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Preamble

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ELECTROMAGNETIC SCATTERING BY
PERFECTLY CONDUCTING BODIES

A thesis submitted to the Faculty of
Mathematical and Physical Sciences of
the University of Surrey, for the degree of
Doctor of Philosophy

by

Jeremy John Gribble

October 1981
ABSTRACT

A number of approaches to the problem of scattering of electromagnetic waves by metallic bodies are examined, in the context of earlier work done at the University of Surrey under a contract from the Royal Aircraft Establishment at Farnborough.

A review of this earlier work is given, followed by some fundamental electromagnetic theory and a discussion of solutions of the two dimensional wave equation.

This leads naturally to the introduction of the Rayleigh hypothesis for scattering from irregularly shaped cylinders.

The analytical continuation method of Wilton and Mittra for overcoming the problem of the invalidity of the Rayleigh hypothesis is given, and its reformulation in matrix notation motivates discussion of the properties of the scattering matrices of bodies which gave a number of symmetries. Because the scattering matrix of a body, depends only on the body and not at all upon the incident field, any symmetries of the scatterer must be reflected in the form of the scattering matrix. These results are used to discuss the boundary condition problem for a perfectly conducting body of rotation.

Chaper eight contains an attempt to extend aperture field theory as used for calculating near-axis fields in, for example, reflector antenna systems to more general scattering problems. Some related work by Bach and his associates is discussed.
In chapter nine an alternative approach to the calculation of the scattering matrix is given which leads to a reformulation of the method of physical optics. The modified method is used to investigate the validity of the Rayleigh hypothesis for a slightly perturbed circular cylinder and the results compared with those due to Van Den Berg and Fokkema.

Finally, in chapter ten, another method of overcoming the invalidity of the Rayleigh hypothesis is examined, but is shown to be impractical to implement numerically.
ACKNOWLEDGEMENTS

The author would like to express his gratitude to Dr. S. Cornbleet for his supervision and assistance during the last three years.

Thanks are also due to Mrs. M. Fortuna who typed the manuscript with great efficiency and considerable patience.

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§GI.1 General Introduction

The purpose of this introduction is to set the work of the following thesis into perspective with its historical background and other work in the field. The work has been concerned with the problem of predicting the radiation patterns of antennas mounted on aircraft. Successful techniques exist for the regions in which the aircraft is small or very large compared to a wavelength. The aim of this study was to investigate methods which could be applied to problems in the intermediate frequency range. Three of the most important of the established techniques will be described in some detail at the end of the introduction.

A contract (Contract No. AT/2064/037RL) was given by the Royal Aircraft Establishment at Farnborough to the University of Surrey at Guildford for the investigation of 'Radiation from antennas mounted on irregularly shaped bodies'. Work was performed in the Physics Department under the supervision of Dr. S. Cornbleet by Dr. R.J. Chignell, Mr. D.H. Munro and Mr. J.J. Gribble. Some of the work of Chignell and Munro was done in collaboration. The best way to describe the contributions of these workers is to describe the contents of the various reports which were produced in chronological order (References 13, 14, 15 and 16).

§GI.2 Chignell, July 1975

This report was mostly concerned with the radiation patterns of aircraft more in the low frequency range, which, it was shown, could be represented as simple combinations of a few multipole fields. Thus a simple model of the scattering properties of an aircraft consisting of crossed dipoles was envisaged. This led to relatively
simple expressions for the radiation pattern in one plane like

$$\cos \theta + a \sin \theta \cos \theta$$

where $a$ was an unknown constant. Unfortunately the theory gave no way of calculating such unknown constants, save by comparison with solutions obtained using other techniques, or by the circular procedure of optimisation by comparison with experiment. The particular problem of a short monopole on an infinite cylinder was considered. The radiation pattern was expanded as a linear combination of spherical harmonics, and the coefficients in the expansion were obtained by comparison with an exact solution due to Wait.

Chignell introduced the idea of a scattering representation of antenna problems, in particular, the concept of the scattering matrix. Thus if the field were expanded as a linear combination of modes of some sort, then the column vector of coefficients describing the scattered field could be obtained from the column vector of coefficients representing the incident field due to the antenna by a matrix multiplication (of course, multiple scattering effects should be accounted for). In this manner he considered the problems of

(a) a line source above an infinite semi-cylindrical boss on an infinite ground plane (a simple model of antenna, fuselage and wings) and

(b) a line source above three infinite semi-cylindrical bosses and a ground plane (a simple model of a twin-engined aircraft). He did not include the effect of multiple reflection by two curved surfaces.
In this report, the multipole expansion method was essentially abandoned, and the most attention diverted to the scattering description of the problem. (The former was dropped because it became necessary to include multipoles of complex dimensions. With the usual accuracy of hindsight, in view of the interest shown in recent years in the use of sources with locations in complex space (for example Felsen in reference 29) this was unfortunate). He calculated the radar-cross-section in terms of the scattering matrix and began to investigate the problem of finding the scattering matrix for infinite cylinders of arbitrary cross-sectional shape. This is essentially a two-dimensional problem. This introduces the problem of the validity of the so-called 'Rayleigh Hypothesis' for the following reasons.

To calculate the scattering matrix, the field is expanded as a linear combination of modes, so that the coefficients in the expansion which describe the scattered field are the unknowns. These coefficients can be found by applying the appropriate boundary conditions at the surface of the scatterer. Now the form of the expansion which is used to calculate the radiation pattern well away from the scatterer may not be the same as the form used to calculate the field close to the scatterer. The Rayleigh Hypothesis states that the same expansion may be used in both cases. In fact, the Rayleigh Hypothesis is not generally valid.

Chignell noted a procedure devised by Wilton and Mittra (reference 11) by which one could 'analytically continue' the expansion of the field valid for points away from the scatterer, up to the surface of the scatterer, so that the boundary conditions could be used. He
applied this procedure to the calculation of the scattering matrix.

He reworked the problem of a line-source above a semi-cylindrical boss on an infinite ground-plane. He treated the ground plane as an image plane, which is an erroneous procedure. (The image by reflection in the semi-cylinder is not the same as the image by reflection in the plane. Also its position depends on the direction of observation). The results that he produced were not compared with experiment, and were really only a study of the numerical effects of truncating the infinite series which appeared in the solution. He extended his computer program to cover the case of the 'twin-engined aircraft'.

He then gave the first discussion of the problem of how to extend the method to three-dimensional problems. The Wilton-Mitra procedure involves the use of results for the transformation of solutions to the wave equation under translations of the co-ordinate system. The discussion was concerned largely with the form that these transformations take in three dimensions.

§GI.4 Chignell, April 1976

The first part of this report was given over to a summary of the two-dimensional scattering theory. A preliminary discussion of a test of the two-dimensional theory was given. The problem chosen was that of a perfectly conducting infinite square cylinder with a parallel line source lying close to one corner, and in the diagonal plane of the square (Figure GI.1). No results were presented in this report, attention being confined to the numerical problems of computational implementation.
Most of the rest of this report was concerned with the extension of the method to three-dimensions. In particular:

(a) It was pointed out that the number of modes needed to represent a scattering problem with a given degree of accuracy depends quadratically on the size of the scatterer in wavelengths for a three-dimensional problem, but only linearly for a two-dimensional problem. So, for a fixed computing capacity, the maximum size of three-dimensional scatterers which could be solved would be much smaller than the maximum size of two-dimensional scatterers.

(b) Thus he was motivated to consider the scattering from bodies with rotational symmetry for which the number of modes required would be a lot less than in the full three-dimensional problem. The idea was that an aircraft fuselage could be treated as such a body and the effect of wings, tail, etc. included as perturbations.

(c) Further reduction of the number of modes required was possible by considering the geometrical symmetries of the antenna/scatterer system thus decreasing the computing power needed.

§GI.5 Munro, February 1977

After a summary of the two-dimensional scattering theory, the bulk of this report is given over to the test problem mentioned above. There is an extensive discussion of the numerical properties of the computer program. In particular:

(a) The stability of the solution as the number of boundary points representing the scatterer is changed.
(b) The convergence properties of the solution as the number of modes used is increased.

The agreement of the results with experiment was good.

§GI.6 Munro, Summer 1978

A proper typed copy of this report was never produced, and the handwritten version that exists is incomplete in some respects. This is particularly unfortunate since it contains the first actual theory of the scattering matrix of a body of revolution. Fortunately his procedure may be described quite accurately verbally. A more detailed mathematical examination of the problem is given in the body of this work.

Munro considered a body of rotation arranged so that its axis coincided with the x-axis of a system of cartesian co-ordinates (Figure GI.2). He postulated that because of the rotational symmetry of the body, it is sufficient to apply the boundary condition only at points on the scatterer which lie in the xy-plane. In what follows, this procedure will be justified, with some reservations.

Then, he set up a system of spherical polar co-ordinates \((r, \theta, \phi)\) such that the z-axis was given by \(\theta = 0\), and the xy-plane by \(\theta = \frac{\pi}{2}\). Following a customary procedure in electromagnetic problems he expressed the field as a vector sum of transverse components (i.e. the \(\theta\) and \(\phi\) components). No generality is lost, since the radial component may be obtained from Maxwell's equations. Unfortunately when he came to apply the boundary condition he said that the total transverse field is zero, at the surface of the scatterer, which is incorrect, and forgot about the radial field component altogether.
The equations that he went on to derive contained no coupling between the \( \theta \) and \( \phi \) components.

The rest of the report contained remarks about the two-dimensional theory, calculation of the fields due to various kinds of incident field sources, notes on the numerical aspects of the computing, and some miscellaneous mathematical results.

\textbf{SGI.7 Gribble, December 1979}

This report was intended as an introduction to, and critique of Munro's work. The point was made that the scattering matrix depends on the scatterer only, and not the incident field.

Explicit consideration was given to the Transverse Electric polarisation case of the two-dimensional problem. This had not been done by either Chignell or Munro, and was probably the reason why Munro used the wrong boundary conditions in his three-dimensional development. This error is equivalent to assuming that locally the scatterer has a constant radius of curvature.

It was shown how in the low-frequency limit a 'wire model' representation of the two dimensional scatterer is compatible with the expansion of the field in terms of modes, and a new method was proposed by which the scattering properties of a two/three dimensional body would be represented by a two/three dimensional array by line/point sources. The unknowns in the problem were the strengths of the line/point sources. Some work was done on this, but it did not get very far, since the location of the sources turned out to be more important than first thought, and no rational scheme could be devised to do this. It is interesting to note that this problem has been
solved by Chow and Chaudhuri in reference (40) who, using a computer routine, varied both the strengths and positions of the sources, to optimise the solutions to various types of boundary condition problem.

§GI.8 Geometrical Optics

There are a number of different approaches to the subject of Geometrical Optics. Up to the mid-nineteenth century, it could have been considered as a self-contained discipline with its own basic postulates and theorems. However, this viewpoint is now inappropriate in a discussion of Geometrical Optics as a technique for performing Electromagnetic Scattering Calculations.

A more rigorous approach is via the so-called Luneberg-Kline expansion (reference (1) p.26) and this will be discussed briefly for the case of a scalar field which obeys the Helmholtz wave equation

$$(\nabla^2 + k^2)u = 0.$$ 

An asymptotic expansion of the field is made of the form

$$u \sim \exp(jk\phi) \sum_{n=0}^{\infty} u_n(jk)^{-n} \quad \text{In the limit as } k \to \infty$$

Substituting this form into the wave equation one obtains the 'Eiconal Equation' for the phase function $\phi$ and a recurrence equation for the $u_n$. The Geometrical Optics field is usually taken to be the first term in this series corresponding to propagation with an infinitesimal wavelength. Although this is undoubtedly the most rigorous way of linking Geometrical Optics with field theory, a more empirical approach will be adopted here. The problem is simplified because we only consider scatterers which are perfect conductors, i.e.
those which are perfect reflectors, in terms of Geometrical Optics. Further simplification will be achieved by restricting attention to a scalar field.

It is postulated that:

1. At each point in free space there exists a value of the complex scalar field, described by an amplitude and a phase.

2. There is an associated energy density, which is proportional to the square of the amplitude.

3. This energy propagates away from source regions along paths of energy flow called 'rays'.

4. In free space the rays are straight lines. The phase associated with a given ray changes uniformly at a rate of $2\pi/\text{wavelength}$ with the distance moved down the ray.

5. The amplitude associated with a given ray is governed by the conservation of energy as applied to a small 'tube' of rays near the ray in question.

6. A ray impinging on a scatterer gives rise to a 'reflected' ray, which travels in a direction given by the law of specular reflection, the amplitude of which at the point of reflection is equal to that of the incident ray there, but the phase of which differs by $\pi$.

7. The field at a given point is the phasor sum of all the associated fields of rays which pass through that point.
The most important point to be noted here is that postulate No. 6 implies the formation of 'sharp' shadows. That is, Geometrical Optics predicts no diffraction phenomena. Postulate No. 5 predicts the inverse first power law for the amplitude of the field due to a highly localised source. Because the exact Geometric Optics solution of a problem is often easy to find compared to the exact solution based on a finite wavelength, it is useful to state a

Correspondence Principle for Electromagnetic Problems

The high frequency limit of an exact solution to an electromagnetic problem must tend to the Geometric Optics solution in that limit. If the two do not agree, then the former is wrong.

§GI.9 The Geometrical Theory of Diffraction

The geometrical theory of diffraction (GTD) is regarded as an attempt to extend Geometrical Optics to take account of diffraction effects. The classical paper on GTD is that of Keller (reference 41). More recent developments may be found in references (4) and (5).

Geometrical Optics recognises only one kind of interaction of rays with scatterers: reflection. GTD introduces other kinds of scattered rays. For example: (Figures GI.3-6).

(i) A ray incident upon an edge on the scatterer gives rise to a cone of 'edge diffracted rays'. The amplitude and phase with which one of these rays is launched is given by the product of the complex amplitude of the incident ray, and the 'edge-diffraction coefficient'.
(ii) A ray which is incident tangentially on a scatterer can become trapped, and travel on the surface of the body, following a geodesic as a 'creeping-ray'. The amplitude of the creeping ray decays, due to the emission of other rays tangentially at each point on its route.

(iii) An incident ray which strikes the tip of a vertex can give rise to 'vertex-diffracted' rays, travelling in directions exterior to the body whose complex amplitudes are given by the product of the complex amplitude of the incident ray with a 'vertex diffraction' coefficient.

The diffraction coefficients, and the decay rate of creeping rays can be obtained by comparison with known exact solutions for problems which contain the appropriate diffracting feature. For example, the edge diffraction coefficient can be obtained from the solution for the problem of a plane-wave incident upon a semi-infinite half plane.

Just as in Geometrical Optics, a ray may suffer multiple reflection, in GTD, a ray may undergo any combination of reflection, edge diffraction, tip diffraction etc. on its travels. Geometrical Optics and GTD are 'Asymptotic Theories', in the sense that when the wave-length is very small compared to the physical scale of the problem, for a given level of accuracy only rays which have undergone a small number of such scattering events are significant. As the wavelength increases rays which have reached the field point via more and more tortuous routes have to be included to maintain the quality of the solution. In recent years, various other asymptotic diffraction theories have been devised, but these will not be discussed.
§GI.10 A Low-Frequency Technique: Integral Equations and the Method of Moments

Geometrical Optics and GTD are asymptotic methods in the high frequency limit. A technique which is more appropriate in the low frequency limit will now be discussed.

There are many variations on this technique, but the general principles will be illustrated by a single example that is taken from a paper by Miller (reference (20)).

A mathematical model is made by a continuous metallic structure as a network of wires. The incident field is considered to induce currents upon the wires which give rise to the scattered field. The boundary condition is that the total tangential field at each point on the original structure is zero. Some simplifying assumptions are made:

1. The circumferential current is negligible

2. The circumferential variation of the longitudinal current can be ignored

3. The wires can be treated as thin in the sense that the boundary condition need be applied at only one point on the circumference, and that the full surface integration may be replaced by a line integration along the wires.

Miller then gives the Pocklington-type integral equation (there is not a unique integral-equation formulation of this type of problem) for a wire structure of contour $C(r)$ as
\[ s \cdot E_{\text{incident}}(s) = \frac{j \omega \mu}{4\pi} \oint_{C(r')} I(s')G_0(s,s')ds' \]

where \( s \) and \( s' \) are points on \( C \), and

\[ G_0(s,s') = \left[ s \cdot s' + \frac{1}{k^2} (s \cdot v)(s' \cdot v) \right] g_0(r,r') \]

\[ g_0(r,r') = \frac{\exp(-jkR)}{R} \]

\[ R = |r - r'| \geq a(r) \]

\[ \hat{s} = \frac{vC(r)}{|vC(r)|} \quad \hat{s}' = \frac{vC(r')}{|vC(r')|} \]

\( g_0(r,r') \) is the free space scalar Green's function (see Chapter 2).

\( G_0(s,s') \) is related to the Dyadic Green's function and gives the electric field of a unit current element in the direction of \( \hat{s'} \), and finds its component along the direction of \( \hat{s} \). \( s \) and \( s' \) are unit tangent vectors to \( C \). \( a(r) \) is the radius of the wire at the point \( r \).

The wire grid is taken to be a collection of straight line segments. The current on the \( i \)th segment is written as

\[ I_i(s') = A_i + B_i \sin[k(s' - s_{i1})] + C_i \cos[k(s' - s_{i1})] \]

with \( (s' - s_{i1}) \) being the distance of the point \( s' \) from the centre of the \( i \)th segment, so that \( A_i + C_i \) is the current at the centre of the segment. This form is substituted into the integral equation, and by evaluating \( s \cdot E_{\text{incident}}(s) \) at the centre of each segment in turn, a system of simultaneous linear equations for the unknowns \( A_i \), \( B_i \) and \( C_i \) (\( i = 1, 2, \ldots \)) is obtained. In conjunction with further
information obtained by applying constraints of continuity of current at segment junctions etc. the system is solved in the normal way.

The technique is described as being most suitable at low frequencies, since the higher the frequency, the more closely packed, shorter, and hence numerous the wire segments must be to represent a structure of given physical size. Thus there will be more unknowns, and more computer time will be required. According to Miller, the total computer time required is well approximated by $AN^2 + BN^3$ where $N$ is the number of wire segments and $A$ and $B$ are constants which depend on the details of the program and the machine being used.
CHAPTER 1: DEFINITION OF THE SCATTERING PROBLEM

\( \text{§1.1} \) We consider the problem of a body placed into a pre-existing electromagnetic field, the incident field. The change in the field produced by the introduction of the scattering body is the scattered field. The determination of the scattered field constitutes the 'scattering problem'. This logical sequence ensures that the definition of the scattered field is unambiguous. As an example of the confusion which can otherwise arise, there is the statement of the principle of 'Physical Optics' (reference 1, p.29):

..... over the shadowed portion of the body the surface field is zero.

This could be taken to imply that in the shadow region the incident field is zero, producing a zero scattered field. This is not so. The incident field is non-zero, but in the shadow region it is assumed that the scattered field is exactly equal and opposite to it.

To represent the effect of the scatterer mathematically, boundary conditions are introduced. A general form of boundary condition is given in reference 1. It is the impedance boundary condition:

\[
E - (E \cdot \hat{n})\hat{n} = \eta Z \hat{n} \times H
\] (1.1)

\( \eta \) and \( Z \) are quantities describing the electrical properties of the scatterer and the surrounding medium. For a perfect conductor \( \eta = 0 \).
We wish to consider the less general problem of the fields produced by antennae mounted on aircraft. We restrict attention to problems which are in the so-called 'resonant region' i.e. the characteristic size of the scatterer is a few wavelengths.

The boundary conditions can now be fixed. The electrical properties of the surrounding medium - air - differ insignificantly from those of vacuum (its refractive index is 1.0003). If we assume that the aircraft is constructed of metallic substance, then this implies an electrical resistivity of the order of $10^{-7}\text{m}$ (CRC Handbook of Chemistry and Physics). The high conductivity of metals suggests that to a good approximation they may be considered as perfect electrical conductors, which would simplify the boundary condition problem considerably. To evaluate this possibility we consider the expression for the skin depth:

$$\delta = \left(\frac{\rho \lambda}{\mu_0 \pi c}\right)^{1/2} \quad (1.2)$$

where $\rho$ = resistivity of metal ($= 10^{-7}\text{m}$)

$\lambda$ = wavelength in vacuum

$\mu_0$ = permeability of free space ($= 4\pi \times 10^{-7}\text{Hm}^{-1}$)

$c$ = speed of light ($= 3 \times 10^8\text{ms}^{-1}$)

For a perfect conductor $\delta = 0$, so we will assume that a metal may be considered to be a perfect conductor to a sufficient approximation if the skin depth is small compared to both the freespace wavelength and the thickness of the aircraft hull. This requires that the skin depth be less than about $10^{-3}\text{m}$. This gives:
\[ \lambda \ll \frac{\mu_0 \pi \epsilon_0^2}{\rho} \ll 4 \times 3 \times 10^{-7} \times 3 \times 10^6 = 10^{-6} \times 10^7 \ll 4 \times 10^3 \text{m}. \]

This condition is certainly satisfied. Thus the boundary condition (1.1) takes the simpler form

\[ E - (E \cdot \hat{n})\hat{n} = 0 \quad (1.2) \]

i.e. the tangential component of the electric field is zero.

This general definition of the scattering problem given in the first paragraph may be difficult to apply to antenna problems because the aircraft fuselage may be considered as part of the field source. This is so, but it is thought that the approach given is the only reasonable one. Nevertheless care must be taken. For example the suggestion has been made that a monopole antenna may be represented by a dipole field. The reasoning behind this is that the monopole image by reflection combines with the original monopole to make a dipole. This is fallacious, because the reflected field is part of the scattered field, and should not be included in the incident field. Furthermore, the position of the image by reflection in a curved surface depends on the direction of observation.

This study is an interpretation of the current techniques used in most scattering configurations and has not been used itself to solve realistic specific problems. Attention has been given to the underlying principles rather than specific application. Thus, the scattering configurations considered are simple geometrical
shapes, and in two dimensions rather than three, allowing the scalar wave equation to be used.
CHAPTER 2: REVIEW OF ELECTROMAGNETIC THEORY

§2.1 In this Chapter we review the basic results of electromagnetic theory which will be used in following chapters. We start with a statement of Maxwell's equations in a form appropriate to an isotropic medium. MKS units are used and the symbols have their usual meaning:

\[
\begin{align*}
\nabla \cdot (\varepsilon \varepsilon_0 E) &= \rho \\
\nabla \cdot (\mu \mu_0 H) &= 0 \\
\n\nabla \times E &= -\frac{\partial B}{\partial t} \\
\n\nabla \times H &= J + \frac{\partial \varepsilon}{\partial t} (\varepsilon \varepsilon_0 E) \\
\end{align*}
\]

(2.1)

§2.2 The Poynting Vector

By using the boundary condition (1.2) we need consider propagation in a vacuum only, so that \( \mu = \varepsilon = 1 \) and \( J = \rho = 0 \). If we form scalar products with the third and fourth of these equations then:

\[
H \cdot \nabla \times E = -H \cdot \frac{\partial B}{\partial t}
\]

and

\[
E \cdot \nabla \times H = E \cdot \frac{\partial D}{\partial t}
\]

Subtracting, we obtain

\[
\nabla \cdot (\varepsilon \varepsilon_0 E) = - \left( \mu_0 H \cdot \frac{\partial H}{\partial t} + \varepsilon_0 E \cdot \frac{\partial E}{\partial t} \right) = - \frac{\partial}{\partial t} \frac{1}{2} \left( \mu_0 H^2 + \varepsilon_0 E^2 \right)
\]

(2.2)

We recognise (2.2) as expressing a conservation law and from dimensional considerations we see that the conserved quantity is energy. We identify \( P = \varepsilon \cdot H \) as the Poynting vector representing energy flux.
§2.3 Time Harmonic Propagation

Following the customary practice, we assume that the fields have a form:

\[ E = E(r) \exp(j \omega t), \quad H = H(r) \exp(j \omega t). \]  \hspace{1cm} (2.3)

where the components of \( E \) and \( H \) may take complex values.

Then it is easily shown that the fields satisfy the Helmholtz wave equation:

\[
\begin{align*}
(\nabla^2 + k^2)E &= 0 \quad \text{with} \quad \omega = ck \\
(\nabla^2 + k^2)H &= 0 \quad \text{and} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}
\end{align*}
\]  \hspace{1cm} (2.4)

We have introduced fields whose components may take complex values. In itself this is physically unrealistic and we must make the convention that we work with either the real or imaginary part of the complex field component. Adapting the former let \( e \) and \( h \) be the 'physical' real fields, then:

\[
\begin{align*}
e(r,t) &= \frac{1}{2}(E \exp(j \omega t) + E^* \exp(-j \omega t)) \\
h(r,t) &= \frac{1}{2}(H \exp(j \omega t) + H^* \exp(-j \omega t))
\end{align*}
\]

So that the real Poynting vector is:

\[ P = e \cdot h = \frac{1}{2}(E \cdot H \exp(2j \omega t) + E^* \cdot H^* \exp(-2j \omega t) + E^* \cdot H^* + E^* \cdot H) \]  \hspace{1cm} (2.5)

If we take the time average of equation (2.5) then

\[ P = \frac{1}{2}(E \cdot H^* + E^* \cdot H) \]  \hspace{1cm} (2.6)
§2.4 Green's Theorem

Consider the field equation

$$(\nabla^2 + k^2)u = s(r)$$

where $u$ is a scalar field, and $s(r)$ is a 'source' term. Using the Dirac delta function the source distribution may be written as

$$s(r) = \int s(r')\delta(r - r')d^3r'$$

where the volume integral is taken over all space. Because $r$ and $r'$ are independent, the solution for $u$ may be undertaken in two steps. First, the field equation is solved for a delta-function source distribution, and the field due to the actual source distribution is obtained by superposition of the point source fields (since the field equation is linear). This is the motivation for defining the Green's functions for the scalar Helmholtz equation in free space:

$$(\nabla^2 + k^2)g(r, r') = -\delta(r - r')$$

The differentiation is with respect to the unprimed co-ordinates.

Now Green's second identity for two general scalar fields $\phi$ and $\psi$ is:

$$\int_V (\phi\nabla^2\psi - \nabla\psi\nabla\phi)d^3r = \int_S (\psi\nabla\phi - \phi\nabla\psi) \cdot dS$$

where $V$ is the volume contained by the surface $S$. For a source free region so that

$$(\nabla^2 + k^2)u = 0$$

$\phi = u(r)$ and $\psi = g(r, r')$. 

