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A STUDY OF THE PLASTIC REGION AROUND A CIRCULAR APERTURE IN A THIN PLATE UNDER TENSION

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ABSTRACT

The problem of a circular aperture in a thin plate under tension, is set briefly in a historical background. Relevant published work is surveyed and various ways of strain measurement are examined with a view to their usefulness in plastic and elastic regions of a metal plate. The method of a thin photoelastic layer, bonded to the metal surface, is selected; its advantages and difficulties are investigated.

The design is given of an apparatus which has been constructed to stretch a thin metal plate in either one direction or in two directions at right angles. A reflection polariscope was arranged above this tensioning table for examining bonded specimens in polarized light, strains being measured from the photoelastic patterns obtained. Plastic and elastic regions, surrounding the central circular hole of each specimen, were examined by this means. The method of oblique incidence has been developed for use with a bonded layer, in order to separate the individual principal strains.

Results are given for a single tension and for two equal tensions, using aluminium alloy L71. The strains obtained for a single tension are compared with an experimental and a theoretical work. The plastic-elastic boundary (using Tresca's yield condition) and the measured values of strain, in the case of equal tensions, are compared with theoretical works.
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NOTATION

Where a symbol is used for more than one quantity, or one other than that given below, its meaning is made clear in the text.

a  radius of the central hole
b, c  constants
ci  plastic-elastic boundary
d  outer radius of a circular slab
i  imaginary unit.
k  shear stress at yield
l  length
n  integer
n, na  number of fringes at oblique incidence
n, n, n  number of fringes corresponding to the principal stresses p, q.
p  larger principal stress in the plane
q  smaller principal stress in the plane
r  radial co-ordinate, or third principal stress
u, v, w  components of displacement
x, y, z  Cartesian co-ordinates
z  complex quantity
A, B, C  constants
D,  displacement
\( E \)  
\( E_s \)  
\( G \)  
\( K \)  
\( L \)  
\( M \)  
\( N \)  
\( T, T_1, T_2, T_3 \)  
\( X \)  
\( X_0 \)  
\( Z \)  
\( \alpha, \beta \)  
\( \alpha(z), \beta(z) \)  
\( \gamma \)  
\( \delta \)  
\( \partial \)  
\( \varepsilon \)  
\( \varepsilon_p \)  
\( \varepsilon_q \)  
\( \eta \)  
\( \Theta \)  
\( \Theta_n, \Theta_l, \Theta_o \)  
\( \text{tensile modulus} \)  
\( \text{secant modulus} \)  
\( \text{shear modulus} \)  
\( \text{photoelastic constant} \)  
\( \text{length} \)  
\( \text{magnification} \)  
\( \text{number of fringes} \)  
\( \text{applied tension (per unit cross section) in } x, y, 1\text{st, 2nd direction respectively.} \)  
\( \text{length on the specimen} \)  
\( \text{length in air} \)  
\( \text{cartesian co-ordinate} \)  
\( \text{angles} \)  
\( \text{complex potentials} \)  
\( \text{Poisson's ratio for the layer} \)  
\( \text{differential} \)  
\( \text{partial differential} \)  
\( \text{strain} \)  
\( \text{larger principal strain} \)  
\( \text{smaller principal strain} \)  
\( \text{Poisson's ratio} \)  
\( \text{angle, angular co-ordinate} \)  
\( \text{angle in layer, prism, air, respectively} \)
\( \lambda \) cutout factor

\( \mu, \mu_c \) refractive index of layer, prism

\( \gamma \) constant

\( \rho \) radius of plastic-elastic boundary

\( \sigma \) stress

\( \sigma_{\infty} \) stress at infinity

\( \sigma_{\text{avg}} \) average stress along the transverse diameter

\( \sigma_x, \sigma_y \) stress in the direction of \( x, y \), etc.

\( \tau_{xy}, \tau_{\text{etc.}} \) shear stress in the \( xy, \theta \) plane, etc.

\( \tau_{\text{max}} \) maximum shear stress

\( \phi \) complex quantity

\( \Omega(z) \) complex potential

\( \nabla^2 \) operator \( = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)

\( \chi \) Airy stress function

\( \Psi \) complex shear stress combination \( = \tau_{xz} + i \tau_{yz} \)

\( \Theta \) stress combination \( = \sigma_x + \sigma_y \)

\( \Phi \) stress combination \( = \sigma_x - \sigma_y + 2i \tau_{xy} \)

\( \mathcal{R}(z) \) complex potential

Abbreviations

- t.p.s.i. tons per square inch.
- p.s.i. pounds per square inch
- 2-D two dimensional
- K.G. kilogramme.
- \( \nu \) versus
- \( \sim \) very approximately.
SECTION I

HISTORICAL OUTLINE

From the time that Robert Hooke announced his law of proportionality between deformation and force, many observers noted the failure of the law under conditions of high stress. However, nearly 200 years elapsed before systematic experiments with metals were carried out to determine the criterion of yielding.

During the 1860's, Tresca subjected a number of metals to different combinations of stress, concluding that yielding occurred when the maximum shear stress reached a critical value. St. Venant used Tresca's results to formulate a theory of plasticity in two dimensions for a plastic-rigid body (i.e. ignoring the elastic component of strain). A year later Lévy (1871) added a basic set of equations for three dimensions.

A number of investigators tried to determine the combination of stresses in two or three dimensions that defined the limits of plasticity or the yield point. Mohr (1900) concluded that the onset of plasticity was governed by limiting combinations of normal stress and shearing stress on any one section. Guest however, from tests on hollow cylinders of steel and brass, concluded that maximum shear stress governed the limits of elasticity. The diverging results
obtained with different materials led to a number of failure
criteria being proposed.

In 1913, von Mises suggested a yield criterion on the basis
of purely mathematical considerations; it was interpreted later
by Hencky as implying that yielding occurred when the elastic shear-
strain energy reached a critical value.

Hencky in 1923 discovered simple geometrical properties of
the field of slip lines and in 1930 Geiringer obtained the equations
governing the variation of the velocity of flow along them. Tresca's
criterion was used by Prandtl who introduced a "soap film" analogy,
and by Nadai who used the "sand-hill" analogy and the "roof-and-
membrane" analogy for torsion of bars in the plastic stage.

In 1925 von Karman analysed the state of stress in rolling
and later Siebel and Sachs put forward similar theories for wire
drawing.

It was not until 1926 that the yield criterion of von Mises
was upheld experimentally. Lode measured the deformation of
tubes of various metals under combined tension and internal pressure,
showing that Lévy's plastic-rigid equations with Mises's yield
criterion were valid to the first approximation. Certain
divergences occurred however and these were confirmed later by
Taylor and Quinney (1931).

The theory was generalised in two important respects from 1930-2.
Reuss made allowance for the elastic component of strain and Schmidt and Odquist showed how strain hardening could be brought within the framework of the Lévy-Mises equations. The first generalisation was broadly confirmed by Hohenemser (1931-2) and the second by Schmidt.

By 1932 a theory had been constructed, reproducing the main plastic and elastic properties of an isotropic material. However from then until the early 1940's little progress was made in the solution of special problems. Further generalisations were formulated, but the mathematical difficulties involved limited their usefulness; a rival theory by Hencky (1924) was favoured for its analytical simplicity in spite of its assumption, contrary to experience, of a one-one relationship between stress and strain.

General.

Theories of plasticity may be divided into two main groups; total strain theories, based on the assumption that for continual loading the state of strain is completely determined by the state of stress; incremental theories, assuming the increment of strain to be uniquely determined by the existing stress and the increment of stress.

It is now generally recognised that the incremental theories are necessary for a complete description of the behaviour of metals under plastic conditions but that mathematical simplification can
sometimes be obtained with little loss of accuracy by using the total strain theories.

If the same yield criterion is assumed, the theories are equivalent provided the principal stress directions remain constant and the principal shear stresses remain in constant ratio.

Most of the problems investigated have been those for which the elastic strains were negligible in comparison to the plastic strains. Little work has been done either theoretically or experimentally on elastic-plastic problems where the strains were of the same order. The two dimensional case of a hole in a plate under tension is a fundamental problem in this field.

**The Problem of a Hole in a Flat Plate under Tension.**

Due to the effects of stress concentration, plastic areas commence on the boundary of the hole and slowly extend with increasing load.

![Diagram of Plastic Regions](image)

A) Single Tension  
B) Two Equal Tensions

**FIG. 1.**
Surrounding the plastic region (or regions) is material still within its elastic limit, so that the resulting plastic strains are only of the same order as the elastic strains. The latter cannot therefore be neglected in any valid theory and the simplifications of a supposed plastic-rigid material may not be used.

Difficulty results from the position of the plastic-elastic boundary being unknown and (except for two roughly equal tensions at right angles) from two separate plastic regions that occur, causing the boundary to be partly in the material and partly along the edge of the hole (see fig. 1A). The theoretical approach to the problem is therefore formidable even under the simplest laws of plasticity and taking no account of strain hardening.

Experimental investigation requires measurement of the magnitude and direction of the principal strains in the plastic and elastic regions and the location of the plastic-elastic boundary. Difficulties arise in the plastic regions where (with the exception of lines of symmetry) the directions of principal strain are unknown.
SURVEY OF PUBLISHED WORK

A number of theoretical papers have been published in recent years on plastic flow around a hole in a flat plate under tension. Among them, Parasyuk deals with the case of constant shear round a circular hole in a flat plate (1); Savin and Parasyuk (2) make one of the few attempts to deal with work hardening. This latter however, is specialised by the assumption of rigidity which is inadmissible for the problem in hand where plastic and elastic strains are of the same order.

The method of limit analysis has been applied to the problem by Hodge (3) and also by Gaydon (4). Hodge defines a cutout factor as the largest multiplier such that if \((T_x, T_y)\) represent the load at failure on a square with no cutout, the square with a (circular) cutout will not fail under any load less than \((\lambda T_x, \lambda T_y)\). Upper and lower bounds are then found for the cutout factor. The analysis is based on the assumptions of plane stress and Tresca's yield condition.

Hodge later published two papers (5)(6) in which he set up the complete elastic-plastic problem for a circular slab with central cutout, subjected to uniform radial external tension. For the case where the slab is large compared with the central hole, the problem is identical to that of a large plate subjected to two equal tensions at right angles and within the scope of the work in hand.

His first paper deals with a perfectly plastic material using
Trosca’s yield condition. The analysis is carried out under plane stress conditions but with possibly finite deformation. The material is assumed to be isotropic, homogenous and incompressible, and to obey the plastic potential law (which states that the strain-rate vector is normal to the yield surface).

The equations are solved by a perturbation method based on the ratio of maximum shear stress ($k$) to shear modulus ($G$), in which each of the significant quantities stress, displacement, and slab thickness, is expanded in a power series in this ratio.

Where $r$ = radial co-ordinate, $\rho$ = radius of plastic-elastic boundary, $d$ = outer radius of slab, $a$ = radius of circular cutout.

The resulting solution in the plastic-elastic case for strains is:

Elastic Region ($\rho < r < d$)

$$
\varepsilon_\theta = \frac{k}{G} \left[ -\frac{a}{3} + \frac{\alpha^2}{6r^2} - \frac{a \sigma^2}{2r^4} \right] - \frac{k^2 a}{8G^2} \left[ 2 + 3 \log \frac{\rho}{a} + \left( \log \frac{\rho}{a} \right)^2 \right] \left( 1 + 3 \frac{\rho^2}{r^2} \right) .
$$

Plastic Region ($d < r < \rho$)

$$
\begin{align*}
\varepsilon_\theta &= \frac{k}{3G} \left[ \frac{1}{r} + \frac{a^2}{r^2} + \frac{2a}{r} \log \frac{\rho^2}{r} \right] + \frac{k^2 a}{G^2} \left[ 2 + \log \frac{\rho^2}{a^2} + \left( \log \frac{\rho^2}{a^2} \right)^2 \right] \\
&\quad \times \left[ -2 \log \frac{\rho}{a} \left( \log \frac{\rho}{a} \right) + \frac{2}{3} \left( \log \frac{\rho^2}{a^2} \right)^2 \right] \\
\varepsilon_r &= \frac{k}{3G} \left[ -\frac{2}{r} + \frac{a^2}{2r^2} \right] - \frac{k^2 a}{G^2} \left[ 2 + \log \frac{\rho^2}{a^2} + \left( \log \frac{\rho^2}{a^2} \right)^2 \right] \left( \log \frac{\rho}{a} \right)
\end{align*}
$$

and the load is given by

$$
\frac{1}{2k} = \left[ \frac{2}{2r^2} \frac{\partial^2}{\partial r^2} \frac{a^2}{a} - \frac{k}{6G} \left[ 4 - \frac{2a}{\rho^2} + \frac{2a^2}{\rho^2} - \frac{2a^2}{\rho^2} + \frac{3a^2}{\rho^2} - \frac{a}{(1 + \rho^2)^2} \log \frac{\rho^2}{a^2} \right] \right],
$$

As the slab width tends to infinity, this simplifies to:

$$
\frac{1}{2k} = 1 - \frac{a}{2r} \quad \cdots \cdots \quad \text{ignoring terms in } \frac{a}{G} \text{ (since } k \sim 0.01) \text{.}
$$
The radial strains obtained, and the radius of the plastic-elastic boundary can be subjected to experimental test. (See Section IV).

Hodge's second paper (6) again deals with the condition of plane stress. He considers the effects of strain hardening, by approximating a stress-strain curve in tension to three straight line segments. For biaxial states of stress the material is assumed to satisfy a flow law based on the maximum shear stress.

General equations are given and then simplified by assuming that the boundary displacements may be neglected for small strains, and that the elastic components may be neglected for large strains.

Fig. 2 shows the form of the segmented stress-strain curve, used in the analysis.

\[
\begin{align*}
\frac{\varepsilon_x}{E} &= n = (1 + \cot \alpha)^{-\frac{1}{2}} \\
\end{align*}
\]

FIG. 2.

For the first plastic stage in the region AB, the solutions are given in closed form and then rearranged as a power series in \( n \) for easier computation. The solutions (up to terms in \( n^2 \)) are, for strains

Elastic
\[
\begin{align*}
(0 \leq r \leq d) \quad & \varepsilon_\theta = \frac{1}{2} \frac{b}{E} \left\{ 1 - \frac{a}{2} \left( 1 + \frac{3a^2}{4r^2} \right) \left[ 1 - \frac{n^2}{a} \left( \log \frac{a}{d} \right)^2 \right] \right\} \quad (5)
\end{align*}
\]

Plastic
\[
\begin{align*}
(a \leq r < b) \quad & \varepsilon_r = \frac{1}{2} \frac{b}{E} \left\{ 1 - \frac{2a}{r} - \frac{n^2}{a} \left[ \log \frac{a}{r} + (\log \frac{a}{r})^2 - (\log \frac{a}{d})^2 \right] \right\} \quad (6)
\end{align*}
\]
Solutions are indicated for the case of plastic regions beyond B (fig. 2) and are given for complete plasticity. The strains and the position of the plastic-elastic boundary, for the case of a contained plastic region, can be subjected to experimental test by choosing the form of fig. 2 to suit the material used.

A different approach has been made by Galin (7) who deals with the condition of plane strain. A hole in an infinite plate is considered under two tensions at right angles ($T_1$ and $T_2$). Limitations are imposed on the ratio of $T_1 : T_2$, as the theory allows only one plastic-elastic boundary to be present, i.e. the plastic region must always completely surround the hole (see fig. 1).

The stresses in the plastic region are assumed to be independent of $\Theta$; although this is correct when the perpendicular tensions $T_1$ and $T_2$ are equal, it is unlikely to be the case if $T_1 \neq T_2$, for which it is put forward.

The displacement (or velocity) components are not considered and therefore no conditions involving their continuity over the plastic-elastic boundary are made.

The solution under these assumptions is biharmonic and is summarised below using complex potentials (see Appendix p.121).
Suffix 1 ...... quantities in the plastic region.

Suffix 2 ...... " " " elastic "

The stress combinations in the plastic region are taken to be

\[ \Phi' = -2k \]
\[ \Omega' = 2k \left(1 + 2 \log \frac{r}{a} \right) \]

Complex potentials are then found so that:

a) in the elastic region the correct stresses at infinity are obtained. i.e. \( \Omega_2 \rightarrow \Phi_1 + \Phi_2 \) & \( \Phi_2 \rightarrow \Phi_1 - \Phi_2 \) as \( |z| \rightarrow \infty \)

b) the stress components are continuous from region 1 to 2 on crossing the plastic-elastic boundary \( c \).

Galin achieves this by a conformal transformation of the curve \( c \) into a unit circle \(|\phi| = 1\) in a new \( \phi \)-plane using the transformation

\[ z = c(\phi + \frac{b}{\phi}) \]

Condition a) is satisfied by taking

and \[ \log \frac{c}{a} = \frac{T_2 - T_1}{2k} \]

Suitable forms for the complex potentials are then specified in the \( \phi \)-plane as:

\[ \alpha_2(\phi) = A + 4k \log \frac{\phi}{\phi} \]

\[ \beta_2(\phi) = 4k \left( 1 + \frac{b \phi^2}{\phi^2 - b} \right) \]

and these give the required continuity of stress.

The predicted shape of the plastic-elastic boundary is an ellipse, becoming a circle when \( T_1 = T_2 \).
Under this condition the radius of the circle is given by the equation

$$T = \frac{1}{r} \left( \log \frac{L}{a} + 2 \right)$$

and can be subjected to experimental test.

Faierberg (8) sets out to give an approximate solution for the case of tension in one direction in a thin plate with a circular hole. The method used is based on the application of a theory of elastic and plastic bending of a curved bar. The equations are built up in stages, first with a plastic region in compression in the bar, then in the fibres in tension and then both together. He next extends the problem to a thin ring, considering the bending moment in one quadrant. Finally the solution is extended to the case of a plate six times the diameter of the hole. It is this section that comes within the scope of the work in hand.

He gives curves for strain along the transverse diameter and a graph of the shape and extent of the plastic region in this direction. These are compared with the present experimental work in Section IV.

Griffith (9) carried out an experimental investigation of the effects of plastic flow in a tension panel with a central circular hole.
Seven uniformly dimensioned 24S - T (B.S.S. DTD 5010) panels, 24" wide x .091" thick x 58" overall length with a 4" diam. central circular hole were subjected to various tensions (along their length) to study the effect of plastic flow at the point of maximum stress concentration. Loading and unloading the panels was repeated up to 100 times, but no observable change occurred in the maximum value of the stress and strain concentration factors.

