., which are needed to deal with the material and which at the same time may be very elementary. It may be advisable to give, in the introduction of the material, only general hints about these definitions, rules, etc., instead of full accounts. In the first version of the booklet on Trigonometry students found 'tedious' a section on Geometry and Co-ordinate Geometry which had been treated with certain depth (see (8) in section 7.2.2.1).

2. If a particular notation is intended to be used throughout the revision material and it cannot be ascertained whether the
student knows it, it is advisable to introduce such notation just before the actual revision material. However, if this is only needed for a particular section of the material, it should be given at the beginning of such section. In the booklet on Trigonometry, the notation needed for each particular topic was introduced by means of augmenting frames (see (5) in section 7.2.2.1) at the beginning of the sequence of frames of the topic; and in the booklet on Algebraic Manipulation it was done by means of revision notes given at the beginning of each section in the booklet (see (4) in section 7.3.1).

(3) Since different students may need to revise different topics, the items in the pre-test should be related to the topics in the material. In addition to this, the number of items in the pre-test should be large enough as to indicate precisely the topics to be revised, thus enabling the student to pinpoint where he really needs revision and, at the same time, not making him revise material that he already masters. Furthermore, the items should have the same level of difficulty as the more complex exercises within each topic. In the evaluation of the booklet on Trigonometry it was found that the number of items in the pre-test was too short and that students had to revise material that they had already mastered (see (8) in section 7.2.2.1).

(4) The selection of the format of the material (i.e. how the material is presented) is influenced by how frequently the students have dealt with the subject matter during their studies, that is, the format of the material comprising subject matter that the students have studied recently for the first time is likely to be different from the format of the material the students have met early in their studies and which they have also met several times
later on. Although in my own case the booklet on Trigonometry
was written using the format of programmed learning because
this had proved to be of particular benefit for revision
purposes (see section 7.2), the format of the booklet on Algebraic
Manipulation was chosen to be different due to the fact that
students' needs for algebra were somehow different from their
needs for trigonometry (see section 7.3).

(5) If the material consists of a programmed text of the branching
type, it is advisable to have one frame on each page and also
have the text unscrambled. I found that when the text is
scrambled the student feels discouraged (see (8) in section
7.2.2.1 and section 7.2.2.2).
CHAPTER 8

DISCUSSION, CONCLUSIONS AND FURTHER WORK
The research and development study presented in this thesis was initiated as a reaction to the existing problem of the interface between school and university in the area of mathematics. I have attempted to learn more about the conditions behind this problem investigating the mathematics prerequisite for engineering courses and the relationship between the students' mathematical needs just before leaving secondary school and at the point of entry to university. At the same time I have developed educational products aimed to reduce the problem, such as approaches to individual diagnosis (including a battery of diagnostic tests based on mathematics prerequisite knowledge and skills for engineering courses) and remedial materials.

I shall present in this chapter a general discussion of the material from different chapters which relate to each other, then a summary of the general conclusions that emerged of the study and finally some recommendations for further work.

8.2 DISCUSSION

It has been recognized that in mathematics learning, one item of knowledge requires the prior mastery of another item (Gagné 1962). For example, in order to succeed in a second year mathematics course one must have mastery of the knowledge of a first year mathematics course, or in order to do Algebra one should first have mastery of certain elements of arithmetic. In order to ensure that a student has the mastery of the knowledge prerequisite for a new course, I have suggested a revised scheme of the process of curriculum development based on the hierarchical theory of Gagné (1970). This revised scheme includes a stage of diagnosis and
remedial instruction (see fig. 1.2 in Chapter 1) in which the student who intends to enter a new course is given remedial work based on individual diagnosis of knowledge prerequisite for the new course before he actually starts it (see fig. 1.3 in Chapter 1).

Engineering departments expect the students to have mastery of certain concepts and skills in mathematics before they come to university, as these will be needed to understand the material taught in the engineering courses. This means that before a newly enrolled engineering student starts his university courses he should have already achieved certain objectives. A list of such objectives is presented in section 4.7. These objectives are based on the topics and level of difficulty given by respondents in the survey carried out in this research (see Chapter 3). It was found in this survey that when one intends to identify the level of difficulty expected in a topic, one should also bear in mind the prerequisites for dealing with the topic in question, as there may be cases in which the student does not even have these.

The fact that engineering students are actually required to have an A-level, or an equivalent, does not ensure uniformity in their mathematical knowledge. Indeed, not all students have met at school certain pieces of mathematics that are regarded as prerequisite for their first year courses (see sections 6.2.1 and 6.2.2). In addition to this staff lecturing first year engineering courses expect their students to know certain mathematical topics which do not appear in the A-level mathematics syllabuses of some Examining Boards (see section 3.4).

Students entering engineering courses argued that they had forgotten certain topics in mathematics due to the break between school and university (see sections 6.2.1 and 6.2.2). It was found in this research
that topics which involve the knowledge of a concept or a numerical value are likely to be forgotten at the transition and that students found difficulty in those topics which involve the use of formulae because they did not actually know the formulae (see section 6.4). Due to all this, students may find themselves in a situation in which they may have to try to assimilate the new material on the engineering courses whilst they revise, or learn, the basic mathematics on which the material is based.

All these facts create concern regarding the interface between school and university, particularly for engineering students, and with the provision of a solution to the problem of how to have, at the beginning of the first year engineering courses, groups of students more homogeneous in their knowledge of mathematics. The revision of all mathematics prerequisite for engineering courses is not necessary for all newly enrolled students (see Chapter 6). Therefore, a system whereby individual needs could be identified and attended to is of great importance. According to the experience gained in this research, three elements should be taken into consideration before deciding on a particular scheme of individual diagnosis. These elements are: (1) the number of students involved, (2) the number of staff involved, and (3) the length of time available for the exercise.

The T.T. method of individual diagnosis (see section 5.2.1) was found to be efficient when the number of students per each staff is small (i.e. 5 students or less per each staff per hour). The experience at the University of Southampton showed that when the proportion students/staff is large, the T.T. method is not to be recommended (see Section 5.2). A system whereby individual diagnosis can be given and which copes with
large numbers of students consists of giving the students a battery of diagnostic tests. These tests enable each student to diagnose his individual needs without having a tutorial discussion after the completion of the tests, as he can mark the tests himself using the correct answers and the criteria associated with the battery.

According to the experience gained in the development of the battery of diagnostic tests for students entering engineering courses (discussed in Chapter 4), a battery of mathematics diagnostic tests for a new course should have the following characteristics: (i) it should cover those aspects in mathematics which are needed to cope with the new course (i.e., prerequisites), and (ii) it should diagnose students' individual difficulties in each of these aspects. In other words, the battery should consist of criterion-referenced mathematics tests which have a placement and a diagnostic function. In addition to these characteristics, it should also be borne in mind that (a) the items in the tests should play a diagnostic role (i.e., the tests should be valid for diagnosis), (b) there should be at least three items for each of the objectives on which the test is based (i.e., the tests should have a certain reliability), and (c) the length of each of the tests in the battery depends on the conditions under which the student will complete them (i.e., the tests should be usable). I should also point out that this last point (i.e., usability) is usually given more importance than the other two due to the constraints of time for the administration of the test. However, it should be borne in mind that if the purpose of giving the test(s) to students is to diagnose their weaknesses in order to give each of them appropriate remedial work, the validity and reliability of the test(s) are also of great importance, as these ensure that the remedial work suggested to each student is the one he actually needs. Otherwise he may be revising material which he actually does not need to revise (see section 4.7.3).
The T.A.O.I. method of individual diagnosis (see section 5.3.2) was found to be very valuable because it enables one to locate specific difficulties students may have, such as ignorance of basic algorithms or formulae, forgetfulness, misunderstanding or careless mistakes. According to my experience with the use of the method with a group of students entering engineering courses, the method requires that the student (i) be presented with one problem at a time, and (ii) be interviewed about the process of his working on the problem either when any difficulty emerges during his working or immediately after he has finished with the problem. Since this method needs a one-to-one contact for the diagnosis, it can only be used when a very small group of students is involved.

After students' weaknesses have been identified, by whatever means, they should be advised to do some work to remedy their deficiencies. In this research, four elements that characterize a scheme of revision work have been identified. These elements are:

(1) **Materials**  The kind of materials which were found to be useful were those specially written for revision purposes, particularly those which had been designed using the principles of programmed learning. When designing materials for revision purposes, the following aspects should be borne in mind: (i) the selection of the format of the material depends on the frequency with which students have studied the subject matter comprised in it (see 4 in section 7.4); (ii) the material should include a pre-test with as many items as possible; and (iii) the topics in the material should be related to the items in the pre-test.

(2) **Management**  The process whereby students select the material relevant for their particular revision work should enable them to find such material quickly.
(3) **Time** The length of time allocated for the revision work influences the structure of the scheme of revision work. It should be borne in mind that the period of time for the actual revision work should be long enough as to allow the weakest students to work over all the subject matter they need to revise.

(4) **Motivation** It was found in this research that the post-tests students took after they had completed the revision work acted as a motivational factor for the carrying out of the revision work. If these are not included in the scheme of revision, some other alternative should be included.

8.3 **GENERAL CONCLUSIONS**

The outcomes of this research can be summarised as follows:

- Mathematical topics which do not appear in the A-level mathematics syllabuses of some Examining Boards were expected to be known by newly enrolled engineering students.

- When the level of difficulty expected in a topic is identified, the prerequisites for dealing with the topic should also be identified.

- The number of students, the number of staff and the time available for the exercise of diagnosis seem to affect the structure of the scheme of diagnosis.

- The T.T. method of individual diagnosis is efficient when there is a small number of students per each staff.
Individual diagnosis in mathematics can be given to a large number of students by means of a battery of diagnostic tests.

A mathematics test for diagnosis should be a criterion-referenced test with placement and diagnostic functions.

A mathematics test for diagnosis should have at least three items per each of the objectives on which it is based.

The T.A.O.I. method of individual diagnosis enables one to locate specific difficulties students may have.

Mathematical topics which involve the knowledge of a concept or a numerical value are likely to be forgotten in the transition between school and university.

The use of formula books seems to be a reason for the difficulty students encounter when dealing with topics which involve the use of formulae.

Materials, Management, Time and Motivation are elements that characterize a scheme of revision work.

Materials for revision purposes should include a pretest which has as many items as possible and a relationship between the items and the topics in the material. The format of the material depends on how frequently students have dealt with subject matter involved.

Any scheme of revision work should take place at the beginning of the new course for which the revision work is intended.
As was shown in this study, the stage of diagnosis in the curriculum planning for first year university courses is an activity that should not be neglected. Although this stage has been used in some universities, its use is still very scarce.

Many of the findings presented in this thesis revealed that there are certain discrepancies between what is taught at school and what is expected from the students when they come to university. This phenomenon raises several questions which need to be answered: Why students at school do not grasp some knowledge, such as solution of linear equations, rules for transposition of factors in an inequality? Is there any relationship between these topics and the examination papers of the different Examining Boards? Why students do not learn trigonometry at school? Is the use of formula books the only reasons for this? Research in the area of diagnosis is something new. In primary school some research has been carried out in the area of reading and arithmetic, but even in these cases, the diagnosis has been used, most of the time, to tell if a student can or cannot do something but not why he cannot do it (Gage and Berliner 1975). At university level, diagnosis only has sense if remedial instruction is being provided, that is, diagnosis means to tell the student what he cannot do and how to remedy the situation. In this area, further research should develop new means to diagnose precisely large numbers of students and to ascertain whether the student has forgotten a topic or if it is new to him.

Since the materials used for revision purposes should be self-teaching materials, there is a great need for research in this area of styles of
self-teaching materials that facilitate the student to regain the knowledge and skills he has lost.

In this study I was concerned with the diagnosis of students at the school-university interface but does a problem of matching exist between the first and second year at university, and between the second and the final year, particularly in those institutions which have industrial years?

Finally, several more detailed problems of different kinds have been noticed, but not yet solved. Can students really use formulae, if they have always to refer to formula books? How can the diagnostic tests be made more reliable? How can one distinguish in a diagnosis between a forgotten topic and a new one? Can the T.A.O.I. method of individual diagnosis be used more generally than has been the case in this thesis?
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The following is a list of possible mathematical topics which has been compiled from previous work done on this matter.

If the mathematical topic is needed in your engineering course, would you please indicate this by giving in column A an example of an engineering topic for which the mathematical topic is needed, and then giving in column B an example of a specific situation, taken from your own engineering course, showing how the mathematical topic is used (see the example below).

The example you give in column B will help us to determine the level of difficulty of the topic you are giving in column A and therefore, will enable us to construct the question of the pre-knowledge test accordingly.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ENGINEERING TOPIC</strong></td>
<td><strong>EXAMPLE OF MATH. NEEDED</strong></td>
</tr>
<tr>
<td>6.- Use of formulas for sin(x+y), cos(x+y), tan(x+y)</td>
<td>JRC circuits sin(x+y), simultaneous equations</td>
</tr>
</tbody>
</table>

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**APPENDIX A**

The SV-questionnaires: pilot version

The purpose of this survey is to gather some information for the revision work in mathematics, to be carried out by lecturers in engineering courses in Universities. This survey is designed to obtain your views and opinions about the basic mathematics definitions, knowledge and skills which are needed to cope with the mathematics needed at the beginning of the engineering course which you yourself are giving to first-year students in your department. This is therefore mathematics which the students are expected to have by the time they arrive at University.

It is hoped that by getting your views you will enable me to design a pre-knowledge test for the students.

I would be most grateful if you could complete the survey and return it to me in the attached envelope to the Institute for Educational Technology of the University of Surrey.

Thank you for your co-operation.

NAME:

UNIVERSITY:

SUBJECT YOU TEACH TO FIRST YEAR ENGINEERING STUDENTS:

EDUARDO GOMES-JEUN
Institute for Educational Technics,
University of Surrey, Guildford, Surrey,
April 1978.
<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ENGINEERING TOPIC</strong></td>
<td><strong>EXAMPLE OF MATHS. NEEDED</strong></td>
</tr>
<tr>
<td>1. Binomial Theorem for positive integral index</td>
<td></td>
</tr>
<tr>
<td>2. Binomial Theorem for fractional or negative exponent</td>
<td></td>
</tr>
<tr>
<td>3. Use of formulae for $\sin(x+y)$, $\cos(x+y)$, $\tan(x+y)$</td>
<td></td>
</tr>
<tr>
<td>4. Use of formulae for $\sin x \sin y$, $\cos x \cos y$, $\tan x \tan y$</td>
<td></td>
</tr>
<tr>
<td>5. Solution of triangles (sine and cosine formulae)</td>
<td></td>
</tr>
<tr>
<td>6. Numerical values for expressions involving sines and cosines of angles expressed as a fraction or multiple of $\pi$</td>
<td><strong>ENGINEERING TOPIC</strong></td>
</tr>
<tr>
<td>7. General solutions of equations such as $\sin x = 1$</td>
<td><strong>EXAMPLE OF MATHS. NEEDED</strong></td>
</tr>
<tr>
<td>10. Integration by parts</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX II

The SV-questionnaire: Final version

PREKNOWLEDGE IN MATHEMATICS OF STUDENTS ENTERING ENGINEERING COURSES

SURVEY

This survey is designed to obtain your views and opinions about the basic mathematics knowledge and skills which are needed in the engineering course which you yourself are giving to first year students in your department. This is therefore mathematics which the students are expected to know by the time they arrive at university.

It is hoped that by giving me your views you will enable me to design a pretest in mathematics for students entering engineering courses at universities.

I would be most grateful if you could complete this questionnaire and return it to me in the attached envelope to:

Mr. Eduardo Gonzalez-Leon,
Institute for Educational Technology,
University of Surrey,
Guildford, Surrey.

Thank you for your cooperation.

December 1978

Eduardo Gonzalez-Leon
INSTRUCTIONS

The questionnaire is to be filled in bearing in mind those students who have a single A-level in mathematics or an equivalent when they enter engineering courses at universities.

The mathematical topics in column A were compiled in the pilot stage of the present questionnaire which was carried out with staff lecturing to first year engineering students. The examples given in column B were given by these staff to illustrate the level of difficulty expected of each topic in column A.

If the mathematical topic in column A is needed in your engineering course, would you please indicate this by ringing in Column C the term(s) in which you need it (1: Autumn term; 2: Spring term; 3: Summer term), and then giving in Column D a specific example which illustrates the level of difficulty expected, similar to the examples given in column B (There are some topics in which no examples are given in column B because I was unable to identify any during the pilot stage).
<table>
<thead>
<tr>
<th><strong>COLUMN A</strong></th>
<th><strong>COLUMN B</strong></th>
<th><strong>COLUMN C</strong></th>
<th><strong>COLUMN D</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MATHEMATICAL TOPIC</strong></td>
<td><strong>EXAMPLE OF THE LEVEL OF DIFFICULTY</strong></td>
<td><strong>TERM IN WHICH THE TOPIC IS NEEDED</strong></td>
<td><strong>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</strong></td>
</tr>
<tr>
<td>Logs &amp; Logarithms</td>
<td>If ( \log_b x = a - b ), what is the value of ( x )?</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Application of properties</td>
<td>Expand ( \log \left( \frac{a}{b} \right) ) using the properties of logarithms</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Vectors</td>
<td>Addition and Subtraction of two vectors</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Composition of two vectors in two dimensions</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Scalar Product of two vectors</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Complex Numbers</td>
<td>Addition and Subtraction of two complex numbers</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Product and Quotient of two complex numbers</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Use of notation ( e^{\text{i} \phi} )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Modulus of a complex number</td>
<td>Find the modulus of the complex number ( 11 - 3i )</td>
<td>1 2 3</td>
</tr>
<tr>
<td></td>
<td>Conversion between rectangular and polar form</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>COLUMN A</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY</td>
<td>COLUMN C</td>
<td>COLUMN D</td>
</tr>
<tr>
<td>------------------</td>
<td>------------------------------------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>MATHEMATICAL TOPIC</td>
<td>OPERATIONS WITH FRACTIONS</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY</td>
<td>TERM IN WHICH THE TOPIC IS NEEDED</td>
</tr>
<tr>
<td>Operations with fractions</td>
<td>( \frac{a-c}{b-d} ) as a single fraction</td>
<td>1 2 3</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Algebra</td>
<td>Expansion of ( (a+b)^2 )</td>
<td>Multiply out and simplify as much as possible: ( (a+b)(a-b)^2 )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Algebra</td>
<td>Rooting algebraic manipulation</td>
<td>Rewrite as a single power of ( x ): ( \frac{(x^2)^2}{(x^3)^3} )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Algebra</td>
<td>Transposition of formuiae</td>
<td>Solve for ( x ): ( W=\frac{\sqrt{TR}}{K}, R \neq 0 )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Algebra</td>
<td>Solution of linear equations</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Algebra</td>
<td>Solution of simultaneous linear equations with two or three unknowns</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Algebra</td>
<td>Solution of quadratic equations</td>
<td>Solve for ( x ): ( 2x^2-5x-1=0 )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Algebra</td>
<td>Meaning of the symbol ( \Sigma )</td>
<td>Evaluate: ( \frac{1}{\Sigma_{x=i}^5(3x+1)} )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>Inequalities</td>
<td>Definition of modulus of a real number</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Inequalities</td>
<td>Solution of inequalities in one variable</td>
<td>Solve for ( x ): ( 3-2x \geq 4 )</td>
<td>1 2 3</td>
</tr>
<tr>
<td>MATHEMATICAL TOPIC</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY</td>
<td>TERM IN WHICH THE TOPIC IS NEEDED</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</td>
</tr>
<tr>
<td>--------------------------------------------------------</td>
<td>-------------------------------------</td>
<td>----------------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td>Definitions of the trigonometric functions in a right angled triangle</td>
<td>Application of the definitions in problems, e.g. ( \frac{y}{x} = \tan \theta ) ( x = \sin \theta ); ( y = \cos \theta )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Trigonometry</td>
<td>Evaluate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>General solution of equations such as ( \sin \theta = \frac{1}{2} )</td>
<td>Expand:</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Use of formulae for: ( \sin(x \pm y) ); ( \cos(x \pm y) ); ( \tan(x \pm y) )</td>
<td>Expand:</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Use of formulae for: ( \sin(x \pm y) ); ( \cos(x \pm y) ); ( \tan(x \pm y) )</td>
<td>Expand: ( \sin(x \pm y) \pm \sin(x \pm y) ( x \pm y )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Recognition of the formula, ( \cos(\alpha)\cos(\beta) = \frac{1}{2} [\cos(\alpha + \beta) \pm \cos(\alpha - \beta)] )</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Sine and Cosine rules</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Identification and sketching of the sine and cosine curves</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Binomial Theorem for positive integral index</td>
<td>Expansion of ( (a+x)^n ) for ( n = 2, 3, 4 )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Binomial Theorem for fractional or negative index</td>
<td>Expansion of ( (1-x^2)^{-\frac{1}{2}} )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>COLUMN A</td>
<td>COLUMN B</td>
<td>COLUMN C</td>
<td>COLUMN D</td>
</tr>
<tr>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>MATHEMATICAL TOPIC</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY</td>
<td>TERM IN WHICH THE TOPIC IS NEEDED</td>
<td>EXAMPLE OF THE LEVEL OF DIFFICULTY EXPECTED</td>
</tr>
<tr>
<td>Differential of algebraic functions</td>
<td>( y = x^2 - 2x ) find ( \frac{dy}{dx} )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of trigonometric functions</td>
<td>( y = \sin^2 x - \cos x ) find ( \frac{dy}{dx} )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of expressions involving exponentials</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of expressions involving logarithms</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of the sum of two functions</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of the product of two functions</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of the quotient of two functions</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Differential of function of a function</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Integration of algebraic functions</td>
<td></td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Integration of expressions involving trigonometric functions</td>
<td>Evaluate ( \int \sin^2 \omega t , dt )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>COLUMN A</td>
<td>COLUMN B</td>
<td>COLUMN C</td>
<td>COLUMN D</td>
</tr>
<tr>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td><strong>MATHEMATICAL TOPIC</strong></td>
<td><strong>EXAMPLE OF THE LEVEL OF DIFFICULTY</strong></td>
<td><strong>TEST IN WHICH THE TOPIC IS NEEDED</strong></td>
<td><strong>EXAMPLE OF THE LEVEL, OF DIFFICULTY EXPECTED</strong></td>
</tr>
<tr>
<td>Integration of expressions involving ( e^x )</td>
<td>Evaluate ( \int e^x , dx )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Integration of expressions involving logarithms</td>
<td>Evaluate ( \int \ln x , dx )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Integration by substitution</td>
<td>Evaluate ( \int \frac{dx}{\sqrt{1-x^2}} )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Integration by parts</td>
<td>Evaluate ( \int x^2 e^x , dx )</td>
<td>1 2 3</td>
<td></td>
</tr>
<tr>
<td>Double integral</td>
<td>Evaluate ( \iint x^2 , dx , dy )</td>
<td>1 2 3</td>
<td></td>
</tr>
</tbody>
</table>

Are there other topics that you would like to add? If so, please name the topic(s) in column A and give, in column D, in the same way as you did before, an example showing the level of difficulty expected.
APPENDIX C

APPLICATION OF THE LAWS OF EXPONENTS

The Amalgamator

The Teacher

The Student

1. Identify the degree of the following:

- $\frac{a}{b^2}$
- $\frac{c}{d^3}$
- $\frac{e}{f^4}$

2. Perform the following multiplications:

- $(a \times b) - (c \times d)$
- $(a \times b) + (c \times d)$
- $(a \times b) - (c \times d)^2$

3. Expand the following expressions:

- $(a + b)^2$
- $(a - b)^2$
- $(a + b)(a - b)$

4. Factor the following expressions:

- $a^2 - b^2$
- $(a + b)^2 - (a - b)^2$
- $(a + b)(a - b)^2$

5. Apply the laws of exponents to simplify the following:

- $a^3 \times a^4$
- $a^5 \div a^2$
- $a^{2+3}$

6. Evaluate the following expressions:

- $a^2 + b^2$
- $a^3 - b^3$
- $a^2 \times b^2$

7. Show the steps for each operation:

- $(a + b)^2$
- $(a - b)^2$
- $(a + b)(a - b)$

8. Identify the type of question in the column:

- $a$ - Algebra
- $b$ - Geometry
- $c$ - Calculus

9. Reflect on the learning process:

- What strategies did you use to solve the problems?
- How did you verify your answers?

10. Discuss the importance of using the laws of exponents in real-world applications.

Thank you for your help.
quotient of two polynomials, e.g. \( \frac{T \times A}{B + 6} \)

8.- Simplify a given fraction such as the example above.

9.- Add rational algebraic fractions.

10.- Subtract rational algebraic fractions.

11.- Multiply rational algebraic fractions.

12.- Divide rational algebraic fractions.

A radical is an expression of the form \( \sqrt[n]{a} \). The positive integer \( n \) is the index of the radical.

13.- Application of the laws for radicals:

\[
\begin{align*}
(\sqrt[n]{a})^n &= a \\
\sqrt[n]{a \cdot b} &= \sqrt[n]{a} \cdot \sqrt[n]{b} \\
\sqrt[n]{a} &= \sqrt[n]{\frac{a}{b}} \\
\sqrt[n]{a^m} &= (\sqrt[n]{a})^m \\
\sqrt[n]{a} &= \sqrt[n]{a}
\end{align*}
\]

14.- Perform operations such as:

\[
\begin{align*}
\sqrt[5]{a} \cdot \sqrt[5]{b} \\
\sqrt[5]{a^2} \div \sqrt[5]{b}
\end{align*}
\]

15.- Rationalization of the denominator of a fraction whose denominator is a binomial quadratic root, as for example:

\[
\frac{5}{2 \sqrt{3} + 2\sqrt{5}}
\]

16.- Solve for \( x \) equations such as:

\[
\begin{align*}
& S = 4x + 2x + 3 \\
& \frac{2}{x} = \frac{1}{3} \\
& 1 = \frac{2}{x} \\
& y = \frac{3}{2} x^3 \\
& x = 3 + \frac{2}{y}
\end{align*}
\]

Are there any other points that you would like to ask? If so please specify.
The purpose of this questionnaire is to get some information about the material concerned with the Pre-knowledge test you had on mathematics at the beginning of the present academic year.

Your answers will help to improve future tests and the programmed material concerned with it.

Please write your name in the space at the bottom of this sheet. The answers you give in this questionnaire will be confidential to me.

Thanks you for your help.

EDUARDO GUZALAS LINO
Institute for Educational Technology
University of Surrey, Guildford
October 1976

1. The Pre-knowledge test was (please tick the appropriate box):
   - very difficult
   - difficult
   - about right
   - easy
   - very easy

2. Which parts of the Pre-knowledge test did you find difficult (please tick the appropriate box): System
   - Inequalities
   - Binomial expansion
   - Limits
   - Trigonometric formulae
   - Trigonometric values
   - Logarithms
   - Quadratic equations
   - Differentiation
   - Integration

3. Which parts of the Pre-knowledge test were new for you or you had forgotten? (Please tick the appropriate box):
   - New
   - Forgotten
   - Inequalities
   - Binomial expansion
   - Limits
   - Trigonometric values
   - Logarithms
   - Quadratic equations
   - Differentiation
   - Integration

4. Have you used the Resource material in the Reading Room? (Please tick the appropriate box):
   - Yes
   - As preparation for Unit '2'
   - For any other purpose
   - Please specify:

6. Please tick those materials you have used in the Reading Room (tick the appropriate box):
   - Booklets
   - The Binomial Theorem
   - Trigonometric Relationships
   - Differentiation
   - Integration
   - Graph Matching
   - Any other material?

7. Did you use any programmed text book? (Please tick the appropriate box):
   - Yes
   - No

8. Did you find the booklets you used (if any) (please tick the appropriate box):
   - very interesting
   - interesting
   - average
   - boring
   - very boring

9. Would you like to say anything else about these materials (i.e. advantages, disadvantages, presentation, difficulty, etc.)?

Thank you for your cooperation and your time.

Please return the questionnaire to me in class or in the reading room.

[Signature]

October 1976
## APPENDIX E

### The OMI-questionnaire: The 1977 version

The purpose of this questionnaire is to get some information about the revision work in mathematics and the material used, which was carried out during the first week of this academic year.

Your answers will help to improve future activities of this kind and the materials used. Please write your name and tick your department below:

**Name:**

(BLOCK LETTERS PLEASE)

**Department:**

- Civil ☐
- Electrical ☐
- Mechanical ☐

The answers you give in this questionnaire will be confidential to me.

Thank you very much for your help.

EDUARDO S. GONZALEZ-LOPEZ
Institute for Educational Technology
University of Surrey

### 2. Which parts of the Pre-knowledge test did you find difficult?

<table>
<thead>
<tr>
<th>Difficulty Level</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very difficult</td>
<td>☐</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>About right</td>
<td></td>
<td>☐</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Easy</td>
<td></td>
<td></td>
<td>☐</td>
<td></td>
</tr>
<tr>
<td>Very easy</td>
<td></td>
<td></td>
<td></td>
<td>☐</td>
</tr>
</tbody>
</table>

- (a) Operations with fractions
- (b) Multiplication of brackets
- (c) Operations involving exponents
- (d) Solution of quadratic equations
- (e) Inequalities
- (f) Binomial expansion with fractional exponent
- (g) Trigonometric formulae
- (h) Graphs of the trigonometric functions
- (i) Trigonometric values
- (j) Logarithms
- (k) Differential Calculus
- (l) Integral Calculus

### 3. Which parts of the Pre-knowledge test did you find difficult?

- (a) Operations with fractions
- (b) Multiplication of brackets
- (c) Operations involving exponents
- (d) Solution of quadratic equations
- (e) Inequalities
- (f) Binomial expansion with fractional exponent
- (g) Trigonometric formulae
- (h) Graphs of the trigonometric functions
- (i) Trigonometric values
- (j) Logarithms
- (k) Differential Calculus
- (l) Integral Calculus

### 4. How did you find the discussion with the tutor after the Pre-knowledge test?

- Very useful ☐
- Not very useful ☐
- Useful ☐
- Useless ☐

Please comment on this below:

__________________________

__________________________
6. How did you find the booklets you used (if any)?

(a) very interesting  
interesting  
average  
boring  
very boring  

(b) very useful  
useful  
average  
not very useful  
useless  

7. Would you like to say anything else about these materials (i.e., advantages, disadvantages, presentation, difficulties, etc.)?

__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________

8. Did you take a test on any of the following after doing the appropriate revision:

   Algebraic Manipulation, Trigonometry
   and/or Inequalities?

   YES  ☐  Go to question 9
   NO  ☐  Go to question 10

9. How did you find the test(s)?

<table>
<thead>
<tr>
<th>ALGEBRAIC MANIPULATION</th>
<th>TRIGONOMETRY</th>
<th>INEQUALITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>very difficult</td>
<td>☐</td>
<td>☐</td>
</tr>
<tr>
<td>difficult</td>
<td>☐</td>
<td>☐</td>
</tr>
<tr>
<td>about right</td>
<td>☐</td>
<td>☐</td>
</tr>
<tr>
<td>easy</td>
<td>☐</td>
<td>☐</td>
</tr>
<tr>
<td>very easy</td>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

10. Would you like to say anything else about the whole revision work?

__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________

Thank you for your co-operation. Please return this questionnaire to me in the icing box, or leave it with the Course Assistant.

October 1977
APPENDIX F

THE QUESTIONNAIRE

The purpose of this questionnaire is to get some information about the Programed text on Trigonometry by S. Gonzalez-Leon (which is available in the Reading Room) in order to improve it for a future occasion.

Would you please fill it in and return it to me before you go?

Thank you very much for your cooperation and your time.

Your specialization is (Please tick the appropriate box):
Acoustical □ Civil □ Electronic □ Electrical □ Mechanical □

1.— Had you used any Programed text before?
   Please tick the appropriate box:
   YES □ NO □

2.— How did you find this particular Programed text on Trigonometry?
   Please tick the appropriate box:
   a) very interesting □ interesting □ average □ boring □ very boring □
   b) very useful □ useful □ average □ not very useful □ useless □

3.— Which answer(s) of the Pre-test of the Programed text on Trigonometry did you get wrong?
   Please tick the appropriate box(s): 1 □ 2 □ 3 □ 4 □ 5 □ 6 □ 7 □ 8 □ 9 □ 10 □ 11 □ 12 □ 13 □ 14 □ 15 □ 16 □ 17 □ 18 □ 19 □

4.— Is there any topic in Geometry that you had not met before and which you needed in the Programed text?
   Please tick the appropriate box:
   YES □ Go to question 5
   NO □ Go to question 6

5.— Name the topics in Geometry that you had not met.
6.- Did you understand the instructions given in the programme? Please tick the appropriate box:
   YES □  Go to question 8
   NO □  Go to question 7

7.- Please comment on the instructions you did not understand.

8.- Is there any wording that you think is confusing? Please tick the appropriate box:
   YES □  Go to question 9
   NO □  Go to question 10

9.- Please comment on the wording you think is confusing.

10.- Would you like to write anything else about the whole Programmed text on Trigonometry (i.e. advantages, disadvantages, presentation, difficulties, etc.)? If yes, please do so below.

EUGENE GONZALEZ-LEON
Institute for Educational Technology
University of Surrey, Guildford
October 1977
The purpose of this questionnaire is to get some information about the booklet on Algebraic Manipulation by E. González-León (which is available in the Reading Room), in order to improve it for a future occasion.

Would you please fill it in and return it to me before you go?

Thank you very much for your co-operation and your time.

Your specialism (Please tick the appropriate box):
- Acoustical
- Civil
- Electronic
- Electrical
- Mechanical

1.- Had you used any booklet or book like this on Algebraic Manipulation before?
   Please tick the appropriate box:
   YES □   NO □

2.- How did you find this particular booklet on Algebraic Manipulation?
   Please tick the appropriate boxes:
   a) very interesting  interesting  average  boring  very boring
   □ □ □ □ □
   b) very useful  useful  average  not very useful  useless
   □ □ □ □ □

3.- Which answer(s) of the pre-test of the booklet did you get wrong?
   Please tick the appropriate box(es):
   1 □  2 □  3 □  4 □  5 □  6 □  7 □  8 □  9 □  10 □

4.- Did you understand the instructions given in the booklet?
   Please tick the appropriate box:
   YES □ - GO TO QUESTION 6
   NO □ - GO TO QUESTION 5

5.- Please comment on the instructions you did not understand.

6.- Would you like to write anything else about the whole booklet on Algebraic Manipulation (i.e. advantages, disadvantages, presentation, difficulties, etc)?
   If yes, please do so below.

EDUARDO GONZÁLEZ-LEÓN
APPENDIX H

Institutions involved in the Main survey

Heriot-Watt University
Imperial College
Middlesex Polytechnic
Thames Polytechnic
University of Aston
University of Bath
University of Birmingham
University of Brunel
University of Exeter
University of Salford
APPENDIX I

The open-ended version of the 1978 mathematics test

1. Express \( \frac{a}{b} - \frac{c}{d} \) as a single fraction.

2. Rewrite as a single power of \( x \): \( \frac{(x^3)^{\frac{1}{3}}}{(x^2)^3} \)

3. Multiply out and simplify as much as possible: \((a + x)(a - x)^2\)

4. Solve for \( T \) the equation:
   \[ W = \frac{V}{K} - \frac{TR}{K^2}, \quad R \neq 0 \]

5. Solve for \( x \) the equation: \( 2x^2 - 5x - 1 = 0 \)

6. Solve for \( x \) the inequality: \( 3 - 2x > 4 \)

7. Solve for \( x \) the inequality: \( |x - 3| < 2 \)

8. Evaluate \( \sin^2 \frac{\pi}{2} - \cos \pi \)

9. Express \( \cos(30^0 - x) \) in terms of \( \sin x \) and \( \cos x \)

10. Write down \( \cos x + \cos y \) in the form of a product of the sine and/or cosine of \( \frac{x + y}{2} \) and/or \( \frac{x - y}{2} \)

11. Express \( \cos(wt + \alpha) \cos wt \) as the sum of two sine or cosine terms.

12. Find the modulus of the complex number \( 4 - 3i \)

13. Expand \( \log \left( \frac{ab}{c} \right) \), using the properties of logarithms.

14. If \( \log_e x = a - b \), what is the value of \( x \)?
15. Find the first three terms of \( \frac{1}{(1 - x)^{\frac{3}{2}}} \) when expanded as a series in powers of \( x \) for \( 0 < x < 1 \)

16. Find the slope of the tangent to the curve \( y = x^3 - 2x \) at the point \( x = -1, y = 1 \)

17. If \( y = \sin^2 x - \cos x \), find \( \frac{dy}{dx} \)

18. Evaluate the integral \( \int e^{kx} \, dx \)

19. What substitution of a new variable for \( x \) would allow you most readily to evaluate the integral \( \int \frac{dx}{x\sqrt{1 - x}} \)

20. What is the value of \( I \) obtained in the integration by parts of:
\[
\int x^2 e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - I
\]
The 1976 Mathematics Pre-knowledge Test

This is a short test designed to help us obtain some idea of your knowledge of a number of topics in mathematics: no penalty is attached to getting an answer wrong.

Only one answer is correct for each question. Please enter the letter A, B, C or D of the answer which you think is correct in the appropriate box on the yellow sheet.

If you have no idea of the answer to a particular question do not guess.

