Microwave cavity measurements
on a magnetoactive plasma

A thesis submitted
for the degree of
Doctor of Philosophy

by

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University of London 1971
Abstract

The interaction of microwaves with a magnetoactive plasma is investigated with a plasma loaded resonant cavity. The cylindrical cavity resonant in the $\text{TE}_{111}$ mode has a coaxial plasma rod which is longitudinally magnetised. When empty the cavity supports two circularly polarised modes with the same resonant frequency and the introduction of the magnetoactive plasma removes the degeneracy. The system is analysed firstly by the perturbation theory which has a number of limitations. Some of these are removed by a second method of analysis in which the propagation characteristics of a waveguide with three coaxial media are determined. This analysis is restricted to a lossless plasma but allows the perturbation theory to be modified and thus applied to the lossy system. The experimental observations on the changes in resonant frequency and Q-value of the cavity caused by the plasma are interpreted with the theory to give values of the plasma characteristics, electron density and collision frequency.
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The origin of plasma physics lies in the studies of electrical discharges through gases around the turn of the century. The most prominent pioneers in the field were J.J. Thomson and I. Langmuir, and it was the latter who introduced the term 'plasma' to describe a macroscopically neutral collection of charged and neutral particles. The existence and fundamental behaviour of this, the fourth state of matter, impinges upon many of the branches of physics and, conversely, plasma research combines many of the basic disciplines of physics into a single study.

The current interest in plasmas derives from research in several of the most advanced fields of physics. Among these are the efforts to control thermonuclear fusion and magnetohydrodynamics for the generation of electricity, the study of astrophysics, the development of gas discharge and semiconductor devices, terrestrial communications and now, in the space age, extraterrestrial communications, involving penetration of the ionosphere and of the plasma sheath surrounding a space vehicle on its re-entry into the Earth's atmosphere. All of these problems demand an increasing knowledge of the behaviour of plasma in its various forms. Much of the basic theory of plasma behaviour is covered by Shkarofsky et al and S.C. Brown.

Electromagnetic theory in its present state stems from the formulation by Maxwell of the equations to describe electromagnetic radiation. Subsequent experimental investigations led to the invention of generators and detectors in hitherto unexplored parts of the electromagnetic spectrum and produced the powerful method of 'wire-less' communication. Techniques advanced rapidly and were extended to higher frequencies from which better communications systems and radar were the principal developments. Research continues in this field with the objectives of increased sophistication in techniques and devices and the extension of its methods towards the infra-red region.

The interaction of electromagnetic radiation with plasmas is particularly important in the fields of communications and electronic devices and the microwave region of the electromagnetic spectrum is important is plasma diagnostics. Many methods of laboratory...
investigation have been used with the following objectives: to check the increasing sophistication of the theoretical descriptions of plasma-wave interactions; to determine the electron density, collision frequency and other important parameters of the plasma, and to examine the behaviour of plasma loaded systems with a view to their possible use in new devices.

Electromagnetic waves propagate through a magneto-active plasma in a variety of modes distinguished by the relative orientations of the electromagnetic field and the magnetising field, the frequency of the radiation and the properties of the plasma. Laboratory wave experiments have been reviewed by Heald and Wharton, Wharton and Trivelpiece and Crawford and extensive theoretical coverage has been given by Ginzburg and Allis, Buchsbaum and Bers.

For laboratory investigations of low density plasmas, plasma loaded resonant microwave cavities have been widely used because of their particular merits. The techniques of the method have been well established for the study of other materials (e.g. ferites, dielectrics) and the analyses of the configurations (generally a plasma column coaxial with a cylindrical cavity) are suited to simplification through a perturbation approach in which the plasma parameters are directly related to the measurements made of shift and resonant frequency and change in $Q$ of the cavity produced by the introduction of a small sample of the material.

The first measurements of this kind on plasmas were made by Adler and other investigations of nonmagnetic plasmas followed. The theories were extended to include magnetised plasmas for many cavity modes.

The perturbation approach however suffers from some serious limitations some of which may be overcome by suitable choice of the mode employed. The problem of ac space charge was removed by Buchbaum and Brown using a $TE_{011}$ mode cavity for which the electric field is perpendicular to the electron density gradients in the plasma. The effect of this space charge can be accounted for with other modes by assuming the plasma column to be in a static uniform electric field which allows the field inside the plasma to be calculated in terms of the field outside. Measurements by Moresco and Zititi with a $TE_{111}$ mode resonant cavity show that this is a good approximation under some conditions which extend the range of application of the perturbation theory.
Other factors which must be taken into account are the presence of an annular glass tube which contains the plasma (if it does not fill the cavity) and holes in the ends of the cavity to accommodate the plasma tube. For TE modes the holes have very little effect because the electric field falls to zero at the end walls. Harris and Balfour, studying ionization decay in cesium vapour, used a TM mode cavity and reduced the effect of the glass walls by placing its cylindrical walls in the region of zero electric field. Also the effect of the holes on the electric field distribution was reduced by measuring the shift in resonant frequency caused by the introduction of a gas of known relative permittivity but this does not take into account further distortions of the field which occur when a plasma is used. Pipiskova and Lukac have examined the field distribution of a TM mode cavity with end holes and a glass tube and indicate that an error 29% will occur in measurements of electron density if these factors are ignored with this system. Kent and Heintz have studied a TE mode cavity filled with plasma. When the electron cyclotron resonant frequency is equal to resonant frequency of the unperturbed cavity.

To advance the theoretical descriptions of plasma loaded cavities the perturbation theory gave way to 'exact' solutions of the problem formulated from Maxwell's equations. Many approaches to the exact solution are possible based upon the choice of simplifying assumptions. If the plasma is unmagnetised then the problem is simply to determine the characteristics of a waveguide filled with three (plasma, glass and air) coaxial isotropic dielectrics. However, the problem is more complex when a magnetic field is applied since the plasma medium is now an anisotropic dielectric. This problem is analogous to that of ferrite loaded waveguides and cavities which have received considerable attention. Solutions have been obtained for waveguides and cavities both filled and partially filled with plasmas and ferrites and show that generally pure TE or TM modes do not exist. (Such designations are however useful and continue to be used to describe modes which in the limit conform to the designation). Kent has solved the problem of a TE mode cavity filled with plasma while Shohet gives solutions for TE and TE mode cavities containing an axial magneto-active plasma surrounded by air.

The most commonly used plasma in this type of experiment is the positive column of a dc discharge which has a radial variation of electron density. An exact type of solution is necessary to take into account this further complication of the problem and the distribution has been described by one of a variety of approximate functions. Kent and Thomas have compared the perturbation formula for the TE modes with exact solutions to determine how a non-uniformity parameter as well as some average density can be obtained from measurements of frequency shift and their experimental results are
"plausible but have no independent confirmation". Bryant has solved the problem of a waveguide filled with isotropic plasma assuming a zero order Bessel function distribution. Agdur and Enander have compared, with good agreement, solutions of a partially filled cavity for several modes with experimental observations. Blevin and Reynolds have analysed the $\text{TE}_{omn}$ and $\text{TM}_{lmo}$ resonances of a cavity containing a non-uniform isotropic annular plasma column and for the $\text{TE}_{qj}$ mode have compared this with the perturbation solution assuming uniform electron density. Thus they related the average electron density obtained from the simpler perturbation formula to the actual density distribution. The problem of a $\text{TM}_{2010}$ mode cavity partially filled with a non-uniform isotropic plasma surrounded by a glass container has recently been studied by Rozwitalski but solutions have only been obtained when the glass container is neglected.

The electron collision frequency is the parameter which characterises losses in the plasma. Such losses can readily be accounted for with the perturbation theory and a relationship between the collision frequency and change in Q-value of the cavity established. In general the exact solutions to the problem are obtained assuming a collisionless plasma. Shohet and Moskowitz have obtained both the real and imaginary parts of the solution to the problem of a cavity partially filled with a non-magnetic but lossy plasma for a number of modes.

The work presented here is concerned with the $\text{TE}_{111}$ mode, in which an essentially plane polarised wave is propagated along the direction of the applied magnetic field. The theoretical analysis of the system is related to its experimental behaviour in order to evaluate the characteristic properties of the plasma.

Following the recognised approach, the basic variables which describe the plasma are defined; the most important of these in this system are the density of electrons and the electron collision frequency. Restriction to a cold plasma allows the behaviour of the plasma in its interaction with an electromagnetic field to be reduced to that of a medium with a tensor permittivity whose components are related to the plasma properties. Propagation of the particular mode through such a medium of infinite extent is considered in order to demonstrate the wave characteristics.

The theory of the perturbation of microwave cavities by the introduction of a small sample of the material under investigation is outlined and its application to a system, suitable for studying the particular mode of interaction, enables the plasma properties to be related to the measured quantities, which are change in resonant frequency and Q-value of the cavity. Some of the limitations of the perturbation theory are removed by reformulating the problem with fewer assumptions and obtaining 'exact' solutions.

A closely related problem is the propagation of electromagnetic waves through ferrites, which have a tensor permeability of the same
form as the tensor permittivity used to describe a plasma. A method developed for the analysis of the propagation of waves through bounded ferrite systems is adapted for application to this plasma loaded system. The less restricted approach means that this exact analysis is applicable over a much wider range of conditions than is probed experimentally. It requires a numerical solution but provides, as before, a relationship between the plasma properties and the measured quantities. Comparison of the two theories shows that the perturbation theory with minor modifications gives an acceptable description of the system well beyond the expected limits of application.

A suitable experimental system was devised in which the positive column of a low pressure, d.c. discharge in argon was the plasma under investigation. The measurements which were made confirmed the expected behaviour of the system and exposed some of the experimental limitations. The results were used to establish the criteria for the design of an optimized system and a second configuration, designed on this basis, was constructed. Measurements of shift in resonant frequency and change in Q were made for a range of pressure and discharge current, these being the variables which control the plasma characteristics. These results were interpreted in terms of the exact theory and the modified perturbation theory to give values of electron density and collision frequency corresponding to each pressure and discharge current.

The results show that the theoretical analyses are based on a suitable description of this type of interaction between electromagnetic-radiation and plasmas and give an adequate account of the principal characteristics of the system. However it is also made apparent that this particular configuration has limited application in the field of plasma diagnostics.
Chapter 1

The Nature of a Plasma in an Electromagnetic Field

1.1 Plasma

A plasma is defined as a medium of positively and negatively charged and neutral particles which is macroscopically neutral.

The name is most commonly applied to a partially ionised gas but has recently found application to the state existing in metals of electrons moving in a lattice of fixed positive charges. Here, it will be reserved to identify the positive column of a low pressure d.c. gas discharge, which is the system for which the term was introduced by Langmuir in 1929. Such a system, being a nearly field-free region containing electrons, ions, atoms and molecules with overall charge neutrality, corresponds well to the above definition of a plasma, in contrast to many of the other systems to which it has since been applied.

1.2 The Equations of Plasma Physics

The behaviour of a plasma may be analysed to a limited extent by considering the interaction of an individual particle with all other particles. This microscopic approach is mathematically impractical. It is sufficient to adopt a statistical approach/
approach and then the plasma can be adequately described by a
number of macroscopic variables.

This method is treated thoroughly in a number of books\(^2, 3, 4\)
and is only briefly outlined here.

For an infinite homogeneous plasma in thermal equilibrium
these variables are:

a) the average electron density \(n_e\);
each electron has a charge \(e\)
and mass \(m_e\).
b) the average ion density \(n_i\);
all ions are assumed to be
of the same type with charge \(-Ze\)
and mass \(m_i\).

Because the plasma has macroscopic neutrality these quantities
are related by the equation:

\[
\bar{n}_e e - \bar{n}_i Ze = 0
\]

c) average velocity of electrons \(v_e\)
da) average velocity of ions \(v_i\)

But since the plasma is in equilibrium

\[
v_e = v_i = 0
\]
e) average electron pressure \(p_e\)
f) average ion pressure \(p_i\)

\(p_e\) and \(p_i\) can be expressed in terms of the electron
temperature/
temperature $T_e$ and ion temperature $T_i$ as

$$\bar{p}_e = \frac{n_e k T_e}{m_e} \quad 1.1$$

$$\bar{p}_i = \frac{n_i k T_i}{m_i} \quad 1.2$$

Where $k$ is the Boltzmann constant. Thermal equilibrium means that $T_e = T_i$.

The application of a magnetic field, $B_0$, causes the electrons and ions to gyrate about the direction of the magnetic field with frequencies given by:

$$\omega_B = \frac{e B_0}{m_e} \quad 1.3$$

$$\omega_B = \frac{Ze B_0}{m_i} \quad 1.4$$

These are known as the electron and ion gyromagnetic frequencies respectively.

It is convenient to express the particle densities in terms of frequencies given by:

$$\omega_p = \left( \frac{n_e e^2}{\varepsilon_0 m_e} \right)^{1/2} \quad 1.5$$

$$\omega_p = \left( \frac{n_i (Ze)^2}{\varepsilon_0 m_i} \right)^{1/2} \quad 1.6$$

Where $\varepsilon_0$ is the permittivity of free space and the two expressions are known respectively as the electron plasma frequency and the ion plasma frequency.
The propagation of an electromagnetic wave through a plasma perturbs the existing equilibrium state. The variables above can then be represented by a combination of the equilibrium variables with a small amplitude perturbation.

\[
\begin{align*}
\tilde{n}_e + n_e & \quad \text{perturbed electron density} \\
\tilde{n}_i + n_i & \quad \text{perturbed ion density} \\
\tilde{v}_e + v_e & = v_e \quad \text{perturbed electron velocity} \\
\tilde{v}_i + v_i & = v_i \quad \text{perturbed ion velocity} \\
\tilde{p}_e + p_e & \quad \text{perturbed electron pressure} \\
\tilde{p}_i + p_i & \quad \text{perturbed ion pressure}
\end{align*}
\]

The charge density is:

\[ \rho = n_e e - n_i Ze \]

1.7

The current vector is:

\[ \mathbf{J} = \left( \tilde{n}_e + n_e \right) e v_e - \left( \tilde{n}_i + n_i \right) Z e v_i \]

\[ = \tilde{n}_e e v_e - n_i Z e v_i \]

1.8

where second order terms are neglected.

Conservation of particles gives the equations:

\[ \dot{n}_e + \nabla \cdot (\tilde{n}_e v_e) = 0 \]

1.9

\[ \dot{n}_i + \nabla \cdot (\tilde{n}_i v_e) = 0 \]

1.10

The equations of conservation of momentum for electrons and ions are:
ions are:

\[ \dot{\nabla}_e m_e \dot{v}_e = Ne(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}_0) - \nabla_{pe} + P_{en} + P_{ei} \quad 1.11 \]

\[ \dot{\nabla}_i m_i \dot{v}_i = -N_i Ze(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}_0) - \nabla_{pi} + P_{in} + P_{ie} \quad 1.12 \]

where second order terms are neglected,

\( \mathbf{E} \) is the electric field of the wave, and

\( \mathbf{B}_0 \), the magnetic field of the wave is neglected in comparison to \( B^o \).

Momentum transfer, between electrons and neutral particles is \( P_{en} \), between ions and neutral particles is \( P_{in} \), between ions and electrons is \( P_{ei} \) or \( P_{ie} \) and these can be written as:

\[ P_{en} = -\nabla_{en} \nabla_{e e} \dot{v}_e \]
\[ 1.13 \]

\[ P_{in} = -\nabla_{in} \nabla_{i i} \dot{v}_i \]
\[ 1.14 \]

\[ P_{ei} = -\nabla_{ei} \nabla_{e i} (\dot{v}_e - \dot{v}_i) \]
\[ 1.15 \]

\( \nabla_{en} \) and \( \nabla_{in} \) are collision frequencies between charged and neutral particles and \( \nabla_{ei} \) is the electron-ion collision frequency.

Assuming that the perturbations are adiabatic

\[ \nabla_{pe} = m_e v_e^2 \nabla_{e e} \]
\[ 1.16 \]

\[ \nabla_{pi} = m_i v_i^2 \nabla_{i i} \]
\[ 1.17 \]

where \( v_e \) and \( v_i \) are the average speeds of thermal motion of electrons and ions.

Maxwell's equations which relate the electric and magnetic fields/
fields of the waves to the current vector and charge density are:

\[ \nabla \times \mathbf{B} = -\mathbf{j} \]
\[ \nabla \times \mathbf{H} = \mathbf{j} + \mathbf{E} \]
\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]

Maxwell's equations together with the equations of conservation of particles and momentum and the expression for charge density and current vector are the equations of plasma physics.

1.3 The Tensor Permittivity of a Magnetoactive Plasma

The above equations will be used to express the behaviour of a magnetoactive plasma in terms of a tensor permittivity. In order to do this it is necessary to introduce a few simplifying assumptions.

Consider the momentum transfer terms \( P_{en} \), \( P_{ei} \) in the equation of conservation of electron momentum. The perturbed velocities \( \mathbf{v}_e \) and \( \mathbf{v}_i \) are produced by the electromagnetic field and because \( m_i \) is much greater than \( m_e \), \( \mathbf{v}_i \) can in general be ignored in comparison to \( \mathbf{v}_e \). Hence

\[ P_{en} + P_{ei} = -n_e m_e \mathbf{v}_e (\mathbf{v}_e \cdot \mathbf{E}) \]
\[ = -n_e m_e \mathbf{v}_e \mathbf{V} \]

where \( \mathbf{V} \) is an effective electron collision frequency which may/
may be assumed to be a constant.³

Further, the restriction that the plasma is 'cold' means that thermal motion is negligible i.e. $\nabla T = 0$. Then Eq 1.11 may be written

$$m_e \frac{dv_e}{dt} = e(E + v_e \times B_o) - m_e \nu v_e$$

1.23

Without loss of generality the uniform static magnetic field $B_o$ may be taken to coincide with the z direction.

The time variation of the electric field of the electromagnetic wave may be taken as $e^{j\omega t}$. Hence $v_{ex}, v_{ey}$ vary similarly.

Under these conditions Eq 1.23 written in component form becomes:

$$v_{ex}(\nabla + j\omega) = \frac{e}{m_e} (E_x + v_{ey} B_0)$$

1.24

$$v_{ey}(\nabla + j\omega) = \frac{e}{m_e} (E_y - v_{ex} B_0)$$

1.25

$$v_{ez}(\nabla + j\omega) = \frac{e}{m_e} E_z$$

1.26

These give specific equations for the components of $v_e$ which may be conveniently written in tensor form:

$$v_e = \frac{e}{m_e} [v_e] E$$

1.27

or:
A similar set of equations for the components of the ion velocity may be derived from the equation of conservation of ion momentum.

However, the contribution made by the ions to the current vector, $\mathbf{J}$, as given by Eq 1.8 is seen to depend upon the ion velocity. The velocity of the ions, as has been stated before, is small compared with the velocity of the electrons; so that neglecting the effect of the ions Eq 1.8 becomes

$$\mathbf{J} = n_e \mathbf{v}_e$$

$$= \frac{n_e e^2}{m_e} [\mathbf{v}_e] \mathbf{E}$$

$$= [\sigma] \mathbf{E}$$

where $[\sigma]$ is the conductivity tensor.

