BENDING EFFICIENCY
OF CROSS SECTIONAL SHAPES

By

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A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF PHILOSOPHY

UNIVERSITY OF SURREY
DEPARTMENT OF CIVIL ENGINEERING

DECEMBER 1980
The thesis describes the derivation and uses of a simple function, "The Bending Efficiency Factor", which may be used to examine the performance of cross-sectional shapes in bending. The function yields a value for efficiency which is independent of the overall dimensions of the cross-section, providing a measure of the effectiveness for the distribution of material within beam cross-sections.

The Bending Efficiency Factor is used to investigate the performance of steel I-sections in bending, this is achieved by considering the variation of bending efficiency with change in cross-sectional shape, the information is displayed graphically in the form of efficiency surfaces.

It is shown how the constraints imposed by the methods of production and design codes of practice can be used to form boundaries to the efficiency surfaces.

The combination of the idea of bending efficiency surfaces and the boundaries formed by the constraints is shown to provide the basis of a simple procedure for the design of structural steel cross-sections for optimum bending strength.
To My Family
I wish to express my thanks to the British Steel Corporation and in particular to Mr. J.E. Dibley of that organisation for allowing me to undertake this project. I would also like to express my gratitude to my supervisor Dr. H. Nooshin for his patience, persistence and encouragement throughout this work. Finally, thanks are due to Mrs. P. Wadsworth for her efficient typing of the manuscript.
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CHAPTER ONE
INTRODUCTION

1.1 The Problem

When faced with the problem of designing the cross section of a structural element, one is confronted with a number of conflicting requirements. The designer has to consider those limitations due to the methods of production in addition to the structural constraints related to the element's usage.

The following work considers the behaviour of beams in bending and shows how efficient shapes may be achieved. It indicates how the various factors which constrain a design may be combined, giving the designer an appreciation of the relative importance of each constraint, and allowing the optimization of cross sectional shape to be made directly.

1.2 Objectives

The principal objectives of this work are listed below:

(1) to show how the material within a cross section may be arranged to most effectively resist bending and illustrate how the bending efficiency of cross sectional shapes may be assessed quantitatively.

(2) To show how the constraints imposed by the production processes and those imposed by the codes of practice may be combined to give the designer a clear picture of the relative importance of each constraint at the outset of the design.

(3) To use the ideas contained in (1) and (2) above to effect the design of a steel beam cross section to be produced by hot rolling or welding.
1.3 RESTRICTIONS

The scope of the ideas contained here are restricted in the main to I-sections.

Sections are assumed to be constructed from a single material, generally steel, aluminium alloy or a similar homogeneous material.

No direct consideration has been given to the effects of torsion, shear, or any design requirement in which the overall length of the section comes into play.

1.4 SURVEY OF PREVIOUS WORK

During the last twenty years a number of papers have been published on the subject of improving the performance of structural steel members subject to bending. One reason for the interest in this subject was due to the advent of higher yield strength steels; and the need to produce section shapes to utilize fully the properties of these materials. In addition, it was necessary to determine where the advantages gained by increased structural efficiency would outweigh the higher material and production costs.

Two notable papers published during this period were those of Schilling (Ref.1) and Haaijer (Ref.2). Both authors and most of the other contributors to the subject approached the problem in a similar way. The objective in each case was to devise a minimum weight or minimum cost design for a plate girder given a value for stiffness or bending strength and web slenderness. A summary of the method used is given below.

To simplify the analysis of an I section the idealized shape shown in Fig.1.1 may be used with only slight loss of accuracy (if the flange thickness is small compared with the depth of the section).

![Fig. 1.1](T→O)
Using Fig.1.1, it is possible to form expressions for the stiffness or bending strength of the section using the following three terms:

- the cross sectional area \( A \),
- the web slenderness \( D/t \),
- and the proportion of web area to total area \( R_w \).

These expressions may then be differentiated with respect to \( R_w \) to give the proportion of web area to total area necessary to maximize the stiffness or bending strength. This yields the following values for \( R_w \):

\[
R_w = \frac{3}{4} \text{ for optimum stiffness},
R_w = \frac{1}{2} \text{ for optimum elastic bending strength},
\text{and } R_w = \frac{2}{3} \text{ for optimum plastic bending strength}.
\]

The method introduced in these papers provides a simple and effective way of proportioning a cross section to ensure a minimum weight design, optimizing either bending strength or stiffness. It can also be used to investigate the sensitivity of a design by examining the bending strength or stiffness for a range of values of \( R_w \). There are however, certain drawbacks to this approach:

1. Although the contribution of flange thickness may be ignored for deep slender beams, it plays a significant role when considering stockier sections.
2. This approach does not indicate the effects of constraints other than web slenderness on the efficiency of the design.
3. The designer is often faced with the problem of producing sections having a fixed depth and bending strength. In this situation the method just described becomes inconvenient.

In the following Chapters an approach to the optimization of cross sectional shapes is described which overcomes these shortcomings. In addition, an extension to Schilling's method is included in Appendix A, which enables the effect of flange thickness to be investigated.
1.5 Structural Efficiency

With the scarcity of material resources and the high cost of materials and manpower it is important that the engineer should consider the needs of a project as a whole when deciding on the form of structure required. This may be considered irrelevant in relation to something as outwardly simple as the production of efficient cross sectional shapes. However, the choice of section may have consequences which outweigh any savings brought about by improved structural efficiency.

This may be illustrated by considering the example of the cost of cladding for a multi-storey building. The engineer's goal is to produce a safe, efficient and cheap structure which corresponds as closely as possible to the needs of the architect. The cost of cladding is usually a higher proportion of the project cost than that of the structure. Using a small section depth, which in structural terms may result in a less efficient and more costly structure could enable the overall cost of the project to be reduced.

It will be noted that the bulk of this work considers how to achieve an efficient design for one or a family of structural elements, to this end it will be shown how aspects related to an efficient design may be rapidly determined; where the word efficient need not necessarily mean least weight.

1.6 Design of Beams

The process of designing a beam consists of determining the best compromise between various conflicting requirements. The influences on the choice of cross sectional shape of some of these requirements is discussed in this section.

1.6.1 Production Methods

Normally, welded plate girders are the only form of steel cross section that the structural engineer is called on to design. There are few restrictions placed on this type of beam by the production process, the obvious ones being the availability of suitable sizes of material and the capacity of the fabricating shop to handle large pieces.
However, if the engineer is required to design shapes to be produced by hot rolling the constraints imposed by the production process place severe restrictions on the shapes which may be achieved. These constraints are discussed in detail in Chapter 3.

1.6.2 BENDING STRENGTH

In ensuring that a beam has sufficient bending strength, the designer must proportion the cross section such that the material is used to its best advantage, i.e. by concentrating the material in the flanges and ensuring that they are placed sufficiently apart. The proportion of the usable material strength is dependent on the overall stability of the beam which requires a knowledge of how the element is to be used.

1.6.3 STIFFNESS

The designer must ensure that a beam does not deflect excessively to the detriment of the cladding, services and floors, etc. This is dependent on the overall depth of the section and its usage within the structure.

1.6.4 SHEAR STRENGTH

Shear strength is mainly dependent on the proportion of the material in the web, and as such conflicts with the requirements for bending strength. The production process for hot rolled sections places a restriction on the ratio of the flange to web thickness which ensures sufficient material in the web for most requirements.

1.6.5 LOCAL BUCKLING

The local buckling of a section is dependent on the flange and web slenderness. The designer is required by the codes of practice to ensure that the stresses in the flange and web are sufficiently low to prevent local buckling. This is achieved by limiting the flange and web slenderness which restricts the range of shapes which may be used.
1.6.6 OVERALL STABILITY

The overall stability of a beam is dependent on the slenderness of the member and the way in which it is restrained within the structure. The main properties of the cross section which affect overall instability are the minimum radius of gyration and the ratio of overall depth to flange thickness. This form of instability tends to preclude the use of narrow flanged deep beams.

1.6.7 TORSION

For torsion the polar moment of inertia plays an important role, and for the case of rolled sections due regard should be paid to the fillet at the web and flange junction.

1.6.8 CONNECTIONS AND USAGE

In designing a beam it is important to consider how it may be connected to other parts of the structure, such that the shape allows the most effective form of connection to be used. This is particularly true when designing ranges of sections, where it is necessary to consider the size of columns which are likely to frame into the beams and whether a simplified connection system can be adopted to reduce the number of different components to be stocked.
1.7 Approach

The normal method of designing a cross-sectional shape to meet the requirements noted in Section 1.6 can often result in a lengthy iterative procedure if a rolled section is to be produced (for welded beams the lack of choice of plate sizes usually results in a design which although fulfilling the design objectives is heavier than if optimum plate thicknesses were available). The design procedure often considers each aspect of the design in isolation, and if a "finely tuned" design is economically feasible this can result in large numbers of iterations before an optimum shape is achieved. The following Chapters discuss techniques whereby many aspects related to the design maybe considered together, allowing the relative effect of each aspect on the system as a whole to be investigated. By considering the process of design in this manner lengthy iterative procedures can be avoided and the designer is given a deeper insight into the factors which affect the choice of cross-sectional shape.
CHAPTER TWO

BENDING EFFICIENCY FACTOR

2.1 INTRODUCTION

This Chapter introduces the "Bending Efficiency Factor", which was originated by Dr. H. Nooshin, University of Surrey and is described in (Ref.3).

The bending efficiency factor is a simple function which may be used to assess the effectiveness of a cross-sectional shape to resist bending. Unlike most other measures of effectiveness used for this purpose the function is independent of the overall dimensions of the cross-section. It can be used to yield a numerical constant related to the efficiency of a single section. It may also be used in its functional form to examine the variation of bending efficiency with change in cross-sectional shape.

2.2 DERIVATION OF THE BENDING EFFICIENCY FACTOR

Consider the beam shown in Fig.2.1a it has a straight longitudinal axis, a constant cross-section and is composed of a single homogeneous material. Let the longitudinal axis be denoted by L and the cross-section be symmetric about both the X and Y axes as shown in Fig.2.1b, with X and Y being the principal axes of the cross-section. The beam is supported and loaded as shown in Fig.2.1c, such that it's central region is in pure bending.

Two forms of bending efficiency factor are investigated:

(1) The bending efficiency factor related to a beam where the material is within the linear elastic range, see Fig.2.1b; This is referred to as the Elastic Bending Efficiency Factor.

(2) The bending efficiency factor related to a beam where the material is in the plastic range. (In the sense used for the simple plastic analysis of plane frames and beams, subject to the usual simplifying assumptions for this form of material behaviour) see Fig. 2.1b. This is referred to as the Plastic Bending Efficiency Factor.
Fig. 2.1a

Fig. 2.1b

Fig. 2.1c
Case (1)

In the region of pure bending for the beam shown in Fig.2.1 the maximum tensile and compressive stresses at the extreme fibres of the cross section due to the bending moment $M$ are given by

$$
\text{maximum tensile stress } \sigma_t = \frac{M}{Z_x}
$$

and maximum compressive stress $\sigma_c = \frac{M}{Z_x}$.

Where $Z_x$ is the section modulus about the $X$ axis, and where due to the symmetry of the section about the $X$ axis $\sigma_t$ and $\sigma_c$ are numerically equal. Therefore, the general equation for the maximum tensile or compressive stress for this system may be given by

$$
\sigma = \frac{M}{Z_x} \quad \text{........ 2.1.}
$$

Where $\sigma$ represents either the maximum tensile or compressive stress. Multiplying each side of equation 2.1 by the cross sectional area $A$ and rearranging gives

$$
A = \frac{MA}{\sigma Z_x} \quad \text{........ 2.2.}
$$

The ratio $Z_x/A$ may be written as a linear function of the overall depth of the cross section $D$. This can be illustrated by considering the ratio for the cross sectional shape shown in Fig.2.1b.

