Random walks exhibiting anomalous diffusion: Elephants, urns and the limits of normality

Michael J Kearney[1] and Richard J Martin[2]

[1]*Senate House, University of Surrey, Guildford, Surrey, GU2 7XH, UK*

[2]*Department of Mathematics, Imperial College London, South Kensington, London, SW7 2AZ, UK*

Abstract

A random walk model is presented which exhibits a transition from standard to anomalous diffusion as a parameter is varied. The model is a variant on the elephant random walk and differs in respect of the treatment of the initial state, which in the present work consists of a given number N of fixed steps. This also links the elephant random walk to other types of history dependent random walk. As well as being amenable to direct analysis, the model is shown to be asymptotically equivalent to a non-linear urn process. This provides fresh insights into the limiting form of the distribution of the walker's position at large times. Although the distribution is intrinsically non-Gaussian in the anomalous diffusion regime, it gradually reverts to normal form when N is large under quite general conditions.

Email: m.j.kearney@surrey.ac.uk; richard.martin1@imperial.ac.uk

Keywords: Diffusion, Stochastic particle dynamics, Stochastic processes

Contents

1. Introduction

The widespread observation of stochastic processes which exhibit non-standard or anomalous diffusion has led to a great deal of analysis of models which show such behaviours, see e.g. [1-3]. For a random walk, the defining characteristic is that the variance of the walker's position grows sub-linearly or super-linearly with time at large times, corresponding to sub-diffusion or super-diffusion respectively. Many different underlying mechanisms have been studied which generate these characteristics, incorporating ideas from the theory of continuous time random walks, Lévy flights, fractional dynamics and so on. Attention in this regard has also been given to non-Markovian walks with long term memory, some recent examples of which may be found in [4-11].

The elephant random walk (ERW), introduced by Schütz and Trimper [12], provides a non-trivial but theoretically tractable model which exhibits a transition from standard to super-diffusion as a parameter α is varied. The model is intrinsically non-Markovian in nature since the walker's next step is influenced by its entire preceding history. Since the original exposition, the model has been analysed and extended in various ways. One important line of enquiry has been to clarify the limiting nature of the distribution (probability density) of the walker's position at large times. Initially, the distribution was thought to be normal (i.e. Gaussian) irrespective of the value of α; subsequently it was found that it is actually non-Gaussian in the super-diffusion regime, although it is difficult to be specific and this is a topic of continuing research [13-19]. Minor modifications to the original formulation lead to sub-diffusion as well as super-diffusion [20, 21], and various other models turn out to have a close connection to the ERW as well, see e.g. [22-25].

In this paper we study another variant of the ERW, which differs from the original in that the initial state now consists of a given number $N > 0$ of *fixed* (i.e. pre-defined) steps. We call this Model I. The primary motivation is to understand how, in the anomalous diffusion regime, the limiting distribution and overall statistics are affected by the early step history. By considering the limit of a large number of free (i.e. random) steps beyond the N steps specified from the outset, and calculating the moments of the walker's position exactly, the nature of the transition from standard to super-diffusive behaviour is made clear, as is the transition of the distribution from being Gaussian to non-Gaussian. A key question of interest is what happens when N is large, i.e. $N >> 1$? It turns out that under such circumstances the anomalous diffusion regime persists, but the limiting distribution gradually reverts to Gaussian form under quite general conditions. This broadens one's understanding and also strengthens the link to other models. We show that a natural way to frame both the discussion and the analysis is to take a continuum limit as $N \to \infty$, in the spirit of how a basic isotropic random walk reduces in the limit to the Wiener process or standard Brownian motion.

Recent studies of the original ERW have benefited from a mapping onto an urn process, which has greatly aided understanding of the limiting behaviour as the number of steps grows large [16]. In other fields, the nature of urn-based random walk models has been explored in parallel [26-32]. We exploit this here by demonstrating asymptotic equivalence of Model I to a non-linear generalization of the standard two-component Pólya urn process [31], which we call Model II. This generalization has found application in many fields such as neuronal development [33], the organization of growing networks exhibiting preferential attachment [34-36], information cascade in voter models [37, 38] and the emergence of macrostructure in economics [39, 40]. Through this connection, we are able to derive exact results for the distribution of the walker's position at large times using an embedding approach based on the properties of continuous-time birth processes. In turn, this demonstrates the various limiting behaviours when N is large and provides a wider context within which the behaviour of the original ERW may be reconciled.

2. A modified elephant random walk

2.1 Preamble

The main model we examine, our Model I, is a variant of the original ERW [12]. We denote the position of the walker by the integer variable Δ_T, where T is the number of steps taken. The evolution follows,

$$\Delta_{T+1} = \Delta_T + \sigma_{T+1}; \qquad \Delta_T = \sum_{k=1}^{T} \sigma_k \tag{1}$$

where $\sigma_k = \pm 1$ is a random step variable. At time $T+1$ with $T \geq N$ the variable σ_{T+1} is chosen as follows: select one of the σ_k from the set $\{\sigma_k\} \equiv \{\sigma_1, \sigma_2, ..., \sigma_T\}$ at random with uniform probability T^{-1}, then with probability p choose $\sigma_{T+1} = \sigma_k$, else with probability $1-p$ choose $\sigma_{T+1} = -\sigma_k$. In this way the walker's next step is influenced by its entire history; an intrinsically non-Markovian characteristic.

Where our model deviates from the original ERW is that we assume the first $N > 0$ steps are *pre-defined*. In other words, the initial state is the set of *fixed* step variables $\{\sigma_{k \leq N}\} \equiv \{\sigma_1, \sigma_2, ..., \sigma_N\}$. How these are generated is not important for now; what is important is that they are fixed and characterized by the two parameters

N and $\Delta_N = \Sigma_{k=1}^{N}\sigma_k$, with $-N \le \Delta_N \le N$ (see figure 1). We exclude the case $N = 0$ which would correspond precisely to the original ERW. There, the initial step is undefined and requires a separate rule, namely with probability β choose $\sigma_1 = +1$, else with probability $1-\beta$ choose $\sigma_1 = -1$.

We introduce for future reference the 'population' variables A and B, which represent the total number of positive $(\sigma_k = +1)$ and negative $(\sigma_k = -1)$ steps taken up to time T (including those fixed steps used to construct the initial state). With reference to figure 1 we note that $A + B = T$ and $A - B = \Delta_T$, or more directly;

$$\begin{aligned} A &= \tfrac{1}{2}[T + \Delta_T]; \qquad B = \tfrac{1}{2}[T - \Delta_T] \\ a &= \tfrac{1}{2}[N + \Delta_N]; \qquad b = \tfrac{1}{2}[N - \Delta_N]. \end{aligned} \tag{2}$$

The initial sate consists of a positive steps and b negative steps, with $N = a + b$ and $\Delta_N = a - b$. We can now write the probability that $\sigma_{T+1} = \pm 1$ for $T \ge N$ as follows;

$$\begin{aligned} \Pr(\sigma_{T+1} = +1 \mid \{\sigma_k\}) &= \left(\frac{A}{A+B}\right)p + \left(\frac{B}{A+B}\right)(1-p) \\ \Pr(\sigma_{T+1} = -1 \mid \{\sigma_k\}) &= \left(\frac{A}{A+B}\right)(1-p) + \left(\frac{B}{A+B}\right)p. \end{aligned} \tag{3}$$

After some elementary algebra using (2) we then have,

$$\Pr(\sigma_{T+1} = \pm 1 | \{\sigma_k\}) = \frac{1}{2}\left[1 \pm \alpha\left(\frac{\Delta_T}{T}\right)\right]; \qquad T \geq N \tag{4}$$

where $\alpha \equiv 2p-1$ and $-1 \leq \alpha \leq 1$. In turn, the expected value of the next step conditional on the history is given by,

$$\left\langle \sigma_{T+1} \,|\, \{\sigma_k\} \right\rangle = \left\langle \sigma_{T+1} \,|\, \Delta_T \right\rangle = \alpha\left(\frac{\Delta_T}{T}\right). \tag{5}$$

A key point is that the step probability conditioned on the history reduces to a dependence on the present position and the number of steps to date. Thus the non-Markovian nature of the process manifests itself as a non-homogeneous Markov chain whose step probabilities depend explicitly on T.

When $p = \frac{1}{2}$ or $\alpha = 0$ the process (4) reduces to a basic isotropic random walk, where Δ_T for $T > N$ increases or decreases each step with equal probability $\frac{1}{2}$. It is obviously well known that the distribution of the walker's position is Gaussian as $T \to \infty$ in this case. More subtly, when $p = 1$ or $\alpha = 1$ the process (4) reduces to a trivial ballistic (i.e. deterministic) walk if $\Delta_N = \pm N$, when Δ_T either increases every step or decreases every step with probability 1. This is automatically the case for $N = 0$ (the original ERW) and also $N = 1$; however, when treated more generally with $|\Delta_N| < N$ the $\alpha = 1$ case is interesting in its own right and far from trivial. In fact, in the extreme case where $N = 2$ and $\Delta_N = 0$ we will show that the distribution of the walker's position when $\alpha = 1$ is actually uniform for all $T \geq 2$.