The time allowed for the test is 30 minutes.

1. The expression $\frac{2}{3} - \frac{5}{3}$ is equal to
   (A) $\frac{-1}{3}$
   (B) $\frac{-2}{3}$
   (C) $\frac{-3}{3}$
   (D) $\frac{-4}{3}$

2. $(\cos x^2 \cos x^2)$. can be expressed as
   (A) $\cos x^2 + \cos x^2$
   (B) $\cos x^2 - \cos x^2$
   (C) $\cos x^2 - 2 \cos x^2$
   (D) $\cos x^2 + 2 \cos x^2$

3. $\frac{\sin x}{(\sin x)^2}$ is equal to
   (A) $\frac{1}{\sin x}$
   (B) $\frac{1}{\cos x}$
   (C) $\sec x$
   (D) $\csc x$

4. If $x$ satisfies the inequality $3x > 4$, then
   (A) $x > \frac{\sqrt{3}}{3}$
   (B) $x > \frac{4}{3}$
   (C) $x > \frac{1}{4}$
   (D) $x < \frac{4}{3}$

5. The first three terms of the expansion of $\frac{1}{(x+2)^2}$, where $0 < x < 1$, are
   (A) $1 - 4x + 8x^2$
   (B) $1 + 4x + 8x^2$
   (C) $1 + 4x + 8x^3$
   (D) $1 + 4x + 8x^2$

6. The value of $\frac{1}{\cos x}$ is
   (A) $\frac{1}{\sin x}$
   (B) $1$
   (C) $\sec x$
   (D) Indeterminate
10. If \( 4 \sin x + 3 \cos x \) is put into the form \( R \sin(x+\phi) \) then
   \[
   \begin{align*}
   (A) & \quad R = 5, \quad \tan \phi = -\frac{3}{4} \\
   (B) & \quad R = 5, \quad \tan \phi = -\frac{3}{4} \\
   (C) & \quad R = 7, \quad \tan \phi = \frac{3}{4} \\
   (D) & \quad R = 7, \quad \tan \phi = \frac{3}{4}
   \end{align*}
   \]

11. The general solution for \( y \) of the equation \( \cot^2 y = 1 \) is
   \[
   \begin{align*}
   (A) & \quad y = \pm \frac{\pi}{4} + 2\pi n \\
   (B) & \quad y = \pm \frac{\pi}{2} + 2\pi n \\
   (C) & \quad y = \pm \frac{\pi}{3} + 2\pi n \\
   (D) & \quad y = \pm \frac{\pi}{6} + 2\pi n
   \end{align*}
   \]
   where \( n \) is any positive or negative integer or zero.

12. The value of \( \sin^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{4} \) is
   \[
   \begin{align*}
   (A) & \quad -1 \\
   (B) & \quad 0 \\
   (C) & \quad 1 \\
   (D) & \quad 2
   \end{align*}
   \]

13. \( \log_a (ax^2) \) can be expressed as
   \[
   \begin{align*}
   (A) & \quad A = 2x \\
   (B) & \quad (\log A) = 2x \\
   (C) & \quad (\log A) = 2 \log x \\
   (D) & \quad (\log A) - A = x
   \end{align*}
   \]

14. The equation \( x^2 + y^2 = 2x + 4y - 14 = 0 \) is
   \[
   \begin{align*}
   (A) & \quad \text{the equation of a circle whose centre is at the origin} \\
   (B) & \quad \text{the equation of a circle whose centre is at the point (1,-2)} \\
   (C) & \quad \text{the equation of a circle whose centre is at the point (-2,1)} \\
   (D) & \quad \text{not the equation of a circle.}
   \end{align*}
   \]

15. The slope of the tangent to the curve \( y = x^2 - 2x \) at the point \( x = 1, y = 1 \) is
   \[
   \begin{align*}
   (A) & \quad 3 \\
   (B) & \quad 1 \\
   (C) & \quad -1 \\
   (D) & \quad -3
   \end{align*}
   \]

16. If \( y = a^{x^2} \), where \( a \) is constant, then \( \frac{dy}{dx} \) equals
   \[
   \begin{align*}
   (A) & \quad 2ax^2 \ln a \\
   (B) & \quad a^{x^2} \\
   (C) & \quad 2ax^2 \ln a \\
   (D) & \quad a^{x^2} \ln a
   \end{align*}
   \]

17. \[
\int \frac{dx}{(x^2 - a^2)} \]
   is equal to a constant plus
   \[
   \begin{align*}
   (A) & \quad \sin^{-1} \frac{x}{a} \\
   (B) & \quad \cos^{-1} \frac{x}{a} \\
   (C) & \quad \tan^{-1} \frac{x}{a} \\
   (D) & \quad \log (x^2 - a^2)
   \end{align*}
   \]

18. \[
\int e^{ax} \ dx \]
   is equal to a constant plus
   \[
   \begin{align*}
   (A) & \quad e^{ax} \\
   (B) & \quad x e^{ax} \\
   (C) & \quad \frac{1}{a} x e^{ax} \\
   (D) & \quad x e^{2ax}
   \end{align*}
   \]

19. Using integration by parts, one can put
   \[
   \int x^2 e^{2x} \ dx = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} \]
   where 1 equals
   \[
   \begin{align*}
   (A) & \quad \int e^{2x} \ dx \\
   (B) & \quad \int x e^{2x} \ dx \\
   (C) & \quad \int 2x e^{2x} \ dx \\
   (D) & \quad \int \frac{1}{2} x e^{2x} \ dx
   \end{align*}
   \]

20. Using the substitution \( x = \cos \theta \) the integral
   \[
   \int (x^2 - 1)^{\frac{3}{2}} \ dx \]
   becomes
   \[
   \begin{align*}
   (A) & \quad \int \sin^3 \theta \ d\theta \\
   (B) & \quad \int \sin^2 \theta \ d\theta \\
   (C) & \quad \int \sin \theta \ d\theta \\
   (D) & \quad \int \sin \theta \cos \theta \ d\theta
   \end{align*}
   \]

5.

7.
APPENDIX \(\text{C}\)

The Mathematics Pre-knowledge Test

This is a short test designed to help us obtain some idea of your knowledge of a number of topics in mathematics; no penalty is attached to getting an answer wrong.

Only one answer is correct for each question. Please enter the letter A, B, C, or D of the answer which you think is correct in the appropriate box on the yellow sheet.

If you have no idea of the answer to a particular question do not guess.

The time allowed for the test is 30 minutes.

1. The expression \(\frac{x}{x} - \frac{3}{x}\) is equal to
   (A) \(\frac{1}{x}\)
   (B) \(\frac{1}{x}\)
   (C) \(\frac{1}{x}\)
   (D) \(\frac{1}{x}\)

2. \((\cos x)^2\) can be expressed as
   (A) \(\cos^2 x + \sin^2 x\)
   (B) \(\cos^2 x - \sin^2 x\)
   (C) \(\sin^2 x + \sin^2 x\)
   (D) \(\cos^2 x - \sin^2 x\)

3. \(\left(\frac{x}{y}\right)^2\) is equal to
   (A) \(\frac{x}{y}\)
   (B) \(\frac{x}{y}\)
   (C) \(\frac{x}{y}\)
   (D) \(\frac{x}{y}\)

4. The solutions of the equation \(2x^2 - 3x - 1 = 0\) are given by
   (A) \(\left(\frac{3 + \sqrt{5}}{2}\right)\)
   (B) \(\left(\frac{3 - \sqrt{5}}{2}\right)\)
   (C) \(\left(\frac{3 + \sqrt{5}}{2}\right)\)
   (D) \(\left(\frac{3 - \sqrt{5}}{2}\right)\)

5. If \(x\) satisfies the inequality \(x - 2x > 1\), then
   (A) \(x < -2\)
   (B) \(x < 1\)
   (C) \(x > -2\)
   (D) \(x < 1\)

6. To which of the following is the inequality \([x - 3] < 2\) equivalent?
   (A) \(x - 1\) and \(x < 3\)
   (B) \(x < 1\) and \(x > 3\)
   (C) \(x - 1\) and \(x > 3\)
   (D) \(x = 1\) and \(x < 3\)

7. The first three terms of the expansion of \(\frac{1}{(1-x)^3}\), where \(0 < x < 1\), are
   (A) \(1 - 3x + x^3\)
   (B) \(1 - 2x + x^2\)
   (C) \(1 - 3x + x^2\)
   (D) \(1 - 2x + x^3\)

8. \(\cos^2 \theta + \sin^2 \theta\) is the same as
   (A) \(\cos \theta\)
   (B) \(-\cos \theta\)
   (C) \(\sin \theta\)
   (D) \(-\sin \theta\)

9. \(\cos 2\theta\) is the same as
   (A) \(\cos^2 \theta + \sin^2 \theta\)
   (B) \(\cos^2 \theta - \sin^2 \theta\)
   (C) \(2 \sin \theta \cos \theta\)
   (D) \(1 - 2 \cos^2 \theta\)
10. The above is a sketch of

(a) sec x
(b) cosec x
(c) tan x
(d) cot x

11. Which of the following pairs of values of \( \theta \) satisfy the equation \( \cos \theta = -\frac{1}{2} \)?

(a) \( \theta = 120^\circ \) and \( -120^\circ \)
(b) \( \theta = 150^\circ \) and \( -150^\circ \)
(c) \( \theta = 120^\circ \) and \( -120^\circ \)
(d) \( \theta = -120^\circ \) and \( 120^\circ \)

12. The value of \( \sin^2 \frac{\pi}{3} - \cos \theta \) is

(a) -1
(b) 0
(c) 1
(d) 2

13. \( \log_a(a^x) \) can be expressed as

(a) \( x \)
(b) \( (\log_a x) - x \)
(c) \( (\log_a x) - \log_a a \)
(d) \( \log_a x - x \)

14. If \( \log_a x = a - b \), then

(a) \( x = a^a - a^b \)
(b) \( x = a^a + a^b \)
(c) \( x = a^{a-b} \)
(d) \( x = a^b \)

15. The slope of the tangent to the curve \( y = x^3 - 2x \) at the point \( x = 1, y = 1 \) is

(a) 3
(b) 1
(c) -1
(d) -3

16. If \( y = ax^2 \), where \( a \) is constant, then \( \frac{dy}{dx} \) equals

(a) \( a^2x^2 \)
(b) \( ax^3 \)
(c) \( ax^2 \)
(d) \( ax \)
The 1978 Mathematics Test

MATHEMATICS TEST

This test is designed to help me to obtain some idea of your knowledge of a number of topics in mathematics at the same time that it helps you to find out where you might have weaknesses.

Only one answer is correct for each question. Please tick the appropriate box to each question on the yellow sheet. If you have no idea of the answer to a particular question, please do not guess.

No penalty is attached to getting an answer wrong. The result of the test will be treated in strict confidence.

The maximum time allowed is 60 minutes, but you may read less.

Thank you for your help.

Eduardo Gonzalez - Leon
Institute for Educational Technology,
University of Surrey, Guildford, Surrey.

APPENDIX

The 1978 Mathematics Test

3. $(a + x)(a - x)$ can be expressed as:

(A) $a^2 + x^2 - 2ax$

(B) $a^2 + ax^2 - x^3$

(C) $a^2 - ax^2 + x^3$

(D) $a^2 - ax - x^3$

6. If $N = x^2$ and $R 
eq 0$, then $T$ is equal to

(A) $V - WR^2$

(B) $V^2 - WR$

(C) $V - WR^3$

(D) $V - WR^4$

2. $\frac{(a^3 + b^3)}{(a^2)^2}$ is equal to

(A) $a^3$

(B) $\frac{a^3}{a^2}$

(C) $\frac{a^3}{a^2} - a^3$

(D) $\frac{a^3}{a^2} + a^3$
6. If \( x \) satisfies the inequality \( 3 - 2x > 1 \), then

- (A) \( x > 0.5 \)
- (B) \( x < 0.5 \)
- (C) \( x > -0.5 \)
- (D) \( x < -0.5 \)

7. The inequality \( |x - 3| < 2 \) is equivalent to:

- (A) \( x < 1 \) and \( x > 5 \)
- (B) \( x > 1 \) and \( x < 5 \)
- (C) \( x < 1 \) and \( x > 5 \)
- (D) \( x > 1 \) and \( x < 5 \)

8. The value of \( \sin^2 \frac{x}{2} - \cos x \) is

- (A) -1
- (B) 0
- (C) 1
- (D) 2

10. \( \cos x + \cos y \) is the same as:

- (A) \( 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} \)
- (B) \( 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} \)
- (C) \( -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \)
- (D) \( 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \)

11. \( \cos (\omega t + \alpha) \cos \omega t \) is the same as:

- (A) \( \cos (\omega t + \alpha) \cos \omega t \)
- (B) \( \cos (\omega t + \alpha) - \cos \omega t \)
- (C) \( \cos (\omega t + \alpha) + \cos \omega t \)
- (D) \( \cos (\omega t + \alpha) - \cos \omega t \)

12. The modulus of the complex number \( 4 - 3i \) is:

- (A) 1
- (B) \( \sqrt{7} \)
- (C) \( \sqrt{7} \)
- (D) 5
1. \( \log \log b \)
2. \( \log a + \log b + \log c \)
3. \( - \log a - \log b + \log c \)
4. \( \log a + \log b - \log c \)

5. \( \sin 2x + \sin x \)
6. \( \cos^2 x + \sin x \)
7. \( 3 \sin x \)
8. \( \sin x \)

9. \( \int e^x \, dx \) is equal to a constant plus
10. \( \int e^x \, dx \)

11. If \( \log_b x = a - b \), then
12. \( x = a^b - b^a \)
13. \( x = a(e - b) \)
14. \( x = a^b \)
15. \( x = (a-b)^b \)

16. If \( \frac{1}{1-x} \) is expanded as a series in powers of \( x \), for \( 0 < x < 1 \), then the first three terms of the series are:
17. \( 1 + \frac{1}{2} x + \frac{1}{3} x^2 \)
18. \( 1 + \frac{2}{1} x + \frac{2}{3} x^2 \)
19. \( 1 + \frac{3}{1} x + \frac{3}{2} x^2 \)
20. \( 1 + \frac{4}{1} x + \frac{4}{3} x^2 \)

12. The slope of the tangent to the curve \( y = x^3 - 2x \) at the point \( x = -1, y = 1 \) is
13. \( 3 \)
14. \( 1 \)
15. \( -1 \)
16. \( -3 \)

18. Using Integration by parts, one can put \( \int x^2 e^x \, dx = \int x^2 e^x \, dx - I \) where \( I \) is equal to:
19. \( \int x^2 e^x \, dx \)
20. \( \int x e^x \, dx \)
21. \( \int e^x \, dx \)
22. \( \int e^x \, dx \)
23. \( \int e^x \, dx \)
24. \( \int e^x \, dx \)
ITEM NO. 4

The distractors in this item were constructed using different types of errors students commit when they apply the rules of transposition of terms and factors. Each alternative is obtained as follows:

(A) is the correct answer

(B) is obtained when the sign of TR (i.e. minus) is omitted when transposing terms, that is,

\[ W = \frac{V}{K} - \frac{TR}{K^2} \quad \Rightarrow \quad WK^2 = VK - TR \]

\[ TR = WK^2 - VK \quad \text{(error)} \quad \Rightarrow \quad T = \frac{WK^2 - VK}{R} \]

(C) is obtained when an error is committed when the denominators are simplified (i.e. \( W \) is multiplied by \( K \) and \( K^3 \)), that is,

\[ W = \frac{V}{K} - \frac{TR}{K^2} \quad \Rightarrow \quad WK^2 = V - TR \quad \text{(error)} \]

\[ TR = V - WK^3 \quad \Rightarrow \quad T = \frac{V - WK^3}{R} \]

(D) is obtained when \( K - K^2 \) is treated as a common denominator, that is,

\[ W = \frac{V}{K} - \frac{TR}{K^2} \quad \Rightarrow \quad W(K - K^2) = V - TR \quad \text{(error)} \]

\[ TR = V - W(K - K^2) \quad \Rightarrow \quad T = \frac{V - W(K - K^2)}{R} \]

ITEM NO. 9

In this item the student should use the formula

\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]

The different distractors were constructed using the formulae
to expand \( \cos (A + B) \) and \( \sin (A + B) \) as follows:

(A) is obtained using the formula for \( \cos (A + B) \)
(B) is the correct answer
(C) is obtained using the formula for \( \sin (A - B) \)
(D) is obtained using the formula for \( \sin (A + B) \)

ITEM No. 10

This item consists of a simple recognition of a formula. The alternatives in this item were constructed using the formula for \( \sin x - \sin y \) (alternative (A)), \( \cos x + \cos y \) (alternative (B), correct answer), \( \cos x - \cos y \) (alternative (C)), and \( \sin x + \sin y \) (alternative (D)).

ITEM No. 11

This item tests the use of the formula for \( \cos x - \cos y \) but reading it from right to left. The alternatives were constructed using the formulae for \( \cos x + \cos y \) and \( \cos x - \cos y \) as follows:

(A) is obtained using the formula for \( \cos x - \cos y \) (correct answer)
(B) is obtained using the formula for \( \cos x + \cos y \)
(C) and (D) are obtained using the formula used for (A) and (B) but taking \( x = wt + \alpha \) and \( y = wt \) instead of \( \frac{x + y}{2} = wt + \alpha \) and \( \frac{x - y}{2} = wt \)

ITEM No. 12

The alternatives of this item were constructed using the following expressions as the modulus for a complex number \( a + bi \):

(A) \( \sqrt{a + b} \)
(B) \( \sqrt{a^2 - b^2} \)
(C) \( \sqrt{(a - b)^2} \)
(D) \( \sqrt{a^2 + b^2} \) (correct answer)

ITEM No. 13

The distractors in this item were made up of misconceptions given to the correct properties of logarithms.
ITEM No 17

Each of the alternatives of this item were constructed as follows:

(A) is the correct answer

(B) is obtained when it is assumed that
\[ \frac{d}{dx} (\sin^2 x) = \cos^2 x \]

(C) is obtained when it is assumed that
\[ \frac{d}{dx} (\sin^2 x) = 2\sin x, \text{ that is,} \]
\[ \frac{dy}{dx} = 2\sin x + \sin x = 3\sin x. \]

(D) is obtained when it is assumed that
\[ \frac{d}{dx} (\sin^2 x) = 2\sin x \quad \text{and} \quad \frac{d}{dx} (\cos x) = \sin x, \]
\text{that is,}
\[ \frac{dy}{dx} = 2\sin x - \sin x = \sin x \]

ITEM No. 19

The distractors of this item were constructed so that they might be possible substitutions to evaluate the integral. If any of these substitutions lead to the evaluation of the integral, it does not do it most readily.
## APPENDIX N

The 1979 Mathematics Test

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### MATHEMATICS TEST

This test is designed to help you to obtain some idea of your knowledge of a number of topics in mathematics at the same time that it may help you to find out where you might have weaknesses.

The material in the test covers basic mathematics needed by virtually all engineering and science students. The test is divided into two parts. Part 1 consists of the questions 1 to 17, and Part 2 of the questions 18 to 47. Instructions about how to answer the questions of each Part are given at the beginning of each part.

The result of the test will be treated in strict confidence. The maximum time allowed is 60 minutes and you have to answer as many questions as you can.

Thank you for your help.

Eduardo González-León
Institute for Educational Technology,
University of Surrey, Guildford, Surrey

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### COLUMN A

1. Expand the following:
   1. \((a+1)^2\)
   2. \((a-1)^2\)
   3. \((a+1)(a-1)\)

   **COLUMN B**
   1. \(a^4+1\)
   2. \(a^4-2\)
   3. \(2a-2\)

### Write as a single power of \(a\):

4. \(x^8\)
5. \(x^\frac{1}{2}\)
6. \((x^2)^3\)
7. \((x^3)^2\)
8. \(x^{\frac{1}{2}}\)
9. \(x^\frac{1}{3}\)
Write as a single fraction:

10. \( \frac{1}{2} + \frac{1}{4} \)
   (a) \( \frac{1}{2} \)
   (b) \( \frac{3}{4} \)

11. \( \frac{1}{2} - \frac{1}{4} \)
   (a) \( \frac{1}{4} \)
   (b) \( \frac{3}{4} \)

12. \( \frac{1}{2} \times \frac{3}{4} \)
   (a) \( \frac{3}{8} \)
   (b) \( \frac{1}{2} \)

13. \( \frac{1}{2} + \frac{1}{4} \)
   (a) \( \frac{1}{2} \)
   (b) \( \frac{3}{4} \)

---

Expand the following:

14. \((1-x)^2\)
   (a) \(1+4x^2+4x^4-x^6\)
   (b) \(1-4x^2+4x^4-x^6\)
   (c) \(1-4x^2+4x^4-x^6\)

15. \((1+x)^2\)
   (a) \(1+4x^2+4x^4-x^6\)
   (b) \(1+4x^2+4x^4-x^6\)
   (c) \(1+4x^2+4x^4-x^6\)

---

In the following binomials where \(0 < x < 1\)
Find the first three terms of the series obtained in the expansion into powers of \(x\) of each binomial.

16. \((1+x\frac{1}{2})\)
   (a) \(1+2x^2+4x^4\)
   (b) \(1+x^2+2x^4\)
   (c) \(1+2x^2+4x^4\)

17. \((1+x\frac{1}{3})\)
   (a) \(1+2x^3+4x^6\)
   (b) \(1+2x^3+4x^6\)
   (c) \(1+2x^3+4x^6\)

---

18. The solutions of the quadratic equation \(3x^2-7x-1=0\) are given by:
   (a) \(x=\frac{1}{3}(7\pm\sqrt{65})\)
   (b) \(x=\frac{1}{3}(7\pm\sqrt{65})\)
   (c) \(x=\frac{1}{3}(7\pm\sqrt{65})\)
   (d) \(x=\frac{1}{3}(7\pm\sqrt{65})\)

19. If \(4x-3y=2\), then:
   (a) \(4x-3y=2\)
   (b) \(4x-3y=2\)
   (c) \(4x-3y=2\)
   (d) \(4x-3y=2\)

20. If \(-2x=3y+4y\), then \(x\) is equal to:
   (a) \(6y\)
   (b) \(-6y\)
   (c) \(-15+2y\)
   (d) \(15-2y\)

21. If \(W=\frac{V}{R}+\frac{70}{N}\) and \(W \neq 0\), then \(V\) is equal to:
   (a) \(\frac{1}{2}(17-VN)\)
   (b) \(\frac{1}{2}(17-VN)\)
   (c) \(\frac{1}{2}(17-VN)\)
   (d) \(\frac{1}{2}(17-VN)\)
23. The set of equations \[
\begin{align*}
2x + 3y &= -2 \\
-x + 2y &= 1
\end{align*}
\]
has:
(A) One solution only
(B) More than one solution but a finite number
(C) An infinite number of solutions
(D) No solutions

24. If \( x \) satisfies the inequality \( 3 - 2x > 4 \), then:
(A) \( x > 1 \)
(B) \( x < 1 \)
(C) \( x > 2 \)
(D) \( x < 2 \)

25. The inequality \( |x - 2| < 3 \) is equivalent to:
(A) \( x < 5 \) and \( x > -1 \)
(B) \( x < 5 \) and \( x > -5 \)
(C) \( x < 3 \) and \( x > -1 \)
(D) \( x < 5 \)

26. If \( z = 3 + 2i \) and \( z_2 = 5 + 4i \), then the product \( z_1z_2 \) is:
(A) \(-23 - 22i\)
(B) \(-23 + 22i\)
(C) \(-3 + 22i\)
(D) \(-3 - 22i\)

27. Expanding \( \frac{3}{x+1} \) we obtain:
(A) \( b^3 + b^2 \)
(B) \( b^3 + b \)
(C) \( b^3 \)
(D) \( b^2 + b \)

28. If \( \log_a x = \frac{a}{b} \), then:
(A) \( x = a^b \)
(B) \( x = b^a \)
(C) \( x = a^b \)
(D) \( x = b^a \)

29. \( \log \left( \frac{x^3}{c^n} \right) \) is the same as:
(A) \( 3 \log x - 4 \log c \)
(B) \( 2 \log x + 3 \log b + 4 \log c \)
(C) \( -2 \log x - 3 \log b + 4 \log c \)
(D) \( 2 \log a + 3 \log b - 4 \log c \)

30. If \( z = 3+i \), then the modulus \( |z| \) of \( z \) is:
(A) \( 1 \)
(B) \( \sqrt{5} \)
(C) \( 7 \)
(D) \( 5 \)

31. If \( z = 1+i \) is expressed in the form \( r (\cos \theta + i \sin \theta) \), then \( \tan \theta \) is:
(A) \( 1 \)
(B) \( \frac{1}{2} \)
(C) \( \frac{\sqrt{3}}{2} \)
(D) \( \frac{1}{3} \)

32. If \( x = 3 + 2i \) and \( x_2 = 5 + 4i \), then the product \( x_1x_2 \) is:
(A) \( 23 + 22i \)
(B) \( -23 - 22i \)
(C) \( -3 + 22i \)
(D) \( -3 - 22i \)

33. If \( x = -\frac{5+1}{2} \) is expressed in the form \( x = a + bi \), then:
(A) \( x = -\frac{1}{2} \)
(B) \( x = -\frac{5}{2} \)
(C) \( x = -2 \cdot \frac{5}{2} \)
(D) \( x = -\frac{5}{2} \)

34. The angle \( \frac{\pi}{6} \) expressed in degrees is:
(A) \( 30^\circ \)
(B) \( 45^\circ \)
(C) \( 60^\circ \)
(D) \( 90^\circ \)

35. Which of the following pairs of values satisfy the equation \( \cos \theta = -\frac{1}{2} \)?
(A) \( 120^\circ \) and \( -150^\circ \)
(B) \( 120^\circ \) and \( -120^\circ \)
(C) \( 150^\circ \) and \( -150^\circ \)
(D) \( 150^\circ \) and \( -120^\circ \)

36. The general solution of the equation \( \cos \theta = -\frac{1}{2} \) is:
(A) \( \theta = 2\pi n \pm \frac{\pi}{3} \)
(B) \( \theta = 2\pi n \pm \frac{2\pi}{3} \)
(C) \( \theta = 2\pi n \pm \frac{\pi}{3} \)
(D) \( \theta = 2\pi n \pm \frac{\pi}{6} \)

37. \( \cos(\beta - \alpha) \) is the same as:
(A) \( \sin \beta \cos \alpha + \cos \beta \sin \alpha \)
(B) \( \sin \beta \cos \alpha - \cos \beta \sin \alpha \)
(C) \( \sin \beta \cos \alpha + \cos \beta \sin \alpha \)
(D) \( \sin \beta \cos \alpha - \cos \beta \sin \alpha \)

38. \( 5 \sin \beta + \sin 2\beta \) is the same as:
(A) \( 2 \sin \beta \cos \beta \)
(B) \( 2 \sin \beta \cos \beta \)
(C) \( -2 \sin \beta \cos \beta \)
(D) \( 2 \sin \beta \cos \beta \)
39. \( \cos(wt+\alpha)\cos(wt+\delta) \) is the same as:

(A) \( \frac{1}{2} \left[ \cos(wt+\delta) - \cos(wt+\alpha) \right] \)

(B) \( \frac{1}{2} \left[ \cos(wt+\delta) - \cos(wt-\alpha) \right] \)

(C) \( \frac{1}{2} \left[ \cos(wt+\delta) - \cos(wt+\alpha) \right] \)

(D) \( \frac{1}{2} \left[ \cos(wt+\delta) + \cos(wt+\alpha) \right] \)

40. If in the figure on the right

\[ \theta = 3, \phi = 4 \text{ and } \theta = 120^\circ, \]

then the value of \( m \) is:

(A) \( \sqrt{13} \)

(B) \( \sqrt{19} \)

(C) \( \sqrt{35} \)

(D) \( \sqrt{37} \)

41. The above is a sketch of:

(A) \( \sin x \)

(B) \( \sin 2x \)

(C) \( \cos x \)

(D) \( \cos 2x \)

42. If \( y = \sin x + \cos x \), then \( \frac{dy}{dx} \) is:

(A) \( \cos x - \sin x \)

(B) \( \cos x + \sin x \)

(C) \( \sin x \)

(D) \( \cos x \)

43. If \( x = \sin^2t \), then \( \frac{dx}{dt} \) is:

(A) \( 2 \sin t \)

(B) \( 2 \cos t \)

(C) \( 2 \sin 3t \cos 3t \)

(D) \( 2 \sin 3t \)

44. If \( y = e^x \), then \( \frac{dy}{dx} \) is:

(A) \( e^x \)

(B) \( \ln e^x \)

(C) \( e^x \) (D) \( x e^x \)

45. If \( y = \frac{3x^4}{2x+4} \), then \( \frac{dy}{dx} \) is:

(A) \( \frac{14x^2}{(2x+4)^2} \)

(B) \( \frac{2x^3}{(2x+4)^2} \)

(C) \( \frac{12x^4}{(2x+4)^2} \)

(D) \( \frac{12x^4}{(2x+4)^2} \)

46. \( \int x^m \, dx \) is equal to a constant plus:

(A) \( \frac{x^{m+1}}{m+1} \)

(B) \( \frac{x^{m+2}}{m+2} \)

(C) \( \frac{1}{m+1} x^{m+1} \)

(D) \( \frac{1}{2} x^{m+1} \)
APPENDIX P

BATTERY OF MATHEMATICS DIAGNOSTIC TESTS
**INSTRUCTIONS**

Below you will find two columns (A and B). Choose the entry in column B which you consider is the correct answer to each question in column A. Write the corresponding code letters in the boxes provided in the answer sheet. You may use each code letter once, more than once or not at all.

To make it easier to answer this part, it has been divided into sections by horizontal lines so that you can work with each section independently from the others, that is, to answer the questions 1 to 3 in column A you only need the entries (a) to (g) in column B; for the questions 4 to 9 in column A, the entries (h) to (r) in column B, and for the questions 10 to 13 in column A, the entries (s) to (z) in column B.

<table>
<thead>
<tr>
<th>COLUMN A</th>
<th>COLUMN B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expand the following:</td>
<td></td>
</tr>
<tr>
<td>1. ((a+1)^2)</td>
<td>(a) (a^2+1)</td>
</tr>
<tr>
<td></td>
<td>(b) (a^2+2a+1)</td>
</tr>
<tr>
<td></td>
<td>(c) (a^2-1)</td>
</tr>
<tr>
<td></td>
<td>(d) (a^2-2a-1)</td>
</tr>
<tr>
<td>2. ((a-1)^2)</td>
<td>(e) (a^2-2a+1)</td>
</tr>
<tr>
<td></td>
<td>(f) (2a+2)</td>
</tr>
<tr>
<td>3. ((a+1)(a-1))</td>
<td>(g) (2a-2)</td>
</tr>
</tbody>
</table>
Write as a single power of \( x \):

4. \( x^6x^2 \)

5. \( \frac{x^6}{x^2} \)

6. \( (x^6)^2 \)

7. \( (x^6)^\frac{1}{2} \)

8. \( \frac{1}{x^6} \)

9. \( \frac{1}{(x^6)^2} \)

Write as a single fraction:

10. \( \frac{7}{2} + \frac{5}{4} \)

11. \( \frac{7}{2} - \frac{5}{4} \)

12. \( \frac{7}{2} \times \frac{5}{4} \)

13. \( \frac{7}{2} \div \frac{5}{4} \)
INSTRUCTIONS

Each of the following questions have four possible answers of which only one is correct. Please tick, on the answer sheet, the box which you consider gives the correct answer to each question. If you have no idea of the answer to a particular question, please do not guess.

14. If $4x-3=5+2y$, then:

(A) $4x-2y=2$  
(C) $4x+2y=2$

(B) $4x-2y=8$  
(D) $4x+2y=8$

15. If $-2x=30-4y$, then $x$ is equal to:

(A) $60-8y$  
(C) $-15+2y$

(B) $-60+8y$  
(D) $15-2y$

16. If $W=\frac{V}{K} - \frac{TR}{K^2}$ and $R \neq 0$, then $T$ is equal to:

(A) $\frac{1}{R} (VK-WK^2)$  
(C) $\frac{1}{R} (V-WK^2)$

(B) $\frac{1}{R} (WK^2-VK)$  
(D) $\frac{1}{R} \left[ V-W(K-K^2) \right]$

17. If $\alpha - w = h + \frac{c_2 - c_1}{2}$, then $c_1$ is:

(A) $2h + c_2 - 2(\alpha - w)$  
(C) $h + c_2 - 2(\alpha - w)$

(B) $2h + c_2 - (\alpha - w)$  
(D) $2h + c_2 - 2\alpha + w$
18. If \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \), then \( R_2 \) is equal to:

(A) \( \frac{RR_1}{R - R_1} \)  
(B) \( \frac{RR_1}{R_1 - R} \)  
(C) \( \frac{R - R_1}{RR_1} \)  
(D) \( \frac{R_1 - R}{RR_1} \)

19. The general formula to solve the quadratic equation

\[ ax^2 + bx + c = 0 \quad (a \neq 0) \]

is given by:

(A) \( x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a} \)  
(B) \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

19. The general formula to solve the quadratic equation

\[ ax^2 + bx + c = 0 \quad (a \neq 0) \]

is given by:

(C) \( x = \frac{-b \pm \sqrt{b^2 + 2ac}}{2a} \)  
(D) \( x = \frac{-b \pm \sqrt{b^2 - 2ac}}{2a} \)

20. The solutions of the quadratic equation \( x^2 + 3x + 1 = 0 \) are

given by:

(A) \( x = \frac{1}{2} (-3 \pm \sqrt{11}) \)  
(B) \( x = \frac{1}{2} (-3 \pm \sqrt{13}) \)

21. The solutions of the quadratic equation \( 3x^2 + 5x + 1 = 0 \)

are given by:

(A) \( x = \frac{1}{6} (-5 \pm \sqrt{19}) \)  
(B) \( x = \frac{1}{6} (-5 \pm \sqrt{37}) \)

22. The solutions of the quadratic equation \( 3x^2 - 7x - 1 = 0 \) are

given by:

(A) \( x = \frac{1}{6} (7 \pm \sqrt{55}) \)  
(B) \( x = \frac{1}{6} (7 \pm \sqrt{43}) \)  
(C) \( x = \frac{1}{6} (7 \pm \sqrt{61}) \)  
(D) \( x = \frac{1}{6} (7 \pm \sqrt{37}) \)
24. The solutions of the set of equations \[ \begin{align*} x + y - z &= 1 \\ x - 2y + z &= -2 \\ 2x + y + z &= 3 \end{align*} \] are:

(A) \( x = -\frac{2}{3} ; \ y = -\frac{1}{3} ; \ z = -2 \)  
(C) \( x = \frac{2}{7} ; \ y = \frac{11}{7} ; \ z = \frac{6}{7} \)

(B) \( x = -\frac{1}{3} ; \ y = \frac{1}{3} ; \ z = -1 \)  
(D) \( x = \frac{1}{7} ; \ y = \frac{9}{7} ; \ z = \frac{3}{7} \)

25. The solution of the set of equations \[ \begin{align*} x + y + z &= 6 \\ 2x + 3y + 4z &= 20 \\ 3x + 5y + 6z &= 31 \end{align*} \] are:

(A) \( x = 3 ; \ y = -2 ; \ z = 5 \)  
(C) \( x = -1 ; \ y = 6 ; \ z = 1 \)

(B) \( x = 1 ; \ y = 2 ; \ z = 3 \)  
(D) \( x = 2 ; \ y = 0 ; \ z = 4 \)

26. The expression \( s = \sum_{r=1}^{3} r^3 \) is equal to:

(A) \( b + b^2 + b^3 \)  
(C) \( b^{1+2+3} \)

(B) \( b \times b \times b \)  
(D) \( b^{1+2+3} \)

27. Expanding \( s = \sum_{r=1}^{3} (r+1)^2 \) we obtain \( s \) equal to:

(A) 27  
(C) 49

(B) 29  
(D) 81

28. Expanding \( s = \sum_{r=3}^{5} (2r - 1) \) we obtain \( s \) equal to:

(A) 16  
(C) 49

(B) 18  
(D) 81
PART 1

INSTRUCTIONS

Each of the following questions have four possible answers of which only one is correct. Please tick, on the answer sheet, the box which you consider gives the correct answer to each question. If you have no idea of the answer to a particular question, please do not guess.