Loads were varied so that the average net section stress (i.e. allowing for the central cutout) ranged from $23.5 \times 10^3$ lbs/sq.in. to $50.6 \times 10^3$ lbs/sq.in. Strains were measured mainly by $\frac{1}{2}$" gauge length electrical resistance strain gauges but some 1" gauges were used. Electromagnetic gauges, specifically designed to be accurate ($\sim 1\frac{1}{2}$\%) at large strains, measured the maximum strain concentrations.

In an elastic strain distribution, the $\frac{1}{2}$" gauge length would give a calculated error of 2.5\% below maximum strain, due to the strain gradient across the gauge. The results are given for $\frac{1}{2}$" gauges but no corrections have been made to the readings.

Purely elastic strains gave measured concentration factors of 3.08 at the hole edge compared to the theoretical factor of 3.00. Griffith deals in terms of net section stress, whereupon the factor becomes 2.50 for a 6 : 1 ratio of plate width to hole diameter and a corresponding value of 2.57 was obtained.
Graphs are given for panel load versus strain at the hole edge, for stress and strain concentration factors versus the average net section stress, and for strain versus distance from the hole edge. The stresses have been simply calculated by reading off from the tensile stress-strain curve of the material.

This method of stress determination can only be approximate since it assumes the stress to depend solely on the final state of strain, and apparently makes no allowance for the effect of transverse stresses.

The strains obtained are compared with the present work in Section IV.

Budiansky and Vidensek (10) published a paper that was concerned, as was an earlier paper by Stowell (11), with the calculation of stress around a circular hole in a plate under a single tension, using the results obtained by Griffith above.

An approximate theoretical solution is presented for the stresses in the plastic range, based on the simple deformation theory of plasticity, and is found by application of a variational principal in conjunction with the Raleigh-Ritz procedure, and the use of a high speed computing machine. (S.E.A.C.) Numerical results are obtained for four different materials, which are characterised by four distinct uniaxial stress-strain curves. These curves were defined using a
form proposed by Ramberg and Osgood (12):-

\[ E = \frac{\sigma}{\epsilon} \left[ 1 + \frac{3}{2} \left( \frac{\sigma}{\sigma_r} \right)^{n-1} \right] \]

where \( \sigma \) is the applied stress, \( \sigma_r \) is the value of stress at which the secant modulus \( E_s = 0.7E \) and \( n \) is a parameter chosen to provide the best fit to the stress-strain curve of the actual material under consideration.

Formulae are given for the resulting values of \( \sigma_r, \sigma_0 \) and \( \tau_r \), and curves of stress distribution are drawn.

The authors conclude that the solution is far from the final answer, but that an extension of the present approach to include more degrees of freedom would overtax the computing machine.
SCOPe OF THE PRESENT WORK

An apparatus has been constructed to stretch a thin plate containing a small hole, in either one direction or in two directions at right angles, with tensions up to 20 tons. Aluminium alloy L71 has been used as the plate material. This approaches the ideal case of linear elastic extension, rapid turn over at yield and then low tangential modulus.

Ratios of plate width to hole diameter from 6:1 to 24:1 have been used. For an infinite plate under elastic extension, the stress concentration falls from three times at the hole edge to within 2% of its infinity value for points at six times the radius from the hole. The cases dealt with therefore approach that of a hole in an infinite plate while the plastic regions remain small.

A number of specimens under a single tension have been investigated and some work has been done on two equal tensions at right angles. Strain measurements have been made in plastic and elastic regions and the plastic-elastic boundary determined using the Tresca criterion.

Use has been made of thin photoelastic layers bonded to the metal to determine the strains. The layers have been viewed in polarised light by means of a reflection polariscope. A number of practical difficulties had to be overcome in order to obtain accurate measurements.
a) Strain gauge bridge  
b) " selector  
c) Cyls. for load measurement  
d) Lamp house  
e) Prism (above specimen)  
f) Levers  
g) Specimens
and the work demonstrates this method for the analysis of two-dimensional problems.

Fig. 3 shows the apparatus, with the optical system arranged for use at oblique incidence (see p. 72).

Results for the case of a single tension are compared with the experimental work on lines of symmetry by Griffith (9) and a theory by Faierberg (8). The case for two equal tensions is compared with theoretical works by Hodges (5)(6) and Galin (7).
Ideally, a method was required that would determine accurately the plastic and elastic strains and their principal directions at all points on a plane metal surface.

With this in view, the advantages and limitations were examined of a number of methods for strain measurement. Some gave only qualitative results or required considerable plastic strain to produce a measurable effect. Certain methods required the metal to have a fine grain structure and some were suitable only for steels; others had more general application.

The methods of strain detection considered were:

1) X-Ray diffraction.
2) Brittle lacquers.
3) Grid Method
4) Mechanical extensometers.
5) Lüders lines.
6) Acid reaction.
7) Hardness changes.
8) Interference between surface and glass plate.
9) Grain size changes.
10) Resistance changes.
11) Magnetic changes.
12) Metallographic methods.
13) Electrical resistance strain gauges.
14) Adhesion of a photoelastic layer.

1) X-ray Diffraction

To measure strains, a fine beam of X-rays of known wavelength is directed on to the specimen while under load. The back reflection method is used and the diffracted cone of rays impinges on photographic film to form a diffraction circle. Only grains having a particular orientation give rise to the circle whose diameter is a measure of the interatomic spacing of these grains. The change in atomic spacing gives the elastic strain for stresses in line with the particular orientation selected by the X-rays.

Such measurements yield only the elastic part of the strain even though considerable plastic flow may have occurred. This has the unique advantage that strains may be converted directly into stresses.

Measurements are confined to the surface layers as the X-ray penetration is small. Under ideal conditions a precision of \(\pm 1,500\) p.s.i. can be obtained, but more usually \(\pm 4,500\) p.s.i. Moreover, metals subjected to severe cold working, or with large grain size or lack of uniform composition cause the accuracy of measurement to be considerably reduced (13). For the work in hand, the precision of measurement required (within 5%) therefore ruled out the method.
2) Brittle Lacquers.

Certain commercial preparations are available for coating the surface of a specimen. If conditions are suitable, a given maximum principal strain causes cracking at right angles to its direction. By varying the composition of the lacquer, a range of values of strain can be made to produce cracks. The method is therefore very useful for determining the directions of principal strain. However, the absolute value of strain, estimated for instance by means of a standard test piece similarly coated, is liable to considerable error due to the many factors influencing the lacquer.

In a paper by Durrelli et al the authors isolate thirty-four variables (14). Such factors as age of coating, distance of spray gun from specimen, humidity, temperature during loading and many others have a considerable effect on the sensitivity of any given grade of lacquer, so that an absolute accuracy of ±20% is difficult to achieve.

3) Grid Method

The surface of a specimen is made flat and polished and then a grid of fine lines, equally spaced, is printed or machined on to it. Strains in the specimen are measured from the change in line spacing.

The method has been used successfully to follow certain forming operations where large plastic flows occur (15), but cannot be used for the problem in hand because of the small strains involved.
4) **Mechanical Extensometers.**

By means of lovers, such extensometers have been made to give a high degree of sensitivity and accuracy. Some are made to deflect beams of light while others use dial gauges or electrical strain gauges to detect the strain.

For single strain measurements they are often ideal instruments, but for a large number of strains to be determined simultaneously, a prohibitive number of such extensometers would be required. Moreover, the gauge lengths are usually large and would give only the average strain over these regions.

5) **Lüders Lines.**

These appear as fine surface markings on the polished surfaces of certain steels when strained beyond the elastic limit. They have been shown to be areas of plastic deformation.

By means of Fry's etching solution (a mixture of hydrochloric acid and copper chloride) the plastically deformed areas can be made to appear dark against the lighter background of the undeformed metal. Nadai (16) has used the markings to show up regions of plastic flow in a number of different configurations.

The lines seem to appear only for materials having an unstable region in their stress-strain curves.
6) **Acid Reaction.**

It was found by Whitely and Hallimond (17) that if iron or steel specimens were subjected to strain, they reacted with nitric acid to evolve gaseous, the quantities of which varied according to the amount of strain. The method used was to cut the specimen, after straining, into small pieces; dissolve it in nitric acid and measure the quantities of nitrous oxide, nitric oxide, ammonia, and nitrogen that were evolved. They found that the sum of the volumes of ammonia and nitrogen were most sensitive to strain. A strain of 1% could be detected by this means, but the method was only suitable for measuring permanent set.

7) **Hardness Changes.**

Change in hardness with strain has been detected in metals, but for small plastic strains the surface needs to be polished and the metal in an annealed state (18)(20). Experiments with a wrought aluminium alloy and a bar of mild steel (not annealed) showed that when subjected to a permanent strain of 1%, any change in hardness was no more than the average scatter that occurred when testing different parts of the same specimen.

8) **Interference between Surface and a Glass Plate.**

The method has been used by Frocht to study a stainless steel ring subjected to a concentrated diametral load (19). One side of the ring was made optically flat and a glass prism placed in contact
with this surface. Monochromatic light was collimated and then
directed through the prism, optical interference taking place
between the metal surface and the lower prism face. Loading the
specimen in a plane parallel to the surface caused thickness changes
in the specimen in proportion to the sum of the principal stresses in
the plane. The changes of thickness were observed as contours of
differing interference orders.

The method is very sensitive to thickness changes, but requires
the specimen surface to be very accurately flat. Any twisting or
bending that occurred on loading a specimen would cause considerable
deterioration of the pattern and this ruled out the method for the
present work with a thin plate under tension.

9) Grain size changes.

When strained mild steels are annealed within suitable limits of
temperature, the growth of certain grains occurs at the expense of
others; C. Chappell (20) suggested that these gave the lines of
plastic flow. However Turner and Jevons (21) have shown that
deformation wedges reach considerably further than the recrystallised
area of abnormal grain growth.

10) Resistance Changes.

An investigation by J. Divis (22) set out to test the practical
utility of the previous discovery by M. Hoore that tensile stress
and resistance are intimately connected. It was found that resistance increased with stress almost in exact proportion up to the breaking point. However, stress measurement in structures by noting the variations of resistance was impracticable, the cross-section of the members being too large to cause appreciable difference in resistance.

Further difficulties are the irregularities of surface causing thickness changes and contact variations for a moving probe.

11) Magnetic Changes.

Changes in the permeability of various steels were investigated by Smith and Sherman Jr. (23). The compensated permeameter as designed by Burrows (24) was used in these tests. Curves of induction vs. applied field were plotted for 0, 500 K.G., 1000 K.G. loads on 3/8 sq.cm. mild steel and the results showed variations in the curves with stress. However considerable difference in sensitivity between compression and tension occurred and in some cases a reversal of effect with increasing load.

12) Metallographic Methods.

Numerous workers have observed under the microscope the slip bands which are produced in individual grains, the elongation of equiaxed grains, the breaking of large crystals into smaller grains and the rupture of grain boundaries, all of which occur at times in
strained metals. Turner and Jevons (21) attempted to correlate the magnitude of deformation with the microscopical changes in structure visible on polished and etched specimens. They found that very severe straining was easily recognised by these means, but that small permanent strain was not readily detected.

13) **Electrical Resistance Strain Gauges**

These are now widely used as a means of strain measurement, being simple to apply, having negligible reaction on specimens and capable of good accuracy.

The gauges are usually made in the form of a grid of fine wire affixed to a paper or plastic base. The wire is manufactured free from strain hysteresis effects and the operating portion of the gauge can be kept down to $\frac{1}{3}'' \times \frac{1}{3}''$ (or less) though usually larger.

The gauge base is attached to the specimen surface and strains on loading are transmitted to the wire grid causing a resistance change. A Wheatstone bridge is used with the active gauge in one arm and a dummy in the opposite arm to minimise temperature drift.

Providing the gauge can be calibrated in position, great accuracy can be obtained in determining strains from the resistance changes, up to the point at which the gauge base starts to yield. Gauges prepared with plastic bases may be reliable up to $\sim 2\%$ strain.

The accuracy of a gauge is limited when no means of calibration is possible, as batches of apparently identical gauges glued under
identical conditions are found to vary in sensitivity by several per cent. Moreover readings obtained give the average strain over the area of the gauge in the direction of the gauge length (with a small effect for cross strains) and this can be considerably below the peak strain where the rate of change of strain is high.

Measurements in the plastic region require a prior knowledge of the directions of principal strains, since the latter cannot be computed from an array of gauges as in elastic regions.

14) **Adhesion of a Photoelastic Layer.**

Mesnager (25) suggested in 1930 that the method of photoelasticity could be applied to measuring strains in metal if a birefringent layer was bonded on to the polished surface of the metal. By using a reflection polariscope a strain pattern would be observed in the layer on subjecting the metal to stress. He applied the method to metals within their elastic range but met with only moderate success due to the difficulties of bonding and the low optical sensitivity of the available materials.

More recently a series of experiments were carried out by Linge (26) using various adhesives and plastics and producing strains beyond the elastic limit of the metal.

With the introduction of epoxy resins, adhesion to metals has become much less formidable (27) and some of these resins are more optically sensitive than previously known plastics. These factors would enable more extensive and accurate measurements to be made.
The method appeared to be the most promising for the measurements that were required. Its advantages, difficulties, and limitations were investigated and a series of experiments commenced. The next chapter deals more fully with the method and its relation to normal photoelastic work.
SECTION IIIB.
THE BONDED PHOTOELASTIC LAYER.

OUTLINE OF PHOTOELASTICITY.
Stress analysis in the elastic range by means of photoelastic models is now well known. The method enables a complete evaluation of stresses and strains in two-dimensional work and has been extended to three-dimensional problems by the use of frozen stress technique and sectioning.

A typical apparatus for the observation and measurement of strain in two dimensions is illustrated in fig. 4.

Transmission Polariscope

FIG.4.
If the polariser is set to transmit horizontally polarised light and the analyser to receive light vertically polarised, no light reaches the screen while the model remains stress free. As the model is loaded the plane of polarisation of the light passing through
any point is rotated by an angle proportional to the difference of the principal stresses \((p - q)\) at the point, and to the thickness of the model. When the plane of polarisation becomes vertical, the analyser transmits a maximum amount of light and this part of the image appears bright.

As loading continues, the plane of polarisation becomes horizontal when again no light is passed by the analyser. Further loading produces light and dark alternately. The number of dark-light-dark cycles (isochromatic fringe number) is a measure of the rotation that has occurred and thus of the difference of the principal stresses. The image of the specimen appears as a contour map and for each contour \(p - q = \text{a constant}\). Using a standard test piece, the sensitivity of the material in number of fringes per unit stress (and thickness) can be determined and from this the value of the constant is obtained.

At free boundaries in a (thin) model, only one principal stress exists, \(p\) say, so \(q = 0\) and the fringe value at a boundary gives not only \(p - q\) but \(p + q\). For materials having linear stress-strain relations, the sum of the principal stresses, \(p + q\), can be shown to obey Laplace's equation for two dimensions.

\[
\frac{\partial^2(p+q)}{\partial x^2} + \frac{\partial^2(p+q)}{\partial y^2} = 0 \quad (14)
\]

This enables a numerical averaging process to be used to determine
p+q at all interior points once the values on the boundary are known.

Combining the values obtained for p - q and p+q, the individual principal stresses are obtained.

The directions of these principal stresses may be found by using a white light source and rotating analyser and polariser together. The light and dark patterns occurring previously with monochromatic light now appear as a pattern of colour changes, and darkness is only observed (except for an isotropic point where p - q = 0) at points where one of the principal stresses lies parallel to the axis of polarisation of the analyser or polariser. The locus of such points is an isoclinic. Since principal stresses at any point are always at right angles to each other, it is only necessary to observe the direction of one of these; this can be done by rotating the analyser and polariser together until the required point is in darkness, then reading off the analyser (or polariser) direction.

When using monochromatic light to produce the isochromatic fringes, the regions having principal stresses parallel to polariser and analyser again appear dark and obscure the isochromatic fringes. These dark regions (isoclinics) may be removed by the use of two quarter-wave plates, one either side of the specimen. The first is arranged to convert the plane polarised light to circularly polarised, and the second to convert back to plane polarised light after passing through the model.
MODIFICATIONS NECESSARY WITH A BONDED LAYER

Light passing through the photoelastic layer is now reflected at the metal surface and the transmission polariscope must be replaced by a reflection system. A typical arrangement is shown in fig. 5.

![Diagram of light path through a polariscope](image)

**FIG. 5.**

Light from the mercury lamp M is collimated by the condenser lens L, passed through the filter F, through the polariser P and is then reflected downwards by a semi-silvered mirror. The rays pass through the photoelastic layer to the surface of the specimen and are then reflected upwards through the semi-silvered mirror, through the analyser A to the camera lens. The rays forming the image of the specimen are then brought to focus on the photographic film.
Extra light losses now occur, due mainly to the semi-silvered mirror and the reflection at the metal surface. These, together with the absorption on passing through the polaroids and filter, make a high source brightness essential. A high pressure mercury arc was used in the present work, with the lamp house aperture considerably larger than the arc dimensions, in order to get a maximum amount of light to the condenser lens. The optical system is described more fully in Section III, p. 47.

The rays pass through the layer twice, thus the fringe sensitivity for a given stress in the layer and for a given thickness, is twice that for a transmission polariscope.

The method of obtaining \( p + q \) by numerical averaging is no longer valid in regions where plastic flow occurs in the metal (see "Limitations and Requirements for Accuracy" p. 37). Thus some other method is required before the individual strains can be determined.
PUBLISHED WORK ON THE LAYER.

Since Mesnager's original suggestion of a bonded layer (25) few published works on the method have appeared.

Linge (26) carried out an investigation at Cranfield College of Aeronautics in 1951 and a report was published in February, 1956.

The work was mainly concerned with techniques developed for using bonded layers on metal. A number of photoelastic materials were tried; these were Catalin 800, CR 39 and Marco Resins SB26C and SB 28C. Specimens of light alloy, magnesium, and mild steel were used.

Photographs were given of plastic deformation for a tapered tensile specimen at various loads, for two lugs under load and for holes in flat bars. Stress concentrations have been plotted in terms of fringe number and also in pounds/sq.inch. Presumably, the individual strains were calculated by the process of numerical averaging, and then converted to stresses using the tensile stress-strain curve of the metal. This method of calculating the individual strains is inaccurate in regions of plasticity (see p.37).

A number of glues were investigated, Araldite 101 with hardener 951 being found the most suitable in conjunction with sandblasting the metal surface and degreasing carefully.

(In the present work it was found necessary to heat the Araldite
101 in order to obtain complete curing. (This resulted in contraction stresses on cooling.)

The effects at the layer boundary were not considered (see p. 35) and no mention was made of the effects due to Poisson's ratio differing for layer and metal (see p. 39).

Dagostino et al. (29) have given an account of the surface preparation of aluminium alloys for bonding, and the use of epoxy resins for photoelastic layers.