2.2 Anomalous diffusion

The signature of anomalous diffusion is that the variance $\mathrm{Var}(\Delta_T) \neq O(T)$ as $T \to \infty$. One can calculate the first two moments of Δ_T as governed by (4) exactly, from which an anomalous diffusion transition may be demonstrated as α varies. Adapting the analysis in [13], as set out in Appendix A, the expected position of the walker obeys the recursion,

$$\langle \Delta_{T+1} \rangle = \left[1 + \frac{\alpha}{T} \right] \langle \Delta_T \rangle \qquad (6)$$

with solution for $T \geq N$,

$$\langle \Delta_T \rangle = \Delta_N \frac{\Gamma(N)}{\Gamma(N+\alpha)} \frac{\Gamma(T+\alpha)}{\Gamma(T)} \qquad (7)$$

where $\Gamma(z)$ is the gamma function. The two cases $\alpha = 1$ and $\alpha = 0$ play an important role in what follows and for $T \geq N$;

$$\begin{aligned} \langle \Delta_T \rangle &= \Delta_N \left(\frac{T}{N} \right); \qquad && \alpha = 1 \\ \langle \Delta_T \rangle &= \Delta_N; && \alpha = 0. \end{aligned} \qquad (8)$$

The result for $\alpha = 0$ is what one expects for an isotropic random walk. Using the known asymptotic expansion as $z \to \infty$ [41],

$$\lim_{z\to\infty} \frac{\Gamma(z+c)}{\Gamma(z+d)} = z^{c-d}\left[1+\frac{(c-d)(c+d-1)}{2z}+O\left(\frac{1}{z^2}\right)\right] \tag{9}$$

one can let $T \to \infty$ in (7) with other parameters fixed to show that,

$$\langle \Delta_T \rangle \sim \Delta_N \frac{\Gamma(N)}{\Gamma(N+\alpha)} T^{\alpha}. \tag{10}$$

When $\alpha > 0$ the path of the walker moves away from zero on average, whereas when $\alpha < 0$ the path of the walker moves towards zero on average. This is intuitively clear, since depending on whether $p > \frac{1}{2}$ or $p < \frac{1}{2}$ one will get positive or negative reinforcement when selecting the next step based on the history of the previous steps.

Regarding the second moment, which is somewhat trickier to evaluate, we have the recursion (see Appendix A),

$$\langle \Delta_{T+1}^2 \rangle = \left[1+\frac{2\alpha}{T}\right]\langle \Delta_T^2 \rangle + 1 \tag{11}$$

and the solution for $T \geq N$ is,

$$\langle \Delta_T^2 \rangle = \left(\frac{1}{2\alpha-1}\right)\frac{\Gamma(T+2\alpha)}{\Gamma(T)}\left[\frac{\Gamma(N+1)}{\Gamma(N+2\alpha)} - \frac{\Gamma(T+1)}{\Gamma(T+2\alpha)}\right] + \Delta_N^2 \frac{\Gamma(N)}{\Gamma(N+2\alpha)}\frac{\Gamma(T+2\alpha)}{\Gamma(T)}. \tag{12}$$

It is apparent from the first term of (12) that the case $\alpha = \frac{1}{2}$ is of interest and marks a point of transition in behaviour. More specifically, if we let $T \to \infty$ with other parameters fixed we get using (9),

$$\left\langle \Delta_T^2 \right\rangle \sim \begin{cases} \dfrac{\Gamma(N)}{\Gamma(N+2\alpha)}\left[\dfrac{N}{2\alpha-1}+\Delta_N^2\right]T^{2\alpha}; & \alpha > \frac{1}{2} \\ \\ \left(\dfrac{1}{1-2\alpha}\right)T; & \alpha < \frac{1}{2}. \end{cases} \tag{13}$$

When $\alpha < \frac{1}{2}$ one has from (13) and (10) that $\mathrm{Var}(\Delta_T) = O(T)$ as $T \to \infty$, which implies that the walker exhibits standard diffusive behaviour. On the other hand, when $\alpha > \frac{1}{2}$ one has $\mathrm{Var}(\Delta_T) = O(T^{2\alpha})$ and this indicates super-diffusive behaviour. To understand what happens at the transition point itself, let $\varepsilon = 2\alpha - 1$ in (12) and take the limit $\alpha \to \frac{1}{2}$ using the known expansion for integer m [41],

$$\lim_{\varepsilon \to 0} \frac{\Gamma(m+\varepsilon)}{\Gamma(m)} = 1 + \varepsilon\psi_0(m) + O(\varepsilon^2); \qquad \psi_0(m) = -\gamma + \sum_{k=1}^{m-1}\frac{1}{k} \tag{14}$$

where $\psi_0(z)$ is the digamma function and $\gamma = 0.577...$ is the Euler-Mascheroni constant. One then has the exact result,

$$\left\langle \Delta_T^2 \right\rangle = T\sum_{k=N+1}^{T}\frac{1}{k} + \Delta_N^2\left(\frac{T}{N}\right); \qquad \alpha = \frac{1}{2}. \tag{15}$$

The summation in the first term of (15) leads to a logarithmic factor as $T \to \infty$ and one finds that $\mathrm{Var}(\Delta_T) \sim T \log T$. The logarithmic correction implies marginally super-diffusive behaviour.

For completeness and future reference, the exact variances for the cases $\alpha = 1$ and $\alpha = 0$ which complement (8) are given for $T \geq N$ by,

$$\begin{aligned} &\mathrm{Var}(\Delta_T) = \frac{(N^2 - \Delta_N^2)}{(N+1)} \frac{(T-N)}{N} \left(\frac{T}{N} \right); && \alpha = 1 \\ &\mathrm{Var}(\Delta_T) = T - N; && \alpha = 0. \end{aligned} \tag{16}$$

As a general observation, $\mathrm{Var}(\Delta_T)$ depends in a relevant way on N and Δ_N as $T \to \infty$ when $\alpha > \frac{1}{2}$, but to leading order is independent of N and Δ_N when $\alpha < \frac{1}{2}$. In other words, the initial state only has a lasting effect on the variance in the super-diffusion regime. For $\alpha = 1$, picking up on the point made at the end of the previous section, in the extreme case where $N = 2$ and $\Delta_N = 0$ this reduces to $\mathrm{Var}(\Delta_T) = T(T-2)/3$. This is what one expects if the distribution of the walker's position is uniform for all $T \geq 2$; a full proof will be given later.

The model as presented is closely related to that studied in [24, 25] in the context of a history-dependent random walk mapped onto a correlated binary string (with the sequence of 1's and 0's depicting the sequence of positive $(\sigma_k = +1)$ and negative $(\sigma_k = -1)$ steps taken). In that work, the role of α is to act as a measure of the strength of the correlations, whilst N is a measure of the length of the original string.

The analysis presented in [24, 25] also shows (based on a continuous-time Fokker-Planck equation) that the dynamics becomes super-diffusive when $\alpha > \frac{1}{2}$; in fact, essentially the same behaviour as given above is observed. As well as providing a unifying perspective, the approximations used here to study the $T \to \infty$ limit are controlled precisely and the analysis is consequently tighter than the Fokker-Planck approach. Similar findings are reported in different settings relating to random walks with slightly modified memory rules [4, 11], bond percolation on random recursive trees [22], two-component urn processes [27, 31, 32] and voting models with two types of voter behaviour [37].

2.3 Skewness and kurtosis

The results so far say little about the form of the distribution of Δ_T as $T \to \infty$. Regarding the original ERW, it was initially thought that the distribution is normal (i.e. Gaussian) for all α; later it became clear through a variety of theoretical approaches that this is true when $\alpha < \frac{1}{2}$ but not true when $\alpha > \frac{1}{2}$ [13-19]. In other words, the distribution is non-Gaussian in the super-diffusion regime. We can illustrate this here for arbitrary fixed N by considering as $T \to \infty$ the limiting skewness γ_∞ and kurtosis κ_∞, which involve the third and fourth moments of mean-shifted displacement $R_T \equiv \Delta_T - \langle \Delta_T \rangle$;

$$\gamma_\infty \equiv \lim_{T\to\infty} \frac{\langle {R_T}^3 \rangle}{\langle {R_T}^2 \rangle^{3/2}}; \qquad \kappa_\infty \equiv \lim_{T\to\infty} \frac{\langle {R_T}^4 \rangle}{\langle {R_T}^2 \rangle^{2}}. \tag{17}$$

For a Gaussian process, $\gamma_\infty = 0$ and $\kappa_\infty = 3$. With reference to the results provided in Appendix A, if we let $T \to \infty$ with other parameters fixed one finds using (9) that,

$$\left\langle \Delta_T^3 \right\rangle \sim \begin{cases} \dfrac{\Gamma(N)}{\Gamma(N+3\alpha)} \left[\Delta_N \left(\dfrac{3N+1+\alpha}{2\alpha-1} \right) + \Delta_N^3 \right] T^{3\alpha}; & \alpha > \tfrac{1}{2} \\ \\ \left(\dfrac{3\Delta_N}{1-2\alpha} \right) \dfrac{\Gamma(N)}{\Gamma(N+\alpha)} T^{1+\alpha}; & \alpha < \tfrac{1}{2} \end{cases} \tag{18}$$

together with

$$\left\langle \Delta_T^4 \right\rangle \sim \begin{cases} \dfrac{\Gamma(N)}{\Gamma(N+4\alpha)} \left[\dfrac{3N(N-1+4\alpha)}{(2\alpha-1)^2} - \dfrac{9N}{(4\alpha-1)} \right. & \\ \qquad\qquad \left. + \Delta_N^2 \dfrac{(6N+4+4\alpha)}{(2\alpha-1)} + \Delta_N^4 \right] T^{4\alpha}; & \alpha > \tfrac{1}{2} \\ \\ 3\left(\dfrac{1}{1-2\alpha} \right)^2 T^2; & \alpha < \tfrac{1}{2}. \end{cases} \tag{19}$$

These results, in conjunction with (10) and (13), show that $\gamma_\infty = 0$ and $\kappa_\infty = 3$ when $\alpha < \tfrac{1}{2}$ for any value of N, consistent with the distribution being Gaussian. However, when $\alpha > \tfrac{1}{2}$ one finds that $\gamma_\infty \neq 0$ and $\kappa_\infty \neq 3$ and hence the distribution is definitively non-Gaussian. As an illustration, in the case $\alpha = 1$ one has the exact results,

$$\gamma_\infty = -\frac{4\Delta_N \sqrt{N+1}}{(N+2)\sqrt{N^2 - \Delta_N^2}}$$

$$\kappa_\infty = 3 - \frac{6}{N+3} + \frac{24\Delta_N^2 (N+1)}{(N^2 - \Delta_N^2)(N+2)(N+3)}. \qquad (20)$$

In the extreme case where $N = 2$ and $\Delta_N = 0$ this reduces to $\gamma_\infty = 0$ and $\kappa_\infty = 9/5$, again consistent with the distribution of the walker's position being uniform. More generally, it will become apparent later that these values are attributed to a beta distribution $\mathrm{Beta}(a,b)$ with shape parameters given by (2), i.e. $a = \frac{1}{2}(N + \Delta_N)$ and $b = \frac{1}{2}(N - \Delta_N)$. Note that $\kappa_\infty \neq 3$ even when $\Delta_N = 0$; thus the non-Gaussian distribution is not simply a consequence of an asymmetrical initial state.