INEQUALITIES

1. If \( x \) satisfies the inequality \(-3x < y\), then:
   
   (A) \( x < y + 3 \)  \hspace{1cm}  \text{(C)} \ x > y + 3
   
   (B) \( x < -\frac{y}{3} \)  \hspace{1cm}  \text{(D)} \ x > -\frac{y}{3}

2. If \( x \) satisfies the inequality \(3-2x > 4\), then:

   (A) \( x > \frac{1}{2} \)  \hspace{1cm}  \text{(C)} \ x < \frac{1}{2}

   (B) \( x > -\frac{1}{2} \)  \hspace{1cm}  \text{(D)} \ x < -\frac{1}{2}

3. The inequality \(|x-2|<3\) is equivalent to:

   (A) \( x < 5 \) and \( x > -5 \)  \hspace{1cm}  \text{(C)} \ x < 5

   (B) \( x < 5 \) and \( x > -1 \)  \hspace{1cm}  \text{(D)} \ x < \sqrt{5}

4. The inequality \(|5-x|<8\) is equivalent to

   (A) \( x < 13 \) and \( x > -3 \)  \hspace{1cm}  \text{(C)} \ x < 3

   (B) \( x < 3 \) and \( x > -3 \)  \hspace{1cm}  \text{(D)} \ x < -\sqrt{3}
5. If \( \log_e x = a - b \), then:

(A) \( x = e^{a-b} \)  \hspace{1cm} (C) \( x = e^a - b \)

(B) \( x = e^{a-b} \)  \hspace{1cm} (D) \( x = (a-b)^e \)

6. \( \log \left[ \frac{a^3b^2}{c^4} \right] \) is the same as:

(A) \( \frac{6 \log a \log b}{4 \log c} \)  \hspace{1cm} (C) \( -2 \log a - 3 \log b + 4 \log c \)

(B) \( 2 \log a + 3 \log b + 4 \log c \)  \hspace{1cm} (D) \( 2 \log a + 3 \log b - 4 \log c \)

7. \( \log \sqrt[3]{ab} \) is the same as:

(A) \( \frac{2}{3} \log a + \frac{2}{3} \log b \)  \hspace{1cm} (C) \( 3 \log a + 6 \log b \)

(B) \( \frac{1}{3} \log a + \frac{2}{3} \log b \)  \hspace{1cm} (D) \( 6 \log a + 6 \log b \)

**COMPLEX NUMBERS**

In the following questions, \( i \) denotes \( \sqrt{-1} \)

8. If \( z = a + bi \), then the modulus \( |z| \) of \( z \) is:

(A) \( \sqrt{a^2 + b^2} \)  \hspace{1cm} (C) \( a^2 - b^2 \)

(B) \( \sqrt{a^2 - b^2} \)  \hspace{1cm} (D) \( a - b \)

9. If \( z = 4 + 3i \), then the modulus \( |z| \) of \( z \) is:

(A) 1  \hspace{1cm} (C) 7

(B) \( \sqrt{7} \)  \hspace{1cm} (D) 5
10. If \( z = -2 + 1 \), then the modulus \( |z| \) of \( z \) is:

(A) \( \sqrt{5} \)  
(B) \(-3\)  
(C) \( \sqrt{3} \)  
(D) \( 3 \)

11. If \( z = 4 - 2i \), then the modulus \( |z| \) of \( z \) is:

(A) \( 6 \)  
(B) \( 2\sqrt{5} \)  
(C) \( 2\sqrt{3} \)  
(D) \( 12 \)

12. If the complex number \( z = \sqrt{2} - i\sqrt{2} \) is expressed in the form \( z = r(\cos \theta + isin \theta) \), then the value of \( \theta \) is:

(A) \( \frac{1}{4}\pi \)  
(B) \( \frac{3}{4}\pi \)  
(C) \( \frac{5}{4}\pi \)  
(D) \( \frac{7}{4}\pi \)

13. If the complex number \( z = 1 + i\sqrt{3} \) is expressed in the form \( z = r(\cos \theta + isin \theta) \), then the value of \( \theta \) is:

(A) \( \frac{1}{6}\pi \)  
(B) \( \frac{5}{6}\pi \)  
(C) \( \frac{7}{6}\pi \)  
(D) \( \frac{11}{6}\pi \)

14. If the complex number \( z = 4i \) is expressed in the form \( z = r(\cos \theta + isin \theta) \), then the value of \( \theta \) is:

(A) \( 0 \)  
(B) \( \frac{1}{2}\pi \)  
(C) \( \pi \)  
(D) \( \frac{3}{2}\pi \)

15. If \( z_1 = 1 + i \) and \( z_2 = 2 - 3i \), then the product \( z_1 \times z_2 \) is:

(A) \(-5 + i\)  
(B) \(5 - i\)  
(C) \(-1 + i\)  
(D) \(-1 - i\)

16. If \( z_1 = 1 - 2i \) and \( z_2 = 3 - 4i \), then the product \( z_1 \times z_2 \) is:

(A) \(-11 + 10i\)  
(B) \(-11 + 10i\)  
(C) \(5 + 10i\)  
(D) \(-5 - 10i\)
17. If \( z_1 = -3 + 2i \) and \( z_2 = 5 + 4i \), then the product \( z_1 \times z_2 \) is:

(A) \(-7 - 2i\)  
(B) \(-7 + 2i\)  
(C) \(-23 - 2i\)  
(D) \(23 + 2i\)

18. If \( z_1 = 1 + i \) and \( z_2 = 2 - i \), then the quotient \( \frac{z_1}{z_2} \) is:

(A) \(\frac{3}{5} + i\)  
(B) \(\frac{1}{2} - i\)  
(C) \(\frac{2}{5} + \frac{3}{5} i\)  
(D) \(\frac{3}{5} + \frac{1}{5} i\)

19. If \( z_1 = 3 + i \) and \( z_2 = 4 + i \), then the quotient \( \frac{z_1}{z_2} \) is:

(A) \(\frac{13}{17} + \frac{7}{17} i\)  
(B) \(\frac{13}{17} + \frac{1}{17} i\)  
(C) \(\frac{11}{17} + \frac{1}{17} i\)  
(D) \(\frac{3}{4} + i\)

20. If \( z_1 = -4 + 5i \) and \( z_2 = 2 - 4i \), then the quotient \( \frac{z_1}{z_2} \) is:

(A) \(-\frac{7}{5} - \frac{3}{10} i\)  
(B) \(\frac{3}{5} - \frac{3}{10} i\)  
(C) \(\frac{3}{5} + \frac{3}{10} i\)  
(D) \(-2 - \frac{5}{4} i\)

### PART 2

**INSTRUCTIONS**

Below you will find two columns (A and B). Choose the entry in column B which you consider is the correct answer to each question in column A. Write the corresponding code letters in the boxes provided in the answer sheet. You may use each code letter once, more than once or not at all.

To make it easier to answer this part, it has been divided into sections by horizontal lines so that you can work with each section independently from the others, that is, to answer the questions and in column A you only need the entries (a) to (c) in column B; for the questions 21 and 22 in column A, the entries (d) to (g) in column B.
BINOMIAL THEOREM

COLUMN A | COLUMN B

Expand the following:

21. \((1+x)^4\) 

(a) \(1+4x+6x^2+4x^3+x^4\)

(b) \(1+4x-6x^2+4x^3+x^4\)

(c) \(1-4x+6x^2-4x^3+x^4\)

22. \((1-x)^4\)

In the following binomials \(0<x<1\)

Find the first three terms of the series obtained in the expansion in powers of \(x\) of each binomial.

23. \((1+x)^{\frac{1}{2}}\)

(d) \(1+\frac{1}{4}x + \frac{3}{32}x^2\)

(e) \(1+\frac{1}{4}x - \frac{3}{32}x^2\)

(f) \(1+\frac{1}{4}x + \frac{5}{32}x^2\)

(g) \(1-\frac{1}{4}x + \frac{5}{32}x^2\)

24. \((1+x)^{-\frac{1}{4}}\)
PART 1

INSTRUCTIONS

Below you will find two columns (A and B). Choose the entry in column B which you consider is the correct answer to each question in column A. Write the corresponding code letters in the boxes provided in the answer sheet. You may use each code letter once, more than once or not at all.

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<table>
<thead>
<tr>
<th>COLUMN A</th>
<th>COLUMN B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Express the following angles in degrees</td>
<td></td>
</tr>
<tr>
<td>1. ( \frac{1}{6} \pi )</td>
<td>(a) 30°</td>
</tr>
<tr>
<td>2. ( \frac{3}{4} \pi )</td>
<td>(b) 60°</td>
</tr>
<tr>
<td>3. ( \frac{3}{2} \pi )</td>
<td>(c) 120°</td>
</tr>
<tr>
<td>4. ( \frac{5}{3} \pi )</td>
<td>(d) 135°</td>
</tr>
<tr>
<td></td>
<td>(e) 200°</td>
</tr>
<tr>
<td></td>
<td>(f) 270°</td>
</tr>
<tr>
<td></td>
<td>(g) 300°</td>
</tr>
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The value of:

| 5. \( \sin \frac{3}{2} \pi \) | (h) \( \sqrt{3}/2 \) |
| 6. \( \sin \frac{11}{6} \pi \) | (i) 1 |
| 7. \( \cos \frac{1}{2} \pi \) | (j) \( 1/2 \) |
| 8. \( \cos \frac{1}{6} \pi \) | (k) 0 |
| | (l) -1/2 |
| | (m) \(-1\) |
| | (n) \(-\sqrt{3}/2\) |
PART 2

INSTRUCTIONS

Each of the following questions have four possible answers of which only one is correct. Please tick, on the answer sheet, the box which you consider gives the correct answer to each question. If you have no idea of the answer to a particular question, please do not guess.

9. From the figure on the right, we have that:
   (A) \( \sin \theta = \frac{p}{m} \) and \( \cos \theta = \frac{n}{m} \)
   (B) \( \sin \theta = \frac{m}{p} \) and \( \cos \theta = \frac{m}{n} \)
   (C) \( \sin \theta = \frac{m}{n} \) and \( \cos \theta = \frac{m}{p} \)
   (D) \( \sin \theta = \frac{n}{m} \) and \( \cos \theta = \frac{p}{m} \)

10. Which of the following pair of values satisfy the equation \( \cos \theta = -\frac{1}{2} \)?
   (A) \( 120^\circ \) and \( -150^\circ \)
   (B) \( 120^\circ \) and \( -120^\circ \)
   (C) \( 150^\circ \) and \( -150^\circ \)
   (D) \( 150^\circ \) and \( -120^\circ \)

11. The general solution of the equation \( \cos \theta = \frac{1}{2} \) is:
   (A) \( \theta = 2n\pi \pm \frac{\pi}{3} \)
   (B) \( \theta = 2n\pi \pm \frac{2\pi}{3} \)
   (C) \( \theta = n\pi \pm \frac{\pi}{3} \)
   (D) \( \theta = n\pi \pm \frac{2\pi}{3} \)

12. The general solution of the equation \( \sin \theta = \frac{1}{2} \) is:
   (A) \( \theta = 2n\pi \pm \frac{\pi}{3} \)
   (B) \( \theta = 2n\pi \pm \frac{2\pi}{3} \)
   (C) \( \theta = n\pi \pm \frac{\pi}{3} \)
   (D) \( \theta = n\pi \pm \frac{2\pi}{3} \)
13. \( \cos (A - B) \) is the same as:

(A) \( \sin A \cos B + \cos A \sin B \)  
(B) \( \sin A \cos B - \cos A \sin B \)  
(C) \( \cos A \cos B + \sin A \sin B \)  
(D) \( \cos A \cos B - \sin A \sin B \)

14. \( \cos(30^\circ-x) \) is the same as:

(A) \( \frac{1}{2} (\sqrt{3} \cos x - \sin x) \)  
(B) \( \frac{1}{2} (\sqrt{3} \cos x + \sin x) \)  
(C) \( \frac{1}{2} (\cos x - \sqrt{3} \sin x) \)  
(D) \( \frac{1}{2} (\cos x + \sqrt{3} \sin x) \)

15. \( \sin (A + B) \) is the same as:

(A) \( \sin A \cos B + \cos A \sin B \)  
(B) \( \sin A \cos B - \cos A \sin B \)  
(C) \( \cos A \cos B + \sin A \sin B \)  
(D) \( \cos A \cos B - \sin A \sin B \)

16. \( \sin (x + 60^\circ) \) is the same as:

(A) \( \frac{1}{2} (\cos x + 3\sin x) \)  
(B) \( \frac{1}{2} (\sin x + 3\cos x) \)  
(C) \( \frac{1}{2} (\sin x + 3\cos x) \)  
(D) \( \frac{1}{2} (\cos x - 3\sin x) \)

17. \( \sin A + \sin B \) is the same as:

(A) \( 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \)  
(B) \( 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \)  
(C) \( -2\sin \frac{A+B}{2} \sin \frac{A-B}{2} \)  
(D) \( 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \)

18. \( \sin 6\omega t + \sin 2\omega t \) is the same as:

(A) \( 2 \cos 4\omega t \sin 2\omega t \)  
(B) \( 2 \sin 4\omega t \cos 2\omega t \)  
(C) \( 2 \cos 4\omega t \cos 2\omega t \)  
(D) \( -2 \sin 4\omega t \sin 2\omega t \)

19. \( -\cos A - \cos B \) is the same as:

(A) \( 2\cos \frac{A+B}{2} \sin \frac{A-B}{2} \)  
(B) \( 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} \)  
(C) \( -2\sin \frac{A+B}{2} \sin \frac{A-B}{2} \)  
(D) \( 2\sin \frac{A+B}{2} \cos \frac{A-B}{2} \)

20. \( \cos 3x - \cos x \) is the same as

(A) \( 2\cos 2x \sin x \)  
(B) \( 2\sin 2x \cos x \)  
(C) \( 2\cos 2x \cos x \)  
(D) \( -2\sin 2x \sin x \)
21. In the figure on the right, we have, according to the cosine rule, that:

(A) \( m^2 = n + p - 2np\cos M \)
(B) \( m^2 = n^2 + p^2 - 2np\cos M \)
(C) \( m^2 = n^2 + p^2 + 2np\cos M \)
(D) \( m^2 = n^2 + p^2 + 2np\cos M \)

22. If in the figure on the right \( n = 3, p = 4 \) and \( M = 120^\circ \) then the value of \( m \) is:

(A) \( \sqrt{13} \)
(B) \( \sqrt{19} \)
(C) \( \sqrt{31} \)
(D) \( \sqrt{37} \)

23. In the figure on the right \( n = 1, p = 5 \) and \( M = 60^\circ \) then the value of \( m \) is:

(A) 1
(B) \( \sqrt{21} \)
(C) \( \sqrt{28.5} \)
(D) \( \sqrt{31} \)

24. If in the figure on the right \( m = 2, n = 3 \) and \( p = 120^\circ \) then the value of \( p \) is:

(A) \( \sqrt{7} \)
(B) \( \sqrt{11} \)
(C) \( 4 \)
(D) \( \sqrt{19} \)

25. The above is a sketch of:

(A) \( \sin x \)  (B) \( \sin 2x \)  (C) \( \cos x \)  (D) \( \cos 2x \)
INSTRUCTIONS
Each of the following questions have four possible answers of which only one is correct. Please tick, on the answer sheet, the box which you consider gives the correct answer to each question. If you have no idea of the answer to a particular question, please do not guess.

DIFFERENTIAL CALCULUS
1. If \( y = \sin x + \cos x \), then \( \frac{dy}{dx} \) is:
   (A) \( \cos x + \sin x \)  
   (B) \( \cos x - \sin x \)  
   (C) \( -\cos x + \sin x \)  
   (D) \( -\cos x - \sin x \)

2. If \( y = (x^2 + 3)^5 \), then \( \frac{dy}{dx} \) is:
   (A) \( 5(x^2 + 3)^4 \)  
   (B) \( 10x(x^2 + 3)^4 \)  
   (C) \( 10(x^2 + 3)^4 \)  
   (D) \( 5x^2(x^2 + 3)^4 \)

3. If \( y = e^{-4t} \), then \( \frac{dy}{dt} \) is:
   (A) \( e^{-4t} \)  
   (B) \( \frac{1}{4} e^{-4t} \)  
   (C) \( -\frac{1}{4} e^{-4t} \)  
   (D) \( -4 e^{-4t} \)

4. If \( y = \ln(a + bt) \), where \( a \) and \( b \) are constants, then \( \frac{dy}{dx} \) is:
   (A) \( \frac{1}{bt} \)  
   (B) \( \frac{1}{a+bt} \)  
   (C) \( \frac{b}{a+bt} \)  
   (D) \( \frac{bt}{a+bt} \)

5. If \( y = \sin^2 t \), then \( \frac{dy}{dt} \) is:
   (A) \( 2 \sin t \cos t \)  
   (B) \( 2 \sin t \cos^2 t \)  
   (C) \( 2 \sin t \)  
   (D) \( 2 \cos^2 t \)

6. If \( y = \sin^2 3t \), then \( \frac{dy}{dt} \) is:
   (A) \( 2 \sin 3t \)  
   (B) \( 6 \sin 3t \)  
   (C) \( 2 \sin 3t \cos 3t \)  
   (D) \( 6 \sin 3t \cos 3t \)
7. If \( y = \sin^2(2wt + \phi) \) where \( \phi \) is a constant, then \( \frac{dy}{dt} \) is:
   (A) \( 2\sin(2wt + \phi) \)    (C) \( 4w\sin(2wt + \phi)\cos(2wt + \phi) \)
   (B) \( 2\cos^2(2wt + \phi) \)    (D) \( 2w\sin(2wt + \phi)\cos(2wt + \phi) \)

8. If \( u \) and \( v \) are functions of \( x \), then \( \frac{d}{dx}(uv) \) is:
   (A) \( \frac{du}{dx} + \frac{dv}{dx} \)    (C) \( u \frac{du}{dx} + v \frac{dv}{dx} \)
   (B) \( u \frac{dv}{dx} \)    (D) \( u \frac{dv}{dx} + v \frac{du}{dx} \)

9. If \( y = xe^x \), then \( \frac{dy}{dx} \) is:
   (A) \( e^x \)    (B) \( 1+e^x \)    (C) \( e^x + xe^x \)    (D) \( x+e^x \)

10. If \( y = t \sin t \), then \( \frac{dy}{dt} \) is:
    (A) \( 1 + \cos t \)    (C) \( t \cos t + \sin t \)
    (B) \( t + \sin t \cos t \)    (D) \( \cos t \)

11. If \( y = x \ln x \) then \( \frac{dy}{dx} \) is:
    (A) \( \frac{1}{x} \)    (C) \( 1 + \frac{1}{x} \)
    (B) \( \frac{1}{x} + \ln x \)    (D) \( x + \ln x/x \)

12. If \( u \) and \( v \) are functions of \( x \), then \( \frac{d}{dx} \left( \frac{u}{v} \right) \) is:
    (A) \( \frac{1}{\sqrt{2}} \left[ v \frac{dv}{dx} - u \frac{du}{dx} \right] \)
    (C) \( \frac{1}{\sqrt{2}} \left[ v \frac{dv}{dx} - u \frac{du}{dx} \right] \)
    (B) \( \frac{1}{\sqrt{2}} \left[ v \frac{dv}{dx} + u \frac{du}{dx} \right] \)
    (D) \( \frac{1}{\sqrt{2}} \left[ v \frac{dv}{dx} + u \frac{du}{dx} \right] \)

13. If \( y = \frac{3x+4}{2x-1} \), then \( \frac{dy}{dx} \) is:
    (A) \( -\frac{11}{(2x-1)^2} \)    (C) \( \frac{12x+5}{(2x-1)^2} \)

14. If \( y = \sin t / t \), then \( \frac{dy}{dt} \) is:
    (A) \( \frac{1}{t} \quad t - \sin t \cos t \)    (C) \( \frac{1}{t} \quad t + \sin t \cos t \)
    (B) \( \frac{1}{t} \quad t \cos t + \sin t \)    (D) \( \frac{1}{t} \quad t \cos t - \sin t \)

15. If \( y = \ln x/2x \), then \( \frac{dy}{dx} \) is:
    (A) \( 1/4x \quad 2 - 2\ln x \)    (C) \( 1/4x \quad 4x + \ln x/x \)
    (B) \( 1/4x \quad 2 + 2\ln x \)    (D) \( 1/4x \quad 4x - \ln x/x \)
16. \( \int (7x^2 + 2x + 3) \, dx \) is equal to
(A) \( \frac{7}{3}x^3 + x^2 + 3x \)  
(C) \( 7x^3 + 2x^2 + 3x \)
(B) \( \frac{7}{2}x^3 + 2x^2 + 3x \)  
(D) \( 14x + 2 \)

17. \( \int e^{\alpha x} \, dx \) is equal to a constant plus:
(A) \( ke^{\alpha x} \)  
(B) \( xe^{\alpha x} \)  
(C) \( \frac{1}{\alpha} e^{\alpha x} \)  
(D) \( \frac{1}{\alpha} e^{\alpha x} \)

18. \( \int \frac{dx}{x} \) is equal to:
(A) \(-\ln x\)  
(B) \(\ln x\)  
(C) \(1/\ln x\)  
(D) \(-1/\ln x\)

19. The integral \( \int \frac{dx}{x\sqrt{1-x}} \) is most readily evaluated with the substitution:
(A) \(u=1-x\)  
(B) \(u^2=1-x\)  
(C) \(\sin u=x\)  
(D) \(\cos u=x\)

20. The integral \( \int \frac{dx}{(3x+2)^2} \) is most readily evaluated with the substitution:
(A) \(u=x\)  
(B) \(u=3x\)  
(C) \(u=3x+2\)  
(D) \(u=(3x+2)^2\)

21. The integral \( \int \frac{x \, dx}{x^2 - 3} \) is most readily evaluated with the substitution:
(A) \(u=x^2\)  
(B) \(u=x^2+3\)  
(C) \(u=\frac{x^2}{3}\)  
(D) \(u=3x^2-9\)

22. The integral \( \int \frac{dx}{(x-3)(x+1)} \) is equal to
(A) \(\frac{1}{4}\left[\int \frac{dx}{x-3} - \int \frac{dx}{x+1}\right]\)  
(C) \(\int \frac{dx}{x-3} - \int \frac{dx}{x+1}\)
(B) \(\frac{1}{4}\left[\int \frac{dx}{x-3} + \int \frac{dx}{x+1}\right]\)  
(D) \(\int \frac{dx}{x-3} + \int \frac{dx}{x+1}\)

23. The integral \( \int \frac{dx}{(x+4)(x+5)} \) is equal to
(A) \(\frac{1}{5}\left[\int \frac{dx}{x+4} + \int \frac{dx}{x+5}\right]\)  
(C) \(\int \frac{dx}{x+4} - \int \frac{dx}{x+5}\)
(B) \(\frac{1}{5}\left[\int \frac{dx}{x+4} - \int \frac{dx}{x+5}\right]\)  
(D) \(\int \frac{dx}{x+4} + \int \frac{dx}{x+5}\)
24. The integral \[
\int \frac{dx}{(x-1)(x+1)}
\] is equal to

(A) \[\frac{1}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1}\]

(B) \[\frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1}\]

(C) \[\frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{x-1}\]

(D) \[-\frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{x-1}\]

25. If \(u\) and \(v\) are functions of \(x\), then to integrate by parts we use the formula:

(A) \[\int u \left( \frac{dv}{dx} \right) dx = uv - \int v \left( \frac{du}{dx} \right) dx\]

(B) \[\int u \left( \frac{dv}{dx} \right) dx = uv - \int v \left( \frac{du}{dx} \right) dx\]

(C) \[\int u \left( \frac{dv}{dx} \right) dx = uv - \int v \left( \frac{du}{dx} \right) dx\]

(D) \[\int u \left( \frac{dv}{dx} \right) dx = uv - \int \left( \frac{du}{dx} \right) dx\]

26. Using integration by parts, one can put:

\[\int x^2 e^x dx = x^2 e^x - M,\] where \(M\) is equal to:

(A) \[\int x^2 e^x dx\]

(B) \[\int x^2 e^x dx\]

(C) \[\int 2x e^x dx\]

(D) \[\int x e^x dx\]

27. Using integration by parts, one can put:

\[\int x^3 \sin x dx = -x^3 \cos x - M,\] where \(M\) is equal to:

(A) \[\int (-x^2 \cos x) dx\]

(B) \[\int (- \cos x \sin x) dx\]

(C) \[\int x^3 dx\]

(D) \[\int x^3 dx\]

28. Using integration by parts, one can put:

\[\int e^x \cos x dx = e^x \sin x - M,\] where \(M\) is equal to:

(A) \[\int e^x dx\]

(B) \[\int e^x \sin x dx\]

(C) \[\int e^x dx\]

(D) \[\int \sin x \cos x dx\]
### Test one: Algebra.

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Test two: Inequalities - Logarithms - Complex Numbers - The binomial Theorem.

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EXAMPLES OF THE USE OF THE T.A.O.I. METHOD

The problems in these examples are:

1. Solve for $x$ the inequality $3 - 2x > 4$

2. Solve for $x$ the inequality $|x - 3| < 2$

3. Evaluate $\frac{\sin \frac{\pi}{2}}{2} - \cos \pi$

4. Expand $\log \left( \frac{ab}{c} \right)$, using the properties of logarithms.

Keys:

a.- Remarks in round brackets ( ) are remarks made by the interviewer

b.- Remarks in square brackets [ ] are explanatory notes.
EXAMPLE No. 1

(a) Solving the problem

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<td>[ He writes down the problem and solves it without saying anything]</td>
<td>[ \begin{align*} 3 - 2x &amp;&gt; 4 \ -2x &amp;&gt; 1 \ 2x &amp;&lt; 1 \ x &amp;&lt; \frac{1}{2} \end{align*} ]</td>
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<tr>
<td>[He goes over the problem again checking what he has done. Then he introduces a minus sign in one of the inequalities in his solution]</td>
<td>[ \begin{align*} 2x &amp;&lt; -1 \ x &amp;&lt; \frac{1}{2} \end{align*} ]</td>
</tr>
<tr>
<td>[pause]</td>
<td>[He crosses out the above inequality and solves the problem again from -2x &gt; 1 ]</td>
</tr>
<tr>
<td>[You have here -2x &gt; 1 and then here 2x &gt; -1 [in his second attempt]. You did not change the direction of the inequality ]</td>
<td>[ \begin{align*} -2x &amp;&gt; 1 \ 2x &amp;&gt; -1 \ x &amp;&gt; -\frac{1}{2} \end{align*} ]</td>
</tr>
</tbody>
</table>

(b) Interview (transcribed audiotape)

(Why did you change your mind at this point.? [What he crossed out]. Did you check with particular values for x? ).

If this was the answer for x, x had to be less than 1/2. In fact, if x is less than 1/2, then 2x is less than one, and 3 - 1 is not bigger than 4, so it is wrong.

(You have here -2x > 1 and then here 2x > -1 [in his second attempt]. You did not change the direction of the inequality )

[pause]

Yes, right. I was not sure about it.

(When you multiply by a negative number... )

Oh, I see. The first time I had got the minus sign but the second time I took it the other way round.
(c) Summary

This student gave the impression that he did not know that he should change the direction of the inequality when both sides of it are multiplied by a negative number. He doubted after he had checked his answer for \( x < \frac{1}{2} \) and apparently thought that something should be negative, as he then introduced a minus sign just before he crossed out what he had done. The interview situation after the completion of the test revealed that in fact he had idea of the rules for transposition (when he interrupted the interviewer) but that he was not sure about such rules.

EXAMPLE No. 2

(a) Solving the problem

<table>
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<th><strong>Student's written answer</strong></th>
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</thead>
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<tr>
<td>[He writes down the problem]</td>
<td>(</td>
</tr>
<tr>
<td>[He thinks for about 45 seconds but does not say anything. Finally he decides to leave the problem and solve another one]</td>
<td>Later</td>
</tr>
<tr>
<td>[He writes]</td>
<td>(</td>
</tr>
<tr>
<td>[pause]</td>
<td></td>
</tr>
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</table>

(b) Interview (transcribed audiotape)

(You attempted to solve this problem but then you decided to leave it for later. What happened in the first attempt?)
I did not know the correct way of doing it. I do not know... I do not know a method for doing that. I just more or less, not necessarily trial and error, but more or less, you know... thinking what the upper limit was going to be and what the lower limit was going to be... as supposed to... knowing a set method to go for it.

(Now, from here \([ x - 3 < 2 ]\) you go to here \([ x < 5]\). How did you obtain the 5 ?)

As I said, you know, not necessarily trial and error, the \(x - 3\) is less than 2, so \(x\) has to be less than 5

(And the one \([x \leq 1]\)?)

Substituting values \([|x - 3| < 2]\).

(c) **Summary**

The difficulty that this student had was that he did not know how to solve inequalities involving modulus. (Before attempting this problem, he had previously solved the inequality \(3 - 2x < 4\) with no difficulty). He knew that there had to be an upper limit and a lower limit for \(x\) but he did not really know how to find them, so he decided to solve the inequality \(x - 3 < 2\) to find one of the limits and then to use trial and error to find the other limit.

**Example No. 3**

Process (transcribed audiotape)

[He writes down the problem]

Evaluate....Evaluate.... what do we mean by evaluate?

Let us convert that [he writes]

I shall use [he writes]

Now you get \(x = \frac{\pi}{2}\), therefore [he writes]

Student's written answer

\[
\sin^2 \frac{\pi}{Z} - \cos \pi
\]

\[
\sin^2 x =
\]

\[
\cos 2x = 1 - 2 \sin^2 x
\]

\[
\cos \pi = 1 - 2 \sin^2 \frac{\pi}{2}
\]

\[
\sin^2 \frac{\pi}{2} = \frac{1 - \cos \pi}{2}
\]
Yes, I will substitute that in [the expression sin^{2} \frac{\pi}{2} - \cos \pi] [he writes]

\[
\frac{1 - \cos \pi}{2} - \cos \pi
\]

\[
= \frac{1}{2} - \frac{1}{2} \cos \pi - \cos \pi
\]

\[
= \frac{1}{2} - \frac{3}{2} \cos \pi
\]

\[
= \frac{1}{2} - \frac{3}{2} (-1)
\]

\[
= 2
\]

[He goes through the solution again to check that everything is correct].

(What is \( \frac{\pi}{2} \) ?)

\( \pi \) is a number, it is 90° which is 1 [i.e. sin 90° = 1]

[He writes]

I should have done it that way, 1 is \( \sin^{2} \frac{\pi}{2} \), \( \cos \pi \) is minus 1, therefore 1 - (-1) = 2. I forgot, I suppose that is what I should have done. I did it that way [what he actually wrote] but........ it would have been easier to do it that way, but I always.... I always tend to do it that long because that.... I mean, you do not doubt when it is 'substitute' the values in and when you actually.... in our mathematics questions we had got things like that and we had actually to do them in this way, that is why I went about.... that long way rather than this... substituting in.

Summary

After this student had read the question, he apparently got confused by the word "Evaluation", and moreover, he said "what do we mean by Evaluate?". Apparently, he has the tendency to do a problem in a particular way because he expects the problem to be of a particular kind. In this case, he expected this problem to do with manipulation of trigonometric identities. Since in his A-level mathematics course he used to solve problems in which he had to use a lot of trigonometric manipulation, he did not hesitate with the present problem and tried to solve it doing the same kind of things he used to do at school.
Process (transcribed audiotape) - Student's written answer

[He writes the question]
[pause]

(What are you going to do?)

I am trying to ... I have done questions like this before... I know that it comes to the power of something...but I actually...
I think I will write it down [he writes] \[ \log(a x \log(c)) \]

I think that is ... I can do something with it but I cannot explain why I have done that...
I am losing myself a little bit here. I guess... to be quite honest, I cannot really answer the question for some reason... I have completely forgotten all my old work.

(Do you remember how to expand the logarithm of a quotient ?).

Logarithm of a quotient [ he writes] \[ \log(a) = \log(a) - \log(b) \]

Oh! It is the same as [he writes]
Is that the same as... let us see, let us have a try. I think I will try... let me explain it here. Let us choose [ he writes] \[ a = 10, b = 100, c = 1000 \]

Then that is equal to [he writes] \[ \log(\frac{10 \times 100}{1000}) = 0 \]

therefore, using now properties of logarithms [he writes] \[ \log(\frac{ab}{c}) = \log(a) + \log(b) - \log(c) \]

\[ = 1 + 2 - 3 \]

\[ = 0 \]

I think that looks like the right answer...
It was very... I do not know, I just... I was doing.... that is not one of the standard ones.
We were doing that earlier, were not we? When
you get $x$ to the power of something times $x$ to the power of something [he writes] $X \times X$

It is the same sort of method.....

(Why did you first write $\log\left(\frac{ab}{c}\right) = \frac{\log a \cdot \log b}{\log c}$?)

Because I ... because in our mathematics course we did that [product of powers with the same base] and that [properties of logarithms] in the same week, and I had got a doubt between the two... I do not know it was something like you multiply by $\log c$ and therefore the $\log$ cancel out or something like that. I cannot quite remember what it was... I had completely gone off the track. It took me a little while to figure what exactly was happening.

Summary

The difficulty that this student had was that he did logarithms and product of powers with the same basis during the same week in his A-level mathematics course and this would appear to have confused him. He tried to sort this out with an example using powers of 10 because he knew that $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$ and $\log 1000 = 3$. He found that $\log\left(\frac{10 \times 100}{1000}\right) = 0$ and he knew that $\log\left(\frac{a}{b}\right) = \log a - \log b$, therefore he concluded that $\log\left(\frac{ab}{c}\right)$ could not be equal to $\frac{\log a \cdot \log b}{\log c}$ because $rac{\log 10 \cdot \log 100}{\log 1000} = \frac{2}{3}$ and not zero.

From this, he concluded that $\log\left(\frac{ab}{c}\right)$ should be equal to

$\log a + \log b - \log c$ and checked it with the values he had chosen for $a, b,$ and $c$, that is, $\log a + \log b - \log c = 1 + 2 - 3 = 0$.

To summarise, if we had only asked the student to solve the problem, he would have written $\log\left(\frac{ab}{c}\right) = \frac{\log a \cdot \log b}{\log c}$ and would therefore have left the problem there, not allowing us to detect what his difficulty was.
Algebraic Manipulation

1. Expand \((a+3b)^{3}\).
2. Simplify \(\frac{(a^2+16)}{(a^2+4)}\).
3. Solve \(x^2 - 6x - 3 = 0\).
4. Factorize \(2x^2 - 11x + 5\).
5. Solve the set of equations
   \[\begin{align*}
   2x - y &= 5 \\
   3x + 2y &= 11
   \end{align*}\]
6. Simplify \(\frac{2x - 1}{3} - \frac{2x - 1}{2}\).
7. Rationalize the denominator in the expression \(\frac{2}{1 - \sqrt{3}}\).
8. Divide \(2x^3 - 6x^2 + 9x - 3\) by \(x^2 - x + 2\).

Trigonometry

1. Express \(\cos 2\theta\) in terms of \(\tan \theta\).
2. Evaluate \(\sin \frac{5\pi}{4} = \cos \theta - \tan \left(\frac{-\pi}{4}\right)\).
3. Evaluate \(\sin 60^\circ\).
4. Sketch the graph of \(\sin \theta\), \(\cos \theta\).

5. In the triangle \(ABC\) find the length \(BC\) given that \(AC = \sqrt{5}\), \(\angle BAC = 30^\circ\) and \(\angle ABC = 120^\circ\).

6. In the triangle \(PQR\) find \(\cos \theta\) given that \(PR = 6, PQ = 3, \theta = 4\).

Inequalities

1. Express the following intervals in terms of inequalities.
   \[\begin{align*}
   &(i) \quad (3, 7) \\
   &(ii) \quad [-2, 0] \\
   &(iii) \quad (1, \infty)
   \end{align*}\]
2. Solve the inequality \(x + 6 \leq 3 - x\).
3. Solve the double inequality \(4 - 3x < 2x + 3 < 2x - 4\).
4. Solve the inequality \(|x - 3| \leq 3\).
5. Solve the inequality \(|\frac{x - 1}{x - 2}| = 2\).
Dear Student,

Programmed Trigonometry

I am designing a programmed text in trigonometry which should be useful to anyone who wishes to revise his knowledge in this subject. At the moment I have got a version on trial and I would like to ask your cooperation in helping me to improve it. At the same time you will be able to improve your knowledge of trigonometry.

Would you therefore please do the following:

(1) Work through the text, writing the answers to each frame into the book. (Ignore the instruction not to write in the book.)

(2) If you have any questions or comments, please write them on the page opposite to the appropriate frame. Anything in connection with the instructions, content of frame, wording, etc., that you can comment on will help me.

(3) As you work through the book, please write down the date and time, each time that you start and finish any part of the book, against the frame where you start and the frame where you finish.

(4) I shall come to see you on (date) at (time) to see how you are getting on and to gather your impressions.

Thank you very much for your help.

E. C. González-León.
MATHEMATICS AT THE SCHOOL - UNIVERSITY INTERFACE,

WITH SPECIAL REFERENCE TO THE NEEDS

OF ENGINEERING STUDENTS

by

EDUARDO GONZÁLEZ-LEÓN

In partial fulfilment of the requirements of the

Ph.D. Degree, University of Surrey, 1979
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VOLUME II

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ALGEBRAIC MANIPULATION

By

Eduardo Gonzalez-Leon

Institute for Educational Technology
University of Surrey
Guildford

July 1977
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To the student

In Science and Engineering one often has to deal with algebraic manipulation; therefore it is very important for you to develop your skill in carrying this out easily and accurately.

This booklet, which is designed to help students acquire these skills, is based on the principle of attempting exercises of increasing order of difficulty.

