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UNIVERSITY OF SURREY

"INSTABILITY OF THIN-WALLED SECTIONS"

by

USAM KHAIRI BUNNI, B.Sc.

A thesis submitted for the Degree of
Doctor of Philosophy
in the Department of Civil Engineering

June 1973
Synopsis

An investigation is made of the instability of thin-walled sections. A versatile computer program based on the finite element method of structural analysis is developed which is capable of determining the critical loads and buckling modes of arbitrarily shaped closed or open thin-walled sections under arbitrary loading. The effectiveness and validity of the program is checked against some classical problems and then applied rigorously to the study of the local buckling of box columns and the overall stability of thin-walled box girders in cantilever. An experimental programme is performed on the buckling of thin-walled plastic box-columns for comparison with the analysis.
TO MY PARENTS
Acknowledgments

It is a pleasure to record my sincere gratitude to Dr. W. J. Supple for all the help, advice and encouragement he has given me.

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Chapter 1

Introduction

1.1 General Introduction

The intricacies faced in the design and analysis of thin-walled structures are offset by the advantages gained from them, namely:

a. Flexibility in the design shape and form;

b. Light weight: weight per covered area;

c. Economy.

In general, such structures may come under either one of two categories:

1. As assemblages of some thin structural forms, e.g. box girder bridges and tubular space structures;

2. Having small ratios of thickness to other dimensions, as in the cases of thin shells and stressed skin structures.

It is this common feature, small wall thicknesses, that may give rise to some form of instability in such structures. The instability being local, overall or a combination of both.

The concept of instability or buckling is not a new phenomenon to the engineering world. It has been the subject of extensive research and studies for over two centuries. The first break through came with the findings of Euler in 1774, on the buckling of columns under axial compression. The reader is referred to Langhaar\(^{(17^* )}\) and Bleich\(^{(2)}\) for a detailed history of the buckling problem.

* Numbers in brackets refer to the references given in Appendix 3.
Despite all this intensive work, the instability analysis still remains one of the most challenging problems in the field of engineering owing to the perplexities and mathematical complexities surrounding such a non-linear problem. It is not, therefore, surprising to find only a limited number of closed form solutions, corresponding to some specific cases, available to the designers.

1.2 The Finite Element Method of Stress Analysis

If the difficulties surrounding the instability problem make it impossible to attain an exact solution, it is, then, logical for designers to turn towards some analytical methods that would yield plausible results with high degrees of accuracy. The finite element method is one method amidst a variety of methods that can be employed in such an analysis.

A resume of the finite element method is given in Chapter 2 of this thesis. The resume describes this method of stress analysis and includes the derivation of a stiffness matrix for a flat quadrilateral shell element using the principle of virtual displacement. The various matrix formulations, involved in such a derivation, are given explicitly in Appendix 1.

1.3 The Instability Analysis

An instability analysis of thin-walled structures is given in Chapter 3 of this thesis. A non-linear analysis computer program has been developed, based on the finite element method using the displacement models discussed in section 3.2.

The non-linear nature of such a study prohibits the use of the classical theories of linear analysis. Hence, some iterative techniques were employed to circumvent this difficulty.
The computer program determines the critical load level that induces some form of instability in a structural system. Such a determination is based on the Newton-Raphson iterative approach. Furthermore, the program detects the critical degrees of freedom and, employing a suggested iterative approach, based on a successive iteration technique, determines the buckling mode for the structure, (section 3.6).

To help the reader in following the basic parts of the program, a general discussion on basic instability theory is given in section 3.3.

1.4 Applications

The computer program has been applied to the instability analysis of various thin-walled structures to investigate the validity or the inadequacy, as the case may be, of some of the classical assumptions and design procedures followed by designers when dealing with such structural systems.

Prior to such applications, the program was tested against some problems whose closed form solutions are well established. The instability analysis of thin flat plates, with different aspect ratios and boundary conditions, and that of curved sheet panels were investigated. The results of this study is given in the earlier part of Chapter 4.

The main applications of the program were on two thin-walled structures, formed as assemblages of some structural elements, which are thought to be of relevant importance to current design and analysis. The first application is that to the instability analysis of box columns and the second is to that of cantilevered box girders.

With regards to the box column problem, the classical assumptions consider the boundary conditions of the longitudinal edges, of the structural element forming the box, to be of simple support type. Hence, the instability analysis of such
a structure is reduced to an instability study of the individual elements constituting the four faces of the box column.

Although plausible results have been obtained for rectangular box sections using such assumptions, little work has been done to investigate the generality of these assumptions, when dealing with box sections formed as assemblages of flat plates and curved sheet panels, or some shell type surfaces.

Such a study was carried out on box columns with the cross-section shown in Fig. App 2-1. No assumptions were made regarding the boundary condition along the longitudinal edges of the flat webs and the curved flanges. The investigation was supplemented by an experimental programme on such box columns. The experimental study is reported in Chapter 5. The specifications and properties of the test models are given in Appendix 2.

The next application was to the current most controversial bridge engineering structures, namely box girders. The work was motivated by the recent collapse of two major bridges: the Milford Haven and the Koblenz bridges. Both collapsed whilst under construction by the cantilever method, so it was to the cantilevered state that attention was directed.

Although the present design procedure for box girder bridges requires a stability analysis of individual plates or stiffened panels, taken in isolation, the present study, as given in section 4.6 and 4.6.1, illustrates the importance of a final check on the global stability of the whole structure.

Past experience indicates that thin-walled structures may adopt shell-type behaviour and as such display their undesirable instability characteristics. Hence, the effects of initial imperfections on the critical load levels can be of a major importance.
An extensive programme was carried out to study the global instability of a cantilevered box girder. The study included the evaluation of the critical load level and the determination of the buckling mode. Initial imperfections, in the form of initially applied root moment, were incorporated and their effects on the critical load levels were recorded. The study is presented in sections 4.6 and 4.6.1.

To supplement the discussion given throughout the thesis, final concluding remarks are given in Chapter 6.
Chapter 2

Resume': The Finite Element Method of Analysis

2.1 General Introduction

Less than twenty years ago, the engineering world witnessed the début of the finite element method as a powerful tool for the analysis of complex structures. The method can be considered as a successful development of the work done by Hrennikoff in 1941\(^1\), McHenry in 1943-1944\(^2\), on lattice analogy and the work of others, such as Newmark in 1949\(^3\), Levy in 1953\(^4\), Parikh and Norris\(^5\) in their idealization of plates and shells into networks of beams which could then be treated as grillage systems.

The actual development of the finite method, as it is known today, can be attributed to Turner, Clough, Martin and Topp\(^6\). Their work published in 1956 is still considered as one of the first classical contributions to the development of this method of analysis.

Since then much work has been carried out to refine this technique of stress analysis against errors associated with it namely: idealization errors, discretization errors and manipulation errors, the explanation of which will be given later in this chapter.

2.2 The Finite Element Method of Analysis

The concept of matrix method of structural analysis for skeletal structures is well established. Basically, a skeletal structure can be viewed as an assemblage of 'two-ended' members (or elements) arbitrarily connected at a finite number of points, usually called "Joints", at which the conditions of equilibrium and compatibility are satisfied. The matrix method of analysis relates the external applied loads to the displacements at the joints by a linear transformation, a form of which is given in equation (2-2).
Unlike a skeletal structure where a finite number of joints define the whole configuration, a true elastic continuum has an infinite number of interconnected points. Hence, to satisfy the conditions of equilibrium and compatibility for all the points of a continuum would be a formidable feat for all but the most trivial situations.

To circumvent these difficulties, the continuum is assumed to have a finite number of points, arbitrarily chosen, at which the equilibrium and/or compatibility requirements are satisfied. These points are called 'Nodes' or 'Nodal Points'. Thus, a continuum can be viewed as an assemblage of non-overlapping discrete (finite) elements formed by arbitrarily joining a number of nodal points by imaginary lines*.

Hence, a finite element is of an arbitrary shape depending largely on the analyst and to a lesser extent on the geometry of the continuum to be analysed. The most commonly used are the triangular, rectangular and quadrilateral flat elements^+. The errors associated with this representation, i.e. a continuum by a number of finite elements, are referred to as 'Discretization errors'. 'Idealization' errors refer to the approximations involved in representing a curved continuum, e.g. shell surfaces, by a number of flat elements. The numerical rounding errors arising throughout the computational sequences are referred to as 'Manipulation' errors.

2.2.1 Choice of a Function

Having decided on the shape of an element, a function is chosen to define uniquely the state of either the displacement

---

* Elements with internal nodes and/or with additional nodes along their boundaries have been suggested and used by a number of authors\(^{(12, 41, 45)}\).

\(^{+}\) Curved shell elements and three dimensional elements have been employed by a number of researchers\(^{(10, 12, 41, 45)}\).
or stress within each finite element*. This is a modification of the well recognised Rayleigh-Ritz method with the difference that in the finite element method the chosen function represents a state of either deformation or stress of individual elements whilst in the Rayleigh-Ritz method the assumed function represents the displacement field in the entire continuum. Thus, the finite element method can be considered as a localized Rayleigh-Ritz technique of analysis.

A perfect and ideal function should satisfy both equilibrium and compatibility requirements within and along the edges of an element. For membrane behaviour the equilibrium requirements are the well established differential equations of two-dimensional elasticity,

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0
\]

To satisfy the compatibility requirements, continuity of deflection over the whole surface of an element and continuity of slopes along and across the edges should be established.

In practice, the assumed functions usually violate one or both of these requirements. A common violation with displacement functions is that of normal slopes across the edges of an element. The disadvantages and consequences

* The concept of using a 'Hybrid' model: a mixed model that uses both displacement and stress function in representing the behaviour of an element, has been considered by Veubeke(43).
involved with this violation are discussed in section 3.2.2. Fully compatible functions for rectangular and triangular elements have been suggested by Melosh\textsuperscript{(21)} and Veubeke\textsuperscript{(44)}, respectively.

2.2.1.1 Upper and Lower Bounds to a Solution

Due to the approximations involved in representing an actual continuum by some mathematical model, being a displacement or an equilibrium model, the exact solution cannot be attained. However, a limit to the exact solution can be reached with such models. The problem, thence, lies in finding whether the approximate solution represents an upper or a lower bound to the exact solution.

The concept of upper and lower bounds to a solution is related to the state of strain energy of the structure. For a structure under the action of an arbitrary loading system, the restrictions imposed by a chosen displacement function, where a finite number of generalised coordinates are assumed to represent an infinite number of degrees of freedom, lead to an underestimate of the strain energy. On the other hand, the restrictions associated with an assumed stress function lead to an overestimate of the strain energy\textsuperscript{(29)},

\[ U_d < U < U_e \]

where, $U$ is the strain energy of the actual continuum; $U_d$ and $U_e$ represent the strain energy of a displacement model and an equilibrium model, respectively. Hence, if a compatible displacement function is employed, a lower bound to the exact solution (nodal displacements) will be obtained. Conversely, an upper bound will be reached with an equilibrium model.

2.2.2 Methods of Analysis

Once a function is chosen to define uniquely the state of either deformation or stress of an individual element, the
original continuum is no longer of any concern to the analyst. The idealized structure is analysed with a suitable method that satisfies the general rules of structural analysis, namely:

a. Equilibrium:-- internal element forces at a node 'i' on the continuum are in equilibrium with the external applied loads at that node;

b. Compatibility:-- upon loading the structure, the elements deform in such a way that they continue to meet along and across the peripheries;

c. Force-Displacement Relationship:-- internal forces and internal displacements within every individual element are related in accordance with the elasticity equations relating stress with strain.

In general, two basic methods that satisfy these general rules can be employed for the analysis:

1. The Equilibrium Method, generally known as the Stiffness Method;

2. The Compatibility Method, generally known as the Flexibility Method.

Both methods relate the forces (external applied loads: forces and moments) with the nodal displacements (both deflections and rotations) of a linear elastic structure by a linear transformation of the form:

\[
[K] \cdot \{d\} = \{w\} \quad \ldots \ldots (2-2)
\]

where, \([K]\) is the stiffness matrix of the structure, \(\{d\}\) is the nodal displacement vector, \(\{w\}\) is the applied load vector.

Assuming that the matrix \(K\) is non-singular, equation (2-2) can be re-written in the form:
\[ \{d\} = [K^{-1}] \cdot \{W\} \quad \ldots \ldots (2-3) \]

or,

\[ \{d\} = [F'] \cdot \{W\} \quad \ldots \ldots (2-4) \]

where \( F \) is the flexibility matrix of the structure. Both matrices, \( K \) and \( F \), are symmetrical and positive definite (the latter is true in the case of stable structures, section 3.3).

The choice between the stiffness method and the flexibility method depends largely on the type of function(s) assumed. In general, stiffness method is employed when a displacement function is assumed whilst the flexibility method is used when a stress function is chosen.

The stiffness method was followed throughout this research work. The following sections explain the method and show the construction of the stiffness matrix \( K \) for a flat quadrilateral shell element.

2.3 Summary of the Finite Element Method Using a Displacement Function and Employing the Stiffness Method of Analysis

1. A displacement function is chosen to represent the displacement behaviour of every individual element on the continuum. The chosen function should satisfy the following requirement suggested by Melosh(21):

   a. The function should be continuous (the derivatives need not be continuous) over the element surface;

   b. The function should assure compatibility of displacements, both deflections and rotations, between adjacent elements. To establish this criterion the displacements along any edge of the element should be functions of the degrees of freedom of the nodal points bounding that edge, hence, the nodal degrees of freedom are chosen as the generalized displacements (section 2.4),
c. Since the relation given in equation (2-2) is of a linear nature, the function should be a linear function of the generalized coordinates.

If these requirements are satisfied, convergence to an exact solution is guaranteed as the mesh size is reduced, i.e. the discretization errors approach zero in the limit.

2. Using either the principle of virtual work (direct approach) or the principle of a stationary total potential (energy approach), the stiffness matrices of every individual element on the continuum can be computed [8, 29].

3. The stiffness matrix of the whole structure is, then, generated. The stiffness matrix is a singular matrix in the absence of boundary conditions, hence, a set of constraints is imposed on the matrix to make it possible for the structure to be analysed.

4. From the force-displacement relationship of equation (2-3), the nodal displacements are calculated. The elements of the load vector W are the external applied loads which are, usually, represented as concentrated forces and moments acting at the nodal points.

5. The stress distribution over each finite element is calculated. Two approaches can be followed:

a. From the chosen function, a stress-nodal displacement relationship can be constructed from which a typical stress distribution can be determined;

b. By a best fit polynomial to the deformed surface; the polynomial being subsequently differentiated in accordance with the elasticity equations relating stress with displacement.
2.4 Basic Relationships

A displacement function is chosen to simulate the behaviour of an element under the action of external applied loads by defining its nodal displacements in terms of some generalized parameters (coordinates), \( \alpha \). In matrix form,

\[
d_e = B \cdot \alpha 
\]  
\[\text{.....(2-5)}\]

where \( d_e \) is the nodal displacement vector of an element \( i-j-k-l \) in Fig. 2-1,

\[
d_e = \begin{bmatrix} d_i \\ d_j \\ d_k \\ d_l \end{bmatrix} 
\]  
\[\text{.....(2-6)}\]

where the sub-vectors \( d_i, d_j, d_k \) and \( d_l \) represent the degrees of freedom of the respective nodal points, i.e.

\[
d_i = \begin{bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} 
\]  
\[\text{.....(2-7)}\]

corresponding to the following load vector,

\[
p_i = \begin{bmatrix} P_x \\ P_y \\ P_z \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} 
\]  
\[\text{.....(2-8)}\]
Fig. 2-1 A Rectangular Element Under the Action of:

a. in-plane forces, and
b. lateral forces.

c. A right handed cartesian coordinate system.
if six degrees of freedom are assumed per nodal point.

The components of the matrix B define the relative positions of the nodal points (i, j, k and l) to a chosen coordinate system.

The common practice is to have the number of general parameters of the chosen function coinciding with the number of degrees of freedom, q, attributed to an element. Hence, the matrix B is a square matrix consisting of \((q \times q)\) components. But, in general, the number of general parameters can be greater than the number of degrees of freedom\(^{(28)}\), hence, the matrix B becomes a rectangular matrix of the form,

\[
B = \begin{bmatrix}
B_a & B_b \\
\end{bmatrix}
\]

where \(n\) is the number of 'dependent' parameters (section 2.5), i.e. the total number of the generalized parameters equals \(q + n\).

