BUSINESS CYCLES AND THE MANAGEMENT OF FINANCIAL RISK

A Thesis Presented for the Doctor of Philosophy

by

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The author explicitly specifies a New Keynesian style model embodying a financial constraint on the availability of equity and a financial market imperfection with regard to the existence of state-contingent assets based upon the published papers of Greenwald and Stiglitz (1988, 1990, and 1993). Using computer based numerical simulation, the author validates the three unproven Propositions found in the Greenwald and Stiglitz 1993 article with regard to the model’s comparative static behaviour. Through the inclusion of a parameter for technology into the production function, the author shows that observations made by Greenwald and Stiglitz with regard to the effect of equity infusions is subject to qualification. Investigation of the model’s inter-temporal behaviour reveals that the claims made by Greenwald and Stiglitz with regard to multiple periodicity are again subject to many qualifications. Linearization around the steady-state equilibrium as suggested by Greenwald and Stiglitz is shown to offer limited insight because of the implied non-linearity of the model’s first order difference equation. Calibrated numerical simulation of the non-linear difference equation reveals the potential for both single and multiple periodicity, period doubling bifurcations, and chaotic trajectories displaying sensitivity to initial conditions. In addition it was shown that the model’s implied random attractor was key to understanding its inter-temporal behavior. In the Greenwald and Stiglitz articles the existence of derivative markets such as futures or options to manage risk are assumed away. The author, in order to investigate the effects of futures or options markets upon business cycles, modifies the explicitly specified model to include the use of state-contingent assets. Introducing the use of derivative financial products to manage risk, using numerical simulation, produces the surprising result that in the aggregate they may lead to slightly greater output instability. In addition to the model’s structure, several intuitive reasons for these results are discussed in depth. The Greenwald and Stiglitz model also assumed that the cost of capital was not risk adjusted. The author modifies the explicitly specified model and using numerical simulation shows that like other unrealistic assumptions concerning dividend distribution, leads to alternative laws of motion. The research is concluded with discussion of possible policy and regulatory implications.
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In praise of God for sanctifying the work of our hands.
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CHAPTER ONE. INTRODUCTION
1.0 Introduction to the Issues

The problem of accounting for cyclic fluctuations in output, recurring booms and busts, falling and rising unemployment, and price instability is as old as economics itself. From the earliest writings the fundamental issue facing theorists has been whether cyclic fluctuations can arise only through external, exogenous events; or whether cyclic fluctuations are inherent to an economy, the result of an endogenous process. In the former, identified with the Neoclassical world and its derivatives, through the maximizing behaviour of rational agents, prices and wages adjust with sufficient speed to ensure continuous market clearing and hence no spare capacity, including unemployment. In the latter, although individual firms and consumers are equally rational, rigidities and constraints in adjustment processes lead to under-utilization of resources, including labour, in the aggregate. These alternative conceptions of cyclic fluctuations have profound implications for economics as a whole because they require distinct micro-economic paradigms and imply different forms of general equilibria.

Taking these alternative conceptions of how business cycles arise somewhat further, we find they hold distinct implications for the rationality and logical consistency of a self-organizing economy. Is the nature of an economy logically consistent with general equilibrium theory? Is it benign, only going awry because of external shocks or poor policy? Are the effects of such shocks self-correcting if left to their own devices? The modern justification for the exogenous and self-correcting conception economic fluctuations was found in the General Equilibrium system conceived by Leon Walras (1954), although its intellectual forbearers stretch back to Smith’s invisible hand argument and the broader teachings of the Enlightenment, notably the Philosophes. Economic cycles induced by exogenous shocks and disturbances may occur, but the utility maximizing behaviour of consumers coupled with the profit maximizing behaviour
of firms in a competitive economy, would return the state variables to equilibrium along a welfare efficient frontier. Idle savings and spare capacity are not permanent problems and may be remedied through speeding the process of adjustment. Today, this conception finds expression in the New Classical and Real Business Cycle school of thought. The alternative conception, embodied in endogenous models of economic fluctuations, is that the self-organizing economy is logically inconsistent with general equilibrium theory. It is flawed, not self-correcting and without intervention, potentially malign. In contemporary research, this conception of how fluctuations arise in a self-organizing economy, uses micro-theoretic imperfections to explain aggregate fluctuations and sub-optimization. It is known as the New Keynesians school.

Reconciling or creating a synthesis between the two conceptions of why fluctuations in output for self-organizing economies may arise would appear insurmountable given their assumptions, structures, and respective short-comings. On the one hand we have New Classical models, consistent with general equilibrium theory and the received principles of micro-economics combining exogenous serially uncorrelated impulses (perturbations) with a propagation mechanism producing serially correlated responses. Such propagation mechanisms involve confusion over nominal values versus real values, requiring changes in output and prices to be unexpected in order for cycles to be generated (Lucas, 1972). Further research into general equilibrium consistent models of economic fluctuations, the Real Business Cycle have relied upon unexpected random supply-side shocks through the production function (Kydland and Prescott, 1982; and Long and Plosser, 1983), although empirical evidence for supply shocks is not supportive (Hall, 1990). Some have questioned the realism of the Real Business Cycle school requiring an external agent, such as technological events as the initial impulse (McCallum, 1986).
On the other hand, we have the equally problematic alternative to the Real Business cycle school, the heirs to Keynesianism. They rely upon micro-theoretic conditions in a disequilibrium setting to generate output fluctuations. Often they involve second-order type arguments. For example, flat profit functions imply income losses are second order in magnitude, making adjustments in prices unjustified. Apart from the problem of logical inconsistency with general equilibrium theory, the shortcomings of New Keynesian research are several-fold. Firstly, in trying to create strong micro-theoretic foundations, at the partial equilibrium level; it places the burden of aggregate demand fluctuations upon the narrow foundations of erratic private expectations of consumption and investment spending, leading rational agents to sub-optimization. Phenomena such as menu costs, mark-up pricing, wage and price inflexibility, or nominal rigidities as found in Mankiw’s collection (1993) might affect a single industry or market, but are they sufficiently pervasive to affect the entire economy ad infinitum? In the Samuelson sense, is there sufficient correspondence between the maximizing behaviour of individuals/firms and the stability conditions relating to the interaction between economic units (page 258, 1983). By comparison, we recall that Keynes, at least, appealed to systemic collapse in the efficiency of capital and systemic wage rigidity. A second criticism of the New Keynesian school is that while their models seek to explain inter-temporal fluctuations in output, their sources of sub-optimization are essentially static, and not correctible through other endogenous mechanisms such as technological improvements and innovation. That systemic endogenous flaws, as Keynes believed were not in any practical sense, self-correcting, is an arguable proposition especially in the conditions of the 1930s; however to believe that the monocausal micro-imperfections found in New Keynesian theories and models, in a modern innovative economy would persist, stretches credibility.
From the above we see that a unified theory embracing both schools of thought is unlikely, however there may be value in investigating the principles of correspondence, between micro-foundations and aggregate stability conditions. Within a given school of thought, it may lead to a richer understanding of how the alternative conceptions may produce a synthesized conception of economic fluctuations. Moreover, in seeking some form of reconciliation between the competing schools, and addressing the highlighted short-comings of the New Keynesian variant, it may suggest ways that such models may be used to address the extent to which endogenous innovations may interact and correct macroeconomic disturbances and alter the path of cyclic fluctuations.

1.1 The Problem

We have seen above that the New Keynesian school of thought utilizes endogenous phenomena, such as rigidities and sub-optimization, to explain fluctuations in aggregate economic output. Such models, like Keynesianism itself, it is argued are logically inconsistent with the tatonnement process of the Walrasian auctioneer. The attainment of Samuelson correspondence is questionable based upon the remarks above. Unemployment and spare resources require some rigidity somewhere, so that all markets simultaneously do not clear and that economic agents remain unaware of disequilibrium (Gale, 1983). Recall, disequilibrium, in the New Keynesian literature, is stable, and not simply a short-run deviation from neo-classical results, for otherwise any form of Keynesian economics would simply involve the slowness of convergence to Walrasian equilibrium, and not the sub-optimization generated through constraints and rigidities.

To create general equilibrium foundations for the Keynesian system a prodigious amount of intellectual effort was undertaken. Beginning with Clower (1965) and later Dreze (1975) an alternative
general equilibrium framework for the Keynesian model was proposed, involving trade in effective demands, that is traders make plans based upon predictions of what they expect to be able to trade at given prices. Bennasy in a series of articles (1975, 1976, 1977) attempted to develop a theory of rationing scheme compatible with non-Walrasian equilibrium in Keynesian style model. But such solutions to the problem of logical inconsistencies inherent to any Keynesian or New Keynesian style model were not without critics, notably Malinvaud (1975), Gale (1983) and others. Faced with seemingly intractable problems of creating non-Walrasian disequilibrium foundations for a Keynesian model, and seeking to explain spare resources and unemployment over sustained periods, we return to the question of adjustment or connection between the Keynesian disequilibrium condition and the Neo-Classical paradigm with its general equilibrium structure. Specifically, in this regard, we consider the role of Arrow-Debreu securities in the form of endogenous state-contingent assets in possibly affecting the convergence to a Walrasian solution in New Keynesian models.

To understand this possibility, we recall that the direction of causation in the two schools of thought are distinctly different. We have explained that New Keynesianism is not simply about tardy adjustment arising from frictions in an essentially classical world, but rather concerns rational agents achieving sub-optimal solutions because of rigidities and constraints. In Keynesianism and its heirs, economic fluctuations, markets not clearing, and the existence of spare resources are the result of imperfections and constraints. In the New Classical and Real Business cycle approach, in contrast, full utilization of resources in equilibrium results from prices and wages adjusting quickly enough to ensure market clearing without spare resources. This causal distinction between the schools of thought, suggests a possible synthesis between the competing schools, although not reconciliation.
We have mentioned the questionable reality of static immutable rigidities as found in the New Keynesian set-up. The non-existence of complete markets for all state-contingent assets has been used to explain the existence of business cycles in the absence of exogenous disturbances and within a general equilibrium setting (Lucas, 1987 and Krainer, 1992). Moral hazard and transaction costs have been posited as reasons why complete Arrow-Debreu markets do not exist. But what if there existence through an endogenous economic process were possible, one as endogenous as the rigidities precluding full-optimization? Would they reduce or eliminate the effects of rigidities and the effects of constraints precluding general equilibrium optimization? How would the existence of state contingent assets affect the dynamics of New Keynesian models? Would problems of moral hazard create new effects? Would they transform New Keynesian style models into Neo-classical type solutions or would combining state contingent assets with rigidities lead to sub-optimal solutions of a new nature? If such securities interacted with a firm’s risks and exposures, could any such results be reconciled with Modigliani-Miller capital structure theory which argues the valuation and hence behaviour based upon it, is invariant to capital structure? These and many related issues we have undertaken in the present research.

1.3 Methodology

In order to pursue the above subject matter we have turned to the New Keynesian style models involving both a financial constraint and a financial market imperfection. Like other New Keynesian models, in such research, rigidities at the micro level become cycles at the macro level because of sub-maximization of social welfare. Such models use capital markets imperfections in alternative forms such as rationing of capital by quantity rather than price (the rate of interest rate), monitoring costs in extending credit, resulting for
example, in not risk adjusting the cost of capital, informational asymmetries between lenders and borrowers, and moral hazards in borrowing (Stiglitz and Weiss, 1981). Typically such conditions lead to one form of finance, such as external capital through borrowing having costs additional to that of funds which are generated internally, through profit retention. In our research we utilize one model in particular by Joseph Stiglitz and Bruce Greenwald (1988, 1990, 1993) (hereafter, GS) embodying both a financial constraint on the availability of equity, a financial imperfection relating to capital borrowed which induces a form of risk aversion and the assumption that contingent assets do not exist. The emphasis in such models, as that by GS, upon imperfections and constraints as sources of cyclic behaviour, make them ideal for analysing whether the introduction of state contingent assets as an endogenous means of correcting such imperfections, may have macroeconomic implications. To this end, our methodology involves including Arrow-Debreu type securities into such a model as a set of specified explicit relationships. To emphasize, by introducing state contingent assets we are not merely making prices more flexible and speeding up the adjustment process so that a New Keynesian model has the general equilibrium properties identified with New Classicism. Rather at micro-theoretic level we are including structures and relationships to possibly off-set the effects arising from capital market imperfections.

Most New Keynesian research involves the creation and analysis of implicitly specified relationships. In contrast, in our methodology we utilize numerical simulation. Analytic methods have limited applications with regard to learning the inter-temporal behaviour of such models of aggregate fluctuations, as the one developed by GS. There are several reasons for such limitations. Such models, as the one by GS, use non-linear difference or differential equations, and as analysis of such equations, with the exception of some first order ones, is either difficult or impossible;
numerical simulation is necessary to examine the inter-temporal dynamics of output fluctuations (Baker & Gollub, 1990 or Ott, 1993). Faced with such limitations, many researchers, such as GS, have resorted to methods of linearization in the neighbourhood of a steady-state solution, including Taylor’s formula or Liapunov’s direct method (Azariadis, 1993). We consider linearization around steady-state equilibrium, but show that such simplifications ignore and obscure the rich dynamics inherent to the GS model, such as higher periodicity, period-doubling bifurcations, and multiple steady-states. We discover, in fact, that the claim by GS that their model generates cycles of multiple periodicity is unwarranted using linearization. In our research, using numerical simulation, we discover that small changes in initial parameter values produce alternative trajectories of varying periodicity, changing the model’s laws of motion. The inter-temporal dynamics of a non-linear models based upon the work of GS may even exhibit aperiodic cycles which are extremely sensitivity to initial conditions, deterministic chaos. Furthermore, through calibration of our numerical method, we find that for any form of cyclic behaviour to occur, a minimal level of technological investment productivity is required, and that an aggregate economy exhibiting greater productivity, appears more prone to oscillatory vibrations.

In order to undertake numerical simulation we naturally require explicit functional forms. Using the various GS models as guidance, we combine the requirements of their implicit functional forms with received micro-economic theory to specify explicit equations and relationships. Although there are software packages for simulation of difference and differential equations, we have chosen to write our own program, using a spread-sheet language. Designing the structure and algorithms of the simulation program de novo, has helped in the research conceptualization. Having problemitized the issues surrounding New Keynesian models of economic fluctuations and shown how we intend to confront such issues, we turn to the
literature review found in Chapter 2, in order to place the subject matter in a formal setting.
2.0 Introduction

In the following literature review, we will examine in depth the two contemporary approaches to analysing and modelling economic activity and disturbances of a cyclic nature. Business cycle theories may be compared and contrasted using three key concepts:

- Perturbation, that is how cycles begin;
- Propagation, that is how economic forces disseminate and interact; and
- Laws of Motion, that is the nature of periodic motion (or limit cycle) into which the system settles after some transient period or the nature of the steady state when motion ceases.

Applying these three concepts to the literature, the differences between contemporary schools of thought will be made clear. These efforts will show how our topic, a theoretical contribution to the business cycle literature which incorporates certain recent financial innovations at the microeconomic level, is related to the field of research and knowledge. In so doing, we will establish both its justification and lineage. Through tracing key themes found in the topic of Business Cycles, the intellectual context of our own work will be examined. By revealing the field’s key themes, we will be able to show how our own research is both derived from and contributes to the body of knowledge.

2.01 Schools of Thought in Retrospect

The evolution of business cycle theory has been a long one, and has involved virtually all schools of economic thought. Surveying the many contributions to the business cycle literature, leads to two related observations: That emphasis upon reconciliation and synthesis with received views has been minimal and attempts to reconcile the many contributions with the paradigms of microeconomics presents a great challenge. Ignoring previous work,
often new models of business cycles have arisen in order to explain the latest events. Advancements in economic and econometric techniques have also spurred the development of new business cycle theories, but with minimal interest in ensuring reconciliation with previous research. Many theories of business cycles have relied upon a particular modification to the basic micro economic paradigm without considering its general equilibrium implications. Although theories of business cycles have evolved with the progress of technique and the desire to explain prevailing events, yielding distinctive modifications to received paradigms, the challenge of producing a unified or general theory embracing competing views and having adequate foundations in microeconomics remains.

Competing theories of business cycles abound, often having distinct micro economic foundations, in part, because what they seek to explain varies. This point is illustrated in the many definitions found of what a business cycle is. There are strongly empirical and theory neutral definitions such as the following: "Business cycles are recurrent sequences of alternating phases of expansion and contraction that involve a great number of diverse economic processes and show up as distinct fluctuations in comprehensive series on employment, income, and trade-aspects of aggregate economic activity." (Sargent, 1992, page 283). In a similar empirical vein, Sargent describes business cycles as the "... tendency of certain economic variables to possess persistent cycles of approximately constant amplitude..." (Zarnowitz, page 215, 1979). On theoretical grounds some definitions of business cycles excluded amplitude criteria, as found in an earlier definition of Burns and Mitchell (1946). According to Mitchell and Burns, expansions and contractions may be strong or weak, sudden or persistent. The definition of Dotsey & King (1987) found in the New Palgrave Dictionary of Economics has strong theoretical flavour. Their definition includes the statistical aspects of business cycles: They define a business cycles as the "...
stochastic components of macroeconomic time series - as stationary stochastic processes." implying that cycles may not arise from deterministic and non-random events. Meanwhile, also on theoretical grounds, Nelson & Plosser (1982) define business cycles as not necessarily stationary and not necessarily arising from stochastic components of time series. The variety of business cycle definitions underscores the difficulties inherent to any coherent classification. With these competing definitions in mind, we now discuss some earlier theories of business cycles in retrospect. Considering them, will permit us to see how in contemporary research, such themes reoccur.

Early economic theories of business cycles relied largely upon exogenous forces as sources of disturbances. For classical economists such as Smith, Ricardo, and Mill, exogenous forces or perturbations producing cyclic behaviour included weather, demographics and technology. Such perturbations alone explained the occurrence of business cycles, because the model embodied among other features, a vertical aggregate supply curve, flexible wages and prices. Any system or model based upon such propositions exhibited neutrality, that is, the equilibrium was invariant with respect monetary effects; and dichotomy, that is real variables are independent of the absolute price level.

More formally, business cycles in the classical system stood upon four pillars: Namely, that i.) Prices were set in order to clear all markets- including the labour market; ii.) There was no money illusion; iii.) Price expectations had unitary elasticity; and iv.) Alternative equilibriums did not have any distribution effects. In the classical system, the independence of output and employment from the price level implied a vertical aggregate supply, movements in which alone can change the level of output. In classical business cycles, changes in aggregate demand only affect the level of prices. Thus exogenous forces were relied upon because under the classical system, by virtue
of Say's Law, market economy's generated stable and efficient equilibria. If markets function in the Walrasian sense, then cycles can only arise from exogenous shocks, or cycles must be equilibrium phenomenon, as Frisch (1933) would later argue. The use of exogenous forces to explain business cycles produced a model fully compatible with micro theory.

Arguably in response to economic conditions and as alternatives to the classical model, endogenous models of business cycles were posited. Such models of business cycles were based upon theories at variance with the axioms and propositions found in the classical system. A subject of discussion was whether an efficient and stable equilibrium may be achieved at minimum cost. In this vein, examples of endogenous models include Hawtrey's gold standard theory (1913) or under-consumption theories of Hobson (1909, 1922). In the 19th Century, Marx gave rise to many theories showing that cycles are inherent to an economic system involving the private ownership of capital and required an excess supply of labour. In the opinion of endogenous cycle theoreticians, the invisible hand was either flawed or non-existent. Later endogenous theories of business cycles utilized investment-based theories; good examples of which are the works of J.M. Clark (1917) and Wicksell (1936). Kalecki's (1935) model used the following dynamic equation of capital accumulation, to produce constant oscillations in investment capital, $K$, for certain but arbitrary values in the parameters $a$, $τ$, $n$.

$$K(t) = \frac{a}{τ} K(t) - (\frac{a}{τ} + n) K(t-τ)$$

According to Goodwin (1951), Kalecki believed that the institutions and internal structures of capitalism inevitably generated business cycles. Endogenous theories of cost-price/profit margin relationships were also called into service in the works of Mitchell (1941). According to Sherman (page 70, 1991), Mitchell saw the business
cycle as inherent and unique to capitalist institutions". Hayek (1933 and 1939) proposed that disturbances expressed as business cycles were equilibrium phenomenon rather than disturbances from equilibrium. Endogenous theories of business cycles challenged the classical structure, posing the possibility that if the system was not self-correcting and if perturbed, would not return to equilibrium through the effects of excess supply and demand upon price.

Some proponents of endogenous economic cycles went so far as to argue for a dropping of key axioms and propositions found in the Classical economic paradigm. Kuznets in a seminal paper, entitled "Equilibrium Economics and Business Cycle Theory", argued that the incorporation of business cycle theory into economic theory may require the abandonment of the rational agent assumption (1930). Kuznets stated that there were two ways of incorporating business cycles into classic microeconomic and equilibrium theory: 1.) as "... a consequence of cycles in outside factors and that this variation in what might be called the economic constants does not necessarily disturb the determinate fundamental relations between economic factors..." (page 396); or 2.) as a "... deviation from a preconceived picture of reality..." (page 399). Today we would call the former Real Business Cycles, and the latter as New Keynesian Business Cycle Models.

In the 1930s, a comprehensive theory of the macro economy and cyclic behaviour circumventing reconciliation with micro economics was tried in the work of Keynes. In the General Theory, Keynes (1936) side-stepped classical propositions discussed above, by replacing the microeconomic division of value theory and monetary theory found in classical economics, with a paradigm based upon output as a whole which was distinct from the theory of the individual consumer or firm, (Hoover, 1991). Keynes attributed the business cycle, and in particular the events of the 1930s, as a
collapse of the marginal efficiency of capital combined with wage and price inflexibility. Instability in aggregate demand was the source of fluctuations in aggregate economic activity. Keynesian business cycles were the result of endogenous flaws, inconsistencies which prevented a full-employment solution. In the General Theory, three alternative endogenous explanations appear: 1.) Wage levels are too high to be consistent with full employment; 2.) The expenditure sector curve never reaches the full-employment level of real income at any positive rate of interest; and 3.) The demand or supply for real cash balances imply a monetary sector curve at too high a rate of interest. The possibility of endogenous innovations modifying such cycles was not considered.

As a tool for business cycle analysis, the Keynesian model or its heir, the textbook IS-LM neo-classical synthesis developed by Hicks (1937), became the source of many debates, mostly involving its micro foundations and the nature of equilibrium. Surveying the Keynesian style models of the post World War II period, we find that many used endogenous relationships such as multipliers and accelerators to generate cycles (Gordon, 1986), and to add a dynamic dimension (Plosser, 1989). Findings faults with the Keynesian system began not long after the publication of the General Theory. There was much debate over whether the equations of the national income model had a full-employment solution (page 47, Bailey 1971). Hicks and Leontieff initiated the attack suggesting the interpretation that the model of Keynes was special case, not a contraposition, and hence reconcilable with the classical system (Hicks, 1937). Later critics such as Clower (1965), observed that the Keynesian business cycle showed neither convergence nor stability, and violated Walrasian law of flows. Moreover, according to Hahn (1977), it was not a general equilibrium model. Whether the synthesis structure was used for Keynesian analysis (fixed money wages) or monetary neutrality positions, it lacked rigour at the micro economic level. It
attempted to identify disequilibrium with inefficient sub-optimal outcomes, such as unemployment; while equilibrium meant full employment.

The frontal challenge to Keynesian economics began in the 1960s with the rise of monetarism, the principle proponent of which was Friedman (Friedman & Schwartz, *A Monetary History of the United States*, 1963). Again using the same basic IS-LM Model, exogenous money supply effects were considered for their nominal effect. Utilizing a quantity-theoretic proposition that changes in the stock of money are the main determinant of changes in nominal income, monetarists argued that the demand for money is relatively stable. Fluctuations in aggregate demand and real economic activity could arise from phases of low and high growth rates in the quantity of money, because of significant lags in wage and price adjustments. Hence, the main source of critical monetary disturbances is outside the private economy in central bank policy actions. The monetarists approach to business cycles claimed to be in harmony with classical micro foundation, because it attributed nominal cycles to exogenous erratic aggregate demand growth caused by unstable monetary growth. Monetarists believed that fiscal policy was ineffective, and that monetary policy alone should be the tool of policy. Like Keynesianism, the level of aggregation, however, obscured many issues.

Considered in retrospect, although the structure of the basic Keynes - Hicks ISLM Model was adaptable to different schools of business cycle research, the problem of ensuring consistency between competing theories, and reconciling such theories with micro economic precepts remained. As tools for analysing business cycles, ISLM/Neo classical synthesis models were weakest where they had to be the strongest, that is in the concept of equilibrium. Were business cycle phenomena such as unemployment and declines in GNP the result of disequilibrium situations? Using short-run non
neutral money, Patinkin tried to explain business cycle phenomenon as short-run disequilibrium phenomenon occurring as the real balance effect (1954). A sluggish real balance effect implied that money supply shocks could have short term real effects. The levels of aggregation in such works, however, obscured critical issues. Moreover, it was only a model of short-run disequilibrium. Clower (1965) questioned its logical consistency, claiming that it lacked adequate micro foundations. According to Hahn (1977, page 26), either such theory was not about equilibrium at all or it depended on certain price rigidities. Such models begged the question of whether agents were out-of Walrasian equilibrium or whether a different notion of equilibrium was required. Such criticisms had significant theoretical implications because business cycles had long been considered equilibrium phenomena. In order to reconcile business cycles with received classical microeconomic theory and to create a cohesive vision, it was felt that the debate needed to change gears. Perhaps better macro economic theory, demanded stronger micro foundations, particularly with regard to such issues as exogeniety and equilibrium. It was felt such efforts would lead to a better understanding of business cycles.

Interest in creating a unified model of the macro economy which explains cyclic behaviour, and which is consistent with received microeconomic theory, led to research into equilibrium and disequilibrium theory. Although the contributions such as Barro and Grossman's disequilibrium model of a macro economy provided inter-temporal adjustment insights, the behaviour depicted did not resemble empirical cycles (Barro & Grossman, Chapter 6, *Dynamics of Aggregate Demand*, 1976). A further criticism of disequilibrium macro models was that at a micro economic level, they implied uncleared markets and non-optimizing behaviour on the part of individuals. Exogenous models, in contrast regard cyclic behaviour as disturbances from equilibrium. The random shocks required for such
models, however, do not produce the persistent movements in output or employment identified with business cycles, unless a propagation mechanism as used by Frisch (1933) was utilized. Such issues have led to research into questions involving the existence, desirability, and stability of equilibrium. Whether expansions and contractions occur as part of an equilibrium process, or whether cyclic disturbances are evidence of the economy being out of equilibrium became the focus of debate.

Equilibrium theories initially utilized monetary misperception mechanisms, as pioneered by Barro (1977) and Lucas (1972,1975), as the source of impulse for cycles, but were replaced more recently by real business cycle theories, often called New Classical theories. Originated by Nelson & Plosser (1982) and Kydland & Prescott (1982), their approach involves equilibrium using unanticipated exogenous changes in technology as a source of cycles. In such models, the shocks arising from changes in technology dissipate over time. The emphasis in real business cycle theory is upon the source of disturbance to equilibrium and generally does not include the propagation mechanism. New Classical models of business cycles work as displacements from equilibrium.

In contrast, contemporary disequilibrium theories, known as the New Keynesian school, use nominal rigidities and inefficiencies at the microeconomic level as the source of aggregate disturbances. Claims of realism at microeconomic level figure strongly in this school of thought, although this may be somewhat disingenuous. According to Greenwald and Stiglitz (hereafter, GS) (page 120, Keynesian, New Keynesian and New Classical Economics, 1987) this school of thought seeks to "... adapt micro-theory to macro-theory." Rather than trying to develop a macroeconomic theory which is consistent with received precepts of microeconomics, NK researchers have turned the issue on its head, that is attempting to re-write the latter. Such research was pioneered by Mankiw (1985), and Akerlof &
Yellen (1985). According to New Keynesians, rigidities at the micro level become cycles because of sub-maximization of social welfare: Flat profit functions imply income losses are second order in magnitude, and hence adjusting prices is not justified, (Krainer 1992 page 48). In addition to the equilibrium versus disequilibrium distinction, other key differences between these two contemporary schools of thought include the dichotomy between nominal variables such as money and real variables such as output. Known as Money Illusion, in the New Keynesian school the distinction is violated. Misperceptions between what is nominal and what is real at the microeconomic level are both a source of perturbation and explain cycle promulgation. In addition to remaining inconsistent with standard micro economics, New Keynesian models, as they involve many "small stories" or competing models, do not lend themselves easily to a synthesis or general view.

2.02 Discussion Plan

By discussing business cycle theories in retrospect, we have seen the emergence of contemporary research from its earliest traditions. In addition, we have seen the varying role played by micro foundations from the original implicit models of business cycles found in the writings of the classical economists to the work of the present day. We have seen the difficulties of reconciling competing schools of thought with one another and with received micro economic theory. In contemporary business cycle literature, the New Keynesian school and the New Classical-Real Business cycle school, the debate over micro economic foundations as we have seen, continues, as manifested in concern over the nature and existence of equilibrium and exogeneity. In order to understand such research in-depth, identify its short-comings and thereby create grounds for the present work, we utilize the mathematical concepts found in Lagrangean dynamics and the theory of chaotic dynamics or sensitivity to initial
conditions, originating in the works of Henry Poincare’ (1913). By applying such concepts to contemporary schools, their strengths and weaknesses in relation to the issues raised above, may be analyzed.

We begin using the concept of *Perturbation* as a tool for contrasting and comparing the Real Business Cycle approach with that of New Keynesianism, followed by *Propagation* and *Laws of Motion* or *dynamical behaviour*. (We use the term, *perturbation* interchangeably with the terms such as, *shocks* or *impulse*.) By discussing how cycles begin, and spread-out, whether they are *conservative* or *dissipative*, that is whether they exhibit *persistence*, and the nature of periodicity (if any); the differences between contemporary schools with respect to micro economic precepts and such notions as equilibrium, duality and dichotomy; and exogeneity will be delineated. This schema will reveal the literature’s limitations and point the way for future research.

### 2.10 SOURCES OF PERTURBATION

#### 2.11 Background

By the term, *perturbation*, we refer to the shocks or disturbances or other forces upon a system, in this case an economy, which are additional to the forces which causes its regular motion. It is also distinct from the *propagation* mechanism. By propagation we refer to how disturbances interact, and spread from one economic variate to another. In the business cycle literature, perturbation as a source of cycles has been distinguished from cycles which arise in of themselves, from a specified model’s structure. In contemporary economics, exogenous perturbation as a source of cycles is generally associated with Real Business Cycles, while endogenous propagation is associated with New Keynesian Cycles. This typology of business cycles, we will show, however, is not rigorous: Lagrangean classical dynamics allows for small oscillations around both a static equilibrium position and steady motion. A model in which perturbations modify
or augment endogenous cycles, through some form of persistence(1,2),(996,988) or propagation may as well be constructed.\(^1\)

Historically, the endogenous and disequilibrium models of business cycles, whether it was Marx, Kalecki, or Keynes have always been the main opposition to the equilibrium classical system, using exogenous shocks. Such dichotomy may however, be artificial. There is no a-priori reason, why model economies might not involve both exogenous impulses along with endogenous dynamic cycles. The seminal work of Frisch (1933) distinguished analytically between perturbation and propagation, however, in his view dynamic laws explain the damped oscillations, that is dissipation of energy through propagation; while impulses or shocks, as forms of perturbation, are the random shocks which by adding energy, maintain the cycle. According to Frisch (1933, page 197), impulses or shocks outside the economy can maintain a cycle generated from within the system. Thus for example:

\[
x_{t+1} = f(x_t, z_t)
\]

(2.2)

Given an exogenous process \{z_t\} and an initial output condition \(x_0\), the equation suffices to uniquely determine the evolution of the predetermined state variables, which in turn predetermines the evolution of the other endogenous state variables. If there are no exogenous shocks the equation for \(x_{t+1}\) reduces to:

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\(^1\) Under Classical Dynamics, with constant coefficients and linear equations of motion, analysis of disturbances and oscillations assumes that the body is rigid, that is the position of any part of the body's mass does not change in position relative to any other part, regardless of the forces acting upon the body. (For example, the ratio of unemployment to imports is constant throughout the cycle, changing proportionately.)
Unless such a system converges asymptotically to some constant value, in the form of a steady state, call it, \( x^* \), we would have an equilibrium with endogenous fluctuations. Endogenous fluctuations may be either deterministically cyclic or deterministically chaotic, that is sensitive to initial states (Guesnerie & Woodford, 1992), a point which we pursue in depth below. Thus, on the basis of mathematical formalism alone, there are no reasons to define endogenous cycles as a concept relevant only to economies without exogenous shocks. There is no a priori reason to dissociate endogenous cycles from models exhibiting steady state equilibria.

2.12 Real Business Cycle Equilibrium Models

Typically in contemporary exogenous shock models the emphasis is upon the source of perturbation. The equilibrium is well defined, and often unique locally, and stable in the sense that in the absence of recurrent exogenous shocks, the economy will tend towards a steady state, but because of shocks a possibly stationary, (that is, having constant phase space) pattern of fluctuations will be observed. Propagation in such models is not emphasized, and as the systems depicted by such models are dissipative, persistence is minimal. The equilibrium and exogenous school of business cycles, falls under the rubric of New Classicism, uses various sources of perturbation to generate cycles. For example, in the new classical approach of Lucas (1972, 1977) and Barro (1976); misperceptions in the money supply process, that is, extracting signal from noise, results in shocks or impulses which disturb the system from equilibrium. Such shocks are exogenous. A subsequent version of new classicism, known as real business cycles, is found in such works of Nelson & Plosser (1982), Kydland & Prescott (1982), Long & Plosser (1983), Barro & King
(1984), King & Plosser (1984), or Cooley & Hanson (1989); in such research changes in aggregate supply through technology or labour are the source of perturbations. The real business cycle model uses exogenous perturbations to an equilibrium system to generate output fluctuations. These shocks may be caused by changes in demography, technology, or other factors. Supply side shocks can be represented with the following production function:

\[ Y_t = f(L_t, K_t, z_t) \]

where \( Y \) is real GNP, \( f \) is the functional form, \( L \) is the labour input, \( K \) is the capital stock, \( z \) is a term that picks up shocks to the production function, all subscripted over time, \( t \). In real business cycle models, the source of perturbations, the shock term is assumed to evolve according to the following process:

\[ z_{t+1} = \alpha + z_t + \epsilon_{t+1} \]

where \( \alpha \) is a constant term, and \( \epsilon \) is a random error term with an expected value of zero. In many real business cycle models, technology increases at a constant rate, \( \alpha \), plus any innovations in the random error term, \( \epsilon \), which may be expressed analytically, thus:

\[ \Delta z_t = \alpha + \epsilon_t \]

Thus changes in the production function are a random walk with drift.

According to real business theory, an economy is constantly receiving exogenous perturbations or shocks to the production function, as expressed above. In this school of thought, such shocks are not necessarily offsetting, that is a positive shock is not necessarily followed by a negative shock. Hence, the effect of shocks may accumulate, leading to some form of persistence. If
consecutive shocks occur in the same direction, then output growth changes. These shocks are viewed as *persisting* over time, for the foreseeable future. This observation has intuitive appeal because in the real world effects upon output activity may persist for several quarters, above the long run growth trend. Although real business cycle models involve the supply side of the economy, interactions with money are not precluded. In some models, there is a correlation although without Granger causation, between output and money: Increases in real GNP lead to higher demand for real money balances, thus linking a perturbation with a method of propagation.

2.13 New Keynesian Disequilibrium Models

As explained above, although there is no reason for endogenous cycles to be only relevant to economies without exogenous shocks. Nonetheless, this is often the tactic employed by researchers using such models. In the New Keynesian literature, shocks, impulses, or perturbation do not play an important role. In the works of R.J. Gordon (1982); Mishkin (1983); or Hall & Fields (1987) deterministic cycles originate in the endogenous rigidities. While in the work of Mankiw & Romer (1988) or Mankiw (1990) weak incentives by individual firms to optimize profits by adjusting output, leads to sub-optimal social welfare in the aggregate, demonstrating our earlier point with regard to New Keynesian research turning the problem of reconciliation with received theory, or leads to *re-writing* micro economics. It also leads to many competing, not necessarily consistent explanations of dynamic behaviour. In New Keynesian models of the economy, the fault is not in the stars, but in themselves.

Let us look at perturbation in the New Keynesian business cycle in depth. A major contributions of New Keynesian research is the prediction that unstable aggregate demand and supply are important
determinants of the business cycle. Aggregate demand instability causes business cycles because wages and prices are assumed to be less than perfectly flexible in the short run. ("Menu Costs" in Mankiw, 1985). Aggregate supply effects cause business cycles for the same reason that the classical model suggests: real changes in the labour market and/or the production function change the quantity of output firms can produce at a given price. Thus in the New Keynesian perspective, the source of the cycle is endogenous to the economy itself. (Around what sort of growth trend the cycles occur, for example static, deterministic, stochastic, is a matter we will address later.)

In the New Keynesian business cycle, the sources of cycles are sometimes similar to those found in traditional Keynesian reasoning. Erratic private expectations of consumption and investment spending lead to aggregate demand uncertainty. In addition to such sources for cycles, New Keynesians also stress unstable money demand. Thus violating classical neutrality, as such phenomenon it is argued, have real effects. These additional theories of cycles are very different from traditional Keynesian arguments. Furthermore, they distinguish the New Keynesian school from dichotomy of the real business cycle school: nominal magnitudes have real effects. How such endogenous perturbations work their way through the economy is also different in the New Keynesian business cycle, and will be discussed in the subsequent section.

2.20 METHOD OF CYCLE PROPAGATION
2.21 Background

Propagation refers to the internal dynamics of a market economy, that is with or without a policy shock or impulse to fundamentals (money supply, tastes, or technologies), how disturbances from a static or steady motion equilibrium by one or more economic variates, for example unemployment or interest rates, lead to disturbances in
other economic variates, such as GNP. Propagation phenomenon may arise from an initial shock, like ripples in a pond from a stone, or from endogenous oscillations. It may be of a dissipative or conservative nature. Propagation, may or may not involve exogenous impulses or disturbances, and may lead to either deterministic or chaotic cycles. Propagation is an important aspect of business cycles because the transmission from one market to another may generate cyclic behaviour which persists long after the initial stimulus is gone, as shown by Rotemberg & Saloner (1986). Moreover, the transmission of energy across the economy may lead to cyclic activity at the aggregate level. We will find below, that contemporary schools of business cycle research take very different perspectives on these issues.

2.22 Propagation in New Classical Business Cycles Models

In New Classical Models of business cycles, whether we are referring to the older monetary misperception models of Lucas and Barro, or the real business cycles of Kydland & Prescott, how a cycle begins has always been more important than how it spreads out. In the Lucas (1975) model of monetary misperceptions, temporary external random shocks are propagated into a cycle (around a growth trend) by a linear propagation model which provides a mechanism for the effects of the shock to persist over time. For Lucas and Barro, the cause of business cycles is outside the economy, while the endogenous structure only serves to propagate the impulses through the system.

In real business cycles, real shocks rather than monetary shocks are transmitted into cycles by, a linear propagation mechanism. Some real business cycle models developed in the late 1980s, (see, Stockman (1988) for a review) included dynamic recursive techniques to solve sole agent optimization problems in linear propagation mechanism. Real business cycle models rely mainly upon the shock
rather than the linear propagation to generate cycles. This approach is a break from the tradition of Wicksell and Frisch, who combined exogenous shocks with endogenous methods of propagation to generate cycles.

Real business cycle propagation is something of a hybrid between damped oscillation models, where new energy is required in the form of repeated shocks, to offset the dampening, as found in the model of Frisch (1933); and the non-oscillatory convergence models where the cycle is driven purely by shocks, as found in Slutsky (1937). Real business cycle propagation does not use persistence (serial correlation) to propagate temporary random shocks (e.g. technology, demographics) as in the work of Slutsky who showed that output resembling economic time series could be generated in this manner. In real business cycles, because linear propagation is not strong enough, shocks must be auto-correlated or persistent to generate cyclic behaviour resembling actual time series. Propagation in the real business cycle model, beginning with Lucas, interprets the propagation mechanisms as a means of transmitting exogenous shocks through an otherwise stable economy. Propagation transforms shocks into cycles. In contrast, for earlier economists, notably Frisch, dichotomy between impulse and propagation was developed to explain how exogenous impulses outside the economy can maintain a cycle generated from within the system. In real business cycles propagation takes something of a back-seat in driving the economy. Serial correlation is not emphasized. Propagation merely explains how exogenous impulses of themselves become cycles. According to Mullineux, Dickinson & Peng (1993) such emphasis, however, may be implausible. A fundamental objection to real business cycle propagation mechanisms, they argue, is that they are linear. Surveys of non-linear business cycle modelling (for example, Scheinkman (1990)) show that the presumption of linearity has little merit, making linearization around the steady-state, as many
widely used, of questionable merit. Non-linear models, as we will show below, exhibit empirically appealing dynamics, resembling business cycle activity, as well bifurcation and deterministic chaos.

2.23 Propagation in New Keynesian Business Cycles Models

We have explained that New Keynesian Disequilibrium Models rely upon erratic private expectations of consumption and investment, along with unstable money demand, as sources of aggregate demand instability. Rigidities act as an endogenous source, although not an exogenous impulse, to business cycles. New Keynesian propagation of such unstable demand when combined with rigidities works in its own way, as we explain below.

In the New Keynesian business cycle, if an economy experiences an increase in aggregate demand, the initial response will be an increase in both prices and output. The increase in demand, even if it is anticipated in such models, causes prices to rise but only modestly relative to the change in demand because New Keynesians argue that private disincentives such as menu costs and mark-up pricing practices keeps prices from being completely flexible in the short run. Rigidities figure strongly. With mark-up pricing, prices rise in the short run by the full amount of the change in aggregate demand only if the costs also rise by the full amount of the change in aggregate demand. New Keynesians, however, claim that short run costs are insensitive to demand changes.

In the works of New Keynesian researchers such as Gordon (1982), Mishkin (1983), Mankiw, Akerlof & Yellen (1985), Rotemberg (1987), Ball, Mankiw & Romer (1988), or Romer (1990) there are several explicit propagation mechanisms. In such models, short-run costs are seen as insensitive to changes in demand. In the short-run nominal wages are also rigid, explaining why costs are insensitive to changes in demand. In the models of Taylor (1980) and Blanchard (1987) wages and prices adjust differently, because of
implicit and explicit labour contracts. Another reason for slow cost adjustment arises from the existence of price contracts between firms and suppliers. Until contracts are renewed, orders are received and filled at old prices. For example, in the model of Caplin & Spulber (1987) the fixed cost of changing prices, "microeconomic frictions", have macroeconomic effects. Failing to adjust to changes in the short-run, costs in the form of nominal rigidities, in some models arise because firms are assumed to operate in imperfectly competitive markets, as in Mankiw (1988) and Startz (1989). A further New Keynesian business cycle propagation mechanism involves accounting practices. First-in, first-out, procedures emphasizes historic rather than replacement costs. If items are not priced at the marginal replacement cost, then perceived costs will not move with demand. The net result of such propagation phenomenon is that an increase in aggregate demand will cause prices to rise in the short run, but not by the full amount of the increase in demand. Moreover, as part of the propagation mechanism, the increase in price level drives down real wages which allows firms at the aggregate level to employ more labour and produce more output. To describe propagation in New Keynesian business cycle models our focus is upon features which retard or prevent classical solutions. New Keynesian propagation is, in a sense a friction, retarding the clock-work like mechanism of the classical system. New Keynesian rigidities act like frictional or viscous forces in classical dynamics, leading to a dissipation in system energy thereby preventing a return to Walrasian equilibrium. Disequilibrium is stable, and not merely a short-run deviation from neo-classical results. Otherwise much of New Keynesian economics would simply involve the slowness of convergence and not sub-optimization. A good example of such New Keynesian propagation mechanisms, is found in the model of Blanchard and Kiyotaki (1987) or Akerlof and Yellen (1985), prices as set by individual firms do not adjust to profit maximizing levels of output, after changes in costs,
because the profit function is relatively flat around the optimal level and therefore, the loss from not adjusting is of a second-order magnitude (by the envelope theorem). Such behaviour collectively, leads to sub-optimization and a loss in welfare.

Consider an example based upon the model of Akerlof & Yellen (1985), in which monopolistically competitive firms have set prices and output so as to maximize profits. Their sales depend upon relative prices (along the demand curve) and aggregate demand (such as real money balances) which shifts the demand curve. In this initial equilibrium, output has been determined so as to maximize profits. Now suppose that there is a small decline in the money supply, and because of rigidities, several firms do not adjust prices downward. In a classical sense, because the profit function is relatively flat, they behave sub-optimally because the profit loss is relatively small from sub-maximization. Such a situation might also arise if firms do not adjust prices because of short-run cost rigidities, as discussed earlier. Now, if a sufficient number of firms behave this way, prices will be higher than they would otherwise be because of the contraction in output, and hence the value of real balances will be lower. Such second order effects lead to overall rigidity of general price level. The implicit propagation mechanism in such cycles is that second order effects, because of flat profit functions, are not sufficient incentive for an individual firm to adjust prices to profit maximizing levels. In the aggregate, such implicit propagation mechanisms, reduce by a first order magnitude the value of the real money balance, eventually shifting the demand curve for all the firms remaining. Krainer calls this phenomenon an "aggregate demand externality." (Krainer, page 48, 1992).

2.30 LAWS OF MOTION

2.31 Background

Above we have discussed the issue of propagation with respect
to contemporary business cycles. Our concern was how either
exogenous shocks or endogenous rigidities spread-out to affect the
larger economy, and thereby generate a business cycle. In this third
section on *Laws of Motion*, using concepts found in classical
dynamics, our concern is now the nature of periodic motion.
Periodicity may be of any order, although if we identify periods with,
say, three month quarterly economic data, cycles in excess of eight
iterates or more are uncommon. Our focus will be upon how
business cycle aggregates, such as output, behave over time about
a position of either equilibrium or steady state motion or even
deterministic chaos. The stability of motion will be a key concern.
In New Classical models and Real Business Cycle models, we will see,
motion arises from recurring shocks to the production function and to
the supply side of the economy. Such perturbations lead to motion
which is either of a periodic nature (a limit cycle) or after a transient
period settles into a steady state, viz motion has ceased. Such
shocks or perturbations are stochastic, they exhibit persistence, and
are not necessarily off-setting.

In contrast, the laws of motion found in New Keynesian business
cycle models arise from the micro economic imperfections specified
in their structure. Such motion is self-generating and endogenous to
the specified structure, including erratic expectations with regard to
investment and consumption spending, but not stochastic
perturbations like the Real Business cycle school. The motion
exhibited by New Keynesian models may involve oscillatory behaviour
(limit cycles) of various orders of periodicity and may even exhibit
deterministic chaos. Some endogenous models involving
disequilibrium theory presents a *special problem* from a laws of
motion standpoint, as measuring displacement without either a static
or equilibrium path coordinate, the degrees of freedom, in a classical
dynamic sense, is not defined. Borrowing from classical dynamics,
if we do not know where a physical mass is supposed to be, or in an
economic setting where an economy's output should be, as in classical equilibrium, how can displacement via a law of motion have any significance? A reference point is required to measure displacement from it.

2.3.2 Laws of Motion in New Classical Real Business Cycle Models

In contemporary equilibrium business cycles, including both the original monetary disturbance models of Lucas and Barro or the newer real business cycle models of Kydland and Prescott, cycles originate with an exogenous perturbation mechanism, in the form of a stochastic shock to the supply side of the economy. Propagation uses a linear mechanism in order for the shocks to persist over time. Such impulse energy, however, is not generally sufficient to produce continuous cycles, resembling economic time series, which as mentioned earlier has been a criticism. Rather, once the energy from such shocks has dissipated, a return to a dynamic stable equilibrium occurs. The laws of motion for real business cycles may be described in the following manner: After an initial shock, the energy from such shocks is dissipated, fluctuations are damped and the time path after a shock is convergent. In most such modes there is insufficient persistence for shocks to generate oscillatory motion. According to Real Business cycle theorists, economies are constantly receiving such shocks, creating the regular oscillations we call business cycles.