Two types of procedure were explored. In the first, the entire assembly of metal and plastic was heated to about 100°C to obtain a low elastic modulus from the plastic. Some lack of linearity between strain and birefringence occurred. The second procedure was to use a flexible plastic which could follow large strains at room temperature.

Surface preparation of the aluminium was given as:

A) Surface scoured with 0 grade emery until smooth.
B) Immersed 3 - 5 minutes in detergent solution, between 160-180°F and rinsed in cold water. Then immersed in a chromic acid bath (5% chromic acid in water) for 1-3 minutes at 140 - 160°F, rinsed in cold water and dried in an air blast.

The very viscous type epoxy resin adhesives (comparable to
Araldite 101) were found to be the most successful; a method of mixing resin and hardener is given that avoids the introduction of air bubbles by means of a rubber sack, inserting the hardener with a hypodermic syringe.

Photographs are given illustrating patterns obtained for notched bars, and the authors point out the need for caution in interpreting strain at the edges of the layer. No attempt is made to determine strains from the photographs or to analyse the effects at the layer edge.

The method of heating the whole assembly to 100°C. appeared to be a considerable undertaking for the large size of specimen required in the present work. The surface preparation described above has been found to work well, but the photographs shown of tests at room temperature have large initial stresses, suggesting that the authors found it necessary to bond at an elevated temperature.
LIMITATIONS AND REQUIREMENTS FOR ACCURACY

The two chief limitations of the bonded layer are its restraining influence on the metal and the effects at the layer edge. Both of these result from the thickness of the layer used in order to obtain adequate sensitivity.

The restraint can be kept low by means of a low tensile modulus for the layer and by keeping the thickness below that of the metal. With Araldite used for the layer, the elastic modulus is about 4% of that of the aluminium alloy and changes the stress-strain curve of the alloy from (a) in Fig.6 for the alloy alone, to (b) for alloy plus Araldite. (The case for equal thickness is shown.) If the stress scale is reduced by 4% curve (c) is obtained, i.e. by allowing for the portion of the total tension taken by the Araldite. The restraining influence is thus of no account in elastic regions, being simply equivalent to a smaller applied tension (except for very small effects, due to Poisson's ratio differing for Araldite and alloy.) In the plastic region, the slope is raised above that of the metal alone by an amount equal to the modulus of the Araldite.

The effects at the edges of the layer are simplified where the boundaries of the layer and metal are identical. Consider, for example, a rectangular metal plate under tension, with a bonded layer and common central hole (fig. 7).
In any elastic extension, the strains of layer and metal are identical at all interior points and at the boundary of the hole (except for a small Poisson's ratio effect).

With increasing tension, regions occur (commencing at the boundary of the hole) where the metal is beyond its yield point. These inelastic strains in the metal try to make the layer follow them by forces across the glue line. The lower surface of the layer will follow the metal strains completely. However, the upper surface of the layer will not follow the inelastic metal strains unless subjected to forces in its own plane.

Fig. 8 (a) illustrates a small element of the layer subjected to forces from the surrounding material which enable the layer to follow the inelastic strains of the metal (providing the strain does not change too quickly along the specimen).

Fig. 8 (b) shows a small element close to a boundary (where contraction strains may occur up to half the strain value along the
boundary). No forces act horizontally across the boundary on the upper surface of the layer and the inelastic metal strains are therefore not transmitted completely through the layer.

(a) An Interior Region  
(b) At a Boundary.

FIG. 8.

The inequality between strains in the layer and metal can be reduced by having the specimen and cutouts large compared with the layer thickness.

Further requirements for accuracy are:

1) A strong uniform adhesion between the photoelastic layer and the metal at all points of their common surface.

2) A uniform thickness for the layer (or variations that are small and measured).

3) A long linear fringe-strain curve for the material of the layer.

4) Some method other than numerical averaging must be used in order to separate the principal strains, since, as shown below, the sum of the principal stresses in the layer in regions where the metal flows plastically, no longer satisfies Laplace's equation in two dimensions.
The layer derives all its stresses and strains initially from its lower surface in contact with the metal, so three-dimensional forces act on the layer and (if the material of the layer is within its elastic limit) the more general equation of Laplace holds:

$$\frac{\partial^2(pq+\tau)}{\partial x^2} + \frac{\partial^2(p+q+r)}{\partial y^2} + \frac{\partial^2(p+q+r)}{\partial z^2} = 0 \quad (\text{15})$$

Consider the case of a rectangular metal plate with a bonded layer as in fig. 7. In an elastic extension the third principal stress $\tau$, and the change in stress across the thickness of the layer, would only be appreciable in regions near $a - a'$ and $b - b'$, since suitable loading on these two boundaries would produce identical strains to those in the metal (except for a small Poisson's ratio effect), even though the layer were not bonded to the metal. Thus in elastic extensions $\gamma$ and $\frac{\partial^2(p+q)}{\partial z^2}$ are both sufficiently small to be neglected and the equation reduces to 2-D form for which the method of numerical averaging is applicable.

Where inelastic strains occur in the metal, strong local forces are required across the glue line to force the layer to follow them; $\gamma$ and $\frac{\partial^2(p+q)}{\partial z^2}$ are now no longer small, and the method of numerical averaging is inaccurate.
EFFECT OF POISSON'S RATIO

PHOTOELASTIC CALIBRATION.

A number of experiments were carried out to determine the fringe-strain characteristics of Araldite alone and when bonded to aluminium. The sensitivity of the bonded layer (viewed at normal incidence) was found to be about 4% lower than for Araldite alone; this was expected as Poisson's ratio for Araldite is higher than that of the metal. The effect is considered in detail below.

In a stressed two-dimensional specimen, the number $N$ of fringes observed, corresponds to the maximum shear stress by a direct proportionality i.e.

$$N = 2K \tau_{\text{max}} \quad \quad \quad (16)$$

When the specimen is subjected to a pure tension $p$, the number of fringes (or reversals of light and dark) is given by

$$n_p = K \cdot p \quad \quad \quad (17)$$

where $n_p$ is the number of fringe units corresponding to the (larger) principal stress $p$.

Thus $K = \frac{n_p}{p}$, or if strain is measured, $K = \frac{n_p}{E \cdot \varepsilon_p}$

$$\text{i.e. } K = \frac{\text{number of fringes observed}}{E \cdot \varepsilon_p} \quad \quad \quad (18)$$

where $E = \text{Young's modulus for the photoelastic material.}$

$\varepsilon_p = \text{The strain in the direction of } p.$

In the more general case when a second principal stress $q$ exists,

$$n_q = K q \quad \quad \quad (19)$$

in which case

$$N = n_p - n_q$$
for light at normal incidence, since
\[ \tau_{\text{max}} = \frac{1}{2}(p - q) \]

Let \( \varepsilon_q \) = the strain in the direction of \( q \).
\( \gamma \) = Poisson's ratio for the photoelastic material.
\( \eta \) = Poisson's ratio for aluminium alloy.

To determine the effects when the photoelastic layer is bonded to aluminium, consider the case of a bonded tensile specimen.

![Diagram of bonded tensile specimen](FIG.9)

The lateral contraction of the layer is restricted to \( \eta \varepsilon_p \) instead of \( \varepsilon_p \) (ignoring the lateral reaction on the metal).

Two effects occur:

Firstly, a value of \( \varepsilon_q \) will now be obtained from measurements in the Araldite (e.g. by oblique incidence) when the transverse stress in the metal is zero. Thus the stresses in the layer are not in direct proportion to the stresses in the metal and it is necessary to develop a strain-optic law, since the photoelastic effects in the layer occur because of the transference of strain from the metal. These strains may be obtained using the stress-strain equations:

\[ E \varepsilon_p = p - \gamma q \quad \text{and} \quad E \varepsilon_q = q - \gamma p, \quad \text{(20)} \]

in conjunction with equations (17) and (19).

Secondly, since the observed number of fringes will not
now be simply a measure of \( n_p \) (but of \( n_p - n_q \)), \( K \) cannot be directly evaluated.

Substituting \(-\eta \varepsilon_p\) for \( \varepsilon_q \) in equation (20),

\[
\begin{align*}
E \cdot \varepsilon_p &= p - q - q \\
-\eta E \cdot \varepsilon_p &= q - p
\end{align*}
\]

Subtracting equations (21) we have

\[
E \cdot \varepsilon_p (1 + \eta) = p - q + \eta (p - q)
\]

Thus \( p - q = E \cdot \varepsilon_p (1 + \eta) \) \(-\ldots-\) (22)

The number of fringes observed at normal incidence = \( K(p - q) \)

so now, \( K = \frac{\text{Number of fringes observed} \cdot (1 + \eta)}{E \cdot \varepsilon_p (1 + \eta)} \) \(-\ldots-\) (23)

By comparing equation (23) with equation (18) it is seen that when measurements are made to determine \( K \) from a bonded layer an extra factor is introduced which depends on the Poisson's ratio of the layer and the metal. The factor is 1.04 for Araldite on aluminium up to the elastic limit of the metal, but this will decrease in regions of large plastic flow.

OTHER EFFECTS.

Considering a bonded tensile specimen again, small transverse stresses are transmitted to the metal from the layer. In elastic regions these are compressions amounting to only 4% of the stress in the layer, which in turn is only about 4% of that in the metal. Thus, for equal thicknesses the effect in the metal is less than 0.2%.
In fully plastic regions, Poisson's ratio for the metal becomes larger than that of the layer and a small transverse tension in the metal is obtained.

Small effects occur in the layer at boundaries. The metal cannot force the layer to follow it completely at the edges (see fig. 8), and the transverse strain of the layer tends to that required by its own Poisson's ratio. In elastic regions viewed at normal incidence, this causes a slight increase in sensitivity at a boundary (i.e. approaching the sensitivity of an unbonded layer).
An apparatus was required that could apply tension to a thin metal plate in one direction or in two directions at right angles, and be suitable for use with a reflection polariscope.

The design was based on a number of requirements as follows:

The strain sensitivity of the photoelastic layer gives a lower practical limit to its thickness. In the case of aluminium alloys at their yield point, the strain amounts to about 0.6% at which 1/16" thick Araldite, if used for the layer, would give approximately five fringes in mercury-green light. This was considered sufficient providing measurements could be made to 1/10 of a fringe.

To keep the restraining influence of the layer low, the metal needed to be at least as thick as the layer i.e. 1/16".

For the metal specimen to approximate to a hole in an infinite plate, the width of the specimen needs to be say six or more times the diameter of the hole. This in turn needs to be large compared with the thickness of the metal (to obtain a two-dimensional stress distribution) and of the layer in order to obtain
accurate measurements close to the hole (see Section II "Requirements for Accuracy"). Thus the larger the specimen the better.

To apply sufficient tension to a large specimen, taking parts of it beyond the elastic limit required a large force and this was to be applied in two directions at right angles.

The tension acting in each direction needed to be continuously variable, independent of the other direction, and uniform across the width of the specimen.

The overall extension of the specimen would be minute (~1/2%) since an elastic region would surround the plastic area near the hole.

A compromise on the size of the specimen was made, allowing a maximum width of twelve inches. A peak load of 20 tons in each direction was thought sufficient, and a simple way of achieving this appeared to be by means of a lever system, as the overall extension would be small.

Fig. no.10 illustrates the system employed using a lever ratio of 10:1.

The levers could be repeated on the other two sides of the specimen for stretching in two directions. To give a measure of adjustment of tension across the width of the specimen, each lever system comprised three separate levers acting in parallel (fig. 11).

The table (fig.12) needed to withstand 20 tons in two
directions without breaking or bowing, and was made from 2" x 1\(\frac{1}{2}\)" x \(\frac{1}{4}\)" mild steel channel section welded together. The weakest points of this design were at places such as a-a' in fig. no. 12, where the stress produced by cross loading would tend to cause buckling along the weld. Steel inserts were fitted into these regions.

Mild steel sections were welded to the ends of the table for the lever fulcrums and set accurately in line.

The upper arms of the levers were kept down to 2" to keep the bending moment, and thus the stresses, down to a minimum. The levers were made from 2\(\frac{1}{2}\)" x 1" section with eye hole and pivot welded on (fig. no. 13). With this thickness of material, and under maximum load, calculation showed the levers would have a maximum tensile stress (in the outer fibres) of approximately 16 t.p.s.i.

Fig. no.14 shows the dimensions of the eye holes, pivots, and the lugs and pins used to connect the levers to the clamping plates, described below.

If the material of the specimen were in sheet form it would be near its yield point under full load, even in parts remote from the hole, and a means of gripping would be required that avoided appreciable stress concentration. The clamping plates were made to accommodate various methods of fixing and experiments were carried out to find a suitable system. Each pair of plates were
made together and drilled together to take seven 1"
diameter high tensile bolts and three 2" diameter steel pins.
Fig. no. 15 shows the arrangement.

Six compression cylinders were made for accurate measurement
of the applied load (Fig. no. 16). These were cut from 2"
diameter solid drawn steel tube of good quality and constant wall
thickness. Strain gauges were attached to the cylinders as
described on p. 51.
THE OPTICAL SYSTEM

A reflection polariscope was made with 6" diameter components so that a large area of the specimen could be photographed with each exposure. Figure no. 18 shows the system when used with monochromatic light.

The lamp house was made to contain a 250 watt tungsten filament projection lamp as a white light source and a 125 watt high pressure mercury discharge lamp for monochromatic light. A high source brightness was essential due to light losses occurring at the semi-silvered mirror, the metal surface, and in passing through the polaroids.

When the optical system was later modified for oblique incidence, less light losses occurred and shorter exposures could be given.

Both lamps were cooled by a convective flow of air in a housing of 1/8 gauge mild steel sheet with brass supports (fig. no. 18).

A brass framework for polariser, analyser, mirrors (and quarter wave plates when required) was made from 1" x 1" x 3/8" brass angle brazed together. Concentric brass rings, marked off in degrees from 0° to 360°, were made for the polariser and analyser. Each polaroid sheet was sandwiched between two thin circular glass sheets, and clamped to a brass ring which could be set at any required angle.

The mirrors were made from plate glass and coated by the
Ross Optical Co. The semi-reflecting surface was coated with aluminium and the fully reflecting surface with a chrome-aluminium deposit.

For the mercury-green filter a 6" square of gelatin was used, protected by glass plates on either side.

Considerable amounts of light from the condenser passed straight through the semi-reflecting mirror and on to the black backing. Reflected light from this surface was found to be superimposed on the image in the camera, causing considerable haze. The effect was eliminated by arranging a backing of black strips at $45^\circ$, deflecting the stray light away from the main beam. This is illustrated in fig. no.17.

![Diagram of light path](Image)
The optical bench was made from two 1" diameter steel tubes, fixed parallel to each other by welded distance pieces at each end, and arranged to pass through the brass framework containing the polaroids and mirrors. The lamp house and condenser lens were fixed to the underside of the optical bench and the camera above, the whole system being supported above the tension table but free to turn in a horizontal plane.

The arrangement of the optical system, after modifying it for oblique incidence (see p.72), is illustrated in fig. 3.
The Casting Oven

Casting temperatures for Araldite range from 100° to 160°C and curing times from 15 to 3 hours, during which time the temperature is kept constant. Lowering the temperature slowly after curing is necessary to avoid residual stresses.

For the specimens, whose overall size could be up to 18" square, a special oven was required as the largest available was limited to an inside floor area of 14" x 14".

The form of the oven made is shown in fig. no. 19. A 1/2" flat steel plate 14" x 14" is heated from underneath by six mica resistance strips totalling 750 watts. Specimens could be clamped to the plate by means of an aluminium square, with their edges projecting outwards from the heated surface. An asbestos lid 3" deep fitted over the plate. Temperature was controlled by a sensitive Sunvic thermostat whose coiled bimetal strip fitted into a steel block clamped to the underside of the heated plate.

A small motor with a spindle geared down to 1 revolution per hour was fitted below the thermostat control and arranged to lower the temperature slowly after curing (in 16 revolutions of the spindle).

The aluminium clamping square was also used for the sides of the mould, being covered with a thin layer of silicone grease to
insure that the Araldite did not adhere to it while curing.
Castings could be made separately by using the heated plate as the mould base, with a similar coating of grease.

**Strain Gauges and Bridge.**

A Tinsley portable strain gauge-bridge was obtained, calibrated for strains of .05%, .1%, .25% and .5% for full scale reading of the slider, which was turned until a null deflection on the internal galvanometer was obtained. The bridge was used with a 10 way selector and is illustrated in fig. no.29.

Each of the six compression cylinders used for measuring the applied tensions had four gauges attached parallel to its axis and spaced symmetrically round the mid plane. This reduced the effect of out of balance loads (28). Four other gauges were placed at right angles to the first set and each group were connected in series but in opposite arms of the bridge. This minimised the effect of change of resistance with temperature.

The gauges were affixed following the manufacturers' instructions, and after drying them in an oven for 24 hours they were covered with a rubber waterproofing compound.

The calibration of the cylinders is described in Section IV.
ALUMINIUM ALLOY L71.

This was the alloy recommended by the Ministry of Supply for use under their contract. It had a stress-strain curve with a straight initial portion, rapid turnover and low strain hardening, and was used throughout the work.

Equivalent specifications are DDT 646B and HS 15 WP, the alloy being solution treated and artificially aged at 175°C to 155°C for 9 to 20 hours. A typical composition for the main alloying elements was:

- Copper 4.1%
- Manganese .75%
- Silicon .76%
- Magnesium .63%

The elastic constants of the alloy were:

- Tensile Modulus $1.04 \times 10^7$ lbs.ins.$^{-2}$
- Poisson's ratio .33
- .1% proof stress 23 t.p.s.i.

The material was obtained in sheets 6 ft. x 3 ft. and thicknesses of .055", .085", .128". Tensile specimens were cut parallel to and at right angles to the final direction of rolling and used in conjunction with a Denison tensile test machine, and a Lamb's roller extensometer to determine stress-strain curves for the material. Fig.46 shows the results obtained with test pieces.
of type B.S.S.485 and a 2" gauge length.

In the process of bonding, as explained on p. 68, some of the alloy was heated for 20 hours at 120°C. This temperature was on the fringe of artificial ageing temperatures so tensile specimens were similarly treated to assess the difference, if any, in the yield point. Only slight changes occurred; the yield point of the .055" material increased by 1% and that of the .085" material by 2%.

The yield point in each stress-strain curve has been taken as the stress at which the initial slope continued upwards, meets the slope of the fully plastic region extended backwards. (See fig.46).
GRIPPING THE SPECIMEN

When gripping tensile test pieces, the ends are usually made considerably thicker or wider than the central testing area to ensure that, in spite of being weakened by the method of gripping, the ends are still stronger than the centre.

The work in hand was concerned with plates very much wider than the central hole, with a plate width of 6" or more but length not greater than 16". This limited any increase in width at the gripping edges since uniform tension was required over the central area.