Thus, equation (2-5) becomes,

\[
d = B_a a_a + B_b a_b
\]

or,

\[
d = B_a a_a + B_b a_b \quad \ldots \ldots (2-10)
\]
where $a$ and $b$ represent the 'independent' and the 'dependent' generalized parameters, respectively.

Re-arranging equation (2-10) and assuming that the matrix $B_a$ is non-singular,

$$a = B_a^{-1} \cdot (d - B_b \cdot b)$$

or,

$$a = M \cdot d'$$

where $I$ is a unit matrix.

The element strains, $e$, can be expressed in terms of the generalized parameters vector $a$,

$$e = D \cdot a$$

The components of matrix $D$ are obtained by appropriate differentiation of the chosen displacement function. For a plane stress-strain analysis,

$$e = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$
where $\varepsilon_x$ and $\varepsilon_y$ are the direct strains and $\gamma_{xy}$ is the shearing strain. For plate bending, the strain vector $\varepsilon$ represents the curvature of the plate, i.e.

$$\varepsilon = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad \ldots \ldots (2-15)$$

Relating stresses to strains, as in the case of plane stress-strain analysis, or moments to curvatures, as in the case of plate bending, by the elasticity matrices $E$ (Appendix 1),

$$\sigma = E \cdot \varepsilon \quad \ldots \ldots (2-16)$$

Substituting for $\varepsilon$ the equality in equation (2-13),

$$\sigma = E \cdot D \cdot \alpha \quad \ldots \ldots (2-17)$$

For plane stress-strain analysis, the stress vector represents direct and shear stresses,

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \ldots \ldots (2-18)$$

whilst for plate bending, the stress vector refers to bending and twisting moments,

$$\sigma = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \quad \ldots \ldots (2-19)$$
2.5 Derivation of the Stiffness Matrix Using the Principle of Virtual Displacement

A necessary and sufficient condition of equilibrium (between forces, \( P \), and stresses, \( \sigma \)) for a structure in state of compatible deformation, \( \Delta \), is that the equality relationship between virtual work, \( dW \), and virtual strain energy, \( dU \),

\[
dW = dU \quad \ldots \ldots (2-20)
\]

holds true for any arbitrary virtual displacement, \( d\Delta \).

The internal work, \( dW_i \), of any differentiable volume of an element is given as,

\[
dW_i = \epsilon^{*T} \cdot \sigma \cdot dV \quad \ldots \ldots (2-21)
\]

where the asterisk * represents virtual quantities and the superscript \( T \) denotes matrix transposition. Integrating over the whole volume of the element,

\[
W_i = \int_{\text{vol}} \epsilon^{*T} \cdot \sigma \cdot dV \quad \ldots \ldots (2-22)
\]

The external work associated with virtual displacement, \( \Delta^* \), is given as,

\[
W_e = \Delta^{*T} \cdot P \quad \ldots \ldots (2-23)
\]

where \( P \) is the equivalent load vector whose components are both the external applied loads and the loads exerted on the element by the adjacent ones,

\[
P^T = \begin{bmatrix} q & n \\ Q & 0 \end{bmatrix} \quad \ldots \ldots (2-24)
\]
Using the relationship of equation (2-20),
\[ \Delta^{*T} \cdot \mathbf{P} = \int_{\text{vol}} \varepsilon^{*T} \cdot \sigma \cdot dV \] ..........(2-25)

and substituting equations (2-12, -13, -17) in equation (2-25) yields,
\[ \Delta^{*T} \cdot \mathbf{P} = \int \{ (D \cdot M \cdot \Delta^*)^T \cdot (E \cdot D \cdot M \cdot \Delta) \} \cdot dV \] ..........(2-26)

noting that the displacement \( \Delta \) is not a virtual quantity but a real one corresponding to a real stress. Re-arranging equation (2-26),
\[ \Delta^{*T} \cdot \mathbf{P} = \int (\Delta^{*T} \cdot M^T \cdot D^T \cdot E \cdot D \cdot M \cdot \Delta) \cdot dV \] ..........(2-27)

and extracting from the integral the terms which are independent on \( x \) and \( y \), yields,
\[ \Delta^{*T} \cdot \mathbf{P} = \Delta^{*T} \cdot M^T \{ \int (D^T \cdot E \cdot D) \cdot dV \} \cdot M \cdot \Delta \] ..........(2-28)

Since the virtual displacement \( \Delta^* \) is an arbitrary quantity, equation (2-28) can be written as,
\[ \mathbf{P} = M^T \{ \int (D^T \cdot E \cdot D) \cdot dV \} \cdot M \cdot \Delta \] ..........(2-29)

Denoting the quantity \( \{ \int_{\text{vol}} (D^T \cdot E \cdot D) \cdot dV \} \) by matrix \( C \), equation (2-29) becomes
\[ \mathbf{P} = (M^T \cdot C \cdot M) \cdot \Delta \] ..........(2-30)

From the linear transformation of equation (2-2), the quantity \( (M^T \cdot C \cdot M) \) represents the element stiffness matrix \( K \), i.e.
\[ K = M^T \cdot C \cdot M \]  

(2-31)

The total energy, \( V(Q_i, P) \), is given as, (37),

\[ V(Q_i, P) = U(Q_i) - P \cdot \varepsilon(Q_i) \]  

(2-32)

where \( U(Q_i) \) is the generalized strain energy,

\( Q_i \) represents some generalized coordinates,

and \( P \) represents some generalized forces acting through

the generalized displacements \( \varepsilon(Q_i) \).

In matrix form,

\[ V = \frac{1}{2} d^T K d - d^T W \]  

(2-33)

A necessary and sufficient condition of equilibrium is a
stationary value of the total potential energy, i.e. the
first variation of \( V \) must be zero (section 3.3):

\[ \delta V = \frac{\partial V}{\partial d_1} \delta d_1 + \frac{\partial V}{\partial d_2} \delta d_2 + \frac{\partial V}{\partial d_3} \delta d_3 + \cdots \frac{\partial V}{\partial d_q} \delta d_q + \right. \]

\[ \frac{\partial V}{\partial d_{b_1}} \delta d_{b_1} + \frac{\partial V}{\partial d_{b_2}} \delta d_{b_2} + \frac{\partial V}{\partial d_{b_3}} \delta d_{b_3} + \cdots \frac{\partial V}{\partial d_{b_n}} \delta d_{b_n} = 0 \]  

(2-34)

where \( d_1, d_2 \ldots d_q \) represent the different degrees of
freedom of the structure whilst \( a_{b_1}, a_{b_2}, a_{b_3}, \ldots a_{b_n} \)
represent the different dependent parameters of the chosen
function (equation 2-10).

For \( \delta V \) to be zero, all the derivatives of \( V \) must be
equal to zero, i.e.,

\[ \frac{\partial V}{\partial d_i} = 0 \ ; \quad \frac{\partial V}{\partial d_2} = 0 \ ; \quad \cdots \quad \frac{\partial V}{\partial d_q} = 0 \ ; \right. \]

\[ \frac{\partial V}{\partial d_{b_1}} = 0 \ ; \quad \frac{\partial V}{\partial d_{b_2}} = 0 \ ; \quad \cdots \quad \frac{\partial V}{\partial d_{b_n}} = 0 \]  

(2-35)
Equation (2-33) can be written explicitly as

\[
V = \frac{1}{2} \left[ d_1 \begin{bmatrix} d_1 & d_2 & \ldots & d_q & d_{b_1} & \ldots & d_{b_n} \end{bmatrix} \begin{bmatrix} \kappa_1 & \kappa_2 & \ldots & \kappa_q & \kappa_0 & \ldots & \kappa_0 \\ \kappa_1 & \kappa_2 & \ldots & \kappa_q & \kappa_0 & \ldots & \kappa_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \kappa_1 & \kappa_2 & \ldots & \kappa_q & \kappa_0 & \ldots & \kappa_0 \\ \kappa_1 & \kappa_2 & \ldots & \kappa_q & \kappa_0 & \ldots & \kappa_0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_q \\ \vdots \\ d_{b_1} \\ \vdots \\ d_{b_n} \end{bmatrix} \right].
\]

\[
= \frac{1}{2} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_q \\ \rho_{b_1} \\ \vdots \\ \rho_{b_n} \end{bmatrix}
\]

\[
\ldots (2-36)
\]

Differentiating with respect to \( d_1 \), only those terms with \( d_1 \) yield a non-zero derivative. Hence, the total potential energy, \( V \), can be expanded while retaining only those terms in \( d_1 \),

\[
V = \frac{1}{2} \left( \kappa_{i_1} d_1^2 + \kappa_{i_2} d_1 d_2 + \ldots + \kappa_{i_q} d_1 d_q + \ldots + \kappa_{i_{b_n}} d_1 d_{b_n} \right) - d_1 \rho_i
\]

\[
\ldots (2-37)
\]

\[
\frac{\partial V}{\partial d_i} = \frac{1}{2} \left( \kappa_{i_1} d_1 + \kappa_{i_2} d_2 + \ldots + \kappa_{i_q} d_q + \ldots + \kappa_{i_{b_n}} d_{b_n} \right) - \rho_i = 0
\]

\[
\ldots (2-38)
\]

Differentiating \( V \) with respect to all degrees of freedom and all the dependent parameters \( d_{b_n} \), yields a set of simultaneous equations which can be expressed and partitioned in matrix form as,
It follows that

\[ Q = K_{aa} d + K_{ab} \alpha_b \]

and

\[ 0 = K_{ba} d + K_{bb} \alpha_b \] \hspace{1cm} \text{(2-39)}

Solving for \( \alpha_b \),

\[ \alpha_b = -K_{bb}^{-1} K_{ba} d \] \hspace{1cm} \text{(2-40)}

Therefore,

\[ Q = K_{aa} d - K_{ab} K_{bb}^{-1} K_{ba} d \] \hspace{1cm} \text{(2-41)}

or,

\[ Q = (K_{aa} - K_{ab} K_{bb}^{-1} K_{ba}) d \]

By definition, the element stiffness matrix

\[ K_e = K_{aa} - K_{ab} K_{bb}^{-1} K_{ba} \] \hspace{1cm} \text{(2-42)}

For the particular case when the number of generalized parameters coincides with the number of degrees of freedom, i.e. \( \alpha_b \) is zero, the element stiffness matrix \( K_e \) is reduced to \( K_{aa} \) only.
2.6 The Instability of Flat Shell Element

The large deflection and non-linear geometry concepts can be employed in the derivation of the stiffness matrix of a flat shell element under the combined action of in-plane and lateral forces. An assumption is made concerning the uncoupling of the membrane and flexural behaviour of the element\(^9\). This uncoupling is in the nature of linear analysis but which can be used in simplifying problems of a non-linear nature, e.g. instability analysis.

The physical interpretations of the assumption can be summarized as follows:

For a plate under the combined action of both in-plane and lateral forces, the total elongation in the \(x\) and \(y\) directions are given by Timoshenko and Gere\(^{39}\):

\[
\varepsilon_x = \frac{\partial U}{\partial x} + \frac{1}{2} \left( \frac{\partial W}{\partial x} \right)^2
\]

\[
\varepsilon_y = \frac{\partial V}{\partial y} + \frac{1}{2} \left( \frac{\partial W}{\partial y} \right)^2
\]

\[
\gamma_{xy} = \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) + \frac{\partial W}{\partial x} \frac{\partial W}{\partial y}
\]  \hspace{1cm} \ldots  (2-43)

where the strains associated with lateral deformations are

\[
\varepsilon_{xf} = \frac{1}{2} \left( \frac{\partial W}{\partial x} \right)^2
\]

\[
\varepsilon_{yf} = \frac{1}{2} \left( \frac{\partial W}{\partial y} \right)^2
\]

\[
\gamma_{xyf} = \frac{\partial W}{\partial x} \frac{\partial W}{\partial y}
\]  \hspace{1cm} \ldots  (2-44)

Assuming that the elongations due to bending are small compared to those due to in-plane forces and that the in-plane
forces remain constant during bending, the total change in strain energy due to bending is

\[ U = \iint \left[ N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xy} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \, dx \, dy + \]

\[ \frac{1}{2} \iint \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \, dx \, dy. + \]

\[ \frac{1}{2} D \iint \left\{ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 - 2(1 - \nu) \left[ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - \left( \frac{\partial w}{\partial y} \right)^2 \right] \right\} \, dx \, dy \]

\[ \ldots \ldots (2-45) \]

where \( N_x, N_y \) and \( N_{xy} \) are the in-plane forces (stresses per unit length).

The third integral on the right hand side of equation (2-45) represents the strain energy of a plate under pure bending. Timoshenko and Gere\(^{(39)}\) have shown that the first integral represents the work done by the in-plane forces acting in its middle plane. Hence, the rest of the expression, i.e. the second integral, represents the work done during bending in stretching the middle surface. Thus, the potential energy for flexure is increased by this second integral,

\[ U' = \frac{1}{2} \iint \left\{ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right\} \, dx \, dy \]

\[ \ldots \ldots (2-46) \]

From equation (2-46) it can be seen that the assumption considers the work done in stretching the middle surface as a function of the lateral and not of the in-plane deformations. In matrix form, the uncoupling assumption partitions the master stiffness matrix \( K \) in the form

\[
\begin{bmatrix}
K_m & 0 \\
0 & K_f
\end{bmatrix}
\begin{bmatrix}
d_m \\
d_f
\end{bmatrix}
=
\begin{bmatrix}
P_m \\
P_f
\end{bmatrix}
\]

\[ \ldots \ldots (2-47) \]
where the subscripts $m$ and $f$ denote membrane and flexural behaviour, respectively; $d$ is the nodal displacements vector and $P$ is the external applied load vector.

The effect of stretching the middle surface is represented by the matrix $N$, usually known as the 'Instability' or 'Incremental' stiffness matrix. This is added to the master or geometric stiffness matrix $K$ (equation 2-47) noting that the matrix $N$ is a function of the lateral deformations only. Hence, the 'Effective' stiffness matrix becomes,

\[
\begin{pmatrix}
K_m & 0 \\
0 & K_f \\
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 \\
0 & N \\
\end{pmatrix}
\begin{pmatrix}
d_m \\
d_f \\
\end{pmatrix}
= 
\begin{pmatrix}
P_m \\
P_f \\
\end{pmatrix}
\]

or,

\[(K + N) \cdot d = W \]  

where $K+N$ is the effective stiffness matrix of the element.

In what follows this uncoupling has been employed in the derivation of the elements stiffness matrices (Appendix 1).
3.1 Introduction

The concept of non-linear analysis can be incorporated into a structural problem in three forms:

a. Material non-linearity: non-linear mechanical properties of the structural material;

b. Geometric non-linearity;

c. Beam-column effects.

All these effects may be considered simultaneously.

Material non-linearity is not in the scope of this research work. But geometric non-linearities and in particular the instability analysis of thin-walled structures constitutes the main area of study. Langhaar \(^{(16)}\) gives a classical definition of the term 'stability' as: "infinitesimal disturbances in stable systems cause only infinitesimal displacements in configuration space". A more general definition of the instability phenomenon is the collapse of a structure due to excessive deformations*. Collapse in this context may be local or global. In this study the instability analysis includes the effects of changing geometry and stretching of the middle surface in the derivation of the effective stiffness matrix of a flat quadrilateral shell element, as discussed in section 2.6.

* It is worth noting that geometric non-linearities and large deflections do not always induce instability, e.g. the behaviour of cable structures.
A major limitation of the finite element method lies in its inability to provide an exact solution for a structural continuum. But such a solution can be approached in the limit (section 2.2.1.1), depending on the accuracy of the chosen functions in simulating the behaviour of the continuum. Much work has been carried out in:

a. Search for the ideal geometric shapes for the structural elements, and

b. Providing more accurate functions for both displacement- and equilibrium models.

In the present study, the chosen functions used are functions which have been suggested and tested by a number of investigators and found to be suitable for such studies.