For this shape $Z_x$ and $A$ may be expressed as

$$
Z_x = \frac{BD^3-(B-t)(D-2T)^3}{6D}
$$

and

$$
A = 2BT + t(D-2T) \; .
$$

Therefore, the ratio $Z_x/A$ may be written as

$$
\frac{Z_x}{A} = \frac{BD^3-(B-t)(D-2T)^3}{6D(2BT+t(D-2T))} \; .
$$
It is convenient to express the vertical dimensions of the cross-section as proportions of the overall depth, and express the horizontal dimensions as proportions of the overall width. For this purpose the terms $\alpha$ and $\beta$ are used where

$$\alpha = \frac{T}{D}$$

and

$$\beta = \frac{t}{B}.$$  

Then in terms of $\alpha$ and $\beta$ the ratio $Zx/A$ may be written as

$$\frac{Zx}{A} = D \left[ \frac{1-(1-\beta)(1-2\alpha)^3}{6(2\alpha+\beta(1-2\alpha))} \right].$$

Where the bracketed terms on the right hand side are independent of the overall dimensions of the cross section, and thus it can be seen that the right hand side may be written as a linear function of the cross-sectional depth $D$.

Therefore, one may write

$$\frac{Zx}{A} = f(D) \quad \ldots \ldots \quad 2.3.$$  

Where $f$ is a function of the ratios between the different dimensions of the cross-section and is independent of the actual dimensions of the cross section. A consequence of this is that the cross-section could be enlarged or reduced in size about either or both principal axes without altering the value of $f$. Therefore $f$ may be thought of as a function of the "shape" of the cross-section.

Substituting for $Zx/A$ from equation 2.3 in equation 2.2 gives

$$A = \frac{M}{gf(D)} \quad \ldots \ldots \quad 2.4.$$  

Equation 2.4 which is an expression of the governing design condition provides a convenient means for investigating the manner in which the cross-sectional area may be minimised.
Of the various items of information required for the simple design of a beam the bending moment $M$ is given, the magnitude of the maximum tensile or compressive stress $\sigma$ is prescribed by the material properties and the manner in which the beam is to be utilized. Therefore, both $M$ and $\sigma$ may be considered as constants. The choice of the overall depth of the cross-section $D$ often depends on criteria other than its effect on the magnitude of bending stresses, such as architectural requirements and the needs of the mechanical and electrical services. The designer usually seeks a compromise with these other interests to ensure that $D$ is large enough to allow a mass produced beam shape to be used, avoiding the need for a purpose built section.

This leaves only two terms which may be considered as variables, namely, the cross sectional area $A$ and the function $f$. From equation 2.4 it can be seen that to achieve a minimum cross-sectional area one must maximize $f$. Therefore, it follows that $f$ may be used as a measure of the effectiveness of a cross-sectional shape to resist bending stresses. That is, to achieve a minimum cross sectional area, and consequently a minimum weight, the function $f$ must be made as large as other factors such as the local buckling requirements will allow, as will be discussed in chapters 3 and 4.

Case (2)

Consider the system shown in Fig.2.1, for case (2) it is assumed that the material behaves in a perfectly plastic manner. Within the region of pure bending the maximum tensile and compressive stresses at the extreme fibres of the section are given by

$$\text{maximum tensile stress } \sigma_t = \frac{M}{\bar{Z}_x}$$

and

$$\text{maximum compressive stress } \sigma_c = \frac{M}{\bar{Z}_x}.$$

where $\bar{Z}_x$ is the plastic modulus for the section about the X axis, and where as for case (1) $\sigma_t$ and $\sigma_c$ are numerically equal. Therefore, a general expression for the maximum tensile or compressive stress may be given by
\[ \sigma = \frac{M}{Z_x} \quad \ldots \ldots \quad 2.5 \]

where \( \sigma \) represents either the maximum tensile or compressive stress.

Multiplying each side of equation 2.5 by the cross-sectional area \( A \) and rearranging gives

\[ A = \frac{MA}{\sigma Z_x} \quad \ldots \ldots \quad 2.6. \]

where as for the elastic case the ratio \( Z_x/A \) is a linear function of the overall depth of the cross-section \( D \) and may be written as

\[ \frac{Z_x}{A} = fD \quad \ldots \ldots \quad 2.7. \]

The function \( f \) has all the attributes of the elastic function and therefore may be used as a measure of effectiveness of a cross-sectional shape to resist bending stresses in the plastic region.

Representation of the Elastic and Plastic Bending Efficiency Factors

It is found that convenient limiting values are achieved by working with a simple function of \( f \) rather than \( f \) itself. This function is referred to as the "Bending Efficiency Factor" and is denoted by \( e \), where \( e \) is given by

\[ e = 2f \quad \ldots \ldots \quad 2.8. \]

Therefore, in terms of equations 2.3 the Elastic Bending Efficiency Factor may be written as

\[ ex = \frac{2Z_x}{D A} \quad \ldots \ldots \quad 2.9 \]

or \[ ex = \frac{4I_x}{D^2 A}. \]

where \( I_x \) is the second moment of area of the cross-section about the \( X \) axis.

In terms of equation 2.7 the Plastic Bending Efficiency Factor may be written as
\[ \epsilon_x = \frac{2 \bar{Z}_x}{DA} \quad \cdots \quad 2:10. \]

Exactly the same expression may be formed with respect to the Y axis giving rise to \( \epsilon_y \) and \( \bar{\epsilon}_y \). In the following Sections where \( e \) is shown unsubscripted it is used to denote either the elastic or plastic bending efficiency factor about either neutral axis.

2.2.1 INVARIANT TRANSFORMATIONS

It is possible to perform certain transformations on a cross section without the value of the bending efficiency factor, these transformations are referred to as invariant transformations. The following are some invariant transformations.

1. Multiplication of all the horizontal dimensions of a cross-section by a constant factor.
2. Multiplication of all the vertical dimensions of a cross-section by a constant factor.
3. Any combination of the first two invariant transformations - an important example of this transformation being photographic enlargement or reduction of the shape.

Fig. 2.2 shows four cross sectional shapes each having the same value for \( \epsilon_x \), shapes 2, 3 and 4 are the result of applying the invariant transformations on shape 1.
2.3 LIMITING VALUES OF THE BENDING EFFICIENCY FACTOR

The limiting values of the bending efficiency factor are zero and unity. In practice neither of these two limiting values can be achieved since they correspond to shapes having impractical dimensions. It is however, useful to consider these limiting values and they are discussed in the sequel.

2.3.1 UPPER LIMIT

Consider a beam in pure bending having a depth D and a cross sectional area A. To realize the maximum bending strength for this section it is necessary to distribute the material in the cross section as far from the neutral plane as the overall depth will allow. This can only be achieved by the shape shown in fig.2.3, which comprises two horizontal strips of material, distance D apart, extending to infinity and having thicknesses approaching zero. These are joined by a vertical strip which has a thickness approaching zero.

\[ \text{Fig. 2.3} \]

For this imaginary cross-sectional shape the second moment of area about the neutral plane can be expressed as

\[ I_n = \frac{AD^2}{4} \]

and the corresponding expression for \( e_n \) is given by

\[ e_n = \frac{4I_n}{D^2A} \]
Substituting for $In$ in this expression yields

$$en = \frac{4AD^2}{4AD^2} = 1.$$  

Thus, the upper limit of $en$ is unity.

### 2.3.2 LOWER LIMIT

Consider a beam in pure bending having a depth $D$ and a cross-sectional area $A$. To realize the minimum bending strength for this section it is necessary to distribute the material in the cross-section as close to the neutral plane as possible. This is achieved by the shape shown in fig.2.4, which comprises a strip of material that extends to infinity and lies along the neutral plane and has a thickness that approaches zero. In addition, there is a vertical strip of depth $D$ having a thickness approaching zero as shown in the fig.2.4.

![Fig. 2.4](image)

Since the length of the vertical strip is finite then it's area approaches zero. Therefore, the only possible contribution to the second moment of area about the neutral plane comes from the horizontal strip. Since the depth of this strip approaches zero then the second moment of area is bound to approach zero. Furthermore, the entities $D$, $A$ and $In$ can never be negative and as $D$ and $A$ are always finite, by considering the expression for $en$

$$en = \frac{4In}{D^2A},$$

it follows that zero is indeed the lower limit of $en$. 
2.4 APPLICATION OF THE BENDING EFFICIENCY FACTOR

From Section 2.3 it can be seen that the bending efficiency factor provides a quantitative measure of the ability of a cross-section to resist bending stresses. Where the larger the value for e the more effectively the material of the cross-section is placed. To illustrate this, values of ex and \( \bar{e}x \) have been calculated for a number of simple shapes, this information is shown in Table 2.1.

Table 2.1

<table>
<thead>
<tr>
<th>Cross Sectional Shape</th>
<th>( e_x )</th>
<th>( \bar{e}x )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Square" /></td>
<td>1/6</td>
<td>1/3</td>
</tr>
<tr>
<td><img src="image" alt="Rectangle" /></td>
<td>1/3</td>
<td>1/2</td>
</tr>
<tr>
<td><img src="image" alt="Circle" /></td>
<td>1/4</td>
<td>( 4/3\pi )</td>
</tr>
<tr>
<td><img src="image" alt="椭圆" /></td>
<td>1/2</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td><img src="image" alt="Rectangular" /></td>
<td>1/3</td>
<td>1/2</td>
</tr>
<tr>
<td><img src="image" alt="Trapezoidal" /></td>
<td>2/3</td>
<td>3/4</td>
</tr>
<tr>
<td><img src="image" alt="I-Section" /></td>
<td>7/9</td>
<td>5/6</td>
</tr>
</tbody>
</table>

T is assumed to be small compared to the depth D.
2.5 The Bending Efficiency of I-Sections

Consider the I-section shown in Fig. 2.5

From equation 2.9 the elastic bending efficiency factor $e_x$ is given by

$$ e_x = \frac{2 \sum_x}{DA} $$

where $\sum_x$ may be expressed as

$$ \sum_x = \frac{BD^3 - (B-t)(D-2T)^3}{6D} $$

and where the cross-sectional area $A$ is

$$ A = 2BT + t(D-2T). $$

Therefore, $e_x$ is given by

$$ e_x = \frac{BD^3 - (B-t)(D-2T)^3}{3D(2BT+t(D-2T))}. $$

This may also be written as

$$ e_x = \frac{1-(1-\beta)(1-2\alpha)^3}{3(\beta+2\alpha(1-\beta))} \quad \ldots \ldots 2.11. $$

where $\alpha$ and $\beta$ are as discussed in Section 2.2, namely

$$ \alpha = \frac{T}{D}, $$

and $\beta = \frac{t}{B},$. 
where \( \alpha \) is referred to as the "relative flange thickness"
and \( \beta \) is referred to as the "relative web thickness."

The relative flange thickness has a range from 0 to \( \frac{1}{2} \). When \( \alpha \) is \( \frac{1}{2} \), the flange thickness is half the section depth and therefore the web, as such, no longer exists and the shape degenerates into a rectangle.

The relative web thickness has a range from 0 to 1. When \( \beta \) is 1, the web thickness equals the flange width and the shape degenerates into a rectangle.

These limiting values are of interest but, do not refer to practical I-sections. By using the values of \( \alpha \) and \( \beta \) between these limits it is possible to use equation 2.11 to examine the variation of the bending efficiency factor over the whole range of possible doubly symmetric I-shapes, as will be shown in the following Sections.