The laws of motion for real business cycles part company with some of their intellectual predecessors: In the model of Frisch (1933) sufficiently close erratic impulses, adding energy to the system, through persistence are sufficient for cycles to continue. This possibility was also proposed by Slutsky (1937) and in the earlier work of Wicksell (1936). In real business cycles there is a little persistence to shocks (no auto correlation), while cycles occur within a dynamic equilibrium process which perturbations briefly effect. In time the economy returns to its trend growth, which may exhibit
oscillations (Hall, page 126). Such oscillations are generated as an optimising agents’ response to productivity shocks. As Prescott (1986) expresses it: "... given the people’s ability and willingness to inter-temporally substitute consumption and leisure, it would be puzzling if the economy did not display these large fluctuations in output and employment..." A model developed from the literature will illustrate this observation analytically.

In order to see how a real business cycle model is able to generate cyclical fluctuations solely through a consequence of the optimising behaviour in response to a technological shock, we review the basic ingredients of the Plosser (1989) model. The utility function of a representative agent from consumption, $C$, and labour, $L$, is of the form:

$$U_t = E_{s=0} B^{ts} \left( C_{t+s}, L_{t+s} \right)$$

(2.7)

The production function, of a single final good $Y_t$ produced under constant returns to scale technology is represented by:

$$Y_t = \Theta_t F(K_t, N_t)$$

(2.8)

Where $K_t$ is the predetermined capital stock at time $t-1$; $N_t$ is the predetermined input in time $t$; and $\Theta_t$ is a temporary shift factor to model total factor productivity. Now, the produced commodity $Y$ can either be consumed or invested. If invested it becomes part of the capital stock, that is available for production in the next period. Thus capital stock will evolve according to the following non-autonomous first order dynamic difference equation, where $I_t$ is investment at time $t$:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

(2.9)
The following resource constraints for each period $t$ are:

$$L_t + N_t \leq 1, C_t + I_t \leq Y_t$$

A representative agent will be used to compute a set of relative prices which are market clearing, viz equilibrium prices. These choices represent at the individual agent level, the outcome of a competitive economy. Optimally, the agent chooses an amount of consumption, work effort, and output over time. The optimization problem faced by the representative agent is specified by the following Lagrangian:

$$L = \sum_{t=0}^{\infty} B^t \left[ u(C_t, 1-N_t) + \sum_{t=0}^{\infty} \lambda_t \left[ \Theta_t F(K_t, N_t) - C_t - K_{t+1} + (1-\delta) K_t \right] \right]$$

The first order conditions give the time paths for the variables using the time-invariant decision rules:

$$(2.12)$$

$$C_t = \Theta_t \left[ \Theta_{t+1} \right]$$

$$N_t = \Theta_t \left[ \Theta_{t+1} \right]$$

$$K_{t+1} = \Theta_t \left[ \Theta_{t+1} \right]$$

To understand the Laws of Motion for this real business cycle model, we begin by paraphrasing Plosser's (1989, page 73-74) observation that "in the absence of changes in technology, that is when $\Theta_t = \Theta$ for all $t$, and given some initial capital stock; the values of consumption, hours worked, capital and output for the representative agent described above, will converge to the constants above, referred to as the steady state, that is where the relevant variables all grow at the same identical rate." As a Law of Motion, the steady state is a generalization of the concept of a stationary
state, in which the relevant variables all remain constant, that is grow at zero rates. Since, according to Plosser, the decision rules are monotonically stable, non-linear decision rules may be approximated using linear equations. King, Plosser, & Rebello (1988) use this assumption to isolate the exogenous shocks as the origin of the business cycle. An example of the steady state for the capital stock equation would be the following:

(2.13)

\[ \hat{K}_{t+1} = \mu_1 \hat{K}_t + \Psi_1 \theta_t \Psi_2 \Sigma_0 \mu_2 \theta_{t-j-1} \]

Where the arguments, \( \mu_1, \mu_2, \Psi_1, \Psi_2 \) are functions themselves of the underlying parameters of tastes and technology. In Plosser’s model, the next period’s capital stock depends on the current capital stock and the current level of productivity and subsequent discounted productivity shifts. According to Plosser, the steady state conditions ensure the following:

(2.14)

\[ \mu_1 < 1; \mu_2 > 1 \]

To formulate a Law of Motion, this result guarantees the stability of the solution. Given equation 2.9 for capital accumulation, then through an iterative method, the cyclical component of the capital stock \( K_t \) can be written as a function of past shocks:

(2.15)

\[ K_t = f \left( \Sigma_0 \theta_{t-j-1} \right) \]

This result means that the cycle is entirely exogenous. As a Law of Motion for this very typical real business cycle model, if exogenous shocks do not exist, then the model reduces to:

(2.16)

\[ \hat{K}_{t+1} = \mu_1 \hat{K}_t \]
which is a stable solution. Thus in the Plosser model, the response of individual agents to productivity shocks, in work, effort, output, consumption, and investment leads to stable cycles being generated which are Pareto optimal (see, Chapter 12, *Inter-temporal Optimality* of Azariadis, 1993). The Laws of Motion in real business cycle models is logically consistent with the macro economic paradigm of equilibrium economics, as Kuznets predicted.

2.33 Laws of Motion for New Keynesian Business Cycle Models

The laws of motion found in New Keynesian business cycle models arise from the micro economic imperfections specified in their structure. Such motion is self-generating and endogenous to the specified structure, including erratic expectations with regard to investment and consumption spending. We have seen that the laws of motion in New Keynesian models, like the original research inspired by the *General Theory*, largely rely upon erratic private expectations of consumption and investment spending as major sources of aggregate demand instability. New Keynesian models which utilize aggregate supply effects are exceptional, but tend to use real changes in labour markets or production functions to alter the quantity of output firms are willing to produce at a given price level. The motion exhibited by New Keynesian models may involve oscillatory behaviour (limit cycles) of various orders of periodicity and may even exhibit deterministic chaos. In the New Keynesian approach, micro economic imperfections in market structure, incomplete rationality, and a sluggish adjustment process produce varieties of inter-temporal behaviour, involving non-Walrasian equilibrium. The emphasis is upon rigidities, frictions, and constraints, which prevent solutions. In some models, the specified structure given certain parameter values leads to regular oscillatory behaviour around a stationary mean which we might refer to as business cycles. In other instances, the same model with different
parameter values, may produce an inter-temporal solution which is chaotic, and anything but regular. Initial conditions may also affect the inter-temporal solution.

In some New Keynesian business cycle models, we find that the notion of equilibrium coordinates are well defined and therefore the laws of motion may be described even in the short run; while others New Keynesian business cycle models lack equilibrium coordinates. Some endogenous models of the New Keynesian variety involving disequilibrium theory present a special problem from a laws of motion standpoint, as measuring displacement without either a static or equilibrium path coordinates, leaves the degrees of freedom, in a classical dynamic sense, undefined. Recall, from the beginning of Chapter Two, to define business cycles as stationary stochastic processes is common, however, many New Keynesian business cycles, exhibit alterative inter-temporal behaviour. Often in such models the assumption is that business cycles are disequilibrium phenomenon arising from rigidities. The static counterpart of a New Keynesian model represents an economic environment which is stationary, without being necessarily in equilibrium. In a dynamic New Keynesian model, a non-stationary environment may not be in dynamic equilibrium. By way of comparison, in equilibrium Real Business cycle models, steady motion implies that with respect to real economic output, (the counterpart to physical coordinates and velocities), the system remained constant. In a Real Business Cycle model, the system of steady motion was such that after disturbance, economic aggregates may either oscillate or because of damping, energy dissipation, slowly return to the steady motion path. In New Keynesian models, the system is not constant with respect to output (velocities and coordinates in the physical sense). Begging the question of whether cyclic behaviour can be described if a stationary, albeit stochastic, process has not been specified? Borrowing from classical dynamics, if we do not know where a physical mass is
supposed to be, even probabilistically, or in an economic setting where an economy’s output should be, as in classical equilibrium, how can cyclic displacement via a law of motion have any meaning? A reference level of output is required to measure displacement from it, cyclic or otherwise.

The above observations suggest analysing the dynamic behaviour for New Keynesian models, that is defining laws of motion, creates complications. In the above literature review, business cycle models have been classified using the concepts of endogenous versus exogenous schemes, equilibrium versus disequilibrium approaches, and how motion begins, how it propagates, and whether it persists. Considering the laws of motion for New Keynesian models, requires, at times, dropping the presumption of underlying stationary process. Cycles of various periodicity may occur, such models may even exhibit as we show in Chapter Five, deterministic chaos. Some New Keynesian models exhibit hysteresis and multiple steady-states. Without shocks (perturbations), a non-stationary time-path may arise as a consequence of the dynamic interaction between economic agents. In considering such inter-temporal behaviour, it is important to remember that the presumption of a reference frame arising from an underlying stationary trend, may not be relevant. Defining the laws of motion for such business cycle models, presents challenges. Looking at two well-known models from the financial imperfections and constraints literature, illustrates the differences in inter-temporal dynamics.

Financial constraints and financial market imperfections figure strongly in the New Keynesian business cycle literature and as we will see shape their laws of motion. We will examine two such models to see what Laws of Motion they obey. The stylized facts behind these business cycle models include the asymmetrical information between borrowers and lenders; the asymmetrical effects of interest rate changes upon the quality of borrowers; the difficulty which lenders
have in monitoring borrowers after credit has been extended; and the practical limit to writing financial contracts which are tied to real performance (such as return on equity or return on sales) rather than nominal amounts and nominal rates of return on debt which may not be risk adjusted. In the next chapter, we discuss the evolution of these concepts in depth, however, for the purposes of understanding business cycle Laws of Motion, we look at two such models.

Consider the model found in Greenwald & Stiglitz (hereafter, GS) (1988, 1990, and 1993), which we fully explore in Chapters 3 and 4, although here we introduce it. We have the objective profit function for the $i^{th}$ firm:

$\max_{q} \Pi \left[ q - (1 + r_t) \left( w_t \phi(q_i) - a_i^t \right) - c_i^t F(v_{t+1}) \right]$  

Where, $q$ is the profit maximizing level of output for the $i^{th}$ firm, $r_t$ is the rate of interest, $w_t$ is the wage paid at time $t$, $\phi(q_i)$ is the labour requirement for an output of $q_i$, $a_i^t$ is the internal equity reserves of the $i^{th}$ firm at time $t$, $c_i^t$ is the cost of bankruptcy of the $i^{th}$ firm at time $t$, $F(v_{t+1})$ is a probability distribution for the risk of bankruptcy arising from the relation between the firm specific threshold level, $v_{t+1}$, and the firm specific price realized at time $t + 1$ when output is sold, which we call $u_{t+1}$, for which the working capital was borrowed to pay wages at time $t$. Maximization of equation 2.17 yields an aggregate supply function of the form:

$q_t = q(w_t, r_t, a_{t}^1, \ldots, a_{t}^n, F)$

(2.18)

In their model, GS add an inelastic labour supply, along with an aggregate demand equation which includes a money market condition for equilibrium. The two equilibrium conditions in the model are the
following (Where $W^*$ is the reservation wage, and $L$ the inelastically supplied quantity of labour):

\[(2.19)\]

\[w_t^e = w^*, \text{ iff } \phi(q^*_c) \leq L; \text{ otherwise } \phi(q^*_c) > L\]

For labour demand at time $t$, $L^d_t$, we have the following, where it is assumed that all firms for a given level of equity, will produce equal levels of output because of identical technology and hence have equal labour requirements:

\[(2.20)\]

\[\sum_{t=1}^{n} \phi(q^*_t)\]

In the GS Model, the crucial state variable is real equity, as it determines the level of output. Its dynamics are specified with the following first-order difference equation:

\[(2.21)\]

\[a_{t+1}^d = q^*_t - [w_t\phi(q^*_t) - a_t^d](1+x_t^d - M_{t+1}^d)\]

Where, $M_t$ is the net dividends paid out of accumulated profits. At the steady state, the given supply of labour, $L$, determines the full employment output level, $q^*$, where $\phi(q^*) = L$. Hence given $q^*$, equilibrium in the goods market determines $r^*$. In this way the steady-state wage which ensures that the full employment is such that aggregate demand equals aggregate supply:

\[(2.22)\]

\[q^* = q[w_t^e, x_t^*, a_t^e, F]\]

Equation 2.22 defines a function on wages, accordingly:

\[(2.23)\]

\[w_t^e = w(a_t^e)\]
Although dividend payments are stochastic, GS specify that they are an increasing function of the amount of equity, GS express real dividends, $M/p$ as $\frac{m_{t+1}}{a_{t+1}}=m(a_{t+1})$, leading to their difference equation.

\[(2.24)\]

\[
a_{t+1} = a^* - \left[ w(a_e) L-a_e \right] (1+r^*) - m(a_{t+1}) = G(a_e)
\]

Where $G(a_e)$ is the generalized function of the autonomous non-linear, first-order difference equation on equity, $a_e$. According to the authors, GS, under suitable assumptions on parameters, this difference equation is globally stable around a full employment, real equity value of $a^*$. The equation describes the Law of Motion for the GS model. According to GS, under suitable assumptions on parameters, this difference equation is globally stable around a full employment, real equity value of $a^*$. The equation describes the Law of Motion for the GS model. According to GS, equation 2.24, describe both paths to equilibrium: "... from an initial equity position below $a^*$ there is generally a path of increasing output... and equity that converges to $a^*$; from an initial equity position above $a^*$, there is a full employment path along which falling equity and wage levels converge monotonically to $a^*$ and $w^*$." (GS, page 118, 1988). According to GS, in their model, perturbations to the steady-state may arise from monetary shocks, demand shocks, and "uncertainty" shocks" with regard to future expected firm specific prices, $P_{t+1}/P_t$ or $u_{t+1}$. Depending upon whether the increase in, for example, the degree of "uncertainty" is permanent, then the drop in output at each level of equity, $a^*$, will be permanent. According to GS, small shocks can have large and persistent effects upon output (page 106-107, GS, 1993), however an exogenous shock on money supply, real demand, or "uncertainty" may have short run effects without altering the steady-state level of output (page 118, GS, 1988). The claims of GS with regard to dynamics of their model, we note, however, combine linearization and analysis. Owing to the model's non-linearity, such claims generally require validation using numerical methods, as we pursue later (Chaos in Dynamical Systems, Chapter 1, Ott, 1993).
We observe that with respect to finding a law of motion for this New Keynesian business cycle model, that it is arguably \textit{classical} even in the short run, because nominal shocks have an effect only if they are \textit{not} anticipated, which may be argued resembles a real business cycle model (page 168, Benassi, Chirco, and Colombo, 1995). If the shocks are anticipated, the effects are manifested through supply considerations. A further point to observe here, from a Law of Motion standpoint is that the financial constraint found in the model, the link between equity and output, implies short run multiplier effects. For example, if the money supply were to fall and its effects were not anticipated, it would lead to a fall in real equity values, which in turn would produce a fall in overall output. Lower output, via a negative multiplier, means lower income for consumers, which has second round effects for employment and income. The propagation mechanism of the GS model relies upon real interest rates: the decrease in real demand translates itself into a drop in real interest rates through the implied increase of demand for real balances, which translates into further decline in price, lower revenues and profits, and hence lower real equity, and output. (Such dynamics work through the demand for working capital to pay real wages to produce next period output and dividends tied to anticipated but not realized profits.) Such effects may produce unemployment as \( L(q_J) \) falls below \( q^* \), the steady state level of output. Similarly, short-run effects are caused by real shocks upon demand or uncertainty.

The laws of motion for the GS model exhibit several classical traits, according to Benassi, et al (Chapter 4, 1995). The economy described by GS is perfectly competitive and exhibits flexible prices. It differs from a classically inspired system, in our estimate, in its use of information assumptions which appear as a bound on the firm's equity sales, from which bankruptcy and related costs arise. This equity constraint produces risk averse behaviour by the firm. Non-neutral nominal shocks occur because prices and wages affect the
firm's asset position and risk taking investment is non-neutral with respect to the positions of debt and equity holders. Moreover, because financial contracts for borrowing are not specified in real terms and are not tied to financial performance, but rather to nominal rates of interest which are not indexed or risk adjusted, aggregate output effects occur. Financial markets, because of the equity constraint and the financial market imperfection, rather than being mere reflections of the real-side of an economy, are in the GS model, the source of cyclic persistence. Surprisingly, however, in combination with these New Keynesian traits which make it worthy for inclusion in the two volume set, New Keynesian Economics, (edited by Mankiw & Romer, 1993); the GS model has classical traits from a business cycle standpoint because of its Laws of Motion and much of its structure. Returning to our earlier observation on the problems inherent to defining laws of motion for many New Keynesian models, in the GS model aggregates obey stochastic stationary processes.

Finding what could be described as a New Keynesian Law of Motion in Farmer's (1984) piece on aggregate supply is somewhat easier. Like the work of GS, the role of bankruptcy is used in output determination, although Farmer's approach provides a contract theoretic framework. In his model, the contract form is endogenous to the model: the rate of interest and the efficiency of the contract are related. Describing his model, a firm contracting with an outside financier must provide him with an expected return at least equal to the available alternative- the real interest rate. There is a perfect competition in contracts, which implies a real relative rigidity. Now, in a risky environment, the firm must commit to payments in good states of nature sufficiently high in order to compensate for lower payments in bad states of nature (Benassi, et al, pages 164-165). The firm's ability to repay its obligations, in bad states, is limited by its wealth- as specified in the default contract. Hence, an increase in
the interest rate becomes an increase in the promised good-state payments: if the contract is signed before (and the employment is set after) uncertainty is resolved, so higher payments mean lower demand. Thus, higher rates of interest increase the frequency of either bankruptcy (Farmer, 1984), or as in his later work, of lay-offs (Farmer, 1985), thereby linking real rates of interest to the aggregate supply schedule. Quoting Farmer (page 923, 1984): "... the important implication of using a bankruptcy constraint to generate risk averse behaviour is that the magnitude of the employment distortion is predicted to vary systematically with the rate of interest."

Interpreting the model from its laws of motion we find that in Farmer's model output and employment depend upon the real rate of interest, so that a shock on the asset market has a direct effect on the goods and labour markets. From a Motion standpoint this leads to two observations. Firstly, that these shocks are real, since "... money may be included in a variety of ways, but any method that has the property that only real balances matter will generate steady-state equilibria in which the rate of inflation equals the rate of monetary expansion" (Farmer, page 928, 1984). Secondly, Farmer's model economy has no natural rate, since its steady state equilibrium rate of unemployment depends upon the sequence of asset market equilibrium (hysteresis) - and hence of real shocks affecting the latter unlike GS. His model's law of motion, cannot be described as New Classical. The equilibrium to which his economy converges is not invariant to the path followed to reach it. The law of motion for this model would be described as path dependent. Reviewing these two New Keynesian business cycle models for the properties of their dynamic behaviour reveals that generalizations about the literature are not made easily. As Fair notes (page 143, edited by Belongia & Garfinkel, 1991) with respect to New Keynesian models of business cycles "...it is hard to get a big picture.... There are many small stories." Accordingly, their laws of motion vary, and even exhibit, as
we have seen, New Classical traits.

2.40 CONCLUSIONS

Contemporary business cycle literature, New Keynesian and Real Business Cycle, can trace its origins to the beginnings of economics. Expressed in a new manner, and using the latest techniques, contemporary business cycle literature confronts the timeless and basic issue of whether a self-organizing economic system in which capital is privately owned and markets direct its allocation, will generate stable output and efficient utilization of resources, including people. In order to explain departures from this ideal, Real Business cycle theorists attributes oscillations to exogenous and possibly persistent, perturbations. If not for such exogenous occurrences, they argue, an economies’s output would grow uninterruptedly along trend, reflecting the marginal productivity of capital. Essentially, this depiction of an economy’s behaviour might be faulted as a *deus ex machina* solution. How, it might be asked, can a model of business cycles make any claim to universality or comprehensiveness, if forces outside the model are needed to generate aggregate supply instability? We ask, why are not unexpected growth in the money supply or improvements in technology endogenous to the system?

New Keynesian models, by contrast, we have seen appeal to micro imperfections to explain changes in aggregate demand, and oscillations along trend. Their research resembles the original propositions of Keynes on the obstacles to a full-employment/output solution arising from any of the following aggregate conditions mentioned previously:

- Wages being set too high;
- The demand or supply for real cash balances having a horizontal range at too high a rate of interest; or
- The expenditure curve never reaching the full-employment level of real income at any positive rate of interest.
But New Keynesians lay claim to macro explanations utilizing essentially partial equilibrium phenomenon, such as imperfections in competition, capital markets or information, and in this way differ from the broad foundations arising from general wage and price inflexibility used by Keynes. New Keynesians, in trying to create robust micro foundations, and overcome the logical inconsistencies found in models derived from the *General Theory*, leave us with many small-stories, none of which would appear strong enough to carry the weight required for a comprehensive model of fluctuations in aggregate economic output as influenced by changes in aggregate demand. As mentioned in Chapter One, the Samuelson principle of correspondence seems to be ignored. A second shortcoming of New Keynesian models is that although the modeled micro-imperfections generate persistent cyclic dynamic behaviour, they remain essentially static and not self-correcting. Keynes appears to have believed that the economy was *eventually* self-correcting, but that the wider social costs were unacceptable, and therefore demanded an activist fiscal policy, especially in view of his distrust of Central Bankers (Leijonhufvud, 1968). In contrast, New Keynesian theorists utilize micro-imperfections to generate dynamic results, but appear to leave unexplained why such imperfections are not correcting, for example, through innovations in how markets function or what products are available. (Would a macroeconomic model ever anticipate cost lowering innovations such as information technology?) Whether such micro economic innovations may remedy or modify the cyclic oscillations depicted in New Keynesians models, and if so, how, is the subject of this dissertation. In order to pursue this subject, we present and explore in Chapter Three in much greater depth, the Greenwald-Stiglitz (GS) model in which financial market imperfections and constraints are the sources of macro economic disturbances.
CHAPTER THREE. MODEL REVIEW AND BACKGROUND

3.0 Introduction

In this chapter we will explain how a New Keynesian business cycle model embodying a financial constraint and a financial market imperfection may be utilized to model and analyze the effects of certain innovations, in the form of futures and options securities, used for the management of financial risk. Our interest concerns the impact of such forms of innovation upon aggregate economic activity and how they may interact with the stylized endogenous specifications found in such models as sources of cyclic behavior.

In the course of this research, we investigate whether there is a theoretical basis for the many stylized assertions that the use of such securities as futures and options have changed business behaviour, and by doing so, have generated desirable macroeconomic effects, as measured by aggregate output, and by dynamic stability.

Conventional wisdom with regard to futures and options and their use for risk management, typically embodies three unproven stylized assertions:

- The growth in the availability and use of options and futures has been as a response to the growth in systemic macro risk arising from secular changes;

- The adoption of risk management practices by business has been beneficial to shareholders; and

- The pervasive use of risk management has yielded macroeconomic gains by reducing the amplitude of business cycle behaviour without a reduction in long-term growth rates.

For example, we find the view that the practice of risk management using options and futures has arisen in response to secular economic
changes expressed by Professor Clifford W. Smith

"Not surprisingly, the financial markets have responded to this increased volatility. The past fifteen years have witnessed the evolution of a range of financial instruments and strategies that can be used to manage the resulting exposures to financial price risk."

Professor Smith, an Editor of the Journal of Finance further alleges that the use of risk management represents sound practice for all concerned. He writes:

"Today, financial price risk can affect not only quarterly profits but may determine a firm's very survival. Unpredictable movements in exchange rates, interest rates, and commodity prices present risks that cannot be ignored. It's no longer enough to be the firm with the most advanced production technology, the cheapest labour supply, or the best marketing team—because price volatility can put even well run firms out of business."

In a similar vein, we find in The Handbook of Currency and Interest Rate Hedging, published by the New York Institute of Finance, that by managing risks, firms may be able to reduce expected taxes, reduce costs of financial distress, increase debt capacity, and reduce a firm's borrowing costs (1990). Turning to the aggregate impact of risk management, we find Professor Smith's implied observation that macro-economic gains arise because the risk bearing parties are able to carry their costs with greater efficiency. He writes:

"... financial instruments now exist that permit the direct transfer of financial price risk to a third party more willing to accept that risk."

1 The Handbook of Interest Rate Risk Management, Robert Schwartz and Clifford Smith, 1990.
It would seem, according to Professor Smith, what is good for the individual firm, must in the aggregate be "desirable". In a similar vein, Professor Schiller in *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks* (1993), argues that it might be possible to reduce or eliminate macro-economic disturbances through appropriate risk management. We find that a logical connection between risk management and the achievement of a firm's financial and commercial objectives is widely asserted among practitioners and academicians within this field, with the implication that such activities contribute to shareholder value, and moreover have desirable aggregate effects. For example, Robert G. Tompkins in *Options Analysis* (1994), writes, "Options function as insurance and in this area the needs of world commerce and services by the world of finance dovetail perfectly." In *Advanced Strategies in Financial Risk Management*, it is claimed that *macro-swap* or *macro-option*, as is currently under development, holds the potential for insulating companies from the risks of general economic down-turns (Chapter 13, *Advanced Strategies*, Marshall, Bansal, Herbst, and Tucker, editors R.J. Smith and C.W. Smith, 1995). The above viewpoints, notwithstanding, as we discuss below, are problematic at many levels.

Looking at the individual firm, the arguments for the management of financial risk are not cut-and-dry. According to some authorities, an accepted and received perspective on why firms should hedge does not exist (Leland, 1998). The conventional viewpoints, as expressed above, adopt the *risk transference* model of options and futures in which derivative markets are likened to insurance. In this school of thought, derivatives exist in order to price and sell systemic risk between hedgers and speculators. A fully hedged position represents the purchase of certainty equivalence. This school of thought has a long lineage, dating at least to Keynes (1923, 1930) and later reinforced by Kaldor (1939) and Hicks (1946).
Outside of the technical requirements, the Keynes/Hicks perspective implies that futures markets are biased downwards with regard to estimates of prices in the future to reflect the risk premium earned by speculators, that such markets behave systemically vis-a-vis the securities markets, and that in the aggregate hedgers are net short.

As we will pursue in depth in Chapter Five, for any and all of these points, empirical support is not uniform. An immediate problem with the perspective of Keynes, Kaldor and Hicks, is that if derivative markets behaved systemically, hedging involves the trade-off of non-diversifiable reward against risk. Under risk neutrality, points found on the security market line have a net present value of zero. Only with positive risk premia will investors pay to avoid systematic risk. Trading-off risk against return through taking long or short positions on state contingent assets produces no gains to shareholders, since market capitalization is risk adjusted and hence invariant to capital structure (Modigliani & Miller, 1958, 1963).

In addition to the micro-theoretic issues surrounding the conventional views on hedging, we have problems regarding the alleged macro benefits. The connection between the use of options and futures for hedging and the alleged benefits at the individual economic agent level, to gains at the macro-economic level, however measured, have yet to be established, although often presumed. With little or no theory, relying upon anecdotes, it is often asserted that the use of risk management can stabilize the economy and smooth cyclic disturbances, (see, for example, Schiller, Chapter 4 1993). The aggregate gains to the use of futures and options for risk management is believed to arise, because the hedger/speculator relationship shifts costs to where it may be borne at lower cost. Such observations assume that risk preference and tolerance is not uniformly distributed; while making many non-trivial presumptions with regard to aggregation (see, Sato, 1975). As Bacchetta and Caminal (page 3, 1996) note, "...the fact that financial factors matter
for individual firms does not imply that they matter at the macroeconomic level." The presumption that if risks could be managed away, the macro costs of mistakes might be reduced or eliminated, lacks theoretical underpinnings and is difficult to establish empirically, for several reasons including secular changes in the economic environment. Among the questions raised by the above positions with regard to the use of futures and options by corporations under the rubric of risk management are the following:

- Are the effects of risk management purely firm specific, or are their aggregate positive externalities, such as the smoothing of business cycles?
- Are there effects at the Aggregate Demand and Supply levels?
- Does the use of RM techniques, in a sense create a public good through the reduction in investment and consumption instability?
- What are the business cycle properties of an economy having contingent asset markets?
- Can an optimal span of state contingent markets be formulated?

The controversial theoretical foundations for risk management may appear surprising in view of the fact that virtually all the World's exchanges offer both simple and complex derivative products, as a means of managing exposures and expressing complex market views, but the role of finance in general in the theory of business cycles has not received adequate attention (Krainer, 1992). According to the author, the implications of how financial contracts of any variety shape supply adjustment in both product and financial markets has been overlooked. A rigorous analysis or modelling of the significance of risk management, involving for example futures markets, for the macroeconomy has yet to be undertaken, while progress with regard to theoretical general equilibrium modelling of futures market is only recent. The literature on General Equilibrium Theory with Incomplete
Asset (GEI) Markets (see, Geanakoplos, 1990, for a survey) is a recent example, however, it has not been extended to the business cycle literature or macro-economics in general. Equally surprising is the small attention which has been given to the modelling of futures markets themselves at even a partial equilibrium level. The economics of futures markets has received limited attention (Goss and Yamey (1976)); while the economic role in the economy of such practices in the aggregate are less well understood (J.L. Stein (1986; and 1992, edited by Goss); Silber, (1984) edited by Peck). The asset/inventory management approach to futures and options promoted by Williams (1994), which we return to in Chapter Five, is intriguing but not widely accepted. From the above, we see that the inclusion of risk management innovations into a macroeconomic framework or a model capable of addressing the unproven assertions mentioned earlier, remains untrodden ground.

To illuminate these issues and address them on a rigorous micro-theoretic basis, we have turned to the business cycle models found in the New Keynesian literature. Their emphasis upon imperfections and constraints as sources of cyclic behaviour, make them potentially ideal for analysing whether hedging as a means of correcting such imperfections, has macroeconomic implications. Such models combine uncertainty in the formation of expectations with micro-theoretic constraints and imperfections. In such models mistakes arising from the combination of uncertainty and imperfections lead to cycles. Introducing risk management into such

GEI, firstly, has provided a framework for understanding the pricing of assets and the significance of asset span; Secondly, GEI has forced economists to develop new methodologies for constructing existence proofs for general equilibrium; thirdly, it has demonstrated a significant difference between real and financial assets; fourthly, it has shown that default and bankruptcy can be understood as equilibrium, and not disequilibrium phenomenon; and fifthly, it has increased the presumption against the Pareto efficiency of the market process. (Geanakoplos, 1990).
models as way of correcting imperfections and reducing uncertainty, may present a means of addressing the issues raised earlier, as well offering further insight into their dynamic behaviour. To adequately conceptualize the above issues, we explore in Chapter Three the theoretical specification of such New Keynesian models, their methods of propagation and laws of motion, including the role of imperfections in dynamic behaviour, after which we turn our attention to one model in particular by Greenwald and Stiglitz (1988, 1990, 1993).

The role of mistakes and imperfections are important for macroeconomic models of business cycles because both New Keynesian and Real Business approaches have in distinct ways used them to motivate aggregate activity, as noted in the previous chapter. Would the signal extraction problem found in a Lucas model or the technology shocks of Kydland & Prescott have the same implications if economic agents, anticipating uncertain outcomes, hedged their exposures? In New Keynesian models, we ask, would risk management correct the aggregate demand instability induced by the effect of erratic expectations upon consumption or investment decisions? In such models, could the impact of second order suboptimization be neutralized through risk management? If economic agents, facing uncertain environments, were to hedge parameters on which their consumption and investment spending plans were based, would the sources of instability be removed, thereby changing the dynamic aggregate behaviour? Risk management, it might be argued, would inter-act with the laws of motion found in such models. Indeed, the use of risk management products might even change an economy’s laws of motion in ways more complex than simple assertions derived from aggregating partial equilibrium results. The fact that risk management has been ignored from a macroeconomic standpoint and business cycle standpoint, notwithstanding the many groundless assertions noted earlier, may be surprising, but
such innovations are fairly recent. To fill this gap, the purpose of our research is to understand the significance of innovations in financial risk management, at the macro level, and shed light upon the many stylized assertions regarding the use of derivative products. To this end, in Chapter Three, with reference to the relevant literature, a theoretical model is presented and discussed with a view to laying the ground-work to address the above issues.

The methodology for our research involves a theoretical contribution to the New Keynesian business cycle literature, which explicitly includes the use of options on futures markets for risk management. As the model may not be solved analytically, and because linearized approximations of non-linear systems may obscure their properties, we rely upon numerical methods to investigate its cyclic and macro-economic properties (Ott, 1993). In the Literature Review, Chapter Two, we discussed the major features of contemporary business cycle models. We have given considerable space to business cycles models known as New Keynesian. In Chapter Three, we discuss in depth the features and properties of New Keynesian models having a financial constraint and financial market imperfection model. Our primary emphasis will be upon business cycle models which incorporate credit market imperfections and financial constraints, to model the role of money or capital.

3.1 NK Models having Credit Market Imperfections

The risks arising from many economic events and trends facing economic agents, may not be risk managed through the use of derivative hedging schemes. Examples of wrong products and wrong markets abound. Several authors have considered the implications of being able to hedge the erratic expectations arising from levels of GNP, inflation, or growth in specific markets (Haar 1993, Schiller 1993, 1995). In contrast, the world's financial markets offer derivatives products connected to virtually all traded securities,
both assets and liabilities, including foreign exchange markets, commodity markets, bills and bonds, share prices, credit to mention but a few. For example, at the company level, hedging the interest rate exposure on a floating rate source of funds, or the foreign exchange revenue from a foreign investment are common fair. If such efforts are beneficial at the company level and whether they have macro and macro-dynamic implications remains to be seen.

To address the question at the business cycle level whether, hedging by economic agents has macro effects such as generating positive externalities, we turn to a class of New Keynesian models which explicitly include a financial market component. In these models, phenomena such as erratic expectations, frictions, constraints, and returns with regard to sources of capital are both the sources of perturbations and result in modifications to the laws of motion for economic cycles. As the dynamic properties of such models may not be learned analytically, we develop and specify such a model and use calibrated simulation to learn its properties. The simulations we use to establish a reference frame with which to compare aggregate dynamic performance if risk management were included, functionally specified, and modeled. Such New Keynesian models are appealing for they include both a financial constraint along with a form of financial market imperfection, both of which might be effected through the use of derivatives. To appreciate the foundations upon which we build our analysis, we begin our discussion by considering two classes of New Keynesian business cycle models, using the classical dynamics schema found in Chapter Two. This approach will allow us to look at their structure in some depth.

New Keynesian Business Cycles Models with credit market imperfections originate from a set of stylized facts which we will illustrate with some examples. The stylized facts behind this field of research include the asymmetrical information between borrowers and
lenders; the asymmetrical effect of interest rate changes upon the quality of borrowers; whether and to what extent financial capital is rationed by price or quantity; the difficulty which lenders have in monitoring borrowers after credit has been extended; and the practical limits to writing financial contracts which are tied to real performance (such as return on equity or return on assets or return on sales) rather than rates of interest, which may or may not be risk adjusted, tied to fixed amounts of capital.

As sources of business cycle perturbation, such models stand in marked contrast to models found in the real business cycle literature. In NK models, there is a break-down in classical duality, reversing the direction of causation between real markets and financial markets. One approach is the so-called financial accelerator, where information imperfections in capital markets exacerbate business cycle fluctuations (Bernanke, Gertler, & Gilchrist, 1994). In the model on which our research is based, the credit market imperfection arises from the de facto lending contract, specified in nominal amounts and at nominal rates of interest, set at a contracted rate rather than being tied to performance. Borrowing in the face of uncertain return on sales and investment, leads to possible insolvency. Thus financial phenomenon produce real side effects, which is a marked departure from the financial structure irrelevancy found in the Modigliani-Miller theorem and Neo-Classical duality.

How credit market imperfection models of the business cycle lead to aggregate demand and supply instability, will now be considered. Our concern is how imperfections, as means of perturbation, works its way into having macro economic effects, that is how it is propagated, and thereby modify an economy’s laws of motion, such as changing the periodicity of cycles. In this respect, credit market imperfection models, exhibit either of two approaches regarding propagation: i) The role of financial markets in affecting aggregate supply and demand through the monetary transmission
mechanism; and ii) whether the effect of lending markets is felt through credit rationing or through higher interest rates. The model on which our research is based relies upon the former as well as the latter, both of which will be discussed below in order to place the research in context.

The monetary transmission mechanism as a means of propagation of credit market imperfections has been fairly well-explored in the NK literature. On the first approach, Fama (1985) and Bernanke and Blinder (1988) challenge the traditional Neo-Classical IS-LM view in which higher interest rates induce consumers and firms to reduce their investments and purchases: Debt capital is rationed by price. In such literature by contrast, contractionary economic policy, selling government bonds to the public, for example, not only raises interest rates but may influence the tendency of banks to lend by reducing the quality of bank reserves. Such phenomenon may be described as a sort of reverse income or giffen good effect: banks lend more to compensate for the decline in loan demand, producing a moral hazard. Moreover, if information on credit market conditions is imperfect, other lenders may not be able to off-set this lending shortfall. Such models are at variance with classical duality, as they have causation running from the financial side to the real side of the economy, rather than the latter being a reflection of the former. This direct effect upon banks leads to reduced credit, which in turn leads to real effects through investment. The impact upon financial intermediation is also examined by Diamond and Boyd (1984), Prescott and Boyd (1986). McCulloch (1975) suggested that financial mis-intermediation, that is mis-matching the tenor of assets and liabilities by the banking system, (for example, lending long and borrowing short), causes uncertainty about nominal interest rates, misdirecting investment decisions in the real sector.

Turning to the second area, that is credit rationing, we find a contractionary economic policy leading to credit rationing, as shown
in the models of Jaffee and Russell (1976) and Stiglitz and Weiss (1981). Capital is not rationed by the price mechanism, the rate of interest, but rather through availability. Mankiw (1986) and Blinder (1987) also demonstrate the role of rationing in affecting the lending process. Empirical research by Hoshi, Kashyap, and Scharfstein (1988) and Fazzari, Hubbard, and Petersen (1988) finds support for these microeconomic effects of credit rationing at an aggregate level. The implications of capital rationing we expand upon our in research, where a constraint on the availability of equity leads firms to borrow, often sub-economically.

Focusing upon the concept of dichotomy in credit market imperfection models of business cycles, for example involving a constraint on the availability of new equity, reveals a fair amount about the method their of propagation. Classical dichotomy rests upon the tenet that purely nominal changes cannot have real effects. Classicism and New Classicism postulate that the economic decisions of firms and consumers with respect to consumption, saving, and investment, are the result of real factors. In New Keynesian Credit Market Imperfection Models, dichotomy is violated and second order effects are insufficient to influence optimizing behavior.

In the relationship between financial markets and their method of propagation, Credit Market Imperfection Models are distinguished from other schools of thought in their rejection of Neo-classical dichotomy. The standard approach of the finance literature has been to analyze an economy where production and investment decisions are separated from financing decisions, and then to determine the impact, if any, that alternative financial structures will have upon firm valuation, and ultimately aggregate activity. Known as the Fisher's Separation Theorem in Capital Market and Structure Theory, it precludes any interaction between real and financial magnitudes, as would be found in garden variety classical economics (I. Fisher, 1930). For any given set of operating decisions, the equilibrium total market
value of any firm, is unaffected by its financing decision. Moreover, according to the Theorem, the firms financing decisions have no effects on either the wealth or the capital market opportunities of its securities holders. The Theorem holds that the optimal operating decisions of firms are not dependent upon financing decisions, (Arrow, 1964). A firm's financing decisions have no effect on its total market value, (Debreu, 1959). Operating and financing decisions are, separable. In the classical paradigm, financial markets merely reflect the operating efficiency, the state of technology and nature facing the firm. The economy may dichotomized.

In NK Credit Market Imperfection Models of business cycles, the Fisher's Separation Theorem is rejected. Rejecting dichotomy means that initial disturbances are propagated from financial markets to real markets. In Credit Market Imperfections models, dichotomy is turned on its head: Financial market problems are the source and means of propagation and not merely the reflection of the real world. They are both the perturbations which lead to aggregate demand and supply effects, and the means through which such effects spread-out. For example, in some models, in response to monetary and real shocks, capital markets initiate supply adjustments in the product and asset market while financial contracts constrain and shape supply adjustments in product and financial markets.

In many NK Credit Market Imperfections Models of Business Cycles the rejection of classical dichotomy is the means of propagation leading to aggregate demand and supply effects. Rejecting dichotomy, credit market imperfections in the form of financial constraints lead to real disturbances. The propagation mechanism behind this phenomenon assumes that second order effects are insufficient incentive for a firm to adjust its behaviour in the face of, for example, a shift in aggregate demand (page 48, Krainer, 1992). In NK models using credit market imperfections, bankruptcy monitoring costs or asymmetric information on the quality
of the loan may induce a lender to reduce or restrict the quantity of
credit, to limit adverse selection, which when combined with limited
access to self-finance may increase the probability of bankruptcy
(page 144, Benassi, et al. 1995). In another model, credit markets
and aggregate demand, for example, may be affected by the fact that
loan contracts are generally written in nominal terms, in which case
nominal events may have real effects, as appears in the models of
Farmer (1984) and Gertler (1988). Shocks which initially might have
reduced output or redistributed wealth between lenders and
borrowers, diminishes the allocative efficiency of credit markets. In
models as Bernanke (1983), Bernake and Gertler (1989), Greenwald
and Stiglitz (1988), and Williamson (1987) have shown how financial
market imperfections can magnify the effects of disturbances
introducing new propagation mechanisms. In Mankiw (1986),
changes in nominal interest rates alter the riskiness of the pool of
borrowers. The social surplus of credit markets might be a
discontinuous function of interest rates, so that if rates change, a
formally efficient market might vanish. All these models, are
examples of reversed dichotomy where financial markets are the
source of perturbation and the means of propagation leading to
aggregate cyclic disturbances. All of these NK credit market
imperfection and capital rationing models contradict "...the basic
tenet of the Modigliani-Miller (hereafter, MM) theorem, according to
which the balance sheet of the decision maker is immaterial to his
profit (or utility) maximizing choices." (page 154, Benassi, et al.,
1995). Moreover they do not attempt to reconcile their findings with
the conditions under MM theory in which capital structure may have
output effects.

Support for the view that credit market imperfections have
macroeconomic effects has been investigated at the empirical level.
Friedman (1983, 1986) finding a high correlation between aggregate
nominal measures of credit and real activity; while Wojnilower (1980)
and Eckstein and Sinai (1986) have investigated the role of credit disturbances in macro fluctuations using data since the Second World War. While the severity of the Great Depression, according to Bernanke (1983) was due to the downward shock to credit supply stemming from the increased riskiness of loans and the concern which banks had over maintaining sufficient liquidity in the event of unexpected withdrawals- runs. King (1983), Bernanke and Blinder (1989) and Romer (1990) have investigated empirically the view that bank lending seems to have had little role in explaining macro fluctuations and cycles, leaving a role for rationing and credit market imperfections as the NK theorists believe. Their work supports the NK credit market imperfection perspective, and is contrary to the orthodox view that the demand for loans is responsive to interest rates and income levels. In their view there is no cyclic impact effect upon GNP from rate changes.

3.2 NK Models having a Financial Constraint

We have discussed the structures and logic found in NK credit market imperfection models. We have reviewed how by reversing duality, cycles may be both generated and propagated. Our own research into the significance of risk management upon business cycles involves both credit market imperfection models and an equity constraint models. Such models are interesting because how loan contracts are specified, might have significance at the aggregate level, if the exposures accepted by the debtor, and the amount of capital and agreed return placed at risk by the creditor, were fixed or limited or could be modified. In addition to market imperfection models of business cycles, the use of risk management also concerns models involving what are known as financial constraints. In the model upon which our research is based, a financial constraint on the availability of equity figures strongly, because firms wishing to expand are forced to borrow when no more new equity may be
raised, and debt capital has a special cost. In order to understand the derivation of our research, we will review the related literature.

Interest in the *Financial Constraint* as a component of NK research into credit market imperfections grew out of the general recognition that the budget constraint and Keynesian notions of liquidity preference lacked logical coherence, (see, Kohn, 1984). To understand this perspective, we begin by recalling the traditional paradigm. In the macroeconomics found in the Keynesian-neoclassical literature, each household owns a variety of assets which compose its total present wealth. Given the household’s present wealth, the problem is how to allocate between cash and other assets. The desire to hold cash is a function of income and the rate of interest, and is known as the *liquidity preference function*. The financial sector in the Keynesian system could be specified and analysed in balance sheet terms, as stocks rather than flows. Among practitioners, the general view was that aggregative analysis of money supply and demand was adequately underpinned by Patinkin’s (1956) integration of monetary and value theory. The budget side was considered sufficiently justified at the microeconomic level. Although other researchers attempted to round-out this paradigm, the general picture seemed sufficiently complete. For example, Tobin (1958) advanced this theory further by proposing that the demand for money is partly a result of the wealth-holder’s desire to diversify their holdings.

Historically, dissatisfaction with the above outlook began with the finding of its logical shortcomings. Clower (1965, 1967), Hahn (1966, 1977, 1978 and 1980) and Tsiang (1966,1969,1977) maintained that because the role of money had not been given a formal expression, the normative and positive implications were specious. Thus, although the NK literature using the financial constraint is a relatively new improvement to the budget constraint school of thought, and it has antecedents dating back to Robertson
Dissatisfaction with the liquidity preference function led to Clower's 1967 model proposing to replace the budget constraint with a general financial constraint, and thereby give more rigour to the liquidity preference schedule. What money actually did needed to be modeled explicitly. Clower's dichotomized budget constraint was designed to capture the role of money in enforcing budget constraints. The timing of his work was opportune. The Keynesian perspective that the endogenous logic of the economy was not perfectly self-coordinating had remained popular; while the view that business cycles were disequilibrium phenomenon, was gaining ascendancy. Moreover, there was a growing dissatisfaction with loose aggregative specification. The financial constraint filled-in the picture. The work of Clower and others provided a structure for a new level of disequilibrium analysis. In combination with the four features common to other NK research, the use of financial constraints, provided an additional logical pillar for NK research.

Some recent applications of models using the financial constraint will illustrate its usefulness as well as show how the features common to other NK models are quite intact. A paper by Howitt (1988), is a good example of this sort of research. In his model, inflation has real effects through the timing and coordination of transactions. In his model, because goods in inventory and money, are held, the effects of inflation are complex. For example, agents whom he terms middle men, because of perfect access to finance, are not affected by inflation, while other agents are. The differing effects of the financial constraint leads to individual and social optimums diverging, with implications for the conduct of monetary policy, and our earlier aggregative externality theme. In an article by Aoki and Leijonhufvud (1988), the micro foundations for investment

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3 See, the survey by Kohn (1981, 1984) along with the defense of the financial constraint on aggregate expenditure, in Kohn (1980).
are modelled. In the model, because the firm typically lacks direct information on the rate of accumulation of capital by its competitors, future rental values of capital goods cannot be imputed. Hence it is unable to calculate its current demand price for additions to its capital stock. In Aoki and Leijonhufvud the information problem facing firms is solved through certain sets of microeconomic assumptions about behaviour.

The role of the financial constraint in NK research also has focused upon inflation when stabilization policies are anticipated. Drazen and Helpman (1985a, 1985b) used a model without labour supply, where different policy instruments have different inflation impacts. In Drazen and Helpman (1988) the model is extended to include an elastic labour supply, and the policy instruments of labour taxation and open market operations. They find, for example, that the anticipated method of taxation, for example, distortionary labour taxation versus lump-sum taxation, generates different inflation paths. Anticipated open market operations may also, in their model, lead to rising inflation. With inelastic interest elasticity of demand for money, stabilization using open market operations, leads to rising inflation, prior to the actual stabilization date; while labour tax stabilization prior to the actual stabilization date leads to declining inflation. This work of Howitt is extended by Kohn (1988) to include how money enters the economy, where he finds that different types of inflation have different redistributive effects, in the presence of financial constraints. First order effects occur in the Kohn model because second order partials with respect to inflation are assumed to be small. Again, we have reversed duality as a means of propagation combined with sub-optimality: Disturbances in financial markets lead to aggregate effects.