### 3.2.1 Membrane Behaviour

The in-plane displacement functions are second-order polynomials, suggested by Gallagher\(^8\) for a quadrilateral element under the action of in-plane forces,

\[
\begin{align*}
  u &= \frac{1}{E} \left[ \kappa_1 x + \kappa_2 xy - \kappa_3 x - \frac{\kappa_4}{2} (\kappa_5 x^2 + y^2) + \kappa_6 x y + \kappa_7 \right] \\
  v &= \frac{1}{E} \left[ -\kappa_4 y - \frac{\kappa_5}{2} (\kappa_2 x^2 + \kappa_3 y^2) + \kappa_9 x y + \kappa_{10} x y + 2(1 + \nu) \kappa_{11} x - \kappa_{12} x + \kappa_{13} \right]
\end{align*}
\]
The functions are derived from an assumed stress distribution that satisfies the pertinent differential equations of elasticity (eqs. 2-1) at every point on the element. The derivation of these functions is given in Appendix 1.

The drawback with such polynomial expressions lies in their violations of the compatibility requirements along the element boundaries. To illustrate this, a rectangular element is chosen as a particular configuration for the quadrilateral element. Along any of the boundaries, \( x = \text{constant} \) or \( y = \text{constant} \), the functions are reduced to parabolic variations in either \( x \) or \( y \). The two end displacements, at the nodal points bounding that edge, are not sufficient to define uniquely such expressions and, hence, compatibility is violated. For an irregular quadrilateral element such a violation is more pronounced.

The arguments for and against employing non-compatible functions are discussed in the next section.

### 3.2.2 Flexural Behaviour

The displacement function employed in this analysis is that presented by Zienkiewicz\(^{45}\) for a rectangular flat element. The function is an incomplete fourth-order polynomial where only two out of the five fourth-order terms have been retained,

\[
w = \beta_1 x^3 + \beta_2 x^2 + \beta_3 x + \beta_4 y^3 + \beta_5 y^2 + \beta_6 y + \beta_7 x^3 y + \beta_8 x^2 y + \beta_9 x y + \beta_{10} x^3 + \beta_{11} x y^2 + \beta_{12} \quad \ldots \quad (3-2)
\]

where \( \beta_1, \beta_2, \ldots, \beta_{12} \) are the generalized coefficients of the function.
Whilst continuity of lateral displacements, \( w \), is guaranteed with such a polynomial, compatibility of normal slopes across the element boundaries is violated. This follows from the nature of the suggested function:

Along the boundaries, \( x = \text{constant} \) or \( y = \text{constant} \), the variation of lateral displacements is of a cubic order with respect to either \( y \) or \( x \). Four constants, the lateral displacements and the rotations at the nodal points, bounding that edge, define uniquely the cubic expression. Hence, continuity of lateral displacements is guaranteed.

Normal slopes vary in a parabolic form across the element but in a cubic manner along the boundaries, \( x = \text{constant} \) or \( y = \text{constant} \). The two end rotations at the nodal points, bounding that edge, are not sufficient to define uniquely the cubic polynomial. Hence, continuity of normal slopes is violated.

For a quadrilateral element both continuity of lateral displacements and of normal slopes could be violated with such a function.

A number of authors have justified the use of non-conforming functions with the stiffness method of analysis since past experience has shown that such functions can save a great deal of computation time and cumbersome effort in analysing complex structures and still yield acceptable results.

Even though the use of a displacement model tends to yield a stiffer idealized structure whilst an equilibrium model tends to give a more flexible structure, combination of such two models should be avoided, ideally. Plausible results can be obtained from such a combination but care must be taken in the interpretations of such results.
Further to this, we may note the observations of Kapur and Hartz (15) and of the author (section 4.2.1) regarding the bounding of the buckling loads. Namely, with assumed displacement functions one would expect convergence from above to the true buckling loads. However, with non-compatible functions convergence from below is usually obtained owing to the associated relaxation of the inter-element boundary conditions. This consequence is normally well behaved. In this respect the findings of Melosh (21) assume importance. He showed that the use of fully compatible functions does guarantee convergence to the correct solution with the reduction in mesh size whereas the use of non-compatible functions, though producing convergence, give no guarantee that the converged solution is the correct one. However, analysis procedures based on this method can be checked against closed form solutions for some simple or classical problems.

To sum up, the following points are worth noting:

1. To assist in convergence towards a correct solution the number of independent coefficients, in the chosen function, should coincide with the number of unit modes corresponding to an element (21, 26);

2. To assure continuity of lateral displacements and slopes along and across an element boundaries, the deformations along any edge should be functions, only, of the degrees of freedom ascribed to the nodal points bounding that edge.

3.3 The Instability Analysis

To aid in the understanding of the procedures used in building up the finite element stability program, which follows in sections 3.5, .6, .7, some discussion on basic instability will now be presented. Terms will be defined when they first appear.
The total change in total potential energy, $\Delta V$, for a system perturbing from a state of equilibrium, due to some virtual displacements, $\delta q$, can be expanded in Taylor series in the form

$$\Delta V = \delta^1 V + \frac{1}{2} \delta^2 V + O(\delta^3) \quad \hdots \quad (3-3)$$

where $\delta^1 V$ and $\delta^2 V$ represent the first and second variations in $V$, respectively. $O(\delta^3)$ includes all the terms of order three and higher. For a system with one displacement coordinate, $q_i$, (29)

$$\delta^1 V = \frac{\partial V}{\partial q_i} \delta q \quad \hdots \quad (3-4a)$$

and

$$\delta^2 V = \frac{\partial^2 V}{\partial q_i^2} (\delta q)^2 \quad \hdots \quad (3-4b)$$

Equations 3-3 and 3-4 led to the formulation of the well established axioms on the conditions of equilibrium and stability (15, 29, 37).

**AXIOM 1**: A necessary and sufficient condition for equilibrium is a stationary value of the total potential with respect to all generalized coordinates, i.e. equation 3-4a must vanish for all values of $i$,

$$\delta^1 V = \frac{\partial V}{\partial q_i} \delta q = 0 \quad \text{for} \quad i = 1, 2, \ldots n \quad \hdots \quad (3-5)$$

But $\delta q_i$ represents some arbitrary virtual displacements, therefore, the gradient $\frac{\partial V}{\partial q_i}$ should vanish,

$$\frac{\partial V}{\partial q_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots n \quad \hdots \quad (3-6)$$

**AXIOM 2**: A necessary and sufficient condition for stability of equilibrium is a complete relative minimum of the total potential, i.e. the second-order variation of the total potential, $\delta^2 V$, should be positive, i.e.,

$$\frac{\partial^2 V}{\partial q_i \partial q_j} > 0 \quad \text{and} \quad \text{for} \quad i = 1, 2, \ldots n \quad \hdots \quad (3-7)$$

and

$$\text{for} \quad j = 1, 2, \ldots n$$
Fig. 3-1 shows a graphical representation of this axiom for a system that is in equilibrium, with one and two displacement coordinate(s).

The total potential energy of a structure that is under the combined action of both membrane and lateral forces, can be expressed as a function of some generalized forces and some generalized displacements, equation 2-32. The second variation, expressed in the form of the Hessian matrix $H$, can be written as,

$$
\delta^2 V = \sum \left( \sum \frac{2V}{\delta q_i \delta q_j} \right) \delta q_i \delta q_j \quad \ldots \quad (3-8)
$$

$$
\delta^2 V = \delta q^T H \delta q \simeq \delta q^T (K + N) \delta q \quad \ldots \quad (3-9)
$$

where the matrix $K+N$ represents the effective stiffness matrix, as given in equation 2-48.

The second variation of the total potential energy, a quadratic form in $\delta q$, can be diagonalized in an infinite number of ways. One way of achieving this diagonalization is by means of orthogonal transformation,

$$
\delta^2 V = c_1 u_1^2 = c_1 u_1^2 + c_2 u_2^2 + \ldots c_n u_n^2 \quad \ldots \quad (3-10)
$$

where,

$$
u_1 = T^{-1} q_1 \quad \ldots \quad (3-11)$$
FIG. 3-1: Graphical Representation of $\delta^2 V$

- $\delta^2 V > 0$ stable equilibrium
- $\delta^2 V < 0$ unstable equilibrium
- $\delta^2 V = 0$ critical equilibrium
T is a non-singular transformation matrix relating the generalized coordinates, \( q_i \), to the 'principal' coordinates, \( u_i \); \( C_i \) are the 'stability' coefficients.

From this diagonalization, the following conditions are established:

1. If the quadratic form is positive definite (all \( C_i > 0 \)) then the equilibrium state is said to be 'thoroughly stable';

2. If the quadratic form is indefinite, admitting at least one negative \( C_s \), then the equilibrium state is said to be 'thoroughly unstable' with respect to the principal coordinate \( u_s \) (degree of freedom). The number of negative coefficients indicates the 'degree of instability';

3. If the quadratic form is positive semidefinite, then at least one \( C_s \) equals to zero and the equilibrium state is said to be critical with respect to the principal coordinate \( u_s \).

The stability coefficients can be represented by the diagonal elements of the decomposed stiffness matrix \( [K+N] \). Hence, a scan of the algebraic signs of the diagonal elements of such a matrix can give a clear indication of the nature of the state of equilibrium.

Arising from the results of the general theory of elastic stability, the forms of instability that may occur are Limit Point and Bifurcational, (Fig. 3-2). These are summarized as follows:

* reference to Thompson\(^{(37)}\) and Chilver\(^{(4)}\) can be made for more detailed exposition.
Fig. 3-2: Forms of Instability, plotted in Two-Dimensional Space

**2. Bifurcational**

- **e.g. Cylindrical shells**
  - Unstable symmetric
  - Stable symmetric

- **e.g. Plates and struts**
  - Unstable symmetric
  - Stable symmetric

- **e.g. Threefold frames**
  - Asymmetric

**1. Limit Point**

- Unstable path
- Stable path
a. Limit Point - in this form of instability the equilibrium path loses its stability on reaching a locally maximum value of $A$, some loading parameter. Examples: shallow arches and domes;

b. Bifurcational buckling occurs when the 'fundamental' path, emerging from the origin, loses its stability on intersecting another distinct and continuous equilibrium path, known as the Post-Buckling path. At the point of intersection, usually called a bifurcation point, an exchange of stability* occurs and instability follows (36, 37).

For bifurcating systems, the study of structural imperfections is of paramount importance, particularly for systems with unstable post-buckling paths. The effects of initial imperfections on the equilibrium paths are shown diagramatically in Fig. 3-3.

3.4 Iterative Techniques for Non-Linear Analysis

The present instability analysis is concerned with determining the critical local buckling load of thin-walled structures. The essence of non-linear behaviour associated with such a study prohibits the use of the classical linear methods of analysis. The so-called iterative techniques are very powerful methods of dealing with the solution of non-linear problems (1, 5, 11, 25). Although the rest of this chapter is devoted, directly or indirectly, to such methods of analysis, other methods have been adopted by some authors in solving non-linear structural problems. The energy search methods are examples of such different approaches (1).
Fig. 3-3 Effects of Initial Imperfections on Equilibrium Paths, plotted in Two-Dimensional Space.
A limited number of iterative techniques is available to the analyst in tracing the non-linear load displacement paths, incorporated in geometrically non-linear structural problems. In this thesis, the Newton-Raphson method has been employed in finding a solution to the non-linear problem of instability. The method is discussed in detail in the next section.

The role of the Newton-Raphson technique in non-linear analysis can be viewed as one method amidst a variety of related methods which are delineated as follows:

1. General incremental techniques:
   a. incremental approach,
   b. perturbation approach,
   c. initial value approach;

2. Self-correcting techniques:
   a. successive iteration approach,
   b. Newton-Raphson approach,
   c. modified Newton-Raphson approach,
   d. self-correcting incremental/modified Newton-Raphson approach,
   e. self-correcting initial value approach.

For a more detailed exposition of these techniques, reference can be made to the work of Benedetti et al.\(^1\), Haisler et al.\(^{11}\) and Oden\(^{25}\).

3.4.1 The Newton-Raphson Method

The Newton-Raphson method is a self-correcting technique of iterative analysis. The approach is suitable for problems with high degrees of non-linearity, yielding accurate results with good rates of convergence.
The method can be summarized as follows:

1. A set of external loads, $W_o$ (Fig. 3-4), is applied to the structure;

2. The geometric stiffness matrix $K$ of equation (2-48) is generated. At this state the instability matrix $N$, representing the stretching of the middle surface, is set to zero (a null matrix);

3. A set of boundary conditions and constraints are imposed on the structure and thence, a first estimate of the nodal displacements is obtained,

$$d_1 = K_o^{-1} W_o \quad \ldots \ldots (3-12)$$

where $K_o$ is based on the original geometry;

4. The membrane forces (stresses) are evaluated and used in generating the incremental stiffness matrix $N$;

5. The geometry of the structure is corrected with respect to the calculated displacements and hence, a new stiffness matrix $K$ is formulated;

6. Using the effective stiffness matrix $[K+N]$ and the nodal displacements, the structure's equilibrium is investigated by calculating the out-of-balance forces, $\Delta W$, given by,

$$\Delta W_i = \Delta W_i - \left[ K+N \right] \Delta d_i \quad \ldots \ldots (3-13)$$

(where $\Delta d_i$ as defined by equation 3-14);

7. The out-of-balance forces are applied to the structure and a set of 'incremental' displacements, $\Delta d$, is obtained. In general,
FIG. 3-4: The Newton-Raphson Iterative Technique.
\[ \Delta d_i = \left[ K + N \right]^{-1} \Delta W_i \quad \ldots \ldots (3-14) \]

8. The iterative process (steps 4-7 inclusive) is repeated, using the incremental displacements, the out-of-balance forces and the accumulated membrane forces (equation 3-18). The procedure is terminated if:

\[
\begin{align*}
\| \Delta d_i \| - \| \Delta d_{i-1} \| & \leq 0.0001 \\
\| \Delta d_i \| & \leq 0.0001 \\
\| \Delta W_i \| & \leq 0.0001
\end{align*}
\]  

Geometry Load

The value 0.0001 is chosen arbitrarily to represent a small quantity termed as zero. The notation \( \| v \| \) represents the Euclidean norm of a vector \( v \):

\[ \| v \| = (v^T v)^{\frac{1}{2}} \quad \ldots \ldots (3-16) \]

Throughout this iterative process, all the calculated quantities are accumulated and stored in the computer (section 3.7), i.e. at any iteration cycle \( i \), the displacement vector \( d_i \) is represented as:

\[ d_i = d_1 + \sum_{j=1}^{j=i-1} \Delta d_j \quad (i > 1) \quad \ldots \ldots (3-17) \]

Similarly, the membrane forces (stresses) can be represented as:

\[ f_i = f_1 + \sum_{j=1}^{j=i-1} \Delta f_j \quad (i > 1) \quad \ldots \ldots (3-18) \]

This makes it possible for the matrices \( K \) and \( N \) to be formed with respect to the latest geometry and total membrane forces.

Thus, the final results, nodal displacements and membrane stresses, represent the true solution for a structure that
a. is in a state of equilibrium;

b. has a geometry defined as

\[ G_f = G_o + d_f \]

where \( G_f \) and \( G_o \) are the final and original geometries, respectively, and \( d_f \) is the final displacements as defined by equation 3-17;

c. incorporates membrane forces as defined by equation 3-18.

3.5 Calculating the Critical Load

Equation 2-48 defines the basic matrix relationship between loads and displacements for a structure that is under the combined action of membrane and lateral forces.

For the particular case where the effects of geometric non-linearity are ignored, the stiffness matrix \( K \) remains constant whilst the matrix \( N \) undergoes a series of corrections with the increase in the intensity of the membrane forces and of the applied loading system. Hence, equation 2-48 becomes:

\[ \begin{bmatrix} K + \lambda N \end{bmatrix} d = W \]

where \( \lambda \) is a correction factor.

With such a relationship in hand, the critical load level can be estimated by an eigenvalue determination. And thus, the problem is reduced to a search for the smallest eigenvalue, \( \lambda \), for a non-vanishing \( d \) in the equation:

\[ (K + \lambda N) d = 0 \]

In the general case where the change in geometry is considered, both matrices, \( K \) and \( N \), are affected and hence, equation 2-48 becomes:
\[
\left[ \lambda_1 \mathbf{K} + \lambda_2 \mathbf{N} \right] \mathbf{d} = \mathbf{W}
\] ....(3-21)

where \( \lambda_1 \) and \( \lambda_2 \) are some non-linear correction factors.