A similar expression may be obtained from \( \bar{\varepsilon}_x \). From equation 2.10 the plastic bending efficiency factor \( \bar{\varepsilon}_x \) is given by

\[
\bar{\varepsilon}_x = \frac{2 \bar{Z}_x}{DA},
\]

where \( \bar{Z}_x \) may be expressed as

\[
\bar{Z}_x = \frac{BD^2 - (B-t)(D-2T)^2}{4},
\]

and the cross-sectional area is given by

\[ A = 2BT + t(D-2t), \]

where the notation is as shown in fig.2.5.

Therefore, \( \bar{\varepsilon}_x \) may be written as

\[
\bar{\varepsilon}_x = \frac{BD^2 - (B-t)(D-2T)^2}{2D(2BT + t(D-2T))}.
\]

Alternatively, using the relative flange and web thicknesses as explained before, \( \bar{\varepsilon}_x \) may be written as
\[ \varepsilon_x = \frac{4a(1-a) + (1-2a)^2}{2(\beta + 2a(1-\beta))} \] \[ \ldots \ldots \quad 2.12. \]

The expressions for \( \varepsilon_y \) and \( \bar{\varepsilon}_y \) may be obtained in a similar fashion, with the resulting equations being

\[ \varepsilon_y = \frac{2a +(1-2a)\beta^3}{3(\beta + 2a(1-\beta))} \] \[ \ldots \ldots \quad 2.13 \]

and

\[ \bar{\varepsilon}_y = \frac{1 - (1-2a)(1-\beta^2)}{2(\beta + 2a(1-\beta))} \] \[ \ldots \ldots \quad 2.14. \]

2.6 RANGE OF VALUES FOR \( \varepsilon_x, \bar{\varepsilon}_x, \varepsilon_y \) AND \( \bar{\varepsilon}_y \)

The upper limit for values of \( \varepsilon_x \) and \( \bar{\varepsilon}_x \) occur at \( \alpha = 0, \beta = 0 \) and this corresponds to the shape shown in fig.2.3. The lower limit for the values of \( \varepsilon_x \) and \( \bar{\varepsilon}_x \) occur at \( \alpha = \frac{1}{3}, \beta = 1 \) and this corresponds to a rectangle.

The upper limit for values of \( \varepsilon_y \) and \( \bar{\varepsilon}_y \) occur at \( \alpha = 0, \beta = 0 \) and this corresponds to a rectangle. The lower limit for the values of \( \varepsilon_y \) and \( \bar{\varepsilon}_y \) occur at \( \alpha = \frac{1}{3}, \beta = 1 \) and this corresponds to the shape shown in Fig.2.4. These limiting values may be obtained by substituting appropriate values of \( \alpha \) and \( \beta \) in equations 2.11 to 2.14. The resulting values are displayed in Table 2.2.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Bending Efficiency Factor</th>
<th>( \alpha = \frac{1}{3}, \beta = 1 ) lower limit</th>
<th>( \alpha = 0, \beta = 0 ) upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>I I I</td>
<td>( \varepsilon_x )</td>
<td>( \frac{1}{3} )</td>
<td>1</td>
</tr>
<tr>
<td>I I I</td>
<td>( \bar{\varepsilon}_x )</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>I I I</td>
<td>( \varepsilon_y )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>I I I</td>
<td>( \bar{\varepsilon}_y )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>
2.7 VARIATION OF ex AND ēx

One can use the expressions 2.11 and 2.12 to plot ēx or ex versus α for a range of values of β to show the variation in bending efficiency with respect to change in cross sectional shape. This is done in Figs. 2.6 and 2.7, the former relating to ēx and the latter to ex.

The curves in these figures show a similar behaviour, initially as the relative flange thickness α is increased the bending efficiency factor is increased until a maximum is reached, after which, any further increase in α results in a reduction in the bending efficiency factor. At $\alpha = \frac{1}{2}$ the value of ex or ēx is the same as that relating to $\alpha = 0$, with both the values corresponding to a rectangular shape.

It will be seen that as the relative web thickness $\beta$ is increased the value of $\alpha$ at which ex or ēx is a maximum becomes progressively larger. In addition as $\beta$ is increased the corresponding maximum value of ex or ēx becomes smaller.

A function may be derived for the elastic case, which, for any $\beta$ yields the value of $\alpha$ at which the bending efficiency factor ex is a maximum. The function may be obtained through the limiting processes of calculus. This is achieved by setting the partial derivative of ex with respect to $\alpha$ to zero and solving the resulting equation.

From equation 2.11 the bending efficiency factor ex is given by

$$ex = \frac{1-(1-\beta)(1-2\alpha)^3}{3(\beta+2\alpha(1-\beta))}.$$

Letting

$$u = 1 - (1-\beta)(1-2\alpha)^3$$

and

$$v = 3(\beta+2\alpha(1-\beta)),$$

then

$$\frac{\partial ex}{\partial \alpha} = \frac{V \frac{\partial u}{\partial \alpha} - U \frac{\partial v}{\partial \alpha}}{v^2}.$$
Fig. 2.6

Fig. 2.7
and by setting \( \frac{\partial \mathbf{v}}{\partial \alpha} \) to zero one obtains

\[
\nu \frac{\partial u}{\partial \alpha} - \mu \frac{\partial w}{\partial \alpha} = 0 \quad \ldots \quad 2.15.
\]

where

\[
\frac{\partial u}{\partial \alpha} = 6(1-\beta)(1-2\alpha)^2
\]

and \( \frac{\partial w}{\partial \alpha} = 6(1-\beta) \).

Substituting these terms into equation 2.15 yields

\[
3(\beta+2\alpha(1-\beta))(6(1-\beta)(1-2\alpha)^2) - (1-(1-\beta)(1-2\alpha)^3)(6(1-\beta)) = 0,
\]

which may be simplified to give

\[
\alpha^3(8(1-\beta)) + \alpha^2(6(2\beta-1)) - \alpha(6\beta) + \beta = 0.
\]

Solving this cubic equation* yields three roots one of which is relevant and is given by

\[
\alpha = \frac{(1-2\beta)-2 \sin \left[ \sin^{-1} \left( \frac{2\beta^2-4\beta+1}{3} \right) \right]}{4(1-\beta)} \quad \ldots \quad 2.16.
\]

Henceforth the optimum value for \( \alpha \) obtained from this equation will be denoted by \( \alpha_0 \).

A similar function may be obtained for the plastic bending efficiency factor \( \bar{e}_x \), where from equation 2.12 \( \bar{e}_x \) is given by

\[
\bar{e}_x = \frac{4\alpha(1-\alpha) + (1-2\alpha)^2\beta}{2(\beta+2\alpha(1-\beta))}
\]

Proceeding in a similar fashion one obtains

*The solution for this cubic equation is shown in Appendix B.
Substituting these terms into equation 2.15 results in

\[2(\beta+2\alpha(1-\beta))(4(1-\beta)(1-2\alpha)) - (4\alpha(1-\alpha)+(1-2\alpha)^2\beta)(4(1-\beta)) = 0,\]

which may be simplified to

\[\alpha^2(4(\beta-1)) - \alpha(4\beta) + \beta = 0.\]

One of the roots of this quadratic equation is relevant and is given by

\[\alpha = \frac{1}{2 + 2\beta^{-1}} \quad \ldots \quad 2.17.\]

Henceforth, the optimum value for \(\alpha\) obtained from this equation will be denoted by \(\overline{\alpha}_0\).

Figs. 2.8 and 2.9 have been plotted using equations 2.11 and 2.12 (with \(\beta\) as the dependent variable and \(\alpha\) as the independent variable), for certain values of the bending efficiency factors \(e_x\) and \(\tilde{e}_x\). Included on each figure is the curve indicating the optimum values of relative flange thickness \(\tilde{\alpha}_0\) or \(\overline{\alpha}_0\) which is shown dotted.

Consider the intersection of the \(\tilde{\alpha}_0\) curve with one of the bending efficiency contours shown in Fig. 2.8. It will be noted that the bending efficiency factor is nearly flat (in relation to the \(\alpha\) axis) either side of the intersection. That is, for any value of \(\beta\) there exists a band of \(\alpha\) values either side of the \(\tilde{\alpha}_0\) curve having similar values of bending efficiency to that at \(\tilde{\alpha}_0\). This is found to be true for both elastic and plastic cases.
Fig. 2:10 Elastic Bending Efficiency Surface $\ell x$. 
Fig. 2.11 Plastic Bending Efficiency Surface $\bar{c}_x$. 
This region of flat response becomes progressively smaller as the \( \alpha_0 \) or \( \tilde{\alpha}_0 \) curve approaches the origin, where, the bending efficiency factor becomes relatively sensitive to changes in \( \alpha \) and/or \( \beta \).

It will be noticed that only a part in either of the figs. 2.8 and 2.9 is of practical significance. This region is bounded by \( 0<\beta<0.2 \) and \( 0<\beta<0.2 \). It is also in this area where the bending efficiency factor \( e \) or \( \tilde{e} \) is most sensitive to changes in \( \alpha \) and \( \beta \).

Figs. 2.10 and 2.11 show a three dimensional representation of Figs. 2.8 and 2.9. From Figs. 2.10 and 2.11 it may be seen that the bending efficiency factor plotted in three dimensions, in the space represented by the axes \( e \), \( \alpha \) and \( \beta \), is a surface, and in the sequel where the phrase bending efficiency surface is mentioned it is used in this sense.

2.8 SIMILARITIES BETWEEN \( \alpha_0 \) AND \( \tilde{\alpha}_0 \)

In Fig. 2.12, the values of \( \alpha_0 \) and \( \tilde{\alpha}_0 \) have been plotted versus \( \beta \), it will be noted that the curves are of similar shape and position.

In Section 2.7 it was shown that the bending efficiency factors \( e \) or \( \tilde{e} \), have a flat response either side of the \( \alpha_0 \) or \( \tilde{\alpha}_0 \) curves. It is found that any point on the \( \alpha_0 \) or \( \tilde{\alpha}_0 \) curves will lie within the region of flat response for the other, and vice versa. From this one can conclude that a cross-section which has an optimum shape (in relation to its \( \beta \) value) for the plastic case will also be near optimum if considered for the elastic case, and vice versa.

![Graph showing similarities between \( \alpha_0 \) and \( \tilde{\alpha}_0 \)](image1)

Fig. 2.12
2.9 Shape Factor

In Section 2.2 it was shown that for the same shape the plastic bending efficiency factor is larger than the corresponding elastic bending efficiency factor. The ratio $\bar{e}_x/e_x$ turns out to be what in simple plastic design is referred to as the "Shape Factor" which is used to denote the ratio $\bar{Z}_x/Z_x$. To illustrate this it is only necessary to note that

$$\frac{\bar{e}_x}{e_x} = \frac{2 \bar{Z}_x}{DA},$$

which implies that

$$\frac{\bar{e}_x}{e_x} = \frac{\bar{Z}_x}{Z_x}.$$

2.10 Variation of $e_y$ and $\bar{e}_y$

One can use equations 2.13 and 2.14 to plot $e_y$ or $\bar{e}_y$ versus $\beta$ for a range of values of $\alpha$ to show the variation in bending efficiency. This is done in Figs. 2.13 and 2.14. Unlike the $e_x$ and $\bar{e}_x$ surfaces which are convex, the $e_y$ and $\bar{e}_y$ surfaces are concave and have the optimum value for bending efficiency occurring along the boundaries. This is also shown in Figs. 2.15 and 2.16, where $\alpha$ is plotted versus $\beta$ for various values of $e_y$ and $\bar{e}_y$. The boundaries correspond to a rectangular shape, therefore, obtaining the optimum bending efficiency about the Y axis results in an impractical shape if a reasonable performance about the X axis is also required.