One way of ensuring adequate strength at the ends of a specimen was to reduce the central thickness of a thick plate. However, to maintain a constant thickness of say 1/16" over an area of up to 12" x 12" was considered a formidable task. Moreover it was thought desirable to use the sheet metal in the thicknesses supplied, if this could be achieved.

A number of methods of gripping the specimen were considered. Bending the ends in a small arc (fig. no. 21) was unsuccessful due to the limited plastic elongation of the alloy, causing cracking.

FIG. 21.
Gripping by means of bolts passing through the material was tried. Seven 1" diam. high tensile bolts were used at each end of the specimen and strengths up to approximately half the ultimate strength of the sheet were obtained. It was thought that with a greater number of smaller bolts, spread out over a larger area and clamping between rough plates, the gripping strength would approach that of the rest of the plate. However, each specimen would require considerable preparation.

The use of rounded knife edges was rejected as the pressure necessary to prevent sliding, combined with the applied tensile load would cause failure by shear.

Friction wedges commonly used in testing machines were considered.
With the system of tensioning used, the upper arms of the levers were limited to approximately 2" between the centres of the fulcrum and the line of tension (d in fig.22); any increase in this distance required a corresponding increase in the strength of the levers. It was estimated that to grip specimens up to 12" wide by means of friction wedges would require a larger distance between fulcrum and line of tension. Moreover small machining errors could cause the tension across the specimen to be non-uniform.

Welding on gripping edges was tried. Aluminium alloy rods \( \frac{1}{2}'' \times \frac{3}{8}'' \times 12'' \) were fillet welded to the ends of test pieces, but failure occurred near the weld at approximately half the normal yield strength of the material. Soldering on edges at a somewhat lower temperature produced the same result. It appeared that this heat treatment of the metal at 400-500\(^\circ\)C caused unavoidable strength deterioration.

**Gripping by Redux Bonding**

To overcome the deterioration of the alloy at high temperatures, experiments on bonding with Redux were tried. This is a phenol formaldehyde resin brushed on to the surfaces to be bonded and then polyvinyl formaldehyde is sprayed over to give strength after curing. Recommended temperatures are 140-180\(^\circ\)C for 15-5 minutes during which time the surfaces must be under high pressure. Surface preparation is important and shear strengths as high as
4 t.p.s.i. have been obtained under optimum conditions. Any tendency to pool has to be avoided as this can cause considerable reduction in strength.

Rectangular sections 12" x 2½" were cut from 5/16" aluminium alloy sheet. A ½" x ½" slot was milled out along the length of each section which was then prepared for bonding.

(The method of gripping after bonding is illustrated in fig.23)

FIG. 23.

Surface preparation for the specimen and the section to be bonded consisted of degreasing the surfaces with acetone, followed by rubbing with coarse emery and degreasing again by scrubbing with detergent in warm water. The surfaces were then rinsed, etched with a warm mixture of 5% chromic acid and 15% sulphuric acid in water, rinsed in water and allowed to dry.
The phenol formaldehyde was then brushed on and the Redux powder sprayed over, any excess being tapped off. This was left to dry for 30 minutes before placing the surfaces to be bonded in contact.

A clamp made from channel section provided the pressure of 50 - 100 lbs. p.s.i. required while curing. Layers of Neoprene were used to even out the pressure (fig. 24).

Heat was supplied from two bunsen burners and the temperature measured approximately using a thermocouple and a calibrated galvanometer. The temperature was kept at about 160°C for ten minutes and then allowed to cool slowly to 80°C before immersing both clamp and specimen in cold water. The clamp was then unbolted.

A tension of 16 tons was obtained using a 12" wide specimen .085" thick, before the Redux bond parted. For a specimen only .055" thick this tension would be sufficient to cause appreciable
plastic flow, so gripping pieces were bonded on to this thickness of plate. The maximum obtainable tension however was found to fall from 16 tons to 11 tons. It was concluded that the shape of the gripping piece was far from ideal and that with its rectangular form the great majority of the stress on the bond was occurring in the first \( \frac{1}{4} \)"; the thinner specimen would cause an even greater stress concentration.

The gripping pieces were then made with a tapered section (fig.25).

![Diagram](image)

**FIG. 25**

The taper was found to increase the bond strength sufficiently to allow large plastic strains to be produced in specimens of .055" and .085" thickness, and all later specimens were bonded in this manner.

Measurements with strain gauges showed that the stress in the central region (with no hole) was substantially uniform when the levers were suitably adjusted.
PRERPARATION OF THE PHOTOELASTIC LAYER.

Araldite (an epoxy resin) has been used throughout for the layer. In casting resin form, both resin and hardener are solids at room temperature, whereas for the cold setting Araldite glues, resin and hardener are in liquid form.

Araldite has a number of advantages over previous photoelastic plastics, having high optical sensitivity, low time-stress effects, a long linear stress-strain curve and a strong bond with metal.

Casting and Machining.

Experiments were first aimed at casting Araldite directly on to the aluminium alloy, using the oven described on p. 50. The surface treatment of the alloy consisted of degreasing with acetone, removing the surface layer with coarse emery, scrubbing in warm detergent solution, then rinsing and allowing to dry.

Various proportions of casting resin to hardener were used for the castings, typical quantities being 220 gm. of Casting Resin B and 80 gm. Hardener 901. The casting resin was heated in a metal
container at 130°C until completely molten. The hardener was placed on a clean surface and all lumps pounded to powder, any foreign matter being removed. Once the resin was molten, the hardener was tipped in and stirred until the mixture became as clear as the resin alone.

The specimen was then clamped to the heated oven plate, silicone grease was smeared around the edges of the mould and the casting mixture poured on. The oven lid was put over and the casting left overnight at 120°C, then cooled slowly.

On removal the specimen was appreciably bowed, and this appeared to be unavoidable even when using the lowest recommended curing temperatures and very slow cooling. Examination in polarised light revealed considerable residual stresses, presumably forming on cooling due to the thermal contraction of the Araldite being far greater than that of the aluminium.

To avoid this initial stress it appeared necessary to produce castings separately, then to bond them to specimens using a cold setting Araldite glue.

For these castings the steel plate of the oven was used as the base of the mould and a thin layer of silicone grease (Releasil No.7) was smeared all over. The method of casting was similar to that described above but the proportions of resin and hardener were standardised at 180 gm. of casting resin B and 60 gm. of hardener 951. Curing temperatures were kept close to 125°C and castings
were again left overnight to cure. Care was taken to ensure that the mould base was accurately horizontal.

After curing and cooling slowly, the casting was removed for machining. Its upper surface was flat and smooth but its underside (in contact with the grease) rough and ridged.

To machine the casting truly flat proved difficult. A surface plate was obtained and the casting was clamped flat face downwards on to this by aluminium bars along two sides. On milling with either horizontal or vertical cutters, ridges occurred between neighbouring cuts, probably because of an imperfect sideways traverse. Further thickness variations resulted from a slight lift of the casting from the surface plate.

Experiments with a number of castings led to the use of a thin layer of grease between casting and surface plate, acting as a vacuum seal and holding the casting down. The conventional milling cutters were discarded in favour of a trepanning tool. This had a cutting tip at the end of a radius arm and was set to revolve in a 10" diameter circle (Fig. 26).

The whole of the required casting area could be cut with one forward traverse using this tool. Approximately 0.01" was removed from the surface with each traverse, using oil as a lubricant.

When a casting had reached its required thickness, the forward traverse was stopped and the tool fed vertically downwards while
rotating, so that a circle was cut out. This was removed for bonding to a specimen.

Castings from 0.13" to 0.04" were obtained by this means with thickness variations of ± 0.001" or less.

When the castings were viewed in the polariscope they were found to transmit considerable light even before machining and appeared but slightly affected by local straining. It was thought that this might be due to a vertical stress caused by a temperature gradient existing across the casting during curing and cooling.

An annealing oven was made by building heavy iron plates into an old oven and surrounding these by copious mica insulation. Asbestos was used to blank off the space between the plates. (Fig. 27).
The castings after machining were placed between two glass sheets, separated by paper, and lifted into the oven between the iron plates. The temperature was raised to 140°C, the current switched off and the oven left to cool, taking about 16 hours to fall to hand warmth. Castings annealed in this manner were flat and substantially free from residual stresses.
BONDING EXPERIMENTS.

It appeared necessary to use some form of epoxy resin adhesive because of the high adhesion strength needed between the casting and the metal. Two recommended resins were Araldite 101 and Araldite 103 used in conjunction with hardener 951. These were all liquids but Araldite 101 was very viscous.

A number of small pieces of alloy and Araldite were cut up to try different methods of glueing and surface preparation.

The alloy, after removing surface grease, was treated with various combinations of:

(1) Coarse emery finish.
(2) Fine emery finish.
(3) Degreasing with acetone (redistilled).
(4) Degreasing by scrubbing with detergent.
(5) Etching with 5% chromic acid in water.

The Araldite surface was varied from (1) to (4), and the consistency of the adhesive ranged above and below the recommended hardener ratio of 1:14 by weight, for Araldite 101 and 1:9 for Araldite 103.

Most specimens were tested after 48 hours but in cases where the glue was still not hard they were left for up to 14 days. The methods of test were by tensioning the metal until the bond failed.
or by drilling holes through the Araldite and metal, when any lifting near the hole produced a lighter area that was easily detected. The larger the holes (up to \( \frac{3}{8}'' \) were used) the more severe the test appeared to be.

The 101 resin seemed considerably superior to the less viscous 103. Little difference in strength was observed between a coarse and a fine emery finish, but degreasing by scrubbing with detergent followed by etching appeared to give the best results.

A full size specimen with a 1'' central hole and an 8'' circle of Araldite (\( \frac{1}{8}'' \) thick) was prepared. Resin 101 and hardener 951 in the ratio of 14:1 were thoroughly mixed in a small polythene sack to avoid air bubbles (29). The mixture was poured from the lower end of the sack in a ring around the 1'' hole, the Araldite casting carefully placed over and then loaded with weights. The specimen was supported on a 9'' square table 4'' high with a central hole to allow excess glue to flow out.

After 48 hours the specimen was stretched and a photoelastic pattern obtained which however, slowly disappeared. On releasing the tension a pattern again formed and slowly diminished. The glue seemed to be slowly slipping.

Further experiments with larger proportions of hardener, more thorough mixing and longer curing times still produced photoelastic patterns that decreased with time, though more slowly. On heating
for 2 hours at 60°C the glue set firmly but considerable contraction stresses appeared on cooling.

It was found that even a 10°C rise in temperature while glueing produced observable stresses when cooled.

The less viscous 103 set hard at room temperature, but its adhesion to metal was not sufficiently good. However it did seem to adhere strongly to the casting surface.

Experiments were tried in which the alloy was coated with a strongly adhering thin film using heat. It was thought that the Araldite 103, sandwiched between the film and casting might give good adhesion, and that if the film were sufficiently thin the effect of the contraction stresses would be small.

This method was found to be successful, and thin films (~0.002") of Araldite 101 and of Araldite casting resin could be formed on the alloy with certain precautions. The methods are set out below.

**Surface Preparation.**

Excess grease from the alloy surface was removed with cotton wool soaked in acetone. Medium emery, followed by a fine grade was used, and the surface then scrubbed with detergent in running warm water, rinsed thoroughly and allowed to dry by warming to 40-50°C. The finer grades of emery were found to give better resolution of fringes near the hole, but increased surface glare and non-uniformity of illumination.
Covering the Alloy with a Film of Casting Resin.

20 gms. of casting resin B were melted at 120° C and then 10 gms. of powdered hardener 901 were mixed in and stirred until the mixture cleared. The specimen, after the surface treatment described above, was placed on the heated oven top (at 120° C). After approximately 1½ minutes the mixture was poured on to the surface and spread out evenly to a thin film using a piece of folded paper. The excess was moved off the specimen and the oven top then put over. The temperature was lowered to 90°C for 6 hours before raising again to 120° C and leaving to cure overnight.

The initial 6 hours at low temperature was necessary, together with the high ratio of hardener, in order that the latter did not all evaporate away from the film before curing was effected.

Covering Aluminium with a Thin Film of Araldite 101.

Araldite 101 and hardener 951 Y/ere mixed in the ratio of 9:1 by weight. The prepared specimen was placed on the oven surface at 60° C and the mixture was poured on and spread out as before. The oven top was replaced and the temperature lowered to 35°C for 6 hours before leaving overnight to cure at 70° C.

Gluing Casting to Specimen using Araldite 103

Any high spots in the Araldite film were lightly emeried and the casting to be used was bevelled around its edge. The specimen
and the casting, already slightly roughened from machining, were scrubbed in warm detergent solution, rinsed and allowed to dry.

30 gms. of Araldite 103 were weighed out into a boiling tube and this was rosted in warm water until all bubbles had risen to the surface. 5 gms. of hardener 951 were then stirred in; the mixture was left for a few minutes until any air bubbles had risen and then poured out in a ring around the hole in the specimen, the latter being supported on a steel table with a \( \frac{3}{4} \)" central hole. The casting was lowered symmetrically over the hole, touching first at one side, then working across to avoid trapping air, and pressed down until excess glue appeared all round, and from the central hole. Metal weights of a few pounds were placed on the surface and the glue left to cure for 24 hours. The specimen was then removed and a central hole made in the Araldite by drilling out with small holes and then filing until the Araldite boundary matched that of the alloy.
In normal two-dimensional photoelastic work the patterns obtained are contours of \( p - q \approx \) Constant. A number of methods are available for the separation of the principal stresses (or strains), e.g. numerical methods can be used to calculate the values of \( p \pm q \) at all interior points of a model once the values are known on the boundaries. However, such methods are no longer valid in a region of plasticity. (see p.37).

Numerical integration of the equilibrium equation and the method of shear differences are used in normal photoelastic work. These are however, limited by the accuracy in determining the isoclinics, which have not been found well defined in the present work with a thin bonded layer.

Methods using the change in thickness of a specimen are well known. In the present work, e.g. for a \( 1/16 \)th layer bonded to \( 1/16 \)" metal under a single tension, the change in thickness at the yield point (of the metal) is only \( \sim 25 \cdot 10^{-6} \) inches, for which an accuracy of within 2% would require measuring down to \( 5 \cdot 10^{-6} \) inches. For the present work with
specimens up to 12" wide this difficulty could only be overcome by having lateral thickness gauges at every required point of measurement. These would cause the photoelastic pattern to be obscured.

Experiments to obtain a value of \( p \) alone were tried, using the photoelastic pattern due to permanent set obtained after stretching a specimen. Narrow channels were machined through the depth of the Araldite, parallel to the direction of stretching. The idea was for the channel to create a free edge for the Araldite so that only one principal stress could exist along it, and to observe the change in the photoelastic pattern. The method appeared moderately successful but could only be applied to the pattern of permanent set.

The use of Small Holes.

A number of experiments were tried to determine the ratio \( p : q \) by means of small holes drilled to the depth of the Araldite.

The metal was left untouched in order to avoid spurious stress concentrations. Each hole was to simulate on a minute scale a
hole in an infinite plate acted on by two tensions at right angles. The two principal stresses acted around the boundaries of the hole to produce a minute pattern orientated about the principal stress directions. As the material around the lower edge of the hole was bonded to the metal, the same stress concentration as for a "complete" hole would not be obtained. However, it was thought that the shape of the pattern produced would vary with the ratio of the tensions near this hole in a similar manner to a "complete" hole.

Fig. 36 illustrates the patterns obtained.

It was found that the method was sensitive when \( p \) & \( q \) were of the same order (and sign) but that for \( psq \) greater than 5:1 the patterns obtained were too similar to those obtained when \( q = 0 \) with \( p \) decreased by an appropriate amount. (Actually by an amount equal to \( q \) so that \( p - q \) remained equal in the two cases).

One feature of the patterns was that in elastic regions, where the number of fringes was low, the directions of principal stress could usually be determined with an accuracy better than that given by isoclinics.

The Method of Oblique Incidence.

A method of separating \( p \) and \( q \) by oblique incidence was suggested for normal photoelastic work by D.C. Drucker in 1943 (30). A photograph of the specimen was taken at normal incidence; the
spacemon was rotated through a known angle (θ) about an axis of symmetry and a second photograph taken. (See fig. 28).

![Diagram showing the setup for principal stress measurement](image)

**FIG. 28.**

The principal stresses \( p, q \) may each be represented by a number of fringes \( n_p, n_q \) - the number (with relevant sign) that would be observed for one if the other were zero.

For normal incidence, the number of fringes observed at any point is \( n \equiv n_p - n_q \)

On rotation about one principal stress \( (p \text{ say}) \) the number of fringes at any point is \( n_2 = \frac{n_p - n_q \cos^2 \Theta}{\cos \Theta} \)

From those two equations, \( n_p \) and \( n_q \) can be separately determined

\[
\begin{align*}
n_p &= \frac{n_2 - n_q \cos \Theta \cos \Theta}{\sin^2 \Theta}, \\
n_q &= \frac{n_2 \cos \Theta}{\sin^2 \Theta} - n\end{align*}
\]  

(24)

The method is immediately applicable to the calculation of stresses along lines of symmetry since one principal stress lies along this line. For other points a prior knowledge of the directions of principal stresses is necessary.

The method was tried for the case of a layer bonded to metal. A specimen with appreciable permanent set was first used, taking photographs at normal incidence and at approximately \( 45^\circ \). The
arrangement is illustrated in fig. 29.

Normal Incidence.

Oblique Incidence.

FIG. 29

No semi-silvered mirror was required for the photo at oblique incidence, but it was necessary to immerse the specimen in water to cut down surface reflection and to keep the light rays at a moderate oblique angle on entering the Araldite.

Definite differences in the two patterns were observed, but difficulty occurred in relating any given point on one photo with that on the other. A further difficulty was the unknown phase difference that occurred between the two photos due to using a semi-silvered mirror for normal incidence only.

To overcome these problems and to take photographs of the specimen while under load, the following system was adopted. (Fig.30).
A large prism was used with its lower surface mated to the Araldite layer by means of a liquid. The camera, lamp house, condenser lens and polaroids were attached to parallel rails so that the whole optical system (including the prism) could be moved round by 90°.

Two photographs at oblique incidence were taken, one with the optical system along (and one at right angles to) the line of tension. From these two photographs, the values of $n_p$ & $n_q$ could be calculated for all points along the lines of symmetry of the specimen and for any other points where one of the principal stresses ($p$ or $q$) was parallel to the direction of tension.