To evaluate the critical load level, an iterative procedure based on the Newton-Raphson method has been employed. The procedure can be summarized as follows:

1. A set of external loads, well below the critical load, is applied to the structure. The Newton-Raphson iterative approach is employed and a state of equilibrium is attained;

2. The intensity of the applied load is increased by some increment. The structure, with the deformed geometry and the membrane forces, is analysed following the same iterative approach;

3. The determinant of the effective stiffness matrix \( [\mathbf{K+N}] \) and the degree of instability (section 3.3) are calculated at the end of each iterative cycle within the iterative process;

4. If
   a. \( |\mathbf{K+N}| > 0 \) and the matrix is positive definite, then the structure is in a state of stable equilibrium. Steps 2-4 inclusive, are repeated;
   b. \( |\mathbf{K+N}| \approx 0 \) then the applied loads represent a critical load level. The structure is in a state of incipient instability and hence, the iterative process is terminated,
   c. \( |\mathbf{K+N}| < 0 \) then the structure is in a state of unstable equilibrium. The intensity of the applied loads is reduced and the iterative process is repeated.
3.6 The Buckling Mode

An iterative procedure is suggested as an approach to determine the buckling mode of a structure that is in a state of unstable equilibrium, with a single degree of instability, neighbouring a state of incipient instability (critical equilibrium). Basically, the iterative scheme is directed towards finding an eigenvector, representing the buckling mode, that corresponds to the lowest eigenvalue. The suggested procedure is as follows:

1. The diagonal elements of the diagonalized stiffness matrix are scanned. A negative element corresponds to a critical degree of freedom, \( u_s \) (section 3.3);

2. A first approximation to the eigenvector is assumed to be a null vector, \( \mathbf{v}_1 \), except for one element, corresponding to \( u_s \), which is set to \( (1.0) \). The figure 1.0 is chosen arbitrarily as to give the vector a unit length;

3. An iterative process in the form of successive substitution is employed to yield better approximations to the eigenvector,

\[
\mathbf{v}_i = \left[ \mathbf{K} + \mathbf{N} \right]^{-1} \mathbf{v}_{i-1}
\]

... (3-22)

The process is terminated when the inequality

\[
0.999 \| \mathbf{v}_i \| < \| \mathbf{v}_{i-1} \| < 1.001 \| \mathbf{v}_i \|
\]

is satisfied. The values 0.999 and 1.001 are chosen arbitrarily.

3.7 The Computer Program

A computer program has been developed by the author for the local instability studies of thin-walled structures using
the finite element method of analysis. The program, written in ALGOL language and used with the ICL 1905F computer installed at the University of Surrey and with the ATLAS computer of the Science Research Council (was installed at Didcot, Berkshire), is based on the finite element method as described in Chapter 2 and Appendix 1 and incorporates the different iterative procedures described earlier in this chapter.

The program was applied to the instability analysis of flat plates, curved sheet panels, thin-walled box sections and cantilevered box girders. These applications are presented and discussed in the next chapter.

A descriptive flow chart for the program is given schematically in Fig. 3-5 and can be summarized as follows:

1. The basic information is read from an input set of data;

2. The matrix formulations of Appendix 1 are followed and the element stiffness matrices $K_e$ are generated;

3. The element stiffness matrices are, usually, formed in the elements own coordinate systems (usually called Local coordinate systems). Hence, a matrix transformation to some common coordinate system (usually called Frame or Global coordinate system) is carried out;

4. The structure's stiffness matrix is assembled. It is in the nature of the stiffness matrix to be square, symmetric and sparse*. Further, the sparsity in a

* A sparse matrix is a matrix with a large number of zero elements.
FIG. 3-5: The Computer Program—Flow Chart
stiffness matrix follows some uniformed pattern where the non-zero elements are grouped around the principal diagonal and hence, the matrix is called a 'banded' matrix. Fig. 3-6b shows a typical stiffness matrix for the hypothetical structure shown in Fig. 3-6a.

It is obvious then, that only half the banded matrix needs to be formulated and stored in the computer. The advantages with this type of formulation are more evident with larger stiffness matrices corresponding to larger and more complex structural systems;

5. The iterative procedures involved in calculating the critical load levels are followed. The convergence criteria are those given in equations 3-15. If the structure is in a state of unstable equilibrium with a single degree of instability, then the buckling mode is determined following the iterative procedure described in section 3.6;

6. The whole analysis is terminated when:

   a. the difference between two critical loads, determined at two different load levels, approach a small quantity termed as zero; and

   b. the ratio between the Euclidean norm of the total displacement vector, \( \delta_i \) (Fig. 3-7), for one full iterative cycle, corresponding to a certain load level, \( W_i \), and that of the total accumulated displacement vector, \( D_i \), approaches a small quantity termed as zero, i.e.

\[
\frac{\| \delta_i \|}{\| D_i \|} \leq 0.0001 \text{ (chosen arbitrarily)}
\]
Fig. 3-6a Some Hypothetical Mesh

Fig. 3-6b A Typical Stiffness Matrix, Where Only the Non-Zero Elements are indicated. (X).
FIG. 3-7: A Hypothetical Load-Displacement Trace.
Applications of the Finite Element Method to the Instability Analysis of Thin-Walled Structures

Chapter 4

4.1 Introduction

The finite element method, as described in chapters 2 and 3, has been employed in the instability analysis of a) thin flat plates, b) curved sheet panels, c) thin-walled box sections, as assemblages of flat plates and curved sheet panels, and d) thin-walled cantilevered box girders.

4.2 The Instability of Thin Flat Plates

Large number of structural systems can be constructed as assemblages of flat plate elements. Box girder construction, folded plate structures and even the idealization of curved continua by a number of finite flat elements, a common practice with the finite element method, are but a few examples demonstrating the versatility of thin flat plates.

A considerable amount of work has been carried out on the subject of 'buckling of flat plates', however, owing to the mathematical complexities surrounding the problem of instability, only a limited number of analytical solutions corresponding to specific cases, e.g. type of supports, loading, etc., are available. The finite element method can serve as an answer when such intricacies arise.

In this research work, the instability analysis of rectangular flat plates, uniformly compressed in one direction, and with varying boundary conditions, was investigated to check the accuracy of the computer program, developed by the author (section 3.7), against some non-linear problems whose closed form solutions are well established\(^{(15, 39)}\). Three different aspect ratios \(\gamma = \frac{a}{b}\) for the plate were considered:
\[ \gamma = 1 \text{ (square plate)} \]
\[ \gamma = 2 \]
\[ \gamma = 10 \text{ (long flat strip)} \]

Fig. 4-1. Rectangular plate uniformly compressed in one direction.

4.2.1 Simply Supported Square Plates Uniformly Compressed in One Direction

Fig. 4-3 shows four finite element models for a simply supported square plate. Symmetry of both geometry and loading has been considered and thus, only one quarter of the plate needs to be analysed. One has to remember that the advantages of symmetry can be employed in non-linear analysis only when the structure is known to behave or deform in a symmetrical fashion. The behaviour of arches, symmetrical with respect to both geometry and loading, is one example where symmetry should not be considered throughout the analysis (Fig. 4-2).

![Unsymmetrical Deformation](image_a)

![Symmetrical Deformation](image_b)

(a) Unsymmetrical Deformation  (b) Symmetrical Deformation

Fig. 4-2 Behaviour of Arches Under Lateral Loads.

The exact solution for a simply supported rectangular plate uniformly compressed in one direction is given by Timoshenko and Gere\(^{(39)}\) as,

\[ \sigma_{cr} = \frac{k \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{b} \right)^2 \]

\[ \cdots \cdots (4-1) \]

(for square plates \(k = 4.0\))
Fig. 4-3 Finite Element Models for a Simply Supported Square Plate.
Comparison between the finite element results and the exact solution is given in Table 4-1.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Finite element solutions, $\sigma_r$ psi</th>
<th>$K_{approx}$</th>
<th>$\frac{K_{approx}}{K_{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x 1</td>
<td>288.55</td>
<td>3.193</td>
<td>0.798</td>
</tr>
<tr>
<td>2 x 2</td>
<td>339.45</td>
<td>3.756</td>
<td>0.939</td>
</tr>
<tr>
<td>3 x 3</td>
<td>352.80</td>
<td>3.903</td>
<td>0.976</td>
</tr>
<tr>
<td>4 x 4</td>
<td>356.00</td>
<td>3.939</td>
<td>0.985</td>
</tr>
</tbody>
</table>

Table 4-1 Finite Element Solutions for a Simply Supported Square Plate Uniformly Compressed in One Direction ($\sigma_{cr}^{exact} = 361.524$ psi).

Fig. 4-5 illustrates the convergence of the finite element solution with the reduction in the mesh size.

Following the iterative procedure, described in section 3.6, the buckling mode was determined. Fig. 4-4 shows a typical buckling mode obtained for model (C) in Fig. 4-3.

Fig. 4-4 Buckling Mode of a Square Plate (3x3 Elements, Fig. 4-3).
Fig. 4-5: Convergence of the Finite Element Solution.

No. of elements

0.1
0.2
0.3
0.4
0.5
1.0

1
2
3
4
12
16

All sides simply supported.

One quarter is analysed
(no. of elements is that per quarter).
a Small Centrally Applied Internal Load on a Circular Pipe with Initial Imperfections

Fig. 4-6 Centrally Deflections of Square Plate Under Axial Compression Together With

\[ \times (1 \times 1) \]
\[ \odot (2 \times 2) \]
\[ \times (4 \times 4) \]

\[ \gamma = 0 \]
\[ \gamma = 0.05 \]
A perfect flat plate exhibits a stable-symmetrical point of bifurcation, a form of which is shown in Fig. 3-2. Hence, the presence of any initial imperfections should induce a behaviour similar to that shown in Fig. 3-3. Initial imperfections were incorporated in the finite element models in the form of a small lateral load acting at the centre of the plate. Fig. 4-6 shows the behaviour of the square plate under various degrees of imperfections. As anticipated, the deformations greatly increase as the critical load level is approached.

4.2.2 Rectangular Plates Uniformly Compressed in One Direction \((\gamma = 2)\)

The instability analysis of a flat rectangular plate, simply supported along the loaded edges, \(x = 0\) and \(x = a\), and with varying boundary conditions along the other two edges, has been investigated using the finite element model shown in Fig. 4-7.

![Finite Element Model for a Rectangular Plate](image)

**Fig. 4-7** Finite Element Model for a Rectangular Plate
\((a=20\ \text{in.}, \ b=10\ \text{in.}, \ t=0.1\ \text{in.}, \ E=10^6\ \text{psi}, \ \nu=0.3)\).

Table 4-2 gives the finite element results compared to the exact solution as given by Timoshenko and Gere\(^{39}\). No attempt was made to study the effects of reducing the mesh size of the model since the object of this exercise was to demonstrate the flexibility of the finite element method in such analyses. Fig. 4-8 illustrates the buckling mode of the particular case where all four sides are simply supported.
Fig. 4-8 Buckling Mode of a Simply Supported Rectangular Plate

Under Axial Compression
### Table 4-2 Finite Element Solutions for a Rectangular Plate Uniformly Compressed in One Direction (aspect ratio -2)

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Finite element solutions; ( \sigma_{cr} ) psi</th>
<th>( K(\text{eq.4-1}) ) ( K_{\text{approx}} )</th>
<th>( K_{\text{exact}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = 0 )</td>
<td>( Y = b )</td>
<td>( K_{\text{approx}} )</td>
<td>( K_{\text{exact}} )</td>
</tr>
<tr>
<td>SS</td>
<td>SS</td>
<td>291.00</td>
<td>3.220</td>
</tr>
<tr>
<td>SS</td>
<td>Free</td>
<td>59.60</td>
<td>0.659</td>
</tr>
<tr>
<td>Fixed</td>
<td>Free</td>
<td>115.20</td>
<td>1.275</td>
</tr>
</tbody>
</table>

* x=0, x=a simply supported
* S.S. = simple support

#### 4.2.3 Rectangular Strip Uniformly Compressed in One Direction \((y = 10)\)

As an extension to the work reported in the previous section, the instability analysis of a rectangular strip (aspect ratio of 10), simply supported along the two loaded edges and free along the other two boundaries, was examined with the use of the finite element models shown in Fig. 4-9.

Comparison between the finite element solutions and the exact one, calculated by treating the flat strip as a hinged strut, is given in Table 4-3.

<table>
<thead>
<tr>
<th>Model</th>
<th>Finite element solutions; ( P_r ) lbs.</th>
<th>( P_{\text{approx}} ) ( P_{\text{exact}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>120.05</td>
<td>1.216</td>
</tr>
<tr>
<td>B</td>
<td>99.80</td>
<td>1.011</td>
</tr>
<tr>
<td>C</td>
<td>99.18</td>
<td>1.005</td>
</tr>
<tr>
<td>D</td>
<td>98.52</td>
<td>0.998</td>
</tr>
<tr>
<td>E</td>
<td>99.80</td>
<td>1.011</td>
</tr>
<tr>
<td>F</td>
<td>98.52</td>
<td>0.998</td>
</tr>
</tbody>
</table>

\( P_{\text{exact}} = \frac{\pi^2 EI}{a^2} \)

\( = 98.696 \text{ lbs.} \)

Table 4-3 Finite Element Solutions for a Rectangular Strip (aspect ratio 10)
a = 10 in; b = 1 in; t = 0.22894 in; E = 10^6 psi; \nu = 0.3

Fig. 4-9 Displacement Models for a Rectangular Strip a/b = 10, Treated as a strut.
Fig. 4-10: Convergence of a Rectangular Strip $a/b = 10$ (Fig. 4-9) Under Axial Compression.
Fig. 4-11 Convergence of Central Deflections of a Rectangular Strip (a/b = 10) Under Axial Compression with Initial Imperfections.
A study was carried out, using model (C) in Fig. 4-9, to investigate the effects of the relative positionings of the starting load level, in the iterative technique described in section 3.5, to the final converged solution. Fig. 4-10 illustrates the results of this study.

To simulate initial loading imperfections, two small lateral loads were applied to the plate at the nodal points A and B in Fig. 4-9c. As anticipated, the results of this analysis, presented in Fig. 4-11, resembled those obtained for a simply supported plate, Fig. 4-6.

4.3 The Instability of Curved Sheet Panels

The instability analysis of curved sheet panels, forming a segment of a shallow cylindrical shell, Fig. 4-12, simply supported along the four edges, was investigated. The cross-sectional dimensions of the panel are meant to represent the curved flanges of the tested box units shown in Fig. App 2-1.

The theoretical instability analysis based on shell theories (39) yields the following equations, to be used in calculating the critical buckling stress of such panels,
if, \[ \beta R \geq 2\pi \sqrt{\frac{R^2 - l^2}{12(1-\nu^2)}} \] \[ \cdots \cdots (4-2) \]

then, \[ \sigma_{cr} = \frac{E t}{R \sqrt{3(1-\nu^2)}} \] \[ \cdots \cdots (4-3) \]

= buckling stress of cylinders under axial compression.

else, \[ \sigma_{cr} = \frac{\pi^2 E t^2}{3(1-\nu^2)(\beta R)^2} + \frac{E \beta}{4 \pi^2} \] \[ \cdots \cdots (4-4) \]

where the first part of the equation defines the buckling stress of a flat plate under axial compression; the second part, \( \frac{E \beta}{4 \pi^2} \), represents the increase in stiffness due to the curvature of the panel.

It is assumed that the present model has the following properties:

\[ \beta = \frac{\pi}{6} ; \quad R = 6.3438 \text{ in.} ; \quad t = 0.0625 \text{ in.} ; \quad L = 12 \text{ in.} ; \]

Modulus of elasticity = \( \frac{0.4085 \times 10^6}{6} \) psi.

Poissons ratio = 0.35.

With such geometrical and mechanical properties, the critical buckling stress for the panel is that given by equation 4-3, yielding a value of 2480.495 lbs per sq.in. (\( P_{cr} = 514.95 \) lbs.) for the theoretical buckling stress.

The idealized finite element models are shown in Fig. 4-13. It is assumed that the panel displays a symmetrical behaviour, i.e. deformations, and hence only one quarter of the panel was considered.