Within the region $0<\alpha<0.2$ and $0<\beta<0.2$ small changes in $\alpha$ and $\beta$ result in large changes in the bending efficiency factors $e_y$ and $\bar{e}_y$, which is similar to the case of $e_x$ and $\bar{e}_x$. It may be seen that in this region, and more particularly the portion of the area below the $\alpha_0$ and $\bar{\alpha}_0$ curves, the direction of increasing efficiency is similar for both X and Y axis cases. That is, given a value for $\alpha$ and $\beta$ within this region, reducing the relative web thickness results in a larger value for bending efficiency for both cases. This is however, not necessarily
the most effective means of obtaining a larger value of bending factor.

Figs. 2.17 and 2.18 show a schematic representation of the $e_y$ and $\bar{e}_y$ surfaces.
Fig. 2.15
Fig. 2.16
Fig. 2.17 Elastic Bending Efficiency Surface $\eta_y$. 
Fig 2.18 Plastic Bending Efficiency Surface $\bar{e}_y$. 
2.11 VARIATION IN SHAPE

So far, the bending efficiency factor has been considered in a strictly non-dimensional form. Such that, for any point \((α, β)\) on the bending efficiency surface corresponds to an infinite number of shapes (related to one another by appropriate invariant transformations). This however, is not the only way in which the concept of bending efficiency factor can usefully be employed.

Consider the equation for the elastic bending efficiency factor

\[
\text{ex} = \frac{2 \cdot Zx}{D \cdot A}.
\]

It will be seen that by fixing the values of two out of the three terms, \(D\), \(A\) and \(Zx\) on the right hand side, \(\text{ex}\) will be related to the third term. Therefore, the bending efficiency surface may be used to show the variation of the efficiency of the cross-sectional shape with respect to one of the terms \(D\), \(A\) or \(Zx\). One consequence of this is that it is possible to obtain a unique shape corresponding to a point on the bending efficiency surface as will be illustrated below.

A useful means of classifying cross-sectional shapes is with respect to the ratio \(A/D^2\). The present set of British Universal Beam sections occur within a small range of \(A/D^2\), namely, from \(A/D^2 = 0.03\) to \(A/D^2 = 0.08\), with the majority of sections occurring between the limits \(A/D^2 = 0.035\) to \(A/D^2 = 0.055\). A choice of values for \(α\) and \(β\) will determine a point on the bending efficiency surface. In addition, a choice of values for \(A\) and \(D\) will firstly determine the value for \(A/D^2\) and consequently the other dimensions of the cross-section, namely \(T\), \(t\) and \(B\) may be obtained using the following relations

- Flange Thickness \(T = αD\),
- Flange Width \(B = A / (D(β+2α(1-β)))\)
- and Web Thickness \(t = βB\).

Having found the dimensions of the cross section in terms of \(α\), \(β\), \(A\) and \(D\) one can obtain the shape related to the ratio \(A/D^2\) at any point on the bending efficiency surface. Figs. 2.19 and 2.20 show the variation in shape using the values of \(A/D^2 = 0.30\) and \(A/D^2 = 0.40\).
respectively. It will be noted that only a small portion of the surface gives rise to practical shapes. This region is bounded by 0<α<0.2 and 0<β<0.2, which confirms the limits suggested in Section 2.7. Fig. 2.21 shows an enlarged view of a part of this area using a value of 0.05 for A/D^2. The shapes corresponding to the β axis should be represented by rectangles, however, to show the extreme shapes that occur as α tends to zero a small positive offset has been added to α=0.

Another useful means of classifying cross-sectional shapes is with respect to the ratio Zx/D^3. The present set of universal beam sections occur within a small range of Zx/D^3, namely, from Zx/D^3=0.01 to Zx/D^3=0.03 with the majority of sections occurring between the limits Zx/D^3 = 0.011 to Zx/D^3 = 0.018. A choice of α and β will determine a point on the bending efficiency surface. In addition, a choice of values for Zx and D will firstly determine the value for Zx/D^3 and consequently the other dimensions of the cross section may be obtained using the following relations.

\[
\begin{align*}
\text{Flange Thickness} & \quad T = \alpha D , \\
\text{Cross-sectional area} & \quad A = \frac{2 \ Zx}{D^{ex}} , \\
\text{Flange breadth} & \quad B = A/(D(\beta+2\alpha(1-\beta)) ) \\
\text{and Web Thickness} & \quad t = \beta B .
\end{align*}
\]

Having found the dimensions in terms of α and β one can obtain the shape related to the ratio of Zx/D^3. Fig. 2.22 shows the variation in shape using the value for Zx/D^3 of 0.017, where the change in ex is related to change in cross-sectional area.

The final pair of terms which may be used are Zx and A. For this case the dimensions of the cross-section are given by

\[
\begin{align*}
\text{Section depth} & \quad D = \frac{2 \ Zx}{ex \ A} , \\
\text{Flange thickness} & \quad T = \alpha D , \\
\text{Flange breadth} & \quad B = A/(D(\beta+2\alpha(1-\beta)) ) \\
\text{and Web thickness} & \quad t = \beta B .
\end{align*}
\]
Fig. 2.23 shows the variation in depth for a constant $Z_x$ and $A$. Although examples have been shown for the elastic bending efficiency factor, any of the other surfaces discussed so far may also be used.
Fig. 2.20
RELATIVE FLANGE THICKNESS

RELATIVE WEB THICKNESS

Fig. 2.21
Fig. 2.22
Fig. 2.23
CHAPTER 3
BOUNDARY CONDITIONS

3.1 RESTRICTIONS TO CROSS-SECTION SHAPES

In Chapter 2 the concept of the bending efficiency factor was introduced as a means of assessing the ability of a cross-section to resist bending. However, the design of a beam involves a compromise between various conflicting requirements. Bending strength, although of primary importance, is only one of many influences on the final shape. Some of the factors which may affect the shape of the cross-section are listed below.

(1) The need to achieve an optimum (least cost or least weight) shape to resist in service stresses.

(2) The need to prevent local and/or overall instability of the beam.

(3) The need to produce the beam in the most economical way, i.e. within the limitations of the production process used.

This Chapter describes some of these other considerations exemplified by (2) and (3) above which may influence the choice of cross sectional shape. It then describes how these constraints may be used to form boundaries to the bending efficiency surface described in Chapter 2. The list of constraints considered here is not exhaustive, but the techniques used in this Chapter are applicable to other forms of constraints.

3.2 PRODUCTION PROCESS CONSTRAINTS

The following describes some constraints imposed by the hot rolling process.
3.2.1 HOT ROLLED SECTIONS

Hot rolling is the process by which a large proportion of the steel I-sections used in the engineering industries are produced, it is also a process which imposes severe restrictions on the profiles that can be achieved.

3.2.2 RESIDUAL COOLING STRESSES

A major limitation on hot rolled shapes relates to the ratio of flange thickness to web thickness, T/t, which should not exceed 1.6 (ref.4). The value of this ratio is a good measure of the magnitude of thermal residual stresses which are "locked in" the section during cooling. If this ratio has a value of 1 which is a practical lower boundary for the production of profiles (some column sections have a ratio of T/t =1) then the residual stresses are of low magnitude and distributed evenly across the profile. If the ratio exceeds 1.6 the distribution becomes more "peaky" and the total locked in stresses are higher. This can result in the web buckling as the section cools and also in poor performance if used as a compression member in a structure.

The reason for this variation in the magnitude and distribution of thermal residual stresses may be explained as follows. After leaving the rolling stands and having been sawn to length the sections are left to cool. To ensure even cooling they are stacked with their webs vertical and with their flange tips touching those of the adjacent sections. The centre of web and the flange tips cool and gain strength faster than the areas of flange around the web junction, thus, when this mass of material finally cools and tries to contract the flange tips and the majority of the web resist this shrinkage and are put in compression and the slowest cooling areas are held in tension. It may be seen that if the value of the ratio for flange thickness to web thickness is high the web will tend to cool quickly and thus the magnitude of the cooling residual stresses will be large. An acceptable maximum value for the ratio T/t has been found to be 1.6, profiles rolled at this value or below are not prone to web buckling on cooling and have an adequate strut behaviour.
3.2.2 MATERIAL PROPERTIES

Another reason for using the constraint $T/t \leq 1.6$ for hot rolled sections is to reduce the variation in material properties through the cross-section. The rapid cooling of the flange tips and web result in the micro structure of the steel varying over the cross section. This results in different material properties at different parts of the cross-section. As a consequence the web material has a different yield strength to that of the flanges.

Tests carried out on series of sections where this ratio varied from 1.4 to 1.78 gave an average flange yield strength of 95.3 percent of the average web yield strength (ref.4).

3.2.3 ROLLING

Additional cooling problems occur during the rolling process, where for certain types of rolling stand, sections having a web slenderness greater than 56 (where web slenderness is given by $(D-2T)/t$) are unable to be rolled, due to web buckles occurring during the latter rolling passes. The web cools so rapidly during the rolling process that unbalanced drafting occurs between the flange and web, resulting in the web forming buckles along its length.

3.2.4 COLD STRAIGHTENING

The uneven cooling of the sections and the methods by which they are transported and cut to length whilst hot results in beams and columns rarely complying with the specifications laid down for overall straightness (ref.5). To bring sections within the straightness tolerance required by B.S.4, they are cold rolled in a roller straightening machine (see Fig.3.1). This consists of seven or nine rolls arranged in two rows in a single housing with either all or the bottom row of rolls driven. The top rolls are placed midway between the bottom rolls and may be adjusted vertically by screws. These are positioned to give a successively smaller offset towards the output end of the machine, each top roll making less deformation than the preceding one. The machine deforms the section about its weaker axis, the rolls applying their loads to the root radii
Section A-A

Fig. 3-1
and part of the web. As a result the flange tips are plastically deformed successively in tension and compression. If the section has wide flanges and a thin web there is a tendency for the web to be torn away from the flanges during the process. To prevent this roller straightening is restricted to sections having a flange width to web thickness less than or equal to 18 (ref 5 and 6). For profiles having a higher value of flange slenderness the sections have to be gag straightened which is a slow operator dependent process and is uneconomic for high output rolling mills.

3.2.5 FEEDSTOCK AND PLANT

In addition to the profile limitations so far mentioned there are also limits to the overall size of section which may be rolled, this being a function of the type of rolling stand, size of billet or continuous cast strand available and capacity of the slab reheating furnaces.

3.3 WELDED SECTIONS

The production of steel I-shapes by welding imposes two main constraints on the shape as explained below.

3.3.1 FEED STOCK

Only a limited range of flat plate thicknesses are rolled and some of these have price extras as a result of being produced in limited quantities. There are therefore a range of preferred thicknesses of material which may be used for flange and web plates. It is usually uneconomic to reduce the width of rolled plate which in consequence dictates the depth and width of the section. Thus, cross-sectional shapes are limited to a discrete set of sizes.

3.3.2 WELDING

It has been shown that weld shrinkage residual stresses cause flanges to turn inwards during the welding process. To avoid taking special measures to prevent this occurring the flange outstand slenderness \((B-t)/2T\) is restricted.
3.4 Design Code Constraints

Some constraints imposed by design codes of practice are described below.

The structural steel design codes of practice limit the slenderness of both web and flanges to prevent the occurrence of local or overall instability. Resistance to local instability is a function of the cross-sectional shape and the material properties. Therefore, the constraints limiting web and flange slenderness may be used directly with the bending efficiency surfaces developed previously.