For rotation $\Theta$, about the $p$ axis the number of fringes observed at any point

$$n_i = \frac{n_p - n_q \cos^2 \Theta}{\cos \Theta},$$

(25)
For rotation $\Theta_2$ about the $q$ axis the corresponding number of fringes is

$$n_2 = \frac{n_0 \cos^2 \Theta_2 - n_q}{\cos \Theta_2} \quad \cdots \quad (26)$$

From these two equations:

$$n_p = \frac{n_1 \cos \Theta_1 - n_2 \cos^2 \Theta_1 \cos \Theta_2}{1 - \cos^2 \Theta_1 \cos^2 \Theta_2} \quad \cdots \quad (27)$$

$$n_q = \frac{n_1 \cos \Theta_1 \cos^2 \Theta_2 - n_2 \cos \Theta_2}{1 - \cos^2 \Theta_1 \cos^2 \Theta_2} \quad \cdots \quad (28)$$

The use of two oblique photographs gives a greater difference in fringe number at any point than one at oblique and one at normal incidence. This increases the accuracy in the determination of $n_p$ & $n_q$ from $n_1$ & $n_2$, and at the same time any unknown phase difference between the two photographs is eliminated.

The 6" prism was first made from glass sheet filled with water, the lower surface of the prism being mated to the Araldite with water. Considerable surface reflection occurred which obscured the photelastic pattern. The refractive index of the Araldite was measured to be 1.637, and as that of water is only 1.33 the difference accounted for the surface reflection.

A second 6" prism was made from perspex; liquid paraffin was used to fill it and to mate the prism-Araldite surface. This cut down the surface reflection to about 1/5 of that previously observed.
The surface of the Araldite was marked out with an accurately spaced system of squares for which Indian ink was found to be suitable. The squares enabled the exact angle of obliquity to be determined and corresponding points on the two photographs to be accurately located.

Since a 6" diameter field lens was not used, the camera had a finite angle of view, which caused changes in magnification and in the angle of obliquity along the specimen in the direction of the camera. Allowance had to be made for these changes before corresponding points on the two photographs could be compared. (See p. 87).
TEST PROCEDURE

Choice of Specimen Dimensions.

It was attempted to test different thicknesses of alloy, different plate widths, and different hole diameters in order to assess:

a) The effect of the finite ratio between plate width and hole diameter.

b) The restraining influence of the Araldite layer on the deformation of the metal.

c) The boundary effect of the layer, i.e. the fall in sensitivity that occurs at a common boundary when inelastic strains are present in the metal and are not fully transmitted through the layer.

Most of the specimens tested were only partially successful because of failure in the bond of the layer or of the gripping pieces at less than full load. This limited the experimental estimation of the above three effects.

Manufacture of the Specimens.

The dimensions were marked out on a sheet of L71 alloy with the length of the specimen either parallel to the final direction of rolling or at right angles to it. A band saw was used to cut the metal slowly, fast cuts causing distortion of the thinner
sheets. The edges of the specimen were ground and filed down to the required size and the central hole, for diameters between \( \frac{1}{2} \)" and 1", formed using a counter-boring tool. Small holes were drilled out, and holes larger than 1" were cut with a radius cutter.

Preliminary trials with 2-way specimens showed that it was necessary to thin down the central region to obtain sufficient stress at the centre at loads within the tensile limit of the arms. The specimen was clamped down on a surface plate on a vertical milling machine. The cutter blades were rounded at their lower edges to avoid any sharp change in thickness, and set to take light cuts of the alloy.

Sequence of Bonding Operations.

The methods described are the outcome of approximately twenty full scale tests and numerous smallscale trials. A number of the techniques have already been set out but are briefly recapitulated to show their sequence.

Surface grease was removed from the specimen and one side emeried. After measuring the thickness of the sheet a thin film of casting resin was put on as explained on p.68. It was necessary to put this film on before bonding the gripping edges, as the temperature used to cure the film was above the safe maximum temperature of 70°C. for an unclamped Redux bond. However, bonding could be carried out first if a film of Araldite 101 was to be put on, due to its lower curing
temperature.

The gripping pieces were shaped and bonded as explained on p. 57.

The Araldite casting was bevelled around its edge before glueing it to the specimen with Araldite 103 (see p. 68). This was to reduce the stress concentration at the casting edge. After curing, the central hole was drilled through the Araldite and filed to size.

When oblique incidence was to be used, accurate squares were marked out on the casting surface in order to key the pairs of photographs together.

The surface of the specimen, from tapered gripping piece right up to the casting edge was covered with black paint. The inside edge of the hole was also covered. These precautions were found to be essential when using liquid paraffin to mate the prism and Alaldite surfaces.

The effect of liquid paraffin on wetting an Araldite-metal junction under stress was found out when a change over was made from a prism mated with water to one with liquid paraffin. The Araldite was found to crack off when the aluminium was at only one third of its required final tension. Several experiments were necessary to isolate the liquid as the cause of this failure. Keeping the liquid away from the Araldite-metal junction was found to eliminate the trouble.

A shallow trough was formed on the surface of the specimen by
using adhesive tape around the sides and under the central hole.

The specimen was fixed into the clamping plates which were bolted tightly together. Current to the compression cylinder strain gauges was switched on and left for half an hour before setting the zero. The specimen was then put under slight tension and the liquid paraffin poured into the trough. A prism of either 6" or 8" base (depending on the specimen size) was immersed in the trough, taking care not to trap air bubbles.

The directions of stress around the hole under a single tension, did not appear to deviate by more than 20° - 30° from the direction of tension, so that no isoclinics obscured the field of view when the analyser and polariser were arranged (crossed) at 45° to the applied tension. This enabled quarter wave plates to be dispensed with.

The camera used was a quarter-plate fitted with a 7" 4.5 lens and had a rack extension and a moveable back. When using oblique incidence it was necessary to tilt the back to maintain a good focus over the whole area of observation. Ilford "Chromatic" plates were used, with the lens set at f/16, and given 30 seconds exposure. The 6" square filter, transmitting the mercury green line, was rested across the analyser, allowing only monochromatic light to the camera.

The 125 watt mercury lamp was switched on and focussed to a roughly parallel beam illuminating an area of about 7" square.
The tension was increased to a desired value and made uniform across the width of the specimen by adjusting the three levers individually. The strain gauge readings were noted, a photograph taken along one axis, and the whole optical system including the prism, was then rotated through $90^\circ$ and a second photograph taken.

The tension was then increased and a further pair of photographs taken. This was repeated until either the Araldite or Redux failed, or the plastic flow was considered sufficiently large.

In cases where the Araldite cracked, the thickness of casting plus glue could be checked accurately by removing sections and measuring with a micrometer. Alternatively, an optical method was used, focussing a travelling microscope at the surface of the metal and then of the Araldite. This apparent depth was multiplied by the refractive index of the Araldite.
PHOTOGRAPHIC MEASUREMENTS

Measurements were usually made from the 3/4 plate negatives, as prints on to photographic paper were found to distort on drying; the sides of a printed square differing by up to 3%.

A travelling microscope was used to view the negatives, which were held on a stand parallel to the microscope traverse and illuminated from behind. Fig. 31 illustrates the arrangement.

A tube was fitted over the microscope objective (usually of 4" focal length) so that only light transmitted through the plate was received. The high contrast obtained was necessary for reliable fringe measurements, especially close to the hole where a 2/3" objective lens was used to resolve the fringes. These readings were keyed to those made with the 4" lens by readings at the hole edge or at a marked reference point. Measurements could be read to 0.001 cms.

With photographs taken at oblique incidence, one set of
surface markings always appeared as a double set of lines on a photograph taken in one direction; the lines perpendicular to these appeared double on the second photograph (at right angles to the first). This was caused by the surface markings casting shadows on the metal surface.

![Diagram](image)

**FIG. 32**

From fig. 32 it can be seen that the ray that passes through the layer symmetrically below the surface line appears half way between the surface line and its shadow. Thus points half way between line and shadow are the points equivalent to the single line on the photograph taken at right angles.

The angle of obliquity of the light in the Araldite was calculated from measurements of the calibrating squares. Fig. 33 shows the path of two parallel rays emerging from the layer at a distance X apart.
After passing through the prism, the rays enter the air at a distance $X_0$ apart and it is this distance that is recorded by the camera.

A formula for the angle of obliquity is deduced below for rays leaving the prism at angles close to 45° from the vertical.

- $\Theta_n$ = angle of obliquity in Araldite.
- $\Theta_l$ = angle of prism.
- $\Theta_0$ = angle of air.
- $X$ = distance between 2 rays on leaving layer
- $X_0$ = distance between 2 rays in the air.
- $\mu_r$ = refractive index of prism.
- $\mu_A$ = refractive index of Araldite.

In fig. 33 $ab = \frac{X_0}{\cos \beta} = \frac{X \cos \Theta_l}{\cos \alpha} \therefore \cos \Theta_L = \frac{X_0}{X} \frac{\cos \alpha}{\cos \beta}$

But $\alpha$ and $\beta$ are both small, so by Snell's law $\frac{\beta}{\alpha} = \mu_L$.

$$\therefore \frac{\cos \alpha}{\cos \beta} = \frac{1 - \frac{1}{2} \alpha^4}{1 - \frac{1}{2} \beta^2} = \frac{1 - \frac{1}{2} \mu_r^2 \alpha^4}{1 - \frac{1}{2} \mu_A^2 \alpha^4}$$

$$= \left[1 + \frac{1}{2} (\mu_r^2 - 1) \alpha^2 \right]$$

ignoring terms in $\alpha^4$ and higher.

$$\therefore \cos \Theta_L = \frac{X_0}{X} \left[1 + \frac{1}{2} (\mu_r^2 - 1) \alpha^2 \right] \quad \text{(29)}$$
Since \( \alpha = 45 - \Theta_c \), \( \Theta_c \) is contained in both sides of the expression. Its value may be obtained by a tabular method working backwards from \( \Theta_c \). The table below was constructed for liquid paraffin in the prism, i.e. \( \mu_L = 1.48 \).

<table>
<thead>
<tr>
<th>( \Theta_c^\circ )</th>
<th>( \cos \Theta_c )</th>
<th>( 1 + \frac{\mu_L^2}{\mu_A^2} \alpha )</th>
<th>Corresponding ( \frac{X_o}{X} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>.7314</td>
<td>1.0007</td>
<td>.7309</td>
</tr>
<tr>
<td>42</td>
<td>.7431</td>
<td>1.0016</td>
<td>.7419</td>
</tr>
<tr>
<td>41</td>
<td>.7547</td>
<td>1.0029</td>
<td>.7525</td>
</tr>
<tr>
<td>40</td>
<td>.7660</td>
<td>1.0045</td>
<td>.7626</td>
</tr>
<tr>
<td>39</td>
<td>.7771</td>
<td>1.0065</td>
<td>.7721</td>
</tr>
<tr>
<td>38</td>
<td>.7880</td>
<td>1.0089</td>
<td>.7810</td>
</tr>
<tr>
<td>37</td>
<td>.7986</td>
<td>1.0116</td>
<td>.7895</td>
</tr>
</tbody>
</table>

Starting now with any measured value of \( \frac{X_o}{X} \), \( \Theta_c \) can be obtained by simple interpolation. This value can then be converted to \( \Theta_A \) (the angle of obliquity in the Araldite,) using:

\[
\frac{\sin \Theta_A}{\sin \Theta_c} = \frac{\mu_c}{\mu_A}, \quad \text{i.e.} \quad \Theta_A = \sin^{-1} \left[ \frac{\mu_c}{\mu_A} \sin \Theta_c \right]
\]

The plate holder of the camera was set at an angle to the lens, in order to get good focussing over the whole of the image. Allowance was made for this tilt (it effectively increased \( X \) slightly) and for the change in specimen dimensions on load, when
determining $\frac{X''}{X}$. The calibrating squares extended by approximately $\frac{1}{2}$% in the direction of tension and contracted by $1/6$% in the transverse direction.

The use of oblique incidence, and the finite angle of view of the camera caused different regions of the specimen to subtend different angles at varying distances from the lens. Perspective effects therefore occurred and it was necessary to allow for the change in magnification & in the angle of incidence along the specimen.

Measurement of the calibrating squares gave the change in magnification for each inch along the specimen, as $4.05\%$.

Let a distance of $x''$ be measured on the photograph from a reference point (where the magnification is $M$), corresponding to a distance $X''$ on the specimen. The relation between $x$ and $X$ is then required.

Magnification at a distance $X''$ from the reference point will be $M(1 + \frac{4.05}{100}X'')$.

The average change in magnification over this distance is $M(1 + \frac{4.05}{100}X)$. Therefore $X = \frac{x}{M(1 + \frac{2.25}{100}X)}$.

Substituting $\frac{X}{M}$ for $X$ in the small term, and ignoring terms beyond the first order of smallness, gives

$$x = \frac{X}{M} \left(1 - \frac{2.25}{100} \frac{X}{M}\right) . \quad (31)$$

The change in angle of incidence was worked out approximately as $5/6^\circ$ per inch. The calculation is given in the appendix p. 126.
A number of specimens were stretched and photographed at normal incidence while trying to find a means of separating the principal strains. Some of these produced interesting results.

Fig. 34 shows the photoelastic patterns obtained for a 3" wide specimen with a 3/4" hole subjected to a range of tension in one direction. Small plastic areas are present in (b) and large plastic areas in (c) and (d). The arrows show the direction of applied tension and some fringes are numbered. Details of the test are given in Table 1.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Details of Specimen</th>
<th>Applied tension</th>
<th>Gross Stress t.p.s.i.</th>
<th>Stress T allowing for layer. t.p.s.i.</th>
<th>T/2K</th>
</tr>
</thead>
<tbody>
<tr>
<td>34(a)</td>
<td>Specimen no.S10 .0835&quot; L71</td>
<td>2.53</td>
<td>10.07</td>
<td>9.75</td>
<td>.362</td>
</tr>
<tr>
<td>&quot; (b)</td>
<td>.0835&quot; wide</td>
<td>4.77</td>
<td>19.0</td>
<td>18.4</td>
<td>.684</td>
</tr>
<tr>
<td>&quot; (c)</td>
<td>.0835&quot; hole</td>
<td>6.14</td>
<td>24.4</td>
<td>23.6</td>
<td>.877</td>
</tr>
<tr>
<td>&quot; (d)</td>
<td>.0835&quot; Araldite layer.</td>
<td>6.71</td>
<td>26.7</td>
<td>25.85</td>
<td>.960</td>
</tr>
</tbody>
</table>

**TABLE 1.**

T = Stress in the alloy remote from the hole.
2k = Tensile stress at the yield point of the alloy.
FOUR STAGES OF TENSION (one direction) IN A 3" SPECIMEN WITH 1/4" HOLE.

Fringe number 5 shows the approx. position of the plastic-elastic boundary.
Under conditions of plane stress, the maximum shear stress is given by $\frac{1}{2}p$ when $q$ (the smaller principal stress) is of the same sign as $p$ (+ve). However, in regions where $q$ is negative or zero, the maximum shear stress is given by $\frac{1}{2}(p - q)$. If the metal is assumed to yield at a given maximum shear stress, then (for $q < 0$) a given fringe number (being proportional to $p - q$) will define the boundary. For the specimen shown, this fringe number is 5, or more accurately 4.85, calculated from the sensitivity and thickness of the layer and the yield point of the alloy.

The stresses in the purely elastic case are shown in fig. 47. It can be seen that in the region $\Theta = 45^\circ$, $q$ is slightly negative; in the region $\Theta = 0$, $q$ is either -ve or close to zero; in the region of $\Theta = 90^\circ$, $q$ tends to zero away from the hole. Machining channels in the Araldite layer of the above specimen (across the pattern of permanent set) also indicated that $q$ was negative (compressive) in the region $\Theta = 30^\circ$ to $60^\circ$. Thus, in fig. 34, fringe number 5 (or 4.85) defines, at least approximately, the plastic-elastic boundary in (c) and (d).

The growth of the plastic area along the transverse diameter is seen to be limited, plastic regions spreading out at approximately $35^\circ$ to this direction.
THREE STAGES OF PERMANENT SET (from three different specimens).

(c) is the residual pattern corresponding to (d) in Fig. 34.
Fig. 35

Each photograph is from a different specimen and shows a stage of permanent set, increasing from (a) to (c). (c) shows the pattern corresponding to fig. 34(d).

The fringe number surrounding each of the patterns of permanent set is very close to zero, indicating that the permanent strains of the plastic areas have not been greatly compressed by the surrounding elastic areas on releasing the load. These patterns thus give a good indication of the extent of the plastic region in each case (allowing for a slight "spread" of the patterns of about half a fringe.)

Table 2 gives details of the specimens.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Ratio of plate width to hole diameter</th>
<th>Max. Stress in alloy remote from hole before release</th>
<th>( \frac{T}{2k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>35 (a)</td>
<td>12:1</td>
<td>15.5 t.p.s.i.</td>
<td>.575</td>
</tr>
<tr>
<td>&quot; (b)</td>
<td>24:1</td>
<td>22.2 &quot;</td>
<td>.81</td>
</tr>
<tr>
<td>&quot; (c)</td>
<td>12:1</td>
<td>25.85 &quot;</td>
<td>.96</td>
</tr>
</tbody>
</table>

**TABLE 2.**

Fig. 36.

This shows one quadrant of a specimen stretched in two directions with different ratios of the two tensions \( T_1 \) and \( T_2 \). The small holes are drilled through the Araldite layer only, and were used (as explained in Section III p. 71) in an attempt to deduce the ratio of the principal strains at any point from the patterns formed. The
ONE QUADRANT OF A SPECIMEN UNDER TWO TENSIONS AT RIGHT ANGLES.

The pattern of the central hole together with those of small holes (through the Araldite layer only) can be seen.

(a) $T_1:T_2 = 1:1$  (b) $T_1:T_2 = 1.6:1$  (c) $T_1:T_2 = 3.5:1$
way in which these small (and "incomplete") holes have patterns imitating those of the central hole can be seen. Their change of pattern with the ratio of $T_1 : T_2$ and their orientation in the direction of stress is distinct.

Table 3 gives details of the specimen.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Approx. Enlargement</th>
<th>$\frac{T_1}{T_2}$</th>
<th>Specimen Dimensions &quot;Small Hole&quot; dimensions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>36 (a)</td>
<td>x 2</td>
<td>1:1</td>
<td>4&quot; x 4&quot; central area .084&quot; thick L71</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>.047&quot; diam.</td>
</tr>
<tr>
<td>&quot; (b)</td>
<td>&quot;</td>
<td>1.6:1</td>
<td>1/2&quot; diam. central hole .05&quot; deep.</td>
</tr>
<tr>
<td>&quot; (c)</td>
<td>&quot;</td>
<td>3.5:1</td>
<td>.05&quot; thick layer of Araldite.</td>
</tr>
</tbody>
</table>

TABLE 3:

If the Araldite layer were extremely thin and followed the strains in the metal completely, the small holes drilled through the layer alone would produce no observable local strain patterns. The small hole patterns therefore rely on the imperfect manner in which the Araldite follows the strains in the metal, and give some idea of the (maximum) boundary error that could occur, and the distance from a boundary at which the error is negligible.