The finite element solutions are given in Table 4-4. Fig. 4-14 shows these results in a graphical representation illustrating the effects of the so-called idealization errors (section 2.2) on the converged solution.
Fig. 4-13 Finite Element Models (shown in developed-plans).
Fig. 4-14 Convergence of the Buckling Load of Axially Compressed Cylindrical Shells.

No. of elements: 30  20  10

No. along the curved edge: 3  2  1

Approx. (b)

Scale (a)

1.0

0.5

Scale (b)

Scale (a)
The longitudinal stress distribution, \( \sigma_y \), throughout the panel, prior to buckling are shown in Figs. 4-15 ... 4-17.

Fig. 4-18 presents a buckling mode for model (C), Fig. 4-13, superimposed on the plots showing the growth of deformations, along the central line and axis of symmetry Al-Al (Fig. 4-13), prior to buckling.

<table>
<thead>
<tr>
<th>Model</th>
<th>Finite element solutions; ( \sigma_{cr} ) psi</th>
<th>( \left( \frac{\sigma_{\text{approx}}}{\sigma_{\text{exact}} \sigma_{cr}} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>610.625</td>
<td>0.246</td>
</tr>
<tr>
<td>B</td>
<td>1971.643</td>
<td>0.795</td>
</tr>
<tr>
<td>C</td>
<td>2285.833</td>
<td>0.922</td>
</tr>
</tbody>
</table>

Table 4-4 Finite Element Solutions for a Curved Sheet Panel.
\( \sigma_{\text{exact}} \sigma_{cr} = 2480.495 \) psi
Stress Distribution in Model B (Fig. 4-13)

<table>
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</tr>
</tbody>
</table>

*Fig. 4-15 Stress Distribution in Model A (Fig. 4-13)*
<table>
<thead>
<tr>
<th>2248</th>
<th>2249</th>
<th>2250</th>
<th>2251</th>
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<th>2255</th>
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<td>2293</td>
<td>2294</td>
<td>2295</td>
<td>2296</td>
<td>2297</td>
</tr>
</tbody>
</table>

**Fig. 4-17** Stress Distribution in Model C (Fig. 4-13)
Under axial compression, buckling mode of the center generator of a curved sheet panel.
4.4 The Instability of Thin-Walled Box Sections

Structural systems formed as assemblages of flat plate elements have been of great interest to engineers owing to their versatility in adapting themselves to various engineering forms of construction. The particular case of box sections is of special interest due to the wide use of such structural sections, especially in the field of bridge engineering where box girder bridges have attracted the attention of a large number of structural and bridge designers.

The probable failure criterion for these highly efficient structural forms is of instability. Although individual components, in isolation, may be found to be stable, the guarantee of the overall stability can only be obtained if the interaction of the components is not retrograde.

The classical assumptions regarding thin-walled structures, when submitted to end compression, can be summed up as follows (3):

1. The common longitudinal edges of the plate elements remain straight;

2. The angles between the adjacent plate elements remain constant during buckling;

3. The number of buckles, which occurs in all plate elements simultaneously, is the same.

In effect, these assumptions simulate support condition along the common longitudinal edges of the plate elements and that the buckling load of the whole section is governed by that of the individual plate elements. Based on these assumptions, Fig. 4-19a shows the classical buckled form of a square box section whilst Fig. 4-19b shows the governing plate element.
Although such assumptions have given plausible results for a number of cases, including square and rectangular box sections, one should not generalize the validity of such assumptions to every thin-walled structure. Two particular cases are studied here. The case of thin-walled box section is investigated in this section and that of a cantilevered box girder in section 4.6.

The local buckling behaviour of a box section, formed as an assemblage of flat plates (constituting the webs) and curved sheet panels (constituting the flanges), Fig. 4-19, was examined by the finite element method of analysis where no assumptions were made regarding the behaviour of the longitudinal edges. The study was supplemented by an experimental programme which is reported in the next chapter.

Although such box sections can display either symmetrical or unsymmetrical form of behaviour, symmetry has been considered in building up the finite element models shown in Fig. 4-20. The accuracy of such models can be deduced from the results obtained for both flat plates and curved sheet panels with similar meshes (Figs. 4-3 and 4-13). There the accuracy of solutions was found to be in the region of 80% and ameliorating with finer meshes. Hence, one may expect similar degree of accuracy with the models shown in Fig. 4-20.
FIG. 4-20 FINITE ELEMENT MODELS.
The critical load levels, as obtained from the finite element analyses, are tabulated in Table 4-5; a comparison with the exact theoretical buckling loads for simply supported rectangular plates and curved sheet panels are included*. The stress distributions, evaluated at load levels neighbouring the critical ones, are given in Fig. 4-21 ... 4-25.

<table>
<thead>
<tr>
<th>Model</th>
<th>Finite element solutions; $\sigma_{cr}$ psi</th>
<th>$(\sigma_{exact})_{cr}$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1992.00</td>
<td>0.803 0.757</td>
</tr>
<tr>
<td>B</td>
<td>1803.95</td>
<td>0.727 0.685</td>
</tr>
<tr>
<td>C</td>
<td>1961.40</td>
<td>0.791 0.745</td>
</tr>
<tr>
<td>D</td>
<td>1704.00</td>
<td>0.687 0.647</td>
</tr>
<tr>
<td>E</td>
<td>1691.00</td>
<td>0.682 0.642</td>
</tr>
</tbody>
</table>

Table 4-5 Finite Element Solutions for a Box Section.

Figs. 4-26 ... 4-29 show the determined buckling modes of the box sections superimposed on plots showing the displacement growths prior to buckling.

A comparison between this analytical study and the experimental results is drawn in the next chapter (section 5.5).

4.5 Discussion

Results have shown that the finite element method of analysis, together with the non-linear iterative techniques, discussed in the earlier chapters, can be used successfully to investigate non-linear problems of instability.

The convergence of the solutions with respect to the mesh size is in agreement with the findings of Kapur and Hartz (15).

* The effects of the ends boundary conditions are not considered owing to the fact that such effects extend only to a distance equal to the width of the flat plate, as in the case of plate elements, and to a distance equal to $\sqrt{R.t}$, as in the case of curved panels (7).
\[ \Delta \sigma = 1991.3 \text{ psi} \]

(Pt. A - 20') Buckling in Model, A', (t=6 in.)

Stress distribution prior to

Pt. A - 21
$P_t = 1.717 \text{ ksi}$
$\Delta P = 1688.8 \text{ psi}$

Fig. 4-20: Buckling in model $D_1$ (T=12 ft)

Fig. 4-24: Stress distribution prior to buckling
\[ P = 777 \text{ psi} \]
\[ t = 1689 \text{ psi} \]

**Figure 4-20** Stress Distribution Prior to Buckling

(Height of 12 feet, E, \(I = 12 \text{ in}^3\))
Fig. 4-26 Behaviour of a 6 in. Box Section Under Axial Compression.

A. Centre line of the curved flange.
B. Centre line of the flat web.
Fig. 4-27 Behaviour of a 9 in. Box Section Under Axial Compression (Model C, Fig. 4-20).

A. Centre line of the curved flange.
B. Centre line of the flat web.
Fig. 4-28 Behaviour of a 12 in. Box Section Under Axial Compression (Model D, Fig. 4-20).

A. Centre line of the curved flange.
B. Centre line of the flat web.
Fig. 4-29 Behavior of a 12 in. Box Section Under Axial Compression (Model B, Fig. 4-20).

(a) Center line of the flat web.
(b) Center line of the curved flange.

---

0 = 240
0 = 160
0 = 80 lbs

---

Buckling mode (10 ft).
Prior to buckling (10 ft).
Growth of deformations.
Figs. 4-4 and 4-14 show that convergence to the final solution, buckling load, was from below rather than from above as expected with displacement functions. The explanation of this behaviour was discussed earlier in section 3.2.2.

The results of the analytical study on the effects of initial imperfections on the behaviour of flat plates, for example (Figs. 4-6 and 4-11), are in agreement with the discussion given in section 3.3 and shown diagramatically in Fig. 3-3.

The iterative technique suggested for determining the buckling mode yields reasonable results especially when compared to the well established buckling modes of flat plates and cylindrical shells under axial compression. It has to be remembered that the elements of the eigenvector, as obtained by the suggested iterative technique, (section 3.6), represent the buckled configuration to an undetermined scale.

A discussion on the instability analysis of thin-walled box sections is given in chapter 6.

4.6 The Instability of Cantilevered Box Girders

Although such structural forms have been adopted by a large number of bridge engineers, as in some erection procedures in the construction of long span box girder bridges, little work has been done to study the global stability of the box structure. Past experience has indicated that structures which are assemblages of thin plate elements begin to adopt the form of thin shell structures and as such may inherit the undesirable buckling characteristics of such configurations (section 3.3). The present design procedure of checking the stability of individual plate elements and stiffened panels is, therefore, inadequate and a final check should be made of the global stability of the box structure.
The inadequacy of the present design procedure came to light following the recent collapse of the Milford Haven bridge in Great Britain, the Koblenz bridge over the Rhein, and the Lower Yarra bridge in Australia and the partial collapse of the Fourth Danube bridge in Austria, during construction. Studies to establish the cause of these failures indicated the presence of high working stress levels, due to a number of imperfections during construction, which abrogated the normal factor of safety. These working stresses were high enough to cause local buckling in some stiffened panels. Such local instability contributed to the total or partial collapse of the whole box girders.

Such is the need for a detailed study of the over-all stability of cantilevered box girders. The author's contribution, thought to be the first of its kind, is presented in this section where the developed finite element program (section 3.7) is used for the non-linear analysis of a cantilevered box girder, with and without initial imperfections. An introduction to such a study is given in the following two paragraphs.

An idealized box is shown in Fig. 4-30a. The live and dead loads of the cantilever can, for this discussion, be represented by a generalized loading system, defined by the parameter $A$, Fig. 4-30 and hence, the most highly stressed region which is the region of interest occurs near the extreme support*. Buckling of the web plates does not necessarily imply collapse since diagonal tension field action can maintain the structural integrity of these components. However an associated or subsequent instability of the support diaphragm and lower plate produces a collapse situation as shown in Fig. 4-30b. Noting that the proposed buckling deformation is

* The region of interest could equally well be at a section where abrupt changes in plate thicknesses occur.
Fig. 4-30

(A) Cantilevered Box
(B) Possible Collapse Mechanism

region of interest
highly localized we may, as a tentative first step, obtain some idea of the type of instability by considering a simple equivalent pin-jointed rigid-link model\(^{(22)}\), representing the region of interest, Fig. 4-31.

Such models have been successfully used in the past to indicate the post-buckling of columns and individual plates. The bending stiffnesses of the diaphragm and bottom flange are represented by linear angular springs and the axial load in the bottom flange is taken as being proportional to the axial load in the diaphragm. Analysis of the model reveals stability characteristics of the shell type and further reveals an unstable post-buckling equilibrium path. Hence, any initial imperfection produces a loading curve which reaches a maximum load value well below the critical value of the perfect model, Fig. 3-3. A detailed study of models of this type was performed by Nay\(^{(22)}\).

A more realistic representation of the box girder is given by the finite element model shown in Fig. 4-32. The model represents one half of a rectangular cantilever box girder, width to depth ratio of 2 and length to depth ratio of 8, with two end diaphragms. A finer mesh was used near the support, the region of interest, to yield better approximations to the final solutions, both nodal displacements and elements stresses. The applied load was meant to describe a uniform load distributed over the top flange and assumed to be acting along the edge B-B', Fig. 4-32. Throughout this study, this loading system is defined by a single parameter, \(A\). The boundary condition, given in Fig. 4-33, was meant to represent that at the region of interest, shown in Fig. 4-30.

Thus, the finite element study corresponded to one specific cantilevered box girder with a particular geometric configuration, loading system and boundary condition. Such a structural system is referred to, in this study, as the actual model*.

* The terms 'perfect' and 'imperfect' models can be misleading in such a study.
Fig. 4.31: Form of instability
Fig. 4.32: Finite element mesh.

walls thickness = 0.0417 in.

free-end diaphragm
The finite element analysis yields a bound to the critical load, \( \Lambda_{cr} \), that induces some form of instability in such a model. The bound was found to be:

\[
197.2 \text{ lbs.} < \Lambda_{cr} < 198.8 \text{ lbs.}
\]

Fig. 4-34 shows the longitudinal stress distribution, \( \sigma_y \), throughout the box girder prior to the state of instability.

The deformations of the free-end with the increase in the load intensity and the over-all deformation of the bottom flange are shown in Figs. 4-35 and 4-36, respectively. The buckling mode was investigated and is shown in Figs. 4-37 and 4-38.

### 4.6.1 Initial Imperfections

Initial imperfections can be an important factor in the design of box girder bridges. Nay's study on rigid-link model (22), described earlier in section 4.6, showed that such structures are highly sensitive to initial imperfections, owing to the nature of their post-buckling equilibrium paths.
THE END-DIA-PHAGRAMS ARE NOT SHOWN.

Poisson's ratio 0.33

GIRDER

BENDING IN A CANTILEVERED BOX

FIG. 4.34 STRESS DISTRIBUTION PRIOR TO

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Fig. 4-35 Lateral Applied Load vs. Vertical Deflections at the Free-End.
Fig. 4.36: Vertical deflections—bottom flange

\(\text{Def} \cdot D \text{ in Fig. 4.32)}\)
Fig. 4-37 BUCKLING MODE FROM FINITE ELEMENT STUDY
FIG. 4-38 BUCKLING MODE FROM FINITE ELEMENT STUDY.
Prior to the publication of the Merrison report, bridge designers used to refer to B.S. 153 for the design of box girder bridges, although the B.S. 153 is meant for plate girder and not box girder bridges. Hence, the recommendations given in B.S. 153 were followed in choosing the different plate elements thicknesses without any separate study of the effects of initial imperfections*.

The Merrison report is more conservative in its recommendations in the design of box girder bridges. This is not surprising since the report is meant entirely for such bridges. And hence, more attention is given to the effects of initial imperfections.

In the present study, the effect of initial imperfections on the general and global behaviour of the cantilevered box, as described in the previous section, was investigated. The imperfections were in the form of initial applied moment, acting at point (D')†, Fig. 4-32. The initial moment was applied as a function of the total root moment, i.e.,

\[ M_{\text{applied}} = \alpha M_{\text{root}} \]

where \( \alpha \) ranged between 0.001 to 0.01, both in the positive and negative directions. Fig. 4-39 illustrates the sensitivity of the cantilever to the presence of this form of initial imperfections. The growth of deformations, prior to instability, of critical degrees of freedom, as indicated by the computer program (section 3.3 and 3.7), for various degrees of imperfections, are shown in Figs. 4-40 ... 4-44.

---

* B.S. 153 includes the effects of initial imperfections in the recommended allowable stresses for various cases.
† Initial imperfections in the form of geometrical imperfections were studied but in these instances problems were experienced in that the basic iterative technique tended to 'lose' such imperfections en route to convergence. The use of equivalent load imperfections obviated this difficulty.
CANTELEVERED BOX GIRDER Fig. 4-32.

Fig. 4.39: Imperfections Sensitivity.
Fig. 4.40: Growth of deformations.

\( \theta_x \) (Prior to buckling)

\( \varepsilon = 0.001 \) to 0.010
Fig. 4.1: Growth of deformations.

Joint b, Fig. 4.32

Prior to buckling.
Fig. 4.4.2: Growth of deformations.

θ4 Prior to buckling.

Joint (a): Fig. 4.32
4.6.2 Discussion on the Instability of Cantilevered Box Girders

A study of the general behaviour of the cantilever indicates that under load, the bottom flange tends to bend upwards at the root, Fig. 4-36, until a local maximum value in the load parameter \( \lambda \) is reached and a snap situation occurs. This snap situation is in the nature of most imperfect systems.

The general stability theory and the findings of Nay\(^{(22)}\) predict sensitivity of such structural systems, cantilevered box girders, for a form of bifurcational type of instability. Hence, the application of any initial imperfections can have a major effect on the critical load levels. This is shown diagramatically in Fig. 3-3.

The present investigation indicates that for the particular box girder studied, with the assigned boundary condition, type of imperfections and their point of application, some variations in the critical load level do occur. This sensitivity is illustrated in Fig. 4-39 by the shallow curve representing the drop in the critical load with the increase in the degrees of imperfections. More drastic effects on the critical load level could correspond to, perhaps, other types of boundary conditions, other types of load imperfections, etc. This could be an interesting topic for further research.