The exercises in the booklet have been divided into separate topics. To see which topics you should attempt, you are advised to proceed as follows:

1. Answer the pre-test provided
2. Check your answers with the ones provided just after the pre-test
3. Go to the "Pretest-Topic Correspondence" section, and use this to find out which topics you should study
4. Read the instructions on the following page
1) Remove brackets and simplify the resulting expression as much as you can:

\[ 7a - \left\{ 5b - (a+7b) + (3-2a+b) - 1 \right\} \]

2) Expand and simplify:

a) \((3xy^3 - 5yz)^2\)  

b) \((xy+8)(xy-7)\)

3) Evaluate each of the following:

a.1) \(-4(-4)^2\)  
a.2) \((0.17)^{2.8}\)

b) Simplify as fully as you can:

b.1) \(e^{3\ln x+5\ln y}\)  
b.2) \(\frac{4ab^2c^{-\frac{3}{2}}}{2a^{-\frac{1}{2}}b^{-1}c}\)

4) a) Evaluate \(-8^{-\frac{1}{3}}\)

b) Remove the square root from the following expression and simplify:

\[ a \sqrt{9b^4c^3} \]

c) Remove the square root from the denominator of:

\[ c.1) \frac{4xyz}{\sqrt{2xyz^5}} \quad c.2) \frac{2\sqrt{3}}{\sqrt{6} + 2} \]

5) a) Subtract \(x-2y+x^2-4y^2\) from \(3y^2+x-x^2\)

b) Multiply \(x^4+1-x^2\) by \(x^3-3x+1\)

c) Divide \(2x^3+x^5-2-3x\) by \(x^2-3x+1\) and write the result in the form \(\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}\)
6) Break the following expressions up into a product of linear factors:
   a) \(4x^2 + 4x + 1\)  
   b) \(6x^2 - 13x + 6\)

7) Simplify as fully as you can:
   a) \(\frac{x-3}{x^2+6x+9} + \frac{x-1}{x^2-9}\)
   b) \(\frac{6x-12}{4xy-4x} \cdot \frac{x^2-1}{x^2+2x+1}\)

8) Solve for \(x\):
   \[\frac{2x-9}{6} = \frac{3x+4}{4}\]

9) Solve the following set of equations for \(x\) and \(y\):
   \[
   \begin{cases}
   2x - 3y = 9 \\
   4x - y = 8
   \end{cases}
   \]

10) a) Solve for \(x\) by using the standard formula:
    \[3x^2 - 5x - 2 = 0\]
    b) Solve for \(x\): \(\frac{x^2}{3} - 27 = 0\)
    c) Solve for \(x\): \(3x^2 - 3x = 0\)
1) \(10a + b - 2\)

2) a. i) \(-\frac{1}{4}\)  
   b. i) \(x^5 y^5\)
   a. ii) 0.007  
   b. ii) \(20a^\frac{7}{5} b^2 c^{-\frac{1}{2}}\)

3) a) \(2x - y^2 - 2y\)
   b) \(x^6 - 3x^5 + 3x^3 - 3x + 1\)
   c) \(\frac{x^5 + 2x^3 - 3x - 2}{x^2 - 3x + 1} = x^3 + 3x^2 + 10x + 27 + \frac{88x + 25}{x^2 - 3x + 1}\)

4) a) \(9x^2 y^6 - 30xy^4 z + 25y^2 z^2\)
   b) \(x^2 y^2 + xy - 56\)

5) a) \((2x + 1)^2\)
   b) \((2x - 3)(3x - 2)\)

6) a) \(\frac{x^2 - 4x + 6}{(x+3)^2(x-3)}\)
   b) \(\frac{3(x-2)(x-1)}{2(y-1)(x+1)}\)

7) a) \(-\frac{1}{2}\)
   c. i) \(2\sqrt[3]{4x^2 y}\)
   b) \(3abc \sqrt{c}\)
   c. ii) \(\sqrt{3} (\sqrt{6} - 2)\)

8) \(x = -6\)

9) \(x = \frac{3}{2}, y = -2\)

10) a) \(x = 2; -\frac{1}{3}\)
   b) \(x = 9; -9\)
   c) \(x = 0; 1\)
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Instructions

Each topic in the booklet consists of some worked examples which you should study first, and then there are some exercises for you to solve.

You can check your answers to the exercises with the correct ones provided at the end of the booklet.

If an answer is wrong, review the relevant worked example in order to determine whether you have used the correct method, and then attempt the question again, taking care with your calculations.
1. Removing Brackets

Removal of brackets is governed by the following rules:

1.1 If a + sign (or no sign) precedes a bracket, this bracket may be removed without affecting the terms inside the brackets.

1.2 If a - sign precedes a bracket, this bracket may be removed if each sign inside the bracket is changed.

1.3 If more than one set of brackets is present, the inner ones are to be removed first.

**Example:**
Remove the brackets in the following and simplify the resulting expressions by combining like terms.

\[ a - 3 - \{ 4b - 2 + (5a - 2b) \} \]

**Solution:**

\[ a - 3 - \{ 4b - 2 + (5a - 2b) \} = a - 3 - \{ 4b - 2 + 5a - 2b \} \text{ (rules 1.1 and 1.3) } \]

\[ = a - 3 - 4b + 2 - 5a + 2b \text{ (rule 1.2) } \]

\[ = -4a - 2b - 1 \text{ (combining like terms) } \]
**Exercise 1:**

Remove brackets in each of the following, and simplify the resulting expression by combining like terms.

1) \((3a^2 - 3b + 2c) + (5b - 7c)\)

2) \((3a^2 - 3b + 2c) - (5b - 7c)\)

3) \(2a + \{3a - (5a + b) + 7b\}\)

4) \(7a - \{5b - (a + 7b) + (3a - 2 + b) - 1\}\)
2. Multiplication of brackets

The following are some common products which occur frequently in mathematics.

2.1 \( a(c+d) = ac + ad \)

**EXAMPLE**

\[
5x^2(3x^4 - 2xy + 8) = 5x^2(3x^4) + 5x^2(-2xy) + 5x^2(8) \]

\[
= 15x^6 - 10x^3y + 40x^2
\]

2.2 \((a+b)(c+d) = ac + ad + bc + bd\)

**EXAMPLE**

\[
(x+3)(y+5) = xy + 5x + 3y + 15
\]

2.3 \((a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2\)

**EXAMPLE**

\[
(5x+3y)^2 = (5x)^2 + 2(5x)(3y) + (3y)^2
\]

\[
= 25x^2 + 30xy + 9y^2
\]

2.4 \((a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2\)
\[(4x^3y - 3x^2)^2 = (4x^3y)^2 - 2(4x^3y)(3x^2) + (3x^2)^2\]
\[= 16x^6y^2 - 24x^5y + 9x^4\]

2.5 \((a+b)(a-b) = a^2 - b^2\)

\[\text{EXAMPLE}\]
\[(xy^2 + 5)(xy^2 - 5) = (xy^2)^2 - (5)^2 = x^2y^4 - 25\]

2.6 \((x+a)(x+b) = x^2 + (a+b)x + ab\)

\[\text{EXAMPLE}\]
\[(3x^2 - 7)(3x^2 + 2) = (3x^2)^2 - (7+2)(3x^2) + (-7)(2)\]
\[= 9x^4 - 15x^2 - 14\]

2.7 \((ax+b)(cx+d) = acx^2 + (ad+bc)x + bd\)

\[\text{EXAMPLE}\]
\[(2x-5)(3x+8) = (2)(3)x^2 + \left[(2)(8) + (-5)(3)\right]x + (-5)(8)\]
\[= 6x^2 + x - 40\]
Exercise 2

Find each of the following products:

1) \(2ab^3 (4a^2b^5 + 7ab)\)

2) \(-5xy (3x^2y - 3xy^2)\)

3) \((x^2 + xy)(y^2 - 1)\)

4) \((2x - y)(3xy - 4y)\)

5) \((4x^2y^3 + 3)^2\)

6) \((3xy^3 - 5yz^2)^3\)

7) \((xy + 8)(xy - 7)\)

8) \((5x + 2y)(7x - 4y)\)

9) \((3e^x + 4)(5e^x - 1)\)
3. Exponents

3.1 Integral exponents

If n is a positive integer we define:

3.1.1 \( a^n = a \cdot a \cdot a \ldots a \) (n times)

3.1.2 \( a^{-n} = \frac{1}{a^n} \) assuming \( a \neq 0 \)

EXAMPLES:

a) \( 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 \) (by property 3.1.1)

b) \((-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32\) (by property 3.1.1)

c) \( 2^{-3} = \frac{1}{2^3} = \frac{1}{8} \) (by properties 3.1.2 and 3.1.1)

d) \((-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}\) (by properties 3.1.2 and 3.1.1)

Exercise 3.1

Evaluate each of the following:

1) \( 5^4 \) 2) \((-\frac{2}{3})^3\) 3) \( 5^{-2} \) 4) \(-4(4)^{-2}\)

3.2 Decimal exponent

To evaluate expressions such as \( 2^{3.7} \) we need to use logarithms.

We first recall the two basic properties of the logarithm; namely

3.2.1 \( \log a^b = b \log a \)
the first of which is needed below. If these are unfamiliar to you then you are advised to look at a book on logarithms elsewhere.

**EXAMPLES:**

a) To evaluate $2^{3.7}$ proceed as follows:
   Set $x = 2^{3.7}$ and take logarithms:
   \[
   \log x = \log 2^{3.7} = 3.7 \log 2 \quad \text{(by property 3.2.1)}
   \]
   \[
   \log x = 3.7 (0.3010) \quad \text{(using four-figure tables of common logarithms)}.
   \]
   Therefore
   \[
   \log x = 1.1137
   \]
   and so
   \[
   x = 12.99 \quad \text{(using four-figure tables of common antilogarithms)}.
   \]

b) To evaluate $(0.17)^{2.8}$ proceed as follows:
   Set $x = (0.17)^{2.8}$ and take logarithms:
   \[
   \log x = \log (0.17)^{2.8} = 2.8 \log (0.17) \quad \text{(by property 3.2.1)}
   \]
   \[
   \log x = 2.8 (1.2304) \quad \text{(using four-figure tables of common logarithms)}.
   \]
   Therefore
   \[
   \log x = 2.8 (-1 + 0.2304)
   \]
   \[
   \log x = -2.8 + 0.6451
   \]
   \[
   \log x = -2.1549
   \]
   \[
   \log x = -2.1549 - 1 + 1
   \]
   \[
   \log x = -3 + 0.8451
   \]
\( x = 0.007 \) (using four-figure tables of common logarithms)

If you have difficulties with this example, you may have to refresh your knowledge of the use of logarithms therefore you are advised to look at a book on logarithms elsewhere.

**Exercise 3.2**

Evaluate each of the following:

1. \( 3^{0.2} \)
2. \( 10^{1.8} \)
3. \( (2.4)^{0.7} \)
4. \( (0.031)^{2.4} \)

**3.3 Zero exponent**

We define \( a^0 = 1 \) if \( a \neq 0 \)

**EXAMPLES**:

a) \( 5^0 = 1 \)

b) \((-4)^0 = 1\)

**3.4 General laws of exponents**

If \( p \) and \( q \) are real numbers, the following laws hold (note that they can be read and use in either direction):

3.4.1 \( a^p \cdot a^q = a^{p+q} \)

**EXAMPLES**

a) \( 2^3 \cdot 2^4 \cdot 2^5 = 2^{3+4+5} = 2^{12} \)

b) \( 3^{-4} \cdot 3^9 = 3^{-4+9} = 3^5 \)
3.4.2 \quad (a^p)^q = a^{pq}

**EXAMPLES**

a) \((x^3)^2 = x^6\)

b) \((5^{-7})^4 = 5^{-28}\)

3.4.3 \quad \frac{a^p}{a^q} = a^{p-q} \quad \text{assuming } a \neq 0

**EXAMPLES**

a) \(\frac{x^6}{x^4} = x^{6-4} = x^2\)

b) \(\frac{5^4}{5^{-3}} = 5^{4-(-3)} = 5^{4+3} = 5^7\)

3.4.4 \quad (a \cdot b)^p = a^p \cdot b^p

**EXAMPLES**

a) \((x \cdot 3)^4 = x^4 \cdot 3^4\)

b) \((4x)^5 = 4^5 \cdot x^5\)

c) \((-2)^5 = (-1 \cdot 2)^5 = (-1)^5 \cdot 2^5 = -32\)
EXAMPLES

a) \( \left( \frac{x}{3} \right)^6 = \frac{x^6}{3^6} \)

b) \( \left( \frac{x}{y} \right)^{-3} = \frac{x^{-3}}{y^{-3}} \)

3.5 Miscellaneous examples

a) \((-5xy^3)^2 = (-5)^2 x^2 (y^3)^2 = 25x^2y^6\)

b) \((-2x)^2 = \frac{1}{(-2x)^2} = \frac{1}{4x^2}\)

c) \((4 \cdot 10^{-6}) (2 \cdot 10^4) = 4 \cdot 2 \cdot 10^{-6} \cdot 10^4 = 8 \cdot 10^{-2}\)

d) \(\frac{8 \cdot 10^2}{2 \cdot 10^6} = 4 \cdot 10^{2-6} = 4 \cdot 10^{-4}\)

e) \(\frac{4a^3 b^{-2} c^{-\frac{3}{2}}}{2a^{-\frac{1}{2}} b^4 c} = \left(\frac{4}{2}\right) a^{3-\frac{1}{2}} b^{-2+4} c^{-\frac{3}{2}-1} = 2a^{\frac{5}{2}} b^2 c^{-\frac{5}{2}}\)

Exercise 3.3

Simplify:

1) \(\frac{x^{m+3}}{x^{m-1}}\)

2) \(\frac{x^3 \cdot x^2 \cdot x^4}{x^{-1} \cdot x^0 \cdot x^{-3}}\)
3) \((x^2 \cdot y^{-2} \cdot z^{x^{-1}})^x\)

4) \([(x^{-1})^{-x}]^x\)

5) \(-3(-1)^5 (-4)^{-2}\)

6) \(\frac{7^4 \cdot 10^{-5}}{7^3 \cdot 10^8}\)

7) \(\left(\frac{x^8 \cdot y^4}{5^4}\right)^{-\frac{1}{4}}\)

8) \(\frac{a^{\frac{3}{5}} \cdot b^{-\frac{6}{7}} \cdot c^{-\frac{3}{11}}}{b^{-\frac{3}{7}} \cdot c^{-\frac{5}{3}} \cdot a^{-\frac{5}{3}}}\)

### 3.6 Logarithmic exponent

There is a useful property which allows us to simplify powers where \(e\) appears as a base and the logarithm to base \(e\) (also called natural logarithm, \(\ln\)) of an expression appears as exponent. The property is:

\[ e^{\ln x} = x \]

**EXAMPLE**

Simplify \(e^{3+2\ln x}\)

We first of all use property 3.4.1 (reading it from right to left) to get

\[ e^{3+2\ln x} = e^3 \cdot e^{2\ln x} \]

\[ = e^3 \cdot e^{2\ln x^2} \quad \text{(by property 3.2.1)} \]

\[ = e^3 \cdot x^2 \quad \text{(by property 3.6.1)} \]
Exercise 3.4

Simplify the following:

1) $e^{\ln 1}$

2) $e^{\ln xy}$

3) $e^{a \ln x}$

4) $e^{\ln x + 4}$

5) $e^{3 \ln x + 5 \ln y}$

6) $e^{\ln x - 2 \ln y}$
A radical is an expression of the form \( \sqrt[n]{a} \) which denotes the principal \( n \)th root of \( a \). We shall be concerned with real roots only.

4.1 If \( m \) and \( n \) are positive integers, and \( a \neq 0 \) and \( n \) is even, we define:

4.1.1 \[ a^{\frac{m}{n}} = \sqrt[n]{a^m} \]

4.1.2 \[ a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \]

**NOTE:** We do not write the \( a \) in square roots 

*e.g.* \( \sqrt{4} = 2 \), not \( \frac{1}{\sqrt{4}} = 2 \)

In additive need to know how to treat fractional powers of \(-1\) the general rules are:

4.1.3 \( (-1)^{\frac{m}{n}} \), no real roots

4.1.4 \( (-1)^{\frac{1}{n}} \), \((-1)\) is the only real root

**EXAMPLES**

a) \( 4^{\frac{3}{2}} = \sqrt[2]{4^3} = \sqrt[2]{64} = 8 \)

b) \( 8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4} \)

c) \((-2)^{\frac{3}{4}}\), no real root
Exercise 4.1

Evaluate each of the following:

1) \(8^{\frac{2}{3}}\)  
2) \((-a^3)^{\frac{1}{5}}\)  
3) \(\left(\frac{1}{16}\right)^{\frac{1}{2}}\)  
4) \(8^{\frac{1}{3}}\)  
5) \(8^{-\frac{1}{3}}\)  
6) \(-8^{\frac{1}{3}}\)  
7) \((-8)^{\frac{1}{3}}\)

4.2 Laws for radicals

If \(m\) and \(n\) are positive integers, the following laws hold:

(NOTE: If \(n\) is even, assume \(a, b \geq 0\))

4.2.1 \(\left(\sqrt[n]{a}\right)^n = a\)

**Examples**

a) \(\left(\sqrt[4]{3}\right)^4 = 3\)

b) \(\left(\sqrt{a+b}\right)^n = a+b\)

4.2.2 \(\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}\)

**Examples**

a) \(\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}\)

b) \(\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}\)
4.2.3 \( \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad b \neq 0 \)

**EXAMPLES**

a) \( \sqrt[5]{\frac{7}{32}} = \frac{\sqrt[5]{7}}{\sqrt[5]{32}} = \frac{\sqrt[5]{7}}{2} \)

b) \( \sqrt[3]{\frac{(a+b)^6}{a^3}} = \frac{\sqrt[3]{(a+b)^6}}{\sqrt[3]{a^3}} = \frac{(a+b)^2}{a} \)

4.2.4 \( \sqrt[n]{a^m} = (\sqrt[n]{a})^m \)

**EXAMPLES**

a) \( \sqrt[3]{27^4} = (\sqrt[3]{27})^4 = 3^4 = 81 \)

b) \( \sqrt[4]{4^3} = (\sqrt[4]{4})^3 = 2^3 = 8 \)

4.2.5 \( \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a} \)

**EXAMPLES**

a) \( \sqrt[4]{5} = \sqrt[4]{5} \)

b) \( 3\sqrt[5]{ab^2} = \sqrt[15]{ab^2} \)

4.3 Miscellaneous examples

In each case remove the radical sign and simplify as much as possible:

a) \( \sqrt[3]{32} \)  

b) \( \sqrt[8]{x^5y^7} \)
Exercise 4.2

Remove the radical signs and simplify as much as possible:
1) \( \sqrt{18} \)  
2) \( 5\sqrt{243} \)  
3) \( a\sqrt{9bc^3} \)  
4) \( 3\sqrt{64x^2y^6} \)
5) \( (7\sqrt{4ab}) \)  
6) \( \sqrt{256} \)  
7) \( \frac{a}{b}\sqrt{75a^3b^2} \)

4.4 Rationalising the denominator

Rationalising the denominator consists in removing the radical sign.

EXAMPLES

a) To rationalize the denominator of \( \frac{4xy}{3\sqrt{2xy^5}} \)

we proceed like this:

We multiply the numerator and denominator by an expression which makes the quantity under the radical sign in the denominator a perfect \( n \)th power (here \( n=3 \)) and then remove the denominator from under the radical sign.

The expression in this case is \( \sqrt[3]{2x^2y} \)

Then

\[
\frac{4xy^2}{3\sqrt{2xy^5}} = \frac{4xy^2}{\sqrt[3]{2xy^5}} \cdot \frac{\sqrt[3]{2^2x^2y}}{\sqrt[3]{2^2x^2y}} = \frac{4xy^2\cdot2x^2y}{\sqrt[3]{2^3x^3y^6}}
\]
\[
\frac{4xy \sqrt{4x^2y}}{2xy^2} = 2 \sqrt[4]{4x^2y}
\]

b) To rationalize the denominator of \( \frac{a}{b + \sqrt{3}} \) we use the fact that \((A+B)(A-B) = A^2 - B^2 \) (see 3.) and proceed as follows:

\[
\frac{a}{b + \sqrt{3}} = \frac{a}{b + \sqrt{3}} \cdot \frac{b - \sqrt{3}}{b - \sqrt{3}} = \frac{a(b - \sqrt{3})}{b^2 - 3}
\]

**Exercise 4.3**

Rationalize the denominator of:

1) \( \frac{3}{\sqrt[3]{6}} \)

2) \( \frac{a\sqrt{b}}{b\sqrt{a}} \)

3) \( \frac{5abc}{\sqrt[6]{3^4a^7b^5}} \)

4) \( \frac{x\sqrt{4y}}{y\sqrt{x}} \)

5) \( \frac{1}{\sqrt{7} - 2} \)

6) \( \frac{-2}{2 - \sqrt{3}} \)

7) \( \frac{x-25}{\sqrt{x} + 5} \)

8) \( \frac{2\sqrt{3}}{\sqrt{6} + 2} \)
5.1 Addition

The addition of polynomials is done by combining like terms. One simple way to carry this out is to write the polynomials under each one with the like terms in the same column and then add these columns.

**EXAMPLE**

Add \( 2x^2 + y^2 - x + y \); \( 3y^2 + x - x^2 \) and \( x - 2y + x^2 - 4y \)

**Solution**

\[
\begin{array}{c}
2x^2 + y^2 - x + y \\
- x^2 + 3y^2 + x \\
x^2 - 4y^2 + x - 2y \\
\hline
2x^2 + 0 + x - y
\end{array}
\]

(adding) \( 2x^2 + x - y \)

**Ans.** \( 2x^2 + x - y \)

5.2 Subtraction

The subtraction of polynomials is done by changing the sign of every term in the expression which is being subtracted and then adding the result to the other expression.

**EXAMPLE**

Subtract \( 5x^2 + x + 2y^2 - 3xy \) from \( 7x^2 + 6xy + 5y^2 \)
Solution

\[
\begin{align*}
7x^2 + 6xy + 5y^2 \\
-5x^2 + 3xy - 2y^2 - x
\end{align*}
\]

(adding) \(2x^2 + 9xy + 3y^2 - x\)

Ans. \(2x^2 + 9xy + 3y^2 - x\)

Exercise 5.1

Given the polynomials:

- \(A = x^2 + y^2 - z^2 + 2xy - 2yz\)
- \(B = y^2 + z^2 + 2xy + 2yz\)
- \(C = 2x^2 + y^2 - x + y\)
- \(D = x - 2y + x^2 - 4y^2\)
- \(E = 3y^2 + x - x^2\)

Find:

1) \(A + B\)  
2) \(A + C\)  
3) \(C + D\)  
4) \(D + E\)  
5) \(A - B\)  
6) \(B - C\)  
7) \(C - E\)  
8) \(E - D\)

5.3 Multiplication

To multiply two polynomials by each other, multiply each of the terms of one polynomial by each of the terms of the other polynomial and combine the results, taking into account the rules of the signs, the laws of the exponents and the usual laws of multiplication.

It is often useful to arrange the polynomials in descending
or ascending powers of one of the variables involved.

**EXAMPLE**

Multiply \( A = x - 5y + x^2 \) by \( B = 3y^2 - x^2 + x \).

**Solution**

Arranging in descending powers of \( x \):

\[
A = x^2 + x - 5y \\
B = -x^2 + x + 3y
\]

\[
\begin{align*}
\text{(Multiplying by } -x^2) & \quad -x^4 - x^3 + 5xy \\
\text{(Multiplying by } x) & \quad x^3 + x^2 - 5xy \\
\text{(Multiplying by } 3y) & \quad 3x^2y + 3xy - 15y^2 \\
\end{align*}
\]

\[
-x^4 + 0 + 8x^2y + x^2 - 2xy - 15y^2
\]

Ans. \( -x^4 + 8x^2y + x^2 - 2xy - 15y^2 \)

**5.4 Division**

To divide one polynomial (called dividend) by another (called the divisor) we proceed as follows:

(a) Arrange the terms of both polynomials in descending (or ascending) power of one of the variables common to both polynomials.

(b) Divide the first term of the dividend by the first term in the divisor. This gives the first term of the quotient (see below)
(c) Multiply the first term of the quotient by the divisor and subtract from the dividend, thus obtaining a new dividend.

(d) Using the dividend obtained in (c), repeat steps (b) and (c) until a remainder is obtained which is either of degree lower than the degree of the divisor or zero.

(e) The result is written in the form

\[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}
\]

**EXAMPLE**

Divide \(3x^4 - 2 + 13x^2 - 7x^3\) by \(3 + x^2 - 2x\)

(\text{Dividend}) \quad \text{(Divisor)}

First term of the quotient (step (b))

\[
3x^3 - x + 2
\]

\[
\begin{array}{c|cc}
\text{Dividend} & 3x^4 - 7x^3 + 13x^2 & -2 \\
-3x^4 + 6x^3 - 9x^2 & -2 & -2 \\
\hline
-7x^3 + 4x^2 & -2 \\
-7x^3 + 2x^2 - 3x & \\
\hline
2x^2 + 3x - 2 & \\
-2x^2 + 4x - 6 & \\
\hline
7x - 8 &
\end{array}
\]

(Step (c))

(Step (d))

then \[
\frac{3x^4 - 7x^3 + 13x^2 - 2}{x^2 - 2x + 3} = 3x^2 - x + 2 + \frac{7x - 8}{x^2 - 2x + 3} \] (Step (e))
Exercise 5.2

Given that the polynomials:

\[ A = x^4 + 1 - x^2 \quad B = 1 - x \quad C = x^2 - 3x + 1 \]
\[ D = x + 4 \quad E = 2x^3 + x^5 - x - 3x \]

then find:

1) \( A \cdot B \)  
2) \( A \cdot C \)  
3) \( C \cdot D \)  
4) \( A \cdot D \)  
5) \( \frac{A}{B} \)  
6) \( \frac{C}{D} \)  
7) \( \frac{E}{C} \)  
8) \( \frac{E}{B} \)
6. Polynomial factorisation

Polynomial factorisation is the process of separating, or resolving a polynomial into factors. It is not always possible to do this in a few simple steps, but when it is the following results frequently prove useful.

6.1 Common factor

Type: \( ac + ad = a(c + d) \)

**Example**

\[ 10xy^2 + 6x^2y - 8x^3y^3 = 2xy(5y + 3x - 4x^2y^2) \]

6.2 Difference of two squares

Type: \( a^2 - b^2 = (a+b)(a-b) \)

**Example**

\[ 4x^2 - 9y^4 = (2x)^2 - (3y^2)^2 = (2x + 3y^2)(2x - 3y^2) \]

6.3 Perfect square

Types: \( a^2 + 2ab + b^2 = (a + b)^2 \)
\( a^2 - 2ab + b^2 = (a - b)^2 \)

**Examples**

a) \( x^2 + 8x + 16 \)
\[ x^2 + 8x + 16 \]
\[ \downarrow \quad \downarrow \]
\[ x \quad 4 \quad \text{(square roots)} \]
\[ 2(x)(4) = 8x \]
therefore
\[ x^2 + 8x + 16 = (x + 4)^2 \]

\[ b) \quad 9x^2 - 12xy + 4y^2 \]
\[ \downarrow \quad \downarrow \]
\[ 3x \quad 2y \quad \text{(square roots)} \]
\[ x(3x)(2y) = 12xy \]
therefore
\[ 9x^2 - 12xy + 4y^2 = (3x - 2y)^2 \]

6.4 Other examples

Types:
\[ x^2 + (a+b)x + ab = (x+a)(x+b) \]
\[ a \times x^2 + (ad+bc)x + bd = (ax+b)(cx+d) \]

a) Factorise \[ x^2 - 4x - 21 \]

Solution

To factorise \( x^2 - 4x - 21 \) we have to find two numbers \( a \) and \( b \) such that their sum \( a+b=4 \) and their product \( a \cdot b = -21 \).

By trial we find that \( a=3 \) and \( b=-7 \), therefore
\[ x^2 - 4x - 21 = (x+3)(x-7) \]
b) Factorise \(6x^2 + 11x - 10\)

**Solution**

To factorise \(6x^2 + 11x - 10\) we have to find numbers \(a, b, c\) and \(d\) such that \(ac = 6, bd = -10\) and \(ad + bc = 11\).

We find by trial that \(a = 3, c = 2, b = -2, d = 5\) satisfies these conditions, therefore

\[6x^2 + 11x - 10 = (3x - 2)(2x + 5)\]

### 6.5 Grouping terms

**Type:** \(ac + bc + ad + bd = c(a+b) + d(a+b) = (a+b)(c+d)\)

**Example**

\[3xy - 2x + 12y - 8 = x(3y - 2) + 4(3y - 2) = (3y - 2)(x + 4)\]

**Exercise 6**

Factorise each of the following expressions:

1) \(3x^2 y^4 + 6x^3 y^3\)
2) \(20x^2 y^5 - 10x^5 y^2 + 5x^3 y^4\)
3) \(4 - x^8\)
4) \(9x^2 y^2 - 16z^2\)
5) \(4x^3 + 4x + 1\)
6) \(9x^4 - 24x^2 y^3 + 16y^6\)
7) \(x^2 - 6x - 7\)
8) \(x^2 - 7x + 12\)
9) \(2x^2 + 10x + 12\)
10) \(6x^2 - 13x + 6\)
A rational fraction is an expression which can be written as the quotient of two polynomials. 

\[ \frac{x^2 - 2x - 15}{x + 3} \]

The rules for manipulation of rational functions are the same as for fractions in arithmetic.

7.1 Simplifying a rational function

To simplify a given expression we factorise its numerator and denominator and then cancel common factors assuming they are not equal to zero.

**EXAMPLE**

\[ \frac{x^2 - 2x - 15}{x + 3} = \frac{(x + 3)(x - 5)}{(x + 3)} = x - 5 \quad \text{provided} \quad x \neq -3 \]

7.2 Addition and Subtraction

7.2.1) To add and subtract rational functions with a common denominator we proceed as we do in arithmetic and then simplify the resulting fraction.

**EXAMPLE**

\[ \frac{5x + 3}{x + 1} + \frac{3x - 2}{x + 1} - \frac{6x - 4}{x + 1} = \frac{(5x + 3) + (3x - 2) - (6x - 4)}{x + 1} \]

\[ = \frac{5x + 3 + 3x - 2 - 6x + 4}{x + 1} = \frac{2x + 5}{x + 1} \]

7.2.2) To add and subtract fractions having different
a) Factorise the denominators of the given fractions.

b) Write each of the fractions as equivalent fractions having a common denominator (The least common denominator of the given set of fractions is the least common multiple of the denominators of the fractions).

c) Proceed as in 7.2.1

**Example**

\[
\frac{3}{x^2 - y^2} - \frac{2x}{x^2 + 2xy + y^2} = \frac{3x}{(x+y)(x-y)} - \frac{2x}{(x+y)^2}
\]

\[
= \frac{3x(x+y) - 2x(x-y)}{(x+y)^2(x-y)} = \frac{3x^2 + 3xy - 2x^2 + 2xy}{(x+y)^2(x-y)}
\]

\[
= \frac{x^2 + 5xy}{(x+y)^2(x-y)}
\]

### 7.3 Multiplication

**Example**

\[
\frac{x^2 - 4x + 3}{x^2 - x - 2} \cdot \frac{x+1}{x-3} = \frac{(x-3)(x-1)}{(x+1)(x-2)} \cdot \frac{x+1}{x-3}
\]

\[
= \frac{(x-3)(x-1)(x+1)}{(x+1)(x-2)(x-3)} = \frac{x-1}{x-2}
\]

### 7.4 Division

The quotient of two rational fractions is obtained by inverting the divisor and then multiplying.
EXAMPLE

\[
\frac{x^3-4}{x+3} : \frac{x-2}{x+1} = \frac{(x+2)(x-2)}{(x+3)} \cdot \frac{(x+1)}{(x-2)} = \frac{(x+2)(x+1)}{(x+3)}
\]

Exercise 7

Perform the following operations:

1) \[
\frac{5(x+1)}{x-2} - \frac{2(x-3)}{x^2-4} + \frac{x}{x+2}
\]

2) \[
\frac{x-3}{x^2+6x+9} + \frac{x-1}{x^2-9}
\]

3) \[
\frac{6x-12}{4xy-4x} \cdot \frac{x^2-1}{x^2+2x+1}
\]

4) \[
\frac{x^2+3x}{4x^2-4} \cdot \frac{2x^2+2x}{x^2-9} \cdot \frac{x^2-4x+3}{x^2}
\]

5) \[
\frac{x^2+3x+2}{x^2+4x-21} : \frac{x^2+4x+4}{x^2-9}
\]
EXAMPLE

Solve for $x$ : \[ \frac{2x-9}{6} = \frac{3x+4}{4} \]

Solution

Firstly we multiply both sides of the equality by the L.C.D (Lowest Common Denominator) of 6 and 4, i.e. 12

\[ 2(2x-9) = 3(3x+4) \]

then

\[ 4x - 18 = 9x + 12 \quad \text{(removing brackets)} \]

\[ 4x - 9x = 18 + 12 \quad \text{(transposing terms)} \]

\[ -5x = 30 \quad \text{(simplifying)} \]

\[ x = \frac{30}{-5} \quad \text{(dividing by -5)} \]

\[ x = -6 \quad \text{(answer)} \]

Exercise 8

Solve for $x$ :

1) $3x - 2 = 7$
2) $4x - 3 = 5 - 2x$
3) $x + 3(x - 4) = 4$
4) $\frac{x}{3} = 5 - \frac{x}{2}$
5) $\frac{x-3}{4} = \frac{2x+4}{10}$
A linear equation in two unknowns $x$ and $y$ is of the form $a_1x + b_1y = c_1$, where $a, b, c$ are constants and $a$ and $b$ are not both zero. If we consider two such equations:

\begin{align*}
  a_1x + b_1y &= c_1 \\
  a_2x + b_2y &= c_2
\end{align*}

we say that we have two simultaneous linear equations in two unknowns, or a set of two linear equations in two unknowns. A pair of values for $x$ and $y$ which satisfies both equations is called a solution of the given equations.

We illustrate how to solve such a set by using the elimination method. This consists of multiplying one or both of the equations by a number which results in one of the unknowns occurring with the same coefficient but opposite signs in the two equations.

**EXAMPLE**

Solve the set of equations $\begin{cases}
4x + 2y = 5 \\
6x - 3y = -2
\end{cases}$

**Solution**

Multiplying the first equation by 3 and the second by 2 we obtain

\begin{align*}
12x + 6y &= 15 \\
10x - 6y &= -4
\end{align*}

\[22x = 11 \quad \text{(adding)}\]
Substituting $\frac{1}{2}$ for $x$ in the first equation we obtain:

$$2 + 2y = 5$$
$$2y = 5 - 2$$
$$2y = 3$$
$$y = \frac{3}{2}.$$

Thus $x = \frac{1}{2}$ and $y = \frac{3}{2}$.

**Exercise 9**

Solve the following set of equations:

(1) \[
\begin{align*}
2x - y &= 4 \\
x + 2y &= -3
\end{align*}
\]

(2) \[
\begin{align*}
3x - y &= -6 \\
2x + 3y &= 7
\end{align*}
\]

(3) \[
\begin{align*}
2x - 3y &= 9 \\
4x - y &= 8
\end{align*}
\]

(4) \[
\begin{align*}
4x - 2y &= 5 \\
5x + 3y &= -2
\end{align*}
\]
10. Solution of quadratic equations

10.1 The simplest kind of quadratic equation is one of the form \( x^2 = k \), where \( k \) is some positive number or zero, and its solution consists of the pair of roots (solutions) \( x = \pm \sqrt{k} \).

**Example**

Solve for \( x \) : \( 9x^2 - 16 = 0 \)

**Solution**

\[
9x^2 = 16 \quad \text{(transposing \(-16\))}
\]

\[
x^2 = \frac{16}{9} \quad \text{(dividing by 9)}
\]

\[
x = \pm \sqrt{\frac{16}{9}} = \pm \frac{4}{3} \quad \text{(extracting square roots)}
\]

**Exercise 10.1**

Solve for \( x \):

1) \( 4x^2 - 25 = 0 \)
2) \( 3x^2 - 7 = 0 \)
3) \( x^2 - 121 = 0 \)
4) \( \frac{x^2}{3} - 27 = 0 \)

10.2 The general solution of the quadratic equation \( ax^2 + bx + c = 0 \) is given by the formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

When \( b^2 - 4ac = 0 \) the roots are equal, when \( b^2 - 4ac < 0 \) the roots are complex.

**Example**

Solve for \( x \) by formula: \( 3x^2 - 5x - 2 = 0 \)
In this case \( a = 3 \), \( b = -5 \) and \( c = -2 \), therefore, we get by formula:

\[
\chi = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(-2)}}{2(3)} = \frac{5 \pm \sqrt{25 + 24}}{6}
\]

so that

\[
\chi = \frac{5 \pm \sqrt{49}}{6} = \frac{5 \pm 7}{6}
\]

then

\[
\chi = 2; -\frac{1}{3}
\]

**NOTE:** This equation could also be solved by factorization

\[(3 \chi + 1)(\chi - 2) = 0\]

\[
\therefore \chi = -\frac{1}{3} \text{ or } 2
\]

However, this method can only be used if the roots are simple numbers.

**Exercise 10.2**

Solve for \( \chi \) by formula:

1) \( 3\chi^2 - 7\chi + 4 = 0 \)
2) \( \chi^2 - 3\chi + 2 = 0 \)
3) \( \chi^2 + 3\chi - 2 = 0 \)
4) \( 4\chi^2 + 12\chi + 9 = 0 \)

10.3 The equation \( a\chi^2 + b\chi + c = 0 \) is easy to solve when \( c = 0 \), i.e. \( a\chi^2 + b\chi = 0 \) since \( \chi = 0 \) is one solution and the remaining solution may be obtained by dividing through by \( \chi \).