Another important area of NK business cycle financial constraint research uses the flow of funds approach. This approach emphasizes micro foundations and again takes the reversed duality
approach found elsewhere in the literature, that is phenomenon on
the financial side of the economy may have real side effects. In a
seminal piece by Tsiang (1982) with wider implications for non-
classical macroeconomic and even monetary theory, the author
adopts the flow approach, and discusses the implications for liquidity
preference theory. To consider a core issue relating liquidity
preference theory, Diamond (1988) develops a financial constraint
model without money, but with credit. He proposes an endogenous
Keynesian mechanism, in which the expectation of insufficient credit
leads to an actual credit crunch with macroeconomic implications.

3.3 The Greenwald-Stiglitz Imperfection and Constraint Model

Having explained and illustrated the lineage of NK credit market
imperfection models and NK financial constraint models with their
implications for both static and dynamic economic behaviour, we now
look at one model in depth, as we propose to use its structure for
modification in our own original research contribution. (We note that
the model cannot be solved analytically, and hence in order to learn
the implications of non-linearity for inter-temporal dynamics, it must
be specified functionally and simulated, a task we leave to Chapter
Four.) Our focus is upon the NK research of Greenwald and Stiglitz
(hereafter GS) as appear in a series of related articles (1988, 1990,
and 1993) all involving a financial constraint and a form of credit
market imperfection. The financial constraint, in these models,
concerns the availability of equity while the credit market imperfection
is in the form of a de facto lending contract, at a interest rate not
adjusted to the risk of default. In addition, there are several other
features making it unique.

The set-up of the GS Models is fairly standard macroeconomic
structure in which the main NK financial constraint modification is
that instead of credit rationing, an equity constraint exists. Firms in
their model are forced, therefore to borrow debt capital, and as inputs
must be paid for before outputs are sold, the decision to produce represents an inherently risky investment decision. The use of borrowed working capita, therefore, involves a credit market imperfection in the form of informationally asymmetry. A borrower might prefer the cost of funds to be tied to returns on sales or, if aware of the risks of default, might adjust to risk, the price of the loan. Since the cost of capital is not risk adjusted to price driven fluctuations in the market value of the firm, neither the borrower nor the lender can be sure whether the decision to borrow is a good idea. All the aforementioned GS articles have identical themes and similar structures, however the 1993 version is simpler and more elegant. It will therefore be the focus of discussion, as well as our point of departure for our research and modelling efforts, although at times reference will be made to the Appendices of earlier article, as GS utilize them to justify various assertions and results found in the latter.

Placing the GS model in context, we begin by noting it has several direct antecedents. Such models use various forms of credit market imperfections and financial constraints to create macro disturbances and effects. The GS paper is similar to that of Bernanke and Gertler (1989), Fazzari, Hubbard and Peterson (1988) and Hubbard (1990), as well as the much older pieces by Lindbeck (1963) and Kuh and Meyer (1957). In some of these works, various measures of financial soundness play a role in determining investment. In Bernanke and Gertler (1989) a costly state verification model constrains the availability of new equity. By comparison, the GS model’s approach to the equity constraint is generalized: Asking the question what are the properties of aggregate behaviour with risk averse firms facing an equity constraint? If firms are forced to borrow at a known cost but for an unknown future return, what are the macro implications? The GS model resembles the earlier literature on dividends, such as Lintner (1971) and Bhattacharya (1980) where
the impact upon a firm’s market valuation from reducing dividends is examined. Again classical duality is turned on its head, with financial decision making having real world implications, rather than being merely a reflection of operating efficiency and technology. This arrangement is also inconsistent with the financial structure irrelevance proposition found in the MM literature.4 Whereas in other NK models, sub-optimal borrowing led to reduced dividends, lower market capitalization, and reduced aggregate supply; in the GS work, the mechanism involves aversion to bankruptcy because of the nature of externally generated capital. In the GS Model, as future output prices may be below a critical level, having borrowed working capital for an uncertain future return because of the equity constraint, dividend and wage commitments may lead to a sharp fall in equity reserves and output. Output prices below anticipated levels have aggregate supply effects, with cyclic propagation through sticky real wages and dividend payments, as paid for through borrowed working capital. The model’s structures and assumptions as discussed below suggest several avenues of research. In particular, the observations about the GS Model suggest various ways in which it may be modified to our purposes. Our objective is to capture the realism of financial innovation, to answer, in particular what are the aggregate business cycle results, if any, of the use of risk management techniques, using futures, options, and other derivatives? How would an economy behave if contingent assets were to available to manage the risks of lower than anticipated prices? Do any of the alleged benefits of managing risk have any macro-economic impacts? A critical and explicit assumption to the GS Model is that firms do not

4Corporate covenants often include limitations on the amount of debt in a capital structure, the realism of an equity constraint is debatable. For lenders to place limits on the amount of additional gearing, is common place. In contrast, returns to equity holders are adjusted for risk through the market place.
have access to futures markets and therefore face an uncertain price in the future for investment/production decisions today (GS, page 79, 1993). Remember, because of credit market imperfections, loan contracts cannot be written to remove the risks inherent to production and future sales. Observations on the GS Model lead to several original innovations involving the use of derivative products for risk management. By modelling financial innovation of derivative markets, such as futures or options, in a NK Financial Constraint model, business cycle questions may be addressed. It will allow us to consider sources of disturbances, methods of propagation, and the laws of motion of a business cycle model which features these important changes in the structure of financial markets. We review the model itself, and afterwards discuss various changes.

The GS Model (1993 version) begins with the aggregate supply side of the model, and addresses the requirements for classical microeconomic system. With output as a function of labour alone,

\[ q = \phi (1), \phi' > 0, \phi'' < 0 \]

(3.1)

\[ P = \frac{\omega}{\phi'} \]

(3.2)

where \( \omega \) is the nominal wage. Dividing both sides by price level \( P \), we have the following definitions in real terms:

\[ \frac{\omega}{P} = w \]

(3.3)

Using these definitions and equation 3.2, GS develop a competitive supply model:
$w/\phi' = \text{Marginal Cost of Production}$

$1 = w/\phi' = MC$

(3.4)
At this point in their discussion, GS discount other Keynesian labour supply relationships in order to justify their new relationship between output, risk bearing, and the probability of bankruptcy. According to GS, other theories of the Keynesian labour supply function, assuming constant real wages in a recession, have neither empirical nor theoretical justification; their theory of aggregate supply does: Rising bankruptcy costs shift the aggregate supply leftward.

A brief digression will bring these matters into focus. Recall, in the Classical Model:

$$N^s = N(w/p), N' > 0$$

(3.5)
and the supply of labour is an increasing function of the real wage (w/p), thus actual employment at the real wage $N^s$, equals the actual employment $N$. Substituting $N$ for $N^s$, yields

$$N = N(w/p)$$

(3.6)
Whereas in the Keynesian System, the equation $N^s = N^*(w/p)$ is eliminated, removing both a supply schedule and an equilibrium condition for the labour market. By appending the labour supply
function in a non-interactive manner, the Keynesian result that $N < N^*$ is produced, that is there is an excess supply of labour-unemployment. The Keynesian model also violates Walras' law as applied to flows: The excess aggregate demand for goods equals the sum of the excess supply of labour weighted by the real wage, holding assets constant. Turning to the GS Model, in their concave to the origin output function specified in equation 1, for labour supply levels of $l_1 > l_2$, the marginal products of labour evaluated at $l_1$ exceeds the real wage evaluated at $l_2$, respectively. This phenomenon occurs because of the continuity of the production function is such that with a sufficiently sharp fall in employment as resulting from a recession, the marginal productivity of labour exceeds the real wage. Although both the second derivatives of output with respect to labour, evaluated at $l_1$ and $l_2$ respectively, are less than zero, they are not equal, as one would expect for small changes in $l$ and with a sufficiently continuity in the production function. As the difference between $l_1$ and $l_2$ goes to epsilon, the derivatives will be equal in the limit, approached from either direction.

$$
\phi''(l_2) - \phi''(l_1), \phi''(l_1) - \phi''(l_2)
$$

(3.7)

$$
I_1 \to I_2, I_2 \to I_1
$$

Such phenomenon as used in the GS model violate classical first order sufficiency condition for optimality that the value of the marginal product equals the wages for all inputs.

A key feature of the GS model is how firms respond to the threat of bankruptcy. In the GS Model, as firms produce more they are required to bear more risk in the form of bankruptcy ($c_t = cq_t$), and this perspective leads the authors to replace equation 3.3 with the following equation where MBC represents the endogenously
determined marginal cost of bankruptcy (see Chapter Four, section 4.3, for detailed discussion).

\[ P = \left( \frac{\omega}{\Phi'} \right) + \text{MBC} \]

(3.8)

Dividing by \( P \), equation 3.3 may be replaced with:

\[ 1 = \frac{w}{\Phi'} + p \]

(3.9)

where we make \( p \) the real marginal cost of bankruptcy, and \( w \) is the real wage. Given \( p \) and \( w \), an equilibrium level of output of the firm may be determined. Including bankruptcy costs, we now have new first order marginal conditions: Production occurs at the level of output where price equals marginal costs of production (including bankruptcy costs). According to GS, it is because of bankruptcy costs that the aggregate supply schedule shifts.

Using the above ground rules, GS next develop the aggregate supply function from firm behaviour. The relationship between input decision making and output results are the heart of the model's dynamics. It is assumed that firms are identical, but are exposed to unique price shocks. They are indexed \( i = 1, \ldots, I \), and decisions occur over discrete time intervals \( t = 1, \ldots, T \). It is assumed that at the beginning of each period, \( t \), a given firm inherits both a nominal level of debt \( B_{t-1} \), and the output of the previous period \( q'_{t-1} \). It is further assumed that there is a one period lag between paying and using inputs in production, and the production of output, which is key to understanding the model's dynamics: We have output \( q'_{t-1} \) only ready for sale at the beginning of period \( t \), although the nominal debt, \( B'_{t-1} \), was incurred at the beginning of period \( t-1 \) in order to pay for the inputs that were required for producing \( q'_{t-1} \). All borrowed capital
must be used to pay wages and dividends out of profits at time \( t \) which has already been stipulated in the previous period. When profits are strong, firms increase output, borrowing capital to pay for new workers, increasing real wages. An increased wage bill, combined with dividend commitments, however, when profits are below anticipated levels has persistent or knock-on effects. In the event that profits at time \( t \) are below what was anticipated at time \( t-1 \) when capital was borrowed to pay wages, it may produce a situation where higher wages lead to lower profits. With the stochastic treatment of net dividend distribution as specified by GS or with dividends either sticky-downward or at a fixed share of profits, the possibility is created for capital reserves of equity to be further affected and reduced.\(^5\) The reduced equity leads to a fall in output. A stylized diagram of the model's dynamics is shown in Figure 3.1, below:

Turning to the mechanics of firm behaviour, we note that associated with the borrowed working capital, the debt, is a nominal contractual rate of interest \( R_{t, t} \), which we note is indexed to the firm, although the GS model does not consider the possibility of different firms facing different rates of interest: All firms are identical and face the same working capital costs. Nominal contractual interest service on debt by \( i'th \) firm, would be:

\[
(3.10) \quad (1 + R_{t, t}^i)B_{t, t}^i
\]

For the \( i'th \) firm, the price of goods \( q_{t, t}^i \), sold at time \( t \), \( P_t \) is randomly distributed with a distribution \( F \) and density function \( f \), and determines the nominal equity of the firm. Profits are assets less liabilities and determine the nominal equity of the firm, \( A_{t, t}^i \):

\(^5\)As will be discussed in Chapter Four, dividends should reflect the firm's cost of equity capital. Adjustments to the market capitalization of firms is more likely to occur through adjustments to the price of a share attached to a dividend stream than through changes in dividends themselves, because of signalling effects (T.A. Marsh & R.C. Merton, 1987).
Naturally, the greater equity the firm has, the more solvent it is, that is the smaller the probability of bankruptcy. When equity, $A'_t < 0$, the firm is declared to be bankrupt. The condition for bankruptcy arises, when the level of debt service on working capital from the payment of wages, adjusted for net dividend distributions, exceeds the value of inventory produced from output. In the GS model, although firms are assumed identical, and we will further assume that at least initially have the same level of equity, the price shocks which they face over time, are random, and unique or firm specific. With the same level equity at time $t$, because of different price shocks at time $t+1$, one firm may be bankrupt and another not. Some firms may be very profitable and others not. Inter-firm effects as might
arise from bankruptcy, however, are not considered in the GS model, except to the extent that the net proceeds of bankrupt firm are assumed in the GS model to be distributed to creditors.\textsuperscript{6} Rounding out the basic set-up, it is assumed that firms face real wage $w$, as per equation 3.9, above. Firms may borrow as much as they want at the prevailing rate of interest, but at terms which must yield the lender an expected real rate of return $r_r$. Critically, it is not assumed that the cost of capital is in any way risk-adjusted to, for example, the firm’s capital structure.

In the GS Model, the problem facing firm managers is one of one period static profit maximization.\textsuperscript{7} The objective function has one instrumental variable, $q_{ft}$, which once workers are paid, leads to a level of debt $B_{t+1}$, and a contractual nominal return $R_{t+1}$, that firm $i$ inherits at the beginning of period $t+1$. The decision of how much working capital to borrow locks firms into a wage bill (paid for with borrowed working capital) and from whence the inter-temporal dynamic properties of the model originate, as discussed above and shown in Figure 3.1. It is assumed at least implicitly, that firms do not borrow for the purpose of creating reserves (Benassi, et al., page 162, 1995). Given the equity constraint, initial conditions for state variable, the level of output determines a level of borrowing. With this set-up, GS now proceed to make the following sets of unique assumptions.

For assumption A1, we specify, following equation 1 above, but written in inverse notation, output is a function of labour alone, which GS assume is the same for all firms:

\begin{itemize}
  \item \textsuperscript{6}For simulation, as discussed in Chapter Four, we will assume that bankrupt firms are refloated at the prevailing average level of equity of firms which are solvent.
  
  \item \textsuperscript{7}Under certain conditions, including risk aversion, multi-period optimization, it may be shown is equivalent to optimization with a single period time horizon (Fama, 1970).
\end{itemize}
\[ l^i_c = \phi (q^i_c) \]  
(3.12)

where:

\[ \Phi^{-1} = \phi \]  
(3.13)

And, because of the inversion we have the following expressions for diminishing returns:

\[ \phi' > 0, \; \phi'' > 0 \]  
(3.14)

For the next assumption, A2, we have the price level faced by an individual firm is determined by the firm specific random variable \( u^i \) and the overall price level \( P \), thus:

\[ \frac{P^i_c}{P} = u^i_c \; \; \; \; \; \; \; \; E(u^i_c) = 1 \]  
(3.15)

and \( u^i \), the firm specific relative price of output of the \( i^{th} \) firm, is identical and independently distributed with a distribution function \( F() \), and density function \( f() \). We note that the stochastic price shock faced by the \( i^{th} \) firm is unique, although the distribution and density functions are not firm specific. Notwithstanding the firm specific price shock, in the GS model, firms are assumed implicitly to be identical in their technology of production. The initial levels of equity may vary, however, for simulation we will assume that they are identical.

For assumption A3, we note as above, if \( \lambda^i_i < 0 \), a given firm will go bankrupt, and their entire proceeds from the sale of \( q^i_{t-1} \)
are disbursed to creditors, that is debt holders, without general
equilibrium implications. (General equilibrium in the GS model, as will
explain, is in the labour market, not the credit market.)

A further result, manipulating the above relates to the level of
inherited nominal debt to nominal wage payments and the level of
nominal equity. Thus we have,

\[ B_t^i = P_t w_t \phi(q_t^i) - A_t^i \]

(3.16)

Given assumptions, A2 and A3, the \( i \)'th firm will earn returns at time
\( t+1 \) according to the distribution on firm specific prices realized only
when prices are revealed. Bankruptcy occurs when what firms have
promised to pay exceeds their income. Using result of equation 4 and
assumption A3, we have the following condition for bankruptcy:

\[ (1+R_t^i) B_t^i \geq P_{t+1} q_t^i \]

(3.17)

Now if we substitute into equations 15 and 16, we have the
following result for the \( i \)th firm:

\[ u_{t+1}^i \leq (1+R_t^i) \left( \frac{P_t}{P_{t+1}} \right) \left( \frac{(w_t \phi(q_t^i) - a_t^i)}{q_t^i} \right) = v_{t+1}^i \]

(3.18)

where, \( a_t^i = A_t^i / P_t \), or the real equity level of firm \( i \) at the beginning
of period \( t \), and \( v_{t+1}^i \) is the price threshold in period \( t+1 \), at which the
\( i \)th firm is just solvent, that is the level equity is positive. Depending
upon the firm specific price realization in relation to this threshold,
bankruptcy may occur.

A key feature of the GS Model is the effect upon firm
behaviour which the possibility of insolvency leading to bankruptcy,
may have. Due to credit market imperfections, loan contracts are not written based upon returns to sales, but rather as a nominal rate of interest on an amount of borrowed working capital. (Moreover, as we will explore in Chapter Six, the cost of capital is not risk adjusted.) Thus returns to sales may be specified for both solvency and insolvency cases as the following, respectively, as returns to lenders. (We note, GS avoid making the general price level random, as opposed to firm specific prices, as a pointless complication.) With regard to the variability faced by decision makers in firm specific prices, $P_{t+1}$, the model's dynamics lies. Remember, $u_t' = P_t'/P_v$ and using equations 3.15 and 3.16 above, we have the following conditions for solvency and bankruptcy respectively: A firm's total net income after dividends must equal or exceed debt obligations.

$$\left(1+R_t^i\right)\left(\frac{P_t}{P_{t+1}}\right) = \left(1+R_t^i\right)\left(\frac{P_t}{P_{t+1}}\right)$$

(3.19)

If, and only if,

$$u_{t+1}^i \geq v_{t+1}^i$$

Bankruptcy occurs however when,

$$\left(1+R_t^i\right)\left(\frac{P_t}{P_{t+1}}\right) = \frac{u_{t+1}^i q_t^i}{w_t \phi (q_t^i) - a_t^i}$$

(3.20)

If, and only if,

$$u_{t+1}^i < v_{t+1}^i$$

(3.21)

Using the price level neutrality assumption stated earlier, namely that,

$$P_{t+1} = E[P_{t+1}]$$

(3.22)
we assume that the future price level is the expected price level, equations 3.17 and 3.18 may be modified to obtain the real rate of return to the ith lender in period t. It is assumed that lenders are perfectly informed and risk neutral, and are enjoined from purchasing equity in the ith firm. This result leads finally to the real rate of interest, \( r_t \). Assumption A4,

\[
E[1+R_t]\left(\frac{p_t}{E(p_{t+1})}\right) = 1 + r_t
\]

(3.23)

Using equations 3.7 and 3.11 can be solved for the equilibrium level of the contractual nominal interest rate \( R_t \), and the solvency threshold \( v_{t+1}^i \), as a function of \( q_{t+1}^i, a_{t+1}^i, w_t, r_t, \) and \( P/E(P_{t+1}) \).

\[
R_t = R_t(q_{t+1}^i, a_{t+1}^i, w_t, r_t, P/E(P_{t+1}))
\]

(3.24)

\[
v_{t+1}^i = v_{t+1}^i(q_{t+1}^i, a_{t+1}^i, w_t, r_t+1, \frac{P_t}{E(P_{t+1})})
\]

(3.25)

Substituting from equation 3.24 into \( F(u) \), results in an expression for the probability of bankruptcy.

\[
\text{Prob. of Bankruptcy} = F\left(v_{t+1}^i(q_{t+1}^i, a_{t+1}^i, w_t, \frac{P_t}{E(P_{t+1})}, 1+r_t)\right)
\]

(3.26)

Bankruptcy occurs when the level of equity of the ith firm including retained profits and the revenues from the sale of output are insufficient to satisfy the debt service or loan commitments. Throughout it is assumed that loans are only serviced without amortization of principle, and borrowing does not occur to create reserves, as mentioned earlier.

Setting forth the optimization problem of the individual firm, involves the instrumental variable \( q_t \). Firms select output to
maximize expected real profits, which is composed of total sales revenues minus repayments to lenders, and minus an expected real cost of bankruptcy. Note, \( c'_t \) is the cost incurred in the event of bankruptcy and \( F(l_{t+1}) \) is its probability of occurring (as per page 88, GS). Thus, we have assumption A5, of firm level profit maximization:

\[
\text{Max} \left( \frac{1}{(P_{t+1})} \right) E \left[ (P_{t+1}^i q_{t+1}^i - (1 + R_t^i) (P_t^i w_t^i (q_t^i) - A_t^i)) - c_t F(l_{t+1}) \right]
\]

(3.27)

It should be noted that in order for firms to be averse to the risk of bankruptcy, the expression \( c'_t F \) must be convex in output, \( q'_t \). A firm cannot "grow" its way out of the fear of bankruptcy. By assumption, risk aversion grows with size, the importance of which we now discuss.

An important assumption found in the GS model is that the cost of bankruptcy is an increasing function of the firm's level of output. The intuition is that larger firms produce larger output, and are therefore more expensive to administer in the event of bankruptcy. (The liquidation of General Motors on a percentage basis costs more than that of the corner green grocer.) Moreover, if bankruptcy costs were merely a fixed amount, firms could escape such threats by increasing sales and output to the point that such costs were nothing but a minor annoyance. Finally, since bankruptcy may occur when managers have selected a high level of output, \( q \) involving wage-bill commitments and because of economic contraction, realized prices next period are low; it underlies that conditions of insolvency in the GS model occur because of poor judgement or the inability to manage exposures. Thus, we have assumption A6,

\[
c'_t = c q'_t
\]

(3.28)

Given assumptions A2 and A3, we can rewrite the objective function,
over the instrumental variable \( q' \), assumption 5, as equation 3.29. Maximizing over subscript \( i \), we have:

\[
\max_{q_t^i} \left[ q_t^i - (1+r_t) (w_t \phi(q_t^i) - a_t^i) - c_t^i F(v_{t+1}^i) \right]
\]

(3.29)

In this expression, a firm’s real output is a function of real wages, real interest rates, real equity holdings, and relative price uncertainty. GS now proceed to derive first order conditions for an interior maximum found in Equation 3.30.

\[
1 - (1+r_t) w_t \phi' = \rho_t^i
\]

(3.30)

(We note that no appeal is made to the Weierstrass Theorem for the sufficient conditions for such a maximum to be a global maximum.) In addition, the authors, solve the marginal bankruptcy costs of the \( i \)th firm in period \( t \).

\[
\rho_t^i = \left( \frac{dc_t^i}{dq_t^i} \right) F + \left[ c_t^i F(v_{t+1}^i) \left( d \frac{v_{t+1}^i}{dq_t^i} \right) \right]
\]

(3.31)

For the second order conditions, GS refers the reader to the appendix of the 1988 paper in which they are derived.
3.4 Extending and Modifying the GS Model

In order to model and analyze whether the use of financial innovations in the form of derivatives, for the management of financial risk have macro-economic implications; the GS model must be extended and modified. We conclude Chapter Three by discussing these modifications from a conceptual standpoint. Our observations and proposals for modifying the GS Model as discussed above, relate to four areas.

1. How to include futures or options upon futures markets in the Model so that their static and dynamic macro-economic effects may be assessed;

2. The micro-theoretic requirements for such a modifications to function;

3. Other changes to the GS model to enhance its realism; and

4. The dynamics of non-linearity and its simulation.

In order to include explicitly risk management into the GS model with a view to examining its macro implications, we adopt the arbitrage principle between futures markets and interest rates. We recall that in the GS Model, firms are forced to borrow because of the equity constraint but are unsure of what the return on future sales of output will be realized in next period. GS deliberately assume away futures markets on the grounds of transaction costs. The absence of futures markets implies that every output decision implies a risky investment decision, given the constraint on the availability of equity. The macroeconomic consequence of this arrangement is that the aggregate supply function, in addition to wages and prices, also depends upon the aggregate equity position of the firm. For example, firms with sufficient equity, avoid the risk of borrowing sub-optimally; while firms with insufficient equity, are forced to borrow and invest, not knowing what prices and thereby what returns will be realized.
Different equity levels determine different output paths, including ones with full-employment. The non-existence of futures markets or the inability to correctly forecast next periods output prices results in a form of aggregate externality in the form of supply mistakes, as mentioned earlier. The absence of futures markets in the GS model points to how the impact of futures markets upon macro variates might be analysed. Although we return to subject of alternative paradigms concerning futures markets in Chapter Five, in order to explain how futures markets and risk management may be incorporated in the GS Model, we provide some further theoretical background on futures markets.

A great deal has been written on whether Futures markets are efficient predictors of spot prices in the future, or whether futures market display a bias. Keynes argued that the risk premium was the source of bias and backwardation. Keynes (1930) and Hicks (1946) argued that the futures price would be less than the expected value of next period’s cash price. They believed that backwardation was normal, that is the futures price will be at a discount to the expected next period cash price. This difference between the expected spot price and the future price available today, represents a risk premium, which the hedger pays to the speculator. Their observation implied that if not for the existence of risk premia, the futures price would be a good predictor of the expected spot price, (Anderson and Danthine (1983)). These theories may provide useful insight with regard to the risk premia, however they do not provide any economic insight into the relation between the spot price today and in the future, and the price quoted in the futures market. For example, futures prices for the same delivery specification change continuously. Can risk premia be continuously unstable? The market efficiency literature, has devoted reams to testing whether futures markets are efficient unbiased predictors of prices in the future (Bray, 1983). Although the Keynes-Hicks insurance paradigm of hedgers and speculators is
appealing, alternative models of risk management have been proposed, as explained below. In Chapter Five we return to one such paradigm using inter-temporal asset allocation.

Other theories on the use of futures markets, such as portfolio approaches offer minimal insight of relevance to the risk management problem. Approaches, using Capital Asset Pricing Theory (eg. Sharpe (1964), Cummins (1983), or Main (1982)) or Arbitrage Pricing Theory (Ross (1976)) have also been applied to futures markets with little realism, Cho (1988). The basic problem with such approaches is that if futures market behave systemically vis-a-vis the market, diversification has no effect upon risk. Portfolio theory involves diversification against idiosyncratic risk. In contrast, Systemic risk cannot be diversified, only idiosyncratic risk. To argue that futures markets as a means of risk management always behave in an idiosyncratic manner in relation to the underlying markets, however, would find little support.

In contrast with the above schools of thought, viewing futures markets as a problem of intertemporal asset allocation provides the theoretical underpinnings we need to inform and advance our research. As discussed earlier, in the GS model, cyclic behaviour arises because of the inter-temporal asset allocation problem faced by firms. Regarding the relationship between spot prices and future prices, as an intertemporal asset allocation problem sheds considerable light upon the relationship between futures prices and interest rates. The work of William's *The Economic Function of Futures Markets* (1994) addresses this question from a fresh perspective. Inspired by the work of Cox, Ingersoll, and Ross (1981), Breeden (1979), Frenkel (1975), and the classic works of Brennan (1958) entitled, "The Supply of Storage", and Working (1948, 1949, 1953a., 1953b.); Williams argues that the relationship between the current spot price and futures price, long or short, reflects an interest rate in commodity terms of the "borrowed" or "lent" position
respectively. According to Williams, futures markets are a means of intertemporal asset allocation to ensure the good use of assets. It is neither a means of pricing and exchanging risk, akin to insurance, nor is it a means of portfolio diversification. Instead, the benefits of risk management arises because futures markets are a form of implicit loan markets, akin to Working's notion of the supply of storage. Seen as a means of allocating resources over time; the relationship between spot and near and distant futures markets prices may be used to create a yield curve and determine implied rates of interest. We see then that the William's paradigm of futures markets is of immediate relevance to the modelling the use of derivatives in the New Keynesian setting of the GS model, as it will serve to explain the inter-temporal dynamic effects thereby arising. If one viewed the use of derivatives as a means of pricing and transferring risk, risk management, as imagined by the many authors discussed above, it might follow that their use would be stabilizing at the macroeconomic level. In contrast, adopting the Williams paradigm that the use of derivatives concerns inter-temporal asset allocation, as we will see, is consistent with the inter-temporal dynamic effects generated in Chapter Five.

Turning to the details of the William's paradigm will help us consider how it may be implemented in the GS setting. Assuming no bias as risk premia, for example, a three month option on a futures contract expiring in six months time implies a rate of interest rate three months hence.\footnote{If a risk premia do exist, the futures markets are biased estimates of short term interest rates. The stability of this bias is favourite area of research: For example, "The Determinants of Hedging and Risk Premia in Commodity Futures Markets", Hirshleifer (1989); and "Macroeconomic Forces: Systematic Risk and Financial Variables: An Empirical Investigation," Young, Berry, Harvey and Page (1991). The view among practitioners is that futures markets are predictors of short-term interest rates, and not biased or otherwise predictors of spot or future spot prices, (Stigum, 1981) which accords with} in Williams' model, interest rates are
predictors of the equilibrium relationship between the spot and futures price, and the value of a position today versus the value of position "tomorrow", ceteris paribus, because futures markets represent an implicit form of borrowing, implying rates of interest in terms of the deliverable. Quoting Williams, (ibid, page 74, (1994)).

"A short hedging operation [selling forward], the spot purchase of a commodity and its simultaneous sale for future delivery, amounts to borrowing a commodity over an interval of time while lending money. Likewise, a typical long hedging operation is often in part an implicit forward loan of a commodity. ..., it follows that a futures market for a commodity is primarily part of an implicit loan market for that commodity."

For example, a spot purchase and a short sale in a futures market, is an implicit loan for a commodity at the rate of interest $i$. Adhering to this paradigm, futures markets are a way of borrowing or lending positions, that is allocating assets on an intertemporal basis. For the $i^{th}$ firm, the relationship between the spot price at time $t$ and the futures price available at time $t$, for delivery at time $t+1$, implies a commodity rate of interest, in the absence of risk premia (Williams, 1994). The "use" of a commodity like the use of money attracts a rate of interest on the commodity itself. If borrowed, the rate of interest represents how much additional amounts of the commodity must be returned, and if lent how much additional commodity one can expect (Chapter 4, Williams, 1994).

$$\frac{p'_{t+1}}{p_t} = (1+i_t)$$

(3.32)

Where $p'_t$ is the future price at time $t$ for deliver the next period and $i_t$ is the rate of interest in commodity terms thereby implied.

Similarly the decision to sell from output today versus promising to deliver via the futures market, may be conceptualized using the same paradigm advanced by Williams.
framework.

In the William's paradigm, the relationship between the cash and the futures market may be used to derive an implied Fischerian forward rate and yield curve in commodity terms. Assuming no risk premia, by selling the cash position in the present, a firm could take the proceeds and invest it at the market rate, say \( r \); or alternatively, the firm could sell in the next period via the futures market earning yield \( i \) on the commodity itself. If the forward rate exceeds the rate \( r \), the firm will use the futures market; while if the implied forward rate is less than \( r \), the firm will sell in the cash market now, and invest at \( r \). In equilibrium, both strategies should be equally profitable, via the presence of arbitrage. Implicitly, futures markets are a means of borrowing and lending, for example, commodities. By comparison, it is the presence of intertemporal arbitrage opportunities with discrete time lags and futures markets assumed away, however, which in the GS model figures strongly in its dynamics of insufficiently risk averse firms over-investing in working capital and driving-up real wages, and with certain dividend assumptions, driving equity down and eventually output. In the GS model the lag between when borrowing/investment decisions are made, combined with variability in firm specific prices affecting what returns on working capital are realized, including the possibility of bankruptcy, which together play a critical role in the model's dynamics. Adopting the William's paradigm, importantly, tells us why using derivatives as a means of inter-temporal asset allocation may have an effect upon the Laws of Motion found in a GS style model. Gains arise from the use derivatives through improved inter-temporal asset allocation, locking in the value of future production. Consistent with the William's paradigm, introducing options into the GS model improves inter-temporal allocation, encouraging investment and output, and would, if a short-side of the market were modelled, eliminate opportunities for arbitrage. As will be shown through simulations of the GS model,
the use of derivatives allows firms to concentrate upon price relationships rather than the absolute level of prices.

The William’s paradigm applied to understanding the GS Equity Constraint Model’s provides insights with regard to critical questions as the impact upon aggregate cyclic behaviour of innovations, such as derivative markets. Incorporating the use of derivatives into the GS model may permit us to learn the extent to which they may interact with the endogenous sources of cycles found in such models. In order to explain the results thereby arising, the William’s paradigm we will see, provides useful insights. To address the general question of the significance of firm level financial factors upon business cycles and the specific question of the use of derivatives upon business cycles, we propose firstly to modify the GS model by including put options which may be used to obtain a minimum level of return on borrowed working capital. In outline, such options will be priced endogenously through the model using, inter alia, the cost of capital and the uncertainty in firm specific prices $F(u')$, without new exogenous assumptions required. The extent of usage of risk management will be endogenized based upon the relationship between the futures price versus cash price and the rate of interest, and variability in firm specific prices.

In accord with the William’s paradigm that futures markets arise endogenously rather than being assumed into existence, and that their usage is an economic problem, we propose that the modifications to the GS model should be of the same spirit (Chapter 6, Williams, 1994). In the actual GS Model, as mentioned, the existence of futures markets is assumed away. Capital markets are highly imperfect, and are limited to debt and equity securities with the latter constrained, but without state-contingent assets. Between the extremes of not existing and being completely available, we introduce options on futures markets into a version of the GS model, as a means of analysing the effects of such risk management practices.
upon business cycles. We have noted that in the GS model, the assumed absence of futures markets are a source of "mis-borrowing", and possible bankruptcy during economic down-turns, leading to dynamic aggregate output effects. In a modified version of the GS model, we propose making a supply of derivative products endogenous to the model and making their use a matter of economic decision making for the individual, optimizing firm.

The above theoretical structure, as outlined, for example, will allow us to ask what would be the business cycle effects if the ability to service borrowed working capital (because of the equity constraint) were ensured via options? Would the risk of bankruptcy be the same if borrowing, per se, did not increase the risk to book value from expanding market value through gearing? Would there be Aggregate Supply and dynamic effects from the use of derivatives priced according to underlying price variability and the cost of capital? The micro-theoretic requirements for such modifications of the GS model along the lines discussed are several-fold. Although we motivate them fully in Chapters Four and Five, we sketch them here for their relevance to conceptualizing the problem at hand, the business cycle implications of futures and options markets. In the GS model, the marginal cost of bankruptcy drives a wedge between traditional marginal cost and wages. Aversion to bankruptcy arises because of the variability surrounding firm specific prices, \( \frac{P_{t+1}}{P_t} \), and their relationship to the threshold level \( \frac{\gamma_{t+1}}{\gamma_t} \). GS make the bankruptcy threshold level and hence the probability of bankruptcy a function of many parameters, about which there is little uncertainty. As we will explain in Chapter Four in depth, firm specific prices for which future output is sold is stochastic and for a given firm, its relationship to its specific threshold level determines when bankruptcy occurs. In the simulated version of the modified GS Model, we will also use the standard deviation for the stochastic process on firm specific prices in the pricing of options on futures used to manage the risks of
investing in working capital. By specifying the modified GS model accordingly, its realism is enhanced, and thereby we make the provision of derivative usage endogenous to the model. We do not assume perfect capital markets, *willing* futures markets into existence along with their use, but rather given variously endogenously determined parameters, we assume that if they are economic to use, a market exists. Lastly, with regard to the use of derivatives, we invoke *small-country* assumptions, that is with regard to the availability of risk management at the prevailing price, representative firms in the modified GS model face a horizontal supply of options with which to manage the value of future inventory.

Other changes to the GS model to enhance its realism concern the cost of capital and profit retention. As we will discover in Chapter Five, the use of derivatives for enhanced inter-temporal allocation of assets, "risk management", does indeed change the GS model's laws of motion, however, two key assumptions found therein may not be realistic. As mentioned earlier, it is common place in the NK literature, for example, Fama (1985) and Bernanke and Blinder (1988) to challenge the traditional Neo-Classical IS-LM view, compatible with received Financial Theory, that higher interest rates induce consumers and firms to reduce their investments and purchases: Debt capital is rationed by price or the cost of capital is risk adjusted so that financial gearing does not change the firm's market capitalization. In Chapter Six, including risk management in the model, we consider the inter-temporal effects of replacing the assumption of a fixed cost of capital with one tied to the quality of credit, specifically the ratio of debt to equity. As we have observed, in the GS model, capital structure is composed of debt and equity. Equity is constrained which leads to possibly sub-economic ex ante borrowing of working capital. Such behaviour implies a ratio of debt to equity. Investigating the aggregate effects of rationing debt capital as well by price, according to the quality of credit, might be revealing.
A further modification to the model, which again may enhance its realism concerns dividend payments. GS avoid any modelling of dividend structure by merely assuming that it is *theory neutral* or stochastic. They only observe that dividend payments must be sufficiently so that the impact firm equity does not grow explosively (page 100, GS 1993). Overall, GS devote little attention to the implications of profit distribution. To rectify this limitation we and implement a version of the GS model as a simulation program, in which we assume that dividends are a fixed percentage of profits. Varying the fixed percentage of profits retained, we will learn has inter-temporal effects upon the model’s dynamics. In the GS model we will see how both rigid real wages and dividend payments as fixed percentage of profits, sharply reduces internal capital, equity, leading to the dynamic swings in output. Like other aspects of the GS model, this effect is a key implicit assumption of their work. We note that it challenges received financial theory on the neutrality of dividends versus capital gains from the standpoint of effects upon valuation, (“Dividend Policy, Growth, and the Valuation of Shares,” Miller & Modigliani, 1961) just as much as allowing capital structure to effect valuation. Arguably, by increasing debt to pay potential dividends, the model creates a bias favouring equity holders at the expense of debt holders. Although equity holders are keen to expand market value, the concern of bond holders is only to preserve book value. What if, however, dividend payment could adjust both up and down to the rate of change in profitability in a manner sensitive to the distinct incentives of debt and equity holders? Would valuation neutrality of dividend structure be restored? Recall, that it is because of the firm’s commitment to a level of dividends, limiting profit retention, which reduces equity and which in turn, leads to a drop in output, and further cyclic phenomenon. We wonder whether it is realistic to believe that firm would endure bankruptcy to pay a dividend? Although incentives to expand market value exist, it is
usually not with complete disregard of the risks to book value during economic contractions. (Although following Friedman’s view of positive economics, a model should be evaluated on the utility of its insights; and not its assumptions, when such assumptions challenge reality, it is may be useful to question their value and implications.) Enhancing valuation through reducing dividends in the manner described above so that they are not sticky downwards or so that they are valuation neutral, to the extent that it protects the firm’s equity, might dampen any oscillations in output generated through the dynamic laws of motion described earlier. Such modifications to the fixed percentage approach and their implications we leave until Chapter Six.

The final aspect of modification to the GS style model concerns simulation of its non-linear dynamics. Although using numerical simulation to examine the non-linear dynamics of the GS style model does not modify the model per se, it does change how we learn about it. As mentioned in Chapter One and explained in detail below in Chapter Four, GS appeal to linearization around the steady-state in order to make analytic observations about the model’s inter-temporal dynamic behaviour. For example, they claim that the model exhibits multi-periodicity, that is repeated observations above or below the steady-state, the 45° line in logistic space, resembling business cycles, although this claim based upon qualitative analysis is difficult to substantiate. As we justify in Chapter Four, linearization as GS utilize, rather than being a helpful technique, obscures much of the model’s rich dynamics, including period doubling bifurcations, multi-periodicity, aperiodicity, and extreme sensitivity to initial parameter values, that is chaos. Through calibration, numerical simulation, we discover, also reveals that certain initial parameter values must exceed certain minimum values in order for the model to cycle, rather than follow a monotonic trajectory. According to many authorities, since analysis of even the most simple first-order difference and
differential equations is generally not possible, numerical simulation is the only choice (Ott, 1993 or Baker & Gollub, 1990). In order to understand the non-linear dynamics of the explicitly specified GS style model, with and without the management of risk, numerical simulation is utilized in both Chapters Four and Five.
CHAPTER FOUR. SPECIFICATION AND SIMULATION OF THE GS STYLE MODEL

4.0 Introduction

In Chapter Four, we explain our approach to specification and computer-based simulation of a New Keynesian macroeconomic model based upon the articles of Greenwald and Stiglitz (hereafter, "GS") (1988, 1993). The heuristic relationships found in their 1988 and 1993 articles form the basis for the functional forms and algorithms developed in the computer simulation program. They are consistent with the class of financial market imperfection and financial constraint models of which the GS work is an example. Using the specified functional forms, as embedded within a software program, various simulations will be conducted which verify the propositions found in the GS article along with the financial constraint and imperfection aspects. The computer-based model uses an optimization algorithm, to obtain partial equilibrium at the firm level, general equilibrium at the aggregate level, and a dynamic output trajectory, using the techniques of phase diagrams and simulation. Further simulations showing the nature of competitive equilibrium and dynamic behaviour will also be conducted, establishing the functional forms and algorithms used as a valid representation of a GS style model. We conclude with various experiments to analyze the sensitivity of the output trajectory to changes in exogenous variables.

Before these tasks, we provide some justification of and background to computer simulation. There are situations when computer simulation of economic systems has an advantage over analysis of reduced form equations, particularly those arising from non-linear systems as found in the class of models of the GS model. Simulation allows us to learn how the model behaves when an analytic solution (i.e. the algebraic reduced form of the structured equations) is difficult to find or analyze from the standpoint of how changes in exogenous variables effect dependent variables over time
(Ott, 1993). In addition, when non-linearities feature in dynamic relationships, analysis may be impossible (page 662, Hoy, Livernoise, McKenna and Rees, 1996). Constraints involving inequalities may also make direct analysis difficult. By using a simulation program, we may improve our understanding of the dynamic behaviour of systems such as those involving a non-linear difference equation model as found in the model developed by GS. Further, simulation allows us to calibrate the model, to answer the question of whether plausible values for exogenous parameters generate fluctuations economic output variables resembling reality.

Exhibit 4.1. Stylized Program Flow Chart
4.1 Program Design

Turning to the design of the computer program used in simulation of a GS style model, it has been written using spread-sheet software. To specify the GS Model as an economic system suitable for simulation, we begin by noting the general top-down structure, as shown in Exhibit 4.1, above. Inputs appear in two different boxes. The ones at the top of the diagram refer to those required by the GS Model as published, and are denoted Basic Model Inputs. Such input variables are exogenous to the model, and represent the sources of system perturbations. We describe both Basic and Additional Model Inputs in Exhibit 4.2, below, noting that simulations may be performed modifying one or more related set of parameters:

Exhibit 4.2

**Basic Model Inputs**

- A contractual rate of interest: Nominal: $R_t$; real: $r_t = R_t/P_t$
- Household rate of time discount $\delta$ which equals $r_t$
- A contractual wage rate at time $t$, $w_t$
- An initial level of equity for the $i$'th firm, $A_i^t$; real: $a_i^t = A_i^t/P_t$
- A profit retention variable, $M_{i,t+1}$, at time $t+1$, for the $i$'th firm, as new equity, net of distributions; real: $m_{i,t+1}^t = M_{i,t+1}^t/P_t$

**Additional Model Inputs**

- A price level at time $t$, $P_t$, and at $t+1$, $P_{t+1}$
- A firm specific price received, at time $t$, $P_{i,t}$
- The ratio of firm specific price increase to the general price level, $P_{i,t+1}/P_t = u_t$
- The standard deviation in the stochastic process on firm specific prices, $\sigma$
- The distribution function and density function, respectively, associated with the stochastic process on firm specific prices: $F(u_{i,t})$; $f(u_{i,t})$
- An implied ratio of debt to equity, $b_i^t/a_i^t$, given an initial level of equity for the $i$'th firm, $a_i^t$
- Production function technology coefficient, $k$
Throughout, the superscript $i$ refers to the $i$'th firm. It is assumed for simulation that the population of firms equals twenty-five. Firms which retain all their profits may increase their level of equity, while firms which distribute all their earnings, may maintain a constant level of equity, unless reduced by financial obligations. The revenues and hence the profits of individual firms are effected by firm specific prices which are assumed to be normally distributed.

The second set of inputs, as shown in Exhibit 4.2, denoted Additional Model Inputs, represent other exogenous variables which are used in the simulation program to incorporate various creative modifications to the GS style model, and allow for the generation of other useful experiments, as well as validation of the unproven observations and Propositions found in their article. Their corresponding variable names also appear in Exhibit 4.2. Like Basic Model Inputs, they feature in the model as sources of perturbation, some of which affect the model's law of motion, such as periodicity. By assumption, the model is calibrated to inputs, both Basic Inputs and Additional Inputs, which are either specified as annual rates, such as the annual real rate of interest, $r_t$, ex ante expected firm specific price increases $P_{t+1}/P_t = \delta_{i_{t+1}}$, or, are empirically realistic, such as the ratio of debt to equity. The real rate of interest $r$, which is set equal to $\delta$, the consumer rate of time discount, which is not subscripted on time, hence making $r$, constant over time. The program has been designed so that such perturbations as exogenous inputs may be used selectively and individually: Unless one decides to conduct an experiment using such exogenous variables, the program uses a default value. If, however, one adjusts such input parameters, as exogenous inputs, new sets of dynamic output solutions will be generated.

The dynamic economic system specified and modelled in the program encompasses all three of the Modules appearing in the middle of Exhibit 4.1. We have Module I which solves the firm level
optimization problem given the various input parameters mentioned earlier. In Module II the model solves for a general equilibrium in labour input space, using the aggregate supply schedule generated from the rising portion of the marginal cost curves found in Module I, to determine an aggregate demand for labour function. Equating the aggregate demand schedule with the aggregate supply of labour, we solve for equilibrium in wages and labour input. Lastly, we have Module III which performs dynamic analysis as suggested in the GS article, through constructing a non-linear phase diagram, computing its slopes, and solving for iterative solutions. Using Module III, we show how changes in exogenous initial conditions impact the dynamic trajectory of equity and output over time. We note that the results from Module II are required to generate solutions for Module III, and similarly, the results of Module I are required for Module II. The outputs of the first and second modules, may be considered for their static results, confirming through numerical simulation behaviour that the stylized features of the GS Model have been reproduced.

The right hand side of Exhibit 4.1 shows the various outputs generated by the software programme. Proceeding from the top to bottom, we find the box labelled, Firm Level Outputs, which are the results of the firm level optimization problem found in Module I. We have for the $i^{th}$ firm, revenues, costs, profits, optimal production, retained profits, and its solvency versus bankruptcy status. Moving down, we have the second box, labelled Aggregate Output and Price Level. Through the solutions found in Module II, an aggregate level of output and prices are generated, along with the implied aggregate demand and supply of labour. As per the GS Model (page 94, 1993), aggregate equity and output are obtained by summation from the individual firms. Having in each period a certain proportion of the assumed population of twenty-five firms become insolvent would imply that eventually, all the firms would disappear because there is no process in the GS model for the creation of new firms. We adopt
the convention of re-floating firms which are bankrupt at the average level of equity of the remaining firms. Lastly, we have outputs found in the box labelled, Dynamic Output Path. Here the results of dynamic solutions for the level of equity, output, and prices appear. Further, the nature of these solutions are described by their qualities, such as stability, convergence, and general laws of motion. A listing of all specified functional forms used for simulation and the corresponding general formulation as appears in the GS article, is found in Appendix 1.

4.2 The Production and Revenue Functions

In this Section we describe the output and revenue relationships found in the Firm Level Optimization Program of Module I. We will explain what is presented in the GS Article, how it has been expressed as a functional form in the program, and how the algorithms found in Module I make use of it. In addition, we will introduce features for the adopted functional form, not required by GS, but which makes validation of one of their results possible as well as allowing for various experiments and simulations.

The problem facing a modeller in selecting specific functional forms to capture the general ones found in an article such as that by GS is that a loss in generality may arise as such features give rise to unique behaviour, begging the question of to what extent the spirit of a model has been captured. An answer to this question cannot be objective. In specifying these functional forms we have sought to satisfy the requirements of the GS model along with meeting the requirements of economic theory. We begin with the GS production and revenue functions, whereafter we explain their use in the program.