2.4 The large *N* limit

A major theme in the present work is to understand the behaviour of the walker in the super-diffusion regime when N is large, i.e. $N >> 1$. The above already affords some insights. Two interesting questions are (i) what happens when the initial state is symmetric, i.e. $\Delta_N = 0$, and (ii) what happens when the initial state symmetry is slightly broken, i.e. $\Delta_N \neq 0$? To this end, we make the additional assumption (which is not too restrictive) that $\Delta_N = O(N^{1/2})$, in other words, $|\Delta_N| << N$. In this case, one may show from (10), (13), (18) and (19) that as $N \to \infty$ for $\alpha > \frac{1}{2}$;

$$\gamma_\infty = -\frac{\Delta_N (2\alpha-1)^{3/2}(1+3\alpha)}{N^{3/2}}\left[1+O\left(\frac{1}{N}\right)\right]$$

$$\kappa_\infty = 3 - \frac{6(2\alpha-1)^2(1+2\alpha)}{(4\alpha-1)N}\left[1+O\left(\frac{1}{N}\right)\right]. \qquad (21)$$

This strongly suggests that the distribution increasingly reverts to normal (i.e. Gaussian) form as $N \to \infty$ for *all* values of α, provided $|\Delta_N| << N$.

This is already an important observation, however care is needed since the procedure leading to (21) is based on one particular way of taking the twin limits $T \to \infty$ and $N \to \infty$. A more general approach should allow for the ratio T/N to be held constant during the limiting process. To do this, we can usefully consider a continuum limit of the discrete process governed by (1) and (4), somewhat analogous to how standard Brownian motion may be viewed as the continuum limit of a basic isotropic random walk. Let us write $\Delta_N = \chi_0 \sqrt{N}$ and $T = (1+\eta)N$, with χ_0 and $\eta \geq 0$ fixed but otherwise arbitrary, and consider the limit $N \to \infty$. Our assertion is that the scaled random variable χ_η defined as,

$$\chi_\eta \equiv \lim_{N\to\infty} \frac{\Delta_{T=\lfloor (1+\eta)N \rfloor}}{\sqrt{N}} \qquad (22)$$

exists and has a well-defined Gaussian distribution for all α. The variable η plays the role of a rescaled time parameter, whilst χ_0 is seen to be a rescaled initial position for the free (random) portion of the path, i.e. the steps of the path for which $T > N$. Given this, based on (7) and (12) it follows that,

$$\left\langle \chi_\eta \right\rangle = \chi_0 \, (1+\eta)^\alpha \tag{23}$$

$$\left\langle \chi_\eta^2 \right\rangle = \frac{1}{2\alpha - 1}(1+\eta)\left[(1+\eta)^{2\alpha-1} - 1\right] + \chi_0^2 \, (1+\eta)^{2\alpha} \, . \tag{24}$$

Moreover, given the results in Appendix A, we also have,

$$\left\langle \chi_\eta^3 \right\rangle = \frac{3\chi_0}{2\alpha - 1}(1+\eta)^{\alpha+1}\left[(1+\eta)^{2\alpha-1} - 1\right] + \chi_0^3 \, (1+\eta)^{3\alpha} \tag{25}$$

$$\begin{aligned} \left\langle \chi_\eta^4 \right\rangle = {} & \frac{3}{(2\alpha - 1)^2}(1+\eta)^2\left[(1+\eta)^{2\alpha-1} - 1\right]^2 \\ & + \frac{6\chi_0^2}{(2\alpha - 1)}(1+\eta)^{2\alpha+1}\left[(1+\eta)^{2\alpha-1} - 1\right] + \chi_0^4 (1+\eta)^{4\alpha} \, . \end{aligned} \tag{26}$$

These results for the first four moments are precisely what one would expect on the basis of χ_η having a normal distribution with density,

$$p(\chi_\eta) = \frac{1}{\sqrt{2\pi \mathrm{Var}(\chi_\eta)}} \exp\left(- \frac{(\chi_\eta - \langle \chi_\eta \rangle)^2}{2\mathrm{Var}(\chi_\eta)} \right) \tag{27}$$

with mean $\left\langle \chi_\eta \right\rangle$ given by (23) and variance based on (23) and (24),

$$\mathrm{Var}(\chi_\eta) = \frac{1}{2\alpha - 1}(1+\eta)\left[(1+\eta)^{2\alpha-1} - 1\right]. \tag{28}$$

This holds true for *all* α. However, it is evident that the super-diffusion regime persists in this rescaled picture; thus as $\eta \to \infty$, for $\alpha < \frac{1}{2}$ so $\mathrm{Var}(\chi_\eta) = O(\eta)$, whilst for $\alpha > \frac{1}{2}$ so $\mathrm{Var}(\chi_\eta) = O(\eta^{2\alpha})$. The key message here (not necessarily obvious) is that one can have anomalous diffusion governed by a Gaussian distribution. At the transition point itself, one can take the limit $\alpha \to \frac{1}{2}$ in (28) to derive,

$$\mathrm{Var}(\chi_\eta) = (1+\eta)\log(1+\eta); \qquad \alpha = \tfrac{1}{2}. \qquad (29)$$

In the next Section, these results are explored from a different perspective which provides a more fundamental justification of (27). A discussion of the limitations of specifically choosing $\Delta_N = O(N^{1/2})$ throughout the analysis will be deferred to Section 4.

3. Urn representations

3.1 Preamble

The form of (3) is that of a generalized two-component urn model, which has as special cases the standard Pólya urn model ($p = 1$ or $\alpha = 1$) and the Friedman urn model ($p = 0$ or $\alpha = -1$) [42]. To make this connection clear, imagine an urn containing A red and B green balls corresponding to the total number of positive $(\sigma_k = +1)$ and negative $(\sigma_k = -1)$ steps taken up to time $T \geq N$. Initially there are a red balls and b green balls corresponding to the fixed steps used to construct the initial state up to time N. At time $T+1$ a ball is selected uniformly at random and replaced in the urn together with either a ball of the same colour (with probability p), or with a ball of the opposite colour (with probability $1-p$). In this manner, the next step of the walk is chosen randomly according to (3), and the analysis of the model is equivalent to the analysis of the corresponding urn process. The variables A and B taken as a pair are derived from a two-dimensional Markov chain.

This connection to urn processes was essentially pointed out in a different contextual setting in [27] and has been the basis of recent work which has shed considerable light on the original ERW [16]. Here we utilise the urn picture but stay close to the idea of studying random paths. Let us define $S=T-N$ such that $S\geq 0$ denotes the number of free (random) steps the walker takes after the initial N given steps. The basic aim is to calculate the end-point probability $P_S(A,B)$ that, after S steps, a walker which starts at (a,b) finishes at the point (A,B). This involves summing over all possible paths accounting for the different probabilistic weight of each path (see figure 1). It is clear from this picture that $P_S(A,B)$ is invariant under the interchange $(a,A)\Leftrightarrow(b,B)$, which later on will be a powerful guiding principle for controlling approximations. Given S, the two the variables A, B are related since they lie on the line $A+B=N+S$, subject to $A\geq a$ and $B\geq b$. Alternatively, given A and B, then S is uniquely determined

We can consider two special cases in isolation. When $\alpha=0$, as noted above, Model I reduces to that of an isotropic random walk. In this case, each contributing path has the same probabilistic weight and it is trivial to write down,

$$P_S^{(\alpha=0)}(A,B) = \frac{S!}{(A-a)!(B-b)!} \times \left(\frac{1}{2}\right)^S . \tag{30}$$

From this one can calculate the mean and variance of, say, the variable A,

$$\langle A \rangle = a + \frac{S}{2}; \qquad \mathrm{Var}(A) = \frac{S}{4}. \tag{31}$$

The mean and variance of $\Delta_T = 2A - T$ are then as given in (8) and (16). The fact that $\mathrm{Var}(A) = O(S)$ as $S \to \infty$ implies the walker exhibits standard diffusive behaviour. One can further analyse (30) using Stirling's approximation to derive as $S \to \infty$ for a, b fixed,

$$P_S^{(\alpha=0)}(A,B) \approx \sqrt{\frac{2}{\pi S}} \exp\left\{ -\frac{[(A-a)-(B-b)]^2}{2S} \right\}. \tag{32}$$

This is a familiar result and the limiting distribution (and hence for Δ_T also) is naturally of Gaussian form.