Now for the wave $\mathbf{D} = \mathbf{E} \mathbf{E}$ so that

$$\mathbf{J} + \mathbf{D} = [\sigma] \mathbf{E} + j\omega \mathbf{E}$$

$$= j\omega \mathbf{E}$$

defining the permittivity tensor $[\mathbf{\varepsilon}]$ which thus written explicitly/
explicitly is:

\[
\left[ \xi \right] = \xi_0 \left\{ \left[ 1 \right] + \frac{[\sigma_T]}{j\omega \xi_0} \right\}
\]

\[
= \xi_0 \left\{ \left[ 1 \right] + \frac{n_e^2}{m_e \xi_0} \cdot \frac{1}{j\omega} \left[ v_e \right] \right\}
\]

\[
= \xi_0 \left\{ \left[ 1 \right] + \frac{\omega_p^2}{j\omega} \left[ v_e \right] \right\}
\]

\[
= \xi_0 \left[ \begin{array}{ccc}
1 + \frac{\omega_p^2 (\nu + j\omega)}{j\omega [(\nu + j\omega)^2 + \omega_b^2]} & -\frac{2\omega_p \omega_b}{j\omega [(\nu + j\omega)^2 + \omega_b^2]} & 0 \\
\frac{2\omega_p \omega_b}{j\omega [(\nu + j\omega)^2 + \omega_b^2]} & 1 + \frac{\omega_p^2 (\nu + j\omega)}{j\omega [(\nu + j\omega)^2 + \omega_b^2]} & 0 \\
0 & 0 & 1 + \frac{\omega_p^2}{j\omega (\nu + j\omega)}
\end{array} \right]
\]

Thus the behaviour of the plasma has been reduced to an effective tensor permittivity and it is in terms of this that the interaction of electromagnetic waves with the plasma will, as far as is possible, be described.

It will be convenient to define:

\[
\xi_{11} = 1 + \frac{\omega_p^2 (\nu + j\omega)}{j\omega [(\nu + j\omega)^2 + \omega_b^2]}
\]

\[
\xi_{12} = \frac{\omega_p^2 \omega_b}{\omega [(\nu + j\omega)^2 + \omega_b^2]}
\]
\[
\xi_{33} = 1 + \frac{\omega_p^2}{j\omega(\gamma + j\omega)}
\]

so that equation 1.30 takes the condensed form:

\[
\begin{bmatrix}
\xi
\end{bmatrix} = \begin{bmatrix}
\xi_0 & j\xi_{12} & 0 \\
j\xi_{12} & \xi_{11} & 0 \\
0 & 0 & \xi_{33}
\end{bmatrix}
\]

While Eqs 1.30, 1.31, 1.32 have been derived in Cartesian co-ordinates they are equally true for circularly cylindrical co-ordinates since both are orthogonal systems.

1.4 Characteristics of Wave Propagation

An infinite magneto-active plasma is able to sustain the propagation of several types of electromagnetic waves. The multiplicity of waves is a consequence of the tensor permittivity. The characteristics of these waves have been treated extensively in the literature.2,3,5,6.

The particular system to be considered here is that in which the direction of propagation coincides with the direction of the magnetic field. Under these conditions, Maxwell's equations 1.19 and 1.20 must be solved simultaneously to show the wave characteristics. The time variation is \(e^{j\omega t}\), so that \(\partial/\partial t = j\omega\) and the spatial variation is \(e^{-j\beta x}\), where the propagation constant, \(\beta\), is related to the wavelength, \(\lambda\), by \(\beta = 2\pi/\lambda\). A uniform plane wave solution is sought for which \(\partial/\partial x = 0\).
Maxwell's equations when expanded in component form are thus:

\[
\begin{align*}
\frac{j}{\beta} H_y &= j\omega \varepsilon_o \left[ \varepsilon_{11} E_x + j \varepsilon_{12} E_y \right] \quad 1.34 \\
-j\beta H_x &= j\omega \varepsilon_o \left[ -j \varepsilon_{12} E_x + \varepsilon_{11} E_y \right] \quad 1.35 \\
j\beta E_y &= -j\omega \mu_o H_x \quad 1.36 \\
-j\beta E_x &= -j\omega \mu_o H_y \quad 1.37 \\
H_x &= E_y = 0
\end{align*}
\]

Eliminating \( H_x \) between Eqs. 1.35 and 1.36:

\[
\begin{align*}
\left[ \beta^2 - \omega^2 \mu_o \varepsilon_o \varepsilon_{11} \right] E_y &= -j\omega^2 \mu_o \varepsilon_o \varepsilon_{12} E_x \\
\frac{E_x}{E_y} &= \frac{-j}{\beta^2 - \omega^2 \mu_o \varepsilon_o \varepsilon_{12}} \quad 1.38
\end{align*}
\]

Eliminating \( H_y \) between 1.34 and 1.37:

\[
\begin{align*}
\left[ \beta^2 - \omega^2 \mu_o \varepsilon_o \varepsilon_{11} \right] E_x &= j\omega \mu_o \varepsilon_o \varepsilon_{12} E_y \\
\frac{E_x}{E_y} &= \frac{j\omega^2 \mu_o \varepsilon_o \varepsilon_{12}}{\beta^2 - \omega^2 \mu_o \varepsilon_o \varepsilon_{11}} \quad 1.39
\end{align*}
\]

Equating Eq. 1.38 to Eq. 1.39 gives:

\[
\beta_x^2 = \omega^2 \mu_o \varepsilon_o \left[ \varepsilon_{11} \pm \varepsilon_{12} \right] \quad 1.40
\]

This/
This shows there to be two possible values of $\beta$ for a given frequency and upon substitution of Eq. 1.40 into Eq. 1.38 or Eq. 1.39.

$\beta_+ \text{ corresponds to } E_x = jE_y$

and $\beta_- \text{ corresponds to } E_x = -jE_y$

For each of these two cases the transverse components of the electric field are equal in magnitude but in phase quadrature, a situation which is known as circular polarisation. The sum of the components is a rotating vector of constant amplitude and the fact that rotation may be right-handed or left-handed gives rise to the two values for the propagation constant.

A circularly polarised wave will propagate through the medium with a single value of $\beta$ given by Eq. 1.40 and the medium acts with an effective scalar permittivity:

$$\varepsilon_{\text{eff}} = \varepsilon_0 (\varepsilon_{11} \pm \varepsilon_{12})$$

A plane polarised wave may be regarded as the sum of two contra-rotating circularly polarised waves, each of which here propagates with a distinct value of $\beta$. The consequence of this is that at any value of $\phi$ subsequent to an initial value the sum of the components is a plane polarised wave which has a phase difference with respect to the initial plane of polarisation. Thus the plane of polarisation rotates as the wave progresses. This is the phenomenon of Faraday rotation more familiar as an optical effect.

The duality of the interaction of the electromagnetic waves with the plasma has a simple physical explanation. The electrons in the magnetic/
magnetic field gyrate about the direction of the field with the frequency \( \omega_b \) and the rotation is clockwise when viewed along the direction of the field.

A right-handed (clockwise) circularly polarised electric field vector is characterised by \( \textbf{E}_x = j \textbf{E}_y \) for which

\[
\beta^2 = \omega^2 \mu_0 \varepsilon_0 \left[ \varepsilon_{11} + \varepsilon_{12} \right]
\]

and a left-handed (anticlockwise) circularly polarised electric field vector is characterised by \( \textbf{E}_x = -j \textbf{E}_y \) for which

\[
\beta^2 = \omega^2 \mu_0 \varepsilon_0 \left[ \varepsilon_{11} - \varepsilon_{12} \right]
\]

Now from Eqs. 1.31 and 1.32 for a collisionless plasma \( (v = 0) \)

\[
\varepsilon_{11} \pm \varepsilon_{12} = 1 + \frac{\omega_p^2}{\omega (\omega_b - \omega)} \tag{1.41}
\]

This, and hence \( \beta \), has a singularity when \( \omega_b = \omega \) for the right-handed wave. There is no corresponding singularity for the left-handed wave.

The singularity occurs when the electromagnetic field and the electrons are rotating in the same direction with the same frequency. The wave is effectively 'pumping' an oscillating system at the resonant frequency of that system. When the rotations are opposed there can be no corresponding effect.

1.5 Effect of the Ions

At this stage the neglect of the effects of ions, in sect. 1.3, can/
can be justified. The density of the plasma to be considered is less than $10^{13}$ electrons/cc and the frequency of the electromagnetic wave to be used is in the X-Band microwave region, where $\omega \sim 5 \times 10^{10}$ rad/sec.

Had the ions not been ignored then equation 1.41 would read

$$(\xi_{11} \pm \xi_{12}) = 1 + \frac{\omega_p^2}{\omega(\omega_b + \omega)} + \frac{\omega_p^2}{\omega(\omega_B + \omega)}$$

1.42

in which the final term on the right-hand side represents the effects of the ions. This can also be written

$$\frac{\omega_p^2}{\omega^2} \left( \frac{\omega_B}{\omega + 1} \right)$$

Taking as an example a neon plasma of the density given above, $\omega_p^2/\omega^2$ has the value of $\sim 10^{-8}$ which is negligible in comparison to unity. The ion term only becomes significant when $\omega_B/\omega \sim 1$, which for the same example would require a magnetic field of $\sim 10^3$ gauss. This is quite obviously an impossibly high figure and the ion term never becomes significant in the system to be used.
Chapter 2

Perturbations in Microwave Cavities

2.1 Introduction

There are several ways in which the properties of materials can be investigated at microwave frequencies. They take the three basic forms of interaction between the waves and the material in free space, waveguides and cavities. Of these, the interaction in cavities lends itself particularly to simplification. When a small sample of the material is introduced into a resonant microwave cavity the resonant frequency of the cavity is altered. The amount of the shift in frequency depends upon the properties of the material and simple formulae can be deduced relating the shift to the properties. The method has found widespread application because of the simplicity of the relationships, although the approximations necessary imply limitations. The properties of such materials as dielectrics, ferrites and plasmas have been investigated in this way.

The concept of perturbation of cavities was introduced by Slater to evaluate the effect of a small distortion of a cavity boundary. There naturally followed from this a theory to cover the introduction of media into a cavity. This theory is given most clearly by Waldron and is outlined here to show the approximations and limitations.

2.2 General Perturbation Formula

The electric and magnetic fields in a cavity in the unperturbed state/
state at resonance are:

\[ E = e^{j\omega t} \]
\[ H = e^{j\omega t} \]

where \( E \) and \( H \) are functions of position dependent on the mode, and \( \omega_r \) is the angular resonant frequency.

The introduction of a small, non-conducting sample into the cavity modifies the fields and the resonant frequency such that they may be expressed in terms of the former functions with additional terms as

\[ (E + E_1)e^{j(\omega_r + \delta\omega)t} \]
\[ (H + H_1)e^{j(\omega_r + \delta\omega)t} \]

Substitution of these into Maxwell's equations 1.18 and 1.19 gives

\[ \nabla \times E = -j\omega E = -j\omega_0 \mu H \quad 2.1 \]
\[ \nabla \times H = j\omega_0 D = j\omega_0 \epsilon_0 E \quad 2.2 \]
\[ \nabla \times (E + E_1) = -j(\omega_r + \delta\omega)(E + E_1) \quad 2.3 \]
\[ \nabla \times (H + H_1) = j(\omega_r + \delta\omega)(D + D_1) \quad 2.4 \]

By subtraction

\[ \nabla \times E_1 = -j\omega E_1 - j\delta\omega(E + E_1) \quad 2.5 \]
\[ \nabla \times H_1 = j\omega_0 D_1 + j\delta\omega(D + D_1) \quad 2.6 \]

Take the scalar products of \( H \) with Eq 2.5 and \( E \) with Eq 2.6 and add:

\[ H \cdot \nabla \times E_1 + E \cdot \nabla \times H_1 = j\delta\omega \left[ E \cdot (D + D_1) - H \cdot (E + E_1) \right] \]
\[ + j\omega_0 \left[ E \cdot D_1 - H \cdot E_1 \right] \]

Use the vector identity \( A \cdot \nabla \times B = B \cdot \nabla \times A - \nabla \cdot (A \times B) \).
substitute from Eqs 2.1 and 2.2 and integrate over the volume of the cavity $\Omega$:

$$j\omega \iiint_{\Omega} \left[ \mathbf{E} \cdot (\mathbf{D} + \mathbf{D}_1) - \mathbf{H} \cdot (\mathbf{B} + \mathbf{B}_1) \right] d\mathbf{v} = \begin{cases} 2.7 \text{a} \\ 2.7 \text{b} \\ 2.7 \text{c} \\ 2.7 \text{d} \end{cases}$$

If the walls of the cavity are taken as perfectly conducting then term 2.7 d is zero.

Now $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$

and outside the sample

$$\mathbf{B}_1 = \mu_0 \mathbf{H}_1 \text{ and } \mathbf{D}_1 = \varepsilon_0 \mathbf{E}_1$$

so that outside the sample

$$\mathbf{E} \cdot \mathbf{D}_1 = \varepsilon_0 \mathbf{E} \cdot \mathbf{E}_1 = \mathbf{D} \cdot \mathbf{E}_1$$

and $\mathbf{H} \cdot \mathbf{B}_1 = \mu_0 \mathbf{H} \cdot \mathbf{H}_1 = \mathbf{D} \cdot \mathbf{H}_1$

The only contribution from 2.7 b and 2.7 c is over the volume of the sample $\Omega$.

If the perturbation is small, then, in term 2.7 a $\mathbf{D}_1$, $\mathbf{B}_1$ may be neglected in comparison with $\mathbf{D}$ and $\mathbf{B}$ so that Eq 2.7 becomes

$$\frac{\delta \omega}{\omega} = \frac{\iiint_{\Omega} \left[ \mathbf{E} \cdot (\mathbf{D} - \mathbf{E} \cdot \mathbf{D}_1) - (\mathbf{H} \cdot \mathbf{B} - \mathbf{H} \cdot \mathbf{B}_1) \right] d\mathbf{v}}{\iiint_{\Omega} (\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B}) d\mathbf{v}} \quad 2.8$$

This formula, relating the shift in frequency to the fields in the cavity, has been derived without reference to a particular system and is therefore general in its application. Some consideration must now/
now be given to the system which is to be used so that the integrals may be evaluated.

2.3 Application to Specified System

2.3 i. To determine the cavity mode required it must be assumed that to a first approximation the introduction of the perturbation does not significantly affect the field pattern.

The only practical shape for a plasma sample is a cylindrical rod. To approach an interaction of the type considered in Sect. 1.4, the mode must be chosen so that along this rod, contained within the cavity, the electric field is transverse and able to be circularly polarised. The two contra-rotating circularly polarised waves propagate through the plasma with different $\omega/b$ relationships. A waveguide of circular cross-section, loaded with an axial magnetoactive plasma rod will correspondingly divide each circularly polarisable mode into two different modes of propagation. Considering a cavity to be a short length of such a waveguide system, for each of the modes of circular polarisation the cavity should have different resonant frequencies. These resonances occur at the frequencies for which the length of the cavity is an integral number of half-wavelengths.

A cylindrical cavity resonant in the $TE_{111}$ mode meets these requirements. The mode may be circularly polarised and the cavity resonates when it is one half wavelength long. The fields associated with this mode in a cavity of length $l$ and radius $r_3$ are/
The electric field is purely transverse and is shown in Fig. 1. Along the axis of the cavity it is plane polarised but a circularly polarised field may be established by feeding the cavity from two ports at $90^\circ$ with signals of equal amplitude and in phase quadrature. Thus, by applying to the cavity the two circularly polarised modes separately, each resonant frequency of the cavity can be investigated without interference from the other mode. If the cavity is fed from a single port then a plane-polarised wave is established within the cavity. Considering this as the sum of two contra-rotating circularly polarised waves it is evident that the two resonant modes will be excited in the cavity together.
The magnetic field pattern is unimportant as the plasma, having permeability \( \mu_0 \), does not directly interact with the wave through the magnetic fields.

The essential features of the system to be employed are a cylindrical cavity resonant in the TE\(_{111}\) mode with a coaxial plasma rod, in a uniform magnetic field directed along the axis.

2.3.ii. For many modes Eq. 2.8 is evaluated assuming that the effect of the plasma on the value of the electric field may be neglected. This is in contrast to the situation when measuring ferrite properties by perturbation methods, where the internal fields must always be determined because they are seriously changed by the presence of the ferrite. For the TE\(_{1mn}\) modes in a plasma loaded cavity with electron densities so high that \( \omega_p^2 \sim \omega (\omega + \omega_p) \) the microwave field configuration is drastically modified by a.c. space charge. It is necessary then to determine the fields inside the medium in terms of the fields of the unperturbed cavity.

Since there is no longitudinal component of the electric field the problem is reduced to two dimensions. The diameter of the plasma rod is
taken to be sufficiently smaller than the wavelength of the
applied field so that it can be considered to be in a uniform field
and the effects of time variations can be ignored\textsuperscript{16}. This quasi-
static approach allows the electric field to be expressed as the
gradient of a scalar potential function\textsuperscript{14}, 7.

Recent experimental work\textsuperscript{3,14} with a plasma loaded TE\textsubscript{111} mode
resonant cavity has shown that the perturbation theory which includes
the effects of space charge is in good agreement with the
experimental results for values of $\frac{\omega^2}{\rho} < 0.4$.

The problem to be solved is the electrostatic system of a
two-dimensional disc of relative permittivity $\varepsilon$ in a uniform

\[
\begin{bmatrix}
\varepsilon_{uu} & j\varepsilon_{uz} \\
-j\varepsilon_{uz} & \varepsilon_{uu}
\end{bmatrix}
\text{unidirectional electric field of}
\begin{align*}
\text{strength } E_{0}/2 \text{ (Fig. 2).}
\end{align*}

The centre of the disc is the origin of sets of coordinates $(x, y)$
and $(r, \theta)$, and the electric field is taken to be in the $x$ direction.

A potential function $\psi$ will be established such that
everywhere

\[ E = - \nabla \psi \]

and

\[ \nabla^2 \psi = 0 \]

and on the boundary, $r = r_1$, the normal component of $\mathbf{D}$ must
be continuous.