Fig. 37

This shows the results obtained from a specimen stretched with two equal tensions at right angles. Curves of fringe number against
Fig. 37

Equal Tensions
(At Normal Incidence)

Prue Número $(p-y)$

Distance from hole centre in radii
distance from the hole centre are plotted for three difference stress levels, the highest two causing plastic flow around the hole. Details of the test are set out in table 4.

The central region of the specimen was thinned down (see fig. 43) in order to keep the stress there higher than in the arms. The stress (in the centre) due to thinning was determined from the known sensitivity and thickness of the Araldite layer, and the number of fringes measured in a purely elastic extension.

The photoelastic patterns obtained were concentric circles, showing that the stresses and strains had radial symmetry and that the plastic-elastic boundary was therefore a concentric circle.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Photo. No.</th>
<th>Applied Load Tons.</th>
<th>Central Stress t.p.s.i.</th>
<th>$T_{2K}$ t.p.s.i.</th>
<th>Details of Specimen</th>
</tr>
</thead>
<tbody>
<tr>
<td>37(a)</td>
<td>91</td>
<td>2.54</td>
<td>9.46</td>
<td>.35</td>
<td>Alloy L71</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4&quot; wide arms</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0844&quot; thick arms</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0405&quot; central thick-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>ness.</td>
</tr>
<tr>
<td>37(b)</td>
<td>92</td>
<td>3.97</td>
<td>14.8</td>
<td>.55</td>
<td>.5&quot; hole diameter.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0548&quot; Araldite</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>thickness.</td>
</tr>
<tr>
<td>37(c)</td>
<td>93</td>
<td>5.415</td>
<td>20.2</td>
<td>.74</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.**
RESULTS AT OBLIQUE INCIDENCE.

The method used to determine the strains from photographic measurements is illustrated below. The calculations for one pair of photographs at one level of stress in a specimen are followed through the various stages of working which has been divided into seven tables. (Fig. 39 shows the two photographs used.)

The first table (no. 5) determines the stress in the alloy remote from the central hole, allowing for the small proportion of the total tension taken by the Araldite.

<table>
<thead>
<tr>
<th>Photo No.</th>
<th>Details of Specimen</th>
<th>Strain Gauge Readings</th>
<th>Total Tension in Tons</th>
<th>Tons by Cross Sect. of Alloy</th>
<th>T.P.S.I. Stress in Alloy Remote from hole</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. S21</td>
<td>L71 alloy .0835&quot; x 6.03&quot; cross-section.</td>
<td>G₁, G₂, G₃</td>
<td>.263 .252 .248</td>
<td>12.63</td>
<td>25.07</td>
</tr>
<tr>
<td>197)</td>
<td>Single tension 2¼&quot; diam. central hole Araldite layer No. C41 .0567&quot; thick.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
FIG. 39

PHOTOGRAPHS AT OBLIQUE INCIDENCE FOR A 6" WIDE SPECIMEN WITH A $\frac{3}{4}"$ HOLE.

(a) Rotation about p-axis  (b) Rotation about q-axis.
The second table (No. 6) is concerned with measurements made on these reference lines and the determination of $X_e$ (see Photographic Measurements P. 85). Allowance is made for the change in dimensions of the specimen on load and for the tilt of the plate holder (which was necessary for accurate focusing).

$$X_e = \frac{\text{dist. on photo in 1 dim.}}{\text{dist. on loaded spec.}} \times \frac{\text{dist. on photo in 2 dim.}}{\text{dist. on loaded spec.}} \times \text{Plate tilt factor}$$

<table>
<thead>
<tr>
<th>Posn.</th>
<th>Mic. Rdg. cms.</th>
<th>Dirn.</th>
<th>Photo dist. for nominal Specimen dist. of 2&quot; 4&quot;</th>
<th>Actual dist. on loaded spec. factor</th>
<th>Allowance for change plate tilt</th>
<th>$\frac{X_e}{X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PHOTO NO. 197</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1&quot;→③</td>
<td>9.676</td>
<td>up</td>
<td>1.737</td>
<td>5.06</td>
<td>.998</td>
<td>nil</td>
</tr>
<tr>
<td>1&quot;→①</td>
<td>11.413</td>
<td>to appl.</td>
<td>3.480</td>
<td>10.16</td>
<td>.998</td>
<td>nil</td>
</tr>
<tr>
<td>2&quot;→⑨</td>
<td>8.504</td>
<td>to</td>
<td>2.685</td>
<td>10.16</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>2&quot;→①</td>
<td>12.284</td>
<td>to</td>
<td>2.685</td>
<td>10.16</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>1&quot;→②</td>
<td>11.248</td>
<td>Parallel</td>
<td>1.341</td>
<td>5.07</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>1&quot;→①</td>
<td>9.907</td>
<td>Parallel to</td>
<td>1.341</td>
<td>5.07</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>2&quot;→②</td>
<td>11.877</td>
<td>applied tension</td>
<td>2.685</td>
<td>10.16</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>2&quot;→①</td>
<td>9.192</td>
<td>applied tension</td>
<td>2.685</td>
<td>10.16</td>
<td>1.005</td>
<td>.9833</td>
</tr>
<tr>
<td>PHOTO NO. 198</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1&quot;→③</td>
<td>12.142</td>
<td>up</td>
<td>1.338</td>
<td>5.06</td>
<td>.998</td>
<td>.9833</td>
</tr>
<tr>
<td>1&quot;→①</td>
<td>10.804</td>
<td>up</td>
<td>2.687</td>
<td>10.16</td>
<td>.998</td>
<td>.9833</td>
</tr>
<tr>
<td>2&quot;→⑨</td>
<td>12.778</td>
<td>applied tension</td>
<td>2.687</td>
<td>10.16</td>
<td>.998</td>
<td>.9833</td>
</tr>
<tr>
<td>2&quot;→①</td>
<td>10.091</td>
<td>applied tension</td>
<td>2.687</td>
<td>10.16</td>
<td>.998</td>
<td>.9833</td>
</tr>
<tr>
<td>1&quot;→②</td>
<td>8.252</td>
<td>Parallel</td>
<td>1.773</td>
<td>5.07</td>
<td>1.005</td>
<td>nil</td>
</tr>
<tr>
<td>1&quot;→①</td>
<td>10.025</td>
<td>Parallel to</td>
<td>1.773</td>
<td>5.07</td>
<td>1.005</td>
<td>nil</td>
</tr>
<tr>
<td>2&quot;→②</td>
<td>7.371</td>
<td>applied tension</td>
<td>3.538</td>
<td>10.16</td>
<td>1.005</td>
<td>nil</td>
</tr>
<tr>
<td>2&quot;→①</td>
<td>10.909</td>
<td>applied tension</td>
<td>3.538</td>
<td>10.16</td>
<td>1.005</td>
<td>nil</td>
</tr>
</tbody>
</table>

**TABLE 6**
From the value of \( \frac{X_0}{X} \) obtained in the above table, the angle of obliquity could be determined from the table on p.86, and from this, the angle of the rays in the Araldite layer. These values, together with the magnification at the hole centre and at a chosen reference point are set out below (Table 7).

<table>
<thead>
<tr>
<th>Photo No.</th>
<th>( \frac{X_0}{X} )</th>
<th>Angle of Central Ray in Prism. (Calc. from Table, p.86)</th>
<th>Angle of Central Ray in Araldite Layer.</th>
<th>Magnification at Hole Centre. Perpendicular to line of tension.</th>
<th>Mag( n ) at Reference Point.</th>
</tr>
</thead>
<tbody>
<tr>
<td>197</td>
<td>.7529</td>
<td>40° 57½'</td>
<td>36° 20'</td>
<td>.3435</td>
<td>.3435</td>
</tr>
<tr>
<td>198</td>
<td>.7502</td>
<td>41° 13'</td>
<td>36° 35'</td>
<td>.2650</td>
<td>.2547</td>
</tr>
</tbody>
</table>

**TABLE 7.**

Measurements of fringe number versus distance along the transverse axis were made using the drawn line at 1" → 3 as a reference point. The actual number to be allotted to any fringe was usually found by counting on the photograph from the minimum fringe number present. This was on the axis parallel to the applied tension and was never found to be more than \( 1\frac{1}{2} \) fringes above zero.

The next table (no.8) converts the distances measured from the reference point on photograph 197 into actual distances on the specimen. Table 9 converts distances from photograph 198 into specimen distances allowing for the changing magnification that occurs.
<table>
<thead>
<tr>
<th>Photo No.</th>
<th>Fringe No. or Position</th>
<th>Readings on Travelling Microscope</th>
<th>Distance from Ref. Point on Specimen in inches $X = \frac{x}{\times}$ (Magn. at Ref. Pt. x 2.54)</th>
</tr>
</thead>
<tbody>
<tr>
<td>197</td>
<td>1&quot; to Ref. Pt.</td>
<td>11.331</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3/8</td>
<td>10.511</td>
<td>-820</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>11.117</td>
<td>-214</td>
</tr>
<tr>
<td></td>
<td>2 1/2</td>
<td>11.371</td>
<td>+0.04</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>11.536</td>
<td>0.205</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>11.615</td>
<td>-2845</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>11.668</td>
<td>0.337</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>11.699</td>
<td>0.368</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>11.722</td>
<td>0.391</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>11.742</td>
<td>0.411</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>11.758</td>
<td>0.427</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>11.775</td>
<td>0.444</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>11.789</td>
<td>0.458</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>11.802</td>
<td>0.471</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>11.813</td>
<td>0.482</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>11.822</td>
<td>0.491</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>11.832</td>
<td>0.501</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>11.840</td>
<td>0.509</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>11.849</td>
<td>0.518</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>11.855</td>
<td>0.524</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>11.863</td>
<td>0.532</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>11.871</td>
<td>0.540</td>
</tr>
<tr>
<td></td>
<td>20 1/2 Hole edge</td>
<td>11.877</td>
<td>0.546</td>
</tr>
<tr>
<td></td>
<td>For Hole edge</td>
<td>12.538</td>
<td>1.207</td>
</tr>
</tbody>
</table>

**TABLE 8**
<table>
<thead>
<tr>
<th>Photo No.</th>
<th>Fringe No. or Position</th>
<th>Rg. on Trav. Mic.</th>
<th>xoms. Dist. on Photo from ref point.</th>
<th>Allowance for Mag. change x = 1 - 0.0313 *</th>
<th>Magn. change x = \frac{xf}{0.0313} x Magn. x 2.54</th>
</tr>
</thead>
<tbody>
<tr>
<td>198</td>
<td>Zero)</td>
<td>12.047</td>
<td>0</td>
<td>-0.048 1.0048 -0.237</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1² -&gt; 3²</td>
<td>12.20</td>
<td>-.153</td>
<td>-0.0201 1.0201 -0.014</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.7</td>
<td>12.69</td>
<td>-.643</td>
<td>-0.0003 1.0003 -0.017</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>12.058</td>
<td>-.011</td>
<td>-0.0021 0.9979 + 0.103</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6³/₂</td>
<td>11.980</td>
<td>+.067</td>
<td>+0.023 0.9968 + 1.555</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>11.946</td>
<td>+.101</td>
<td>+0.046 0.9954 + 2.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>11.901</td>
<td>+.146</td>
<td>+0.023 0.9931 + 3.40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>11.8705</td>
<td>+.1765</td>
<td>+0.046 0.9939 + 3.40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>11.842</td>
<td>+.205</td>
<td>+0.064 0.9936 + 3.15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>11.826</td>
<td>+.221</td>
<td>+0.064 0.9936 + 3.15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>11.807</td>
<td>+.240</td>
<td>+0.064 0.9936 + 3.15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>11.792</td>
<td>+.2545</td>
<td>+0.080 0.9920 + 3.91</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>11.780</td>
<td>+.267</td>
<td>+0.080 0.9920 + 3.91</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>11.768</td>
<td>+.279</td>
<td>+0.087 0.9913 + 4.28</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>11.7575</td>
<td>+.2895</td>
<td>+0.091 0.9909 + 4.43</td>
<td></td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>11.7445</td>
<td>+.3025</td>
<td>+0.095 0.9905 + 4.63</td>
<td></td>
</tr>
<tr>
<td>Hole edge</td>
<td></td>
<td>11.668</td>
<td>+.379</td>
<td>+0.0118 0.9882 + 5.79</td>
<td></td>
</tr>
</tbody>
</table>

"objective now used to resolve the fringes.

Readings are keyed to hole edge:

<table>
<thead>
<tr>
<th>Hole edge</th>
<th>3.1695</th>
<th>.379</th>
<th>.0118</th>
<th>0.9882</th>
<th>5.79</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>3.237</td>
<td>.3115</td>
<td>.0097</td>
<td>0.993</td>
<td>.477</td>
</tr>
<tr>
<td>19</td>
<td>3.228</td>
<td>.3205</td>
<td>.0100</td>
<td>0.990</td>
<td>.490</td>
</tr>
<tr>
<td>20</td>
<td>3.219</td>
<td>.3295</td>
<td>.0103</td>
<td>0.987</td>
<td>.504</td>
</tr>
<tr>
<td>21</td>
<td>3.2115</td>
<td>.337</td>
<td>.0105</td>
<td>0.985</td>
<td>.515</td>
</tr>
<tr>
<td>22</td>
<td>3.204</td>
<td>.3445</td>
<td>.0108</td>
<td>0.982</td>
<td>.527</td>
</tr>
<tr>
<td>23</td>
<td>3.197</td>
<td>.3515</td>
<td>.0110</td>
<td>0.980</td>
<td>.537</td>
</tr>
<tr>
<td>24</td>
<td>3.1905</td>
<td>.358</td>
<td>.0112</td>
<td>0.988</td>
<td>.546</td>
</tr>
</tbody>
</table>

**TABLE 2**
A graph of fringe number versus distance from the hole centre could now be plotted for the two sets of fringes \( (n_1, n_2) \) and corresponding values read off at selected distances. Fig. 40 shows \( n_1 \) and \( n_2 \) obtained from the photographs.

The formulae derived on p76 were then used to determine \( n_1 \) and \( n_2 \):

\[
\begin{align*}
n_1 &= \cos \Theta_1 (n - n_2 \cos \Theta_1 \cos \Theta_2) \\
n_2 &= \frac{\cos \Theta_2 (n - n_1 \cos \Theta_1 \cos \Theta_2)}{1 - \cos^2 \Theta_1 \cos^2 \Theta_2}
\end{align*}
\]

The values were computed by means of a 15 column table shown here in two halves, allowing for the change in angle with distance for the "\( n_1 \)" fringes.

<table>
<thead>
<tr>
<th>DISTANCE</th>
<th>Radii Inches from hole ref. centre</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( \Theta_1 )</th>
<th>( \Theta_2 )</th>
<th>( \cos \Theta_1 )</th>
<th>( \cos \Theta_2 )</th>
<th>( \cos \Theta_1 \cos \Theta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.87</td>
<td>.45</td>
<td>5.60</td>
<td>3.38</td>
<td>37°</td>
<td>47°</td>
<td>.7903</td>
<td>.8056</td>
<td>.6366</td>
</tr>
<tr>
<td>3.07</td>
<td>.15</td>
<td>5.77</td>
<td>3.01</td>
<td>37°</td>
<td>32\frac{1}{2}°</td>
<td>.7930</td>
<td>&quot;</td>
<td>.6389</td>
</tr>
<tr>
<td>2.537</td>
<td>.05</td>
<td>6.20</td>
<td>2.47</td>
<td>37°</td>
<td>22\frac{1}{2}°</td>
<td>.7947</td>
<td>&quot;</td>
<td>.6401</td>
</tr>
<tr>
<td>2.003</td>
<td>.25</td>
<td>8.50</td>
<td>3.13</td>
<td>37°</td>
<td>12\frac{1}{2}°</td>
<td>.7964</td>
<td>&quot;</td>
<td>.6415</td>
</tr>
<tr>
<td>1.602</td>
<td>.4</td>
<td>13.41</td>
<td>5.37</td>
<td>37°</td>
<td>5°</td>
<td>.7978</td>
<td>&quot;</td>
<td>.6427</td>
</tr>
<tr>
<td>1.336</td>
<td>.5</td>
<td>19.72</td>
<td>9.48</td>
<td>37°</td>
<td>0°</td>
<td>.7986</td>
<td>&quot;</td>
<td>.6433</td>
</tr>
<tr>
<td>1.203</td>
<td>.55</td>
<td>24.31</td>
<td>12.94</td>
<td>36°</td>
<td>57\frac{1}{2}°</td>
<td>.7990</td>
<td>&quot;</td>
<td>.6436</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

| TABLE 10 |
|----------|----------------------------------|
| 1 - (\cos^2 \Theta_1 \cos^2 \Theta_2) | \( n_1 \cos \Theta_1 \cos \Theta_2 \) | \( n_2 \cos \Theta_1 \cos \Theta_2 \) | \( n_1 \cos \Theta_1 \cos \Theta_2 \) | \( n_2 \cos \Theta_1 \cos \Theta_2 \) | \( n_1 \) | \( n_2 \) | \( (6) \times (13) \) | \( (7) \times (12) \) |
| .5947    | 3.565                            | 2.15    | .185    | 3.45    | 4.59    | .25      |
| .5918    | 3.685                            | 1.92    | .675    | 3.85    | 5.155   | .92      |
| .5901    | 3.97                             | 1.58    | 1.50    | 4.62    | 6.22    | 2.05     |
| .5885    | 5.45                             | 2.01    | 2.32    | 6.49    | 8.79    | 3.17     |
| .5870    | 8.61                             | 3.45    | 3.245   | 9.96    | 13.52   | 4.45     |
| .5862    | 12.69                            | 6.10    | 3.21    | 13.62   | 18.53   | 4.41     |
| .5858    | 15.64                            | 8.33    | 2.70    | 15.98   | 21.76   | 3.71     |
|         |                                   |         |         |         | 25.41   | 0        |
Fig. 40

SINGLE TENSION

Observed Fringe Numbers along the Transverse Diameter at Oblique Incidence

(a) Rotation about the axis parallel to applied tension (giving $n_v$)
(b) Rotation about perpendicular axis (giving $n_h$)

DST in Radii from Hole Centre
The values of \( n_p \) and \( n_q \) are shown plotted in fig. 4. They were converted into strains using equations (32) and (33) below, derived from equations (17), (19), (20):

\[
\begin{align*}
E & = P - \delta_q \quad (32) \\
\delta & = \frac{1}{E}(n_p - \delta n_q) \\
\delta & = \frac{1}{E}(n_q - \delta n_p) \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>DISTANCE in radii</th>
<th>( n_p )</th>
<th>( \gamma n_p ) (( \gamma = .39 ))</th>
<th>( n_q )</th>
<th>( \gamma n_q )</th>
<th>( \frac{1}{E} ) Strain</th>
<th>( \delta ) Strain</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.87</td>
<td>4.59</td>
<td>1.790</td>
<td>.25</td>
<td>.098 1.44</td>
<td>.514</td>
<td>-.176</td>
<td></td>
</tr>
<tr>
<td>3.07</td>
<td>5.15</td>
<td>2.010</td>
<td>.92</td>
<td>.359 &quot;</td>
<td>.549</td>
<td>-.125</td>
<td></td>
</tr>
<tr>
<td>2.537</td>
<td>6.22</td>
<td>2.426</td>
<td>2.05</td>
<td>.800 &quot;</td>
<td>.620</td>
<td>-.043</td>
<td></td>
</tr>
<tr>
<td>2.003</td>
<td>8.79</td>
<td>3.429</td>
<td>3.17</td>
<td>1.236 &quot;</td>
<td>.865</td>
<td>-.030</td>
<td></td>
</tr>
<tr>
<td>1.602</td>
<td>13.52</td>
<td>5.273</td>
<td>4.45</td>
<td>1.735 &quot;</td>
<td>1.348</td>
<td>-.094</td>
<td></td>
</tr>
<tr>
<td>1.336</td>
<td>18.53</td>
<td>7.228</td>
<td>4.41</td>
<td>1.720 1.170</td>
<td>1.965</td>
<td>-.330</td>
<td></td>
</tr>
<tr>
<td>1.203</td>
<td>21.76</td>
<td>8.487</td>
<td>3.71</td>
<td>1.446 1.184</td>
<td>2.406</td>
<td>-.566</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>25.41</td>
<td>9.910</td>
<td>0</td>
<td>1.22</td>
<td>3.10</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 11.**

The principal strains \( \varepsilon_p \) and \( \varepsilon_q \) are shown in fig. 42, d, d'.