Fig. 4-39 indicates that the actual or so called perfect model exhibits a load-equilibrium configuration of the limit type. Application of initial moments at the root gives a family of curves, one of which displays a bifurcational form of instability.

Such a behaviour is well established for flat plate instability analysis. This is demonstrated diagramatically in Fig. 4-45.
If one considers the plate shown in Fig. 4-45b as that of a perfect case, then the effect of any restraining moments tends to induce a membrane state of stress leading to bifurcational type of instability.

Going back to the cantilevered box where an unstable form of bifurcation is expected to arise, Fig. 4-39 can now be studied in the light of the preceding discussion. The classical form of an unstable symmetric point of bifurcation can be shown as,
Similar structural systems but with some built-in initial imperfections can now be represented by the equilibrium path corresponding to the classical path for an imperfect system. And hence, any applied imperfections acting against the initial imperfections can lead to a membrane state of stress, i.e. pushing the equilibrium path towards that of a classical perfect system, Fig. 4-48.

The different plots showing the growth of deformation, prior to buckling, of some critical degrees of freedom, Figs. 4-40 ... 4-44, give some indication of the general
behaviour. But care should be taken in interpreting such 2-dimensional plots when considering the general behaviour of the whole structure in an n+1 dimensional space. Reference to Supple\(^{(31, 32)}\), Supple and Chilver\(^{(33)}\) and Thompson\(^{(37)}\) can be made for more detailed exposition.

The problem of large initial imperfections is more complex than that of small imperfections. This is due, partly, to the fact that such imperfections can create a completely different structural system and hence, a complete change in the behaviour pattern. This might explain the rise in the critical load level, Fig. 4-39; at higher degrees of imperfections. The effects of such imperfections on a 'perfect' structural system is worth following and as such is recommended for future research work.

The foregoing discussion clearly shows that any imperfections-sensitivity of the shell-type may erode the apparent factors of safety assumed in ignorance of these effects.
Chapter 5

An Experimental Study on the Local Instability of Thin-Walled Box Sections

5.1 Introduction

The versatility of the finite element method, when applied to problems of non-linear analysis, was demonstrated in the previous chapter where the method was applied to the local instability studies of various thin-walled structures. One example of such structures was the thin-walled box section examined in section 4.4. There the box section, formed as assemblages of curved sheet panels and thin flat plates, was an idealized version of an actual box section, Fig. App 2-1; the finite element idealization disregarded the protruding 'lips' of the actual box section.

The forementioned box sections are commercially produced structural units, made of polyvinylchloride (a plastic material), which by virtue of their protruding lips can be interconnected to form multi-cell panels.

A study on the mechanical properties of the plastic material was carried out and is reported in Appendix 2. As a result of this study, the following properties were determined:

Modulus of Elasticity ........ 0.4085 x 10^6 psi
Maximum Tensile Strength ...... 7540.0 psi
Poissons Ratio ................. 0.35

5.2 The Test Units

Tests were carried out on such box units, varying in lengths from 6 in. up to 56 in. (slenderness ratio 6.27 - 58.55) to study, experimentally, their local buckling behaviour when submitted to end compression. To secure uniform distribution
of the applied compressive load to the units cross-sections, the boxes had their machined ends fixed to ½ in. steel end-plates. Hence, the columns boundary condition was assumed to be approaching that of fixed end columns.

Interaction between local and overall Euler-type instability modes is possible in compressed thin-walled columns, if sufficiently long (38,46). The longer units were studied closely to detect the possible presence of such interaction. The findings of this study are discussed in section 5.4.

5.3 Test Procedure

The axial compressive load was applied through the Instron machine shown in Fig. 5-1. A test unit is centred on a load cell, which is bolted to the base of the machine. The load is applied to the top end of the unit through the Instron's crosshead; the load cell reactive forces represent the load applied to the bottom end of the unit.

The intensity of the applied load was plotted against the shortening of the units on a chart plotter synchronized with the movement of the cross-head.

The lateral deformations of the test unit were measured by means of mechanical dial gauges (0.0001 in. per division), mounted on a special rig enabling one to observe and measure the deformations along the four sides of the box simultaneously. And thus, the deformations prior to buckling were recorded and so were the buckling modes.

5.4 Comments and Experimental Results

The local buckling load of a test unit was taken as the load at which sizable surface waves first appeared along the curved sheet panels. Such an appearance was usually accompanied
Fig. 5-1  A Typical Test Unit Under Axial Compression.
by a 'banging sound'. This is in the nature of shell buckling owing to their non-linear behaviour\(^{(46)}\). At this load level the tested unit displayed a loss of energy which was demonstrated by a drop in the load intensity, as recorded on the chart plotter. Fig. 5-2 illustrates such a behaviour.

The shorter units (L ≤ 18 in.) tended to buckle into a definite configuration with a definite number of half wavelengths (buckles) and thus, with an increase in the applied load the same mode of deformation and the number of half wavelengths remained the same. On the other hand, longer units tended to buckle into a definite configuration with a certain number of half wavelengths, but with an increase in the applied load, a second and, in some cases, a third mode of deformations was obtained (Figs. 5-11 to 5-14). In general, the buckling patterns tended to approach a sinusoidal configuration as in the cases of flat plates and cylindrical shells under axial compression.

Supple\(^{(31)}\) has investigated the mechanics of the change in buckle pattern in the post-buckling regime. The general conclusion being that changes are brought about by the non-linear interaction of buckling modes. The local critical loads for long plate units are much closer together than those for short units and so one would expect changes of buckle form to be more probable for the longer units. As stated earlier, this was indeed observed.

The tests were in agreement with the findings of Fung and Sechler\(^{(7)}\) on the instability of thin elastic shells. Fung and Sechler state ".... it is possible to buckle a cylinder, stop the test, completely release the loading and let the cylinder return to an unstrained condition. Most likely some visible marks of initial buckles will remain on the cylinder. Now, upon reloading the cylinder, quite often it is found that new buckles appear and develop at places other than the old locations ....". This behaviour was noted in the work carried out in this thesis. Figs. 5-11 to 5-14 demonstrate this agreement.
Fig. 5-2 Axial load vs. end-shortening of the test units.
Although the local buckling stresses, as given in Table 5-1, were lower than the Euler buckling stresses, calculated for fixed end columns, Fig. 5-19 indicates the possible presence of an interaction between local and overall buckling. Such an interaction is possible in the presence of some initial imperfections in, both, the structural components forming the box unit (local imperfections) and/or in the alignment of the column axis. Furthermore, the reduction in the stiffnesses of the columns due to local buckling can be, in some cases, high enough to induce an Euler type of instability.

5.5 Experimental vs. Analytical Results

The results of the experimental study are tabulated in Table 5-1. Based on the comparison for individual plates and curved panels, the mesh sizes used here in the finite element models were expected to yield solutions with, about 80% degree of accuracy. However, owing to the perplexities surrounding a) the buckling behaviour of shell-type structures, b) the boundary condition of the longitudinal edges of the flat plates and curved sheet panels, forming the box sections, c) the effects of the protruding 'lips' which were not included in the finite element models, and d) the non-linear relation between buckling stresses and the elements thicknesses, it is, therefore, unwise to endeavour into a direct comparison between the experimental and the analytical results. Nevertheless, the finite element solutions, for the cases investigated, together with the corresponding experimental results are reproduced in Table 5-2. Consequently, Table 5-2 should be viewed as a confirmation of what was discussed earlier in section 4.4 regarding the care that must be taken in interpreting the local buckling stress of a whole section based on an instability study of the individual elements, forming the section.

It is worth noting that while the buckling formulae for flat plates can be used in calculating the buckling loads of box sections, formed from joining a number of flat plate
elements, the introduction of curvature introduces the instability phenomenon which, by the general theory of elastic instability, introduces the concept of imperfection sensitivity which is closely related to its post-buckling behaviour\(^4, 37\), Fig. 3-3. Hence, more scatter in the results is expected. This, as Table 5-1 shows, was indeed observed.

Furthermore, past experience has shown that for thin-walled columns with an Euler load much higher than the local buckling loads of the components forming the section, failure loads can be lower than the local buckling loads\(^46\). Hence, more scatter in the results.

Further to the forementioned discussion, one may include other factors on which buckling in general may depend, namely:

1. The initial imperfections that make the behaviour of an actual column different from an ideal one:
   a. unavoidable eccentricity in the loading,
   b. initial curvature of the column,
   c. non-homogeneity of the material;

2. The rigidity of the testing machine.

The experimental buckling modes of the box columns are shown in Fig. 5-11 to 5-14 whilst the analytical ones, for the cases studied in section 4.4, are shown in Figs. 4-26 to 4-29. Any comparison between the experimental and theoretical modes should be studied in a qualitative form and not in a quantitative form. This follows from the nature of the iterative procedure suggested in section 3.6 to determine the buckling modes of thin-walled structures, i.e. the mode amplitude has an undetermined multiplier. However, the general
forms of the buckling modes observed in the experimental programme were predicted by the finite element theoretical analysis. In particular, we might refer to the large amplitude of deformation at the mid-length of the shorter specimens.
### TABLE 5-1
EXPERIMENTAL RESULTS

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* equation 4-3

* equation 4-1  b=1.57 in

* treated as fixed-end columns
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**Table 4-5**
Fig. 5-3 Behaviour of a 6in. box column.
(specimen 0-6-1)
Fig. 5-4 Behaviour of a 6in. box column (specimen 0-6-2)
Fig. 5-5 Behaviour of a 6in box column (specimen 0-6-3)
Fig. 5-6 Behaviour of a 12 in box column
(specimen 1-0-1)
Fig. 5-7 Behaviour of a 12 in box column (specimen 1-0-2)
Fig. 5-8 Behaviour of a 12 in box column  
(specimen 1-0-3)
Fig. 5-9 Buckling mode for an 18in box column
(specimen 1-6-1)
Fig. 5-10 Buckling mode for an 18 in box column (specimen 1-5-2)
Fig 5-11 BUCKLING MODES FOR A 24 in BOX COLUMN. (specimen 2-0-1)
Fig. 5-12a  BUCKLING MODE FOR A 36 in. BOX COLUMN  
(specimen 3-0-1)
Fig. 5-12b  A SECOND BUCKLING MODE FOR SPECIMEN 3-0-1
Fig. 5-13  BUCKLING MODES FOR A 48 in BOX COLUMN
(specimen 4-0-1)
Fig. 5-14 BUCKLING MODES FOR A 56in BOX COLUMN
(specimen 4-8-1)
Fig. 5-15 A Buckling Mode for a 6 in. Box Column Under Axial Compression

Fig. 5-16 A Buckling Mode for a 12 in. Box Column Under Axial Compression
Fig. 5-17 A Buckling Mode for a 24 in. Box Column Under Axial Compression
Fig. 5-18 A Buckling Mode for an 18 in. Box Column Under Axial Compression

Fig. 5-19 A Buckling Mode for a 48 in. Box Column Under Axial Compression
Fig. 5-20  A 56 in. Box Column Under Axial Compression (general set-up)
Concluding Remarks

A recapitulation of the findings of the present work is given in this chapter:

The results of this study illustrate the power and potentiality of the finite element method of stress analysis when applied to non-linear structural analyses, an example of which is the instability analysis of thin-walled structures.

It is not the aim of this research work to develop the finite element technique in the sense of developing or suggesting a new geometrical element or some element functions. Hence, the chosen displacement functions, used in this work, are those which have been suggested by a number of researchers and found to be suitable for this type of study.

A discussion on the limitations of these functions is given in sections 3.2.1 and 3.2.2. The accuracy of the functions has been demonstrated by calculating the critical buckling loads of thin flat plates and curved sheet panels, submitted to axial compression. The results of this investigation indicate that plausible results with high degrees of accuracy can be obtained with such functions (Chapter 4).

An observation, regarding the bounding of the solutions, is worth noting here. Although assumed displacement functions tend to converge from above towards the true buckling loads, the results of this investigation and the findings of Kapur and Hartz (15), on similar studies, indicate convergence from below. This is in the nature of non-compatible displacement functions, owing to the associated inter-element boundary condition (section 3.2.2).

Having realized the versatility of the finite element method in stress analysis, a large number of theoreticians have suggested some compatible displacement functions (21, 44).
Hence, the introduction of such functions into this type of investigation can be recommended as a rigorous check, and, hopefully, an improvement on the method employed herein.

Furthermore, the chosen displacement functions (used in this thesis) yield five degrees of freedom per nodal point of an element in its own coordinate system; the in-plane rotation, $\theta_z$, has been omitted. A study on the effects of $\theta_z$ on the final solutions might be of value.

The computer program, with the incorporated iterative techniques, has been found to be well behaved in all the problems to which it has been applied. The convergence criteria are, in a sense, arbitrarily chosen by the author. Hence, it is possible that some other criteria could be suggested to improve the rate of convergence.

One of the main applications of the computer program has been to the instability analysis of thin-walled box columns to investigate the 'validity' of the classical assumptions, regarding the boundary condition along the longitudinal edges of the structural components, forming the four faces of the box section. The classical assumptions treats such boundaries as being of simple support condition and hence, the problem of determining the critical buckling load of the whole box columns is reduced to an instability study of the individual structural components.

The analytical results for the studied box section, Fig. 4-20, yield critical load levels lower than those for, both, the flat webs and the curved sheet panels, taken in isolation and assumed to be simply supported along the four edges. One may associate such variations to the finite element models in representing an actual box section; the accuracy of the displacement models, given in Fig. 4-20, was merely a 'guess' following the instability studies on flat plates and curved sheet panels, with similar finite element mesh sizes (section 4.2 and 4.3).
However, past experience and the theory of instability indicate that the instability behaviour of thin-walled structures resembles that of shell-type structures. Hence, a form of bifurcational instability, which is highly sensitive to initial imperfections, can be expected. Consequently, a stability study on individual components, taken in isolation, does not guarantee a global stability of the box column.

The important concept of interaction between local and overall modes of instability of thin-walled columns was not included in the brief of the present study.

The experimental study on box columns, formed as assemblages of flat plates and curved sheet panels, Fig. App 2-1, is given in Chapter 5. No direct comparison, between the results of this study and those obtained from the finite element analyses on the idealized box sections, Fig. 4-20, can be drawn owing to:

1. The presence of the 'lips' in the actual test models. The effects of 'lips' or end reinforcement on the overall stability of structural systems are given by Bulson (3);

2. The unavoidable presence of initial imperfections in the test models, whilst the finite element models were assumed to be 'perfect'.

The effects of initial imperfections on the global stability of thin-walled structures can be of major significance in varying the critical load levels. Instability theories predict such variations to be in order of $\varepsilon^{1/2}$ to $\varepsilon^{2/3}$ (4, 37), where $\varepsilon$ represents initial imperfections. Hence, it is not surprising to find the scatter in the experimental results (Table 5-1). However, such an experimental study may give some indication to the general behaviour of such structural systems.
The controversial problem of cantilevered box girders was investigated in this research study to illustrate the inadequacy of the design procedure, where the stability of structural elements and stiffened panels, taken in isolation, is assumed to guarantee a global stability of the whole box girder.

The results of an intensive analysis on a cantilevered box girder is given in sections 4.6 and 4.6.1. The investigation includes a study on the effects of initial imperfections on the global instability of the cantilevered box girder. One form of imperfections was investigated; the imperfection being an initially applied root moment (section 4.6.1). Results show variations in the critical load level indicative of unstable post-buckling characteristics (Fig. 4-39). Hence, for design purposes, a final check on the global instability of the cantilevered box should be carried out.

A discussion on the form of instability, as detected by the computer program, is given in section 4.6.2. Results indicate that the 'perfect' structure tends to display a limit type form of instability. However, the application of initial imperfections gives a family of curves, one of which exhibits a bifurcational form of instability.

Although some variations in the critical load levels were reported for this form of imperfection, it is feasible that more drastic effects may arise with some other forms of initial imperfections. Hence, the effects of random imperfections may be suggested as a topic for future research. A number of authors have emphasized the effects of such imperfections on the instability behaviour of structural systems.