On the other hand, with the choice $\alpha = 1$ Model I reduces to the standard Pólya urn process [42]. For this case there is also a reasonably elementary derivation of $P_S(A,B)$ based on the fact that the probabilistic weight of each contributing path, although more complicated, is again the same, see e.g. [28];

$$P_S^{(\alpha=1)}(A,B) = \frac{S!}{(A-a)!\,(B-b)!} \times \frac{(a+b-1)!\,(A-1)!\,(B-1)!}{(a-1)!\,(b-1)!\,(A+B-1)!}. \tag{33}$$

With more effort, one can calculate mean and variance of, say, the variable A [43],

$$\langle A \rangle = a + \frac{aS}{a+b}; \qquad \mathrm{Var}(A) = \frac{abS(a+b+S)}{(a+b)^2(a+b+1)}. \tag{34}$$

Once again, the mean and variance of Δ_T are then as given in (8) and (16). This time the fact that $\mathrm{Var}(A) = O(S^2)$ as $S \to \infty$ implies the walker exhibits super-diffusive behaviour. Using Stirling's approximation one may show that as $S \to \infty$ with $a > 0$ and $b > 0$ fixed,

$$P_S^{(\alpha=1)}(A,B) \approx \frac{(a+b-1)!}{(a-1)!(b-1)!} \left(\frac{A-a}{S} \right)^{a-1} \left(\frac{B-b}{S} \right)^{b-1} \frac{1}{S} \tag{35}$$

where the approximated mean and variance of A according to (35) are now given by,

$$\langle A \rangle \approx a + \frac{aS}{a+b}; \qquad \mathrm{Var}(A) \approx \frac{abS^2}{(a+b)^2(a+b+1)}. \tag{36}$$

In this case the limiting distribution (and hence for Δ_T also) is clearly non-Gaussian; it is in fact the distribution $\mathrm{Beta}(a,b)$. When the shape parameters a and b are small the deviation from normality is significant; e.g. in the extreme case $a = b = 1$ one has from (33) that $P_S^{(\alpha=1)}(A,B) = (S+1)^{-1}$ and the distribution is uniform. With reference to (2), this proves the statements made earlier about the extreme case where $N = 2$

and $\Delta_N = 0$. However, as a and b get progressively larger so the distribution gradually reverts to Gaussian form under quite general conditions, which is a central message in what follows for the entire super-diffusion regime $\alpha > \frac{1}{2}$. To see how this works when $\alpha = 1$, recall once more that $a = \frac{1}{2}(N + \Delta_N)$, $b = \frac{1}{2}(N - \Delta_N)$, then as in Section 2.4 let $N \to \infty$ with $\Delta_N = O(\sqrt{N})$ and $S = O(N)$. Note that this implies $|a - b| << a + b$, which corresponds to $|\Delta_N| << N$. One may then show from (33) that,

$$P_S(A, B) \approx \sqrt{\frac{2(a+b)}{\pi(a+b+S)S}} \exp\left(-\frac{2(bA - aB)^2}{(a+b)(a+b+S)S}\right) \qquad (37)$$

where the approximated mean and variance of A according to (37) are now given by,

$$\langle A \rangle \approx a + \frac{aS}{a+b}; \qquad \text{Var}(A) \approx \frac{(a+b+S)S}{4(a+b)}. \qquad (38)$$

These can be deduced from the exact result (34) in the stated limit.

3.2 A non-linear extension

Unlike the two special cases discussed above, the calculation of the end-point probability $P_S(A, B)$ for general α, wherein each directed lattice path between (a, b) and (A, B) has a different probabilistic weight, is much more challenging. To make progress we introduce another urn model, which we call Model II, defined via modified transition probabilities based on a different function than (3) of the population variables A and B. Thus at time $T+1$ for $T \geq N$, the variable σ_{T+1} is now chosen as follows;

$$\Pr^{\mathrm{II}}(\sigma_{T+1} = +1) = \frac{A^{\alpha}}{A^{\alpha} + B^{\alpha}}$$
$$\Pr^{\mathrm{II}}(\sigma_{T+1} = -1) = \frac{B^{\alpha}}{A^{\alpha} + B^{\alpha}}. \tag{39}$$

This is a non-linear variant of the standard Pólya urn model, see e.g. [31, 35, 36]. Remarkably, it turns out one can evaluate $P_S^{\mathrm{II}}(A,B)$ exactly for this model.

Of key importance to the discussion is the fact that (3) and (39) are *identical* when $\alpha = 1, 0, -1$, and also asymptotically equivalent in the appropriate limit for all α. To see this directly, using (2) we can rewrite (39) as follows;

$$\Pr^{\mathrm{II}}(\sigma_{T+1} = \pm 1) = \frac{(1 \pm \Gamma_T)^{\alpha}}{(1+\Gamma_T)^{\alpha} + (1-\Gamma_T)^{\alpha}}; \qquad \Gamma_T \equiv \frac{\Delta_T}{T} \tag{40}$$

and the equivalence to (4) for the values $\alpha = 1, 0, -1$ follows immediately. Moreover, if $|\Delta_T| << T$ such that $|\Gamma_T| << 1$, one has from (40) that,

$$\Pr^{\mathrm{II}}(\sigma_{T+1} = \pm 1) = \frac{1}{2}\left[1 \pm \alpha \Gamma_T\right] + O(\Gamma_T^3). \tag{41}$$

The first term in (41) corresponds precisely to (4) and as $|\Gamma_T| \to 0$ is progressively a better approximation for any α. It seems plausible to assume that $|\Gamma_T| \to 0$ almost surely as $T \to \infty$; this intuition turns out to be correct for $\alpha \neq 1$ (the case $\alpha = 1$ is special but has already been solved exactly in Section 3.1). Thus, provided $N >> 1$ and $|\Delta_N| << N$ so that initially $|\Gamma_N| << 1$, it follows almost surely that $|\Gamma_T| << 1$ for all $T \geq N$. One can therefore view (40) as an effective extension of (4) which will have the same behaviour as $N \to \infty$ for all α, provided Δ_N is suitably small.

The calculation of $P_S^{\text{II}}(A, B)$ for Model II may be carried out by adapting the embedding technique discussed in [31]. By way of preliminaries, let $X(t)$ be a birth process on the non-negative integers j with state-dependent transition rates j^{α}, with $X(0) = a > 0$. Let $P(a, k; t)$ denote the probability of $X(t)$ being in state $k \geq a$ at time t, with $P(a, a; t) = e^{-a^{\alpha} t}$. Further, let $p(a, k; \tau)$ denote the probability density for the first passage time τ for $X(t)$ to reach state $k > a$. This is simply the transition rate from state $k - 1$ multiplied by the probability of being in state $k - 1$ at time τ; in other words, $p(a, k; \tau) = (k-1)^{\alpha} P(a, k-1; \tau)$, with $p(a, a+1; \tau) = a^{\alpha} e^{-a^{\alpha} \tau}$. Now let $Y(t)$ be an independent version of the same process with $Y(0) = b > 0$. Modifying the arguments set out in [31], one can calculate the probability $P_S^{\text{II}}(A, B)$ by integrating over all possible realisations of $X(t)$ and $Y(t)$ for which at the time the process $Y(t)$ makes the transition to state B, the process $X(t)$ is in a given state A, or at the time the process $X(t)$ makes the transition to state A, the process $Y(t)$ is in a given state B. This gives for $S > 0$,

$$P_S^{\,\mathrm{II}}(A,B) = \int_0^\infty p(b,B;\tau)\,P(a,A;\tau)\,d\tau + \int_0^\infty p(a,A;\tau)\,P(b,B;\tau)\,d\tau. \quad (42)$$

The fact that this procedure correctly captures the precise probabilistic weight of each possible directed path relies on the memoryless nature and the independence of the birth processes $X(t)$ and $Y(t)$, coupled with the observation that for a given step, say $(m,n) \to (m+1,n)$; c.f. (39),

$$\begin{aligned}\Pr^{\mathrm{II}}((m,n) \to (m+1,n)) &= \int_0^\infty p(m,m+1;\tau)P(n,n;\tau)\,d\tau \\ &= \int_0^\infty m^\alpha\, e^{-m^\alpha \tau} \times e^{-n^\alpha \tau}\,d\tau = \frac{m^\alpha}{m^\alpha + n^\alpha}\end{aligned} \quad (43)$$

and correspondingly $\Pr^{\mathrm{II}}((m,n) \to (m,n+1)) = n^\alpha / (n^\alpha + m^\alpha)$.

To make further progress, we note that for $\alpha \neq 0$ (see [31] for a full discussion),

$$P(a,k;t) = \frac{1}{k^\alpha} \sum_{j=a}^{k} j^\alpha \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{k} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} e^{-j^\alpha t} \quad (44)$$

$$p(a,k;\tau) = \sum_{j=a}^{k-1} j^\alpha \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{k-1} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} e^{-j^\alpha \tau}. \quad (45)$$

Let us consider the first integral I_1 in (42), recognizing that the second integral I_2 is the same as I_1 after the exchange $(a,A) \Leftrightarrow (b,B)$. Using (44) and (45) and carrying out the integral one obtains,

$$I_1 = \frac{1}{A^\alpha} \sum_{j=a}^{A} \sum_{j'=b}^{B-1} \frac{j^\alpha j'^\alpha}{j'^\alpha + j^\alpha} \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{A} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} \left\{ \prod_{\substack{\ell'=b \\ \ell' \neq j'}}^{B-1} \frac{\ell'^\alpha}{\ell'^\alpha - j'^\alpha} \right\}. \tag{46}$$

This may be simplified in two ways using the following partial fraction identity, which holds for any z [31],

$$\sum_{j=a}^{k-1} \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{k-1} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} \frac{j^\alpha}{j^\alpha + z} \equiv \prod_{j=a}^{k-1} \frac{j^\alpha}{j^\alpha + z} \tag{47}$$

to carry out either the summation over j' or j;

$$I_1 = \frac{1}{A^\alpha} \sum_{j=a}^{A} j^\alpha \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{A} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} \left\{ \prod_{\ell'=b}^{B-1} \frac{\ell'^\alpha}{\ell'^\alpha + j^\alpha} \right\}$$

$$I_1 = \frac{1}{A^\alpha} \sum_{j'=b}^{B-1} j'^\alpha \left\{ \prod_{\substack{\ell'=b \\ \ell \neq j'}}^{B-1} \frac{\ell'^\alpha}{\ell'^\alpha - j'^\alpha} \right\} \left\{ \prod_{\ell=a}^{A} \frac{\ell^\alpha}{\ell^\alpha + j'^\alpha} \right\}. \tag{48}$$

One can now evaluate the second integral I_2 in (42) using the second form of I_1 in (48) and making the exchange $(a,A) \Leftrightarrow (b,B)$. Adding this to the first form of I_1 in (48) gives, after some straightforward algebraic manipulations,

$$P_S^{\,\mathrm{II}}(A,B) = \frac{A^\alpha + B^\alpha}{A^\alpha B^\alpha} \sum_{j=a}^{A} j^\alpha \left\{ \prod_{\substack{\ell=a \\ \ell \neq j}}^{A} \frac{\ell^\alpha}{\ell^\alpha - j^\alpha} \right\} \left\{ \prod_{\ell'=b}^{B} \frac{\ell'^\alpha}{\ell'^\alpha + j^\alpha} \right\}. \qquad (49)$$

This is the *exact* solution of Model II for all $\alpha \neq 0$ given $a > 0$ and $b > 0$, i.e. $N \geq 2$. As written, the exchange symmetry $(a, A) \Leftrightarrow (b, B)$ is no longer obvious; other equivalent forms can be written down but this one is relatively compact. The result for $\alpha = 0$ was previously given in (30). It should be possible, in principle, to take the limit $\alpha \to 0$ in (49) to recover (30), although we do not pursue that here. Deriving (49) addresses one of the lines of enquiry left open in [31].