GS assume that aggregate output $q$ is a function of labour $L$ alone, and define it accordingly:
The partial derivatives of their general function obey classical economic theory. We note that they have specified that their general function with only one argument, labour. By specifying their equation accordingly, GS have ruled-out consideration of factor shares and factor substitution, as would be found, for example in a CES production function. The specific functional form which we have selected to represent 4.1, to map from labour input space to output space, is logarithmic. For the \(i^{th}\) firm at time \(t\), we have the following specified functional form, which like the GS general form has only one argument, labour, although a parameter for the productivity of technology \(k\), as we explain, has been added. The specified form obeys the signs required of the partial derivatives:

\[
g_{t}^{i}=\ln(L_{t}^{iK})
\]

Moreover, as required by the Model, it is convex. An additional reason for specifying this functional form is that the GS Model requires that the cost function be inversely related to the production function, making labour a function of output. Inverting a logarithmic function into one obeying the requirements of the cost function (as discussed in 4.3) is easy. Like the GS model, we note that the production function has no parameters for distribution, as there is only one input to production, labour, and consequently, no parameter for the elasticity of substitution.

Although not included in the GS article, we have introduced a technology coefficient (the degree \(k\) in equation 4.2) to parameterize the productivity of labour. In addition to allowing us to make experiments of the effects of change in productivity upon dynamics,
the coefficient is useful to validate the assertion by GS of when cycles occur. The technology coefficient is similar to the parameter found in a Cobb-Douglas Production function or CES function.\(^1\) As the production function has only one input, labour, the level of productivity corresponds to an output per man-hour, although this is not reflected in the wage level, \(w.\)\(^2\) The degree of the production function, \(k,\) may be set as a constant of any value. Experiments showing the effect of changes in \(k\) upon the output trajectory appear in Figures 32 and 33, at the end of the chapter. This degree coefficient serves to represent the state of technology of an economy, which determines productivity of labour. Throughout all simulations it is constant and non-random making it distinct from the Kydland & Prescott (1982) style random technology shock approach, as appears in the literature (eg. M.O.Ravin, 1997).\(^\)\(^\)

With the production function specified, formulation of the revenue function is a straightforward matter, and follows the \(P_{t+1}q_t^i\) expression found in the GS article. Thus the specific revenue function found in the computer simulation program is the following:

\[
\text{Revenue} = P_{t+1}^i * q_t^i
\]  
\[(4.3)\]

Using the production and revenue functional forms to generate a revenue curve, as required by Module I, for a given level of technology coefficient (for example, degree 5) and set of prices, at a single time \(t,\) for the \(i'\)th firm, involves the construction of a simple algorithm. Using a schedule of labour inputs, the corresponding level of output, \(q,\) and the corresponding level of revenue is computed.

\(^1\)We specify technology as the power to which the one parameter \(l\) is raised; unlike the CES or Cobb-Douglas production functions where capital is included and adjusted by a power.

\(^2\)The marginal product does not determine the factor income. Factor income is determined by the value of the marginal product through the interaction of supply and demand.
Plots of revenue functions, for a given set of prices and technology, appears in Figure 2, seen below. As may be observed the explicit revenue function, along with other functions which we develop below,

![Figure 2: Cost, Revenue and Profit Functions for given Interest Rates, Prices, Technology](image)

obeys the requirements of the GS Model as specified in the partial derivatives, and displays concavity. Having explained the specification of the production and revenue functions found in the GS model, we turn to the cost function, which is key to the financial constraint and financial market imperfections found in the GS model.

4.3 The Cost Function

It is within the specification of the cost function, in particular the inclusion of a cost of bankruptcy, that the unique properties found in the GS Model derive. GS have added to a classical style cost function involving borrowed working capital, a feature to represent the behaviour of firms when a possibility of bankruptcy exists. This property gives the model its unique dynamic behaviour as well as sets the stage for our modifications of it. We divide our discussion of
explaining and specifying the GS cost function as a simulation program, into three sections: We first look at the basic cost related to borrowing working capital, thereafter we turn to the inclusion of expected costs of bankruptcy parameters, making observations on its functionality, and lastly we conclude with the specification as developed and calibrated in the computer simulation program. Throughout, our approach is to work within the requirements of the GS model in combination with received economic theory in order to specify explicit relationships suitable for calibration and numerical simulation.

We begin by explaining the derivation of the GS cost function at an abstract level. The justification for deriving a cost function from a production function, as GS do, is found in the Theory of Duality between Cost and Technology functions. Since the technology of production, firstly, is regular (closed and non-empty) and; secondly, since it is monotonic; the construction of a cost function from a production function is a valid exercise (Varian, 1992). In this light, the cost function found in the GS Model is derived from expression 4.1 found above. Using \( w \) for real wage, the marginal cost of production in real terms is defined as:

\[
\frac{w}{\phi'} = MC \text{ of Production}
\]  

(4.4)

Applying the concept of duality, we explain how the cost function including the arguments for expected cost of bankruptcy may be developed and specified. Referring first to how GS have presented their objective function, in general form, in expression 4.5

---

Although GS’s interest is in the effect of investment uncertainty upon the dynamic trajectory of aggregate output, from a micro-theoretic standpoint, little here is new: Research into the effects of increasing risk upon firm behaviour include Nickell’s Investment Decisions of Firms (1978), Hirshleifer’s Investment, Interest, and Capital (1970).
below:

$$\text{Max}_{\sigma^t} \left[ q_t^i - (1+r) \left( w_t \phi(q_t^i) - a_t^i \right) - c_t^i F(V_{t+1}) \right]$$

(4.5)

As may be observed in the GS Objective Function, there are two aspects of cost. Beginning with the non-bankruptcy component, we have:

$$(1+r) \left( w_t \phi(q_t^i) - a_t^i \right)$$

(4.6)

Expression 4.6 tells us that firms borrow financial capital to pay labour production costs as determined by the labour requirement function multiplied by the wage level. The cost of such borrowing to the $i^{th}$ firm is reduced by the level of equity, $a_t^i$, which arises, in the GS model, from profits in the form of retained earnings and new equity sold, $m(a_{t+1}^i)$, as appears in their difference equation (page 99, GS 1993). As profits are determined by firm specific prices which are stochastic, we see that equity $a_{t+1}^i$ is determined by both $a_t^i$ and $u_t^i$.

With regard to new equity sold net of dividends, $m(a_{t+1}^i)$, in the numerical simulation program, we assume rather that the dividend structure of firms does not change in a random manner, an approach involving minimal theoretical content. In order to avoid introducing another random variable, rather, in the simulation program we make dividends a fixed proportion of profits. By adopting this approach we are able to investigate the dynamic effects in the GS model of changes to a firm’s capital base. (Arguably, by modelling dividends

\[\text{Accounting and not theoretic relationship exists between retained profits, dividends, and investment (page 553, Copeland & Weston). Dividend yields should reflect the cost of equity as derived from Capital Asset Pricing theory. Empirical tests of dividend theory models supports the view that firms are loathe to increase dividends unless they can be supported with adequate profits (Lintner, 1965 and 1965, and Fama & Babiak, 1968). Similarly, downward rigidity in dividend structures may arise for various reasons including "signalling effects".}\]
as a fixed percentage of profits which itself is a function of the firm specific change in the stochastic process on prices, one random dimension has already been introduced.) As an alternative to either the fixed percentage approach or the stochastic approach of GS, a dividend structure which is consistent with received financial-economic theory is considered in Chapter Six. With greater proportion of equity in the capital structure, the effective cost of borrowing should be reduced, even though real interest rates may be unchanged. For now, however, following the GS model, we ignore the effect capital structure may have upon the cost of debt and equity.

To specify an explicit expression as required for the algorithm of the computer program, as found in Module I, we supplement the GS model with received theory, in the following manner. Firstly, we note that the left-hand expression in (4.5) uses the inverse of the production function: Where production was a function of labour, as in 4.1 above; one component of cost in the GS Model depends upon the amount of labour employed at the wage rate $w$:

$$w \phi (q^*)$$

(4.7)

Where we note that small $\phi$ is the inverse function of the large $\Phi$ found in equation 4.1, as used by GS. To represent the GS Model, the cost function found in equation 4.7 must be the reciprocal of 4.1 the production function. The cost function, GS specify as a function of the sole argument, labour. Although GS only require that the cost function be derived from inverting the production function, since the production function is concave, the cost function must be convex in $q$. Since the production function was specified in Module I as a logarithmic function, to represent the left-hand side of the cost function, we use an exponential function, as shown in equation 4.8 below, as it is convex over the relevant region. This satisfies received
theory requiring a cost function be convex and rising at an increasing rate (M. Intrilligator, 1971).

\[(1+x) \left[ w_t \exp \left( \frac{q_t^i}{k} \right) - a_t^i \right] \]

(4.8)

We now turn to the bankruptcy component of the cost function, as found in the right-most portion of expression 4.5. We will consider the role of firm specific prices in determining the event of bankruptcy, along with the nature of risk aversion in the GS model. We conclude with how these arrangements are specified in a manner suitable for numerical simulation, which explicitly includes the variability of firm specific prices leading to the event of bankruptcy.

We begin by referring to the right-most term of expression 4.5 above, parameter \( cF(\nu_{t+1}^i) \). It is the product of the cost incurred in the event of bankruptcy and \( F(\nu_{t+1}^i) \), the probability distribution for the threshold level of its occurrence.\(^5\) The event, bankruptcy, occurs when firms are unable to meet their financial obligations, debt exceeds sales revenues, expressed in nominal terms:

\[(1+R) B_t^i \geq P_{t+1}^i Q_t^i \]

(4.9)

where \( B \) is the firm's nominal debt. As we see, it is the firm specific relative price at time \( t+1 \) or ex post, at which the \( i \)'th firm sells its output in relation to the threshold criteria, which determines solvency. We can express bankruptcy as the threshold solvency level, \( \nu_{t+1}^i \), and the ex post price experienced by the \( i \)'th firm. To do so we first define the firm specific change in relative prices in relation to the general price level:

\(^5\)No assumptions are made by GS with regard to the distribution \( F \), and its density function, although we will assume that they are normally distributed.
Bankruptcy occurs when what the firm has promised to repay in working capital borrowed at time $t$ exceeds the firm's income at time $t+1$, as determined by firm specific prices, which are stochastic. A firm remains solvent when the market value of the firm at time $t+1$ combining revenues plus equity reserves from the previous period, exceeds debt obligations contracted at time $t$. Unanticipated change in nominal prices is not neutral because it affects the firm's asset position. The firm specific price level at time $t+1$ at which bankruptcy occurs, we define in relation to the threshold level introduced earlier, accordingly:

$$u_{t+1}^i < v_{t+1}^i$$

(4.11)

Expression 4.11 shows when bankruptcy occurs as the relationship between the firm specific price relative at time $t$, having distribution $F$, to the solvency threshold price of the $i$'th firm at time $t+1$, $v_{t+1}^i$. Alternatively, we can now express bankruptcy as:

$$u_{t+1}^i < (1+x) \left( \frac{P_t}{P_{t+1}} \right) \frac{(w_t^i q_t^i - a_t^i)}{q_t^i} = v_{t+1}^i$$

(4.12)

Where, $a_t^i = A_t^i / P_t$ represents real equity of the $i$'th firm. Referring to the above expressions based upon the GS model, permits us to see how bankruptcy arises when the stochastic firm specific relative price fails to equal or exceed the solvency threshold level.

Relating firm specific price variability to the event of bankruptcy will facilitate understanding the nature of risk aversion in the GS model. The cost of bankruptcy $c$ is a function of $q_t^i$, and
assumes decreasing returns to scale. GS identify convexity in output with induced risk aversion (1988, page 103). Recall, it is the expected firm specific prices at time \( t+1 \), the ratio expressed by \( \frac{\Delta q}{t+1} \), to which firms, ex ante, respond to avoid bankruptcy occurring at time \( t+1 \), inducing risk aversion. GS propose, as per their assumption, A6, that the cost of bankruptcy increases with the quantity of a firm's output, for three intuitive reasons, which together are the basis for the ex ante induced risk aversion, as manifested by convexity in output \( q \). On this important point, we expand because understanding the source of risk aversion in the model is key to its dynamic behaviour. As per equation 4.1 above, GS show (1988, page 108) that \( \phi'' > 0 \) implies elasticity of output with respect to equity, \( \eta < 0 \), although it need not be a constant. According to GS decreasing returns to scale (labour), implies risk aversion because output increases with a firm's equity at a decreasing rate. These relationships justify the assertions by GS such as, "Bankruptcy is costly and firms take these costs into account in their production decisions." (page 82, 1993); or, "... firms act to avoid bankruptcy.", and "... bankruptcy avoidance behaviour induces a kind of risk aversion." (page 88, 1993). Thus induced risk aversion by firms in the GS model arises from the convexity of the output function.

Now returning to the right-most expression found in expression 4.5,combining the guidance found in the GS article with received theory, to express the bankruptcy in the computer program, the functional specification must be convex in output and display risk aversion to the threat of bankruptcy. As explained in footnote 7, risk

\[ \text{footnote 7: Namely, 1). The second derivative of bankruptcy cost with respect to firm size is } > 0; \text{ 2.) The greater the absolute size of the potential mistake, viz. output versus prices received, the more averse to risk managers should be; and 3.) bankruptcy costs are not a fixed amount, i.e. firms cannot "grow-out-of" worrying about them. These three assumptions justify footnote 16 of GS (1993), risk aversion is only achieved if the } c_F \text{ function is convex in } q. \]
aversion only occurs when $cF(v)$ is appropriately convex in $q$, and referring to the GS assumption A6, bankruptcy costs increases with the level of the firm’s output. To ensure that the cost of bankruptcy does rise increasingly with the level of output and that risk aversion is displayed, $cF(v)$ is explicitly specified in the computer program as:

$$cF(v^i_{t+1}) = \left[ \exp \left( \frac{q^i_t}{k} \right) \right] F(v^i_{t+1})$$

(4.13)

We note that this specification is convex in $q$ as required by GS (page 88, 1993). The function for $c$ cannot be an exogenous constant because it must be convex in output to induce ex ante risk aversion. The probability distribution function for bankruptcy in 4.13 $F(v'_{t+1})$ for determining the threshold of bankruptcy in the GS Model is not stated as an explicit distribution or density function, but as a function having the following arguments interacting in a complex manner:

$$F[v^i_{t+1}(q^i_t, a^i, w_t, \frac{P_t}{E(P_{t+1})}, 1+r)]$$

(4.14)

Where $E(P_{t+1})$ is the expected increase in the general price level. To specify this equation functionally and make it operational as a simulation algorithm, we begin with some observations about its arguments and nature.

In the GS model the probability of bankruptcy, $F$, for the individual firm depends upon the arguments shown in equation 4.14. As mentioned, an explicit probability distribution and density function for $F$, GS do not specify, although for purposes of simulation we will adopt the assumption of normality. Equity and output are firm specific, while other arguments found in 4.14, as state variables and exogenous parameters, are the same for all firms, which are assumed identical. In the model, GS use both a general price level as well as
a sectoral price level without mentioning how, as random variables, they might be distributed or how firms form expectations on their underlying distribution. According to GS there is less uncertainty about the general price level than there is with regard to firm specific relative prices (GS, page 86, 1993). For purposes of simulation and research, we will assume that the general price level is invariant. Bankruptcy arises when the realized firm specific prices, normally distributed by assumption, at time $t+1$, falls below the critical value determined through equation 4.14. Hence the probability of bankruptcy refers to the chances of this event occurring. When the ratio of the firm specific relative price in period $t+1$ to the general price level in period $t$, $P_{t+1}/P_t$ or $u_{t+1}^t$, is equal to or exceeds $\sqrt{t+1}$, as determined by the arguments of equation 4.14, bankruptcy is avoided. We note that although the determinants of the probability of bankruptcy are firm specific, the $F$ distribution is not so. (Firms face the same $F$ distribution, however, the joint probability of the occurrence of bankruptcy is independent of one another.) For a given set of firm specific state variables and exogenous parameters, determining the probability of bankruptcy, because of firm specific price shocks, some firms may go bankrupt and others not. The differing outcomes arise because of the varying firm specific price shocks experienced by the population of firms, even assuming that all firms have the same initial level of equity as we have for purposes of simulation, although GS do not. In this regard, GS assume only symmetry.

Placing the solvency criteria and the probability of bankruptcy in a risk aversion context we note the following. Bankruptcy is important in the GS model because it is the source of risk aversion leading to modifications in investment, borrowing and eventually output behaviour. A time $t$, firm specific output $q_t^i$, and equity $a_t^i$, and wages $w_t^i$, determine the critical value of $\sqrt{t+1}$, the solvency threshold, and the probability of bankruptcy thereby occurring (page 88, GS
1993). Uncertainty over the relationship between firm specific prices $u'_{t+1}$ to the critical value as expressed by $\nabla_{t+1}$ leads to risk aversion on the part of firms. Low prices, and low returns, as expressed by $P_{t+1}/P_t$ or $\alpha'_{t+1}$ may lead to insufficient market value including equity, $a_{t+1}$, and hence insolvency. Bankruptcy is avoided because the stochastic process on $u'_{t+1}$ is such that it equals or exceeds the threshold value $\nabla_{t+1}$. Or in economic terms, in the GS Model bankruptcy occurs because there is insufficient reserves of equity plus net income to meet financial obligations. When the received revenue from next period output is insufficient to repay the debt borrowed to pay wages in the present period because of the New Keynesian style financial constraint on the availability of equity, the firm is bankrupt. Further with regard to the nature of bankruptcy and risk aversion in the GS model, we note that the borrowing decision is irreversible and involves an ex ante anticipation of the relationship between the firm specific randomly distributed relative price $u'_{t+1}$, which we will assume are normally distributed as explained below, and the threshold solvency price $\nabla_{t+1}$. As mentioned earlier, according to GS, the major source of uncertainty is with regard to firm specific prices and not the general price level (page 86, GS, 1993). The probability that firm specific prices in the next period may be below the critical level induces a form of risk aversion. When and if bankruptcy does occur, it is the ex post result of an ex ante decision, namely of whether to fund labour cost through borrowed working capital to produce output. The discrete time ex ante/ex post relationship underlies the New Keynesian features of the model: There is a constraint on new equity being raised so firms must borrow to fund working capital, the uncertain returns to which occur in the future.

New Keynesian features appear in the asymmetry between lenders and borrowers as well, since presumably the latter are unable to adequately monitor the density function on the solvency threshold price $\nabla_{t+1}$, and its relation to a uncertainty surrounding firm specific
prices. We note that this ex ante/ex post relationship requires the use of a discrete, rather than continuous time framework. By including the cost of bankruptcy in the cost function, GS seek to address the question of the properties of aggregate dynamic behaviour for risk averse firms operating under financial constraints and imperfections. Arguably, their work is inconsistent with the Modigliani-Miller theorem of the irrelevance of asset composition (risky debt and less risky equity) upon real variables, namely valuation and output. The model's set-up suggest several ways for improvement. For example, we note that the interest rate at which working capital is borrowed does not include a risk premium reflecting the risk of bankruptcy arising from greater debt, there is a financial market imperfection in that obligations are not tied to real performance, and that the future value of contingent assets and liabilities, such as borrowed working capital, may not be hedged. In the GS model, futures markets are assumed not to exist because of transaction costs.

To develop and calibrate a model in the style of GS suitable for simulation, building upon the above, we make the following points. Recalling expression 4.14, both firm specific and non-firm specific exogenous variables determine the probability that $\frac{\lambda}{\lambda r}$ is below $\frac{\lambda}{\lambda r}$, that is, of bankruptcy occurring. The firm specific variables determining the solvency relative price and hence the probability of bankruptcy are firm equity and output. But since output, itself is a concave and function of equity, deterministic except for the variability in firm specific prices, then to the extent that the solvency relative price is determined by firm specific variables as equity, it is the level of equity, inter alia, which ex post at time $t+1$, determines the threshold for the probability of bankruptcy. As the level of equity is firm specific in the GS model, and although firms are subject to the same constraints and produce output according to the same production function, because of the relationship between random firm specific prices and the solvency relative price, some firms will go
bankrupt while others will survive. (According to GS, the revenues of bankrupt firms are divided equally among debtors (page 86, GS 1993), although we will adopt the convention that firms are "re-floated" at the average level of equity of firms surviving.) Relating the threshold for solvency to the random variable of firm specific prices, GS tell us that there is uncertainty regarding firm specific relative prices, although not how they are distributed: According to GS, the stochastic process on firm specific relative prices, $P_{t+1}/P_t$ or $u_{t+1}^i$, has distribution function $F$ and is i.i.d., (page 85, 1993). Together the relationship between the threshold for solvency relative price $v_{t+1}^i$ and firm specific prices $u_{t+1}^i$, determine when and if the $i^{th}$ firm becomes bankrupt. Formally this relationship may be represented as the following distribution function:

$$\text{Prob. (Bankruptcy)} = P(u_{t+1}^i < v_{t+1}^i) = \int_{-\infty}^{v_{t+1}^i} f(u) \, du$$

(4.15)

which is equivalent to:

$$P(\text{Bankruptcy}) = P(u_{t+1}^i q_{t+1}^i < \left[ (1+r) \left( \frac{P_t}{P_{t+1}} \right) (w_t \phi(q_t^i) - a_t^i) \right])$$

(4.15')

Given a $F(u_{t+1}^i)$, which we have assumed normally distributed, we can say that bankruptcy occurs with probability $F' = F(u_{t+1}^i)$, where we define $u_{t+1}^i$ as the following:

(4.16)

$$u_{t+1}^i = \left[ (1+r) \left( \frac{P_t}{P_{t+1}} \right) (w_t \phi(q_t^i) - a_t^i) \right]$$

price realization for the $i^{th}$ firm below which default takes place.

\[7\]In the simulation program, a polynomial approximation of the normal distribution is used. (As found, page 932, expression 26.2.18 of Handbook of Mathematical Functions, 1968.)
To specify and model the above arrangements regarding firm behaviour and structure in the simulation program, we assume the following: To numerically simulate a version of the GS model, in our program we assume a normal distribution for the stochastic process on firm specific prices $u_{t+1}$. Inputting the mean and standard deviation for such firm specific random price shocks, using a normal distribution on such shocks, the simulation program exposes each of the assumed population of twenty-five firms to a unique random price shock. The density function for the normal distribution on firm specific prices, $p_{t+1}/p_t$, is specified accordingly:

$$f(u_{t+1}) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(u_{t+1} - \mu)^2}{2\sigma^2}\right) \quad -\infty \leq u_{t+1} \leq \infty$$

Where $\mu$ and $\sigma$ are the mean and standard deviation respectively. In the simulation program, using the normal $F(u_{t+1})$, distribution and assuming the price shocks as approximated using a polynomial function, faced by individual firms are i.i.d. with mean of zero and a standard deviation varied by assumption, leads to a cost of bankruptcy in its event, $c_{t+1}F(\sqrt{u_{t+1}})$, as per equation 4.5 above. As we would expect from an intuitive standpoint, and as we validate through simulation, because of ex ante risk aversion, potentially greater marginal cost of bankruptcy, $c_{t+1}$, leads to reduced employment and output (page 92, GS 1993). Ex ante, induced risk aversion changes with the variance in the $F$ distribution on firm specific prices, $\sigma^2$, and the cost incurred in the event of bankruptcy. If perfect foresight existed on prices, variance zero, borrowing decisions and leading to equity-output decisions would be such that the chance of bankruptcy would be zero. In combination with parameters for the uniform distribution, according to which firm specific price shocks occur, the developed algorithm for numerical simulation requires the use of the
marginal bankruptcy cost function, as shown above in equation 4.13 leading to the event of bankruptcy, based upon the stochastic realisation of firm specific prices, that is when \( u_{t+1} \) for the \( i^{th} \) firm falls below the threshold value for \( \sqrt{r+1} \).

To model expressions 4.16 in the simulation program and to calculate the probability that a given firm is bankrupt we have assumed, as mentioned above, that its firm specific price realization for a given firm is normally distributed. Across the population of firms, and hence at a given time \( t \), the relationship between the distribution on firm specific price \( u_{t+1} \) and the threshold level determined by expression 4.14 determines the risk of bankruptcy for the \( i^{th} \) firm sampled from the same population. Or, as per expressions 4.16, the individual firm's price realization is such that in relation to \( \sqrt{r+1} \), that if liabilities exceed revenues less equity, bankruptcy has occurred. If the price realization is such that revenues plus assets exceed liabilities, the assumed normally distributed random processes \( u \) in relation to \( v \), is such that the \( i^{th} \) firm is solvent while if the latter exceeds the former, bankruptcy occurs for the \( i^{th} \) firm. In the simulation program, the threshold criteria \( \sqrt{r+1} \) is computed according to the various arguments appearing in equation 4.14 and then compared with the realized random price \( u_{t+1} \) from the distribution \( F \). Values for \( u_{t+1} \) below a certain threshold level, measured in standard deviations from \( \sqrt{r+1} \), are declared bankrupt. If ex post, prices at time \( t+1 \) are below the critical level as determined by equation 4.14 above, total equity at time \( t+1 \) is may be insufficient, resulting in insolvency, as per equation 4.9 above. At a given time \( t \), firms form expectations of the chances of bankruptcy given the uncertainty on firm specific prices, \( P_{t+1}/P_v \) and so adjust their behaviour which has dynamic implications. A final point on modelling the occurrence of bankruptcy in the economy represented by the GS model and the behaviour it occasions. In the simulation program, at a given point in time \( t \), given
the assumed properties with respect to the distribution on firm specific prices and the threshold criteria of $v_{t+1}$, a proportion of the assumed population of twenty-five firms will go bankrupt. In the GS article, no process on the creation of new firms is introduced. From the standpoint of a simulation program and assuming a fixed quantity of firms, this would lead to the absurdity of the economy over-time disappearing. By assumption, as mentioned above, in the simulation program, and with some degree of realism, in the period after which bankruptcy has occurred, the firm is re-floated with the average level of equity of firms which are solvent. Its debt are cancelled.

We conclude our discussion of the cost function found in the GS model with how it relates to subject of dynamic behaviour. Although we pursue the laws of motion found in the GS model in sections below, it is important to note that the source of dynamics in the GS model is not the occurrence of bankruptcy, per se. In the GS model, it is the ex ante expectations and adjustments to the variability of firm specific prices which through the calculus of firm level optimization determines the dynamic path on equity and output, including whether bankruptcy may occur. As mentioned above, $a'_{t+1}$ is a function of $a'_t$ and $u'_t$, with the latter an exogenous stochastic process which is normally distributed. By changing initial conditions in exogenous parameters, the performance of representative firms through the business cycle are affected. It is important to note that the model's dynamics are driven by both a random component and one which is deterministic. The model's dynamics are driven by bankruptcy arising because random firm specific prices are below the threshold level, and the aversion to bankruptcy modifying behaviour in reaction to the borrowing-investment-wage and profit spiral, as depicted in Figure 3.1 of Chapter Three. The $i^{th}$ representative firm forms ex ante expectations on firm specific stochastic future prices in relation to the threshold, $v'_{t+1}$, given the probability $F(lu'_{t+1})$, thereby affecting inter-temporal dynamics during economic expansions and
contractions. Ex ante expectations of the event of bankruptcy formed by agents causes adjustments in behaviour as determined by the probability distribution on firm specific prices of which they have knowledge. Such stochastic perturbations arising from the relationship between firm specific prices and their relationship to the threshold level, are one of the sources of dynamic behaviour, as is found in the Real Business literature. The dynamics of the model are derived from risk aversion and the effects of investment under conditions of firm specific price uncertainty. Random perturbations to the system, which result in bankruptcy, along with the nature of borrowing and investment, generate the model's dynamic behaviour, along with perturbations which effect it. The GS model is essentially deterministic in nature, notwithstanding the uncertainty over firm specific prices and consequent revenues which modifies the one-dimensionality of the implied non-linear autonomous first order difference map, making it effectively second order as we will discuss below. The modification of firm behaviour through risk aversion, as we will discuss in Section 4.6, has inter-temporal implications. We now turn to Section 4.4, where we use the revenue and cost functions to solve for firm level optimization.

4.4 Firm Level Optimization

In this section we will review the GS approach to firm level profit maximization, and how this is specified in Module I of the computer program. As this specification process does not involve the introduction of any new functional forms, we limit our discussion to validating through calibrated simulation, the three unproven propositions on the comparative static relationships between equity, production and output, and price variance, found in the GS article (1993, page 102). Such simulations, we note allow for analysis of the sensitivity of an optimal solution to changes in various exogenous variables. The computer algorithms and program for
solving the classical profit maximization problem for the \( i^{th} \) firm uses the explicit revenue and cost functions specified in Sections 4.2 and 4.3, the calibrated exogenous inputs at time \( t \), and the assumed level of equity \( a_i \) of the \( i^{th} \) firm, to maximize profits over quantity, \( q \), equity. The result of solving the profit maximization problem over \( q \), is an optimal output, \( q_i \), and a next period equity \( a_{i+1} \), which may be used in the subsequent iteration of the profit maximization to find an optimal output for the \( i^{th} \) firm, of \( q_{i+1} \), at time \( t+1 \), and so on, for however many iterations. Micro phenomena, determine the level of equity and output, through changes or shifts in the revenue and cost functions from which the level of profit maximization is determined. Determining the aggregate level of equity, \( a_t \) and \( q_t \) over time requires maximization of the individual firm profit maximization problem through the simulation program, because as we will explain no closed form solution exists for an expression, such as, \( a_{t+1} = G(a_t) \), as per GS (page 100, 1993).

Reviewing these simulations, we see the effects upon firm level profit maximization of changes in exogenous variables. To show that Module I follows the GS Model, we recall their Propositions 1, 2, and 3 found (1993, page 92).

- Proposition 1. The higher the level of equity, the lower the marginal bankruptcy cost (risk premium) \( \rho_i \), and hence the higher the level of production.

- Proposition 2. Increases in the degree of uncertainty result in an increase in the marginal bankruptcy cost (risk premium) and hence in a lower level of output.

- Proposition 3. At least near the capacity level, output is a concave function of the equity level.

To validate Proposition 1, we perform two experiments changing the initial level of equity, for the \( i^{th} \) firm. As we can see from Figures 3 and 4, changes in the initial level of equity, at time \( t \), through
changes in the expected cost of bankruptcy have an affect upon the level of output $q$, at time $t+1$. The change between Figures 3 and 4, involves decreasing the amount of equity available so that the firm is forced to borrow more, and thereby increase its marginal bankruptcy cost, for a specified normally distributed $F$ distribution on firm specific prices, $u'_{t+1}$. We see, the cost curve and the marginal cost curve shift leftwards between Figure 3 and 4. Proposition 2, we verify through the simulations reported in figures 5 and 6, involves changing the standard deviation on firm specific prices, $u'_{t+1}$, and thereby affecting the marginal bankruptcy cost through the probability of bankruptcy, $F(u'_{t+1})$, as will be discussed further in Section 4.3. The two experiments in Figures 5 & 6 for the $i^{th}$ firm, concern the variability of firm specific prices, $u'_{t+1}$. In Figure 5, the standard deviation on firm specific prices is 10%, while in Figure 6, the standard deviation is increased to 20%. We observe that an increase in the variability of firm specific prices, leads to a decrease in output as firms seek to avoid bankruptcy, that is falling below the threshold level, $v''$, as presented in the GS Proposition 2. Like the Figures 3 and 4, a change in the optimal level of output occurs through a shift in the cost function, as shown in equation 4.5 above. Together, the four experiments demonstrate the effects upon firm behaviour of changing the marginal cost of bankruptcy through the variance of firm specific prices and the level of equity. The specified parameter for the cost of bankruptcy, as included in the program through the specified expression 4.13, because of induced risk aversion, influences the level of both output and investment, in the fashion specified in the GS model.\(^8\)

\(^8\) In the GS Model, investment refers to the expenditure on working capital to pay the wage bill.
FIRM LEVEL PROFIT MAXIMIZATION

Figure 3: D/E = 2

Figure 4: D/E = 4
Figure 5. Standard Dev. in Prices 5%.

Figure 6. Standard Dev. in Prices 20%.
We conclude Section 4.4 on Firm level optimization by showing the effect, changes in real wages and the rate of interest have upon the level of equity and output for a representative firm. In this way we further show how the simulation program captures the principal features of the GS model. To understand this effect, we refer again to the non-stochastic exogenous variables found in expression 4.14, namely wages and interest rates. According to GS, increases in real wages and the real interest rate reduce equity formation and therefore output (1993, pages 90-91). Although there are no general equilibrium effects upon interest rates in the GS model, wages play an important role. In the GS model, firms borrow working capital to pay wages, increasing real wages. When profits fall because of low firm specific prices, the commitment to greater real wages, reduces profits and equity, lowering output. Comparative static Figures 7 to 10 show such behaviour, further demonstrating that Module I of the Program, as explicitly specified and simulated, embodies the spirit of the GS model. The program results, graphed in Figures 7 and 8, show the effect of increasing the level of wages in period \( t \) by 50%. While the program results, graphed in Figures 9 and 10, show what effect the rate of interest at which the firm is forced to borrow to pay for working capital, i.e. to cover wages, has upon static output. Between the simulations found in Figures 9 and 10, the rate of interest charged is decreased from 10% to 3%. As we see, the results predicted in the GS model and article may be simulated using Module I of the computer programme. Having demonstrated that in Module I, we have expressed in our specified model and program the features and requirements of the financial constraint-financial market imperfection model of the style of GS Model, our next step involves aggregation. The modelling and simulation of aggregation is explained in the following section.

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9 In Chapter Six, we consider the effect of having interest rates change through an endogenous mechanism.
Figure 7: Low Wages

Figure 8: High Wages
FIRM LEVEL PROFIT MAXIMIZATION

Firm, Period 1

Figure 9: Interest Rate of 10%

Figure 10: Interest Rate of 3%
4.5 Aggregation and General Equilibrium

Having explained the functional specifications and algorithms used in Module I, we may now turn to Module II, where the problem of aggregation and general equilibrium are addressed. Our plan is to review the general set-up of the GS model, along with its broader business cycle implications. We will explain its specification in the simulation algorithm, and how it relates to its behaviour. Like the previous sections, our approach is to combine the stylized general specifications found in the GS articles with received economic theory to specify functional relationships suitable for the construction of algorithms and calibration. We apply this approach over three sub-sections. Firstly, we begin with aggregation and consider how it may be specified in a simulation program to create an aggregate supply function. In this respect, we also consider the implications of its shape. Secondly, we discuss how the equilibrium adjusts to changes in the variability of firm specific prices, \( \sigma \) on \( F(u_{t+1}) \), and other exogenous model parameters. In this context we offer comments upon the nature of equilibrium in the GS model, its existence, uniqueness and stability and whether it is properly New Keynesian. As general equilibrium in the GS model occurs in labour markets, in the third sub-section, we use the aggregate supply of output function to create an aggregate requirement for labour function which we solve simultaneously against the aggregate labour supply function which is assumed in the GS model to be an increasing function of wages. In addition to formalism, closing the system in general equilibrium against an upward sloping labour supply function has a role in the model's dynamics, because increases in aggregate output lead to adjustments in real wages, equity, profits, borrowing, investment, and the labour requirement function and so on.

4.5.1 Aggregation. In the GS Model it is assumed that all firms are identical, having the same cost function and production functions. In addition we have assumed that the distribution of initial
equity levels across firms are identical, and not merely symmetrical as GS assume. Importantly, the price shocks to which the assumed population of twenty-five firms are exposed are firm specific. Such shocks we assume are normally distributed. We begin with the GS general function for individual output before explaining how the simulation program works to determine output in the aggregate.

\[ q_t^i = q(w_t, r_t, a_t^i, \sigma, \gamma) \]

(4.18)

Where \( \sigma \) is the variability in firm specific prices corresponding to distribution, \( F(t_{i+1}) \), as explained in Section 4.3 above. Subscripted at time \( t \), \( w \) is the level of wages and \( r \) the rate of interest, which are both identical to all firms. Furthermore, as mentioned above, we assume, as per GS, that the rate of interest is time invariant, a constant. The distribution in equity across all firms, over time, as per GS (page 94) is symmetric, and its dispersion is specified by the parameter, \( \gamma \), although in the simulation program, over time, the dispersion of equity results endogenously. The level of equity, \( a_t^i \) is firm specific and together with the other parameters determines the level of output of the individual firm. These exogenous variables found in equation 4.18, may be investigated as sources of perturbation. According to GS, they uniquely determine the level of output of the individual firm, as shown for a representative firm, previously in Figures 3 to 10.

In the simulation program, to implement the above set-up is straightforward and involves summing the outputs of the assumed population of the individual twenty-five firms, as expressed in equation 4.19 below:

\[ q_t = \sum_i q_t^i \]

(4.19)
Summing up the outputs of individual firms as per equation 4.19 above, yields a quantity of aggregate output, along with the labour requirement given that output. In the simulation program, the GS parameter, $y$, for equity dispersion results endogenously, over time, although initially it is assumed to be identical for all firms. In the simulation program, individual firm specific price realizations are drawn from the same distribution. The normal distribution of such drawings combined with the size of the population, together satisfy the GS requirements with regard to symmetry in the distribution of equity across all firms. Such individual price realisations serve to vary the levels of equity across the population of firms over-time, however as per the GS assumption (page 94, GS 1993) around the average level of equity, the distribution is symmetrical.

Although knowing the aggregate quantity of output is sufficient to determine the required amount of labour (in terms of $\phi(q)$ or $l$ and wages), in the simulation program we solve for general equilibrium so as to allow other experiments, and dynamic simulation. Although strictly only one corresponding point on the aggregate supply of labour (in terms of wages and quantity of labour) must be found to achieve general equilibrium, we have created in the simulation program, an aggregate demand for labour from the aggregate supply of output, as per equation 4.19, by summing horizontally the rising portions of the respective marginal cost curves, of the assumed population of twenty-five firms. Creating the complete aggregate output and aggregate demand for labour functions allows further insights into the nature of the GS model and its behaviour. The algorithm to accomplish this is found in Module II. Before moving to sub-section 2, we show how the specification and algorithm for Aggregate Supply behaves with respect to exogenous perturbations as suggested by GS in their article; we consider its neutrality (equilibrium is invariant with respect to changes in monetary units) arising from a vertical aggregate supply curve.
In a neo-classical macroeconomic model, the quantity of labour employed is independent of the price level. The assumptions of perfectly flexible prices (including wages), no money illusion, and unit elastic price expectations ensure that the quantity of labour employed is invariant to any change in the level of prices. In the GS Model, the quantity of output is driven by the amount of equity which results from individual firm level optimization, given the wage rate and interest rates. In the GS Model, the level of output is independent of the general price level, making it vertical as shown in GS Figure III, (page 91, 1993). Independence between output and the price level makes the aggregate supply function vertical, although shifts in aggregate supply can occur, for example through real changes in interest rates, wages, or changes in technology or the demographics of labour.

The shape of the aggregate supply function reveals a lot about a GS style model featuring a financial constraint and borrowing imperfections. The shape of the aggregate supply function indicates the effect that changing various exogenous variables along with various modifications involving financial innovation may have upon system dynamics. In the GS model, "uncertainty shocks" with regard to firm specific prices effects the returns to borrowed working capital directly, and over time, the equity levels of firms indirectly. The expected cost of bankruptcy arising from the variability in firm specific prices influences production decisions, leads to shifts in aggregate supply because firm specific price variability affects ex ante optimization behaviour involving the avoidance of the cost of bankruptcy. Bankruptcy arises ex post because of unanticipated changes in the firm specific stochastic price process. By implication, according to GS, for a given level of variability in firm specific prices, however, such shocks do not alter the long run steady-state in equity and output, as we will verify in Section 4.7 below (page 118, GS 1988). Changes in real wages and real interest rates, however, do
shift the aggregate supply curve as we show below. Although according to GS, uncertainty with regard to the general price level does not play an important role in their model (page 86, GS, 1993), we see that the model behaves classically. In a manner typical of neo-classical models, changes in the price level, propagated through production and output decisions, as might result from an increase in the money supply, according to GS (page 107, 1993), would have no effect on real magnitudes. A nominal, monetary shock is a short-term effect which does not move the economy from the steady-state level of output of the economy, leaving the aggregate supply function unchanged.

Turning to the short-run, we see that the GS models behaves in a largely classical manner, notwithstanding its method of persistence. Shocks in the form of nominal price variability have only short-run effects, leaving unchanged the steady-state output level of the economy and differing only in the level of prices (page 119, GS, 1988). Thus although the model is described as New Keynesian, it has classical features even in the short run because nominal shocks may only have an effect if they are not anticipated (hence the term "shocks"). Secondly their effect is propagated through the supply side. Thirdly in the long-run they do not affect the steady-state level of output. According to GS, only unanticipated shocks, for example in firm specific price variability, real wages, and real interest rates, have an effect and they arise through the supply side.

More generally the GS Model has New Classical features because it is through changes in aggregate supply that results are generated. Shocks from increased uncertainty surrounding the return on future investment in working capital resulting from changes to the variability in firm specific have long term effects upon the trajectory of investment and equilibrium output. Increased uncertainty surrounding firm specific prices have a supply side effect because of imperfections in the way financial contracts are written, that is they
are not indexed to real performance and they are in nominal terms. Unexpected lower prices do not, in the GS Model, have an aggregate demand effect. When firm specific prices are low, given the wage-bill commitment, equity falls. The net result is that the GS aggregate supply is vertical but is subject to non-neutral shocks because of the effect upon the firm’s asset position. Due to the "wedge" which the fear of bankruptcy drives between the expected prices and "traditional" marginal costs, the modified marginal cost function rises at an increasing rate. The aggregate supply function thereby derived from the sum of the rising portion of the individual marginal cost curves, is vertically asymptotic in the simulation program and as per GS Figure III (page 91, 1993).

In light of the above, it is interesting to compare briefly the GS model with New Classical and Rational Expectations literature. The price shocks in the GS model result from unanticipated shifts in the demand curve and in some way resembles the short and long-run behaviour found in the "price surprise" literature of new classical macroeconomics, as for example in the Lucas "signal extraction problem" (Froyen & Waud, 1988). According to such literature, only unanticipated price surprises, operating through a redistribution of assets, have output effects, leaving anticipated policy ineffectual. A Lucas-Sargent aggregate supply curve is derived solely from equilibrium in the labour market. In a Lucas-Sargent model, the inability of agents to extract signal from noise originating from a price surprise, affects production decisions directly. In the GS model, aggregate supply is determined through equilibrium conditions in the labour and credit markets. In the GS model real effects arise from the effect of different claims, debt and equity, upon the behaviour of firms.

4.5.2. To see how a version of the GS style model behaves in the aggregate various experiments are performed, notably changing the level of equity, the rate of interest, and the variability in firm
specific price shocks $F(u_{t+1})$, all of which effect the risk of bankruptcy. In Figures 11 to 16, we find graphs generated by the results of simulating the program with different sets of state variables. The results, as appear in the Figures, verify that aggregate supply computed in Module II behaves as conjectured by GS in their article with respect to changes in such variables, as shown in their Figure III, (page 91, 1993). As may be seen, by changing these variables, relating to the initial level of equity, slightly shifts the aggregate supply function in Figures 11 and 12, through its effect upon marginal bankruptcy cost, as shown for time $t$. (Comparing the shift in Aggregate Supply against the dotted vertical line assists in interpretation.) In Figures 13 and 14, the role of interest rates charged on borrowed working capital are shown to have a small effect on the aggregate supply according to the stylized predictions made by GS. According to GS "...the higher real interest rate means that firms must pay back more..." (page 91, 1993). Increasing the real rate of interest has an effect upon the Aggregate Supply function, shifting it slightly to the left, because it increases the cost at which firms borrow to pay the working capital required for the wage bill. Lastly, in Figures 15 and 16, the aggregate supply function responds to changing the risk in firm specific price variability $F(u_{t+1})$, increasing the risk of bankruptcy, as suggested by GS in their article. We see that a perturbation in the form of changing the variance in firm specific price shocks, as we did in Figures 5 and 6 above, as it increases marginal bankruptcy costs, raising the risk of bankruptcy, has an impact, because it shifts the aggregate supply function to the left. A rightward shift, in contrast, would occur because for a given level of investment using borrowed working capital, the returns would be greater and the chance of bankruptcy reduced. These results simulate the comparative statics discussed in the GS Model. Like Figure III found in the GS article (page 91, 1993), the aggregate supply functions resulting from the simulations, rises at an increasing
rate into a vertical asymptote. Like GS Figure III, for different levels of equity, price variability, and interest rates, at time $t$, families of aggregate supply curves may be generated.

Figure 11. $D/E = 2$

Figure 12. $D/E = 4$
Figure 13. Interest Rate = 20%

Figure 14. Interest Rate = 5%
Figure 15. Standard Deviation in Prices = 5%.

Figure 16. Standard Deviation in Prices = 20%.
4.5.3 Having specified in the simulation model an aggregate supply function of the style found in the GS article and verified that it behaves as they propose, we now consider the inclusion of a general equilibrium solution in the simulation program. As mentioned previously, in the GS model the labour requirement function is solved against the aggregate supply of labour schedule which is assumed in the GS model to be an increasing function of wages. Although aggregate output is simply the sum of the outputs of the individual firms, ensuring general equilibrium between the aggregate amount of labour required and the upward sloping supply of labour, over time, plays a role in the model’s dynamics, because business cycle effects are propagated through the borrowing-investment-wages-equity-profits spiral. Unless an upward sloping aggregate supply of labour function is present, changes in the requirement of labour, given output, will not produce real wage effects.

In the final sub-section of Section 5.3, we explain how the aggregate supply of output function is used to solve for a general equilibrium solution in labour markets. To solve for general equilibrium, GS transform their aggregate supply function from output space into input space, as an implied requirement for labour. Using the labour required for a given level of output, the GS articles (1988, 1993) formulate a general equilibrium solution using the aggregate supply of labour. The labour supply function it is assumed derives from the utility maximization calculus of consumers in the aggregate. Hence, general equilibrium in the GS Model arises from the interaction between microeconomic units of firms and households.

As per the GS article, we require in the simulation program a simultaneous solution in the labour market to solve for the equilibrium quantity of labour supplied and the consequent wage. Explaining the stylized structure found in the GS model will help to explain how it may be modelled in the simulation program. As mentioned, the aggregate supply of output function is transformed into input space,
as an "aggregate labour requirement function." In equations 4.2 and 4.8 above, the production function and the cost function for the i'th firm at time t, were explicitly modeled as logarithmic and exponential, respectively. According to GS, the demand for labour is the sum of the demands of the individual firms. In the aggregate, a level of output given a functional relationship leads to an aggregate labour requirement function accordingly:

\[ l_t = \sum l_t^i = \sum \exp\left(\frac{a_t^i}{k}\right) \]

4.20)

Using the labour requirement function, an equilibrium level of wages may be solved for by setting equation 4.20 equal to the supply of labour. For simplicity we assume that the supply of labour is an increasing linear function of wages with an intercept of zero and a slope calculated, as explained below:

\[ l = s(w_t) \]

4.21)

The shape of the labour requirement function is not stated in either 1993 piece nor the appendices of the 1988 article to which the authors refer, leaving only the GS Figure V (page 95, 1993) as a guide. To generate an Aggregate Demand for labour, in Module II, we iterate the program found in Module I itself to calculate the quantity of labour which would be required at different levels of wages, \( w_t \).

