A STUDY OF DEUTERON STRIPPING REACTIONS ON
LIGHT AND HEAVY TARGETS

BY

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DEDICATED TO

MY FATHER

AND

THE MEMORY OF MY MOTHER
This thesis is concerned with the study of the spin-dependence of low-energy deuteron stripping on light and heavy target nuclei.

In the case of a heavy target, the relative importance of the folding model spin-orbit interaction and (d,p) channel coupling effects upon the elastic vector analyzing power for deuteron scattering from \(^{90}\text{Zr}\) at 5.5 MeV is studied. Coupled channels calculations are performed in which the elastic channel and neutron transfer to the strongly populated states of \(^{91}\text{Zr}\) are treated explicitly. The results of these calculations indicate significant corrections to the elastic \(^{1T_{11}}\) and (d,p) reaction observables and provide a consistent qualitative description of the data.

In case of a light target nucleus, it has been suggested that the spin degree of freedom in plasma fusion reactions could be utilised to enhance or suppress fusion reaction rates. For example if the \(d(d,n)^{3}\text{He}\) reaction were suppressed then a 'neutron-free' \(d-^{3}\text{He}\) fusion reactor is likely to work. This suppression is thought to occur at low energies when the deuterons are polarized with spin parallel. Calculations by Hofmann and Fick showed that the \(d-d\) reaction initiated by low energy polarized deuterons is not suppressed and hence neither is the neutron production. These results were contradicted by Zhang et al. In order to throw further light on the question an approximate RGM calculation has been formulated in which the role of \(D\)-states of \(^{3}\text{He}\) and the deuteron were studied. In these calculations the initial state is described by distorted wave generated by i) non-local separable potentials, and ii) the orthogonality condition model (OCM). In each case the \(d-d\) phase shifts from the RGM were fitted. Nuclear wave functions consistent with a realistic nucleon-nucleon interaction were
used throughout. The distorted wave in the exit channel has been generated by a local neutron optical potential which fits the $n-^3\text{He}$ scattering data.

In marked contrast with the previously reported DWBA calculations, which claim to reproduce the data, it is found that the data is overestimated by a factor of 150 at a centre of mass energy at 55 KeV. Repeating the calculations using another neutron optical potential, which also fits the $n-^3\text{He}$ scattering data, yields a good agreement with the data.
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CHAPTER 1

INTRODUCTION

1.1 The Origin of the Spin-Dependence at Low Energies

Until recently it was thought that the measured analyzing powers in low energy composite projectile-nucleus scattering were generated entirely by, and were largely consistent with, the spin dependent terms of the projectile-target interaction derived from the folding model calculations [Kn 73a, We 76]. The accuracy of these model spin dependent interactions is however once again the subject of detailed investigations. There are now examples of both polarized deuteron [To 83] and light heavy ion [Ni 82, Wi 84] elastic scattering analysis in which the explicit inclusion of one or more strongly coupled channels have been shown to generate large spin-orbit effects in the projectile-target effective interaction. A result is that the magnitude of the spin-orbit effects arising from the folding model $L,S$ potential are such that they play only minor role in the description of the experimental data.

In one such study, Tostevin and Johnson [To 83] showed that in sub-Coulomb barrier deuteron elastic scattering from $^{208}$Pb, the observed elastic vector analyzing power, $\mathcal{M}_1^E$, could be explained quantitatively in terms of the strong coupling which exists, in that case, between the elastic and $(d,p)$ transfer channels. Additional contributions, arising from the non-orthogonality of these channels,
which are neglected in conventional coupled reaction channels calculations, have also been shown to be significant for this system [Th 83], whereas the folding model spin-orbit interaction provides only a minor perturbation to the coupled channels effect upon the calculated elastic $iT_1$.

In fact, as originally predicted by Johnson [Jo 66, Jo67], the tensor analyzing powers are completely dominated by the deuteron D-state. The measurements of the tensor analyzing powers provide the best available experimental observation of deuteron D-state effects. According to DWBA calculations [Jo 67, De 69], non-zero tensor analyzing powers can arise not only from the spin-orbit interactions in the deuteron and the proton channels, but also from the deuteron D-state. The availability of polarized deuteron beams has greatly enhanced the body of acquired experimental data. The most stringent test posed, is the application of a reaction theory to the measurement of polarization analyzing powers. No longer is the main concern of the theorist to fit the differential cross sections. Now, emphasis is placed on the understanding of how this fit is obtained by studying the various contributions from each particular analyzing power. DWBA calculations for $(d,p)$ reaction, initiated by a polarized deuteron beam, have shown that the deuteron D-state has a very small effect on the reaction cross section, but can produce large effects on the polarization observables [Br 71, Kn 73b].

1.2 Plan of Thesis

For reference purpose, an outline of the development of this thesis
CHAPTER 1: The preceding section briefly reviewed the historical development and the resulting philosophical motivation for the work described in this thesis. The remaining section is concerning the formal theory of scattering and deuteron stripping reactions.

CHAPTER 2: Discussion of the techniques, DWBA and CRC, which describe the direct reaction theory.

CHAPTER 3: In this chapter the relative importance of the spin-orbit interaction predicted by the folding model and (d,p) transfer channel coupling effects upon the calculated elastic $^1T_{11}$ is investigated in the case of deuteron scattering from $^{90}$Zr at 5.5 MeV. The Q-values of the (d,p) reactions are large and positive, the energies of the outgoing proton exceed the Coulomb barrier, and as a result the reaction is sensitive to the nuclear potential. Three nuclear optical potentials are involved in the calculations. The calculations have been performed for deuteron elastic scattering and for transitions to the states in $^{91}$Zr at excitation energies 0.0, 1.2, 2.04, 2.56 MeV. The calculations have been done with both DWBA and coupled reaction channels.

CHAPTER 4: In this chapter a model (one-step DWBA) is formulated and used to calculate the polarized and unpolarized $d(d,n)^3$He fusion reaction cross sections at centre of mass energies of 55, 100, and 150 KeV. The same calculations have been performed at higher energies (centre of mass energy of 12.65 MeV). The model is an approximate RGM calculation. It includes the important aspects of antisymmetrization,
treated exactly in the RGM, and an accurate treatment of the structure of the light nuclei, including all internal D-states of both the deuteron and \(^3\)He.

CHAPTERS 5, 6: These two chapters concern the numerical calculations of the distorted wave in the incident channel for the \(d(d,n)^3\)He reaction. In chapter 5, the distorted wave has been generated by local (phenomenological imaginary + Coulomb) plus finite non-local separable potentials, while in chapter 6, the distorted wave was generated within the orthogonality condition model. In both cases the parameters of the potentials were chosen to reproduce the \(d-d\) elastic phase shifts of the one channel RGM calculations of Thompson [Th 70] for centre of mass energies of 0-20 MeV.

CHAPTER 7: Numerical results are presented for \(d(d,n)^3\)He at energies relevant to fusion reactors using the formalism described in chapter 4, together with the distorted waves of chapters 5 and 6. This chapter also discuss the results.

1.3 Formal Theory of Scattering and Reactions

The basic reaction to be investigated can be represented by \(A(d,p)B,\) or more descriptively by

\[
A + d(=p+n) \rightarrow B(=A+n) + p ,
\]  

(1.1)
Definition of the (d,p) reaction channel variables
where A and B are the target and the residual nuclei in the reaction. \( d \) and \( p \) stand for the incident deuteron and the outgoing proton, respectively. It is assumed that the residual nucleus B is a bound state of the \( n+A \) system. Use is made of the coordinate system of fig. (1.1) in which \( R(=r) \) represents the coordinate of the centre of mass of the deuteron relative to the target A, and \( r \) is the relative position vector of the neutron from the proton. If \( r_n \) represents the displacement between the nucleus A and the neutron from the deuteron, the following relations may be obtained.

\[
\begin{align*}
\mathbf{r}_1 &= \frac{1}{2} \mathbf{r} + \mathbf{r}_n \\
\mathbf{r}_2 &= \mathbf{r} + \gamma \mathbf{r}_n
\end{align*}
\]

Where \( \gamma = M_A / (M_A + M_n) \approx A/(A+1) \) with \( M_A \) the mass of the target nucleus and \( M_n \) of the captured neutron. Asymptotically, in a \( (d,p) \) reaction, the physical situation in which we are interested is the one in which the incident beam, the two nuclei \( d \) and \( A \) are in their ground states and moving in a plane wave state with relative momentum \( k_1 \), but will have outgoing spherical waves of protons in the exit channel. The initial state wave function for the scattering event can be written as:

\[
\chi_{k_1} = \exp(i k_1 \cdot r_1) \ \phi_{s_1} (r,p,n) \ \psi^A_{\alpha \alpha} (\xi) \quad (1.4)
\]

Where the functions \( \phi (r,p,n) \) and \( \psi (\xi) \) represent the deuteron and the target internal wave functions. \( \xi \) denotes the set of the
internal coordinates belonging to the nucleus A.

In a similar manner the final state wave function can be represented by

\[ \chi = \exp(ik_2 \cdot r_2) \sum_{s_2} \Phi^{\alpha}_{s_2}(p) \psi^{B}_{b\beta}(f,n), \]  

where \( k_2 \) is the momentum of the free proton, and \( \Phi(p) \) is the internal spin wave function of the proton. The function \( \psi(f,n) \) represents the wave function of the residual nucleus.

The total Hamiltonian of the system can be written as:

\[ H = H^A + H_{np} + K_1 + V_{dA}, \]  

and in the exit channel

\[ H = H^B + K_2 + V_{pB}, \]  

where

\[ V_{dA} = V_{nA} + V_{pA}, \]  

and

\[ V_{pB} = V_{pA} + V_{np}, \]  

\( H^A, H^B, H_{np} \) are the internal Hamiltonians of the nuclei A and B, and of the deuteron respectively, and \( K_i \) are the appropriate channel kinetic energy operators. \( V_{np} \) is the neutron-proton interaction, \( V_{nA} \) and \( V_{pA} \)
are the neutron-target and proton-target interactions respectively. Asymptotically, \( \chi_{k_1} \) and \( \chi_{k_2} \) satisfy the Schrödinger equations

\[
(H_{np} + K) \chi_{k_1} = (E_1 - e_{np} - e_{d}) \chi_{k_1},
\]

and

\[
(H_{np} + K_2) \chi_{k_2} = (E_2 - e_{B}) \chi_{k_2},
\]

where

\[
E_i = \hbar^2 k_i^2 / 2 \mu_i, \quad i=1,2,
\]

\( \mu_1 \) and \( \mu_2 \) are the reduced masses in the incident and the exit channels. \( e_{A}, e_{B}, \) and \( e_{d} \) are the binding energies of the nuclei \( A, B, \) and the deuteron respectively. The \( Q \)-value of the \((\text{d},\text{p})\) reaction is defined by

\[
Q = E_2 - E_1 = e_n - e_d,
\]

where \( e_n \) is the separation energy of the neutron from nucleus \( B. \) Conservation of energy requires that
\[ E = E_1 - e_A - e_d \]

\[ = E_2 - e_B , \quad (1.14) \]

The total wave function \( \Psi^{(+)}(\xi, p, n) \) for the full many body system \((A+2)\) obey the complete Schrödinger equation

\[ (E - H) \Psi^{(+)}(\xi, p, n) = 0 . \quad (1.15) \]

\( \Psi^{(+)}(\xi, p, n) \) denotes the exact solution to the problem involving an incident deuteron on the target nucleus \( A \), and as such, it contains information on all the processes that can occur, such as elastic and inelastic scattering, break-up of the deuteron, the \((d,p)\) reaction itself, as well as other reactions and the distortion of the deuteron as it moves close to or in the target nucleus \( A \). The asymptotic behaviour of \( \Psi^{(+)} \) in the proton channel for large \( r_2 \) is a radially outgoing wave. This is given in the asymptotic region by

\[ \Psi^{(+)}(\xi, p, n) = \frac{-\nu_2}{2\pi \hbar^2} \phi \beta_s \sigma_{s_2} s_2 k_1 T |a, \alpha s, \sigma_{s_2} k_1, \rangle \]

\[ \Psi^{(\beta)}(\xi, n) \phi_{s_2}^{2}(p) e^{-i k r_2/r_2} . \quad (1.16) \]

The evaluation of the transition matrix element appearing in equation (1.16) is the objective of approximate stripping theories, since the T-matrix determines all the observable quantities in the reaction.
For an incident polarized deuteron beam, it is possible to measure not only the cross section, but also the sensitivity to the incident beam polarization. Polarization of a beam of particles is conveniently expressed by the density matrix [Fa 57] when observables such as the differential cross section, polarization and the analyzing powers are required to be calculated. The problem is to calculate the final density matrix \( P_f \) from the known initial density matrix \( P_i \) [La 55]. It can be shown [Me 66] that \( P_f \) is in fact related to \( P_i \) through the scattering matrix \( T \) via the transformation

\[
P_f = TP_i T^+ \tag{1.17}
\]

If the incident deuteron is unpolarized, then \( P_i = \frac{1}{3} I \), and

\[
P_f = TT^+/3 \tag{1.18}
\]

For any spin operator \( \hat{O} \) the averaged expectation value is given in terms of the density matrix \( P \) by

\[
\langle \hat{O} \rangle = \frac{\text{Tr}(P \hat{O})}{\text{Tr}(P)} \tag{1.19}
\]

The differential cross section in a \((d,p)\) reaction is then

\[
\sigma = \frac{\mu_1 \mu_2}{(2\pi h^2)^2} \frac{k_2}{k_1} \frac{\text{Tr}(P_f)}{\text{Tr}(P_i)} \tag{1.20}
\]
The unpolarized cross section becomes [Au 70]

\[
\sigma_0 = \frac{\mu_1 \mu_2 k_2}{(2\pi h^2)^2} \frac{1}{k_1} \frac{\text{Tr}(T^+)}{(2s_1+1)(2a+1)}
\]  

(1.21)

When considering a polarized deuteron beam it is convenient to employ the irreducible tensor operator \( \tau_{kq}^{(s)} \) [Ro 63], for arbitrary spin \( s \) with the matrix elements in the spin representative given by

\[
\langle s\sigma' | \tau_{kq}^{(s)} | s\sigma \rangle = \hat{s}(-)^{s-q} \langle s\sigma' s-\sigma | q \rangle
\]  

(1.22)

These obey the relations

\[
\text{Tr}\{ \tau_{kq}^{(s)} \tau_{k'q'}^{+(s)} \} = \hat{s}^2 \langle \tau_{kk'} \tau_{qq'} \rangle
\]  

(1.23)

where \( \hat{s}^2 = (2s+1) \), and

\[
\tau_{kq}^{(s)} = (-)^{q} \tau_{k-q}^{+(s)}
\]  

(1.24)

as stipulated by the Madison Convention [Ba 71]. The expectation values of these tensors, defined by eq. (1.19), gives the statistical spin tensors \( t_{kq}^{(s)} \) in the incident and the final states, i.e
The density matrix for the initial state can be expanded in terms of the tensors $t_{k_1 q_1}$ by [La 55]

$$\hat{s}_1^2 f_1(s_1) = \sum_{k_1 q_1} t_{k_1 q_1}(s_1) \tau_{k_1 q_1}^+ , \quad k_1 = 0, 1, 2 \quad (1.27)$$

The differential cross section for a polarized deuteron beam in a $(d,p)$ reaction eq. (1.20) can then be written as

$$\sigma = \sigma_0 \{ 1 + \sum_{k_1 q_1} t_{k_1 q_1}(s_1) \tau_{k_1 q_1}^+ \} , \quad k_1 = 1, 2 \quad (1.28)$$

where

$$T_{k_1 q_1} = \frac{\text{Tr}(T \tau_{k_1 q_1} T^+)}{\text{Tr}(T T^+)} \quad (1.29)$$
is the analyzing power of the reaction. With the choice of a right-handed coordinate system in which the positive $Z$-axis is along the direction of the incident deuteron $k_1$, and the $Y$-axis is along $k_1 \wedge k_2$ (for $(d,p)$ reaction, Madison Convention), then

$$T_{kq} = (-)^{k-q} T_{k-q}$$

(1.30)

if parity is conserved. Thus $T_{10} = 0$, $T_{11} = T_{20} = \text{pure imaginary}$, and $T_{2q} = (-)^{q} T_{2-q} = \text{pure real}$, then the expression for the cross section (1.28) may be written explicitly ($k \leq 2$) as

$$\sigma = \sigma_0 \{1 + 2T_{11} \text{Re}(t_{11}) + T_{20} t_{20}$$

$$2T_{21} \text{Re}(t_{21}) + 2T_{22} \text{Re}(t_{22})\}$$

(1.31)

The analyzing powers therefore determine the sensitivity of the differential cross section to the polarization of the incident deuteron beam.
CHAPTER 2

METHODS OF DIRECT REACTION THEORY

Direct reactions have been of great importance as a source of information about nuclear structure. One may say [Sa 83] that direct reactions describe a good overlap between the wave functions in the incident and the exit channels. This means that the collision may occur quickly with a minimum of rearrangement of the constituent nucleons. This situation has been said to be one of that involves only a few internal degrees of freedom of the colliding systems, whether these are best described in terms of single-particle or collective modes of motion. Most of the current theories further assume weak coupling, that is the elastic scattering is the most important process that occurs and that inelastic or rearrangement events can be treated as perturbations.

The two techniques most often used to interpret and analyze experimental data on direct reactions are the so-called distorted wave and coupled channels methods. These methods have been applied with considerable success, yielding quantitative as well as qualitative information about nuclear structure and dynamics.

2.1 Distorted Wave Born Approximation

The DWBA is a widely-used and very successful method for the analysis of nuclear reactions. This subject has been discussed
extensively in the literature [To 61, Au 64, Sa 64, Au 70, Gl 75, Sa 83]. In the DWBA method, the incoming deuteron in a (d,p) reaction is represented by an initially plane wave which has been distorted by the potential in the vicinity of the target nucleus. Strictly speaking, the distorting potential should be identical to the deuteron optical potential which describe the elastic scattering from the same target. The wave function of the outgoing proton in a (d,p) reaction is generated in the same fashion.

The process by which the neutron is transfered from the deuteron to the target nucleus is also treated in an approximate way by assuming that the interaction responsible for the transition occurs just once, that is, the transfer takes place in a single one-step process, without any change in the structure of the core to which the neutron is added. This is equivalent to the neglect of coupled channels (multi-step) and compound nucleus effects. The cross section, analyzing powers, and polarization of the proton in a (d,p) reaction can be calculated once the transition matrix for the process is known [Da 71]. In the DWBA method, a transition amplitude corresponding to no spin-dependent distortion has the form [Au 70]

$$T_{DWBA} = \int \int \chi_{-}^{(*)} (k_{2},l_{2}) <s_{2}s_{2}^{*}b_{\beta}|V_{np}|s_{1}s_{1}^{*}a_{\alpha}> \chi_{+}^{(+)} (k_{1},l_{1})$$

$$= <b_{\beta}s_{2}s_{2}^{*}s_{2}^{*}|V_{np}|a_{\alpha}s_{1}s_{1}^{*}|k_{1}>$$

(2.1)

and this is the starting point for DWBA calculations. If $V_{np}$ is central, then it will be a scalar in the separation $r$, and the proton
and the neutron will be in an S-state of relative motion within the incident deuteron.

The functions \( \chi^\pm \) are the distorted waves which describe the elastic scattering of deuteron on A (asymptotically with momentum \( k_1 \)) before the collision, and of the proton on B (with \( k_2 \)) after collision. Asymptotically, the \( \chi \) have the form of a plane wave plus scattered outgoing waves; in the absence of a Coulomb field

\[
\chi^\pm(k_1, r_1) \rightarrow \exp(\text{i} k_1 \cdot r_1) + f(\theta) \exp(\text{i} k_1 \cdot r_1) / r_1 \tag{2.2}
\]

The superscripts (+) or (-) denote the usual outgoing and ingoing wave boundary conditions; the two are related by the time reversal, which in the absence of spins has the form

\[
\chi^{(-)}(k_1, r_1) = \chi^{(+)}(\text{-}k_1, r_1) \tag{2.3}
\]

In DWBA, the distorted waves are chosen to be the wave functions of the optical potentials whose parameters are chosen to reproduce the elastic cross section.

\[
\left[ -\frac{\hbar^2}{2 m_i} \nabla^2 + U(r_i) + U_{\text{Coul}}(r_i) \right] \chi(k_1, r_1) = 0 \tag{2.4}
\]

where \( U(r_i) \) is the optical potential (which will be discussed in chapter 3). \( U_{\text{Coul}}(r_i) \) is the Coulomb potential. If the optical potential \( U \) includes a spin-orbit potential, the function \( \chi \) becomes a
matrix in spin space $\chi_{\sigma'\sigma}$, and eq. (2.1) can be written as

$$T_{DWBA} = J \sum_{\sigma_2 \sigma_1} \int d\mathbf{r} \int d\mathbf{r}' \chi_{\sigma_2' \sigma_2}(\mathbf{k}, \mathbf{r}, \mathbf{k}', \mathbf{r}') \langle s_2 \sigma_2' | b \beta | V_{np} | s_1 \sigma_1' a \alpha \rangle$$

where $J$ is the Jacobian of the transformation to the relative coordinates. $\sigma_i$ is the $z$-component of the spin. Terms with $\sigma_i' \neq \sigma_i$ allow the possibility of the spin-flip during the elastic scattering. The time reversal (2.3) then becomes

$$\chi_{\sigma_1' \sigma_1}^{(-)}(\mathbf{k}, \mathbf{r}, \mathbf{k}', \mathbf{r}') = (-)^{\sigma_i - \sigma_i'} \chi_{\sigma_1' \sigma_1}^{(+)}(-\mathbf{k}, \mathbf{r}, \mathbf{k}', \mathbf{r}')$$

(2.6)

In a similar way, coupling to the nuclear spin would make $\chi$ dependent upon those spins.

2.1.1 The Form Factor

The remaining factor in the amplitude (2.1) is the matrix element of the interaction causing the transfer event, taken between the internal states of the colliding pairs.
\[
\langle s_2 \sigma_2' b_\beta | V_{np} | l s_1' a_\alpha \rangle = \sum_{np} \int d \xi' \sigma_2' \Phi_{s_2}^* (p) \psi_{b_\beta}^* (\xi', n) V_{np} \phi_{s_1}^*(\xi, p, n) \psi_{a_\alpha}^* (\xi) \tag{2.7}
\]

where \( \sum \) denotes the summations over the spin coordinates of the neutron and proton in the incident deuteron, and \( \int d \xi \) denotes integration over all internal coordinates of \( \Lambda \).

Expression (2.7) is a function of \( \xi_1 \) and \( \xi_2' \), it plays the role of an effective interaction for transition between elastic states \( \chi^{(+)} \) and \( \chi^{(-)} \). This expression contains all information on nuclear structure and angular momentum selection rules.

In eq. (2.7), \( V_{np} \) does not depend on the \( \xi \) coordinates of the target, then

\[
F^{AB}_{\xi, n} = \int d \xi' \psi_{b_\beta}^* (\xi', n) \psi_{a_\alpha}^* (\xi) \tag{2.8}
\]

where \( \xi_n \) is the position vector of the neutron relative to the target. \( F^{AB}_{\xi, n} \) may be expanded [Pi 65], exactly, in states of total angular momentum of the captured neutron \( j_n \) as

\[
F^{AB}_{\xi, n} = \sum_{j_n m_n} \langle a_\alpha j_n m_n | b_\beta > \Phi^{AB}_{j_n m_n} (\xi_n) \tag{2.9}
\]
The usual interpretation of $\Phi$ is the product of a (normalized) single particle wave function times a spectroscopic amplitude [Fr 60],

$$
\Phi_{j_n m_n}^{AB} (r_n) = \{S_{n+1}^{AB} / N+1\}^{1/2} \psi_{n+l}^{AB} (r_n). \quad (2.10)
$$

The $\psi$ are eigenstates of some single-particle potential well. The orbital angular momentum $l_n$ is restricted to one of the values $l_n = j_n + 1/2$ according to the parity change $\pi_{AB}$ in the transition, since $\pi_{AB} = (-)^n$, the expansion is then over principle quantum number $n$. $N$ is the number of neutrons in $A$. The spectroscopic factor $S_{n+l}^{AB}$ measures the probability that the nuclear state produced in $(d,p)$ reaction has its parentage based on the ground state of the target, with a single particle in the shell-model state $nlj$ [Fr 60].