Then
\[
\int \int_S (u(r)v g(r,r') - g(r,r')v u(r)) \, dS
\]
\[
= \int_V u(r)[- \delta(r - r') - k^2 g(r,r')] - g(r,r')[- k^2 u(r)] d^3r
\]
\[
= - \int_V u(r) \delta(r - r') d^3r
\]
\[
\int \int_S (u(r)v g(r,r') - g(r,r')v u(r)) \, dS
\]
\[
= 0 \text{ if } r' \text{ is outside } V
\]
\[
= - u(r') \text{ if } r' \text{ is inside } V
\] (2.7)

It is customary to be rid of the minus sign by working with the inward pointing surface vector in the integral. This convention will be adopted in what follows.

Equation 2.7 may be regarded as the mathematical realisation of Huygens principle. In three dimensions
\[
g(r,r') = (\exp(-jk|r - r'|))(4\pi|r - r'|)^{-1}
\] (2.8)
and in a space of two dimensions
\[
g(r,r') = - \frac{3}{4} H_0^{(2)}(k|r - r'|)
\] (2.9)

In this case, the integral in (2.7) becomes an integral round a closed curve in a plane and the surface element \(dS'\) becomes a line element \(dl'\). With the time convention which we have chosen (eq. 2.3), equations (2.8) and (2.9) represent outgoing spherical and cylindrical curves respectively.
§2.5 Two Dimensional Problems

We will often use the concept of a scatterer whose cross-section in a $z = \text{constant}$ plane does not depend on the value of $z$, a cylindrical scatterer. If the electromagnetic field is such that it does not contain a $z$ dependence, then $\frac{\partial}{\partial z} = 0$ in Maxwell's equations. A cylinder scattering a $z$ independent field constitutes a two-dimensional problem. The importance of this concept lies in the fact that Maxwell's equations and the boundary conditions are separable giving the two sets:

\[
\begin{align*}
H_x, H_y, E_z \text{ equations:} & & E_x, E_y, H_z \text{ equations:} \\
- \mu_0 \frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} & \frac{\partial E_x}{\partial t} &= \frac{\partial H_z}{\partial y} \\
- \mu_0 \frac{\partial H_y}{\partial t} &= - \frac{\partial E_z}{\partial x} & \frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial x} \\
+ \varepsilon_0 \frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} & - \mu_0 \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\
\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} &= 0 & \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} &= 0
\end{align*}
\]

$E_z = 0$ at the scatterer $E_x - (n_x E_x + n_y E_y)n_x = 0$

$E_y - (n_x E_x + n_y E_y)n_y = 0$

at the surface.

Thus we can identify two states of polarisation which are decoupled. The $H_x, H_y, E_z$ equations are said to describe Transverse Magnetic (TM) waves and the $E_x, E_y, H_z$ equations are said to describe Transverse Electric (TE) waves. It will be seen that the two-dimensional problem can be completely framed in terms of $E_z$ and $H_z$, both of which satisfy the wave equation, although with somewhat different boundary conditions.
The concept of the two-dimensional problem is capable of some extension. Suppose we have a scalar field $E$ which is deemed to depend on $z$, but can be written in the form:

$$E(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e(x,y,K) \exp(jKz) dK$$

Then

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] E(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (k^2 - K^2) \right) \exp(jKz) e(x,y,K) dK = 0$$

or

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (k^2 - K^2) \right] e(x,y,K) = 0 \quad (2.10)$$

Equation (2.10) expresses an essentially two-dimensional situation with the wavevector $k$ replaced by $\sqrt{k^2 - K^2}$. Of course, two types of solution will be required according to whether $K$ is greater than, or less than $k$. This type of analysis was used by Wait in his investigation of radiation from cylinders (reference 2). This shows that problems involving cylinders and an obliquely incident field may be considered as two dimensional problems.

§2.6 Units

In most of what follows, we will use units of length such that the wavelength is one. For example, the wavenumber, $k$, will usually take the value $2\pi$. It is felt that this procedure gives a better physical insight than would the use of 'real' units. Insofar as electromagnetic quantities appear, SI units are used.
CHAPTER 3: A CLASSIFICATION OF TECHNIQUES AND CRITIQUE OF THE RAY CONCEPT

§3.1 As already stated, attention will be restricted to scatterers which are perfect conductors. The field is thus deemed not to penetrate the scatterer at all, that is, we need consider only the 'exterior' problem. Also we shall be interested principally in scatterers whose characteristic size is greater than about one wavelength and less than about twenty. There are numerous techniques in existence to tackle problems in either the high, or the low frequency range, but they tend to become cumbersome when applied to this middle ground.

The concept of resonance needs some clarification. In this connection one should be careful to distinguish between interference effects and true resonance effects. As an example of this, consider the problem of a plane wave incident on a slit (Figure 3.1). Consider how the solution would proceed using the Geometrical Theory of Diffraction (GTD) for a field point which lies in the Geometrical Optics shadow region. If the slit is wide then a good solution may be obtained by considering the field as a sum of two diffracted rays only: UAP and VBP. The sum of these two rays will exhibit interference effects as P is moved, but no resonance. This only happens when the slit becomes narrower, and rays which are diffracted many times back and forth from one edge to another must be counted in order to maintain the quality of the solution: UA(BA)P, UA(BA)(BA)P etc., VB(AB)P, VB(AB)(AB)P etc. This interaction between different points of the scattering system constitutes Resonance.
With this distinction in mind, we may classify techniques for solving scattering problems into two classes which can be designated as 'Global' and 'Local'. The former class is found to be most useful at low frequencies, and the latter at high frequencies.

The Global approach treats the scattering body as a whole. A specific example would be to represent the scatterer as an assembly of conducting wires. There is coupling between currents on different wires. This leads to an integral equation which may be solved numerically by the 'method of moments' (in itself only a mathematical technique). Another example would be the modal expansion of the field which will be discussed later (i.e. the representation of the field as a linear combination of modes). The characteristic of the Global approach is that it leads to large systems of simultaneous linear equations. This is a distinct disadvantage from the computational point of view. As the size of the scatterer increases, more unknowns are required. However, Global methods have the great advantage that, since they consider all the scatterer at once, the resonance effects are automatically accounted for.

Local methods tend to depend on the 'principle of local field'. This was first due to Fock in the context of the 'Penumbral field' (reference 3). A good statement of the principle is due to Bach in reference 4:

'... in the high frequency limit, processes such as reflection and diffraction depend only on the properties, electrical and geometrical, of the scatterer in the immediate neighbourhood of the point of reflection and diffraction'.

If we accept the validity of the principle, then it means that we can divide a scatterer into sections and consider each section separately. This has two further implications:

a) The different parts of the scatterer can be assumed to interact with the field as if they were the corresponding parts of different scatterers, for which the exact solutions are known. These are solutions of so called 'Canonical problems'. The canonical example of this is the use of the exact solution for the perfectly conducting half plane to calculate the diffraction due to edges on finite bodies (reference 5, p.115).

b) The use of rays to keep track of the contributions to the scattering due to the various 'sub scatterers'. An example of this is the previously mentioned problem of the slit, where one uses rays to keep track of the multiple diffractions from the two edges.

The chief disadvantage of the local approach is that resonance effects must be explicitly taken into account. Once again, an example is afforded by the GTD solution of the slit problem. One must make a conscious decision to include rays which are diffracted more than once.

Local field methods tend to be ray theories. (The class of ray theories of diffraction).

It is appropriate at this point to discuss the ray concept in more detail. Rays provide a good framework for the discussion of scattering processes. They simplify calculation. Their great significance is, that coupled with the laws of geometrical optics
they provide what is probably the only a priori design technique for optical systems. A ray is defined as a local trajectory of energy flow, so that its tangent is the Poynting vector. Since the significant quantity is the flow of energy this is a 'better' definition than one involving normals to phase-fronts, which may disagree with the former definition for propagation in an anisotropic medium. (See Felsen in reference 4).

However, there are problems with ray calculations. If there is an infinity of rays passing through a point, or travelling in the same direction, then ray optics cannot give a sensible answer. (This is not to say that such calculations are impossible: see reference 6). Also, in GTD, isolated rays are considered to be incident on diffracting features of infinitesimal size, which are supposed to scatter a finite amount of energy.

Another example of the incompatibility of the ray concept, compared with rigorous field theory arises in connection with creeping rays. Consider a plane wave incident convex scatterer. For high frequencies, the lowest degree of approximation is afforded by geometrical optics, which says that there is no field at the surface of the scatterer in the shadow region. According to GTD, the next approximation gives some field on the surface in the shadow region due to a sum of creeping rays. This violates the boundary condition, but may still give appropriate results in the far field.

The point to be made is that rays are essentially symbolic. They are a shorthand for more complex, but more rigorous field calculations. (A related view was expressed by Shiller in reference (38)). The fact that infinitesimal scattering features can have a finite scattering cross-section, which is absurd in a strictly local scattering theory, is accounted for by the diffraction coefficient.
Another conceptual difficulty manifests itself in the confusion between rays and 'plane waves'. These are not the same thing. One may choose to represent a plane wave as a ray, or a number of rays, but there is no unique choice. A ray has a specific location in space. A plane wave does not.

It has been suggested by Cornbleet (in reference 19) that for a given direction of scattering, all rays, including those not travelling in that direction, must be taken into account. (This is not exotic as it sounds. In the problem of the plane wave and slit, the Geometrical Optics rays transmitted travel in only one direction, but they can be used to calculate a non-zero diffracted field in other directions). The method of doing so is to use ray theory to obtain the field on a specified surface. This field is combined with a Green's function, and then integrated. This is the basis of the 'Aperture Field Method' which will be discussed in more detail later.

§3.2 The Physical Interpretation of Mathematics

The problems with the ray concept raise the question of how much freedom we have in the interpretation of the mathematics which arises in physical problems. The Kirchoff diffraction integral is a field concept, but it is shown in Appendix A how the Kirchoff result for a plane wave incident on a slit may be interpreted in terms of edge diffracted rays in the style of GTD. The answer seems to be that Nature does not care what language we use to describe her, but some languages are more appropriate than others.
CHAPTER 4: TWO DIMENSIONAL WAVES

§4.1 The Helmholtz scalar wave equation in a medium which is not necessarily vacuum, is

\[(\nabla^2 + k^2)E = \text{source term} \]  \hspace{1cm} (4.1)

\[= 0 \text{ in vacuo} \]

We now consider the question whether solutions of the source free equation may be used if there are sources present. This is the condition required for the solution of the exterior problem for isolated scatterers. It is well known that solutions may be found for the source free equation (4.1) of the form:

\[A_m H_n^{(1)}(\rho)\exp(jm\theta) \]  \hspace{1cm} (4.2)

Here \(m\) is a constant of separation. The general solution is a sum of such terms (or an integral with respect to \(m\), if \(m\) is considered to form a continuous spectrum). However there is nothing in the mathematics to say what values \(m\) may take. These are governed by the character of the physical problem concerned.

What is usually done is to say that in physical space, points whose angular co-ordinates differ by \(2\pi\) are identical, so that \(m\) must be an integer for the field to be single valued. This is clearly wrong as may be seen from the example of the perfectly conducting half plane. MacDonald in reference 8 found that half-integral values of \(m\) in eq. (4.2) were appropriate for the half-plane problem.

A solution of the half plane problem was presented by Sommerfeld in 1896, by constructing solutions to the wave equation on a Riemann surface. (See for example reference 9). It is interesting
to note that a more physical approach was given by Lord Rayleigh in his 'Theory of Sound' section 207. If the field has no singularities then eq. (4.2) may be rewritten as:

\[ E = \sum_{m} J_{m}(kr)(A_{m} \sin m \theta + B_{m} \cos m \theta) \]  \hspace{1cm} (4.3)

Impose the boundary conditions that \( E = 0 \) everywhere on two fixed radii \( \theta = 0 \) and \( \theta = \beta \). This may be accomplished by choosing \( B_{m} = 0 \) and \( \sin m \beta = 0 \) i.e. \( m = n\pi/\beta \) with \( n \) an integer. Now open out the radii so that \( \beta = 2\pi \), then eq. (4.3) becomes:

\[ E = \sum_{n=0}^{\infty} A_{n} J_{n}(kr) \sin \left( \frac{n\theta}{2} \right) \]  \hspace{1cm} (4.4)

Thus, half integral 'quantum-numbers' have been introduced in a physical manner. It will be objected that it would be unreasonable to expect eq. (4.2) to be valid in the above case, since it is only everywhere valid in a completely source free space, but for scattering problems we are interested in situations where sources are confined to quite small areas and it may be that eq. (4.2) is of use away from these regions.

To illustrate this point, reconsider eq. (4.4) which was derived with the origin of co-ordinates at the edge of the half-plane, which was lying along \( \theta = 0 \). Suppose we translate the origin a distance \( d \) to the left, as in Figure 4.1. Using the Graf addition theorem (see reference 7 eq. 9.1.79) we can express the general solution in terms of the new co-ordinates \((\rho, \phi)\). We find that there are two distinct cases:
If $p < d$ then
\[
E = \sum_{m=-\infty}^{\infty} J_m(kp) \left\{ P_m \sin(m\phi) + Q_m \cos(m\phi) \right\}
\]
(4.5a)

with
\[
P_m = \sum_{n=0}^{\infty} - J_{m+\frac{n}{2}}(kd) \cos\left(\frac{n\pi}{2}\right)
\]
and
\[
Q_m = \sum_{n=0}^{\infty} - J_{m+\frac{n}{2}}(kd) \sin\left(\frac{n\pi}{2}\right)
\]

and in the second case $p > d$ giving:
\[
E = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{+\infty} A_n J_m(kd) J_{m+\frac{n}{2}}(kp) \exp\left(j\left(m + \frac{n}{2}\right)\phi\right)
\]
(4.5b)

Equation (4.5b) still has cylindrical wavefunctions of half-integral order. Equation (4.5a) now has cylindrical wavefunctions of integral order only. The point to notice is that this latter equation applies to a region which is source-free inside a circle centred at the origin.

A similar situation prevails when we consider the field due to a cylinder as in Figure 4.2. Since $E = 0$ at its surface we may use Green's theorem in the form:
\[
E(P) = - \int G \left(\frac{\partial E}{\partial n}\right)_{\text{Surface}} \nabla \cdot d\mathbf{l}
\]
(4.6)

with
\[
G = - \frac{j}{4} H_0^{(2)}(kr) = - \frac{j}{4} \sum_{m=-\infty}^{+\infty} H_m^{(2)}(kr) J_m(kp) \exp(jm(\phi - \theta))
\]
(4.7)
where we have assumed harmonic time dependence. The justification for using the theorem for a field point outside the closed surface of integration is given in a later chapter.

The transformation of the Green's function in eq. (4.7) shows that the scattered field consists entirely of outgoing cylindrical waves of integer order. This form is only valid if \( R > \rho \). This will only certainly be the case for all points on the scatterer if the field point lies outside the smallest circle enclosing the scatterer and centred at the origin. Otherwise, the scattered field may contain incoming cylindrical waves. (See next section). It should be noted that all the modes in eq. (4.7) are of integer order.

Therefore, we conclude that the representation (4.2) of the general solution to eq. (4.1) may be used in source-free regions bounded by circles, even when there are sources present elsewhere.

§4.2 The Rayleigh Hypothesis

In equation (4.6) the normal derivative of the field depends on the surface current on the scatterer. Equation (4.7) indicates that outside the enclosing circle, the field which the current produces may be represented as a sum of purely outgoing cylindrical waves. If one wishes to solve the boundary condition problem, then one requires an expression for the scattered field at the surface of the body. The supposition that the form:

\[
E^{\text{scat}} = \sum_{n=-\infty}^{+\infty} B_n H^2(\text{k}r) \exp(j \text{n} \theta)
\]

(4.8)

is valid not only outside the enclosing circle, but also right up to the surface of the scatterer is known as the Rayleigh hypothesis.
In fact it is not always valid. The conditions under which it is valid have been given by Van den Berg and Fokkema in reference 10, who also give an extensive bibliography of work on the Rayleigh hypothesis.

Those conditions may usefully be summarised. In cylindrical polar co-ordinates the cross-section of the scatterer is given by \( \rho = \rho(\theta) \) with both \( \theta \) and \( \rho \) real. The radius of the largest inscribed circle (i.e. the minimum value of \( \rho \) for \( \theta \) in the range 0-2\( \pi \)) is \( r_{\text{min}} \).

Van den Berg and Fokkema allow \( \theta \) to take complex values and by an argument concerning the convergence of series they find that the Rayleigh hypothesis holds if and only if

\[
|\left(r_{\text{min}}/\rho(\theta_p)\right)\exp(j\theta_p)| > 1
\]

where \( \theta = \theta_p \) if

1. \( \frac{d\rho}{d\theta} = j\rho \) with \( \text{Im}(\theta) < 0 \)
2. \( \theta_p \) is a non-analytic point of \( \rho(\theta) \)
3. \( \theta_p \) is a zero of \( \rho(\theta) \)

There is no immediately obvious physical interpretation of this condition.

It is easy to see physically why the Rayleigh hypothesis is not valid. Refer to Figure 4.3. If the field point is at \( P \) then there are parts of the scatterer such as \( S \) which are 'outside of' \( P \) - i.e. further from the origin. Any radiation at \( P \) due to current at \( S \) will be incoming and not representable in the form of eq. (4.8). It has been shown by Millar (reference 42 and its bibliography) that the Rayleigh hypothesis is valid if the singularities of the scatterer field lie within the largest circle enclosed by the scattering contour.
and centred upon the origin (shaded in Figure 4.3). Since in general we know only that the singularities are contained within the scattering contour, we can only be sure that eq.(4.8) is a valid form outside the largest enclosing circle.

Furthermore, according to Van Den Berg and Fokkema, the Rayleigh hypothesis is never valid for a body with edges. Again, this is easily seen physically, at least at high frequencies. According to the GTD concept the edge will radiate a diffracted field. Topologically there will be points for which the edge diffracted rays will point inward, from which the conclusion follows.

Thus, we have to consider the problem that we may wish to deal with scattering cross-sections for which the Rayleigh hypothesis does not hold. One way of overcoming this problem is to use the analytic continuation method of Wilton and Mittra (reference (11)).
§5.1 The method was first outlined in a paper (reference 11) published in 1972. The work had its roots in a paper published on the inverse-scattering problem in 1970 (reference 12). It takes as its unknowns the coefficients describing the fields, which is a more direct approach than calculating surface currents as is done in, for example, wire modelling techniques.

The incident and scattered fields are expressed as a sum of the fundamental modes (eq. 4.2). The coefficients of the scattered field are the unknowns. The series, which contain an infinite number of terms, are truncated so that only the finite number of significant field coefficients remain. A system of simultaneous linear equations is obtained by applying the boundary conditions at a number of points on the surface of the scattering body. However, as was indicated in the previous chapter, the series of modes may not converge near or at the surface of the scatterer. The method described above can only be used for a scatterer whose cross-section does not depart significantly from being circular. (See reference 11 on 'Mode Matching of fields').

Following Wilton and Mittra, we will now show how a Bessel function addition theorem may be used to describe the field relative to a new origin in a form which will be valid at the surface of the scatterer. This having been done, the boundary conditions may be used. Wilton and Mittra gave two versions of the procedure. What follows is the second version which they call the 'inside expansion'.
§5.2 Transverse Magnetic Polarisation

For the transverse magnetic case we can work with the $z$-component of the electric field only, which must be zero at the surface of the scatterer. We write:

$$E = \sum_{m=-\infty}^{\infty} (A_m H^{(1)}_m(k\rho) + B_m H^{(2)}_m(k\rho)) \exp(j m \phi)$$  \hspace{1cm} (5.1)

With the time dependence of $\exp(j \omega t)$ the $B$ coefficients describe the outgoing part of the field, and the $A$ coefficients the incoming part. It should be noted that these cannot be directly identified with the scattered and incident fields. It is true that the scattered field outside the enclosing circle is entirely outgoing, but the incident field will usually consist of equal incoming and outgoing parts. For the moment however, we will regard the $B$ coefficients as entirely unknown.

Using the Graf addition theorem we may transfer the origin of co-ordinates from 0 to $\bar{0}$: see Figure 5.1. If $\bar{0}$ lies outside the enclosing circle then $R$ is always greater than $\rho$, and the form of the addition theorem which is appropriate is:

$$H^{(1)}_m(k\rho) \exp(j m(\phi - \bar{\phi})) = \sum_{n=-\infty}^{+\infty} H^{(1)}_{m+n}(kR) J_n(k\rho) \exp(j n(\theta + \pi - \bar{\phi}))$$

$$+ \sum_{n=-\infty}^{+\infty} H^{(2)}_{m+n}(kR) \exp(j (m+n)\phi) \exp(j n(\pi - \bar{\phi}))$$  \hspace{1cm} (5.2)

$$H^{(1)}_m(k\rho) \exp(j m \phi) = \sum_{n=-\infty}^{+\infty} H^{(1)}_{m+n}(kR) J_n(k\rho) \exp(j (m+n)\phi) \exp(j n(\pi - \bar{\phi}))$$  \hspace{1cm} (5.3)
Substituting eq. (5.3) into eq. (5.1) the boundary condition at the
ith boundary point may be written:

\[ \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( A_m H_n^{(1)}(kR_i) + B_m H_n^{(2)}(kR_i) \right) \]
\[ \cdot \left( J_n(k\rho_i) \exp(j(m+n)\theta_i) \exp(jn(\pi - \phi_i)) \right) = 0 \]  

The equations (5.4) are set up for a sufficient number of boundary
points to determine or overdetermine the unknown Bs (in a least
squares sense).

Note that the radial part of the field in the new co-ordinate
system is

\[ J_n(k\rho_i) = \frac{1}{2} \left\{ H_n^{(1)}(k\rho_i) + H_n^{(2)}(k\rho_i) \right\} \]

which consists of an equal mixture of incoming and outgoing parts.
This is because the field can have no singularities in free space
and the \( H_n^{(1)}(k\rho_i) \) and the \( H_n^{(2)}(k\rho_i) \) must occur in the combination
which prevents this. This will be expressed later by saying that
the scattering matrix of free space is the identity matrix.

Consider the largest circle centred on the new origin and
excluding the scatterer. If the circle just touches the scatterer
at the ith boundary point, then the circle must be tangential to the
scattering surface at that point, i.e. the translated origin must
lie on a normal to the scattering surface. The value of \( R \) must be
sufficiently large that some part of the 'excluding' circle lies
outside the enclosing circle. Again \( R \) must not be so large that
the convergence of the series is poor. Within these constraints,
the position of the translated origin is arbitrary.
§5.3 Transverse Electric Polarisation

This was not explicitly considered in reference (11) nor in Munro's work (references (13) and (14)). A mention was made in reference 15. As before, we need work with a scalar field only, but the application of the boundary condition is much more complicated than in the TM case. We write for the z component of the magnetic field:

\[
H = \sum_{m=-\infty}^{+\infty} (A_m H_m^{(1)}(k\rho) + B_m H_m^{(2)}(k\rho))\exp(j m \phi) \tag{5.5}
\]

Since we have framed the boundary conditions in terms of the electric field we must obtain \(E\) by differentiating eq. (5.5) i.e.

\[
E = \frac{1}{J\omega \varepsilon_0} \nabla \cdot H \quad \text{with} \quad H = H_k
\]

i.e.

\[
j\omega \varepsilon_0 E_\rho = \frac{1}{\rho} \frac{\partial H}{\partial \phi} \quad \text{and} \quad E_\phi = -\frac{\partial H}{\partial \rho}
\]

so that

\[
E_\rho = \frac{1}{J\omega \varepsilon_0} \sum_{m=-\infty}^{+\infty} \frac{jm}{\rho} (A_m H_m^{(1)}(k\rho) + B_m H_m^{(2)}(k\rho))\exp(j m \phi) \tag{5.6}
\]

and

\[
E_\phi = -\frac{1}{J\omega \varepsilon_0} \sum_{m=-\infty}^{+\infty} k(A_m H_m^{(1)}(k\rho) + B_m H_m^{(2)}(k\rho))\exp(j m \phi) \tag{5.7}
\]

(The dot \(\cdot\) implies differentiation with respect to argument).

For a perfect conductor, the boundary condition 1.1 can be written \(\vec{E} \cdot \hat{t} = 0\) where \(\hat{t}\) is the unit tangent vector. That is, the tangential component of the field zero at the scatterer, not the transverse component (i.e. \(E_\phi\)). This is not a trivial point: as will be seen in the extension of the method into three dimensions. Now:
\[ \hat{r} = \frac{1}{\rho} \frac{d\rho}{d\phi} \rho + \hat{\phi} \]

So the boundary condition becomes

\[ \frac{1}{\rho} \frac{d\rho}{d\phi} E_\rho + E_\phi = 0 \quad (5.8) \]

If the scatterer were 'large' in terms of wavelengths, and with a surface with a low curvature, then \( \frac{1}{\rho} \frac{d\rho}{d\phi} \ll 1 \) and we would be able to use the simpler boundary condition \( E_\phi = 0 \). However, in general:

\[ \sum_{m=\pm \infty} \frac{j m}{\rho^2} \frac{d\rho}{d\phi} (A_m H_m^{(1)}(\rho) + B_m H_m^{(2)}(\rho)) \exp(j m \phi) \]

\[ = - \frac{k}{\rho} (A_{m-1} H_{m-1}^{(1)}(\rho) - H_{m+1}^{(1)}(\rho)) + B_{m-1} H_{m-1}^{(2)}(\rho) - H_{m+1}^{(2)}(\rho)) \exp(j m \phi) \]

\[ = 0 \quad (5.9) \]

where we have used:

\[ H_m^{(1)(2)}(\rho) = \frac{1}{\rho} (H_{m-1}^{(1)(2)}(\rho) - H_{m+1}^{(1)(2)}(\rho)) \]

Equation (5.9) can now be put into a convergent form by the use of eq. (5.3) and applied at a number of points on the scatterer. Two points about the numerical implementation should be noted here:

1. In the algebra the infinite limits have been left on the summation signs. In practice the sums must be truncated. Wilton and Mittra state that terms up to about order \( k(R + \rho_{\text{circ}}) \) should be included. This is easy to understand physically if we consider scattering by an object whose largest dimension is \( D \). Elementary diffraction theory tells us that the smallest detail
on the resulting pattern will be of order \( \frac{\text{wavelength}}{D} = \frac{1}{D} \) radians. A mode of order \( N \) can register detail of order \( \frac{2\pi}{N} \) from which \( N_{\text{max}} \approx 2\pi D \), with \( R + \rho_{\text{circle}} \) as a measure of the maximum extent of the system in the translated co-ordinate system.