Furthermore, it has to be remembered that the effects of initial imperfections on a structure may be related to the shape of the post-buckling path, the structure tends to follow. Hence, a study regarding the post-buckling behaviour of cantilevered box girders could be of great value.
The nature of the present study limits the investigation to one static loading parameter. A further study into the effects of more than one loading parameter, and, possible, the inclusion of dynamic loading system could be of some interest. Furthermore, no mention is given, in this thesis, to creep and plastic buckling since these phenomena were considered beyond the scope of this non-linear elastic analysis.
Appendix 1

Matrix Formulations

App 1.1 Membrane Behaviour

App 1.1.1 Derivation of the In-plane Displacement Functions

A direct integration of the differential equations of equilibrium (2-1) yields a set of polynomials which can represent the in-plane displacement functions.

A stress distribution that satisfies the equations of equilibrium can be in the form\(^{(8)}\):

\[
\begin{align*}
\sigma_x &= \alpha_1 + \alpha_2 y \\
\sigma_y &= \alpha_3 + \alpha_4 x \\
\tau_{xy} &= \alpha_5
\end{align*}
\]

..... (App 1-1)

From the theory of two-dimensional elasticity, the stress-strain relationship for an isotropic material can be expressed as:

\[
\begin{align*}
\sigma_x &= \frac{E}{1-\nu^2} \left( \epsilon_x + \nu \epsilon_y \right) \\
\sigma_y &= \frac{E}{1-\nu^2} \left( \epsilon_y + \nu \epsilon_x \right) \\
\tau_{xy} &= \frac{E}{2(1+\nu)} \chi_{xy}
\end{align*}
\]

\[
\epsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) \\
\epsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) \\
\gamma = \frac{2(1+\nu)}{E} \tau_{xy}
\]

..... (App 1-2)

In matrix form:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\chi_{xy}
\end{bmatrix}
\]

\[
\sigma = E_m \cdot \epsilon
\]

..... (App 1-3)
Substituting equations App 1-1 into App 1-2:

$$
\varepsilon_x = \frac{1}{\varepsilon} \left( \alpha_x + \alpha_z Y - \nu \alpha_3 x - \nu \alpha_4 x \right) = \frac{\partial \varepsilon_x}{\partial x}
$$

$$
\varepsilon_y = \frac{1}{\varepsilon} \left( -\nu \alpha_1 - \nu \alpha_2 Y + \alpha_3 + \alpha_4 x \right) = \frac{\partial \varepsilon_y}{\partial y}
$$

$$
\gamma_{xy} = \frac{2(1+\nu)}{\varepsilon} \alpha_x = \frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_y}{\partial x}
$$

(\text{App 1-4})

Integrating equation App 1-4a with respect to x:

$$
\mu = \frac{1}{\varepsilon} \left( \alpha_x x + \alpha_x Y - \nu \alpha_3 x - \nu \alpha_4 x^2 + \nu \alpha_5 \right)
$$

(\text{App 1-5})

where \( f(y) \) is an arbitrary function of \( y \).

Similarly, integrating equation App 1-4b with respect to y:

$$
\nu = \frac{1}{\varepsilon} \left( -\nu \alpha_1 Y - \nu \alpha_2 Y^2 + \alpha_3 y + \alpha_4 x Y + \nu \alpha_6 \right)
$$

(\text{App 1-6})

where \( g(x) \) is an arbitrary function of \( x \).

Differentiating and substituting into equation App 1-4c:

$$
\alpha_2 x + \nu' y + \alpha_4 x + \alpha_4 y + \alpha_2 y = 2(1+\nu) \alpha_x
$$

Letting,

$$
\alpha_4 y + \nu' y = \alpha_x
$$

$$
\alpha_2 x + \nu' x = 2(1+\nu) \alpha_x - \alpha_4
$$

Upon integrating,

$$
\nu' y = -\alpha_x \frac{y^2}{2} + \alpha_x y + \alpha_7
$$

$$
\nu' x = -\alpha_x \frac{x^2}{2} + 2(1+\nu) \alpha_5 x - \alpha_6 x + \alpha_8
$$

Therefore,

$$
\mu = \frac{1}{\varepsilon} \left( \alpha_x x + \alpha_x Y - \nu \alpha_3 x - \nu \alpha_4 x^2 + \nu \alpha_5 \right)
$$

(\text{App 1-7})

$$
\nu = \frac{1}{\varepsilon} \left( -\nu \alpha_1 Y - \frac{\alpha_2}{2} (\nu Y^2 + x^2) + \alpha_6 Y + \alpha_4 x Y + 2(1+\nu) \alpha_x x - \alpha_6 x + \alpha_8 \right)
$$

(\text{App 1-8})
### App 1.1.2 The Membrane Stiffness Matrix $K_m$

Equations App 1-5 and 1-6 define the displacement functions $u$ and $v$ in terms of eight generalized coordinates $α_1$, $α_2$, $α_3$, ..., $α_8$. The number of the generalized coordinates coincides with the eight degrees of freedom of a quadrilateral element where two degrees of freedom, $u(δx)$ and $v(δy)$, are ascribed to each nodal point. Hence, matrix $B$ in equation 2-5 is a square matrix that can be written explicitly in the form:

| $\begin{array}{c|cccc|c|c|c|c|c}
\hline
u_i & x_i & x_i y_i & -x_i & -(\omega x_i^2 y_i^2) & 0 & y_i & 1 & 0 & \alpha_1 \\
\hline
v_i & -y_i & -(\omega y_i^2 x_i^2) & y_i & x_i y_i & 2(1+\omega)x_i & -x_i & 0 & 1 & \alpha_2 \\
\hline
u_j & \frac{1}{E} & & & & & & & & \\
\hline
v_j & & & & & & & & & \\
\hline
u_k & x_i & x_i y_i & -x_i & -(\omega x_i^2 y_i^2) & 0 & y_i & 1 & 0 & \alpha_3 \\
\hline
v_k & -y_i & -(\omega y_i^2 x_i^2) & y_i & x_i y_i & 2(1+\omega)x_i & -x_i & 0 & 1 & \alpha_4 \\
\hline
u_l & & & & & & & & & \\
\hline
v_l & & & & & & & & & \\
\hline
\end{array}$

\[ Δ_m = B_m α \]  \hspace{1cm} (App 1-9)

where subscript $m$ refers to membrane behaviour.

The derivatives of the displacement functions are given in equations App 1-4.
Hence, equation 2-13 can be written as:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{\delta}

\begin{bmatrix}
1 & -\nu & -\nu x & 0 & 0 & 0 & 0 \\
-\nu & 1 & -\nu y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1-\nu) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & d_2 & d_3 \\
0 & 0 & 0 & 0 & d_4 & d_5 & d_6 \\
0 & 0 & 0 & 0 & d_7 & d_8 \\
\end{bmatrix}
\]

or

\[
\varepsilon = D_m \alpha \quad \text{......(App1-10)}
\]

The principle of virtual displacement is employed in deriving the membrane stiffness matrix \( K_m \) (equation 2-47). The procedure is as presented in section 2.5 yielding the following matrix equation:

\[
K_m = B_m^{-T} C_m B_m^{-1} \quad \text{......(App1-11)}
\]

where,

\[
C_m = \int_{vol} (D^T E D)_m dV \quad \text{......(App1-12)}
\]

For an element with constant thickness, \( t \), the matrix \( C_m \) becomes:

\[
C_m = t \int_{area} (D^T E D)_m dA \quad \text{......(App1-13)}
\]

The matrix \( C_m \) is generated and given explicitly in Fig. App 1-1.
where  

\[ A = \text{surface area of the element} \]

\[ x_1, x_2, y_1 \text{ and } y_2 - \text{the limits of integration} \]

**Fig. App 1-1 Matrix \( C_m \).**
The assumed displacement function for plate bending is that presented by Zienkiewicz (45), and reproduced in equation 3-2:
\[ \omega = \beta_1 x^3 + \beta_2 x^2 + \beta_3 x + \beta_4 y^2 + \beta_5 y + \beta_6 x^2 y + \beta_7 x y^2 + \beta_8 x^2 + \beta_9 x y + \beta_{10} y^2 + \beta_{11} x + \beta_{12} \]
\[ \cdots \quad \text{(App 1-14)} \]
where \( \beta_1, \beta_2, \beta_3, \ldots, \beta_{12} \) are the generalized coordinates.

The expressions for bending and twisting moments, per unit length, for a plate under pure bending, are given by Timoshenko and Woinowski-Krieger (40):

\[ M_x = -D \left( \frac{\partial^2 \omega}{\partial x^2} + \nu \frac{\partial^2 \omega}{\partial y^2} \right) \]
\[ M_y = -D \left( \frac{\partial^2 \omega}{\partial y^2} + \nu \frac{\partial^2 \omega}{\partial x^2} \right) \]
\[ M_{xy} = D (1 - \nu) \frac{\partial^2 \omega}{\partial x \partial y} \]
\[ \cdots \quad \text{(App 1-15)} \]
where \( D = \frac{E t^3}{12(1 - \nu^2)} \) is the flexural rigidity of the plate, \( t \) is the thickness of the plate, \( E \) and \( \nu \) are the modulus of elasticity and Poisson's ratio, respectively.

In matrix form, equation App 1-15 can be written as
\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \frac{E t^3}{12(1 - \nu^2)}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 \omega}{\partial x^2} \\
\frac{\partial^2 \omega}{\partial y^2} \\
\frac{\partial^2 \omega}{\partial x \partial y}
\end{bmatrix}
\quad \text{(App 1-16)}
\]

\( \sigma_f = E_f \epsilon_f \) \quad \text{(App 1-16)}

where the subscript \( f \) refers to flexural behaviour.
Differentiating the displacement function \( w, f(x,y) \), with respect to \( x \) and \( y \):

\[
\frac{\partial w}{\partial x} = 3\beta_1 x^2 + 2\beta_2 x + \beta_3 + 3\beta_7 x^2 y + 2\beta_8 xy + \beta_4 y + \beta_{10} y^2 + \beta_{11} y^2 - \\
\frac{\partial w}{\partial x^2} = 6\beta_1 x + 2\beta_2 + 6\beta_7 y + 2\beta_8 \]

\[
\frac{\partial w}{\partial y} = 3\beta_4 y^2 + 2\beta_5 y + \beta_6 + \beta_7 x^2 + \beta_8 x^2 + \beta_9 x + 3\beta_{10} y^2 + 2\beta_{11} xy \\
\frac{\partial w}{\partial y^2} = 6\beta_4 y + 2\beta_5 + 6\beta_{10} y + 2\beta_9 x \\
\frac{\partial w}{\partial x \partial y} = 3\beta_7 x^2 + 2\beta_8 x + \beta_9 + 3\beta_{10} y^2 + 2\beta_{11} y
\]

..... (App 1-17)

The out-of-plane displacements at any point \( i \), on the continuum, due to lateral loadings are given as:

\[
d_i = \begin{bmatrix} -w \\ \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} w \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}
\]

Hence, equation 2-5 can be re-written explicitly as shown in Fig. App 1-2.

\[
d_f = B_f \beta
\]

..... (App 1-19)

The moments-curvature relationships of equation 2-16 can be re-written in the form:

\[
\begin{pmatrix}
-6x & -2 & 0 & 0 & 0 & 0 & -6xy & -2y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -6y & -2 & 0 & 0 & 0 & -6xy & -2x & 0 \\
0 & 0 & 0 & 0 & 0 & 6x & 4x & 2 & 6y & 4y & 0
\end{pmatrix}
\]

\[
E_f = D_f \alpha
\]

..... (App 1-20)
\[ \text{Fig. APP 1-2 Matrix} \]

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<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{\varepsilon} )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{x} )</td>
<td>( \frac{k}{y} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The principle of virtual displacement is followed in deriving the stiffness matrix $K_f$ of equation 2-47, yielding

$$K_f = B_f^{-T} C_f B_f^{-1} \quad \ldots \quad (\text{App 1-21})$$

where,

$$C_f = (D^T E D)_f dA \quad \ldots \quad (\text{App 1-22})$$

The matrix $C_f$ is generated and is given explicitly in Fig. App 1-3.
\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} & \frac{e}{\varepsilon} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

Limits of integration:
- \( x_1 \) and \( x_2 \), \( x_1, x_2 \)
- The element area of

where:

\[ X = \text{surface area of} \]

\[ A \]
The incremental stiffness matrix $N$ represents the effect of stretching the middle surface due to bending (section 2.6). The matrix $N$ is represented by the following expression for the potential energy, equation 2-46:

$$U = \frac{1}{2} \int \int \left\{ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2 N_{xy} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right\} dx \, dy$$

With reference to equations App 1-14 and 17, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ can be expressed in matrix form as:

$$\begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 & 2x & 1 & 0 & 0 & 0 & 3xy & 2xy & x^3 & x^2 & x & 3xy^2 & 2xy & 0 \\ 0 & 0 & 0 & 3y^2 & 2y & 1 & x^3 & x^2 & x & 3xy^2 & 2xy & 0 & \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{12} \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = G \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

or,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = GB^{-1} \Delta$$

The stress tensor $N_e$ can be represented as:

$$N_e = \begin{pmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{pmatrix}$$
Hence, the potential energy of equation 2-46 can be written in matrix form as:

\[ U = \frac{1}{2} \iint \left\{ \theta^T N_x \theta \right\} \, dx \, dy \]  

...... (App 1-27)

Substituting equation App 1-25 into equation App 1-27 and extracting from the integral the terms which are not functions of \( x \) and \( y \):

\[ U = \frac{1}{2} \Delta_f \left\{ B_f^T \left[ \iint \left\{ G^T N_e G \right\} \, dx \, dy \right] B_f \right\} \Delta_f \]  

...... (App 1-28)

The strain energy can be expressed as

\[ U = \frac{1}{2} d^T K d \]  

...... (App 1-29)

Comparing equation App 1-28 with equation 1-29, yields

\[ N = B_f^T C_n B_f^{-1} \]  

...... (App 1-30)

where,

\[ C_n = \iint (G^T N_e G) \, dx \, dy \]  

...... (App 1-31)

The matrix \( C_n \) is generated and is given explicitly in Figs. App 1-4, 1-5, 1-6, as a function of \( N_x, N_y \) and \( N_{xy} \) where:

\[
N_x = \frac{1}{4} \sum_{i=1}^{i=4} (\sigma_x)_i \cdot t \\
N_y = \frac{1}{4} \sum_{i=1}^{i=4} (\sigma_y)_i \cdot t \\
N_{xy} = \frac{1}{4} \sum_{i=1}^{i=4} (\sigma_{xy})_i \cdot t
\]  

...... (App 1-32)

Hence

\[ C_n = C_{nx} + C_{ny} + C_{nxy} \]  

...... (App 1-33)
where,

\[ A = \text{surface area of the element} \]
\[ x_1, x_2, y_1 \text{ and } y_2 = \text{limits of integration} \]

Fig. App 1-4 Matrix \[ [C_n]_x \]
where,
\[ A = \text{surface area of the element} \]
\[ x_1, x_2, y_1 \text{ and } y_2 \text{ - limits of integration} \]

Fig. App 1-5 Matrix \([C_n]_y\)
where,

\[ A = \text{surface area of the element} \]

\[ x_1, x_2, y_1 \text{ and } y_2 - \text{limits of integration} \]

Fig. App 1-6 Matrix \([C_n]_{xy}\)
Appendix 2

Test Specimen: Material, Mechanical Properties and Flexural Behaviour

App 2.1 Introduction

Tests were carried out on hollow extruded box units, made of rigid polyvinylchloride plastic (PVC), as an experimental study on the local instability of thin-walled structures. A typical cross-section of such a unit is shown in Fig. App 2-1.

App 2.2 Material and Mechanical Properties

Being a thermoplastic material (a material that can be re-softened through the application of heat and re-hardened through cooling), the mechanical properties of PVC are functions of temperature, time, rate of strain and other factors which are beyond the scope of this study.

Softening point of PVC is about 82°C\(^{19}\) whilst its service temperature is considered to be 20°C\(^{14}\). Imperial Chemical Industries (ICI) assumes that PVC has a constant working stress between -30°C and +20°C. At 20°C the working stress starts to drop reaching zero working stress at temperatures varying between 55 and 60°C.