In practice, equation (2.9) is used in most calculations. The correct asymptotic behavior of $\psi_{n+l}^{AB}$ can be guaranteed by the use of a wave function corresponding to a Woods-Saxon potential whose parameters are adjusted so that the neutron binding energy in the well is equal to the experimental separation energy. In general, only one value of $l_n$ and $j_n$ contribute to the form factor $F_{AB}^{AB} (r_n)$, and we shall assume the approximation

$$
F_{AB}^{AB} (r_n) = \sum_{m_n} <a_{\alpha j_n m_n} b_{\beta} >_{l_n j_n}^{n+1} \psi_{n+l}^{AB} (r_n). \quad (2.11)
$$

where the spectroscopic factor has been written as $\alpha_{n+l}^{AB}$, and
\[ \psi_{1n}^{m_n}(r_n) = R_{j_n}^{n} (r_n) \sum_{l_n s_3 j_n} \psi_{1n}^{m_n} (\hat{r}_n, \lambda_n) \Phi_{s_3}^{\sigma_3}(n). \tag{2.12} \]

\( s_3 \) is the spin of the captured neutron, and the principal quantum number, \( n \), is no longer shown explicitly.

\[ \psi_{1n}^{m_n} (\hat{r}_n, \lambda_n) = \sum_{l_n s_3 j_n} (l_n \sigma_3 s_3 | j_n m_n) \psi_{1n}^{\lambda_n} (\hat{r}_n) \Phi_{s_3}^{\sigma_3}(n). \tag{2.13} \]

where \( \Phi_{s_3}^{\sigma_3}(n) \) is the intrinsic spin wave function of the neutron.

The remaining terms in eq. (2.7) contain the wave function of the incident deuteron \( \Phi_{s_1}^{1}(\xi, p, n) \). This function is an admixture of states with orbital angular momentum 0 and 2 and may be written as

\[ \Phi_{s_1}^{1}(\xi, p, n) = \sum_{L=0,2} U_L^{\lambda} (r) \psi_{Ls_1 s_1}^{1} (\hat{r}, p, n) \tag{2.14} \]

where \( U_0 \) and \( U_2 \) are respectively, the S- and D- state radial components of the deuteron wave function. As with eq. (2.13)

\[ \psi_{Ls_1 s_1}^{1} (\hat{r}, p, n) = \sum_{\Lambda \sigma_1' \Lambda \sigma_1} (L \Lambda s' \sigma_1' | s_1 \sigma_1) \psi_{L}^{\Lambda} (\hat{r}) \Phi_{s_1'}^{\sigma_1'}(p, n). \tag{2.15} \]

where \( \Phi_{s_1'}^{\sigma_1'}(p, n) \) represents the triplet spin wave function of the n-p system.
The product \( V_{np}(r) \Phi_{s_1}(r,p,n) \) appearing in eq. (2.7) can be written as

\[
V_{np}(r) \Phi_{s_1}(r,p,n) = \left\{ \frac{\hbar^2}{2\mu} \nabla^2 - e_d \right\} \Phi_{s_1}(r,p,n)
\]

\[
= \sum_{L=0,2} V_L(r) \sum_{s' s} \Phi_{s_1}(r,p,n) \Phi_{s_1}(r,p,n) \tag{2.17}
\]

where \( \mu \) is the deuteron reduced mass, \( \nabla \) is the gradient operator in the relative \( n-p \) coordinate \( r \). The radial functions \( V_L(r) \) are defined by [Jo 67]

\[
V_L(r) = \left[ \frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L(L+1)}{r^2} \right) - e_d \right] U_L(r) \tag{2.18}
\]

Substituting from equations (2.11-18) into (2.7), using the relation

\[
Y^A_L(r) = (-)^A Y^{-A}_L(r)
\]

yields
\[
\langle s_2 \sigma_2 \tau \beta | V_{np} | s_1 \sigma_1 \alpha \alpha \rangle = \sum_{lj} i^{-1} C_{lsj, \lambda} (r_2, r_1)(-s_2^2 - \sigma_2^2)(a \alpha | jm | b \beta ) (s_1 \sigma_1 s_2 \sigma_2 | \lambda \sigma \lambda)(l \lambda \sigma \lambda | jm) \quad (2.19)
\]

where

\[
G_{lsj, \lambda}(r_2, r_1) = \sum_{jn} \sum_{L=0,2} A_{lsj, n}^L \int_{lsj, \lambda}(r_2, r_1) \quad (2.20)
\]

where using the properties of Racah Coefficients

\[
A_{lsj, n}^L = \alpha_{jn}^L = \sum_{s_1 s_2 s_3} W_{LS_l s_{s_2} s_{s_3}} W(l_3 s_{s_2} s_{s_3}) W(l_n s_{s_2} j_n L) \quad (2.21)
\]

and

\[
\int_{lsj, \lambda}(r_2, r_1) = i^{-1} n^{-1} \sum_{jn} JR_{jn} \langle r_n | V_L | \rangle \int_{jn} \lambda (\tilde{r}_n, \tilde{r}) \quad (2.22)
\]

and \( T_{jn} \lambda \) is given by [Br 62]

\[
T_{jn} \lambda (\tilde{r}_n, \tilde{r}) = \sum_{\lambda n^L} (l_n \lambda \lambda L \lambda | 1 \lambda \lambda ) Y_{\lambda n}^\wedge (\tilde{r}_n) Y_{\lambda n}^\wedge (\tilde{r}) \quad (2.23)
\]

The separation of \( G_{lsj, \lambda} \) in (2.20) into a spectroscopic factor \( A_{lsj, n}^L \) and a form factor \( \int_{lsj, \lambda} \) is one of convenience, so that standard types of form factors with a simple normalization may be used in computation.
2.1.2 The DWBA Transition Amplitude

The amplitude (2.5) is then given by (using (2.19))

\[ T_{\text{DWBA}} = \sum_{l s j m} \hat{j}^{(a \alpha jm | b \beta)} \hat{\beta}^{l \rho \sigma_2 \sigma_1} (k_2, k_1) \]  \hspace{1cm} (2.24)

The reduced amplitude \( \hat{\beta} \) is given by

\[ \hat{j}^{l \rho \sigma_2 \sigma_1} \hat{\beta}^{s_{2}^{-} \rho \sigma_1} = \sum_{\sigma_2 \sigma_1} (-s_{2}^{-} \rho \sigma_1) \chi_{l \rho \sigma_1}^{s_{2}^{-} \sigma_1} (k_1, s_2, s_2^{-}, s_1, \lambda_{1}s_1, \lambda_{2}) \]  \hspace{1cm} (2.25)

The reduced amplitude \( \hat{\beta} \) involves a 6-fold integration over \( r_1 \) and \( r_2 \), and the numerical calculation of such integral is quite difficult. For this reason, the "zero-range" approximation is often introduced.

In order to compute these amplitudes (2.25), an explicit expression for the reduced amplitude is needed. To obtain such expression we use the partial wave expansion of the distorted waves \( \chi_{l \rho \sigma_1}^{(+)} (k_1, r_1) \) and \( \chi_{l \rho \sigma_2}^{(-)} (k_2, r_2) \).
\begin{equation}
\chi^{(+)}_{\sigma_i} (k_i, r_i) = \frac{4\pi}{k_i r_i} \sum_{l'_i} (1_{l'_1}^l \sigma_{l'_1} | j_i, m_i \rangle (1_{l'_1}^l \sigma_{l'_1} | j_i, m_i \rangle \chi_j^{l'_1} (k_i, r_i)
\end{equation}

where \(l\) represents the orbital angular momentum for the relative motion between the deuteron and the target nucleus, \(j_i\) is the total angular momentum.

In eq. (2.6), the time reversal involves reversing the direction of rotation of the spins, that is \(\sigma_i \rightarrow -\sigma_i\) (for example see [Ro 67]). Explicitly \(\chi^{(-)}\) is given by the same expression (2.26) as \(\chi^{(+)}\), except for the replacement of \(\chi_j^{l_1}\) by its complex conjugate. It satisfies the Schrödinger equation as \(\chi^{(+)}\) except for the replacement of the potential by its time-reverse.

When \(l_i \neq l'_i\) is allowed, the radial function \(\chi_{l_i}^{j_i} (k_i, r_i)\) satisfy sets of coupled equations, one for each value of \(j\).

\begin{equation}
\left[ \frac{d^2}{dr_i^2} + k_i^2 - \frac{l_i^2 (l_i^2 + 1_i)}{r_i^2} - \frac{2\mu_i}{\hbar^2} (1_i^j | U(r) | 1_i^j) \right] \chi_{l_i}^{j_i} (k_i, r_i)
\end{equation}

\begin{equation}
= \sum_{l_i'' \neq l_i'} \frac{2\mu_i}{\hbar^2} (1_i^{j_i} | U(r) | 1_i^{j_i}) \chi_{l_i}^{j_i} (k_i, r_i) \tag{2.27}
\end{equation}
If the nature of the potential is such that $l \neq l'$ is not allowed, then
\[ \chi_{l_{i}l_{i}'}^{j_{i}} = 0 \text{ for } l \neq l' \text{ and, if } \chi_{l_{i}l_{i}'}^{j_{i}} = \chi_{l_{i}l_{i}'}^{j_{i}} \text{ then eq. (2.27) becomes} \]
\[
\left[ \frac{d^2}{dr_i^2} + k_i^2 - \frac{l_i(l_i + 1)}{r_i^2} - \frac{2\mu_i}{\hbar^2} (l_i,j|U(r)|l_i,j) \right] \chi_{l_{i}l_{i}'}^{j_{i}}(k_i,r_i) = 0 \quad (2.28)
\]

At radii so large that the nuclear potentials can be neglected, these radial functions go over to the form
\[
\chi_{l_{i}}^{j_{i}}(k_i,r) = i e^{i \sigma_{l_{i}}} \{ R_{l_{i}}(kr) - S_{l_{i}} R_{l_{i}}(kr) \}/2 \quad (2.29)
\]

\( \sigma_{l_{i}} \) is the Coulomb phase shift
\[
\sigma_{l_{i}} = \arg \Gamma(l+1+in)
\]
\[
n = \frac{ZZ'e^2}{\hbar^2 k^2}
\]

In eq. (2.29), \( S_{l_{i}}^{j_{i}} \) is the scattering matrix element for the \((l,j)\) wave. The function \( R_{l_{i}} \) is defined by Hull and Breit [Hu 59, Br 62] to be the Coulomb analogue of \( kr h^{(1)} \), where \( h^{(1)} \) is the outgoing spherical Hankel function [Sc 49]. In terms of the regular and the irregular radial Coulomb functions, this function \( R_{l_{i}} \) is
\[
R_{l_{i}} = G_{l_{i}} + iF_{l_{i}} \quad (2.30)
\]

then eq. (2.29) becomes
\[ \chi^j_1(k,r) = e^{i\sigma} \left\{ F_1^j(kr) + \frac{1}{2} i (1-S^j_1) H_1^j(kr) \right\} \] (2.31)

In practice (2.28) is usually solved by numerical integration from \( r=0 \) and matching the value and the slope of the result onto the form of (2.31) at \( r > r_{\text{int}} \). This procedure then gives a value for \( S^j_1 \) [Me 66].

2.2 Coupled Reaction Channels Formalism (CRC)

Coupled-channels calculations have been widely used for inelastic scattering for many years, but until recently, application to transfer (rearrangement) reactions have been infrequent. This application introduces a new feature; the initial and the final channels in a transfer reaction are not fully orthogonal, and this leads to additional terms (non-orthogonality corrections).

In \((d,p)\) reaction, if the transition is too strong then a first-order treatment will not be a good approximation for the reaction. A way to allow contributions from higher order terms is to solve explicitly the coupled equations of the CRC.

Fig. (2.1) illustrates some possible transitions for rearrangement collision. It shows the direct coupling between the incident channel 1 and the transfer channel 2. Each arrow in fig. (2.1) represents a matrix element of the interaction acting once (one such matrix is equivalent to DWBA for that transition). Solving the corresponding CRC equations is equivalent to summing all possible contributions of arrows.
A model wave function which describes the processes in fig. (2.1) can be written as

\[
\Psi_{\text{model}} \left( x_1, x_2, k_1, k_2, r_1, r_2 \right) = \chi_{\lambda_1, \sigma_1, \alpha} \left( k_1, r_1 \right) \psi_1 (x_1)
\]

\[
\Psi_{\text{model}} \left( x_1, x_2, k_1, k_2, r_1, r_2 \right) = \chi_{\lambda_1, \sigma_1, \alpha} \left( k_1, r_1 \right) \psi_1 (x_1)
\]

\[
(E - H) \Psi_{\text{model}} \left( x_1, x_2, k_1, k_2, r_1, r_2 \right) = 0
\]

In terms of the notation in section 1.3, \( \psi_1 \) and \( \psi_2 \) describe the
internal states of the incident and the exit channel respectively.

\[ \Psi_1(x_1) = \Phi_{s_1} \sigma_1(r) \Psi_{a \alpha}^{A}(t) \]  \hspace{1cm} (2.34)

\[ \Psi_2(x_2) = \Phi_{s_2} \sigma_2(p) \Psi_{b \beta}^{B}(t, n) \]  \hspace{1cm} (2.35)

The \( \chi_i(k_i, r_i) \) functions appearing in eq. (2.32) describe the relative motion in the incident and the exit channel \((i=1, 2)\). It is important to mention that in a rearrangement collision, the channel coordinate is no longer the same in all rearrangement channels. Asymptotically, and in the absence of the Coulomb forces

\[ \chi_1(k_1, r_1) \rightarrow \exp(ik \cdot r_1) + f(\theta)\exp(ik \cdot r_1)/r \]  \hspace{1cm} (2.36)

\[ \chi_2(k_2, r_2) \rightarrow f(\theta)\exp(ik \cdot r_2)/r \]  \hspace{1cm} (2.37)

From the projection of eq. (2.33) onto the incident and the exit channels,

\[ (\Psi_1^{\text{model}} | (E-H) | \Psi_1^{\text{model}}) = 0 \]  \hspace{1cm} (2.38)

\[ (\Psi_2^{\text{model}} | (E-H) | \Psi_2^{\text{model}}) = 0 \]  \hspace{1cm} (2.39)

with the two equivalent forms of \( H \) given by equations (1.6) and (1.7).
The coupled reaction channels equations for the transfer reaction can be obtained by substituting from (1.6,7) and (2.32) into (2.38,39)

\[
[(E-e_{\text{dA}}) - K_1 - \langle \Psi_1 | V_{\text{dA}} | \Psi_1 \rangle] C_1(k_1, r_1)
\]

\[
= \left( \Psi_1(x_1) \right) \left( E - E \right) \left( \Psi_2(x_2) \right)
\]

\[
= \left( \Psi_2(x_2) \right) \left( E - E \right) \left( \Psi_1(x_1) \right)
\]

These coupled reaction channels equations may be written explicitly as

\[
[(E-e_{\text{eB}}) - K_2 - \langle \Psi_2 | V_{\text{pA}} + V_{\text{np}} | \Psi_2 \rangle] C_2(k_2, r_2)
\]

\[
= \int d\tau_2 K_{12}(r_1, \tau_2) \left( \Psi_1(x_1) \right) \left( E - E \right) \left( \Psi_2(x_2) \right)
\]

\[
= \int d\tau_1 K_{21}(r_2, \tau_1) \left( \Psi_2(x_2) \right) \left( E - E \right) \left( \Psi_1(x_1) \right)
\]

where the kernels are

\[
K_{12}(r_1, \tau_2) = J_{12} \int d\xi_1 \left( \Psi_1^*(x_1) \right) \left( E - E \right) \left( \Psi_2(x_2) \right)
\]

\[
K_{21}(r_2, \tau_1) = J_{21} \int d\xi_2 \left( \Psi_2^*(x_2) \right) \left( E - E \right) \left( \Psi_1(x_1) \right)
\]
where $J_{12}, J_{21}$ are the Jacobians for transforming the internal coordinates $x_1$ and $x_2$ into $(\xi_1, \xi_2)$ and $(\xi_2, \xi_1)$ respectively. For further insight into the structure of the kernels, we may consider a particular partition of $\mathcal{H}$, for example prior representation eq. (1.6).

Then (2.44) becomes

$$K_{12}(\xi_1, \xi_2) = I'_{12}^{\text{(prior)}} + N'_{12}^{\text{(prior)}}$$

(2.46)

where

$$I'_{12}^{\text{(prior)}} = J_{12} \int d^2 \xi_1 \psi^*(x_1) \psi^*(x_2) dA_{12}$$

(2.47)

$$N'_{12}^{\text{(prior)}} = J_{12} [K - (E - e - e')] \int d^2 \xi_1 \psi^*(x_1) \psi^*(x_2)$$

(2.48)

The term $I'_{12}$ is called the interaction kernel, while the additional term $N'_{12}$ is known as the non-orthogonality kernel. So these kernels (coupling terms) are partly due to interactions and partly due to non-orthogonality effects.

Equations (2.47,48) may be generalized by introducing an auxiliary potential $U_{1}(r_{1})$ which is diagonal in the internal states, then we have

$$I_{12}^{\text{(prior)}} = J_{12} \int d^2 \xi_1 \psi^*(x_1) (V - U_{1}) \psi^*(x_2)$$

(2.49)
The alternative 'post' interaction expression for $K_{12}$ may be obtained by using eq. (1.7), and introducing an analogous auxiliary potential $U_2(r_2)$ in channel 2.

$$I_{12}^{(\text{post})} = \int \frac{d^3 \psi^*(x_1)}{J} \left( \frac{\psi(x_1)}{p^A} + \frac{U_2}{p^B} \right) \psi_2(x_2)$$

and

$$N_{12}^{(\text{post})} = \int \frac{d^3 \psi^*(x_1)}{J} \psi_2(x_2) [K_{2} + U_2 - (E - e)]$$

In the same way it is possible to obtain the corresponding expressions for $K_{21}$.

Either the post or the prior form of the kernel may be used in CRC equations. The separation of $K$ into $I$ and $N$ terms is arbitrary to the extent that the auxiliary potential $U_1$ and $U_2$ are arbitrary. If these potentials are chosen so that

$$[K_1 + U_1 - (E - e)] \chi_1(k_1, \Gamma_1) = 0$$

and

$$[K_2 + U_2 - (E - e)] \chi_2(k_2, \Gamma_2) = 0$$

then the non-orthogonality terms would vanish. However, the choice of
\[ U_1 = (\psi_1 | V_{dA} | \psi_1) \]  \hspace{1cm} (2.55)

and

\[ U_2 = (\psi_2 | V_{pB} | \psi_2) \]  \hspace{1cm} (2.56)

already comes close to making NO terms vanish \[ [Au 70] \]. This choice has been seen by others \[ [Co 75, Co 76, Co 81] \] to reproduce the importance of NO terms in actual calculations.

In many applications of CRC method the non-orthogonality terms are simply neglected \[ [St 66, Ra 67, Oh 70, Co 73] \]. At the time \( U_1 \) and \( U_2 \) are equated to (2.55,56) respectively, while these are chosen to be like the local, complex optical potentials that fit observed elastic scattering for these channels.
CHAPTER 3

COUPLED CHANNEL CALCULATIONS FOR $^{90}_{\text{Zr}}(d,p)^{91}_{\text{Zr}}$

The work we will present in this chapter is an extension of the work of Tostevin and Johnson which concerns $(d,p)$ scattering on $^{208}_{\text{Pb}}$ at sub-Coulomb energy [To 83]. In their work they employed the coupled channels formalism and assumed pure Coulomb distortion between the deuteron and proton and the target (the Q-values of the reaction were very small). In the present work, $(d,p)$ on $^{90}_{\text{Zr}}$ at sub-Coulomb energy, the reduced charge number of the target nucleus and the increase of the scattering energies (Q-values of the reaction are large and positive) relative to the Coulomb barrier makes the present calculations more sensitive to the nuclear potentials than the earlier calculations [To 83].

In this chapter we shall investigate the relative importance of the folding model spin-orbit interaction [Wa 58] and the transfer channel coupling mechanism through the study of deuteron elastic scattering and $(d,p)$ reaction upon $^{90}_{\text{Zr}}$ at an incident energy of 5.5 MeV, the energy at which the experiment has been performed.

It has been shown that the calculations using the folding model central potential do not satisfactorily reproduce the measured differential cross sections [Pe 67, Ke 73]. In order to obtain good agreement it is necessary to change the strength of the real central potentials by 10-20% and to increase the strength of the imaginary
Such changes in the central potential produces large changes in the calculated cross section and analyzing powers. Consequently, the usefulness of the folding model is limited. However, the model may still be useful to predict the spin-dependent parts of the potential and thus to reduce the ambiguities in the optical model analysis which result from the large number of parameters. Less accuracy is required for the spin-dependent terms, since they are relatively weak compared to the central potential. Knutson and Haeberli [Kn 75a] have shown that the spin-orbit and the tensor potentials predicted by the folding model are quantitatively consistent with observed vector and tensor analyzing powers for deuteron elastic scattering near sub-Coulomb energies [Kn 75a, Ka 76].

More recent and complete theoretical analysis [To 83, Ni 82, Wi 84, Oh 84] have shown, however that these earlier studies of the spin-orbit interaction were too phenomenological in their approach to allow conclusions to be drawn as to the importance and the correctness of the spin-orbit interaction arising out of the folding model. In fact it was shown that projectile excitation to discrete excited states, in the case of low energy $^6$Li and $^7$Li scattering [Ni 82] and coupling to (d,p) transfer channels, in the case of sub-Coulomb deuteron scattering [To 83], are the dominant mechanisms by which the observed elastic vector analyzing power, $i_T$, are produced. These physical processes are not taken into account in any way in the conventional single-folding model prescription for projectile-target interaction.

Tostevin and Johnson [To 83] have shown that the folding model spin-orbit potential alone did not reproduce the measured elastic $i_T$. 
for deuteron scattering from $^{208}$Pb at 8 MeV. The calculated elastic $iT_{11}$ fails in predicting the correct sign or magnitude of the experimental data. They demonstrated that the mechanism which produces the observed elastic $iT_{11}$ at 8 and 9 MeV, is the coupling of the elastic channel to the weakly bound (Q-values $\approx 0$) neutron transfer channels of $^{209}$Pb. Their results also showed that the deuteron D-state produced no additional effects upon the cross section and vector analyzing power, $iT_{11}$, but its effect upon the elastic tensor analyzing powers, $T_{2q}$, is large. Similarly, deuteron D-state effects upon the calculated (d,p) cross section and $iT_{11}$ are also small, as can be seen from the calculations and the figures of reference [Kn 77].