2. The theory gives no indication of how to choose the 'best' locations of the boundary points.

Heuristically, an even spacing of points with a higher concentration on regions of the scatterer where \( \frac{d\rho}{d\phi} \rho^{-1} \) is not small, would seem reasonable. A first step to a rigorous solution would be the formulation of a criterion to decide what is implied by 'best' in this context.
CHAPTER 6: COMMENTS ON THE MODAL EXPANSION METHOD

AND INTRODUCTION TO THE THREE DIMENSIONAL PROBLEM

§6.1 The High Frequency Limit

In this limit, the scatterer becomes extremely large (in terms of wavelengths) and in the limit an infinite number of boundary points will be required to specify a scatterer of fixed physical size. Clemmow and Weston (references 17 and 18) considered the diffraction of a plane wave by circular and almost-circular cylinders. In reference (18) it is shown how for a scatterer whose surface was specified in terms of a continuous function it was possible to identify shadow zones, creeping rays etc. as leading terms in the asymptotic expansion of the solution. Thus, in this sense, the high frequency limit of the modal expansion method is Geometrical Optics/GDT.

It is fallacious to believe that: (Chignell reference (16) 1976) '... at high frequencies the plane wave spectrum approach naturally becomes the asymptotic method generally known as the Geometrical Theory of Diffraction. In this manner the modal scattering description of the resonant frequency range naturally turns into the high frequency optical technique'.

for the reasons given in Chapter Three: 'A plane wave is not the same as a ray. A plane wave has no specific location in space, only a specific direction, whereas a ray has both. While a plane wave spectrum theory will give the same results as GTD in regions where both theories are valid, it is believed that the correspondence is by no means as direct as the above remark would suggest.'
$\S 6.2$ The Low Frequency Limit

A commonly used technique for solving antenna problems is the wire grid modelling technique, where the structure is replaced by a mesh of wires and the unknowns are the currents carried by the wires. The customary assumption is that the circumferential variation of the longitudinal current can be ignored (reference 20). The implication of this for a model of a cylindrical TM scatterer is that if the longitudinal current is constant around the circumference, then the radiation of a single wire is isotropic and depends on a single (complex) constant. The field due to a collection of $N$ such wires for the Transverse Magnetic Polarisation is

$$E_{\text{scatt}}(r) = \sum_{n=1}^{N} A_n H_0^{(2)}(kr) \left( r - r_n \right)$$

(6.1)

where $(r_n, \theta_n)$ are the co-ordinates of the centre of the $n$th wire. Using the Graf addition theorem this can be written as:

$$E_{\text{scatt}} = \sum_{m=-\infty}^{+\infty} H_m^{(2)}(kr) \exp(j m \theta) \left( \sum_{n=1}^{N} A_n J_m(k r_n) \exp(-j m \theta_n) \right)$$

(6.2)

which is of the form of eq. (4.8). The addition theorem is valid if $r > r_n$ for all $n$. Thus in this simple instance the wire grid model is consistent with the concept of a scattered field which can be written as a sum of outgoing cylindrical waves for field points outside the smallest enclosing circle.

$\S 6.3$ The Three Dimensional Problem

Scattering from a general three dimensional body is an extremely complicated problem. General solutions to Maxwell's equations
in spherical polar co-ordinates can be found in terms of Spherical Hankel functions and spherical harmonics, and modal solutions can be set up. It can be postulated that a three dimensional analogy to the invalidity of the Rayleigh hypothesis exists, which could in principle be treated by the analytic continuation procedure, as suggested by Chignell in 1976. This has not been considered here. A further complication with the three dimensional problem is that the number of modes needed increases quadratically with the maximum order of Hankel function used, rather than linearly. Thus for a fixed computing capacity the largest three dimensional problem will be much smaller than the largest two dimensional problem. For this reason, Chignell, Munro and Gribble restrict their attention to scattering from (perfectly conducting) bodies of revolution. It could be anticipated that as one dimension is 'redundant' in determining the shape of such a body, then the problem could be reduced to a quasi-two dimensional one.

Chignell based his work on the following hypothesis (quoted from reference 16(c)):

'Here the assumption of a cylindrical scatterer or quasi-cylindrical scatterer is most important. The point is that if a mode is incident upon the origin, then the energy incident on a particular \( \phi \) plane is scattered in that plane. This means that the problem can essentially be reduced to two dimensions...'

The justification for this statement seems to lie in the work of Garbacz and Turpin, and Harrington and Mautz (references 21 and 22). There it is shown that there exists a set of characteristic currents and corresponding radiated fields for a given perfectly conducting
scatterer. To each incoming characteristic mode corresponds an 
outgoing characteristic mode which is simply its complex conjugate. 
An incoming mode will be scattered into its outgoing mode only, 
suffering a phase-shift in the process. As an example, the 
characteristic modes for the circular cylinder are just the terms 
\( H_m^{(1)}(kr) \exp(j m \phi) \). It is easily seen (from Appendix B) that the 
'scattering coefficients' are phase shifts since:

\[
\left| \frac{H_m^{(1)}(ka)}{H_m^{(2)}(ka)} \right| = \left| \frac{H_m^{(1)}(ka)}{H_m^{(1)}(ka)} \right| = 1 .
\]

Since the incoming and outgoing modes are complex conjugates, their 
variation with angle will be the same, giving rise to Chignell's 
hypothesis. However, it will be true only if the field is expressed 
in terms of the characteristic modes and not in general if ordinary 
vector mode functions are used.

The procedure was continued by Munro in the following manner: 
In reference (14) it is stated: 'As the scatterer is a body of revolu-
tion, the scattering in each plane containing the axis of revolution 
will be identical, and only the scattering matrix in one of those 
planes need be considered' - and again -

'As the scatterer has an axis of revolution on the X-axis, 
only the energy incident in the XY plane will be scattered into the 
XY plane, and into no other plane....'

The previous two statements are rather vague. The second 
will be true on the Geometrical Optics limit, but at any finite 
frequency, each surface element will in general radiate in all 
directions exterior to the scatterer. His actual procedure is to 
apply the boundary conditions only for points lying in the XY plane 
(setting \( \theta = \frac{\pi}{2} \) in polar co-ordinates).
In fact, it turns out that this is correct, but in order to justify the procedure we will have to consider the problem with the axis of rotation lying along the $z$-axis, beforehand.

6.4 Transverse and Tangential Boundary Conditions

A more serious objection to the procedure is the misapplication of the boundary conditions. For convenience Munro has followed Kahn and Wasylikowski (reference 23) in expanding in Spherical modes only the Transverse electric field. This is perfectly permissible as the radial component of the field can be easily obtained by applying Maxwell's equations. However, in reference 14, the radial component is never mentioned. Along the contour of the scatterer in the $XY$ plane he applies the boundary conditions:

$$ r \mathbf{E}_{\text{scat}} = - r \mathbf{E}_{\text{inc}} $$

where $\mathbf{E}_{\text{scat}}$ and $\mathbf{E}_{\text{inc}}$ are the transverse electric fields (i.e. the vector sum of $\hat{\theta}$ and $\hat{\phi}$ components) but the correct boundary condition is that the total tangential field is zero at the surface, i.e.

$$ E_{\theta}^{\text{total}} = 0 \text{ and } \frac{1}{\rho} \frac{d}{d\phi} E_{\phi}^{\text{total}} + \frac{1}{\rho} E_{\phi}^{\text{total}} = 0, \quad -\theta \to \theta, \quad -\phi \to \phi $$

Munro omits the $\frac{1}{\rho} \frac{d}{d\phi}$ term. Since $E_{\rho}$ depends on both $E_{\theta}$ and $E_{\phi}$ this omission means that his equations provide no way of mixing $E_{\theta}$ and $E_{\phi}$. He finds that the $\hat{\theta}$ and $\hat{\phi}$ equations decouple so that there is no mechanism for cross-polarisation.

A priori, it might be that the inaccuracy introduced by this error is negligible. After all, the discrepancy vanishes for generators which are (nearly) circular. We take as a measure of the error the quantity

$$ \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{\rho^2} \left( \frac{d\rho}{d\phi} \right)^2 d\phi \right)^{1/2} $$

(5.3)
where the integral is taken around a cross-section which includes the axis of symmetry. If this quantity is much less then one, then the 'transverse' boundary conditions are applicable. In the problem considered by Munro the cross-section was rectangular. In reference (15) it was shown that if the ratio of the lengths of the sides was $x$ then the value of eq. (6.3) is:

$$\left[\frac{2}{\pi} \left( x + \frac{1}{x} - \frac{\pi}{2} \right) \right]^{1/2}$$

The value of $x$ was 5 giving a fractional error of about 1.5, so in this case we would not expect the effect of the error to be negligible.

In Chapter 7 a derivation of the correct method for extending the two dimensional theory to bodies of revolution will be given.
§7.1 The problems that we are mainly concerned with are linear. The Maxwell vacuum equations are linear, and dealing with perfect conductors, the scattering material does not directly enter the problem at all. This is because the current induced by the incident field is confined to a limitingly thin surface layer. Since the superposition of two fields which have zero tangential components at the surface, itself has a zero tangential component, the boundary condition is in this sense linear. If we expand the total field in terms of incoming and outgoing fields then for a given scattering geometry it is obvious that there is a linear transformation between them. This was shown in more formal terms in reference 23. If we write the fields in terms of the coefficients in the modal expansion, then these coefficients may be conveniently written as column vectors. It follows that these column vectors are related by a Scattering Matrix.

This idea is not new. An early, well known reference is due to Montgomery and Dicke (reference 24). Some of what follows may be found there. Scattering matrices seem to be more familiar in the context of Network and Waveguide theory rather than isolated scatterers. This last application is more familiar in Quantum scattering problems. See(for example) reference 25.

A more recent piece of work which does apply the scattering matrix concept to isolated scatterers is that of Waterman in reference 39. He considers both the interior and exterior problems using a procedure which he says is equivalent to the moment method. He applies the constraint of a zero field within the 'inscribed
sphere' inside a solid scatterer, and by the continuation properties of the functions involved, the interior field is made zero everywhere.

We will discuss first the scattering of scalar waves in two dimensions, then scalar waves in three dimensions, and then the full problem of electromagnetic scattering from a three dimensional body of revolution. Here we will present the solution to the problem of the reduction of the three-dimensional problem to two dimensions.

§7.2 On the Two Definitions of the Scattering Matrix

A possible source of confusion can arise from there being two reasonable definitions of the scattering matrix, neither of which is more fundamental than the other. Confusion is avoided by consistency of use. In two dimensions we already know that the total field outside the minimum circle enclosing a scatterer may be written:

\[ E = \sum_m (a_m H_m^{(1)}(kr) + b_m H_m^{(2)}(kr)) \exp(j m \phi) \]  

or, in vector notation

\[ a = \begin{bmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_{+1} \\ \vdots \end{bmatrix} \quad b = \begin{bmatrix} \vdots \\ b_{-1} \\ b_0 \\ b_{+1} \\ \vdots \end{bmatrix} \]  

(7.2) and (7.3)
Now $a$ is uniquely determined by the given incident field, since by hypothesis, the scattered field is entirely outgoing. This implies that we can write:

$$b = b_{\text{incident}} + b_{\text{scattered}} \quad (7.4)$$

Here $b_{\text{incident}}$ is the outgoing part of the incident field. $b_{\text{scattered}}$ is a complete description of the purely scattered field which consists of outgoing waves only. We define the $S$ matrix:

$$b_{\text{scattered}} + b_{\text{incident}} = S a \quad (7.5)$$

and the 'D' matrix:

$$b_{\text{scattered}} = D a \quad (7.6)$$

This point has been stressed in order to prevent identification of $a$ with the complete incident field and $b$ with the complete scattered field. In an exterior scattering problem there can be no sources of the incident field inside the scatterer. This being so, the incident field in the region of the origin must consist equally of incoming and outgoing parts, i.e.

$$a_m = b_m^{\text{incident}} \quad \text{for all } m \quad (7.7)$$

equivalently:

$$b_{\text{incident}} = I a \quad (7.8)$$

where $I$ is the identity matrix. In free space, $b_{\text{scattered}}$ is necessarily zero. We see from eq. (7.5) then that the scattering matrix of free space is the identity. We see also that:

$$S = I + D \quad (7.9)$$
The matrix D represents the difference in scattering power between 'free space' and 'free space plus scatterer'. Whether one works with S or D is purely a matter of choice and we will normally use S. Let us give a specific example. From Appendix B, for TM scattering from the circular cylinder

\[ b_m = - \frac{H_m^{(1)}(ka)}{H_m^{(2)}(ka)} a_m \]  

(B2):

These are total field coefficients so that:

\[ S_{nm} = - \frac{H_m^{(1)}(ka)}{H_m^{(2)}(ka)} \delta_{nm} \]

i.e. the S matrix is in this case diagonal. Using eq. (7.9) we obtain

\[ D_{nm} = S_{nm} - I_{nm} = - \left( \frac{H_m^{(1)}(ka)}{H_m^{(2)}(ka)} + 1 \right) \delta_{nm} \]

The fields derived using the S matrix are complete and will satisfy the boundary conditions when analytically continued. The fields derived using the D matrix will not.

§7.3 Reformation of the Wilton and Mittra Method

Following Munro (reference 13) we may use the analytic continuation method to calculate the scattering matrix of a two dimensional body. Refer to equation (5.4) and define matrices

\[ X_{im} = \sum_{n=-\infty}^{+\infty} H_{m+n}^{(1)}(kR_i) (J_n(kp_i) \exp j(m+n)e_i \exp jn(\nu - \phi_i)) \]
\[ Y_{im} = \sum_{n=-\infty}^{+\infty} H_{m+n}^{(2)}(kR) \cdot (J_n(k\rho_i) \exp j(m + n)\theta_i \exp \frac{jn(\pi - \phi_i)}{2}) \]

then the boundary condition may be written

\[ X_a + Y_b = 0 \quad \Rightarrow \quad b = -Y^{-1}X_a \quad \text{(7.10)} \]

In practice, \( a \) and \( b \) will be truncated as indicated earlier. It may happen that the system of equations is 'overdetermined' in the sense that there are more boundary points than there are unknowns. If this happens the solutions may be found in a least squares sense. See Wilton and Mittra reference (11). For the case of the transverse electric polarisation

\[ X_{im} = jm \left[ \frac{1}{\rho^2} \frac{\partial}{\partial \phi} \right] H_{m}^{(1)}(k\rho_i) \exp(jm\phi_i) \]

\[ - \frac{k}{2} \left( H_{m-1}^{(1)}(k\rho_i) - H_{m+1}^{(1)}(k\rho_i) \right) \exp(jm\phi_i) \quad \text{(7.11)} \]

with a similar expression for the \( Y_{im} \). In the interests of clarity, eq. (7.11) has not been analytically continued. The expressions for the matrices \( X \) and \( Y \) depend only on the positions of the boundary points. By the definition of the \( S \) matrix:

\[ b = Sa = -Y^{-1}Xa \quad \text{(7.12)} \]

If eq. (7.12) holds for arbitrary \( a \), then \( S = -1 Y^{-1}X \) for all \( a \), and the scattering matrix depends only on the scatterer and not on the details of the field.

If the scatterer is invariant under some transformation of the scattering problem, then the scattering matrix will not change.
This will place constraints on the form of the scattering matrix. In what follows, we will derive some of these constraints.

§7.4 Rotation of Co-ordinate Axes

Suppose the same field may be represented in two different co-ordinate systems:

\[ E = \sum_m (a_m H_m^{(1)}(k\mathbf{r}) + b_m H_m^{(2)}(k\mathbf{r}))\exp(j m \phi) \]
\[ = \sum_m (\tilde{a}_m H_m^{(1)}(k\tilde{\mathbf{r}}) + \tilde{b}_m H_m^{(2)}(k\tilde{\mathbf{r}}))\exp(j m \tilde{\phi}) \]

If the two systems are related by a rotation about a mutual z-axis then \( \tilde{\mathbf{r}} = \mathbf{r} \) and \( \tilde{\phi} = \phi + \chi \) with \( \chi \) as the angle of rotation. Hence

\[ a_m = \tilde{a}_m \exp(j m \chi) \]
\[ b_m = \tilde{b}_m \exp(j m \chi) \]

Define a matrix \( R \) such that \( R_{\mu\nu} = \delta_{\mu\nu} \exp(j \chi) \), then:

\[ a = Ra \quad \text{and} \quad b = R\tilde{b} \]

By definition: \( b = S a \) and \( \tilde{b} = S\tilde{a} \) from which

\[ S = R^{-1} S R = R^* S R \]

The asterisk indicates the complex conjugate. Let us suppose that the scatterer possesses \( n \)-fold rotational symmetry. Then the \( R \) matrix becomes:

\[ R_{\mu\nu} = \delta_{\mu\nu} \exp\left(j \nu \frac{2\pi}{n}\right) \]

The geometrical invariance of the scatterer under this rotation gives that: \( S = R^* S R \) for the \( S \) matrix. Explicitly:
\[ S_{\alpha \rho} = \sum_{\beta} \sum_{\gamma} R^*_{\alpha \beta} S_{\beta \rho} R_{\gamma \rho} \]

\[ = \sum_{\beta} \sum_{\gamma} \delta_{\alpha \beta} \delta_{\beta \gamma} S_{\beta \rho} \exp \left[ j \frac{2\pi}{n} (\rho - \beta) \right] \]

\[ = S_{\alpha \rho} \exp \left[ j \frac{2\pi}{n} (\rho - \alpha) \right] \quad (7.13) \]

If \( \rho = \alpha \) then eq. (7.13) is automatically satisfied, so this constraint says nothing about the leading diagonal of the scattering matrix. If \( \alpha \neq \rho \), then eq. (7.13) can only be satisfied if \( (\rho - \alpha) = n \times \text{integer} \). Otherwise \( S_{\alpha \rho} = 0 \). Thus for a body with \( n \)-fold rotational symmetry only the leading diagonal and every \( n \)th diagonal above and below it may be non zero. In the limiting case of the circular cylinder, \( n = \infty \), and only the leading diagonal may be non-zero. Another particular case is the square cylinder. Chignell on p.30 of his 1976 report noted this effect of the four-fold symmetry.

\subsection*{7.5 Reflection of Co-ordinate System}

As before let the field be represented in two different coordinate systems:

\[ E = \sum_{m} (a_m H^1_m(kr) + b_m H^2_m(kr)) \exp(j m \phi) \]

\[ = \sum_{n} (\tilde{a}_n H^1_n(k\tilde{r}) + \tilde{b}_n H^2_n(k\tilde{r})) \exp(j n \phi) \quad (7.14) \]

Let the two sets of co-ordinates be related by: \( \tilde{\phi} = -\phi \), \( \tilde{r} = r \).

This is a reflection in the \( \phi = 0 \) axis. Substitute for \( \tilde{r} \) and \( \tilde{\phi} \) in eq. (7.14) to obtain:
\[ E = \sum_n \left( a_n(-1)^{nH_1}(kr) + b_n(-1)^{nH_2}(kr) \right) \exp(-jn\phi) \]

(7.15)

where we have used the well-known result connecting Bessel functions of equal but opposite order. If now we put \( m = -n \) in eq. (7.15), we obtain

\[ E = \sum_m \left( \tilde{a}_m(-1)^{mH_1}(kr) + \tilde{b}_m(-1)^{mH_2}(kr) \right) \exp(jm\phi) \]

Thus:

\[ a_m = \tilde{a}_m(-1)^m \quad \text{and} \quad b_m = \tilde{b}_m(-1)^m. \]

Define a matrix \( M \) such that:

\[ M_{\mu \nu} = \delta_{-\mu \nu}(-1)^{\mu} = \delta_{\mu,-\nu}(-1)^{\nu}. \]

(7.16)

Then:

\[ a = MA \quad \text{and} \quad b = MB. \]

It is easily shown that the transformation of the scattering matrix is:

\[ S = M^{-1}SM \]

(7.17)

In a similar manner we can consider reflections in the \( \phi = \pm \pi/2 \) axis. The co-ordinates transform as \( \tilde{r} = r \tilde{\phi} = \pi - \phi \). Then the modal coefficients become

\[ a_m = \tilde{a}_{-m} \quad \text{and} \quad b_m = \tilde{b}_n. \]

Define a matrix \( W \) such that:

\[ W_{\mu \nu} = \delta_{\mu,-\nu} = \delta_{-\mu \nu} \]

(7.18)

and

\[ S = W^{-1}SW \]

(7.19)
A simplification can be achieved because M and W are self inverse. For example:

\[(M^2)_{\alpha r} = \sum_\beta M_{\alpha \beta} M_{\beta r} = \sum_\beta \delta_{\alpha \beta} (-1)^8 \delta_{\beta r} = \delta_{\alpha r} = I_{\alpha r}\]

Suppose that the scatterer is symmetrical about the \(\phi = 0\) axis. Then:

\[S = M^{-1} S M = M S M \quad (7.20)\]

(An alternative expression of this is that S and M commute). If eq. (7.20) is written in full, then

\[S_{\alpha \rho} = \sum_\beta \sum_r M_{\alpha \beta} S_{\beta r} M_{r \rho} = \sum_\beta \sum_r \delta_{\alpha \beta} (-1)^8 S_{\beta r} \delta_{r \rho} = (-1)^{\rho - \alpha} S_{-\alpha, -\rho}\]

If the scatterer has a symmetry under reflection in the other axis then

\[S_{\alpha \rho} = \sum_\beta \sum_r \delta_{\alpha \beta} S_{\beta r} \delta_{r \rho} = S_{-\alpha, -\rho}\]

As a check of consistency, if the scatterer possesses both kinds of reflection symmetry, then the matrix elements are necessarily zero unless \((\rho - \alpha)\) is even. This is the same condition as would be derived from rotational invariance with \(n = 2\). This is because the product of the two reflections is a rotation of 180°.

§7.6 Complex Conjugation, or Time Reversal

This section corresponds to reference (24) section (9.19). However we deal with the simpler two dimensional case. It has
already been noted that with an assumed time dependence $\exp^{+\mathbf{j} \omega t}$ that outgoing waves contain $H_n^{(2)}$ functions and incoming waves contain $H_n^{(1)}$ functions. Since these are very nearly the same, being complex conjugates of each other, it is possible to regard the former as incoming and the latter as outgoing. This is just time reversal. Mathematically since the operator $(\nu^2 + k^2)$ is real, then the complex conjugate of a solution to the Helmholtz wave equation is also a solution i.e. $(\nu^2 + k^2)E = 0$ implies that $(\nu^2 + k^2)E^* = 0$.

If the boundary conditions do not mix up the real and imaginary parts of the field then both $E$ and $E^*$ will solve the same boundary condition (subject of course to the need for analytical continuation). Taking the complex conjugate of the usual modal solution:

$$E^* = \sum_m (a_m^* H_m^{(2)}(kr) + b_m^* H_m^{(1)}(kr)) \exp(-\mathbf{j} m \phi)$$

Replace $m$ by $-n$ in the summation:

$$E^* = \sum_n (b_{-n}^* H_n^{(1)}(kr) + a_{-n}^* H_n^{(2)}(kr)) (-1)^n \exp(\mathbf{j} n \phi)$$

Define a new set of modal coefficients:

$$a_{-n} = b_{-n} (-1)^n \quad \text{and} \quad b_{-n} = a_{-n}^* (-1)^n$$

or

$$\mathbf{a} = \mathbf{M} \mathbf{b}^* \quad \text{and} \quad \mathbf{b} = \mathbf{M}^* \mathbf{a}$$

Here we have used the reflection matrix $\mathbf{M}$ defined in eq. (7.16). The effect of the complex conjugate has been to exchange the roles of the incoming and outgoing modal coefficients. We will assume that the new column vectors are connected by the same scattering
matrix as before. This is invariance under time reversal. Then by definition of the S matrix:

\[ \mathbf{S} \mathbf{a} = \mathbf{S} \mathbf{a} \]
giving

\[ \mathbf{M} \mathbf{a}^{\ast} = \mathbf{S} \mathbf{m}^{\ast} = \mathbf{S} \mathbf{M} \mathbf{S} \mathbf{a} \]
i.e.

\[ \mathbf{M} \mathbf{S} \mathbf{M} \mathbf{S}^{\ast} = \mathbf{I} \quad (7.17). \]

§7.7 The Question of Energy Conservation

If the scatterer is perfectly conducting then it will absorb no energy. That is, in the steady state the time average Poynting vector associated with the incoming field must be equal and opposite to the Poynting Vector associated with the outgoing field. A less direct approach will be used here. Compute for the TM case the vector \( (\mathbf{E} \wedge \mathbf{H}^{\ast}) \). In polar co-ordinates let \( \mathbf{E} = \mathbf{E}_k \). Then Faraday's Law of induction gives:

\[ \mathbf{H} = \frac{j}{\omega \mu_0} \mathbf{\nabla} \wedge \mathbf{E} \]

Written in terms of components this gives

\[ H_r^{\ast} = -\frac{j}{\omega \mu_0} \frac{1}{r} \frac{\partial E_{\phi}^{\ast}}{\partial \phi} \quad \text{and} \quad H_\phi^{\ast} = \frac{j}{\omega \mu_0} \frac{\partial E_r^{\ast}}{\partial r} \]

Therefore

\[ \mathbf{E} \wedge \mathbf{H}^{\ast} = \mathbf{E}_k \wedge (H_r^{\ast} \mathbf{\hat{r}} + H_\phi^{\ast} \mathbf{\hat{\phi}}) = -\frac{j}{\omega \mu_0} \frac{1}{r} E \frac{\partial E_{\phi}^{\ast}}{\partial \phi} \mathbf{\hat{\phi}} - \frac{j}{\omega \mu_0} E \frac{\partial E_r^{\ast}}{\partial r} \mathbf{\hat{r}} \]

Integrate the Poynting Vector over a circular surface centred at the origin. This means we can ignore the \( \phi \) component of the Poynting vector. Now form the integral:
Written out in full, this is:

\[
\int_{0}^{2\pi} E^* \frac{\partial E}{\partial r} - E \frac{\partial E^*}{\partial r} \, d\phi
\]

Only the terms for which \( m = n \) will survive the integration, and the term which remains is proportional to:

\[
\sum_{m} a_{m}^{*} a_{m} H_{m}^{(1)}(kr) H_{m}^{(1)}(kr) + a_{m}^{*} b_{m} H_{m}^{(2)}(kr) H_{m}^{(2)}(kr)
\]

\[
+ b_{m}^{*} a_{m} H_{m}^{(1)}(kr) H_{m}^{(1)}(kr) + b_{m}^{*} b_{m} H_{m}^{(2)}(kr) H_{m}^{(2)}(kr)
\]

\[- a_{m}^{*} a_{m} H_{m}^{(1)}(kr) H_{m}^{(2)}(kr) - a_{m}^{*} b_{m} H_{m}^{(1)}(kr) H_{m}^{(1)}(kr)
\]

\[- b_{m}^{*} a_{m} H_{m}^{(2)}(kr) H_{m}^{(1)}(kr) - b_{m}^{*} b_{m} H_{m}^{(2)}(kr) H_{m}^{(1)}(kr)
\]

\[= \sum_{m} (H_{m}^{(2)}(kr) H_{m}^{(1)}(kr) - H_{m}^{(1)}(kr) H_{m}^{(2)}(kr))(a_{m} a_{m}^{*} - b_{m} b_{m}^{*})
\]

By working with the complete field the cross terms in the field have cancelled i.e. terms like \( E_{\text{incoming}} \cdot H_{\text{outgoing}} \) do not contribute to the Poynting Vector. This is as well since it is not obvious what they mean physically. We can directly and separately equate the Poynting Vectors associated with the incoming and out-
going parts of the field, and this will be done automatically subsequently. This happens because only the real part of the Poynting Vector is significant, and the complex Poynting Vector contains an imaginary factor which introduces a minus sign when the complex conjugate is added.