For the case of the standard Pólya urn $\alpha = 1$, where Model I and Model II coincide precisely, one can obtain by rewriting (49) more explicitly;

$$P_S^{(\alpha=1)}(A,B) = \frac{(A+B)(A-1)!(B-1)!}{(a-1)!(b-1)!} \sum_{j=a}^{A} \frac{(-1)^{j-a}(j+b-1)!}{(j-a)!(A-j)!(B+j)!}. \qquad (50)$$

This is quite different in appearance from (35), but their equivalence can be verified numerically. In figure 2, a comparison is made between $P_S^{\,\mathrm{II}}(A,B)$ for $\alpha = 1$ based on numerical simulations of the process (39) and the exact result (49), in this instance plotted as a function of A with the parameters S, a, b fixed. The agreement is excellent. The asymmetry with respect to the line $A = B$ due to the small initial state asymmetry $a > b$ grows with S when $\alpha > 0$ and is clearly visible. When $\alpha = 1$ the skewness derived from simulations is $\gamma_S \approx -0.034$, which compares favourably with the theoretical value $\gamma_\infty = -0.0329...$ derived from (20), whereas the kurtosis derived from simulations is $\kappa_S \approx 2.85$, which compares very well with a theoretical value $\kappa_\infty = 2.8552...$ derived from (20). Also shown in figure 2 is the close agreement between numerical simulations of the process (39) and the exact result (49) for the critical case $\alpha = \frac{1}{2}$.

For completeness, for the Friedman urn case $\alpha = -1$, where Model I and Model II also coincide precisely, one has by rewriting (49),

$$P_S^{(\alpha=-1)}(A,B) = (A+B)\sum_{j=a}^{A} \frac{(-1)^{A-j}\, j^{A+B-a-b}(j+b-1)!}{(j-a)!(A-j)!(B+j)!}. \qquad (51)$$

In figure 3 we compare the exact result against the results of numerical simulations of the process (39), again with excellent agreement. For $\alpha < 0$ the initial asymmetry with respect to the line $A = B$ decreases with S and is therefore practically invisible.

3.3. The large *N* limit revisited

The limiting process previously discussed in Section 3.1 is based on letting $N \to \infty$ with $\Delta_N = O(\sqrt{N})$ and $S = O(N)$. When applied to Model II, starting from the exact result (49) is not especially easy; however, one can derive an asymptotically precise approximation for $P_S^{\,\mathrm{II}}(A,B)$ based on the integral representation (42). This will then also be asymptotically precise for Model I. In what follows, the invariance under the exchange $(a,A) \Leftrightarrow (b,B)$ is preserved at each stage of approximation.

The key idea is that the first passage time τ for the birth process $X(t)$ is asymptotically normal. This means that in the limit of interest one can approximate the first passage time probability density (45) as follows,

$$p(a,k;\tau) \approx \frac{1}{\sqrt{2\pi V(a,k)}} \exp\left\{-\frac{(\tau - M(a,k))^2}{2V(a,k)}\right\} \tag{52}$$

where $M(a,k)$ and $V(a,k)$ are the mean and variance of τ respectively;

$$M(a,k) = \sum_{j=a}^{k-1} \frac{1}{j^{\alpha}}; \qquad V(a,k) = \sum_{j=a}^{k-1} \frac{1}{j^{2\alpha}}. \tag{53}$$

The latter follow from the fact that the time τ_j spent in state j has probability density $p(j, j+1; \tau_j) = j^{\alpha} e^{-j^{\alpha}\tau_j}$. The proof of (52) is essentially that of the central limit theorem for independent but non-identically distributed random variables [31].

One can then use (52) in conjunction with (42) to approximate $P_S^{\mathrm{II}}(A,B)$. Noting that $p(a,k+1;T) = k^{\alpha} P(a,k;T)$, one can carry out the integrals in (42) analytically after extending the lower limits to $-\infty$ to derive,

$$P_S^{\mathrm{II}}(A,B) \approx \frac{A^{-\alpha} + B^{-\alpha}}{\sqrt{2\pi(V(a,A)+V(b,B))}} \exp\left(-\frac{(M(a,A)-M(b,B))^2}{2(V(a,A)+V(b,B))}\right). \tag{54}$$

One can also approximate (53) using the Euler-Maclaurin theorem;

$$M(a,k) \approx \frac{k^{1-\alpha} - a^{1-\alpha}}{1-\alpha}; \qquad V(a,k) \approx \frac{k^{1-2\alpha} - a^{1-2\alpha}}{1-2\alpha}. \tag{55}$$

The special nature of the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$ is reflected in (55). It follows after some straightforward algebra that (54) further simplifies to,

$$P_S^{\,\mathrm{II}}(A,B) \approx \sqrt{\frac{2(2\alpha-1)N^{2\alpha-1}}{\pi(N+S)[(N+S)^{2\alpha-1} - N^{2\alpha-1}]}} \times \exp\left(-\frac{(2\alpha-1)[(A-B)N^{\alpha} - \Delta_N (A+B)^{\alpha}]^2}{2N(N+S)[(N+S)^{2\alpha-1} - N^{2\alpha-1}]}\right). \tag{56}$$

As a check, these agree with (32) and (37) when $\alpha = 0$ and $\alpha = 1$. The approximated mean and variance of, say, A as derived from (56) are given by,

$$\langle A \rangle \approx \frac{N+S}{2} + \frac{\Delta_N}{2}\left(\frac{N+S}{N}\right)^{\alpha}; \qquad \mathrm{Var}(A) \approx \frac{1}{4}\left(\frac{N+S}{2\alpha-1}\right)\left(\left(\frac{N+S}{N}\right)^{2\alpha-1} - 1\right). \tag{57}$$

As a further check, these also agree with (31) and (38) when $\alpha = 0$ and $\alpha = 1$. In figure 4, one sees that for $\alpha = -1$ and $\alpha = \frac{1}{2}$ the approximate result (56) is in very good agreement with the exact results shown previously in figure 2 and figure 3, even on a logarithmic scale which exposes the tail behaviour. Thus the anticipated Gaussian form is evident. More pertinently, this is also true in the super-diffusive regime when $\alpha = 1$, although here the approximation is not quite as good in the tails reflecting the persistent influence of the initial state.

In conjunction with (2), the stochastic behaviour of the variable Δ_T as $T \to \infty$ may be determined from $P_S(A, B)$ through the normalised probability density;

$$p(\Delta_T) = \frac{1}{2} P_{T-N}\left(\frac{T+\Delta_T}{2}, \frac{T-\Delta_T}{2}\right). \tag{58}$$

One can therefore write down based on (56) and (58) an estimate for the normalised probability density;

$$p(\Delta_T) \approx \frac{1}{\sqrt{2\pi \mathrm{Var}(\Delta_T)}} \exp\left(-\frac{(\Delta_T - \langle \Delta_T \rangle)^2}{2\mathrm{Var}(\Delta_T)}\right) \tag{59}$$

where the approximated mean and variance according to (59) are given by,

$$\begin{aligned} \langle \Delta_T \rangle &\approx \Delta_N \left(\frac{T}{N}\right)^{\alpha} \\ \mathrm{Var}(\Delta_T) &\approx \frac{T}{2\alpha - 1}\left(\left(\frac{T}{N}\right)^{2\alpha-1} - 1\right). \end{aligned} \tag{60}$$

The variance in (60) does not depend on Δ_N because of the assumption $|\Delta_N| << N$ made in the derivation. At the transition point, one can take the limit $\alpha \to \frac{1}{2}$ in (60) to derive,

$$\mathrm{Var}(\Delta_T) \approx T \log\left(\frac{T}{N}\right); \qquad \alpha = \tfrac{1}{2}. \qquad (61)$$

It follows from (59) and (60) that $|\Gamma_T| \equiv |\Delta_T / T| \to 0$ almost surely as $T \to \infty$ for $\alpha \neq 1$, which supports the earlier assumption. By taking the full limit $N \to \infty$, one can now confirm that the random variable χ_η discussed in Section 2.4 is Gaussian in form and satisfies (27).

3.4. Recurrence and transience

It is known in the context of Model II that if $\alpha < \frac{1}{2}$ both populations A, B tend to infinity almost surely, but in doing so alternate as to which is the largest arbitrarily many times [35, 36]. This means that Δ_T fluctuates around zero infinitely often (recurrent behaviour). Conversely, if $\frac{1}{2} < \alpha \leq 1$ both populations A, B tend to infinity almost surely, but beyond a certain number of steps (which is realization specific) one is always larger than the other [35, 36]. This means that beyond a certain number of steps Δ_T is either permanently positive or negative (transient behaviour). By extension of the previous analysis, these findings also apply to Model I. To study this further, based on (59) one can calculate the distribution of the first passage time to zero, T_0, namely the time at which a given walker first reaches zero. Following the arguments given in [31], one has $\Pr(T_0 > T) = 1 - 2\Pr(+/-)$. Here, $\Pr(+/-)$ is the probability that the position of the walker after $T > N$ steps has the opposite sign from its initial value. One can obtain an asymptotically precise estimate for $\Pr(+/-)$

by integrating the probability density (59) over the half interval $(-\infty,0]$ if $\Delta_N > 0$, or the half interval $[0,\infty)$ if $\Delta_N < 0$;

$$\Pr(+/-) \approx \Phi[-\theta_T]; \qquad \Phi(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi \tag{62}$$

where,

$$\theta_T = \frac{|\Delta_N|}{\sqrt{N}} \times \begin{cases} \dfrac{(2\alpha-1)^{1/2}}{\left[1-\left(\dfrac{N}{T}\right)^{2\alpha-1}\right]^{1/2}} & \alpha > \tfrac{1}{2} \\[2ex] \dfrac{(1-2\alpha)^{1/2}}{\left[\left(\dfrac{T}{N}\right)^{1-2\alpha}-1\right]^{1/2}} & \alpha < \tfrac{1}{2}. \end{cases} \tag{63}$$