3.1 Zero-Range Approximation

The DWBA expression for the transition amplitude (2.24-25) involves an integration over the space of both $r_1$ and $r_2$. This 6-fold integration has been discussed by Austern and Sawaguri [Au 64, Sa 67]. In the zero-range assumption the deuteron is assumed to exist purely in the S-state ($L=0$). The physical meaning of zero-range approximation is that, the proton is emitted at the same point at which the neutron is deposited, so that

$$
\begin{align*}
 &r_n \rightarrow r_1 \\
 \end{align*}
$$

(3.1)

$$
\begin{align*}
 &r \rightarrow \frac{A}{A+1} \frac{r}{l} = \gamma r \\
 \end{align*}
$$

(3.2)

Considering the zero-range approximation for the function $G$ as expressed in eq.(2.20). This, together with the partial-wave expansion
(2.26), yields an expression for $\beta$ when inserted in eq.(2.25)

$$\hat{j}_j \frac{1}{2} (2R) = D_{0} \sum_{\sigma_{1}^2 \sigma_{2}^2} (-)^{s_{1}^2 - s_{2}^2} (s_{1}^2 \sigma_{1}^2 s_{2}^2 \sigma_{2}^2) (1 \lambda \sigma \mid jm)$$

$$\frac{1}{\gamma} \int d\vec{x}_1 \chi_j^j (k_2 \gamma R_{1}) R_{1} (r) \chi_j^j (k_1, r_1)$$

$$\int d\vec{x}_1 \chi_j^j (k_2 \gamma R_{1}) R_{1} (r) \chi_j^j (k_1, r_1)$$

$$\int d\vec{x}_1 \chi_j^j (k_2 \gamma R_{1}) R_{1} (r) \chi_j^j (k_1, r_1)$$

The integration over the three spherical harmonics in (3.3) gives

$$(4\pi)^{1/2} \frac{\hat{l}_{1} \hat{l}_{2}}{\hat{l}_{n}} (1 \lambda_1 \lambda_2 \lambda_1 \lambda_2 | l \lambda | n \lambda) (1 \theta \theta | l \theta | n \theta)$$

Eq. (3.3) becomes
\[
\beta_{lj}^{1m\sigma_2\sigma_1(2R)} = 4\pi \sum (-)^{s_2-s'_2} (s_1\sigma_1, s'_2, \sigma_2, 1, \lambda, \sigma | jm)
\]

\[
(1_1, \lambda_1 s_1, \sigma_1 | J_1 M_1) (1_1, \lambda_1 s'_1, \sigma_1 | J_1 M_1)
\]

\[
(1_2, \lambda_2 s_2, -\sigma_2 | J_2 M_2) (1_2, \lambda_2 s'_2, \sigma_2 | J_2 M_2)
\]

\[
i^{l_1+1_2} (-)^{\sigma_2-\sigma'_2} \frac{i^{l_1+1_2}}{i_n} (1_1, \lambda_1^{l_1} \lambda_2^{l_2} | 1_n \lambda_n)
\]

\[
(1_1^\lambda l_2^\lambda | 1_n^\lambda) \frac{\lambda_1^*}{Y_{1}^{\lambda_1}(k_1)} Y_{1/2}^{l_2}(-k_2) \int_{1_2}^{1s_j} (ZR) (3.5)
\]

with the summation over \(1_2, \lambda_2, l_2, J_2, M_2, \sigma_2; 1_1, \lambda_1, l_1, J_1, M_1, \sigma_1; \lambda, \sigma \) and \( \lambda \). The radial integrals are defined as

\[
\int_{1_2}^{1s_j} (ZR) = \frac{(4\pi)^{1/2}}{k_1 k_2} D_0 \int \frac{dr}{\gamma} \frac{\chi_{1_2}^{J_2}(k_2, r_1) R_{1_2}^{J_1} (r_1) \chi_{1_1}^{J_1}(k_1, r_1)}{\gamma} (3.6)
\]

By using the various symmetry properties of the Clebsch-Gordan coefficients, and the contraction relation, eq. (3.5) becomes
\[ \beta_{sj}^{1m\sigma, \sigma'}^{(2R)} = 4\pi \sum_{l_1}^{1-l_1-1} \sum_{j_1}^{1-l_1-1} \sum_{j_2}^{2} \sum_{l_2}^{2} \sum_{s_1}^{j_1} \sum_{s_2}^{j_2} \sum_{l_1}^{l_1} \sum_{J_1}^{J_1} \sum_{M_1}^{M_1} \sum_{l_2}^{l_2} \sum_{J_2}^{J_2} \sum_{M_2}^{M_2} \sum_{1}^{1} \sum_{(2)}^{2} \sum_{1}^{2} \sum_{s_1}^{s_1} \sum_{s_2}^{s_2} (l_1, \lambda_1, s_1, J_1, M_1) (l_2, \lambda_2, s_2, J_2, M_2) \]

\[ (-\lambda_2)^* \gamma_1^{(l_2)} \gamma_1^{(l_1)} \hat{f}_{1}^{1s} \] 

\[ (l_2 \otimes l_1 | 1 \otimes 1) \]

\[ \left\{ \begin{array}{c}
    j & 1 & s \\
    J_1 & 1 & s_1 \\
    J_2 & 1 & s_2 \\
\end{array} \right. \] 

(3.7)

with summation over \( l_2, \lambda_2, J_2 \) and \( l_1, \lambda_1, J_1 \) and

\[ \sum_{s_1,s_2,s_1'} \sum_{s_2,s_2'} \sum_{l_1,l_1'} \sum_{J_1,J_1'} \sum_{l_2,l_2'} \sum_{J_2,J_2'} \sum_{s_1,s_1'} \sum_{s_2,s_2'} \sum_{l_1,l_1'} \sum_{J_1,J_1'} \sum_{l_2,l_2'} \sum_{J_2,J_2'} (l_1 \lambda_1 s_1 \sigma_1 | J_1 M_1) (l_2 \lambda_2 s_2 \sigma_2 | J_2 M_2) (l_1 \lambda_1' l_2 \lambda_2' | 1 \lambda) \]

\[ (-\lambda_2)^* \gamma_1^{(l_2)} \gamma_1^{(l_1)} \hat{f}_{1}^{1s} \] 

\[ (s \sigma_1 s_1 \sigma_1' | s \sigma_2 s_2' | j m) \]

\[ = \hat{s} (-\lambda_2)^* \gamma_1^{(l_2)} \gamma_1^{(l_1)} \] 

\[ (l_1 \hat{J} l_2 \hat{J} | J_1 J_2 M_1 M_2 | j m) \]

\[ \left\{ \begin{array}{c}
    l_1 & s_1 & J_1 \\
    l_2 & s_2 & J_2 \\
    1 & s & j \\
\end{array} \right. \] 

(3.8)

where we have used the Wigner 9-j symbol [Br 62]
3.2 Deuteron Optical Potential

The optical model has been used to describe deuteron elastic scattering from a wide range of target nuclei. Watanabe [Wa 58] has suggested that the deuteron-nucleus potential might be calculated from the empirically known nucleon-nucleus optical potentials. In this so-called folding model, the deuteron potential is taken to be the sum of the neutron-nucleus and proton-nucleus potentials, averaged over the internal motion of the deuteron. When the nucleon-nucleus potentials are made up of central and spin-orbit terms, the deuteron potential will likewise contain a central and a spin-orbit term [Wa 58]. In addition the folding model will contain a tensor potential provided that the effects of the deuteron D-state are included [Sa 60]. The spin-orbit and tensor potentials have only small effect on the elastic scattering cross section, but do affect the analyzing powers [Da 71]. When the spin-dependent potentials are small enough to be treated as perturbations, the spin-orbit potential produces non-zero vector analyzing power, $i T_{11}$, while the tensor potential primarily affects the tensor analyzing powers, $T_{20}$, $T_{21}$, and $T_{22}$, [Jo 71]

There have previously been several optical model studies in which acceptable fits to vector analyzing power data were obtained by using phenomenological spin-orbit potentials [Sc 69, Gr 70, Gr 71]. It has been stated [Ha 72] that the phenomenological spin-orbit strengths are consistent with those expected from the folding model.

The original analysis of the deuteron elastic scattering data from $^{90}$Zr [Kn 73a, Kn 75a] made use of the conventional one-channel deuteron
optical potential. This potential is of a common form, consisting of a Woods-Saxon real central potential, a derivative Woods-Saxon surface imaginary potential and an \( L.S \) spin-orbit potential of Thomas form. The analytic form of the potential [Sc 68] is

\[
U(r) = V_c(r) + V_{so}(r)
\]

where

\[
V_c(r) = -V_0(e^{x}+1)^{-1} + 4iW_0 \frac{d}{dx'} (e^{x'}+1)^{-1}
\]

with

\[
x = (r-r_0^{1/3})/a_0
\]

and

\[
x' = (r-r_W^{1/3})/a_W
\]

\( A \) is the nuclear mass number. To this potential is added the Coulomb potential of a uniformly charged sphere of radius \( 1.3A^{1/3} \) fm.

The spin-dependent potential consists of a real vector spin-orbit coupling terms of the Thomas form

\[
V_{so}(r) = -V_{so} \frac{2}{r} \frac{d}{dr} (e^{x''}+1)^{-1} L.S
\]

where

\[
x'' = (r-r_{so}^{1/3})/a_{so}
\]

and \( S \) is the spin operator for the deuteron and \( L \) the orbital angular
momentum operator. The spin-orbit potential predicted by the folding model can be reproduced quite accurately by the analytic expression (3.13) [Kn 75a].

It was found [Kn 73a] that by adjusting the real and imaginary central terms of the deuteron potential, in the presence of the real folding model spin-orbit interaction, a very good simultaneous fit to the elastic cross section and iT\textsuperscript{11} data could be obtained, as shown by the solid curves in fig. (3.1). The values of the potential depth parameters V\textsubscript{0}, W\textsubscript{D}, V\textsubscript{S0}, and the geometrical parameters r\textsubscript{0}, r\textsubscript{W}, r\textsubscript{S0} and a\textsubscript{0}, a\textsubscript{W}, a\textsubscript{S0} for the potential (3.9) are listed in table 3.1. Unfortunately, if this interaction, fitted to the elastic data (potential I of [Kn 77] and table 3.1) is now used in a distorted wave Born approximation calculation of the corresponding observables in (d,p) reaction, very poor agreement is obtained with the measured values in the four transfer channels of interest [Kn 77] (solid curves in figs. 3.2,3). In particular, the calculated iT\textsubscript{11} for the two populated 1/2\textsuperscript{+} states in \textsuperscript{91}Zr (1.2 Mev and 2.56 Mev), which would vanish in the absence of the spin-orbit interaction and thus provide a stringent test of the deuteron channel potential, are in no way reproduced. A much improved fit to the transfer reaction cross section (dashed curves in figs. (2.2,3) can be obtained by reducing the range of the absorptive term of the original interaction (potential II of [Kn 77] and table 3.1). However, this potential generated in this way is no longer reproduces the measured elastic cross section or iT\textsubscript{11} (dashed curves in fig. 3.1), and should not therefore be used, without additional theoretical justification, within the context of DWBA.
An underlying requirement of this original analysis [Kn 75a] was that the elastic $iT_{11}$ should be fitted by a deuteron potential, the spin-orbit term of which is that predicted by the folding model. If this condition is relaxed, that is we require only that the adjusted central potential should reproduce the elastic cross section, alternative central interactions are possible which improve the agreement between the $(d,p)$ reaction data and DWBA calculations. Two such potentials are denoted potential III and IV in table 3.1. These interactions together with the folding model spin-orbit potential, do not reproduce the elastic $iT_{11}$, as is shown by the dotted curve in fig. (3.1). These $iT_{11}$ arising from the folding model $L_S$ potential alone, are seen to be substantially smaller and out of phase with the experimental data, particularly at backward angles. We shall therefore investigate whether, when the present potential prescription is used, and the dominant $(d,p)$ transfer channels are included explicitly within the coupled channel formalism, improved agreement is obtained with elastic $iT_{11}$ data.

3.3 Details of Coupled Channels Calculations

The calculations presented here were performed using the computer program CHUCK3 [Co ]. For an incident deuteron energy of 5.5 MeV upon $^{90}$Zr, the elastic channel and the neutron transfer to the $5/2^+$ (g.s.), $1/2^+$ (1.2 MeV), $3/2^+$ (2.04 MeV), and $1/2^+$ (2.56 MeV) states of $^{91}$Zr are treated explicitly. In all cases the proton optical potentials and neutron bound state geometry are taken from the global nucleon potential analysis of Becchetti and Greenlees [Be 69] (see Appendix A). For each state the parameters of the proton optical
potentials together with the state energy, number of nodes, \( j_n \), \( l_n \), Q-value, proton incident energy, and the separation energy of the neutron are listed in table 3.2.

The calculations are carried out within zero-range approximation (i.e. using eq. 3.7) with the usual local energy approximation correction [Bu 64] for finite range effects, which constitutes an extremely accurate procedure at these energies. The deuteron S-state strength and the range parameters, \( D_0 \), \( \beta \), [Kn 77] were given the values of \(-122.5 \text{ MeV}\cdot\text{fm}^{3/2}\) and 1.341 fm, respectively, appropriate to the Reid soft core interaction [Kn 77, Re 68].

As already outlined, the major uncertainty in the proposed coupled channels study is thought to be the deuteron channel distorting potential to be used. As the incident deuteron is below the Coulomb barrier, the elastic scattering data does not allow decisive conclusions to be drawn as to the deuteron-target interaction. However, one result of the present coupled reaction channels calculations is that, as the fraction of the reaction cross section attribute to the transfer channels treated is so small, the coupled reaction channels effects do not change the elastic cross section from the calculated using the single channel optical potential. A prerequisite of any deuteron potential used is therefore that it should reproduce the elastic cross section data.

As the measured elastic cross section is rather structureless, by adjusting the parameters of the imaginary central potential, and the real potential depth, the data can be accurately reproduced for the essentially any reasonable choice of the real potential geometry. The
imaginary terms obtained in this way show little dependence however upon the chosen real geometry and are thus rather well determined by the data. Potential III, in table 3.1, has been generated such that the real potential radius and diffuseness parameters are those obtained from the global analysis of Daehnick, Childs, and Vrcelj [Da 80]. Potential IV was obtained by fitting the elastic cross section only, adopting the real potential geometry of Knutson and Haebeli [Kn 77]. The latter choice is significantly different to that of potential III and we used these two derived phenomenological interactions to assess the sensitivity of the presented calculations to the deuteron optical potential assumed.
Table (3.1)

Deuteron Optical Potentials

<table>
<thead>
<tr>
<th>Potential</th>
<th>$V_0$ (MeV)</th>
<th>$r_0$ (fm)</th>
<th>$a_0$ (fm)</th>
<th>$W_D$ (MeV)</th>
<th>$r_w$ (fm)</th>
<th>$a_w$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>94.8</td>
<td>1.2</td>
<td>0.58</td>
<td>6.54</td>
<td>1.77</td>
<td>0.72</td>
</tr>
<tr>
<td>II</td>
<td>94.8</td>
<td>1.2</td>
<td>0.58</td>
<td>8.00</td>
<td>1.30</td>
<td>1.00</td>
</tr>
<tr>
<td>III</td>
<td>90.37</td>
<td>1.17</td>
<td>0.75</td>
<td>9.54</td>
<td>1.66</td>
<td>0.714</td>
</tr>
<tr>
<td>IV</td>
<td>90.85</td>
<td>1.2</td>
<td>0.58</td>
<td>8.35</td>
<td>1.66</td>
<td>0.771</td>
</tr>
</tbody>
</table>

$V_{s0} = 5.63$ MeV, $r_{s0} = 0.98$ fm, $a_{s0} = 1.0$ fm, and $r_c = 1.3$ fm
<table>
<thead>
<tr>
<th>( E_x ) (MeV)</th>
<th>( j_n )</th>
<th>( l_n )</th>
<th>Q-value (MeV)</th>
<th>( E_p ) (MeV)</th>
<th>( S_n ) (MeV)</th>
<th>Proton Optical Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 ) (MeV)</td>
<td>( W_D ) (MeV)</td>
<td>( V_{so} ) (MeV)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>1</td>
<td>5/2</td>
<td>2</td>
<td>4.965</td>
<td>10.450</td>
<td>7.145</td>
</tr>
<tr>
<td>1.2</td>
<td>2</td>
<td>1/2</td>
<td>0</td>
<td>3.765</td>
<td>9.246</td>
<td>5.985</td>
</tr>
<tr>
<td>2.04</td>
<td>1</td>
<td>3/2</td>
<td>2</td>
<td>2.925</td>
<td>8.396</td>
<td>5.145</td>
</tr>
<tr>
<td>2.56</td>
<td>2</td>
<td>1/2</td>
<td>0</td>
<td>2.405</td>
<td>7.871</td>
<td>4.625</td>
</tr>
</tbody>
</table>

\( r = 1.17 \text{ fm} \) \( r = 1.32 \text{ fm} \) \( r = 1.01 \text{ fm} \)
\( a = 0.75 \text{ fm} \) \( a = 0.595 \text{ fm} \) \( a = 0.75 \text{ fm} \)
Fig. (3.1): Angular distributions of the cross section, (ratio to Rutherford) and the vector analyzing power for deuteron elastic scattering from $^{90}$Zr at 5.5 MeV. The solid curves show the results of the calculations using potential I, which fits the experimental data, while the dotted and dashed curves show the results of the calculations using potentials II and III respectively.
Fig. (3.1)
Figs. (3.2,3): Angular distributions of the cross section and iT_1^T for $^{90}$Zr(d,p)$^{91}$Zr reaction at 5.5 MeV. The solid and dashed curves show the results of DWBA calculations with potentials II and I respectively. The results given by the solid curve are obtained by adjusting the absorptive term in the deuteron potential I in order to fit the experimental data for the cross section.
3.4 Results of Coupled Channels Calculations

In figs. (3.4-7) we present the calculated cross sections and the vector analyzing powers, \(iT_{\|}\) (dashed curves) for the four transitions under consideration which result from the two-channel coupled reaction channels calculation, in which only the elastic channel and the transfer channel in question are included. The deuteron potential used in these and all following calculations is potential III unless stated otherwise. These two-channel results (dashed curves) are very similar to those obtained from DWBA (dotted curves). The transfer channel spectroscopic factors are determined such that the magnitude of the experimental \((d,p)\) cross section are reproduced within these two-channels calculations. The values of these spectroscopic factors obtained by using potential III and other previous DWBA calculations [Ra 73, Bi 70, Kn 77] are listed in table 3.3. It is these values which are used in the subsequent full (five-channel) calculations. These full coupled reaction channels results for \((d,p)\) observables are the solid curves.

Immediately apparent are the relatively large effects induced in each transfer channel when the remaining three \((d,p)\) transitions are included, whereas as stated earlier, the coupled channel corrections to the elastic cross section are essentially zero. This effect is attributed to a small number of partial waves near \(l=0\), which contribute actively to all transfer matrix elements, being changed significantly by the coupling. These few low \(l\)-partial waves, while significant for the evaluation of the elastic channel \(iT_{\|}\), are
relatively much less important in the calculation of the elastic cross section.

We also observe that the (d,p) reaction \( iT_1 \) for the two \( 1/2^+ \) states, figs.(3.5,7), which would vanish in the absence of the deuteron channel spin-orbit interaction in DWBA calculations, receive significant corrections from the coupling to the other transfer channels. The dot-dashed curves in figs.(3.5,7) are obtained when the strength of the folding model spin-orbit interaction is set to be zero in the coupled channels calculation and thus reflect the magnitude of the spin-orbit effects arising from the channel coupling effect alone. It is apparent that both the folding model spin-orbit interaction and the coupled channels effect play comparable roles in the full calculations, however, it is the channel coupling which is responsible for the negative trend in the calculated \( 1/2^+ \) states \( iT_1 \) at extreme backward angles, which is required by the data. In the \( 5/2^+ \) and \( 3/2^+ \) transitions on the other hand large \( iT_1 \) arise simply due to the non-zero orbital angular momentum of the transferred neutron. These transitions are therefore rather insensitive to details of the deuteron channel spin-orbit interactions, whether this originates from the folding model or transfer channel coupling.

In all transitions the agreement between the calculated (d,p) cross sections and the data is only qualitative. However, the essential features of the angular distributions are reproduced. Little variation is observed when potential IV is used rather than potential III. The agreement obtained was deemed sufficiently good to ensure that the essential features of the transfer channels effects upon the calculated
elastic vector analyzing power, $iT_{11}$, are included.

The full coupled channels predictions for the elastic $iT_{11}$ are shown by the solid curve in fig.(3.8). The two peaked appearance of the data is reproduced by the calculations, but with a somewhat reduced magnitude. The dashed curve in fig.(3.8) is obtained when the folding model spin-orbit interaction is removed from the calculation. The calculated $iT_{11}$ are approximately halved in magnitude throughout the angular range. It is clear that the folding model spin-orbit interaction is important to reproducing the data.

In order to demonstrate that these $iT_{11}$ angular distributions are in fact a coherent effect of the four $(d,p)$ channels we present, in fig.(3.9), the calculated elastic $iT_{11}$ which result from the two-channel calculations in which each transfer channel is included individually. Clearly no one transfer channel alone is responsible for the double-humped $iT_{11}$ distribution obtained in the full channel calculations, and required by the data.
Table 3.3

Spectroscopic Factors for States of $^9_{\text{Zr}}$

<table>
<thead>
<tr>
<th>State</th>
<th>Spectroscopic Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_x$</td>
</tr>
<tr>
<td></td>
<td>(MeV)</td>
</tr>
<tr>
<td>[Ra 73]</td>
<td>[Bi 70]</td>
</tr>
<tr>
<td>0.0</td>
<td>5/2$^+$</td>
</tr>
<tr>
<td>1.2</td>
<td>1/2$^+$</td>
</tr>
<tr>
<td>2.04</td>
<td>3/2$^+$</td>
</tr>
<tr>
<td>2.56</td>
<td>1/2$^+$</td>
</tr>
</tbody>
</table>
Figs. (3.4-7): Angular distributions of the cross section and $iT_{jj}$ for (d,p) from $^{90}$Zr with $j'=5/2^+, 1/2^+, 3/2^+$, and $1/2^+$ respectively. The dotted curves show the results of DWBA calculations. The dashed and solid curves show the results of the two-channel and five-channel coupled channels calculations. The dotted-dashed curves in figs. (3.5,7) show the calculated elastic $iT_{jj}$ predicted by the five-channel coupled channels calculations in the absence of the folding model spin-orbit potential.
Fig. (3.8): Full coupled channels predictions for the calculated elastic $iT_{\parallel}$ are shown by the solid curve when the folding model spin-orbit potential is included in the calculations and by the dashed curve in the absence of the folding model spin-orbit potential. The dotted curve shows the calculated elastic $iT_{\parallel}$ predicted by the folding model spin-orbit potential alone.

Fig. (3.9): The calculated elastic $iT_{\parallel}$ which result from the two-channel calculations in which the elastic channel and the neutron transfer to $5/2^+$ (solid curve), $1/2^+$ (dashed curve), $3/2^+$ (dotted curve), and $1/2^+$ (dotted-dashed curve) states of $^{91}$Zr are treated explicitly.
3.5 Discussion and Conclusions

In this chapter, we have presented a coupled reaction channels study of the spin-dependence of low energy deuteron elastic scattering, associated with the strong coupling to (d,p) reaction channels and the folding model spin-orbit interaction.