We now demand that the last expression is zero for all values of \( r \). Then \( a_m a^*_m = b_m b^*_m \) for arbitrary \( a_m \), i.e.

\[
a^*Ta = b^*b = (Sa)^*Sa = a^*(ST*S)a
\]

therefore

\[
ST*S = I \quad (7.21)
\]

and energy is conserved if the scattering matrix is unitary.

§7.8 Translation Matrices

So far, the only geometrical transformations of co-ordinates that have been considered are rotations and reflections. In order to complete the set of transformations in the plane which leave the scattering contour basically unchanged we must consider the effect of translations. This can easily be done using the Graf addition theorem. However, care must be taken to distinguish between two distinct cases.

1. The field point is distant from origins so that incoming and outgoing waves retain their respective characters under the transformation.

2. The field point is near to one of the origins. This will cause a mixing of wave characters so that for example, a wave which
is outgoing with respect to the first origin will be seen as a mixture of incoming and outgoing waves at the second origin.

Case No. 1

Refer now to Fig. 7.1. This case corresponds to \( r_2 > R \).

The addition theorem gives:

\[
H_n^{(1)(2)}(kr_1)\exp(jnx_1) = \sum_{m=-\infty}^{\infty} H_m^{(1)(2)}(kr_2)J_m(kR)\exp(jmx_2)
\]

\[
\dot{\cdot} H_n^{(1)(2)}(kr_1)\exp(jn(\phi_2 - \phi_1)) = \sum_{m} H_m^{(1)(2)}(kr_2)J_m(kR)\exp(jm(\pi - \phi_2 + \alpha))
\]

\[
\dot{\cdot} (-1)^n H_n^{(1)(2)}(kr_1)\exp(-jn\phi_1) = \sum_{m} H_m^{(1)(2)}(kr_2)J_m(kR)(-1)^m
\]

\[\cdot \exp(-j(m + n)\phi_2)\exp(jm\alpha)\]

\[
\dot{\cdot} H_n^{(1)(2)}(kr_1)\exp(+jn\phi_1) = \sum_{m} H_{m-n}^{(1)(2)}(kr_2)J_m(kR)(-1)^{m-n}
\]

\[\cdot \exp(-j(m - n)\phi_2)\exp(jm\alpha)\]

In the last equation \( n \) has been changed to \(-n\). Putting \( r = m - n \).

\[
H_n^{(1)(2)}(kr_1)\exp(jn\phi_1) = \sum_{m} H_r^{(1)(2)}(kr_2)J_{n-r}(kR)\exp jr_\phi exp(j(n-r)\alpha)
\]

(7.22)

We note in eq. (7.22) that since \((kr_2)\) stays as the argument of a single Hankel function on the right hand side, that the transformed field retains its character. Defining a T matrix element
the transformation of the field coefficients can be obtained.

Consider for example a purely incoming field:

$$\sum_n a_n^{(1)} h_n^{(1)}(kr_1)\exp(jn \phi_1) = \sum_n a_n^{(1)} \sum_r T_{rn} h_r^{(1)}(kr_2)\exp(jr \phi_2)$$

Thus

$$a_r^{(2)} = \sum_{n=-\infty}^{\infty} T_{rn} a_n^{(1)}$$

In matrix notation: $a^{(2)} = Ta^{(1)}$ and $b^{(2)} = Tb^{(1)}$.

Case No. 2

This corresponds to $r_2 < R$. Thus:

$$H_n^{(1)(2)}(kr_1)\exp(jn \phi_1) = \sum_m H_m^{(1)(2)}(kR)J_m(kr_2)\exp jm(\pi - \phi_2 + \alpha)$$

$$\Rightarrow H_n^{(1)(2)}(kr_1)\exp(jn \phi_1) = \sum_m H_m^{(1)(2)}(kR)J_m(kr_2)(-1)^m$$

$$\cdot \exp(-jm \phi_2)\exp(j(m + n)\alpha)$$

Replacing $m$ by $-m$ in the last equation:

$$H_n^{(1)(2)}(kr_1)\exp(jn \phi_1) = \sum_m H_{n-m}^{(1)(2)}(kR)J_m(kr_2)\exp jm \phi_2 \exp((j(n - m)\alpha)$$

By substituting this last result into the standard representation of the field, the transformation for the modal coefficients is obtained. The incoming and outgoing waves cannot be treated separately (as before) because $(kr_2)$ appears as the argument of a Bessel function of the first kind, and not as the argument of a Hankel function. As before
\[ E = \sum_a (a_n^{(1)} H_n^{(1)}(kr_1) + b_n^{(1)} H_n^{(2)}(kr_1)) \exp(jn\phi_1) \]

\[ = \sum_m \left( \sum_n a_n^{(1)} H_n^{(1)}(kR) \frac{1}{2}(H_m^{(1)}(kr_2) + H_m^{(2)}(kr_2)) \right) \exp(jn\phi_2) \exp(j(n - m)\alpha) \]

\[ + b_n^{(1)} H_n^{(2)}(kR) \frac{1}{2}(H_m^{(1)}(kr_2) + H_m^{(2)}(kr_2)) \exp(jn\phi_2) \exp(j(n - m)\alpha) \]

To obtain eq. (7.24), the well known result

\[ J_m(kr_2) = \frac{1}{2}(H_m^{(1)}(kr_2) + H_m^{(2)}(kr_2)) \]

has been used. Thus the modal coefficients in the second co-ordinate system are:

\[ b_m^{(2)} = a_m^{(2)} = \sum_n a_n^{(1)} H_n^{(1)}(kR) \frac{1}{2} \exp(j(n - m)\alpha) \]

\[ + b_n^{(1)} H_n^{(2)}(kR) \frac{1}{2} \exp(j(n - m)\alpha) \]

We can define matrices \( Q^{(1)} \) and \( Q^{(2)} \) such that:

\[ Q_{mn}^{(1)(2)} = \frac{1}{2} H_{n-m}^{(1)(2)}(kR) \exp(j(n - m)\alpha) \]

Then

\[ b^{(2)} = a^{(2)} = Q^{(1)} a^{(1)} + Q^{(2)} b^{(2)} \]

Since, in eq. (7.27) the \( a \) and \( b \) column vectors in the transformed co-ordinate system are the same, the transformed field consists equally of incoming and outgoing parts. This can be seen as an expression of the scattering matrix of free space being the identity.
§7.9 The Two Body Scattering Problem

Attention will now be given to consider the simplest possible example of a multiple scattering problem: the two body problem. In principle, no generality will be lost by considering only two bodies, as one can treat the \( N + 1 \) body problem as a two body problem if the solution for \( N \) bodies is known. Thus one can ascend the ladder of physical complexity by a process of induction.

Wilton and Mittra did consider the two body problem in reference \((11)\), but, what follows is a more complete treatment. It will be assumed that the scattering matrix of each body is known with respect to an associated co-ordinate system, and that the co-ordinate axes of the two bodies, and the axes for the problem as a whole are parallel, as shown in Fig. 7.2.

Let:

\[ S^{(1)}, D^{(1)} \] be the scattering matrices of the scatterer with respect to its associated co-ordinate system. Similarly for \( S^{(2)} \) and \( D^{(2)} \).

Let \( S \) and \( D \) be the scattering matrices of the composite system with respect to the origin 0. Let \( a \) and \( b \) denote total incoming and outgoing fields, and let \( \beta \) represent the scattered field only. For the system as a whole:

\[ B = S A \] \hspace{1cm} (7.28)

The total outgoing field \( B \) is made up of three contributions:

(i) The outwardgoing part of the incident field.

(ii) The outward part of the scattered field from the first body

(iii) The corresponding contribution from the second body.
Therefore:

\[ B = I A + T(10)\beta^{(1)} + T(20)\beta^{(2)} \]  

(7.29)

We use T-matrices because the field point must lie outside the minimum circle enclosing the composite system. Now consider the scattering process from the locality of body No. 1. The scattered field is related to the net incoming field in its vicinity by its D matrix. There are contributions to this incoming field from

(i) The incoming part of the incident field (A)

(ii) The outgoing part of the incident field (IA)

(iii) The scattered field of body No. 2.

Therefore:

\[ \beta^{(1)} = D^{(1)}(Q^{(1)}(01)A + Q^{(2)}(01)IA + Q^{(2)}(21)\beta^{(2)}) \]  

(7.30)

The same reasoning may be applied to the scattering process in the vicinity of the second scatterer and the equation corresponding to Eq. (7.30) may be written: it is:

\[ \beta^{(2)} = D^{(2)}(Q^{(1)}(02)A + Q^{(2)}(02)IA + Q^{(2)}(12)\beta^{(1)}) \]  

(7.31)

Equations (7.28) and (7.29) may be combined to give:

\[ DA = T(10)\beta^{(1)} + T(20)\beta^{(2)} \]  

(7.32)

The coupled matrix equations (7.30) and (7.31) have to be solved to obtain the scattered field vectors which are to be substituted in eq. (7.32). Some simplification may be achieved by using the result \( Q^{(1)} + Q^{(2)} = T \). (This is possible because the incident field...
is by definition the field present before any scatterers are introduced. That is, it is a field in free space which consists equally of incoming and outgoing parts. This remains true even when the origin of co-ordinates is moved). This may be seen from the definitions (7.23) and (7.26). Rewrite eq. (7.30) and eq. (7.31) as:

\[
\beta^{(1)} - D^{(1)} Q^{(2)}(21) \beta^{(2)} = D^{(1)} T(01) A
\]

(7.33)

\[
- D^{(2)} Q^{(2)}(12) \beta^{(1)} + \beta^{(2)} = D^{(2)} T(02) A
\]

(7.34)

Multiply equation (7.33) by \( D^{(2)} Q^{(2)}(12) \) and add:

\[
(I - D^{(2)} Q^{(2)}(12) D^{(1)} Q^{(2)}(21)) \beta^{(2)} = (D^{(2)} Q^{(2)}(12) D^{(1)} T(01)
+ D^{(2)} T(02)) A
\]

that is:

\[
\beta^{(2)} = (I - D^{(2)} Q^{(2)}(12) D^{(1)} Q^{(2)}(21))^{-1}(D^{(2)} Q^{(2)}(12) D^{(1)} T(01)
+ D^{(2)} T(02)) A
\]

(7.35)

Similarly, multiply eq. (7.34) by \( D^{(1)} Q^{(2)}(21) \) and add:

\[
(I - D^{(1)} Q^{(2)}(21) D^{(2)} Q^{(2)}(12)) \beta^{(1)} = (D^{(1)} T(01)
+ D^{(1)} Q^{(2)}(21) D^{(2)} T(02))
\]

that is:

\[
\beta^{(1)} = (I - D^{(1)} Q^{(2)}(21) D^{(2)} Q^{(2)}(12))^{-1}(D^{(1)} T(01)
+ D^{(1)} Q^{(2)}(21) D^{(2)} T(02))
\]

(7.36)

If equations (7.35) and (7.36) are written as
\[ \beta^{(2)} = E^{(2)} A \quad \text{and} \quad \beta^{(1)} = E^{(1)} A \]

then

\[ D = T(10)E^{(1)} + T(20)E^{(2)} \quad \text{(7.37)} \]

In the process of building assemblies of scatterers by adding one sub-scatterer at a time, one would take origins 0 and 1 to be the same. This would result in some simplification of the algebra. For example $T(01)$ would become the identity matrix.

§7.10 Covariance of Field Conditions Under Translation and Rotation

The properties of the scattering matrix which result from applying conservation of energy and invariance under time reversal are properties of the field rather than of the scatterer. These properties should not change when the scatterer is described by a different co-ordinate system (covariance of the scattering matrix properties). The most general such co-ordinate transformation is a combination of a rotation and a translation.

Conservation of Energy: The property of the scattering matrix which this implies is (7.21) $S^* S^* = I$. Under a rotation of co-ordinates, the $S$-matrix transforms as:

\[ S = R^* S R \]

\[ \therefore S^T = R^T S^T R^* T \]

\[ \therefore S^* = R S^* R^* \]

From which $S^* S^T = R S^* R^* R S^T R^* = I$ where we have used $R = R^T$, $R^* = R^{-1}$, and $S^T S^* = I$ by hypothesis.
Under a translation

\[ S = TST^{-1} \]

\[ S^T = T^{-1}ST^T \]

\[ S^* = T^*S^*T^{-1} \]

From which \( S^*S^T = T^*S^*T^{-1}ST^T = I \) as before. We conclude that the conservation of energy is automatically covariant under translations and rotations.

**Time reversal:** The property of the scattering matrix which this implies is (7.17): \( M S(MS)^* = I \).

Under a rotation of co-ordinates the S-matrix transforms as

\[ S = R^*SR \]

\[ (MS)(MS)^* = MR^*S(RM^*)R^*RS^*R^* = MSR^*(MR^*)RS^*R^* \]

\[ = MRSMSR^* \]

\[ = TMSTM^*R^* = I \]

For a translation:

\[ S = TST^{-1} \]

\[ \therefore (MS)(MS) = MTST^{-1}(MT^*)S^*T^{-1} = MT(T^{-1}TM^*)S^*T^{-1} \]

\[ = T^*MSM^*T^{-1} \]

\[ = I \]

It is concluded that if the scattering matrix is invariant under time reversal in one co-ordinate system, it is invariant in all.
Although many of the results which have been presented for the proportion of the scattering matrix of a two dimensional body are standard, it is felt that they are useful as an introduction to the corresponding results in three dimensions or at least to the simplified versions of the three dimensional problem which are presented here.

Another reason why the two dimensional solution remains useful is because the full three-dimensional development is not known. It may be possible to represent a cross section of a three-dimensional problem fairly realistically in two dimensions, whereas it is hard to think of many realistic three dimensional scatterers which are bodies of revolution.

We will examine the problem of the body of revolution from first principles, first scattering of a scalar wave which is the acoustic problem and then the full electromagnetic problem.

§7.12 Scalar Wave Scattering

Assuming a time dependence of \( \exp + j \omega t \) the general solution of the scalar wave equation in free space may be written as:

\[
 u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} h^{(1)}_{\ell}(kr) + B_{\ell m} h^{(2)}_{\ell}(kr)) Y_{\ell m}(\theta, \phi) \tag{7.38}
\]

(we will sometimes omit the limits on summation signs, when this can be done unambiguously). Here, the \( h^{(1)}_{\ell} \) and the \( h^{(2)}_{\ell} \) are spherical Hankel functions and the \( Y_{\ell m} \) are Spherical Harmonics:
\[ Y_{\lambda m} = \left( \frac{(2\lambda+1)(\lambda-m)!}{4\pi(\lambda+m)!} \right)^{1/2} p_{\lambda m}^m(\cos \theta) \exp(j m \phi) \]

In the work on scalar waves, we will use the functions as defined in Jackson (reference 26). It will be assumed that the form of eq. (7.38) is valid even when an isolated scatterer is present, and that the coefficients of the incoming and outgoing modes, the As and Bs respectively are connected by a linear relationship:

\[ B_{\lambda m} = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} S_{st} A_{st} \quad (7.39) \]

§7.13 Rotation of the Co-ordinate System

It will be postulated that there exists a system of 'barred' co-ordinates in which

\[ u = \sum_{\lambda} \sum_{m} (\bar{A}_{\lambda m} h_{\lambda}^{(1)}(k\bar{r}) + \bar{B}_{\lambda m} h_{\lambda}^{(2)}(k\bar{r})) Y_{\lambda m}(\theta, \phi) \quad (7.40) \]

which is connected with the first set of co-ordinates by:

\[ \bar{r} = r \quad \bar{\theta} = \theta \quad \bar{\phi} = \phi + \chi. \]

Rewrite eq. (7.40) as:

\[ u = \sum_{\lambda} \sum_{m} \left( (\exp(j m \chi)\bar{A}_{\lambda m} h_{\lambda}^{(1)}(kr) \right. \]

\[ + (\exp(j m \chi)\bar{B}_{\lambda m} h_{\lambda}^{(2)}(kr)) Y_{\lambda m}(\theta, \phi) \quad (7.41) \]

By comparing equations (7.40) and (7.41):
\[ A_{\alpha m} = \exp(jm \lambda) \bar{A}_{\alpha m} \quad \text{and} \quad B_{\alpha m} = \exp(jm \lambda) \bar{B}_{\alpha m} \]

we assume that \( \bar{B}_{\alpha m} \) is connected to \( \bar{A}_{\alpha} \) by a relation of the form (7.39) so

\[ \sum_{s, t} (S_{st} \exp(jm \lambda) - S_{st} \exp(j \lambda)) \bar{A}_{st} = 0 \]

If the scatterer is a body of revolution then \( S = \bar{S} \) and the angle of rotation is arbitrary. Since the incoming field \( A \) is arbitrary then

\[ S_{\alpha m} = 0 \quad \text{for} \ t \neq m, \ \text{for all} \ s \ \text{and} \ \varphi. \]

This result may be made more explicit in the notation by defining a three index scattering matrix:

\[ S_{s,t}^{st} = \delta^t_m S_{s,m}^{st}. \quad (7.42) \]

The physical interpretation of this result is that there is no coupling between modes which have a different azimuthal variation.

§7.14 Reflection of the Scatterer in the \( \phi = 0 \) Plane

A body of rotation is left unchanged by reflection in a plane containing the axis of symmetry. Without loss of generality we may restrict attention to the \( \phi = 0 \) plane of reflection. Let \( \bar{\phi} = -\phi \), then
$$u = \sum_{\ell} \sum_{m} (A_{\ell m} h_{1}^{(1)}(kr) + B_{\ell m} h_{2}^{(2)}(kr)) Y_{\ell m}(\theta, \phi)$$

$$= \sum_{\ell} \sum_{m} (A_{\ell m} h_{1}^{(1)}(kr) + B_{\ell m} h_{2}^{(2)}(kr)) Y_{\ell m}(\theta, \phi)$$

$$= \sum_{\ell} \sum_{m} (A_{\ell m} h_{1}^{(1)}(kr) + B_{\ell m} h_{2}^{(2)}(kr)) (-1)^{m} Y_{\ell-m}(\theta, \phi)$$

where the relations $Y_{\ell m}(\theta, \phi) = Y_{\ell m}^*(\theta, \phi) = (-1)^{m} Y_{\ell-m}(\theta, \phi)$ have been used. The summation index $m$ can be replaced by $-n$ and since the azimuthal mode summation goes over equal ranges of positive and negative values:

$$u = \sum_{\ell} \sum_{n} (A_{\ell m} h_{1}^{(1)}(kr) + B_{\ell m} h_{2}^{(2)}(kr)) (-1)^{n} Y_{\ell n}(\theta, \phi)$$

from which by comparison with the standard form (7.38) is obtained:

$$A_{\ell n} = (-1)^{-n} A_{\ell,-n} \quad \text{and} \quad B_{\ell n} = (-1)^{-n} B_{\ell,-n}$$

Substitution of these results into eq. (7.39) gives

$$(-1)^{-n} B_{\ell,-n} = \sum_{r} \sum_{t} S_{\ell n}^{r t} (-1)^{-t} A_{r,-t} \quad (7.43)$$

but

$$B_{\ell,-n} = \sum_{u} \sum_{v} S_{\ell n}^{uv} A_{uv} \quad (7.44)$$

The relations (7.43) and (7.44) must hold for an arbitrary choice of the incoming field. As an aid to manipulation choose the particular values

$$A_{uv} = \delta_{u w} \delta_{v x}$$
i.e. the incoming field consists of a single mode only. Then from eq. (7.44)

\[ \bar{b}_{\xi,-n} = \sum_u \sum_v s_{\xi,-u}^{uv} s_{u,v}^{wu} = s_{\xi,-n}^{wx} \]

Substitute this last result into eq. (7.43) then:

\[ (-1)^{-n} s_{\xi,-n}^{wx} = s_{\xi,n}^{w-x} (-1)^x \]

\[ s_{\xi,-n}^{wx} = (-1)^{x+n} s_{\xi,n}^{w-x} \] \hspace{1cm} (7.48)

but we have already shown that both sides of the equation are zero unless \((x + n) = 0\). This gives:

\[ s_{\xi,-n}^{w} = s_{\xi,+n}^{w} \] \hspace{1cm} (7.46)

That is, the scattering matrix elements depend only on the absolute value of the azimuthal order, and not upon its sign.

§7.15 The Conservation of Energy

Once again it is assumed that the scatterer does not absorb energy so that in the steady state the power associated with the incoming field must equal that associated with the outgoing field. For a strictly scalar field the idea of a Poynting vector is not applicable, and it will be assumed that the power flux density is given by the modulus squared of the complex field.

The incoming flux density is given by

\[ \sum_{\xi} \sum_{\xi'} \sum_{m} A_{\xi m} A_{\xi' m}^* h^{(1)}_{\xi}(kr) h^{(2)}_{\xi'}(kr) Y_{\xi m}(\theta, \phi) Y_{\xi' m}^*(\theta, \phi) \]
Now integrate this flux density over the surface of a sphere centred at the origin. The surface element is: $r^2 \sin \theta \, d\theta \, d\phi$.

Then the incoming power is

$$\sum_{\ell} \sum_{m} A_{\ell m} A_{\ell, m}^{*} \cdot h^{(1)}_{\ell} (kr) h^{(2)}_{\ell} (kr) r^2 \times \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta \, Y_{\ell m}^*(\theta, \phi) Y_{\ell, m}^{*} (\theta, \phi) d\theta$$

There is an orthogonality relation for the spherical harmonics (see reference 26, eq. 3.55) which gives the value of the double integral as: $\delta_{\ell \ell}' \delta_{mm}'$. Performing the equivalent calculation for the outgoing energy:

$$\sum_{\ell} A_{\ell m} A_{\ell m}^{*} = \sum_{p} B_{pq} B_{pq}$$

or

$$\sum_{\ell m} A_{\ell m} A_{\ell m}^{*} = \sum_{p q} \sum_{rt} s_{rt} s_{uv}^{*} A_{rt} A_{uv}^{*} \quad (7.47)$$

By reversing the order of the summation signs in (7.47) we obtain

$$\sum_{p} \sum_{q} s_{pq} s_{pq}^{*} = \delta_{ru} \delta_{tv}$$

Using eq. (7.42) to reduce 4-index matrices to 3-index matrices

$$\sum_{p} s_{pt} s_{pt}^{*} = \delta_{ru} \quad (7.48)$$
§7.16 Invariance of the Scattering Matrix Under Time Reversal

Following the now familiar reasoning if

\[ u = \sum_{\ell} \sum_{m} (A_{\ell m} h_{\ell}^{(1)}(kr) + B_{\ell m} h_{\ell}^{(2)}(kr))Y_{\ell m}(\theta, \phi) \]

is a solution of a scattering problem, then so is

\[ u^* = \sum_{\ell} \sum_{m} (B_{\ell m}^* h_{\ell}^{(1)}(kr) + A_{\ell m}^* h_{\ell}^{(2)}(kr)(-1)^m)Y_{\ell m}(\theta, \phi) \]

As usual we may replace \( m \) by \(- m\) in the summation and comparing coefficients obtain:

\[ A_{\ell m} = (-1)^m B_{\ell, -m}^* \]

and:

\[ B_{\ell m} = (-1)^m A_{\ell, -m}^* \]

By the definition of the scattering matrix, and assuming its invariance under the transformation:

\[ B_{\ell m} = \sum_{rt} S_{\ell m}^{rt} A_{rt} \]

or

\[ (-1)^m A_{\ell, -m}^* = \sum_{rt} S_{\ell m}^{rt} (-1)^{-t} B_{r, -t}^* \quad (7.49) \]

but

\[ B_{uv} = \sum_{rt} S_{uv}^{rt} A_{rt} \]

As before, since the incoming field is arbitrary the following particular case may be chosen:

\[ A_{rt} = \delta_{rw} \delta_{tx} \]
Then
\[ B_{uv}^* = S_{uv}^{wx*} \]

Substitute these results for the As and Bs into eq. (7.49):
\[ (-1)^{-m} \delta_{lw} \delta_{mx} = \sum_r S_{lm}^{*} (-1)^{-t} S_{rm}^{wx*} \]

Recall that the 4-index scattering matrix element is only non-zero when the azimuthal indices are equal and perform the sum over the index t.
\[ (-1)^{-m} \delta_{lw} \delta_{mx} = \sum_r S_{lm}^{r,t} (-1)^{-t} S_{rm}^{wx*,r,-t} \]

Let \( x = -m \) then
\[ (-1)^{-m} \delta_{lw} = \sum_r S_{lm}^{r,m} (-1)^{-m} S_{rm}^{w,-m*} \]

Reverting to 3-index matrix elements we have:
\[ \sum_r S_{lm}^{r,m} S_{rm}^{w*} = \delta_{lw} \quad (7.50) \]

Comparing eq. (7.50) with eq. (7.48)
\[ \sum_r S_{lm}^{r,m} S_{rm}^{w*} = \delta_{lw} \]

We find that
\[ S_{lm}^{r,m} = S_{rm}^{w*} \quad (7.51) \]
$7.17$ The Scattering Matrix of Free Space

In the two dimensional case we found that this was the identity matrix. Considering the three dimensional scalar case, we examine the standard form \((7.38)\). In free space there can be no singularity at the origin and this will be the case if: \(B_{\lambda m} = A_{\lambda m}\).