**Results for All Specimens under a Single Tension.**

All the strain measurements at oblique incidence made at different stress levels and for different specimens were worked out in the way illustrated in the previous 7 tables. The resulting strains are shown in graphical form and the table 12 gives the main features of the specimens and stresses corresponding to each graph.
**Fig. 41**

**Single Tension**

Values of $N_a$, $N_o$ along the Transverse Diameter

(See Table 12)
Fig. 42

Single Tension

Principal Strains along the Transverse Diameter

(See Table 12)
In fig. 41 are plotted the values of \( n_p \) and \( n_q \) measured along the transverse diameter from the tests in Table 12, but scaled for a layer thickness of \( .057" \). The fall of sensitivity at the hole boundary can be seen by the change of slope in these regions. The dotted lines are continuations of the highest slopes obtained before the fall off, and probably give a more correct value in this region. (d), (e) and (f) are from different tests well into the plastic region with closely equal loads. The nearness of these three curves gives a good idea of the accuracy of the experiments.

In fig. 42 are plotted the strains corresponding to (a), (b), (c) and (d) of table 12. The dotted lines are again continuations of the highest slopes obtained. The fall in sensitivity at the boundary is seen to be slight for curve a, and to increase progressively in b, c, d, as the amount of plastic flow increases.
Tension in Two Directions.

The specimen used was of .128" L71 alloy, thinned down in the central region to .053" (see fig. 43). The arms were 6" wide and the central hole 0.4" diameter.

The two tensions were kept approximately equal as the load was applied and photographs were taken along both axes at oblique incidence. It was found necessary to put in the quarter wave plates to remove small isoclinics that occurred.

The stress in the central region was determined from the values of $n_p$ and $n_q$ obtained (remote from the hole), together with the known sensitivity and thickness of the Araldite layer. Details of the test are given in table 13.
Fig. 44

Equal Tensions
(At Oblique Incidence)

Dist. from Hole Centre in Radii

\( \eta_p \), \( \eta_q \) - Fringe Numbers
Fig. 45
Principal Strains $\varepsilon_p, \varepsilon_q$
Under Equal Tensions

Dist. in Radii from Hole Centre
### TABLE 13.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Photo No.</th>
<th>Applied Load $T_1$</th>
<th>Layer Thickness</th>
<th>Average Central Stress t.p.s.i.</th>
<th>$\frac{T}{2k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>44) 45)</td>
<td>185) 186)</td>
<td>10.85 10.2</td>
<td>.0499&quot;</td>
<td>20.2</td>
<td>.702</td>
</tr>
<tr>
<td></td>
<td>177) 178)</td>
<td>8.9 8.5</td>
<td>&quot;</td>
<td>16.7</td>
<td>.58</td>
</tr>
</tbody>
</table>

Fig. 44 shows the values of $n_p$ and $n_q$ obtained at maximum load and fig. 45 shows the corresponding strains. If the yield point is measured as explained on p. 53 and the metal is assumed to fail at a given maximum shear stress, the number of fringes at which this occurs (obtained from the layer sensitivity and thickness) is $n_p = 4.7$. The plastic-elastic boundary thus occurs at $\rho = 1.67a$ for $\frac{T}{2k} = .702$. Photographs 177, 178, give a value of $\rho = 1.15a$ at $\frac{T}{2k} = .58$. 
FIG. 46

TENSILE TEST OF L71

(a) \(\frac{1}{2}\)8" gauge cut parallel to dim. of final rolling.
(b) \(\frac{1}{2}\)8" gauge cut 45° to dim. of final rolling.
(c) 0.055" " "
(d) 0.085" gauge, cut in either dim.
**Fig. 47**

Elastic Solution for a Single Tension

**Fig. 48**

Elastic Solution for Equal Tensions
CALIBRATION

The Photoelastic Layer.

In order to determine the sensitivity of each layer bonded to the alloy, a tensile specimen was cut from the same Araldite casting. This was usually between 5" and 8" long with a parallel central portion of between $\frac{3}{4}''$ and 1". Specimens were mounted in a Hounsfield Tensometer with a miniature polariscope to measure fringe number and a Lindley extensometer to measure strain.

A typical set of results is shown in Table 14 and the corresponding graph given in fig. 49.

<table>
<thead>
<tr>
<th>Details of Test</th>
<th>Fringe No.</th>
<th>Extensometer Reading in .0001&quot;</th>
<th>Reading from zero (+110)</th>
<th>% Strain = $\frac{1}{2}$x rdg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile specimen from Araldite</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>from Araldite 4.73*5 138.5</td>
<td>4</td>
<td>73.5</td>
<td>183.5</td>
<td>0.459</td>
</tr>
<tr>
<td>Casting C.34 166.1 276.0</td>
<td>6</td>
<td>166.0</td>
<td>276.0</td>
<td>0.690</td>
</tr>
<tr>
<td>.510&quot; wide 384.5 5.921</td>
<td>8</td>
<td>258.5</td>
<td>386.5</td>
<td>0.921</td>
</tr>
<tr>
<td>.055&quot; thick 466.5 1.166</td>
<td>10</td>
<td>356.5</td>
<td>466.5</td>
<td>1.166</td>
</tr>
<tr>
<td>Viewed in mercury 967.5 2.160</td>
<td>14</td>
<td>545</td>
<td>655</td>
<td>1.638</td>
</tr>
<tr>
<td>green polarised 756.5 1.891</td>
<td>16</td>
<td>646.5</td>
<td>756.5</td>
<td>1.891</td>
</tr>
<tr>
<td>light by reflection at normal</td>
<td>18</td>
<td>754</td>
<td>864</td>
<td>2.160</td>
</tr>
<tr>
<td>incidence 967.5 2.419</td>
<td>20</td>
<td>857.5</td>
<td>967.5</td>
<td>2.419</td>
</tr>
<tr>
<td>Lindley gauge 1073 2.693</td>
<td>22</td>
<td>967</td>
<td>1077</td>
<td>2.693</td>
</tr>
<tr>
<td>2&quot; length 1192 2.958</td>
<td>24</td>
<td>1073</td>
<td>1183</td>
<td>2.958</td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>1192</td>
<td>1302</td>
<td>3.255</td>
</tr>
</tbody>
</table>
Fig. 49

Fringe Number - Strain Characteristic for Araldite

Tensile Specimen 0.05 in. thick, viewed at normal incidence by reflection. (See Table 14)
All the castings had the same nominal ratio of hardener to casting resin, but small variations occurred in fringe sensitivity. Table 15 gives the constants for the various castings used in the results.

<table>
<thead>
<tr>
<th>Specimen No.</th>
<th>Casting No.</th>
<th>Tensile Specimen Dimensions (inches)</th>
<th>No. of Fringes for 1% strain</th>
<th>No. of Fringes for 1&quot; thickness &amp; 1% strain</th>
<th>Young's Modulus E lbs. ins. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S12</td>
<td>C25</td>
<td>-</td>
<td>-</td>
<td>77</td>
<td>-</td>
</tr>
<tr>
<td>S13</td>
<td>C28</td>
<td>.715</td>
<td>.0982</td>
<td>7.64</td>
<td>77.8</td>
</tr>
<tr>
<td>S15</td>
<td>C31</td>
<td>.894</td>
<td>.0510</td>
<td>3.953</td>
<td>77.5</td>
</tr>
<tr>
<td>S19</td>
<td>C40</td>
<td>1.035</td>
<td>.0511</td>
<td>3.935</td>
<td>77.0</td>
</tr>
<tr>
<td>S20</td>
<td>C38</td>
<td>-</td>
<td>.0689</td>
<td>5.253</td>
<td>76.3</td>
</tr>
<tr>
<td>S21</td>
<td>C41</td>
<td>.735</td>
<td>.0530</td>
<td>4.07</td>
<td>77.0</td>
</tr>
</tbody>
</table>

**TABLE 15**

Refractive Index of Araldite.

This was determined by using pieces of Araldite approximately 3/8" thick in conjunction with a short focus travelling microscope. Focussing on the upper surface of the Araldite and then on the lower surface gave the optical thickness. The actual thickness was measured by a micrometer and the ratio of actual thickness to optical thickness gave the refractive index. The average value from a number of measurements was 1.637 (for a casting resin to hardener ratio of 3:1).
The Strain Gauge Cylinders

The cylinders and the associated strain gauge bridge have been described, in Section III, p. 51. A calibration was carried out shortly after making the cylinders and repeated on two further occasions, at 12 monthly intervals. Only small changes in calibration occurred, but it was found necessary to load the cylinders centrally in order to obtain consistent results. Compression up to 7 tons was applied to each cylinder in turn, with strain readings taken at 1 ton intervals. The tensile test machine was then checked with a proving ring (calibrated by N.P.L.)

Fig. 50 shows the results obtained. The corresponding measurements are given below, all gauge readings being converted to equivalent readings on the .05% scale of the strain gauge bridge.

<table>
<thead>
<tr>
<th>Details of Test</th>
<th>Tons Compression</th>
<th>Strain Readings on Bridge</th>
<th>Dial. Cyl.1</th>
<th>Cyl.2</th>
<th>Cyl.3</th>
<th>Cyl.4</th>
<th>Cyl.5</th>
<th>Cyl.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denison No. 23260</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10 ton range</td>
<td>1</td>
<td>.114</td>
<td>.125</td>
<td>.109</td>
<td>.137</td>
<td>.141</td>
<td>.110</td>
<td></td>
</tr>
<tr>
<td>Gauge Factor</td>
<td>2</td>
<td>.230</td>
<td>.248</td>
<td>.223</td>
<td>.267</td>
<td>.273</td>
<td>.230</td>
<td></td>
</tr>
<tr>
<td>on Bridge</td>
<td>3</td>
<td>.351</td>
<td>.367</td>
<td>.342</td>
<td>.398</td>
<td>.406</td>
<td>.352</td>
<td></td>
</tr>
<tr>
<td>at 1.98</td>
<td>4</td>
<td>.474</td>
<td>.490</td>
<td>.462</td>
<td>.529</td>
<td>.538</td>
<td>.474</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>.597</td>
<td>.612</td>
<td>.584</td>
<td>.660</td>
<td>.670</td>
<td>.596</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>.720</td>
<td>.733</td>
<td>.704</td>
<td>.789</td>
<td>.801</td>
<td>.718</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>.839</td>
<td>.853</td>
<td>.824</td>
<td>.920</td>
<td>.936</td>
<td>.840</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 16.
Fig. 50

Calibration of the Compression Cylinders

(See Table 16)
ESTIMATION OF ACCURACY

The Alloy

The yield point was found to vary with the different gauge sheets, but appeared consistent to ± 1% over each sheet for specimens cut in the same direction. (The extreme edges of the sheet were not used.) Differences occurred with the .055" and .128" alloy for specimens cut parallel to and at right angles to the final direction of rolling, but the .085" alloy was found to give closely similar stress-strain curves in the two directions.

The .055" alloy was used only for the case of a single tension, where stresses in regions of maximum stress were mainly in the direction parallel to the applied tension (i.e., the cross-stresses were small). The difference in the yield point in the two directions was thus of no great account. The .128" alloy was used for the case of equal tensions, but the difference in yield point for the two directions amounted to only 4%.

Some of the alloy tested was heated for 20 hours at 120°C while forming thin films of casting resin on the surface. The small change in yield point was allowed for in each case by testing similarly heat-treated tensile specimens.

The average thickness of each specimen was measured by a micrometer to within ± 0.2% and variations were less than ± 0.3% over a 3" length. The width of the sheets was held constant to ± 0.3% and
the average width measured to within ± 0.3%.

**Load Measurement.**

Three compression cylinders, with approximately equal loads, were used for each direction of tension. The strain gauges on the cylinders used in the single tension tests were found to give measurements repeating to within ± 0.3% on recalibration. Those on the other three cylinders repeated to within ± 2%.

Unevenly distributed loads were found to cause errors when calibrating. However, the error in the tests from this cause, averaged over three cylinders, is unlikely to be more than ± 1%.

The strain gauge bridge was found to give consistent readings on all ranges (within ± 0.2% of full scale reading). The load measurement for each single tension test is therefore expected to be within ± 1½% of the true value. Measurement of the perpendicular tension is estimated to be within ± 3%.

**The Layer.**

Fringe-strain sensitivity was found to repeat to within ± 1½% for all the castings tested. The majority of castings used in the tests had their own calibration so that the error in sensitivity is unlikely to be more than ± 1% from the true value.

Variations in the glue and film thicknesses made it necessary
to measure the total Araldite thickness in regions where strain measurements were made. Usually, a travelling microscope was used to measure the apparent depth which was multiplied by the refractive index of the Araldite (known to within ± 0.2%). The error in total thickness is estimated to be within ± 1%.

Initial stresses after bonding were rarely more than 0.2 fringes and appeared close to zero over the majority of each casting.

**Fringe Measurement.**

Nearly all measurements have been made at fringe maxima or minima; graphs were plotted to find fringe value at intermediate points. Measurements to within 0.1 fringes can be made by this method even where the fringes are packed closely, unless too faint to be resolved by the travelling microscope. Fringe numbers rose to beyond 20 in highly stressed regions under a single tension and up to 8 for the tests with equal tensions.

**Oblique Incidence.**

Factors that enter into the calculation of the individual principal strains at any point are:-

1. The refractive index of the Araldite, which is known to ± 0.2%
2. The refractive index of prism material. The prism usually
has had perspex sides containing liquid paraffin, whose refractive index was obtained from tables and checked experimentally to within ±0.4%.

3. The angle of the prism, which has been kept at 45° within ±1°.

4. The reference lines on specimen and photographic negative which have been measured to within ±0.2%.

5. The angle of obliquity of the light in the layer, which has been determined from the above four factors. Calculation of the strains depends to a marked extent on the two angles of the two perpendicular photographs; both angles being ±1° in error can cause a 3½% error in strain values.

6. Location of the fringes. By means of a travelling microscope and the use of the reference lines, it has been possible to locate fringes near the hole to within .01" on the specimen.

At oblique incidence, the rays enter and leave by different paths, giving an average over the total path length. For an angle of refraction of 45°, the effective gauge length is $\sqrt{2} \times$ thickness of the layer, i.e. about .08" for most of the specimens used.

In regions where plastic flow occurs in the metal, a vertical stress may exist in the layer. This has the effect of reducing the values of $p$ and $q$ obtained by an amount equal to the average vertical stress through the thickness of the layer (30).
General.

The results obtained appear to have an individual accuracy of within ±5% and collectively, better than this. They apply to the composite stress-strain curve of alloy plus layer but may be applied to the case of alloy alone where the plastic strains are of the same order as the elastic strains.

A good idea of the overall accuracy may be obtained from curves d, e, f, of fig. 41. These are from three different specimens under closely equal loads (i.e. equal ratios of T: 2k). The specimens were with different prisms, different angles of obliquity, different gauges of material and different ratios of plate width to hole diameter, yet they gave closely equal curves well into the plastic region.
Comparison with Griffith's Experiments under a Single Tension.

Griffith's experimental work has been described on p.11. Figs. 51, 53, 54, are tracings from figs. 4, 3, 6, in his paper, with comparable results of the present work superimposed in broken line.

Fig. 51 shows the stress-strain curves of:

a) 24S-T alloy
b) L71 alloy
c) L71 with an equal thickness of Araldite.

Curve c) is plotted with the stress reduced by 4% (see p.35).

The stress-strain curves are replotted in fig. 52 with the stress $T$ as a fraction of the yield stress $2k$, and the strain as a fraction of the strain at the yield point ($\varepsilon_y$). The curves for 24S-T alloy and L71 (unbonded) now coincide to form a single curve. This is used as the basis of comparison for the two sets of results.

Fig. 53 shows the strains at the point of maximum strain concentration. At these points, Griffiths used $\frac{1}{2}$" electromagnetic gauges, specially constructed to be accurate at high strain, placed inside the 4" diameter hole.

Faierberg (8), in his theoretical paper, makes reference to an experimental determination of the maximum strain concentration with a plate size of 800 x 180 x 5mm and a hole of 30 mm diameter, using mechanical extensometers with a base equal to 2mm. His results agree well with Griffiths, confirming the accuracy of the measurements.
Comparison with Griffith's Results
The broken line in fig. 53 shows the results obtained in the present work with bonded L71 (the strains being reduced by the ratio \( \frac{\sigma_{L71}}{\sigma_{L71}} \) for the purposes of comparison). The restraining influence of the layer is seen in the higher stress required to produce the same strain concentration at the hole edge.