For (d,p) reactions on $^9\text{Be}$, the energy of the outgoing protons is above the Coulomb barrier. The distorted wave calculations for these transitions are found to be sensitive to the nuclear optical potential. The central potential of the deuteron optical potential is adjusted to reproduce the elastic cross section, while the spin-orbit term is predicted by the folding model. The central potential together with the folding model spin-orbit interaction do not reproduce the elastic $i_T^{11}$ (neither the magnitude nor the angular distribution). Performing the full coupled reaction channel calculations using the present potential, and the dominant (d,p) transfer channels, an improved agreement is obtained with the elastic $i_T^{11}$ data (The coupled reaction channel corrections to the elastic cross section are essentially zero). Our full coupled reaction channels calculations have shown that the magnitude of of the calculated $i_T^{11}$ is twice its magnitude, throughout the angular range, if the spin-orbit potential is removed from the full calculations. This shows the importance of including the spin-orbit interaction in the calculations. The calculated elastic $i_T^{11}$ which result from the two-channel coupled reaction channels calculations (each transfer channel is included individually) show that no one transfer channel alone is responsible for reproducing the angular shape.
required by the data.

The coupled reaction channel formalism produce a significant reduction to the transfer reaction cross section, in particular at backward angles. The vanishing \((d,p)\) reaction \(i_{T_{11}}\) for the two \(l/2^+\) states \((l=\emptyset)\) in the absence of the folding model spin-orbit interaction predicted by DWBA, receive significant corrections from the coupling to the other transfer channels. For the \(1/2^+\) state \((E_x=1.2 \text{ MeV})\), the calculated transfer \(i_{T_{11}}\) fig. (3.9) obtained by the full coupled channel calculation in the absence or deuteron spin-orbit provides good fitting with the experimental data, in particular at angles greater than \(60^0\). For the other \(1/2^+\) state \((E_x=2.56 \text{ MeV})\) such effect is very small throughout the angular range. Like the findings from the case of \((d,p)\) on \(^{208}\text{Pb}\), our calculations show that the calculated \(i_{T_{11}}\) with \(l \neq \emptyset\) are reasonably large in magnitude and predominantly of one sign. For transitions with \(l = \emptyset\), the vector analyzing power arises primarily from the spin-orbit terms in the proton-nucleus and deuteron-nucleus potentials. Since the deuteron energy is below the Coulomb barrier, the spin-orbit potentials have a little influence, and the \(l = \emptyset\) vector analyzing powers are rather small in low energy deuteron scattering.

In conclusion, we can say that both the spin-orbit and the transfer channel coupling effects play comparable parts in reproducing the elastic vector analyzing power in the low energy deuteron scattering.
4.1 Nuclear-Spin Polarized Plasma

In fusion reactions, light ions are brought together sufficiently energetically to overcome their natural repulsion. At the energies required for fusion the atoms are fully ionised, creating what is called a 'plasma'. The light ions fuse to yield larger ions, releasing energy in the process. Most fusion research aims to create the conditions under which the deuteron and $^3\text{H}$ ($^3\text{He}$) can fuse to yield neutron (proton) and an ion of $^4\text{He}$. All particles created in a single fusion reaction carry energy that can be turned into heat and, ultimately, into electricity.

Recent technological development [Bh 82, Si 80] has made possible the generation of polarized gases in quantities of practical interest for the production of polarized fusion plasmas. If one could keep a fusion plasma highly polarized long enough, several very desirable consequences could be exploited. Kulsrud et al [Ku 82] set out to calculate the depolarization rates that would result from various mechanisms (such as collisions, spatial field variation and ionizations and recombinations) in a toroidal or mirror fusion reactor, and to examine the benefits one could expect from various polarization schemes. The results of these calculations showed that the
depolarization time is estimated to be longer than the reaction time.

The benefits to be expected from the polarization of the nuclei participating in plasma fusion reactions are (i) the enhancement of the desired fusion cross sections (ii) the suppression of unwanted reactions, and (iii) the control of the direction of emergence of fusion products. In most fusion reactions of light nuclei, some spin states contribute much more strongly to the cross section than others. Examples are given for the reactions $^3\text{H}(d,n)^4\text{He}$ and $^3\text{He}(d,p)^4\text{He}$ [Ku 83, Ta 82]. The cross sections of these reactions at low energies are largest at the $j=3/2^+$ resonances of the compound nucleus $^5\text{He}$ and $^5\text{Li}$. At such low energies (107 keV for the 1st reaction and 430 keV for the 2nd), the reactions occur only in the $l=0,1$ states, so that the angular momentum must be supplied by the spin of the deuteron and $^3\text{H}$ ($^3\text{He}$) nuclei. Since deuteron has spin-1 and $^3\text{H}$ ($^3\text{He}$) spin-1/2, their possible combined spin states are $s=3/2$ and 1/2. The reaction is due almost entirely to interacting pair of deuteron and $^3\text{H}$ ($^3\text{He}$) nuclei with $s=3/2$. Hence if one polarizes the spin of the incoming nuclei to be parallel ($s=3/2$), these two reaction cross sections will be enhanced by 50%.

The $d-^3\text{H}$ fusion is not ideal, because the deuteron fuses with $^3\text{H}$ to generate $^4\text{He}$ plus a hot, highly energetic, neutron. The neutrons would bombard the reactor walls, making them radioactive and causing structural damage. Furthermore the triton presents radiation hazards. The ideal reaction would be fusing a deuteron with $^3\text{He}$ to produce $^4\text{He}$ plus a hot proton. Since protons are electrically charged, they could probably be used to generate electricity directly, without the
thermodynamic and mechanical losses inherent in using steam. Nor would there be the heavy neutron production of the other reaction. Another advantage of d-{}^3\text{He} reaction is that the reaction rate has a maximum (around 200 keV) which is about a factor of 5 higher than the maximum for the d-{}^3\text{H} reaction rate[Go 87].

The effect of the spin polarization on fusion does not end with the d-{}^3\text{He} reaction. This may be the first fusion process used in the reactor, but there can also be fusion between deuterons alone. The deuteron-deuteron fusion with its two branches

\[
d + d \rightarrow {}^3\text{He} + n
\]

\[
\rightarrow {}^3\text{H} + p
\]

produces neutrons as well as radioactive tritons, both of which are undesirable. At energies relevant to a fusion reactor, the d-d reaction rate is higher than the d-{}^3\text{He} reaction rate [Go 87]. Thus, if the fusion of two deuterons and its undesired neutron production could be suppressed, then the idea of 'neutron-free' d-{}^3\text{He} fusion reactor would be likely to work [Kn 85]. A straightforward way to accomplish this might seem to be to polarize the deuterons such that they have parallel spins [Ko 48]. Historically [Be 50, Ro 61] this polarized d(d,n)\text{He} reaction cross section \(\sigma_1\) (at low energies) is thought to be suppressed when the deuterons are polarized in parallel direction (i.e. \(s=2\)), because the Pauli principle prevents the two deuterons from approaching each other to initiate the reaction. In addition, since the total spin of the final state n+\text{He} is \(s=0\) or \(1\), conservation of
angular momentum and parity dictates that the exit channel must be in a relative D-wave which yields additional suppression.

Recent calculations carried out by Hofmann and Fick [Ho 84] within the Resonating Group Model (RGM) concluded however that, in the centre of mass energy range 20-150 keV, this s=2 entrance channel amplitude is not suppressed and that $\sigma_{\text{11}}$ is approximately equal to the corresponding unpolarized cross section $\sigma_0$ at the lowest energy considered, 20 keV. Further, they suggest that the large s=2 amplitude has its origin in the intrinsic D-state component in the $^3$He ground state (the only internal D-state in their calculation) allowing the strong central component of the nucleon-nucleon interaction to connect the s=2 entrance channel to the final state without spin flip. Their numerical results at 20 keV agree with those of a comprehensive R-matrix analysis of the data [Ha 83]. Despite this agreement, both theoretical approaches had some weak points. The R-matrix analysis did not include the data of Ad'yasevich [Ad 81], and the RGM calculation [Ho 84] did not allow for D-state admixtures in the deuterons.

A very recent calculation [Ho 87, Ha 86] has been done in order to settle the above discrepancies and to explain why the fusion of deuterons is not suppressed in the spin channel s=2. In these calculations [Ho 87] the authors have used realistic nucleon-nucleon forces and three Gaussians in order to describe the internal motion of the deuteron and $^3$He. The deuteron D-states have been included in the calculations. The point of importance in their calculations is whether the s=2 matrix element exists without spin flip or not. The conjecture of Hofmann [Ho 87] is that, due to the D-state admixture in the
deuterons, s=2 matrix elements are possible without requiring a spin flip. They claim that if one of the two deuterons is in its D-state and the other in its S-state with s=2 and zero relative angular momentum (noting that the spins of the individual nucleons are opposite in the two deuterons), then this state does not feel Pauli repulsion, and this leads to no suppression of the fusion of the deuterons.

The results of these calculations [Ho 87] agree totally with the R-matrix analysis [Ha 86] for the S-matrix elements for polarized fusion. Both approaches reproduce the Ad'yasevich experimental data [Ad 81] and predict no suppression of the fusion of deuterons.

The large effect upon the reaction cross section due to the $^3$He D-state is nevertheless contrary to expectations, based upon experience of light particle D-state effects at tandem accelerator energies (few MeV) [Kn 75b]. These higher energy calculations are based almost exclusively upon the distorted waves Born approximation (DWBA) model for the reaction amplitude.

Calculations aimed at assessing the quality of distorted wave calculations at low energies have recently been published by Zhang et al [Zh 86]. These authors conclude that their DWBA model is able to reproduce the measured reaction cross section, and in contradiction with the RGM calculations [Ho 84], they find a very small $^3$He D-state effect (s=2 amplitude) and large suppression of $\sigma_{11}$. However, in common with the RGM calculations [Ho 84], the models used by these authors for nucleon-nucleon interaction $V_{np}$, the deuteron bound state $\phi_d$ and the $d-^3$He overlap function are not realistic. This is
particularly true at large separations of the transferred proton from the cores to which it binds in the initial and the final states. Both analysis use Gaussian potentials and wave functions as the basis of the calculations.

4.2 Formulating The Problem

In view of the conflicting conclusions of the results of the RGM [Ho 84] and DWBA [Zh 86], it is thought that the problem deserves more attention. In order to throw further light on this problem we have chosen to undertake an approximate calculation, without the complexity of the RGM, but which allows us to include both the important aspects of antisymmetrization, treated exactly in the RGM, and an accurate treatment of the structure of the light nuclei, including all internal D-states. The implications of the full antisymmetrization for the DWBA formalism is usually neglected, but this will not be satisfactory in the present context.

In chapter 2, an appropriate one-step DWBA amplitude was presented, in which the distorted waves are generated by optical potentials which include spin-orbit coupling terms (spin-orbit representation). In this chapter, our formalism of one-step DWBA will be described in the channel-spin representation. Following the notations of chapter 2, the coupling schemes for this representation are given by

\[ s_1 + a = s, \quad \text{and} \quad l + s = J \]  \hspace{1cm} (4.1)

An exact expression for the transition amplitude for \( A(d,n)B \)
reaction can be written as

\[ T_{\sigma_2 \sigma_1 \alpha} (k_2, k_1) = \langle \chi_{\sigma_2 \sigma_1 \alpha}^{(-)} (k_2, r_2), \chi_{\sigma_1 \alpha}^{(-)} (1, \ldots, N-1) n | W | \psi_{\sigma_1 \alpha}^{(+)} (k_1, \ldots, N), \xi \rangle \]  \hspace{1cm} (4.2)

where \( N \) is the number of nucleons in the d-A system and \( n \) denotes the capture neutron. The transition interaction \( W \) is given by

\[ W = V_{nA} + V_{np} - V(r_2) \]  \hspace{1cm} (4.3)

where \( V_{nA} - V(r_2) \) is usually neglected in the DWBA calculations of stripping and pick-up processes on heavy targets.

In eq. (4.2), \( \chi^{(-)} \) is a final state distorted wave for n-B system generated by the potential \( V(r_2) \), and \( \psi^{(+)} \) is the exact antisymmetrized wave function for the d-A system. The channel variables \( r_1 \) and \( r_2 \) are defined in chapter 1 (egs. 1.2,3). To the extent that the RGM wavefunction is a good approximation of the system, it can replace \( \psi^{(+)} \) on the right hand side of the matrix element (4.2). In the following \( \psi^{(+)} \) will be replaced by its elastic component \( \chi^{(+)} \), defined by

\[ \psi_{\sigma_1 \alpha}^{(+)} (k_1 (1, \ldots, N), \xi) = \mathcal{A} \left[ \chi_{\sigma_1 \alpha}^{(+)} (k_2 (1, 2 \ldots, N), \xi) \right] \]  \hspace{1cm} (4.4)

If the neutrons and protons are treated as distinguishable particles, then the normalized antisymmetrizer \( \mathcal{A} \) [Au 70] will have the form
\[ \mathcal{A} = \sum_{n}^{N} \sum_{p}^{N} (-)^{P_{n}P_{p}} \frac{(-1)^{2}}{2} \frac{(-1)^{2}}{2} \]  

where \( N_{n} \) and \( N_{p} \) are the total number of permutations of the neutrons and protons respectively.

\[ N_{n} = \frac{(\text{no. of neutrons in } A +1)!}{1! \times (\text{no. of neutrons in } A)!} \]  

\[ N_{p} = \frac{(\text{no. of protons in } A +1)!}{1! \times (\text{no. of protons in } A)!} \]

and \((-)^{P_{i}}\) is positive if the permutation operator \( P_{i} \) exchanges an even number of particles between \( d \) and \( A \), and it is negative if \( P_{i} \) exchanges an odd number of particles. \( \mathcal{A} \) antisymmetrizes with respect to neutrons and protons, and

\[ \chi^{(+)\sigma_{1}^{\alpha_{1}}(12),(3,..N),\xi_{1}} = \sum \chi^{(+)\sigma_{1}^{\prime\alpha_{1}^{\prime}},\sigma_{1}^{\prime\alpha_{1}^{\prime}}, (12), \Phi_{\sigma_{1}^{\prime}}, (3,..N) \]  

where
\[
\chi^{(+)}_{\sigma_l^1 \alpha' \sigma_l^1}(k, r) = \frac{4\pi}{k r} \sum \langle s_1 \sigma_1 \alpha \mid s \sigma \rangle <l m s' \sigma' \mid JM>
\]

\[
= \langle s_1 \sigma_1 \alpha' \mid s' \sigma' \rangle <l m s' \sigma' \mid JM>
\]

\[
i = \text{e}^{i \sigma} \left[ Y_{lm}(k) \right] _{\sigma'} \left[ Y_{lm}(k) \right] _{\sigma} \hat{U}^J_{l's'1s}(k, r)
\]

(4.8)

Here \( m' = m + \sigma' = m + \alpha + \sigma_l - \sigma_l' \), and the sum is over \( l, l', s, s', J, \) and \( M \).

The time reversed solution \( \chi^{(-)} \) can be written in the same form as \( \chi^{(+)} \) with

\[
\chi^{(-)}_{\sigma_l^1 \alpha' \sigma_l^1}(k, r) = (-)^{\alpha - \sigma_l - \alpha' - \sigma_l'} \chi^{(+)}_{-\alpha' - \sigma_l' - \alpha - \sigma_l'}(-k, r)
\]

(4.9)

Coupled equations for the radial functions \( \hat{U}^J_{l's'1s}(k, r) \) may be derived as shown in section 2.2, they are

\[
\frac{d^2}{dr^2} \left[ \frac{d^2}{dr^2} + \frac{l'^2(l'^2 + 1)}{r^2} - \frac{2\mu}{\hbar^2} (l's'J \mid V \mid l's'J) \right] \hat{U}^J_{l's'1s}(k, r)
\]

\[
= \sum \frac{2\mu}{\hbar^2} (l's'J \mid V \mid l''s''J) \hat{U}^J_{l''s''1s}(k, r)
\]

(4.10)

Outside the range of the potential the radial functions \( \hat{U}^J_{l's'1s}(k, r) \)

will have the form
$$U_{l's'ls}(kr) = \frac{i}{2} e^{i\sigma_1} \left[ H_1(kr) \ast - S^J_{l's'ls} H_1(kr) \right]$$  \hspace{1cm} (4.11)$$

where $H_1$ and $H_1^*$ are the outgoing and the ingoing Coulomb functions. $S^J_{l's'ls}$ are the scattering matrix elements. If the interaction in (4.10) is diagonal in $s$ and $l$, and does not depend on $J$, then

$$S^J_{l's'ls} \rightarrow S_{ls} \delta_{l'l} S_{s's} \hspace{1cm} (4.12)$$

The techniques which have been used to solve eq. (4.10) with the assumption (4.12) will be presented in chapters 5 and 6.

The transition amplitude (4.2) will have the form

$$T_{\sigma' \beta' \sigma_1 \alpha}(k_2, r_2) = \sum \int \delta \int \chi_{\sigma_2 \beta' \sigma_2 \beta}(k_2, r_2)$$

$$\langle s_2 \sigma_2 \beta' | V_{np} | s_1 \sigma_1 \alpha \rangle \chi_{\sigma_1 \alpha \sigma_1 \alpha}(k_1, r_1) \hspace{1cm} (4.13)$$

with the sum over $\sigma_1 \ldots, \sigma_1', \alpha', \beta'$. The form factor $\langle s_2 \sigma_2 \beta' | V_{np} | s_1 \sigma_1 \alpha \rangle$ has been discussed in section 2.1.1.

The differential cross section can be written as
\[
\frac{d\sigma}{d\Omega} = \frac{\mu_1 \mu_2 k_2 N_2}{(2\pi \hbar)^2 \alpha k_1 N_1 (2s_1+1)(2a+1)} \sum_{\sigma_2 \beta \sigma_1 \alpha} |T_{\sigma_2 \beta \sigma_1 \alpha}(k_2' k_1')|^2 \quad (4.14)
\]

where \( N_2 \) is the number of permutation arises from antisymmetrizing the final state and \( N_1 = N_n N_p \).

### 4.3 Model For \( d(d,n) \) He Reaction

In the case if the reaction is initiated with two identical particles, then eq. (4.4) becomes

\[
\Upsilon^{(+)}_{\sigma_1 \alpha}(k_1(1234), \xi_1) = \mathcal{A} \left[ \chi^{(+)}_{\sigma_1 \alpha}(k_1(12)(34), \xi_1) \right] \quad (4.15)
\]

where

\[
\mathcal{A} = (1-\rho_{13})(1-\rho_{24})/2 \quad (4.16)
\]

\( \mathcal{A} \) antisymmetrizes with respect to neutrons \((1,3)\) and protons \((2,4)\), and

\[
\chi^{(+)}_{\sigma_1 \alpha}(k_1, (12)(34), \xi_1) = \sum_{\sigma_1' \alpha'} \chi^{(+)}_{\sigma_1' \alpha'}(k_1, \xi_1) \Phi^{(12)} \Phi^{(34)} \quad (4.17)
\]
The symmetrized transition amplitude is thus (for d-d initial state)

\[
T^{\text{sym}}_{\sigma_2\sigma_1}(k',k) = T_{\sigma_2\sigma_1}(k',k) + T_{\sigma_2\sigma_1}(k',-k) \quad (4.18)
\]

Each amplitude in the right hand side of eq. (4.18) is given by (4.13).

The differential cross section in terms of the symmetrized transition amplitude (4.18) is given by (4.14), with

\[
\frac{d\sigma}{d\Omega} = \frac{\mu_1 \mu_2}{(2\pi \hbar^2)^2} \frac{k_2 N_2}{k_1 N_1} \frac{1}{(2s_1+1)^2} \sum_{\sigma_2\sigma_1} |T^{\text{sym}}_{\sigma_2\sigma_1}(k',k)|^2 \quad (4.19)
\]

If the reaction is initiated with polarized deuterons, then the polarized cross section will be given by eq. (1.31), with \(\sigma_0 = d\sigma/d\Omega\) of eq. (4.19).

4.4 Describing The Calculations

The constituents of the transition amplitude (4.13) can be described as follow

4.4.1 The Distorted Waves

A model has been used in which the initial state is described by distorted wave generated by
a) Non-local but separable potential (see chapter 5)
b) Orthogonality condition model (see chapter 6)
c) A phenomenological deuteron optical potential

In both cases (a) and (b), the parameters of the potentials were chosen to reproduce the d-d phase shift of the one-channel RGM calculation of Thompson [Th 70] for centre of mass energies of 0-20 MeV. The phase shifts are tabulated by Thompson according to d-d relative orbital angular momentum l and the channel spin s.

In case (c), we did find a deuteron optical potential of the form (3.9), which fits the d-d elastic scattering at centre of mass energy of 12.65 MeV [Va 63a]. The parameters of this potential are listed in table 4.1. Fig.(4.1) shows the comparison between the calculated cross section (solid curve) and the data [Va 63a].

This local deuteron optical potential has been used to generate the distorted wave in the incident channel at very low energy ($E_{CM} = 55$ keV). In all cases, the Coulomb interaction for finite size deuterons is included. The idea of using the local deuteron potential in these calculations is to assess the results due to the use of both non-local separable potentials and the orthogonality condition model. These results will be discussed in chapter 7.

The neutron-He final state distortion was calculated using two different neutron optical potentials. The first potential (Pot. 1) was taken from the optical model analysis of neutron-triton scattering by Sherif and Podmore [Sh 72]. The second potential (Pot. 2) was chosen to reproduce low energy n-He scattering data (cross section and
polarization) [Se 72]. The neutron potentials used here have the same form as equation (3.9). The parameters of these potentials are listed in table 4.2.

The results of the calculation of the n-^3^He elastic scattering are presented in fig.(4.2,3). The solid curves represent the calculated elastic differential cross section, fig.(4.2), and polarization, fig.(4.3), using Pot. 1. The dashed curves describe the same observables due to Pot. 2. The experimental data (differential cross section and polarization) were taken from the references [Sa 61, Se 72, Ka 86].

Since the model (Pot.2) has proved successful in fitting the cross section and polarization at one energy (6 MeV), one might ask the question: is this model able to fit the n-^3^He data over a range of energies with reasonable energy dependence of the parameters? To answer this question we have looked to the optical model analysis of Sherif and Podmore [Sh 72] which fits the n-^3^He elastic scattering data in the energy range from 6 to 23 MeV (six energies have been considered). An inspection of this analysis reveals the following trends in the potential depths.

a) The real depth decreases with increasing energy, which is consistent with results obtained for heavier targets [Ho 71]
b) The imaginary well depth increases with increasing energy.
c) The spin-orbit depth appears to increase with increasing energy.