From this it is easy to see that:

\[
S(\text{free space}) : S_{\lambda m}^{U V} = \delta_{\lambda}^{U} \delta_{m}^{V}
\]

In tensor analysis this would be known as the outer product of the two Kronecker deltas. Often, scattering matrices are made 'two dimensional' by designating each combination of modal indices as a single index. We have not done this because it is felt that it obscures understanding at a detailed level.

$7.18$ The Boundary Condition Problem for the Scalar Wave Equation

To evaluate the scattering matrix in detail, one must solve the boundary condition problem. The boundary condition is that the total field at the surface of the scatterer must be zero.

Because it has been shown that there is no coupling between different orders of azimuthal mode, without loss of generality, attention may be restricted to a field which has only a single azimuthal order. Then eq. \((7.52)\) has to be solved:

\[
\sum_{\lambda=0}^{\infty} (A_{\lambda m} h_{\lambda}^{(1)}(kr) + B_{\lambda m} h_{\lambda}^{(2)}(kr)) Y_{\lambda m}(\theta, \phi) = 0 \quad (7.52)
\]

for all points on the surface of the scatterer. However, the \(\phi\) dependence of all the terms in eq. \((7.52)\) is the same, and may be cancelled. This leaves an equation which must be satisfied for
boundary points along a particular generator of the scatterer. This is effectively a two dimensional problem. Unfortunately, there will be the same convergence problem as there was in the strictly two-dimensional case. That is although outside the smallest sphere centred upon the origin which encloses the scatterer, we expect the scattered field to consist entirely of outgoing waves, inside the sphere, the scattered field may contain an incoming part. It should be possible to analytically continue the solution (eq. 7.52) to the surface of the scatterer as in the two dimensional case, except that the addition theorems for the wave functions are more complicated. A very brief discussion of this point is given in Appendix E.

§7.19 The Relation of this Procedure to that of Munro

Munro's procedure may be justified by the following argument:

1. If B is a body of revolution then in the co-ordinate system C which has the symmetry axis of B as the z-axis, then we have shown that there is no coupling between spherical modes of different azimuthal order. We might express this by saying that the scattering matrix is 'azimuthally diagonal'.

2. If such is the case, then the boundary condition problem may be solved for each individual value of m in turn.

3. For each such sub-problem all the functions concerned have the same dependence on the angle φ.

4. Therefore for each sub-problem, if the boundary condition is satisfied for one generator of B, then it is automatically satisfied everywhere else on B.

Consider a co-ordinate system C' which is obtained from C by an otherwise arbitrary rotation about an axis through the origin.
5. Then there must exist linear combinations of the modes in \( C' \)
which correspond to modes of a single azimuthal order in \( C \). For
example, we might express

\[
E_{tm} = \sum P SP_{tm} h_p^{(2)}(kr)Y_{pm}(\theta, \phi) + h_c^{(1)}(kr)Y_{tm}(\theta, \phi)
\]
as a linear combination of the \( h_{\lambda n}^{(2)}(kr')Y_{\lambda n}(\theta', \phi') \)

6. It must be possible to write the total field in \( C' \) as

\[
E(C') = \sum a_{tm}E_{tm}
\]

7. It is known that each individual \( E_{tm} \) has the property that if
it is zero on one generator of \( B \), then it is zero all over \( B \).

8. So \( E(C') \) must have the same property.

9. Therefore in \( C' \) it is sufficient to apply the boundary condition
along one generator of \( B \). However the scattering matrix obtained
now will not necessarily be 'azimuthally diagonal'.

If this argument is accepted, then it leads to great simplification
in the analytical continuation procedure. For further
details see Appendix E.

§7.20 Scattering of an Electromagnetic Field by a Perfectly
Conducting Body of Rotation

In order to facilitate comparison with references (14) and (23)
spherical vector harmonic functions will be used with real azimuthal
variation (i.e. \( \sin m\phi \) and \( \cos m\phi \) rather than \( \exp(j m\phi) \)). It should
also be noted that the definition of the spherical Hankel function
used in reference (23) is slightly non-standard. This will not matter as we are not greatly concerned here with the detailed radial variation of the fields. What is more significant is that in reference (14) Munro has taken the $\vartheta = \pi/2 \phi = 0$ axis as the axis of rotation of the scatterer. (This greatly simplifies the application of the addition theorem of the wave functions). This convention will not be adopted here because it complicates the mathematics when the rotation of the co-ordinate system is considered. If we were to follow Munro's procedure we would have to say

$$\sin \vartheta \cos \phi = \sin \vartheta \cos \phi$$

and

$$\begin{pmatrix} \sin \vartheta \sin \phi \\ \cos \vartheta \end{pmatrix} = \begin{pmatrix} \cos \chi - \sin \chi \\ \sin \chi \cos \chi \end{pmatrix} \begin{pmatrix} \sin \vartheta \sin \phi \\ \cos \vartheta \end{pmatrix}$$

This leads to 'mixing' of the angular co-ordinates and the transformation of the angular part of the wave function would be very complicated. (see for example reference 28, Appendix 1).

Following Kahn and Wasylkiwskyj in reference (23) we write for the **transverse** electric field:

$$r E_t = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e}^{o} j_t \hat{F}_{nmp}(\alpha, \beta) \hat{e}_{nmp} + F_{nmp}(\gamma, \delta) \hat{e}_{nmp} \cdot \hat{r}$$

(7.53)

where

$$F_{nmp}(\gamma, \delta) = \gamma_{nmp} h_n^{(1)}(kr) + \delta_{nmp} h_n^{(2)}(kr)$$

and

$$\hat{F}_{nmp}(\alpha, \beta) = \alpha_{nmp} \hat{h}_n^{(1)}(kr) + \beta_{nmp} \hat{h}_n^{(2)}(kr)$$

The dot indicates differentiation with respect to argument and the
summation index $p$ (for parity) can take the values $e$ or $0$ (even or odd). The circumflex indicates a unit vector and:

$$\hat{e}_{nmp} = \frac{1}{N_{nm}} (- r_t v^m_n (\cos \theta) \cos m\phi)$$

$$\sin m\phi$$

Here $t^\gamma$ is the transverse part of the gradient operator. $N_{nm}$ is a normalising factor. Further details on the $e_{nmp}$ are given in Appendix F. It is supposed that there is a linear relation between the incoming and outgoing field coefficients, which shall be written as:

$$\beta_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\beta\alpha)_{uvw} + S_{nmp}^{uvw} (\beta\gamma)_{uvw}$$

$$\delta_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\delta\alpha)_{uvw} + S_{nmp}^{uvw} (\delta\gamma)_{uvw}$$

or, more compactly:

$$\begin{bmatrix} \beta \\ \delta \end{bmatrix} = \begin{bmatrix} S(\beta\alpha) \\ S(\beta\gamma) \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$$

We combine eq. (7.53) written out in full with the transformation rules (F5) and (F6) for the vector modes which are derived in Appendix F, to find the relationships between the field coefficients in the two co-ordinate systems:

$$rE_t = \sum_{nm} j \zeta (\tilde{\alpha}_{nme} h_n^{(1)}(kr) + \beta_{nme} h_n^{(2)}(kr)) (\cos mx \hat{e}_{nme} - \sin mx \hat{e}_{nmo})$$

$$+ j \zeta (\tilde{\alpha}_{nmo} h_n^{(1)}(kr) + \beta_{nmo} h_n^{(2)}(kr)) (\sin mx \hat{e}_{nme} + \cos mx \hat{e}_{nmo})$$

$$+ a \text{ similar expression involving the } \tilde{\gamma} \text{ and } \tilde{\delta}.$$
\[
\begin{align*}
&= \sum_{nm} j \zeta (\alpha_{nme} \hat{h}_n^{(1)}(kr) + \beta_{nme} \hat{h}_n^{(2)}(kr)) \hat{e}_{nme} \\
&\quad + j \zeta (\alpha_{nmo} \hat{h}_n^{(1)}(kr) + \beta_{nmo} \hat{h}_n^{(2)}(kr)) \hat{e}_{nmo} \\
&\quad + \text{a similar expression involving the } \gamma \text{ and } \delta.
\end{align*}
\]

By comparing the coefficients of the vector mode functions we obtain:

\[
\begin{pmatrix}
\alpha_{nme} \\
\alpha_{nmo}
\end{pmatrix} =
\begin{pmatrix}
\cos mx & \sin mx \\
-\sin mx & \cos mx
\end{pmatrix}
\begin{pmatrix}
\alpha_{nme} \\
\alpha_{nmo}
\end{pmatrix}
\]

\hspace{1cm} (7.54)

Corresponding relationships hold for the \(\beta\), \(\gamma\) and \(\delta\) coefficients.

Defining

\[
M_{pp'}^{M} =
\begin{pmatrix}
\cos mx & \sin mx \\
-\sin mx & \cos mx
\end{pmatrix}
\]

\hspace{1cm} (7.55)

The suffices on this matrix element are parity suffices. Now, by definition:

\[
\beta_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\beta_\alpha)_{\alpha_{uvw}} + S_{nmp}^{uvw} (\beta_\gamma)_{\gamma_{uvw}}
\]

\hspace{1cm} (7.56)

Assuming the invariance of the \(S\) matrices we may rewrite eq. (7.56) in terms of the transformed coefficients:

\[
\sum_{pp'} \bar{\beta}_{nmp} = \sum_{uvw} \bar{\alpha}_{uvw} S_{nmp}^{uvw} (\beta_\alpha)_{\alpha_{uvw}} + S_{nmp}^{uvw} (\beta_\gamma)_{\gamma_{uvw}}
\]

\[
= \sum_{pp'} \sum_{uvw} \bar{\alpha}_{uvw} S_{nmp}^{uvw} (\beta_\alpha)_{\alpha_{uvw}} + S_{nmp}^{uvw} (\beta_\gamma)_{\gamma_{uvw}}
\]
(V is a matrix of the same form as in equation (7.55)).

Now, the \( \alpha \) and \( \gamma \) coefficients describe the incoming field. They are independent and arbitrary, hence:

\[
\sum_w S_{\text{nmp}}^{uvw} V_{\text{ww}} = \sum_p M_{\text{pp}}^{uvw} S_{\text{nmp}}^{uvw} \quad (7.57)
\]

Where eq. (7.57) holds for \( S(\beta \alpha), S(\beta \gamma), S(\delta \alpha) \) and \( S(\delta \gamma) \) all independently. Changing \( p \) to \( w \) on the right hand side of eq. (7.57):

\[
\sum_w (S_{\text{nmp}}^{uvw} V_{\text{ww}} - S_{\text{nmp}}^{uvw} M_{\text{pw}}) = 0 \quad (7.58)
\]

Equation (7.58) is valid for all \( u, v, n \) and \( m \). We can identify four distinct cases which depend on the values of the 'free' parity indices \( w \) and \( p \). These are:

(i) \( p = w = e \)

\[
S_{\text{nme}}^{uve} V_{\text{ee}} - S_{\text{nme}}^{uve} M_{\text{ee}} + S_{\text{nme}}^{uvo} V_{\text{oe}} - S_{\text{nmo}}^{uve} M_{\text{eo}} = 0
\]

(ii) \( p = \bar{w} = o \)

\[
S_{\text{nme}}^{uve} V_{\text{eo}} - S_{\text{nme}}^{uve} M_{\text{oe}} + S_{\text{nmo}}^{uvo} V_{\text{oo}} - S_{\text{nmo}}^{uvo} M_{\text{oo}} = 0
\]

(iii) \( p = e \quad \bar{w} = o \)

\[
S_{\text{nme}}^{uve} V_{\text{eo}} = S_{\text{nme}}^{uvo} M_{\text{ee}} + S_{\text{nme}}^{uvo} V_{\text{oe}} - S_{\text{nme}}^{uve} M_{\text{eo}} = 0
\]

(iv) \( p = o \quad \bar{w} = e \)

\[
S_{\text{nme}}^{uvo} V_{\text{ee}} - S_{\text{nme}}^{uvo} M_{\text{oe}} + S_{\text{nme}}^{uvo} V_{\text{oe}} - S_{\text{nme}}^{uve} M_{\text{eo}} = 0
\]
Now, the two diagonals of the $M$ and $V$ matrices are linearly independent. Furthermore if $v \neq m$ then the $M$ and $V$ matrices will be independent of each other. In this case then, all the matrix elements will be zero.

If $v = m$ then

$$M_{ee} = M_{oo} = V_{ee} = V_{oo}$$

and

$$M_{eo} = V_{eo} = -M_{oe} = -V_{oe}$$

Case (i) gives:

$$S_{nme}^{umo} = -S_{nmo}^{ume}$$

Case (ii) gives:

$$S_{nme}^{ume} = -S_{nmo}^{umo}$$

Case (iii) gives:

$$S_{nme}^{ume} = \pm S_{nmo}^{umo}$$

Case (iv) gives:

$$S_{nme}^{ume} = \mp S_{nmo}^{umo}$$

Therefore

a) Scattering does not take place between different azimuthal orders.

b) Even-Even scattering is the same as Odd-Odd scattering.

c) Odd-Even scattering is the negative of Even-Odd scattering.

§7.21 Invariance of the Scattering Matrix under Reflection

With our particular choice of vector mode functions, this is not a complicated invariance to investigate. As in the scalar case we may without loss of generality consider reflection in the $\phi = 0$ plane. This is equivalent to: $\phi \rightarrow -\phi$ and $\hat{\phi} \rightarrow -\hat{\phi}$. 
and it is easy to show that:

\[ \hat{e}_{nme} \rightarrow \tilde{e}_{nme} \quad \text{and} \quad \hat{e}_{nmo} \rightarrow -\tilde{e}_{nmo}. \]

The relation between the two sets of field coefficients in the two co-ordinate systems connected by such a reflection is:

\[
\begin{pmatrix}
\alpha_{nme} \\
\alpha_{nmo}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\tilde{\alpha}_{nme} \\
\tilde{\alpha}_{nmo}
\end{pmatrix}
\quad \text{and vice versa.}
\]

Define the Matrix

\[
M_{pp} = \begin{cases}
1 & \text{if } p = \bar{p} = e \\
-1 & \text{if } p = \bar{p} = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Now

\[
\beta_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\beta\alpha)\alpha_{uvw} + S_{nmp}^{uvw} (\beta\gamma)\gamma_{uvw}
\]

\[
\therefore \sum_{\bar{p}} M_{pp} \beta_{nmp} = \sum_{uvw} \sum_{\bar{p}} M_{pp} S_{nmp}^{uvw} (\beta\alpha)\alpha_{uvw} + S_{nmp}^{uvw} (\beta\gamma)\gamma_{uvw}
\]

but

\[
\bar{\beta}_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\beta\alpha)\bar{\alpha}_{uvw} + S_{nmp}^{uvw} (\beta\gamma)\bar{\gamma}_{uvw}
\]

from which

\[
\sum_{\bar{p}} M_{pp} \beta_{nmp} = \sum_{uvw} S_{nmp}^{uvw} (\beta\alpha) \sum_{\bar{w}} M_{\bar{w}} - \alpha_{uvw}
\]

\[+ S_{nmp}^{uvw} (\beta\alpha) \sum_{\bar{w}} M_{\bar{w}} \gamma_{uvw}. \]

The \( \alpha \)s and \( \gamma \)s are independent. Equating coefficients:
\[ \sum_{p} M_{pp} S_{nmp}^{uvw} = \sum_{w} S_{nmp}^{uvw} M_{ww} \]

or

\[ \sum_{w} M_{pw} S_{nmw}^{uvw} - M_{ww} S_{nmp}^{uvw} = 0 \]

Once again we can identify four cases:

(i) \( p = w = e \)

\[ M_{ee} S_{nme}^{uve} - M_{ee} S_{nme}^{uve} + M_{eo} S_{nmo}^{uve} - M_{oe} S_{nme}^{uve} = 0 \]

(ii) \( p = \bar{w} = 0 \)

\[ M_{oe} S_{nme}^{uvo} - M_{eo} S_{nmo}^{uvo} + M_{oo} S_{nmo}^{uvo} - M_{oo} S_{nmo}^{uvo} = 0 \]

(iii) \( p = e \bar{w} = 0 \)

\[ M_{ee} S_{nme}^{uvo} - M_{eo} S_{nmo}^{uvo} + M_{oo} S_{nmo}^{uvo} - M_{oe} S_{nme}^{uvo} = 0 \]

(iv) \( p = 0 \bar{w} = e \)

\[ M_{oe} S_{nme}^{uve} - M_{ee} S_{nme}^{uve} + M_{oo} S_{nmo}^{uve} - M_{oe} S_{nme}^{uve} = 0. \]

From eq. (7.59) we see that \( M_{oo} = -M_{ee} \) and \( M_{oe} = M_{eo} = 0 \); we see that:

\[ S_{nme}^{uvo} = S_{nme}^{uve} = 0. \]

Even modes scatter into even modes only
Odd modes scatter into odd modes only.

The application of the conservation of energy is not in the direct line of argument. A brief treatment is given in Appendix I.
$\S 7.22$ The Boundary Condition Problem

According to Kahn and Wasylykowskyj in reference (23)

\[ rE_t = \sum_{nmp} j \zeta F_{nmp}(\alpha \delta) \hat{e}_{nmp} + F_{nmp}(\gamma \delta) \hat{e}_{nmp} - \hat{r} \quad (7.60) \]

\[ rH_t = \sum_{nmp} F_{nmp}(\alpha \delta) \hat{r} - \hat{e}_{nmp} + j n F_{nmp}(\gamma \delta) \hat{e}_{nmp} \quad (7.61) \]

with \( \zeta = 1/n = \sqrt{\frac{\mu_0}{\varepsilon_0}} \)

The boundary condition is that the total tangential electric field at the surface of the scatterer is zero. In reference (14) Munro took the total transverse field to be zero. Because we are dealing with a body of revolution, these two procedures will agree for the \( \hat{\phi} \) component, but we will obtain a different condition from that implied by the vanishing of the \( \hat{\phi} \) component alone. The radial component of the field must be considered.

Without loss of generality we may consider a field of a single azimuthal order \( m \). (It has already been shown that the scattering process does not mix different azimuthal orders). Similarly we need consider a field of a single 'parity' only since we have already shown that there is no odd-even or even-odd scattering, and that even-even scattering is the same as odd-odd scattering.

The boundary conditions will be examined in two stages. First, the vanishing of the \( \hat{\phi} \) component. Secondly the vanishing of the tangential field component perpendicular to \( \hat{\phi} \). This will require the manipulation of eq. (7.61) to obtain the radial field component.
§7.23 The Vanishing of the $\Phi$-Component

From eqs. (7.60), (F2) and (F7) this gives:

$$\sum_n j \xi \hat{F}_{nmp}(\alpha \beta) \left[ - \frac{1}{N_{nm}} \left( \frac{p_m^{(\cos \theta)}}{\sin \theta} \right) - m \sin m\phi \right]$$

$$\left( - \frac{1}{N_{nm}} \right) \frac{dp_m^{(\cos \theta)}}{d\theta} \left( \frac{\cos m\phi}{\sin m\phi} \right) = 0$$

This equation must hold for the full range of values of $\phi$. Thus it is easy to see that the $\alpha$s and $\beta$s decouple from the $\gamma$s and $\delta$s. The $\cos(m\phi)$ and $\sin(m\phi)$ factors in the two equation may be cancelled and:

$$\sum_n j \xi \hat{F}_{nmp}(\alpha \beta) \left[ - \frac{1}{N_{nm}} \frac{p_m^{(\cos \theta)}}{\sin \theta} \right] = 0 \quad (7.62)$$

and

$$\sum_n F_{nmp}(\gamma \delta) \left[ - \frac{1}{N_{nm}} \frac{dp_m^{(\cos \theta)}}{d\theta} \right] = 0 \quad (7.63)$$

§7.24 The Vanishing of the 'other' field component

Along a line of constant $\phi$, the vector tangent to the surface which lies in the $\phi = \text{constant}$ plane is

$$\hat{t} = \frac{1}{r} \frac{dr}{d\theta} \hat{r} + \hat{\theta}$$

Here $r = r(\theta)$ is the equation of the generator of the scatterer.

Given that $E_\phi$ vanishes, then the boundary condition becomes

$$E_\phi \cdot \hat{t} = 0 \Rightarrow E_\phi = - \frac{1}{r} \frac{dr}{d\theta} E_r$$

$E_r$ may be determined for the expression (7.61) for the transverse
magnetic field using the Maxwell equation:

\[
E = \frac{1}{j\omega e_0} \nabla \times H \\
+ E_r = \frac{1}{r \sin \theta} \left( \frac{3}{\partial \phi} (\sin \phi H_x^{\phi}) - \frac{\partial H_x^{\phi}}{\partial \phi} \right) \times \frac{1}{j\omega e_0} 
\]  

(7.64)

The substitution of eq. (7.61) in eq. (7.64) gives rise to complicated algebra as shown in Appendix G. Here we merely quote the results (eq. G.1) and (eq. G.4):

\[
\sum_n j \xi F_{nmp}(\alpha\beta) \left[ - \frac{1}{N_{nm}} \frac{dP_m^m(\cos \theta)}{d\phi} \right] \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix} \\
+ F_{nmp}(\gamma\delta) \left[ - \frac{1}{N_{nm}} \frac{p_n^m(\cos \theta)}{\sin \theta} \right] \begin{bmatrix} -m \sin(m\phi) \\ +m \cos(m\phi) \end{bmatrix} \\
= - \frac{1}{r} \frac{dr}{d\phi} \cdot \frac{j\xi}{(kr)} \sum_n \left[ - \frac{1}{N_{nm}} \right] F_{nmp}(\alpha\beta)n(n+1)p_n^m(\cos \theta) \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix} 
\]

(7.65)

If we examine equation (7.65) we see that once again, because of the linear independence of \(\cos m\phi\) and \(\sin m\phi\) the \((\alpha\beta)\) equations are decoupled from the \((\gamma\delta)\) equations. This implies that the \(S(\beta\gamma)\) and the \(S(\delta\alpha)\) matrices vanish. Once again, the azimuthal factors may be cancelled to leave:

\[
\sum_n F_{nmp}(\gamma\delta) \left[ - \frac{1}{N_{nm}} \frac{p_n^m(\cos \theta)}{\sin \theta} \right] = 0 
\]

(7.66)

and
\[ \sum_{n} \hat{F}_{nnp}(\alpha \beta) \left( - \frac{1}{N_{nm}} \right) \frac{d p_{n}^{m}(\cos \theta)}{d \theta} \]

\[ + \frac{1}{(kp)^{2}} \frac{d(kp)}{d \theta} \left( - \frac{1}{N_{nm}} \right) F_{nnp}(\alpha \beta)n(n + 1)p_{n}^{m}(\cos \theta) = 0 \quad (7.67) \]

Equations (7.62), (7.63), (7.66) and (7.67) constitute a quasi-two-dimensional problem. Of course they will be subject to the usual convergence difficulties which must be overcome by the analytical continuation procedure (see comments for the scalar problem).

§7.25 Summary of Chapter Seven

1. The linearity of the scattering problem for perfect conductors implies that a scattering matrix may be used to relate the column vectors of the coefficients of the field modes.

2. There are two reasonable definitions of the scattering matrix. The S-matrix relates the incoming field to the total outgoing field. The D matrix relates the incoming field to the scattered field.

3. The quantity (S-D) may be identified with the scattering matrix of free space, which is the identity matrix.

4. The Wilton and Mittra analytic continuation method may be written in a form which provides an expression for the scattering matrix of a two-dimensional body.

5. The scattering matrix is a property of the scattering body only. If a transformation of the co-ordinate system on the field leaves the scatterer unchanged, then the scattering matrix is also unchanged.
6. When applying the condition of conservation of energy, the energy associated with the incoming field may be directly equated to that of the outgoing field. For a perfect conductor in the steady state, there is no absorption of energy, and the two will be equal.

7. By introducing 'Translation Matrices', the scattering matrix of a composite two body system may be found in terms of the scattering matrices of the individual bodies.

8. Properties of the two dimensional matrix which depend on the conservation of energy and invariance under time reversal, are 'covariant' under co-ordinate transformations in the plane which leave the shape of the scatterer unchanged.

9. For the scattering of a scalar field from a body of revolution:

   (i) There is no coupling between different orders of azimuthal mode.

   (ii) Scattering depends only on the absolute value of the azimuthal order, and not upon its sign.

   (iii) The boundary condition problem may be put in a Quasi-two-dimensional form.

10. For the scattering of an electromagnetic (vector) field from a body of revolution:

    (i) There is no coupling between different orders of azimuthal modes.

    (ii) Working with 'even' and 'odd' vector modes, even-even scattering is the same as odd-odd scattering.
(iii) There is no 'odd-even' or 'even-odd' scattering.

(iv) The boundary condition problem may be put in a Quasi-two-dimensional form.

(v) The coefficients in the expansion of the field decouple into two sets.

Much of this chapter has been devoted to deriving what are called 'selection rules' in Quantum Mechanics. However there is not really a direct comparison since Quantum Mechanical calculations of the probability of transition from state \(i \rightarrow j\) depend on the expression:

\[
<j|\hat{H}|i>
\]

which is derived from the 'Golden rule' of time dependent perturbation theory with \(\hat{H}\) as the operator of the perturbation of the Hamiltonian of the system. In our work, there is nothing which corresponds to \(\hat{H}\).
CHAPTER 8 APERTURE FIELD THEORY AND THE CANONICAL NEAR FIELD METHOD

§8.1 It is well known that Green's theorem for the calculation of the unknown field at a point by integration of a known field over a surface may be made the basis of an approximate method for calculating a scattered field. This is the basis of the so-called aperture field method, described as such by Silver in reference (31). It is not therefore a new technique, but its use until recent years seems to have been confined to calculations of the field near the main beam axis of highly directive systems. (For a recent application of this type, see reference (36)). On the more experimental side the technique has seen an increase of interest because of the attention being paid to antenna measurements in the near field, where measurements made directly in the far field are not practicable. See for example, the papers in reference (32). If the far field is taken to lie beyond distances $2D^2/\lambda$ ($D =$ diameter of antenna, $\lambda =$ wavelength) then for a large antenna, this may simply be too big to fit onto a range. What is fairly novel (about the theory) is the use of the technique for scatterers which are not reflector antennas as such, and away from the main reflected field.

There are two immediate advantages of the Aperture field Approach:

(i) The field in the 'aperture' may be approximate, but the far field calculated using Green's theorem is guaranteed to be a solution of the wave equation at the appropriate frequency.
(ii) Any discontinuities present in the aperture field will be 'smoothed out' by the integration.