In cylindrical co-ordinates/
co-ordinates

\[ \frac{\partial^2 \psi}{\partial \nu^2} + \frac{1}{\nu} \frac{\partial \psi}{\partial \nu} + \frac{1}{\nu^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0 \]

and the boundary condition is

\[ \frac{\partial \psi_{\text{outside}}}{\partial \nu} \bigg|_{\nu = \nu_1} = \varepsilon_{\text{II}} \frac{\partial \psi_{\text{inside}}}{\partial \nu} \bigg|_{\nu = \nu_1} + j \varepsilon_{12} \frac{\partial \psi_{\text{inside}}}{\partial \phi} \bigg|_{\nu = \nu_1} \]

These are satisfied by the potential

\[ \psi = \begin{cases} -\frac{E_0}{2} x + P_1 \frac{x}{\nu^2} + P_2 \frac{y}{\nu}, & \nu > \nu_1, \\ A_1 x + A_2 y, & \nu < \nu_1 \end{cases} \]

where

\[ A_1 = \frac{-(\varepsilon_{\text{II}} - 1)}{(\varepsilon_{\text{II}} + 1)^2 - \varepsilon_{12}^2} \cdot E_0 \]

\[ A_2 = \frac{-j \varepsilon_{12}}{(\varepsilon_{\text{II}} + 1)^2 - \varepsilon_{12}^2} \cdot E_0 \]

\[ P_1 = (A_1 + E_0/2) r_1^2 \]

\[ P_2 = A_2 r_1^2 \]

So that the fields inside the disc are

\[ E_x = \frac{\varepsilon_{\text{II}} + 1}{(\varepsilon_{\text{II}} + 1)^2 - \varepsilon_{12}^2} \cdot E_0 \]

\[ E_y = \frac{j \varepsilon_{12}}{(\varepsilon_{\text{II}} + 1)^2 - \varepsilon_{12}^2} \cdot E_0 \]

If the cavity is excited in the circularly polarised mode there is a second field \( + j E_0/2 \) in the \( y \) direction which gives rise to internal
The total internal fields due to a uniform circularly polarised field are Eq 2.10 plus Eq 2.11:

\[ E_x^{\text{TOT}} = \frac{+E_o}{(\varepsilon_{11} + 1) \pm \varepsilon_{12}} \]
\[ E_y^{\text{TOT}} = \frac{-jE_o}{(\varepsilon_{11} + 1) \pm \varepsilon_{12}} \]

The vector sum of these is a circularly polarised field in phase with the original field and of magnitude

\[ \frac{E_o}{(\varepsilon_{11} + 1) \pm \varepsilon_{12}} \]

2.3 iii. Now, returning to Eq 2.8, the magnetic fields do not contribute to the numerator since the permeability of the plasma is \( \mu_0 \). The contribution of the electric fields may be evaluated using the result above for the internal fields produced by a circularly polarised field.

\[ E_1 \cdot D - E_2 \cdot D = \varepsilon_o E \cdot E_1 - E \cdot \left[ \varepsilon_{\text{eff}}(E + E_1) - \varepsilon_o E \right] \]
\[ = (\varepsilon_o - \varepsilon_{\text{eff}}) E \cdot (E + E_1) \]
\[ = - \varepsilon_o \frac{(\varepsilon_{11} - 1 \pm \varepsilon_{12})}{(\varepsilon_{11} + 1 \pm \varepsilon_{12})} E_o^2 \]
So that the numerator has the value
\[
\iiint (\mathbf{E} \cdot \mathbf{D} - \mathbf{E} \cdot \mathbf{D}) dV = -\varepsilon_0 \left( \varepsilon_{n1} - 1 \pm \varepsilon_{12} \right) \varepsilon_0^2 \pi \gamma_1^2 \int_0^L \sin \pi \frac{\varphi}{C} d\varphi
\]
\[2.12\]

The denominator is
\[
\iiint (\mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B}) dV = 2\iiint_{V_o} \varepsilon_0 \mathbf{E} \cdot \mathbf{E} dV
\]
and using the field expressions of Eq 2.9 this is, for the circularly polarized case:
\[
= 2 \varepsilon_0 E_0^2 \int_0^{2\pi} \left[ \frac{\mathbf{J}_1^2(kr)}{(kr)^2} + \frac{1}{(kr)^2} \mathbf{J}_1^2(kr) \right] r dr \int_0^L \sin \pi \frac{\varphi}{C} d\varphi
\]
\[2.13\]

This may be integrated\(^\dagger\) to give the result
\[
= 2 \varepsilon_0 E_0^2 \pi \gamma_1^2 \left[ 1 - \left( \frac{1}{1.841} \right)^2 \right] \mathbf{J}_1^2(1.841) \int_0^L \sin \pi \frac{\varphi}{C} d\varphi
\]
\[2.14\]

Hence on dividing Eq 2.12 by Eq 2.13 an expression is obtained for the frequency shift in terms of the properties of the plasma.

\[
\frac{\Delta \omega}{\omega r} = -\frac{\left( \varepsilon_{n1} - 1 \pm \varepsilon_{12} \right) \gamma_1^2}{\varepsilon_0 \left( \varepsilon_{n1} + 1 \pm \varepsilon_{12} \right) 2 \gamma_3^2} \frac{1}{1 - \left( \frac{1}{1.841} \right)^2} \mathbf{J}_1^2(1.841)
\]
\[2.14\]

The empty cavity resonant in the TE\(_{111}\) mode is degenerate in that it has the same resonant frequency for both right- and left-handed circularly polarised modes. The introduction of the magnetoactive plasma removes this degeneracy, and each mode has a resonant frequency given by Eq 2.14. The cavity, when excited in its plane polarised mode, will/
will exhibit a resonance at each of these frequencies.

2.3 iv. The shift in resonance can now be directly related to the plasma variables.

First put
\[
\frac{\gamma_1^2}{2r_3^2} \frac{1}{\left[1 - \frac{1}{(1.841)^2}\right]} = R
\]

Two values for \( r_1 \) were used:

for \( r_1/r_3 = 0.2333, R = 0.1140 \)

for \( r_1/r_3 = 0.1667, R = 0.0582 \)

So
\[
\frac{\delta \omega_\pm}{\omega_r} = - \frac{(\xi_1 - 1 \pm \xi_{12})}{(\xi_1 + 1 \pm \xi_{12})} R
\]

Using Eq 1.31 and Eq 1.32 to substitute for \( \xi_{11} \) and \( \xi_{12} \), and at the same time normalising so that \( \omega_p/\omega = \Omega_p \) and \( \omega_b/\omega = \Omega_b \), Eq 2.15 takes the form:
\[
\frac{\delta \omega_\pm}{\omega_r} = \frac{\Omega_p^2}{(2 - \Omega_p^2 \mp 2 \Omega_b) - \frac{2 \nu}{\omega_r}} R
\]

It can be seen that this shift is complex; the real part represents the true change in resonant frequency while twice the imaginary part is the change in \( 1/q \) for the cavity (Ref. 12 p.115).

The change in frequency is then:
\[
\frac{\delta \omega_\pm}{\omega_r} = \frac{\Omega_p^2 \left(2 - \Omega_p^2 \mp 2 \Omega_b\right)}{(2 - \Omega_p^2 \mp 2 \Omega_b)^2 + (2 \nu/\omega_r)^2} R
\]

and/
and the change in $\frac{1}{Q}$:

$$
\delta \left( \frac{1}{Q} \right) = \frac{4 \Omega_p^2 \nu/\omega_r}{(2 - \Omega_p^2 / 2 \Omega_b^2)^2 (2 \nu/\omega_r)^2} \Omega_b^2
$$

2.17

For a collisionless plasma there will be no change in $\frac{1}{Q}$ and since the $Q$ value is a measure of cavity losses a collisionless plasma is a lossless plasma. However, a collisionless plasma is only a theoretical concept; in all experimental plasmas there are collisions. From Eq 2.17 it can be seen that the effect of collisions is to reduce the $Q$-value of the cavity; collisions are thus a loss mechanism.

Eq 2.16 and Eq 2.17 are shown graphically in Figs. 3 and 4. Fig. 3 shows how, for small $\omega_p$, the resonance is split by the application of a magnetic field. The resonant frequencies of the two modes diverge as $\Omega_b$ is increased from 0 to 1 and then converge to $\omega_b$, the resonant angular frequency of the unperturbed cavity, from opposite sides as $\Omega_b$ becomes very large. When there is some loss in the system, the shift in the right-handed mode remains finite in the region $\Omega_b > 1$. The shape of this curve is typical for the lossy case where the maximum value of the shift is $\pm \frac{\Omega_p^2 \Omega_b^2 \omega_r}{2(\nu/\omega_r)}$ when $(1 - \Omega_b) = \pm \nu/\omega_r$.

The left-handed mode with small loss does not depart significantly from the lossless case.

Fig. 4 shows the change in $\frac{1}{Q}$ as a function of $\Omega_b$ for the lossy case. For the right-handed mode it has a peak value of $-\frac{\Omega_p^2 R}{(\nu/\omega_r)}$ at $\Omega_b = 1$ reducing to half this when the shift in frequency is maximum. For the left-handed mode the change in $\frac{1}{Q}$ is not large enough to show on the scale/
Fig. 3. Resonant frequency against $\sigma_b$ for zero and small loss $\sigma_p \ll 1$

\[ \omega_L = \omega_0 \left( 1 + \frac{\sigma_p^2 R}{2} \right) \]

Fig. 4. Change in $(VQ)$ against $\sigma_b$

Right handed mode

\( \frac{\sigma_p}{\sqrt{\omega_0}} \)
scale of this figure.

At constant $\Omega_p$, the changes in resonant frequency and $(1/Q)$ are directly proportional to $\Omega_P^2$, provided that $\Omega_P \ll 1$.

The application of the perturbation formula has shown how the properties of the plasma are related to the shift in resonant frequency of the cavity. The measurement of the resonant frequencies of the two modes at a known value of $\Omega_p$ enables the plasma density and collision frequency to be determined, while the measurement of the change in Q provides an alternative method or a check.

2.4. Limitations on the Application of the Perturbation Theory

It is difficult to establish quantitative limitations on the method because the extent to which variations in the field may be tolerated is a factor which cannot be precisely defined. Persson has however made some estimations from which the plasma density is restricted to values corresponding to $\Omega_P \ll 1$. In order to extend the upper limit of electron density which can be measured the frequency of the applied fields should be as high as is practicable. For this reason it was decided to use a cavity resonant in the X-band frequency range i.e. of the order $10^{10}$ Hz.

The plasma rod should be in a uniform circularly polarised field. This is produced only along the axis of the cavity so over the cross-section of a rod of any finite size there will be variations in magnitude of the field and the wave will be elliptically polarised. A small diameter/
diameter rod is thus desirable, but in practice a narrow plasma is more difficult to sustain. For the two chosen values of \( r \), the variations in magnitude of the electric field are 6.8% and 3.7%. The radius of the plasma is important from another aspect. The plasma extends through the ends of the cavity and it is desirable to have the holes as small as possible to minimise their effect.

The method assumes the fields to be effectively unchanged but it can be shown\(^1\) that a cavity with two concentric media will not support a pure TE mode. The actual changes in the fields will be different from those which the perturbation theory allows inside the plasma although they can be expected to have some similarity.

The theory has been formulated on the basis of a uniform plasma but it is well known that an experimental plasma has a radial variation of electron density.

The effect of the glass tube needed to contain the plasma has not been an integral part of the analysis but it will be considered later when the results are interpreted (Sect 5.3).

The resonant frequency of the TE\(_{111}\) mode in the empty cavity is sufficiently removed from the resonant frequency of any other mode that no confusion should occur. However, the introduction of the glass has a different effect on each mode and it was necessary to check experimentally that no other mode interfered with the required TE\(_{111}\) mode. This could impose some limitations on the measurable shift.

It can be seen from Eq. 2.17 that the Q of the cavity decreases as the electron density increases. This in practice was found to limit some measurements.
While the emphasis is on extending the measurement of electron density to the highest possible value it is as well to note that there is a lower limit determined by how accurately a difference in frequency can be measured.

This list includes the practical limitations of the system as well as the limitations of application inherent in the perturbation theory. In the next chapter the system is analysed more fully so as to remove some of the limitations of the theory and the practical limitations are discussed later.
3.1 Introduction

Some of the limitations imposed in Sect. 2.4 can be overcome by the exact analysis of a system of three coaxial media in a cylindrical waveguide (Fig. 5) following the method used by Epstein. The centre medium of radius \( r_1 \) is the plasma whose permittivity \( \begin{bmatrix} \varepsilon \end{bmatrix} \) is defined by Eq 1.33. This is surrounded by a glass annulus of permittivity \( (\varepsilon_0 \varepsilon_z) \) and external radius \( r_2 \). Between this and the metal wall of the waveguide, radius \( r_3 \), the medium is air of permittivity \( (\varepsilon_0 \varepsilon_z) \).

No restriction is placed upon the fields which may exist in the system so that the limit to \( \Omega_p \) is removed. The electric field in the plasma is not assumed to be uniformly circularly polarised and thus the plasma region can be of any size. Further, longitudinal electric fields are allowed and the glass tube is made an integral part of the analysis.

Epstein developed the method for the propagation of waves through bounded ferrites, whose permeability is a tensor of the same form as \( \begin{bmatrix} \varepsilon \end{bmatrix} \). The symmetry of Maxwell's equations allows the analysis to be readily applied to a plasma. The wave equations are used to evaluate a wave number \( k \) in terms of the components of the plasma permittivity and \( \omega \) and \( \beta \). The field components in the plasma are derived and the application of boundary conditions gives a set of equations, for which the compatibility condition determines permissible values of the propagation constant.
3.2 Propagation Characteristics in the Plasma

The plasma is assumed to be homogeneous, and in a uniform magnetic field in the z direction so that the permittivity is uniform and is defined by Eq 1.33. The field components of the electromagnetic wave vary as $e^{j\omega t}$ so that $\partial / \partial t = j\omega$. Maxwell's equations take the form:

\[ \nabla \times \mathbf{H} = j\omega \varepsilon \mathbf{E} \quad 3.1 \]
\[ \nabla \times \mathbf{E} = -j\omega \mu_0 \mathbf{H} \quad 3.2 \]
\[ \nabla \cdot (\varepsilon \mathbf{E}) = 0 \quad 3.3 \]
\[ \nabla \cdot \mathbf{H} = 0 \quad 3.4 \]

\( \mathbf{H} \) can be used as a vector potential from which \( \mathbf{E} \) can be derived. For this Eq 3.1 is written:

\[ \mathbf{E} = -\frac{j}{\omega} \varepsilon^{-1} \nabla \times \mathbf{H} \quad 3.5 \]

where \( \varepsilon^{-1} \) is the reciprocal tensor to \( \varepsilon \) given by:

\[ \varepsilon^{-1} = \frac{1}{\varepsilon_0} \begin{bmatrix} \mathbf{M} & jK & 0 \\ -jK & \mathbf{M} & 0 \\ 0 & 0 & M_3 \end{bmatrix} \quad 3.6 \]

and

\[ \mathbf{M} = \frac{\varepsilon_{11}}{\varepsilon_{11} - \varepsilon_{12}} \quad ; \quad K = -\frac{\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{12}} \quad ; \quad M_3 = \frac{1}{\varepsilon_{33}} \]

The wave equation for \( \mathbf{H} \) is obtained by taking \( \nabla \times \) Eq 3.5 and substituting from Eq 3.2:

\[ \omega^2 \mu_0 \mathbf{H} = \nabla \times (\varepsilon^{-1} \nabla \times \mathbf{H}) \quad 3.7 \]

The expansion of this Cartesian co-ordinates gives three equations for the components of \( \mathbf{H} \):
for the components of $\mathbf{H}$:

$$
\omega^2 \mu_0 \varepsilon_0 H_x = \frac{\partial}{\partial y} \left[ M_y \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \right] - \frac{\partial}{\partial z} \left[ j K \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + M_z \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \right]
$$

$$
\omega^2 \mu_0 \varepsilon_0 H_y = \frac{\partial}{\partial z} \left[ M_z \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + j K \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \right]
- \frac{\partial}{\partial y} \left[ M_y \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) + j K \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \right]
$$

$$
\omega^2 \mu_0 \varepsilon_0 H_z = \frac{\partial}{\partial x} \left[ - j K \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + M_z \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \right]
- \frac{\partial}{\partial y} \left[ M_y \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) + j K \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \right]
$$

Using Eq 3.4 and putting $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ these become:

$$
\left[ \omega^2 \mu_0 \varepsilon_0 + M_3 \nabla^2 + M \frac{\partial^2}{\partial z^2} \right] H_x + j K \frac{\partial^2 H_y}{\partial z^2}
- \frac{\partial}{\partial z} \left[ \left( M - M_3 \right) \frac{\partial}{\partial x} + j K \frac{\partial}{\partial y} \right] H_z = 0 \tag{3.8}
$$

$$
\left[ \omega^2 \mu_0 \varepsilon_0 + M_3 \nabla^2 + M \frac{\partial^2}{\partial z^2} \right] H_y - j K \frac{\partial^2 H_x}{\partial z^2}
- \frac{\partial}{\partial z} \left[ \left( M - M_3 \right) \frac{\partial}{\partial y} - j K \frac{\partial}{\partial x} \right] H_z = 0 \tag{3.9}
$$

$$
\left[ \omega^2 \mu_0 \varepsilon_0 + M \left( \nabla^2 + \frac{\partial^2}{\partial z^2} \right) \right] H_z - j K \frac{\partial}{\partial z} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = 0 \tag{3.10}
$$

Eq 3.9 is multiplied by $\pm j$ and added to Eq 3.8 and the field components.
components are combined as \( Q_{1,2} = H_x + jH_y \). To Eq 3.10 is added:

\[
\nabla K \frac{\partial}{\partial z} \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) = 0
\]

with the result that two equations are produced in \( Q \) and \( H_z \):

\[
\left[ \omega^2 \mu_0 \varepsilon_0 - \nabla^2 \right] \left[ (M + M_3) \frac{\partial^2}{\partial z^2} \right] Q - \frac{\partial}{\partial z} \left[ (M - M_3) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y} \right) \right] H_z = 0 \tag{3.11}
\]

\[
\left[ \omega^2 \mu_0 \varepsilon_0 + M_3 \nabla^2 \right] H_z + \nabla \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y} \right) Q = 0 \tag{3.12}
\]

From these it is possible to obtain equations in either \( H_z \) or \( Q \) alone, in the following way:

Operating on Eq 3.11 with \( +K \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y} \right) \) gives:

\[
L(H_z) = 0
\]

and operating on Eq 3.12 with \( (M - M_3) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y} \right) \) gives:

\[
L(Q) = 0 \tag{3.13}
\]

where \( L \) is the operator:

\[
L = \omega^2 \mu_0 \varepsilon_0 \nabla^2 + \omega^2 \mu_0 \varepsilon_0 \left[ (M + M_3) \nabla^2 \right] \frac{\partial^2}{\partial z^2} + M M_3 \nabla^4
\]

\[
+ \left[ (M - M_3) \nabla^2 \right] \frac{\partial^2}{\partial z^2} + (M^2 - K^2) \frac{\partial^4}{\partial z^4}
\]

The fields \( H_x \) and \( H_y \) can be expressed in terms of \( Q \) as

\( H_x = \frac{1}{2} (Q_1 + Q_2) \) and \( H_y = -\frac{1}{2} j (Q_1 - Q_2) \). Now since the operator \( L \) is the same for both \( Q_1 \) and \( Q_2 \), the Eq 3.13 must also be satisfied by \( H_x \) and \( H_y \). Eq 3.7 for \( H \) has thus been transformed to:

\[
L(H) = 0 \tag{3.14}
\]