The same trends have been applied to our model (Pot.2) to fit the n-^3^He cross section at 21 MeV. The resulting potential is listed in
table (4.2) as Pot.4. The agreement between the model and the data [Se 72] is satisfactory (dashed curve in fig. (4.4)). The solid curve in fig. (4.4) shows the result due to Pot.3 [Sh 72].
Table (4.1)

Deuteron Optical Potentials at $E_d = 25.3$ MeV

<table>
<thead>
<tr>
<th>Pot.</th>
<th>$V_0$ (MeV)</th>
<th>$r_0$ (fm)</th>
<th>$a_0$ (fm)</th>
<th>$W_D$ (MeV)</th>
<th>$r_W$ (fm)</th>
<th>$a_W$ (fm)</th>
<th>$r_C$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>31.0</td>
<td>1.587</td>
<td>0.87</td>
<td>14.526</td>
<td>0.173</td>
<td>1.051</td>
<td>1.3</td>
</tr>
</tbody>
</table>
Table (4.2)

Neutron Optical Potentials

<table>
<thead>
<tr>
<th>Pot.</th>
<th>$E_n$ (MeV)</th>
<th>$V_0$ (MeV)</th>
<th>$r_0$ (fm)</th>
<th>$a_0$ (fm)</th>
<th>$W_D$ (MeV)</th>
<th>$r_W$ (fm)</th>
<th>$a_W$ (fm)</th>
<th>$V_{SO}$ (MeV)</th>
<th>$r_{SO}$ (fm)</th>
<th>$a_{SO}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pot.1</td>
<td>6</td>
<td>50.35</td>
<td>1.488</td>
<td>0.144</td>
<td>0.52</td>
<td>1.501</td>
<td>0.378</td>
<td>1.73</td>
<td>1.049</td>
<td>0.289</td>
</tr>
<tr>
<td>Pot.2</td>
<td>6</td>
<td>135.50</td>
<td>1.698</td>
<td>0.700</td>
<td>7.25</td>
<td>1.501</td>
<td>0.378</td>
<td>5.25</td>
<td>1.049</td>
<td>0.289</td>
</tr>
<tr>
<td>Pot.3</td>
<td>21</td>
<td>42.00</td>
<td>1.488</td>
<td>0.144</td>
<td>6.19</td>
<td>1.501</td>
<td>0.378</td>
<td>3.77</td>
<td>1.049</td>
<td>0.289</td>
</tr>
<tr>
<td>Pot.4</td>
<td>21</td>
<td>115.50</td>
<td>1.698</td>
<td>0.700</td>
<td>39.5</td>
<td>1.501</td>
<td>0.378</td>
<td>9.85</td>
<td>1.049</td>
<td>0.289</td>
</tr>
</tbody>
</table>
Fig. (4.1): The differential cross section for d-d elastic
scattering at 12.65 MeV calculated with the deuteron optical potential
of table 4.1. The experimental data is taken from [Va 63a].
Figs. (4.2,3): The differential cross section and polarization for n-^3^He elastic scattering at 6 MeV. The solid curves show the results when Pot.1 (table 4.2) is used in the calculations, while the dashed curves show the results due to Pot.2 (table 4.2). The experimental data are taken from [Sa 61, Ka 86].
Polarization

Fig. (4.2)

Fig. (4.3)
Fig. (4.4): The differential cross section for $n^3\text{He}$ elastic scattering at 21 MeV. The solid and dashed curves show the results of using Pot.3 and Pot.4 (table 4.2) respectively.
4.4.2 The Transition Interaction

The contribution of $V_{np}$ in $W$ (see eq. 4.3) to the transition matrix is well-adapted to a full finite-range treatment, including D-state effects, using techniques developed for DWBA calculations of (d,p) reactions. In this term all contribution can be expressed in terms of the ground state wave functions of the deuterons and $^3$He. The remaining contribution to $W$ from $V_{nA} = V_{13} + V_{23} - V(r_2)$ are usually neglected in DWBA (where 1 and 3 refer to neutrons and 2 and 4 refer to protons). Although we have no quantitative justification for this step in the present context, it should be born in mind that our primary purpose is to assess the role of different models for the internal deuteron and $^3$He wave function in the transition matrix element. This can be achieved in a well-tested and understood framework for the $V_{np} (= V_{34})$ terms but not so easily for the others. The resulting amplitude actually calculated here has obvious connection with that calculated by Zhang et al. [Zh 86], although their work does not make clear which of the terms of $W$ they have included.

* 4.4.3 The Overlap Functions

To discuss the accuracy of the absolute value of the calculated

* In this section, $s, \sigma$ will be used to denote the spin and the projection of $^3$He. $r_1$ and $r_2$ will define the relative n-p distances in each of the two deuterons.
cross section for the transfer reactions induced by light ions, it is essential to employ realistic wave functions for the light ions and interactions which cause the transfer. The consistency between the reaction interaction and the interaction used to derive the light ion wave function is particularly important in the case where the interaction has a hard or soft core part [Do 74, Ku 81].

In these calculations the deuteron ground state wave function of the Reid-soft core interaction [Re 68] was used.

The deuteron-He overlap function \( \rho = \frac{r_1 + r_2}{2} \)

\[
\begin{align*}
\langle \rho, \Phi^p_4(4) \Phi_2^m(12) | \Phi_h(124) \rangle \\
= \sum_{L \sigma \sigma \sigma} \left( L \sigma \sigma | \frac{1}{2} \sigma \sigma \right) \left( \text{Im} \frac{1}{2} \sigma \sigma | \sigma \sigma \right) U_L(f) \mathcal{Y}_L(\hat{\rho})
\end{align*}
\]

\[ (4.20) \]

was obtained from a five channel Faddeev calculation for He of Ishikawa, Sasakawa and Sawada [Is 86]
5.1 Definition of the Separable Potential for d-d Reaction

The work presented here concerns the incident channel of the \(d(d,n)^3\text{He}\) reaction. In this channel, the elastic scattering of deuterons by deuterons has been calculated within an approximate RGM calculations. Here, a finite-rank non-local separable potential has been used to represent the d-d scattering. A separable potential has been chosen to simulate in an approximate way the non-local features of the exact RGM kernels.

Let us consider the two body non-relativistic Schrodinger equation

\[
\frac{-\hbar^2}{2m} \nabla^2 \psi (r) + \int \! dr' \langle r | V | r' \rangle \psi (r') = E \psi (r) \quad (5.1a)
\]

where the kinetic energy is

\[
E = \frac{\hbar^2 k^2}{2m}
\]

and \(k\), \(m\) being, respectively the incident momentum and the reduced mass. \(\langle r | V | r' \rangle\) is the nuclear potential which, in general, is non-local. In momentum space eq. (5.1a) can be written as
\[
\frac{\hbar^2 k^2}{2m} \Phi(k) + \int dk' \langle k | V | k' \rangle \Phi(k') = E_0 \Phi(k) \quad (5.1b)
\]

If \( \langle k | V | k' \rangle \) depends only on \( k-k' \), then the potential is local, i.e \( \langle r | V | r' \rangle \) must be a function of \( r \) multiplied by \( \delta(r-r') \). Otherwise the potential becomes necessarily non-local.

A rank-N separable potential which acts on states of orbital angular momentum \( l=0, 1, 2, \ldots L \) in the spin channels \( s=0, 1, 2, \ldots \), has the form

\[
V = \sum_i \lambda_{i l s l' s'} \langle l m s | \sigma \rangle \langle J M | (1' m' s' \sigma') | J M \rangle \\
\langle \psi_{i l s l' s'} | V | \psi_{i l s l' s'} \rangle (5.2)
\]

with the summation over \( l, s, l', s', J, m', \sigma', m, \sigma \), \( M \). \( \lambda_{i l s l' s'} \) are constants depending on both \( l \) and \( s \). If \( V \) is independent of \( J \), and \( \lambda \) is diagonal in \( l \) and \( s \) then

\[
V = \sum_{i l s \sigma} \lambda_{i l s} \langle \psi_{i l s} | V | \psi_{i l s} \rangle (5.3)
\]

This simplification is sufficient for the purpose of simulating the RGM calculations of Thompson [Th 70].

The ket in eq. (5.3) has the form
$$\langle r | V |_{l, s, m, \sigma} = V_{l, s, m, \sigma} \chi_{s, \sigma}$$  \hspace{1cm} (5.4a)$$
or alternatively

$$\langle k | V |_{l, s, m, \sigma} = V_{l, s, m, \sigma} \chi_{s, \sigma}$$  \hspace{1cm} (5.4b)$$

5.2 Analytic Solution For The Scattering Problem

5.2.1 Rank-1 Nonlocal Separable Potential

Assuming the potentials given in eqs. (5.1a,b) are rank-1 separable potentials which act only in S-waves then according to (5.4a,b), these potentials can be written as

$$\langle r' | V | r \rangle = \lambda V(r)V(r')$$  \hspace{1cm} (5.5a)$$
or equivalently

$$\langle k' | V | k \rangle = \lambda V(k)V(k')$$  \hspace{1cm} (5.5b)$$

where $V$ is assumed to be real and $\lambda$ is negative for an attractive potential, and

$$V(k) = \int dr e^{ik \cdot r} V(r) = \frac{4\pi}{k} \int dr r \sin(kr) V(r)$$  \hspace{1cm} (5.6)$$

Substitute from (5.5b) into (5.1b) yields
\( (E_0 - \frac{\hbar^2 k^2}{2m}) \Phi(k) = \lambda V(k) \int dk' V(k') \Phi(k') \)

\[ = C \cdot V(k) \quad (5.7) \]

where

\[ C = \lambda \int dk' V(k') \Phi(k') \quad (5.8) \]

In such a case, one can easily find the exact solution for the scattering case

\[ \Phi(k) = N \delta(k-k_o) + \frac{CV(k)}{E_0 - E + i\epsilon} \quad (5.9) \]

where \( N \) is the normalization constant of the incident plane wave, \( E = \hbar^2 k^2 / 2m \) and \( \epsilon \) is a real positive infinitesimal quantity which makes the scattered wave of \( \Phi(k) \) only an outgoing wave.

Then from (5.8) and (5.9)

\[ C = \frac{N \lambda V(k_o)}{\{1 - \lambda \int dk V^2(k)/(E_0 - E + i\epsilon)\}} \quad (5.10) \]

In configuration space

\[ \Psi(r) = \frac{1}{(2\pi)^{3/2}} \int dk e^{i\mathbf{k} \cdot \mathbf{r}} \Phi(k) \quad (5.11) \]

and from (5.9)
\[ \psi(r) = e^{i k_0 \cdot r} + \frac{C}{(2\pi)^{3/2}} \int \frac{dk}{E_0 - E + i\epsilon} \frac{V(k)}{\frac{i k \cdot r}{E_0 - E + i\epsilon}} \]

\[ = e^{i k_0 \cdot r} + \frac{C}{(2\pi)^{3/2}} \int \frac{dk}{E_0 - E + i\epsilon} \frac{e^{i k \cdot (r - r')}}{E_0 - E + i\epsilon} V(r') \quad (5.12) \]

where we have chosen \( N = (2\pi)^{3/2} \)

The \( k \) integral in eq. (5.12) is standard and gives

\[ \frac{1}{(2\pi)^3} \int \frac{dk}{E_0 - E + i\epsilon} \frac{e^{i k \cdot (r - r')}}{E_0 - E + i\epsilon} \rightarrow \frac{2m}{\hbar^2} \left( \frac{-1}{4\pi} \right) \frac{e^{i k_0 |r - r'|}}{|r - r'|} \quad (5.13) \]

and the function (5.12) becomes for \( r \rightarrow \infty \)

\[ \psi(r) = e^{i k_0 \cdot r} + \frac{\frac{2m}{\hbar^2} \left( \frac{-1}{4\pi} \right) e^{i k_0 |r - r'|}}{r} \{ \frac{2m}{\hbar^2} \left( \frac{-1}{4\pi} \right) CV(k') \} \quad (5.14) \]

Eq. (5.14) defines the scattering amplitude \( f(k_0 \rightarrow k'_0) \) as the amplitude of the outgoing wave in the outgoing channel induced by a plane wave of unit amplitude in the incident channel. If the initial and the final channels are the same, then \( f \) refers to the elastic scattering. An expression for \( f \) can be obtained from (5.10) and (5.14)
An exact expression for the phase shift $\delta$ can be obtained in two different ways,

i) - in terms of the partial-wave expansion of the scattering amplitude

\[
 f(\theta) = \frac{1}{2i k_0} \sum_{l} (2l+1) (e^{2i \delta_l - l}) P_l(\cos \theta) \tag{5.16}
\]

ii) - in terms of the transition amplitude, where the phase shift is defined as [Mo 68]

\[
 \tan \delta_1 = \frac{\text{Im} T_1(k_0)}{\text{Re} T_1(k_0)} \tag{5.17}
\]

An expression for the S-wave phase shift can be obtained by considering the assumption (5.5b), where $V(k) = 1/(k^2 + \gamma^2)$ [Ya 54], in (5.15a) and matching the result with (5.18) for $l=0$, we get

\[
k_0 \cot \delta_0 = \frac{1}{2 \gamma} (k_0^2 - \gamma^2) - \frac{\hbar^2}{2m} \frac{1}{\lambda \Pi} (k_0^2 + \gamma^2)^2 \tag{5.18}
\]
5.2.2 Rank-2 nonlocal separable potential

It has been found that expression (5.18) did not reproduce the deuteron deuteron phase shift [Th 70] for centre of mass energy from 0-20 MeV. In an attempt to obtain more precise expression for the phase shift in this case (l=0), a rank-2 separable potential is considered.

Mongan [Mo 68, Mo 69] has proposed three separable potential models of the nucleon-nucleon interaction (cases I-III [Mo 69]). These potentials are rank-2 separable interactions of the form

\[
V (k,k') = g (k)g (k') - h (k)h (k')
\]

parametrized in the momentum space for each l, s state of the two nucleon system. g and h are the repulsive and the attractive parts of the interaction.

The exact solution of (5.1b), with the potential given by (5.19), yields [Mo 68]

\[
T (k \rightarrow k') = N (k \rightarrow k')/D (k)
\]

where
\[ N_1(k_\rightarrow k') = g_1(k_\rightarrow k') \left[ 1 + \frac{2m}{\hbar^2} \int \frac{dk k^2 g_1^2(k)}{k_0^2 - k^2 + i\varepsilon} \right] \]

\[- h_1(k_\rightarrow k') \left[ 1 - \frac{2m}{\hbar^2} \int \frac{dk k^2 g_1^2(k)}{k_0^2 - k^2 + i\varepsilon} \right] \]

\[ - \left\{ g_1(k_\rightarrow k') h_1(k_\rightarrow k') + h_1(k_\rightarrow k') g_1(k_\rightarrow k') \right\} \]

\[ \frac{2m}{\hbar^2} \int \frac{dk k^2 g_1^2(k) h_1(k)}{k_0^2 - k^2 + i\varepsilon} \]  \( (5.20b) \)

and

\[ D_1(k_0) = \left[ 1 - \frac{2m}{\hbar^2} \int \frac{dk k^2 g_1^2(k)}{k_0^2 - k^2 + i\varepsilon} \right] \]

\[ \left[ 1 + \frac{2m}{\hbar^2} \int \frac{dk k^2 h_1^2(k)}{k_0^2 - k^2 + i\varepsilon} \right] \]

\[ + \left[ \frac{2m}{\hbar^2} \int \frac{dk k^2 g_1(k) h_1(k)}{k_0^2 - k^2 + i\varepsilon} \right]^2 \]  \( (5.20c) \)

To get an exact expression for the S-wave phase shifts, the case II parametrization [Mo 68] is the most suitable case in treating \( l=0 \) states. Explicitly, for the case II the form factors \( g_1, h_1 \) are
where $c_{r,0s}^\prime$, $a_{r,0s}$ are the parameters which describe the repulsive term, and $c_{a,0s}^\prime$, $a_{a,0s}$ are the corresponding parameters for the attractive part. These parameters are dependent on $l$ and $s$. The subscripts $A$ and $R$ refer respectively to the attractive and the repulsive parts of the interaction.

Substituting from (5.21) and (5.20) into (5.17), yields

$$g(k) = \frac{c_{r,0s}^\prime}{(k + a_{r,0s}^2)}$$

and

$$h(k) = \frac{c_{a,0s}^\prime}{(k + a_{a,0s}^2)}$$

where

$$\tan \delta_{0s} = \frac{B_{0s}(k)}{A_{0s}(k)}$$

$$B_{0s}(k_0) = \frac{2m}{\hbar^2} \frac{\pi k}{2} \left\{ \frac{c_{r,0s}^2(k_0^2 + a_{r,0s}^2)^2}{c_{a,0s}^2(k_0^2 + a_{a,0s}^2)^2} - \frac{c_{r,0s}^2(k_0^2 + a_{r,0s}^2)^2}{c_{r,0s}^2(k_0^2 + a_{r,0s}^2)^2} \right\}$$

$$- \frac{2m}{\hbar^2} \frac{\pi}{4} \frac{C_{a,0s}^2 C_{r,0s}^2}{A_{a,0s} R_{r,0s}} (k_0^2 - a_{r,0s}^2)$$

and

$$\left\{ \frac{a_{a,0s}^2}{a_{a,0s}^2 R_{r,0s}} \right\}^2 \left\{ \frac{a_{a,0s}^2 R_{r,0s}}{a_{a,0s}^2} \right\}^2$$

(5.22b)
In a similar manner an exact expression for the P-waves phase shift can be obtained. The case I parametrization [Mo 68] is the suitable one for the derivation. For the case I, the form factors are given by

\[ A_{0s}(k) = \frac{2m}{\hbar^2} \pi \left\{ \frac{c^2}{A_{0s}} \left( k^2 - a^2 \right) \left( k^2 + a^2 \right) / a_{R,0s} \right\} \]

\[ - c^2 \left( k^2 - a^2 \right) \left( k^2 + a^2 \right) / a_{R,0s} \}

\[ - \left( \frac{2m}{\hbar^2} \pi \right) \left( c_{A,0s} c_{R,0s} \right)^2 \left( a_{A,0s} a_{R,0s} \right) / \left( a_{A,0s} a_{R,0s} \right) \}

\[ + \left( k^2 + a^2 \right) \left( k^2 + a^2 \right) / \left( k^2 + a^2 \right) \]

\[ \tan \frac{\delta}{\pi} = \frac{B(k)}{A_{0s}(k)} \]  

Collecting the results of (5.23), (5.20) and (5.17), we get

\[ \tan \frac{\delta}{\pi} = \frac{B(k)}{A_{0s}(k)} \]

Where
Expressions (5.22) and (5.24) describe the S and P-waves phase shifts. The interaction used in deriving these expressions was rank-2 non-local separable interaction of the form (5.19). In case of D-wave \((l=2)\), it has been found that a rank-1 non-local separable interaction (the attractive part of (5.19)) is sufficient to describe the scattering in this state. The chosen form factor in this case has the form
From (5.15), (5.25) and (5.17), the expression for D-waves phase shifts has the form

\[
\tan \delta_{2s} = \left\{ \frac{2 \hbar^2}{\pi m} \frac{c_{A,2s}}{a_{A,2s}} \left( k + a_{A,2s} \right)^2 \right\} \left\{ \frac{1}{16k^5 a_{A,2s}^5} \right\} + \left( 5k - 15k^3 a_{A,2s}^2 - 5k^2 a_{A,2s}^4 - a_{A,2s}^6 \right) \left( 16k^5 a_{A,2s}^5 \right) 
\]

5.3 Fits to Deuteron-Deuteron Scattering Data

The expressions (5.22, 24, 26) have been used in fitting d−d phase shifts of Thompson [Th 72] for centre of mass energies from 0–20 MeV. The fitting was done using a least square minimization program. For each l,s combination, we used 11 data points. The program searched for the values of the interaction parameters which minimize the sum of the square of the residuals

\[
\sum_{R}^2 = \sum_{i=1}^{11} \left[ \sum_{l,s}^{\text{exp}} (E_i) - \sum_{l,s}^{\text{cal}} (E_i) \right]^2 (5.27)
\]

All our fits were made with the basic separable potential form (5.19). The forms of the separable potentials were chosen so that it leads to a scattering amplitude, that satisfies some conditions, one of these
conditions is, the off-energy shell amplitude must reduce to the correct on-shell amplitude. The potential parameters for each value of 1(=0,1,2) in every spin channel s(=0,1,2) are listed in table 5.1. These parameters have been used to calculate the phase shifts given by expressions (5.22,24,26). The calculated phase shifts for each 1 and s (solid curves) together with Thompson phase shifts [Th 72] (cross points) are shown in figs.(5.1-5)

5.4 Numerical Solutions

In our formalism of solving the scattering problem of d-d analytically we have used a real nuclear potential, which has been considered as non-local separable potential only, so that the resulting phase shifts are real. The RGM phase shift, we have fitted in section 5.3, were due to a real nuclear plus Coulomb potentials. We have therefore inserted a Coulomb potential in our formalism (5.1). That is in addition to the presence of the non-local separable potential. The equation we have to solve is then

\[(E - T - V - U_{Coul}) \psi = 0\]  \hspace{1cm} (5.28)

In section 5.2, analytic solutions of the second order differential equations were obtained, but many differential equations are difficult to solve analytically. In such cases, for example (5.28), it is necessary to turn to numerical methods of solutions.
5.4.1 Rank-1 Separable Interaction

In order to clarify our technique we first describe a numerical approach to the simplest case. The fact that this case can be solved analytically also gives a good check of the numerical methods.

In this case the potential $V(r)$ is non-local separable of rank-1. The corresponding Schrödinger equation for a given $l$ and $s$ may be written explicitly as

$$ (E - T)U_1(r) = CV_1(r) $$  \hspace{1cm} (5.29)

where

$$ C = \lambda_1 \int dr' r' V_1(r')U_1(r') $$  \hspace{1cm} (5.30)

Considering eq. (5.29) with $C=1$, and call $U_1(r)$ to be any solution of it, which is regular at $r=0$, i.e

$$ (E - T_1)\bar{U}_1(r) = V_1(r) $$  \hspace{1cm} (5.31)

$V_1(r)$ is defined by (5.3). A program has been written in order to solve (5.29), using Runge-Kutta method (see appendix C).

From (5.29) and (5.31), we see that

$$ (E - T_1)\{U_1(r)/C - \bar{U}_1(r)\} = 0 $$  \hspace{1cm} (5.32)
i.e., \( U_1(r)/C - \tilde{U}_1(r) \) must be a regular solution of (5.32), i.e

\[
\{U_1(r)/C - \tilde{U}_1(r)\} = A j_1(kr)
\]  

(5.33)

and hence

\[
U_1(r) = C \tilde{U}_1(r) + B j_1(kr)
\]  

(5.34)

where \( C \) and \( B \) are not known (yet).