The implication is that $\lim_{T\to\infty} \Pr(+/-) = \tfrac{1}{2}$ if $\alpha < \tfrac{1}{2}$. This means the first passage time is finite with probability one (i.e. the walk is recurrent). Moreover, as $T \to \infty$ one has the accurate estimate if $|\Delta_N| << N$;

$$\Pr(T_0 > T) \approx |\Delta_N| \sqrt{\frac{2(1-2\alpha)}{\pi N}} \left(\frac{N}{T}\right)^{1/2-\alpha} \qquad \alpha < \tfrac{1}{2}. \tag{64}$$

At the transition point itself $\alpha = \tfrac{1}{2}$ one also has $\lim_{T\to\infty} \Pr(+/-) = \tfrac{1}{2}$ and so the first passage time is still finite with probability one (the walk is marginally recurrent). More specifically, regarding the first passage time as $T \to \infty$,

$$\Pr(T_0 > T) \approx |\Delta_N| \sqrt{\frac{2}{\pi N}} \left[\log\left(\frac{T}{N} \right) \right]^{-1/2} \qquad \alpha = \tfrac{1}{2}. \qquad (65)$$

On the other hand, if $\alpha > \frac{1}{2}$ so $\lim_{T\to\infty} \Pr(+/-) < \frac{1}{2}$. This means the first passage time is no longer finite with probability one (the walk is transient), i.e. the first passage time for a given realization of the walker's path may be infinite.

4. Discussion

In this paper we have studied the statistics of a random walker within the elephant random walk framework, given the first N steps of the walk are fixed (Model I). By calculating the first two moments of the walker's position as $T \to \infty$, it is shown that there is a clear demarcation between a standard diffusion regime and an anomalous diffusion regime, depending on the parameter α. Further calculation of the skewness and kurtosis in the limit $T \to \infty$ supports the idea that in the former regime the limiting distribution is normal (i.e. Gaussian), whilst in the latter regime the distribution is non-Gaussian. However, as N becomes large the deviation from Gaussian behaviour in the anomalous diffusion regime becomes less and less pronounced, assuming Δ_N is sufficiently small. Such statements have been made precise by studying a non-linear urn model (Model II) which can be solved exactly and is asymptotically equivalent to Model I in the relevant limit. Where the twin limits $T \to \infty$ and $N \to \infty$ are taken, the basic approach has been to fix the ratio $T/N = 1+\eta$ during the limiting process, where $\eta \geq 0$ is arbitrary.

Throughout the paper, the condition on Δ_N being small has been interpreted by looking at the specific case where $\Delta_N = O(\sqrt{N})$ as $N \to \infty$. Numerically one finds that this leads to very accurate approximations for a fixed value of $N >> 1$, provided that $|\Delta_N| << N$. A greater challenge would be to explore the more general case $\Delta_N = o(N)$ as $N \to \infty$. Technically this is hard to do in the super-diffusion regime, and it would be an interesting avenue for further analysis. We speculate, however, that the overall conclusions will remain largely unaltered, i.e. only in the case where $\Delta_N = O(N)$ is the reversion of the distribution to normal form as $N \to \infty$ fundamentally questionable. The limiting form (20) for the skewness and kurtosis when $\alpha = 1$ support this hypothesis.

The choice $\Delta_N = O(\sqrt{N})$ highlights another point. Throughout the analysis we have said nothing about how the initial state, specified by the given set of variables $\{\sigma_{k \le N}\} \equiv \{\sigma_1, \sigma_2, ..., \sigma_N\}$, is generated. This is because the only parameters that matter are N and Δ_N, and the details beyond that are irrelevant. However, one could imagine a scenario where the initial N steps are generated by a (quenched) realization of a basic isotropic random walk. When N is large, one will then automatically be able to say that $\Delta_N = \chi_0 \sqrt{N}$, where χ_0 is now a normally distributed random variable of mean zero and variance one. This suggests another interesting process to study, one where the first N steps result from a basic isotropic random walk ($\alpha = 0$) and then the memory process is 'switched on' ($\alpha \neq 0$). By appropriate averaging of results in this paper with respect to the random variable χ_0 one could study the walker's eventual position. The conclusion that seems safe to draw is that the distribution at large times will tend to normal form if $N >> 1$, even if the nature of the diffusion is anomalous.

Finally, the introduction of Model II, primarily as a technical device to make asymptotically precise statements about Model I, also provides another random walk with memory process in its own right whose solution (49) has been given exactly. In this case, unlike Model I, one can treat α as an arbitrary parameter. This means one can relax the constraint that $-1 \leq \alpha \leq 1$ and consider, for example, the situation where $\alpha > 1$. Here it is known that one enters a monopolistic regime where, in terms of the two populations A and B, only one will eventually tend to infinity whilst the other will remain strictly finite [31, 35, 36]. This implies unusual statistical behaviour for Δ_T which would be interesting to study further. There may also be a deeper link to the random walk model featuring extreme value memory discussed in [9].

Appendix A: Evaluating moments for Model I

In relation to Model I described in Section 2, for $T \geq N$ the first moment or expected value of the position obeys the recursion,

$$\langle \Delta_{T+1} \rangle = \langle \Delta_T + \sigma_{T+1} \rangle = \left[1 + \frac{\alpha}{T}\right] \langle \Delta_T \rangle \qquad \text{(A1)}$$

where we have used (5) and the fact that $\langle \sigma_{T+1} \rangle = \langle \langle \sigma_{T+1} | \Delta_T \rangle \rangle$. To solve (A1) one can simply iterate to derive,

$$\langle \Delta_T \rangle = \Delta_N \left(1 + \frac{\alpha}{N}\right)\left(1 + \frac{\alpha}{N+1}\right) \ldots \left(1 + \frac{\alpha}{T-1}\right) = \Delta_N \prod_{j=N}^{T-1} \left(1 + \frac{\alpha}{j}\right). \qquad \text{(A2)}$$

Alternatively, one can write

$$\langle \Delta_T \rangle = \Delta_N \frac{\Pi_{j=N}^{T-1} (j + \alpha)}{\Pi_{j=N}^{T-1} j} = \Delta_N \frac{\Gamma(N)}{\Gamma(N+\alpha)} \frac{\Gamma(T+\alpha)}{\Gamma(T)} \qquad \text{(A3)}$$

where $\Gamma(z)$ is the gamma function which satisfies $\Gamma(z+1) = z\Gamma(z)$.

For the second moment we have the recursion,

$$\langle \Delta_{T+1}^2 \rangle = \langle (\Delta_T + \sigma_{T+1})^2 \rangle = \left[1 + \frac{2\alpha}{T}\right] \langle \Delta_T^2 \rangle + 1 \qquad \text{(A4)}$$

where we have used (5) and the fact that $\langle \sigma_{T+1}\Delta_T \rangle = \langle \langle \sigma_{T+1} | \Delta_T \rangle \times \Delta_T \rangle$ and ${\sigma_{T+1}}^2 \equiv 1$.

To solve this it is helpful to note that the linear difference equation,

$$M_{T+1} = a_T M_T + b_T \tag{A5}$$

has the solution for $T \geq N$ (which can be proved by direct substitution),

$$M_T = \sum_{k=N}^{T-1} \left(\prod_{j=k+1}^{T-1} a_j \right) b_k + M_N \prod_{j=N}^{T-1} a_j . \tag{A6}$$

With reference to (A4) and (A6), this gives,

$$\begin{aligned} \langle \Delta_T^2 \rangle &= \sum_{k=N}^{T-1} \left\{ \prod_{j=k+1}^{T-1} \left(1 + \frac{2\alpha}{j} \right) \right\} \times 1 + \Delta_N^2 \prod_{j=N}^{T-1} \left(1 + \frac{2\alpha}{j} \right) \\ &= \frac{\Gamma(T+2\alpha)}{\Gamma(T)} \sum_{k=N}^{T-1} \frac{\Gamma(k+1)}{\Gamma(k+1+2\alpha)} + \Delta_N^2 \frac{\Gamma(N)}{\Gamma(N+2\alpha)} \frac{\Gamma(T+2\alpha)}{\Gamma(T)} . \end{aligned} \tag{A7}$$

To simplify further one can use the following identity (which can be proved by induction on the summation variable k, see also [19]),

$$\sum_{k=N}^{T-1} \frac{\Gamma(k+c)}{\Gamma(k+d)} \equiv \frac{1}{d-c-1} \left[\frac{\Gamma(N+c)}{\Gamma(N-1+d)} - \frac{\Gamma(T+c)}{\Gamma(T-1+d)} \right] \tag{A8}$$

which then gives the final result,

$$\left\langle \Delta_T^2 \right\rangle = \left(\frac{1}{2\alpha - 1} \right) \frac{\Gamma(T+2\alpha)}{\Gamma(T)} \left[\frac{\Gamma(N+1)}{\Gamma(N+2\alpha)} - \frac{\Gamma(T+1)}{\Gamma(T+2\alpha)} \right]$$

$$+ \Delta_N^2 \frac{\Gamma(N)}{\Gamma(N+2\alpha)} \frac{\Gamma(T+2\alpha)}{\Gamma(T)} . \tag{A9}$$

The evaluation of higher order moments becomes progressively more involved. For the third moment we have as the logical extension of (A4),

$$\left\langle \Delta_{T+1}^3 \right\rangle = \left[1 + \frac{3\alpha}{T} \right] \left\langle \Delta_T^3 \right\rangle + \left[3 + \frac{\alpha}{T} \right] \left\langle \Delta_T \right\rangle . \tag{A10}$$