According to GS the supply of labour is determined by the utility maximizing behaviour of an infinitely lived representative consumer who can borrow and lend at the rate of interest, \( r_t \), which is equal to the subjective rate of inter-temporal trade-off, \( \delta \), and hence constant over time. (The assumption conveniently assumes away saving for
the future or borrowing against a future income stream.) The assumption allows GS to make the supply of labour an increasing function only of the wage in the current period, \( w_t \) (page 96, GS 1993). The wage level at time \( t \), \( w_t \), along with the optimal quantity of labour to which the optimal level of output corresponds, are used to calculate the slope of the aggregate supply of labour function. Consumer behaviour in the GS Model requires that at a wage level of zero, no labour is supplied, making the slope intercept for aggregate supply of labour function the origin \((0,0)\). General equilibrium in the GS model is specified as the simultaneous solution of the aggregate labour requirement function and the aggregate supply of labour function. In the simulation program, in Module ii, given the two equations and the two unknowns, real wages, \( w_r \) and the required quantity of labour, \( l_r \) markets clear. The quantity of labour supplied by consumers corresponds to the amount required given the aggregate output at the real wage \( w_r \) or as expressed in general form by GS:

\[
\begin{align*}
 w_c &= w(q_c) \\
(4.22) \\
\text{The first derivative shows that as output rises, so does wages:} \\
 w' &= \left( \frac{\Phi'}{s} \right) > 0 \\
(4.23) \\
\text{Finally, by substituting from the labour and capital market equilibria into the aggregate supply of output, } q_c, \text{ connection to the level of equity is made:} \\
q_c &= q\left(w(q_c), r_c, a_c\right) \\
(4.24) \\
\text{which can be solved for equilibrium output, } q_r = H(a_r).
The algorithm used in Module II of the Program, ensures that at the optimal level of $q_t$, the aggregate requirement for labour function intersects the aggregate supply of labour at wage level corresponding to the optimal level of output at time $t$, thereby solving for general equilibrium. In labour input space, this result is explicitly modelled in Module II and results, at time $t$, are illustrated in Figures 17, 18 and 19. As we can see, changes in the real wage level shift both the supply and demand for labour. As wage levels are increased, ceteris paribus, general equilibrium occurs at a lower level of labour consumption (demand) by firms. Increasing wages leads to a shifts in both the aggregate labour supply curve, as per the GS article, as well as the aggregate demand for labour curve. In Figure 19, we leave the real wage unchanged but assume less initial debt in the capital structure. Naturally with less chance of bankruptcy, investment is encouraged and the demand for labour increased. The algorithm found in Module II by using the profit maximizing level of individual outputs of Module I, to determine the aggregate labour requirement, ensures consistency between aggregate supply and demand for output and general equilibrium within the labour market.

![General Equilibrium: Aggregate Supply & Demand for Labour](image)

Figure 17. Low Wages, D/E = 4
Figure 18. High Wages, D/E = 4

Figure 19. Low Wage, D/E = 3
4.6 System Dynamics: Background

In the last four sections of Chapter 4, we turn to the problem of implementing and simulating the GS model dynamically as specified in Module III. In Section 4.6, as background, we discuss the role of initial conditions in the form of micro-economic assumptions in determining the model's cyclic behaviour and dynamic qualities. In considering such dynamics, we find that initial conditions with regard to micro-economic assumptions, play a critical role in determining the steady-state stability, the inter-temporal trajectory, and the existence of attractors. Considering the GS model's micro-assumptions, we cover four connected sets of observations in Section 4.6. We begin with the role played by induced risk aversion arising from the stochastic process in prices. Thereafter we look at the role played by dividends and profit retention and the initial level of equity in determining the dynamic behaviour of the version of the GS model implemented for simulation along with the role played including whether oscillatory cycles will occur. In this context we mention the issues raised from the standpoint of capital structure theory of such dynamics. We conclude Section 4.6, with two sets of observations on the implications of non-linearity found in the model's first order deterministic autonomous difference equation and whether the model has random attractors. Section 4.6 forms the back-drop to Section 4.7 where we explain how the simulation algorithm was implemented. As an introduction, it involves dynamic aggregation: In the GS Model, changes to the level of output, including cycles are propagated through the level of equity. GS construct their inter-temporal equilibrium in equity space rather than in output space, making the time path on real equity. For a given firm, according to GS and as specified in the various simulation Modules, the level of equity uniquely determines the level of output over time, as shown in their Figure VI.A (page 98, 1993). For dynamic numerical simulation, as we will explain, the outputs and equity levels of individual firms
are aggregated in Module II, for time $t$, and as we explain below, are used in Module III, to construct an inter-temporal trajectory on the level of equity and output. In this procedure, we will use their assumptions of symmetry in the distribution of equity across the population of firms (page 94, GS 1993). In Section 4.8, we conduct some experiments to show that the computer program behaves according to the general requirements of the GS model, recognizing that the "results" found in their articles, are merely general descriptions of how such a system might behave if it were specified and simulated, as we have done. Lastly in Section 4.9, using numerical simulation, we explore such issues relating to inter-temporal behaviour, including chaos and bifurcations, as may exist in the GS style model.

We begin with some general observations on the dynamic implications of the aversion to bankruptcy assumed in the GS model and as specified in the modelled algorithm. As explained in Section 4.3, bankruptcy arises because of the relationship between firm specific price realizations and the threshold level, $\mathcal{V}_{t+1}$, and aversion to bankruptcy is attributable, as per GS Proposition II, to the variance around firm specific prices. The induced aversion to bankruptcy arising from firm specific price variability, we will learn, has inter-temporal effects. By understanding it, we can better see the effect its changes have upon the model's dynamics. Risk aversion is a key feature of the model because of the distinction made by GS between the risks of using equity and debt capital.\footnote{Although productive capital, per se, does not enter the production function, GS do use the term equity capital and working capital to denote the financial assets used to pay wages (pages 124-125, 1988).} To understand how risk aversion operates in the GS model, and how it effects inter-temporal dynamics, we consider the decreasing returns to scale exhibited by the relationship between equity and output. Through the one period lags between output decisions affecting real wages and dividends, a
firm's equity position is affected. Under such dynamics, a firm's asset composition has non-neutral output effects. Output increases less than proportionately with increases in assets (debt plus equity), or risk aversion increases with the scale and profitability of the representative firm, as specified in the following condition:

\[
\frac{dq}{da} \frac{a}{q} < 1
\]

(4.25)

The intuition behind assumption equation 4.25 is that the liquidation of a multi-national corporation costs more on a percentage of valuation basis than the corner grocer. As we recall from Section 4.3, equation 4.13, the cost of bankruptcy increases with output. The reason risk aversion has inter-temporal implications is because of the dynamic connection between output and equity. Both the initial condition of equity and the increases in equity arising from profit retention (net of dividends paid), affect the trajectory of long term output. According to the GS Model, the greater the proportion of equity in the capital structure, the less chance of bankruptcy occurring. (Naturally, without debt, a firm can not go bankrupt.) Aversion to bankruptcy, the possibility that \( u_{r+1}^t < v_{r+1}^t \), induces risk averse behaviour discouraging borrowing of capital for investment thereby reducing the level of output. Increasing the level of equity, or increasing the level of profit retention over time, increases the level of output, but less than proportionately, because of risk aversion. Thus from an inter-temporal standpoint, induced risk aversion, arising from the variance on firm specific prices, is important because it dampens the positive relationship between equity in the firms capital structure and output. Output is convex in equity.

The mathematics of risk aversion used in the GS model explain further our observations on the inter-temporal implications of equity levels. In the GS model (page 108, 1988), by assuming decreasing
returns to scale, the second derivative of output is negative.
It follows that the elasticity of equity with respect to output is less than unity, as per equation 4.25. The value of equity in reducing the threat of bankruptcy must be decreasing as the scale of production rises, which is tantamount to increasing induced risk aversion.² A given increase in equity increases output, but reduces the threat of bankruptcy at a decreasing rate. The dynamic implication of the above observations is that risk aversion serves to dampen the effect of greater equity on the time path of output: reducing the size of oscillations when and if they occur. Dynamically, greater equity means greater output, but at a decreasing rate, because of the induced, ex ante, risk aversion to bankruptcy displayed by firms arising from the possibility that firm specific prices may be below the threshold level, as modeled in the cost function. Simulation experiments, appearing below in Section 4.8, on variability of firm specific prices, verify this relationship, showing the inter-temporal nature of risk aversion.

The importance of the above results are shown by briefly contrasting them with how the GS model would behave if constant returns to scale prevailed, a point which is made in the footnotes to the 1988 GS article. With constant returns to scale, risk neutrality and constant risk aversion prevail, the GS Model generates different solutions. Specifically, output would rise proportionately with increases in equity.

\[
\frac{a}{q} \frac{dq}{da} = 1
\]

(4.26)

². We use the terminology in the sense of J.W. Pratt (1964), although strictly speaking risk aversion requires a utility function, and none per se is suggested by GS. If we identify the profit function as the utility function of management, we must assume that management and ownership are synonymous.
With constant returns to scale and no risk aversion, the optimal output is a linear function of equity, \( a \), with a constant of proportionality \( q/a \), (GS, page 133, 1988). A model with constant returns to scale and no risk aversion would display different dynamics.

Turning to some observations on the dynamic properties of the GS model, we commence with the role played by dividends. At the micro-economic level, dividends together with wages lead to changes in equity and hence determine output, as depicted in Figure 3.1 of Chapter Three. Dividends and profit retention have inter-temporal implications in the GS model through their effect upon the level of equity. As GS caution, unless a sufficient amount of profits are distributed equity and output may grow explosively. To examine this conjecture, in the simulated version of the GS model, as explained in Chapter Three, we adopt a fixed percentage rule to investigate the effects of different levels of profit retention. In the GS Model, equity in time \( t+1 \), consists of equity in period \( t \) plus earnings on that equity plus new equity sales less dividends paid, the latter being an unspecified random process. These arguments appear in the non-linear, autonomous, first order difference function, as specified by GS for nominal equity of the \( i^{th} \) firm:

\[
A_{t+1}^i = P_{t+1}^i q_t^i - (1+R) (P_t W_c \phi (q_t^i) - A_t^i) - M_{t+1}^i
\]

(4.28)

In the GS model, \( M_{t+1}^i \) is a random variable representing the nominal value of dividends paid less new equity issued. The Equation shown in 4.28 is autonomous because time does not enter as a separate
argument. Although for simulation we have assumed a constant for $M_{t+1}$, we note that having an additional stochastic influence upon the model, through dividends, as per GS would introduce an additional dimensionality in addition to the additional dimensionality from the variability on firm specific prices. The effect of higher dimensionality is to transform the behaviour of the model away from that of a first order process. Equations 4.28 along with 4.24, and the ex ante anticipated behaviour in relation to the stochastic firm specific price process, according to GS, completely describe the dynamic behaviour of the Model. The influence of these other variables we look at in subsequent sections. Modifying the model’s dividend structure, we will see, reveals its interesting dynamic properties.

We will validate the GS observation concerning the dynamic implications of dividend structure using the calibrated model in Section 4.8. For now we offer some observations upon the systems dynamics and how they may be interpreted in a broader context. On the issue of convergence versus divergence when oscillations are present, with reference to the arguments of equation 4.28 we consider the effects of changes in equity arising from alternative dividend pay-out schemes. Although not proven by GS, when a positive margin exists between the cost of funds and the implied return on financial capital or "working capital" (page 125, 1988 GS) $(P_{t+1}/P_t > r_d)$, increasing profitability and/or increasing the retention of net dividends, will render the system less convergent. Increasing the capital base encourages investment. A second set of observations concerns the nature of equilibrium. At a point in time, the GS economy is in equilibrium both on and off the steady state: Oscillations are from the steady state, not from equilibrium, giving the model its neo-classical flavour. (Although as we will discover, multiple steady-states are possible.) At each point in time, markets clear in equilibrium. With regard to the model’s cyclic dynamics and consequent laws of motion, the firm’s internally generated capital lies
at the heart of its dynamics. With greater profits and possibly unrealistic, unchanged profit distribution, dividends in absolute terms, increase. Conversely, when profits decline, because of poor firm specific prices, with the dividend pay-out rate unchanged, internal funds fall, reducing output. In the GS model and in the numerical simulation program, such behaviour arises because the level of equity affects the "base" from which the dividend dynamics may occur. Increases in equity lead to increases in output, with greater employment and real wages. Fixed dividends and greater real wages reduce profits, decreasing equity, leading to a decline in output... to wit, cycles. Equity increases the level of output, while with greater gearing (more debt), because of induced risk aversion found in the cost function, the result is to render the system more stable. Increasing equity relative to debt, reduces the fear of bankruptcy, reduces risk aversion, and, if sufficient, allows the dynamic trajectory of equity and output, to grow explosively, as expressed in Figure 3.1 of Chapter 3. Cycles in the GS model, and the consequent laws of motion, are autonomous, and are not the result of forcing variables, although changes in the latter may alter the trajectory. Such cycles are deterministic, autonomous, and self-driven oscillatory phenomena.

To summarize the second set of observations found in Section 4.6, the dynamic trajectory and model’s laws of motion are very sensitive to initial conditions in the dividend pay-out ratio, profitability, the initial level of equity for a given firm, and firm specific price variability. In aggregation the effects may be seen. Using the symmetry assumption of GS in the distribution of equity across firms, the difference equation for the $i^{th}$ firm shown in equation 4.28 may be aggregated to the following:

$$a_{t+1} = g_t - (1+x) (w_t \phi (q_t) - a_t) - m(a_t) = G(a_t)$$

(4.29)

If the slope of first order non-linear difference equation, found in
equation 4.29, as it crosses the 45° line found in a phase diagram (logistic space), is negative, the economy represented by the model may exhibit single or multiple period cyclic behaviour in output under any or all of the following circumstances:

- if the firm is insufficiently profitable;
- if an insufficient level of profits are retained; and
- if the initial level of equity is sufficiently low and thereby the ratio of debt to equity sufficiently high;

In the GS model, when oscillations do occur, they are more likely to be convergent, under the following circumstances:

- if the ratio of debt to equity is high, rendering the phase line less steep absolutely;
- if initial debt is sufficiently great, or if sufficiently large profit distribution, occurs; and
- if, because of induced ex ante risk aversion, firm specific price variability is sufficiently large.

Hence, if no new equity were raised and profits were fully distributed, the effect on firms forced to borrow increasing amounts of debt to support the assets necessary for output growth, would be to dampen oscillations, when they are present. (Whether such oscillations are of a single period or display multiple periodicity depends upon the non-linearity displayed by equation 4.29.) Owing to induced risk aversion on the part of firms, the existence of debt dampens oscillations, or makes the system more convergent when the trajectory is monotonic, a point we show in simulations of Section 4.8. In the model developed from the GS articles, we find that low profit retention and greater gearing (the ratio of debt to equity) are associated with oscillations, and they have a dampening effect upon such dynamic behaviour when it occurs because of aversion to bankruptcy. With sufficiently greater equity or low conditional probabilities of bankruptcy, if oscillations are present, cycles may be explosive. Although the dynamics of the GS style model are deterministic, increasing the initial condition on the mean of firm specific prices
(F(u')) or decreasing the variance in such prices, leads to a dynamic trajectory in output at a higher level, rendering the system less convergent, and possibly divergent in an explosive manner, ceteris paribus. A further result which we show below in Section 4.8.

Reconciling such results with received financial theory of Modigliani and Miller (1958) (hereafter, MM) and neo-classical economics, presents a variety of interesting issues relevant to our larger concerns of the effect of risk and its management upon macro dynamics. In the literature, such issues have received limited attention (see, R.Krainer, Chapter 4, 1992). According to MM theory, a firm’s valuation is unrelated to how its assets are funded. Relatedly, in Neoclassical theory dichotomy holds, money is a veil. Whereas, in the GS model, the level of equity and the threat to changes in valuation, viz bankruptcy, drive output. Secondly, according to MM theory a firm’s valuation is unrelated to how its returns to shareholders are divided between capital gains and dividends. The valuation of a firm is determined by the riskiness of its net operating income, not its return structure. Increasing debt within a firm’s capital structure increases systematic (the β of Capital Asset Pricing theory), and hence the rate at which enhanced earnings are capitalized. To the extent that such “discounting” occurs, the effect upon firm valuation of gearing-up, adjusted for taxes and interest deductibility, is minimal or non-existent. In the GS model, the special "cost" of debt, versus equity, leads to an adjustment in behaviour adding to the model’s convergent qualities, stabilizing profits and returns. If the stylized result of GS merely derive from a reduction in systematic risk, moving downward along the security market line; we have a problem from the standpoint of received financial theory: Since no change in valuation has occurred, there can be no change in the threat of bankruptcy. In contrast, in the literature, several explanations for MM non-neutrality arise: In one article, it lies with the higher cost of internally generated funds
(Nance, Smith, and Smithson, 1993); while in another it is seen as means to lower the variability of the shadow value of internal funds (Froot, Sharfstein, and Stein 1993).³ None of these MM non-neutrality perspectives sits well with the GS model. An alternative construction might be that the risk aversion in the GS model leads to a reduction in company specific risk, such as a change in the cost of financing (to the extent that its assets are valued more highly) lessening the threat of bankruptcy. Management of risk, through the use of options and futures for example, to reduce or eliminate the downside implications of the stochastic behaviour of firm specific prices, may be viewed as a company specific action to protect company specific assets, although it involves market risk, and requires certain assumptions regarding the behaviour of such markets. Arguably, a construction of risk aversion and risk management as applied to the GS model which is consistent with financial theory and offers useful insights is found in the William’s inter-temporal asset allocation paradigm, already mentioned in Chapter Three. As described, using the paradigm of William’s, risk management might be viewed as inter-temporal asset allocation, leading to improved use of assets and greater output. In the William’s paradigm, risk management is not about reducing risk, but rather concerns ensuring that returns are commensurate with risk inherent to the firm and its business. We explore these issues further in Chapter Five.

The last set of observations found in Section 4.6 concern the dynamics of the GS model. We begin with model’s "conditions" for oscillations to occur and thereafter address the issue of non-linearity as found in the first-order nature of the difference equation on equity. We compare how GS handle non-linearity analytically versus how it may be evaluated using dynamic simulation. In the GS model, several

³ Such arguments are in addition to the follow-up work by Modigliani and Miller on the effect of taxes upon capital structure (1963).
"conditions" are presented in order for oscillations to occur. Some of these are mentioned by GS while others merit clarification. The non-linearity of the GS difference equation, we will see, makes the path of equity over time, \( a_t \), sensitive to the initial value, \( a_0 \), along with other initial conditions as found in equation 4.29. The slope of the difference equation, drawn in phase space, may change, through an inflection, one or more times, being positive or negative in one or more regions. Furthermore, the point(s) of inflection may be to left or right of the 45° line. Depending upon the initial condition on the level of equity, for a given firm (which we assume to be equal), will tell us whether we are in a region of the phase line where oscillations may occur. The second condition relates to the growth in equity, arising from retained profits. If the initial amount of equity is small, for example, and net dividend distribution, \( m \), is sufficiently large, or conversely profit retention is small, and when because of the first order conditions on profit maximization the following holds:

\[
(1+r) \phi / \omega < 1
\]  

(4.30)

an oscillatory trajectory over time, may arise. For such cycles to occur, the increased output must lead to greater wages, leading to a sharp fall in profits on the back of a fall in realized firm specific revenues. With less equity, and hence more debt, investment in working capital falls, leading to less labour, and a fall in real wages. In the generation of cycles profit distribution is also important. Over time, \( m \), as GS explain, the level of dividends, net of new equity issued, must be sufficiently large, so that the firm’s equity base does not grow too fast, leading to an explosive growth in output, and producing a trajectory which is non-cyclic, and monotonically
divergent. Such a condition may also arise when there is little debt in the firm's capital structure. These observations we confirm in the simulations of Section 4.8.

Understanding merely when oscillations occur as per the GS article, however, tells us little about the nature of its dynamic trajectory because the system modelled is non-linear. By way of comparison, for linear systems, a negative sloped phase line implies that oscillations will occur, and they will be divergent, if the slope is less than minus one in the neighbourhood of the steady-state; and convergent if greater than minus one. For a linear phase line, for example, a uniform oscillatory path can be produced iff the slope of the phase line is -1. Such cycles would oscillate with one iterate below and one iterate above the steady-state, and not exhibit the multiple periodicity (page 100, GS, 1993) which they claim. But the first order difference equation model presented in the GS article, as per equation 4.29, is not linear as may be shown. Although GS begin their discussion of the models dynamics with reference to the total derivative of equation 4.22 above (as found, page 100, GS 1993), and as shown in equation 4.31, below, mentioning that its slope as it crosses the 45° line must be negative for the trajectory to be oscillatory, this statement is somewhat mis-leading, and is only true as a tautology. Using $G(a_t)$ referring to equation 4.29, we have the total differential:

$$G' = (1+r) - m' - ((1+r)(\psi'\phi' + \phi') - 1) H'$$

(4.31)

Inspecting the slope of equation 4.31 using a linear approximation

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3 An alternative to allowing growth in equity to generate an explosive trajectory in output, would be to make the net dividend pay-out ratio an endogenous function related to capital structure and the opportunities and risks facing shareholders and debt holders. Such an approach would be grounded in a considerable body of theory (Modigliani & Miller, 1961; John Lintner, 1961; or Gordon & Gould, 1979).
around the equilibrium steady state, as GS suggest, provides only limited insight. With a non-linear difference equation the dynamic behaviour may be much more complex than merely convergent or divergent one period oscillations around the steady-state. Cycles of two or more periods may occur. Before explaining why, we review the approach adopted by GS.

There are two method of qualitative interpretation of the dynamic qualities of non-linear systems, when calibrated simulation is not available both involving linearisation in the neighbourhood of the steady state. The two most well known are: 1) Applying Taylor’s formula to linearise the function within the neighbourhood of the steady state; and 2) Liapunov’s direct method which defines an energy like function for the dynamical system (Azariadis, pages 58-62, 1993). Method one may be used to determine whether or not the steady-state is stable. Their use are justified when the system’s equilibrium is asymptotically stable (pages 469-471, Intrilligator, 1971).

The mathematics to support the linearisation approach, however has several requirements regarding stability which itself can be defined in various ways. In Azariadis (pages 58 -59, 1994), we note that for systems involving steady long-run equilibrium, there are three definitions of stability for a fixed point ($a^*$ in the GS article), namely:

1. A stationary state is **stable** of a dynamical system if all orbits that start near it, stay near it;
2. A stationary state is **asymptotically stable** if all orbits tend to the steady state asymptotically; and
3. A stationary state is **structurally stable** if the qualitative properties of orbits are invariant to small perturbations in the continuously differentiable map ($G(a^*)$ in GS). Structural stability is concerned with perturbations to initial system conditions.

Although GS do not suggest in their article into which definition of stability their system would correspond, it is the case that for all three
possibilities, a linear approximation of $G$ in the neighbourhood of an equilibrium steady state, may be constructed, so long as the said equilibrium is hyperbolic (the generic case), an issue also not pursued by GS in either of their articles (1988, 1993). According to such theory, in the vicinity of the steady state, a Taylor expansion in the form of a linear polynomial of equation 4.29 is a good approximation, provided the Jacobian $[D g'(a')]$ matrix is invertible. Topologically equivalent systems have the same qualitative dynamical properties.

Before discussing how numerical simulation of the GS model may be undertaken, we consider further the limitations of their analytic method, and thereby justify the approach we will utilize. Consider the nature of the dynamics found in the GS model and as will be performed through numerical simulation. In Section 4.5, we explained how expression 4.18, the aggregate supply of output, $q$, and expression 4.29, the first order non-linear difference equation on the level of equity, completely determine the dynamic behaviour of the model. According to the GS article, in order for their model to generate cycles, that is not converge to a monotonic state "...the level of net equity outflows must be sufficiently large so that the real level of equity in the economy does not increase without bound." (page 100, GS 1993). Sufficiently large net equity outflows from firms may arise because of individual firm level profit distribution policies. Cycles may also arise, that is non-monotonic convergence to a steady-state may occur, because of sufficient price variability, through the risk of bankruptcy, discouraging investment and leading

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4We refer here to the Hartman-Grobman Theorem (Azariadis, page 59, 1994) which states that if a steady state is a hyperbolic fixed point of a nonlinear system, then the nonlinear system is equivalent up to a continuous change of coordinates, to a linear system with a coefficient matrix from the Taylor expansion.
to unbounded growth in the aggregate level of equity.\textsuperscript{5} In the GS Model, when cycles do take place, with sufficient equity levels, such cycles may be divergent and explosive. For deterministic periodic cycles to occur requires the slope of the equation 4.29 to be negative, for some values of \(a_t\). GS base these analytic results on the signs found in the total differential of equation 4.31 repeated below:

\[
G' = (1+\delta) - m' - ((1+\delta) (\psi' \phi + \psi' w) - 1) H' 
\]

(4.33)

This expression, as a differential, represents the slope of the first-order difference equation on the level of equity. Evaluated at the steady-state equilibrium, if it is negative, a necessary but not sufficient condition for the model to generate cycles, arises when the following holds:

\[
(1+\delta) \psi' w < 1 
\]

(4.34)

Where, the individual rate of time discount \(\delta = r\), the constant real rate of interest. (We note in this context, the sign of derivative evaluated at a given point is not, in of itself, a sufficient condition for determining whether a system will generate cycles.) Secondly, it requires that the profit distribution \(m'\) to be sufficiently large or that the increased output increases real wages \(w_t\) sufficiently. According to GS, these are the necessary analytic conditions for cycles to occur in their model. (The conditions for when cycles are convergent or explosive and whether they exhibit multiple-periodicity is discussed in Section 4.7.)

Without clarifying any of the above issues, GS in their article appeal to linearisation around the equilibrium steady-state as an

\textsuperscript{5}Aversion to risk arises from the relationship between firm specific prices and the threshold bankruptcy level of prices, \(u_{t+1}^i\) and \(v_i^t\), respectively. Money neutrality is not considered.
approximation of the phase slope, and thereby as a qualitative method to describe dynamic behaviour and the possible path to a steady-state solution. The asymptotically steady-state interpretation for linearisation of the model, as used in the 1993 article (page 100-101). By referring to the slope of the total differential, GS appeal to the technique of approximating the behaviour of a non-linear system in the neighbourhood of the asymptotically steady state by a closely related linear system (in footnote 28, page 100, 1993 in which they refer to the technique adopted by Grandmount, 1985). Furthermore, they note (page 135, 1988) that when \( \ldots c_f(u_{t+1}) \) is linear in \( a_t \) [equity] and that the valuation function for the multi-period decision problem is linear in \( a_t \); then the \( \ldots \)multi-period decision problem is qualitatively identical to the single-period problem..." If this holds, the initial level of equity and the next period level of equity, may be used as an approximation of the long term dynamic solution. Or, formally, Bellman’s dynamic programming applies and the principle of optimality holds (Chiang, 1992). In this fashion, GS draw their observations on the model’s dynamics from studying the stability of a linear system corresponding to their non-linear system. We replace this approach in Section 4.7, explaining how proper numerical simulation was undertaken, to examine the model’s dynamics by calculating the values for equity and output over time. Using the developed methodology, in Section 4.8, alternative trajectories are computed to reveal general laws of motion along with the sensitivity of dynamic solutions to initial conditions. In Section 4.9, we explore the dynamic implications of non-linearity still further, showing how and when aperiodicity may arise.

In order to appreciate the merits of numerical simulation of non-linear models, it helps to further discuss the short-comings of the linearisation method we have described as used by GS. Linearization obscures whether the slope of the non-linear phase line as plotted by equation 4.29 changes slope before or after it crosses the steady
state line. Sharp inflection points in logistic maps, so called "tent diagrams", may depict chaotic trajectories (Chapter 2, Baker and Gollub, 1990). A phase line may cross the steady-state 45° line more than once, in a series of piece-wise linear mappings (Section 1.4, Chapter 1, Gulick, 1992). Moreover, non-linearity means that with different initial condition, \( a_o \), alternative dynamic trajectories may be generated. Considering the slope of the phase line as it crosses the 45° obscures these issues. For example, difference equations, for example, besides those having a slope of -1, may still give rise to uniform oscillations (Hoy, et al. pages 672-675, 1996). A phase line of sufficient degree, may also change slope more than once. Thus the slope of the total differential, as per equation 4.31 above, as shown by GS (page 100, 1993) offers insight, subject to many qualification: The dynamics implied by a non-linear difference equation, for example whether a path to a steady state exists, whether multiple steady-states solutions exist, bifurcations occur, and whether the trajectories are aperiodic or chaotic have several requirements. These requirements include the slope it displays before or after it crosses the steady state 45° line \( (da/dt = 0) \), how often it changes slope, and the initial conditions, our \( a_o \), are all relevant to dynamic numerical simulation. As a necessary condition, a one dimensional logistic mapping, as a first-order non-linear difference equation must be non-invertible in order to generate chaotic solutions (Chaos in Dynamical Systems, page 24, Ott, 1993), but such dynamic behaviour demands simulation in order to investigate it. Informally, chaos is associated with sensitive dependence to initial conditions and aperiodic motion (page 9, Kelsey, 1988).\(^6\) Notwithstanding the complex non-linear dynamics, as noted above, it is questionable whether the GS system should be described as first

\[ ^6 \text{There are several definitions of chaos in use including positive topological entropy, positive Liapunov exponents, and the existence of strange attractors (page 9, Kelsey, 1988).} \]
order, given the existence of random inputs, notably, the variability in firm specific prices. (In which case, the dynamic behaviour of the system will exhibit neither steady-states nor periodic cycles.) These observations, we will see, imply that in the GS model, the path to divergence or convergence may not be uniform, the amplitude of oscillations may vary, a steady-state (one period cycle) may never be reached, and the dynamic behaviour may resemble, at least, a second order system. To understand the models potential dynamic behaviour requires specification and simulation of the non-linear difference equation to learn how it responds to various forcing variables, and initial conditions. Numerical simulation is required to understand the dynamic behaviour of non-linear difference and differential equations of even first order, as equation 4.29, because analytic solutions, generally, do not exist and linearization is of limited interest.

Using the linearization around a single steady-state solution, and appealing to the slope found in equation 4.31, as GS have done, therefore, is somewhat misleading. As the difference equation model is non-linear, equation 4.31 as a differential, is only an approximation of the phase line, at a point in time, and depending upon from what initial conditions it is evaluated, may not be in the neighbourhood of the steady state, or there may be multiple steady-state solutions. Linearization around the steady-state, as GS invoke may obscure the possibility of slope changes and whether they occur before or after the phase line crosses the 45°. The slopes of tangent lines evaluated in the neighbourhood of the steady state may be very different, especially if one or more inflections occur (Chapter 1, Gulick, 1992). Depending upon where the linearized form is evaluated, the implied dynamics may be very different. As may be shown through numerical simulation, given certain initial conditions, sharp sign-changing inflections in the phase line may occur, which is the motivation for using numerical simulation rather than analysis. Equation 4.31 as a differential of equation 4.29, depending upon
where it is evaluated, tells us about a slope at a point along the curve, not about the slopes of all tangent lines along the curve, especially when a discontinuity or sign-changing inflection exists. The only information that a derivative evaluated at a steady-state gives is whether that particular state is stable or not. Using equation 4.30 to tell us about the general nature of the system's trajectory when it crosses the 45° degree line, is justifiable subject to extreme qualification: It may be only an approximation of the slope of equation 4.29 in the neighbourhood of a stationary state, it ignores the issue of whether a sign change, if any, has occurred above or below the 45°, and it ignores the importance of initial conditions. Moreover, given the implied higher-dimensionality of the GS model, it simplifies matters drastically. As is well-known, other deterministic trajectories and even multiple equilibria may arise, depending upon initial conditions and changes in forcing parameters.\(^7\) Introducing random inputs introduces further complications.

Apart from the higher order issues, the problem with the linearization method adopted by GS is that it ignores the rich dynamics found in such non-linear models. Non-linearity and initial values affect the GS style model's cyclic periodicity, that is the number of period in a cycle and whether such iterations occur above or below the steady state. The qualitative linearisation approach used by GS side-steps the rich dynamics revealed through numerical simulation, because, as mentioned, investigating the dynamic properties of non-linear difference equations using analytic techniques, is generally not possible. GS assert, without substantiation, that their model may generate cycles of multiple periodicity (page 100, 1993). Although the GS claim that their model exhibits such patterns, is not knowable from the general functions

\(^7\)According to Azariadis, multiple equilibria may arise from "...missing initial conditions; subjective state or parameter spaces; .... or lack of homeomorphism." (page 449, 1993)
presented and the method proposed. According to GS their first order difference equation on equity, as shown in equation 4.29 above, is non-linear (page 100, 1993); and accordingly the specified functions used to model it in the simulation program, involving two transcendental functions on wage costs and bankruptcy costs, as explained in Section 4.3, are also non-linear. Complex dynamic trajectories involving multiple periodicity are important because they give the model its business cycle flavour. According to GS, rather than the usual single-period, oscillatory type motion with alternating values above and below the steady state, they allege that more than one observation, over time, may occur above or below the steady state. Achieving business cycles featuring multiple periodicity, as GS suggest, that is repeated solutions above or below the steady state, however, are not observable using linearisation, as the implied trajectory merely would oscillate around the steady-state 45° line, precluding the possibility of multiple period cycles as they claim for their model.\(^8\) While an abstract model as GS present, without specified functional forms, may exhibit such behaviour in theory, linearisation as they appeal to for analytic results, makes it unprovable. In point of fact, a key implication of non-linearity of the phase line is that even if the implied trajectory were unstable, the path on equity, \(a_t\), does not necessarily diverge endlessly to infinity or zero. Instead, the trajectory may never converge to the steady-state, oscillating within a bounded range or even converge to regular multiple periodic behaviour, as we show and discuss below in Figure 20. Such oscillations, neither convergent nor divergent, may lead to fluctuations around the steady state of varying amplitude, producing stable limit cycles (Hoy, et al. pages 670-672, 1996).

\(^8\)For comparison, we note that for second order difference equations such dynamic behaviour may occur when the roots are complex. The periodicity may be determined using a modified circular function (Sargent. pages 177-179, 1979).
Stylised Logistic Phase Plot and Orbit

Figure 20
Expanding, a further problem with the claims made by GS using the linearization method relates to the role of initial values. Using numerical simulation allows us to see how, for a given phase line, the periodicity of the cycles produced depends upon the initial value, that is $a_0$ from which the trajectory commences. For certain initial values, the dynamic system may bifurcate, that is switch from, say, two period oscillation around the steady-state to, for example, a four period cycle between bounded values (Azariadis, *Intertemporal Macroeconomics*, pages 98-99, 1993). Figure 20 below, depicting a stylized Phase Diagram we use to illustrate such observations and possibilities. Looking at the stylized phase portrait of Figure 20, the trajectory is oscillatory explosive for initial values in proximity of the steady-state point, since it is less than minus one. Plotting the phase line shows us what happens globally, rather than merely making observations around the steady-state. From a very small initial condition the plotted trajectory first exhibits monotonic convergence, however as the neighbourhood of a steady-state is approached, bounded oscillations occur. In the Figure, we initially see convergence behaviour followed by a limit cycle with upper and lower bounds. Following the arrow around the phase portrait reveals the following pattern of solutions above and below the steady-state (denoting "A" for above, and "B" for below), as shown in Exhibit 4.3 below. For the quadratic family as depicted in Figure 20, for certain initial values, the trajectory may oscillate around the steady-state later bifurcating into an aperiodic or chaotic orbit (Hoy et al., pages 674-675, 1996). Moreover, sixteen iterations are required until we return to where we began. Formally, we would say that such a phase plot exhibits a cycle of sixteen periods. For similar phase plots, but with different initial conditions we may not see convergence to the steady-state but rather a stable limit cycle, with oscillations between these two iterates occurring. Cycles having a different number of periods may arise. We can see from Figure 20 that by changing the initial
condition, very different trajectories may be generated.

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<th>ITERATE NUMBER</th>
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Exhibit 4.3 Iterates in relation to Steady-State

Since a non-linear phase line may change direction, and because it may change direction above or below the 45° line, the possibility for repeated equilibrium solutions above or below the steady state is created... multiple periodicity, that is cycles of varying length. Moreover, it is possible, given certain functional forms as those specified for use for the first-order autonomous difference equation
in the GS style model, and appropriate initial values, \( a \), to generate cycles which are bounded around the steady-state and are of varying length, or even aperiodic. Such apparently random but actually deterministic trajectories, generated by bounded cycles of varying length, are known as \textit{chaos}. Chaotic solutions are extremely sensitive to initial values and can be described as exhibiting aperiodic motion. Chaos is opposite to periodic motion. A requirement for chaos is that the first order difference equation, as a one-dimensional map, be \textit{non-invertible} (page 24, Ott, 1993). The difference equation found in equation 4.31, as we see from the above discussion may change sign, implying that equation 4.29 is not monotonic, and the necessary condition for chaotic solutions, non-invertibility exists. Although as we will amplify upon below, in addition to the potential from deterministic chaos, the GS model contains a random attractor, which further modifies its dynamic behaviour. Applying such theory to understanding a GS style model in which micro-economic imperfections and constraints are investigated for their effect upon dynamic economic behaviour in the aggregate, we see that numerical simulation of a non-linear dynamic business cycle model is necessary in order to capture the model’s rich dynamics, including aperiodicity and chaos.

4.7 Modelling Dynamic Simulation

In Sections 4.7 and 4.8 of Chapter 4, we explain how the stylized analytic observations found in the GS article relating to dynamic behaviour, may be used to explicitly model and simulate dynamic solutions. In these Sections, to simulate the GS model, as introduced earlier, we iterate the optimization and aggregation specifications of Modules I and II, to compute aggregate equity and output over time. These procedures and algorithms constitute Module III. By simulation of the \( G(a) \) function over time, as appears in equation 4.29, we improve upon the linearization approach
suggested by GS, for the reasons discussed in Section 4.6. As explained, we know multiple periodicity cannot be determined using linearization around the steady-state, that is, from merely knowing the slope of the phase line as it crosses the 45°. Rather to learn the behaviour of the entire trajectory, in contrast, a simulation method using Modules I, II, and III of the computer program must be used in order to learn about the model's rich dynamics. The computer program and algorithms used, as we will show, behave according to the requirements specified in their article, and reveals the model's sensitivity to initial conditions. We leave to Section 4.9, investigation of the model's order and dimensionality, as well as period doubling and aperiodicity as arise from its non-linearity.

Simulating a version of the GS model dynamically, we implement the procedures specified in Module III which relies upon the results of Modules I and II to find aggregate solutions for equity and output over time. Iterative computation of the simulation program of Module I is necessary because no closed form solution to the $G(a_t)$ exists. Rather, it must be computed by re-optimizing Module I, using for example $G(a'_1)$ to compute $a'_2$, and using $G(a'_2)$ to obtain $a'_3$, and so on. Although this could be undertaken for all twenty-five firms over time, for computational simplicity, we have relied upon the GS assumptions of symmetry in the distribution of equity across the population of firms (page 94, GS 1993), to determine an average $G(a'_j)$ relationship over time, for a given firm, based upon the average level of equity among the population of twenty-five firms in period two. For the selected firm alone, successive $a'_j$ are calculated through re-optimizing Modules I for forty-eight periods further, giving a total trajectory of fifty iterates. The result is a vector of $G_t$ values which are then applied forward from period two to the remaining population of twenty-four individual firms. Although the dynamic trajectory for all firms uses a functional average $G(a_j)$, price shocks remain unique to each firm. The dynamic
results for the twenty-five firms are aggregated in Module II. The Module III algorithm also plots the iterates of $a_t$ and $q_t$ in the manner described above, as per the difference equation 4.29, to generates a trajectory over time as well as the phase line in logit space. As required for simulation, the intercept on the $a_t$ axis represents the initial amount of equity in period one, which is assumed. After period one it is computed internally through the profit maximization process. In the simulations, for all periods, we assume a net level of profit retention over time, a standard deviation in firm specific prices, and a level of technology. As we will discover, the model’s dynamics are very sensitive to changes in initial values. As we change the aforementioned exogenous variables inputs to Module I, so changes the aggregate trajectory on equity and output. Using these results, we will verify the GS claim that their model’s dynamics exhibit cycles of exceeding two periods ("multiple periodicity"), that is repeated solutions above or below the steady state.

A final point regarding to procedures concerns aggregation. According to GS, the level of equity over time, uniquely determines the level of output. As introduced in Section 4.5 above, to determine the aggregate level of output, we summed the outputs of the individual firms. Although the initial level of equity in the simulation program is assumed to be the same for all firms, and the successive $G(a_t)$ are based upon that of the average firm, over-time with individual price shocks, individual equity and output levels per firm vary. To simulate the model dynamically, levels of individual firm equity and output are summed. Although this procedure arises from the aggregation of the outputs of the individual firms, the result resembles the functional form found in Figure 6A, of the GS article, mapping equity to output. As we will observe in the figures below, the dynamic trajectory of output follows that of equity, with a slight dampening arising from the functional relationship between equity and output at the individual firm level. Lastly, in this context, we note
that the dynamic trajectories shown are not unique. As we will show, exposing the population of twenty-five firms to a new set of price shocks will generate similar but different trajectories, in terms periodicity and convergence. Moreover, different combinations of exogenous parameters may produce similarly behaved trajectories.

Before turning to Section 4.8 and showing the dynamic behaviour of the GS Model with respect to key parameters, we conclude Section 4.7 by ensuring that the approach explained above generates dynamic trajectories and such trajectories responds to exogenous variables in the manner required by the GS article. Figures 21, 22, and 23, using different levels of profit distribution over time are illustrative. Although GS assume that net dividend outflows are stochastic, they still must be sufficiently large to produce cycles. We have used, as explained in Section 4.6, a fixed distribution rule on dividend polices to see their effect upon the level of equity, as it drives output. Although the authors conjecture that higher dividend pay-outs net of new equity sales, may determine when and how the model will generate cycles, this observation is verified using the simulation model. Reviewing the results below, we see that simulation validates the analytic claims made by GS on the role of dividends in their model. We observe that for particular state variables assumed, and as noted in each figure respectively, some of the cyclic trajectories generated by the algorithm are of multiple periodicity (cycles are greater than two periods) as conjectured by GS in their article (page 100, 1993). In Figure 21, we see a quick convergence to a one period cycle, i.e. the steady-state, however, in Figure 23, we observe a six period cycle. Figure 24, also shows a six period cycle, which if iterated further might converge to an attractor. These four simulations validate the observation by GS of the effects of dividend retention upon the systems dynamics: Through sufficiently increasing the level of profits retained, the amplitude of cycles will increase, and may even become explosive, because with
greater equity firms are keen to borrow and invest. Greater equity leads to a reduction in ex ante aversion to bankruptcy. Greater output leads to higher real wages and with stochastic firm specific prices, sudden and unexpected falls in profits and investment. If dividend payments are sufficiently small, the oscillations in the system may even become explosive. In contrast, increasing profit distribution, because it reduces the level of firm equity, stabilizes the system because firms exhibit ex ante aversion to bankruptcy. Less borrowing, leads to smaller increases in real wages, stabilizing profits when prices are lower than anticipated. Paying dividends from the profits of investment in working capital over time, serves to reduce the level of firm equity, and because of risk aversion, has a dampening effect upon output fluctuations. Lastly, in Figure 24, we maintain the same level of profit distribution, but expose the population of twenty-five firms to a new price shock. As we can see, the trajectory is similar in nature, with the largest differences during the transient portion, in the beginning. Over-time its path is toward the same steady-state solution. Such results resemble the stylized trajectory found in Figure 20. The affect of changing other exogenous variables, which GS suggest in their article, but do not analyze, upon the dynamic trajectory, we consider in the simulations found in section 4.8. The simulations reveal the effects of changes in the exogenous variables mentioned by GS and which account for the model’s dynamic characteristics, and show that the developed simulation program behaves according the general requirements specified in the GS article.
Fig. 21. 15% Profit Distribution, Initial D/E = 3.5, Tech = 4.5, Sigma = 20%

Fig. 22. 10% Profit Distribution, Initial D/E = 3.5, Tech = 4.5, Sigma = 20%
Fig. 23. 8% Profit Distribution, Initial D/E = 3.5, Tech = 4.5, Sigma = 20%

Fig. 24. 8% Profit Distribution, Initial D/E = 3.5, T = 4.5, Sigma = 20%
4.8 Simulations with a GS Style Model

In this next section of Chapter Four, we show further results of using the computer program for simulation, relying upon the dynamic numerical simulation procedure introduced in Section 4.7. In this way we verify that the program responds as claimed by the analysis found in the GS article. Again, we note that these simulations are not unique because of the variability in firm specific prices. Dynamic behaviour which is similar, for example convergent or of a certain periodicity, but which follows a slightly different dynamic trajectory may be generated through new simulations, as will be shown.

Turning to the simulations performed, three different sets of trajectories of output are computed from the proposed dynamic simulation algorithm of Module III, each relies upon changing initial conditions of a different exogenous variables, while holding all others constant. In addition for some we have shown the effect of a firm specific price variability, leaving the parameters unchanged. By modifying such state variables, different dynamic trajectories on equity and output may be generated. We will see how under certain conditions, oscillatory patterns, or cycles, exhibiting both convergence and divergence, may be generated. According to GS, the rate of profit distribution (dividend retention), uncertainty on firm specific prices, and the initial level of equity, all account for cyclic fluctuations in the model (page 90, 1993). To verify their Propositions, the three sets of simulations conducted, for each of which a graph of the dynamic trajectory from simulation is shown. The three sets of simulations are the following:

- The Variability in Firm Specific Prices, Figures 25, 26, 27 & 28;
- The Initial Amount of Equity, Figures 29, 30 and 31; and
- The Level of Technology (Figures 32 and 33).

The three sets of simulations are instructive of how changes in various initial conditions for state variables affect the system’s
dynamics as well as illustrating the role of calibration in successful simulation. Firm specific price variability, we note is important because it affects the anticipated relationship between realized prices and the threshold level of bankruptcy. According to GS, the "...degree of uncertainty..." is one of the state variables having an effect upon the level of investment and output (page 93, 1993), and which they suggest "...account for cyclic fluctuations in the model". When price variability is greater, it should discourage investment, reducing dynamic output. Although the initial conditions on equity levels of individual firms will vary over time with individual price realizations, the initial level of equity may have an affect which we investigate in Figures 29, 30 and 31. Technology is not mentioned by GS, however, we have added it via the production function, and the results of simulations appear in Figures 32 and 33. Critically, the technology parameter allows us to validate one of the GS claims of when cycles occur as shown in equation 4.34 above. We will also discover that by including technology in New Keynesian financial market imperfection model, a useful insight will be gained regarding when an economy is prone to cyclic behaviour.

We begin our observations on these experiments with those relating to the variability in firm specific prices of Figures 25 to 27. As suggested, as firm specific price variability increases, investment and equity growth is discouraged, dampening the economic cycles. The simulations show what happens when the variability in firm specific prices is changed. Although all three simulations involve oscillatory convergence, we see that the larger the variability in firm specific prices, the more quickly the convergence to a steady-state transpires. Holding all other exogenous parameters constant in the

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9According GS, in order for oscillatory cycles to occur, the condition $\phi'w < 1/(1+r)$, that is value of the marginal productivity of labour must be less than the reciprocal of the rate of interest plus one. The marginal productivity, $\phi'$, cannot be investigated without a parameter for technology.
three simulations we see the effect of increasing the standard deviation from 15% to 17.5% to 20% has a dampening effect upon the cyclic oscillations. With greater variability in firm specific prices, risk aversion leads firms to reduce their output, rendering the trajectory increasingly stable. Although investigated in detail later, for interest, Figure 28 show the effect of a firm specific price shock, holding price variability unchanged from 20%. Given a new set of firm specific price shocks across the population of twenty-five firms a new unique trajectory may be generated, displaying similar Laws of Motion, however.

Figure 25. 15% Sigma, Tech = 5, Profit Distribution = 8%, Initial D/E = 3.5
Figure 26. 17.5% Sigma, Tech = 5, Profit Distribution = 8%, Initial D/E = 3.5

Figure 27. 20% Sigma, Tech = 5, Profit Distribution = 8%, Initial D/E = 3.5
Experiments on the initial amount of equity, Figures 29 and 30 confirm the effects of greater equity on investment and cyclic periodicity. In Figure 29, after the transient phase, we see cycle of six periods in length. In Figure 31, we use the same state variables as Figure 30, but with a different firm specific price shock, that is a different drawing of normally distributed firm specific prices was used. Importantly, we see that the trajectory is similar, but not exactly. (As we take-up in Section 4.9, even for a given trajectory, the variability in firm specific prices serves to introduce a higher dimensionality to the GS system.) With insufficient levels of equity, the risk of bankruptcy from greater borrowing discourages investment, dampening equity creation, as the results show. The initial level of equity is important because it is from this base, optimising over time, that the model solves for an new optimal amount of debt of i'th firm, given the initial level of equity and the firm specific price shocks. Over-time because of firm specific price variability, levels of equity of individual firms will vary and hence no assumptions may be made, but as we see because the cycles are
deterministic (apart from the additional dimensionality introduced from firm specific price variability), initial conditions play an important role. In systems exhibiting aperiodic motion, that is deterministic chaotic behaviour, the trajectory is very sensitive to changes in initial conditions. The initial conditions on the population of twenty-five firms has persistent effects upon output and employment decisions which affect wages and ultimately profitability which may reduce output in the next phase of the business cycle. We see that equity increases the level of output, while with greater gearing (more debt), because of induced risk aversion found in the cost function, the result is to render the system more stable. Increasing equity relative to debt, reduces the fear of bankruptcy which may occur because of the relationship between firm specific prices and the threshold level, as explained earlier in Chapter Four. With sufficiently greater equity, risk aversion may be reduced, allowing the dynamic trajectory of equity and output, to grow explosively, as shown in Figure 3.1 of Chapter Three.