The local instability constraints discussed in the sequel relate either to an "elastically" designed system or a "plastically" designed system. In the latter the limits are such as to allow a plastic hinge to fully develop in the element without instability occurring and thus enable the sections full rotation capacity to be achieved. They are therefore more severe than the limits for elastic design.

3.4.1 Local Instability - Flange Slenderness

Flanges are subject to a limit to prevent local buckling occurring. The limit depends whether the section is produced by hot rolling or welding. In the latest codes of practice welded sections are subject to more severe limits due to the high residual stress present around the web to flange junction. The limits presently in use are shown in Fig. 3.2a.

3.4.2 Local Instability - Web Slenderness

The web local buckling limit is also dependent on the production process used. In the latest codes of practice welded sections again being subject to more severe limits than hot rolled sections. The limits for web slenderness presently in use are shown in Fig. 3.2b.
Flange Constraints

<table>
<thead>
<tr>
<th>Grade</th>
<th>43</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{B}{T} )</td>
<td>18</td>
<td>15.25</td>
<td>13.75</td>
</tr>
<tr>
<td>( \frac{B-t}{2T} )</td>
<td>16</td>
<td>14</td>
<td>12</td>
</tr>
</tbody>
</table>

Plastic

Web Constraints

<table>
<thead>
<tr>
<th>Grade</th>
<th>43</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{D-2T}{t} )</td>
<td>53</td>
<td>45</td>
<td>40</td>
</tr>
<tr>
<td>( \frac{D-2T}{t} )</td>
<td>85</td>
<td>75</td>
<td>65</td>
</tr>
</tbody>
</table>

Plastic

Elastic
### 3.5 Tabulation of Constraints

The constraints mentioned previously are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Design Code of Practice</th>
<th>Production Process</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Flange Slenderness</strong></td>
<td>Limit for compression flange slenderness, plastic design (local stability)</td>
<td>limit to prevent 'turn in' of flanges during welding</td>
</tr>
<tr>
<td><strong>B/T ≤ const.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Flange Outstand</strong></td>
<td>Limit for compression flange slenderness, elastic design (local stability)</td>
<td>limit to prevent web buckles forming during the rolling process</td>
</tr>
<tr>
<td><strong>(B-t)/2T ≤ const.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Web Slenderness</strong></td>
<td>Limit for web slenderness elastic and plastic design (local stability)</td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
<tr>
<td><strong>(D-2T)/t ≤ const.</strong></td>
<td>Overall stability criteria for strut behaviour and lateral torsional buckling</td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
<tr>
<td><strong>D/T ≤ const.</strong></td>
<td></td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
<tr>
<td><strong>B/t ≤ const.</strong></td>
<td></td>
<td>limit to prevent web tear during the roller straightening of hot rolled sections.</td>
</tr>
<tr>
<td><strong>Aspect Ratio</strong></td>
<td></td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
<tr>
<td><strong>D/B ≤ const.</strong></td>
<td></td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
<tr>
<td><strong>Flange thickness to web thickness ratio</strong></td>
<td>limit to prevent web buckles on cooling of hot rolled sections due to high residual stresses.</td>
<td>limit imposed by feedstock and/or production plant.</td>
</tr>
</tbody>
</table>
3.6 FORMULATION OF BOUNDARIES

This Section indicates how the constraints discussed so far may be expressed in terms of $\alpha$ and $\beta$, and used to form boundaries to the bending efficiency surfaces described in Chapter 2. They are derived to suit two modes of use of the bending efficiency factor:

(1) Non-Dimensional Mode

In this mode, any point $(\alpha, \beta)$ on the bending efficiency surface corresponds to an infinite number of possible cross-sectional shapes (related to one another by appropriate invariant transformations). To rearrange the constraints to suit this mode it is necessary to fix two of the dimensions $D$, $B$, $T$ or $t$. In this respect the following two ratios have been found useful.

The ratio of flange thickness to web thickness $T/t'$ denoted by $\delta$, and the aspect ratio, $D/B$ denoted by $\gamma$.

(2) Dimensional Mode

In this mode, any point $(\alpha, \beta)$ on the bending efficiency surface represents a unique cross-sectional shape related to one of the ratios $A/D^2$, $Zx/D^3$ or $Zx/D^3$. In this mode it is necessary to fix the value of $A$ and $D$ or $Zx$ and $D$ or $Zx$ and $D$, where:

The ratio $A/D^2$ is denoted by $\psi$, the ratio $Zx/D^3$ is denoted by $\omega$ and the ratio $\bar{Zx}/D^3$ is denoted by $\bar{\omega}$.

Two examples are now given to show the formulation of the web slenderness constraint in terms of $\alpha$ and $\beta$.

Example 1 - Non-Dimensional Mode

The web slenderness inequality is given by

$$\frac{D-2T}{t} \leq C.$$  

Where, $D$, $T$ and $t$ are as shown in Fig.2.5, and where $C$ is a numeric
constant having a range dependent on the grade of material as discussed in Section 3.4.

The inequality may be written as

\[ \frac{D}{t} - \frac{2T}{t} \leq C. \]

The ratio \( \frac{D}{t} \) may be written as

\[ \frac{\gamma}{\beta} , \]

and the ratio \( \frac{T}{t} \) may be written as

\[ \frac{\alpha \gamma}{\beta} . \]

Therefore, the inequality may be written as

\[ \frac{\gamma}{\beta} - \frac{2\alpha \gamma}{\beta} \leq C , \]

or \[ \beta \geq \frac{\gamma(1-2\alpha)}{C} \]

...... 3.1.

Equation 3.1 is the web slenderness constraint expressed in terms of \( \alpha \), \( \beta \) and \( \gamma \).

A similar process may be adopted to express the constraint in terms of \( \alpha \), \( \beta \) and \( \theta \).

The ratio \( \frac{D}{t} \) may be written as

\[ \frac{\theta}{\alpha} \]

and \[ 2\theta = \frac{2T}{t} . \]

Therefore, the inequality may be written as
\[ \frac{\theta}{\alpha} - 2\theta \leq C, \]
or \[ \alpha \geq \frac{\theta}{2\theta + C} \] \hspace{1cm} \text{.... 3.2.}

Equation 3.2 is the web slenderness constraint expressed in terms of \( \alpha \), \( \beta \) and \( \theta \).

**Example 2 - Dimensional Mode**

The inequality is given by

\[ \frac{D-2T}{t} \leq C, \]

where \( T \) and \( t \) may be expressed as

\[ T = D\alpha \]
and \( t = B\beta \)

Therefore, the inequality may be written as

\[ \frac{D(1-2\alpha)}{B\beta} \leq C \]

or \[ B \geq \frac{D(1-2\alpha)}{C\beta} \] \hspace{1cm} \text{.... 3.3.}

The cross-sectional area \( A \) is given by

\[ A = 2BT + t(D-2T) \]
or \[ A = 2BD\alpha + B\beta D(1-2\alpha) \]

Consequently,

\[ B = \frac{A}{2D\alpha + B\beta D(1-2\alpha)} \] \hspace{1cm} \text{.... 3.4.}
Substituting for $B$ from equation 3.4 into equation 3.3 gives

$$\frac{A}{2D\alpha + B\beta(1-2\alpha)} \geq \frac{D(1-2\alpha)}{CB},$$

or

$$B \geq \frac{2\alpha D^2 (1-2\alpha)}{AC-D^2 (1-2\alpha)^2}$$

or

$$B \geq \frac{2\alpha (1-2\alpha)}{\psi C - (1-2\alpha)^2} \ldots \ldots \text{3.5}. $$

Equation 3.5 is the web slenderness constraint in terms of $\alpha$, $\beta$ and $\psi$.

A similar process may be adopted to express the constraint in terms $\alpha$, $\beta$ and $\Omega$.

The inequality is given by

$$\frac{D-2T}{t} \leq C,$$

from the last formulation

$$B \geq \frac{D(1-2\alpha)}{CB} \ldots \ldots \text{3.6}$$

and

$$B = \frac{A}{D(\beta(1-2\alpha)+2\alpha)} \ldots \ldots \text{3.7}.$$  

The cross sectional area $A$ may be written as

$$A = \frac{2Zx}{Dex},$$

where from equation 2.11 $ex$ is given by

$$ex = \frac{1-(1-\beta)(1-2\alpha)^3}{3(2\alpha + \beta(1-2\alpha))}.$$  

Therefore, equation 3.7 may be rewritten as
Substituting for $B$ from equation 3.8 into equation 3.6 gives

$$\frac{6 \, Zx}{D^2(1-(1-\beta)(1-2\alpha)^3)} \geq \frac{D(1-2\alpha)}{C\beta},$$

or

$$\beta \geq \frac{1-(1-2\alpha)^3}{\frac{6 \, Zx C}{D^3(1-2\alpha)} - (1-2\alpha)^3}$$

or

$$\beta \geq \frac{1-(1-2\alpha)^3}{\frac{6 \, \Omega C}{(1-2\alpha)} - (1-2\alpha)^3} \quad \ldots \quad 3.9.$$

Equation 3.9 is the web slenderness constraint in terms of $\alpha$, $\beta$ and $\Omega$.

A similar process may be adopted to express the constraint in terms of $\alpha$, $\beta$ and $\Omega$.

The inequality is given by

$$\frac{D-2T}{t} \leq C,$$

From the last formulation

$$B \geq \frac{D(1-2\alpha)}{C\beta} \quad \ldots \quad 3.10$$

and

$$B = \frac{A}{D(\beta(1-2\alpha)+2\alpha)} \quad \ldots \quad 3.11.$$

The cross-sectional area $A$ may be written as

$$A = \frac{2 \, Zx}{D \, \bar{e}x},$$

where from equation 2.12 $\bar{e}x$ is given by
\[
\varepsilon_x = \frac{4a(1-a)+(1-2a)^2\beta}{2(\beta(1-2a)+2a)}.
\]

Therefore, equation 3.11 may be rewritten as

\[
B = \frac{4\bar{Z}x}{D^2(4a(1-a)+(1-2a)^2\beta)} \quad \ldots \quad 3.12.
\]

Substituting for B from equation 3.12 into equation 3.10 gives

\[
\frac{4\bar{Z}x}{D^2(4a(1-a)+(1-2a)^2\beta)} \geq \frac{D(1-2a)}{C\beta}
\]

or

\[
\beta \geq \frac{4\alpha(1-a)}{4\bar{Z}x C - (1-2a)^2} D^3(1-2a)
\]

or

\[
\beta \geq \frac{4\alpha(1-a)}{4\bar{n} C - (1-2a)^2} \quad \ldots \quad 3.13.
\]

Equation 3.13 is the web slenderness constraint in terms of \( \alpha \), \( \beta \) and \( \bar{n} \).

Tables 3.2 and 3.3 show all the constraints considered in this Thesis formulated in terms of \( \alpha \), \( \beta \) and one of \( \theta \), \( \gamma \), \( \psi \) or \( \Omega \). Table 3.2 relates to the non-dimensional cases and Table 3.3 relates to the dimensional cases.

To conclude this Chapter each of the constraints is plotted (Figs.3.3 to 3.14) on the bending efficiency surfaces. Appropriate values for the constant \( C \) are used, each constraint being plotted in three forms.