Application of (A6) with (A3) gives,

$$\left\langle \Delta_T^3 \right\rangle = \sum_{k=N}^{T-1} \left\{ \prod_{j=k+1}^{T-1} \left(1 + \frac{3\alpha}{j} \right) \right\} \left(3 + \frac{\alpha}{k} \right) \left\langle \Delta_k \right\rangle + \Delta_N^3 \prod_{j=N}^{T-1} \left(1 + \frac{3\alpha}{j} \right)$$

$$= \Delta_N \frac{\Gamma(N)}{\Gamma(N+\alpha)} \frac{\Gamma(T+3\alpha)}{\Gamma(T)} \sum_{k=N}^{T-1} \frac{\Gamma(k+1)}{\Gamma(k+1+3\alpha)} \left(3 + \frac{\alpha}{k} \right) \frac{\Gamma(k+\alpha)}{\Gamma(k)} \tag{A11}$$

$$+ \Delta_N^3 \frac{\Gamma(N)}{\Gamma(N+3\alpha)} \frac{\Gamma(T+3\alpha)}{\Gamma(T)} .$$

Rearranging and simplifying the summand one has,

$$\left\langle \Delta_T^3 \right\rangle = \Delta_N \frac{\Gamma(N)}{\Gamma(N+\alpha)} \frac{\Gamma(T+3\alpha)}{\Gamma(T)} \sum_{k=N}^{T-1} \left[3 \frac{\Gamma(k+1+\alpha)}{\Gamma(k+1+3\alpha)} - 2\alpha \frac{\Gamma(k+\alpha)}{\Gamma(k+1+3\alpha)} \right]$$

$$+ \Delta_N^3 \frac{\Gamma(N)}{\Gamma(N+3\alpha)} \frac{\Gamma(T+3\alpha)}{\Gamma(T)} \tag{A12}$$

whereupon using (A8) one can derive,

$$\left\langle \Delta_T^3 \right\rangle = \left[\Delta_N F(N,\alpha,T) + \Delta_N^3 \frac{\Gamma(N)}{\Gamma(N+3\alpha)} \right] \frac{\Gamma(T+3\alpha)}{\Gamma(T)} \tag{A13}$$

$$\begin{aligned} F(N,\alpha,T) = &\left(\frac{3}{2\alpha-1} \right) \frac{\Gamma(N)}{\Gamma(N+\alpha)} \left[\frac{\Gamma(N+1+\alpha)}{\Gamma(N+3\alpha)} - \frac{\Gamma(T+1+\alpha)}{\Gamma(T+3\alpha)} \right] \\ &- \frac{\Gamma(N)}{\Gamma(N+\alpha)} \left[\frac{\Gamma(N+\alpha)}{\Gamma(N+3\alpha)} - \frac{\Gamma(T+\alpha)}{\Gamma(T+3\alpha)} \right]. \end{aligned} \tag{A14}$$

Regarding the fourth moment we have,

$$\left\langle \Delta_{T+1}^4 \right\rangle = \left[1 + \frac{4\alpha}{T} \right] \left\langle \Delta_T^4 \right\rangle + \left[6 + \frac{4\alpha}{T} \right] \left\langle \Delta_T^2 \right\rangle + 1. \tag{A15}$$

Application of the above methodology eventually gives (leaving out the details),

$$\left\langle \Delta_T^4 \right\rangle = \left[G(N,\alpha,T) + \Delta_N^2 H(N,\alpha,T) + \Delta_N^4 \frac{\Gamma(N)}{\Gamma(N+4\alpha)} \right] \frac{\Gamma(T+4\alpha)}{\Gamma(T)} \tag{A16}$$

where,

$$G(N,\alpha,T)=6\left(\frac{1}{2\alpha-1}\right)^2\frac{\Gamma(N+1)}{\Gamma(N+2\alpha)}\left[\frac{\Gamma(N+1+2\alpha)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+1+2\alpha)}{\Gamma(T+4\alpha)}\right]$$

$$-4\left(\frac{1}{2\alpha-1}\right)\frac{\Gamma(N+1)}{\Gamma(N+2\alpha)}\left[\frac{\Gamma(N+2\alpha)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+2\alpha)}{\Gamma(T+4\alpha)}\right]$$

(A17)

$$-3\left(\frac{1}{2\alpha-1}\right)^2\left[\frac{\Gamma(N+2)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+2)}{\Gamma(T+4\alpha)}\right]$$

$$-\left(\frac{1}{4\alpha-1}\right)\left(\frac{2\alpha-5}{2\alpha-1}\right)\left[\frac{\Gamma(N+1)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+1)}{\Gamma(T+4\alpha)}\right]$$

$$H(N,\alpha,T)=\left(\frac{6}{2\alpha-1}\right)\frac{\Gamma(N)}{\Gamma(N+2\alpha)}\left[\frac{\Gamma(N+1+2\alpha)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+1+2\alpha)}{\Gamma(T+4\alpha)}\right]$$

(A18)

$$-4\frac{\Gamma(N)}{\Gamma(N+2\alpha)}\left[\frac{\Gamma(N+2\alpha)}{\Gamma(N+4\alpha)}-\frac{\Gamma(T+2\alpha)}{\Gamma(T+4\alpha)}\right].$$

References

[1] Klafter J, Blumen A and Shlesinger M F 1987 *Phys. Rev. A* **35** 3081

[2] Metzler R and Klafter J 2000 *Phys. Rep.* **339** 1

[3] Metzler R and Klafter J 2004 *J. Phys. A: Math. Gen.* **37** R161

[4] Kim H -J 2014 *Phys. Rev. E* **90** 012103

[5] Boyer D and Solis-Sala C 2014 *Phys. Rev. Lett.* **112** 240601

[6] Boyer D and Romo-Cruz J C R 2014 *Phys. Rev. E* **90** 042136

[7] Kazimierski L D, Abramso G and Kuperman M N 2105 *Phys. Rev. E* **91** 012124

[8] da Silva M A A, Viswanathan G M and Cressoni J C 2015 *Physica A* **421** 522

[9] Harris R J 2015 *New J. Phys.* **17** 053049

[10] Budini A A 2016 *Phys. Rev. E* **94** 052142

[11] Budini A A 2017 *Phys. Rev. E* **95** 052110

[12] Schütz G M and Trimper S 2004 *Phys. Rev. E* **70** 045101

[13] Paraan F N C and Esguerra J P 2006 Phys. Rev. E **74** 032101

[14] Cressoni J C, Viswanathan G M, Ferreira AS and da Silva M A A 2012 *Phys Rev E* **86** 022103

[15] da Silva M A A, Schütz G M , Viswanathan G M and Trimper S 2013 *Phys. Rev. E* **88** 022115

[16] Baur E and Bertoin J 2016 *Phys. Rev. E* **94** 052134

[17] Coletti C F, Gava R and Schütz G M 2017 *J. Math. Phys.* **58** 053303

[18] Coletti C F, Gava R and Schütz G M 2017 2017 *J. Stat. Mech.: Theor. Expt.* **2017** 123207

[19] Bercu B 2017 2018 *J. Phys. A: Math. Theor.* **51** 015201

[20] Kumar N, Harbola U and Lindenberg K 2010 *Phys. Rev. E* **82** 021101

[21] Harbola U, Kumar N and Lindenberg K 2014 *Phys. Rev. E* **90** 022136

[22] Kürsten R 2016 *Phys. Rev. E* **93** 032111

[23] Usatenko O V and Yampol'skii V A 2003 *Phys. Rev. Lett.* **90** 110601

[24] Hod S and Keshet U 2004 *Phys. Rev. E* **70** 015104(R)

[25] Keshet U and Hod S 2005 *Phys. Rev. E* **72** 046144

[26] Menshikov M and Volkov S 2008 *Electron. J. Probab.* **13** 944

[27] Huillet T 2008 *J. Phys. A: Math. Theor.* **41** 505005

[28] Antal T, Ben-Naim E and Krapivsky P L 2010 *J. Stat. Mech.: Theor. Expt.* P07009

[29] Jiang B, Sun L, Figueiredo D R, Ribeiro B and Towsley D 2015 *J. Stat. Mech.: Theor. Expt.* P11022

[30] Krenn D, Mahmoud H and Ward M D 2016 arXiv:1608.01233

[31] Kearney M J and Martin R J 2016 *J. Stat. Mech.: Theor. Expt.* **2016** 123407

[32] Huillet T 2017 *J. Stat. Mech.: Theor. Expt.* **2017** 093211

[33] Khanin K and Khanin R 2001 *J. Math. Biol.* **42** 26

[34] Krapivsky P L and Redner S 2001 *Phys. Rev. E* **63** 066123

[35] Oliveira R and Spencer J 2005 *Internet Math.* **2** 121

[36] Oliveira R 2008 *Combin. Prob. Comp.* **17** 87

[37] Hisakado M and Mori S 2010 *J. Phys. A: Math. Theor.* **43** 315207

[38] Mori S and Hisakado M 2015 *Phys. Rev. E* **92** 052112

[39] Arthur W B, Ermoliev Yu M and Kaniovski Yu M 1987 *Eur. J. Oper. Res.* **30** 294

[40] Arthur W B 1989 *The Economic Journal* **99** 116

[41] Abramowitz M and Stegun, I A 1972 *Handbook of mathematical functions: with formulas, graphs, and mathematical tables* (New York: Dover)

[42] Johnson N L and Kotz S 1977 *Urn Models and Their Application: An Approach to Modern Discrete Probability Theory* (New York: Wiley)