However there are two major problems:

(i) It is well known that such an approach does not lead to a self-consistent field. That is, when the field point approaches the surface of integration, one does not obtain the original aperture field. The inconsistencies of the approach can be removed by the use of a proper Green's function - see Jackson, section 9.8 - but specialised Green's functions are only available for a few geometries, so we will retain the primitive approach using the free space Green's function. (This was why the Kirchoff method is used in Appendix A - partly to be consistent with what follows, partly because the Appendix is only intended as an illustration).

(ii) The shape and position of the surface of integration are both arbitrary. If the field on the intermediate surface was exact then this would not matter. However, of necessity the field on the surface of integration is approximate, and a priori, it may be that the far field obtained is highly dependent on the integrating surface.

The choice of the shape of the integrating surface is partly determined by convenience. Green's theorem applies to a surface enclosing source-free space. If we consider localised scatterers then the contribution to the integral from parts of the surface distant from the system will vanish as the distance of the surface becomes very large. If we consider the two dimensional situation
as shown in Figure 8.1, we expect the contribution to the integral from FED to go inversely as the square of its radius. The contribution from AF and CD cancel each other if the two sections are made to coincide (this happens because the normals to the surfaces point in opposite directions). We are left with the loop ABC, a surface of integration which encloses the system.

The obvious choice - dictated by simplicity - of ABC is a circle which is centred at the origin of co-ordinates. A suitable radius for this circle is at this stage still an unknown quantity but this problem was investigated using the program GOLD, which is described later.

An alternative geometry was suggested by Cornbleet (as shown in Figure 8.2). It consists of a plane AA'B'B and a semicircle BCDA. The contribution from the semicircle vanishes as the radius becomes infinitely large and the integral is taken over a finite section A'B' of the infinite plane AB. The advantage of this arrangement is that since aperture field methods work best in a direction of observation normal to the surface of integration (QQ'), one could always arrange this by rotating the plane A'B' around so that the normal points are in the desired direction.

This approach has not been tried, for the following reasons:

(i) One would have to recompute the values of the field on A'B' for each distinct angle of observation. Even if one position of A'B' is used for a (small) range of observation directions near QQ' this problem would still exist.

(ii) Refer now to Figure (8.3). We have already said that we wish to 'count rays' which are not travelling specifically in
the direction of observation, i.e. we want to use as much of the
information which is available as possible. By integrating over
the plane we throw away half the information to start with simply
because it is travelling away from the plane. Of the information
which crosses that surface, the fraction which can be captured
depends on the width W of the surface used. Suppose the perpendi-
cular distance of the plane from the origin is 'a' and we wish to
capture a fraction 'f' of the information remaining to us then:

\[ W = 2a \tan\left(\frac{\pi f}{2}\right) \]

Now, with \( f = 95\% \) and \( a = 1 \) (wavelength), then \( W = 25.4 \). The
corresponding circular contour would capture all the information
for a total length of surface of about six wavelengths.

The use of the 'Huyghens-plane' may be appropriate for a
reflector antenna system where most of the energy travels in a
limited range of angle. It is expected to be less useful for more
general kinds of scatterers for which the scattering is in all
directions.

Finally we may mention a psychological disadvantage of the
circular contour: for an observation angle \( \theta \) there will be contribu-
tions to the far field from parts of the surface with angular co-
ordinates near \((\theta + \pi)\) i.e. 'invisible' points.

§8.2 The Problem of the Perfectly Conducting Square

The problem considered to illustrate the approach was that of
a perfectly conducting square, subject to the field due to a line
source lying on a diagonal of the square. The side of the square
was 1.8 wavelengths and the line source was 1.3 wavelengths from
the corner of the square, so that the distance between source and scatterer was extremely small. The choice was dictated largely by convenience. Munro and Chignell had used the same problem and experimental data produced by RAE were already available (see reference 33).

In 1980 a simple program called GOLD-2 was written to compute the scattering pattern for this problem, using the aperture field technique. Briefly, an approximate value for the field on a circle centred at the origin was calculated assuming that the only contributions to the field were 'direct' and 'reflected' radiation. (In the geometrical Optics sense of the words). A fuller description of the program, and a listing are given in Appendix J.

§8.3 Discussion of the Results

In a sense, the actual agreement or otherwise between the experimental and calculated results is here a side issue only. We are more concerned with the behaviour of the program as the parameter R(2) - the radius of the surface of integration - is changed. The calculated points agree with the experimental curve quite well considering the highly idealised calculation of the intermediate fields. That the agreement deep in the shadow region is poor is not surprising since for simplicity we considered no edge-diffracted fields from the 'upper' and 'lower' corners of the square which would have penetrated into the shadow region on the integrating surface (Figure 8.4). Also the effect of edge diffraction at the corner nearest the source was omitted which one would expect to be rather important.
Now a priori the choice of the radius of the intermediate circle is arbitrary except that it should not be too large. (This would be equivalent to going directly to the far field, which would be no improvement on the old technique). Heuristically the surface of integration should be 'close' to the scatterer - say within a few wavelengths. How close? If the field near the scatterer was known to a very good approximation then the size and shape of the intermediate surface would not matter, but then there would be no motivation to do the aperture field calculation in the first place. Our purpose is to improve a (relatively) poor approximation by the integration. It was thought beforehand that there would be a range of values of the intermediate circle radius for which the scattering pattern would not change much, but the graphs obtained for a modest range of circle radius show that this is not the case. (Graphs 8.1-8.4).

In determining the value of the radius for which there is the best agreement of theory with experiment, most attention should be paid to observation angles less than about 135°, since, for the reasons given above, this is where one would expect the results to be most reliable. Bearing this in mind, it is seen that the smallest radius is the best.

§8.4 The Work of Bach, Frandsen and Larsen

This is an appropriate point at which to discuss the results given by Bach et al. in a paper presented in the 1981 IEE conference on Antennas and Propagation (reference 34).

In this paper, the authors presented an analysis of a focus reflector antenna. Using direct, reflected and GTD edge rays they
calculated the field at points on the minimum sphere enclosing the scatterer. By evaluating 'Orthogonality Integrals' they represented the field as an expansion in Spherical modes. This could then be taken directly to the far field. In this way, they had a definite a priori reason for taking the near field surface to be the smallest sphere: it would minimise the number of modes needed.

The agreement with fiducial results calculated using the moment method was excellent, especially in the 'shadow region'. This can be attributed to two factors:

(i) The inclusion of GTD edge-diffracted rays. This would improve the approximation to the field everywhere on the sphere, but especially in the shadow region.

(ii) The system which they considered was much larger than ours. The focal length was $8\lambda$, the antenna diameter $20\lambda$ and the radius of the sphere was $16\lambda$. The use of geometrical optics criteria would be more valid under these circumstances, than for our rather small system.

§8.5 The Canonical Near Field Method

To return to our problem, the improving agreement between experiment and theory with the decreasing size of the intermediate surface strongly suggested that the intermediate surface should be wrapped as closely round the scatterer as possible. However, in the approximation that the intermediate field is that given by Geometrical Optics, this is just the same as the method of Physical Optics.(see for example reference 1). With the intermediate field surface 'away' from the scatterer, there was the option of improving the intermediate field by including edge diffracted rays etc. This
concept breaks down near the scatterer. However we can still invoke the Principle of Local field by postulating that on points of the intermediate surface which are close to diffracting features the intermediate field can be obtained from the exact solution for the appropriate canonical problem. This approach will be called the Canonical Near Field Method, or CNFM.

§8.6 A Simple Application of the Canonical Near Field Method

The CNFM was applied at two levels of sophistication to the test case problem of the square cylinder and the line source. At the first level, the method is essentially identical with Physical Optics, and will be identified as such. In this first approximation it is assumed that there is a non-zero normal derivative of the field on those surfaces of the square which are 'illuminated' (in the sense of Geometrical Optics). On the shadowed faces of the square it is assumed that the normal derivative of the field is zero. On the illuminated surface, the field is taken to be that due to the original source and its image by reflection. Since the total field actually on the surface is zero, the total field at an observation point $P$ is:

$$u_p = u_p^{\text{incident}} + \int - G\left[\frac{\partial u}{\partial n}\right]_{\text{surface}} \, d\mathbf{z}.$$  

where the integral is taken over the two illuminated sides of the square.

The slightly more sophisticated version makes the following assumption to calculate the normal derivative of the field. Refer now to Figure (8.5). As before, the normal derivative of the field on BCD is taken to be zero. However, on DAB it is taken to have the
same value as it would on the infinite right angled wedge FDABE. Since SA is in fact small, the approximate form of the exact solution can be used as in reference (1) eq. 6.69. (Refer to Figure 8.6). In the following:

\[ v = \frac{3}{2}, \quad \Omega = \frac{\pi}{4}, \quad \phi = \frac{\pi}{4}, \quad \rho_0 = 1.8(2)^{-1/2} \]

\[ E_z = \frac{4}{\pi} \left( \frac{1}{2} k \rho_0 \right)^{1/v} \frac{H_1^{(1)} (k(\rho^2 + \rho_0^2)^{1/2})}{((\rho^2 + \rho_0^2)^{1/2})^{1/v}} \frac{\sin \phi - \Omega}{\sin \phi - \Omega} \]

The normal derivative of the field was calculated from

\[ \left( \frac{\partial E_z}{\partial \phi} \right)_{\phi = \frac{\pi}{4}} = \pi \]

(In chapter 6 of reference (1), there are several misprints of the value of \( v \). This piece of work was done using the \( \exp(-j\omega t) \) time convention).

The expression is valid for \( k \rho_0 \ll 1 \). A brief discussion of the program written to perform the calculations is given in Appendix K. We conclude this section with a note on the validity of the approximation of regarding the illuminated part of the square as a truncated infinite wedge. A yet more sophisticated approximation would be to regard:

- \( D'AB' \) as part of the infinite wedge FDABE
- \( B'BC \) as part of the infinite wedge HABCG
- \( D'DC \) as part of the infinite wedge JADCI.

The points \( B' \) and \( D' \) are those where, roughly speaking, the influence of the corners \( B \) and \( D \) respectively are of the same magnitude as the influence of the corner \( A \). Although we can arbitrarily choose to ignore the contribution from BC and CD, this is unreasonable unless...
the distances BB' and DD' are small compared with the side of the square. We will now show that this assumption is plausible using a simple calculation which admits the ideas of GTD. Although this is not strictly valid for the small distances concerned, order of magnitude accuracy should be obtainable. By associating a diffraction coefficient with the corners A, B, C and D, we obtain

\[ \text{Diffracted field at B' via corner A} \]
\[ = \frac{\text{(Diffraction coefficient of A)}}{(SA)(AB')} \]

\[ \text{Diffracted field at B' via corner B} \]
\[ = \frac{\text{(Diffraction coefficient of B)}}{(SB)(BB')} \]

Assuming that the diffraction coefficients are approximately equal, then the point B' is determined by:

\[ (SA)((AB) - (BB')) = (SB)(BB') \]

where we have used \((AB') = (AB) - (BB')\). Thus

\[ (BB') = \frac{(SA)(AB)}{(SA)+(SB)} \quad (SA) = 0.03. \]

(since \((SA)\) is small and \((AB) \approx (SB)\)). Thus the assumption is justified. This is equivalent to counting a corner diffracted ray from the corner nearest to the source, which would be the largest of the diffracted fields. However as no diffraction effects due to the corners B and D have been explicitly included in the calculations, the results in the shadow regions cannot be expected to be very good.
§8.7 Discussion of Results

The conventional physical optics method is a standard technique used in treating reflector antenna systems where most of the power tends to be concentrated in the main beam. The accuracy of the method is poor away from the direction of specular reflection (reference 1) and in the shadow zone where the effects of current which has 'leaked' around the edges of the system become important. (For a detailed analysis for some of the more mathematical problems see reference (35)).

It is not unexpected that the results obtained from our first approximation share these characteristics (Graph 8.5). Where the agreement between the experimental and theoretical results is good, it is unspectacular because the 'main-beam' is wide and flat by reflector antenna standards. The experimental curve does not have much structure.

The most important point to note is the range of agreement with experiment of the two approximations. The first approximation agrees with experiment to within a decibel out of about 60° off axis. The CNFM curve (Graph 8.6) agrees well out to 110° which is almost double the range. Furthermore the CNFM shadow is much deeper and more realistic. It is seen that by making a simple improvement in the value of the assumed field at the surface of the scatterer a considerable improvement is made relative to the solution which neglects the effect of the corner of the square nearest the source. No doubt, the solution could be further improved by including the two other illuminated corners of the square, especially for the solution in the shadow region.
§8.8 General Discussion and Conclusion

In this chapter we have briefly examined two related methods for performing scattering calculations: the aperture field method and CNFM. Both are based on the near field to far field approach but differ in where and how the near fields are calculated.

(i) As presented, both methods have ignored resonance effects. However the aperture field method, for which rays can be used to calculate the intermediate field, can easily be made to include resonance effects (by counting multiply diffracted rays etc.). The CNFM uses the canonical problem concept in a more basic way by partitioning the scatterer into different sections, and there is no obvious way that interactions between the sections may be taken into account.

(ii) The CNFM retains a basic distinction between illuminated and shadowed regions of the scatterer. There is no obvious way that diffracting features which are deep in the shadow zone may be accounted for, whereas in the aperture field approach, energy may migrate into the shadow by means of creeping rays etc.

(iii) CNFM presents one with the problem of partitioning the scatterer into a number of simple extended sections which must be matched to corresponding canonical problems. This may not be easy with very irregular scatterers. A ray-based approach can treat the scatterer as an assembly of edges, vertices etc. so that the 'decomposition' problem is very much easier.
(iv) CNFM does not treat diffracting features as points. This may be important for diffracting features which are close together.

(v) The CNFM is essentially equivalent to assuming edge currents, as is sometimes done to improve the physical solution, and it is probably best regarded as an improvement of that method.

(vi) With essentially the same method for calculating the intermediate field the aperture field results are far better than the corresponding CNFM results.

It is concluded, therefore that the CNFM approach is most useful for reflector-antenna-like systems for the field in the shadow zone is unimportant. For more general scatterers where the field is required over a wide range of angles, the aperture field approach is better.
CHAPTER 9 AN ALTERNATIVE APPROACH TO THE CALCULATION OF THE SCATTERING MATRIX

The following work originated as a result of trying to find alternative methods (to Wilton-Mitra analytically continued mode-matching) for the calculation of the scattering matrix. One possibility would be to pick out the linear dependence of scattered field upon incident field from a knowledge of the exact solution. However, exact solutions are known for only a few finite scatterers, and those that are, may not be directly expressible in terms of cylindrical wave functions. For example, the fields due to an elliptical cylinder have their most natural expression in terms of Mathieu functions. There is nothing 'sacred' about cylindrical (or spherical) waves, but they have a relatively simple form and one should be consistent in usage of modes.

It seemed possible that the Physical Optics Solution would be useful for the calculation of the scattering matrix, except that Physical Optics is poor in the shadow region which is only a symptom of the major problem which is that the method completely ignores the geometry of the scatterer in that region. This is unfortunate, since the scattering matrix is supposed to give the scattered field for any possible incident field, so that any part of the scatterer could be illuminated.

The following proposal was made: to split the incident field into incoming and outgoing parts and use the Physical Optics method on the incoming part only. Thus because a given incoming mode has no particular 'shadow' region, then all parts of the scatterer would be involved in the calculation of the scattered field. Further, although the Physical Optics Ansatz does not include resonant effects
(i.e. the sense of chapter 3) one might expect for a convex scatterer at least that this may not be too important. In such scatterers resonance effects are associated with 'creeping rays' and the Poynting Vector associated with a given incoming cylindrical mode will not normally be tangential to the scatterer at its surface.

As previously noted, Physical Optics is based on the canonical problem of a plane wave incident on an infinite plane. For a given scattering surface it would be expected that the method would work best for fields which most closely resemble a plane wave (at least, locally). For cylindrical wave functions this is found if the magnitude of the order of the Bessel function is somewhat less than its argument. In our case this means that the Physical Optics Ansatz will be valid for most of the modes which are significant (see chapter five). Difficulties will arise for modes in the 'transition region' where the order becomes equal to \((2\pi \times \text{scatterer size})\) and the solution may have to be truncated.

§ 9.2 A Simple Example - The Circular Cylinder

Consider a field which consists of a single incoming mode

\[
U = A_m H_m^{(1)}(kr) \exp(jm\theta) \tag{9.1}
\]

Calculate the quantity

\[
J = 2 \frac{\partial U}{\partial n} \bigg|_{r=a} = 2kA_m H_m^{(1)}(ka) \exp(jm\theta) \tag{9.2}
\]

Here \(a\) is the radius of the cylinder (see Figure (9.1)). Now calculate the quantity

\[
V = - \int_{\theta=0}^{2\pi} G \, J \, \, d\theta \, \text{with} \, G = - \frac{j}{4} H_0^{(2)}(kR) \tag{9.3}
\]
The Green's function $G$ may be expanded using (as before) the Graf addition theorem. For field points $P$ which lie exterior to the cylinder:

$$-\frac{j}{4} H_0^{(2)}(kR) = -\frac{j}{4} \sum_{t=-\infty}^{+\infty} H_t^{(2)}(kR)J_t(ka)\exp(jt(\phi - \theta))$$  \hfill (9.4)

If eqs. (9.4) and (9.2) are substituted into eq. (9.3) the only term which survives the integration is the one for which $t = m$. Therefore:

$$V = -\int_0^{2\pi} -\frac{j}{4} 2H_m^{(2)}(kR)J_m(ka)\exp(jm(\phi - \theta))kA_m\tilde{H}_m^{(1)}(ka)\exp(jm\phi)\,d\phi$$ \hfill (9.5)

$$= \frac{j}{4} 2j_m(ka)\tilde{H}_m^{(1)}(ka)2\pi A_mH_m^{(2)}(kR)\exp(jm\phi)$$ \hfill (9.6)

To obtain a physical interpretation of this field which has just been calculated, pick out from equation (9.6) the coefficient:

$$B_m = 2j\frac{\pi}{2} J_m(ka)\tilde{H}_m^{(1)}(ka)A_m$$ \hfill (9.7)

As noted above, the method will be most valid when $m \ll (ka)$. If this is the case then the asymptotic form of the Bessel functions may be used:

$$B_m = j\left[\frac{\pi ka}{2}\right](H_m^{(1)}(ka) + H_m^{(2)}(ka))\tilde{H}_m^{(1)}(ka)A_m$$

but

$$H_m^{(1)}(2)(ka) = \left[\frac{2}{\pi ka}\right]^{\frac{3}{2}} \exp\left[-j\left(ka - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right]$$
But eq. (9.8) is just the asymptotic form of eq. (B.4). We have calculated the purely scattered field. We see then that for low values of m the expression (9.7) agrees with the corresponding exact expression for the scattered field coefficient. However, proper comparison should be made using the full scattered field due to a superposition of outgoing modes.

The scattered fields due to plane waves incident on circular cylinders (see Figure 9.2) of varying radii were calculated and are shown in graphs (9.1) to (9.6). (These correspond to Figures (2.4(a) (b) and (c) in Chapter 2 of reference 1). The solutions using the new technique are denoted as MPO (Modified Physical Optics). The MPO solution excludes modal functions of order greater than (ka).

§9.3 Discussion of Results

Near the back scattering angle the PO (Physical Optics) solution is better than the MPO solution although the latter is not in error by more than about 2 decibels in this region, which is the one in which PO traditionally works well. Deeper into the shadow region both solutions deteriorate a little, but whereas the PO solution consistently underestimates the scattered power, the MPO solution oscillates about the true value for angles > 90°, MPO describes the
structure of the pattern better than PO, especially for the smaller
cylinder radii. It should be remembered that because the incident
field is a plane wave, these graphs show PO at its best, and MPO
at its worst. This is because the expansion of a plane wave in
terms of incoming and outgoing fields has modal coefficients whose
magnitude do not decrease with increasing order. Thus the higher
order MPO coefficients which are the least accurate are given
unusual prominence. Furthermore, for a plane wave, the 'illuminated
area' of the cylinder has its maximum extent.

Consider the simulation of an antenna problem by using a
localised source close to the scatterer:

(i) A smaller portion of the surface would be 'illuminated'
which would be detrimental to PO, but would not effect
MPO at all.

(ii) There might be parts of the 'illuminated' region for
which the complete incident field might not closely
resemble a local plane wave. Again this would be
detrimental to PO, but not to MPO.

(iii) For a local source, the incoming field coefficients
would tend to fall off with increasing order which
would lessen the effect of inaccuracy in the MPO
calculations of higher order.

These three points, taken in conjunction with the graphs
suggest that MPO solution would be better than the PO solution for
problems involving a source in the vicinity of a scatterer.
Line Source Near a Circular Cylinder

In order to test these ideas, the method was applied to the problem of a line-source near to a perfectly conducting circular cylinder. (The author would like to thank Mrs. Helen Walker for doing the computing for this section). The results for a radius \( a = 8/2\pi \) are plotted in graphs 9.7-9.12. The parameter 'd constant' is the distance between the cylinder and the line-source. The results are seen to be disappointing. The accuracies of the MPO and PO curves (compared with the exact solution) are comparable, the main difference being that the Physical Optics solution deviates systematically, whereas the MPO solution appears to deviate in random manner. In fact, the MPO solution approximates to a truncated Fourier series of the exact solution. Unfortunately it is not the case that the MPO solution is better in the shadow region. It should be remembered that the exact solution varies over quite a small range so that an error of 2 or 3 decibels appears more significant than it might if the exact solution had deep nulls. It would be interesting to make a comparison of the methods for such a problem. It is curious to note that MPO apparently works better than PO for the larger source-cylinder separations. This is the opposite of what was expected.

§9.4 A More General Theory, and the Scattering Matrix

Consider an irregularly shaped body whose surface is given by \( r = r(\theta) \). Consider an incoming field:

\[
U_m = A_m H_m^{(1)}(kr)\exp(j m \theta)
\]  

(9.9)
Define the quantities

\[ J_m = 2 \frac{\partial N_m}{\partial n} = \frac{A_m}{\gamma} \left[ kH_m^{(1)}(kr) - \frac{j m}{r} \frac{dH_m^{(1)}(kr)}{d\phi} \right] \exp(j m \theta) \]  

(9.10)

\[ \gamma = \left[ 1 + \left( \frac{1}{r} \frac{d\gamma}{d\phi} \right)^2 \right]^{\frac{1}{2}} \]  

(9.11)

Then the normal to the surface is given by:

\[ \hat{n} = \left( \hat{r} - \frac{1}{r} \frac{dr}{d\phi} \hat{\phi} \right) \gamma^{-1} \]  

(9.12)

The element of arc length is

\[ dl = \gamma r d\phi \]  

(9.13)

Then

\[ J_m = \hat{n} \cdot \nabla N_m \]

\[ = \frac{2A_m}{\gamma} \left[ kH_m^{(1)}(kr) - \frac{j m}{r} \frac{dH_m^{(1)}(kr)}{d\phi} \right] \exp(j m \theta) \]  

(9.14)

If \( r(\theta) \) is a single valued function, then it must be possible to expand functions defined along the surface of the scatterer as Fourier series, in particular let:

\[ J_t(kr)[(kr)H_m^{(1)}(kr) - \frac{j m}{r} \frac{dH_m^{(1)}(kr)}{d\phi}] \exp(j m - t)\theta = \sum_{s=-\infty}^{+\infty} K_{m}^{s} \exp(js\theta) \]

(9.15)

Then

\[ \text{Scattered field} = \int_{\theta=0}^{2\pi} -\frac{j \pi}{4} \sum_{t=-\infty}^{+\infty} H_m^{(2)}(kr) \exp(j t \phi) 2A_m \sum_{s=-\infty}^{+\infty} K_{m}^{s} \exp(js\theta) d\theta \]

\[ = \sum_{t=-\infty}^{+\infty} (-j \pi k_{0}) A_m H_t^{(2)}(kr) \exp(j t \phi) \]
By comparison with the definition of the scattering matrix it is seen that

\[ D_{tm} = -j \pi K_{m}^{t}. \]  

(9.16)

In eq. (9.16) we have an expression for the elements of the \( D \) matrix in terms of an integral of an essentially geometrical quantity (the left hand side of eq. 9.15). The MPO approximation is valid for a range of \(|m|\) less than about \((2\pi \times \text{size of scatterer})\). The actual value of \(|m|_{\text{max}}\) is probably best found by visual inspection, with the \( D \) matrix padded out with zeroes for higher values of \(|m|\). The index \( t \) was introduced via the expansion (9.4) of the Green's function with no approximation so there is no upper limit on the value of \(|t|\). Because of the essential symmetry between incoming and outgoing waves it ought to be possible to 'do the calculation backwards' and derive values of the scattering matrix for all \(|m|\) and a limited range of \(|t|\) so that only the 'corners' of the \( D \) matrix will have to be padded out with zeroes. This however has not yet been done.

§9.5 Present Status of the Method

No attempt has yet been made to apply the method to problems other than the circular cylinder. However the circle is such a simple geometry that the agreement with the exact solution may be fortuitous. It is necessary to understand the method better before going onto a more complicated application, in particular how it is to be reconciled with conventional Physical Optics.

§9.6 A Proposed Restatement of the Principle of Physical Optics

At the centre of Physical Optics is the distinction between 'illuminated' and 'shadowed' regions of the scatterer. These words
have the meaning given by Geometrical Optics which really is a separate discipline from scattering theory proper, being essentially frequency independent. A redefinition of these regions in terms of quantities actually present in the problem leads naturally to the following:

A point on the surface of the scatterer will be deemed to be illuminated if the Poynting Vector of the incident field evaluated at that point, does not lie on the same side of the tangent plane as the outward unit vector normal to the surface. Otherwise, the point is said to be shadowed.

The observation is made that since the scattering problem is linear in the sense described in Chapter 7, then it must always be possible to decompose the incident field into a sum of component fields, and consider the scattering due to each component field individually. The Poynting vector of a sum of fields is not the same as the sum of the individual Poynting Vectors, so that where the 'shadow' lies depends very much on how the incident field is decomposed.

Now it is possible to understand MPO. If the incident field is decomposed into 'incoming' and 'outgoing' fields then the outgoing fields never contribute to the scattering, since their Poynting Vectors are always directed outwards. The incoming modes are to be treated as we have done. At this level, how well the method works depends on how closely the incoming modes locally resemble plane waves at each point on the scatterers' surface.
§9.7 A Tentative Investigation of the Conditions Needed for the Validity of the Rayleigh Hypothesis

We shall use the physical ideas suggested by the MPO technique in order to derive conditions under which the Rayleigh hypothesis is valid. The results obtained agree approximately with those obtained by Van den Berg and Fokkema in reference (10) for a perturbed circular cylinder.