**Example**

Solve for \( \chi \): \( 3\chi^2 - 6 = 0 \)

**Solution**

This may be rewritten in factored terms as:
\[ 3x(x-2) = 0 \]

Since the product on the left-hand side of this equation is \(0\) when either factor equals \(0\), the condition of the equation is satisfied if we set:

\[ 3x = 0 \]

which means \(x = 0\),

and we set

\[ x - 2 = 0 \quad \text{(equating the second factor to 0)} \]

then \(x = 2\).

Therefore, the answers are \(x = 0; 2\).

**Exercise 10.3**

Solve for \(x\):

1) \(x^2 - 7x = 0\)
2) \(4x^2 + 5x = 0\)
3) \(3x^2 - 3x = 0\)
Answers to Exercises

Exercise 1:
1) \(3a^2 + 2b - 5c\)  3) \(6b\)
2) \(3a^2 - 8b + 9c\)  4) \(10a + b - 2\)

Exercise 2:
1) \(8a^2 b + 14a^2 b^4\)  6) \(9x^2 y^6 - 30xy^4z + 25y^2 z^2\)
2) \(-15x^3y^3 + 15x^2 y^3\)  7) \(x^2 y^3 + xy - 56\)
3) \(x^2 y^2 - x^2 + xy^3 - xy\)  8) \(35x^2 - 6xy - 8y^2\)
4) \(6x^2 y - 8xy - 3xy^2 + 4y^2\)  9) \(10e^{2x} + 18e^x - 4\)
5) \(16x^4 y^9 + 24x^2 y^5 + 9\)

Exercise 3.1
1) \(6 \times 25\)  2) \(-\frac{8}{27}\)  3) \(\frac{1}{25}\)  4) \(-\frac{1}{4}\)

Exercise 3.2
1) \(1.246\)  2) \(12.02\)  3) \(1.845\)

Exercise 3.3
1) \(x^4\)  2) \(x^9\)  3) \(x^{4n} y^{-1} z^{2n-2}\)  4) \(x^{-6}\)
5) \(\frac{3}{16}\)  6) \(7 \cdot 10^3\)  7) \(\frac{4}{15}\)  8) \(a^\frac{5}{6} b^{-\frac{7}{6}}\)

Exercise 3.4
1) \(1\)  2) \(xy\)  3) \(x^a\)  4) \(xe^4\)
5) \(x^3y^5\)  6) \(\frac{x}{x^n}\)
Exercise 4.1
1) 4  2) -a  3) $\frac{1}{4}$  4) $z$  5) $\frac{1}{2}$  6) $-\frac{1}{2}$  7) $-\frac{1}{2}$

Exercise 4.2
1) $3\sqrt{a}$  2) $6\sqrt{9}$  3) $3abc\sqrt{c}$  4) $4x^2y^2\sqrt{x}$  5) $98\sqrt{2ab}$  6) $x^6\sqrt{4}$  7) $5a^2\sqrt{3a}$

Exercise 4.3
1) $\frac{3\sqrt{36}}{a}$  2) $\frac{\sqrt{ab}}{b}$  3) $\frac{5bc\sqrt{9a^5b}}{3a}$  4) $\frac{2\sqrt{2x^3y}}{y}$  5) $\frac{\sqrt{7+a}}{3}$  6) $-2(z+\sqrt{3})$
7) $\sqrt{x} - 5$  8) $\sqrt{3}(\sqrt{6} - z)$

Exercise 5.1
1) $x^2 + 4xy + 2y^2$  5) $x^2 - 2z^2 - 4yz$
2) $3x^2 - x - 2xy - 2yz + 2y^2 - z^2$  6) $-2x^2 + x + 2xy + 2yz + y + z^2$
3) $3x^2 - 3y^2 - y$  7) $3x^2 - 2x - 2y^2 + y$
4) $2x - 2y - y^3$  8) $-x^2 + 7y^2 + 2y$

Exercise 5.2
1) $-x^5 + x^4 + x^3 - x^2 - x + 1$  2) $x^6 - 3x^5 + 3x^3 - 3x + 1$
3) $x^3 + x^2 - 11x + 4$  4) $x^5 + 4x^4 - x^3 - 4x^2 + x + 1$
5) Quotient: $-x^3 - x^2$
   Remainder: 1
6) Quotient: $x - 7$
   Remainder: $-27$
7) Quotient: $x^3 + 3x^2 + 10x + 27$
   Remainder: $x - 2$
8) Quotient: $-x^4 - x^3 - 3x^3 - 3x$
Exercise 6
1) \(3x^2y^3(y+2x)\)  
6) \((3x^2-4y^3)^2\)
2) \(5x^2y^3(4y^3-2x^3+xy)\)  
7) \((x+1)(x-7)\)
3) \((x+x^3)(x-x^3)\)  
8) \((x-4)(x-3)\)
4) \((3xy+4z)(3xy-4z)\)  
9) \(z(x-3)(x-2)\)
5) \((2x+1)^2\)  
10) \((2x-3)(3x-2)\)

Exercise 7
1) \(\frac{6x^2-11x+16}{x^2-4}\)  
2) \(\frac{2(x-1)}{(x+3)}\)  
3) \(\frac{3(x-2)(x-1)}{2(y-1)(x+1)}\)
4) \(\frac{1}{x}\)  
5) \(\frac{(x+1)(x-3)}{(x-7)(x+2)}\)

Exercise 8
1) \(3\)  
2) \(\frac{4}{3}\)  
3) \(4\)  
4) \(6\)  
5) \(23\)

Exercise 9
1) \(x=1, y=-2\)  
3) \(x=\frac{3}{2}, y=-2\)
2) \(x=-1, y=3\)  
4) \(x=\frac{1}{2}, y=-\frac{3}{2}\)

Exercise 10.1
1) \(\frac{4}{3}; 1\)  
2) \(2; 1\)  
3) \(-\frac{3\pm\sqrt{17}}{2}\)  
4) \(-\frac{3}{2}\) (a double root)

Exercise 10.2
1) \(\pm \frac{5}{2}\)  
2) \(\pm \sqrt{\frac{7}{3}}\)  
3) \(\pm 11\)  
4) \(\pm 9\)

Exercise 10.3
1) \(0; 7\)  
2) \(0; -\frac{5}{4}\)  
3) \(0; 1\)
APPENDIX V

PROGRAMMED TRIGONOMETRY

The programme presented in this appendix was designed in a programmed text with "through-the-book" format with one frame on each page and the answer corresponding to a frame on the next page. The student, after having read and responded a frame, turns the page in order to find the correct answer and instructions of what to do next. For easy inspection, the programme is presented here in "down-the-page" format.
PROGRAMMED TRIGONOMETRY

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This programmed booklet is designed to fill in the gaps in your understanding of trigonometry and to aid its revision.

A general feature of programmed texts is that the material being taught is broken down into small steps which are called frames. Each frame presents a certain amount of information and it will also require you to use what you have learnt by asking you to fill a blank, solve a problem, derive a formula, select the correct answer from among several alternative answers, or make some other active response before you go on to the next frame.

It is extremely important that you actually write the answer to each frame before proceeding further because it will allow you to decide whether you have understood a point or whether you require additional explanations and examples according to your grasp of the material.

Avoid the temptation to race ahead because you "think you could write" and answer correctly. Take as much time as you need with each frame and do not write your answer until you understand what is being asked. There is no prize for completing this booklet in record time.

This booklet consists of the following sections:

1. A list of objectives, which describes what you should be able to do upon completing the programme
2. A pre-test
3. Answers to the pre-test
4. Objective-Pretest-Frame Correspondence
5. Plan of the programme
6. Introduction
7. Further instructions
8. The programme
9. A post-test
10. Answers to the post-test
11. Supplementary problems
12. Answers to the supplementary problems
13. A list of formulae
14. Appendices (A, B, C, D)

Before starting the programme attempt the pre-test and check your answers with the ones immediately provided after the pre-test, and then go on to the section "Objective-Pretest-Frame Correspondence" to see which frames of the programme you should consult.

If you like to read further information about the development of this booklet, read Appendix D, but it is not necessary for dealing with the booklet.
1. To be able to define the trigonometric functions sine, cosine and tangent for an acute angle in a right angled triangle.

2. Given one trigonometric function (sine, cosine or tangent) for an angle $\theta$ in a right angled triangle, to be able to calculate the other two by using Pythagoras' Theorem and the definitions of the trigonometric functions.

3. To be able to define cotangent, secant and cosecant for an acute angle in a right angled triangle.

4. Without using any tables, to be able to give the values of the six trigonometric functions for 30°, 45° and 60°.

5. To be able to define the six trigonometric functions for any angle and to be able to state the signs of the trigonometric functions for angles (in standard position) in the different quadrants.

6. Without using any tables, to be able to give the six trigonometric function values for 0°, 90°, 180°, 270° and 360°.

7. To be able to use the tables of Natural Trigonometric Functions to find the trigonometric functions for an acute angle measured in degrees or radians.

8. To be able to use the tables of Natural Trigonometric Functions to find an acute angle whose trigonometric function value is given.

9. Given an angle measured in radians, to be able to express it as $\theta + 2\pi n$, where $0 \leq \theta < 2\pi$ and $n$ is an integer, i.e. $n = 0, \pm 1, \pm 2, \ldots$, and to express its trigonometric functions in terms of the trigonometric function of $\theta$.

10. Given an angle measured in degrees, to be able to express it as $\theta + n \times 360^\circ$, where $0 \leq \theta < 360^\circ$ and $n$ is an integer, i.e. $n = 0, \pm 1, \pm 2, \ldots$, and to express its trigonometric functions in terms of the trigonometric functions of $\theta$.

11. To be able to express the trigonometric functions of $-\theta$ in terms of the trigonometric functions of $\theta$.

12. To be able to find the sine, cosine and tangent of $(\alpha + \beta)$ and $(\alpha - \beta)$ by using the addition and subtraction formulae.

13. To be able to use the tables of Natural Trigonometric Functions to find the trigonometric functions for an angle greater than 90°.

14. To be able to draw the graphs of the trigonometric functions and describe their behaviours.

15. Given $\sin \theta$ (cos $\theta$) to be able to find $\cos \theta$ (sin $\theta$) by using the identity $\sin^2 \theta + \cos^2 \theta = 1$. 
17. To be able to express \( \sin A + \sin B, \sin A - \sin B, \cos A + \cos B \) and \( \cos A - \cos B \) as a product by using their respective formulae.

18. To be able to apply the sine and/or cosine rules to find the unknown side(s) and angle(s) of a triangle in each of the following cases:
   (a) Given one side and two angles.
   (b) Given two sides and the angle opposite one of them.
   (c) Given two sides and the included angle.
   (d) Given the three sides.

19. To be able to relate \( \sin \theta, \theta \) and \( \tan \theta \), when \( \theta \) is a small angle measured in radians.

---

**PRE-TEST**

1. According to the triangle in Fig. 1 \( \sin \theta = \frac{m}{p} \)

   Figure 1

   \[ \begin{array}{c}
   \text{Complete the following in the same way:} \\
   \cos \theta = - \\
   \tan \theta = -
   \end{array} \]

2. Given that \( \sin \theta = 1/3 \), where \( \theta \) is an acute angle in a right-angled triangle, calculate \( \cos \theta \) and \( \tan \theta \) by using Pythagoras' Theorem and the definitions of the trigonometric functions.

3. According to the triangle in Fig. 3 \( \cot \theta = \frac{4}{3} \)

   Figure 3

   \[ \begin{array}{c}
   \text{Complete the following in the same way:} \\
   \sec \theta = - \\
   \cosec \theta = -
   \end{array} \]
5. Complete the table below by writing + or − as in the example.

<table>
<thead>
<tr>
<th>θ in quadrant</th>
<th>sinθ</th>
<th>cosθ</th>
<th>tanθ</th>
<th>cotθ</th>
<th>secθ</th>
<th>cosecθ</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
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<td>−</td>
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<tr>
<td>III</td>
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<td>−</td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td></td>
<td>−</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Complete the following, as in the examples:

(a) sin 0° = 0  
(b) cot 180° = −∞  
(c) sin 270° = −1  
(d) cos 180° = −1  
(e) sec 0° = ±∞  
(f) tan 360° = 0  
(g) cot 90° = 0  
(h) cosec 360° = ±∞

7. Using four-figure tables of Natural Trigonometric Functions, find the values of:

(a) cos 16° 8'  
(b) sin \( \frac{2\pi}{7} \)

8. Using four-figure tables of Natural Trigonometric Functions, find the values of θ (0° < θ < 90°) in each of the following cases:

(a) Given that sinθ = 0.3507  
(b) Given that cosθ = 0.4355

9. Express \( \frac{11\pi}{3} \) as θ + 2πn where 0° < θ < 2π and n is an integer, i.e. n = 0, ±1, ±2,... and then express sin \( \frac{11\pi}{3} \) in terms of sinθ.

10. Express 840° as θ + n x 360° where 0° < θ < 360° and n is an integer, i.e. n = 0, ±1, ±2,... and then express tan 840° in terms of tanθ.

11. Complete the following, as in the example:

(a) sin (−734°) = −sin 734°  
(b) cos (−17°) =  
(c) cot (−α) =  
(d) cosec (−55°) =
13. Using four-figure tables of Natural Trigonometric Functions, find the values of:

(a) \( \cos 283° 5 \)  
(b) \( \sin 965° \)

14. Sketch the graph of \( \tan \theta \) for \( 0 < \theta < 180° \) and describe the behaviour of \( \tan \theta \) as \( \theta \) approaches 90° through values less than 90° and through values more than 90°.

15. Find \( \sin \theta \) given that \( \cos \theta = \frac{1}{2} \) and \( \theta \) is in the fourth quadrant.

16. Find \( \sec \theta \) given that \( \tan \theta = \frac{\sqrt{3}}{3} \) and \( \theta \) is in the third quadrant.

17. Complete the following:

\[
\sin 50° + \sin 40° = 2 \sin 45° \times \ldots \ldots .
\]

18. Find the unknown side (to one decimal place) and angles of the triangle in the figure below, given that: \( a = 4, \ c = 5 \) and \( B = 45° \)

19. Complete the following by filling in with > or <:

\[
\sin \theta \ldots \ldots \theta \ldots \ldots \tan \theta
\]

(\( \theta \) is a small angle measured in radians)
1. \( \cos \theta = \frac{n}{p} ; \tan \theta = \frac{m}{n} \)

2. \( \cos \theta = \frac{2\sqrt{2}}{3} ; \tan \theta = \frac{1}{2\sqrt{2}} \)

3. \( \sec \theta = \frac{5}{4} ; \cosec \theta = \frac{5}{3} \)

4. \[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\theta & \sin \theta & \cos \theta & \tan \theta & \cosec \theta & \sec \theta & \cot \theta \\
\hline
30^\circ & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & 2 & \frac{2}{\sqrt{3}} & \sqrt{3} \\
45^\circ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \sqrt{2} & \frac{\sqrt{2}}{2} & 1 \\
60^\circ & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{3} & 2 & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\hline
\end{array}
\]

5. \[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{sin quadrant} & \sin \theta & \cos \theta & \tan \theta & \cot \theta & \sec \theta & \cosec \theta \\
\hline
I & + & + & + & + & + & + \\
II & + & - & - & - & - & + \\
III & - & - & + & + & - & - \\
IV & - & + & + & - & + & - \\
\hline
\end{array}
\]

6. c) \(-1\)  
   d) \(-1\)  
   e) \(1\)  
   f) \(0\)  
   g) \(0\)  
   h) \(\infty\)

7. a) \(0.9607\)  
   b) \(0.7817\)

8. a) \(20^\circ 32'\)  
   b) \(64^\circ 11'\)

9. \(\sin \frac{14}{3} \pi = \sin \frac{2}{3} \pi\)

10. \(\tan 840^\circ = \tan 120^\circ\)
11. \( \text{b) } \cos 17^\circ \quad \text{c) } -\cot \infty \quad \text{d) } -\csc 55^\circ \)

12. \( \text{a) } \frac{1 + \sqrt{3}}{2\sqrt{2}} \quad \text{b) } 2 \sin \theta \cos \theta \quad \text{c) } \cos \theta \quad \text{d) } - \cos \theta \quad \text{e) } - \sin \theta \)

13. \( \text{a) } 0.2264 \quad \text{b) } -0.9063 \)

14. The tangent of an angle can be made larger than any chosen positive number by taking the angle close enough to, yet less than 90°.

The tangent of an angle can be made less than any chosen negative number by letting the angle exceed 90° by a sufficiently small amount.
15. \( \sin \theta = -\frac{\sqrt{15}}{4} \)

16. \( \sec \theta = -\frac{5}{4} \)

17. \( \cos 5^\circ \)

18. \( b = 3.6 \); \( A = 51^\circ47' \); \( C = 83^\circ13' \)

19. < ; <

Now go to the section "Objective-Pretest-Frame correspondence" to see which frames of the programme you should consult.

<table>
<thead>
<tr>
<th>OBJECTIVES</th>
<th>PRE-TEST QUESTION</th>
<th>FRAMES</th>
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</thead>
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</tr>
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<td>2</td>
<td>8 - 13</td>
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<tr>
<td>3</td>
<td>3</td>
<td>14 - 17</td>
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<td>4</td>
<td>4</td>
<td>18 - 22</td>
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<tr>
<td>5</td>
<td>5</td>
<td>23 - 28</td>
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<tr>
<td></td>
<td>7b</td>
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<td>8a</td>
<td>47 - 55</td>
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<tr>
<td></td>
<td>8b</td>
<td>56 - 62</td>
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<td>OBJECTIVES</td>
<td>PRE-TEST QUESTION</td>
<td>FRAMES</td>
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<td>12</td>
<td>12a</td>
<td>70 - 73</td>
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<tr>
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<td>12b</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td>12c</td>
<td>75 - 76</td>
</tr>
<tr>
<td></td>
<td>12d</td>
<td>77 - 78</td>
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<td>12e</td>
<td>79 - 81</td>
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<td>13</td>
<td>13a-13b</td>
<td>82 - 89</td>
</tr>
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<td>99 - 102</td>
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<td>103 - 106</td>
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<td>107 - 112</td>
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<td>113 - 129</td>
</tr>
<tr>
<td>19</td>
<td>19</td>
<td>130 - 137</td>
</tr>
</tbody>
</table>

NOTE: Before consulting any of the frames of the programme read the sections "Introduction" and "Further instructions", which are just before frame 1.

### PLAN OF THE PROGRAMME

| Sine, cosine and tangent of an acute angle in a right angled triangle | 1 - 13 |
| Cotangent, secant and cosecant of an acute angle in a right angled triangle | 14 - 17 |
| Trigonometric function values for 30°, 45° and 60° | 18 - 22 |
| Generalization of the definitions of the trigonometric functions to any angle | 23 - 28 |
| Trigonometric function values for 0°, 90°, 180°, 270° and 360° | 29 |
| Tables of Natural Trigonometric Functions for acute angles | 30 - 62 |
| Trigonometric Functions for angles greater than 2π (360°) | 63 - 66 |
| Trigonometric Functions for negative angles | 67 - 69 |
| Trigonometric Formulae | 70 - 81 |
INTRODUCTION

At this stage I would like to introduce to you some terms and symbols which are concerned with the idea of an angle and which will be used throughout the booklet.

Imagine the half line OA (figure 1) which is rotated about a point O in the direction indicated by the arrow.

Let it take up the position indicated by OB. In rotating from OA to OB an angle AOB is described, so we have an angle formed by the rotation of a half line about a fixed point.
OA is called the initial side, OB the terminal side, and O the vertex of the angle AOB.
The"angle AOB" is denoted by $\angle AOB$ or $\hat{A}OB$.

It should be noted that the first letter, in this case A, always indicates a point on the initial side, the middle letter the vertex of the angle and the last letter a point on the terminal side of the angle.

An angle may also be denoted by the vertex letter if there is only one angle having this vertex, as O in the figure I.

So that we can avoid confusion and more easily describe the direction of the rotation of an angle in the plane, we refer to a clock and say that anticlockwise rotation is positive, and clockwise rotation is negative.

In figure II, $\angle BOA$ and $\angle RTV$ are positive, whereas $\angle MNP$ and $\angle ZYX$ are negative.
In figure III, \( \angle DEF \) is a positive angle, its initial side is ED, its terminal side is EF and its vertex is E.

FURTHER INSTRUCTIONS

To go through the programme you must proceed like this:

Read the frame, then write down the answer to the question asked at the end of it.

Remember that it is extremely important that you actually answer each frame before proceeding further.

After you have written your own answer, check it with the correct one which will be found on the next page with instructions of what to do after you have checked it.

The numbers of the frames are ringed, for example \( \Box \) means "frame 5"
The symbol \( \xrightarrow{5} \) means "go to frame 5", and the symbol \( \xrightarrow{} \) at the end of a page means "go on to the next page".

The structure of the programme is shown in the figure below.

![Diagram showing programme structure]

After you have completed the frames which you have been asked to consult, go to page xxvi.
The right angled triangle is a figure of special importance because it met in many of our ordinary everyday experiences and has numerous applications in Science and Engineering. Hence the right angled triangle is given special attention in trigonometry courses.

We shall begin our study of the right angled triangle by learning the definitions of the trigonometric functions of an acute angle.

Mark all the right angles on the figure below with the conventional sign.

FIGURE 1

Remember that the answers to all questions are given on the next page. Always check your answers before continuing.
Let us consider the right angled triangle in figure 2, and let \( \theta \) be the particular angle shown in it.

![Figure 2](image)

The trigonometric functions of \( \theta \) which we are going to define are:

<table>
<thead>
<tr>
<th>NAME OF THE FUNCTION</th>
<th>ABBREVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine of ( \theta )</td>
<td>( \sin \theta )</td>
</tr>
<tr>
<td>cosine of ( \theta )</td>
<td>( \cos \theta )</td>
</tr>
<tr>
<td>tangent of ( \theta )</td>
<td>( \tan \theta )</td>
</tr>
</tbody>
</table>

They are defined by the following ratios:

\[
\sin \theta = \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}}
\]

\[
\cos \theta = \frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}
\]

\[
\tan \theta = \frac{\text{length of the opposite side}}{\text{length of the adjacent side}}
\]
\[
\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}
\]
\[
\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}
\]
\[
\tan \theta = \frac{\text{opposite side}}{\text{hypotenuse}}
\]

According to the notation in figure 2, we have that:
\[
\sin \theta = \frac{a}{c}
\]

Complete the following according to the figure 2:
\[
\cos \theta = \ldots \ldots \quad \tan \theta = \ldots \ldots
\]

Answer \( \square \)
\[
\cos \theta = \frac{b}{c} \quad \tan \theta = \frac{a}{b}
\]

All right \( \Rightarrow \) \( \square \)
Otherwise \( \Rightarrow \) \( \square \)
Perhaps you made a mistake in identifying the opposite side, the adjacent side or the hypotenuse in figure 2, so let us identify them:

In figure 3(a) we have with respect to \( \theta \) that:
- \( a \) is the opposite side
- \( b \) is the adjacent side
- \( c \) is the hypotenuse

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}
\]

Now, try the definition of sine, cosine and tangent with the triangle in figure 3 (b)

\[
\sin \alpha = ........... \quad \cos \alpha = ........... \quad \tan \alpha = ...........
\]

Answer \( 3 \)

\[
\sin \alpha = \frac{y}{z} \quad \cos \alpha = \frac{x}{z} \quad \tan \alpha = \frac{y}{x}
\]

All right \( \rightarrow 1 \)
Otherwise \( \rightarrow 2 \)
We can remember the definitions of the trigonometric functions by using the mnemonic rules SOH - CAH - TOA, where:

SOH: Sine is the Opposite divided by the Hypotenuse
CAH: Cosine is the Adjacent divided by the Hypotenuse
TOA: Tangent is the Opposite divided by the Adjacent

Having learnt the definitions of the trigonometric functions of an acute angle in a right angled triangle, let us apply them to particular right angled triangles.

From the triangle in figure 5 we have that sin is:
(ring the appropriate answer)

a) $\frac{2}{3}$  
b) $\frac{2}{\sqrt{13}}$  
c) $\frac{3}{\sqrt{13}}$  
d) don't know
Your answer was c) \( \frac{3}{\sqrt{13}} \) but notice that 3 is the adjacent side and \( \sqrt{13} \) is the hypotenuse (figure 6), hence, you have:

\[
\frac{3}{\sqrt{13}} = \frac{\text{adjacent side}}{\text{Hypotenuse}} = \cos \theta \quad \text{not} \sin \theta \quad \text{as required}
\]

(remember the mnemonic rules SOH-CAH-TOA)

FIGURE 6
Try now with the tangent. In the triangle of figure 6 we have that \( \tan \theta \) is:

(ring the appropriate answer)

a) \( \frac{2}{3} \)  

b) \( \frac{2}{\sqrt{13}} \)  

c) \( \frac{3}{\sqrt{13}} \)  

d) don't know

Answer  \( \textcircled{6} \)

If your answer is

- a) \( \frac{2}{3} \) \( \rightarrow \) \( 6 \)
- b) \( \frac{2}{\sqrt{13}} \) \( \rightarrow \) \( 2 \)
- c) \( \frac{3}{\sqrt{13}} \) \( \rightarrow \) \( 6 \text{ again} \)
- d) don't know \( \rightarrow \) \( 2 \)
Your answer was a) 2/3 but notice that 2 is the opposite side and 3 is the adjacent side (figure 7), hence, you have:

opposite side = tan θ not sin θ as required.
adjacent side

(remember the mnemonic rules SOH-CAH-TOA)

Try now with the cosine. In figure 7 we have that cos θ is:

(ring the appropriate answer)

a) 2/3 b) 2/√13 c) 3/√13 d) don't know

Answer 7

If your answer is  a) 2/3 → 7 again
b) 2/√13 → 2
c) 3/√13 → 8
d) don't know → 2
Correct.

Now, let us solve a more general problem.

Find $\cos \theta$ from the triangle in figure 8

Hint: In the first instance find the value of $X$,

Answer

$x = h$

$\cos \theta = \frac{4}{5}$

All right $\rightarrow$ (10)

Otherwise $\rightarrow$ (9)
In the first instance we have to find the value of $b$ by using Pythagoras' Theorem:

$$x^2 + 3^2 = 5^2$$
$$x^2 + 9 = 25$$
$$x^2 = 25 - 9 = 16$$
$$x = \sqrt{16} = 4$$

then by the definition of cosine we have that:

$$\cos \theta = \frac{x}{5} = \frac{4}{5}$$

Now try this problem:
Find $\tan \alpha$ from the triangle in figure 10
Again in the first instance we have to find the value of $X$ by using Pythagoras' Theorem.

\[ X^2 + 2^2 = 5^2 \]
\[ X^2 + 4 = 25 \]
\[ X^2 = 25 - 4 = 21 \]
\[ X = \sqrt{21} \]

Then by the definition of tangent we have that: \[ \tan \alpha = \frac{X}{2} = \frac{\sqrt{21}}{2} \]
Now, let us try another problem, a little bit different.

In the triangle in figure 12(a), \( \sin \theta = 1/2 \), let us try to find the value of \( \cos \theta \).

![Triangle diagram](image)

We know that \( \sin \theta = 1/2 \) and \( \sin \theta = \frac{a}{c} \) (figure 12(a)), so for simplicity we choose \( a = 1 \) and \( c = 2 \) in a right angled triangle, as shown in figure 12(b). Now we can proceed as we did in the previous problem,

\[
\begin{align*}
&b^2 + 1^2 = 2^2 \\
b^2 + 1 &= 4 \\
b^2 &= 3 \\
b &= \sqrt{3}
\end{align*}
\]

then by the definition of cosine we have that

\[
\cos \theta = \frac{b}{2} = \frac{\sqrt{3}}{2}
\]

Cont'..... 12

NOTE: We could also choose \( a = 2, c = 4 \) or \( a = 5, c = 10 \), etc., but this would lead to the same answer.

In fact, suppose we had chosen \( a = 5 \) and \( c = 10 \), then \( \sin \theta = \frac{a}{c} = \frac{5}{10} = \frac{1}{2} \)
the same would have happened if we had chosen \( a = 2, c = 4 \) or \( a = 3, c = 6 \), etc.

In the triangle in figure 12(c) \( \tan \alpha = 3/4 \), find the value of \( \sin \theta \).

![Triangle diagram](image)
We know that \( \tan \alpha = \frac{3}{4} \) and \( \tan \alpha = \frac{n}{p} \) (figure 13(a)), so we can choose \( n = 3 \) and \( p = 4 \) (figure 13(b)).

\[
m^2 = p^2 + n^2
\]
\[
m^2 = 4^2 + 3^2 = 16 + 9 = 25
\]
\[
m = \sqrt{25} = 5
\]

Then by the definition of sine we have that:

\[
\sin \alpha = \frac{3}{m} = \frac{3}{5}
\]
We can now define three other trigonometric functions:

<table>
<thead>
<tr>
<th>NAME OF THE FUNCTION</th>
<th>ABBREVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>secant of $\theta$</td>
<td>sec $\theta$</td>
</tr>
<tr>
<td>cosecant of $\theta$</td>
<td>cosec $\theta$</td>
</tr>
<tr>
<td>cotangent of $\theta$</td>
<td>cot $\theta$</td>
</tr>
</tbody>
</table>

They are defined by the following:

$$\sec \theta = \frac{1}{\cos \theta}$$
$$\csc \theta = \frac{1}{\sin \theta}$$
$$\cot \theta = \frac{1}{\tan \theta}$$

Having defined these new trigonometric functions, we obtain from the triangle in figure 14 that:

$$\sec \theta = \frac{1}{\cos \theta} = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

Complete the following:

$$\csc \theta = \frac{1}{\sin \theta} = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$
$$\cot \theta = \frac{1}{\tan \theta} = \frac{b}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

Answer

$$\csc \theta = \frac{1}{\sin \theta} = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$
$$\cot \theta = \frac{1}{\tan \theta} = \frac{b}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

If all right $\rightarrow$ (15)
Otherwise $\rightarrow$ (14) again
Having learnt the definitions of these new trigonometric functions, let us apply them for acute angles in particular right angled triangles. From the triangle of figure 15, we obtain that \( \cot \theta \) is:

(ring the appropriate answer)

a) \( \frac{4}{3} \)
b) \( \frac{5}{3} \)
c) \( \frac{5}{4} \)
d) don't know

Answer: \( 15 \)

If your answer is a) \( \frac{4}{3} \) \( \rightarrow \) \( 18 \)
b) \( \frac{5}{3} \) \( \rightarrow \) \( 17 \)
c) \( \frac{5}{4} \) \( \rightarrow \) \( 16 \)
d) don't know \( \rightarrow \) \( 14 \)
Your answer was \( \frac{5}{4} \) but notice that 5 is the hypotenuse and 4 is the adjacent side (figure 16), hence you have:

\[
\frac{5}{4} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \sec \theta , \text{ not cot} \theta \text{ as required}
\]

Let us now try the cosecant. From figure 16 we have that cosec \( \theta \) is:

(ring the appropriate answer)

a) \( \frac{4}{3} \)  
b) \( \frac{5}{3} \)  
c) \( \frac{5}{4} \)  
d) don't know

Answer 16

If your answer is

a) \( \frac{4}{3} \)  \( \rightarrow 14 \)
b) \( \frac{5}{3} \)  \( \rightarrow 18 \)
c) \( \frac{5}{4} \)  \( \rightarrow 16 \text{ again} \)
d) don't know  \( \rightarrow 14 \)
Your answer was 5/3 but notice that 5 is the hypotenuse and 3 is the opposite side (figure 17), hence you have:

\[
\frac{5}{3} = \frac{\text{hypotenuse}}{\text{opposite side}} = \text{cosec} \theta \text{, not cot} \theta \text{ as required}
\]

Let us now try the secant. From figure 17 we have that sec \theta is:

(\text{ring the appropriate answer})

a) \ \frac{4}{3} \quad b) \ \frac{5}{3} \quad c) \ \frac{5}{4} \quad d) \ \text{don't know}

Answer \ \boxed{17}

If your answer is

a) \ \frac{4}{3} \quad \rightarrow \quad 15 \\

b) \ \frac{5}{3} \quad \rightarrow \quad 17 \text{ again} \\

c) \ \frac{5}{4} \quad \rightarrow \quad 16 \\

d) \ \text{don't know} \quad \rightarrow \quad 15
Correct.
There are certain special angles whose trigonometric function values can be
determined geometrically. Among these are the angles $30^\circ$, $45^\circ$ and $60^\circ$.
The function values for $45^\circ$ may be read directly from a right angled triangle
whose acute angles are each equal to $45^\circ$.
For convenience, we let the length of the equal sides be 1. (figure 18)

Then by using Pythagoras' Theorem:

\[ c^2 = 1^2 + 1^2 = 1 + 1 = 2 \]
\[ c = \sqrt{2} \]

Then \( \sin 45^\circ = \frac{1}{c} = \frac{1}{\sqrt{2}} \)

Complete: \( \cos 45^\circ = \frac{\sqrt{2}}{2} = \frac{1}{2} \) \( \tan 45^\circ = \frac{1}{1} = 1 \)

Answer

\[ \cos 45^\circ = \frac{1}{c} = \frac{1}{\sqrt{2}} \]

\[ \tan 45^\circ = \frac{1}{1} = 1 \]
The same trigonometric function values for $45^\circ$ would have been obtained if we had chosen 2, 3, 4 etc. as the length of the equal sides of the triangle.

In fact, if we had chosen 3 for example, we would have obtained (see figure 19)

\[
c^2 = 3^2 + 3^2 = 9 + 9 = 18
\]
\[
c = \sqrt{18} = 3\sqrt{2}
\]

then $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

\[
\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}
\]

\[
\tan 45^\circ = \frac{3}{3} = 1
\]

NO RESPONSE REQUIRED

---

To obtain the trigonometric function values for $30^\circ$ and $60^\circ$, we use one of the right angled triangles formed by bisecting an angle (dividing into two equal angles) of an equilateral triangle (triangle which has its three sides of equal length).

Let us consider the triangle in figure 20(a) and suppose, for convenience, that the length of its sides are 2.

If we bisect one of the angles of the triangle we will form two right angled triangles with angles of $30^\circ$ and $60^\circ$ (figure 20(b)).

So we have, in figure 20(b) that $b = 2$ and $c = 1$ because $M$ is the midpoint of the side $AB$.

Then by using Pythagoras' Theorem we obtain:

\[
a^2 = b^2 - c^2 = 2^2 - 1^2 = 4 - 1 = 3 \Rightarrow a = \sqrt{3}
\]

then we have that:

\[
\sin 30^\circ = \frac{c}{b} = \frac{1}{2}
\]

\[
\cos 30^\circ = \frac{a}{b} = \frac{\sqrt{3}}{2}
\]

\[
\tan 30^\circ = \frac{c}{a} = \frac{1}{\sqrt{3}}
\]
Complete the following:

\[
\begin{align*}
\sin 60^\circ &= \quad = \\
\cos 60^\circ &= \quad = \\
\tan 60^\circ &= \quad = \\
\end{align*}
\]

**Diagram:**

- **Figure 20(a):**
  - Triangle with angles labeled 60°, 60°, and 60°.
  - Side labeled 'a' opposite 60°.

- **Figure 20(b):**
  - Triangle with angle labeled 60° and side labeled 'b'.
  - M is the midpoint of AB.

**Answer:**

\[
\begin{align*}
\sin 60^\circ &= \frac{a}{b} = \frac{\sqrt{3}}{2} \\
\cos 60^\circ &= \frac{c}{b} = \frac{1}{2} \\
\tan 60^\circ &= \frac{a}{b} = \frac{\sqrt{3}}{1} = \sqrt{3}
\end{align*}
\]

**NOTE:** The same values would have been obtained if we had chosen any other number for the length of the sides of the triangle.
Complete the table below:

<table>
<thead>
<tr>
<th>θ</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td></td>
<td>1/√2</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td></td>
<td></td>
<td>√3</td>
</tr>
</tbody>
</table>

Answer

<table>
<thead>
<tr>
<th>θ</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>1/2</td>
<td>√3/2</td>
<td>1/√3</td>
</tr>
<tr>
<td>45°</td>
<td>1/√2</td>
<td>1/√2</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
</tr>
</tbody>
</table>
Now, complete the table below:

<table>
<thead>
<tr>
<th>θ</th>
<th>cot θ</th>
<th>sec θ</th>
<th>cosec θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>( \sqrt{3} )</td>
<td>2/( \sqrt{3} )</td>
<td>2</td>
</tr>
<tr>
<td>45°</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>60°</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>2</td>
<td>2/( \sqrt{3} )</td>
</tr>
</tbody>
</table>

Answer: 22
We are now ready to generalize the definitions of trigonometric functions for any angle.

Let us consider a Rectangular Cartesian Coordinate System. First let be an angle in standard position with its terminal side in the first quadrant. Take any point $P$, other than the origin, on the terminal side of $\theta$. Let us call the coordinates of $P$ $(x, y)$. Then in figure 23(a) $OM = x$ and $MP = y$; $x$ is called the abscissa and $y$ the ordinate of $P$.