Fig. 54 shows the strains along a transverse line from the hole centre. The original figure from which this curve was traced was drawn in a "three dimensional" manner with the average net-section stress \( \sigma_{Av} \) as its third parameter. The highest curve has been reproduced (\( \sigma_{Av} = 45.10^3 \) lbs./sq.in.). The topmost point in fig. 54 is correct, since it was determined by the electromagnetic gauges. However, the other points were measured by wire resistance gauges. Griffith estimates these to be correct within 2% up to only 0.24% strain, above which the accuracy falls. The low end of his curve (approaching \( \gamma = 6a \)) appears to be about 8% above the correct value determined from the values of \( \Omega \) and the tensile modulus.

A comparable curve from the present work is shown as a broken line in fig. 54. This is about 4% below the correct value of at the low end (\( \gamma = 6a \)). It remains below Griffith's curve but follows its shape closely.

Comparison with Faierberg's Theory for a Single Tension.

The method Faierberg (8) used to obtain a solution has been outlined on p. 11. In figs. 19 and 20 of his paper he plots
**Fig. 55**

Comparison with Faierberg's Predicted Strains

Faierberg:
- \( T_{\text{max}} = 6a \) (\( \frac{T}{2a} = -7 \))
- \( \varepsilon_{\text{max}} = 7a \) (\( \frac{T}{2a} = -7 \))

Present Work (\( \frac{T}{2a} = -7 \))

**Fig. 56**

Faierberg's Predicted Plastic-Elastic Boundary
applied load (as $\frac{F}{2k}$) against the strain (as $\frac{\varepsilon}{\varepsilon_s}$) at several points along the transverse diameter for ratios of plate width to hole diameter of (a) 6:1 and (b) 7:1. Readings taken from these curves are shown below.

(a) $(r_{max} = 6a)$

<table>
<thead>
<tr>
<th>$\frac{r}{2k}$</th>
<th>$\frac{\varepsilon}{\varepsilon_s}$ for: $r = 4a$, $r = 2a$, $r = a$</th>
<th>$r = a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.8</td>
<td>1.35, 3.7, 8.7</td>
<td></td>
</tr>
<tr>
<td>.7</td>
<td>.85, 2.35, 5.4</td>
<td></td>
</tr>
<tr>
<td>.6</td>
<td>.6, 1.4, 3.13</td>
<td></td>
</tr>
<tr>
<td>.5</td>
<td>.4, .9, 1.9</td>
<td></td>
</tr>
</tbody>
</table>

(b) $(r_{max} = 7a)$

<table>
<thead>
<tr>
<th>$\frac{r}{2k}$</th>
<th>$\frac{\varepsilon}{\varepsilon_s}$ for: $r = 7a$, $r = 3a$, $r = 2a$, $r = a$</th>
<th>$r = a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.8</td>
<td>.5, 2.5, 4.33, 9.6</td>
<td></td>
</tr>
<tr>
<td>.7</td>
<td>.35, 1.57, 2.6, 5.75</td>
<td></td>
</tr>
<tr>
<td>.6</td>
<td>.29, .9, 1.45, 2.85</td>
<td></td>
</tr>
<tr>
<td>.5</td>
<td>.3, .6, .85, 1.7</td>
<td></td>
</tr>
</tbody>
</table>

In fig. 55, values of strain for $T/2k = .7$ are plotted together with those of the present experimental work at $T/2k = .77$. This higher load more than compensates for the restraining influence of the layer and the theoretical strain values should be smaller. However, it can be seen that though the theory agrees approximately with experiment at $r = a$ and $r = 5a$, the theoretical strains are considerably larger in the region between. This difference is probably due to the shape and extent of the plastic-elastic boundary deduced by Faierberg, on extending his solution from a thin ring to a hole in a plate. Fig. 56 shows the plastic-elastic boundary obtained by Faierberg.
The photoelastic patterns from the present work with a single tension at loads of \( T/2k = .8 \) or more are shown in fig. 34. The plastic-elastic boundary does not extend far along the transverse diameter but spreads out at approximately 35° to this direction.

Another feature of Faierberg's theory which is unlikely to be correct is that he obtains higher strains for the larger plate width.

Comparison with Theory for Two Equal Tensions

Hodge, in both of his papers (5), (6), deals with the condition of generalised plane stress which is also valid for the present experimental work. In order to compare his second paper (allowing for work hardening), the composite stress-strain curve of L71 + Araldite has been used to determine his parameter "n" (see fig. 2). Hodge indicates the method of solution for a stress-strain curve of three segments, but gives the full equations only in the case of two segments (i.e. linear strain hardening). The curve for L71 + Araldite has therefore been approximated to two segments AB, BC, as shown in fig. 57.

The maximum strain in the tests occurs at C (approximately 1%); line BC has been arranged to form equal areas above and below the true stress-strain curve, meeting the initial slope at B. This leads to \( n = .289 \).
**Fig. 57**

Strain hardening curve used for Hodge's Theory

**Fig. 58**

Comparison with Hodge's Strains

(\( \sigma = \text{experimental point} \))
In fig. 58, $a_1$ and $a_2$ show the strains ($\varepsilon_0$ and $\varepsilon_T$) from the theory for $T/2k = 1.4$; $b_1$ and $b_2$ are the strains from Hodge's former theory assuming the Tresca yield condition. The experimental points (from p.102) are shown as small circles. They lie above the theoretical values of $\varepsilon_T$, but are in good agreement with the curves of $\varepsilon_0$, particularly $a_1$.

In fig. 59, (a) shows the plastic-elastic boundary (of radius $\rho$) predicted by the strain hardening theory. (b) shows the boundary given by his previous paper. The experimental points are again shown as small circles. It can be seen that the predicted plastic-elastic boundary, on allowing for work hardening, agrees well with experiment.

Galin (7) deals with the case of plane strain, where a third principal stress exists equal to $\eta (p + q)$. Thus in regions where $q < \eta \frac{r}{k + r} \rho$, maximum shear occurs in the $x$, $y$ plane and is given by $\frac{1}{2}(p-q)$. Yield under the Tresca condition will therefore occur at $(p-q) = 2k$.

If we were to assume the present experiments to be performed under the condition of plane strain, the (Tresca) yield criterion would be $p-q = 2k$ or, in terms of strain $\varepsilon_P - \varepsilon_q = \text{constant}$. Viewing the results of the tests on p.92 and p.102 under this assumption, the plastic-elastic boundary appears as $\rho = 1.06a$ for $T/2k = 0.55$, $j\rho = 1.35a$ for $T/2k = 0.74$, $\rho = 1.24a$ for $T/2k = 0.70$, shown as dotted circles together.
FIG. 59

Comparison with Hodge's predicted plastic-elastic boundary

FIG. 60

Galin's predicted plastic-elastic boundary

Experimental points giving an upper bound.
with Galin's predicted plastic-elastic boundary in fig. 60.

Since the tests were under the condition of (generalised) plane stress, it is to be expected that the points would lie above the true curve, deviating further and further as the plastic region spreads. The points thus give an upper limit to the extent of the plastic-elastic boundary in the case of plane strain and it can be seen that Galin's solution lies close to, and under these points.
The method of a bonded photoelastic layer has been used successfully to measure strains in elastic and plastic regions around a hole in a thin aluminium alloy plate. Clear and symmetrical photoelastic patterns were obtained using a reflection polariscope situated above the specimen; the latter being loaded in either one or two directions by means of a specially constructed tension table.

The problem of gripping wide specimens in sheet form has been overcome by bonding tapered end pieces on to the surfaces of the specimens. This has enabled the applied stress to approach closely to the tensile limit of the alloy, with uniform tension in the central region.

By careful mixing, machining and annealing, thin stress free castings of Araldite have been produced for the photoelastic layer. A method of bonding that gave low initial stresses have been developed, covering the alloy with a thin film of heated Araldite resin, then bonding on the casting with a cold setting Araldite glue.

The advantage of a bonded layer is that it enables the magnitude and direction of strains to be measured at all points on the surface of a specimen even though plastic strains are present in the metal. Thus more information is obtained than from an array of strain gauges, which cannot determine the magnitude and direction
of principal strain in regions where stress and strain are not linearly related.

An accuracy of within ±5% has been obtained (using normal incidence and oblique incidence) for the composite stress-strain curve of the alloy and layer. The layer exerts a restraining influence on the metal in regions of plastic flow, but the results obtained may be applied to the case of alloy alone (i.e. unrestricted) with little loss of accuracy for small plastic strains.

Measurements in the layer fall below the true value at free boundaries where large plastic strains occur. However, measurements appear reliable up to a distance from the boundary which is approximately equal to the thickness of the layer. Thus measurements are reliable up to 1/16" from a boundary in the present work with layers of 1/16" or less. Where the plastic strains at a boundary are small (e.g. for the tests with equal tensions) the error is small even at the boundary edge.

The method of oblique incidence has been modified for use with a bonded layer, in order to separate the principal strains. The measurements at any point rely only on a small area surrounding the point, and thus avoid the cumulative errors of an integrating process. The method gives good accuracy in elastic and plastic regions providing a number of precautions are observed with respect to fringe location and measurement of the angle of obliquity.
Tests on specimens photographed at normal incidence, have given the strain distribution (as contours of $\epsilon_r - \epsilon_0$), and the shape of the plastic-elastic boundary with increasing load. A radially symmetrical pattern was obtained under equal tensions, showing the plastic-elastic boundary to be a circle. In the case of a single tension, plastic areas commenced at the hole edge at points along the transverse diameter, spreading slowly along and away from the boundary. At large values of stress, plastic regions spread out at approximately $35^\circ$ from the transverse diameter, and thereafter increased rapidly in this direction.

Little difference has been observed with ratios of plate width to hole diameter which ranged from 6:1 to 24:1, it therefore appears that a hole in a plate of six times its diameter or more will behave in a closely similar way to a hole in an infinite plate, at least until the stresses are sufficiently high to cause the plastic regions to spread out from the transverse diameter.

Measurements at oblique incidence on specimens under a single tension have been concentrated mainly along the transverse diameter. Comparison with Faierberg's theory shows considerable disagreement both as to the strains obtained and the shape and extent of the plastic-elastic boundary.

The tests with equal tensions have been compared with two
theories by Hodge under the condition of generalised plane stress, and have been used as an upper limit for Galin's theory of plane strain. Good agreement has been obtained, especially with Hodge's second theory allowing for work hardening.
APPENDICES

A. Summary of Two-Dimensional Elasticity in Complex Co-ordinates.

Complex potentials have been used for Galin's solution for equal tensions. A summary of two-dimensional elasticity in complex co-ordinates is therefore given below.

The usual equations of equilibrium and stress-strain relations in cartesian co-ordinates \((x, y, z)\) are simplified by transforming to new independent variables \((z, \bar{z}, \bar{z})\) given by

\[
\begin{align*}
\bar{z} &= x + iy \\
\bar{z} &= x - iy
\end{align*}
\]

(34)

The cartesian stress components may be combined to form the stress combinations

\[
\begin{align*}
\sigma &= \sigma_x + \sigma_y \\
\Phi &= \sigma_x - \sigma_y + 2i \tau_{xy} \\
\Psi &= i \gamma_{yz} + \gamma_{xz}
\end{align*}
\]

(35)

With these substitutions, the equations of equilibrium (for no body force) take the form

\[
\begin{align*}
\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial \bar{z}} + \frac{\partial \Psi}{\partial z} &= 0 \\
\frac{\partial \Psi}{\partial z} + \frac{\partial \Phi}{\partial \bar{z}} + \frac{\partial \Psi}{\partial \bar{z}} &= 0
\end{align*}
\]

(36)

Where bars denote conjugate complex quantities.
If \( D = u + iv \) is the complex displacement, the stress-strain relations take the form

\[
(1-z\eta)\psi = 2G\left\{ \frac{\partial D}{\partial z} + \frac{\partial D}{\partial \bar{z}} + 2\eta \frac{\partial w}{\partial z} \right\}, \quad \phi = 4G \frac{\partial D}{\partial \bar{z}}, \quad (37)
\]

\[
(1-z\eta)\phi = 2G\left\{ \eta \left( \frac{\partial D}{\partial z} + \frac{\partial D}{\partial \bar{z}} \right) + (1-\eta) \frac{\partial w}{\partial z} \right\}, \quad \psi = G \frac{\partial D}{\partial z} + 2G \frac{\partial w}{\partial \bar{z}}, \quad (38)
\]

**Plane Strain**

The condition of plane strain is defined by \( \omega = 0 \), \( \frac{\partial D}{\partial \bar{z}} = 0 \) and the above equations may be solved in terms of two functions of the complex variable \( z \). Let these complex potentials be \( \mathcal{R}(z) \) and \( \omega(z) \). Then, where \( K = 3-4\eta \):

\[
8GD = K \mathcal{R}(z) - z \mathcal{R}'(\bar{z}) - \omega'(z) \quad (39)
\]
\[
2\phi = \mathcal{R}'(z) + \mathcal{R}'(\bar{z}) \quad (40)
\]
\[
-2\phi = z \mathcal{R}''(\bar{z}) + \omega''(z)
\]

Since only derivatives of the complex potentials occur in eqn (40), the stress combinations may also be written

\[
2\phi = \alpha(z) + \bar{\alpha}(\bar{z})
\]
\[
-2\phi = z \alpha'(\bar{z}) + \bar{\alpha}'(\bar{z})
\]

where \( \alpha(z) = \mathcal{R}'(z) \) and \( \beta(z) = \omega''(z) \)

These equations are required for Galin's solution (see p.9).
For solutions in polar co-ordinates the stress combinations used are

\[ \theta' = \sigma_r + \sigma_\theta \quad \phi' = \sigma_r - \sigma_\theta + 2i \tau_{r\theta} \quad \ldots \ (41) \]

and these are related to \( \theta \) and \( \phi \) by

\[ \theta' = \theta \quad \text{and} \quad \phi' = e^{-2i\theta} \phi \quad \ldots \ (42) \]

In the Airy stress function approach

\[ \sigma_x = \frac{\partial^2 \chi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \chi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \quad \ldots \ (43) \]

and \( \chi \) is a solution of the equation \( \nabla^4 \chi = 0 \quad \ldots \ (44) \)

Since

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4}{\partial z^2} \]

the Airy stress function is related to the complex potential method by:

\[ 4 \chi = \text{real part of } \left\{ z \sqrt{z}(\bar{z}) + \bar{z}(\bar{z}) \right\} \quad \ldots \ (45) \]

A stress system satisfying equations (43) and (44), thus allowing representation in complex potential form, is referred to as a "biharmonic stress system".

**Generalised Plane Stress**

Under this condition, the stresses and displacements are averaged across the thickness of the plate. Filon showed that if Poisson's ratio is modified to \( \eta' \) such that \( \eta' = \frac{\eta}{1+\eta} \), then the
equations of generalised plane stress are identical to those of plane strain.

B. Elastic Solution for a Hole in a Flat Plate.

The solution for an unstressed hole in a large plate under tension may be obtained by taking

\[ \mathcal{R}(z) = \frac{A}{z} \quad \omega(z) = -\frac{B}{z^2} + c \log z \]  

To satisfy the boundary conditions around the hole

\[ \mathcal{R}(\bar{z}) + \bar{z} \mathcal{R}'(\bar{z}) + \omega'(\bar{z}) = 0, \quad \bar{z} = a^2 \]  

Substitution in this boundary condition gives:

\[ A = 2T\alpha^2, \quad B = -T\alpha^2, \quad C = -2T\alpha^2 \]

This leads to

\[ \mathcal{H} = T\left( 1 - \frac{2\alpha^2}{\pi^2} \cos 2\theta \right) \quad (47) \]

and

\[ \mathcal{\Phi} = T\left\{ C_\alpha \phi \left( \frac{2\alpha^2}{\pi^2} - \frac{3\alpha^2}{\pi^4} \right) - 1 + \frac{\alpha^2}{\pi^2} \cos 2\theta \right\} + \bar{\gamma} \left[ \left( \frac{2\alpha^2}{\pi^2} - \frac{3\alpha^2}{\pi^4} \right) \sin 4\theta + \frac{\alpha^2}{\pi^2} \sin 2\theta \right] \]  

(48)
On separating the stress components

\[ \sigma_r = \frac{T}{2} \left\{ (1 - \frac{a^2}{r^2}) + (1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}) \cos 2\theta \right\} \]

\[ \sigma_\theta = \frac{T}{2} \left\{ (1 + \frac{a^2}{r^2}) - (1 + \frac{3a^4}{r^4}) \cos 2\theta \right\} \]  

\[ \gamma_{r\theta} = -\frac{T}{2} \left( 1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2} \right) \sin 2\theta \]  

For two tensions at right angles, the solution is simply obtained by the superposition of two "single tensions" of appropriate magnitude, one being rotated by 90°.

In the case where the tensions are equal they are equivalent to an all round tension and the equations simplify to:

\[ \sigma_r = \left( 1 - \frac{a^2}{r^2} \right) T \]

\[ \sigma_\theta = \left( 1 + \frac{a^2}{r^2} \right) T \]

\[ \gamma_{r\theta} = 0 \]  

The solutions for a single tension and for two equal tensions are shown graphically in figs. 47, 48.
G. Approximate Calculation for the Change in Angle

Of Obliquity along the Specimen.

The change in the angle of obliquity in the layer \( \Theta_A \) is worked out below for rays incident on the prism at an angle \( \Theta \) close to \( 45^\circ \). Thus in fig. 61\( \alpha \) and \( \beta \) (in radians) are both small.

In fig. 61 \( AB = L \alpha + AD \beta = L \alpha \), where \( L = \frac{A \beta}{\alpha} \). Then \( \beta = \mu L \).

\[
L = \frac{A \beta}{\mu} \quad \text{and} \quad L \text{ is approximately constant as in practice} \quad l >> AD.
\]

Now if \( BC = X \) then \( X = AB \sqrt{2} = L \alpha \sqrt{2} \). \( \delta X = L \sqrt{2} \delta \alpha \)

Since \( \frac{\delta \alpha}{\beta} = \mu L \) and \( \Theta_L = 45^\circ - \beta \), \( \delta \alpha = \mu L \delta \beta = -\mu L \delta \Theta_L \)

\[
\delta X = -\mu L \sqrt{2} \delta \Theta_L \quad \text{and} \quad \delta \Theta_L = -\frac{1}{\mu L \sqrt{2}} \delta X.
\]

The angle in the layer \( \Theta_A \), is given by \( \sin \Theta_L / \sin \Theta_A = \mu A / \mu L \)

\[
\delta \Theta_A = \frac{\mu A}{\mu A \cot \Theta_A} \delta \Theta_L = -\frac{1}{\mu A \cot \Theta_A} \sqrt{2} L \delta X
\]

Thus the change in the angle of obliquity in the layer with distance along the specimen is given by:

\[
\frac{\delta \Theta_A}{\delta X} = -\frac{1}{\sqrt{2} L \mu A \cot \Theta_A} \cot \Theta_L
\]
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