**Figure 29.** D/E = 3.75, Tech = 4, Sigma = 15%, Profit Distribution = 8%
Figure 30. D/E = 4, Tech = 4, Profit Distribution = 8%, Sigma = 15%

Figure 31. D/E = 4, Tech = 4, Profit Distribution = 8%, Sigma = 15%
From the simulations we see that the computer program behaves according to the analytic claims made by GS using general functional forms. All of these simulation results correspond with and validate what GS claim in their articles as to how the model should respond, while calibration shows the sharp sensitivity of inter-temporal dynamics to initial conditions, as well as those held constant.
over the trajectory, such as the variability in firm specific prices and the percentage of profits distributed. Particularly interesting is how firm specific price shocks modify the trajectory without changing its over-all dynamic quality. An important intuition throughout is that with greater equity, because the fear of bankruptcy is reduced, greater investment leads to higher real wages and increasing employment which in the next phase of the business cycle, reduces profits and output. With greater equity and less fear of bankruptcy, investment is encouraged, creating a demand-pull effect upon real wages. Under such dynamics, the size of oscillations increase. With very little debt in the capital structure, the growth in equity may even render the dynamic system, explosive.

Turning lastly to experiments on the level of technology, Figures 32 and 33, we report some interesting results which initially may appear counter-intuitive, but upon reflection make considerable sense. In these two experiments the only change is the technology parameter used in the production function as explained in Section 4.2. Although GS do not include a technology parameter in their production function, one was required for calibration and it is interesting for its effects upon the model’s dynamic qualities. Using various coefficients on the technology coefficient, holding other state variables constant, the model was simulated to determine the effect upon the model’s output trajectory (see, footnote 8 above). Between Figures 32 and 33, the technology coefficient was increased from 4 to 5 as shown. Surprisingly, we see that with a higher level of technology or productivity, the economy is more stable. How is this possible? One might have thought that as the economy as represented by the model, becomes more productive, that is producing greater levels of output from the same inputs, accumulating equity, it would become less stable absolutely causing greater amplitude in equilibrium output, but we find that this is not the case. Other effects, however, appear to dominate.
Recalling the nature of deterministic cycles in the GS economy, as depicted in Figure 3.1, will help to explain why. Cycles in the GS model, and the consequent laws of motion, are autonomous, self-driven, deterministic oscillatory phenomena. They are not the result of forcing variables or exogenous stochastic perturbations, although changes in the former may alter the trajectory. In the GS economy all capital borrowed in time $t$ must be used as working capital to pay wages and dividends out of profits which has already been committed to in the previous period. The time lag between the borrowing of working capital and its use plays a critical role in the GS Model. When profits are strong, firms are encouraged to increase their borrowing of working capital to hire and pay new workers, increasing real wages. An increased wage bill, however when profits are below anticipated levels has persistent effects. If profits at time $t$ are below what was anticipated when the capital was borrowed to pay wages, it may create a situation where higher real wages lead to lower profits. Thus cycles in the GS model arise because of the combination of sufficiently large distribution of profits with greater investment leading to greater real wages. Greater real wages reduces profits, decreasing equity, especially when realized firm specific prices are low, leading to a decline in output... to wit, cycles. But why should more productive technology stabilize the economy? The reason appears to lie with the implied labour - technology trade-off arising from alternative levels of productivity. We understand that GS model cycles are driven by the lag and uncertainty with regard to borrowing working capital and using it to pay real wages. Under such conditions, greater growth and output will raise real wages when realized profits are below anticipated levels. The fall in profits leads to less borrowing of working capital and investment, reducing real wages in the next period. If the technology level of an economy were very productive, achieving the same level of output with less labour would be possible. Given the level of equity, with better technology,
the same level of output may be achieved with less resort to borrowing. Reducing the need to borrow working capital in order to pay a wage bill because of more productive technology, lessens the threat of raising real wages possibly when it is unsustainable. In contrast, an economy with a low level of productive technology must borrow that much more in working capital to pay its wage bill, increasing real wages and entering into commitments which may not be sustainable if the economy turns sour. For these reasons, we see that with more productive technology, the economy is more stable; and with less productive technology, the economy is less stable, as shown above by comparing Figures 32 and 33. Although more productive technology does increase the return to borrowed working capital, it would appear that the labour productivity trade-off described above appears to dominate, as shown in the Figures. By adding the technology coefficient to the production function found in the GS Model, specifying it explicitly and including it in the dynamic simulations, we add to knowledge about the dynamics of such New Keynesian models and show its relevance to other contemporary research.

The results, for example, may explain why a highly productive economy such as the United States appears to exhibit less oscillations from around trend, while the labour intensive economies, for example, in the developing world are more prone to cyclic oscillations. A further implication of the above results would be that cycles in a developing country with low levels of productivity have endogenous sources which may be in addition to or interact with exogenous shocks. Such observations relating to how levels of productivity influence economic stability have counterparts in contemporary research. Referring to developing economies, according to the research summary on the topic of stability and aggregate economic activity found in Lal & Myint (page 216, 1996); "...instability in growth rates does not necessarily lead to poor overall growth
performance." The technology parameter results are also interesting from the standpoint of the New Classical Kydland-Prescott "technology shocks" literature, as it allows one to consider the effect of a "shock" or perturbation in the level of technology, but here applied in a New Keynesian setting, involving micro-market imperfections and endogenous cycles. In a financial market imperfection New Keynesian model, such shocks might have asymmetrical effects. A "shock" to a New Keynesian economy of the GS formulation, involving an increase in productivity, might render the economy more stable, as it reduces the importance of the borrowing-investment - real wage bill - profit spiral. While if the productivity of technology suddenly fell, necessitating more labour intensive technology to sustain output, the economy may be rendered less stable and more prone to cyclic oscillations of greater amplitude. We see from these results that including the technology parameter in the simulated GS model provides several insights with respect to the dynamic behaviour of this New Keynesian model.
4.9 Cycles, Chaos and Dimensionality in a Non-Linear System

In the last section of Chapter Four we examine further the rich dynamics found in the GS style model which has been modelled and simulated using the computer program. As explained in Section 4.6 above, analysis relying upon linearization around the steady state, as adopted by GS in their article has its uses, but ignores such possibilities as sign changes in the phase line, multiple equilibria, the importance of initial conditions, and such phenomenon as bifurcation and chaos. In addition, in Section 4.9 we address the question of order and dimensionality in the GS model, considering whether it is properly first order, and if not what the implications are for dynamic behaviour. We begin by enlarging upon our previous discussion of the dynamics of non-linear first-order difference equations. Whereafter, we apply our remarks to the dynamics generated by simulation of the GS style model.

In Section 4.6, we noted the stability conditions for steady-state equilibrium are that the slope of the phase line in absolute value as evaluated for the derivative, $G'$, as per equation 4.31 above, must be less than unity at the steady-state. An unstable trajectory arises if the absolute value of the derivative, $G'$, as per equation 4.31 above, is greater than unity at that point. While this result is useful for qualitative analysis of linearized systems, as it allows one to determine the existence and assess the stability of a steady-state equilibrium; it has its limitations. Firstly, as a steady-state equilibrium, it represents only local stability not global stability. Secondly, linearization ignores a model's complex dynamics, in particular such phenomenon as period doubling (bifurcations), aperiodic trajectories and chaos. Further, linearization ignores the importance of initial conditions outside the neighbourhood of the steady-state. For example, a system may display oscillatory convergence to a steady-state point, call it $a^*$, from a point in the neighbourhood of it, but may not necessarily converge to it from all
initial conditions. Deciding whether linearization is justified, however, is complicated by the fact that no test exists for global stability exists (Hoy, page 668, 1996). Linearization around the steady-state of non-linear first-order difference equation, as appealed to by GS, obscures such possibilities.

To expand further upon this theme, a key problem with linearization around a steady-state equilibrium is that it obscures the possibility of sign changes in the derivative, $G'$, found in expression 4.31 of the non-linear difference equation, 4.29. A change in sign may lead to very complex dynamics where the system displays both oscillatory and monotonic phase line segments (Hoy, page 674, 1996), as we first showed above in Figure 20. Non-linear difference equations which produce parabola shapes may display dynamic behaviour, such as cycles which repeat themselves after two or more periods, and even ergodic chaos, in which there is no apparent regularity in the behaviour of the trajectory, as we saw. Such phase lines which changes slope may display a peak either to right or to the left of the 45° axis. An example of the former we showed in the stylized Phase Diagram of Figure 20 in Section 4.6, which begins with monotonic convergence, followed by multiple period oscillations in the neighbourhood of the steady state. In Figure 20 of Section 4.6, we saw it took sixteen iterations until the cycle was completed. One may also construct other non-linear phase lines exhibiting oscillations which never converge to the steady-state, but instead oscillate within a bounded range or limit cycle of two periods or longer. Dynamic simulation of the GS Model, may generate plots in phase space featuring limit cycles or alternatively plots which do not display cycles of any length but remain bounded. When such orbits and trajectories display no regularity and are very sensitive to changes in initial values $a_0$, the trajectory may be chaotic, (page 42, Baker &
A final reason why linearization is inadequate relates to the non-invertibility of the map generated by $G(a_t)$. As may be shown from our description of the GS system, although, apart from the variability in firm specific prices (as we explain below), one may predict $a_{t+1}$ from $a_t$, there is ambiguity in trying to retrodict $a_t$ from $a_{t+1}$. A necessary condition for any one dimensional map to exhibit chaotic behaviour is that it be non-invertible (page 90, Baker & Gollub, 1990).\[11\] As a useful illustration, such chaotic trajectories may occur with the so-called tent-map, as explained below.

We can see from Figure 20 of Section 4.6 how the orbit around the steady-state would be very sensitive to changes in the initial condition on equity. The complex dynamics arose from the parabolic shape described by the phase line. In the so-called tent-map we see even greater sensitivity to initial conditions in which chaos is readily observed, as shown below in Figure 34. Analytically, the tent family may be said to consist of functions $T_\mu$ defined accordingly:

\[
\begin{align*}
T_\mu(a_t) &= 2\mu a_t \text{ for } 0 < a_t < 0.5 \\
T_\mu(a_t) &= 2\mu (1-a_t) \text{ for } 0.5 < a_t < 1
\end{align*}
\]

As may be shown, increasing $\mu$ lifts the height of the tent graph. In Figure 34, the represented tent graph has a $\mu$ of 5/6 and has two fixed points, one at zero and the second where the phase line crosses the $45^\circ$ line. For a given tent style phase line, with piece-wise linear segments, depending upon from where the orbit begins, very different trajectories may be plotted, with varying number of periods per cycle.

\[10\] We are referring to observable or ergodic chaos as contrasted with topological chaos in that we consider the fraction of all possible initial conditions generating chaotic orbits (Li & Yorke, 1975). See, pages 106-107, Azariadis, 1994.

\[11\] See Footnote 6 above on the definitions and conditions for chaos based on Kelsey (1988).
Other phase lines of the tent style may display not one but many sharp inflection points, with consequently many fixed points, that is where the $45^\circ$ is crossed (Chapter 2, D. Gulick, 1992). Stretching and folding, as is found in the tent style map, is common to many non-linear difference equations and is associated with chaotic behaviour, and the sensitivity of trajectories to initial conditions (pages 33-36, D. Gulick, 1992). From consideration of such phenomenon, we see that linearization around the steady-state obscures such possibilities as global instability, multiple steady-state equilibria, and chaos (page 91, D. Gulick, 1992). Deterministic dynamical systems can generate chaotic dynamics, appearing very irregular (Boldrin & Woodford, 1990). Phase portraits with sharp inflection points, as the tent map, are useful in understanding the dynamic behaviour derived from simulating the GS style model, because the plot of $G(a)$ is not smooth and changes signs. The method developed in Sections 4.7 and 4.8 for simulation of the non-linear difference equation will allow us to examine some aspects of the non-linear, dynamic behaviour found in the specified version of the GS model. Below, we will show that the simulated GS model is...
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capable of generating, such dynamics as aperiodic chaotic orbits around a steady state and sensitivity to initial conditions.

A further set of observations about the dynamics of non-linear phase which will assist us in interpreting the simulated results concerns bifurcations of dynamic trajectories. Formally, we can describe bifurcations accordingly: If we consider a family of dynamical systems as those generated by changing the initial values or exogenous parameters as found for example, in equation 4.31 above, we have the following general form:

\[ x_{t+1} = F(x_t; \alpha) \]

\[ F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \]

(4.36)

Equation 4.36 is indexed upon the parameter \( \alpha \in \Omega \). As \( \alpha \) varies, so will the dynamic solutions of the system. Small changes in initial values or other parameters may not affect the qualitative structure of the system's orbits, although as we showed in Section 4.8, for sufficiently large changes, qualitative changes in dynamics will occur. When sufficiently large changes which lead to qualitative changes in the system's orbit structure and dynamic trajectory, such as the periodicity of the cycle doubling, a bifurcation is said to have occurred. Bifurcations of dynamic trajectories arising from changes in structural parameters or initial values, may produce changes in the number of steady states, and the nature of orbits in the neighbourhood of a given equilibrium. Even if a steady-state solution continues to exist and respond smoothly to changes in structural parameters, as the exogenous variables used for simulation in Section 4.8, there can still be changes in the type of steady-state equilibria. Whether such changes occur, depends upon the modulus of its eigenvalues (Azariadis, page 63, 1994). Three well known types of bifurcations are the following:
\begin{itemize}
  \item The Saddle-Node Bifurcation;
  \item The Flip Bifurcation; and
  \item The Hopf Bifurcation.
\end{itemize}

Illustrating them, in Figures 35, 36, and 37 we show some stylized examples of these bifurcations involving one-parameter families of discrete systems. The Saddle-Node Bifurcation of Figure 35, we see two hyperbolic equilibria, one stable and one not, at the bifurcation point $a^0$, producing a fold pattern. The Flip Bifurcation of Figure 36 has a single real eigenvalue on the boundary of the unit circle with value of -1. For small perturbations to the parameter $a$, the stability type of the dynamical system changes. In Figure 37 we illustrate the Hopf Bifurcation which arises when eigenvalues are complex conjugates with modulus 1.\(^\text{12}\) As we have seen from plots of the trajectories of the GS model, changes in the number of periods per cycle may occur, with a change in a parameter value. Such phenomenon will be shown below in the phase portraits arising from further simulation.

\(^\text{12}\)That is, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ with $det = \lambda_1 \lambda_2 = a^2 + b^2 = |\lambda|^2 = 1$ which requires that $|a| < 1$, and therefore that the trace $= \lambda_1 + \lambda_2 = 2a \in (-2,2)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figs.png}
\caption{Fig. 35 Saddle Node Bifurcation}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figs.png}
\caption{Fig. 36 The Flip Bifurcation}
\end{figure}
As presented, the GS model is a first order autonomous difference equation system, which because of non-linearity may exhibit quite complicated dynamics. In the 1993 article, GS present their difference equation as the following, from equation 4.29:

$$a_{t+1} = G(a_t)$$

(4.37)

But is this an accurate specification of their system? We know that the functional relationship $G(a_t)$ is non-invertible, but are there other reasons why a non unique mapping does not exist. In the GS model, no unique transformation from $a_t$ to $a_{t+1}$ exists because of the variability in firms specific prices, $\epsilon_t$, making another reason why it cannot be invertible. Properly understood, the GS system as modelled in the simulation program, is really a two dimensional system, which may be specified in general form as the following:

$$a_{t+1} = G(a_t, \epsilon_t)$$

(4.38)

In the GS model, because of the exogenous stochastic parameter, $\epsilon_t$, the phase plot cannot lie on a single curve. No unique transformation from time $t$ to time $t+1$ is possible. The random parameter affects
the convergence to the steady-state. Although there may exist a $G(a^*, u')$ such that $u'$ has no effect, this would appear unlikely as we still have a random perturbation capable of disturbing the steady-state.\footnote{Investigation of such possibilities would require at least double precision (fourteen decimal places) and many more iterations than undertaken for simulation of the GS model.} In such circumstances to speak of the behaviour of the GS model as exhibiting either periodic cycles or chaotic cycles is only half the story. The presence of a \textit{random attractor} in the form of firm specific price variability means that the GS system should behave like a second order system, rather than one which is first order. Support for this point is found in the 1988 GS article where, in the version of the model having firm specific prices and dividends net of new equity sales exogenous and random, the authors remark, "...the behaviour of the model is described by a third-order nonlinear difference equation." (page 118, 1988). Replacing the random behaviour of dividends from the model and using a time invariant constant as we have, still leaves greater dimensionality than would be implied if the model were first order and had no stochastic parameters. Presenting simulations of the GS model as dynamic trajectories over time obscures the higher dimensionality of the GS system. Properly a vector space of $\mathbb{R}^3$ is necessary to show the complete picture, although graphing the iterates of the simulated GS style model $a_t$ against $a_{t+1}$, as appear below in the phase portraits below is very useful, as they show that no unique transformation from time $t$ to time $t+1$ exists. As we will see, without changing parameters, allowing only random perturbations to the system, should produce trajectories resembling a deterministic system in which a range of initial parameter values had been selected (page 12, Kelsey, 1988). Without changing initial parameter values, such as equity and interest rates, through repeated random perturbations should generate a cyclic orbit which is broad band rather than points. The points at which
changes in parameter values will produce different periodicities, points of bifurcation, will also be less evident. Finally, we can expect by specifying the random attractor, chaotic behaviour should become more common in simulation results (ibid, 1988).

The above observations show that the variability in firm specific prices in the GS model works in two ways, as we will reveal through simulation. On the one hand, firm specific price variability through marginal bankruptcy cost enhances the convergent qualities of the GS style system, while on the other hand, it increases the variability of orbits, should they arise. In addition it increases the chances of chaotic behaviour. As the variability in random firm specific price perturbations is increased, chaotic behaviour can be expected to be more common (page 14, Kelsey, 1988). The interplay in the GS model between endogenous propagation mechanism for cycles working in conjunction with random exogenous perturbations, adding energy to the system, in the form of firm specific price variability, harkens back to the works of Frisch (1933), as mentioned in Chapter Two.

Having set the stage with the above, we consider the complex dynamics found in the simulated version of the GS style model by examining several figures in which the *orbits* are plotted in phase space. The depiction of orbits such appears in the literature and are referred to as a *phase-portrait* (see, S. Krasner, editor, Chapter 11, "Chaos and the Business Cycle", *The Ubiquity of Chaos*, 1990 or A. Mullineux, et al. Chapter 5, *Business Cycles*, 1993). Such portraits are useful in seeing the role of random attractors on the behaviour of the GS system and its effective dimensionality. The orbits are the plotted iterates generated by simulation, through the model, of the difference equation 4.29 and resemble the phase line, shown above in Figure 20. We observe the richness of the model's dynamics. A given non-linear phase line, itself, may display various periodic structures depending upon initial conditions,
as discussed above with reference to the tent family of functions. With a phase diagram having sufficiently sharp and high inflection, using the simulation program, we can show that by varying the initial conditions on various parameters, a great variety of phase portraits may be constructed, often with quite different dynamic behaviour. In fact, a non-linear phase line exhibiting ergodic chaos, possesses an infinite variety of different cycles, according to Sarkovskii’s 1964 theorem (see, Collet & Eckmann, 1980). As we show below, for very small changes, for example, in the variability of firm specific prices, dividend policy, affects the model’s dynamic behaviour sharply. In addition to the chaotic dynamics found in the GS model, its sensitivity to initial conditions and aperiodic trajectories, we have the presence of random inputs. Within the context of the GS model, even holding initial conditions the same, the orbit may be changed by exposing the populations of firms to new random and individual price perturbations, further illustrating the effect of random attractors. To illustrate such observations, we show the results of seven simulations using the phase portrait-orbit technique which are summarized in the following table, Exhibit 4.3. (We note, that in each phase-portrait, for reference, the 45° is shown.

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>TECH.</th>
<th>DIVID.</th>
<th>Debt/Equity</th>
<th>SIGMA, P^i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 38</td>
<td>4</td>
<td>10%</td>
<td>3.75</td>
<td>11%</td>
</tr>
<tr>
<td>Figure 39</td>
<td>4</td>
<td>8%</td>
<td>3.75</td>
<td>11%</td>
</tr>
<tr>
<td>Figure 40</td>
<td>4</td>
<td>8%</td>
<td>4.0</td>
<td>11%</td>
</tr>
<tr>
<td>Figure 41</td>
<td>4.5</td>
<td>8%</td>
<td>4.0</td>
<td>11%</td>
</tr>
<tr>
<td>Figure 42</td>
<td>4.5</td>
<td>8%</td>
<td>4.0</td>
<td>8%</td>
</tr>
<tr>
<td>Figure 43</td>
<td>4.5</td>
<td>9%</td>
<td>4.0</td>
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</tr>
<tr>
<td>Figure 44</td>
<td>4.5</td>
<td>10%</td>
<td>4.0</td>
<td>6%</td>
</tr>
</tbody>
</table>

Exhibit 4.4
Looking at the first two sets of diagrams, Figures 38 and 39, we see that the orbits display quite chaotic dynamics, not behaving in a regular manner. Comparing the two, decreasing the amount of profit distribution over time, from 10% to 8%, that is allowing each of the assumed population of twenty-five firms to retain a greater percentage of profits over time, in the aggregate, leads to slower convergence to the steady-state. As supposed by GS, the result occurs because with greater equity, firms are encouraged to be less risk averse, favouring borrowing and investment, increasing exposure to the business cycle. Both phase portraits show that while the first argument in $G(a_j, u_j)$ is convergent, the orbits retain their random component, because of $u_j$. Holding everything else the same except the initial level of equity and the implied ratio of debt to equity, we see in Figure 40, that when the initial condition on the amount of equity, and the ratio of debt to equity is increased, the orbits are tighter and the tendency to convergence is greater because of the greater threat of bankruptcy from more debt in the capital structure. Although the capital structure will vary over time with profitability and dividend retention, changing the initial conditions with respect to having greater debt in the capital structure, has a persistent effect because it determines from where the dynamic trajectory begins. As we have seen, very different dynamic behaviour can arise from such small changes. By increasing the threat of bankruptcy arising from firm specific prices being below the threshold level for solvency, a dampening of the orbits occurs, while still leaving them subject to the random component shown in equation 4.38. Risk aversion, leads firms to reduce their borrowing of working capital to fund the wage bill, with the result that the system is stabilized and the convergence to the steady-state is hastened. Dynamics, we see, are dependent upon both initial conditions and the random input, $u_j$. In Figure 41, again holding other parameters constant, we change the level of technology used by the population of twenty-five firms from a value
of 4 to a value of 4.5. As discussed in Section 4.8 above, with better technology, firms are less labour intensive. With a smaller exposure to the wage-profit spiral as discussed in Chapter Three, the economy is more stable, converging to a steady-state much more quickly. Like other changes in state variables, reducing the variability of firm specific prices, as shown in Figures 42 and 43, has the effect of making the economy less stable. Uncertainty is reduced with regard to the returns on borrowed working capital, encouraging investment and increasing exposure to cyclic effects. In these figures, rather than convergence to a steady-state, we see a limit cycles which behave randomly because of the stochastic component of expression 4.38. With reduced firm specific price variability, the threat of bankruptcy as arising from the relationship between firm specific prices and the threshold level, as per equations 4.15 and 4.15', is lessened, encouraging firms to expand borrowing and invest in working capital to pay wages, thereby accumulating equity and increasing output. Figures 42 and 43 only differ by the percentage of profits distributed as dividends. We see that by changing dividends from 8% of profits to 9% of profits, a very small change, the dynamics are affected. As per our earlier observation of the effect of increasing equity through retaining profits continues to hold: It lessens the stability of the GS style model, because with greater equity accumulation over time, borrowing and investing is encouraged, increasing the exposure to business cycle effects. The stochastic component of equation 4.38, in this simulation continues in Figure 43 to make the orbits random. In the final Figure, 44, we increase dividends while at the same time decreasing the risk surrounding firm specific prices, $u'$ as an initial condition. By reducing the variability in firm specific price still further to 6%, risk aversion to bankruptcy is reduced, encouraging the borrowing of working capital and investment. Such practices increase the exposure to the business cycle dynamics. Increasing dividend
distribution from profits over time, however, is stabilizing to firm output, because with less equity, borrowing working capital and investment is discouraged, creating greater risk of bankruptcy. As we see comparing Figures 43 and 44, by a small change to an initial condition through adjusting a dividend policy and firm specific price variability, the number of periods per cycle may be changed. Combined and sometimes amplified by the random component of equation 4.38, we see small changes in initial conditions have a large effect upon dynamic trajectories and orbits. We see that the phase portrait now does not display convergence, but rather appears to oscillate in a two or three period cycle around the steady state. As observed, because of the random component, the amplitude in fluctuations remains. In an economy in which the marginal cost of bankruptcy enters into the firm’s optimal decision making, reducing risk has the potential for increasing oscillatory behaviour, while at the same time, in the GS system, decreasing the role of the random attractor. The results serve to further validate GS Proposition 2, (page 92, 1993) that increased uncertainty leads to lower investment.

Figure 38. Tech. = 4, Divid. = 10%, Initial D/E = 3.75, Sigma = 11%
Figure 39. Tech. = 4, Divid. = 8\%, Initial D/E = 3.75, Sigma = 11\%

Figure 40. Tech. = 4, Divid. = 8\%, Initial D/E = 4, Sigma = 11\%
Figure 41. Tech. = 4.5, Divid. = 8%, Initial D/E = 4, Sigma = 11%

Figure 42. Tech. = 4.5, Divid. = 8%, Initial D/E = 4, Sigma = 8%
In comparing the phase portraits, resulting from simulations of the GS style model, we see that numerical simulation rather than linearization around the steady-state, is needed to reveal the effects of combining complex non-linear dynamics with higher dimensionality, arising from a random attractor. Through simulations involving at a minimum single precision (seven decimal points), and many more
iterations, such dynamics could be investigated in depth. The simulation program has allowed us to study the dynamics of a GS style model with respect to sensitivity of initial conditions, changing periodicity per cycles, aperiodic cycles or chaos, and the effect of greater dimensionality in the form of a random input. As we have observed through simulation, for example, very small changes in the assumed parameter for the variability of firm specific prices which enters into the firm optimisation problem through the marginal bankruptcy aspect of the cost function, has the effect of changing the dynamic behaviour of the model. Higher dimensionality ensures, however, that the relationship between a given $a_t$ and $a_{t+1}$ is not a bijection. Even if, for example, $a_5 = 10$ and $a_{15} = 10$, there is no reason to believe that $a_6 = a_{16}$. In addition to the dimensionality issue, we have also seen that other small changes in initial conditions have large effects upon dynamic behaviour.

Chaotic dynamics in which the orbits are either acyclic or aperiodic, we have shown, can arise in the GS model. Orbits around the steady state(s) $(da/dt = 0)$, we have seen are sensitive to initial conditions. Although there are necessary conditions for chaotic trajectories to occur in first order non-linear systems, non-invertibility as discussed above and in Section 4.6; there are no sufficient conditions telling when chaotic solutions may arise. Hence, numerical simulation is necessary in order to learn the laws of motion for such systems (Azariadis, page 107, 1994 and Ott, 1993). As we have learned from the above discussion, simulation of the GS system, generates phase portraits or orbits displaying, at times, aperiodic behaviour which is influenced by the presence of random attractors, as per expression 4.38. The dynamics of the implemented GS style system, may appear random because of the non-linear dynamics producing chaotic iterations as well as the truly random processes arising from firm specific price variability. If not for the random attractor, the dynamics of the GS model would be completely
determined by the difference equation found in equation 4.29 and simulated through the model. As we know, non-stochastic, non-linear dynamics may produce chaotic orbits, and display sensitivity to small changes of initial conditions. Changes in initial conditions, change the periodicity of the GS style model's dynamics, as we have seen, by varying the number of periods per cycle. Although GS depict their cycles as four-period cycles (see, GS Figure VI, page 99, 1993), we have shown through simulation of the model, that a greater number of iterations per cycle may occur, confirming our earlier observations in Section 4.6 on multiple-periodicity. Both convergent and divergent trajectories and orbits may be created as well those having a chaotic nature occur, by varying the parameters of the simulated GS model. From the above we can see that the inter-temporal behaviour the GS model is significantly richer than as supposed in their article. It may display chaotic dynamics and behave according to greater dimensionality than first order.
CHAPTER FIVE. STATE CONTINGENT ASSETS IN A MODEL WITH FINANCIAL IMPERFECTIONS

5.0 Introduction

In this chapter we undertake modifications to the simulation program explained in Chapter Four to address the questions raised in Chapter Three of whether and how the existence and use of state contingent assets may effect business cycles. We build upon the results of Chapter Four where we saw how a GS style Model may be specified in a computer simulation program. In order to simulate the model, a population of twenty-five firms having initially the same level of equity was assumed. Individual firms were exposed to unique price shocks, over-time, which were assumed to be normally distributed. As per the GS model, the probability of bankruptcy was determined by the relationship between firm specific price realizations, $u_i^{t+1}$, and the threshold level, $v_i^{t+1}$. Bankruptcy occurs when $u_i^{t+1}$ is below $v_i^{t+1}$. We saw through its affect upon the dimensionality of the model, that random firm specific price variability, has the effect of making a first order model behave as one which is of a second order. We further assumed for purposes of simulation, given the relatively small population of firms, that firms are re-floated when bankruptcy occurs, as explained in Chapter Four. To simulate a version of the GS style model, a variability in firm specific prices was assumed. Lastly, in order to learn about the effects of dividend structure upon the model's dynamics, and reduce the dimensionality of the system, the stochastic approach to profit distribution assumed in the GS 1993 article was replaced with a fixed rate of profit retention. Using these assumptions in the specified model, experiments were performed upon exogenous variables to verify the stylized observations made in the GS article relating to comparative statics. In addition, the informal remarks made by GS in their article regarding dynamics were investigated, shedding light upon the model's complex inter-temporal behaviour. Importantly,
using simulation, experiments performed to investigate the model's non-linear dynamics, showed the claim by GS (page 100, 1993) that the model may exhibit cycles of multiple periodicity, is subject to many qualifications. Using certain initial parameter values, cycles of varying periodicity, along with chaotic behaviour, were observed. Numerical analysis showed that the dynamic behaviour around the steady-state solution(s) were particularly sensitive to assumptions on initial conditions for exogenous parameters, including the level of equity, the variability in firm specific prices, the level of technology and the dividend pay-out ratio. Finally by plotting the phase portraits of the simulated GS model, along with related discussion, it was shown that because of random attractors, the dynamic behaviour can best be described as a second order system rather than one which is first order, as it is presented. Building upon the above results we turn to Chapter Five.

In Chapter Five, we begin by explaining modifications to the GS model from an abstract perspective, discussing state contingent assets and risk management. In the next section, we discuss and review both the theoretical basis for risk management, along with specification of the functional forms required for its modelling in the simulation program. In Section 5.3, we turn to the pricing of derivatives, a requirement of the previous sections's risk management specification. Having specified the modified GS style model with the risk management module, in Sections 5.4, and 5.5 computer simulation is used, like Chapter Four, to determine the modified model's comparative static and dynamic behaviour. The results are discussed in Section 5.6. The exogenous variables investigated include financial structure, interest rates, dividend policy, and firm specific price variability. Experiments with regard to initial conditions upon the trajectory reveal that the laws of motion with regard to the wage-dividend spiral found in the GS model are modified when risk management is available to perfect capital markets. The random
attractor firm specific price variability, is particularly interesting in the modified model, as adjusting exposure through risk management, relates directly to the model’s dynamics. Like Chapter Four, we use numerical simulations rather than inspection of the analytic results of reduced form equations to learn how the model’s laws of motion are effected by changes in both initial conditions and constant.

5.1 Risk Management in the Abstract

While explicitly specifying and simulating a GS style model has provided useful insights into its dynamic behaviour, it is useful to recall our original purpose in selecting it. Embodying a New Keynesian perspective in the form of imperfect capital markets, the GS model displays a financial constraint in the availability of new equity and a capital market imperfection: Namely, that the cost of borrowed working capital is not performance related but at a fixed rate of interest. Faced with uncertainty about the firm specific prices to be received and whether they will exceed the threshold level to avoid bankruptcy, individual agents borrow working capital to pay workers, unsure of its future return. Further, it is explicitly assumed by GS in their model that futures markets, or other means of hedging do not exist (page 79, 1993). The combination of the constraint, the imperfection and the assumption of no state contingent assets, options or futures to manage risks, are together the sources of the comparative statics and interesting dynamics investigated in Chapter Four. But what if state contingent assets were introduced into the model in order to manage risk, how would the model behave? Or, more broadly, what are the macroeconomic implications, if any, of risk management? Paraphrasing Bacchetta and Caminal (page 3, 1996), that financial factors matter for individual firms does not imply that they matter at the macroeconomic level. We return to the different schools of thought related to risk management or hedging, first discussed in Chapter Three, in order to see whether and how,
from a micro perspective they may be incorporated within NK assumptions used in the GS model and thereby investigate their macroeconomic implications, if any.

In Chapter Three, we found that risk management has many proponents. According to some authorities, using derivatives for risk management is beneficial to the firm and moreover, through simple aggregation, must be beneficial to the economy. Surprisingly, however, according to other authorities, an accepted and received perspective on why firms hedge, does not exist (page 1216, Leland, 1998). Without understanding the use of derivatives at the micro-theoretic level, it is doubtful whether sense may be made of its macro implications. Looking at these micro-theoretic explanations, we find that in the earliest schools of thought, risk management using derivative markets was likened to insurance, or properly risk transfer. In this school of thought, futures markets and options on futures markets exist in order to price and sell systematic risk between risk averse hedgers and risk loving speculators. A fully hedged position represents the purchase of certainty equivalence using futures or options. This school of thought has a long lineage, dating at least to Keynes (1923, 1930), and later reinforced by Kaldor (1940), and Hicks (1946). In the Keynes/Hicks perspective, the requirements for derivative markets are three fold: Firstly, there must be risk averse hedgers; secondly, in order to transfer risk, there must be sufficient correlation between futures/options prices and spot prices, i.e. the basis must not be too large; and thirdly, that there be sufficient variability to make hedging and speculation justified (Houthakker, New Palgrave Dictionary of Economics, 1987). The essence of the Keynes/Hicks perspective is two-fold. Firstly, the notion that futures markets are biased downwards with regard to estimates of prices in the future to reflect the premium earned by the speculator for bearing risk; and secondly, that futures markets behave systemically vis-a-vis the greater securities market.
Although the Keynes/Hicks perspective offers an elegant and appealing paradigm; it is inadequate for several reasons. In order to identify speculation with the acceptance of a risk premium, futures prices along with options on futures prices must be correlated systematically with the market portfolio. If this were the case, then speculation would represent the acceptance of *systematic* or market risk in exchange for a risk premia.\(^1\) Evidence of futures markets or options on futures markets being correlated to a market portfolio, however, is very ambiguous (Dusak, 1973). A further requirement of the Keynes/Hicks paradigm is that aggregate hedging positions must be short and aggregate speculative positions long, for which there is no evidence. It is impossible, moreover, to identify the use of a particular product, for example buying futures, with taking a particular market position, such as *hedging* or *speculating* (Williams, 1986). What derivative product is used, and how it is used are not necessarily implied by an underlying activity, such as hedging versus speculating (Haar, 1998). In the aggregate, hedgers need not be short. In fact, long asset positions may be hedged through both buying and selling derivatives. Furthermore, participants may operate across products and markets, writing options in a foreign exchange market and selling options in an agricultural commodity market; or finding arbitrage between futures against synthetic futures created out of options. Altogether such observations undermine a paradigm of futures markets based upon the Keynes/Hicks perspective.

The paradigm of Keynes/Hicks presents problems from the standpoint of other aspects of received financial theory, notably the Capital Asset Pricing Model (CAPM) and Modigliani-Miller (MM) capital structure Theory. From the perspective of such theories, the hedging and risk transference view cannot be explained readily: From

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\(^1\) In fairness, the distinction between systematic risk and idiosyncratic risk found in the Finance literature gained currency several decades after the Keynes-Hicks contributions to understanding futures markets.
a Capital Asset Pricing perspective, if the risk being hedged, presumably through the purchase of a derivative, were purely market or systematic risk, a movement along the security market line, as it trades-off risk and return, does not change the market capitalization of the user, and hence generates no gain to shareholders, leaving the question- why hedge? From the standpoint of MM capital structure theory, the micro-theoretic motivation to use derivatives is also troubling. Derivatives, because they are a form of lending and borrowing asset positions, may be utilized to change the composition of capital structure. Market capitalization, according to the theory, adjusted for interest rate deductibility and non-neutrality of taxes, however, should be invariant to changes in the composition of capital structure. In light of MM theory, we again face the question- why hedge? From the perspective of received financial theory, in equilibrium, hedging, as a means of risk transference should not occur as no gains to shareholders arise. Such a conclusion suggests an alternative view that hedging is not about movements along the security market line but a means by which firms remain at a point along the line, by ensuring that they generate returns commensurate with their market risk.²

Interest in reconciling risk management or hedging with the MM Theory on the irrelevance of financial structure policy, has led to other paradigms to explain hedging, notably the research by Johnson (1960), Schrock (1971), and Jerome Stein (1986). Their contributions have relied principally upon portfolio theory to justify hedging as a risk off-setting activity. These critics of the Keynes-Hicks model reject the risk transference perspective in favour of a diversification perspective. Related justifications for hedging include the convexity of tax schedules, the minimization of the expected

²In this context, the article "Is the Risk of Bankruptcy a Systemic Risk" by I.D. Dicheu, The Journal of Finance, vol 53, No. 3, June 1998, is of interest.
costs of financial distress, or even to reduce stockholder-bondholder conflict (Mayers and Smith, 1982; Smith and Stulz, 1985). In Froot, Sharfstein, and Stein (1993), the purpose of hedging is to reduce the cost of external financing. Adopting a portfolio approach to modelling futures markets, they argue that asymmetries with respect to risk tolerance arise because of the different cost of internal versus external finance. In other research, managerial risk aversion is seen as a motive in both Smith and Stulz (1985) and Tufano (1996), although it is doubtful whether adding a quasi-utility theory to financial issues has much practical application. Hedging leads to greater leverage and greater tax advantages in Ross and Otto (1996). Although such models provide insights under limited circumstances, none of them are sufficiently robust to provide a general theory of why futures markets exist. Portfolio theory has intuitive appeal as means of explaining hedging, however, it requires, as a means of diversification, that derivative and underlying asset positions to be inversely correlated and to constitute non-diversifiable, idiosyncratic risk. Systematic risk itself cannot be diversified. Benefits to diversification only arise through the combination of assets which are inversely correlated from an idiosyncratic standpoint. Market risk, itself, cannot be diversified away. For hedging to be a form of portfolio diversification would imply that derivatives, options or futures, behave idiosyncratically vis-a-vis the market. Like our observations above on systematic behaviour above, there is no evidence to this effect (Marcus & Modest, 1983). Moreover returning to MM theory, even if derivative instruments behaved idiosyncratically, investors may just as easily purchase and use such instruments themselves to modify exposure. Why should owners of equity *reward* or place any positive value on what they may do for themselves as easily? Based upon the above, risk management within a corporation, as opposed to investor level, would appear to have no merit. In addition to the theoretical short-comings posed by
most models of future markets, the empirical support for any of the above paradigms is weak (Mian, 1996). Summarising the above, it is not surprising that the macro economic implications of derivatives and risk management are inadequately understood.

A third paradigm regards futures and options on futures markets as a form of inventory management and has been vigorously promoted by Williams (1982, 1984, 1994). Rejecting the risk management paradigm, Williams builds upon the ignored research of Halbrooke Working (1948, 1949, 1953a, 1953b, 1962). Rather than risk transference or the portfolio approach, a not necessarily incompatible third paradigm describes futures markets and options on futures markets as means of borrowing and lending inventory positions to ensure profitable utilisation of assets. According to Williams, futures markets are a means of improving inter-temporal asset allocation. Williams begins from the perspective of why firms hold inventories. In his view, precautionary demand along with transaction demand explains why businesses may use derivatives as a form of synthetic inventory borrowing. In his line of reasoning, derivatives are a form of inventory management, the purpose of which is to ensure good asset utilization, improving profits and increasing output. By improving inter-temporal asset allocation, firms may increase outputs and profits. In this light, like the carrying cost arising from physical storage of inventory, a derivatives position, as a form of lending or borrowing inventory has an implied interest cost in terms of the commodity itself, known as the basis spread. Acknowledging the observation first made by Sraffa (1932) on commodity specific rates of interest, Williams argues that the basis in futures markets, that is the difference between the cash price and future delivery price implies a rate of interest which is related to the

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opportunity cost of physical storage and cannot necessarily be identified with a risk premia.

In the model developed by Williams, derivative markets exist to borrow and lend positions, be they in commodities, foreign exchange, equities, or Treasury Bills. According to Williams, the motives for holding an inventory position explains the use of derivatives for hedging. His model also gives insight into the depth and breadth of futures and options on futures markets. In his model, economic viability determines the extensiveness of derivative markets. Markets do not fail, rather at times, they may be uneconomic. For example, if the opportunity cost of physical storage were very low, having access to synthetic storage via futures or options, may not be justified. Hence, a market for all state contingent assets, at all times, is not necessarily optimal. It should be noted that the William’s model is compatible with CAPM and MM theory, unlike other theories of futures markets: Combining a long position in options combined with a short position in the underlying security or commodity, for example, may be used to produce a risk free rate of return. Empirical investigation of a model based upon the William’s paradigm waits to be done. Like other theories of futures markets, generating testable propositions based upon the William’s paradigm remains non-trivial. Evidently, a general theory, capable of reconciling the different theoretical perspectives has yet to be created and tested. (Lelland, 1998). Nonetheless, we will argue below that the William’s paradigm is a source of useful insights relevant to explaining the macroeconomic impact of risk management in a NK business cycle model.

5.2 Risk Management in a GS Style Model

With the above background on risk management in mind, we turn to how it may be incorporated into the GS style model presented in Chapter Four. In this section we discuss specification of the risk
management module along with its potential impact from the standpoint of the model’s comparative statics and dynamic behaviour. We discuss as well what the modifications mean from the standpoint of risk aversion and its management. To investigate the inter-temporal effects of risk management, or asset allocation in the William’s sense, and thereby address the financial market imperfections found in the GS style model of Chapter Four, we have added a module to represent the use of derivative products for hedging in the form of put options. As described above, different motives for using futures markets have appeared in the literature, however, we adopt the paradigm promoted by Williams, that futures markets or options-on-futures resemble a precautionary means of borrowing or lending inventory and as such represent a means of improving inter-temporal asset allocation (pages 118-130, Williams, 1994). In exploring the behaviour of the modified GS model, we will find that the William’s paradigm is useful in explaining why risk management yields micro-economic gains and has macro-economic effects. In the modified GS style model, we use risk management to reduce or eliminate down-side exposure by ensuring the return to borrowed working capital, at the margin, is at least sufficient to make the firm solvent. In the remarks below, we explain how the modification was undertaken.

The stylized diagram found in Figure 45 is a good place to begin as it shows how a put option operates in the modified GS model by ensuring a minimum return. The inflection point in the diagram corresponds to the respective strike price of the put option which we make equal to the bankruptcy threshold, of $\sqrt{v_{t+1}}$. As we observe in the diagram, the option ensures that a firm specific price realisation will at least equal the bankruptcy threshold level.\(^4\) The

\(^4\) In practise, a firm would settle the option contract for its intrinsic value, if any, and use the amount gained to off-set the loss in the cash market. Risk is not eliminated but rather off-setting risk has been undertaken.
pay-off line for the put option begins below the break-even line, as
the firm managing risk owns or is long the contract, and has a cost.
The larger the cost, the further the vertical distance of the horizontal
section of the put line from the horizontal axis. With the put option,
as prices fall, once the initial cost of the option has been recovered,
the contract becomes increasingly in-the-money. We show the
underlying asset position as the straight diagonal line rising to the
right. In the modified the GS model, the net exposure of the firm, if
fully hedged is the vertical sum of the two lines. By purchasing the
put option, firm specific prices at least equal to the threshold level are
ensured. Given uncertainty over the relationship between expenditure
(on wages, for example) and the receipts from future sales, the put
position ensures a minimum difference between the cost of borrowed
working capital and the revenues from future sales, at time \( t + 1 \). In
the module added to the program, the strike price on the purchased
put option (the right to sell) is set in order that firm specific prices
received at least equal the bankruptcy threshold level, \( u'_{t+1} = \sqrt{v'_{t+1}} \).\(^5\)
In effect, the put option creates a hedge against insolvency, and as
such represents a means of synthetically adjusting the cost of
borrowed working capital, thereby improving the dynamic allocation
of assets.

In including a risk management module into the simulated GS
style model, we adopt the approach favoured by Williams with regard
to the decision to hedge. According to Williams (1994), the decision
of whether to hedge and of what extent to hedge is an economic one
determined by market forces: We argue that in equilibrium, the
marginal benefits of hedging should equal the marginal bankruptcy
cost arising from not hedging. In the added module, risk management
is purchased up to the point where the value of the last currency unit

\(^5\) Recalling equations (4.11) and (4.12) of Chapter Four, by setting the strike
price of the put option at the threshold level, options are at-the-money, that is have
no intrinsic value upon settlement.
spent upon it to ensure the return to the "inventory" of capital equals the expected and discounted downside cost of inadequate equity, viz bankruptcy. Since the occurrence of bankruptcy arises from firm specific prices below the threshold levels, $v_{t+1}$, by purchasing risk management down-side exposure is reduced to the extent that the exposure is completely covered, while the up-side for returns in excess of those anticipated remain, with their consequent cyclic effects via real wages and dividends. As modelled, risk management is determined endogenously by first solving for the price of risk

![Figure 45: Put Option](image-url)

For purposes of comparison, to understand the modifications to the GS model in its structural context, we begin by recalling equation 4.5 of Chapter Four, used for the profit maximization problem, and with its first order condition in the model without the risk management module. Our Chapter Four results on the bankruptcy parameter were.
where, \( c' \) is the marginal bankruptcy cost of firm \( i \) at time \( t \). If the marginal cost of bankruptcy were zero, the first order condition would revert to a standard micro economic result of equating the marginal product to wage costs, \( w(1+r) \). We further note for purposes of comparison that the marginal cost of bankruptcy equals:

\[
\frac{1-c'_t}{\phi'}
\]

**5.2)**

Now, let us introduce parameters for expenditure upon risk management into the profit maximization problem, equation 5.1 above. We include in equation 5.3 below, the price of the put option \( \Psi \), the hedging ratio \( \lambda \), that is the required position in the derivative security needed to manage a given exposure, and \( q \) the exposure. (The strike price of the put option is set so that future revenues satisfy the threshold criteria of equation 4.12 of Chapter Four.) Fully hedged, when \( \lambda \) equals unity, the level of return is sufficient to ensure solvency is guaranteed. The expression \((1-\lambda)\) is determined from the hedge ratio, as further explained in Section 5.3. It is computed from the option pricing formula, and tells us by how much risk management has reduced the Marginal Bankruptcy Cost, according to the exposure which has been covered. It modifies the reduction in the risk of bankruptcy which results from using put options, so that in a fully hedged firm the threat of specific prices falling below the threshold level is eliminated. We have the new objective function for profit maximization including risk management:
As modelled for simulation, the cost of bankruptcy is modified by the introduced parameters for hedging ratio, $\lambda$, so that as the firm's expenditure upon risk management increases, the former declines. Fully hedged, the locked-in strike price equals the threshold price level, $\nu'$, and is sufficient to meet the wage bill as well as the expenditure on the put given the option $\lambda$ ratio, thus eliminating the cost of bankruptcy as appears in equation 5.3. The new first order condition, corresponding to the new maximization problem of 5.3, for an interior maximum, is:

$$w(1+r) = \frac{1-c'-\lambda\psi'}{\phi'}$$

The most right hand term in the numerator gives the marginal cost of risk reduction as output is increased. Under the new specification, if the marginal bankruptcy cost were zero, the incentive to risk manage would disappear and we would have the standard result that the marginal product is increased to the point where it equals the wage.