However,
However, this is a fourth order equation and will thus produce solutions that do not satisfy the second order Eq 3.7. In order that such spurious solutions are eliminated a subsidiary equation is required in addition to Eq 3.14. Eq 3.12 is chosen for this purpose and will be put into a form convenient for later use. The equation is differentiated w.r.t. \( z \) and \( \frac{\partial H}{\partial z} \) is substituted for from Eq 3.4. The result is:

\[
\frac{\partial}{\partial x} \left[ \left\{ \omega^2 \mu_0 \varepsilon_0 + M \left( \nabla^2 + \frac{\partial^2}{\partial z^2} \right) \right\} H_x + jK \frac{\partial^2}{\partial z^2} H_y \right] \\
+ \frac{\partial}{\partial y} \left[ \left\{ \omega^2 \mu_0 \varepsilon_0 + M \left( \nabla^2 + \frac{\partial^2}{\partial z^2} \right) \right\} H_y - jK \frac{\partial^2}{\partial z^2} H_x \right]
\]

3.15

Returning to Eq 3.14 the field \( H \) is a function of the spatial co-ordinates, \( x, y \) and \( z \). For propagation in the preferred \( z \) direction, \( H \) may be taken as a separable function with the \( z \) variation of the form \( e^{-j\beta z} \), whence:

\[
H = H(x, y)e^{-j\beta z}
\]

and \( \frac{\partial}{\partial z} = -j\beta \), \( \frac{\partial^2}{\partial z^2} = -\beta^2 \)

This allows \( L \) to be written in the form:

\[
L \equiv M M_3 \nabla_t^4 + \left[ \left( M + M_3 \right) \left( \omega^2 \mu_0 \varepsilon_0 - M \beta^2 \right) + K^2 \beta^2 \right] \nabla_t^2 \\
+ \left[ \left( \omega^2 \mu_0 \varepsilon_0 - M \beta^2 \right)^2 - K^2 \beta^4 \right]
\]

or alternatively as:

\[
L \equiv M M_3 \left( \nabla_t^2 + \kappa^2 \right) \left( \nabla_t^2 + \kappa^2 \right)
\]

3.16

where/
where:

$$k' = \left( \frac{1}{2 M M_3} \left[ \left( \omega^2 \mu_0 \varepsilon_0 - M \beta^2 \right) \left( M + M_3 \right) + M \beta^2 \pm \sqrt{q} \right] \right)^{3.17}$$

and

$$q = \left[ \left( M - M_3 \right)^2 \left( \omega^2 \mu_0 \varepsilon_0 - M \beta^2 \right)^2 + 2 \left( M + M_3 \right) \left( \omega^2 \mu_0 \varepsilon_0 - M \beta^2 \right) K^2 \beta^2 + K^2 \left( k^2 + 4 M M_3 \right) \beta^4 \right]^{1/2}$$

From Eq 3.16 it is clear that \( \mathbf{H} \) must satisfy the two wave equations:

$$\left( \nabla_t^2 + k^2 \right) \mathbf{H}_1 = 0$$

$$\left( \nabla_t^2 + k^1 \right) \mathbf{H}_2 = 0$$

where \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) are different since they have different wave numbers \( k \) and \( k^1 \). The general solution of Eq 3.14 is then the sum of the two partial fields:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

and it may be taken that \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) satisfy the divergence Eq 3.4 separately.

### 3.3 Field Components in the Plasma Region

Using the abbreviations:

$$\sigma = k \beta^2 \quad \text{and}$$

$$\tau = \mathbf{H} \left( k^2 + \beta^2 \right) - \omega \mu_0 \varepsilon_0$$

where

$$\tau' = \tau \quad \text{or} \quad \tau^1 \quad \text{when} \quad k_1 = k \quad \text{or} \quad k^1$$

the/
the subsidiary Eq 3.15 may be written:

\[
\frac{\partial}{\partial x} \left[ \tau_i H_x + j \sigma H_y \right] + \frac{\partial}{\partial y} \left[ -j \sigma H_x + \tau_i H_y \right] = 0
\]

This suggests that there is some function from which \( H_x \) and \( H_y \) may be derived by differentiation. This function will be written as

\[(\tau_i^2 - \sigma^2)\Pi\]

where:

\[
\begin{align*}
\tau_i H_x + j \sigma H_y &= (\tau_i^2 - \sigma^2) \frac{\partial \Pi}{\partial y} \\
-j \sigma H_x + \tau_i H_y &= -(\tau_i^2 - \sigma^2) \frac{\partial \Pi}{\partial x}
\end{align*}
\]

From these:

\[
\begin{align*}
H_x &= j \sigma \frac{\partial \Pi}{\partial x} + \tau_i \frac{\partial \Pi}{\partial y} \\
H_y &= -\tau_i \frac{\partial \Pi}{\partial x} + j \sigma \frac{\partial \Pi}{\partial y}
\end{align*}
\]

Substituting these into the divergence equation, Eq 3.4 gives:

\[-j \beta H_z = -j \sigma \nabla_t^2 \Pi\]

\(\Pi\) will satisfy the same equation as its derivatives, so

\[I(\Pi) = 0\] and

\[\nabla_t^2 \Pi = -k_i^2 \Pi\]

\[\therefore H_z = -\frac{\sigma}{\beta} k_i^2 \Pi\]

\[= -j \sigma k_i^2 \frac{\partial \Pi}{\partial z}\]

The/
The components of $\mathbf{H}$ may thus be written in the tensor form

$$\mathbf{H} = [S_{\mathbf{H}}] \nabla \pi$$  \hspace{1cm} \text{3.19}

where the tensor

$$[S_{\mathbf{H}}] = \begin{bmatrix} j\sigma & \tau_i & 0 \\ -\tau_i & j\sigma & 0 \\ 0 & 0 & -j\sigma k_z^2 \end{bmatrix}$$

The components of $\mathbf{E}$ can now be derived from Eq 3.5 which becomes:

$$\mathbf{E} = -\frac{j}{\omega} [S_{\mathbf{E}}]^{-1} \nabla \times ([S_{\mathbf{H}}] \nabla \pi)$$

This may be simplified to an equation of the same form as Eq 3.19:

$$\mathbf{E} = -\beta[S_{\mathbf{E}}] \nabla \pi$$  \hspace{1cm} \text{3.20}

where

$$[S_{\mathbf{E}}] = \begin{bmatrix} a_i & jb & 0 \\ -jb & a_i & 0 \\ 0 & 0 & g \end{bmatrix}$$

and

$$a_i = \left[ a_i c_i - k^2 (\beta^2 + k_i^2) \right] \frac{1}{\omega \varepsilon_0}$$

$$b = -\beta \omega \mu_0$$

$$c_i = \frac{-M_3 c_i}{\omega \varepsilon_0} \frac{k_i}{\beta^2}$$

The system described in Sect. 3.1 has a cylindrical plasma rod, so it is now convenient to represent Eq 3.18 in the cylindrical polar co-ordinates $r$, $\phi$ and $z$:

$$\frac{\partial^2 \pi}{\partial Y^2} + \frac{1}{Y} \frac{\partial \pi}{\partial Y} + \frac{1}{r^2} \frac{\partial^2 \pi}{\partial \phi^2} + k_i^2 \pi = 0$$  \hspace{1cm} \text{3.21}

This/
This is Bessel's equation for which the solution is:

$$\Pi_n = C_n (k_i r) e^{j(n\theta - \beta z)}$$

$C_n (k_i r)$ is the general Bessel function of order $n$ (an integer) and can be identified with the Bessel function of the first kind, $J_n (k_i r)$, in this system where the medium is along the axis $r = 0$. $\Pi$ has a value for both $k$ and $k'$ and the general solution is:

$$\Pi_{gen} = A_1 \Pi + A_2 \Pi'$$

where $A_1$ and $A_2$ are arbitrary constants. This can be substituted into Eq 3.19 and Eq 3.20 to obtain the field components in the plasma which are thus:

$$E_Y = -\beta \left[ A_1 [k_0 J_n'(k r) - \frac{n b}{V} J_n(k r)] - A_2 [k_0' J_n'(k'r) - \frac{n b'}{V} J_n(k'r)] \right]$$

$$E_\phi = j\beta \left[ A_1 [k b J_n'(k r) - \frac{n a}{V} J_n(k r)] - A_2 [k' b J_n'(k' r) - \frac{n a'}{V} J_n(k' r)] \right]$$

$$E_z = j\beta^2 \left[ A_1 gj_n J_n(k r) + A_2 gj_n' J_n(k r) \right]$$

$$H_Y = j \left[ A_1 [k_0 J_n'(k r) + \frac{n c}{V} J_n(k r)] - A_2 [k_0' J_n'(k' r) + \frac{n c'}{V} J_n(k' r)] \right]$$

$$H_\phi = - \left[ A_1 [k c J_n'(k r) + \frac{n\sigma}{V} J_n(k r)] + A_2 [k' c J_n'(k' r) + \frac{n\sigma'}{V} J_n(k' r)] \right]$$

$$H_z = - \frac{c}{b} \left[ A_1 k^2 J_n(k r) + A_2 k'^2 J_n(k' r) \right]$$

where $J_n'(k_i r)$ is the derivative of $J_n (k_i r)$ w.r.t. the argument $(k_i r)$.

In a waveguide filled with a single isotropic dielectric the modes for $\pm n$ are essentially the same. Since $J_{-n}(x) = (-1)^n J_n(x)$ the ratio
of the magnitudes of the transverse electric field components $\frac{E}{r/b}$ is
the same for $\pm n$, while the components have a phase difference of either
$+\frac{n}{2}$ or $-\frac{n}{2}$. This means that the $E$ vector in the transverse plane
rotates either clockwise or anticlockwise. For the field components
above, however, the ratios of $\frac{E}{r/b}$ are radically different for $\pm n$.
Two distinct modes are characterised by $\pm n$, which correspond with the
two modes expected from the previous chapter. The value of $n = 1$ was
used in the perturbation theory but must here be given the values $\pm 1$ to
obtain all the solutions.

Since $k$ and $k^\perp$, as expressed by Eq 3.17, are dependent on even powers
of $\beta$ they will have the same values for $\pm \beta$. The characteristics of the
wave are not influenced by the direction of propagation. The standing
wave which exists in a cavity has the same $\omega/\beta$ relationship as the
progressive wave in a similar propagation system and the solutions which
will be found are thus applicable to either type of system.

3.4 Application of Boundary Conditions

Maxwell's equations for the isotropic media, glass and air,
where the fields vary as $e^{j(\omega t - \beta z)}$ are:

$$\nabla \times \mathbf{E} = -j\omega \mu \sigma \mathbf{H}$$  \hspace{1cm} (3.22)

$$\nabla \times \mathbf{H} = j\omega \varepsilon_0 \varepsilon_i \mathbf{E}$$  \hspace{1cm} (3.23)

$$\nabla \cdot \mathbf{E} = 0, \hspace{1cm} \nabla \cdot \mathbf{H} = 0$$  \hspace{1cm} (3.24)

Eqs 3.22, 3.23 give the wave equation:

$$\nabla^2 \mathbf{V} = \omega^2 \mu_0 \varepsilon_0 \varepsilon_i \mathbf{V}$$  \hspace{1cm} (3.25)

$\mathbf{V} = \mathbf{E}$ or $\mathbf{H}$
The expansion of this in circular cylindrical co-ordinates gives

the wave equation for the \( z \) components of \( E \) and \( H \):

\[
\gamma^2 \frac{\partial^2 V_z}{\partial z^2} + \gamma \frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial \phi^2} + \gamma^2 (\omega^2 \epsilon_0 \mu_0 \epsilon_1 - \beta^2) V_z = 0 \tag{3.26}
\]

Introducing the constant \( k_1^2 = (\omega^2 \mu_0 \epsilon_0 \epsilon_1 - \beta^2) \) this takes the form of Bessel's equation for which the general solutions can be written as:

\[
E_z = j \left[ B_1 J_n(k_2 r) + B_2 N_n(k_2 r) \right] e^{j \mu \phi} e^{-j(\omega t - \beta z)} \tag{3.27}
\]

\[
H_z = \left[ B_3 J_n(k_2 r) + B_4 N_n(k_2 r) \right] e^{j \mu \phi} e^{-j(\omega t - \beta z)} \tag{3.28}
\]

where \( N_n \) is the Bessel function of the second kind and these solutions apply to glass, while for air the arbitrary constants \( B_1 \) are replaced by \( C_1 \) and \( k_2 \) becomes \( k_3 \).

The \( E_\phi \) and \( H_\phi \) components are obtained in terms of \( E_z \) and \( H_z \) from the expansion of Eqs 3.22 and 3.23:

\[
E_\phi = \frac{1}{k_2} \left[ - \frac{j \beta}{\gamma} \frac{\partial E_z}{\partial \phi} + j \omega \mu_0 \frac{\partial H_z}{\partial \gamma} \right] \tag{3.29}
\]

\[
H_\phi = -\frac{1}{k_2} \left[ j \omega \epsilon_0 \epsilon_1 \frac{\partial E_z}{\partial \gamma} + \frac{j \beta}{\gamma} \frac{\partial H_z}{\partial \phi} \right] \tag{3.30}
\]

Substituting for \( E_z \) and \( H_z \) and omitting the common variation \( e^{j(\mu \phi + \omega t - \beta z)} \),

\[
E_\phi = \frac{j}{k_2} \left[ B_1 J_n(k_2 r) + B_2 N_n(k_2 r) \right] + \omega \mu_0 \epsilon_1 \left[ B_3 J_n(k_2 r) + B_4 N_n(k_2 r) \right] \tag{3.31}
\]
where again these apply to glass with a corresponding pair in $G, k$ for air.

The equations for the field components in the three media involve the ten arbitrary constants, $A_1, A_2, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4$. The field components must be matched according to the following ten independent boundary conditions:

at the metal wall the electric field must be normal:

i.e. $E_z = E_\phi = 0$ at $r = r_3$

at the interface of two media the tangential field components are continuous:

i.e. $E_z, E_\phi, H_z, H_\phi$ are continuous at $r = r_2$ and $r_1$.

These lead to a set of ten simultaneous equations in the arbitrary constants. For these equations to be compatible the determinant formed by the terms associated with each constant must vanish. This condition provides the relationship between the angular frequency $\omega$ and the propagation constant $\beta$.

The first two boundary conditions are:

$$C_j \tilde{J}_n(k_3 r_3) + C_j' \tilde{N}_n(k_3 r_3) = 0$$

and these are used to eliminate $C_2$ and $C_4$ from the other equations, reducing the number of equations to eight.

Noting
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<th>( C_{\sigma} )</th>
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<th>( A_{\alpha} )</th>
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**Table I.**
Noting that \( n = \pm 1 \), the determinant of the terms of these eight equations is presented in Table I. At the head of each column is shown the constant with which the terms are associated and to the left of each row is shown the condition from which the equation is produced.

From inspection of Eq 3.17 it seems possible that under certain conditions \( k \) and \( k^1 \) may become imaginary or even complex. Epstein, in his analysis of a ferrite system, states that the compatibility condition would only allow real values for \( k \) and \( k^1 \). However, for the plasma case situations were found in which a purely imaginary value of other \( k \) or \( k^1 \) gave sensible solutions in accord with experimental results. The changes to Table I which are necessary to produce these results are obtained by identifying \( C_n(k_r) \) in the plasma region with \( I_n(k_r) \), the modified Bessel Function of the first kind. Whence, in the case when \( k \) is imaginary, the first four components of the column in the determinant headed by \( A_1 \) become as shown in Table 2. Corresponding changes are required to the column below \( A_2 \) when \( k^1 \) is imaginary. The notation in Table 2 is such that while \( k^2 \) is negative, \( k_i \) is used to mean \((-k^2)^{1/2}\). All the rest of Table I remains unchanged and again, to obtain the two/

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<td>(-\beta^2 g I_1(kr))</td>
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<td>(\sigma k^2 / \beta I_1(kr))</td>
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<tr>
<td>(-\beta k b I'_1(kr) + \beta ma / \gamma I_1(kr))</td>
</tr>
<tr>
<td>(\tau k I'_1(kr) + \mu \sigma / \gamma I_1(kr))</td>
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</table>
two solutions \( n \) takes the values \( \pm 1 \).

3.5 Numerical Solution

The determinant is too complicated to provide any clear analytical expression for the propagation characteristics and it therefore becomes necessary to use numerical methods of solution. The problem was programmed for a Sirius computer such that the permissible values of \( \omega \) were obtained for given values of \( \beta \) and the other parameters of the system.

These parameters with the values used were:

Relative permittivity of glass, \( \varepsilon_2 = 4.62 \).
Relative permittivity of air, \( \varepsilon_2 = 1.00 \).
Radius of the plasma, \( r_1 = 0.0035 \text{ m}, 0.0025 \text{ m} \).
Outer radius of the glass, \( r_2 = 0.0050 \text{ m}, 0.0035 \text{ m} \).
(The alternatives are for the two different systems used).
Radius of the cavity, \( r_3 = 0.015 \text{ m} \).
\( \Omega_\rho \) for the range of values, 0 to 0.6.
\( \Omega_\beta \) for the range of values, 0 to 2.
\( /3 \) for the range of values, 0 to 250 m\(^{-1}\).

Typical results obtained from this analysis are shown in Figs. 6 and 7. Fig. 6 is a pair of \( \omega/\beta \) curves for a propagation system with the second set of radii, \( \Omega_\rho^2 = 0.2 \) and \( \Omega_\beta = 0.6 \). It shows how the non-magnetic mode, for the same conditions is split by the application of the magnetic field. By considering the permissible values of \( \omega \) at/
Curves for: \( r_1 = 0.025 \text{ m} \), \( r_2 = 0.035 \text{ m} \), \( r_3 = 0.015 \text{ m} \)
\( \varepsilon_1 = 4.62 \), \( \varepsilon_2 = 1.00 \), \( \Lambda_p = 2 \), \( \Lambda_b = 6 \).  

Positive mode

Negative mode.

\( \omega = \text{velocity of light} \)

\( \beta \)  

Fig. 5. Coaxial System.

Fig. 6. Typical \( \omega/\beta \) curves.
Fig. 7 Resonant frequency against $\Omega_b$. $\gamma_0 = 0$.

$\gamma_1 = 0.0025 \text{m}$. $\gamma_2 = 0.0035 \text{m}$. $\beta = 149.5 \text{m}^2$.
at a particular value of $\beta$, the resonant frequencies of a cavity are
determined with this theory. In Fig. 7 the resonances of the second
system used, in which $\beta = 149.5 \text{ m}^{-1}$, are shown as a function of $\Omega_b$
for various values of $\Omega_\rho^2$. Comparison of Fig. 7 with Fig. 3, in which
corresponding results, obtained from the perturbation theory are
presented, shows that the results from the two theories do not differ
significantly.