App 2.2.1 Modulus of Elasticity (E) and Poisson's Ratio (\(\nu\))

Tests were carried out to determine the values of E and \(\nu\) of PVC at room temperatures (between 20 and 25°C). Two different procedures were followed in determining the modulus of elasticity:

1. Twenty seven strips, having a nominal width of 1.0 in. with an average thickness, varying, for different specimens, between 0.0449 and 0.0595 in., and with
an average gauge length of 5.24-5.70 in., were tested under axial tensile load using an Instron machine model TT-CM-L-M4, Fig. 5-1. The load vs. extension curves were plotted on a standard chart plotter synchronized with the movement of the Instron's cross-heads. An average value of $0.375 \times 10^6$ psi $\pm 4.87\%$ was calculated for the modulus of elasticity (Table App 2-1).

2. Fifteen strips, having a nominal width of 0.5 in. with an average thickness, varying, for different specimens, between 0.0506 and 0.0637 in., were tested under axial tensile load using the Instron machine together with an Instron strain gauge extensometer, model G-51-11. The load vs. strain curves were plotted on an X-Y chart plotter synchronized with the extensometer, mounted on the specimen under test. Average values of $0.416 \times 10^6$ psi $\pm 3.34\%$ for loads up to 55 lbs. and $0.401 \times 10^6$ psi $\pm 4.5\%$ for tensile loads higher than 55 lbs. were calculated for the modulus of elasticity (Table App 2-2).

For the Poisson's ratio, six strips, having a nominal width of 1.0 in. with an average thickness, varying, for different specimens, between 0.0485 and 0.0558 in., were tested under axial tensile loads together with electric strain gauges, Tinsley Type 16A, fixed on both sides of the specimen under test. Strains were read from a Strain Indicator (Peekel T200) through an Extension Box (Peekel 23U). An average value of 0.35 was calculated for the Poisson's ratio ($\nu$) of PVC.

Some differences were expected to rise between the values of $E$ and $\nu$ obtained experimentally and those found elsewhere in literature. Such a comparison is given in Table App 2-3.
Fig. App 2-1 Test Specimen: Typical Cross-Section.

(scale 1 : 1)
Table App 2-1

**Modulus of Elasticity (E) of PVC**

*(Procedure 1)*

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Area (in.²)</th>
<th>Gauge Length (in.)</th>
<th>E x 10^6 psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0525</td>
<td>5.245</td>
<td>0.361</td>
</tr>
<tr>
<td>2</td>
<td>0.0529</td>
<td>5.245</td>
<td>0.350</td>
</tr>
<tr>
<td>3</td>
<td>0.0595</td>
<td>5.245</td>
<td>0.341</td>
</tr>
<tr>
<td>4</td>
<td>0.0509</td>
<td>5.245</td>
<td>0.346</td>
</tr>
<tr>
<td>5</td>
<td>0.0509</td>
<td>5.700</td>
<td>0.401</td>
</tr>
<tr>
<td>6</td>
<td>0.0497</td>
<td>5.450</td>
<td>0.427</td>
</tr>
<tr>
<td>7</td>
<td>0.0509</td>
<td>5.250</td>
<td>0.369</td>
</tr>
<tr>
<td>8</td>
<td>0.0526</td>
<td>5.250</td>
<td>0.378</td>
</tr>
<tr>
<td>9</td>
<td>0.0558</td>
<td>5.720</td>
<td>0.399</td>
</tr>
<tr>
<td>10</td>
<td>0.0529</td>
<td>5.270</td>
<td>0.365</td>
</tr>
<tr>
<td>11</td>
<td>0.0588</td>
<td>5.270</td>
<td>0.349</td>
</tr>
<tr>
<td>12</td>
<td>0.0512</td>
<td>5.270</td>
<td>0.393</td>
</tr>
<tr>
<td>13</td>
<td>0.0511</td>
<td>5.270</td>
<td>0.361</td>
</tr>
<tr>
<td>14</td>
<td>0.0502</td>
<td>5.270</td>
<td>0.375</td>
</tr>
<tr>
<td>15</td>
<td>0.0559</td>
<td>5.270</td>
<td>0.368</td>
</tr>
<tr>
<td>16</td>
<td>0.0510</td>
<td>5.270</td>
<td>0.369</td>
</tr>
<tr>
<td>17</td>
<td>0.0497</td>
<td>5.270</td>
<td>0.387</td>
</tr>
<tr>
<td>18</td>
<td>0.0504</td>
<td>5.280</td>
<td>0.373</td>
</tr>
<tr>
<td>19</td>
<td>0.0509</td>
<td>5.270</td>
<td>0.378</td>
</tr>
<tr>
<td>20</td>
<td>0.0506</td>
<td>5.270</td>
<td>0.380</td>
</tr>
<tr>
<td>21</td>
<td>0.0464</td>
<td>5.280</td>
<td>0.368</td>
</tr>
<tr>
<td>22</td>
<td>0.0462</td>
<td>5.280</td>
<td>0.369</td>
</tr>
<tr>
<td>23</td>
<td>0.0462</td>
<td>5.270</td>
<td>0.383</td>
</tr>
<tr>
<td>24</td>
<td>0.0476</td>
<td>5.270</td>
<td>0.380</td>
</tr>
<tr>
<td>25</td>
<td>0.0464</td>
<td>5.270</td>
<td>0.380</td>
</tr>
<tr>
<td>26</td>
<td>0.0472</td>
<td>5.280</td>
<td>0.395</td>
</tr>
<tr>
<td>27</td>
<td>0.0463</td>
<td>5.280</td>
<td>0.382</td>
</tr>
</tbody>
</table>
Table App 2-2

**Modulus of Elasticity (E) of PVC**

(Procedure 2)

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Area (in.(^2))</th>
<th>E x 10(^6) psi(*)</th>
<th>E x 10(^6) psi(**)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0272</td>
<td>0.409</td>
<td>0.381</td>
</tr>
<tr>
<td>2</td>
<td>0.0286</td>
<td>0.418</td>
<td>0.414</td>
</tr>
<tr>
<td>3</td>
<td>0.0318</td>
<td>0.424</td>
<td>0.402</td>
</tr>
<tr>
<td>4</td>
<td>0.0252</td>
<td>0.436</td>
<td>0.410</td>
</tr>
<tr>
<td>5</td>
<td>0.0261</td>
<td>0.422</td>
<td>0.409</td>
</tr>
<tr>
<td>6</td>
<td>0.0328</td>
<td>0.422</td>
<td>0.407</td>
</tr>
<tr>
<td>7</td>
<td>0.0301</td>
<td>0.413</td>
<td>0.398</td>
</tr>
<tr>
<td>8</td>
<td>0.0309</td>
<td>0.422</td>
<td>0.407</td>
</tr>
<tr>
<td>9</td>
<td>0.0304</td>
<td>0.407</td>
<td>0.388</td>
</tr>
<tr>
<td>10</td>
<td>0.0310</td>
<td>0.396</td>
<td>0.387</td>
</tr>
<tr>
<td>11</td>
<td>0.0263</td>
<td>0.400</td>
<td>0.396</td>
</tr>
<tr>
<td>12</td>
<td>0.0254</td>
<td>0.431</td>
<td>0.403</td>
</tr>
<tr>
<td>13</td>
<td>0.0267</td>
<td>0.403</td>
<td>0.397</td>
</tr>
<tr>
<td>14</td>
<td>0.0283</td>
<td>0.419</td>
<td>0.404</td>
</tr>
<tr>
<td>15</td>
<td>0.0253</td>
<td>0.419</td>
<td>0.415</td>
</tr>
</tbody>
</table>

* maximum applied load = 55 lbs.

** maximum applied load = 232.3 lbs.
The second procedure was expected to yield more accurate results since the recorded strains correspond to the actual extensometer's strains independent of any external factors. Hence, throughout this research work, average values of $0.4085 \times 10^6$ psi and 0.35 are used for the modulus of elasticity and Poisson's ratio, respectively.

<table>
<thead>
<tr>
<th></th>
<th>Modulus of Elasticity $E \times 10^6$ psi</th>
<th>Poisson's Ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shell (19)</td>
<td>0.350</td>
<td></td>
</tr>
<tr>
<td>ICI (14)</td>
<td>0.450</td>
<td></td>
</tr>
<tr>
<td>Sayigh (30)</td>
<td>0.420</td>
<td>0.32-0.37</td>
</tr>
<tr>
<td>Author</td>
<td>Proc.1 0.375</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>Proc.2 0.4085</td>
<td></td>
</tr>
</tbody>
</table>

Table App 2-3 Comparison Between Different Values of $E$ & $\nu$

App 2.2.2 Tensile Strength

Eighteen specimens, of the type specified in the British Standards B.S. 2782 part 3/301 (1957), Fig. App 2-2, with an average thickness varying, for different specimens, between 0.044 and 0.055 in., were tested at room temperature under axial tensile stress.

An average maximum tensile strength of $7540$ psi $\pm 6.6\%$, with an extension of 3-4%, was obtained for PVC. Fig. App 2-3 shows that for strains $\leq 1\%$, PVC can be considered to be Hookean material.

An intensive work on the subject was carried out, at the Department of Mechanical Engineering of the Imperial College, London, University, by Sayigh (30) on sandwich beams using PVC as a skin material. Sayigh drew similar conclusions regarding the stress-strain curve of PVC. Further, he noted that the shape of the specimens has little or no effect on the values of the modulus of elasticity.
Fig. App 2–2 Tensile Specimen.

* w was taken as 1.0 in. to enable the specimen to fit inside the Instron machines' jaws.
### Table App 2-4

**Maximum Tensile Strength of PVC**

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Area (in.(^2))</th>
<th>Max. Tensile Strength (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01194</td>
<td>7441</td>
</tr>
<tr>
<td>2</td>
<td>0.01500</td>
<td>7496</td>
</tr>
<tr>
<td>3</td>
<td>0.01452</td>
<td>7333</td>
</tr>
<tr>
<td>4</td>
<td>0.01380</td>
<td>7349</td>
</tr>
<tr>
<td>5</td>
<td>0.01427</td>
<td>7416</td>
</tr>
<tr>
<td>6</td>
<td>0.01322</td>
<td>7171</td>
</tr>
<tr>
<td>7</td>
<td>0.01201</td>
<td>7435</td>
</tr>
<tr>
<td>8</td>
<td>0.01238</td>
<td>7319</td>
</tr>
<tr>
<td>9</td>
<td>0.01275</td>
<td>6969</td>
</tr>
<tr>
<td>10</td>
<td>0.01287</td>
<td>6938</td>
</tr>
<tr>
<td>11</td>
<td>0.01302</td>
<td>7061</td>
</tr>
<tr>
<td>12</td>
<td>0.01417</td>
<td>7219</td>
</tr>
<tr>
<td>13</td>
<td>0.01439</td>
<td>8488</td>
</tr>
<tr>
<td>14</td>
<td>0.01473</td>
<td>8711</td>
</tr>
<tr>
<td>15(\d\d\d)</td>
<td>0.01285</td>
<td>8115</td>
</tr>
<tr>
<td>16</td>
<td>0.01285</td>
<td>7714</td>
</tr>
<tr>
<td>17</td>
<td>0.01501</td>
<td>8108</td>
</tr>
<tr>
<td>18</td>
<td>0.01436</td>
<td>7446</td>
</tr>
</tbody>
</table>
FIG. APP. 2-3: STRESS STRAIN CURVE OF PVC

0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0

% Strain

Tensile Stress (x10^3 psi)

3750 psi; 1% strain

7540 psi; 3% strain
In order to determine, experimentally, the flexural behaviour of the test units, optical cathetometers and mechanical dial gauges were used to measure the vertical deflections of single units subjected to transverse loads.

The units, with spans of 3 ft., were tested under:

a. Pure bending, usually called four point loading (Fig. App 2-4) with applied moments of 10, 20 and 30 ft.-lbs.

b. Point loads, usually called three point loading (Fig. App 2-5) with applied loads of 20.53, 30.53 and 40 lbs.

The deflection of a beam under pure bending is governed by the equation

\[ \delta_x = \frac{P}{2EI} (L - x^2) \]  

..... (App 2-1)
Table App 2-5 gives the theoretical and experimental deflection of a beam with:

- Modulus of elasticity ........... \(0.4085 \times 10^6\) psi
- Average Thickness ............. 0.0493 in.
- Average Moment of Inertia ... 0.466 in.\(^4\)

<table>
<thead>
<tr>
<th>(X / L)</th>
<th>Moments (ft.-lbs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>(\frac{1}{16})</td>
<td>0.045 in.</td>
</tr>
<tr>
<td></td>
<td>0.0565 **</td>
</tr>
<tr>
<td>(\frac{1}{3})</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>0.0904</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0.096</td>
</tr>
<tr>
<td></td>
<td>0.1017</td>
</tr>
</tbody>
</table>

Table App 2-5 Deflections of a beam under four-point loading.
* see Fig. App 2-4
** theoretical deflection (equation App 2-1)

A comparison between the theoretical and the experimental deflections is shown in Fig. App 2-6. The average experimental value for the flexural rigidity, EI, was found to be \(2.12 \times 10^5\) lb.-in.\(^2\) for a unit with an average walls thickness of 0.0493 in., compared with the expected value of \(1.90 \times 10^5\) lb.-in.\(^2\).

App 2.3.2 Three Point Loading (3PL)

The deflection of a beam under a point load,
FIG. APR 2-6: Deflections of a beam under four-point loadings.
is governed by the following equations:

\[ \sigma_x = \frac{P b x (l^2 - b^2 - x^2)}{6 E I L} \quad x < a \]

\[ \sigma_a = \frac{P a^2 b^2}{3 E I L} \quad x = a \]

\[ \sigma_{max} = \frac{P a b (a + 2b) \sqrt{3a(a + 2b)}}{27 E I L} \quad x = a + \frac{(a + 2b)}{3} \quad a > b \]

...(App 2-2)

Table App 2-6 gives the theoretical and experimental deflections of a beam similar to that used in section App 2.3.1. A comparison between the theoretical and experimental deflections is shown in Fig. App 2-7.

App 2.3.3 Discussion

The results of four-point loading tests are, usually, more reliable than those obtained from the three-point loading tests. This is due to:

a. The absence of shear in the former case. Hence, the measured deflections are due to pure bending. Further, the deflected shape of a beam under pure bending is a circular curve, i.e. constant radius of curvature, R.

\[ \frac{M}{E I} = \frac{1}{R} \]

Thus, with a constant moment, M, and a constant radius of curvature, the resulted EI is more accurate than that calculated with varying M and R as in the case with three point loading tests;
### Table App 2-6

**Deflection of a Beam Under Three-Point Loadings**

<table>
<thead>
<tr>
<th>$X$</th>
<th>Applied Load $P$ (lbs.)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>20.53</td>
<td>30.53</td>
<td>40.00</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>0.049 in.</td>
<td>0.077</td>
<td>0.076</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>0.051 **</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>0.081</td>
<td>0.144</td>
<td>0.179</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.090</td>
<td>0.134</td>
<td>0.175</td>
<td></td>
</tr>
<tr>
<td>$X = a$ (=16.5 in.)</td>
<td>***</td>
<td>0.103</td>
<td>0.153</td>
<td>0.201</td>
</tr>
<tr>
<td>max.</td>
<td>***</td>
<td>0.1035</td>
<td>0.154</td>
<td>0.202</td>
</tr>
<tr>
<td></td>
<td>0.1033</td>
<td>0.1537</td>
<td>0.2013</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.087</td>
<td>0.159</td>
<td>0.211</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1033</td>
<td>0.1537</td>
<td>0.2013</td>
<td></td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>0.098</td>
<td>0.135</td>
<td>0.177</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.079</td>
<td>0.118</td>
<td>0.154</td>
<td></td>
</tr>
<tr>
<td>$\frac{5}{6}$</td>
<td>0.058</td>
<td>0.083</td>
<td>0.102</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.046</td>
<td>0.068</td>
<td>0.089</td>
<td></td>
</tr>
</tbody>
</table>

* see Fig. App 2-5  
** theoretical deflections (equations App 2-6)  
*** values unavailable
FIG. APR 2-7: Deflection of a beam under three point loadings.
b. The local disturbances around the loading points, which in the former are outside the spane under consideration.

The three point loading tests were carried out as a matter of interest and not for evaluating the flexural rigidity of the test specimens. The results of these tests are given in section App 2.3.2 for general consultation.
Appendix 3

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