1) Non-dimensional mode with \( \gamma = 2 \).
2) Non-dimensional mode with \( \theta = 1.5 \).
3) Dimensional mode with \( \psi = 0.05 \).
<table>
<thead>
<tr>
<th>BASIC CONSTRAINT</th>
<th>EQUIVALENT NON-DIMENSIONAL CONSTRAINT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For $\gamma = D/B$</td>
</tr>
<tr>
<td>$\frac{B}{T} = C$</td>
<td>$\alpha = \frac{1}{C\gamma}$</td>
</tr>
<tr>
<td>$\frac{B-t}{2T} = C$</td>
<td>$\alpha = \frac{1-\beta}{2C\gamma}$</td>
</tr>
<tr>
<td>$\frac{D-2T}{t} = C$</td>
<td>$\beta = \gamma \frac{(1-2\alpha)}{C}$</td>
</tr>
<tr>
<td>$\frac{D}{T} = C$</td>
<td>$\alpha = \frac{1}{C}$</td>
</tr>
<tr>
<td>$\frac{B}{t} = C$</td>
<td>$\beta = \frac{1}{C}$</td>
</tr>
<tr>
<td>$\frac{D}{B}$</td>
<td>$\alpha = \frac{\theta \beta}{\gamma}$</td>
</tr>
<tr>
<td>$\frac{T}{t}$</td>
<td>$\alpha = \frac{\theta \beta}{\gamma}$</td>
</tr>
</tbody>
</table>

The constraints are shown as equalities in this table for use in figs. 3.3 to 3.14.
### TABLE 3.3

<table>
<thead>
<tr>
<th>BASIC CONSTRAINT</th>
<th>EQUIVALENT DIMENSIONAL CONSTRAINT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B \leq C )</td>
<td>( \beta = \frac{\psi - 2a^2}{C(1-2a)} )</td>
</tr>
<tr>
<td>( \frac{B-t}{2T} \leq C )</td>
<td>( \beta = \frac{\psi - 4C^2}{\psi + 2Ca(1-2a)} )</td>
</tr>
<tr>
<td>( \frac{D-2T}{t} \leq C )</td>
<td>( \beta = \frac{2a(1-2a)}{\psi C - (1-2a)^2} )</td>
</tr>
<tr>
<td>( \frac{D}{T} = C )</td>
<td>( \alpha = \frac{1}{C} )</td>
</tr>
<tr>
<td>( \frac{B}{t} = C )</td>
<td>( \beta = \frac{1}{C} )</td>
</tr>
<tr>
<td>( \frac{D}{B} = C )</td>
<td>( \beta = \frac{C\psi - 2a}{1-2a} )</td>
</tr>
<tr>
<td>( \frac{T}{c} = C )</td>
<td>( \beta = \frac{2a^2}{\psi C - a(1-2a)} )</td>
</tr>
</tbody>
</table>

The constraints are shown as equalities in this table for use in figs. 3.3 to 3.14.
Fig. 3.3 shows the flange slenderness constraint plotted in dimensional mode. The graph at the top of the page indicates the boundary which the constraint forms to elastic bending efficiency surface $ex$. The region above the constraint line satisfies the condition that the flange width to flange thickness ratio does not exceed 15.25.

The lower graph shows three typical values of the inequality constant plotted to a larger scale. It is in this section of the surface bounded approximately by $\alpha$ from 0.0 to 0.15 and $\beta$ from 0.0 to 0.15 that the majority of useful steel I shapes lie.

Although Figs. 3.3 through 3.14 show the constraints as boundaries to the $ex$ surface any of the other surfaces relating to $\tilde{ex}$, $ey$ and $\tilde{ey}$ described in Chapter 2 could also be used.
FIG. 3.3 FLANGE SLENDERNESS (B/T<sub>const.</sub>)
NON DIMENSIONAL MODE α = \( \frac{1}{\gamma^2} \), γ = 2
FIG. 3.4 FLANGE SLENDERNESS (B/T<const)

NON DIMENSIONAL MODE \( \epsilon = \frac{1}{L_0} \), \( \epsilon = 1.5 \)
FIG. 3.5 FLANGE SLENDERNESS (B/T=const)

DIMENSIONAL MODE \( b = \frac{\psi - 2Ca^2}{Ca(1-2a)} \), \( \psi = 0.05 \)
FIG. 3.6 - FLANGE OUTSTAND SLENDERNESS \((\frac{B}{2T})_{\text{const}}\)

NON DIMENSIONAL MODE \(= \frac{1-B}{26} \), \(\gamma = 2\).
FIG. 3.7 FLANGE OUTSTAND SLENDERNESS $\frac{B-t}{2T}<$const

NON DIMENSIONAL MODE $\delta = \frac{1}{1+2\beta}$, $\delta = 1.5$
FIG. 3.8 FLANGE OUTSTAND SLENDERNESS ($\frac{B-t}{2T} \leq \text{const}$)

NON DIMENSIONAL MODE $\psi = \frac{\phi - 4a^2}{\psi + 2Ca(1-2a)}$, $\psi = 0.05$
FIG. 3.9 WEB SLENDERNESS (\( \frac{D-2T}{t} \approx \text{const} \))

NON DIMENSIONAL MODE \( \delta = \frac{\gamma(1-2\gamma)}{\mu} \), \( \gamma = 2 \)
Fig. 3.10 Web Slenderness \( \left( \frac{D-2T}{t} \right)_{\text{const}} \)

Non-dimensional mode \( \alpha = \frac{t}{c + z_0} \), \( e = 1.5 \)
Fig. 3.11 Web Slenderness \( \frac{D-2t}{t} \leq \text{const} \)

Non dimensional mode

\[ \beta = \frac{2(1-\phi)}{6(1-\phi)} \quad , \quad \phi = 0.05 \]
FIG. 3.12 ASPECT RATIO (D/B = const)

NON DIMENSIONAL MODE \( a = \frac{B}{c} \), \( e = 1 \).
FIG. 3.13 ASPECT RATIO (D/B ≥ const)

DIMENSIONAL MODE \( \delta = \frac{C_4 - 2\psi}{1 - \psi} \), \( \psi = 0.05 \)
FIG. 3.14 FLANGE TO WEB THICKNESS (T/t const)
DIMENSIONAL MODE $b = \frac{2a^2}{\psi(C-a(L-a))}, \psi = 0.05$
CHAPTER FOUR

SHAPE SELECTION

4.1 INTRODUCTION

The hot rolled beam shapes have evolved over many years. As production techniques improved and the rolling process became better understood it was possible to roll sections having thinner webs and to reduce flange tapers and the size of root radii. These gradual changes to the geometry of rolled shapes resulted in sections having improved ratio's of bending strength to weight. In addition the distribution of section properties within the range of rolled beams also changed. The least popular sizes (which also tended to be the least structurally efficient shapes) being dropped from the range, and new sections added to fill the gaps in the range of properties or providing a greater choice of flange widths for the most popular section depths and weights.

With the advent of new structural design procedures, for example plastic design and the introduction of higher grades of steel it has become necessary to reappraise the current range of rolled beam and column sections. To this end an I.S.O. committee has been formed to investigate the production of a new range of sections to give greater scope for the use of rolled sections with the new design methods, and also to gain the maximum advantage from the use of higher grades of steel. This committee has deliberated over many years but as yet has been unable to reach a decision.

One of the problems encountered when designing a new range of rolled shapes is that of reconciling the various conflicting requirements of the production process and the needs of structural design. Normally each of these requirements is considered in isolation and then an iterative procedure is invoked to reach a satisfactory compromise. In this Chapter a useful aid to this process is discussed, whereby many of the aspects of the design may be considered together enabling their relative effect on one
another and on the design as a whole to be seen. The technique involves the use of the bending efficiency factor discussed in Chapter 2 and the constraint boundaries introduced in Chapter 3.

The word "design" is used in the present context to imply the proportioning of an optimum rolled structural shape, and not in its usual form relating purely to the design of an element of a structure.

The techniques discussed in this Chapter may also be used in other contexts. For example, to proportion stiffened plate girders, with the added advantage of being able to consider many aspects of the design together.

4.2 COMBINATION OF CONSTRAINTS

By combining two or more of the constraint boundaries introduced in Chapter 3 it is possible to partition the bending efficiency surfaces \( \text{ex, } \bar{\text{ex}}, \text{ey or } \bar{\text{ey}} \) into feasible and infeasible regions for design purposes, where,

- A feasible region is the segment of the bending efficiency surface within which none of the constraints are violated.
- An infeasible region is the part of the bending efficiency surface where one or more of the constraints are violated.

Plotting the combination of constraints on the bending efficiency surfaces, in addition to showing the feasible and infeasible regions, allows one to assess the relative importance of each constraint. That is, one can gain an idea of the relationship between the various constraints, and also between the constraints and the bending efficiency surface. The remainder of this Section is devoted to discussing the combination of constraints in relation to the two modes of use of the constraint boundaries and the bending efficiency surfaces as discussed in Chapters 2 and 3.

4.2.1 CONSTRAINT COMBINATION IN TERMS OF THE ASPECT RATIO \( \gamma \)

Figs. 4.1 to 4.4 indicate the constraint boundaries in terms of a constant aspect ratio \( \gamma \) (using the equations shown in table 3.2), these are plotted...
on the bending efficiency surface $e_x$. Each figure shows the feasible region which yields shapes suitable for both elastic and plastic design shaded in brown, and the region suitable for elastic design shaded in red. Note that the feasible region for plastic design always forms part of the feasible elastic region. In each of Figs. 4.1 to 4.4 the following constraints are used:

\[
\frac{D-2T}{t} \leq C_1 \text{ (elastic and plastic design constraint)}
\]

\[
\frac{B-t}{2T} \leq C_2 \text{ (elastic design constraint)}
\]

\[
\frac{B}{t} \leq C_3 \text{ (plastic design constraint)}
\]

\[
\frac{T}{t} \leq C_4 \text{ (production constraint for hot rolled sections)},
\]

where $C_1$ to $C_4$ are constants related to a particular constraint as discussed in Chapter 3. Included in Figs. 4.1 to 4.4 are the positions on the bending efficiency surface $e_x$ of the Universal Beam and Universal Column sections.

The limits for the constraints used in Figs. 4.1 to 4.4 are as follows:

**Fig. 4.1**
- $\gamma=3$ for production constraint $\theta = 1.6$
- $\gamma=3$ for web slenderness (elastic), grade 43 steel
- $\gamma=3$ for web slenderness (plastic), grade 43 steel
- $\gamma=3$ for flange outstand slenderness, grade 43 steel
- $\gamma=3$ for flange slenderness, grade 43 steel.

**Fig. 4.2**
- $\gamma=1,2,3$ and 4 for production constraint $\theta = 1.6$
- $\gamma=1,2,3$ and 4 for web slenderness (elastic), grade 43 steel
- $\gamma=1,2,3$ and 4 for flange outstand slenderness, grade 43 steel.

**Fig. 4.3**
- $\gamma=1,2$ and 3 for production constraint $\theta = 1.6$
- $\gamma=1,2$ and 3 for web slenderness (elastic), grades 43, 50 and 55 steels.
$\gamma = 1, 2, \text{ and } 3 \text{ for flange outstand slenderness, grades 43, 50 and 55 steels.}$

Fig. 4.4

$\gamma = 1, 2, \text{ and } 3 \text{ for production constraint } \theta = 1.6$

$\gamma = 1, 2, \text{ and } 3 \text{ for web slenderness (plastic), grades 43, 50 and 55 steels}$

$\gamma = 1, 2, \text{ and } 3 \text{ for flange slenderness, grades 43, 50 and 55 steels.}$

In certain cases it will be noticed that the web slenderness boundary is not shown, this occurs when the boundary does not intersect the feasible region. However, the direction of reducing values of web slenderness is such that within the feasible region the web slenderness is always lower than its limiting value and therefore the boundary may sensibly be omitted. Each constraint in Figs. 4.1 to 4.4 has been formulated in terms of a constant aspect ratio, therefore, each point on a constraint boundary corresponds to a shape having a limiting value for the particular constraint and an aspect ratio of 1, 2, 3 and 4, depending on the particular figure. Elsewhere within a feasible region each point corresponds to a shape having a smaller than limiting value for every constraint, and therefore, none of the constraints which enclose a feasible region are violated.