3.6 Assessment of this theory

Many of the limitations set out in Sect. 2.4 have been overcome.
A comprehensive theory to cover all values of $\Omega_\rho^2$ has been established
and the limitation due to the choice of field components has been
removed. Any size of plasma is allowed and the effect of the glass has
become an integral part of the theory. This has been achieved at the
expense of restricting the theory to the analysis of a lossless plasma
system. However it will be shown in Sect. 5.3 how this theory allows
the perturbation theory to be modified and applied to the experimental
system in order to account for the effects of plasma losses.

Two difficulties remain: neither the radial inhomogeneity of the
plasma density nor the effect of the holes in the ends of the cavity has
been considered. A major change in the theory would be necessary to
accommodate the former and it will be assumed that the application of
this theory to the experimental measurements provides an average electron
density for the plasma. The latter is of more consequence since some
account/
account must be taken of it in order to match the theoretical and experimental results: this is considered in Sect. 5.3 ii.
CHAPTER 4

The First Resonant Cavity

4.1 Design of the Experimental System

It has been shown that the electron density and collision frequency of a plasma can be related to the propagation characteristics of a waveguide loaded with the plasma. Sect. 2.3 and 3.3 show that the characteristics of the mode considered theoretically can be examined experimentally with a cavity resonant in the TE_{111} mode containing a coaxial plasma rod. To make measurements on high electron densities high frequency waves are required, and in order to make measurements in the region of $\Omega_b = 1$ these high frequency waves mean that a magnetic field of several kilogauss is also needed.

An experimental system compatible with these requirements was designed. The microwave system was in the x-band and the cavity had a resonant frequency of about 9 GHz. The magnetic field of up to 6.5 kilogauss, for which $\Omega_b \approx 2$, was provided by an electromagnet with a pole gap sufficient to contain the resonant cavity and discharge tube. A system operating at a higher frequency was considered unsuitable for two reasons: the required magnetic field could not be so easily produced; and it would be more difficult to make a satisfactory smaller discharge tube necessary to fit into the smaller resonant cavity.

The resonant frequency, $f_r$, Hz, of an air-filled cavity of length 1 cm and/
and radius \( r_3 \) cm is given for the \( \text{TE}_{111} \) mode by

\[
f_r = 2.998 \times 10^6 \left[ \frac{1}{4} l^2 + \left( \frac{l - 84}{4\pi r_3^2} \right)^2 \right]^{1/2}
\]

In order to make the most accurate measurements the cavity should have maximum Q-value and this is dependent on the ratio of radius to length. Maximum Q for a cavity resonant in this mode is obtained when \( r_3 = 0.75l \). A cavity with values of \( r_3 = 1.5 \text{ cm} \) and \( l = 2 \text{ cm} \) has a resonant frequency \( f_r = 9.485 \text{ kMc/s} \) from Eq 4.1.

To excite the cavity with circularly polarised waves a pair of inputs are required from mutually perpendicular directions. The coupling should be weak so that the Q of the cavity is not greatly reduced by losses through these ports. Fig. 8 shows the method of coupling adopted: the magnetic fields in the waveguide and cavity are coupled through a circumferential slot closer to one end of the cavity than the other.

The discharge tube, to be coaxial with the cavity, must pass through the end plates and to minimise losses these holes should be no larger than necessary. As a variety of tube diameters were to be tried the end plates were made detachable. Further, one end plate was made rotatable so that the existence of circularly polarised waves within the cavity could be checked.

The size of the plasma tube required was difficult to determine. Assuming a few per cent of the atoms in the gaseous discharge to be ionized the electron density for a pressure of 1 torr could be found. Using/
Fig. 8  Coupling of the magnetic fields at the slot.

Magnetic field.

Electric field.
Using the perturbation theory, this allowed the diameter of the discharge tube required to give a suitable shift in frequency to be determined. Such calculations gave no more than an unreliable estimate of the order of magnitude necessary, but did indicate that a measurable shift should be obtained with a discharge tube diameter less than 1 cm.

Some experimental difficulties in producing a stable discharge were experienced at this time. One was obtained with an external diameter of 1 cm. This was rather large, giving rise to high radiation losses through the holes and a low Q for the cavity. However it was decided to make such measurements as were possible with this plasma tube so that a second system could be produced later using the experience of the behaviour of this system as a more stable basis for the design.

4.2 Experimental System and Preliminary Measurements

The cavity shown in Plate I was made with values of $r_3 = 1.50$ cm and $l = 1.980$ cm having a resonant frequency $f_r$ as given by Eq 4.1 of 9.570 GHz. It was fed from two mutually perpendicular waveguides. At each port the magnetic fields in the waveguide and cavity were coupled through a slot 1.0 cm long by 0.15 cm wide, parallel to the broad dimension of the waveguide. Facility was provided in each arm for an adjustable probe to vary the coupling to the cavity should this have been necessary. The end plates were separate so that they could be readily changed. At one end a square plate was held on by four bolts at the corners of the cavity block; at the other end a circular recess
in the block allowed a rotatable connection to be made. The circular
end plate was held in position by a square clamping plate.

With end plates without holes and fed from either arm the resonant
frequency of the cavity was 9.545 GHz and its Q about 2000. When end
plates with holes were fitted and the plasma tube introduced the
resonant frequency of the cavity fell below the x-band region. The
square end plate was replaced by a tight fitting plate which penetrated
into the cavity and reduced the length to 1.680 cm; this brought the
resonant frequency of the cavity with the plasma tube to 8.805 GHz.

The plasma tube, Plate I, was 11 cm overall length with the diameter
of the positive column region 0.70 cm inside and 1.00 cm outside. The
cathode was an oxide coated coiled-coil of the type used in fluorescent
tubes and the anode a molybdenum cylinder of diameter 0.50 cm, designed
to spread the discharge over the full cross-section of the tube. When
the tube had been cleaned and the cathode activated it was sealed
containing argon at a pressure of 1.2 torr. The part of the tube
subsequently to be used in the cavity contained a visibly uniform
positive column for currents up to 20 mA. Plate II shows the
assembled cavity with the plasma tube held in position by two clamps
at the extreme ends of the block.

The magnetic field was produced by a Newport Instruments Type A
electromagnet with 4" pole pieces which was used without the pole tips.
A pole gap of 7.3 cm was needed to accommodate the cavity and discharge
tube system and this allowed fields of up to 4.0 k gauss to be obtained
with a current of 10 amps supplied from batteries. Over the region to
be/
be used the field was uniform to better than 1%. Continuous measurement, with a Hall probe, of the field strength was required so, before the introduction of the cavity, the field strength at the centre was calibrated against the field strength at a point below the position of the cavity.

The cavity and sealed discharge tube were arranged in the magnetic field as shown in Fig. 9. Preliminary investigations were carried out with the cavity fed from one arm only, the other being terminated with a matched load. Q-curves were obtained for several discharge currents with no magnetic field to ascertain the order of magnitude of the shift in resonant frequency and Q-values of the cavity. The Q of the loaded cavity was found to be of the order of 150 for low currents and the value decreased as the current was raised. It was also found that the shift in resonant frequency was generally less than the half-power width of the Q-curve. (fig 10). This inevitably leads to large percentage errors in measurements of the resonance shift. Further, the application of a magnetic field splits the resonance, but because the cavity was being fed from a single arm, the two resonant modes were present in the cavity together. If the two Q-curves cannot be separated by more than the half-power width of the Q-curves then the two resonances cannot be easily distinguished.

The measurements made indicate that this was unlikely to be realized and under these circumstances it would be preferable to excite the cavity with only one mode at a time so that the resonances could be individually observed. The cavity had been provided with two input ports/
Fig. 9 The first resonant cavity and plasma tube.
Fig. 10. Q-curves of first system; \( \Omega_b = 0 \).

Shift of resonant frequency.
Half-power width of Q-curve.
ports so that by feeding with two signals of equal amplitude but in phase quadrature the two circularly polarised modes could be established separately.

Attempts were then made to achieve this in practice. The method designed for checking the existence of circularly polarised waves within the cavity involved coupling a piece of circular waveguide to the rotatable end plate of the cavity so that the waveguide and the cavity had a common axis. Subsequently a circular to rectangular transition was made and a crystal detector terminated the line. Rotation of these components allowed the power in any chosen plane of polarisation within the cavity to be measured. The necessary lengthening of the system could not be done within the restricted gap between the pole pieces of the magnet. By dismantling the arrangement circularly polarised waves could be set up and checked but not over a sufficiently wide range of frequencies. With the arrangement of Fig. 9 no other satisfactory method of checking circularly polarised waves could be devised. The second system was designed to overcome this failing. It was decided that for the first system such measurements as were possible would be made on the cavity by feeding from one arm only and observing the two resonances together.

A block diagram of the microwave system used for measurements on this system is shown in Fig. 11. Power was fed to the cavity through one input arm while the other arm was terminated with a matched load. The microwave power level incident at the input port was monitored via a/
Fig.11. Block diagram of first microwave system.
a 20 db coupler in the input arm immediately prior to the cavity. This power level was maintained at an arbitrary but constant level by varying the attenuation in the line whenever the klystron frequency was adjusted.

The power level in the cavity was measured by an open ended waveguide probe which picked up power radiated through the hole at one end of the cavity. Repeated measurements of some Q-curves for physical variations in the output coupling showed the coupling to be independent of frequency over the typical range required for a Q-curve. The coupling itself was thus non-resonant and was also sufficiently weak that the power measured could be taken as directly proportional to the power in the cavity.21

To obtain a Q-curve the discharge current and magnetic field were fixed, and the frequency of the klystron was set at a number of values around the resonant frequency. For each of these values the input level was adjusted to the required value and the relative output measured. Q-curves were obtained for a range of discharge currents from 3 to 15 mA and a range of magnetic fields from 0 to 4 kilogauss.

4.3 Results

A selection of the Q-curves obtained are shown in Figs. 12 to 15. Fig. 12 shows the resonances of the cavity for various discharge currents with zero magnetic field. As the current increases the shift of resonant frequency is seen to increase while the Q of the cavity decreases.
Resonances of the first cavity for various discharge currents with no magnetic field.
decreases. For the highest current shown, 18 mA, the resonant frequency and Q of the cavity are both indefinite and no other measurements were made at this current.

Using the values of shift in resonant frequency and $\frac{1}{Q}$ from these Q-curves and applying the results of the perturbation theory estimates of the electron density and collision frequency were made in the following way.

Eqs 2.16 and 2.17 giving the shift in resonant frequency and $\frac{1}{Q}$ from the perturbation theory reduce for the non-magnetic case to:

$$\frac{\Delta \omega}{\omega} = \frac{\Omega_\rho^2 (2 - \Omega_\rho^2)}{(2 - \Omega_\rho^2)^2 + \left(\frac{2 \omega}{\omega_\gamma}\right)^2} R$$

$$\delta\left(\frac{1}{Q}\right) = \frac{4 \Omega_\rho^2 \frac{\nu}{\omega_\gamma}}{(2 - \Omega_\rho^2)^2 + \left(\frac{2 \nu}{\omega_\gamma}\right)^2} R$$

Fig 16 shows the shift in resonant frequency and $\frac{1}{Q}$ as functions of discharge current taken from Fig. 12; and both are approximately linear relationships. Values taken from these graphs for a representative current of 10 mA and substituted into the equations above give $\Omega_\rho^2 = 0.12$ and $\frac{\nu}{\omega_\gamma} = 0.29$.

From Fig. 12 it can also be seen that the peak power monitored diminishes as the discharge current is increased. This is due to the reduction in Q of the cavity. Slater's analysis of resonant cavities shows that the power coupled, at resonance, through a loosely coupled port is proportional to the Q of the cavity. For this system where both the input and output are loosely coupled and the incident power is maintained/
maintained at a constant level, the peak output power should be proportional to $Q^2$. However, it is found that as the $Q$ of the cavity is reduced for higher currents so the peak power measured falls off more rapidly than $Q^2$. The output coupling is a short cut-off plasma loaded waveguide for which an increase in the electron density will result in an increase in the cut-off frequency and greater losses in the plasma. Thus a reduction in the power transmitted beyond that caused by the change in $Q$ is to be expected.

Figs. 13, 14 and 15 show the effect of imposing a magnetic field on the system. The two modes present have different resonant frequencies and it can be seen how the degenerate non-magnetic resonance curve is split by the magnetic field giving two superimposed $Q$-curves. It can also be seen that it is only for the higher discharge currents that the curves are sufficiently separated to be observed as individual peaks.

Fig. 13 shows the resonances of the cavity for various discharge currents when a magnetic field is applied such that $\Omega_b = 0.315$. The two resonant frequencies of the cavity, for the two circularly polarised modes can be clearly seen. The left-hand mode is only slightly affected by the magnetic field and its resonant frequency can be measured. The resonant frequency of the right-hand mode is increased by an amount rather less than the half-power width of the $Q$-curves and can be measured only for some of the higher currents. Because of the considerable overlap of the $Q$-curves meaningful measurements of the $Q$-values are not practicable. However it can be seen that the power in the two modes and the $Q$-values are of the same order of magnitude, indicating that the two modes have similar losses.
Fig. 13
Resonances of the first cavity for various discharge currents with magnetic field of 10 kgauss
Fig. 14: Resonances of the first cavity for various discharge currents with magnetic field of 4.0 kgauss.

Fig. 15: Resonances of the first cavity for discharge current of 9.2 mA for various magnetic fields.
A typical set of Q-curves as the magnetic field is increased for a fixed current of 9.2 mA is shown in Fig. 15. It can be seen that, as the magnetic field is increased, the resonant frequency of the left-handed mode decreases towards the resonant frequency of the cavity with no plasma, while, for the positive mode the resonant frequency increases and the two resonances diverge. The Q value and power in the positive mode decline rapidly as the field is increased above \( \Omega_b \sim 0.5 \), showing that the losses in this mode are much greater. Accurate measurements of the resonant frequency of this mode thus become increasingly difficult.

Measurements on the resonant frequencies of the cavity for values of \( \Omega_b > 1 \) were limited by two experimental factors: the difficulty of maintaining a stable discharge and the inability to obtain magnetic fields greater than 4.0 kilogauss with the apparatus used. Fig. 14 shows the resonances of the cavity for various currents with this maximum magnetic field. The resonant frequency of the positive mode is now seen to be below that of the negative mode. As the magnetic field is increased above 3.14 kilogauss (\( \Omega_b = 1 \)) the two resonances converge.

From these Q-curves the typical behaviour of the plasma-loaded cavity for variations in discharge current and magnetic field is apparent. The observed effects are in general agreement with the variations expected from the theoretical analysis. However, the accuracy and significance of individual Q-curves is limited. This was not unexpected in view of the experimental difficulties of simultaneously monitoring and maintaining magnetic field, discharge current, applied frequency/
frequency and input power level in order to make the measurements.

To obtain as accurate an indication of the behaviour of the cavity as the measurements allow, a graph showing change in resonant frequency as a function of magnetic field was constructed in the following way. For each value of magnetic field, the change in resonant frequency of both modes was plotted as a function of discharge current. Assuming the changes to be proportional to current, straight line graphs were drawn, the slopes of which are change in resonant frequency per mA. In Fig. 17 the values of these slopes are plotted as a function of magnetic field. For comparison, the curves obtained from the perturbation theory are drawn for two cases: for a lossless plasma with $\Omega_P^2 = 0.0105/\text{mA}$ and with $\omega'/\gamma' = 0.28$ and $\Omega_P^2 = 0.0115/\text{mA}$.

It can be seen that the frequency shifts measured are much greater than are compatible with a value of $\omega'/\gamma' = 0.28$. The consistency of these results for the shift in resonant frequency indicates that the losses in the plasma have been overestimated. In Sect. 5.3 the interpretation of the experimental measurements in terms of the exact theory and the perturbation theory in a modified form leads to some adjustment in the evaluation of $\Omega_P^2$ and $\gamma/\omega_r$. However, at this point it is sufficient to acknowledge that the perturbation theory gives a useful indication of the behaviour of this type of plasma loaded cavity. The results obtained with this first system allow an assessment of the practical arrangement to be made which provides the basis for the design of an improved system.
Resonant frequency and \( \sqrt{Q} \) against discharge current.
Fig. 17. Change in resonant frequency/m_A against \( \Omega_b \).

- Right-handed mode
- Left-handed mode

- Experimental points
- Curve for \( \frac{1}{\Omega} \) = 0.32 (perturbation theory)
- Curve for \( \frac{1}{\Omega} \) = 0
CHAPTER 5

The Second Resonant Cavity and Plasma Tube

5.1 Design of the Experimental System

Measurements made with the first system were limited by three factors:

a) The loaded cavity had a low Q-value caused mainly by radiation losses through the holes in the end plates;

b) The output coupling was imprecise and lacked a suitable method for checking the presence of circularly polarised waves within the cavity;

c) A magnetic field greater than 4.0 kilogauss could not be maintained with the pole gap and magnet power supply used.

The consideration given to these factors in order to produce a satisfactory design for the second system is presented in the following sections.

5.1 (i)

The results obtained for the frequency shift and Q of the first cavity enable estimates to be made of these quantities for other cavities and a system to be designed which minimizes the limitations imposed by a low Q. If measurements are to be made with both circularly polarised modes present then it is essential for the difference in resonant frequency to be greater than the half-power width of the Q-curve so that the two resonancies may be resolved. Alternatively, if the cavity is excited with a single mode the ability to make measurements on higher electron densities/
densities and in the region of \( \Omega_b \sim 1 \) must be considered.

The accuracy to which measurements of the frequency shift may be made is limited either by the accuracy of the wavemeter or by the accuracy to which the peak of the resonance curve can be specified. For Q-curves such as those in Fig. 12 the resonant frequency may be determined to within 10\% of the half-power width. A cavity of similar construction and dimensions to that used can be expected to have a similar unloaded Q; i.e. \( \sim 2000 \). Thus, the best that is possible is a half power width \( \sim 5 \) MHz, giving an accuracy of \( \pm 0.5 \) MHz. Since the wavemeter is able to be read to be better than \( \pm 0.5 \) MHz and the Q value of the cavity will be much less than 2000, the accuracy of the measurements will always be limited by the Q of the cavity.

The results of Chapter 4 show that the perturbation theory gives a reasonable analysis of the behaviour of the cavity and thus will be used, because of its direct and simple involvement of the parameters, to indicate the behaviour of similar systems. However, the results obtained when values of \( \Omega^2 > 0.1 \) are used must be viewed with some scepticism as the range of application of the perturbation theory is probably exceeded.

The loaded Q-value of the cavity, \( q_L \), may be related to contributions due to the unloaded cavity, \( q_0 \), the holes and the glass, \( q_H \), and the plasma, by the expression:

\[
\frac{1}{q_L} = \frac{1}{q_0} + \frac{1}{q_H} + \delta(\frac{1}{q})
\]

5.1

where/
where \( \delta \left( \frac{1}{q} \right) \) is the change in \( \frac{1}{q} \) caused by the plasma as given in Eq 2.17.