To understand the potential effects of introducing the risk management module into the GS Model, we rely upon their stylized graph, shown below in Figure 46. The graph resembles GS Figure II (page 83, 1993), except that a third curve has been drawn to represent the Marginal Cost of Production when the risk management module has been added to the program. The relative strength of the two effects, the cost of bankruptcy and the cost of risk management will depend upon the sensitivity of their parameters. If the marginal cost to the firm of risk reduction using put options to ensure a minimum return to borrowed working capital were always less than
reducing the expected marginal cost of bankruptcy via output reduction and reduced borrowing, expenditure upon risk reduction would, on net, result in a greater equilibrium output, and would always be justified. Conversely, although unlikely, if the marginal cost of risk reduction via lowering the optimal scale of output, as it reduces the need for borrowed working capital, were uniformly less than the marginal costs of risk reduction, it would never pay to purchase options and manage risk. If neither of these situations may be assumed, a cross-over point exists, such that a profit maximising optimum will arise when the marginal benefit of risk reduction equals the reduction in expected marginal cost of bankruptcy. Or, in standard micro-economic framework, that the marginal rate of substitution between expenditure on marginal bankruptcy costs and expenditure upon marginal risk management cost as used to ensure minimum returns on borrowed working capital must equal the ratio of their respective costs. (Although in all likelihood, the cost of risk
management to a firm is less than the cost of going bankrupt, hence making it optimal to be fully hedged.) To formalize these points, we consider below how they may be specified analytically.

In the modified GS model of Chapter Five, there are three aspects to cost: The cost of borrowed working capital, the expected cost of bankruptcy, and the cost of risk management using put options. Understanding their respective arguments relate to overall cost, will help us understand how they may interact in the GS model when risk management has been added. We recall our remarks of Chapter Four, where we saw that the expected cost of bankruptcy is convex in output because of risk aversion. In addition, we have validated the GS assertion that increasing the variability of firm specific prices raises the threat of bankruptcy and cost, thereby reducing the level of output. In Section 4.4 we noted that the probability of bankruptcy, \( F(\nu_s) \) was a function of the following parameters, repeated from equation 4.14, as per GS (page 88, 1993) and its relation to the density function on firm specific prices \( \nu_{t+1} \).

\[
F[v_{t+1} (q_t, t, w_{t}, \frac{P_t}{\hat{P}_{t+1}}, 1+r) ]
\]

5.5) We have noted that uncertainty surrounding the relationship between firm specific prices and the threshold level, was the source of risk aversion because lower than expected firm specific prices are what drives the risk of bankruptcy, as per Equation 4.14 of Chapter Four. As specified in the GS model and as used in the simulation program, the cost of bankruptcy is a function of output \( (c_t = cq_t) \). The threat of bankruptcy is influenced by the initial level of equity and working capital arising from retained profits, but its generation is risky because of the one period lag between the time when capital is borrowed and committed to a wage-bill, and when the returns are realized in the sale of output. Now turning to the risk management module added
to the program, we specify the price of risk reducing puts, $\Psi$, as a function of four parameters, namely the *asset price, exercise price, time to expiration, interest rates, and volatility or variability in firm specific prices* (as we explain below in Section 5.3 on option valuation theory). We note that all of these parameters are endogenous to the program and are already utilized for other results.

Having reviewed the various aspects of cost in the modified GS style model, we return to the question, of how introducing risk management will affect optimal firm behaviour. From an analytic perspective, the question involves the three aspects of cost. It involves how they inter-act and depends upon the relative strength of the total differentials of marginal bankruptcy costs and that of the expression for expenditure upon risk management. We have the bankruptcy component:

$$c_{',t} = \left( \frac{dc_t}{dq_t} \right) F + c_t'(V_{t+1}) \frac{dv_{t+1}}{dq_t}$$

5.6)

and the total differential of the risk management parameter $\theta$:

$$\theta' = \frac{\partial \theta}{\partial \psi} \frac{\partial \psi}{\partial \lambda} + \frac{\partial \theta}{\partial \lambda}$$

5.7)

Returning to our earlier remarks, the size of the relative strength of the two effects depends upon how they respond to their respective cost arguments. Practically, the cost of risk management to a given firm is likely to be less than the cost of going bankrupt. A *cross-over* point would only occur under unlikely conditions relating to the price of put options, for example *under extreme* volatility. It pays to manage risk because the price of put options which enters into the

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6 Recalling from GS, there is little uncertainty about the general price level.
cost of risk management ensuring the return on borrowed working capital is much less than the cost of going bankrupt. Options positions confer a significant amount of gearing, that is for a relatively small price ("premium", in industry parlance), a position having a large value, may be controlled. Compared to reducing output in the face of bankruptcy risk, or going bankrupt, managing risk is very attractive.

With positive real bankruptcy costs, depending upon firm level optimisation, the equations as specified within the simulation model determine to what extent the underlying asset positions will be hedged. Such decisions occur at the firm level and they are dynamic, as expressed in the equation. As will verify through simulation, since the cost to the firm of risk management, using derivatives is nearly always below the cost of bankruptcy, it pays to be fully hedged. Expenditure upon risk management, that is ensuring the return on borrowed working capital, through purchasing options at prices determined endogenously by the economy represented by the simulated GS style model, increases the level of output and affects the economies laws of motion under certain conditions. Risk management, may reduce or eliminate the "wedge" which the ex ante expected marginal bankruptcy costs drove between wages and traditional marginal cost, leading to an increase in output. In order for the level of macro output to increase, via increases in equity, a net cost saving must arise, through improved inter-temporal allocation of assets, in the spirit of the William's paradigm. Since the marginal expected cost of bankruptcy generally exceeds the additional cost of ensuring a return to borrowed working capital via put options, a net gain in output should arise from risk management.

The above discussion is useful in seeing the direction of effect from including risk management in the GS model. The size of the effect upon equilibrium output of risk reduction along with the impact upon the model's dynamics, however, cannot be determined
analytically. It requires considering how firm level profit maximization involving the marginal expected cost of bankruptcy and how the marginal cost of risk management respond to their respective arguments, and inter-act to determine the net gain in output, along with changes to dynamic behaviour.

Discussing the mechanics of how risk aversion operates in the modified GS style model will help to understand its likely dynamic impact. To this purpose, we compare it with the specification of Chapter Four. In the GS model of Chapter Four, risk aversion takes the form of reduction in output to reduce the threat of bankruptcy. In the modified model, we retain the fear of bankruptcy arising from firm specific prices below the threshold level, but firms may purchase, in addition, risk management. Rather than reduce the optimal level of output as a precaution against realized firm specific prices below the bankruptcy level, that is returns, below those anticipated when working capital was committed to wage expenditures, firms purchase the right to a precautionary sale of output, at the threshold price ensuring solvency. In adopting the above specification we argue that the approach to risk management remains consistent with the GS model as well as the adopted paradigm of why derivative markets exist. Although risk aversion continues to drive the modified GS style model of Chapter Five, it is not a requirement for the added module per se.

We adopt the William's paradigm of risk management, rather than the insurance paradigm of Keynes and Hicks to inform why risk management occurs along with its likely effects. Risk management as we have noted in Chapters Three and Four, is an essential property of investor behaviour in the GS model (page 88, 1993). The modified GS style model retains the assumption of bankruptcy aversion, although this only affects risk management indirectly, that is through the model. We emphasize this point because only the Keynes/Hicks model of hedging requires risk aversion per se, and not the
precautionary capital inventory management approach of Williams. In adding the risk management module to the GS style model, we thus remain faithful to its original features. The William's paradigm does not need the assumption of aversion to bankruptcy, per se, however in the present GS style model, such behaviour does lead indirectly to risk management. Although risk aversion is utilized in the GS style model, it is not a requirement, per se, for risk management to occur in the added module. Thus risk aversion affects risk management only indirectly through aversion to bankruptcy as per the GS model: It is neither a necessary nor a sufficient assumption for risk management itself to occur, as found in the backwardation/risk premia literature. Even if firms knew the probability that firm specific prices might be below the threshold level, effectively having perfect foresight on the future returns to their borrowed working capital, (that is, possessed Muth Rational Expectations), they would be unwilling to accept actuarially fair gambles if they were risk averse.

In the manner described above, we have adopted the perspective that risk management is a form of *dynamic asset allocation or capital inventory management*, undertaken to ensure the returns from borrowed working capital. By purchasing risk management, firms may reduce or eliminate the threat of insolvency, when firm specific prices are below the threshold requirements. The approach we have adopted to risk management behaviour is dictated by the economics of firm profit maximization. As assumed and specified, if it is *economic* to hedge to whatever degree, a market for risk management exists and further. Only through the GS model does the existence of a positive risk premia justify indirectly expenditure to reduce or eliminate the expected cost of bankruptcy, while the economics of such expenditure depends upon endogenous parameters. Using an *economic* approach to hedging in which state-contingent assets enter into the firm level optimization problem, allows for asset positions to be partially as well as completely
hedged. As a marginal decision rule, a fully hedged position ensuring the return on borrowed working capital and completely eliminating the expected cost of bankruptcy, depending upon the cost of risk management, may not be justified. As we discover, however, through simulation, for reasons mentioned earlier, under normal market conditions, as the cost to the firm of risk management in the form of options, is less than the expected marginal cost of bankruptcy, eliminating completely the cost of bankruptcy, \( \text{pays} \).

Lastly, with regard to the specification of risk management, for realism in the modified GS style model, because the value of an exposed position arising from borrowed working capital, may not respond to a change in the rate of interest charged at the same rate at which the value of the put option ensuring the value of working capital does (as per, Equations 5.3 and 5.4 of Chapter 5), the added module determines the size of the risk management position, \( \lambda \), required to adequately hedge the position, will be specified below in Section 5.3.\(^7\)

Further insights into the potential effects of modifying the GS style model by introducing a risk management module may be gained through looking at firm level decision making. Introducing risk management into the GS style model, we will see, produces effects as demonstrated through comparative statics at the firm level along with having aggregate dynamic impact. In the GS Model, the expected cost of bankruptcy drove a wedge between the value of the marginal product and wages. In Chapter Four, we verified the GS assertion that as marginal cost of bankruptcy grew larger, the optimal output declined. In terms of neo-classical investment theory, a capital constraint along with uncertain returns to borrowed working capital combined with the threat of bankruptcy, as found in the GS model,

\(^7\)In the Derivative's literature such practices are known as constructing a neutral hedge. The size of the derivatives position required to be neutral, is computed using the option pricing model presented in Section 5.3.
creates imperfect capital market conditions, that is the span of securities available does not correspond to all states of the world. According to Nickel, the notion of capital market perfection in a world of uncertainty does not apply and is without meaning (page 159, 1978). The effect of introducing the risk reduction parameter, $\theta$, into the GS model, is to reduce the size of the wedge according to the relationship between the cost of bankruptcy function and the cost of the risk reduction function. Within the context of investment theory, removing uncertainty from investment decision making reduces capital market imperfection: Firms can borrow increased amounts at a given rate of interest without greater risk of default (Layard & Walters, page 327, 1978). In the GS model, since greater borrowing given the equity constraint, is necessary to expand output, such effects should impact both its comparative statics as well as its dynamic behaviour.

In addition to the comparative static results arising from firm level decision making, introducing risk management into the GS model also has implications with regard to dynamic behaviour and stability which are less readily interpreted. In the modified GS model, the size of the output effect depends upon the cost of risk management relative to the expected marginal cost of bankruptcy and how they together affect the accumulation of equity. Although the inter-temporal relationships between investment in working capital, the wage bill and dividend payments remain, expenditure upon risk management, in the aggregate, may change the economies laws of motion, such as the periodicity, and whether the oscillations are explosive, damped, or chaotic, that is sensitive to initial conditions. Such implications for the laws of motion, it has been noted, however, generally, cannot be determined analytically. As discussed in Section 4.9 of Chapter Four, non-linear difference and differential equations are either difficult or impossible to solve analytically, and hence require numerical simulation (page 3, Baker &
Moreover, in the GS model no closed form expression to equation 4.29 may be specified, necessitating firm level optimization over time and aggregation. To facilitate dynamic simulation, we note with regard to specification of the new module algorithm, in the modified model, we have made the price of put options used for risk management endogenous to the GS style model, in a manner explained below in Section 5.3. We adopt "small-country" assumptions of an infinitely elastic supply of "risk management" available to firms. Each of the population of twenty-five firms is assumed to be a price-taker with respect to purchasing put options. A general equilibrium perspective on risk management is not considered.

From all of the above we see that introducing the risk management parameter $\theta$, depending upon its arguments, may, on net, reduce the expected cost of bankruptcy, and thereby ceteris paribus, increase the equilibrium level of output, through facilitating the accumulation of equity, along with changing the model's dynamics. With the added module, under realistic assumptions with regard firm specific price variability, returns should be greater and through equity accumulation, lead to greater output. In the William's paradigm, inter-temporal asset allocation has been improved. In this light, perfecting the GS style model, rectifying its New Keynesian framework, should positively impact the quantity of aggregate output through changes in firm level decision making. As we investigate below through numerical simulation, this observation has both comparative static firm-level implications along with aggregate dynamic effects. (The effects however, may to some extent be overstated because of the lack of realism in some of the GS assumptions. Correcting the model of some of its unrealistic assumptions we postpone until Chapter Six.) The effect upon borrowing, investment, and aggregate output found in the modified GS model are important because, as it is generally cheaper to be fully hedged than it is to face
the expected marginal costs of bankruptcy, introducing the risk management module should lead to increases in output along with other dynamic effects. The quantifiable size and nature of such effects, however, may not be determined analytically from inspection of reduced form equations derived from the first order maximization shown in equation 5.7 above. It requires instead investigation through numerical simulation, as undertaken in Sections 5.4 and 5.5 and as discussed in Section 5.6.

5.3 Risk Management Specification

In order to include risk management in the GS style model, we adopt standard option pricing theory and add a module to the program specified and simulated in Chapter Four. Importantly, the new module uses the exogenous inputs and endogenous results found in the model of the previous chapter, to solve for the price of put options. No new exogenous assumptions are needed. The put options computed in the introduced module represent the state contingent assets used to ensure the returns to borrowed working capital which reduce or eliminate the expected marginal cost of bankruptcy, and threat of insolvency. They are used as a means of ensuring the return or future value of the borrowed inventory of working capital, by locking-in firm specific prices equal to the bankruptcy threshold level. To price such state-contingent assets, the established body of theory has been used.

Usefully, option pricing theory, we note, is not inconsistent with William’s paradigm of futures markets and risk management. As mentioned, earlier, option pricing theory is embedded within CAPM theory since the combination of long options and short assets, or the opposite, may be used to construct a risk-free rate of return. CAPM, however, concerns the general equilibrium pricing of financial securities, whereas the William’s model is a partial equilibrium explanation of why firms ensure an ex ante value for assets or
inventory, improving inter-temporal asset allocation, in this case, of borrowed working capital, given various motives, including the precautionary one. The CAPM perspective on hedging involves movements along the security market line, which we found were difficult to justify from the perspective of economic rationality. The Williams paradigm concerns how firm ensure that they remain on the line, i.e. generate returns according to their level of risk, because put options may reduce or eliminate the fluctuations in return to borrowed working capital. In addition to consistency, the Williams model we will show offers an explanation of why micro-financial behaviour has aggregate effects. We will find also that the adoption of the Williams model within the New Keynesian framework found in the GS model, presents some insights to the systematic risk issues raised earlier arising from a CAPM and MM perspective.

Turning to the mechanics of option pricing, we recall that a put option, as explained, gives the owner the right to sell a given quantity of assets, at a certain strike or exercise price, within or at the end of a specified time frame. Several key economic variates are required to compute a theoretical price of the put option, namely its relationship to the underlying market asset price, the rate of interest rate and the price volatility of the underlying asset. The formulation utilized relies upon the well-known Put-Call Parity theorem.8 Depending upon one’s underlying asset/inventory position, one type of derivative contract over another might be favoured. In the modified the GS model, the concern of firms is with protection from down-side exposure of a long asset position. Their strategy is to buy a put option to ensure firm specific prices, in effect, at least equal to the threshold level, as explained above in Section 5.2. Purchasing a put option confers the right to sell output at the minimum price \( v \), which

8There are many useful discussion of option pricing theory. The works of J.C. Cox and M. Rubenstein, Option Markets, (1985) or R.M. Bookstaber, Option Pricing and Strategies in Investing, (1981) are particularly thorough.
ensures solvency, allowing the firm to break-even. (For comparison, long call options give one the right to purchase a given quantity of assets, at an agreed strike price on or within a specified time frame and may be theoretically priced in a similar manner.)

Understanding the mechanics of risk management, we turn to the theoretical pricing of an option. In received financial theory, five parameters are needed for the option pricing formulas all of which are found in the model presented and simulated in Chapter Four, as exogenous inputs or endogenous results. The parameters and their logic are as follows. Their validation through simulation appears in Section 5.4.

1. \( P_{t+1} \): The Firm Specific Output Price: The greater the price of the underlying quantity, the greater the intrinsic value of a call option with a given strike price (that is, the right to purchase). Conversely, the lower the price of the underlying quantity, the greater the intrinsic value of a put option with a given strike price (that is the right to sell).

2. \( X_{t+1} \): Exercise Price: For a call option which is in the money\(^9\), the lower the exercise price, the greater the intrinsic value of the option. For a put option, the greater the exercise price, the greater the intrinsic value of the option.

3. Time to Expiration: The longer an option has to run, the greater the probability that it will be possible to exercise the option profitably, hence the greater the value of the option. As the GS model assumes a first-order dynamic process, the time to expiration is a single period.

\(^9\)For a Call Option, in-the-money would put the strike price below the market price. For a Put Option, in-the-money would put the strike price above the market price.
4. *r*: Interest Rates: As options involve a right to buy or sell the underlying asset at the discounted value of the future spot price: the greater the degree of discount, the more valuable is the right. The rate of interest rate determines the size of the discount. Analogously, as options are a means of borrowing and lending positions, the value of such lending and borrowing rises with the rate of interest.

5. *σ*: Volatility: Volatility is computed as a moving average from the assumed standard deviation of firm specific prices, as assumed for simulation of the GS model. Variability in firm specific prices affects return on investment. Firm specifically, if prices in the future were known with certainty, there would be no need to purchase options. (Also uncertainty over the threat of bankruptcy would be eliminated.) Ceteris paribus, if the variability of firm specific prices increases, so will the price of its option.

We observe that all of these parameters appear directly or indirectly in the model specified and simulated in Chapter Four, so that no new exogenous parameters are required to endogenously solve for the price of risk management. Relying upon the model’s results to solve for the price of risk management, the module will as well determine the optimal size of the option position required to manage the firm specific price exposure. In the added module, specifically, we use the following standard equation based upon the Put-Call Parity Theorem, as mentioned above, derived from Black-Scholes option pricing formula (F. Black & M. Scholes, 1973), to solve for the price of a put option, \( \Psi_t \):

\[
\Psi_t^I = \Psi(P_t^I, X_{t+1}^I, t, r) = C(P_t^I, X_{t+1}^I, t, r) + X_{t+1}^I e^{-r(t-t')} - P_t^I
\]

where, \( C(P_t^I, X_{t+1}^I, t, r) = P_t N(d_1) - X_{t+1}^I e^{-r(t-t')} N(d_2) \)

5.8)

In the option valuation model to determine the price of risk
management: Where \( d_1 \) and \( d_2 \) are defined accordingly:

\[
5.9) \quad d_1 = \frac{\ln(\frac{P_t}{x_{t+1}}) + (r_t \sigma^2) t}{\sigma \sqrt{t}}
\]

Further, we define the other parameters accordingly:

- \( \Psi \): The put option price
- \( C \): The call option price
- \( P_{t+1} \): Firm specific output price
- \( X_{t+1} \): Exercise price
- \( t \): Time to expiration - one period
- \( \sigma^2 \): The instantaneous variance of the asset price
- \( N( ) \): The cumulative normal distribution function
- \( r_t \): The rate of interest at time \( t \)

Using the above formula for the price of a put option, we incorporate risk reduction and down-side risk elimination into the new model, in the following manner. In Chapter Four we explained how the Marginal Cost of Bankruptcy drove a wedge between the traditional wage/marginal cost relationship. With the added risk management module, the purchase of put options, at a cost calculated endogenously, may serve to reduce the marginal cost of bankruptcy. A fully hedged neutral position eliminates the possibility of bankruptcy, by ensuring that the price received by a given firm, at a minimum, is \( u_{t+1} = v_{t+1} \), that is firm specific prices at least equal the threshold level for solvency. Using the above formula, the program endogenously determines both the price of puts, \( \Psi \), and the size of the position required to full or partially hedge the exposure, that is \( \lambda \). The optimal expenditure on the option position, that is the economics of firm level optimization, hereafter \( \theta \), as mentioned has
as arguments the price of puts, the computed $\lambda$, and the cost of bankruptcy. Put options contracts are purchased up to an optimal point, that is, where the value of the last currency unit spent just equals the expected discounted cost of not having hedged one's positions. Fully hedged, the probability of bankruptcy is zero; while if the firm is fully unhedged, the probability of bankruptcy reverts to the model of Chapter Four, with the consequent effect on firm behaviour. As mentioned earlier, given the efficiency of using options, it nearly always pays to be fully hedged, except for example under conditions of extreme volatility. We assume that the risk management option has an exercise or strike price at-the-money, that is, at the time of purchase, time $t$, without intrinsic value. Firm specifically, the strike price of the put option equals the price of the underlying quantity to be sold at time $t+1$, such that $U_{t+1} > V_{t+1}$. It guarantees a price at time $t+1$ at least equal to the bankruptcy threshold. (Hence, the inequality criteria above, $u_{t+1} > v_{t+1}$. The put option ensures firm specific prices at least equal to the threshold level.) Options, we assume, are of the European style, that is may only be exercised at expiration. We assume that options allow the firm to hedge one period into the future, and if not utilized, as when $u'_{t+1} > v'_{t+1}$, they expire worthless. Fully hedged, a firm may receive a price exceeding $v'_{t+1}$, but never below.

5.4 Comparative Static Simulations of the Modified GS Model

In Section 5.3 above we made some preliminary observations about the modified model's comparative statics. Before looking at how the dynamic behaviour of the GS style model with the added risk management module behaves, we turn to the results of comparative static simulation to see the effects upon output and profit. In this way we ensure that the modified model continues to behave in the same micro-economic fashion as explained in Chapter Four, including that it obeys the three Propositions concerning capital structure, price
uncertainty and output convexity as appears in the GS articles (1988, 1993). We show that the modifications to the GS style model behave as expected with regard to the arguments found in the option pricing module. Recalling some of the simulation experiments of Chapter Four, making comparisons with the modified versus unmodified GS style model, are instructive in this regard. After considering comparative statics, we turn to the dynamics of risk management. With reference to the GS Propositions, the results of six simulation experiments for the $i^{th}$ firm at time $t = 2$, are reported to show how perturbations to exogenous variables effect intertemporal behaviour when risk management is available. Looking first at comparative statics, we recall the equity experiments from Chapter Four. Below, in Figures 47 to 50, we compare the effect upon firm level comparative statics of changing the initial level of equity with and without risk management assumed ("no RM" and "RM"). Simulations, we will see, shows how changes in capital structure affect the use of risk management. In Figures 47 and 48, we see that for the reasons discussed above, the availability of risk management, as it reduces or eliminates the threat of bankruptcy, sharply expands output. For purposes of comparison, the scales of the graphs are the same. We see that under risk management, the revenue and cost curve now rises much more gradually, as shown in much flatter marginal curves, rising beyond the scale of the horizontal axis. In Figures 49 and 50, we have increased the initial level of debt in the capital structure, making risk management that much more important, because the threat of bankruptcy is now greater. Although greater debt makes individual firms more cautious with respect to borrowing working capital and hiring workers, the effect is sharply reduced with risk management purchased in the form of put options. Gearing-up on a small level of equity makes sense when a minimum prices are ensured. In the four comparative static simulations, we note, that all other parameters, notably the variance
in firm specific prices and profit retention are the same (10% and 80%, respectively). The four simulations show the efficiency of risk management using derivatives. In view of their relatively low cost, because of their gearing like effect, for a small premium a large exposure may be controlled.

Figure 47: D/E = 3.5, No RM

Figure 48: D/E = 3.5, RM

Figure 49: D/E = 4, No RM

Figure 50: D/E = 4, RM

Turning to the last two sets of experiments for the $i^{th}$ firm, we see the effect of changing the variability in firm specific prices. It is quite interesting, because on the one hand, increased uncertainty with regard to firm specific prices, $u'_{t+1}$, affects the threat of bankruptcy viz-a-viz the threshold level, $v'_{t+1}$, and thereby reduces
borrowing and eventually aggregate output as we saw in Chapter Four. With greater variability, however, in firm specific prices increases the cost of managing risk, as we explained in the last section. Given the gearing-like efficiency of risk management, however, a net gain in output occurs even with greater firm specific price risk. In Figures 51 and 52 the variability in firm specific prices is increased to 20%, from the previously assumed 10% of the previous four figures. We see, without risk management greater firm specific price variability leads firms to cut-back on investment and produces reduced aggregate output. When the risk management is operational, using the module added to the GS style program, a gain in output occurs, yet the effect, is diminished by its greater cost. Although from the standpoint of profitability, using risk management is superior to reducing investment and output, its benefits are reduced because of the effect greater firm specific price volatility has upon its price.

As we can see from the six comparative static simulations for the $i^{th}$ firm, the availability of risk management boosts output considerably. With greater initial equity in the capital structure, as the ex-ante expected cost of bankruptcy is reduced, the gains in output from managing risk are smaller. As firms purchase put options because of the uncertainty over future firm specific prices in relation
to the threshold bankruptcy criteria, experiments on price volatility are quite interesting. Comparing Figures 51 and 52 with the four previous ones, we see the gains in output when risk management is an option. As price volatility is a key parameter in the risk management pricing module of the simulation program, the above experiments show that risk management does boost the level of output, but the benefit is slightly reduced with greater volatility. As measured by output, when risk is managed, a gain occurs. Under risk management, however, because the cost of managing exposure increases with greater volatility, that is, firm specific price variability, as per the objective function, equation 5.3, the efficiency of avoiding risk is reduced. In sum, we see from comparative static simulations shown above, that with risk management available in the form of a put option, the profit maximising level of output is greater because of the firm’s net reduction in cost. Reducing or eliminating marginal bankruptcy cost, through eliminating uncertainty around firm specific price variability, and hence its relation to the threshold for bankruptcy, encourages the borrowing of working capital, growth in equity, and the expansion of output. Lastly, we note that all of the above comparative static results are intuitively consistent with the Propositions of the GS model.

5.5 Dynamic Simulations of the Modified GS Model

We now turn to how the GS style simulation program behaves over time if risk management were available to firms in their optimisation calculus. Using the method explained in Chapter Four, we have conducted several simulation experiments to investigate the inter-temporal effect of risk management upon the GS style model. Resembling the six experiments used to demonstrate the model’s comparative statics, found in Figures 47 to 52, we now show the effect of risk management upon the model’s dynamics. Under various scenarios with respect to initial values for various parameters, we
report in Figures 53 to 58, the dynamic trajectories with and without risk management to facilitate comparison. For each set of parameters, we have two simulations: In Figure 53a for example, we find the dynamic trajectory and in Figure 53b the phase portrait. As we recall from Chapter Four, the phase portraits were necessary in order to explore the role of random attractors and higher dimensionality in the GS model. Using the same parameters, in Figure 54a and 54b, we have the trajectory and phase portrait plotted but under the assumption that risk management ("RM") is available.

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>D/E</th>
<th>SIGMA</th>
<th>PROFIT RETENTION</th>
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<tr>
<td>Fig. 53a. - b. No RM</td>
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<td>20%</td>
<td>25%</td>
<td>3.5</td>
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<tr>
<td>Fig. 54a. - b. RM</td>
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<td>20%</td>
<td>25%</td>
<td>3.5</td>
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<tr>
<td>Fig. 55a - b. No RM</td>
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<td>20%</td>
<td>25%</td>
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<tr>
<td>Fig. 56a. - b. RM</td>
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<td>20%</td>
<td>25%</td>
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<tr>
<td>Fig. 57a. - b. No RM</td>
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<td>Fig. 58a. - b. RM</td>
<td>6.0</td>
<td>22%</td>
<td>25%</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Table 5.1

The parameters for each simulation as appear in Table 5.1 are noted for the individual figures. The simulation results show that the availability of risk management sharply increases output, transforming the dynamics of the system. As we can see, introducing risk management changes the Laws of Motion from oscillatory
convergence to oscillatory divergence. The trajectory generated when risk management is available features sharply greater levels of aggregate output, because firms are now not afraid to borrow and invest. Compared to simulations 53 and 54, in Figures 55 and 56, we reduce slightly the level of technology, holding other parameters unchanged, which should as we have seen in Chapter Four, make the economy less convergent, in combination with reducing the initial amount of equity in the capital structure. We see that the effects of decreasing the initial level of equity in combination with lowering the level of technology aggregate output are largely off-setting. Having initially less equity in the capital structure, has the effect of making risk management slightly more important, as the risk to insolvency arising from firm specific prices not meeting the threshold level is enhanced. In Figure 56a, we see, introducing risk management transform the trajectory from one which is convergent to one which displays oscillatory divergence. Looking at the phase portrait in Figure 56b, we see that with risk management in place, the oscillatory divergent dynamics are now more effected by the random attractor. In the phase portrait we see that although down-side exposure on firm specific prices have been eliminated, because a given firm now has greater incentive to invest and borrow working capital, its levels of equity are more affected by the random attractor, \( u^i \), than they would be otherwise, if investment and borrowing had been discouraged. Price variability above the threshold level, now plays a greater role. Gearing-up on the level of equity, sharply expands output when the threat of prices below the bankruptcy threshold have been eliminated, notwithstanding the lower level of productive technology. Thus by introducing the risk management into the model, with the assumed exogenous parameters, the system is rendered divergent and the periodicity of cycles increased. With risk management, as evidenced by the phase portraits, the random attractor, \( u^i \), has a greater influence upon the model’s dynamic
behaviour, because although down side price exposure has been reduced or eliminated, the spiral of borrowing- investing- wages and profit, are encouraged. Without risk management available, firms facing the threat of firm specific prices, \( v'_f \), below \( v''_f \), borrow less, with a smaller impact upon real wages, and ultimately profits and output, thereby leaving a smaller role for the random input in the model's dynamics.

Making a comparison between Figures 56 and 54, we see that by introducing risk management, the economy in both cases is changed from oscillatory convergence to divergence. As measured by aggregate output, the benefits of risk management are large. Gearing up on a lower level of initial equity is particularly beneficial when risk management is available, especially when because of a lower level of technology, production is more labour intensive. Under such circumstances, the random input is more important and has a strong effect upon the system's dynamic trajectory. In the last two sets of simulations, we change only the variability in firm specific prices from 20% to 22%. Increasing the variability in firm specific prices even very slightly make the economy more convergent as shown by comparing Figures 55 and 57, when risk management is not an option. When risk management is available, however, increasing the variability in firm specific prices, increases the influence of the random input. Price variability effects risk aversion through the relationship between firm specific prices \( v'_f \), and the threshold \( v''_f \), as we know from our simulations in Chapter Four. Changing price variability is particularly important when risk management is considered, because firstly increasing it, makes hedging more important, secondly, it slightly raises its price through the option formula, as computed endogenously by the model, and lastly, as we have noted it makes the model more subject to the random input. Comparing Figures 56 and 58, we see that the model as calibrated is fairly sensitive to the price of risk management.
Figure 53a. No RM: Initial D/E = 5.75, Tech = 3.5, Dividends = 25%, Sigma = 20%

Figure 53b. No RM: Initial D/E = 5.75, Tech = 3.5, Divid. = 25%, Sigma = 20%
Figure 54a. RM: Initial D/E = 5.75, Tech = 3.5, Divid. = 25%, Sigma = 20%

Figure 54b. RM: Initial D/E = 5.75, Tech = 3.5, Divid. = 25%, Sigma = 20%
Figure 55a. No RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 20%

Figure 55b. No RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 20%
Figure 56a. RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 20%

Figure 56b. RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 20%
Figure 57. No RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 22%

Figure 57b. No RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 22%
Figure 58a. RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 22%

Figure 58b. RM: Initial D/E = 6, Tech = 3, Divid. = 25%, Sigma = 22%
While output is expanded with risk management in Figure 58, the effect is smaller because of the greater cost of managing risk, making the importance of the random attractor somewhat smaller. Explaining further, risk management, as we have seen so far increases output, but at the expense of increasing aggregate instability, and making the model's dynamics more sensitive to the random attractor.

Relating the effect of increasing firm specific price variability to the broader issues, we see that without risk management, it renders the system more stable, as firms become more risk averse to the possibility of firm specific prices, $u_{t+1}$ below the threshold $v_{t+1}$, that is insolvent. Although risk management facilitates and encourages investment and output swings, when firm specific price variability is increased, it has the effect of raising the cost of risk management. Compared to the output and dynamics when price variability is smaller, under such conditions, investment and growth in equity is discouraged, as shown in the levels of attained in Figure 58a compared to that obtained in 56a. The result arises because greater price variability makes it more dear for the firm to hedge its position. More expensive risk management has the same effect as increasing costs, upon system stability. Investment and borrowing are slightly discouraged and output is reduced. Although down-side exposure may be reduced or eliminated, aggregate instability remains because borrowing of working capital, relative to borrowing without risk management available, is encouraged, exaggerating the wage-dividend spiral found in the model. Such effects are shown in the corresponding phase-portrait or orbit where we see that the dynamics are subject to the random attractor. Although the dynamics of the GS model arise from the output-wage-profit and dividend spiral, as depicted in the stylized Figure 3.1 of Chapter Three, they are randomly effected by the variability in firm specific prices through its impact upon profits and retained earnings. As explained earlier, following the William's paradigm of futures markets, as options are
a means of borrowing or lending a position in order to improve the inter-temporal allocation of assets, it is not surprising the greater growth in equity with risk management leads to greater aggregate instability.

In all three sets of simulations, we see that introducing risk management, increases output sharply relative to not managing risk. Comparing Figures 58a and 58b with 56a and 56b, we see however that because risk management is slightly more expensive, the gains in output are reduced, within the time frame shown, recalling that the trajectory is divergent, and many more iterations with greater precision are often useful in numerical analysis. Predictably, increasing firm specific price variability has a dampening effect upon cyclic oscillations. The plotting of phase portraits, moreover, shows that with risk management, although the down-side of price exposure has been eliminated, the "exposure" of the model's dynamics to the effects of the random input are enhanced. With and without risk management, aggregate output declines in comparison to the levels with lower firm specific price variability because of the greater risk of bankruptcy. Risk management still has a role in increasing output, but the gain is smaller because of its greater cost. As we recall, the price of put options is positively related to price variability.

5.6 Risk Management and Non-Linear Dynamics

In the final section of Chapter Five, the inter-temporal behaviour of the modified GS style model is discussed from standpoint of how risk management influences dynamic behaviour. We consider the sensitivity to changes in initial conditions, and how it effects the role of the random input the model's dynamics. In this way, we may round-out the subjects of non-linear dynamics and chaotic behaviour, in light of the simulations seen in Section 5.6. In Chapter Four, we explained that linearization around the steady state ignores such possibilities as sign changes in the phase line as typified
In the tent family of functions which display multiple equilibria, sensitivity to initial conditions, and such phenomenon as period doubling or bifurcations (Chapter 2, Gulick, 1992). In Section 4.9 of Chapter Four, such phenomena were examined using numerical simulation. It was also shown that properly, the GS model should behave as a second order system, because of the random input, $u'$. In Section 5.6, returning to the subject of non-linear dynamics, we discuss the laws of motion for the modified GS style model relative to the laws of motion for the unmodified one, that is without risk management. We will interpret further the meaning of how the modified model behaves in response to changes in initial conditions and perturbations to the exogenous parameters which drive the cost of risk reduction.

Building upon the observations made in Section 5.5, we have found that introducing risk management renders the system's dynamics less stable and leads to increased periodicity, that is the number of iterations per cycle. In addition, with risk management encouraging investment, the dynamics are more effected by the random input, $u'$. The phase portraits shown in Section 5.5, are typical of the a tent-style logistic map, indicating chaotic oscillations (see, page 23-25, Ott, 1993 or Chapter 1, Gulick, 1992). The realism of such results are subject to qualification, however, in view of some of the model's unrealistic structural specifications, as we will explain in Chapter Six. In the six simulation experiments detailed above in Table 5.1, a perturbation to a particular exogenous initial parameter was performed to determine its effect upon the inter-temporal dynamics of the GS style model when risk management is assumed. As a point of reference we showed how the model performs when there is no management of risk and firms adjust output to reduce the expected cost of bankruptcy. Although some of these parameters have been investigated before, our purpose was to consider in particular the effect upon periodic behaviour arising
from the model’s non-linear dynamics when it is assumed that risk is managed from the standpoint of initial condition sensitivity.

For each of the simulations we have seen the dynamic trajectories and the corresponding phase portraits. The changes to inter-temporal dynamics between each figure reveals the great sensitivity of the GS style model to varying the initial conditions. Such sensitivity is particularly important when the system is rendered explosive and divergent under risk management. The trajectories and orbits for each experiment demonstrate how the model behaves when risk management is available to the firm. We have seen from the phase portraits and the dynamic trajectories, for example, as shown in Figure 55 and 56, the effect of reducing the initial equity in the capital structure for each of the population of twenty-five firms. With decreased initial equity, there is more risk to manage, because of greater borrowing. Such circumstances, without risk management, would render the system more stable as it raises marginal bankruptcy cost, inducing risk aversion. Assuming risk management, under the assumption of reduced initial equity, as shown in the dynamic trajectories and phase portraits, renders the system oscillatory divergent, and more subject to the random input. Gearing-up on reduced equity is attractive, especially when the down-side of borrowing, having insufficient profits to service debt, has been eliminated, however, there are macro economic implications.

Surveying the above results, an intuitive interpretation is that the availability of risk management creates an aggregate externality in the form of increased systemic instability. In a sense, it enhances the role of the random input argument in expression 4.38 while making the other argument of expression 4.38 less important. In the experiments, introducing risk management to the GS style system, unambiguously leads to sharply greater output, as we can see from the vertical scale, but also greater dynamic instability in the aggregate. As mentioned earlier, managing risk is much cheaper than
avoiding it, hence the net gain in output. Like the other simulations, it appears that the micro-economic gains at the firm level generate greater systemic instability at the macro-economic level. Arguably, the costs in macro-economic output variability constitutes a form of externality created by risk management. While before the effects of the random attractor were internalized, they are now externalized into aggregate output instability. The results are consistent with the William’s paradigm in that using options improves the inter-temporal allocation of assets at the firm level, improving profits, and growing equity and output. Properly understood, risk management is not about eliminating or reducing risk. Rather, by allowing the firm to concentrate upon price relationships rather than their absolute level, it yields improved performance as measured by profits and output. At the macro level, however, greater cyclic instability is created.

The simulations found in Section 5.5 serve to verify that the dynamic behaviour of the GS style model with risk management is consistent with option pricing theory, and our general observations on its inter-temporal nature. The simulations provide several new insights into the dynamics of a New Keynesian style model when the assumed financial market imperfection is correctable. We have shown that risk management leads to greater firm output, and greater profitability, but at "price". With risk management, unambiguously, cyclic periodicity increases, as would arise from a sign-changing inflection point in the phase line becoming sharper. (Sharp, sign-changing inflection points are important in logistic maps because may give rise to chaotic trajectories (Chapter 2, Ott, 1993).) Aperiodic behaviour is also observed along with extreme sensitivity to initial conditions, typical of chaos. With risk management, the role of the random attractor in the model is enhanced. Arguably, based upon the above simulations, removing or sharply reducing the threat of bankruptcy arising from the original stylized assumptions on financial market imperfections through risk management, transforms firm
specific risk into a form of systemic macro-externality. The macro-externality takes the form of reduced cyclic stability, and possibly explosively, divergent and chaotic trajectories in aggregate output. Risk management *magnifies* the model’s borrowed working capital - wage - dividend spiral, and hence aggregate instability, enhancing the role of the random attractor. With downside exposure reduced or removed, borrowing for positive but possibly modest returns is encouraged. With the threat of bankruptcy removed, the real wage - profit spiral is enhanced. Recalling the results of Section 5.5 we see that a net gain in output with risk management occurs, along with a greater cyclic instability. In effect, it appears that in the aggregate risks cannot be eliminated, only transformed: The gains in macro output come at the "expense" of greater systemic volatility through the GS model’s structural dynamics, as shown in the phase portraits. Consistent with the William’s paradigm, the use of derivatives improves the inter-temporal allocation of assets, increasing equity and output. Although firm specific exposures have been managed, the "cost" has been increased aggregate instability. We see, that within the confines of the GS style model, adding the risk management module has both aggregate and dynamic output implications.

The above effects, however may be over-stated, because of the lack of realism found in some of the GS specifications, particularly with regard to unlimited borrowing at a fixed rate of interest which does not include the risk of default. Even with the returns on borrowed working capital risk managed, lenders may still be reluctant to provide unlimited amounts, as per perfect capital market theory. Arguably, the model implicitly assumes an informational asymmetry between lenders and borrowers, in the sense that the former are unaware of the economies’s deep dynamic structure in the Lucas sense, particularly with regard to the borrowed working capital - wage bill externality. Not risk adjusting the cost of capital in the GS model also creates a curious relationship between the providers of equity
and the providers of debt. Risk managed or not, the model would appear to be biased in favour of expanding market value, notwithstanding the risks, placed upon book value. We explore this point in Chapter Six. The questions of market versus book value bias is arguably exacerbated by the dividend pay-out regime required for the GS model. Stochastic as GS propose in their article or on a fixed percentage basis as used for simulation, not applying financially rational behaviour to dividend structure, and assuming that lenders are unaware, further impacts the output-dynamics. Even without risk management in place, the dividend approach specified in the GS model imposes certain biases in the models dynamic behaviour, for reasons we will explain, and hence merits further inquiry. Valuation under Neo-classical financial economics should be invariant to how expected returns are distributed. Returns to share holders, dividend yields, should reflect the risk adjusted cost of equity capital. A commitment to pay dividends rather than retaining capital for precautionary motives exposes debt holders to the risk of market value falling below book value, however, as might occur during an economic contraction. For a firm to pay dividends facing the risk of default or which are unrelated to the cost of equity capital, lacks realism. In Chapter Six, we conclude our analysis of the GS style model with the risk management module by addressing such issues and deficiencies.
SIX: SOME REMAINING TECHNICAL ISSUES

6.0 Introduction

In Chapter Five we investigated the dynamic implications of introducing risk management into the GS style model. Using both linearization around the steady-state and non-linear techniques, we found that the possibility of hedging against price uncertainty increases the magnitude of macro-economic fluctuations. Period doubling bifurcations and even aperiodic chaotic deterministic dynamics exaggerated by the model's stochastic features may arise because firms become less risk averse during the expansionary phase of the economic cycle. We cautioned, however, that some of these results may be subject to qualification, raising issues with respect to the possibly unrealistic specifications found in the GS style model upon which the numerical simulation program is based. In this Chapter, our focus is not upon adding new complexities to the GS style model, but rather upon rectifying these problems in specification, by modifying the specification somewhat further. Like Chapters Four and Five, we use numerical simulation to understand how such changes in specification of the GS style model may alter its Laws of Motion.

We will consider first the issue of the rate charged for loanable funds, and secondly will focus upon the return structure, looking again at the role played by dividends as specified in the GS model. In the GS model, the cost of loanable funds which provide working capital is invariant with respect to firm's capital structure, \( r_t = \delta \), the consumer rate of time preference which is a constant over time. Although with changes in profits, the ratio of debt to equity of a given firm may change, in the GS model, what they are charged for
borrowed working capital does not. Effectively, the GS model does not risk adjust the cost of capital, and hence the return required by the individual time discount, $\delta$. Given the constraint on equity, making the cost of borrowed working capital constant, would seem to encourage borrowing, as gearing is increased. We redress this feature in Chapter Six by making changes to the cost of capital endogenous to the model, as we explain below. The second issue we take-up in Chapter Six concerns the structure of returns in the GS model. The return structure of the GS model, the proportion of profits distributed as dividends versus retained as profits, is very important to its dynamic behaviour. Recalling from Chapter Four, as examined in Figures 22 to 24, and as mentioned by GS (page 100, 1993), profit distribution through dividends must be sufficiently large in order to inhibit unbounded growth in equity. If growth in equity is unbounded, because of diminishing risk aversion, a trajectory in output which is monotonic and divergent may arise. Such dynamic behaviour, however, raises questions with regard to the realism of either a fixed dividend rule as employed in the model or a purely stochastic approach as presented by GS; and secondly, challenges received theory with regard to the impact of return structure upon valuation. As per the Modigliani-Miller theorem (hereafter, MM), the valuation of a firm depends upon the riskiness of it’s net operating income, and not upon it’s mix of dividends and capital gains. Under such a framework, aversion to bankruptcy, a sudden loss in valuation, should be unchanged. In the GS model, however, the structure of returns affects valuation and leads to real effects, including the dynamics of output as we showed in Chapter Four. Rather than financial magnitudes being merely the reflection of real
economic activity as presented in the Neo-Classical system, financial arrangements have real effects. In Chapter Six, we take-up this issue, examine its theoretical aspects, considering how the model might behave if changes to the profit distribution rule were made endogenous to the model's structure.

6.1 The Cost of Loanable Funds

Financial matters, in particular the role of interest rates, have played distinctly different roles in the various business cycle theories. In Neo-Classical macroeconomic theory and its derivatives such as Monetarism, little importance was given to financial considerations such as the cost of loanable funds for two principle reasons. Firstly, because of the principle of dichotomy, namely that real variables are independent of the absolute price level and secondly, that the long-run rate of output, including fluctuations in it, are governed by fundamental trends relating to technology, labour supply, and institutional structures, as embodied in the models of Friedman (1968) and Phelps (1970) using the natural rate hypothesis. In addition to the dichotomy principle, in finance, this perspective finds support in Fisher's Separation principle showing that investment, production and consumption decisions may be undertaken independently from financing decisions (Fisher, 1930), along with the Modigliani-Miller (hereafter, MM) theorem showing that the firm's valuation is independent of how it funds its assets.

In contrast to the above reasoning, investment plays a central role in the Keynesian system and its derivatives because of its effect upon aggregate demand. In Keynes' theory of the business cycles, the importance of the investment function arises through its effect
upon aggregate demand (Hall, Chapter 4, 1990). By comparing the marginal efficiency of capital with the interest rate on borrowed funds, firms decide how best to adjust their capital stock. In Keynesian theory and its derivatives, the market rate of interest, the supply price of capital and expectations of prospective yields together determine aggregate investment (page 51, Hall). Investment volatility, as we explained in Chapter Two is the source of perturbations leading to cyclic variations in output. By contrast, in Neo-Classical economics with its emphasis upon aggregate supply as mentioned above, variations in the rate of interest are not relevant to explaining fluctuations in economic output. Similarly, Monetarism ascribed fluctuations to short-run movements in aggregate demand originating with monetary disturbances. Changes in the rate of interest were not the source of variability, but rather a result of changes in the monetary base. In contemporary research into cyclic fluctuations, the channel of causation between interest rates and aggregate demand tends not to be emphasised. In Real Business Cycle theory, shocks to technology along with strong assumptions regarding the pro-cyclic nature of consumption and leisure, are required to generate patterns empirically consistent with cyclic fluctuations; while the research agenda in New Keynesian research places emphasis upon nominal rigidities such as menu costs, informational assymetries with respect monitoring, and insufficient second order effects precluding optimizing adjustments (see, Mankiw, Volumes I & II, 1993). In view of the above, the GS model with its fixed rate of interest unrelated to the contribution to risk made by investments in working capital funded through debt, features a form of informational asymmetry, as used in the NK literature, at the same
time embodying Neo Classical features as mentioned earlier in Chapter Two, Section 2.33. In the remarks below we pursue this point, examine its implications, and propose an alternative specification.