In MPO it is assumed that a cylindrical wave locally resembles a plane wave. Consider the Poynting Vector associated with the field:

\[ E = H_m^{(1)}(kr) \exp(j m \phi) \hat{k} \]  

(9.17)

The associated magnetic field is given by:

\[ H = \frac{j}{\omega \mu_0} \nabla \times E = \frac{j}{\omega \mu_0} r \times (j m H_m^{(1)}(kr) \hat{r} - kr H_m^{(1)}(kr)) \exp(j m \phi) \]  

(9.18)

The transverse and radial parts of the Poynting Vector are proportional to:

\[ \text{Re} \left( E \times H^* \right)_\phi = \frac{-m}{\omega \mu_0 r} \left( J_m^0(kr) + Y_m^0(kr) \right) \]  

(9.19)

\[ \text{Re} \left( E \times H^* \right)_r = \frac{1}{\omega \mu_0 r} \cdot kr \cdot \left( Y_m(kr) J_m(kr) - J_m(kr) Y_m(kr) \right) \]  

(9.20)

The ratio of the transverse to the radial part of the Poynting Vector gives the tangent of the angle which the incoming Poynting Vector makes with the radial direction. Define:
for the modes which contribute most to the scattering $kr > m$ and it is possible to use the asymptotic form of the Bessel functions to approximate:

$$T_m = \frac{m}{(kr)} = 1$$

for the highest order of significant modes. It is assumed that the effect of the scattering is to reflect the Poynting Vector at the point of incidence specularly (Geometrical Optics!). The Rayleigh hypothesis will be valid if the reflected Poynting never has a negative radial component. At the point of reflection we define (Fig. 9.3):

$$\tan \alpha = \frac{1}{r} \frac{dr}{d\phi} \quad \text{and} \quad \tan \beta = T_m$$

It is seen that the Rayleigh hypothesis will be valid if

$$\alpha \leq \frac{1}{2} \left( \frac{\pi}{2} - \beta \right) \quad \text{for all points.} \quad (9.22)$$

As an illustration of the theory, a specific example will be worked out. Van den Berg and Fokkema consider the case of a perturbed circular cylinder

$$r = a + e \sin n \phi$$

for which

$$\tan \alpha = n \left[ \frac{e}{a} \right] \cos n \phi \cdot \left( 1 + \frac{e}{a} \sin n \phi \right)^{-1} \quad (9.23)$$

Setting $\frac{d}{d\phi} (\tan \alpha) = 0$, eq. (9.23) has a maximum when $\sin n \phi = -\frac{e}{a}$. 

$$\frac{\text{Re}(E^{H*})}{\text{Re}(E^{H*})} = \frac{m\pi}{2} \left( J^2_m(kr) + Y^2_m(kr) \right) \quad (9.21)$$
Let \( \tan \frac{1}{2} \left( \frac{\pi}{2} - \beta \right) = F \). Then the Rayleigh hypothesis will just be valid when

\[
\frac{\left( e / a \right)}{1 - \left( e / a \right)^2} F^{-1/2} = n
\]

or

\[
\frac{e}{a} = F \left( n^2 + F^2 \right)^{-1/2} \left( 1 + \left( n / F \right)^2 \right)^{-1/2}
\]

For simplicity it has been assumed here that \( T_m \) is a constant, whereas in fact eq. (9.21) shows that it varies as the point of reflection is changed. However, it will be seen that \( (e/a) \) is quite a small quantity so that this assumption is not unreasonable. As suggested above, we will take as a working value \( T = 1 \) so that

\[
F = \tan \frac{22}{2} 1/2° = 0.414
\]

In reference (10) values of \( (e/a) \) are given for \( n = 1, 2, 3, 4, 5 \) and 13. It can be deduced that as \( n \to 0 \) \( (e/a) \to \infty \), since in this limit the cross-section is a circular cylinder for which the Rayleigh hypothesis is valid, and the shape does not depend on \( e \). Putting \( a = 1 \) the model gives:

\[
\frac{1}{e^2} = 1 + \frac{n^2}{F^2}
\]

Graph (9.13) is a plot of \( 1/e^2 \) against \( n^2 \) for \( n = 1 \ldots 5 \), using the rigorous values. The points appear to be asymptotic to a straight line. Assuming that the value of \( (e/a) \) for \( n = 13 \) which would be calculated by eq. (9.24) is essentially accurate, then

\[
F = 0.459
\]

which is not too far from the working value and on the graph the straight line given by eq. (9.24) using this value of \( F \) has been
drawn in. The facts that

a) The rigorously derived points fall roughly on a straight line

b) The gradient has roughly the value that was expected

suggest that the ideas of this section are essentially correct.

§9.8 Summary and Conclusions of Chapter 9

(1) Conventional Physical Optics

(a) Does not take account of the complete structure of the scatterer.

(b) Depends on the complete incident field locally resembling a plane wave.

(2) A modified version of the method is proposed. The incoming part of the incident field is isolated, and the Physical Optics procedure applied to this field alone, to obtain the scattered field.

(a) This procedure takes the complete scattering surface into account.

(b) Does not depend on the complete incident field being locally plane

(c) but does break down for orders of incoming cylindrical modes higher than about \((2\pi \times \text{scatterer size})\)

(d) However, such modes are not the most significant ones, especially if the incident field source is close to the scatterer.
(3) Assuming that MPO as given is essentially correct, then it is possible to derive an explicit expression for the scattering matrix elements of an irregularly shaped two dimensional body.

(4) A better understanding of the method is needed before application to more complicated geometries; in particular, the reconciliation of PO and MPO at high frequencies.

(5) A reinterpretation of the conditions necessary for the Rayleigh Hypothesis to hold for a perturbed circular cylinder lend some support to the MPO idea.
§10.1 In the usual two-dimensional scattering problem, we consider an incident field which has no sources near the scatterer so that the total field may be written as:

\[ E(r,\phi) = \sum_m (a_m H_m^{(1)}(kr) + (a_m + b_m^{\text{scat}}) H_m^{(2)}(kr)) \exp(j m \phi) \]

\[ = \sum_m (a_m H_m^{(1)}(kr) + b_m^{\text{scat}} H_m^{(2)}(kr)) \exp(j m \phi) \quad (10.1) \]

with \( b_m = a_m + b_m^{\text{scat}} \). If \( E_{\text{total}}(C) \) is the total field at point \( C \) on the scattering body, then the usual boundary condition is that \( E_{\text{total}}(C) = 0 \) for all \( C \). However the form (10.1) is only certainly valid outside the enclosing circle and cannot directly be used to apply the boundary condition over most of the surface. As has already been seen, this problem can be overcome by the analytical continuation method, but another approach will be examined here. Equation (10.1) will be valid at the point \( P \) where the radius of the scatterer is greatest i.e. where the scatterer is tangential to the enclosing circle (if it has no edges). This is because there will be no radiating elements which lie 'outside' of \( P \), which could give rise to incoming fields there. Application of the boundary condition as it stands gives only one equation. To overcome this, the boundary condition will be generalised. Regard \( E_{\text{total}}(C) \) as a function of the angular co-ordinate \( \phi \) of the point \( C \). If the total field is zero for all \( \phi \), then all its derivatives with respect to \( \phi \) must also be everywhere zero. In particular, at the point \( P \) (for which we may without loss of generality take \( \phi = 0 \))
\[ \left( \frac{d}{d\phi} \right)^q \mathbb{E}_{\text{total}} = 0 \quad \text{for } \phi = 0, \ q = 0, 1, 2, \text{etc.} \]

Assuming that eq. (10.1) is valid at \( P \) even when differentiated, then:

\[ \sum_m a_m \frac{d^q}{d\phi^q} \left[ H_m^{(1)}(kr)\exp(jm\phi) \right] + b_m \frac{d^q}{d\phi^q} \left[ H_m^{(2)}(kr)\exp(jm\phi) \right] = 0 \quad \text{when } \phi = 0 \text{ and } q = 0, 1, 2, \ldots. \] (10.2)

At first sight, this seems to be a promising approach, but numerical implementation of it is impractical. The basic problem is the evaluation of

\[ \frac{d^q}{d\phi^q} \left[ H_m^{(1)}(kr)\exp(jm\phi) \right] \quad \text{for } \phi = 0. \]

One possibility would be to calculate the functions to be differentiated at a series of discrete points in the scatterer. Then differentiation could be carried out using a finite difference method such as:

\[ \frac{dy}{d\theta} (\theta = \theta_n) = \frac{y_{n+1} - y_{n-1}}{2\Delta\theta} \] (10.3)

with \( \Delta\theta \) the angular spacing between data points. However, this procedure will break down for values of \( m \) greater than about \( (2\pi/\Delta\theta) \) because the sampling would not be fine enough to pick up the variations in the \( m \)th order exponential function i.e. \( \Delta\theta \) would be much less than \( \frac{2\pi}{m} \). Even if this condition is satisfied, numerical differentiation would be unsatisfactory (quite apart from rounding errors etc.) because for higher orders of differentiation the most significant contributions will come from the modes of high order which will have
been treated the least accurately. Another problem is that this procedure gives most weight to parts of the scatterer which are near P. With the particular finite difference formula which we have quoted, information about a given boundary point 'migrates' toward P at the rate of $\Delta \theta$ radians per differentiation. Thus information about regions of the scatterer well away from P will appear only in those equations given by the higher order of differentiation which will be the least accurate. Even if the finite difference formula uses more than two data points, this problem still exists.

The scattering matrix exists independent of any particular incident field, so that its evaluation should take into account all parts of the scatterer. The following procedure does this, but again is not suitable for numerical computation.

A Fourier expansion is made of the cylindrical wavefunctions around the surface of the scatterer:

$$H^{(1)(2)}_m(kr(\phi))\exp(j\ m\ \phi) = \sum_n A^{(1)(2)}_{mn} \exp(j\ n\ \phi)$$  \hspace{1cm} (10.4)

so that

$$A^{(1)(2)}_{mn} = \frac{1}{2\pi} \int_0^{2\pi} H^{(1)(2)}_m(kr(\phi))\exp(j(m - n)\phi)d\phi$$  \hspace{1cm} (10.5)

(Only one of the two matrices needed to be calculated using the integral. It is easy to show that $A^{(2)}_{mn} = (-1)^{m-n}A^{(1)*}_{m-n}$. Define another two matrices:

$$L^{(1)(2)}_{qm} = \left[ \left( \frac{d}{d\phi} \right)^q H^{(1)(2)}_m(kr(\phi))\exp(j\ m\ \phi) \right]_{\phi = 0}$$
If the field coefficients are written as column vectors, the boundary condition becomes: 
\[ L^{(1)}a + L^{(2)}b = 0 \]
from which 
\[ b = - L^{(2)}^{-1}L^{(1)}a, \]
so that the scattering matrix is given by: 
\[ S = - L^{(2)}^{-1}L^{(1)}. \]
The \( A_{mn}^{(1)(2)} \) take into account all parts of the scatterer since the Fourier integral goes over all values of \( \phi \).

However there are difficulties with eq. (10.5). The 'mathematical' sum over \( n \) goes from \( -\infty \) to \( +\infty \) but for actual calculation it must be truncated. The factor \( n^q \) increases very rapidly with increasing \( n \) while the elements of the \( A \)-matrices do not fall off very rapidly. The values of \( |A_{on}| \) were computed for the square cylinder and plotted on a logarithmic scale (see graph 10.1) (These matrix elements were non zero for every 4th \( n \) because of the 4-fold rotational symmetry). The graph suggests that for large \( n \) the matrix elements fall off by a factor of only 10 if \( n \) is increased by about 54. (Numerical experimentation suggested that to a first approximation \( A_{mn}^{(1)(2)} \) depends only on \( (m - n) \) for this scatterer). Consider the equation derived from the first differentiation, i.e. \( q = 1 \). Then in the sum over \( n \) the terms will only begin to die away when \( n \) is about 100. For higher values of \( q \) the situation will be much worse.

The reason that so much emphasis has been laid upon this approach and upon the 'modified Physical Optics' of the previous chapter is the following.
It was shown earlier that the scattering matrix reflects the geometrical symmetries of the scatterer. For example, n-fold rotational symmetry of the object implied that the non-zero diagonals of the scattering matrix occur with gaps of \((n - 1)\) zero diagonals in between. Now if the radius of the scatterer considered as a function of angle (or any function of that radius) and were expanded as a Fourier series, then the coefficients in the expansion would show the same sort of pattern.

It is postulated that the scattering matrix of a two dimensional body depends in some relatively simple way on the Fourier Coefficients of some function of the radius. MPO and the work of this chapter can be seen as attempts to illustrate this point. Furthermore since it has already been shown that the scattering problem for a body of revolution can be reduced to a quasi-two dimensional one, then a similar idea should hold for such problems.

It was thought at one stage that the \(L^{(1)}\) and \(L^{(2)}\) matrices would be unitary, which would have been consistent with the unitariness of the S-matrix derived by demanding conservation of energy. Then in the expression for the S-matrix it would have been possible to replace \(L^{(2)-1}\) by \(L^{(2)*T}\) in the expression for the S-matrix. Unfortunately, this does not seem to be the case.

From the numerical point of view, it might be possible to get over the series truncation problem by expanding the cylindrical wave function in terms of a different set of orthogonal functions. This would however conceal the relationship postulated above between the scattering matrix elements and the Fourier coefficients.

If the scattering matrix can be expressed in terms of integrals over the scatterer, then the problem of the 'best' location of the boundary points no longer arises.
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Figure GI.1: an infinite three dimensional scattering geometry and its two dimensional representation.
Figure GI.2: A body of revolution (short circular cylinder) with axis of rotation coincident with x axis of Cartesian system, with spherical polar coordinate system superimposed.
Figure GI.4: edge diffraction in GTD. Axis of 'Keller cone' is tangent to edge at Q. The half-angle of the cone equals the angle of incidence measured from the tangent to the edge at Q.

Figure GI.3: specular reflection of rays in geometrical optics.
Figure GI.5: leaking creeping rays in GTD.

Figure GI.6: vertex diffraction from conical object in GTD.
Figure 3.1: illustrating GTD solution of diffraction by an aperture for the field at point P, including multiple edge diffraction.
Figure 4.1: geometry for transformation of coordinate system for the problem of the perfectly conducting infinite half plane.

Figure 4.2: geometry for the scattering due to an irregularly shaped cylinder.
Figure 4.3: illustration of invalidity of Rayleigh hypothesis, and Millar's criterion.
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Figure 7.1: geometry for calculation of translation matrices.

Figure 7.2: two body scattering geometry.
Figure 8.1

Figure 8.2: application of Green's theorem.
Figure 8.3: integration over a finite Huygens plane.

Figure 8.4: penetration of edge diffracted rays into shadow zone.
Figure 8.5: application of the CNFM to scattering by a square cylinder.

Figure 8.6: geometry for line source and infinite wedge.
Graph 8.1: output from program GOLD-2. Radius=3.0 wavelengths.
Solid line - experimental curve.
Points - theoretical values.
Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Graph 8.2: output from program GOLD-2. Radius=2.0 wavelengths. Solid line - experimental curve. Points - theoretical values. Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Graph 8.3: output from program GOLD-2. Radius = 1.6 wavelengths. 
Solid line - experimental curve. 
Points - theoretical values. 
Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Graph 8.4: output from program GOLD-2. Radius=1.4 wavelengths.
Solid line - experimental curve.
Points - theoretical values.
Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Graph 8.5: output from program POSQ.
Solid line - experimental curve.
Points - theoretical values.
Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Graph 8.6: output from program EOSQ-MK6.
Solid line - experimental curve.
Points - theoretical values.
Scattered power (decibels) from square cylinder as function of angle of observation (degrees)
Figure 9.1.

Figure 9.2.

Geometry for scattering by a circular cylinder.
Figure 9.3: geometry of 'geometrical optics' condition for the validity of the Rayleigh hypothesis. The reflected Poynting vector has a zero radial component.
Graph 9.1: scattered field of a circular cylinder due to an incident plane wave.

horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution
ka = 10
Graph 9.2: scattered field of a circular cylinder due to an incident plane wave.

horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution

$ka = 8$
Graph 9.3: scattered power of a circular cylinder due to an incident plane wave.

horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution

\( ka = 6 \)
Graph 9.4: scattered field of a circular cylinder due to an incident plane wave.
horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution
\( ka = 5 \)
Graph 9.5: scattered field of a circular cylinder due to an incident plane wave.

- horizontal scale - angle of observation (degrees)
- vertical scale - scattered power (decibels)
- solid line - exact solution
- solid points - physical optics solution
- light points - modified physical optics solution
- $ka = 4$
Graph 9.6: scattered field of a circular cylinder due to an incident plane wave.

horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution

$ka = 3$
Graph 9.7: scattered field of a circular cylinder due to a nearby parallel line source.

- Horizontal scale: angle of observation (degrees)
- Vertical scale: scattered power (decibels)

- Solid line: exact solution
- Solid points: physical optics solution
- Light points: modified physical optics solution

$\text{dconstant} = 1.6$
Graph 9.8: scattered field of a circular cylinder due to a nearby parallel line source.

- Horizontal scale - angle of observation (degrees)
- Vertical scale - scattered power (decibels)
- Solid line - exact solution
- Solid points - physical optics solution
- Light points - modified physical optics solution

\[ \text{dconstant} = 0.8 \]
Graph 9.9: scattered field of a circular cylinder due to a nearby parallel line source.
horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution
d\text{constant} = 0.4
Graph 9.10: scattered power of a circular cylinder due to a nearby parallel line source.
horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution
dconstant = 0.2
Graph 9.11: scattered field of a circular cylinder due to nearby parallel line source.

horizontal scale - angle of observation (degrees)
vertical scale - scattered power (decibels)
solid line - exact solution
solid points - physical optics solution
light points - modified physical optics solution
dconstant = 0.1
Graph 9.12: scattered field of a circular cylinder due to a nearby parallel line source.

- Horizontal scale - angle of observation (degrees)
- Vertical scale - scattered power (decibels)
- Solid line - exact solution
- Solid points - physical optics solution
- Light points - modified physical optics solution

$\text{dconstant} = 0.05$
Graph 9.13: investigation of validity of Rayleigh hypothesis for slightly perturbed circular cylinder.  
vertical scale - $e^{-2}$ with $e$ the maximum perturbation for which Rayleigh hypothesis is just valid.  
horizontal scale - $n^2$ where $n$ is the periodicity of the perturbation. (see text)  
Points - exact values for $n=1...5$ from Ref. 10.  
Straight line fitted to $(0,1)$ and $(169,8\pm2.51)$
Graph 10.1: variation of matrix element $A_{on}$ with $n$.
vertical scale $- \log_{10}|A_{on}|$
horizontal scale $- n$
limiting gradient approximates to $-0.5/(54-27)$
Kirchoff Diffraction Theory for the Slit

Refer to Fig. A.1, we consider a plane wave whose value is given by:

$$E^{inc} = \exp(jk(x \cos \theta + y \sin \theta))$$

For points in the plane of the slit, $x = 0$, thus:

$$E_0^{inc} = \exp(jky \sin \theta)$$

Assuming the plane wave to be propagating towards the right then the Green's function is:

$$G = \frac{j}{4} H_0^{(1)}(kr)$$

where:

$$r = R - y \sin \phi \quad \text{and} \quad \frac{\partial r}{\partial n} = - \cos \phi.$$

When the field point is distant from the slit, the Green's function is given to a sufficient degree of accuracy by:

$$G = \frac{j}{4} \sqrt{\frac{2}{\pi kr}} \exp(j(kr - \frac{\pi}{4})) = \frac{j}{4} \sqrt{\frac{2}{\pi kr}} \exp(j(kR - \frac{\pi}{4})) \exp(-jky \sin \phi)$$

$$B(R) = \frac{j}{4} \sqrt{\frac{2}{\pi kr}} \exp(j(kR - \frac{\pi}{4}))$$

and we note that $\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} \frac{\partial r}{\partial n} = -jk \cos \phi G$.

Using the Green's theorem:
\[ E(R, \phi) = \int_{\text{slit}} \left[ E_0^{\text{inc}} \frac{3G}{\partial n} \right] dy - G \frac{\partial E_0^{\text{inc}}}{\partial n} \]

\[ = - B(R) \left[ \frac{\cos \theta + \cos \phi}{\sin \theta - \sin \phi} \right] \exp(jky(\sin \theta - \sin \phi)) \]

\[ \left|_{d_1}^{+} \right| - \left|_{d_2}^{-} \right| \]

\( d_1 \) and \( d_2 \) are distances of the upper and lower slit from the \( x \)-axis.

Now consider the upper limit of the bracket only:

\[ E_1^{\text{inc}} = \exp(jk_1 \sin \theta) \]

is the incident field at the upper edge. Also:

\[ \rho = R - d_1 \sin \phi \]

so that

\[ B(R) \cdot \exp(-jkd_1 \sin \phi) = B(\rho) \]

It is seen that the upper limit of the integral gives rise to a contribution to the diffracted field which is:

\[ E_1^{\text{dif}} = \frac{\cos \theta + \cos \phi}{\sin \theta - \sin \phi} B(\rho) E_1^{\text{inc}} \quad (A.1) \]

We change our notation slightly so as to agree with reference 1, Chapter 8. Put

\[ \phi_0 = \theta + \frac{\pi}{2} \]

\[ \tilde{\phi} = \phi + \frac{3\pi}{2} \]

The diffracted field due to edge No. 1 becomes (dropping the bar on the \( \phi \)):

\[ E_1^{\text{dif}} = \frac{(\sin \phi_0 - \sin \phi)}{(\cos \phi_0 + \cos \phi)} \sqrt{\frac{2}{\pi k_\rho}} \exp(j(k_\rho + \frac{\pi}{4})) E_1^{\text{inc}} \quad (A.2) \]

A deficiency of this simple theory is that it gives the same result for both TE and TM waves because it does not take into account the
details of the boundary conditions. Equation (A.2) should be compared with:

Reference 1, eq. 8.20:

\[ E_{\text{diff}} = \frac{(\sin \phi/2 \sin \phi_0/2)}{(\cos \phi + \cos \phi_0)} \sqrt{\left( \frac{2}{\pi kp} \right)} \exp(j(kp + \frac{\pi}{4}))E_{\text{inc}} \]

and reference 1, eq. 8.37:

\[ E_{\text{diff}} = -\frac{(\cos \phi/2 \cos \phi_0/2)}{(\cos \phi + \cos \phi_0)} \sqrt{\left( \frac{2}{\pi kp} \right)} \exp(j(kp + \frac{\pi}{4}))E_{\text{inc}} \]

Clearly there is a similarity between these expressions, although the resemblance is not exact. From equation (A.2) we see that it is possible to put the results of Kirchoff diffraction theory into a GTD-like form. The full diffraction pattern can be obtained from a sum of two terms like eq. A.2, corresponding to rays from the upper and lower edge. Multiple diffraction could (in principle) be taken into account in the usual way, but the angular dependence of eq. (A.2) is probably not sufficient to permit more than order of magnitude accuracy.
APPENDIX B

The Perfectly Conducting Circular Cylinder

Consider a scalar field $U$ satisfying the two dimensional Helmholtz equation. The scatterer is a circular cylinder of radius $a$.

Working in polar co-ordinates with the origin at the centre of the circle, the general solution for the field outside and on the cylinder may be written as:

$$U = \sum_{m=-\infty}^{+\infty} \left( A_m H^{(1)}_m(kr) + B_m H^{(2)}_m(kr) \right) \exp(j m \phi) \quad (B.1)$$

If the boundary condition is that $U = 0$ for $r = a$, then:

$$B_m = -\frac{H^{(1)}_m(ka)}{H^{(2)}_m(ka)} A_m \quad (B.2)$$

If the boundary condition is that $\frac{\partial U}{\partial n} = 0$ for $r = a$ then:

$$B_m = -\frac{H^{(1)}_m(ka)}{H^{(2)}_m(ka)} A_m \quad (B.3)$$

Equations (B.2) and (B.3) give the total outgoing field. The part of the outgoing field which is 'scattered' is

$$B_m^{\text{scattered}} = -\frac{H^{(1)}_m(ka)}{H^{(2)}_m(ka) + 1} A_m \quad (B.4)$$

and

$$B_m^{\text{scattered}} = -\frac{H^{(1)}_m(ka)}{H^{(2)}_m(ka) + 1} A_m \quad \text{respectively} \quad (B.5)$$
APPENDIX C

Miscellaneous Properties of the Translation Matrix and Rotation Matrix

From equation (7.24) it is easily deduced that if co-ordinate systems 1 and 2 have scattering matrices $S^{(1)}$ and $S^{(2)}$ respectively then they are related by:

$$S^{(2)} = T^{(12)} S^{(1)} T^{(12)-1}$$  \[\text{(C.1)}\]

Similarly

$$S^{(3)} = T^{(23)} S^{(2)} T^{(23)-1} = T^{(23)} T^{(12)} S^{(1)} T^{(12)-1} T^{(23)-1}$$

$$= T^{(13)} S^{(1)} T^{(3)-1}$$

It is seen that the translation matrices are transitive (this can be shown explicitly). As consequence of this, we must have

$$T^{(12)-1} = T^{(21)}$$  \[\text{(C.2)}\]

Now:

$$T_{mn}^{(12)} = J_{n-m}(kR)\exp(j(n - m)\alpha)$$  \[\text{(7.23)}\]

From the geometry of the situation $T^{(21)}$ may be obtained from $T^{(12)}$ by replacing the angle $\alpha$ by $(\alpha + \pi)$

$$T_{mn}^{(12)-1} = T_{mn}^{(21)} = J_{n-m}(kR)\exp(j(n - m)\alpha)\exp(j(n - m)\pi)$$

$$= J_{m-n}(kR)(\exp(j(m - n)\alpha))^* = T_{nm}^{(12)*} = T_{mn}^{(12)*T}$$

Thus:

$$T^{-1} = T^{*T}$$  \[\text{(C.3)}\]
Multiplication by the M Matrix

Consider the products

\[ (TM)_{\alpha \gamma} = \sum_{\beta} T_{\alpha \beta} M_{\beta \gamma} \]

\[ = \sum_{\beta} J_{\beta - \alpha} (kR) \exp(j(\beta - \alpha) \varphi) \delta_{\beta - \gamma} (-1)^{\gamma} \]

\[ = J_{-\gamma - \alpha} (kR) \exp(-j(\alpha + \gamma))(-1)^{\gamma} \]

and

\[ (MT)_{\alpha \gamma} = \sum_{\beta} M_{\alpha \beta} T_{\beta \gamma} \]

\[ = \sum_{\beta} \delta_{\alpha - \beta} (-1)^{\beta} J_{-\gamma - \beta} (kR) \exp(j(\gamma - \beta) \varphi) \]

\[ = J_{\gamma + \alpha} (kR) \exp(j(\gamma + \alpha)) \]

It is easily seen that

\[ (MT) = (TM)^* \] \hspace{1cm} (C.4)

Using the definition \( R_{\mu \nu} = \delta_{\mu \nu} \exp(j \varphi x) \)

\[ (RM)_{\alpha \gamma} = \sum_{\beta} R_{\alpha \beta} M_{\beta \gamma} = \sum_{\beta} \delta_{\alpha \beta} \delta_{\beta, -\gamma} \exp(j \beta \varphi)(-1)^{\gamma} \]

\[ = \delta_{-\alpha \gamma} \exp(j \alpha \varphi)(-1)^{\gamma} \]

and

\[ (MR)_{\alpha \gamma} = \sum_{\beta} M_{\alpha \beta} R_{\beta \gamma} = \sum_{\beta} \delta_{-\alpha \beta} (-1)^{\beta} \delta_{\beta \gamma} \exp(j \gamma \varphi) \]

\[ = \delta_{-\alpha \gamma} \exp(-j \alpha \varphi)(-1)^{\gamma} \]

hence

\[ (RM) = (MR)^* \] \hspace{1cm} (C.5)
APPENDIX D

The Two Dimensional Radar Cross-Section

A special case of the general scattering problem occurs when the incident field is a plane wave and has some relevance to the identification of objects by Radar. Following Munro and Chignell. The calculation for the two dimensional case is presented as a simple example of the use of the scattering matrix.