The main reason for the low Q-value of the loaded cavity was radiation loss through the holes and so a reduction in this loss offers the best possibility for improving the value of \( Q_L \). Consequently, the Q-values of the cavity with four different sizes of hole and glass tube were measured.

Measurements were made with the cavity used in the first system with different end plates and containing a glass tube of internal radius \( r_1 \) cm and external radius \( r_2 \) cm, this being also the radius of the holes in the end plates and \( r_2 = (r_1 + 0.1) \) cm. The results obtained are tabulated below and shown graphically in Fig. 18.

<table>
<thead>
<tr>
<th>( r_1 ) cm</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.35 (as used in 1st system)</td>
<td>220</td>
</tr>
<tr>
<td>0.25</td>
<td>600</td>
</tr>
<tr>
<td>0.15</td>
<td>1050</td>
</tr>
<tr>
<td>0 (glass rod ( r_2 = 0.1 ) cm)</td>
<td>1900</td>
</tr>
</tbody>
</table>

Table 3

The thickness of the end plates also affects the Q but it was found that provided the thickness was greater than about 0.4 cm any further increase had a negligible affect upon the Q. The results above are all for plates 0.5 cm thick.

As the Q of the cavity is improved by reducing radiation losses so the/
Fig. 18. Q-value against $r_i$.

Fig. 19. Calculated $\%$ error / $r_i$. 
the effect of plasma losses becomes more important. For example, in the first system with no applied magnetic field the Q of the cavity was reduced from 145 to 96 by the plasma of a 10 mA discharge. Thus the contribution of $\frac{1}{Q}$ to $\frac{1}{Q_L}$ is $\frac{1}{290}$, which is comparable with the contribution of $\frac{1}{220}$ from the holes and glass when $r_1 = 0.35$.

A quantitative assessment of the effect of the plasma on the Q of the cavities with discharge tubes of various radii may be obtained most conveniently for a constant value of $\Omega_p^2$. With the first system it was shown that measurements with $\Omega_p^2 \approx 0.1$ are possible in the region of $\Omega_b = 1$. Thus $\Omega_p^2 = 0.115$ will be taken as a suitable value with which to make the necessary calculations.

Eqs 2.16 and 2.17 show that, for constant $\Omega_p^2$, both shift in resonant frequency and $\delta (\frac{1}{q})$ are proportional to the area of cross-section of the discharge tube (through the constant $R$). In the case where $\Omega_b = 0$ and $r_1 = 0.35$, $\delta (\frac{1}{q})$ has a value of $\frac{1}{290}$ and the shift in resonant frequency is 50 MHz; corresponding values in proportion to area have been calculated for other radii, and similar calculations for the two modes when $\Omega_b = 0.5$ have also been made. Eq 5.1 then allows $Q_L$ to be found and the values obtained are shown in Fig. 18. Typically the results give that the separation of the resonant frequencies is 30% greater than the half-power width of the Q-curve for the negative mode but is less than 50% of the half-power width of the Q-curve for the positive mode. The considerable overlap of the Q-curves, which will have different amplitudes, means that the resonances cannot be easily distinguished/
distinguished and it is necessary that they should be observed separately. Under these circumstances the expected accuracy of the measurements, as defined above, must be considered. In Fig. 19 the error, taken as one tenth of the half-power width of the Q-curve is shown as a percentage of the calculated frequency shift for the non-magnetic case and also for the two magnetoactive modes. From this graph it can be seen that the error is minimum for \( r_1 = 0.25 \) cm. For radii less than the optimum the error increases because the losses due to the holes and the glass are again dominant.

From this analysis it can be expected that the cavity with a discharge tube of internal radius 0.25 cm and plasma of density such that \( \Omega_0^2 = 0.115 \) should have \( Q = 290 \) and a shift in resonant frequency of 26 MHz when unmagnetized and when a magnetic field is applied such that \( \Omega_b = 0.5, Q = 125 \) and 410 with shifts in resonant frequency of 50 MHz and 20 MHz for the two modes.

Now the possibility of making measurements at higher electron densities may be considered. A crude calculation, assuming that \( \Omega_0^2 \) is small enough that the shift in resonant frequency and \( \frac{1}{Q} \) are still proportional to \( \Omega_0^2 \), is sufficient to indicate the effect of an increase in \( \Omega_0^2 \). For \( r_1 = 0.25 \) and \( \Omega_0^2 = 0.230 \) the shifts in resonant frequency will be doubled while the Q-values become 190, 85 and 310 for the unmagnetized mode and the two modes when \( \Omega_b = 0.5 \) respectively. The errors are thus of the order of 8, 10 and 7% for the three resonances, representing a small improvement over the corresponding cases when \( \Omega_0^2 = 0.115 \).
Although measurements may be made on plasmas of greater density with an improvement in accuracy, the Q of the positive mode in this case is less than 100 and it has been seen that measurements on such broad Q-curves are experimentally more difficult.

The behaviour of the optimized system has been predicted for a particular value of $\Omega_p^2$. If this is to be checked experimentally $\Omega_p^2$ must be translated into the measurable variables current and pressure. The theoretical analysis of a positive column involves some empirically determined variables and it is only through these that the current can be determined.

In the positive column of a d.c. discharge

$$i \propto v_d n r_1^2$$

where $i$ is the discharge current and $v_d$ is the electron drift velocity. Empirical data exists relating $v_d$ to $E/P$ the ratio of the longitudinal electric field in the positive column to the pressure. Other data relates $E/P$ to $pr_1$ ($pr_1 \equiv 1$ Torr cm). If any validity can be given to the extrapolation of these curves for values of $pr_1 \approx 0.3$ Torr cm it may be said that to obtain an electron density such that $\Omega_p^2 = 0.115$, with $p = 1.2$ Torr and $r_1 = 0.25$ cm, a discharge current of between 5 mA and 20 mA is required.

The uncertainty in this estimate arises from the action of the boundaries on the plasma and the consequential doubt about the applicability of the expression for current. Forrest and Franklin
amongst others\textsuperscript{56,57,58} have done much theoretical work on the behaviour of both magnetic and non-magnetic positive columns. The range of discharge radius for which the theories are valid has been extended but there is a lack of experimental work to confirm the theories.

The action of the boundary on the plasma also becomes important when a magneto-active plasma is considered. A longitudinal magnetic field inhibits the flow of electrons to the walls because of their cyclotron motion. So that at equilibrium there will be a reduction in the ionisation rate, effected by a decrease in the longitudinal electric field. This means that $v_d$ is also reduced and if the current is maintained at a constant level the electron density must increase. As the magnetic field is increased, diffusion to the walls is progressively diminished and the electron density rises.

A comparison of the two theories in Sect. 5.3(ii) shows that some modification of the perturbation theory is required, the detailed application of which is given in Sect. 5.3(vi). The optimum radius of the plasma is unaltered but small adjustments must be made to the shifts in resonant frequency and the $Q$-values expected.

5.1 (ii)

The other changes made to the system were as follows:

A probe which may be rotated about the axis of the cavity was needed to check circular polarisation within the cavity. This was achieved with a circular end plate, as for the first cavity, to which, now,/
now, the probe was attached, thereby making the output coupling more definitive.

The system was made more compact by reducing the overall width of the cavity and also by putting the anode end of the plasma tube into the hole in the pole piece. In this way the magnetic field was increased sufficiently to give values of $\Omega_B$ up to 2.0.

It was also decided that, instead of a sealed discharge tube, one connected to a vacuum system would be used in order to give greater flexibility in the production of the discharge and to allow measurements to be made at various pressures.

5.1 (iii)

The modifications to the design of the system described in this section minimise the practical limitations which emerged from the first system. At the same time other limitations have been uncovered which effectively restrict the extension of this method of investigating plasmas. Values of electron densities such that $\Omega_\rho^2 \gtrsim 0.35$ produce $Q$-values too low for reasonably accurate measurements to be made.

5.2 The Experimental System

The improved resonant cavity and plasma tube system is shown in Fig. 20 and Plate III.

The cavity block was essentially unchanged, the cavity of length 2 cm and diameter 3 cm being machined from a brass block. The two input/
Fig. 20. The second resonant cavity and plasma tube.
Plate III
input arms were X-band waveguides recessed into the block so as to be symmetrically placed with respect to two perpendicular lines passing through the axis of the cavity and having their broad faces flush with one end of the block. The input ports were slots 1 cm by 0.1 cm situated centrally at the ends of the input arms.

One end plate fitted flush to the cavity block and was held in position by four bolts. At the other end there was a circular recess in the block with a narrow ridge at the inner edge so as to provide good contact with the end-plate. The circular end plate was connected to the broad face of a length of X-band waveguide which was to be used as the detecting probe. A clamping plate between the end-plate and the probe was held by four spring loaded bolts which were tightened from the opposite end of the block. This allowed the end plate and probe to be rotated while maintaining good contact with the end of the cavity. The holes in the end-plate were 0.7 cm in diameter and the end plate had a thickness of 0.5 cm in the region of the holes.

The discharge tube had an internal radius of 0.25 cm and an external radius of 0.35 cm in the region which passed through the cavity and probe. The anode was a molybdenum cylinder of radius 0.25 cm, which thus spread the discharge over the whole X-section of the tube. This was contained within a bulb which fitted into the recess of the pole piece and the vacuum system was connected to the discharge tube at the bulb. The heated impregnated cathode needed a tube of internal diameter greater than 0.5 cm, so that the glass should not become overheated, yet small/
small enough to fit within the half-inch diameter hole in the pole piece. The overall length of the discharge tube was 40 cm and the cathode and anode ends were connected together in situ next to the anode bulb. The discharge tube could be isolated, by means of a tap, from the conventional vacuum system consisting of a rotary pump, oil diffusion pump and cold trap. A needle valve allowed argon to be admitted to the tube and the pressure was measured with a Pirani gauge calibrated for argon. The discharge tube was outgassed to a pressure of $10^{-5}$ Torr and when the cathode had been activated a d.c. source of 1000 V gave a visibly uniform discharge of up to 60 mA over a pressure range of 0.4 to 1.2 Torr.

The reduction in the overall width of the cavity and probe enabled a smaller pole-gap to be used. A greater reduction would have been possible but for the joint in the glass, however a field of 6 kgauss could be obtained with the gap of 6 cm. Before the cavity and tube were introduced the uniformity of the field was checked and it was found that the field varied by less than 1% over the region to be used. The centre was calibrated against a point below the position of the cavity so that continuous monitoring of the field was possible.

The microwave system used with the second cavity and plasma tube is shown in the block diagram of Fig. 21. It differs from the first system in that the cavity was fed from both input arms.

The klystron oscillator was modulated by a 3.2 kc/s square wave. The microwave frequency was measured with the calibrated cavity wavemeter and the output of the crystal detector in this arm was fed to an/
an oscilloscope so that accurate setting of the wavemeter was possible. A hybrid-T junction divided the signal between two arms to feed the two inputs of the cavity. One arm had an attenuator and two phase-shifters to alter the amplitude and phase of the signal and thus enable circularly polarised waves to be set up in the cavity. In the other arm an attenuator and voltage standing wave indicator were provided, and a directional coupler fed power to a crystal detector so that the amplitude of the input signal could be monitored. The detecting probe coupled to the cavity had a crystal detector and matched load. A square law crystal was used so that the output signal was directly proportional to power. The input and output signals were monitored on a transistor amplifier tuned to 3.2 kc/s.

Measurements in one arm with the V.S.W.I. showed that the voltage standing wave ratio was greater than 10; this is due to the weak coupling. It was decided that measurements of the Q-curve using the V.S.W.I. would not be made but that as before the output power would be measured for a fixed level of input. As neither the input nor the output was critical the coupling was frequency insensitive and this method would provide a reliable measurement of the Q-curve of the cavity.

The application of a magnetic field to the cavity containing plasma split the resonance and in order to obtain a Q-curve for both resonances the analysis of Sect. 5.1 shows it is necessary to observe them individually. When the Q-curves overlapped, to allow measurements to be made on one mode, the other was suppressed by feeding the cavity from both arms so that circularly polarised waves were set up. As the
Q of the cavity was low measurements were needed over a wide frequency range. It was thus necessary to adjust the phase shifter and attenuator at each frequency setting to ensure circular polarisation. However, this was a laborious process and even the improved method of checking could not ensure a purely circularly polarised wave because of asymmetries in the rotatable probe. A simpler but equally satisfactory method of producing approximately circularly polarised waves was devised.

It was found that if the phase shifter was adjusted to suppress a mode at its resonant frequency then over a bandwidth of some 40 MHz the power in that mode was reduced by more than 14 dB. This was adequate to allow measurements to be made on the other mode. It was found that this could be best set up by using saw-tooth modulation on the klystron, thus sweeping the frequency. The output of the cavity was displayed on an oscilloscope and the phase shifter adjusted to remove one mode (Fig. 30). As the Q-curve generally covered a wider frequency range than could be swept by this method measurements were made at fixed frequencies with the klystron modulated by the square wave.

(This method of producing circularly polarised waves was not suitable for the first system as the Q-curves covered a wider frequency range and suppression over the mode with one setting of the phase shifter was not satisfactory).

When the Q-curves did not overlap measurements were made with the cavity fed from a single arm.

Q-curves were obtained for a range of currents from 0 to 25 mA with magnetic/
magnetic fields up to 6 kgauss and pressures from 0.4 to 1.2 Torr. A comprehensive set of results were obtained for the shift in resonant frequency of the positive mode and some measurements were also made on the negative mode.

5.3 Results

5.3 (i)

From the design of the system on the basis of the perturbation theory given in Sect. 5.1 a discharge current of between 5 to 20 mA in argon at a pressure of 1.2 Torr can be expected to give a value of $\Omega_f^2 = 0.115$. For the non-magnetic mode the shift in resonant frequency should be 26 MHz and when $\Omega_b = 0.5$ the two modes should have shifts of 50 MHz and 20 MHz, with corresponding Q-values of 290, 125 and 410. Applying the modifications of Sect. 5.3 vii means that for the same pressure and current the corresponding value of $\Omega_f^2$ is 0.155, while the shifts should be 23, 45 and 18 MHz and the Q-values 310, 137 and 430, for the modes with $\Omega_b = 0$ or 0.5 as before.

Preliminary measurements were made to establish that the experimental behaviour of the cavity agreed with the designed performance. For the cavity in situ containing the discharge tube, but with no plasma, the resonant frequency was $8.809 \pm 0.001$ GHz and the Q was $600 \pm 10$, for both input arms. A discharge current of 9 mA through argon at 1.2 Torr gave a shift in resonant frequency of 23 MHz for $\Omega_b = 0$ and shifts of/
of 58 MHz and 19 MHz for $Q_b = 0.5$. The $Q$-values of these three modes were found to be 250, 75 and 370 respectively.

The shifts in resonant frequency are of the right order although that for the positive magnetic mode is rather high. The $Q$-values are all low but inspection of the $Q$-curves shows them to be asymmetric. If this is taken into account by the method explained in Sect. 5.4 the $Q$'s approach the expected values. These results are sufficient to show that the design value of $\Omega_{\rho}^2$ falls within the predicted range of discharge currents.

5.3 (ii)

A comparison of the results obtained from the perturbation theory and from the exact theory will be made before the experimental results are interpreted in terms of electron densities. To obtain results from the two theories the number of parameters which need to be specified differs. Results from the perturbation theory for a lossless plasma (i.e., $\mathcal{U}_y = 0$) only will be considered since no account has been taken of losses in the exact theory.

The perturbation theory gives the ratio of the shift in resonant frequency to the resonant frequency of the unperturbed cavity, $\frac{\Delta \omega}{\omega}$, when $r_1, r_3, \xi_3, \Omega_p$ and $\Omega_b$ are specified. The exact theory gives a resonant frequency when, in addition to these parameters $r_2, \xi_2$ (to take into account the effect of the glass) and $\beta$ are specified: from these results the same ratio can be found.

In
In the second system the radii $r_1$, $r_2$, and $r_3$ have values of 0.25, 0.35 and 1.50 cm respectively and the relative permittivities $\varepsilon_2$ and $\varepsilon_3$ values of 4.62 and 1.0. For the non-magnetic situation, $\frac{\delta \omega_{\pm}}{\omega_r}$ was evaluated for a range of $\Omega_\rho$ and $\beta$ using the exact theory. The ratio was found to be approximately independent of $\beta$ for constant $\Omega_\rho$ while $\beta$ was greater than 100 m$^{-1}$. The resulting curve of $\frac{\delta \omega_{\pm}}{\omega_r}$ against $\Omega_\rho^2$ is compared in Fig. 22 (page 111) with the results from the perturbation theory obtained using Eq 2.16. The graphs have the same general shape, diverging from approximately linear for large $\Omega_\rho^2$ where the perturbation theory cannot be justifiably applied. The two lines differ in respect of their slopes by a factor $R_\omega = 0.66$ with the perturbation theory indicating the greater shift. This difference may be qualitatively related to the effect of the glass.

In evaluating the denominator of the perturbation formula, Eq 2.8, the integral $\int_E D$ is taken over the volume of the cavity when filled with a medium of permittivity $\varepsilon_o$. The introduction of the glass annulus, with a relative permittivity 4.62 into the cavity causes a redistribution of the fields but it is reasonable to expect that the volume integral will be increased. Such an increase in the magnitude of the denominator means that the plasma will have a proportionately smaller effect upon the resonant frequency of the cavity. The comparison above with the exact theory indicates that the effect of the glass may, in the perturbation theory, be approximately accounted for by an additional constant factor in the formula for resonance shift Eq 2.16. The numerical value of/
of this factor will depend upon the radii of the media and may be obtained from a comparison of the results from the two theories.

When, to compare the theories for non-zero magnetic field, the Eq 2.16 is multiplied by the factor above the two curves of $\frac{\Delta \omega}{\omega}$ against $\Omega_b$ agree to within 3% provided $\Omega_b < 0.2$ and for values of $\Omega_b < 0.9$. For $\Omega_b > 1.1$ a different multiplying factor, 0.72, is appropriate to give similar agreement. For $0.9 < \Omega_b < 1.1$ the curves from the two theories diverge. However, the results of these lossless analyses in this region have no application to the experimental system, where the large resonance shifts are not practically obtained because of plasma losses. When $\Omega_b > 0.2$ the differences in the two theories are more than can be accounted for by a simple multiplying factor.

There is one remaining factor which must be considered before the experimental results from this second system are interpreted with the exact theory. The theory enables the resonant frequency of the cavity defined by the eight parameters $(r_1 r_2 r_3 \xi_2 \xi_3 \Omega \rho \Omega_b \beta)$ to be evaluated. The ratio of shift in resonant frequency for any $\Omega_b$ to the resonant frequency, $\frac{\delta \omega}{\omega}$, can thus be found. By comparing the experimental and theoretical conditions for the same $(\delta \omega)$, the electron density of a given plasma can be determined. However, because the theory does not include the effect of the holes through which the tube passes, the actual resonant frequencies are not directly comparable.