Substitute (5.34) into (5.30) gives

\[
C = \lambda \int dr' r^2 \frac{V_1(r')}{U_1(r')} \{C \tilde{U}_1(r') + B j_1(kr')\}
\]  

(5.35)

and

\[
\frac{C}{B} = \lambda \int dr' r^2 \frac{V_1(r') j_1(kr')}{\{1 - \lambda \int dr' r^2 \frac{V_1(r') \tilde{U}_1(r')}{U_1(r')}\}}
\]  

(5.36)

The right hand side of eq. (5.36) involves quantities we know. If we define \( \alpha = C/B \), then eq. (5.34) will be written as

\[
U_1(r) = B \{\alpha \tilde{U}_1(r) + j_1(kr)\}
\]  

(5.37)

Now we can determine the phase shift \( \alpha \) and the normalization constant \( B \) by matching between equation (5.37) with the asymptotic solution of (5.29) at two points \( r_1, r_2 > r_{\text{int}} \), in the absence of the Coulomb potential.
\[ f_1 = \{ j_1(kr_1)\bar{u}_1(r_2) - j_1(kr_2)\bar{u}_1(r_1)\} \]

\[ \left[ \alpha_1 \{ \bar{u}_1(r_1)h_1(kr_2) - \bar{u}_1(r_2)h_1(kr_1)\} \right. \]

\[ \left. - \{ j_1(kr_1)h_1(kr_2) - j_1(kr_2)h_1(kr_1)\} \right]^{-1} \] (5.38)

if we put \( [\text{Ab 65}] \)

\[ h_1(kr) = n_1(kr) + ij_1(kr) \]

an expression for the phase shift can be obtained

\[ \cot \delta_1 = \left[ \alpha_1 \{ \bar{u}_1(r_1)n_1(kr_2) - \bar{u}_1(r_2)n_1(kr_1)\} \right. \]

\[ \left. - \{ j_1(kr_1)n_1(kr_2) - j_1(kr_2)n_1(kr_1)\} \right] \]

\[ [ j_1(kr_1)\bar{u}_1(r_2) - j_1(kr_2)\bar{u}_1(r_1) ]^{-1} \] (5.39)

eq. (5.38) ensures that \( \text{Im}(1/f) = -1 \)

5.4.2 Local+Rank-1 Separable Potential

In this case the local potential is considered as a Coulomb, then the Schrödinger eq. (5.28) becomes

\[ \left[ E - T_1 - U_{\text{Coul}}(r) - V_1(r) \right] \psi = 0 \] (5.40)
where

\[ U_{\text{Coul}}(r) = \begin{cases} \frac{ZZ'e^2}{r} & r > R_c \\ \frac{ZZ' e (3 - r^2/R_c^2)}{2R_c^2} & r < R_c \end{cases} \] (5.41a)

\[ = \frac{ZZ'e (3-r^2/R_c^2)}{2R_c^2} \]

\[ r < R_c \quad (5.41b) \]

\[ z \text{ and } z' \text{ are the charges of the colliding nuclei, and } R_c \text{ is the radius of the charge density. The Coulomb potential changes only the outgoing wave. In analogy with 5.4.1, we have} \]

\[ (E - T - U_{\text{Coul}}(r))\bar{u}_0(r) = V_1(r) \]

\[ (E - T - 0_{\bar{u}}(r))\{0_{\bar{u}}(r)/C - \bar{u}(r)\} = 0 \]

so we have

\[ 0_{\bar{u}}(r)/C - \bar{u}(r) = \bar{u}_0(r) \]

\[ \bar{u}_0(r) \text{ satisfies} \]

\[ (E - T - \bar{u}_0(r))\bar{u}_0(r) = 0 \]

proceeding as in section 5.4.1,

\[ \bar{u}_0(r) = CU_1(r) + BU_1(r) \]

and

\[ C/B = \lambda_1 \int dr' r'^2 v_1(r')u_0(r')/\{1 - \lambda_1 \int dr' r'^2 v_1(r')u_1(r')\} \]

and therefore eq.(5.46) will be
\[ U_1(r) = B \beta \bar{U}_1(r) + U_1^0(r) \]  

(5.48)

where \( \beta_1 = C/B \). Now it is possible to calculate both the phase shift and \( B \). This can be done by matching (5.48) with the right hand side of eq.(2.31) at two points outside the range of the nuclear potential. Upon doing that we find

\[
\tan \delta_1 = \left[ \beta_1 \{ \bar{U}_1(r_1)F_1(kr_1) - \bar{U}_1^0(r_1)F_1^0(kr_1) \} \right. \\
+ \left. \{ U_1^0(r_1)F_1^0(kr_1) - U_1(r_1)F_1(kr_1) \} \right] \\
\left[ \beta_1 \{ \bar{U}_1(r_2)G_1(kr_2) - \bar{U}_1^0(r_2)G_1^0(kr_2) \} \right. \\
+ \left. \{ U_1^0(r_2)G_1^0(kr_2) - U_1(r_2)G_1(kr_2) \} \right]^{-1} 
\]  

(5.49)

The numerical method of solving the Schrödinger eq. for rank-2 separable potential and for local+rank-2 separable potentials are presented in appendicies C and D respectively.

5.5 Numerical Calculations

A program has been written in order to solve the Schrödinger equation (5.29) for the sum of a local Coulomb (5.41) and a finite rank non-local separable potential (5.3). We have used this program to calculate the phase shift for each partial-wave (S,P,D) in the spin channels \( s=0,1,2 \). This program has also been used to calculate the d-d
elastic differential cross section, and the distorted wave functions described in the spin channel representation.

To check the program we have done the following:

i) Removing all potentials from the Schrödinger equation, yields for each partial-wave distorted waves given by a Bessel function, and a phase shift of zero.

ii) For each partial wave, if we consider only the non-local separable potential with parameters given in table 5.1, we were able fit Thompson phase shifts \[\text{Th 70}\]. Thus the analytic expressions (5.22,24,26) agree with those calculated numerically.

iii) The distorted waves generated numerically for the non-local separable potential alone are the same as those of expression (5.14) in the asymptotic region.

Since the RGM phase shifts included the effects of both nuclear and Coulomb potentials, we inserted the Coulomb potential in our numerical calculations, in addition to the separable potential, and then repeated the fitting process. The potential parameters which give the best fit with the data \[\text{Th 70}\] are given in table 5.2.

It is clear from tables 5.1 and 5.2, that the inclusion of the Coulomb potential affects the S- and P-wave form factors parameters, especially the coupling strengths \(C_{\text{R,ls}}, C_{\text{A,ls}}\). In the two states \(l=0, s=0\) and \(l=1, s=1\), we found that each state is described by a form factor which has two sets of parameters. The first set of parameters fits the phase shift in centre of mass energy range from 0-8 MeV, while the 2nd set fits from 8-20 MeV. The inclusion of the Coulomb potential in the calculations for the partial wave \(l=2\) does not affect the
parameters of the form factor which describe this state.

5.5.1 Elastic Differential Cross Section

If the two colliding nuclei are unpolarized, then the differential cross section has the form

\[
\frac{d\sigma}{d\Omega} = \sum_{\sigma_1\sigma\alpha} \frac{1}{(2s_1+1)(2\alpha+1)} \sum_{\sigma_1'^*\sigma_1^*} |f_{\alpha'\sigma_1'^*\sigma_1^*}(\theta)|^2
\]  

(5.50)

where

\[
f_{\alpha'\sigma_1'^*\sigma_1^*}(\theta) = f_C(\theta) \sum_{\sigma_1\alpha} \frac{f_{\sigma_1\sigma_1^*}}{(2s+1)(2\alpha+1)} \]

(5.51)

and \(f_C(\theta)\) is the Coulomb scattering amplitude,

\[
f_C(\theta) = \frac{-\text{ln} \sin^2(\theta/2) + 2\text{i}\sigma}{2\text{k}\sin^2(\theta/2)}
\]

(5.52)

\(\sigma_0 = \text{arg} \Gamma(1+\text{i}\sigma)\)

\(f_{\alpha'\sigma_1'^*\sigma_1^*}(\theta)\) is the nuclear scattering amplitude,

\[
f_{\alpha'\sigma_1'^*\sigma_1^*}(\theta) = \frac{1}{2iK} \sum (a\alpha s_1\sigma_1|sm)(1\alpha sm|m) (2l+1)^{1/2} (a\alpha's_1's'|s'm')(l'm-m's'm'|jm)
\]

(5.53)

\[e^{2i\sigma_1 j} (\delta_1's'_1s' - \delta_1's's')(4\pi)^{1/2} Y_{l'm-m'}(\theta)\]

summed over \(l,l',s,s',\) and \(j\)
Since the phase shift depends only on $l$ and $s$, then

$$s_{l's's} = S_{ll's's}e^{2\bar{i}S_{l's}}$$  \hspace{1cm} (5.54)

and

$$\sum_j (10sm|jm)(1m'-sm'|jm) = S_{mm'}$$  \hspace{1cm} (5.55)

Using the results of (5.54,55) into (5.53), yields

$$f_{\alpha'\alpha'}(\theta) = \sum_{\sigma_1} (1\alpha 1\sigma_1|\sigma_1)(1m'1\sigma_1'|\sigma_1)f_s'\theta)$$  \hspace{1cm} (5.56)

where $s_j=a=1$, and

$$f_s'(\theta) = \sum_{\sigma_1} \frac{(2\bar{i}+1)}{2ik} e^{2i\sigma_1} e^{2i\bar{i}_{l's}} P_1(\theta)$$  \hspace{1cm} (5.57)

The cross section (5.50) then becomes

$$\frac{d\sigma}{d\Omega} = \sum_s \frac{(2s+1)}{(2a+1)(2s+1)} \left|f_s'(\theta)+f_s(\theta)\right|^2$$  \hspace{1cm} (5.58)

When the colliding particles are identical, $d-d$ case, then

$$\frac{d\sigma}{d\Omega} = \sum_s \frac{2s+1}{(2a+1)^2} \left|f_s'(\theta)+f_s(\theta)\right|^2$$  \hspace{1cm} (5.59)

The elastic scattering of deuterons by deuterons has been calculated (eq.5.59) at $E_d=25.3$ MeV. In these calculations we used our calculated real phase shifts (real potential). The results of these calculations
showed that the magnitude of the calculated cross section is larger than the data [Va 63a], in agreement with the RGM calculations for this reaction at energies below 20 MeV [Th 70]. In an attempt to improve the agreement we have included a local phenomenological imaginary potential in our formalism (5.29). The idea of introducing this potential is to take into account the effect of the reaction channels. This imaginary potential was chosen to be spin-dependent and have the form [Ch 72]

\[ W_s(r) = -W_0 s \left[ \frac{1}{1+\exp\{ (r-R)/a \}} + \frac{4\exp\{ (r-R)/a \}}{1+\exp\{ (r-R)/a \}} \right] \quad (5.60) \]

which has both a volume and a surface term. We used this particular form in this calculation, since it has been found in the case of \( \alpha + \alpha [Br 71] \) that the complex phase shifts determined empirically by Darriulat et al [Da 65] can be fitted quite well by using an imaginary potential of this type.

In this calculation we have used the same geometrical parameters \( (R \) and \( a) \) and the imaginary potential depths \( (W_{00}, W_{01}, \text{and} W_{02}) \) as used in [Ch 72]. These parameters are listed in table 5.3, together with the calculated complex phase shifts at 25.3 MeV.

The differential cross section calculated with imaginary potential depths given in table 5.3 are compared with experimental data [Va 63a]. It seems that the agreement between theory and experiment is quite satisfactory. In fig. (5.6) the solid curve shows the calculated differential cross section, while the squares represent the
5.5.2 The Distorted Waves

The numerical techniques to calculate the partial wave-radial functions appearing in eq. (4.8) have been presented in sections 5.4 and in appendices B, C. The calculations have been performed at centre of mass energy of 55, 100, 150 keV, and 12.65 MeV. The results at 55 keV are shown in figs. (5.7-11). In this case the interaction between the two deuterons is described by non-local separable and Coulomb potentials. In fig. (5.7) the ground state wave function 1=0, s=0 (α-particle) which has a node at 0.95 fm is given. The state 1=0, s=2 also has a node at a radius of about 3 fm (fig. (5.8)). This node is due to the repulsive part of the form factor which describes this state. Figs. (5.8-11) show that for partial waves greater than zero, the relative wave functions have no nodes.

Once these radial functions are calculated, then it is easy to calculate the distorted wave in the channel spin representation required by eq. (4.8).
Table 5.1

The parameters of the nuclear non-local separable potential which give the best fit between the calculated phase shift and Thompson phase shift in the centre of mass energy range (CMS-E.R) 0-20 MeV in the absence of the Coulomb potential

<table>
<thead>
<tr>
<th>$S_{1s}$</th>
<th>$a_R$ (fm$^{-1}$)</th>
<th>$C_R$</th>
<th>$a_A$ (fm$^{-1}$)</th>
<th>$C_A$</th>
<th>$\sum R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{00}$</td>
<td>0.926</td>
<td>147.90</td>
<td>0.869</td>
<td>86.490</td>
<td>6.21</td>
</tr>
<tr>
<td>$S_{02}$</td>
<td>0.575</td>
<td>7.64</td>
<td>0.236</td>
<td>0.615</td>
<td>0.46</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>2.935</td>
<td>11.27</td>
<td>1.019</td>
<td>7.020</td>
<td>2.05</td>
</tr>
<tr>
<td>$S_{20}$</td>
<td>1.056</td>
<td>11.720</td>
<td>5.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_{22}$</td>
<td>1.137</td>
<td>11.070</td>
<td>0.45</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The units of $C_R$ and $C_A$ for $l=0,2$ are defined as (MeV.fm$^2$)$^{1/2}$, while for $l=1$, they are (MeV.fm)$^{1/2}$.
Table 5.2

The parameters of the nuclear non-local separable potential which give the best fit between the calculated phase shift and Thompson phase shift in the centre or mass energy range (CMS-E.R) 0-20 MeV when the Coulomb potential is included.

<table>
<thead>
<tr>
<th>$s_{ls}$</th>
<th>$a_R$ (fm$^{-1}$)</th>
<th>$C_R$</th>
<th>$a_A$ (fm$^{-1}$)</th>
<th>$C_A$</th>
<th>(CMS-E.R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$00$</td>
<td>1.082</td>
<td>19.91</td>
<td>0.511</td>
<td>3.517</td>
<td>0 - 5</td>
</tr>
<tr>
<td></td>
<td>1.033</td>
<td>31.54</td>
<td>0.203</td>
<td>2.120</td>
<td>5 - 20</td>
</tr>
<tr>
<td>$02$</td>
<td>0.541</td>
<td>7.04</td>
<td>0.323</td>
<td>0.430</td>
<td>0 - 20</td>
</tr>
<tr>
<td>$11$</td>
<td>2.279</td>
<td>9.82</td>
<td>0.998</td>
<td>7.395</td>
<td>0 - 5</td>
</tr>
<tr>
<td></td>
<td>2.801</td>
<td>12.49</td>
<td>1.049</td>
<td>7.597</td>
<td>5 - 20</td>
</tr>
<tr>
<td>$20$</td>
<td></td>
<td></td>
<td>1.024</td>
<td>11.235</td>
<td>0 - 20</td>
</tr>
<tr>
<td>$22$</td>
<td></td>
<td></td>
<td>1.104</td>
<td>10.589</td>
<td>0 - 20</td>
</tr>
</tbody>
</table>

Where $C_R$ and $C_A$ have the same units as those given in table 5.1.
Table 5.3

The calculated phase complex phase shift using the parameters given in table 5.1 for the non-local nuclear separable potential at centre-of-mass energy of 12.65 MeV.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>Our calculation</th>
<th>Thompson phase shift</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(deg.)</td>
<td>(deg.)</td>
</tr>
<tr>
<td>$\delta_{1s}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_{00}$</td>
<td>(-54.41,1.400)</td>
<td>(-55.6,0.80)</td>
</tr>
<tr>
<td>$\delta_{02}$</td>
<td>(-91.71,2.040)</td>
<td>(-91.4,2.30)</td>
</tr>
<tr>
<td>$\delta_{11}$</td>
<td>( 89.96,12.10)</td>
<td>( 89.3,13.8)</td>
</tr>
<tr>
<td>$\delta_{20}$</td>
<td>( 49.23,2.400)</td>
<td>( 48.1,1.30)</td>
</tr>
<tr>
<td>$\delta_{22}$</td>
<td>( 22.68,2.300)</td>
<td>( 22.3,2.30)</td>
</tr>
</tbody>
</table>

$W_{00} = 0.1$ MeV, $W_{02} = 0.2$ MeV and $W_{01} = 2.4$ MeV
$R = 3.75$ fm, $a = 0.5$ fm
Figs. (5.1-5): The phase shifts $\delta_{ls}$ obtained from equations (5.22,24,26) are plotted against the centre of mass energy (0-20 MeV). The cross points are the results of the one channel RGM calculations of Thompson [Th 70].
$l = 1, \ s = 1$

Fig. (5.3)

Phase Shift (deg.)

$k (\text{fm}^{-1})$

$-1$
Fig. (5.6): The differential cross section for $^d_2$-$^d_2$ elastic scattering at 12.65 MeV calculated using the complex phase shifts of table 5.3.
Figs. (5.7-10): The relative wave functions $\psi_{l,s}$ between the two deuterons at 55 keV, the solution of eq.5.28, plotted against the relative distance $r$ for $l=0,1,2$ and channel spin $s=0,1,2$. 
6.1 Forbidden States (FS)

In few body system, a major task of the resonating group method (RGM) is to provide effective interactions between composite particles or clusters. These interactions contain information about the internal structure of the clusters, the force between their constituents and most importantly about the antisymmetrization of these constituents.

Consider the one-channel case, a two body collision where both nuclei remain in their ground states. The number of nucleons in the nucleus, \( a \), is denoted by \( n_a \) and in the nucleus \( A \) by \( n_A \). The internal wave functions of \( a \) and \( A \), \( \phi_{a}^{(1,2,\ldots,n_{a})} \) and \( \phi_{A}^{(n_{A}+1,\ldots,n_{A}+n_{A})} \), which are normalized and antisymmetrized. In the RGM, the total wave function can be antisymmetrized by introducing the relative wave function \( U(r) \) between \( a \) and \( A \) as follow

\[
\Psi = \mathcal{A} \left[ \phi_{a}^{(1,2,\ldots,n_{a})} \phi_{A}^{(n_{A}+1,\ldots,n_{A}+n_{A})} U(r) \right] \quad (6.1)
\]

where \( \mathcal{A} \) is the normalized antisymmetrizer [Au 70]

\[
\mathcal{A} = N^{-1/2} \sum_{p} (-1)^{p} \quad (6.2)
\]

\( N \) is the total number of permutations
\begin{equation} N = \frac{(n + n_A)!}{n! n_A!} \tag{6.3} \end{equation}

In eq. (6.2), \((-)^P\) is positive if the permutation operator \(P\) exchanges an even number of particles between \(a\) and \(A\), and it is negative if \(P\) exchanges an odd number of particles.

In eq. (6.1), \(r\) is the relative coordinate between the centre of mass of \(a\) and \(A\).

\[
\xi = \frac{1}{n_a} \sum_{i=1}^{n_a} \xi_i - \frac{1}{n_A} \sum_{j=1}^{n_A} \xi_j
\]

where \(\xi_i\) is the coordinate of the \(i\)th nucleon. To determine \(U(r)\) is the main problem in the RGM.

In certain cases [Ta 65, Ok 66], the RGM equation brings about an almost energy-independent inner oscillation for \(U(r)\). This is one of the important effects of nonlocal potential originating from antisymmetrization of the nucleons, between \(a\) and \(A\). This behavior was first observed in the calculation of \(\alpha - \alpha\) scattering [Ta 65, Ok 66]. The nodal behavior was shown to be the origin of the phenomenological repulsive core [En 64]. The nodal points in the S- and D-wave in the relative wave function agree well with the radius of the repulsive core of the phenomenological \(\alpha - \alpha\) potential. The nodal behavior agrees with that derived from the shell model consideration [Ta 68].

If the relative wave function is given by the eigenfunction of the RGM
kernel, $U_1$, with eigenvalue of unity, we will have null states which is the consequence of the total antisymmetrization; conversely, $U_1$ is forbidden by the Pauli exclusion. Such states are called the forbidden states (FS), which satisfy

$$\mathcal{A} \left\{ \Phi_a \Phi A U_1(r) \right\} = 0 \quad (6.4)$$

6.2 Orthogonality Condition Model (OCM)

The orthogonality condition model has been proposed by Saito as a simplified version for the treatment of the effect of the Pauli principle [Sa 69, Sa 77]. In this model, the inter-cluster wave function is required to be orthogonal to the FS, which give rise to the null states if the complete antisymmetrization is performed, so that the interaction takes place only in the physical space orthogonal to the FS.

It was shown [Ta 65, Ok 66] in the $\alpha-\alpha$ relative wave function calculated within RGM possesses an almost energy-independent-inner oscillation. This behavior is not a particular result for $\alpha-\alpha$ interaction, and appears universally. For example, detailed investigation on this point has been carried out for $^3\text{He}-\alpha$ interaction by Okai and by Tang [Ok 76, Ta 69]. The outermost nodal point agrees well with the radius of the repulsive core; thus the repulsive core is to be a substitute for the characteristic behavior obtained microscopically, and it is called the structure core [Ot 65].

The characteristic inner oscillation corresponds precisely to that expected from the orthogonality to the FS which are the oscillator
functions with the number of quanta less than 4. The exact FS, however, appears only for cases with the equal oscillator parameters of the shell model clusters. According to this idea, $\alpha-\alpha$ scattering was solved by using the so-called direct (folding) potential but imposing the orthogonality to the FS, and was shown to be reproduced well by this model (OCM). Thus, in case of $\alpha-\alpha$ scattering, the effect of the complicated kernels in the RGM is considered to be well approximated by the orthogonality to the FS.

The OCM has been used widely by many authors. The structure of $^{20}$Ne nucleus was analyzed in terms of $^{16}$O+$\alpha$ model [Ma 73], and the result of the OCM was successfully compared with that of RGM [Ma 75].

6.3 Formulation of OCM

The relative motion of the $a+A$ system can be described by the equation

$$\Lambda(E - T - V_D)\Lambda |\psi\rangle = 0$$

(6.5)

where $E$ is the relative energy in centre-of-mass system, $T$ denotes the kinetic energy operator and $V_D$ is the local potential which is regarded as the average potential between $a$ and $A$. In eq. (6.5) $\Lambda$ is the projection operator which gives the effect of the Pauli principle and excludes the FS

$$\Lambda = 1 - |\psi\rangle \langle \psi|$$

(6.6)
where \( \langle U_1 | U_1 \rangle = 1 \), \( |U_1 \rangle \) is the forbidden state. Any solution of \( U \) satisfies

\[
\langle U_1 | U \rangle = 0
\]

Eq. (6.5) then becomes

\[
(E - T - V_D) A |U\rangle = |U_1\rangle \langle U_1 | (E - T - V_D) A |U\rangle
\]

\[
= |U_1\rangle \{- \langle U_1 | (T + V_D) A |U\rangle\}
\]

(6.7)

where

\[
\langle U_1 | E A |U\rangle = E \langle U_1 | A U\rangle = 0
\]

(6.8)

this means that \( A |U\rangle \) also satisfies

\[
(E - T - V_D) |\phi\rangle = - |U_1\rangle \langle U_1 | (T + V_D) |\phi\rangle
\]

(6.9)

where \( \langle \phi | = A |U\rangle \), and \( \phi \) agrees with \( U \) asymptotically.

The advantage of the form (6.9) is, any solution of it automatically satisfies \( \langle U_1 | \phi \rangle = 0 \), where \( E \neq 0 \). This can be proved easily by taking the inner product of both sides of eq. (6.9) by \( |U_1\rangle \).

In order to solve (6.9), we proceed as follows

a) Let \( U_{in} \) to be any regular solution of

\[
(E - T - V_D) |U_{in}\rangle = g |U_{in}\rangle
\]

(6.10)

where \( g \) is arbitrary (=1 for example).
b) \( U_h \) to be a regular solution of

\[
(E - T - V_D) |U_h> = 0
\]  

(6.11)

normalized in a standard way at large \( r \). In the absence of the Coulomb force and \( l = 0 \),

\[
U_h \rightarrow \frac{\sin(kr + \xi_0)}{kr}
\]  

(6.12)

Then the required solution is

\[
|\phi> = aU_h + bU_{1n}
\]  

(6.13)

where \( a \) and \( b \) are chosen so that

i) \( |\phi> \) is orthogonal to \( |U_{1>}, i.e \)

\[
b/a = -\frac{<U_{1} |U_{1}>}{<U_{1n} |U_{1}>}
\]  

(6.14)

ii)

\[
\phi \rightarrow \frac{\sin(kr + \xi_0)}{kr}
\]  

(6.15)

These conditions determine \( a, b, \xi \) and \( \phi \).

The general regular solution of (6.10) is

\[
U_{1n} = \alpha U_h + \beta (E - T - V_D)^{-1} g |U_{1}> 
\]  

(6.16)

and hence (6.13) becomes
\[ |\phi\rangle = (a + b\alpha)U_h + b\beta (E - T - V_D)^{-1} g|U_f\rangle \]  \hspace{1cm} (6.17)

condition (ii) means that \((a + b\alpha) = 1\), so that \(|\phi\rangle\) and \(|U_f\rangle\) have the same ingoing spherical wave amplitude (the second term of 6.17 is purely outgoing wave).

condition (i) gives

\[ b\beta = - \frac{\langle U_f |U_h\rangle}{\langle U_f |(E - T - V_D)^{-1} |U_f\rangle} g \]  \hspace{1cm} (6.18)

6.3 Calculation of \(d-d\) Scattering

In the \(d-d\) system, there is just one forbidden state. This state describes the two deuterons at low energy when they are in their relative ground state with spin parallel \(s=2\).