In Fig. 4.1 the most efficient cross-sectional shape within the feasible region occurs at the point indicated by B in the figure (that is, the intersection of the production constraint and the elastic web slenderness constraint). At point B the bending efficiency factor $ex$ is approximately 2/3. Considering the other elastic constraint shown in the figure, it will be seen that at point B the flange slenderness will be smaller than its limiting value. If a section corresponding to point B were produced, it would be only suitable for elastic design, because although the flange slenderness may be seen to be below its limiting value, the plastic web slenderness has an unacceptable value. However, it is interesting to note that the bending efficiency contour relating to $ex = 0.65$ which passes through point B runs nearly parallel to the $\theta$ boundary. Therefore, by using the point marked A (that is, the intersection of the production constraint and the plastic web slenderness constraint) the corresponding cross-sectional shape would be suitable for both elastic and plastic...
design, and has only a slightly smaller value of bending efficiency. It is noticeable that there are a number of Universal Beam shapes clustered around point B where ex is approximately 0.64. For both points A and B the production constraint and web slenderness constraints are the governing factors for the design, the flange slenderness and flange outstanding slenderness being both lower than their limiting values for a section made from grade 43 steel and having an aspect ratio of 3.

Fig. 4.2 shows the production constraint and elastic design constraints for grade 43 steel in terms of the aspect ratio's 1, 2, 3 and 4. It can be seen from this figure that each of the Universal Column sections lie close to the production constraint boundary for an aspect ratio of 1. As the aspect ratio is increased the web slenderness constraint becomes more of a governing factor for the design and the flange slenderness constraint becomes progressively less critical. The position of maximum bending efficiency occurs in all but one case at the intersection of the e and web slenderness boundaries. For an aspect ratio of 1 the web slenderness constraint does not intersect the feasible region and for this case the position of maximum bending efficiency corresponds to the intersection of the e and flange outstanding slenderness boundaries.

It will be noticed that the Universal Beam sections lie approximately between the aspect ratios 2 and 4 in positions which give slenderness values lower than either of the limiting values for the web or flange outstanding constraints. In all cases the most critical constraint is e. Fortuitously, the bending efficiency contours run nearly parallel to the production constraint boundaries for aspect ratios greater than 1. Therefore, one can move from the position of maximum bending efficiency along the lines of constant e without significantly reducing the value of ex, this also applies to the case of the ex surface.

Figs. 4.3 and 4.4 show the elastic and plastic constraint boundaries for the three grades of steel, with a production constraint e=1.6 for γ=1, 2 and 3. Fig.4.3 indicates that all the Universal Beam and Column sections may be used for elastic design when rolled from grade 55 steel. Fig.4.4 shows that only some universal Beams are suitable for plastic design if rolled from grades 50 and 55 steels.
In Figs. 4.1 to 4.4 the most significant boundary is the production constraint which relates only to hot rolled sections. For sections such as stiffened plate girders this constraint is unnecessary. The value of the web slenderness limit for a stiffened plate girder rises to around 200 depending on the grade of steel used and the stiffening arrangement. For this type of section the position of maximum bending efficiency occurs either at the intersection of the web and flange slenderness boundaries, or at some position along the web slenderness boundary. In addition, as the web slenderness limit is raised, the corresponding boundary moves closer to the origin and therefore a larger value for bending efficiency may be achieved.
Fig. 4.1
Fig. 4.2
Fig. 4.3
RELATIVE FLANGE THICKNESS

RELATIVE WEB THICKNESS

Fig. 4.4
4.2.2 CONSTRAINT COMBINATION IN TERMS OF θ

A similar process to that discussed in Section 4.2.1 may be adopted when using the constraints formulated in terms of the production constraint θ (using the equations shown in table 3.2). Figs. 4.5 to 4.8 show the feasible regions relating to the constraints formulated in terms of a constant θ. The following limits for the constraints are used:

Fig. 4.5
θ=1.6 for γ=3
θ=1.6 for web slenderness (elastic), grade 43 steel
θ=1.6 for web slenderness (plastic), grade 43 steel.

Fig. 4.6
θ=1,1.6 and 2 for γ=3
θ=1,1.6, and 2 for web slenderness (elastic) grade 43 steel.

Fig. 4.7
θ=1.6 for γ=2
θ=1.6 for web slenderness (elastic), grade 43, 50, and 55 steels
θ=1.6 for web slenderness (plastic), grades 43, 50 and 55 steels.

Fig. 4.8
θ=1.6 for γ=3
θ=1.6 for web slenderness (elastic), grades 43, 50 and 55 steels.
θ=1.6 for web slenderness (plastic), grades 43, 50 and 55 steels.

The flange slenderness and flange outstand slenderness boundaries are not shown in Figs. 4.5 to 4.8, these constraint boundaries do not intersect the feasible regions. However, the direction for reducing values of these two constraints is such that at any point within the feasible regions the flange and flange outstand slendernesses are lower than their limiting values.

Using the constraints formulated in this manner allows the designer to investigate the effects of varying the production constraint. The position of maximum bending efficiency occurs in each case at the intersection of the aspect ratio and web slenderness boundaries. Using the constraints in this form does not provide as much information on
the relative effects of each constraint as the previous example. But it does provide a quick method for assessing the design of hot rolled sections and their sensitivity to changes in the value of the production constraint.
RELATIVE WE b THICKNESS

1 \( \sigma = C_1 \)
2 WEB (plastic) = C_2
3 WEB (elastic) = C_3

Fig. 4.5
Fig. 4.6
Fig. 4.7
Fig. 4.8
4.2.3 CONSTRAINTS COMBINATION IN TERMS OF $\psi$

The constraints may also be formulated in terms of a constant cross-sectional area and section depth (using the equations shown in table 3.3, where $\psi = A/D^2$). For this case, each point on a boundary relates to a shape having a constant area and depth and a limiting value for the particular constraint. Feasible regions for the constraints formulated in terms of $\psi$ are shown in Figs. 4.9 to 4.11 where the following limits for the constraints are used:

Fig. 4.9
- $\psi=0.045$ for web slenderness (elastic), grade 43 steel
- $\psi=0.045$ for web slenderness (plastic), grade 43 steel
- $\psi=0.045$ for flange slenderness, grade 43 steel
- $\psi=0.045$ for flange outstand slenderness, grade 43 steel
- $\psi=0.045$ for aspect ratio's 1, 2, 3 and 4
- $\psi=0.045$ for production constraint of 1, 1.6 and 2.

Fig. 4.10
- $\psi=0.040$ for web slenderness (elastic), grade 43 steel
- $\psi=0.040$ for web slenderness (plastic), grade 43 steel
- $\psi=0.040$ for flange slenderness, grade 43 steel
- $\psi=0.040$ for flange outstand slenderness, grade 43 steel
- $\psi=0.040$ for aspect ratio's 1, 2, 3 and 4
- $\psi=0.040$ for production constraints 1, 1.6 and 2.

Fig. 4.11
- $\psi=0.040$ for web slenderness (elastic), grades 43, 50 and 55 steels
- $\psi=0.040$ for web slenderness (plastic), grades 43, 50 and 55 steels
- $\psi=0.040$ for singly stiffened webs slenderness, grades 43, 50 and 55
- $\psi=0.040$ for doubly stiffened web slenderness, grades 43, 50 and 55
- $\psi=0.040$ for flange slenderness, grades 43, 50 and 55 steels
- $\psi=0.040$ for flange outstand slenderness, grades 43, 50 and 55
- $\psi=0.040$ for aspect ratio's of 1, 2, 3 and 4.
- $\psi=0.040$ for production constraints of 1, 1.6 and 2.

Where the constraint boundaries are formulated in terms of $\psi=A/D^2$ or in the case of Section 4.2.4 in terms of $\eta=Zx/D^3$ (that is, corresponding
to the dimensional mode of the bending efficiency surfaces) it has been found useful to form the feasible elastic and plastic regions using the following boundaries:

- feasible region for elastic design: using the elastic web slenderness and flange outstand slenderness boundaries,
- feasible region for plastic design: using the plastic web slenderness and flange slenderness boundaries.

The remainder of the constraints in Figs. 4.9 to 4.11 (being the production constraint and aspect ratio boundaries) may then be used to further divide the feasible regions, or more usefully, they may be used to assess the relative effects of these constraints.

For example, consider Fig.4.9 where $\psi = 0.045$. Given the requirement for a hot rolled section, made from grade 43 steel, suitable for plastic design and having $\psi = 0.045$, which position on the bending efficiency surface would correspond to this shape. The shape must lie somewhere within the feasible plastic region and be close to the production constraint $T/t = 1.6$. Position A in the figure conforms to all these requirements (being the intersection of the $T/t = 1.6$ boundary and the flange slenderness boundary). It will be noticed from the figure that the shape would have an aspect ratio of approximately 2, and its web slenderness would be lower than its limiting value. It can also be seen that by increasing the value of the production constraint, and keeping the same value of flange slenderness, a larger value of bending efficiency could be achieved. This would also bring the web slenderness closer to its limiting value. Using the constraint boundaries in this manner enables one to assess the relative importance on bending efficiency for each constraint.

Fig.4.11 indicates the constraint boundaries for singly and doubly stiffened webs, for stiffened plate girders the production constraint for hot rolled sections may be ignored, and it can be seen from the figure that a much higher value of bending efficiency may be achieved.
RELATIVE WEB THICKNESS

1. WEB (elastic) = C1
2. WEB (plastic) = C2
3. B/T = C3
4. (B-t)/2T = C4

Fig. 4.9
RELATIVE FLANGE THICKNESS

RELATIVE WEB THICKNESS

5  Singly stiffened web
6  Doubly stiffened web

Fig. 4:11
4.2.4 COMBINATION OF CONSTRAINTS IN TERMS OF £

The final method in which the constraints may be formulated is in terms of a constant section depth and either section modulus or plastic modulus, using \( \Omega = Zx/D^3 \) or \( \tilde{\Omega} = \bar{Z}x/D^3 \) (using the equations shown in table 3.3). This method of usage approximates to the design process, where normally the bending strength and section depth are known. For this case each point on a boundary relates to a shape having a constant value for D and Zx or \( \bar{Z}x \) and a limiting value for the particular constraint. The constraints formulated in this manner may be used as shown in Section 4.2.3, the boundaries being very similar in shape to those formulated in terms of \( \psi \). Figs. 4.12 and 4.13 illustrate the constraint boundaries formulated in terms of \( \Omega \) and use the following limits for the constraints:

**Fig. 4.12**
- \( \Omega = 0.017 \) for web slenderness (elastic), grade 43 steel
- \( \Omega = 0.017 \) for web slenderness (plastic), grade 43 steel
- \( \Omega = 0.017 \) for flange slenderness, grade 43 steel
- \( \Omega = 0.017 \) for flange outstand slenderness, grade 43 steel
- \( \Omega = 0.017 \) for aspect ratio's 1, 2, 3 and 4
- \( \Omega = 0.017 \) for production constraints 1, 1.6 and 2.