It was found more convenient to be able to compare the resonant frequencies/
frequencies directly since it was these quantities which were experimentally measured and theoretically calculated. To do this, the value of $\beta$ was established which gave a theoretical resonance of the cavity with $\Omega_p, \Omega_b = 0$ the same as the experimental value, $f_0 = \omega_0/2\pi$. This procedure gave $\beta = 149.49$ m$^{-1}$ as a suitable value, as compared to the value $\beta = 157.08$ m$^{-1}$ appropriate to the cavity of length 2 cm. This value of $\beta$ was then used to find all subsequent resonant frequencies of the second system with which the experimental results were compared.

Since it has been found that $\frac{d\omega}{\omega_r}$ is roughly constant with respect to changes in $\beta$, and certainly the differences in the ratios for the two relatively close values of $\beta$ such of those above are not significant, the comparison of frequencies on this basis is equivalent to the comparison of the ratio $\frac{d\omega}{\omega_r}$. However, this approach reveals a significant physical interpretation of the effect of the holes. By the use of a reduced value of $\beta$ the holes are accounted for with an increase in the effective length of the cavity. This corresponds with the result given by considering Slater’s theory of the perturbation of cavity boundaries. A hole in a region of predominantly magnetic fields, such as is the case here, allows the field to extend into the hole thus decreasing the resonant frequency.

In order that comparisons may now be made between the theory and experiment it must be assumed that the practical effect of the holes does not vary with changes in the plasma parameters. While it cannot be expected that this assumption has any strict validity no quantitative assessment/
assessment is made of such changes in the effects of the holes. However, there is sufficient evidence from the results of the first system to indicate that the changes are not great.

5.3 (iii)

Some typical Q-curves obtained with the second system at a pressure of 0.4 Torr are shown in Fig. 23. At the bottom are shown the resonances of the unmagnetized system for various discharge currents and at the top the resonances of the positive mode for various magnetic fields. The general character of these curves is the same as is described in Sect. 4.3 for corresponding variations in parameters for the first system. An additional variable in the second system is the pressure of the gas. When other parameters are kept constant an increase in pressure has the same effect as an increase in the discharge current; i.e. the shift in resonant frequency increases while the Q-value and peak output power decrease.

From Q-curves such as those in Fig. 23 the resonant frequency of the cavity was found. The results are presented as graphs of resonant frequency against $\Omega_p$ in Figs. 24, 25, 26 and 27 for gas pressures of 0.4, 0.6, 0.9 and 1.2 Torr respectively. At each pressure the results for different discharge currents are shown. Curves from the exact lossless theory were fitted to the experimental points by minimizing $\sum (\text{deviation}/\omega)^2$ and hence the most appropriate value of $\Omega_p^2$ and thus $n_e$ was found for the plasma at each value of discharge current and gas/
Fig. 23. Q-curves of second system at $p = 0.4$ Torr.

Varying magnetic field, $i = 10$ mA.

Varying discharge current, $\omega_b = 0$.

Output (vibrational scale) vs. Frequency (GHz)
gas pressure. These results are tabulated below:

<table>
<thead>
<tr>
<th>( P \text{(Torr)} )</th>
<th>( i \text{(mA)} )</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.068</td>
<td>0.110</td>
<td>0.166</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.085</td>
<td>0.161</td>
<td>0.221</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.130</td>
<td>0.232</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.152</td>
<td>0.278</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>( P \text{(Torr)} )</th>
<th>( i \text{(mA)} )</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.66</td>
<td>1.06</td>
<td>1.60</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.82</td>
<td>1.55</td>
<td>2.13</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1.25</td>
<td>2.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>1.47</td>
<td>2.68</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Typical error in the measurement of a resonant frequency was 10-15% of the shift and these are shown on the 20 mA curve in Fig. 24. Consideration of the r.m.s. deviation of the experimental points from the best fit curves indicates only 2-4% error in the curves. A change of 4% in the values of \( \Omega_p^2 \) given above increases the r.m.s. deviation by a factor greater than 2.

No conclusive relationships can be established between electron density and pressure or current although it would appear that neither is linear.
Fig. 24. Resonant frequency $f_r$ against $\Delta_b$. 0.4 lov.
Fig. 25. Resonant frequency, $f_r$, against $x_b$. 0.6 torr.
Fig. 26. Resonant frequency, $f_r$, against $\Omega_b$. 0.96"w.
Fig. 27. Resonant frequency $f_r$ against $n_b$. 10-12 torr.
5.3 (iv)

In order to establish a value for the collision frequency use must be made of the perturbation theory. A consideration of the change in $Q$ of the cavity caused by the introduction of the plasma will provide, through the application of Eq. 2.17 multiplied by $F(r)$, a value of $\frac{\nu}{\omega_r}$ when $\Omega_r^2$ is known. At $\Omega_b = 0$ Eqs 2.16 and 2.17 give

$$\delta \left( \frac{1}{Q} \right) = - \frac{\delta \omega}{\omega_o} \cdot \frac{4 \frac{\nu}{\omega_r}}{(2 - \Omega_r^2)}$$

$$\frac{\nu}{\omega_r} \approx \delta \left( \frac{1}{Q} \right) \frac{(2 - \Omega_r^2)}{4}$$

But $\delta \left( \frac{1}{Q} \right) = \frac{1}{\omega_r}$ (half-power width of $Q$-curve for $\Omega_r^2 = 0$)

- half-power width of $Q$-curve being considered

$$\frac{\nu}{\omega_r} = 2 - \Omega_r^2 \times \frac{\text{difference in widths of } Q\text{-curves}}{\text{shift in resonant frequency}}$$

The $Q$-curves obtained experimentally were asymmetric because of time variations in the plasma density. This is discussed further in Sect. 5.4. Direct measurement of the half-power width of the $Q$-curves is unrealistic. A more relevant estimate of the half-power width of the $Q$-curve corresponding to the plasma density and $\Omega_r^2$, evaluated from the shift in resonant frequency, can be obtained by taking twice the difference between the resonant frequency and the half-power frequency, on the side nearer the unloaded resonant frequency $\omega_o$, which is less affected by the variations in plasma density. Using the value/
value thus obtained, together with the measured shift in resonant frequency and evaluated, Eq. 5-2 above provided values of $\frac{\nu}{\omega_r}$ for all Q-curve measurements made at $\Omega_b = 0$.

The collision frequency $\nu$ for momentum transfer between electrons and neutral particles is defined by the ratio

$$\nu = \frac{V_e}{\Lambda}$$

where the mean thermal speed of electrons, $V_e = (8 kT_e/\pi m_e)^{1/2}$ assuming a Maxwellian distribution, and $\Lambda$ is the mean free path. Now $\frac{1}{\Lambda}$ is the product of the number of neutral particles per unit volume, $n_n$, with an experimentally determined collision cross-section for momentum transfer $Q_m$. If the gas pressure and temperature are $p$ in torr and $T_g$ in °K then $n_n = 9.7 \times 10^{18} \frac{p}{T_g}$ particles/cc.

Hence $\nu = \left(\frac{8 kT_e}{\pi m_e}\right)^{1/2} \frac{9.7 \times 10^{18}}{T_g} p \cdot Q_m$

$Q_m$ is a function of the electron energy and therefore of the electron temperature. Analysis of the positive column shows $T_e$ to be a slowly varying function of $p r$ and slightly current dependent. But $T_g$ and $T_e$ do not vary greatly over the range of currents used here so that at a fixed pressure $\nu$ is a constant. Experimentally, the values of $\frac{\nu}{\omega_r}$ as found above show no smooth variation with current so that the wide variations may be considered random and an average value found for each/
each pressure. The results of this averaging are given here and shown graphically in Fig. 28.

<table>
<thead>
<tr>
<th>Pressure (Torr)</th>
<th>0.4</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\nu}{\omega_r} )</td>
<td>0.19±0.06</td>
<td>0.28±0.07</td>
<td>0.38±0.08</td>
<td>0.39±0.04</td>
</tr>
<tr>
<td>( \nu \text{ (sec}^{-1}) \times 10^{10} )</td>
<td>1.05</td>
<td>1.55</td>
<td>2.10</td>
<td>2.16</td>
</tr>
</tbody>
</table>

Table 5

5.3 (v)

The crude experimental results, values of \( n_e \) and \( \nu \) for the plasma, have been presented in Sect. 5.3 (iii) and 5.3 (iv). The validity of these results will now be discussed since they differ in some respects from the results expected.

Eq 5.2 allows a theoretical value of \( \frac{\nu}{\omega_r} \) to be found. Data from Cobine\(^2^2\) relates \( T_e \) to \( P_r \) and experimental values of \( Q_m \) as a function of electron energy were obtained from Shkarofsky et al.\(^2^3\). Assuming \( T_g = 350^\circ K \), values of collision frequency, \( \nu \), and \( \frac{\nu}{\omega_r} \) were calculated and these are tabulated below:

<table>
<thead>
<tr>
<th>Pressure (Torr)</th>
<th>0.4</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu \text{ (sec}^{-1}) )</td>
<td>( 1.57 \times 10^8 )</td>
<td>( 2.00 \times 10^8 )</td>
<td>( 2.52 \times 10^8 )</td>
<td>( 2.76 \times 10^8 )</td>
</tr>
<tr>
<td>( \frac{\nu}{\omega_r} )</td>
<td>0.0028</td>
<td>0.0036</td>
<td>0.0045</td>
<td>0.0050</td>
</tr>
</tbody>
</table>

Table 6
Fig. 22. Shift in resonant frequency, $\Delta f / \Delta f_p$.

Fig. 23. $\delta^2(a) / \Delta b$. (0.4 mTorr, 10 mA).

0 Experimental points. — Perturbation theory.
These values are two orders of magnitude less than those obtained experimentally.

This theory may also be used to find $V_e$ and, hence the mean free path, $\Lambda$. $\Lambda$ is found to be of the same order of magnitude as the tube dimensions, thus invalidating the application of this analysis. It would seem that the collision frequency obtained experimentally represents not loss by collisions between electrons and neutral particles but indicates loss by some other mechanism. Since $\Lambda$ is of the order of $r_1$, collisions with the walls of the discharge tube will be an important factor in the loss of energy by the electrons; this effect will be considered first.

There is various evidence to support this. Firstly, the magnitudes of the shifts in resonant frequency recorded around $\Omega_b = 1$ are inconsistent, on the basis of the perturbation theory, with the values of $\frac{\nu}{\omega_r}$ determined experimentally. In Fig. 3 it can be seen how a finite value of $\frac{\nu}{\omega_r}$ modifies the shape of the frequency shift against $\Omega_b$ curve and, as an example, when $\frac{\nu}{\omega_r} = 0.19$ or 0.38 the maximum shifts of the positive mode which can be obtained for $\Omega_p^z \leq 0.2$ are respectively 2.6 or 1.5 times the shift at $\Omega_b = 0$. The experimental points on Figs. 24 and 27 around $\Omega_b = 1$ exhibit shifts well in excess of these limits.

As an alternative to rejecting the accuracy of the measured $\frac{\nu}{\omega_r}$, the anomalously high shifts may be accounted for by assuming that $\Omega_p^z$ is a function of $\Omega_b$ with a maximum at $\Omega_b = 1$. However, the limited evidence from the shifts of the negative mode indicates that $\Omega_p^z$ does/
does not vary as the magnetic field is changed. Having discounted this possibility the inconsistency can only be accounted for by a misinterpretation of the significance of the measured \( \frac{\nu}{\omega_r} \).

Secondly, the expected Q-values for the second system do not agree with those obtained experimentally. The expected Q-values are given in Sect. 5.1 as modified by the factor \( F(r) \) and the preliminary experimental results given in Sect. 5.3 (i) are lower. Further comparisons were made for other currents between experimentally obtained Q-values and values calculated for corresponding \( \Omega_p^2 \) by the method of Sect. 5.1. It was found that the experimental Q-values were again lower than the calculated values. Therefore, a reduction in the size of the discharge tube appears to increase the losses, indicating that the walls of the tube are involved in the loss mechanism.

Thirdly, the experimentally determined values of \( \frac{\nu}{\omega_r} \) for the two systems at 1.2 Torr are 0.28 \((r_1 = 0.35 \text{ cm})\) and 0.39 \((r_1 = 0.25 \text{ cm})\). This confirms that a reduction in the tube radius increases the loss.

Finally, there is evidence from the Q-curves of the positive magnetic modes. The experimentally determined values of \( \mathcal{E}(\frac{1}{Q}) \) may be compared with a curve of \( \mathcal{E}(\frac{1}{Q}) \) against \( \Omega \) obtained from the perturbation formula Eq 2.17 modified by \( F(r) \). A representative example of such a comparison (Fig. 29) is that for the Q-curves in Fig. 23 of the 10 mA discharge at 0.4 Torr. To obtain the curve from Eq 2.17 both \( \frac{\nu}{\omega_r} \) and \( \Omega_p^2 \) must be specified. The best values that can be used for this are \( \Omega_p^2 = 0.068 \), from a comparison of the shifts of this mode with the exact theory,
and \( \frac{\nu}{\omega_p} = 0.19 \), the averaged value for a pressure of 0.4 Torr.

Fig. 29 shows the experimental values of \( \delta \left( \frac{1}{Q} \right) \) to be consistently less than the theoretical curve. The \( Q \) of the cavity for this mode is thus higher than is consistent with the value of \( \frac{\nu}{\omega_p} \).

The curve of best fit to the experimental points gives \( \frac{\nu}{\omega_p} = 0.11 \); this is less than evaluated from the non-magnetic system but does not approach the theoretical value of the collision frequency for momentum transfer. Similar results were obtained from analysis of the \( Q \)-curves at other currents and pressures to give the values of \( \frac{\nu}{\omega_p} \), averaged at each pressure, shown in Table 7.

<table>
<thead>
<tr>
<th>Pressure (Torr)</th>
<th>0.4</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\nu}{\omega_p} )</td>
<td>0.11</td>
<td>0.15</td>
<td>0.20</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Table 7

In a simple physical picture of the magnetic plasma, the electrons with any transverse velocity component are constrained to gyrate about the axis of the magnetic field rather than travel towards the walls of the tube. The typical cyclotron radius, \( r_b \), is of the order 0.05 cm for a magnetic field of 1 kgauss and electron speed = \( V \), corresponding to \( T_e = 22,000^\circ \text{K} \). The majority of the electrons are within the body of the discharge at a distance \( > r_b \) from the walls so that they have a much enhanced opportunity to travel their true mean free path before collision.
collision. The application of the magnetic field thus inhibits collisions with the walls and the measured collision frequency $v$ should tend to the theoretical collision frequency for momentum transfer. While it is clear that the walls have some effect upon the measured $v$, the large reduction expected upon the application of a magnetic field is not experimentally observed and the measured $v$ cannot thus be due solely to collisions with the walls.

Another possibility is that losses are occurring in the glass tube whose conductivity is changed as the plasma current and pressure and hence glass temperature are increased. The perturbation theory does not include the affects of the glass and the exact theory would require reformulation to take account of such losses. The glass temperature was not measured during the experiment but some indication of whether this could account for the high values of $v$ found experimentally can be obtained from the results.

The glass losses can be expected to increase with current and hence the values of $v$ determined ignoring this factor should increase with current. For the results obtained with the first system over a range of current from 3 to 15 mA (Fig. 16) $v$ does not increase uniformly with current but does show a 30% difference between the average values at high and low currents.

The most significant evidence of the effect of the glass comes from the magnetic results of $v$ in the second system. For the more limited ranges of current used here no consistent variation of $v$ with $i$ was obtained over the pressures used either for the non magnetic results alone or when the results with a magnetic field are included. The
changing conductivity of the glass means that $Q_H$ in Eq. 5.1 is current and pressure but not magnetic field dependent. The measured shift in $(1/Q)$ contains a component which is pressure and current sensitive and a second component which also varies with the magnetic field.

Eqs. 2.16 and 2.17 give

$$\Delta \left( \frac{1}{q} \right)_{\text{measured}} = \delta \left( \frac{1}{q} \right)(\text{fn of } i, p, \alpha_b) + \delta \left( \frac{1}{q_H} \right)(\text{fn of } i, p)$$

Thus the difference between $\nu/\omega_{\nu}$ obtained from the nonmagnetic results (Table 5) and the magnetic results (Table 7), assuming $\delta \left( \frac{1}{q_H} \right) = 0$, is to be expected since the measured $\delta \left( \frac{1}{q} \right)$ for $\Omega_H \neq 0$ are consistent only for a lower value of $\nu$.

The magnitude of $\delta \left( \frac{1}{q_H} \right)$ cannot be obtained from the measurements made since its dependence on $i$ and $p$ is not known as the temperature of the glass was not monitored.

However, the measured shifts in $(1/Q)$, both above and below $\Omega_b = 1$ are large and are only consistent with a much larger value of $\nu/\omega_{\nu}$ than is given by positive column theory (Table 5). The best that can be done with the measurements made is to assume $\delta \left( \frac{1}{q_H} \right) = 0$ and to obtain values of $\nu$ which are indicative of its order of magnitude only.

The change in conductivity of the glass may also affect the value of $\Omega_p^2$ obtained but since no systematic variation was obtained between the magnetic and nonmagnetic values of $\Omega_p^2$ the values in table 8 are taken to be the most accurate obtainable.
In Sect. 5.1 it was suggested that maintaining the discharge current constant when a magnetic field was applied should lead to an increase in $n_e$. Because $r_b \ll r_1$ when the magnetic field is only ~ 100 gauss any significant change in $\tilde{n}_e$ produced by the application of a magnetic field should cause all of the measured shifts in resonant frequency at $\Omega_b = 0$ to be below the curves of best fit. No such systematic deviation was observed experimentally, the shifts in general fitting the curves within the limits of their errors. Thus if any such increase in $n_e$ is caused it was not of sufficient magnitude to be measured.

The exact lossless theory curves fit the experimental points well with no obvious systematic deviations. However, since it has been shown that the two theories correspond closely, it is permissible to fit perturbation theory curves to the experimental points so as to include the effects of losses. Applying this theory to the curves of $f_r$ against $\Omega_b$ and $\left( \frac{1}{Q} \right)$ against $\Omega_b$ allows two pairs of values of $\Omega_b^2$, $\nu/\omega_r$ to be calculated. These results are not consistent in that from the $f_r/\Omega_b$ graphs a low value of $\nu/\omega_r$ is obtained, while the other graphs which/
which are a more sensitive indicator of this parameter, give \( \frac{1}{\omega_r} \) a value \( \sim 0.1 \) to 0.2.