In order for the value of the firm to be independent of the ratio of debt to equity, as per MM theory, the excess expected rates of return arising from investments earning above the risk free rate must be discounted according to their contribution to portfolio risk (Sargent, Chapter 7, 1979). In the GS model, however, the rate charged at time $t$, $r_t$, as explained in Chapter Four, Sections 1 and 2, is fixed and invariant to changes in capital structure (It is assumed invariant to time, a constant). As the available equity is fixed, all additions to working capital must be financed through debt. Effectively, the GS models allows a given firm to gear-up its performance at a rate of interest unadjusted for risk, thereby, encouraging investment, increasing the market value of the firm during the expansionary phase of the business cycle while increasing the risk of bankruptcy during the contractionary phase of the cycle when market value tumbles, possibly below that of book value. Such results are very controversial in light of received financial-economic theory. Arguably, the optimizing behaviour of firms in the GS model resembles the investment decision rule of Keynes (1936) and Tobin’s $Q$ ratio (1969), in that the criterion for capital expansion involves making ex ante comparisons of what firms may earn in the product market with what savers require in the lending/securities market. Effectively in the GS model, when the ratio of the ex ante expected rate of return on the book value of their assets exceeds the cost of capital ($Q > 1$), then expansion should occur. Such an approach has
implications for how we interpret both the analysis found in their article along with the results of simulation.

By specifying the cost of capital as invariant to the ratio of debt to equity, that is without adjustment for risk, as the total capital base expands, increasing valuation, the GS model violates Fisher’s principle of Separation, is at odds with MM theory, and distances itself from Neo-Classical principles of neutrality and dichotomy. Without risk adjusting the cost of capital, gearing adds to the present value of the firm, growing market value but increasing the exposure to book value, especially during the contractionary phase of a business cycle. Although ex ante risk aversion enters into the decision by firms with respect to their capital borrowing behaviour, by allowing, in effect, valuation to increase with gearing, (because the cost of capital is not risk adjusted), a logical inconsistency it might be argued is created between how firm’s regard risk versus how supplier’s of capital regard it. The GS approach resembles the structure specified in the NK informational asymmetry literature (Mankiw, 1985). For example, although the representative firm increases the ratio of debt to equity in its capital structure, because of monitoring costs, financial intermediaries continue to lend working capital at the fixed rate of interest. Lenders are systematically unaware of the economies’s laws of motion. By not risk adjusting the cost of capital, the model is enhancing the firm’s market valuation, thereby reducing the necessity for risk averse behaviour. In contrast, theoretically, if the valuation of a firm’s return across different states of nature were discounted using a rate which is risk adjusted over time to changes in capital structure, unlike the GS model, because of its effect upon borrowing and investing behaviour, the threat of
insolvency during the contractionary phase of the business cycle might be reduced. Risk adjusting the cost of capital should be stabilizing to the simulated GS style model, but can this observation with regard to dynamic behaviour be validated using numerical simulation? The question of risk adjusting the cost of capital is critical because of its effect upon the firm’s book value. Although equity providers are keen to see the market value of the firm increase through borrowing, especially at a fixed rate of interest; the concern of lenders, however, is to preserve book value, informational asymmetry notwithstanding. Risk adjusting the cost of capital as debt is increased would hold constant or even reduce market capitalization, potentially transforming the model’s dynamics.

In light of the above commentary, below we re-specify the model accordingly and discuss the results of numerical simulation. There are many possible ways of adjusting the cost of debt capital in the GS model to a firm’s capital structure. We have chosen, without loss of generality, to specify this relationship by adjusting the cost of capital in proportion to the change in the ratio of debt to equity over time, calibrated according to slope of a Security Market Line, that is the rate at which the market trades-off risk and reward. (We use contemporary empirical market data for the security market line.)

These changes in specification were introduced into Module I of the simulation program with consequent effects for its dynamic iterations. Like Chapters Four and Five, we compute $G(a_t, u_t)$ for the firm with the average level of equity in period two, as the representative firm,

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1The slope of the security market line is quite robust to changes in systemic market factors. A slope of 0.7 was assumed for calibration which is consistent with the body of empirical work (see, Copeland & Weston, Chapter 7, 1992).
and then apply the results to the assumed population of twenty-five firm, using their individual firm specific price shocks. We use the same procedures for aggregation as explained in Chapter Four. In order to learn as much as possible from this change in specification to the GS style model, we present both inter-temporal trajectories and the phase portraits, as found below appear in Figures 59 to 62. The effects of the change in specification when risk management are also discussed.

Turning to Figures 59a and 59b and 60a and 60b, we see two sets of simulations with and without the cost of capital adjusted over time to changes in the capital structure, as per the discussion above. In the first, resembling the results of Chapter Four, we see first a dynamic trajectory and secondly a phase orbit which is slightly convergent. In Figures 60a and 60b, holding all parameters the same, we see that the effect of assuming that the cost of capital is adjusted over-time to changes in the capital structure. In the simulations, over time, assuming that the cost of capital is risk adjusted, we see that convergence is quicker, and the level of aggregate output falls by roughly one-third. Risk adjusting the cost of capital serves to dampen the oscillations. When profits grow such that the ratio of debt to equity falls, the cost of borrowed working capital declines. If, under such circumstances, although random firm specific prices are low and profits fall, in the next period working capital loan obligation is reduced. The interaction lessens the effect of the fall in profits upon investment, required labour and real wages. Under risk adjusted cost of capital, if profits fall because of low firm prices and the ratio of debt to equity increases, the increased cost of capital in the next period will dampen investment. Rather than continuing to borrow at
the same rate when profits fall, firm respond by reducing investment
and borrowed working capital. With less debt capital exposure, if
prices should be low again, the cyclic impact is reduced. From the
above description of the model's mechanics, we can see why risk
adjusting the cost of capital leads to lower levels of investment and
output. The introduced change in specification has reduced the
extent to which the dynamics are driven by the random input $u_t$ in
$G(a, u_t)$. Although not shown, simulations of the model assuming
risk management is available but making the cost of capital sensitive
to capital structure, also reveal a dampening of the oscillations, which
as we recall from Chapter Five grew sharply, often in an explosive
manner, when hedging was introduced. The results show that our
findings in Chapter Five with respect to the impact upon aggregate
stability of risk management are to some extent over-stated.
Adjusting interest rates to changes in capital structure when risk
management is available leads, tends to stabilize oscillations when
they are convergent, and if they are divergent may change the system
to one which is not. We see that introducing risk adjusted rates of
interest may reduce the number of periods per cycle.

Appraising the above results, we observe that after a manner,
firms in addition to being risk averse to bankruptcy, are now using
profitability as arises from price variability, to gauge the investment
environment. Although firm specific prices are normally distributed,
and a good price is as likely to be followed by a bad price, as any
price, through its impact upon the cost of capital, the firm as an
optimizing agent, in effect, uses it as leading indicator of what might
follow, in the fashion of the accelerator principle of J.M. Clark
(1917), wherein changes in output causes changes in investment.
Re-specifying the GS style model to make the cost of capital continuously adjustable to changes in equity and capital structure resembles the accelerator literature, where we find postulated a great sensitivity of investment to changes in the rate of growth in income (Hall, 1990). Making the cost of capital sensitive to capital structure has the net effect we have seen of dampening the economic cycle with and without risk management available.
Figure 59a. Cost of Capital Constant, Debt/Equity = 4, Tech = 4, Divid. = 6%, Sigma = 20%

Figure 59b. Cost of Capital Constant, Debt/Equity = 4, Tech = 4, Divid. = 6%, Sigma = 20%
Figure 60a. Cost of Capital Gearing Sensitive. Debt/Equity = 4, Tech. = 4, Divid. = 6%, Sigma = 20%

Figure 60b. Cost of Capital Gearing Sensitive. Debt/Equity = 4, Tech. = 4, Divid. = 6%, Sigma = 20%
6.2 Dividends and Valuation

As we first noted in Chapter Four, the rate at which profits are distributed plays a very important role in the dynamics of the GS model through its effect upon the firm’s level of equity. Without some distribution of profits, although the firm may not raise new equity because of the constraint, its growth may allow the capital base to increase monotonically, leading to an explosive oscillations in aggregate output. With large equity reserves, the fear of bankruptcy would be reduced, and output would grow sharply in an explosive manner, as we saw in Figures 21 to 24 of Chapter Four. With proportionately less debt, risk taking investment is encouraged, resulting in larger and steeper declines in aggregate output during contractionary phase of the economic cycle, because of the relationship between real wages and profits. Such observations, however, beg the question of the model’s underlying logic with regard to the effect of dividends upon market value or capitalization. Although we have explained how the model behaves, from the standpoint of comparative statics as well as dynamics, questions remain with regard to the effect of dividends upon the risk of bankruptcy. At the end of Chapter Five, we raised the objection that using either a fixed dividend rule or a purely stochastic variable as proposed by GS, is unrealistic since firms faced with the choice of insolvency or not to pay dividends, would surely choose the latter. We also mentioned in Chapter Three, Section 3.3, that the model’s downward rigidity with respect to profit distribution. More formally, the approach taken by GS involves the fundamental issue of whether valuation should be effected by how profits are distributed. According to Modigliani-Miller (MM) Theory, adjusted for risk and
taxation issues, valuation is invariant to the nature of earnings. Given a level of return, capitalization does not change with mix of dividends and capital gains. In the simulated version of the GS model, however, we see that the risk of insolvency is reduced and real output effects occur the less profits are distributed, (that is the more they are retained), leading to increased dynamic instability. Retaining earnings through reduced dividends, lessens the risk of bankruptcy, encouraging investment, by enhancing market valuation. This result arises because of the implicit assumptions adopted by GS in their model, particularly with regard to why investment occurs and the constraint on the availability of new equity. At variance with received financial-economic theory, real effects upon the dynamics of economic output arise, in this context, from changes in dividend distribution by way of valuation. Structured in this manner, the GS model with respect to the dividend structure involves several logical inconsistencies. We begin with these inconsistencies and their implications, before we consider how their approach to dividends might be reformulated.

As mentioned, according to received neo-classical economic and financial theory of Modigliani and Miller, firm market valuation should be invariant to how returns are made to holders of equity. Although GS present a general form for the profit maximization problem, as repeated in equation 4.5 of Chapter Four, they are reticent with regard to what underlies the investment decision. GS describe the firms as risk averse, without considering the perspectives of investors and lenders. We only know that firms issue new claims,

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debt, against the book value of their assets, in order to increase market value. Their approach resembles the Q-theory of productive investment as first introduced by Keynes (1936) and developed by Tobin (1968, 1969), wherein firms commit themselves to investment using borrowed working capital, but with profits possibly below anticipated levels. Adopting the Q Theory, investment occurs when firms as profit maximizers, acquire new productive assets (along with contractual obligations to workers in the GS model), and sell claims against their productive proceeds in the capital market for a greater money value than their cost. Conversely, disinvestment, from an initial level of equity, occurs when firms find that productive assets have greater money value than their cost. Under such circumstances, it will be profit maximizing to sell those assets, and distribute the proceeds upon liquidation to claimants, bond holders and to equity holders. Although not stated by GS, the model implies that the criterion for expansion or contraction, is comparing what managers can earn in the product market with what savers require in the securities market. In order to implement such a structure properly, however, it should recognize the trade-off between risk and reward of various claimants, notably holders of debt and equity. GS, we note, only make mention of equity holders under circumstances of liquidation and ignore the risks to debt holders of greater debt (GS, page 86, 1993). The combination of an equity constraint with a financial market imperfection, requiring expansion to be debt financed but using a fixed dividend pay-out rule as we have used or one which is purely random, leads to several inconsistencies.3

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3According to Capital Asset Pricing theory, regardless of individual subjective attitudes towards risk taking, the marginal rate of substitution between risk and reward, in general
To appreciate this point, we draw upon the theoretical contributions of Merton (1974) and Krainer (1992).

In the GS model, funding investment in assets by issuing new equity is not possible. By having a constraint on the availability of new equity, the optimal trade-off for income sharing between debt and equity holders, adjusted for risk, is affected. Generally, bond holders, as is well known, have a prior claim to the income and assets of the firm, while the claims of equity holders are riskier since they are residual. Taking such observations one step further, Merton (1974) has likened shareholders to owners of call options on the underlying assets of the firm while the loan guarantee of bond holders is viewed as a put option. Owners of call options are pleased to see the value of their underlying assets increase in value, especially if it is financed by bonds. Owners of put options, in contrast, merely wish to protect the book value of their claims, not wishing to see the assets expanded, especially if it involves taking on greater exposure with possibly adverse consequences, during the contractionary phase of the business cycle. According to Krainer (chapter 3, 1992), who uses the Merton model to develop an optimal trade-off rule between the different claimants, a risky strategy from the perspective of bond holders is one in which it finances its future assets with ever greater debt. Using the Merton model, and assuming different degrees of risk aversion between the two types of investors, Krainer formulates a globally optimal static equilibrium in which the book value (or product market) value of the company equals its capital market value (Krainer, pages 70-75, 1992). We repeat these results using the graphs below.
Figure 61: Return/Risk Trade-offs for Debt & Equity Holders

<table>
<thead>
<tr>
<th>VARIABLES</th>
<th>SPECIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of Income from Debt and Equity</td>
<td>$X(b)/X(s)$</td>
</tr>
<tr>
<td>Input Contribution from Debt and Equity</td>
<td>$K(b)/K(s)$</td>
</tr>
<tr>
<td>Risk-Reward Trade-Off Rate Debt Holders</td>
<td>$dd$</td>
</tr>
<tr>
<td>Risk-Reward Trade-Off Rate Equity Holders</td>
<td>$ee$</td>
</tr>
</tbody>
</table>

Table 6.1: Variable Specifications
Looking first at Figure 61, we see the optimal trade-off rates for transforming inputs into returns for debt holders and for equity holders. As we can see, equity holders are less risk averse than bond holders and are keen to see market value grow. The schedules, \( dd \) and \( ee \), in both graphs, represent all the points along which Tobin’s \( Q = 1.0 \). Above, the schedules, market value exceeds book value, while below, book exceeds market value. Superimposing the graphs from Figure 61, we represent in Figure 62, the optimal equilibrium of debt and equity in this economy. For reasons of risk aversion, the required rate of return on debt is lower than the return on leveraged equity, and hence the \( ee \) schedule intersects the \( dd \) from below. Without leverage or gearing, the above analysis collapses to the optimal trade-off rule between return to debt and equity holders, and inputs to return, namely:
which is equivalent to the equation of Marginal Rates of Substitution along the contract curve in an Edgeworth Box diagram. The above analysis shows how risk affects the two types of investors who fund factor inputs (labour) to a given firm. According to Krainer, the difference between the two types of investors arises from their different attitudes to taking risks. Debt holders are more risk averse and are keen to preserve book value, while leveraged equity holders, are less risk averse and would like to see market value grow (Krainer, page, 79, 1992).

The above analysis shows the implications of the financial market imperfection and constraint assumptions found in the GS model. The GS model by constraining the amount of equity, and forcing new capital to be financed by debt, favours growing market value at the expense and risk to book value. Nominal assumptions notwithstanding, the GS model favours equity holders over bond holders, even though under accepted business practice, under conditions of insolvency and liquidation, debt-holders have prior claim. The contractionary phase of the business cycle holds greater risk for debt holders than it does for owners of equity when market value may fall below that of book value, however, in the GS model risk aversion relates to the behaviour of equity owners and not holders of debt. (Liquidation occurs when holders of debt do not receive their expected payments, not when equity owners endure poor returns.) Recalling Figures 61 and 62 above, we observe that the GS model favours solutions above the ee and dd schedules. Although the focus
of the GS model is how changes in firm equity affect aggregate output, abstractly, from a general equilibrium perspective, an implication of the GS model, is that debt holders should gain from cyclic expansions, but lose out during economic contractions when market value falls below that of book value. As laid out in the GS model, such results resemble the model of Allen & Gale in which there is under-investment in safe assets when certain classes of financial assets exist (1997).

Such circumstances are heightened by the dividend pay-out rule available to equity holders. Under received financial-economic theory, market valuation should be invariant to the nature of returns. The GS model is constructed so that retained earnings increase the firm’s equity base, encouraging risk taking as it adds to market value. As we saw in Chapter Four, Figures 21 to 24, retention of profits, rather than protecting returns based upon book value owed to bond holders or compensating for the greater risk in the capital structure from new borrowing, dynamic volatility in equity and output is increased. In the GS model, lenders receive a fixed rate as explained in Section 6.1, while shareholders see their returns preserved as dividends plus the potential for growing market value for their investment with greater risk. Although received financial theory tells us that market valuation should not change with the proportion of profits distributed, in the GS model, by retaining earnings—adding to equity, risk taking is encouraged, increasing exposure should firm specific prices fall below the threshold level of bankruptcy. Effectively in the GS model, the implicit call owned by shareholders increases in intrinsic value when profits are retained and borrowing grows, while the implicit put held by bond holders loses intrinsic value.
through greater risk of bankruptcy, as may occur during the contractionary phase of the business cycle. Following option pricing theory, increased firm specific price volatility after capital structure has been decided would benefit equity holders at the expense of debt holders.

How might these inconsistencies be addressed? With a debt financed expansion, and an equity constraint, one possible way to improve the logical consistency of the GS model, so that bond holders are compensated for the increasing risk of bankruptcy as assets are expanded through borrowing, would be if, over-time, dividends were reduced through greater profit retention, and used to reward bond holders for the increased risk to their claims against book value. (A Floating Rate Note would be an example of such an arrangement.) In the simulations below, we further modify the model, just as we introduced risk adjustment to the cost of capital of Section 6.1 above, so that a further endogenous structure and specification yields a gain in the returns to debt holders which is financed by reduction in dividend yields to equity holders. In this way, we equate the risk-reward trade-off between the market value held by equity owners and that of debt holders, as per the optimal trade-off rule of Kraine, discussed above. Like Section 6.1, we continue to use as a proxy for the risk-reward trade-off, an assumed slope of the Security Market Line to determine at what rate returns to debt holders are increased as firm gearing is increased, but now use a portion of retained earnings as the source of the gain. By using retained earnings to modify and risk adjust the returns to bond holders, their exposures during the contractionary phase of the business cycle, as we see in Figures 63 to 64, are reduced. The proposed modification to the
dividend structure, rather than assuming a constant as we have previously or making it stochastic as GS propose, modifies retained earnings over-time. Over time, the random attractor, firm specific price shocks serve to vary equity and profits over-time, altering the capital structure so that the introduced endogenous structure to adjust returns to debt holders at the expense of equity holders. The assumptions for the exogenous parameters used in the simulation appear in each of the graphs.

From Chapter Six we see that some of the earlier results from Chapters Four and Five are subject to further qualification. By risk adjusting the cost of capital and the structure of dividends, that is making these two key parameters endogenous to the model and specified according to received financial theory, we see that some of the general claims made by GS based upon the behaviour of their model are over-stated. Effectively, the original specification of the GS model places greater risk upon bond holders than it does upon equity holders, encouraging borrowing to grow market value, given the equity constraint. If the cost of capital, as per MM theory were risk adjusted to changes in capital structure, the oscillatory trajectory might be dampened and output reduced. Discouraging borrowing even to a small extent, would reduce the incentive and hence the output effects of risk management, as explored in Chapter Five. Dividend structure we have also argued, as per received financial theory, should reflect the cost of equity capital, which is the return required given a level of systematic risk, $\beta$, and the position of the security market line. Making the cost of equity capital endogenous to the system and adjusting the dividend structure also serves to dampen the oscillations observed along with the level of output.
Figure 63a. No Risk Adjustment in Capital Cost, Divid. = 10%, Sigma = 20%, D/E = 4, Tech = 4

Figure 63b. No Risk Adjusted Capital Cost, Divid. = 10%, Sigma = 20%, D/E = 4, Tech. = 4
Figure 64a Risk Adjusted Capital Cost. Divid. = 10%, Sigma = 20%, D/E = 4, Tech = 4

Figure 64b. Risk Adjusted Capital Costs, Divid. = 10%, Sigma = 20%, D/E = 4, Tech = 4
CHAPTER SEVEN. CONCLUSIONS

7.0 Key Results and Findings

In the course of our research we have explored in considerable depth the nature and behaviour of a New Keynesian style model featuring a financial market imperfection and a constraint on the availability of equity. The research has produced several useful findings, which may have a wider significance. In the remarks below, our final chapter, we will review these findings, interpret their meaning from the standpoint of research into macroeconomic modelling, and explore their policy implications, if any. To review, using numerical simulation we have verified all of the unproven Propositions made by GS with regard to comparative static behaviour of their model. The effect of perturbations in key parameters upon the model’s comparative static results, namely the initial level of equity and the degree of uncertainty with regard to firm specific prices, through their effect upon marginal bankruptcy costs, are as GS have claimed. According to GS their research suggests that economies which are highly leveraged may gain substantially from infusions of equity capital (page 92, 1993). Our research into the model’s dynamics shows that this is the case, but there are qualifications: Infusions of equity may increase output, however, they may also result in destabilizing swings in aggregate economic activity. A further qualification relates to the productivity of capital. Although GS do not include a technology parameter in their model, we have, as specified in our simulation program. We have found that the productivity of the economy, as specified in our technology coefficient plays an important role. Perhaps counter-intuitively, when an economy is more productive, because it requires less borrowing of
working capital to pay wages, decreasing exposure to low firm specific prices, it is more stable and less prone to cyclic oscillations. An economy with a low level of productivity, in contrast, as it is forced to borrow greater amounts to fund working capital to pay wages, is more exposed when firm specific prices are low and profits fall.

Our research into understanding the dynamic behaviour of a GS style model, has been very enlightening. Investigation of the model's intertemporal behaviour has shown that the assertions by the authors with regard to the model's dynamic behaviour are subject to many qualifications. We have found that the invocation of linearization around the steady-state, as adopted by GS to support the claim that their model generates multi-period cycles is of little value. We have shown that linearization ignores the model's complex dynamics, in particular such phenomenon as period doubling (bifurcations), aperiodic trajectories and chaos which arise because of the model's non-linear nature. Through numerical simulation we have learned that phase lines with sharp inflections in logistic space, critical-points, may occur (the so-called tent diagram), indicating dynamic trajectories of various limit cycles as well as aperiodicity and chaos. Linearization around the steady-state may only be used to plot single period cycles and ignores the importance of initial conditions outside the neighbourhood of the steady-state. Using numerical simulation, our research has shown that distinctly different dynamic trajectories may be generated making very small changes in initial conditions. As shown through simulation, the model may generate a phase line implying a dynamic trajectory which begins with a region of monotonic convergence followed by cycles of various lengths. A
phase line may cross the steady-state, \( \frac{da}{dt} = 0 \), more than once, implying multiple steady-state equilibria, or display both convergent portions followed by oscillatory portions, separated by a sharp inflection. In addition to these observations, we have shown that although the GS model is an autonomous first order difference equation, it really behaves as a second order equation because of the presence of a random input, \( u_t \) in \( G(a_t, u_t) \). By plotting orbits in phase space of \( a_t \) against \( a_{t+1} \), we saw the second order style behaviour, allowing us to understand the importance of the random attractor in altering behaviour. As mentioned in Chapter Four, this finding supports the assertion by GS in their 1988 article that with two random attractors, the model should behave as a third order difference equation. Together the above results and findings, underscore the value of numerical simulation to investigate non-linear, higher order, dynamic behaviour. Non-linear New Keynesian models have an important role to play in macroeconomic research, but require appropriate techniques.

In addition to gaining a better understanding of the model's complex dynamics, we have also learned something about the implications of its specified structure. In Chapter Five, we highlighted the role of fixed rates of interest and dividend policy in creating a bias in the model's results. The presence of an equity constraint in combination with a cost of borrowed capital which is not risk adjusted to capital structure, as per the GS model, imposes greater risk to book value during the contractionary phase of the economic cycle and hence disadvantages holders of debt. By ignoring the theory of capital structure, we saw in Chapter Six, that the GS model effectively favours borrowing by ignoring the exposure of debt
holders to the risk of bankruptcy. Moreover, the implemented dividend rule, because it favours the distribution of profits to equity holders without compensating bond holders for greater risk to book value, creates a further bias in the model’s dynamic solution. Faced with a constraint on the availability of new equity, to assume that market value may be expanded by borrowing ever greater amounts of debt is difficult to justify in view of received financial theory. In a sense, the GS model by ignoring received financial theory, sets in place a mechanism to increase the chances of market value falling below that of book value, that is bankruptcy, as might occur during a wage-profit downward economic spiral. By not having an endogenous mechanism to correct a New Keynesian rigidity, such as risk adjusting the cost of capital or making profits distributions detrimental to non-equity claimants, as we have done, it might be argued that unrealistic model solutions are favoured. Appropriately re-specifying the model to adjust for these biases, as we accomplished in Chapter Six, to some extent corrects such limitations, subject to the values used for calibration and the assumed specification. Such effects are seen in terms of the model’s dynamic behaviour. Risk adjusting the cost of capital and ensuring that profit distribution does not increase the risks of bankruptcy faced by bond holders, and reflects the cost of equity capital, renders the system somewhat more stable.

Our interest in a financial market imperfection model grew out of our concern with the subject of risk management, and in particular the question of what the macro economic implications were of using derivatives to hedge exposures. To answer this question we turned to a New Keynesian business cycle model featuring financial market
imperfections, to see what effect introducing an endogenous mechanism to manage risk might have upon its aggregate dynamic behaviour might be. To explore the implications of risk management and what its aggregate dynamic implications might be, we introduced an endogenous mechanism to a New Keynesian business cycle model to correct its NK rigidities and imperfections. The 1993 model of Greenwald and Stiglitz is well suited to exploring the effects upon aggregate output dynamics of risk management innovation given its NK rigidities and imperfections. The NK rigidities and imperfections, explicitly and implicitly, are embodied in several features of the GS model. Firstly, the constraint on the raising of new equity forcing firms to borrow debt capital. Secondly, a lag between when working capital is borrowed and when stochastic returns, unique to each firm, are known. Thirdly, that the cost of capital is not tied to performance or risk adjusted to changes in capital structure. And fourthly, the returns to equity holders are not adjusted to the required returns to share holders given the risk arising from changes in capital structure.

By introducing risk management as an innovation to the GS model, rather than reducing output in response to firm specific price uncertainty to reduce the cost of potential bankruptcy, firms may now modify their exposure, but at a cost. In the model as GS claim and as we have shown using numerical simulation, increasing price volatility (or lowering expected prices) reduces aggregate output and makes the economy more stable. In their model, the lag between the decision to borrow working capital to enter into wage contracts and when returns are realized, we saw, creates a threat of bankruptcy because firm specific prices are stochastic. The relation between uncertain prices and a bankruptcy threshold creates a form of risk
aversion through marginal bankruptcy costs driving a wedge into traditional cost structure. Under the model’s lag structure, working capital is borrowed and workers are retained without knowledge of future returns which are stochastic and firm specific. In the GS model, the fear of bankruptcy serves as a deterrent to borrowing, making the issue of how the model might behave if state-contingent assets were introduced, an interesting one. The question is especially interesting in view of the many unsubstantiated assertions, as we have noted, that risk management has macro implications and that these same implications are stabilising and desirable. GS assume that no means exists, such as futures markets, to ensure that prices received at least equal the bankruptcy threshold, begging the question of how the model may behave if an endogenous mechanism to manage risk were available as well as offering related insights.

In this context, we considered both the model’s comparative static and dynamic behaviour if an endogenous mechanism for risk management were to exist, allowing us to answer the question of what the macro implications of its use are, if any. The question is particularly illuminating in light of the many casual and unsupported claims that the management of financial risk at the firm level is necessarily beneficial in the aggregate. Remarkably, we have found through simulation of the GS model that the opposite proposition holds. Namely, that although firms individually may gain from the management of risk, leading to expanded output and greater profits, in the aggregate such behaviour creates a macro externality in the form of cyclic instability. With down-side exposure eliminated, firms are encouraged to borrow working capital and commit to a wage-bill. When prices are low, the effect in reducing profits, output and
workers required, is enhanced. Moreover, as per the GS model, with down-side price exposure eliminated, the firm, because of its greater borrowing and wage bill, now has greater exposure to the random attractor, $u_t$. The intuition behind this insight lies in the nature of derivative markets. Although the firm hedging its revenue exposure has eliminated down-side exposure, it has come at a price. More generally, further intuition is found in the nature of derivative markets. With greater capital at risk, the effects of economic contraction are more severe because although the representative firm has hedged its price and revenue risk, a party opposite to the transaction, who sold the Put Contract faces huge losses. (Although we have not modelled general equilibrium markets for options contracts, per se, in the simulated GS model, such observations based upon casual empiricism inform our discussion of these results, providing intuition.) Recall, derivative markets are only zero sum games in a tautological sense, for every winner there is a loser, but are not so in a quantitative sense. In purchasing a Call Contract, for example, the maximum the buyer may lose is limited to the commission paid to the seller, however the latter's potential loss, should prices increase, is unbounded. In the case of modified GS style model, although the representative firm has ensured a minimum value of its future sales receipts by purchasing put contracts, a party selling the put may face huge losses, which would be exacerbated during an economic contraction as he finds himself holding inventories for which there are no markets. Hence, introducing risk management by encouraging commitments to a greater wage-bill and greater dividends, renders the economic down-turn more severe, because state-contingent assets in the form of futures and options are not zero sum games in the
quantitative sense. The party writing the put has accepted an asymmetric exposure which is exacerbated during an economic contraction. (The liquidity crisis in the Autumn of 1998, in which firms were unable to close-out derivative positions because of the absence of counter-parties, is a recent example.) The management of risk at the firm level may have its attractions, but it also has destabilizing implications in the aggregate when all parties are taken into consideration.

7.1 Implications for Research

What value have our findings for the science of economic modelling in general and the use of New Keynesian models with a financial constraint or financial market imperfection? With regard to the evaluation of models featuring non-linear dynamic relationships, as is well-known in the physical sciences, their ready interpretation using either analytic methods or linearization approaches impose many limitations upon research (Chapter 1, Baker & Gollub, 1990). This point is illustrated in much of the contemporary research into non-linear dynamics, particularly involving endogenous cycles. For example, the survey by Boldrin and Woodford (1990) of non-linear equilibrium models highlights the effect upon dynamic behaviour of endogenous mechanisms and underscores the importance of numerical simulation to reveal both periodic and chaotic behaviour. Non-linear growth models as mentioned in the survey by Scheinkman (1990) and over-lapping generation models as surveyed by Kelsey (1988) present many examples of such fruitful research. The availability of numerical simulation as means of investigating non-linear endogenous cycle models also invites the potential for
revisionism, especially with regard to the rational expectations hypothesis. As noted by Mulllineux, Dickinson and Peng (page 45-47, 1993), much of New Classical modelling has made extensive use of log-linear formulations (eg. Sargent & Wallace, 1975) with minimal a priori justification. Grandmont’s article (1985) in which he demonstrated that a deterministic model which is non-linear and assumes rational expectations, may produce complex dynamics but with non-neutral money supply effects, lends support to the view that the assumption of log-linearity should be abandoned. Given the widespread availability of software capable of numerical simulation, non-linear methods of modelling, arguably, should be accepted as a standard of practice to learn the dynamic behaviour of both New Keynesian as well as New Classical style models. The availability of techniques for modelling complex dynamics also highlights the limitations of comparative static analysis as a tool for macro economic modelling. As is accepted in engineering and physics, in order to understand the laws of motion for persistent phenomenon, appropriate techniques are required, including non-linearity and a higher dimensional systems.

The second set of observations with regard to the research implications of our findings relates back to the remarks in Chapter One on the Samuelson principle of correspondence. We have discussed that New Keynesian endogenous business cycle models are not simply about tardy adjustments to Walrasian equilibrium, but rather concern constraints and imperfections precluding optimization. Sub-optimization and second order arguments figure strongly in the NK literature, however the interaction and connection between economic units and aggregate stability remain weak. The GS model
like many models found in NK research uses partial equilibrium micro imperfections to achieve general macroeconomic imbalances and perturbations from the equilibrium steady-state. In doing so, it places great weight upon narrow foundations. In creating models which rely upon imperfections of a single industry or a market to motivate an entire economy, the role of other economic agents are ignored. In the view of Zarnowitz (1992), New Keynesian research leaves one with many small stories, as opposed to one which is general. (Including parties to the other side of futures transactions or debt and equity holders, might be some examples.) Based upon our research, the view that either sub-optimization must be modelled in the aggregate or that greater micro-market interaction between economic agents should be considered, in order to achieve correspondence, would appear both desirable and sensible.

Further implications of our research relate to the subject of persistence. The GS model, resembling much of New Keynesian macro theory, relies upon endogenous specifications to shift Aggregate Supply, allowing cycles to persist even without outside influences. In relying upon endogenous mechanisms as the source of shifts in Aggregate Supply and persistence, New Keynesians have accepted a weighty challenge: Exogenous influences in the form of systematic perturbations are rejected as unable to produce the recurrent sequences of expansions and contractions known as business cycles. (Even the random attractor found in the GS model, of itself, would not be sufficient to generate cycles.) On the other hand, considering Real Business cycle models, we ask, how much economics is there, after all, in appealing to technological shocks to the production function to create shifts in Aggregate Supply?
Properly, apart from natural disasters or meteorological processes, is not all economic activity and events somehow endogenous? It might be argued that such perturbations as technological shocks are not different, in spirit, from Jevon's sun-spots model of 1884 or Moore's weather driven cycles of 1914 (see, chapter one, Morgan, 1992). Historically some endogenous models, such as those of Frisch (1933) as discussed in Chapter Two, have used endogenous propagation mechanisms to convert exogenous perturbations into cycles. In contemporary research like the GS model, the internal dynamics of credit and capital formation alone interact through wages and profits to have persistent effects upon real economic activity. For how long the cycle may continue, that is for how long the effects of sub-optimization may persist, appear indefinite. Perhaps such observations argue for the introduction of further endogenous mechanisms into NK models as we have done. Or maybe, in the manner of Frisch, perhaps a model of business cycles should include exogenous forces, in the form of financial market innovations or new technology interacting and correcting persistent endogenous defects. We need not always be limited to a model involving only one story.

As indicated by the results of our research, the inclusion of additional endogenous innovations or mechanisms to inter-act with a persistent sub-optimization mechanism, may be rewarding. New Keynesian modelling such as that of GS avoids the issue of how long such endogenous mechanisms may persist. In financial constraint and financial market imperfection models such as that of GS, endogenous mechanisms only contribute to economic instability. Much of New Keynesian literature, by concentrating upon endogenous sources of defects, would appear to ignore the possibility
of autonomous innovations, endogenously solving problems in the manner once suggested by Schumpeter (1939). To disregard the possibility that endogenous mechanisms might lead to further autonomous innovations correcting faults is to adopt an essentially static world-view. To disregard such possibilities is to reject the notion that economic systems are ultimately self-optimizing. In this spirit, we introduced risk management into the GS Model to examine its effects upon cyclic behaviour. New technology or innovation, although difficult to specify, remains a reality, and may be important for its ability to correct or modify the results of sub-optimization and second order effects persisting through into economic cycles. As we have found, with the potential for unanticipated aggregate dynamic effects. In view of our finding that economies with greater productivity are more stable, modelling technological innovation should be a particularly useful topic. (According to some analyst the continuation of economic expansion over the last two decades in the United States, appearing to replace traditional cycles, has been facilitated by the savings in cost arising from information technology.) In light of the above, a productive line of inquiry in such research, might be to consider endogenous mechanisms and autonomous forces which both contribute to cycles in the manner of Frisch, as well as dampen them and inter-act with the phenomenon of persistence. Such lines of reasoning may also hold suggestions for the connections between business cycle theory and the theory of economic growth which perhaps has been obscured by the tendency to assume little or no coherence between very low frequency oscillations (trend) and oscillations of a medium frequency which we call business cycles.
Apart from the broader business cycle issues, our research also holds interesting implications for the macro economic implications of risk management. As a fallacy of aggregation, we have shown the identification of micro-economic gains with macro benefits is without merit. Financial innovations, such as risk management are a reality of the economic landscape, however their ready interpretation and wider economic significance is not easily analyzed. Our modified version of the GS model invoked "small-country" assumptions with regard to risk management. At the prevailing price, if it is economic, firms may purchase as much of it as required to manage exposure. The supply of Put Contracts was made horizontal. In the modified GS model, risk management as it encourages investment has consequences during the contractionary phase of the business cycle. The intuition behind these results relates to the nature of derivative markets, as mentioned above in Section 7.0, and the fact that hedging may occur only one period into the future, while down-turns in the economic cycle may persist for many quarters. Continued hedging may occur, but often it is into a bear market. In this light, methods to establish long-term or perpetual claims on aggregate income, although they do not exist, may be beneficial (Schiller, 1995). All of these results point to the potential value of research into modelling the aggregate dynamic effects of the use of derivative markets such as futures and options. Although considerable effort has been made into the understanding of futures markets at a micro and firm level, general equilibrium modelling in a macro context may represent useful lines of inquiry.
7.3 Implications for Public Policy and Regulation

Our research into the dynamic behavior of a New Keynesian business cycle model having a financial constraint and imperfections along with our research into its behavior when endogenous innovations are introduced may have implications for financial market regulation relating to the use of derivative products as well as the general conduct of macro economic policy. Although a comprehensive discussion of such subjects is beyond the scope of a concluding chapter, we will in Section 7.3, comment on what our findings into the aggregate dynamic effect of derivatives and risk management may hold for the regulation of financial markets, in particular such issues as systemic risk, transparency, and macro disclosure arising from their usage. As discussed in Chapter Five, through various forms of aggregation fallacies, a considerable amount of writing has given support to the view that as risk management, using futures and options, appears to be desirable for individual economic agents, it must be good for the economy as a whole. Contemporaneously, fuelled by recent economic crises, considerable attention has been given at the macro level to concern over the possibly destabilizing effects of derivative products in a world combining capital mobility and floating exchange rates (see Crockett, 1996, for a useful survey) because they combine great financial leverage or gearing with the possibility for a sudden change in value. Situations of illiquidity may be amplified by derivative markets, precisely because of gearing, and when counter-parties to risk management positions may be unable to close-out open positions. Looking at some of the reasons why derivatives may have destabilizing effects in the aggregate is revealing.
The possibility that the use of derivative markets may actually contribute to systemic risk as a form of moral hazard has involved appeals to a number of stylized phenomena. According to some authorities, the Basle rules with regard to capital adequacy interacts with credit to reinforce economic cycles by increased lending during expansions followed by sharply reduced lending during cyclic contractions (Plender, J. "Taming Wild Money", Financial Times. page 21, October 21, 1998). Moreover, according to the same author, widely used Value at Risk models create further problems by reducing credit as volatility increases.... precisely when greater credit might be useful because of the changes in market value.\footnote{Such views are echoed by C.A.E. Goodhart of the Financial Markets Group of the London School of Economics in an unpublished paper entitled, Financial Globalization, Derivatives, Volatility and the Challenge for the Policies of Central Banks.} In this context, derivative markets with their great power for financial gearing are seen as potentially destabilizing. Such viewpoints harken back to the works of the Classical Economists on the effects of gearing, causing rapid changes in the availability of credit, as in Hawtrey's *drain-of-cash* from the banking system model (1913). Making matters worse, trading activities involving derivative positions by financial institutions are often held off-balance sheet, making it difficult to judge the exposure of profits, and hence overall market value, to such positions (Foikerts-Landau, Ito, et al. Chapter II, "International Capital Markets: Initiatives Dealing with Derivatives", IMF: World Economic and Financial Surveys. 1995). According to the same study, accounting standards do not adequately record over-the-counter positions in derivative markets. Such standards do not adequately ensure sufficient over-sight by regulatory bodies and do
not allow investors to adequately assess the risks to market value. Although value-at-risk and capital-at-risk modelling is widely utilized, there is no guarantee that banks and other institutions taking positions in derivative markets indeed use the same model. Complicating the situation is that the cumulative risk of various derivative positions may be less or more than the risks arising to positions in the underlying individual securities. Unlike normal securities, strict additivity does not hold. Matters are made worse by the fact that positions in derivative markets inter-act. A simple linear relationship between exposures arising from derivative positions cannot be assumed. The value-at-risk of derivative positions, with movements in underlying securities and markets, may change dramatically in a matter of minutes. Typically, institutions using derivatives, as a matter of policy, and as encouraged by the Bank for International Settlements and the US Financial Accounting Standards Board, mark-to-market positions on a daily basis. If, however, investors and traders from other institutions, correspondent institutions for example, even using on-line, real-time information technology such as Reuters or Bloomberg are only up-dated on a quarterly, or worse, annual basis, it may reduce the value of such information as a positive externality. Transparency may not be improved. Without transparency the potential for diminishing the severity or reducing the frequency of crises may be limited. According to the Fisher report as published by the BIS in 1994, the reporting of comparisons between ex ante value-at-risk and actual losses, would be beneficial (Fisher, 1994) however it has yet to occur. Unless compelled to report such comparisons, one must wonder what the commercial incentives would be (Folkerts-Landau,
Ito, et al. 1995). In addition, the US Financial Accounting Standards Board has recommended the reporting of individual positions rather than the common practice of netting-out positive and negative derivative positions, because the exposures arising to such positions may not symmetric and off-setting (SFAS No. 119, 1994). Of course, however valuable such recommendations and initiatives may be at the microeconomic level of individual banks and institutions; they do not tell us anything of their benefit in the aggregate or over-time. When one banks illiquidity or credit crunch will transmogrify into a systemic financial crisis is anyone’s guess.

In order to appreciate the aggregate benefit in terms of macroeconomic stability of reporting and understanding value-at-risk arising from the use of derivative products, we need to take-on-board the issue of financial instability. The literature on macro financial crises and instability is old as economics itself and is beyond the scope of the present discussion, although we can say, that critical synthesis of various views and models has yet, if ever, to arrive, as we may gather from selected research. The monetarist view was that real effects arise when monetary disturbances lead to banking panics (M. Friedman & A. Schwartz, 1963). Minsky (1977), as a Neo-Keynesian, linked potential crises to over-accumulation of short-term debt. Blanchard and Watson took a rational expectations approach to the modelling of speculative bubbles (1982). While in a model using a dynamic feedback specification, De Long, Bradford, Shleifer, Summers and Waldmann (1990), argued that rational speculation leads to destabilization. According to Spotton, traditional models which price assets according to the present discounted value of expected income are inadequate because they do not include how
institutional factors shape the formation of expectations (Spotton, 1994). According to Spotton, credit financed speculation may play a critical role (page 183, ibid, 1994). In an article by Bernanke (1981) crises occur because of real effects from changes in the cost of credit. From the above sketch, we see, stretching back to the classical economists, virtually every school of business cycle theory has attempted to explain financial crises. Looking over their models of business cycles and their attempts to explain the existence of financial crises and markets collapses, we observe that their empirical support is usually limited to the examination of a particular crisis. Quoting Zarnowitz (page 110, 1992), "There is no consensus on the nature of financial crises and their role in business cycles."

Where does this leave us with regard to the macroeconomic impact of derivative products, and whether their usage is potentially destabilizing? In addition to the micro-phenomenon described above, are there reasons why derivatives merit special concern from the standpoint of systemic risk to the financial system? Are further policy initiatives called for in order to facilitate macro disclosure and transparency, in order to prevent illiquidity in one derivative market or one of class of security leading to a general crises? We have seen from our discussion of risk management in Chapter Five, that from the standpoint of the party hedging, the notion that either market or idiosyncratic risk is actually reduced through using derivatives is questionable. As we argued, derivatives have more to do with ensuring that we earn returns commensurate with our level of market

risk, $\beta$, by ensuring good asset utilization, rather than reducing market risk which adds nothing to market capitalization. Through our dynamic modelling of the GS model and modifying it by adding state-contingent assets, derivatives actually lead to greater risk taking because down-side exposure was limited, with aggregate implications as measured by the stability of output. In Chapter Four, risk aversion because of price uncertainty discouraged investment, dampening oscillatory behaviour. Risk taking encouraged by the use derivatives, during an economic expansion leads to expanding the market value of the firm, however, during the contractionary phase of the cycle, there are still commitments with respect to working capital borrowed to pay wages and dividends to distribute. Even though the firm’s revenues are guaranteed through having purchased Put Options, prices are still below their upside potential. The result is that the taking of derivative positions as it leads to expansion of the capital base through borrowed working capital, is destabilizing, creating greater risks to book value during an economic contraction.

Further intuition for the macro destabilizing effects of these results were found in the threat to book value faced by debt holders, as discussed in Chapter Six, and the observation, noted earlier that derivative markets for both hedgers and speculators may involve highly asymmetric outcomes. Aggregating across different parties, positions may not be netted-off. At a macro level, one may not assume that hedgers are not net-short and speculator are not net-long (Williams, 1994). During a sharp economic down-turn, because of the asymmetry of exposure, although a firm having hedged, may only lose the price of the option(s), the counter-party to the derivative transaction may lose far greater amounts. Their value-at-risk is not
equal. Under conditions of financial market imperfections, further research and modelling of derivative markets in a general equilibrium setting may be useful to learn greater of their systemic dynamic effects and may be instructive in the formulation of regulation and policy.
APPENDIX 1

GENERAL AND SPECIFIC FUNCTIONAL FORMS
AS USED IN SIMULATION PROGRAM
<table>
<thead>
<tr>
<th>FUNCTION NAME</th>
<th>GENERAL FUNCTIONAL FORM</th>
<th>SPECIFIC FUNCTIONAL FORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production Function</td>
<td>$q_t^i = \Phi(1_t^i) \Phi' &gt; 0 \Phi'' &lt; 0$</td>
<td>$q_t^i = \ln(1_t^i)$</td>
</tr>
<tr>
<td>Basic Cost Function</td>
<td>$(1 + r_t) (w_t \Phi(q_t^i) - a_t^i)$</td>
<td>$(1 + r_t) [w_t \exp\left(\frac{q_t^i}{k}\right) - a_t^i]$</td>
</tr>
<tr>
<td>Bankruptcy Cost Function</td>
<td>$c_t^i F(v_t^i)$</td>
<td>$(\exp\left(\frac{q_t^i}{k}\right)) F(v_t^i)$</td>
</tr>
<tr>
<td>Bankruptcy Threshold</td>
<td>[ P[v_{t+1}^i(q_t^i, a_t^i, w_t, \frac{P_t}{E(P_{t+1})}, 1+r)] ]</td>
<td>[ \text{Prob. (Bankruptcy)} = P(u_{t+1}^i &lt; v_{t+1}^i) = \int_{-\infty}^{v_f(u)} f(u) , du ]</td>
</tr>
<tr>
<td>----------------------</td>
<td>-------------------------------------------------</td>
<td>-------------------------------------------------</td>
</tr>
<tr>
<td>Bankruptcy Occurrence</td>
<td>[ u_{t+1}^i = \frac{P_{t+1}^i}{P_t} \leq v_{t+1}^i ]</td>
<td>[ u_{t+1}^i &lt; (1+r_t) \left( \frac{P_t}{P_{t+1}} \right) \frac{w \exp(q_t^i) - a_t^i}{q_t^i} = v_{t+1}^i ]</td>
</tr>
<tr>
<td>Aggregate Labour Demand</td>
<td>[ l = \Sigma l_t^i = \phi(q_t^i) ]</td>
<td>[ l_t = \Sigma l_t^i = \Sigma \exp\left(\frac{q_t^i}{k}\right) ]</td>
</tr>
<tr>
<td>Aggregate Labour Supply</td>
<td>[ l = s(w_t) \quad s &gt; 0 ]</td>
<td>[ \text{Linear function of wages, origin}(0,0) ]</td>
</tr>
<tr>
<td>Aggregate Output</td>
<td>$q_t = \sum q^i_t w(q_t), r_t, a^i_t, \sigma, \gamma)$</td>
<td>$q_t = \sum q^i_t$</td>
</tr>
<tr>
<td>------------------</td>
<td>-----------------------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>Dynamic Simulation</td>
<td>$a_{t+1} = q_t - (1+r) (w \phi(q_t) - a_t) - m(a_t)) = G(a_t)$</td>
<td>Iterative dynamic numeric simulation.</td>
</tr>
</tbody>
</table>
REFERENCES


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Slutsky, E., "The summation of random causes as the source of cyclical processes", Econometrica, 5, 1937, 105-146.