[43] Zhang P, Chen C and Mahmoud H 2015 *Stat. Prob. Lett.* **96** 149

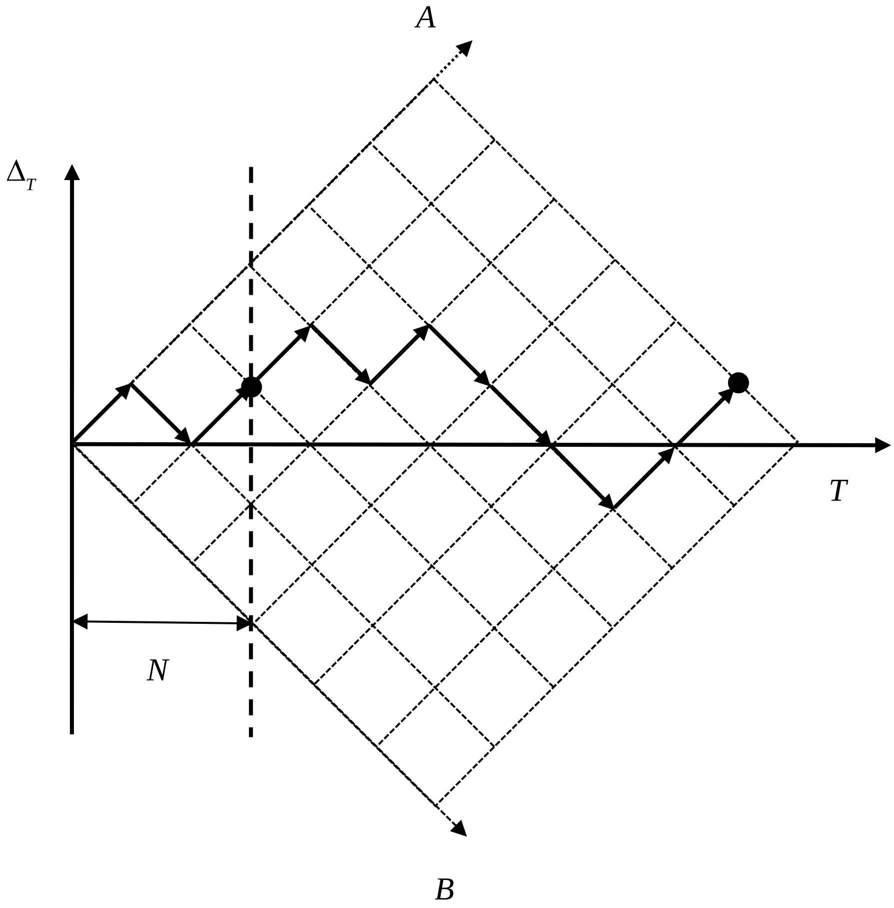

Figure. 1. A realisation of the random walk process Δ_T, with $N = 3$ fixed steps such that $\Delta_N = 1$ and $S = 8$ free (random) steps such that the walker ends after $T = N + S = 11$ steps at $\Delta_{11} = 1$. In terms of the underlying population variables (A, B), the free portion of the directed lattice path starts at $(a, b) = (2,1)$ and ends at $(A, B) = (6,5)$. At any point on the path one has $A + B = T$.

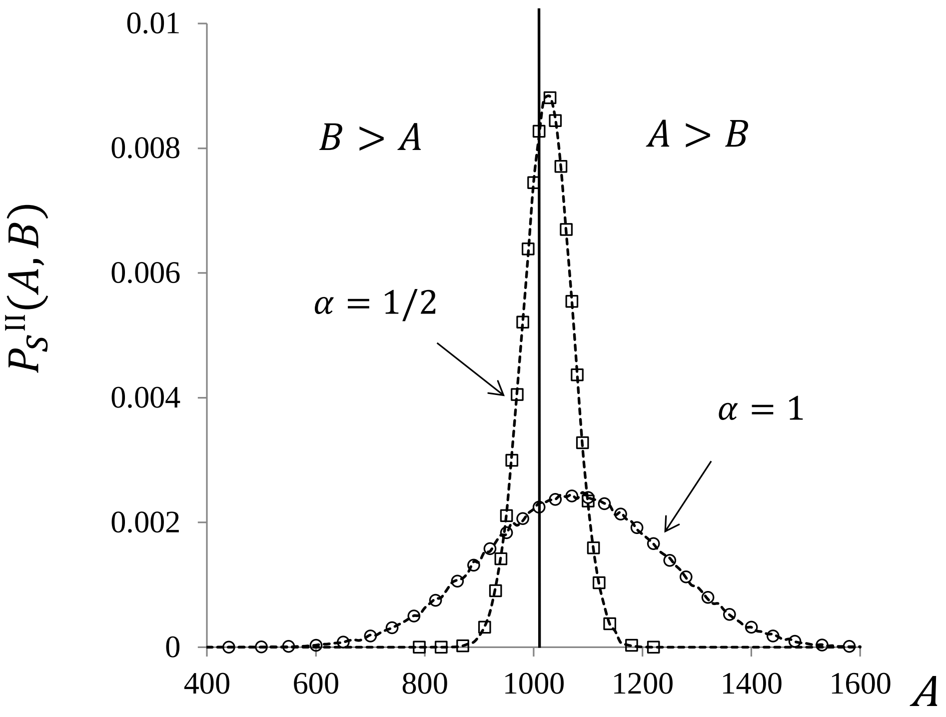

Figure. 2. Comparison of the exact probability $P_S^{\mathrm{II}}(A,B)$ (symbols) with normalized histograms derived from simulations (dotted lines) for $\alpha = 1$ and $\alpha = \frac{1}{2}$. The initial position $(a,b) = (20,18)$ and $S = 2000$ in each case. The solid vertical line marks the point where $A = B$.

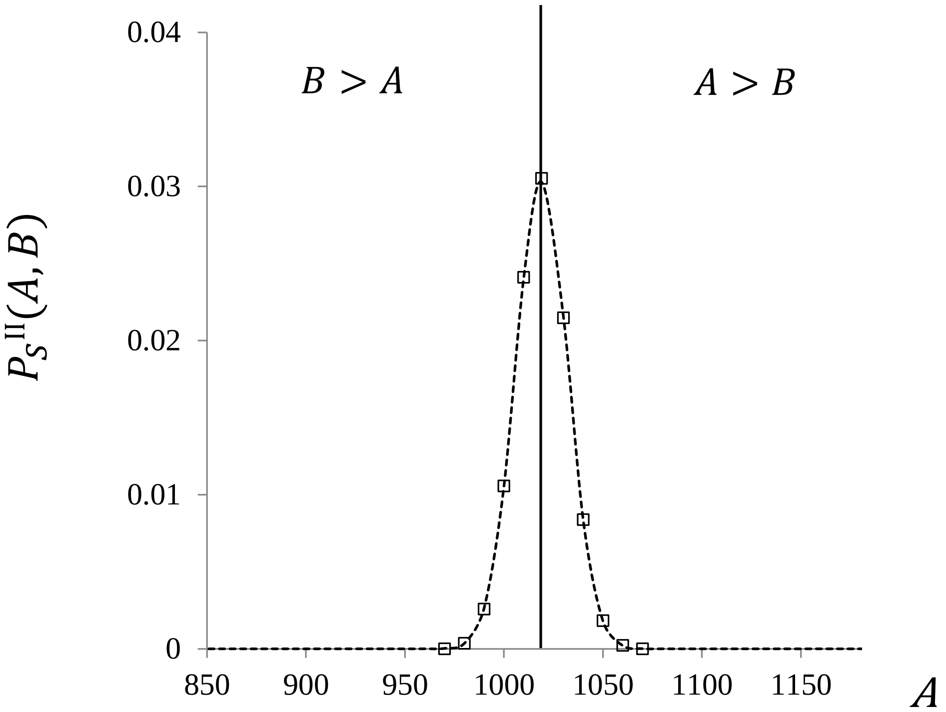

Figure. 3. Comparison of the exact probability $P_S^{\mathrm{II}}(A,B)$ (symbols) with a normalized histogram derived from simulations (dotted line) for $\alpha = -1$. The initial position $(a,b) = (20,18)$ and $S = 2000$. The solid vertical line marks the point where $A = B$.

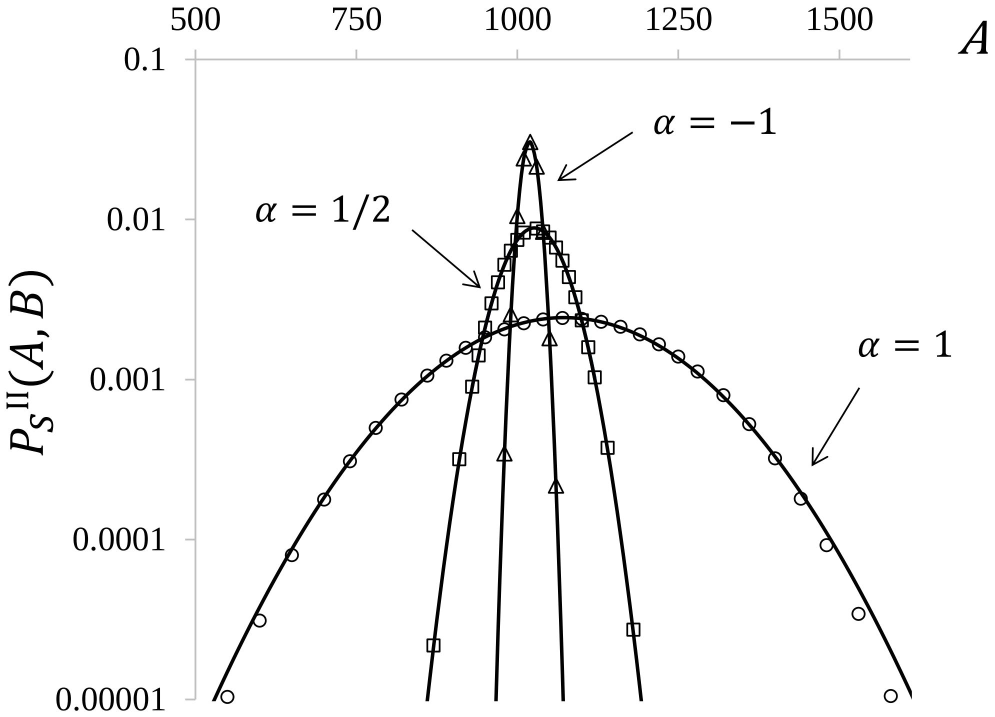

Figure. 4: Comparison of the exact probability $P_S^{\mathrm{II}}(A,B)$ (symbols) against the approximated Gaussian probability (solid lines) for $\alpha = 1$, $\alpha = \frac{1}{2}$ and $\alpha = -1$. The initial position $(a,b) = (20,18)$ and $S = 2000$ in each case.