**Fig. 4.13**
- \( \Omega = 0.017 \) for web slenderness (elastic) grades 43, 50 and 55 steels.
- \( \Omega = 0.017 \) for web slenderness (plastic), grades 43, 50 and 55 steels
- \( \Omega = 0.017 \) for singly stiffened web slenderness, grades 43, 50 and 55 steels
- \( \Omega = 0.017 \) for doubly stiffened web slenderness, grades 43, 50 and 55 steels
- \( \Omega = 0.017 \) for flange slenderness, grades 43, 50 and 55 steels
- \( \Omega = 0.017 \) for flange outstand slenderness, grades 43, 50 and 55 steels
- \( \Omega = 0.017 \) for aspect ratios 1, 2, 3 and 4
- \( \Omega = 0.017 \) for production constraints of 1, 1.6 and 2.
RELATIVE WEB THICKNESS

1  WEB (elastic) = C1
2  WEB (plastic) = C2
3  B/T = C3
4  (B-t)/2T = C4

Fig. 4.12
Fig. 4.13
4.3 DESIGN STRATEGY

In the previous Section a limited number of constraints were used to illustrate the manner in which a feasible design space may be created. It is possible to further reduce the size of this space by considering the effects of other relevant constraints, such that the choice of an optimum shape for a particular design is restricted to a small area. By using the design space in conjunction with a bending efficiency surface it is a simple task to choose the most efficient shape in bending, which will also be valid in respect to each of the constraints considered. Although the bending efficiency surface ex was shown in each of the examples, the ex, ey and ey surfaces could also be used. The procedure outlined above provides a simple method for proportioning a cross-sectional shape and allows the designer to assess the relative effect of each of the chosen constraints.

The problem of choosing the optimum section properties for a range of hot rolled I-sections is outside the scope of this work. However, the techniques discussed in the Thesis form a useful aid for this type of investigation allowing the designer to optimize individual sections for maximum performance within a group produced from one set of rolls.

In addition the proposed method can be used to study the optimum distribution of section properties within a group. For example, to provide certain sections suitable for both elastic and plastic design, and at the lighter end of the group, sections suitable for elastic design only.
4.4 Advances in I-Section Manufacture

Until recently, there was no process which could produce steel I-sections in the variety of sizes and at a low enough cost to compete with hot rolling. Hot rolled sections, although relatively cheap, have inconvenient dimensions making the detail design of a steel structure a lengthy process. In addition, the severe constraints on design of the rolling and associated processes means that only marginal improvements can be made to the bending efficiency of individual sections.

A new manufacturing process which overcoming most of these problems has recently been tried in pilot plants in Japan. The process automatically produces welded beams from coiled strip, the strip is slit to the required width and fed to a roller welding machine to produce a continuously welded I-section. Although at present the heavier types of section cannot be produced, costs are thought to be competitive for light and medium weight sections, the plant being considerably smaller and less expensive than a rolling mill. This manufacturing technique makes possible the use of sections having a higher value of bending efficiency than those constrained by the limitations of hot rolling. In addition sections can be formed having convenient overall dimensions and material thicknesses, simplifying the detail design of steel structures and fabrication procedures.
CONCLUDING REMARKS

The ideas discussed in this Thesis form a useful addition to the section designers "toolkit", providing a simple and effective means of proportioning cross-sectional shapes for optimum bending efficiency.

The bending efficiency factor forms the basis of this approach, yielding a measure of the efficiency in bending of a cross-section which is independent of the overall dimensions of the shape. The function allows the designer to assess the efficiency in bending of a single shape, or to investigate the effects on bending efficiency of change in cross-sectional shape. Combining the concept of the bending efficiency factor with the ability to express the constraints as boundaries to the bending efficiency surfaces enables the designer to assess the effects of each constraint on the design as a whole. In addition, the sensitivity of the design to changes in production techniques or the use of new material characteristics can also be investigated.

The main advantage of using these techniques is that they allow the designer to adopt a systematic approach to the design of cross-sectional shapes. That is, instead of considering each aspect of the design in isolation and then attempting to iterate to a solution, the designer can consider each aspect of the design in relation to the design as a whole.

The concept of the bending efficiency factor also provides a useful teaching aid, allowing the student to investigate where material should be placed within a cross-section to give the optimum bending performance without having to consider the overall dimensions of the cross-section.

In this Thesis the bending efficiency factor has only been considered in relation to doubly symmetric sections, composed of a material in the linearly elastic or fully plastic condition. However, it is possible to derive the bending efficiency factor for non-symmetric shapes provided that the simple engineering theory of bending is acceptable, in addition one can consider materials which are non-linear, or sections composed of more than one material. As a topic for further research
it would be useful to provide an all-embracing definition for the bending efficiency factor to cover the various aspects referred to above.
REFERENCES

CHAPTER 1


CHAPTER 2


CHAPTER 3


APPENDIX A.

Relationship between Schillings Work on Section Optimization and the Bending Efficiency Factor.

This Appendix is included to illustrate the relationship between these two techniques and also introduces an extension to Schillings work. This extension enables the effect of flange thickness to be investigated.

Extended Schilling Formulation for Section Modulus $Z_x$

The following terms are used -

Cross sectional area $A$,
Relative Web Area (Area of web/A) $a$,
Web Slenderness $(D-2T/t)$ $b$,
Flange to Web thickness ratio $(T/t)$ $e$

Putting these terms together, the web thickness may be expressed as

$$ t = (\frac{Aa}{B})^{\frac{1}{3}} $$

and the depth between flanges as

$$ Dp = (Aab)^{\frac{1}{3}}. $$

The other dimensions of the cross-section can then be obtained from

$$ T = e(\frac{Aa}{B})^{\frac{1}{3}}, $$

$$ D = (Aab)^{\frac{1}{3}} + 2e (\frac{Aa}{B})^{\frac{1}{3}} $$

and

$$ B = A(1-a)/2e (\frac{Aa}{B})^{\frac{1}{3}}. $$

Using these terms the section modulus $Zx$ may be expressed as

$$ Zx = \frac{a^{\frac{1}{3}} A^{\frac{1}{3}} b^{\frac{1}{3}} (e(6 + \frac{4e}{B}) + 3b) - a^{\frac{1}{3}} A^{\frac{1}{3}} b^{\frac{1}{3}} (e(6 + \frac{4e}{B} + 2b))}{6(b + 2e)} $$
Differentiating $Z_x$ with respect to the relative web area $a$, setting the result to zero and solving the resulting equation yields the following equation for the optimum value of relative web area $a$.

$$a = \frac{\theta(6b + 4\theta) + 3b^2}{3\theta(6b+4\theta)+6b^2}$$

If $\theta$ is set to zero the value of $a$ is $\frac{1}{2}$ i.e. the standard result obtained by Schilling. However, when $b$ and $\theta$ have values appropriate for universal beam sections $a$ has a range from 0.46 to 0.5 (0.5 being applicable to sections such as stiffened plate girders). Although this range is small, the variation is significant when considering the design of a range of hot rolled sections.

**Extended Schilling Formulation for $Z_x$**

Using the dimensions of the cross-section in the form used in the last example the plastic modulus may be expressed as

$$Z_x = \frac{Aa}{4} (Aab)^{1/2} + \frac{A}{2} (1-a)(Aab)^{1/2} + \frac{A}{2} (1-a) \theta \left(\frac{Aa}{b}\right)^{1/2},$$

which simplifies to

$$Z_x = a^{1/2} \frac{A^{3/2}}{4} \left(2b^{1/2} + \frac{2\theta}{b^{1/2}}\right) - a^{3/2} \frac{A^{3/2}}{4} \left(b^{1/2} + \frac{2\theta}{b^{1/2}}\right).$$

Differentiating $Z_x$ with respect to the relative web area $a$, setting the result to zero and solving the resulting equation yields the following equation for the optimum value of the relative web area $\tilde{a}$.

$$\tilde{a} = \frac{2b + 2\theta}{3b + 6\theta}.$$
range of hot rolled sections. However, for the design of a single beam, where discrete plate sizes must be used the variation has no real significance.

**Schilling Optimum in Terms of the Elastic Bending Efficiency Factor \( \varepsilon_x \).**

Substituting the optimum \( a \) expression into \( Z_x \) and using this expression in the equation for the elastic bending efficiency factor \( \varepsilon_x \).

\[
i.e. \quad \varepsilon_x = \frac{2Z_x}{DA}
\]

yields the following equation

\[
\varepsilon_x = \frac{6b^2 + 2\theta(6b + 4\theta)}{9(b+2\theta)^2}.
\]

If \( \theta \) is set to zero \( \varepsilon_x \) takes its maximum value of \( \varepsilon_x = 2/3 \). For values of \( \theta \) and \( b \) appropriate for universal beam section \( \varepsilon_x \) ranges from 0.64 to 0.66. It is interesting to note that the majority of Universal beams have a value of \( \varepsilon_x \) close to the value of 0.66.

**Schilling Optimum in Terms of the Plastic Bending Efficiency Factor \( \bar{\varepsilon}_x \).**

Substituting the Schilling optimum \( \bar{Z}_x \) expression into the equation for the plastic bending efficiency factor \( \bar{\varepsilon}_x \)

\[
i.e. \quad \bar{\varepsilon}_x = \frac{2\bar{Z}_x}{DA},
\]

yields the following equation

\[
\bar{\varepsilon}_x = \frac{2b + 2\theta}{3b + 6\theta}.
\]

This also happens to be the equation for the relative web area \( \bar{a} \) for this case. If \( \theta \) is set to zero then \( \bar{\varepsilon}_x \) takes its maximum value of 2/3 which is the same value as the elastic case.
APPENDIX B

Solution Of The Cubic Equation To Give The Optimum Value of \( \alpha \) For The Elastic Bending Efficiency Factor

The cubic equation is given by -

\[
\alpha^3(8(1-\beta)) + \alpha^2(6(2\beta-1)) - 6\alpha\beta + \beta = 0
\]

or

\[
\alpha^3 + \alpha^2\left(\frac{3(2\beta-1)}{4(1-\beta)}\right) - \frac{3\beta}{4(1-\beta)} + \frac{\beta}{8(1+\beta)} = 0 \quad \text{...... B.1.}
\]

Using Cardan's Solutions, the equation is transformed into the form

\[
\alpha^3 + p\alpha + q = 0,
\]

where

\[
p = -\frac{1}{3} \left[\frac{3(2\beta-1)}{4(1-\beta)}\right]^2 - \frac{3\beta}{4(1-\beta)},
\]

which simplifies to

\[
p = -\frac{3}{(4(1-\beta))^2},
\]

and

\[
q = \frac{2}{27} \left[\frac{3(2\beta-1)}{4(1-\beta)}\right]^3 + \frac{3(2\beta-1)}{4(1-\beta)} \cdot \frac{2\beta}{8(1-\beta)} + \frac{\beta}{8(1-\beta)}
\]

which may be written as

\[
q = \frac{4\beta-1-2\beta^2}{32(1-\beta)^3}
\]

The solution for equation B.1 is given by

\[
\alpha = y - \frac{(2\beta-1)}{4(1-\beta)},
\]

where \( y \) is a root from the transformed equation.

The relevant root \( y \) is obtained using the trignometrical method from
\[ y = - 2 \sqrt{\frac{p^3}{3}} \cos \left( \frac{\alpha}{3} + 60^\circ \right), \]

where \( \cos \alpha = \frac{q}{2\sqrt{-(p^3)^3}} \)

and where \( y \) may be written as

\[ y = - \frac{2}{4(1-\beta)} \sin \left[ \sin^{-1} \left( \frac{2\beta^2 - 4\beta + 1}{3} \right) \right]. \]

Therefore \( \alpha \) is given by

\[ \alpha = \frac{(1-2\beta) - 2 \sin \left[ \sin^{-1} \left( \frac{2\beta^2 - 4\beta + 1}{3} \right) \right]}{4(1-\beta)}. \]