The following procedure was adopted to provide the values of \( \Omega_\rho^2 \) and \( \frac{1}{\omega_r} \) most consistent with the measurements of both resonant frequency and Q-value. The values of \( \Omega_\rho^2 \) in Table 4 were obtained from a comparison of the measured resonant frequencies with the lossless theory. Perturbation theory curves of \( \delta \left( \frac{1}{Q} \right) / Q_b \) were then fitted to the experimental points obtained from Q-value measurements by using the above lossless values of \( \Omega_\rho^2 \) and thus the values of \( \frac{1}{\omega_r} \) in Table 7 were found. Now, perturbation theory curves may be fitted to the experimental shifts using these values of \( \frac{1}{\omega_r} \). This gives values of \( \Omega_\rho^2 \) differing by 15 to 20% from the values obtained with the lossless theory and makes reconsideration of the values of \( \frac{1}{\omega_r} \) necessary. By repeated application of the perturbation theory to the two sets of experimental results, each time fixing either \( \Omega_\rho^2 \) or \( \frac{1}{\omega_r} \) in order to determine the best value for the other, the most consistent pairs of values of \( \Omega_\rho^2 \) and \( \frac{1}{\omega_r} \) are converged upon.

The results obtained by this method are given in Table 8 and the values of \( \frac{1}{\omega_r} \) are also shown in Fig. 28. The perturbation theory curves from these results are shown with the experimental measurements in Figs. 24 to 27. The shifts in resonant frequency closest to \( \Omega_b = 1 \) exhibit a consistent deviation from the perturbation theory curves and were thus omitted from the above procedure. Other than this the experimental points fit well to the theoretical curves. While the values of/
of \( \gamma \) obtained by this analysis do not approach the theoretical values for collision frequency for momentum transfer; they are indicative of the order of magnitude of the losses in the system.

<table>
<thead>
<tr>
<th>( p ) (Torr)</th>
<th>( \frac{\gamma}{\omega_r} )</th>
<th>( i ) (mA)</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
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<tbody>
<tr>
<td>0.4</td>
<td>0.10</td>
<td>0.078</td>
<td>0.120</td>
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<tr>
<td>0.6</td>
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<td></td>
<td></td>
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<tr>
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<td>0.23</td>
<td>0.165</td>
<td>0.330</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8 \( \frac{\gamma}{\omega_r} \) and \( \Omega_{p}^2 \) 
\( (\pi_0 = 9.65 \times 10^{11} \Omega_{p}^2 \) per cc)

5.3 (vi)

The results obtained were over a very limited range of pressure and current, the limits being imposed by two factors: the ability to obtain a stable discharge; and the ability to make satisfactory measurements on broad Q-curves. A stable discharge could not be obtained with the tube used at gas pressures outside the range 0.4 to 1.2 Torr nor for currents less than 10 mA. At currents higher than those for which results have been given the low Q-value, aggravated by the asymmetry caused by fluctuations in the electron density (Sect. 5.4) prevented measurements being made. Measurements in the region close to \( \Omega_{b} = 1 \), which would have shown whether the exact theory or the perturbation theory
theory fitted better the experimental points, were again prevented
by broad asymmetric Q-curves.

5.3 (vii)

In Sect. 5.3 (ii) a multiplying factor \( F(\rho) \) was introduced into
the perturbation theory so that it was possible to equate the two analyses
of the system. It now becomes necessary to consider the effect of this
modification on the analysis of Sect. 5.1 to determine the optimum
radius of the plasma for the second system.

The perturbation formula Eq 2.16 was applied in Sect. 4.2 to
determine the relationship between \( \Omega_\rho^2 \) and discharge current for the
first system. These calculations must be modified by \( F(\rho) \) which has a
value of 0.74 for the first system. Now, the multiplication factor
derived from the need to modify the evaluation of the denominator of the
general perturbation formula Eq 2.8 to account for the effect of the
glass. It is apparent then that Eq 2.17 for \( \frac{\partial}{\partial \rho} \), which is also
derived from Eq 2.8, must be multiplied by the same factor.

When this is done a discharge current of 10 mA in the first system
corresponds to \( \Omega_\rho^2 = 0.155 \) and the value of \( \frac{\partial}{\partial \rho} \) is 0.28, the same as
that obtained in Sect. 4.2 and unaltered by the inclusion of \( F(\rho) \).

Calculations were made on the expected accuracy to be obtained
from systems with different radii and 'constant' \( \Omega_\rho^2 \). With the
inclusion now of \( F(\rho) \) it is apparent that \( [F(\rho) \Omega_\rho^2] \) was held constant
and since \( F(\rho) \) depends on the radius the value of \( \Omega_\rho^2 \) varied.

The/
The expected accuracy is proportional to the ratio of $\left(\frac{1}{Q_L}\right)$ to the shift in resonant frequency. If now the calculations are repeated for constant $\Omega^2_p = 0.155$ then the results at $r = 3.5$ cm are the same but at other radii are different. The values obtained in Sect. 5.1 for shift in resonant frequency and $\delta \left(\frac{1}{Q}\right)$ at radius $r$ must be multiplied by $F(r)/F(3.5)$. But since $\delta \left(\frac{1}{Q}\right)$ is only part of $\frac{1}{Q_L}$ the expected accuracy will be changed. It can be seen that where plasma losses are high, and thus $\delta \left(\frac{1}{Q}\right)$ dominates $\frac{1}{Q_L}$, the effect of the modifications will be small. The results of making these adjustments are also shown in Fig. 19 and do not significantly alter the optimum radius.

The estimate in Sect. 5.1 of the $Q$ of the 2nd cavity will also be modified by the inclusion of $F(r)$. As above, $\delta \left(\frac{1}{Q}\right)$ must be multiplied by $F(r)/F(3.5)$ to find $Q_L$. Since the factor is less than 1, $\frac{1}{Q_L}$ is reduced and the $Q$ is higher than previously estimated.

5.4 Oscillations of Plasma Density

For the first system an oscilloscope display of the $Q$-curve showed a fairly smooth line with only minor modulation, indicating a uniform electron density. Observation of the discharge current confirmed this although there was 5% modulation at a varying frequency around 20 kHz.

However in the second system the $Q$-curves showed a very much greater degree of modulation. The density was much less stable under all conditions and the discharge current showed about 50% modulation at a rapidly varying frequency again around 20 kHz. In Fig. 30, typical oscilloscope/
Fig. 30. Oscilloscope display of Q-curves.

(a) Frequency swept klystron output.

(b) Cavity output. $\beta_0 < 1$

(c) Negative mode suppressed.
oscilloscope traces of the Q-curves of the second cavity show the modulation and also indicate the effectiveness of suppressing the negative mode. (Sect. 5.2). Fig. 30. a is the klystron output when the klystron is voltage, and therefore frequency, swept. In Fig. 30. b the probe pick-up is displayed for $\Omega_b < 1$ when the cavity is fed from one arm only so that both modes are present. Because of the fluctuations in plasma density the trace, which is the time average of many sweeps, shows both Q-curves as well as the sum. By feeding the cavity from the second arm with a signal of the correct amplitude and phase the negative mode may be suppressed. (Fig. 30 c).

An oscilloscope trace of the discharge current, Fig. 31, shows the form of the fluctuations in current. The density of the trace is indicative of the time for which the current has a particular value and it can be seen that for the majority of the time the current varies about some normal value while at times it shows transient increases of up to double the normal value.

The plasma density can be expected to follow the fluctuations in current and the modulated Q-curves of Fig. 30 can be interpreted in terms of these variations. The fluctuations around the normal $n_e$ lead to displacement of the Q-curve along the frequency axis and the power picked-up will thus vary at any particular frequency. The bursts of high $n_e$ displace the positive mode beyond the frequency range of the klystron, while causing a small shift in the negative mode. At these times the power observed at any frequency is due solely to the negative mode.
Fig. 31. Discharge current.

Fig. 32. Oscilloscope traces of cavity output; 3.2 kHz modulation. $\Delta f < 1$

2.4f is the experimentally measured half-power width.

Fig. 33. Addition of Q-curves.
mode and so the high frequency side of this mode can be seen.

The fluctuations were found to have a greater effect on power levels of both modes for frequencies greater than the resonant frequency. This was also evident when the probe pick-up was examined at particular frequencies. For making measurements on the cavity output at individual frequencies, the klystron was square wave modulated at 3.2 kHz. Oscilloscope traces of the output, Fig. 32, show that at frequencies $< f_r$ the degree of modulation was $\sim 10\%$ while at frequencies $> f_r$ the modulation increased to $\sim 50\%$. That the modulation was caused by fluctuations in electron density was confirmed by observations on the positive mode with $\Omega_b > 1$ when it was found that the degree of modulation was greater for frequencies $< f_r$; i.e. again on the side of the Q-curve further from $f_0$.

The probe output was measured by feeding it to a meter which rectified the variations and indicated a time averaged output at the given frequency. A Q-curve was constructed from such measurements of output level at spot frequencies over a suitable range and the asymmetry of this curve derives from the presence of many values of $n_e$ over the time interval used for averaging.

The Q-curve can be considered as the sum of a Q-curve corresponding to the normal $n_e$ together with curves, for all other values of $n_e$ occurring, whose magnitudes are related to the time for which the particular value of $n_e$ exists. The situation can be approximated to by the addition of a finite number of Q-curves, Fig. 33, and the asymmetric curve/
curve thus formed is similar to those observed experimentally. Since the peak of the curve corresponds to the resonance for the normal $n_e$, it is necessary to measure the corresponding half-power width to obtain an accurate and consistent $Q$-value. Direct measurement of the half-power width gives too low a $Q$, while measurement of the difference between the resonant frequency and the half-power frequency on the side nearer $f$ can be doubled to give, fairly accurately, the true half-power width of a $Q$-curve for normal $n_e$, Fig. 33. The $Q$-values used in Sect. 5.3 were obtained in this way.

It was found that the degree of modulation, particularly on the side of the $Q$-curves further from $f$, increased as the current and $n_e$ were increased. This made the asymmetry more pronounced and flattened the peak of the $Q$-curve thus making measurements more difficult and preventing the extension of results to higher values of electron density.

The observed oscillations have been interpreted in terms of variations in plasma density. The subject of plasma oscillations has been reviewed by Crawford and Kino\textsuperscript{52} who indicate the possible origins of such oscillations.
CHAPTER 6

Conclusion

The purpose of the experiment has been to investigate the interaction of microwaves with a magnetoactive plasma when the direction of propagation of the waves coincides with the applied magnetising field. Theoretical analysis of a cylindrical waveguide containing a coaxial, longitudinally magnetized plasma rod shows that a cavity resonant in the TE\_\_\_\_\_\_\_11\_\_\_\_\_\_ mode is suitable for experimental observation of the interaction. The results from a provisional experimental arrangement enabled a better system to be designed. Measurements of the changes in resonant frequency and Q-value of the cavity caused by the introduction of the plasma are interpreted with the theory to provide values of the main plasma characteristics, electron density and collision frequency.

With the behaviour of the plasma reduced to that of a medium with a tensor permittivity, theories were developed to analyse the experimental system in an idealised form. The plasma was assumed homogeneous and the effect of the holes in the ends of the cavity through which the discharge tube passed was ignored. The established techniques of the perturbation theory of cavities were applied to relate the plasma properties to the experimental measurements. The main limitation in the application of this theory to the system is the omission of any consideration of the glass sheath surrounding the plasma. To overcome this the problem was reformulated and the propagation characteristics of a waveguide containing/
containing three media were determined. The new approach also removed the assumption made in the perturbation theory of a uniform field in the central plasma region. On the other hand this exact theory was restricted to consideration of lossless plasmas. A comparison of the two theories then showed that the inclusion of a multiplying factor in the perturbation theory suitably accounted for the main differences and further showed that the perturbation theory in its modified form was applicable for values of $n_e$ previously excluded. In order to take into account the effect of plasma losses, which were found to be large in this system, the experimental observations were interpreted in terms of the modified perturbation of theory.

Measurements of the shift in resonant frequency alone were not sufficient to enable the collision frequency to be found. While the measurements of $Q$-value were not so reliable the change in $\frac{1}{Q}$ was a more sensitive indicator of plasma losses and thus gives a more accurate estimate of the collision frequency. Some doubt remains as to the applicability of the theories to the experimental system since the effects of the holes and of the changing conductivity of the glass have not been assessed. An attempt is made to take the holes into account when applying the exact theory by assuming they produce an effective lengthening of the cavity. This is not a complete answer as the effect of the holes will depend upon the plasma properties.

There are two indications from the results that the variable effects of the holes might be significant. As the electron density increases the peak power output should be proportional to $Q^2$. It is/
is found that the power falls more rapidly than $Q^2$ suggesting that the coupling is reduced as is thus the effective length of the cavity. The effective unloaded resonance is higher than used so that the shift in resonant frequency and therefore $\bar{n}_e$ would be overestimated. This effect occurs for both current and magnetic field variations so that it cannot be simply accounted for by changes in the conductivity of the glass. In the provisional system $\bar{n}_e$ was found to be approximately proportional to $i$, the discharge current, while in the second system $\bar{n}_e$ is found to rise more rapidly than $i$ over the limited range. This may be due to the effects of the glass but one would expect these variations to be more noticeable over the wider current range of the first system. It is more probably due to misinterpretation caused by the effects of the holes. Also, as $\Omega_b$ approaches 1 the cut-off frequency of the plasma loaded waveguides constituting the holes will increase. The attendant reduction in coupling leads to a similar overestimate of the shift in resonant frequency and it was noted that the shifts of the positive mode recorded around $\Omega_b = 1$ were too large to be consistent with the calculated values of $1/\omega_r$. The evidence from the negative mode that the unloaded resonance is constant, is not significant since the coupling through the holes will differ for the two modes.

The effects of the glass have been considered in para. 53 vi and for the reasons stated have been ignored. Because of this it is clear that the measured values of $\nu$ are overestimated although apparently of the right order of magnitude. This could account for the high shifts of resonant frequency around $\Omega_b = 1$. 

The
The observed oscillations of electron density are manifestations of longitudinal inhomogeneity evidenced by the similar variations of tube current. This will affect the propagation characteristics of the system and leads to a further indeterminate effect on the measured parameters.

Several authors have investigated the problem of a $\text{TE}_{111}$ mode resonant cavity containing a cold magneto-active plasma. The approaches to the problem have been varied; obtaining either perturbation or exact solutions of a filled or partially filled cavity and taking into account some of the following factors: the glass container; end holes in the cavity; plasma losses and non-uniformities of the plasma density. Buchsbaum et al.\textsuperscript{9} give a brief statement of the perturbation solution taking account of a.c. space charge and this has been experimentally investigated by Moresco and Zilli\textsuperscript{34}. Shohet\textsuperscript{41} has given an exact solution of the partially filled cavity but this does not include the glass container. Agdur and Enander\textsuperscript{10} consider the non-uniform case but only as a perturbation of the exact solution for a cavity partially filled with isotropic plasma and the theoretical and experimental results are limited to low $\omega_b$. Kent and Heintz\textsuperscript{37} have investigated a filled cavity, applying the perturbation theory while Kent\textsuperscript{40} subsequently obtained an exact solution. Later Kent and Thomas\textsuperscript{41} considered the same system taking into account non-uniformities of the plasma and modified the perturbation formula by comparison with an exact solution to allow a non-uniformity parameter as well as the average electron density to be obtained from the application/
application of the perturbation formula.

In this investigation an exact solution for a system with three coaxial media has been obtained. It was used to modify the perturbation formula to take account, to some extent, of the presence of the glass and losses in the plasma. The effects of the end holes have also been considered.

The results are in agreement with Moresco and Zilli that the perturbation formula of Buchsbaum et al represents the system for the range of $\Omega_e$ covered provided that the effects of glass and end holes are considered. If a useful indication of the plasma losses is to be obtained then a more sophisticated consideration of these effects is necessary.

Despite the omissions of the theory and recognising the doubt as to the interpretation of the measurements, the experimental evidence showed that the theory described the system fairly well and that the final value of $\bar{n}_e$ was a reasonable estimate. The unexpectedly high value of $\nu$ might be accounted for as some of the indeterminate effects were more rigorously investigated.

The exact theory developed is, within the limitations previously discussed, descriptive of a propagating system as well as of a resonant cavity. The use of this arrangement as a plasma diagnostic method is limited in the range of electron density which can be measured, particularly by the low value of the unloaded $Q$. 
### List of Principal Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D, B )</td>
<td>Electric and magnetic induction</td>
</tr>
<tr>
<td>( B_0 )</td>
<td>Applied magnetic induction</td>
</tr>
<tr>
<td>( e, m_e )</td>
<td>Electronic charge and mass</td>
</tr>
<tr>
<td>( E, H )</td>
<td>Electric and magnetic fields</td>
</tr>
<tr>
<td>( f_r = \frac{\omega}{2\pi} )</td>
<td>Resonant frequency</td>
</tr>
<tr>
<td>( f_0 = \frac{\omega_0}{2\pi} )</td>
<td>Resonant frequency of unloaded cavity</td>
</tr>
<tr>
<td>( \delta f )</td>
<td>Shift in resonant frequency</td>
</tr>
<tr>
<td>( i )</td>
<td>Discharge current</td>
</tr>
<tr>
<td>( j = (-1)^{\frac{3}{2}} )</td>
<td></td>
</tr>
<tr>
<td>( k, k_1, k_2, k_3 )</td>
<td>Wave numbers</td>
</tr>
<tr>
<td>( k )</td>
<td>Boltzmann's constant</td>
</tr>
<tr>
<td>( l )</td>
<td>Length of resonant cavity</td>
</tr>
<tr>
<td>( n_e )</td>
<td>Average electron density</td>
</tr>
<tr>
<td>( R )</td>
<td>Constant</td>
</tr>
<tr>
<td>( r_1, r_2, r_3 )</td>
<td>Radius of plasma, tube and cavity</td>
</tr>
<tr>
<td>( T_e )</td>
<td>Electron temperature</td>
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<tr>
<td>( v_d )</td>
<td>Drift velocity of electron</td>
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<tr>
<td>( V_e )</td>
<td>Mean thermal speed of electrons</td>
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<td>( \beta )</td>
<td>Propagation constant</td>
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<td>Tensor permittivity</td>
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<td>( \varepsilon_2, \varepsilon_3 )</td>
<td>Relative permittivities</td>
</tr>
<tr>
<td>( \varepsilon_0 / \mu_0 )</td>
<td>Permittivity and permeability of free space</td>
</tr>
</tbody>
</table>
\( \lambda \)  Wavelength

\( \Lambda \)  Mean free path of electrons

\( \nu \)  Electron collision frequency

\( \omega \)  Angular frequency

\( \omega_b, \omega_B \)  Cyclotron frequency of electrons and ions

\( \omega_p, \omega_P \)  Plasma frequency of electrons and ions

\( \Omega_b = \frac{\omega_b}{\omega} \),  \( \Omega_p = \frac{\omega_p}{\omega} \)
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Acknowledgements

The work for this thesis was carried out in the Microwave Laboratory of the Physics Department at Battersea College of Advanced Technology (now the University of Surrey). The author would like to thank Dr. K. W. H. Foulds for his help and encouragement throughout the conduct of this research, Professor L. R. E. Elton, Head of the Physics Department, and Battersea College for its financial support.