For a particular partial wave, eq. (6.9) can be solved numerically, for \(d-d\) system, as the superposition of the solutions of the following differential equations (for \(l=0\))

\[ (E - T_o - V_D)U^{(1)} = 0 \]  \hspace{1cm} (6.19)

\[ (E - T_o - V_D)U^{(2)} = U_f \]  \hspace{1cm} (6.20)

where as we have shown in the previous section the coefficients of the superposition can be determined by the condition \(\langle U_f |\phi\rangle = 0\).
The radial form of the FS, $U_1$, is assumed as follows

$$U_1 = \frac{4a}{\pi} \frac{3}{4} \exp(-2ar^2), \quad (6.21)$$

where $r$ is the relative coordinate between the centre of masses of the two deuterons, and $a=0.12$ fm$^{-2}$ [Sa 73].

The direct potential $V_D$ was chosen to have the form

$$V_D = -V_0/[1 + \exp((r-R_0)/a_0)] \quad (6.22)$$

The parameters $V_0$, $R_0$, $a_0$ are chosen so that the direct potential of the RGM in each spin channel can be given by the form (6.22). The search to fit the one channel RGM phase shifts has been started using these parameters. The parameters obtained which give the best fit with the data [Th 70] are listed in table 6.1. Figs.(6.1-5) show the calculated phase shifts together with the RGM phase shifts (cross points) in the centre of mass energy range between 0 and 15 MeV. The RGM features of the phase shifts are well reproduced by the OCM for each partial wave.

Including the phenomenological imaginary potential which has been discussed in chapter 5 within eq. (6.9), yields a complex phase shifts. The calculated phase shifts (at CMS energy of 12.65 MeV) are listed together with the complex RGM data in table 6.2.

The relative wave functions obtained by the OCM for the partial waves $l=0,1,2$ in the spin channels $s=0,1,2$ are shown in figs.(6.6-10).
These wave functions have been calculated at centre of mass energy of 55 keV. Fig. (6.6) represents the ground state wave function $l=0, s=0$ which has a node at 0.95 fm. The wave function of the state $l=0, s=2$ is given in fig. (6.7), where there appears a node at 2.15 fm. This node originates from the orthogonality to the FS, which is just the effect of Pauli principle. To check that the node, which appears in this case, is due to the orthogonality to the forbidden state, the calculations have been repeated with the use of different direct potentials. The results of each case showed that the relative wave function $U_0^2$ has a node at the same radius (2.15 fm). The orthogonality to the forbidden state means

$$<U_0^2|U_1> = 0$$  \hspace{1cm} (6.23)

Equation (2.23) has been calculated, and the obtained result was found to be $-0.00001$. This is demonstrated in fig.(6.11), where the function $r^2U_0^2(r)U_1(r)$ has been plotted against $r$. 

Table 6.1

The parameters of the nuclear local potential $V_D$ which give the best fit between the calculated phase shift and Thompson phase shift.

<table>
<thead>
<tr>
<th>$S_{ls}$</th>
<th>$a_0$ (fm)</th>
<th>$R_0$ (fm)</th>
<th>$V_0$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{00}$</td>
<td>0.7423</td>
<td>3.957</td>
<td>22.17</td>
</tr>
<tr>
<td>$S_{02}$</td>
<td>0.3700</td>
<td>3.650</td>
<td>13.22</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>0.7907</td>
<td>2.297</td>
<td>32.43</td>
</tr>
<tr>
<td>$S_{20}$</td>
<td>0.8151</td>
<td>2.601</td>
<td>25.09</td>
</tr>
<tr>
<td>$S_{22}$</td>
<td>0.7800</td>
<td>1.573</td>
<td>39.94</td>
</tr>
<tr>
<td>$S_{31}$</td>
<td>0.8630</td>
<td>1.749</td>
<td>41.18</td>
</tr>
<tr>
<td>$S_{40}$</td>
<td>0.6840</td>
<td>2.683</td>
<td>38.92</td>
</tr>
<tr>
<td>$S_{42}$</td>
<td>0.9060</td>
<td>1.270</td>
<td>64.17</td>
</tr>
<tr>
<td>$S_{51}$</td>
<td>0.7640</td>
<td>1.699</td>
<td>47.28</td>
</tr>
<tr>
<td>$S_{60}$</td>
<td>0.9100</td>
<td>1.213</td>
<td>39.94</td>
</tr>
<tr>
<td>$S_{62}$</td>
<td>0.5577</td>
<td>3.489</td>
<td>31.40</td>
</tr>
</tbody>
</table>
Table 6.2

The calculated complex phase shift using the parameters given in table 6.1 for the local nuclear potential at centre of mass energy of 12.65 MeV

<table>
<thead>
<tr>
<th>$\delta_{1s}$</th>
<th>OCM calculation</th>
<th>Thompson phase shift</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(deg.)</td>
<td>(deg.)</td>
</tr>
<tr>
<td>$\delta_{00}$</td>
<td>(-53.31,0.812)</td>
<td>(-55.6,0.80)</td>
</tr>
<tr>
<td>$\delta_{02}$</td>
<td>(-88.12,1.672)</td>
<td>(-91.4,2.30)</td>
</tr>
<tr>
<td>$\delta_{11}$</td>
<td>( 89.99,14.14)</td>
<td>( 89.3,13.8)</td>
</tr>
<tr>
<td>$\delta_{20}$</td>
<td>( 48.07,1.252)</td>
<td>( 48.1,1.30)</td>
</tr>
<tr>
<td>$\delta_{22}$</td>
<td>( 22.24,2.256)</td>
<td>( 22.3,2.30)</td>
</tr>
<tr>
<td>$\delta_{31}$</td>
<td>(  7.82,8.925)</td>
<td>(  7.8,9.10)</td>
</tr>
<tr>
<td>$\delta_{40}$</td>
<td>(  2.84,0.073)</td>
<td>(  2.8,0.10)</td>
</tr>
<tr>
<td>$\delta_{42}$</td>
<td>(  1.49,0.134)</td>
<td>(  1.5,0.10)</td>
</tr>
<tr>
<td>$\delta_{51}$</td>
<td>(  0.59,0.280)</td>
<td>(  0.6,0.30)</td>
</tr>
<tr>
<td>$\delta_{60}$</td>
<td>(  0.13,0.002)</td>
<td>(  0.2,0.00)</td>
</tr>
<tr>
<td>$\delta_{62}$</td>
<td>(  0.11,0.003)</td>
<td>(  0.1,0.00)</td>
</tr>
</tbody>
</table>
Figs. (6.1-4): The phase shifts $\delta_{ls}$ obtained from the numerical
solution of eq. (6.9) are plotted against the centre of mass energy
(0-15 MeV). The parameters of the direct potentials $V_D$ are chosen so
that they reproduce the phase shifts of the one channel RGM
calculations of Thompson [Th 70] which are shown by the cross points.
Fig. (6.1) 

Phase Shift (deg.) 

$E$ (MeV) 

Fig. (6.2) 

Phase Shift (deg.) 

$E$ (MeV)
Figs. (6.6-10): The relative wave functions $U_{l,s}$ between the two deuterons at 55 keV, the solution of eq.6.9, plotted against the relative distance $r$ for $l=0,1,2$ and channel spin $s=0,1,2$. 
Fig.(6.7)
Fig. (6.8)  

kr U(r)  

Fig. (6.9)  

kr U(r)  

kr U(l) (r)
Fig. (6.11): The product $r^2 U_0 U_1$ is plotted against $r$ to demonstrate the effect of the orthogonality condition eq. (6.23).
The finite-range $d(d,n)^3\text{He}$ transfer reaction calculations described in chapter 4 were performed using an extensively modified version of the computer code TWOFNR [Ig], which has been rewritten in channel spin formalism in order to accommodate the distorted waves described by eq. (4.8). This program calculates the unpolarized and polarized cross sections for deuteron stripping reactions and the inverse pick-up reactions using the distorted wave theory and including the contributions from the S- and D-states of the deuteron wave function [Jo 71]. The distorted wave in the incident channel, eq.(4.8), has been calculated separately (see chapters 5 and 6) and was read by the program from an external file. The program also reads the calculated realistic form factors, described in section 4.4.2-3, from another external file.

The program has been checked as follows:

The cross section of $d(d,n)^3\text{He}$ reaction has been calculated within plane wave Born approximation (see appendix D). In these calculations, the internal states of the deuteron and $^3\text{He}$ were considered as simple Gaussian [Ch 72, Br 56], and the transition interaction $V_{np}$ is considered to be constant. The calculations have been repeated at different energies, and the result of each case agrees extremely well, throughout the angular range, with that of TWOFNR (using the same form factors and in the absence of the distorting potentials). The ratio of
the results of the two calculations at $E_{\text{CM}}=55$ keV is 1.002.

In section 4.1 the validity of DWBA at energies of few MeV [Kn 75b] was mentioned. An attempt was made to investigate this in the case of $d(d,n)^3$He reaction at centre of mass energy of 12.65 MeV. This has been done by evaluating the symmetrized transition amplitude (4.18), with the distorted wave in the exit channel generated by neutron optical potential of Sherif and Podmore [Sh 72] (Pot.3 of table 4.2). The results of these calculations show that the calculated cross sections are of the same order of magnitude as the data [Va 63b]. These results are presented in fig. (7.1), where the solid and the dashed curves represent the differential cross section when the distorted wave in the incident channel generated by non-local separable potential and by deuteron optical potential of table 4.1 respectively. The dotted curve in fig. (7.1) represents the results when the distorted waves in the incident and exit channels are generated by the non-local separable potential and by the neutron optical Pot.4 of table 4.2 respectively. This result shows that the use of the neutron Pot.4 in the calculations yields a transfer cross section of the same order of magnitude as the data [Va 63b]. As was discussed in section 4.4.1, Pot.4 is generated from Pot.2 in the same way in which Pot.3 is generated from Pot.1.

Since the results predicted by the model described for $d(d,n)^3$He reaction at $E_{\text{CM}}=12.65$ MeV are of the same order of magnitude as the data [Va 63b], it is reasonable to assume that the theory can be used to extrapolate to very low energies (energies relevant to fusion reactors). For these low energies there are experimental data for the
The transition amplitude calculated by Zhang et al. [Zh 86] has the form of eq. (4.18). In their calculations they have used effective potentials (including the folded Coulomb potential) to generate the distorted waves between the two deuterons in the incident channel and between the neutron and $^3\text{He}$ in the exit channel. They reported that these potentials are fitted the d-d elastic scattering phase shifts of Thompson [Th 70] and the n-$^3\text{He}$ elastic cross sections [Sa 61]. In their calculations they have used Gaussian forms to describe the internal wave functions of the deuteron [Ch 72] and $^3\text{He}$ [Br 56]. Their transition interaction $V_{np}$ was taken from Eikemeier and Hackenbroich [Ei 71]. They reported that their results are in agreement with the experimental data [Ja 85].

In this section the symmetrized transition amplitude (4.18) have been evaluated with the distorted wave in the exit channel generated by the neutron optical potential Pot.1 of table 4.2 [Sh 72]. In this case the resulting amplitude actually calculated here has obvious connection with that calculated by Zhang et al. [Zh 86], but with the use of realistic form factors. In agreement with Zhang et al., the results of
the present calculations show a very small $^3\text{He}$ D-state (relative to S-state) effect upon the reaction cross section; however these calculations, unlike theirs, overestimate the cross section data [Ja 85] by a factor of about 150 [Ab 87]. The results of the present calculations are shown in fig. (7.2), where the solid, dashed, dotted curves represent the differential cross sections when the d-d initial state distortion was calculated using the non-local separable potential, orthogonality condition model and the deuteron optical potential of table 4.1 respectively. Fig. (7.2) does not include the experimental data.

To study the discrepancy in the cross section magnitude, calculations in which the S- and P-waves of the entrance channel have been performed individually. The results of these calculations have shown that the transfer cross section is dominated totally by the P-wave, as shown by the dotted curve of fig. (7.3). The S-wave contribution to the reaction cross section is about 8% of its magnitude. This result is shown by the dashed curve in fig. (7.3). Fig. (7.4) shows the same results of fig. (7.3) when the distorted wave in the incident channel generated by the Coulomb potential only. In this case, it is clear that the S-wave contribution dominates the contribution from P-wave. This means that the absence of the distorted nuclear potential (local d-d potential of table 4.1) in the entrance channel reduces the P-wave contribution to the transfer cross section by about one order of magnitude, throughout the angular range. The absence of the nuclear potential also affect the S-wave contribution but not as large as P-wave case. The magnitude of the S-wave contribution to transfer cross section is found to be decreased by
about 0.85 when the nuclear potential is removed from the calculations.

Given the large discrepancy in the cross section magnitude, it is not possible to draw conclusions as to the importance of the D-state from these results. How then are Zhang et al. able to reproduce the magnitude of the cross section?

To study this disagreement, the calculations of Zhang et al. have been investigated carefully. Upon doing this it was found that those authors have been inconsistent in their use of the deuteron wave function and nucleon-nucleon interaction. In the treatment of a transfer reaction amplitude, where the combination $V_{np} \phi_d(12)$ arises, it is vital that $\phi_d(12)$ should be the eigenfunction of the n-p Hamiltonian containing the chosen interaction $V_{np}$. Zhang et al. have not in fact done this. They have taken their deuteron wave function, a simple two term Gaussian form, from one source [Ch 72] and their n-p interaction, also a Gaussian expanded interaction, from another source [Ei 71]. The result is that the deuteron wave function used does not have, in detail, the appropriate radial form in the region of the soft core of the interaction. Thus, the subtle balance between the magnitudes of $V_{np}$ and $\phi_d$ in this region of space where $V_{np}$ is large, and which are treated exactly when the Schrodinger equation is solved, are missing. This effect is demonstrated in figure (7.5). The solid curve shows the relative S-state part of the product

$$D_0(r) = (4\pi)^{1/2} \int r^2 V_{np}(r) \phi_d(r)$$
for the choice of deuteron wave function and $V_{np}$ of Zhang et al. The dashed curve shows the same quantity for the Reid wave function and $V_{np}$ of the present work. This function $D_0(r)$, the integrand of the S-state zero-range normalization constant $D_0$ [Jo 71], integrates to $D_0 = 25 \text{ MeV fm}^{3/2}$ for the Zhang et al. functions rather than the accepted value $D_0 = -122.5 \text{ MeV fm}^{3/2}$ of the Reid interaction. The reason, as stated above, is that the two Gaussian deuteron wave function is an inadequate representation of $\phi_d(r)$ for the purposes of evaluating $V_{np}\phi_d(r)$. As the cross section for the transfer process continues to be dominated by the low momentum transfer parts of the amplitude ($D_0$), this effect alone leads us to expect at least a factor of 25 between the calculations of Zhang et al. and ourselves.

The other major difference between our calculation and that of Zhang is our use of a realistic (Faddeev) deuteron-$^3\text{He}$ overlap function. Figure 7.6 shows the S-state component ($\int \phi_{0}(\bar{\phi})$) of the overlap deduced from the Zhang et al. $^3\text{He}$ [Br 63] and deuteron [Ch 72] wave functions (solid curve) and the Faddeev $^3\text{He}$ and Reid deuteron wave functions (dashed curve). The normalizations of these overlaps are 0.57 and 0.67, respectively. The differences in the derived overlaps are self evident, in particular the much longer range of the realistic calculation. The effect of these formfactors upon the numerical results have been investigated by retaining our prescription for the incident channel distortion, derived from the RGM. Replacing the Reid soft core formfactor $V_{np}\phi_d(r)$ by the Zhang et al. values reduces the calculated cross section by a factor of almost 68! This gives an indication of the importance of finite range effects in our calculations as the reduction expected on the basis of the zero-range
normalization constant $D_0$ was 25. Changing the $d^3$He overlap from the Faddeev values to the Zhang et al. values while retaining our Reid form for $V_{np} \phi_d(r)$ had, by comparison, a relatively small effect reducing the cross section by about a factor of 2.4. When both of these form factors were included simultaneously the magnitude of our calculations agreed very well with the work of Zhang et al [Zh 86].
Fig. (7.1): The differential cross section of $d(d,n)^{3}\text{He}$ reaction at 12.65 MeV predicted by eq.(4.18). The solid and dashed curves show the results when the distorted wave in the incident channel are generated by the non-local separable potential (5.19) and by the deuteron optical potential (table 4.1), and the distorted wave in the exit channel is generated by the neutron Pot.3 of table 4.2. The dotted curve shows the results when the distorted waves in the incident and exit channel are generated by the non-local separable potential (5.19) and by the neutron Pot.4 respectively.
Fig. (7.1)
Fig. (7.2): The differential cross section of d(d,n)$^3$He reaction at 55 keV predicted by eq.(4.18). The solid, dashed and dotted curves show the results when the distorted wave in the incident channel are generated by non-local separable potential (5.19), within the orthogonality condition model and by the deuteron optical potential (table 4.1) respectively, and the distorted wave in the exit channel is generated by the neutron Pot.1 of table 4.2.

Fig. (7.3): The dashed, dotted, and solid curves show the results due to the S-waves only, P-waves only, and total, respectively, contributions to the reaction cross section from incident channel.
Fig. (7.4): This figure shows the same results as fig.(7.3) when the distorted wave in the incident channel is generated by the Coulomb potential only. The dotted-dashed curve is the same as the dotted of fig.(7.2).
\[ \frac{d\sigma}{d\Omega} \] (mb/sr)

\[ \Theta_{cm} \]

Fig. (7.4)
Fig. (7.5): Plot of the function $D_0(r)$ (p.132) obtained using the n-p interaction and deuteron wave function of Zhang et al. [Zh 86] (solid curve) and of Reid [Re 68] (dashed curve).

Fig. (7.6): The deuteron-$^3$He S-wave radial overlap $\int u_0(r)$ calculated using the Gaussian $^3$He and deuteron wave functions of Zhang et al. [Zh 86] (solid curve) and that obtained from the Faddeev calculation for $^3$He [Is 86] using the Reid interaction [Re 68].
7.2 Results with the neutron Pot.2

The calculations presented in section 7.1 have been performed again with the distorted wave in the exit channel generated by the neutron pot.2 (table 4.2). In these calculations the distorted wave in the incident channel has been generated by non-local separable potentials. Like the finding in the previous section the result of these calculations showed very small $^3\text{He}$ D-state effects, but unlike them the present calculations are in agreement with the experimental data [Ja 85] (solid curve in fig. (7.7)). In this situation, where the data has been reproduced, it is meaningful to draw conclusions about the importance of D-state effects of the deuteron and $^3\text{He}$. The presence of the $^3\text{He}$ D-state allows an $s=3/2$ wave function in $^3\text{He}$ which gives rise to an $s=2$ matrix element by the central force. This matrix element produces very small (relative to S-state) effects upon the reaction cross section. This result is shown by the dashed curve in fig (7.7).

The dotted curve shows the results due to the deuteron D-state.

These results, unlike the previous ones (section 7.1) also show that in the low energy regime ($E_{CM}=55$ keV), the cross section is indeed dominated by the S-wave in the entrance channel, in particular around $90^0$ (the dashed curve in fig.(7.8)). The dotted curve in fig.(7.8) shows the P-wave contribution to the transfer cross section. This P-wave produce significant correction to the transfer cross section particularly at forward and backward angles. The same effect has been seen when the distorted wave in the incident channel is generated by Coulomb potential only.
The dashed curve in fig. (7.9) shows the results of the calculations for the \(d(d,n)\) reaction at 55 keV, including all contributions from the D-states of the deuteron and \(^3\)He, while the solid curve shows the result when these contributions are ignored. The dotted curve represents the polarized cross section predicted by the full calculations (the polarized cross section is defined as the cross section when the spins of the two deuterons in the incident channel are polarized in parallel to the direction of the incident momentum, \(k_z\), \(Z\)-axis). In these calculations the polarized cross section arose as a result of the D-state components in the deuteron and \(^3\)He. The dashed curve in fig. (7.10) shows the polarized cross section due to the D-state component of the deuteron, while the dotted curve is the result of the \(^3\)He D-state component. These results show that the polarized cross section (solid curve) is totally dominated by the inclusion of the deuteron D-state. This is contradictory to the findings of RGM calculations [Ho 84] which predicted that the central force is responsible for a 1:1 ratio of the polarized to unpolarized cross sections at centre of mass energy of 20 keV.

The ratio of the polarized to unpolarized cross section (attenuation factor) is seen to be highly-angular dependent (see fig. (7.9)). This is because the polarized cross section increases rapidly with the centre of mass angle, while the angular dependence of the unpolarized cross section is very small. So in order to compare the present results with the results reported using RGM [Ho 84], the total polarized and unpolarized cross sections have been calculated at centre of mass energies of 55, 100, 150 keV. The results of these calculations showed that both the total polarized and unpolarized cross
section are energy dependent. The total unpolarized cross section predicted by these calculations are within 20% of the experimental data [Ru 63]. These results are listed in table 7.1 together with the experimental data [Ru 63]. It is worthwhile to point out that the present results on the unpolarized cross sections represent an improvement over those of the RGM calculation [Ho 84], which are about a factor of 2 smaller than the data [Ru 63]. The results of the present calculations also show that the calculated attenuation factors corresponding to the energies mentioned above to be 0.174, 0.1304, 0.135 respectively, which is approximately constant.

The anisotropy factors have also been calculated at the prescribed energies. The results of the calculations are smaller than the data [Th 66] by a factor of 2, but they are in agreement with the RGM calculation [Ho 84]. The calculated anisotropy factors are listed in table 7.1 together with the data [Th 66].

An interesting feature of using Pot.2 in calculating the $^3\text{He}$ reaction appears when the d-d distortion is described by the deuteron optical potential of table 4.1. In this case the calculated transfer cross section agrees very well with the experimental data [Ja 85] without the inclusion of the D-states of both the deuteron and $^3\text{He}$. This result is shown by the dotted curve in fig. (7.11). The dotted-dashed curve represents the result when the distorted wave in the incident channel is generated by the Coulomb potential only. The solid and the dashed curves in fig. (7.11) are the results when the distorted waves in the incident channel are generated by the non-local separable potential and within the orthogonality condition model.
<table>
<thead>
<tr>
<th>E (MeV)</th>
<th>Unpolarized Cross Section (mb/sr)</th>
<th>Anisotropy Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calculated</td>
<td>Data[Ru 63]</td>
</tr>
<tr>
<td>55</td>
<td>15.4</td>
<td>19.0 ± 1.4</td>
</tr>
<tr>
<td>100</td>
<td>32.9</td>
<td>40.1 ± 2.1</td>
</tr>
<tr>
<td>150</td>
<td>37.9</td>
<td>54.5 ± 3.9</td>
</tr>
</tbody>
</table>
Fig. (7.7): The differential cross section of $d(d,n)^3\text{He}$ reaction at 55 keV predicted by eq.(4.18). The dotted and dashed curves show the contributions from the D-state of the deuteron and $^3\text{He}$ respectively. In these calculations, the distorted waves are generated by the non-local separable potential (5.19) (in the incident channel) and by the neutron Pot.2 of table 4.2 (in the exit channel). The experimental data are taken from [Ja 85].

Fig. (7.8): The results presented in this figure are the same as those described in fig.(7.3), but with the distorted wave in the exit channel generated by the neutron Pot.2 of table 4.2.
Fig. (7.9): Full calculations (including all contributions from the D-state of the deuteron and $^3$He) for the differential cross section of d(d,n)$^3$He reaction at 55 keV are shown by the dashed curve. The solid curve shows the results in the absence of the D-state contributions. The dotted curve shows the results for the polarized cross section.

Fig. (7.10): This figure demonstrates that the polarized cross section (solid curve) is completely dominated by the deuteron D-state (dashed curve). The dotted curve shows the results of the He D-state contribution to the polarized cross section.
Fig. (7.11): The results presented in this figure are the same as those described in fig.(7.2), but with the distorted wave in the exit channel generated by the neutron Pot.2 of table 4.2. The dotted-dashed curve shows the results when the distortion in the incident channel is caused by the Coulomb potential alone.
The subject of this thesis was the study of the effects of spin-dependence in low-energy deuteron stripping on light and heavy target nuclei.