We can see in figure 23(a) that the triangle $OMP$ is a right angled triangle, so according to the definitions of the trigonometric functions in a right angled triangle, we have that:

\[
\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r} = \frac{\text{ordinate of } P}{\text{length of } OP}
\]

\[
\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} = \frac{\text{abscissa of } P}{\text{length of } OP}
\]

\[
\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x} = \frac{\text{ordinate of } P}{\text{abscissa of } P}
\]

where length of $OP = r = x^2 + y^2$

**NOTE:** These values depend on the angle $\theta$ and not on the position of the point $P$, on the terminal side. To show that this is true, let $P'$ be another point on the terminal side of $\theta$ with $x', y'$ the coordinates and $r'$ the length of $OP'$ (figure 23(b)).
Then the right angled triangles $\triangle CMP$ and $\triangle OMP'$ are similar (i.e. they are identical in shape but not in size) and their corresponding sides are proportional.

We have the equal ratios:

\[
\frac{y'}{r'} = \frac{y}{r} \quad \text{which we showed equals } \sin \theta
\]

\[
\frac{x'}{r'} = \frac{x}{r} \quad \text{which we showed equals } \cos \theta
\]

\[
\frac{y'}{x'} = \frac{y}{x} \quad \text{which we showed equals } \tan \theta
\]

This shows that the same values of the trigonometric functions are obtained by choosing $P$ or $P'$.

The definitions of the trigonometric functions given in frame 23 for an acute angle can be used as the definitions of the trigonometric functions for an angle of any magnitude, no matter in which quadrant it lies. For instance, if the angle is in the second or third quadrant (figures 24(a), 24(b)), we proceed as before, to take a point $P(x, y)$ on the terminal side of the angle and then define the sine of $\theta$ as

\[
\sin \theta = \frac{y}{r} = \frac{\text{ordinate of } P}{\text{length of } OP} \quad \text{where } r = x^2 + y^2
\]

Note that $y$ (and also $x$) can be both positive and negative but that $r$ is always taken positive.
Complete the definitions of cosine and tangent according to figures 24(a) and 24(b).

\[ \cos \theta = \frac{X}{r} = \text{abscissa of } P \quad \text{length of } OP \]

\[ \tan \theta = \frac{Y}{X} = \frac{\text{ordinate of } P}{\text{abscissa of } P} \]

If all right \(\rightarrow\) 26

Otherwise \(\rightarrow\) 25
Perhaps you did not realize that we said that the definitions of the trigonometric functions are always the same for whatever angle in whatever quadrant. So in our case (figures 25(a), 25(b)) we have that:

\[ \cos \theta = \frac{X}{r} = \text{abscissa of } P \]
\[ \tan \theta = \frac{Y}{X} = \frac{\text{ordinate of } P}{\text{abscissa of } P} \]

Complete the following according to figure 25(c):
\[ \sin \theta = \frac{Y}{r} = \frac{-2}{\sqrt{13}} \]
\[ \cos \theta = \text{_______} = \text{_______} \]
\[ \tan \theta = \text{_______} = \text{_______} \]

Answer (25)

\[ \cos \theta = \frac{X}{r} = \frac{3}{\sqrt{13}} \]
\[ \tan \theta = -\frac{Y}{X} = \frac{-2}{3} \]

If all right \(\rightarrow\) (26)
Otherwise \(\rightarrow\) (24)
As we have seen, the values of the trigonometric functions of an angle are ratios of numbers which have signs, hence the values of the trigonometric functions can be positive or negative.

Since \( r \) is always positive (i.e. \( r = \sqrt{x^2 + y^2} \)) the signs of the trigonometric functions depend on the signs of the coordinates \( x \) and \( y \) of any point on the terminal side of \( \theta \).

Hence the signs of the trigonometric functions will depend on the quadrant in which the angle is.

For instance, for an angle in the second quadrant (figure 26) we have, for any point \( P(x, y) \) on the terminal side, that \( x < 0 \) and \( y > 0 \), so we have that

\[
\sin \theta = \frac{y}{r} > 0, \quad \cos \theta = \frac{x}{r} < 0, \quad \tan \theta = \frac{y}{x} < 0
\]

We can put all these results in a table:

<table>
<thead>
<tr>
<th>( \theta ) in quadrant</th>
<th>( x )</th>
<th>( y )</th>
<th>( r )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>III</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete the table above by writing + or - as in the example.
Notice that for an angle in the third quadrant, for example, for any point \( P(x, y) \) on the terminal side, \( x < 0 \) and \( y < 0 \), so we have that:

\[
\sin \theta = \frac{y}{r} < 0, \quad \cos \theta = \frac{x}{r} < 0, \quad \tan \theta = \frac{y}{x} < 0
\]

Now, go to frame 26 and try to fill in the given table.
The definitions of the cotangent, cosecant and secant for an angle $\theta$ of any magnitude are taken to be the same as in frame 18, i.e.

$$\cot \theta = \frac{1}{\tan \theta} \quad \cosec \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta}$$

Thus, for example, if $\theta$ is in the third quadrant, $\tan \theta$ is positive and hence $\cot \theta \quad (= \frac{1}{\tan \theta})$ is also positive; $\cos \theta$ is negative, hence $\sec \theta (= \frac{1}{\cos \theta})$ is also negative $\sin \theta$ is negative, hence $\cosec \theta (= \frac{1}{\sin \theta})$ is also negative.

Complete the table below as in the example.

<table>
<thead>
<tr>
<th>$\theta$ in quadrant</th>
<th>$\cot \theta$</th>
<th>$\sec \theta$</th>
<th>$\cosec \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>II</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>III</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>IV</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Answer 28

<table>
<thead>
<tr>
<th>$\theta$ in quadrant</th>
<th>$\cot \theta$</th>
<th>$\sec \theta$</th>
<th>$\cosec \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>II</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>III</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>IV</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

If all right $\rightarrow$ 29
Otherwise $\rightarrow$ 28 again
We wish now to apply the definitions of the trigonometric functions to angles whose terminal sides are along a coordinate axis, particularly $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$ and $360^\circ$.

In finding the values of the trigonometric functions for these angles we are confronted with a new situation—namely, one of the coordinates of the points on the terminal side is equal to zero. If the terminal side is along the $x$-axis, $y = 0$; if the terminal side is along the $y$-axis, $x = 0$. This means, graphically, that if $P(x, y)$ is a point on the terminal side of the angle, the quantities $x$, $y$ and $r$ do not yield a right angled triangle, but the relation $r = \sqrt{x^2 + y^2}$ still holds.

We shall find the values of the trigonometric functions for $0^\circ$ to illustrate the method for determining the trigonometric functions for any angle whose terminal side is along a coordinate axis.

The terminal side of $0^\circ$ is along the positive $x$-axis (figure 29a). Let $P(x, 0)$ be a point on the terminal side of $0^\circ$, then $r = \sqrt{x^2 + 0^2} = \sqrt{x^2} = x$.

Applying the definitions of the trigonometric functions of an angle we obtain:

$$
\sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0 \\
\cos 0^\circ = \frac{x}{r} = \frac{x}{x} = 1 \\
\tan 0^\circ = \frac{y}{x} = \frac{0}{x} = 0 \\
\sec 0^\circ = \frac{r}{x} = \frac{x}{x} = 1
$$

Notice that we have omitted $\cot 0^\circ$ and $\csc 0^\circ$. If we tried to obtain either of these functions in the usual way, zero would appear in the denominator. Division by zero is not defined and hence $0^\circ$ has no cotangent value and no cosecant value. Let us see what actually happens.
Let us consider a Rectangular Cartesian Coordinate system. First let \( \theta \) be a small positive angle in standard position with its terminal side in the first quadrant. Take any point \( P(x, y) \) other than the origin, on the terminal side (figure 29(b)).

![Figure 29(b)](image)

Let \( r = \text{length } OP \). Now \( x \) is slightly less than \( r \) and \( y \) is positive and very small; then \( \cot \theta = \frac{x}{y} \) and \( \cosec \theta = \frac{r}{y} \) are positive and very large.

Next let \( \theta \) decrease toward \( 0^\circ \) (i.e. \( OP \) turns toward \( x \)-axis) with length \( OP = r \). Now \( x \) increases but is always smaller than \( r \) while \( y \) decreases but remains greater than 0. Then \( \cot \theta \) and \( \cosec \theta \) become larger and larger (to see this, take \( r = 1 \) and calculate \( \cosec \theta \) when \( y = 0.1, 0.01, 0.001 \), etc.). Frequently the symbol \( \infty \) (read "infinity", which is not a number) is used to express this, i.e. we write

\[
\cot 0^\circ = \infty \quad \text{and} \quad \cosec 0^\circ = \infty
\]

Complete the table below as in the examples:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
<th>( \sec \theta )</th>
<th>( \cosec \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( 90^\circ )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 180^\circ )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 270^\circ )</td>
<td>-1</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>-1</td>
</tr>
<tr>
<td>( 360^\circ )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The determination of numerical values of trigonometry functions is required in many problems of trigonometry. With few exceptions, the values of the functions cannot be given exactly. Approximate values for acute angles are given in Tables called Tables of Natural Trigonometric Functions to distinguish them from the Tables of Logarithms of the functions.

Some tables give the values of the six functions, others are restricted to the functions sine, cosine, tangent and cotangent; some give the values to four decimal places while others give values to four significant figures (four-figure tables). We shall use the four-figure tables here (see figure 30).
Some examples will illustrate how the tables for the Sine and Cosine functions are used (the tables of the other trigonometric functions are used in a similar manner). To work through the next frames you need a four-figure table. In this table, down the left hand side you will see angles from $0^\circ$ to $89^\circ$ ($0^\circ$ to $4^\circ$ in figure 30).

### Figure 30

<table>
<thead>
<tr>
<th>deg</th>
<th>0'</th>
<th>5'</th>
<th>10'</th>
<th>15'</th>
<th>20'</th>
<th>25'</th>
<th>30'</th>
<th>35'</th>
<th>40'</th>
<th>45'</th>
<th>50'</th>
<th>55'</th>
<th>1'</th>
<th>2'</th>
<th>3'</th>
<th>4'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0015</td>
<td>0.0029</td>
<td>0.0011</td>
<td>0.0058</td>
<td>0.0073</td>
<td>0.0087</td>
<td>0.0102</td>
<td>0.0116</td>
<td>0.0131</td>
<td>0.0145</td>
<td>0.0160</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>0.0175</td>
<td>0.0189</td>
<td>0.0201</td>
<td>0.0216</td>
<td>0.0233</td>
<td>0.0247</td>
<td>0.0262</td>
<td>0.0276</td>
<td>0.0291</td>
<td>0.0305</td>
<td>0.0320</td>
<td>0.0334</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0.0349</td>
<td>0.0364</td>
<td>0.0378</td>
<td>0.0393</td>
<td>0.0407</td>
<td>0.0422</td>
<td>0.0436</td>
<td>0.0451</td>
<td>0.0465</td>
<td>0.0480</td>
<td>0.0494</td>
<td>0.0509</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>0.0523</td>
<td>0.0533</td>
<td>0.0552</td>
<td>0.0567</td>
<td>0.0581</td>
<td>0.0596</td>
<td>0.0610</td>
<td>0.0625</td>
<td>0.0640</td>
<td>0.0654</td>
<td>0.0669</td>
<td>0.0683</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>0.0698</td>
<td>0.0712</td>
<td>0.0727</td>
<td>0.0741</td>
<td>0.0753</td>
<td>0.0770</td>
<td>0.0785</td>
<td>0.0799</td>
<td>0.0814</td>
<td>0.0828</td>
<td>0.0843</td>
<td>0.0857</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

At the top you will see vertical columns headed in minutes in 5' intervals from 0' to 55'. Finally to the right of the 55' column are 4 columns which are headed "Mean differences".

The layout of tables varies slightly between different books and, in particular, some tables list every 6' and not every 5'.

**NOTE:** At the moment we will consider $0 < \theta < 90^\circ$
Example: Find the value of $\sin 15^\circ$

$\sin 15^\circ = \phantom{0.2588}$

Answer

$\sin 15^\circ = 0.2588$

If right $\rightarrow \boxed{31}$

Otherwise $\rightarrow \boxed{32}$
To find the value of \( \sin 15^\circ \) we look down the extreme left-hand column of the table headed "deg" until we come to the number 15. It is the first number on the left in the horizontal row. We then lay our ruler along the row. The number in the column immediately to the right of the 15, the column headed 0, is 0.2588 which is the value of \( \sin 15^\circ \), i.e.

\[
\sin 15^\circ = 0.2588
\]

Now find the value of \( \sin 68^\circ \)

\[
\sin 68^\circ = \text{__________}
\]

Answer \( \boxed{32} \)

\[
\sin 68^\circ = 0.9272
\]

If right \( \rightarrow \boxed{34} \)

Otherwise \( \rightarrow \boxed{33} \)
Example 2:

Find the value of $\sin 15^\circ 40'$

$\sin 15^\circ 40' = \ldots\ldots\ldots\ldots\ldots$

Answer

\[ 34 \]

$\sin 15^\circ 40' = 0.2700$

If right $\rightarrow 37$

Otherwise $\rightarrow 35
To find \( \sin 15^\circ 40' \) we lay our ruler along the horizontal row at \( 15^\circ \) and then look down the column headed \( 40' \) until we come to the ruler. The number we see in that column is 0.2700 which is the value of \( \sin 15^\circ 40' \), i.e.

\[
\sin 15^\circ 40' = 0.2700
\]

Find \( \cos 20^\circ 10' \).

\( \cos 20^\circ 10' = \quad \)

Answer (35)

\( \cos 20^\circ 10' = 0.9387 \)

If right --------------\( \rightarrow \) (37)

Otherwise --------------\( \rightarrow \) (36)
To find the value of $\cos 20^\circ 10'$ lay your ruler along the horizontal row at $20^\circ$ and then look down the column headed $10'$ until you come to the ruler. The number you should see in that column is $0.9387$ and is the value of $\cos 20^\circ 10'$, i.e.

$$\cos 20^\circ 10' = 0.9387$$

Find $\cos 35^\circ 25'$

$$\cos 35^\circ 25' = \begin{array} \hline \hline \end{array}$$

Answer 36

$$\cos 35^\circ 25' = 0.8150$$

If right $\rightarrow$ 37
Otherwise $\rightarrow$ 35
Now we come to another task: when the number of minutes of the given angle is not a multiple of 5, as in $24^\circ 43'$.

In this case we can find \( \sin 24^\circ 43' \) by using the Mean difference columns.

**Example 3:** Find \( \sin 24^\circ 43' \).

To find this value we lay our ruler along the horizontal row at $24^\circ$ and write down $\sin 24^\circ 40'$ i.e., $\sin 24^\circ 40' = 0.4173$.

The angle $24^\circ 43'$ is 3' more than the angle here. The adjustment for the extra 3' is given in the Mean difference columns.

We look along our ruler until we come to the Mean difference columns. The number we see under the 3' column is 8.

The value of $\sin 24^\circ 43'$ is now found as follows:

\[
\begin{align*}
\sin 24^\circ 40' &= 0.4173 \\
\text{Mean difference for 3'} &= 8 \\
(\text{Adding}) \sin 24^\circ 43' &= 0.4181
\end{align*}
\]

If you wanted to find $\sin 70^\circ 2'$ you would proceed like this (complete):

1. Firstly, find $\sin$.
2. Secondly, find the Mean difference for $\ldots$.
3. Thirdly, add the adjustment for the extra 2' to the value of $\sin$.
Firstly \( \sin 70^\circ 0' \)
Secondly \( \text{for } 2' \)
Thirdly \( \sin 70^\circ 0' \)

If all right \( \rightarrow 38 \)
Otherwise \( \rightarrow 37 \) again

\[
\sin 70^\circ 2' = \quad \text{___________}
\]
To find $\sin 70^\circ 2'$ lay your ruler along the horizontal row at $70^\circ$ and write down $\sin 70^\circ 0'$.

You must have found $\sin 70^\circ 0' = 0.9397$.

The angle $70^\circ 2'$ is $2'$ more than the angle here. The adjustment the extra $2'$ is given in the Mean difference columns. Look along your ruler until you come to the Mean difference columns. The number you will see under the $2'$ column is $2$.

The value of $\sin 70^\circ 2'$ is now found as follows:

\[
\sin 70^\circ 2' = (\text{Adding}) \quad \sin 70^\circ 0' + \text{Mean difference for } 2' = 0.9397 + 2 = 0.9399
\]

Find $\sin 10^\circ 4'$.

\[
\sin 10^\circ 4' = \quad \text{---------------------}
\]
\[
\sin 10^\circ 4' = 0.1747
\]

If right \[\rightarrow 40\]

Otherwise \[\rightarrow 37\]
Before going on let us point out something very important about the Mean differences. When we use Mean differences from a smaller angle to the larger:

a) The adjustment (mean difference) is added in finding sine, tangent and secant.

b) The adjustment (mean difference) is subtracted in finding cosine, cotangent and cosecant.

So far we have worked examples where the mean difference has been added, let us now work some examples where it has to be subtracted.

Example 4: Find \( \cos 47^\circ 7' \)

To find this value we lay our ruler along the horizontal row at \( 47^\circ \) and write down \( \cos 47^\circ 5' \), i.e. \( \cos 47^\circ 5' = 0.6809 \)

The angle \( 47^\circ 7' \) is 2' more than the angle here.

The adjustment for the extra 2' is given in the Mean difference columns. We look along our ruler until we come to the Mean difference columns. The number we see under the 2' column is 4.

The value of \( \cos 47^\circ 7' \) is now found as follows:

\[
\begin{align*}
\cos 47^\circ 5' & = 0.6809 \\
\text{Mean difference for 2'} & = 4 \\
\text{(Subtracting)} \quad \cos 47^\circ 7' & = 0.6805
\end{align*}
\]
If you wanted to find \( \cos 16^\circ 8' \) you would proceed like this (complete):

1. Firstly, find \( \cos \) ______

2. Secondly, find the Mean difference for ______

3. Thirdly, subtract the adjustment for the extra 3' from the value of \( \cos \) ______

Answer ______

Firstly ............... \( \cos 16^\circ 5' \)

Secondly ............... for 3'

Thirdly ............... \( \cos 16^\circ 5' \)

If all right \( \rightarrow \) 42

Otherwise \( \rightarrow \) 41 again
Find $\cos 16^\circ 8'$

$$\cos 16^\circ 8' = \_\_\_\_\_$$

Answer 42

$$\cos 16^\circ 8' = 0.9607$$

If right $\rightarrow$ 14

Otherwise $\rightarrow$ 13
To find $\cos 16^\circ 8'$ lay your ruler along the horizontal row at $16^\circ$ and write down $\cos 16^\circ 5'$.

You must have found $\cos 16^\circ 5' = 0.9609$.

The angle $16^\circ 8'$ is $3'$ more than the angle here.

The adjustment for the extra $3'$ is given in the Mean difference columns. Look along your ruler until you come to the Mean difference columns. The number you must see under the $3'$ column is $2$.

The value of $\cos 16^\circ 8'$ is now found as follows:

\[
\begin{align*}
\cos 16^\circ 5' &= 0.9609 \\
\text{Mean difference for } 3' &= 2 \\
(\text{Subtracting}) \cos 16^\circ 8' &= 0.9607
\end{align*}
\]

Find $\cos 57^\circ 18'$

\[
\cos 57^\circ 18' = \_\_\_\_\_\_\_\_
\]

Answer $43$

\[
\cos 57^\circ 18' = 0.5403
\]

If right $\Rightarrow 44$

Otherwise $\Rightarrow 41$
When we use mean differences from a smaller to a larger angle, the adjustment is a) added in finding ________ and ________ and b) it is subtracted in finding ________, ________ and ________.

Answer: ________

a) sine, tangent and secant

b) cosine, cotangent and cosecant

If all right ————> 4.5
Otherwise ————> Re-read 4.0 and then ————> 4.5
If we want to find the trigonometric function values of an angle measured in radians, we proceed like this:

a) Express the angle in degrees
b) Proceed as in the previous frames:

Example: To find \( \sin \frac{3\pi}{8} \) we proceed like this:

\( \frac{\pi}{2} = 90^\circ \), so \( \frac{3\pi}{8} = \frac{3\times90^\circ}{8} = 67.5^\circ = 67^\circ 30' \)

then

\( \sin \frac{3\pi}{8} = \sin 67^\circ 30' = 0.9239 \)

Find \( \sin \frac{2\pi}{7} \)

\( \sin \frac{2\pi}{7} = \) 

Answer \( 45 \)

\( \sin \frac{2\pi}{7} \approx \sin 51^\circ 25' = 0.7817 \)

If all right \( 47 \)
Otherwise \( 46 \)
To find $\sin \frac{2\pi}{7}$ we proceed like this:

$$\pi = 180^\circ \quad \therefore \quad \frac{2\pi}{7} = \frac{2 \times 180^\circ}{7} \approx 51.42^\circ \approx 51^\circ 25'$$

then

$$\sin \frac{2\pi}{7} \approx \sin 51^\circ 25' = 0.7817$$

Find $\sin \frac{3\pi}{7}$

$$\sin \frac{3\pi}{7} = \ldots$$

**Answer** 46

$$\sin \frac{3\pi}{7} \approx 71^\circ 8' = 0.9749$$

If right $\rightarrow$ 47

Otherwise $\rightarrow$ 45
We have learnt to find, using Tables of Natural Trigonometric Functions, the trigonometric function values for a given angle between 0° and 90°.

We now come to the converse process: find an angle between 0° and 90° whose trigonometric value is given.

A few examples will illustrate the procedure to use in this case.

**Example 1:** Find the value of $\theta$ given that $\sin \theta = 0.3584$

\[
\theta = \text{__________} \\
\sin \theta = 0.3584
\]

**Answer** 47

\[
\theta = 21^\circ
\]

If right $\rightarrow$ 50

Otherwise $\rightarrow$ 48
To find the value of \( e \) we run our ruler down the column headed 0' in the table.

The angle shown to the left of the row, in the column headed "deg" is 21°, and this is the value of \( e \), i.e. \( \theta = 21° \). (Remember that we are only working with \( 0 < \theta < 90° \))

Find the value of \( \theta \) given that \( \sin \theta = 0.8090 \)

\[ \theta = \ldots \]

Answer \( \theta = 54° \)

If right \( \rightarrow \) \( 50 \)
Otherwise \( \rightarrow \) \( 49 \)
To find the value of $\theta$ given that $\sin = 0.8090$ we look down the 0' column of the table until we come to the number 0.8090. The angle shown to the left of the row, in the column headed "deg" is 54, i.e. $\theta = 54^\circ$. 

Find the value of $\theta$ given that $\cos \theta = 0.5150$

Answer (49)

$\theta = 59^\circ$

If right $\rightarrow$ (50)
Otherwise $\rightarrow$ (48)
Example 2:

If the given number is not in the 0' column.

Find the value of $\theta$ given that $\sin \theta = 0.1636$

$$\theta = \ldots \ldots \ldots \ldots \ldots \ldots$$

Answer $\text{50}$

$\theta = 9^\circ 25'$

If right $\rightarrow 53$
Otherwise $\rightarrow 51$
If we look again in the 0° column we will notice that the number 0.1636 is not in this column, in this case we must proceed like this: look down the 0° column until we come to the number nearest to \textit{still less} than 0.1636 (i.e. 0.1564), now we lay our ruler along the horizontal row in which 0.1564 occurs and look along it until we come to the number 0.1636.

Now we read off the number of degrees at the extreme left of the ruler, in the column headed "deg", and the number of minutes vertically above the number 0.1636.

\[
\begin{array}{c|c}
\text{deg} & (\text{minutes}) \\
\hline
\end{array}
\]

\text{(degrees)} \quad 0.1636

\[
\begin{array}{c}
\text{(degrees)} \\
\hline
\end{array}
\]

Answer \( 51 \)

\[ \theta = 39^\circ 55' \]

If right \( \rightarrow 53 \)

Otherwise \( \rightarrow 52 \)
The exercise which you were asked to do was exactly like the worked example in the frame (50). You should therefore check that you have followed the method of the worked example.

Example 3:

If the given number is not in the table. Find the value of \( \theta \) given that \( \sin \theta = 0.3507 \) (The number 0.3507 is not in the table).

\[ \theta = \]
$\theta = 20^\circ 32'$

If right $\rightarrow 56$

Otherwise $\rightarrow 54$
To find the value of $\theta$ given that $\sin \theta = 0.3507$ we proceed as before looking down the column headed $\theta'$ until we come to the number nearest to but still less than $0.3507$ ($0.3420$).

Now we lay our ruler along the horizontal row in which $0.3420$ occurs and look along it until we come to the number nearest but still less than $0.3507$, this number is $0.3502$.

If the sine value were $0.3502$ the angle would be $20^\circ 30'$ but this number ($0.3502$) differs in its $4$th decimal figure from the number we want ($0.3507$) by $5$ units, so we keep our ruler still and look along its edge at the Mean difference columns. As the difference $5$ occurs under the $2'$ column, $2'$ must be added to the angle we found before ($20^\circ 30'$) because the angle increases as the sine value increases, i.e.

$$\theta = 20^\circ 30' + 2'$$

$$\theta = 20^\circ 32'$$

In the same way, find the value of $\theta$ given that $\sin \theta = 0.8686$

$$\theta = \underline{\underline{\text{ }} \text{ }}$$

Answer $\underline{\underline{54}}$

$$\theta = 60^\circ 18'$$

If right $\rightarrow \underline{\underline{56}}$

Otherwise $\rightarrow \underline{\underline{55}}$
To find θ given that \( \sin \theta = 0.8686 \) we proceed like this:

Look down the \( \theta \) column of the table until you come to the number nearest to but still less than 0.8686 (0.8660). Now lay your ruler along the horizontal row in which 0.8660 occurs. Look along it until you come to the number nearest but still less than 0.8686 (0.8682).

If the sine value were 0.8682, the angle would be 60° 15' but this number differ in the 4th decimal figure from the number we want (0.8686) by 4.

Keep your ruler still and look along its edge at the mean difference column. You will find that 4 is under the 3' column.
Then 3' more must be added to 60° 15' as the angle increases as the sine value increases, i.e.
\[
\theta = 60° 15' + 3'
\]
\[
\theta = 60° 18'
\]

In the same way, find the value of \( \theta \) given that \( \sin \theta = 0.9302 \)
\[
\theta = \text{__________}
\]

Answer (55)

\( \theta = 68° 28' \)

If right \( \rightarrow (55) \)

Otherwise \( \rightarrow (53) \)
Let us try some examples with the cosine function.

Example 4:

Find the value of $\theta$ given that $\cos \theta = 0.4384$

\[
\theta = \ldots
\]

Answer 56

$\theta = 64^\circ$ (Remember that we are working with $0 < \theta < 90^\circ$)

If right $\rightarrow$ 58

Otherwise $\rightarrow$ 57
The exercise which you were asked to do was exactly like the worked example in the frame 48.

You should therefore check that you have followed the method of the worked example.

Example 5:

If the given number is not in the 0' column.

Find the value of θ given that \( \cos \theta = 0.9830 \)

\( \theta = \) ___________
If you look again in the 0° column you will notice that the number 0.9830 is not in this column, in this case you must proceed like this: look down the 0° column until you come to the number nearest to and greater than 0.9830 (i.e. 0.9848).

Now lay your ruler along the horizontal row in which 0.9848 occurs and look along it until you come to the number 0.9830.

Now read off the number of degrees at the extreme left of the ruler, in the column headed "deg", and the number of minutes vertically above the number 0.9830.
Now in the same way find the value of $\theta$ given that $\cos \theta = 0.8843$.

\[ \theta = \boxed{59} \]

$\theta = 27^\circ \, 50'$

If right $\implies \boxed{60}$
Otherwise $\implies \boxed{58}$
Example 6:

If the given number is not in the table.

Find the value of $\theta$ given that $\cos \theta = 0.4355$ (The number 0.4355 is not in the table).

$\theta =$  

Answer 60

$\theta = 64^\circ 11'$

If right $\rightarrow$ 63

Otherwise $\rightarrow$ 51
To find the value of $\theta$ given that $\cos \theta = 0.4355$ we proceed as before looking down the column headed 0' until we come to the number nearest to and greater than 0.4355 (i.e., 0.4384).

Now we lay our ruler along the horizontal row in which 0.4384 occurs and look along it until we come to the number nearest to but still greater than 0.4355, this number is 0.4358.

If the cosine value were 0.4358 the angle would be $64^\circ 10'$ but this number (0.4358) differs in its 4th decimal figure from the number we want (0.4355) by 3 units, so we keep our ruler still and look along its edge at the Mean difference columns.

As the difference 3 occurs under the 1' column, 1' must be added to the angle we found before ($64^\circ 10'$) because the angle increases as the cosine value decreases, i.e.,

$$\theta = 64^\circ 10' + 1'$$

$$\theta = 64^\circ 11'$$

In the same way, find the value of $\theta$ given that $\cos \theta = 0.6392$
To find the value of $\theta$ given that $\cos \theta = 0.6392$ we look down the column headed $0'$ until we come to the number nearest to and greater than 0.6392 (i.e. 0.6428).

Now we lay our ruler along the horizontal row in which 0.6428 occurs and look along it until we come to the number nearest to but still greater than 0.6392, this number is 0.6394.

If the cosine value were 0.6394 the angle would be $50^\circ 15'$ but this number (0.6394) differs in its 4th decimal figure from the number we want (0.6392) by 2 units, so we keep our ruler still and look along its edge at the Mean difference columns.

As the difference 2 occurs under the 1' column, 1' must be added to the angle we found before ($64^\circ 10'$) because the angle increases as the cosine value decreases, i.e.

$$\theta = 50^\circ 15' + 1'$$

$$\theta = 50^\circ 16'$$

In the same way, find the value of $\theta$ given that $\cos \theta = 0.7406$.

$$\theta = \boxed{\ldots}$$
If right \[63\] Otherwise \[60\]

Now we come to a new point: How to express a trigonometric function of an angle, measured in radians, in terms of the same trigonometric function but of an angle between 0 and 2\(\pi\).

If we draw the angles \(\theta\) (measured in radians) and \(\theta + 2\pi\) in standard position (i.e. when its vertex is at the origin and its initial side coincides with the positive \(x\)-axis), we can appreciate that they have the same terminal sides (figure 63). Therefore, for any point \(P\) on it, \(x\), \(y\) and \(r\) will be the same for both angles.

The figure is drawn for the first quadrant but the result is clearly valid for all.

From figure 63 we obtain that:

\[
\begin{align*}
\sin (\theta + 2\pi) &= \sin \theta \\
\cot (\theta + 2\pi) &= \cot \theta \\
\cos (\theta + 2\pi) &= \cos \theta \\
\sec (\theta + 2\pi) &= \sec \theta \\
\tan (\theta + 2\pi) &= \tan \theta \\
\cosec (\theta + 2\pi) &= \cosec \theta
\end{align*}
\]
Similarly, we can see that if we have an angle $\theta$ (measured in radians), adding or subtracting $2\pi$ radians any number of times does not change the value of any of the trigonometric functions because they will have always the same terminal sides. We can express this by writing:

\[
\sin (\theta + 2n\pi) = \sin \theta, \quad \cot (\theta + 2n\pi) = \cot \theta
\]
\[
\cos (\theta + 2n\pi) = \cos \theta, \quad \sec (\theta + 2n\pi) = \sec \theta
\]
\[
\tan (\theta + 2n\pi) = \tan \theta, \quad \csc (\theta + 2n\pi) = \csc \theta
\]

Where $0 \leq \theta < 2\pi$ and $n$ is an integer, i.e. $n = 0, \pm 1, \pm 2$, this enables us to find the trigonometric function values for angles greater than $2\pi$, or less than 0, by first reducing the problem to the corresponding trigonometric function of an angle between 0 and $2\pi$.

Example:

Express $\sin \frac{14}{3}\pi$ in terms of the sine of an angle between 0 and $2\pi$.

To do this we proceed like this:

\[
\frac{14}{3}\pi = \frac{2}{3}\pi + \frac{4}{3}\pi
\]

therefore

\[
\sin \frac{14}{3}\pi = \sin \left( \frac{2}{3}\pi + \frac{4}{3}\pi \right) = \sin \frac{2}{3}\pi
\]

Express $\cos \frac{25}{4}\pi$ in terms of the cosine of an angle between 0 and $2\pi$.

\[
\cos \frac{25}{4}\pi = \underline{___________}
\]
To express \( \cos \frac{25}{4} \pi \) in terms of the cosine of an angle between 0 and \( 2 \pi \), we proceed as follows:

\[
\frac{25}{4} \pi = \frac{1}{4} \pi + 6
\]

Therefore

\[
\cos \frac{25}{4} \pi = \cos \left( \frac{1}{4} \pi + 6 \pi \right) = \cos \frac{7}{4} \pi
\]

Express \( \tan \frac{8}{3} \pi \) in terms of the tangent of an angle between 0 and \( 2 \pi \).

\[
\tan \frac{8}{3} \pi = \text{__________________}
\]
So far we have seen how to express a trigonometric function of an angle, measured in radians, in terms of the same trigonometric function but of an angle between 0 and \(2\pi\).

If the angle \(\theta\) is measured in degrees, the equivalent expressions to those given in frame (63) are:

\[
\begin{align*}
\sin (\theta + n \times 360^\circ) &= \sin \theta \\
\cos (\theta + n \times 360^\circ) &= \cos \theta \\
\tan (\theta + n \times 360^\circ) &= \tan \theta \\
\cot (\theta + n \times 360^\circ) &= \cot \theta \\
\sec (\theta + n \times 360^\circ) &= \sec \theta \\
\cosec (\theta + n \times 360^\circ) &= \cosec \theta
\end{align*}
\]

where \(0^\circ \leq \theta < 360^\circ\) and \(n\) is an integer, i.e. \(n = 0, \pm 1, \pm 2, \ldots\).
Example:

Express \( \tan 840^\circ \) in terms of the tangent of an angle between \( 0^\circ \) and \( 360^\circ \).

To do this we proceed like this:

\[
840^\circ = 120^\circ + 2 \times 360^\circ
\]

therefore

\[
\tan 840^\circ = \tan (120^\circ + 2 \times 360^\circ) = \tan 120^\circ
\]

Express \( \cos 1000^\circ \) in terms of the cosine of an angle between \( 0^\circ \) and \( 360^\circ \)

\[
\cos 1000^\circ = __________
\]

Answer: \( 65 \) \( \cos 1000^\circ = \cos 280^\circ \)

If right \( \rightarrow 67 \)

Otherwise \( \rightarrow 66 \)
To express \( \cos 1000^\circ \) in terms of the cosine of an angle between \( 0^\circ \) and \( 360^\circ \), we proceed like this:

\[ 1000^\circ = 280^\circ + 2 \times 360^\circ \text{ therefore} \]
\[ \cos 1000^\circ = \cos (280^\circ + 2 \times 360^\circ) = \cos 280^\circ \]

Express \( \cot 450^\circ \) in terms of the cotangent of an angle between \( 0^\circ \) and \( 360^\circ \).

\[ \cot 450^\circ = \]

Answer (66)

\[ \cot 450^\circ = \cot 90^\circ \]

If right \( \rightarrow \) (67)

Otherwise \( \rightarrow \) (65)
In figure 67, $\theta$ and $-\theta$ are constructed in standard position. On their respective terminal sides the points $P(x, y)$ and $P_1(x_1, y_1)$ are located so that $OP = OP_1$.

The triangles $OCP$ and $OCP_1$ are congruent (i.e. one can be superimposed on the other), therefore we have: $x_1 = x$, $y_1 = -y$ and $OP = OP_1$.

Applying the definitions of the trigonometric functions we obtain:

\[
\sin (-\theta) = \frac{y_1}{OP_1} = \frac{-y}{OP} = -\frac{y}{OP} = -\sin \theta
\]

\[
\cos (-\theta) = \frac{x_1}{OP_1} = \frac{x}{OP} = \cos \theta
\]

\[
\tan (-\theta) = \frac{y_1}{x_1} = \frac{-y}{x} = -\frac{y}{x} = -\tan \theta
\]

We could have obtained the same results if we had considered the angle $\theta$ in any of the other quadrants.

This enables us to find the trigonometric function values for negative angles by first reducing the problem to the corresponding trigonometric function of a positive angle.

Complete the following, according to these equalities:

\[
\sin (-73^\circ) = \quad \cos (-145^\circ) = \quad \tan (-\alpha) = \quad
\]
\[
\sin (-73^\circ) = -\sin 73^\circ \\
\cos (-145^\circ) = \cos 145^\circ \\
\tan (-\alpha) = -\tan \alpha
\]

If all right \(\rightarrow\) \(68\)

Otherwise \(\rightarrow\) \(67\) again

---

To express \(\cot (-\theta)\) in terms of \(\cot \theta\) we use the definition of cotangent and the results obtained in frame 69. In fact:

\[
\cot (-\theta) = \frac{1}{\tan (-\theta)} = \frac{1}{-\tan \theta} = -\frac{1}{\tan \theta} = -\cot \theta
\]

In the same way show that:

\[
\sec (-\theta) = \sec \theta \quad \text{and} \quad \csc (-\theta) = -\csc \theta
\]

To check your answers \(\rightarrow\) \(69\)
By the definitions of the trigonometric functions we know that, for the angle in figure 70:

\[ \sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x} \]

(The figure is drawn for the first quadrant but the results are clearly valid for all).

Now note that:

\[ \frac{\sin \theta}{\cos \theta} = \frac{y/x}{x/x} = \frac{y}{x} = \tan \theta \quad \text{hence} \quad \tan \theta = \frac{\sin \theta}{\cos \theta} \]
This result enables us to find the value of the tangent of an angle if we know the sine and cosine of the angle. For example, we know that \( \sin 30^\circ = \frac{1}{2} \) and \( \cos 30^\circ = \frac{\sqrt{3}}{2} \).

Therefore,

\[
\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}
\]

Find, in the same way, \( \tan 60^\circ \)

Answer

\[
\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}
\]
If we have two angles \( \alpha \) and \( \beta \), we can express the sine, cosine, and tangent of the angles \((\alpha + \beta)\) and \((\alpha - \beta)\) in terms of the sine, cosine, and tangent of \( \alpha \) and of \( \beta \) in formulae which are called addition and subtraction formulae.

These formulae are:

**Addition formulae:**

\[
\begin{align*}
\sin (\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\cos (\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\tan (\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\end{align*}
\]

**Subtraction formulae:**

\[
\begin{align*}
\sin (\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
\cos (\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\tan (\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\end{align*}
\]

(If you want to see the proofs of these formulae see Appendix A)

These formulae enable us to find the sine, cosine, and tangent of angles which can be expressed as a sum or difference of two angles whose sine, cosine, and tangent we know.

Let us look at an example.

**Example:** Using the addition and/or subtraction formulae, find \( \sin 105^\circ \)

We can express \(105^\circ\) as \(105^\circ = 60^\circ + 45^\circ\) as we know the trigonometric function values of \(60^\circ\) and \(45^\circ\) therefore.

\[
\begin{align*}
\sin 105^\circ &= \sin (60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\
\sin 105^\circ &= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{3}}{2} + \frac{1}{2} \right) = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{3} + 1}{2} \right)
\end{align*}
\]

Using the addition and/or subtraction formulae, find \( \cos 75^\circ \)

Note: do not evaluate square roots

\[
\cos 75^\circ = \frac{\sqrt{3} + 1}{2}
\]
\[
\cos 75^\circ = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right)
\]

If right \( \rightarrow 74 \)

Otherwise \( \rightarrow 72 \)

We can express 75° as 75° = 30° + 45° as we know the trigonometric function values of 30° and 45°, therefore \( \cos 75^\circ = \cos (30^\circ + 45^\circ) = \cos 30^\circ \times \cos 45^\circ - \sin 30^\circ \times \sin 45^\circ \)

\[
\cos 75^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{3} - 1}{2} \right)
\]

Using the addition and/or subtraction formulae,

find \( \sin 75^\circ \) \( \sin 75^\circ = \)
\[ \sin 75^\circ = \frac{1}{\sqrt{2}} \left( \frac{1 + \sqrt{3}}{2} \right) \]

If right \[ \rightarrow 74 \]
Otherwise \[ \rightarrow 73 \]

We can express \( 75^\circ \) as \( 75^\circ = 30^\circ + 45^\circ \) as we know the trigonometric values of \( 30^\circ \) and \( 45^\circ \), therefore

\[ \sin 75^\circ = \sin (30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \]

\[ \sin 75^\circ = \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}} \left( \frac{1 + \sqrt{3}}{2} \right) \]

Using the addition and/or subtraction formulae, find \( \sin 15^\circ \)

\[ \sin 15^\circ = \text{___________} \]
Using the addition formulae we can determine formulae for \( \sin 2\theta \), \( \cos 2\theta \) and \( \tan 2\theta \). These formulae are called the double-angle formulae.

For example, for \( \sin 2\theta \)

\[
\sin 2\theta = \sin (\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta
\]

For \( \tan 2\theta \):

\[
\tan 2\theta = \tan (\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

Obtain, in the same way, a formula for \( \cos 2\theta \)

\[
\cos 2\theta = \quad
\]
\[ \cos 2\theta = \cos (\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta \]

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]

Now let us find other trigonometric formulae which are very useful and can be easily obtained from the addition and subtraction formulae.

Let us evaluate \( \sin (90^\circ - \theta) \) and \( \cos (90^\circ - \theta) \)

\[ \sin (90^\circ - \theta) = \sin 90^\circ \cos \theta - \cos 90^\circ \sin \theta \]
\[ \sin (90^\circ - \theta) = 1 \times \cos \theta - 0 \times \sin \theta = \cos \theta \]
\[ \cos (90^\circ - \theta) = \cos 90^\circ \cos \theta + \sin 90^\circ \sin \theta \]
\[ \cos (90^\circ - \theta) = 0 \times \cos \theta + 1 \times \sin \theta = \sin \theta \]

then

\[ \sin (90^\circ - \theta) = \cos \theta \text{ and } \cos (90^\circ - \theta) = \sin \theta \]

Evaluate, in the same way, \( \sin (90^\circ + \theta) \) and \( \cos (90^\circ + \theta) \)

\[ \sin (90^\circ + \theta) = \]
\[ \cos (90^\circ + \theta) = \]
\[
\sin (90^\circ + \theta) = \cos \theta \\
\cos (90^\circ + \theta) = -\sin \theta \\
\]

If all right \rightarrow \(77\)

Otherwise \rightarrow \(76\)

\[
\sin (90^\circ + \theta) = \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta \\
\sin (90^\circ + \theta) = 1 \times \cos \theta + 0 \times \sin \theta = \cos \theta
\]

\[
\cos (90^\circ + \theta) = \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \\
\cos (90^\circ + \theta) = 0 \times \cos \theta - 1 \times \sin \theta = -\sin \theta
\]

NO RESPONSE \rightarrow \(77\)

REQUIRED
In the same way evaluate:

a) \( \sin (180^\circ - \theta) = \) 

b) \( \cos (180^\circ - \theta) = \) 

Answer 

\( \sin (180^\circ - \theta) = \sin \theta \)

\( \cos (180^\circ - \theta) = -\cos \theta \)

If right \( \Rightarrow \) 

Otherwise \( \Rightarrow \)
\[ \sin (180° - \theta) = \sin 180° \cos \theta - \cos 180° \sin \theta \]
\[ \sin (180° - \theta) = 0 \times \cos \theta - (-1) \sin \theta = \sin \theta \]
\[ \cos (180° - \theta) = \cos 180° \cos \theta + \sin 180° \sin \theta \]
\[ \cos (180° - \theta) = (-1) \cos \theta + 0 \times \sin \theta = -\cos \theta \]

Then \[ \sin (180° - \theta) = \sin \theta \]
\[ \cos (180° - \theta) = -\cos \theta \]

These two formulae:
\[ \sin (180° - \theta) = \sin \theta \]
\[ \cos (180° - \theta) = -\cos \theta \]

are particularly useful when \( 0 \leq \theta < 90° \) because they enable us to find the trigonometric function values for angles between 90° and 180°, by first reducing the problem to the corresponding trigonometric function of an angle between 0° and 90°.

These results can also be obtained from the figure 79 below.
\[\sin (180^\circ - \theta) = \frac{y}{OQ} = \frac{y}{OP} = \sin \theta\]

\[\cos (180^\circ - \theta) = \frac{-x}{OQ} = \frac{x}{OP} = -\cos \theta\]

Evaluate \(\sin (180^\circ + \theta)\) and \(\cos (180^\circ + \theta)\) by using the addition formulae.

\[\sin (180^\circ + \theta) = \quad \quad \cos (180^\circ + \theta) = \]

\[\text{Answer} \quad 79\]

\[\sin (180^\circ + \theta) = -\sin \theta\]

\[\cos (180^\circ + \theta) = -\cos \theta\]

If right \(\rightarrow 81\)

Otherwise \(\rightarrow 80\)
\[
\sin(180^\circ + \theta) = \sin 180^\circ \cos \theta + \cos 180^\circ \sin \theta
\]
\[
\sin(180^\circ + \theta) = 0 \times \cos \theta + (-1) \sin \theta = -\sin \theta
\]
\[
\cos(180^\circ + \theta) = \cos 180^\circ \cos \theta - \sin 180^\circ \sin \theta
\]
\[
\cos(180^\circ + \theta) = (-1) \cos \theta - 0 \times \sin \theta = -\cos \theta
\]

Then \(\sin(180^\circ + \theta) = -\sin \theta\)
\(\cos(180^\circ + \theta) = -\cos \theta\)

These two formulae: \(\sin(180^\circ + \theta) = -\sin \theta\) and \(\cos(180^\circ + \theta) = -\cos \theta\) are particularly useful when \(0 < \theta < 90^\circ\) because they enable us to find the trigonometric function values for angles between \(180^\circ\) and \(270^\circ\), by first reducing the problem to the corresponding trigonometric function of an angle between \(0^\circ\) and \(90^\circ\).

These results can also be obtained from the figure 81 below.

\[
\sin(180^\circ + \theta) = \frac{-y}{OQ} = -\frac{y}{OP} = -\sin \theta
\]
\[
\cos(180^\circ + \theta) = \frac{-x}{OQ} = -\frac{x}{OP} = -\cos \theta
\]
Functions for angles between $0^\circ$ and $90^\circ$, and nothing was said about the trigonometric function values of angles greater than $90^\circ$ and angles less than $0$. This is what we are going to do in the next frames. Some examples will illustrate how to proceed to find the trigonometric function values of such angles in the Tables.

CASE 1:

If the angle is between $90^\circ$ and $180^\circ$

Example 1:

Find $\sin 147^\circ$

We can express $147^\circ$ as $147^\circ = 180^\circ - 33^\circ$ so

$\sin 147^\circ = \sin (180^\circ - 33) = \sin 33^\circ$

(Remember that in frames 77-79 we saw that $\sin (180^\circ - \theta) = \sin \theta$)

Example 2:

Find $\cos 120^\circ 40'$

We can express $120^\circ 40'$ as $120^\circ 40' = 180^\circ - 59^\circ 20'$

therefore

$\cos 120^\circ 40' = \cos (180^\circ - 59^\circ 20') = -\cos 59^\circ 20'$

(Remember that in frames 77-79 we saw that $\cos (180^\circ - \theta) = -\cos \theta$)

Now we look for $\cos 59^\circ 20'$ in the table

$\cos 120^\circ 40' = -\cos 59^\circ 20' = -0.5100$

Find $\sin 94^\circ 50'$

$\sin 94^\circ 50' = _______
\( \sin 94^\circ 50' = \sin 85^\circ 10' = 0.9964 \)

If right \rightarrow (82)

Otherwise \rightarrow (82) again

CASE II: The angle is between \(180^\circ\) and \(270^\circ\)

Example 3: Find \(\sin 255^\circ\)

We can express \(255^\circ\) as \(255^\circ = 180^\circ + 75^\circ\) so

\(\sin 255^\circ = \sin (180^\circ + 75^\circ) = -\sin 75^\circ\)

(Remember that in frames 79-81 we saw that \(\sin (180^\circ + \theta) = -\sin \theta\))

Now we look for \(\sin 75^\circ\) in the table.

\(\sin 255^\circ = -\sin 75^\circ = -0.9659\)

Find \(\cos 199^\circ 40'\)

\(\cos 199^\circ 40' = \)
cos 199° 40' = -0.9917

If right

Otherwise Remember that cos (180° + θ) = -cos θ
Now try 83 again

CASE III: The angle is in the fourth quadrant.
An angle which is in the fourth quadrant can be written as (360° - θ)

(figure 84)

In figure 84 we can appreciate that (360° - θ) and -θ have the same terminal sides, so sin (360° - θ) = sin (-θ), cos (360° - θ) = cos (-θ), etc.

but in frame 67 we have shown that

sin (-θ) = -sin θ and cos (-θ) = cos θ
therefore

sin (360° - θ) = -sin θ
cos (360° - θ) = cos θ
Let us look at some examples.

Example 4: Find $\sin 305^\circ$  

We can express $305^\circ$ as $305^\circ = 360^\circ - 55^\circ$ so  

$\sin 305^\circ = \sin (360^\circ - 55^\circ) = -\sin 55^\circ$  

Now we look for $\sin 55^\circ$ in the table:  

$\sin 305^\circ = -\sin 55^\circ = -0.8192$

Example 5: Find $\cos 296^\circ 45'$  

We can express $296^\circ 45'$ as $296^\circ 45' = 360^\circ - 63^\circ 15'$ so  

$\cos 296^\circ 45' = \cos (360^\circ - 63^\circ 15') = \cos 63^\circ 15'$  

Now we look for $\cos 63^\circ 15'$ in the table  

$\cos 296^\circ 45' = \cos 63^\circ 15' = 0.4501$

Find $\cos 283^\circ 5'$  

Answer  

$\cos 283^\circ 5' = 0.2264$

If right  

Otherwise  


To find \(\cos 283^\circ 5'\) we proceed like this:

We can express \(283^\circ 5'\) as \(360^\circ - 76^\circ 55'\) so

\[
\cos 283^\circ 5' = \cos (360^\circ - 76^\circ 55') = \cos 76^\circ 55'
\]

Now we look for \(\cos 76^\circ 55'\) in the table

\[
\cos 283^\circ 5' = \cos 76^\circ 55' = 0.2264
\]

And \(\sin 273^\circ 25'\)

\[
\sin 273^\circ 25' = -0.9982
\]
To find \( \sin 273° 25' \) we proceed like this:

We can express \( 273° 25' \) as \( 273° 25' = 360° - 86° 35' \) so

\[
\sin 273° 25' = \sin (360° - 86° 35') = -\sin 86° 35'
\]

Now we look for \( \sin 86° 35' \) in the table.

\[
\sin 273° 25' = -\sin 86° 35' = -0.9982
\]

Find \( \sin 280° 30' \)

\[
\sin 280° 30' = \underline{-0.9833}
\]

Answer 86

\[
\sin 280° 30' = -0.9833
\]

If right \( \rightarrow \) 87

Otherwise \( \rightarrow \) 86 again
CASE IV: The angle is negative.

To find the trigonometric function values of negative angles we use the expressions we studied in frame 67, i.e. \( \sin(-\theta) = -\sin \theta \)
\( \cos(-\theta) = \cos \theta \) etc., to reduce our problem to one of the cases we have just studied.

Example 6: Find \( \sin(-140^\circ) \)

In this case we proceed like this:

\[
\sin(-140^\circ) = -\sin 140^\circ
\]

Now we proceed like we did in Case I, i.e.
\[
\sin(-140^\circ) = -\sin 140^\circ = -\sin (180^\circ - 40^\circ) = -\sin 40^\circ = -0.6428
\]

Find \( \cos(-200^\circ) \)

\[
\cos(-200^\circ) = \ldots
\]

Answer \[
\cos(-200^\circ) = -0.9397
\]
To find \( \cos (-200°) \) we proceed like this:
\[
\cos (-200°) = \cos 200° \quad \text{(see frame ©)}
\]
\[
\cos (200°) = \cos (180° + 20°) = -\cos 20° \quad \text{(see frame (©))}
\]
therefore
\[
\cos (-200°) = -\cos 20° = -0.9397
\]

End \( \sin (-325°) \)

\[
\sin (-325°) = -\sin 35° = 0.5736
\]

If right \( \rightarrow \) 89

Otherwise \( \rightarrow \) 87
CASE V: If the angle is greater than $360^\circ$.
In this case we use the expressions we obtained in frame 65 to reduce the
problem to one of the previous cases.

Example 7: To find $\sin 965^\circ$ we proceed like this:

$$\sin 965^\circ = \sin (245^\circ + 2 \times 360^\circ) = \sin 245^\circ \text{ (see frame 65)}$$

Now we proceed like we did in case II.

$$\sin 965^\circ = \sin (245^\circ) = -\sin (180^\circ + 65^\circ) = \sin 65^\circ$$

then

$$\sin 965^\circ = -\sin 65^\circ = -0.9063$$

Find $\cos 2000^\circ$

$$\cos 2000^\circ = \cos 200^\circ = -\cos 20^\circ = -0.9397$$

If right $\rightarrow 90$

Otherwise $\rightarrow 89 \text{ again}$
We will now consider the graphical representation of the trigonometric functions. Complete the following table, to two decimal places, by using a Table of Natural Trigonometric Functions and then plot the graph of the sine function.

<table>
<thead>
<tr>
<th>θ</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinθ</td>
<td>0</td>
<td>0.87</td>
<td>0.50</td>
<td>-0.87</td>
<td>-0.87</td>
<td>-0.50</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Graph of the sine function](image)

Answer

<table>
<thead>
<tr>
<th>θ</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinθ</td>
<td>0</td>
<td>0.50</td>
<td>0.87</td>
<td>1</td>
<td>0.87</td>
<td>0.50</td>
<td>0</td>
<td>-0.50</td>
<td>-0.87</td>
<td>-1</td>
<td>-0.87</td>
<td>-0.50</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph of the sine function](image)
As we have seen in frame 65, sin θ = sin (θ + n x 360°) where 0 ≤ θ < 360° and n is an integer, i.e. n = 0, ±1, ±2, ....

This means that the curve you plot in frame 90 may be repeated in the intervals -360° to 0°, 360° to 720°, and so on (see figure 91) The graph is called the sine curve.

As the sine function repeats its values after each interval of 360°, we say that the sine function is a periodic function and its period is 360°.

The largest value of the sine is ____ and the smallest value is ____.

Answer 91 1, -1

If right 92

Otherwise, check these values in figure 91 92
Complete the following table, to two decimal places, by using a Table of Natural Trigonometric Functions and then plot the graph of the cosine function.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>0.50</td>
<td>-0.87</td>
<td>-0.87</td>
<td>0.50</td>
<td>0.87</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answer

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>0.87</td>
<td>0.50</td>
<td>0</td>
<td>-0.50</td>
<td>-0.87</td>
<td>-1</td>
<td>-0.87</td>
<td>-0.50</td>
<td>0</td>
<td>0.50</td>
<td>0.87</td>
<td>1</td>
</tr>
</tbody>
</table>
As we have seen in frame 65, \( \cos \theta = \cos (\theta + n \times 360^\circ) \) where \( 0 \leq \theta < 360^\circ \) and \( n \) is an integer, i.e. \( n = 0, +1, +2, \ldots \).

This means that the curve you plot in frame 92 may be repeated in the intervals \(-360^\circ \) to \(0^\circ\), \(360^\circ \) to \(720^\circ\), and so on (see figure 93). The graph is called the cosine curve.

As the cosine function repeats its value after each interval of \(360^\circ\), we say that the cosine function is a periodic function and its period is \(360^\circ\).

The largest value of the cosine is _____ and the smallest value is _____.

Answer  93  1, -1

If right  \(\rightarrow\) 94

Otherwise, check these values in figure 93  \(\rightarrow\) 94
Complete the following table, to two decimal places, by using a Table of Natural Trigonometric Functions and then plot the graph of the tangent between 180° and 360°.

<table>
<thead>
<tr>
<th>θ</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>tan θ</td>
<td>0.00</td>
<td>0.57</td>
<td>1.73</td>
<td>∞</td>
<td>-1.73</td>
<td>-0.57</td>
<td>0</td>
<td>1.73</td>
<td>∞</td>
<td>-1.73</td>
<td>-0.57</td>
<td>0</td>
<td>1.73</td>
</tr>
</tbody>
</table>

Answer

<table>
<thead>
<tr>
<th>θ</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
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<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>tan θ</td>
<td>0.00</td>
<td>0.57</td>
<td>1.73</td>
<td>∞</td>
<td>-1.73</td>
<td>-0.57</td>
<td>0</td>
<td>1.73</td>
<td>∞</td>
<td>-1.73</td>
<td>-0.57</td>
<td>0</td>
<td>1.73</td>
</tr>
</tbody>
</table>
As we have seen in frame 65, \( \tan \theta = \tan (\theta + n \times 360^\circ) \) where \( 0 \leq \theta < 360^\circ \) and \( n \) is an integer, i.e. \( n = 0, \pm 1, \pm 2, \ldots \).

This means that the curve you plot in frame 94 may be repeated in the intervals 
\(-360^\circ \) to \( 0^\circ \), \( 360^\circ \) to \( 720^\circ \), and so on (see figure 95).

The graph is called the tangent curve.

![Tangent Curve](image)

Frequently the symbol \( \infty \) (read "infinity"). which is not a number) is used in describing the behaviour of the tangent function as the angle approaches \( 90^\circ \) through values less than \( 90^\circ \) and through values more than \( 90^\circ \) (see frame 29).

Thus

\[
\begin{align*}
\tan \theta &\rightarrow + \infty \quad \text{as} \quad \theta \rightarrow 90^\circ - \\
\tan \theta &\rightarrow - \infty \quad \text{as} \quad \theta \rightarrow 90^\circ +
\end{align*}
\]

The first line here indicates that the tangent value of an angle can be made larger than any chosen positive number by taking the value of the angle close enough to, but less than \( 90^\circ \).

The second line indicates that the tangent value of an angle can be made less than any chosen negative number by letting the value of the angle exceed \( 90^\circ \) by a sufficiently small amount.

Describe the behaviour of \( \tan \theta \) as \( \theta \) approaches \(-90^\circ \) through values less than \(-90^\circ \) and through values more than \(-90^\circ \).
In frames 91 and 93 we said that the sine and cosine functions are periodic. Now we are going to see the precise definition of a periodic function. A function \( f \) is a periodic function if it repeats its values at regular intervals of the variable. If this interval is, say, \( p \), then for a periodic function we have:

\[
 f(\theta + p) = f(\theta) \quad \text{for all } \theta
\]

You should always use the smallest interval, which makes the function periodic, to define the period \( p \).

In the case of the sine and cosine functions the period is \( 360^\circ \) because \( 360^\circ \) is the smallest positive value such that

\[
 \sin(\theta + 360^\circ) = \sin\theta \quad \text{and} \quad \cos(\theta + 360^\circ) = \cos\theta
\]

From figure 95, it can be seen that the tangent function is a periodic function. Its period is (ring the appropriate answer):

a) \( 90^\circ \)  
   b) \( 180^\circ \)  
   c) \( 360^\circ \)
Note that in figure 95, the curve between $0^\circ$ and $180^\circ$ is repeated in the intervals $-180^\circ$ to $0^\circ$, $180^\circ$ to $360^\circ$, etc. Since the curve repeats its values after each interval of length $180^\circ$, we see that the tangent function is a periodic function whose period is $180^\circ$, i.e.

$$\tan (\theta + 180^\circ) = \tan \theta$$

Notice that it is true that $\tan (\theta + 360^\circ) = \tan \theta$ but the period is the smallest number of degrees, and in the case it is $180^\circ$ (see frame 96).

Correct.

You should be familiar with the graphs of the trigonometric functions so the graphs of the cotangent, secant and cosecant functions are shown below.
The cosecant, cotangent and secant are also periodic functions. The cotangent has a period of $180^\circ$ and the secant and cosecant have a period of $360^\circ$, so (complete)

a) $\cot \theta = \cot (\theta + \ldots^\circ)$

b) $\sec \theta = \sec (\theta + \ldots^\circ)$

c) $\csc \theta = \csc (\theta + \ldots^\circ)$

Answer

a) $180^\circ$

b) $360^\circ$

c) $360^\circ$

If all right $\rightarrow 99$

Otherwise, review $\rightarrow 99$
Now we come to a very important formula connecting $\sin \theta$ and $\cos \theta$

In frame 23 we have seen that for all $\theta$:

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \text{where} \quad x^2 + y^2 = r^2$$

hence

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

then $\sin^2 \theta + \cos^2 \theta = 1$

This means that if we know $\sin \theta$ we can find $\cos \theta$, and conversely, if we know $\cos \theta$, we can find $\sin \theta$.

Example: To find $\cos \theta$ given that $\sin \theta = 1/2$ and $\theta$ is in the first quadrant, we proceed as follows

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \text{therefore} \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos^2 \theta = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Cont'/.....

then $\cos \theta = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}$

since $\cos \theta > 0$ for $\theta$ in the first quadrant, $\cos \theta = \frac{\sqrt{3}}{2}$

Find $\sin \theta$ given that $\cos \theta = 1/4$ and $\theta$ is in the fourth quadrant.

$$\sin \theta = \ldots$$
\[ \sin \theta = -\frac{\sqrt{15}}{4} \]

We know that \( \cos \theta = \frac{1}{4} \) and \( \theta \) is in the fourth quadrant, so:

\[ \sin^2 \theta + \cos^2 \theta = 1 \rightarrow \sin^2 \theta = 1 - \cos^2 \theta \]

then

\[ \sin^2 \theta = 1 - \left( \frac{1}{4} \right)^2 = 1 - \frac{1}{16} = \frac{15}{16} \]

then

\[ \sin \theta = \pm \sqrt{\frac{15}{16}} = \pm \frac{\sqrt{15}}{4} \]

since \( \sin \theta \neq 0 \) for \( \theta \) in the fourth quadrant,

\[ \sin \theta = -\frac{\sqrt{15}}{4} \]

Answer

(100)

We know that \( \cos \theta = \frac{1}{4} \) and \( \theta \) is in the fourth quadrant, so:

\[ \sin^2 \theta + \cos^2 \theta = 1 \rightarrow \sin^2 \theta = 1 - \cos^2 \theta \]

then

\[ \sin^2 \theta = 1 - \left( \frac{1}{4} \right)^2 = 1 - \frac{1}{16} = \frac{15}{16} \]

then

\[ \sin \theta = \pm \sqrt{\frac{15}{16}} = \pm \frac{\sqrt{15}}{4} \]

since \( \sin \theta \neq 0 \) for \( \theta \) in the fourth quadrant,

\[ \sin \theta = -\frac{\sqrt{15}}{4} \]

Answer

(100)

We know that \( \cos \theta = \frac{1}{4} \) and \( \theta \) is in the fourth quadrant, so:

\[ \sin^2 \theta + \cos^2 \theta = 1 \rightarrow \sin^2 \theta = 1 - \cos^2 \theta \]

then

\[ \sin^2 \theta = 1 - \left( \frac{1}{4} \right)^2 = 1 - \frac{1}{16} = \frac{15}{16} \]

then

\[ \sin \theta = \pm \sqrt{\frac{15}{16}} = \pm \frac{\sqrt{15}}{4} \]

since \( \sin \theta \neq 0 \) for \( \theta \) in the fourth quadrant,

\[ \sin \theta = -\frac{\sqrt{15}}{4} \]

Answer

(100)

We know that \( \cos \theta = \frac{1}{4} \) and \( \theta \) is in the fourth quadrant, so:

\[ \sin^2 \theta + \cos^2 \theta = 1 \rightarrow \sin^2 \theta = 1 - \cos^2 \theta \]

then

\[ \sin^2 \theta = 1 - \left( \frac{1}{4} \right)^2 = 1 - \frac{1}{16} = \frac{15}{16} \]

then

\[ \sin \theta = \pm \sqrt{\frac{15}{16}} = \pm \frac{\sqrt{15}}{4} \]

since \( \sin \theta \neq 0 \) for \( \theta \) in the fourth quadrant,

\[ \sin \theta = -\frac{\sqrt{15}}{4} \]
Find $\cos \theta$ given that $\sin \theta = -\frac{1}{3}$ and $\theta$ is in the third quadrant.

Answer 101

$$\cos \theta = -\frac{2\sqrt{2}}{3}$$

If right $\Rightarrow$ 103

Otherwise $\Rightarrow$ 102
We know that \( \sin \theta = -\frac{1}{3} \) and \( \theta \) is in the third quadrant, so:

\[
\begin{align*}
\sin^2 \theta + \cos^2 \theta &= 1 \\
\cos^2 \theta &= 1 - \sin^2 \theta \\
\cos^2 \theta &= 1 - \left(-\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}
\end{align*}
\]

then \( \cos \theta = \pm \sqrt{\frac{8}{9}} = \pm \frac{2\sqrt{2}}{3} \)

Since \( \cos \theta < 0 \) for \( \theta \) in the third quadrant

\[ \cos \theta = -\frac{2\sqrt{2}}{3} \]

Another useful identity can be derived from \( \sin^2 \theta + \cos^2 \theta = 1 \) by dividing both sides of the identity by \( \cos^2 \theta \).  

\[
\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad \left(\frac{\sin \theta}{\cos \theta}\right)^2 + 1 = \left(\frac{\tan \theta}{\sec \theta}\right)^2 
\]

but remember that

\[
\frac{\sin \theta}{\cos \theta} = \tan \theta \quad \text{and} \quad \frac{1}{\cos \theta} = \sec \theta
\]

therefore

\[ \tan^2 \theta + 1 = \sec^2 \theta \]

Show, in the same way, that \( 1 + \cot^2 \theta = \cosec^2 \theta \)

Hint: Divide both sides of the initial identity by \( \sin^2 \theta \)

TO CHECK

YOUR ANSWER
To show that $1 + \cot^2 \theta = \csc^2 \theta$, we divide both sides of $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta$, then:

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \cdot \left( \frac{\sin \theta}{\sin \theta} \right)^2 + \left( \frac{\cos \theta}{\sin \theta} \right)^2 = (\frac{1}{\sin \theta})^2$$

but remember that

$$\frac{\cos \theta}{\sin \theta} = \cot \theta \quad \text{and} \quad \frac{1}{\sin \theta} = \csc \theta$$

therefore

$$1 + \cot^2 \theta = \csc^2 \theta$$

Find $\sec \theta$ given that $\tan \theta = \frac{3}{4}$ and $\theta$ is in the third quadrant.

$$\sec \theta =$$
We know that \( \tan \theta = \frac{3}{4} \) and \( \theta \) is in the third quadrant so:

\[
1 + \tan^2 \theta = \sec^2 \theta
\]

\[
\sec^2 \theta = 1 + \left( \frac{3}{4} \right)^2 = 1 + \frac{9}{16} = \frac{25}{16}
\]

\[
\sec \theta = \pm \frac{5}{4}
\]

Since \( \sec \theta \) is negative for \( \theta \) in the third quadrant,

\[
\sec \theta = -\frac{5}{4}
\]
In Frame 75 we have seen that $\cos^2 \theta = \cos^2 \theta - \sin^2 \theta$ for all values of $\theta$, but $\sin^2 \theta = 1 - \cos^2 \theta$ (from $\sin^2 \theta + \cos^2 \theta = 1$), therefore

$$\cos^2 \theta = \cos^2 \theta - (1 - \cos^2 \theta) = \cos^2 \theta - 1 + \cos^2 \theta = 2\cos^2 \theta - 1$$

On the other hand $\cos^2 \theta = 1 - \sin^2 \theta$ (from $\sin^2 \theta + \cos^2 \theta = 1$), therefore:

$$\cos^2 \theta = 1 - \sin^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta$$

Summarizing

$$\cos 2\theta = 2\cos^2 \theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2 \theta$$

These formulas are very useful when dealing with calculus and you should be familiar with them.

Cont' /....

Other formulas which you should be familiar with are:

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

These formulae are called Sum and Difference of Sines and Cosines (if you want to see the proofs of these formulae see Appendix B).

Let us apply them in some examples.
Example 1: Express \( \sin 50^\circ + \sin 40^\circ \) as a product of suitable sines and/or cosines by using the sum or difference of sines and cosines formulae.

We know that \( \sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2} \)

If we put \( A = 50^\circ \) and \( B = 40^\circ \) we obtain:

\[
\sin 50^\circ + \sin 40^\circ = 2 \sin \frac{50^\circ + 40^\circ}{2} \cos \frac{50^\circ - 40^\circ}{2}
\]

then \( \sin 50^\circ + \sin 40^\circ = 2 \sin 45^\circ \cos 5^\circ \)

Example 2: In the same way, express \( \cos 65^\circ + \cos 15^\circ \) as a product of suitable sines and/or cosines.

\[
\cos 65^\circ + \cos 15^\circ = 2 \cos \frac{65^\circ + 15^\circ}{2} \cos \frac{65^\circ - 15^\circ}{2}
\]

\( \cos 65^\circ + \cos 15^\circ = 2 \cos 40^\circ \cos 25^\circ \)

In the same way, express \( \sin 25^\circ - \sin 5^\circ \) as a product of suitable sines and/or cosines.

\( \sin 25^\circ - \sin 5^\circ = \) 

Answer \( 107 \)

\( \sin 25^\circ - \sin 5^\circ = 2 \cos 15^\circ \sin 10^\circ \)

If right \( \rightarrow 109 \)

Otherwise \( \rightarrow 108 \)
\[ \sin 25^\circ - \sin 5^\circ = 2 \cos \frac{25^\circ + 5^\circ}{2} \sin \frac{25^\circ - 5^\circ}{2} \]

then \( \sin 25^\circ - \sin 5^\circ = 2 \cos 15^\circ \sin 10^\circ \)

\[ \text{NO RESPONSE} \quad \text{REQUIRED} \]

\[ \cos 35^\circ - \cos 75^\circ = \boxed{\text{??????????????}} \]

In the same way, express \( \cos 35^\circ - \cos 75^\circ \) as a product of suitable sines and/or cosines.
We know that

\[ \cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2} \]

If we put \( A = 35^\circ \) and \( B = 75^\circ \) we obtain:

\[ \cos 35^\circ - \cos 75^\circ = -2 \sin \frac{35^\circ + 75^\circ}{2} \sin \frac{35^\circ - 75^\circ}{2} \]

\[ \cos 35^\circ - \cos 75^\circ = -2 \sin 55^\circ \sin (-20^\circ) \]

but \( \sin (-20^\circ) = -\sin 20^\circ \) \( \) (see frame 69)

therefore:

\[ \cos 35^\circ - \cos 75^\circ = 2 \sin 55^\circ \sin 20^\circ \]
In the same way express $\cos 12^\circ + \cos 30^\circ$ as a product of suitable sines and/or cosines.

**Answer**

$\cos 12^\circ + \cos 30^\circ = 2 \cos 21^\circ \cos 9^\circ$

If right $\rightarrow 113$

Otherwise $\rightarrow 112$
The exercise which you were asked to do was exactly like the worked examples in frame 107. You should therefore check first that you have followed the method of the worked example and then check for mistakes in your calculation.

We will now study basic rules used in dealing with a triangle. These rules are called sine and cosine rules.

Sine rule
Let us consider the triangle ABC in which all angles are less than 90° (figure 113(a)).

We draw CD perpendicular to AB. We have now two right angled triangles: ACD and BCD. Let CD = h (figure 113(b))

In triangle ACD: \( h = b \cdot \sin \hat{A} \)
In triangle BCD: \( h = a \cdot \sin \hat{B} \)
Instead of denoting the angle like \( \hat{A} \) and \( \hat{B} \) we will denote them simply by \( A \) and \( B \), respectively, as the majority of the books use this notation.

So

\[
h = b \cdot \sin A \quad \text{and} \quad h = a \cdot \sin B
\]

Thus the terms on the right-hand side of the two equations above are equal:

\[
a \sin B = b \sin B
\]

therefore

\[
\frac{a}{\sin A} = \frac{b}{\sin B}
\]

In a similar manner (by drawing a perpendicular from \( B \) to \( AC \) or a perpendicular from \( A \) to \( BC \)), we obtain:

\[
\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}
\]

This expression is called the Sine rule.
We could use similar arguments to prove the Sine rule for a triangle which has an obtuse angle (figure 114)

State the Sine rule for the triangle of figure 114

\[
\frac{m}{\sin N} = \frac{n}{\sin M} = \frac{p}{\sin P}
\]

Answer (114)

\[
\frac{m}{\sin M} = \frac{n}{\sin N} = \frac{p}{\sin P}
\]

If right \(\rightarrow\) (115)

Otherwise \(\rightarrow\) (113)
There are two cases in which we can apply the sine rule.

**CASE I:** Given one side and two angles

**Example:**

Find the unknown sides and angle of the triangle of the figure 115 (a) given \( b = 10 \), \( A = 30^\circ \) and \( B = 45^\circ \).

To find \( C \) we use

\[
C = 180^\circ - (A + B)
\]

\[
C = 180^\circ - (30^\circ + 45^\circ) = 180^\circ - 75^\circ = 105^\circ
\]

To find \( a \) we use

\[
\frac{a}{\sin A} = \frac{b}{\sin B}
\]

because

1) We require \( a \), so we put it in the numerator
2) We know \( b \), \( A \) and \( B \) (we do not use \( \frac{c}{\sin C} \) as we do not know \( c \))

Then

\[
a = \frac{b \sin A}{\sin B} = \frac{10 \times \sin 30^\circ}{\sin 45^\circ} = \frac{10 \times 0.5}{0.7071} = 7.07
\]

so \( a = 7.1 \) (to one decimal place)

To find \( c \) we use

\[
\frac{c}{\sin C} = \frac{b}{\sin B}
\]

(we can also use \( \frac{a}{\sin A} \) as we have found \( a \) and \( A \))

Then

\[
c = \frac{b \sin C}{\sin B} = \frac{10 \times \sin 105^\circ}{\sin 45^\circ} = \frac{10 \times 0.9659}{0.7071} = 13.66
\]

So \( c = 13.7 \) (to one decimal place)
NOTE:

Every value we get from the Tables of Natural Trigonometric Functions is an approximation, and the errors made in successive approximations can build up. We cannot be accurate to more than about $\pm 3'$ in the value of some angles.

Find the unknown sides (to one decimal place) and angle of the triangle in figure 115(b) below, given that $B = 100^\circ$, $C = 30^\circ$ and $c = 6$.