8.1 Deuteron Stripping on Heavy Target

In chapter 3, a careful study of the spin-dependence of low-energy deuteron elastic scattering associated with strong coupling to (d,p) reaction channels was presented. Earlier work (deuteron scattering from $^{208}$Pb) [To 83] had shown that the observed elastic vector analyzing power, $T^v$, could be explained quantitatively in terms of the strong coupling which exists between the elastic channel and the weakly bound ($Q$-values$=0$) neutron transfer channels of $^{209}$Pb. This reference employed the coupled reaction channels formalism and assumed pure Coulomb distortion between the deuteron and proton and the target. In particular, it did not include phenomenological or short-ranged spin-dependence as would be predicted, for example, by the folding model. The reduced charge number of the target nucleus and the increase in the scattering energy, relative to the Coulomb barrier energy, makes the present work more sensitive to the nuclear potentials than were the earlier calculations. Coupled reaction channels formalism has been applied consistently in the present work to study the relative importance of the folding model spin-orbit interaction and
the (d,p) channel coupling effects upon the calculated elastic $i_T$ in $^{90}$Zr at 5.5 MeV.

In contradiction to the original analysis by Knutson [Kn 75a] of deuteron elastic scattering, it is shown that the folding model spin-orbit interaction is unable to reproduce either the magnitude or the angular distribution of the experimental elastic vector analyzing power. The spin-dependence produced by the two channel coupled reaction channels calculations, in which the elastic channel and an individual transfer channel is included, are unable to reproduce the elastic $i_T$ data. The results presented in chapter 3 show however that the elastic $i_T$ receive large contributions from both the effective spin-dependence induced by the inclusion of all transfer channels and the folding model spin-orbit interaction. These results indicate also that the spin-dependence from the coupled reactions channel calculations is responsible for reproducing the angular shape seen in the experimental elastic $i_T$ data. This is seen by performing the coupled channels calculations in the absence of the folding model spin-orbit interaction. The inclusion of the folding model spin-orbit interaction increases the calculated elastic $i_T$ by about a factor of 2, and improves the fit to the experimental data.

In conclusion, it is possible to say that both the folding model spin-orbit interaction and transfer channel coupling play comparable roles in reproducing the elastic vector analyzing power. The coupling produces no significant corrections to the elastic differential cross section, however.
The results of chapter 3 also show that the higher order couplings of the elastic and transfer channels (rather than DWBA) produced effects on both the transfer cross sections and the vector analyzing powers. These effects produced a significant reduction in the transfer cross sections, particularly at backward angles. For the transfer channel vector analyzing powers, this effect is very significant. For instance, in the two $1/2^+$ states ($l=0$), the vanishing $(d,p)$ reaction $iT_{11}$ predicted by DWBA in the absence of spin-orbit distortion, receive modifications due to the coupling to the other transfer channels. The channel coupling effects are seen to be responsible for the negative trend of the calculated $iT_{11}$ in the two $1/2^+$ states at extreme backward angles, which is required by the data.

8.2 Deuteron Stripping on Light Target

In chapter 4, a model to calculate the $d(d,n)^3$He reaction was formulated. The model, a one-step calculation, is intermediate in complexity between the RGM (where the important aspects of antisymmetrization are included) and the simplest DWBA (where an accurate treatment of the structure of light nuclei including all internal D-states is possible). It was shown that the results of these calculations are highly sensitive to the model used for final state distortion. As was discussed in chapters 4 and 7, two neutron optical potentials were considered in the calculations. It was found that, if the light nuclear wave functions and interactions are treated carefully and the distorted wave in the exit channel is generated by neutron Pot.1, then the presented model seriously overestimated the magnitude of the transfer reaction cross section in the low energy regime ($E_{CM} = 55$
keV) relevant to fusion reactions. The results of these calculations have shown that the large magnitude of the transfer cross section is due to the P-wave contribution in the incident channel. Given this discrepancy in the cross section magnitude, it was not meaningful to draw conclusions as to the importance of the D-state from these results. As was discussed in chapter 7, the results of the present calculations indicate that the agreement with the data reported by Zhang et al. was the result of the use of an inadequate representation of the n-p vertex in the transfer amplitude. The analysis of the present calculations also shows that the long range behaviour of the d-\(^3\)He overlap, as deduced from realistic wave functions, is an important but less vital ingredient to the reaction amplitude at these energies, in contrast to calculations at tandem accelerator energies. This observation, together with the noted importance of the finite range effects, indicate that when the entrance channel is far below the Coulomb barrier this reaction is less surface and more volume dominated than at higher energy. This would indicate that the use of Gaussian based forms for the d-\(^3\)He overlap in the RGM calculations of Hofmann and Fick [Ho 84] is not likely to be a serious source of error in their analysis. Such volume dominance would also increase the importance of multistep processes involving the n-\(^3\)He and p-\(^3\)H partitions which are automatically included in the RGM calculations. An indication that this may be the case is also suggested by the observation that, if we apply our prescription, used here for fusion energies, to the \(^2\)H(d,n)\(^3\)He reaction at a centre of mass energy of 12.65 MeV, where multi-step processes are presumably less important, the calculated cross sections are of the same magnitude as the data [Va 63b].
The results of section 7.2 show however that if the prescribed model is quantitative applied again with the distorted wave in the exit channel generated by the neutron Pot.2, then the results of the calculations are in agreement with the data. These results have shown that transfer cross section is dominated by the S-wave contribution in the incident channel. In this situation it is possible to draw conclusions about the importance of the D-state effect. The full calculations showed a very small $^3\text{He}$ D-state effect ($s=2$ amplitude) and large suppression of the polarized cross section. The inclusion of deuteron D-state in the calculation improved the agreement with the data (unpolarized cross section). These calculations have shown that the polarized cross section arises as a result of the inclusion of the D-state of the deuteron.

These results have also shown that the calculated polarized and unpolarized $d(d,n)^3\text{He}$ reaction cross sections increase rapidly with the centre of mass energy in the entrance channel. These results show that the total unpolarized cross sections represent an improvement over those in the RGM calculation. The anisotropy factors ($C_2/C_0$) predicted by the present calculations are smaller than the experimental data by a factor of about 2.

The ratio of the polarized to unpolarized cross sections (attenuation factors) is found to be approximately a constant of about 15%. This reflects the small contributions of the $s=2$ amplitude at low energies. This conclusion supports the idea of suppressing the cross section of $d(d,n)^3\text{He}$ reaction by using polarized projectile and target nuclei and therefore the concept of a 'neutron-free' $d-^3\text{He}$ fusion reactor.
APPENDIX A

INFORMATION ABOUT $^{90}$Zr(d,p)$^{91}$Zr REACTION

(1) The $Q$-values of the $^{90}$Zr(d,p)$^{91}$Zr are defined as

$$ Q_i = E_p - E_d $$

$$ = \left[ M_d + M_{^{90}Zr} - M_p - M_{^{91}Zr} \right]^2 C - E_x $$

where $\left[ M_d + M_{^{90}Zr} - M_p - M_{^{91}Zr} \right]^2 C$ is the G.S. $Q$-value, $E_p$ and $E_d$ are the CMS kinetic energies of the outgoing proton and the incident deuteron. $E_x$ are the energies of the excited states in the outgoing partition.

(2) The separation energies for the reaction are

$$ S = Q + B $$

where $B$ is the deuteron binding energy ($B_d = 2.225$ MeV)

(3) The proton optical potentials were obtained from the global analysis of Becchetti and Greenlees [Be 69]. These potentials are defined as
\[ U(r) = V - V f(x) + \frac{2}{r} \frac{d}{dr} f(x) V_{so}(\sigma_{1}) \]

\[-i \left[ W f(x) - 4W \frac{d}{dx_D} f(x_D) \right] \]

where \( V \) is given by eq. (5.41), and

\[ f(x) = \left\{ 1 + \exp(\frac{x}{i}) \right\}^{-1} \]

where

\[ x = \left( \frac{r - r_A}{a} \right)^{1/3} \]

The systematic analysis of Becchetti and Greenlees fitted well a large number of elastic differential cross section and polarization data for \( A>40 \) and \( E<50 \) MeV. The parameters are

\[ V = 54.0 - 0.32E(N-Z)/A + 0.4(Z/A)^{1/3} \]
\[ r_0 = 1.17, \quad a_0 = 0.75 \]

\[ W = 0.22E - 2.7, \quad \text{or zero whichever is greater} \]
\[ W_D = 11.8 - 0.25E - 2.7 + 12(N-Z)/2, \quad \text{or zero whichever is greater} \]
\[ r_w = r_D = 1.32, \quad a_w = a_D = 0.51 + 0.7(N-Z)/A \]

\[ V_{so} = 6.2, \quad r_{so} = 1.01, \quad a_{so} = 0.75 \]

All the quantities given in table 3.2 have been calculated from this appendix.
NUMERICAL APPROACH TO LOCAL+RANK-2 SEPARABLE POTENTIAL

In this case we have

\[
(E - T - U_1)U(r) = C V_1(r) + C V_2(r)
\]

\[\text{Coul 1 1 2 2}\]  

B.1

where

\[
C_1 = \gamma \int dr' r^{2} V_1(r') U(r')
\]

B.2

\[
C_2 = \gamma \int dr' r^{2} V_2(r') U(r')
\]

B.3

and

\[
V_i(r) = \int dk_i \frac{e^{ik_i \cdot r}}{2\pi} V_i(k) / i^{3/2}
\]

B.4

where \(V_1(k)\) and \(V_2(k)\) are given by \(g(k)\) and \(h(k)\) of eq.(5.23) when \(l=0\) and of eq.(5.25) when \(l=1\). In eq.(B.1) \(U_1\) is the Coulomb potential defined in chapter 5 (eq.5.41). In order to solve B.1, we have to define two wave functions \(U_1(r)\) and \(U_2(r)\) which are any regular solutions of

\[
(E - T - U_1)U_1(r) = V_1(r)
\]

\[\text{Coul 1 1}\]  

B.5

\[
(E - T - U_2)U_2(r) = V_2(r)
\]

\[\text{Coul 2 2}\]  

B.6
from B.5,6 and B.1

\[(E - T - U)\{U(r) - C_1 U_1 (r) - C_2 U_2 (r)\} = 0\]  \hspace{1cm} \text{B.7}

and then

\[U(r) = C_1 U_1 (r) + C_2 U_2 (r) + BU(r)\]  \hspace{1cm} \text{B.8}

\(o\)

and \(U\) satisfies the solution

\[(E - T - U)U(r) = 0\]  \hspace{1cm} \text{B.9}

Inserting D.8 into D.2,3, yields

\[
\begin{pmatrix}
1 - I(VU) & -I(VU) \\
1 1 & 1 2 \\
I(VU) & 1 - I(VU) \\
2 1 & 2 2
\end{pmatrix}
\begin{pmatrix}
C \\
1 \\
I(VU) \\
2
\end{pmatrix}
= B
\begin{pmatrix}
1 \\
I(VU) \\
2
\end{pmatrix}
\]  \hspace{1cm} \text{B.10}

where

\[I(VU) = \sum_{ij} \int dr' r' V(r)U(r')\]

From B.10

\[
\begin{pmatrix}
C \\
1 \\
C \\
2
\end{pmatrix}
= B
\begin{pmatrix}
-I(VU) \\
1 \\
-I(VU) \\
2
\end{pmatrix}
\]  \hspace{1cm} \text{B.11}
and

\[
M^{-1} = \frac{1}{\det(M)} \begin{pmatrix}
1 & -I(V\ U) & -I(V\ U) \\
2 & 2 & 2 \\
2 & 1 & 1 \\
I(V\ U) & 1-I(V\ U) & 1-I(V\ U)
\end{pmatrix}
\]

It is possible to calculate each element in \( M^{-1} \), and hence from B.1 we have

\[
C/B = \beta_1 \quad \text{and} \quad C/B = \beta_2
\]

Substituting in B.8, we get

\[
U(r) = B_1 \left[ \beta_1 U_1 (r) + \beta_2 U_2 (r) + U(r) \right]
\]

The phase shift and the normalization constant B can be determined by matching between B.14 and the solution of B.9 at two different points, \( r_1, r_2 \geq r_{\text{int}} \). The technique for the numerical approach to rank-2 separable potential, in the absence of the Coulomb potential, is exactly the same as that described above but without the inclusion of the Coulomb potential in equation B.1. The wave function in this case can be written as

\[
U(r) = B_1 \left[ \beta_1 U_1 (r) + \beta_2 U_2 (r) + j(kr) \right]
\]
where \( j(kr) \) is the Bessel function, which corresponds the solution of eq. (B.9) in the absence of the Coulomb potential.
APPENDIX C

NUMERICAL SOLUTION OF EQUATION (5.29)

The explicit form of equation (5.31) after inserting both the Coulomb and the phenomenological potentials can be written as:

\[
\left[ -\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \left\{ k - \frac{1(l+1)}{r^2} - \frac{2m \text{ local}}{\hbar^2} \right\} \right] U = V \quad \text{C.1}
\]

where

\[
V = U + iW \quad \text{C.2}
\]

\( U \) and \( W \) are given by equations 5.41 and 5.60 respectively. In equation C.1, \( V \) is the non-local separable potential.

Consider the \( l=2 \) case where the d-d scattering is described by a rank-1 separable potential (see section 5.2.2). In this case \( V(k) \) was chosen to have the form (5.25). Then from equations (5.25) and (B.4)

\[
V(r) = -\sqrt{\frac{\pi}{8}} C_A (a_A r + 1) \{ \exp(-a_A r) \}/r \quad \text{C.3}
\]

Then equation C.1 becomes
\[
\left[ \frac{\text{d}^2}{\text{d}r^2} + \left\{ k - \frac{1(1+1)}{r^2} - \frac{2m}{\hbar^2} V_{\text{local}} \right\} \right] R =
\]
\[-\sqrt{\pi/8} C_A (a_A r+1)\exp(-a_A r)\]

where

\[ U = R/r \]

Equation C.4 has been solved numerically using the Runge-Kutta method. In this method, the starting values of the wave function and its first derivative must be specified. To determine these values, \( R \) has been written

\[ R = \sum_{i} A_i r^i \]

and the right hand side of equation C.4 has also been expanded as a power series of \( r \). Writing equation C.4 in terms of these series, it is possible to determine the coefficients \( A_i \) and hence the starting points of the wave function and its first derivative. This technique has been applied again to solve eq. (B.1) for \( l=0,1 \).
APPENDIX D

PLANE-WAVE CALCULATION FOR d(d,n)³He REACTION

If the distorting potentials are neglected, then the symmetrized transition amplitude (4.18) reduces to

\[
\mathcal{T}^{\text{sym}}_{\alpha \beta \sigma_3 \sigma_3} (k_2, k_1) = \int \frac{d^4r}{(2\pi)^4} e^{-i \mathbf{k}_2 \cdot \mathbf{r}} \langle \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) | \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) \rangle
\]

\[
\mathcal{V}_{np} \left[ \Phi_{\frac{1}{2} M_1, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) \Phi_{\frac{1}{2} M_2, \frac{1}{2} \sigma_3} (\mathbf{r}^{'}, 34) \right] \]

\[
i \mathbf{k}_1 \cdot \mathbf{r}_1, \quad \gamma_{\alpha \beta} = \{ e^{i \mathbf{k}_1 \cdot \mathbf{r}_1} + (-)^{\gamma_{\alpha \beta}} e^{-i \mathbf{k}_1 \cdot \mathbf{r}_1} \}
\]

D.1

where 1,3 denote the neutrons and 2,4 denote the protons, and

\[
\left[ \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) | \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) \right] = \sum_{M_1, \sigma_3} \frac{1}{2} M_3 \frac{1}{2} \sigma_3 | S_{\alpha \beta} \rangle \langle S_{\alpha \beta} | \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2}) \Phi_{\frac{1}{2} M_3, \frac{1}{2} \sigma_3} (\mathbf{r}, 124, \frac{3}{2})
\]

D.2

If we don't use the isospin formalism the correct ³He wavefunction is simply

\[
\Phi_{\frac{1}{2} M_3} (\mathbf{r}, 124, \frac{3}{2}) = U(\mathbf{r}, \frac{3}{2}) \chi_{00} (42) \Phi_{\frac{1}{2} M_3} (1)
\]

D.3

In these calculations the spatial part, \( U(\mathbf{r}, \frac{3}{2}) \), of the ³He wavefunction was considered as Gaussian form [Br 56]. The spin part of D.3 can be written as
\[ \chi_{00}^{(42)} (\frac{1}{2} M_3) \Phi_{\frac{1}{2} M_3} (1) = \begin{pmatrix} \begin{array}{c} \frac{1}{2} (4) \\ \frac{1}{2} (2) \end{array} \end{pmatrix} \theta, \begin{pmatrix} \frac{1}{2} (1), \frac{1}{2} M_3 \end{pmatrix} > \\
= \sum_{s} \frac{1}{2} (4) \begin{pmatrix} \frac{1}{2} (2) \frac{1}{2} (1) \end{pmatrix} s; \begin{pmatrix} \frac{1}{2} M_3 \end{pmatrix} > \\
< \frac{1}{2}, \begin{pmatrix} \frac{1}{2} \frac{1}{2} \end{pmatrix} s; \begin{pmatrix} \frac{1}{2} \frac{1}{2} \end{pmatrix} \theta, \begin{pmatrix} \frac{1}{2} \frac{1}{2} \end{pmatrix} > \\
= \sum_{s} \frac{1}{2} (4) \begin{pmatrix} \frac{1}{2} (2) \frac{1}{2} (1) \end{pmatrix} s; \begin{pmatrix} \frac{1}{2} M_3 \end{pmatrix} > S W(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \theta) S \\
\text{D.4} \\
\text{from D.1} \\
[ \begin{pmatrix} \phi_{1 M_1} (r', 34) \phi_{1 M_2} (r, 12) \end{pmatrix} ]_{s} = \sum_{M_1 M_2} (1 M_1 1 M_2 | S_{\alpha} \alpha) \phi_{1 M_1} (1) \phi_{1 M_2} (2) \\
= U(r) U(r') \sum_{M_1 M_2} (1 M_1 1 M_2 | S_{\alpha} \alpha) \begin{pmatrix} \frac{1}{2} \sigma'_3 \frac{1}{2} \sigma'_4 \end{pmatrix} \\
\phi_{1 M_1} (1, 2) \phi_{1 \frac{1}{2} \sigma'_3} (3) \phi_{1 \frac{1}{2} \sigma'_4} (4) \text{ D.5} \\
\text{The spatial functions } U \text{ are considered as Gaussian form [Ch 72]} \\
\text{From D.4 and D.5} \\
< \phi_{1 M_1} (12) | \chi_{00}^{(42)} \phi_{\frac{1}{2} M_3} (1) > = \\
< \frac{1}{2} \sigma_4 1 M_1 | \frac{1}{2} M_3 > \sqrt{3} W(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \theta) \phi_{1 \frac{1}{2} \sigma'_4} (4) \\
= - \frac{\sqrt{3}}{2} < 1 M_1 \frac{1}{2} \sigma'_4 | \frac{1}{2} M_3 > \phi_{\frac{1}{2} \sigma'_4} (4) \text{ D.6} \\
\text{Collecting the results or (D.2-6) and using the relation} \\
\sum_{1} \phi_{\frac{1}{2} \sigma'_i}^{*} (i) \phi_{\frac{1}{2} \sigma'_i} (i) = \delta_{i \sigma'_i} \sigma'_i
then eq. (D.1) becomes

$$\text{sym} \sum_{S,M} \langle k_2', k_1' \rangle = -\frac{\sqrt{3}}{2} \sum_{M_1, M_2, M_3} \langle \frac{1}{2} M_1 \frac{3}{2} \frac{1}{2} M_3 | 1 M_2 | S \rangle \langle 1 M_2 | S \rangle \langle \frac{1}{2} M_1 \frac{1}{2} \frac{1}{2} M_3 \rangle$$

$$\int dr_1 dr_2 e^{-i k_2 \cdot r_2} U(r, \rho) \ast V U(r) U(r')$$

Substitute in eq. (D.1), yields

$$\text{From the symmetry properties of Clebsch-Gordon and Racah Coefficients}$$

$$\langle \frac{1}{2} \frac{3}{2} \frac{1}{2} M_3 | 1 M_2 | S \rangle \langle 1 M_2 | S \rangle \langle \frac{1}{2} M_1 \frac{1}{2} \frac{1}{2} M_3 \rangle$$

$$= \sqrt{3} \sum_{\sigma_4} (-)^{\frac{3}{2} - M_3} \langle \frac{1}{2} M_1 \frac{1}{2} \frac{1}{2} - \sigma_4 \rangle \langle \frac{1}{2} \sigma_4 \frac{1}{2} - M_3 \rangle$$

$$= \sqrt{3} \sum_{\sigma_4} (-)^{\frac{3}{2} - M_3} \hat{f} \langle 1 M_1 \frac{1}{2} M_2 \rangle \langle \frac{1}{2} \sigma_4 \frac{1}{2} - M_3 \rangle$$

$$W(\frac{1}{2} \frac{1}{2} 1; \frac{1}{2} f)$$

The summation part of eq. (D.7) then becomes

$$= -\sqrt{6} \sum_{S} \sum_{S} \sum_{\sigma_3} \sum_{\sigma_4} W(\frac{1}{2} \frac{1}{2} 1; \frac{1}{2} S)$$

Substitute in eq. (D.4), yields
A calculation to test the results of eq. (D.10) was performed using the computer program TWOFNR. The calculation was performed with no Coulomb or distorting potentials present. The result of these calculations was compared against the result obtained by evaluating eq. (D.10). The ratio of the results of the two calculations at centre of mass energy of 55 keV is 1.002. This ratio remains approximately constant throughout the angular range.
REFERENCES

Ab 65 :  M. Abramowitz and I.A. Stegun, 'Handbook of Math. Functions',

Ab 87 :  S. Abu-Kamar, M. Igarashi, R. C. Johnson and J. A. Tostevin,
         The d(d,n)³He Reaction at very low energies, J. Phys G: Nucl.
         Phys., in the Press.


         Rev. 113B (1964), 3.


         on Polar. Phenom. in Nuclear Reactions', Univ. of Wisconsin

Be 50 :  F.M. Beiduk, J.R. Pruett and E.J. Konopinski, Phys. Rev. 77
         (1950), 622.


         49 (1982), 25.


Br 62 :  G. Breit, M.H. Hull, Jr., K.E. Lassila, K.D. Pyatt, Jr, and


Co : J.R. Comfort, Computer Program CHUCK3, Univ. of Pittsburg, Unpublised.


Fa 57 : U. Fano, Rev. Mod. Phys. 29 (1957), 74.


Go 87 : B. Goel, 'Notes on The Fusion of Polarized Deuterons', Institut für Neutronenphysik und Reaktortechnik, private communication.


Ha 72 : W. Haeberli, Rev. Brasil. Fis., 2 (1972), 187.


La 55 : W. Lakin, Phys. Rev. 98 (1955), 139.
 devastated by  
Ro 60 : E.S. Rost and N. Austern, Phys. Rev. 120 (1960), 1375.
Sa 77 : S. Saito, Supplement of the progress of Theoretical Physics, No. 62, 1977.


Wa 58 : S. Watanabe, Nucl. Phys. 8 (1958), 484.


Ya 54: Y. Yamaguchi Phys. Rev. 95 (1954), 1628;
Nucl. Phys. 201 (1954), 455.