Answer $\text{115}$

\[ A = 50^\circ, b = 11.8, a = 9.2 \]

If all right $\rightarrow \text{117}$

Otherwise $\rightarrow \text{116}$
The exercise which you were asked to do was exactly like the worked example in frame 115. You should therefore check first that you have followed the method of the worked example and then check for mistakes in your calculation.

CASE II: Given two sides and the angle opposite one of them.
Suppose \( b, c \) and \( B \) are given (figure 117)

![Figure 117](image)

Difficulties can arise in this case and you must remember the various possibilities.

From \( \frac{\sin C}{c} = \frac{\sin B}{b} \), \( \sin C = \frac{c \sin B}{b} \)
We get two angles $C$ and $(180° - C)$, where $C$ is the acute angle for which $\sin C = \frac{c \sin B}{b}$

(i) If $B + (180° - C) > 180°$, the second solution is impossible and only the first solution is valid (figure 117(a)).

(ii) If $B + (180° - C) < 180°$, there are two possible solutions (figure 117(b)).

It will be seen that in (i) we have $b > c$ and in (ii) $b < c$

(iii) If $\sin C > 1$, there is no solution
Example 1: Find the unknown side (to one decimal place) and angles of the triangle in figure 118(a) given that \( a = 7 \), \( c = 4 \) and \( A = 20^\circ \).

From \( \frac{\sin C}{c} = \frac{\sin A}{a} \), we obtain

\[
\sin C = \frac{c \cdot \sin A}{a}
\]

as \( a = 7 \), \( c = 4 \) and \( A = 20^\circ \).

\[
\sin C = \frac{4 \times \sin 20^\circ}{7} = \frac{4 \times 0.3420}{7} = 0.1954
\]

thus \( \sin C = 0.1954 \).

By using a Table of Natural Trigonometric Functions we obtain:

\( C = 11^\circ 16' \) or \( C = 180^\circ - 11^\circ 16' = 168^\circ 44' \).

The value \( C = 168^\circ 44' \) is impossible since it would make \( A + C = 180^\circ \).

Then, for \( C = 11^\circ 16' \)

\[
B = 180^\circ - (A + C)
\]

\[
B = 180^\circ - (20^\circ + 11^\circ 16') = 148^\circ 44'
\]

To find \( b \) we use

\[
\frac{b}{\sin B} = \frac{a}{\sin A}
\]

then

\[
b = \frac{a \sin B}{\sin A} = \frac{7 \times 0.5190}{0.3420} = 10.62
\]

then \( b = 10.6 \) (to one decimal place).
Find the unknown side (to one decimal place) and angles of the triangle in figure 118(b) given that \( a = 8 \), \( b = 12 \) and \( B = 52^\circ \)

**Answer**

\[
\begin{align*}
c &= 15.2 \\
A &= 3^\circ 25' \quad \text{if all right} \\
C &= 87^\circ 35' \quad \text{otherwise}
\end{align*}
\]
Example 2:

Find the unknown side (to one decimal place) and angles of the triangle in figure 119(a) below, given that $A = 18^\circ$, $a = 3$ and $c = 4$.

From $\frac{\sin C}{c} = \frac{\sin A}{a}$ we obtain $\sin C = \frac{c \cdot \sin A}{a}$

due to $\sin C = \frac{4 \cdot \sin 18^\circ}{3} = \frac{4 \cdot 0.3090}{3} = 0.4120$

By using a Table of Natural Trigonometric Functions, we obtain:

$C = 24^\circ 20'$ or $C = 180^\circ - 24^\circ 20' = 155^\circ 40'$

In this case both values of $C$ are valid as $A + C < 180^\circ$, and it is not possible without further information (such as $C$ is acute) to decide which is the correct one. For this reason this case is known as the "ambiguous case".

So we will solve the problem for both values of $C$ by using the expressions

$B = 180^\circ - (A + C)$ and $b = \frac{a \cdot \sin B}{\sin A}$ to calculate $B$ and $b$, respectively.

a) For $C = 24^\circ 20'$

$B = 180^\circ - (18^\circ + 24^\circ 20') = 137^\circ 40'$
\[
\begin{align*}
    b &= \frac{3 \times \sin 137^\circ 40'}{\sin 18^\circ} = \frac{3 \times 0.6734}{0.3090} = 6.53 \\
    \text{then } b &= 6.5 \text{ (to one decimal place)} \\

    b) \text{ For } C = 155^\circ 40' \\
    B &= 180^\circ - (18^\circ + 155^\circ 40') = 6^\circ 20' \\
    b &= \frac{3 \times \sin 6^\circ 20'}{\sin 18^\circ} = \frac{3 \times 0.1103}{0.3090} = 1.07 \\
    \text{then } b &= 1.1 \text{ (to one decimal place)} \\
\end{align*}
\]

Find the unknown side and angles of the triangle in figure 119(b) given that 
\( a = 7, \ c = 4 \) and \( C = 20^\circ \)
There are two solutions:

i) \( A = 36^\circ 45', B = 123^\circ 15', b = 9.8 \)

ii) \( A = 143^\circ 15', B = 26^\circ 45', b = 5.3 \)

If all right \( \rightarrow \) (120)

Otherwise \( \rightarrow \) (119) again

Find the unknown side (to one decimal place) and angles of the triangle in figure 120 given that \( a = 5, b = 9 \) and \( B = 49^\circ \)
\[ A = 19^\circ 36' \]
\[ C = 111^\circ 2^4' \]
\[ c = 7.1 \]

If all right \[ \rightarrow \]

Otherwise, review 117 \[ \rightarrow \] again

---

\[ 121 \]

**Cosine rule**

Let us consider a triangle ABC in which all angles are less than 90° (figure 121(a)).

We draw CD perpendicular to AB. This divides side c into two parts:

Let \( AD = x \), \( DB = c - x \) and \( CD = h \) (figure 121(b)).
We now have two right angled triangles: \( \triangle ACD \) and \( \triangle BCD \).

Using Pythagoras' Theorem for both triangles we find:

In triangle \( \triangle ACD \):
\[
h^2 = b^2 - x^2
\]
In triangle \( \triangle BCD \):
\[
h^2 = a^2 - (c - x)^2
\]

Thus the terms on the right-hand side of the two equations are equal:
\[
b^2 - x^2 = a^2 - (c - x)^2
\]

By algebraically manipulating this equation, we find that
\[
a^2 = b^2 + (c - x)^2 - 2x
\]
\[
a^2 = b^2 + c^2 - 2cx
\]

but from triangle \( \triangle ACD \): \( x = b \cos A \) where \( A \) is the angle \( \angle BAC \) (i.e., \( \angle BAC \)).

therefore:
\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

This expression is called the Cosine rule. Similarly we could prove that:

\[
b^2 = a^2 + c^2 - 2ac \cos B
\]
\[
c^2 = a^2 + b^2 - 2ab \cos C
\]
We could use very similar arguments to prove the Cosine rule for a triangle which has an obtuse angle (figure 122)

\[ m^2 = n^2 + p^2 - 2np \cos M \]
\[ n^2 = m^2 + p^2 - 2mp \cos N \]
\[ p^2 = m^2 + n^2 - 2mn \cos P \]

If all right \[ \rightarrow 123 \]
Otherwise \[ \rightarrow 121 \]
There are two cases in which we can apply the cosine rule.

**CASE I:** Given two sides and the included angle.

**Example:**

Find the unknown side (to one decimal place) and angles of the triangle in figure 123(a) given that

\[ a = 4, \ c = 5 \text{ and } B = 45^\circ \]

To find \( b \) we use

\[ b^2 = a^2 + c^2 - 2ac \cos B \]

then

\[ b^2 = 4^2 + 5^2 - 2(4)(5) \cos 45^\circ \]
\[ b^2 = 16 + 25 - 40(0.7071) \]
\[ b^2 = 41 - 28.28 = 12.72 \]

then

\[ b = \sqrt{12.72} = 3.56 \]

so

\[ b = 3.6 \text{ (to one decimal place)} \]

To find \( A \) we use

\[ \frac{\sin A}{a} = \frac{\sin B}{b} \]

we know \( a, b \) and \( c \)

then

\[ \sin A = \frac{a \sin B}{b} = \frac{4 \cdot \sin 45^\circ}{3.6} \]

\[ \therefore \sin A = \frac{4(0.7071)}{3.6} = 0.7856 \]
By using a Table of Natural Trigonometric Functions we obtain $A = 51^\circ 47'$

To find $C$ we use $C = 180^\circ - (A + B)$

(we can also use $\frac{\sin C}{c} = \frac{\sin B}{b}$ or $c^2 = a^2 + b^2 - 2ab \cos C$)

then $C = 180^\circ - (51^\circ 47' + 45^\circ) = 83^\circ 13'$

Find the unknown side (to one decimal place and angles of the triangle in figure 123(b) given that:

$b = 3$, $c = 6$ and $A = 120^\circ$

\[ \begin{align*}
\text{FIGURE 123(b)}
\end{align*} \]

Note: Remember the note given in frame (115)

\[ \text{Answer (123)} \]

\[ a = 7.9 \]

\[ B = 19^\circ 12' \]

\[ C = 41^\circ 8' \]

If all right $\rightarrow (125)$

Otherwise $\rightarrow (124)$
The exercise which you were asked to do was exactly like the worked example in frame 123. You should therefore check first that you have followed that method of the worked example and then check for mistakes in your calculation.

CASE II: Given the three sides.

Example:

Find the three angles of the triangle in figure 125(a) given that $a = 5$, $b = 6$ and $c = 7$.

To find $A$ we use $a^2 = b^2 + c^2 - 2bc \cos A$.
then \( \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36 + 49 - 25}{84} = 0.7142 \)

then \( A = 44^\circ 25' \)

In the same way we could find \( B \) by using

\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} \]

but it is easier to use

\[ \sin B = \frac{b \sin A}{a} \]

as we have found \( A \)

then

\[ \sin B = \frac{6 \sin 44^\circ 25'}{5} = \frac{6 \cdot (0.6998)}{5} = 0.8397 \]

so \( B = 57^\circ 7' \)

To find \( C \) we use \( C = 180^\circ - (A + B) \)

then \( C = 180^\circ - (44^\circ 25' + 57^\circ 7') = 78^\circ 28' \)

Find the three angles of the triangle in figure 125(b)

given that \( a = 6 \), \( b = 10 \) and \( c = 9 \)

*Read also the note given in frame 115*
\[ A = 36° 20' \]
\[ B = 80° 56' \]
\[ C = 62° 44' \]

If all right \[ \rightarrow 127 \]
Otherwise \[ \rightarrow 126 \]

The exercise which you were asked to do was exactly like the worked example in frame 125. You should therefore check first that you have followed the method of the worked example and then check for mistakes in your calculation.
The Sine and Cosine rules can be applied to:
(ring the appropriate answer).

a) Any triangle
b) Any triangle unless it is a right angled triangle
c) Right angled triangles, only.

Answer : (127)

a) Any triangle

If right → (128)

Otherwise → Read → (113) and then → (128)
In which of the cases below should you start by using the Sine rule (you can ring more than one answer).

a) Given one side and two angles
b) Given two sides and the angle opposite one of them
c) Given two sides and the included angle
d) Given the three sides

Answer 128

a) and b)
Decide, according to the figure 129, which rule you should use first and which angle or side you should find initially.

(ring the appropriate answers)

a) Sine rule  b) Cosine rule  c) R  d) S  e) t

Answer 129

a) and d)

If all right → 130

Otherwise → 128
We will now examine an approximation which is very useful.

<table>
<thead>
<tr>
<th>DEGREES</th>
<th>RADIANS</th>
<th>Sin θ</th>
<th>tan θ</th>
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<tbody>
<tr>
<td>1°</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
</tr>
<tr>
<td>2°</td>
<td>0.035</td>
<td>0.035</td>
<td>0.035</td>
</tr>
<tr>
<td>5°</td>
<td>0.087</td>
<td>0.087</td>
<td>0.088</td>
</tr>
<tr>
<td>10°</td>
<td>0.175</td>
<td>0.174</td>
<td>0.176</td>
</tr>
<tr>
<td>15°</td>
<td>0.262</td>
<td>0.259</td>
<td>0.268</td>
</tr>
<tr>
<td>30°</td>
<td>0.524</td>
<td>0.500</td>
<td>0.577</td>
</tr>
<tr>
<td>45°</td>
<td>0.785</td>
<td>0.707</td>
<td>1.000</td>
</tr>
<tr>
<td>60°</td>
<td>1.047</td>
<td>0.860</td>
<td>1.732</td>
</tr>
</tbody>
</table>

The table above shows values of θ (in degrees and in radians), sinθ and tanθ.

Note that if θ is a small angle and θ is measured in radians, θ and sinθ approximately equal when θ is a small angle measured in degrees (check with the table).
If right \[\text{YES} \quad \text{NO}\]

Otherwise, check this answer in the table of frame 130, and the

According to the table given in frame 130, are \( \theta \) and \( \tan \theta \) approximately equal when \( \theta \) is a small angle measured in radians?
Yes

If right $\rightarrow \boxed{132}$

Otherwise, check this answer in the table of frame 130, and then $\rightarrow \boxed{132}$

According to the table given in frame 130, are $\theta$ and $\tan \theta$ approximately equal when $\theta$ is a small angle measured in degrees?

YES ☐ NO ☐
If right \( \rightarrow \) 133

Otherwise, check this answer in the table of frame 130 and then \( \rightarrow \) 133

Then, \( \theta \), \( \sin \theta \) and \( \tan \theta \) are approximately equal only when \( \theta \) is a \underline{\text{_______}} angle and \( \theta \) is measured in \underline{______}.
Let us illustrate the last result obtained.

Note that in figure 134(a), where \( \theta \) is small, the lengths of the lines \( a \) and \( c \), and the arc \( b \) are more nearly equal to each other than in the figure 134(b), where \( \theta \) is large; in both cases \( \theta \) is measured in radians.
From either figure we have that:

\[ a = r \sin \theta \]
\[ b = r \theta \]
\[ c = r \tan \theta \]

If we put \( r = 1 \) we obtain:

\[ a = \sin \theta \]
\[ b = \theta \]
\[ c = \tan \theta \]

Thus from geometry it would appear that when \( \theta \) is a small angle measured in radians; \( \theta, \sin \theta \) and \( \tan \theta \) are approximately equal.

It can be actually be shown that for small angles \( \sin \theta < \theta < \tan \theta \)

If you want a proof of this, see Appendix C.
Is 1 a good approximation to \( \sin 1^\circ \)?

Answer \( \text{135} \)

No

If right \( \rightarrow \text{137} \)

Otherwise \( \rightarrow \text{136} \)
Your answer is YES, but remember that we saw in frame 125 that θ and sin θ are approximately equal only when θ is a small angle and it is measured in radians, therefore 1 is not a good approximation to sin 1°.

You have completed this programme. If you like to see how you have improved, attempt the post-test that follows.
The numbers of the questions in the post-test are directly related to the ones in the pre-test, e.g.

Question 1 in the post-test the same thing as question 1 in the pre-test.

Attempt the questions which you failed in the pre-test.
1. According to the triangle in Fig.1: \( \cos \theta = \frac{\sqrt{3}}{2} \)

![FIGURE 1]

Complete the following in the same way:

\[
\sin \theta = \quad \tan \theta = 
\]

2. Given that \( \cos \theta = \frac{3}{4} \), where \( \theta \) is an acute angle in a right angled triangle, calculate \( \sin \theta \) and \( \tan \theta \) by using Pythagoras' Theorem and the definitions of the trigonometric functions.

3. According to the triangle in Fig.3: \( \csc \theta = \frac{5}{3} \)

![FIGURE 3]

Complete the following in the same way:

\[
\cot \theta = \quad \sec \theta = 
\]

4. Without the aid of printed tables complete the table below:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td></td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \sqrt{3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td></td>
<td>( \sqrt{2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td></td>
<td>( \frac{2}{\sqrt{3}} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Complete the table below by writing + or − as in the example:

<table>
<thead>
<tr>
<th>( \theta ) in quadrant</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
<th>( \sec \theta )</th>
<th>( \cosec \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td>( + )</td>
<td></td>
<td>( + )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td></td>
<td>( - )</td>
<td></td>
<td>( - )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td></td>
<td>( - )</td>
<td></td>
<td>( + )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td>( - )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. Complete the following, as in the example:
   a) cosec $0^\circ = \infty$
   b) cot $90^\circ = 0$
   c) tan $270^\circ =$
   d) sin $180^\circ =$
   e) cos $90^\circ =$
   f) sec $360^\circ =$
   g) cot $90^\circ =$
   h) cos $270^\circ =$

7. Using four figure tables of Natural Trigonometric Functions, find the values of:
   a) cos $27^\circ 16'$
   b) sin $\frac{3\pi}{8}$

8. Using four figure tables of Natural Trigonometric Functions, find the values of $\theta$ ($0 \leq \theta < 90^\circ$) in each of the following cases:
   a) Given that sin$\theta = 0.4561$
   b) Given that cos$\theta = 0.6341$

9. Express $\frac{16\pi}{5}$ as $\theta + 2n\pi$ where $0 \leq \theta < 2\pi$ and $n$ is an integer, i.e. $n = 0, \pm 1, \pm 2, \ldots$ and then express cos $\frac{16\pi}{5}$ in terms of cos$\theta$.

10. Express $1020^\circ$ as $\theta + n \times 360^\circ$ where $0 \leq \theta < 360^\circ$ and $n$ is an integer, i.e. $n = 0, \pm 1, \pm 2, \ldots$ and then express sec $1020^\circ$ in terms of sec$\theta$.

11. Complete the following as in the example:
   a) cos $(-37^\circ) = \cos 37^\circ$
   b) sin $(-83^\circ) =$
   c) tan $(-\theta) =$
   d) sec $(-100^\circ) =$
12. Apply the addition and/or subtraction formulae to:
   a) \( \cos 15^\circ \) (Hint: consider \( \cos(A-B) \)).
   b) \( \cos 2\theta \)
   c) \( \cos (90^\circ + \theta) \)
   d) \( \sin (180^\circ - \theta) \)
   e) \( \cos (180^\circ + \theta) \)

13. Using four-figure tables of Natural Trigonometric Functions find the value of:
   a) \( \sin 241^\circ 10' \)
   b) \( \cos 650^\circ \)

14. Sketch the graph of \( \sec \theta \) for \( 0 < \theta < 180^\circ \) and describe the behaviour of \( \sec \theta \) as \( \theta \) approaches \( 90^\circ \) through values less than \( 90^\circ \) and through values more than \( 90^\circ \).

15. Find \( \cos \theta \) given that \( \sin \theta = \frac{3}{4} \) and \( \theta \) is in the second quadrant.

16. Find \( \tan \theta \) given that \( \sec \theta = \frac{5}{4} \) and \( \theta \) is in the third quadrant.

17. Complete the following:
   \[
   \cos 50^\circ + \cos 40^\circ = 2 \cos 45^\circ \]

18. Find the unknown sides (to one decimal place) and angle of the triangle in the figure below given that:
   \( B = 100^\circ \), \( C = 30^\circ \) and \( c = 6 \)

19. Complete the following by filling in with > or <
   \[ \sin \theta \quad > \quad \theta \quad < \quad \tan \theta \]

(\( \theta \) is a small angle measured in radians)
1. \( \sin \theta = \frac{x}{z} \); \( \tan \theta = \frac{x}{y} \)

2. \( \sin \theta = \frac{\sqrt{7}}{4} \); \( \tan \theta = \frac{\sqrt{7}}{3} \)

3. \( \cot \theta = \frac{4}{3} \); \( \sec \theta = \frac{5}{4} \)

4. \[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\theta & \sin \theta & \cos \theta & \tan \theta & \csc \theta & \sec \theta & \cot \theta \\
\hline
30^\circ & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & 2 & \frac{2}{\sqrt{3}} & \sqrt{3} \\
45^\circ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \sqrt{2} & \sqrt{2} & 1 \\
60^\circ & \frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3} & \frac{2}{\sqrt{3}} & 2 & \frac{1}{\sqrt{3}} \\
\hline
\end{array}
\]

5. \[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\theta \text{ in quadrant} & \sin \theta & \cos \theta & \tan \theta & \cot \theta & \sec \theta & \csc \theta \\
\hline
I & + & + & + & + & + & + \\
II & + & - & - & - & - & + \\
III & - & - & + & + & - & - \\
IV & - & + & - & - & + & - \\
\hline
\end{array}
\]

6. a) \( c) \infty \) \hspace{1cm} e) 0 \hspace{1cm} g) 0 \\
    d) 0 \hspace{1cm} f) 1 \hspace{1cm} h) 0 

7. a) 0.8889 \hspace{1cm} b) 0.9239

8. a) 27^\circ 8' \hspace{1cm} b) 50^\circ 39'
9. \( \cos \frac{16}{5} \pi = \cos \frac{6}{5} \pi \)

10. \( \sec 1020^\circ = \sec 300^\circ \)

11. b) \(-\sin 83^\circ\)  c) \(-\tan \theta\)  d) \(\sec 100^\circ\)

12. a) \(\frac{1+\sqrt{3}}{2 \sqrt{2}}\)  b) \(\cos^2 \theta - \sin^2 \theta\)

  c) \(-\sin \theta\)  d) \(\sin \theta\)  e) \(-\cos \theta\)

13. a) \(-0.8760\)  b) \(0.3420\)

The secant of an angle can be made larger than any chosen positive number by taxing the angle close enough to, yet less than 90°.

The secant of an angle can be made less than any chosen negative number by letting the angle exceed 90° by a sufficiently small amount.
15. \( \cos \theta = -\frac{\sqrt{7}}{4} \)

16. \( \tan \theta = \frac{3}{4} \)

17. \( \cos 5^\circ \)

18. \( a = 9.2; \quad b = 11.8; \quad A = 50^\circ \)

19. < ; <

If you want to revise again any of the frames of the programme go to the section "Objective-Pretest-Frame correspondence" you consulted after you answered the pre-test. The numbers of the questions of the post-test correspond to the same numbers in the pre-test.

SUPPLEMENTARY PROBLEMS

1) Find the values of the trigonometric functions of \( \theta \) given that:
   a) \( \tan \theta = -\frac{3}{4} \) and \( \theta \) is in the second quadrant
   b) \( \cos \theta = \frac{5}{6} \) and \( \theta \) is in the first quadrant
   c) \( \sec \theta = -\sqrt{5} \) and \( \theta \) is in the third quadrant
   d) \( \csc \theta = -\frac{2}{\sqrt{3}} \) and \( \theta \) is in the fourth quadrant

2. Find, by using Tables of Natural Trigonometric Functions, the following:
   a) \( \sin 855^\circ10' \)
   b) \( \cos 19^\circ45' \)
   c) \( \cot 54^\circ27' \)
   d) \( \tan 27^\circ28' \)
   e) \( \csc (-48^\circ6') \)
   f) \( \sin 580^\circ43' \)
   g) \( \sin \frac{9\pi}{7} \)
   h) \( \cos \frac{9\pi}{4} \)
   i) \( \sin \frac{20\pi}{11} \)
3) Without using any Tables, find the values of sine, cosine and tangent of:
   a) 150°   b) -120°   c) 210°   d) -315°

4) Complete the following according to the formulae of sum and difference of sines and cosines:
   a) sin 25° + sin 15° =
   b) sin 16° - sin 6° =
   c) cos 17° + cos 3° =
   d) cos 10° - cos 2° =
   e) cos 10° + cos 20° =

5) Find, by using Tables of Natural Trigonometric Functions, the values of θ given that: (0 < θ < 360°)
   a) sinθ = -0.3035, θ in the third quadrant
   b) cosθ = 0.8499, θ in the fourth quadrant
   c) tanθ = -0.3680, θ in the second quadrant
   d) tanθ = 0.9971, θ in the first quadrant
6) Find the unknown sides and angles of the following triangles (the same figure is valid for all)

\[ \triangle ABC \]

a) Given that \( c = 25 \), \( A = 35^\circ \) and \( B = 68^\circ \)

b) Given that \( b = 480 \), \( c = 628 \) and \( C = 55^\circ 10' \)

c) Given that \( b = 2 \), \( c = 10 \) and \( B = 14^\circ \)

d) Given that \( a = 132 \), \( b = 224 \) and \( C = 28^\circ 40' \)

e) Given that \( a = 25.2 \), \( b = 37.8 \), and \( c = 43.4 \)

ANSWERS TO SUPPLEMENTARY PROBLEMS

1) \[
\begin{array}{ccccccc}
\sin \theta & \cos \theta & \tan \theta & \cot \theta & \sec \theta & \csc \theta \\
\hline
a) & \frac{3}{5} & -\frac{4}{5} & -\frac{3}{4} & -\frac{4}{3} & -\frac{5}{4} & \frac{5}{3} \\
b) & \frac{\sqrt{11}}{6} & \frac{5}{6} & \frac{\sqrt{11}}{5} & \frac{5}{\sqrt{11}} & \frac{6}{5} & \frac{6}{\sqrt{11}} \\
c) & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 2 & \frac{1}{2} & -5 & -\frac{\sqrt{5}}{2} \\
d) & -\frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3} & -\frac{1}{\sqrt{3}} & 2 & -\frac{2}{\sqrt{3}} \\
\end{array}
\]

2) \[
\begin{array}{ccccccc}
a) & 0.7050 & d) & 0.5200 & g) & -0.7819 \\
b) & 0.9412 & e) & 1.3436 & h) & 0.7071 \\
c) & 0.7148 & f) & -0.6524 & i) & -0.5407 \\
\end{array}
\]
3) \[
\begin{array}{ccc}
\text{sine} & \text{cosine} & \text{tangent} \\
\text{a)} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{3}} \\
\text{b)} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \sqrt{3} \\
\text{c)} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} \\
\text{d)} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1
\end{array}
\]

4) \[
\begin{align*}
\text{a)} & \quad 2 \sin 20^\circ \cos 5^\circ \\
\text{b)} & \quad 2 \cos 11^\circ \sin 5^\circ \\
\text{c)} & \quad 2 \cos 10^\circ \cos 7^\circ \\
\text{d)} & \quad -2 \sin 6^\circ \sin 4^\circ \\
\text{e)} & \quad 2 \cos 15^\circ \cos 5^\circ
\end{align*}
\]

ANSWERS TO SUPPLEMENTARY QUESTIONS continued

5) \[
\begin{align*}
\text{a)} & \quad \theta = 197^\circ 40' \\
\text{b)} & \quad \theta = 328^\circ 12' \\
\text{c)} & \quad \theta = 159^\circ 48' \\
\text{d)} & \quad \theta = 44^\circ 55'
\end{align*}
\]

6) \[
\begin{align*}
\text{a)} & \quad a = 15, b = 24, C = 77^\circ \\
\text{b)} & \quad a = 764, A = 86^\circ, B = 38^\circ 50' \\
\text{c)} & \quad \text{There is no solution because \( \sin C > 1 \)} \\
\text{d)} & \quad c = 125, A = 30^\circ 30', B = 120^\circ 40' \\
\text{e)} & \quad A = 35^\circ 20', B = 60^\circ 10', C = 84^\circ 30'
\end{align*}
\]
LIST OF FORMULAE

Definition of the trigonometric functions

\[ \sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x} \]
\[ \cot \theta = \frac{x}{y} \quad \sec \theta = \frac{r}{x} \quad \cosec \theta = \frac{r}{y} \]

\( \theta \) is an angle of any magnitude and can be in any quadrant.

RECIPROCAL RELATIONS

\[ \sec \theta = \frac{1}{\cos \theta} \quad \cosec \theta = \frac{1}{\sin \theta} \quad \cot \theta = \frac{1}{\tan \theta} \]
\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \]

TRIGONOMETRIC FUNCTIONS OF ANGLES GREATER THAN 360° (2\pi rad)

\[ \sin (\theta + n \times 360°) = \sin \theta \quad \sin (\theta + 2n\pi) = \sin \theta \]
\[ \cos (\theta + n \times 360°) = \cos \theta \quad \cos (\theta + 2n\pi) = \cos \theta \]
\[ \tan (\theta + n \times 360°) = \tan \theta \quad \tan (\theta + 2n\pi) = \tan \theta \]
\[ \cot (\theta + n \times 360°) = \cot \theta \quad \cot (\theta + 2n\pi) = \cot \theta \]
\[ \sec (\theta + n \times 360°) = \sec \theta \quad \sec (\theta + 2n\pi) = \sec \theta \]
\[ \cosec (\theta + n \times 360°) = \cosec \theta \quad \cosec (\theta + 2n\pi) = \cosec \theta \]

where \( \theta \) is measured in degrees

NEGATIVE ANGLES

\[ \sin (-\theta) = -\sin \theta \quad \cot (-\theta) = -\cot \theta \]
\[ \cos (-\theta) = \cos \theta \quad \sec (-\theta) = \sec \theta \]
\[ \tan (-\theta) = -\tan \theta \quad \cot (-\theta) = -\cot \theta \]
ADDITION AND SUBTRACTION FORMULAE

\[ \sin (a + b) = \sin a \cos b + \cos a \sin b \]
\[ \cos (a + b) = \cos a \cos b - \sin a \sin b \]
\[ \sin (a - b) = \sin a \cos b - \cos a \sin b \]
\[ \cos (a - b) = \cos a \cos b + \sin a \sin b \]

DOUBLE ANGLE

\[ \sin 2\theta = 2 \sin \theta \cos \theta \]
\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]
\[ \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \]

ANGLES WHOSE SUM IS 90°

\[ \sin (90° - \theta) = \cos \theta \]
\[ \cos (90° - \theta) = \sin \theta \]
\[ \tan (90° - \theta) = \cot \theta \]

ANGLES WHOSE SUM IS 180°

\[ \sin (180° - \theta) = \sin \theta \]
\[ \cos (180° - \theta) = -\cos \theta \]
\[ \tan (180° - \theta) = -\tan \theta \]

TRIGONOMETRIC IDENTITIES

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
\[ 1 + \tan^2 \theta = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \cosec^2 \theta \]
SUM AND DIFFERENCE OF SINES AND COSINES

\[
\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}
\]

\[
\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}
\]

\[
\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}
\]

\[
\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}
\]

SINE RULE

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

COSINE RULE

\[
a^2 = b^2 + c^2 - 2bc \cos A \\
b^2 = a^2 + c^2 - 2ac \cos B \\
c^2 = a^2 + b^2 - 2ab \cos C
\]

SMALL ANGLES

If \( \theta \) is a small angle measured in radians:

\[
\sin \theta < \theta < \tan \theta
\]
APPENDIX A

ADDITION AND SUBTRACTION FORMULAE

Let \( \alpha \) and \( \beta \) be acute angles constructed so that \( \alpha + \beta < 90^\circ \) and draw them as in Figure 1 below.

Then \( \sin (\alpha + \beta) = \frac{AP}{OP} = \frac{AD + DP}{OP} \)

but \( AD = CB \) because \( ADBC \) is a rectangle,
therefore \( \sin (\alpha + \beta) = \frac{CB + DP}{OP} = \frac{CB}{OP} + \frac{DP}{OP} \)

APPENDIX A continued

But, on the other hand:
\( CB \) can be written as \( CB = \sin OB \) (for triangle \( OBC \) and \( DP \) can be written as \( DP = \cos PB \) (for triangle \( PBD \))
then:
\[
\sin (\alpha + \beta) = \frac{\sin \alpha \cdot OB}{OP} + \frac{\cos \alpha \cdot PB}{OP} = \frac{\sin \alpha \cdot OB}{OP} + \frac{\cos \alpha \cdot PB}{OP}
\]
but \( \frac{OB}{OP} = \cos \beta \) and \( \frac{PB}{OP} = \sin \beta \)
therefore \( \sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \)
Furthermore:

\[ \cos (\alpha + \beta) = \frac{OA}{OP} = \frac{OC-AC}{OP} \]

but AC = DB because ADBC is a rectangle,

therefore \[ \cos (\alpha + \beta) = \frac{OC-DB}{OP} = \frac{OC}{OP} - \frac{DB}{OP} \]

But, on the other hand:

OC can be written as \( OC = \cos \alpha OB \) (for triangle OBC) and

OB can be written as \( DB = \sin \alpha PB \) (for triangle PBD)

then:

\[ \cos (\alpha + \beta) = \frac{\cos \alpha OB}{OP} - \frac{\sin \alpha PB}{OP} = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

It is possible to show that these formulae of \( \sin (\alpha + \beta) \) and \( \cos (\alpha + \beta) \) are correct for any value of \( \alpha \) and \( \beta \), but we will not do it here.

To find the formula for \( \tan (\alpha + \beta) \) we proceed like this:

\[ \tan (\alpha + \beta) = \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} \]

\[ \tan (\alpha + \beta) = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \]

dividing every term by \( \cos \alpha \cos \beta \) we obtain

\[ \tan (\alpha + \beta) = \frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta} \]

\[ \tan (\alpha + \beta) = \frac{\cos \beta}{\cos \alpha} - \frac{\sin \alpha}{\cos \alpha} \]

\[ \tan \alpha + \tan \beta \]

\[ \frac{1}{1 - \tan \alpha \tan \beta} \]
To obtain the subtraction formulae we evaluate \( \sin (\alpha - \beta) \), \( \cos (\alpha - \beta) \) and \( \tan (\alpha - \beta) \) by using the addition formulae.

For example, for \( \cos (\alpha - \beta) \) we proceed like this:

\[
\cos (\alpha - \beta) = \cos [\alpha + (-\beta)] = \cos \alpha \cos (-\beta) - \sin \alpha \sin (-\beta)
\]

but \( \cos (-\beta) = \cos \beta \) and \( \sin (-\beta) = -\sin \beta \) (see frame 67)

then

\[
\cos (\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

In the same way can be shown that

\[
\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad \text{and}
\]

\[
\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

You can prove them as an exercise.

---

**APPENDIX B**

**SUM AND DIFFERENCE OF SINES AND COSINES**

In frame 71 we have seen that

\[
\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta
\]

\[
\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

Evaluating \( \sin (\alpha + \beta) + \sin (\alpha - \beta) \) we obtain:

\[
\sin (\alpha + \beta) + \sin (\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

therefore

\[
\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta
\]
In a similar manner we obtain that
\[
\sin (\alpha + \beta) - \sin (\alpha - \beta) = 2\cos \alpha \sin \beta
\]
\[
\cos (\alpha + \beta) + \cos (\alpha - \beta) = 2\cos \alpha \cos \beta
\]
\[
\cos (\alpha + \beta) - \cos (\alpha - \beta) = -2\sin \alpha \sin \beta
\]

Let \( \alpha + \beta = A \) and \( \alpha - \beta = B \)

then
\[
A + B = (\alpha + \beta) + (\alpha - \beta) = \alpha + \beta + \alpha - \beta = 2\alpha
\]
\[
A - B = (\alpha + \beta) - (\alpha - \beta) = \alpha + \beta - \alpha + \beta = 2\beta
\]

therefore
\[
A + B = 2\alpha
\]
\[
\therefore \alpha = \frac{A + B}{2}, \quad \beta = \frac{A - B}{2}
\]

\[
A - B = 2\beta
\]

Then the expressions we obtained before become:
\[
\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}
\]
\[
\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}
\]
\[
\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}
\]
\[
\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}
\]
THE INEQUALITY $\sin \theta < \theta < \tan \theta$

Let us consider the situation where an arc of a circle subtends an angle $\theta$ ($0 < \theta < \frac{\pi}{2}$) at the centre of the circle (figure below).

AT is the tangent at A to the circle with centre O and radius r.

From $\triangle OAT$ we obtain:

$$AT = OA \tan \theta = r \tan \theta$$

On the other hand:

Area of $\triangle AOAT = \frac{1}{2} OA \times AT$ (because $\triangle AOAT$ is a right angled triangle) therefore

$$\text{Area of } \triangle AOAT = \frac{1}{2} r \cdot r \tan \theta = \frac{1}{2} r^2 \tan \theta$$

Also

$$\text{Area of } \triangle AOB = \frac{1}{2} OB \cdot h = \frac{1}{2} rh$$

but $h = OA \sin \theta = r \sin \theta$

then Area of $\triangle AOB = \frac{1}{2} r \cdot r \sin \theta = \frac{1}{2} r^2 \sin \theta$
APPENDIX C continued

and, provided $\theta$ is measured in radians:

\[
\text{Area of sector AOB} = \left(\frac{\theta}{2\pi}\right) \pi r^2 = \frac{1}{2} r^2 \theta
\]

From the figure, we see that, so long as $\theta$ is acute:

\[
\text{Area } \triangle AOB < \text{Area of Sector AOB} < \text{Area of } \triangle AOT
\]

i.e. \[\frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta\]

as $\frac{1}{2} r^2$ is positive, we can divide each term of the inequality by $\frac{1}{2} r^2$, therefore

\[
\sin \theta < \theta < \tan \theta
\]

APPENDIX D

ADDITIONAL INFORMATION ABOUT THE BOOKLET

This programmed booklet is designed as a result of the evaluation of an activity developed by the Department of Mathematics of Southampton University for the incoming students in Engineering at the beginning of the academic year 1976-1977. The main purpose of this activity was to detect any weaknesses of the students in their knowledge of mathematics by giving them a test comprising 20 multiple choice questions. The result of this evaluation showed that some material on trigonometry should be produced to help the students and this booklet is the result.
The first version of the booklet was tried out with 30 students from three Secondary Schools a few weeks before they took the A-level examination and revised according to the results of that trial.

The diagram of the sequence of the frames in the programme is shown on the next pages.