Consider an incident field which is a plane wave of unit amplitude whose propagation vector \( \mathbf{k} \) makes an angle \( \theta \) with the positive x-axis

\[
E(\text{incident}) = \exp(j \mathbf{k} \cdot r) = \exp(j kr \cos(\phi - \theta))
\]

\[
= \sum_{m=-\infty}^{+\infty} j^m \exp(j m(\phi - \theta)) J_m(kr)
\]

\[
= \sum_{m} j^m \exp(-j m \theta) \frac{1}{2} (H_m^{(1)}(kr) + H_m^{(2)}(kr)) \cdot \exp(j m \phi)
\]

The expansion of the plane wave in terms of Bessel functions is easily derived from the generating function of the Bessel functions. The definition of the Bistatic Radar Cross-Section is

\[
\sigma(\phi) = \lim_{r \to \infty} 2\pi r \left| \frac{E(\text{scattered})}{E(\text{incident})} \right|^2
\]

From eq. (D.1) the incoming field coefficients are given by:

\[
a_m = \frac{1}{2} j^m \exp(-j m \theta).
\]
Therefore:

\[ E(\text{scattered}) = \sum_n b_n H_n^{(2)}(kr) \exp(j n \phi) \]

\[ = \sum_n \left[ \sum_m D_{nm} \frac{1}{2} j^m \exp(-j m \theta) \right] H_n^{(2)}(kr) \exp(j n \phi). \]

and since the incident field is of unit modulus:

\[ \sigma(\phi) = \frac{4}{k} \left| \sum_n \sum_m D_{nm} \frac{1}{2} j^m \exp(-j m \theta) \exp(j n \phi) \right|^2 \quad \text{(D.3)} \]

the asymptotic form of the Hankel function has been used. The monostatic cross-section is obtained by putting \( \phi = \theta + \pi \) and is a measure of the power that is scattered back towards the source of the plane wave. However, to quote Jones p.447 (reference 27):

'If the incident wave is not truly plane the radar cross-section may misrepresent the back-scattering because of the influence of the curvature of the wavefront'.

This was a case where it was appropriate to use the D-matrix since the interest lay in the purely scattered field.
APPENDIX E

The Addition Theorem for Spherical Wave Functions

The transformation properties of the spherical wave functions under a translation of the co-ordinate system will be discussed. It is assumed that:

(a) The corresponding axes of the translated and untranslated systems are parallel.

(b) The field points lie in the same plane as the vector connecting the two origins.

(c) The field point lies nearer the translated origin than to the untranslated origin.

The various forms of the transformation are given in the appendices of a paper by Stein (reference 28(f)). In an \((R, \theta, \phi)\) co-ordinate system the point \(O'\) (Figure E.1) is taken as the origin of the second co-ordinate system \((R', \theta', \phi')\) and oriented so that the translation connecting the systems is represented by the vector \(\overrightarrow{R_0} = (R_0, \theta_0, \phi_0)\).

Condition (b) implies that \(\phi_0 = 0\) and \(\phi' = \pi\). Condition (c) means that \(R' < R_0\).

So, for example, Stein's equation (A.2-1) now reads:

\[
h_{\nu}^{(1)(2)}(kR)p_{\nu}^m(\cos \theta) = j^{-\nu} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \sum_{p} \cdot
\]

(1) See also Chignell's 1976 report, Appendix B.
\[ \cdot (j^{n+p}(2n+1)(-1)^m a(\mu,m|\rho,\nu,n)h_p^{(1)}(2)(kR_0). \]

\[ \cdot p^{m+1}(\cos \theta_0)j_n(kR')P_{-m}^m (\cos \theta')(1)^m) \]

Here \( a(\mu,m|\rho,\nu,n) \) is a quantity related to the Wigner 3-j symbols. The factor \( j_n(kR') \) indicates that the transformed field consists equally of incoming and outgoing parts.

The Analytic Continuation Procedure Used by Munro

The expression quoted above for the addition theorem is complicated. If the argument justifying Munro's procedure in Chapter 7 is valid, then considerable simplification results. As a specific example, consider the case for scalar waves. The boundary condition problem is that

\[ j \cdot j \cdot j \cdot (A f + f = 0 (E > > > > \text{for all points lying on the surface of the body of rotation which now has its axis of rotation coincident with the x-axis). Munro holds that we can, without making the problem underdetermined set} \]

\[ \text{thus applying eq. (E.1) only to those points which lie in the xy plane. Let:} \]

\[ Y_{\pm m}(\frac{\pi}{2},\phi) = Z_{\pm m} \exp(j m \phi) \]

(The values of the \( Z_{\pm m} \) can be calculated from formulae given in, for example, Abramovitz and Stegun). It will now be indicated how to express eq. (E.1) in terms of co-ordinates \((\tilde{r},\tilde{\phi})\) where \( \tilde{r} = 0 \) lies at the point \((R,\theta)\) in the \((r,\phi)\) system. The Graf addition theorem may
be written, for \( r < R \), as:

\[
H^{(1,2)}(kr)\exp(\imath \varphi - \phi) = \sum_{n=-\infty}^{+\infty} H^{(1,2)}_{n+\ell} (kR) J_n(kr) \exp(\imath n(\varphi + \pi - \phi))
\]

Hence:

\[
H^{(1,2)}(kr)\exp(\imath \varphi - \phi) = \sum_{n=-\infty}^{+\infty} H^{(1,2)}_{n+\ell} (kR) J_n(kR) \exp(\imath (n + \ell)\theta) \exp(\imath n(\pi - \phi))
\]

Multiplying by \( \exp(\imath (m - \ell)\phi) \):

\[
H^{(1,2)}_{\ell} (kr) \exp(\imath m \phi) = \sum_{n=-\infty}^{+\infty} H^{(1,2)}_{n+\ell} (kR) J_n(kR) \exp(\imath (n + \ell)\theta) \cdot \exp(\imath n(\pi - \phi)) \exp(\imath (m - \ell)\phi)
\]

Replacing \( \ell \) by \( \ell + 1/2 \) and invoking the definition of the Spherical Hankel functions gives:

\[
h^{(1,2)}_{\ell} (kr) \exp(\imath m \phi) = \left( \frac{R}{r} \right)^{3/2} \sum_{n=-\infty}^{+\infty} h^{(1,2)}_{n+\ell} (kR) J_n(kR) \cdot \exp(\imath (n + \ell + \frac{1}{2})\theta) \cdot \exp(\imath n(\pi - \phi)) \cdot \exp(\imath (m - \ell - \frac{1}{2})\phi)
\]

Equation (E.2) may be substituted into eq. (E.1) in the same way as in the purely two dimensional case. The result is that in the xy plane, spherical waves may be expressed in terms of cylindrical waves in the transformed co-ordinate system. The corresponding calculation for the Vector wave case is more complicated, but similar in principle as may be seen from reference (14). It has not been included for those reasons, and also because the general validity of the Munro procedure is uncertain. Furthermore, in equation (E.2) the co-ordinates \( r \) and \( \phi \) appear on both sides of the equation which is unsatisfactory for a transformation formula.
APPENDIX F

The Vector Mode Functions

The unit vector in spherical polar co-ordinates form an orthonormal triad such that (Fig. F.1)

\[ \hat{r} \wedge \hat{\theta} = \hat{\phi} \quad \text{(and cyclically)} \]

By definition the vector modes are:

\[ \hat{e}_{n\ell m} = \frac{1}{N_{n\ell m}} \left[ -r_{n\ell m} (\cos \theta) \left( \begin{array}{c} \cos m \phi \\ \sin m \phi \end{array} \right) \right] \quad \text{(F.1)} \]

Where two forms of an equation are given, the upper one will correspond to the 'even' vector mode.

\[ \hat{e}_{n\ell m} = -\frac{1}{N_{n\ell m}} \left[ \hat{\theta} + \frac{\phi}{\sin \theta} \hat{\phi} \right] p_n^{m} (\cos \theta) \left( \begin{array}{c} \cos m \phi \\ \sin m \phi \end{array} \right) \]

\[ = -\frac{1}{N_{n\ell m}} \left[ \frac{\hat{\theta}}{\cos \ell} p_n^{m} (\cos \theta) \left( \begin{array}{c} \cos m \phi \\ \sin m \phi \end{array} \right) + \frac{\phi}{\sin \ell} p_n^{m} (\cos \theta) \left( \begin{array}{c} -m \sin m \phi \\ +m \cos m \phi \end{array} \right) \right] \]

\[ \quad \text{(F.2)} \]

The problem is to find out how the \( \hat{e}_{n\ell m} \)'s transform under the rotation of co-ordinates such that \( \hat{\phi} = \phi + \chi \). The direction of the unit vectors \( \hat{r} \), \( \hat{\theta} \) and \( \hat{\phi} \) at a point are defined with respect to the radius vector and the z-axis. Since we are dealing with the field at a given point in the two different co-ordinate systems which share the same z-axis, the unit vectors are not changed by the transformation. It is easy to see (for example) that a radial vector remains a radial vector after rotation of co-ordinates even though its \( \hat{i} \), \( \hat{j} \) and \( \hat{k} \) components will have changed.
Equation (F.2) may be solved for \( \hat{e} \) and \( \hat{j} \). Let:

\[
K_1 = -\frac{1}{N_{nm}} \frac{d}{d\theta} \hat{e}_n^m = \cos(\theta) \hat{e}_{nme} + \sin(\theta) \hat{e}_{nmo} 
\]

\[
K_2 = -\frac{1}{N_{nm}} \frac{m\hat{j}}{\sin \theta} \hat{e}_n^m = -\sin(\theta) \hat{e}_{nme} - \cos(\theta) \hat{e}_{nmo} 
\]

Since the angle \( \theta \), and the vectors \( \hat{e} \) and \( \hat{j} \) do not change it is possible to write:

\[
\hat{e}_{nmp} = K_1 \begin{bmatrix} \cos m \hat{\phi} \\ \sin m \hat{\phi} \end{bmatrix} + K_2 \begin{bmatrix} \sin m \hat{\phi} \\ \cos m \hat{\phi} \end{bmatrix} 
\]

whence:

\[
\hat{e}_{nme} = \cos m \times \hat{e}_{nme} - \sin m \times \hat{e}_{nmo} 
\]

\[
\hat{e}_{nmo} = \sin m \times \hat{e}_{nme} + \cos m \times \hat{e}_{nmo} 
\]

The following two vector products are given for reference:

\[
\hat{e}_{nmp} \times \hat{p} = -\frac{1}{N_{nm}} \left( -\frac{dP_n^m}{d\theta} \begin{bmatrix} \cos m \hat{\phi} \\ \sin m \hat{\phi} \end{bmatrix} \hat{e}_n^m + \frac{P_n^m}{\sin \theta} \begin{bmatrix} -m \sin m \hat{\phi} \\ +m \cos m \hat{\phi} \end{bmatrix} \right) 
\]

\[
\hat{p} \times \hat{e}_{nmp} = -\frac{1}{N_{nm}} \left( \frac{dP_n^m}{d\theta} \begin{bmatrix} \cos m \hat{\phi} \\ \sin m \hat{\phi} \end{bmatrix} \hat{e}_n^m - \frac{P_n^m}{\sin \theta} \begin{bmatrix} -m \sin m \hat{\phi} \\ +m \cos m \hat{\phi} \end{bmatrix} \right) 
\]
APPENDIX G

Derivation of the Radial Component of the Electric Field

Given that:

\[ \hat{r}_E = \sum_n \int \mathcal{F}_{nmp}(\alpha \beta) \left[ \frac{1}{N_{nm}} \frac{dP_m}{d\theta} (\cos \theta) \right] \left( \cos m \phi \right) \sin m \phi \]

\[ + F_{nmp}(\gamma \delta) \left[ \frac{1}{N_{nm}} \left( \frac{p_m(\cos \theta)}{\sin \theta} \right) \right] \left( - m \sin m \phi \right) + m \cos m \phi \]

\[ \hat{r}_H = \sum_n \int \mathcal{F}_{nmp}(\alpha \beta) \left[ \frac{1}{N_{nm}} \frac{dP_m}{d\theta} (\cos \theta) \right] \left( \cos m \phi \right) \sin m \phi \]

\[ + j \eta \mathcal{F}_{nmp}(\gamma \delta) \left[ \frac{1}{N_{nm}} \left( \frac{p_m(\cos \theta)}{\sin \theta} \right) \right] \left( - m \sin m \phi \right) + m \cos m \phi \]

Maxwell's equations give:

\[ \hat{r}_E = \frac{1}{\omega \varepsilon_0} \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} \left( \sin \theta \cdot r H_\phi \right) - \frac{\partial}{\partial \phi} \left( r H_\theta \right) \right\} \]

From which

\[ \hat{r}_E = \frac{1}{\omega \varepsilon_0} \frac{1}{r \sin \theta} \sum_n \left( - \frac{1}{N_{nm}} \right). \]
\[
\cdot F_{nmp}(\alpha\beta) \left\{ \frac{d}{d\theta} \left[ \sin \theta \left( \frac{dP_n^m(\cos \theta)}{d\theta} \right) \right] \right\} \begin{bmatrix} \cos m \phi \\ \sin m \phi \end{bmatrix} - \left\{ \frac{p_n^m(\cos \theta)}{\sin \theta} \right\} \begin{bmatrix} -m \cos m \phi \\ -m \sin m \phi \end{bmatrix}
\]

\[
+ j n \cdot F_{nmp}(\gamma\delta) \left\{ \frac{dP_n^m(\cos \theta)}{d\theta} \right\} \begin{bmatrix} -m \sin m \phi \\ +m \cos m \phi \end{bmatrix} - \left\{ \frac{dP_n^m(\cos \theta)}{d\theta} \right\} \begin{bmatrix} -m \sin m \phi \\ +m \cos m \phi \end{bmatrix}
\]

\[
rE_r = \frac{1}{\omega_0 r} \left[ r \sin \theta \sum_n \left( \frac{1}{N_{nm}^2} \right) F_{nmp}(\alpha\beta) \cdot \left( \frac{d}{d\theta} \left[ \sin \theta \left( \frac{dP_n^m}{d\theta} \right) \right] - m^2 \frac{P_n^m}{\sin \theta} \right) \begin{bmatrix} \cos m \phi \\ \sin m \phi \end{bmatrix}
\]

\[
= \frac{1}{\omega_0 r} \left[ r \sin \theta \sum_n \left( \frac{1}{N_{nm}^2} \right) F_{nmp}(\alpha\beta)(-n(n+1)P_n^m \sin \theta) \begin{bmatrix} \cos m \phi \\ \sin m \phi \end{bmatrix}
\]

\[
= \frac{j\xi}{(kr)} \sum_n \left( \frac{1}{N_{nm}^2} \right) F_{nmp}(\alpha\beta)n(n+1)P_n^m(\cos \theta) \begin{bmatrix} \cos m \phi \\ \sin m \phi \end{bmatrix} \tag{G.4}
\]
APPENDIX H

Complex Source Positions and the Modal Expansion Method

In recent years, a body of techniques have arisen, which depend on the generation of new solutions to Maxwell's equations by the replacement of real arguments with complex values i.e. assigning sources a complex position vector (see for example Felsen, reference 29). The importance of this lies in:

(i) The functions generated are rigorous solutions of the wave equation, so that existing techniques may be applied to them.

(ii) The solutions obtained are highly directional so that they provide a simple way of simulating beams, for example from the feed of a transmitting reflector antenna. An example of this technique is given by Hasselman and Felsen (reference 30).

We shall give a brief discussion of the technique in relation to expansions of the field in cylindrical waves.

The field due to a line source located at $r_s$ is

$$ F = H_0^{(2)}(k|r - r_s|)$$

(H.1)

Let $r_s$ be the complex vector $jb$ where $b$ makes an angle $\beta$ with the $\phi = 0$ axis. Then

$$ |r - r_s|^2 = (x - jb_x)^2 + (y - jb_y)^2$$

$$ = r^2 - b^2 - 2jr \cdot b$$

(H.2)
Equations (H.1) and (H.2) will be evaluated when the field point is far from the origin i.e. $r >> b$. Let

$$\bar{r} = (r^2 - b^2 - 2jr \cdot b)^{\frac{1}{2}}$$

$$= r \left[ 1 - \frac{b^2}{r^2} - 2j \frac{r \cdot b}{r^2} \right]^{\frac{1}{2}} = r - jb \cos(\phi - \beta) \quad (H.3)$$

For points distant from the origin, the asymptotic form of the Hankel function may be used:

$$H_0^{(2)}(kr) = \left( \frac{2}{\pi kr} \right)^{\frac{1}{2}} \exp(-j(kr - \frac{\pi \beta}{4})) \quad (H.4)$$

Now, the general form of the outgoing field is:

$$\lim_{r \to \infty} \sum_{m} B_m H_m^{(2)}(kr) \exp(jm \phi) = \sum_{m} B_m \left( \frac{2}{\pi kr} \right)^{\frac{1}{2}} \exp(-j(kr - \frac{m\pi}{2} - \frac{\pi}{4})) \exp(jm \phi) \quad (H.5)$$

By comparing eq. (H.4) and (H.5) it is seen that:

$$\sum_{m} B_m \exp(-\frac{m\pi}{2}) \exp(jm \phi) = \exp(-kb \cos(\phi - \beta)) \quad (H.6)$$

from which

$$B_m = \frac{1}{2\pi} \exp(-\frac{j\pi}{2}) \int_{0}^{2\pi} \exp(-j \phi) \exp(-kb \cos(\phi - \beta)) d\phi. \quad (H.7)$$

Thus the field due to line source with a complex position may be represented in terms of cylindrical modes. Hence the scattering of such a field can be calculated using the scattering matrix. Furthermore, the expansion in cylindrical waves in eq. (H.5) will not just
be valid at infinity. In effect, we have 'continued' the field (H.4) into the domain of finite \( r \), valid for all \( r > b \), since \( b \) is the only characteristic length associated with the source.

The angular dependence of the far field is given by the right hand side of eq. (H.6). Let \((\phi - \beta) = \pi + \alpha\), then

\[
\exp - (kb \cos(\phi - \beta)) = \exp(-kb \cos(\pi + \alpha))
\]

\[
= \exp(+kb \cos(\alpha))
\]

\[
\propto \exp(kb)\exp\left(-\frac{kb\alpha^2}{2}\right) \quad (H.8)
\]

From eq. (H.8) it is seen that the far field has a Gaussian profile for angles deviating by a small quantity from \((\beta + \pi)\).
APPENDIX I

The Conservation of Energy

In spherical polar co-ordinates:

\[ \mathbf{E}_t = \sum_{\text{nmp}} j_\ell \hat{F}_{\text{nmp}}(\alpha\beta) \hat{e}_{\text{nmp}} + F_{\text{nmp}}(\gamma\delta) \hat{e}_{\text{nmp}} \cdot \hat{r} \quad (I.1) \]

\[ \mathbf{H}_t^* = \sum_{\text{nmp}} F_{\text{nmp}}^*(\alpha\beta) \hat{e}_{\text{nmp}}^* - j_\ell \hat{F}_{\text{nmp}}^*(\gamma\delta) \hat{e}_{\text{nmp}}^* \quad (I.2) \]

It is shown in the main body of the text that the \((\alpha\beta)\) equations decouple from the \((\gamma\delta)\) equations. Thus, we may without loss of generality consider, say, only the \(\alpha\) and \(\beta\) terms. Similarly, azimuthal modes of a single order only (either even or odd) will be considered. Then the incoming fields are:

\[ \mathbf{E}_t = \sum_{\text{n}} j_\ell \alpha_{\text{nmp}} \hat{h}_{\text{n}}^{(1)}(kr) \hat{e}_{\text{nmp}} \]

\[ \mathbf{H}_t^* = \sum_{\text{n}} \alpha_{\text{nmp}}^* \hat{h}_{\text{n}}^{(2)}(kr) \hat{e}_{\text{nmp}}^* \]

By integrating the Poynting Vector over a sphere centred at the origin only the transverse field components need be considered. By invoking an orthogonality property of the \(\hat{e}_s\) and equating incoming to outgoing power we obtain:

\[ \sum_{\text{n}} \alpha_{\text{nmp}}^* \alpha_{\text{nmp}} = \sum_{\text{n}} \beta_{\text{nmp}}^* \beta_{\text{nmp}} \]

(Because the azimuthal and parity indices are fixed, this is essentially the same situation as lead to eq. (7.21) in the two dimensional case). It will give:
\[ \sum_{r} \tilde{S}^{r}_{\tilde{r}} \tilde{S}_{\tilde{r}m}^{T*} = \delta^{r}_{t} \] (1.3)
APPENDIX J

Description of Program GOLD-2

Program GOLD-2 is a 'cleaned-up' version of earlier programs called SILVER and GOLD. Its purpose is to calculate the scattering from an infinite perfectly conducting square cylinder due to a line source parallel to one edge and lying on the extension of the diagonal of the square through that edge (see Figure J.1). It calculates the field on an intermediate circular cylindrical surface enclosing the system, whose centre coincides with that of the square. This 'intermediate' field is calculated as the sum of a field directly from the source and a field due to the images by reflection in the surface of the cylinder. Then the field on the intermediate surface is integrated using Green's theorem to obtain the far field.

The program was written on the University of Surrey's PRIME system in 1980, for use at a terminal. It uses the NAG library routines S17EF and S17ACF to calculate the Bessel functions of order zero, of first and second kinds respectively.

For the purposes of calculation the program sets up 3 concentric circles with radii R(1), R(2) and R(3) related by:

\[ R(3) = R(2) + DR \quad \text{and} \quad R(1) = R(2) - DR \]

where DR is the 'intermediate circle spacing'. Upon each circle are set up M1 evenly spaced points, where M1 can be up to 1000. The Cartesian co-ordinates of these points are held in the arrays XC(3,1000) and YC(3,1000), where the index indicates upon which of the three circles the point lies. Similarly the arrays FRC and FIC hold the real and imaginary parts respectively of the calculated values of the field on the circles.
In order to calculate the field on the intermediate circles, tests are made which classify the intermediate field points into four groups: (Fig. J.2).

Region 1: Points which receive only direct radiation from the source.

Region 2: Points which receive direct and reflected radiation.

Region 3: Points which receive only direct radiation from the source.

Region 4: Shadow Region - no field.

The regions are arrived at using geometrical optics - a very questionable procedure when the scatterer is less than 3 wavelengths wide.

The reflected radiation is assumed to be due to the images of the source in the upper and lower faces of the square, facing the source. An angle of observation AOBS measured from the plane of symmetry of the system is specified via the terminal. The program uses a finite distance from the system to the point of observation. Originally it was possible to vary this 'Radius of Observation', but in GOLD-2 it has been fixed at ROBS = 99.9. According to the criterion mentioned early in Chapter 8, this is safely in the far field.

Normal derivatives of both the intermediate field and the Green's function are obtained by subtracting the value of the quantity at the Jth point on circle 1 from the value at the corresponding point on circle 3. The Green's theorem integral is performed by a simple cumulative sum of the values of the integrand for different points.
round the circle. This is then multiplied by the arc length between two points on circle 2 in order to allow legitimate comparison between results obtained from runs of the program using different circle radii.

**Note on Interpretation of Results**

The important quality in these results is the shape of the radiation pattern rather than the absolute values of the power flux density. Hence the use of the decibel scale. Since the total scattered power for the experimental results will be different from the total power in the calculated case, the two patterns will differ by a constant number of decibels. There are two obvious methods of 'normalisation':

(i) Adjust curves so as to agree at some fixed angle of observation. For example, define the scattered power for \( AOBS = 0 \) to be zero decibels.

(ii) Adjust curves so that they have the same average value for some range of angles. This is equivalent to minimising the sum of the squared differences between experimental and theoretical values in this range. For discussing the output from GOLD-2 we have adopted the second procedure. The motivation for this is that we have neglected the diffracted field from the corner nearest the source, which will be an important source of error around \( AOBS = 0 \).
C GOLD-MODIFICATION OF SILVER
C CALCULATES SCATTERING FROM SQUARE USING APERTURE FIELD METHOD

DOUBLE PRECISION X1, X2, S17AEF, S17ACF
DIMENSION XC(3,1000), YC(3,1000), FRC(3,1000), FIC(3,1000)
DIMENSION FR(1000), FI(1000), R(3)
PI =3.1415927
WRITE(0,410)
010 FORMAT('INPUT RADIUS OF INTERMEDIATE CIRCLE PLEASE')
READ(0,200) R(2)
WRITE(0,200)
020 FORMAT('THANKS')
WRITE(0,410)
030 FORMAT('INTERMEDIATE CIRCLE SPACING PLEASE')
READ(0,200) DR
WRITE(0,200)
040 FORMAT('NUMBER OF POINTS IS.....')
READ(0,600) M1
WRITE(0,600)
050 FORMAT('INTERMEDIATE CIRCLE RADIUS')
READ(0,700) R(1)
WRITE(0,700)
060 FORMAT('INTERMEDIATE CIRCLE RADIUS')
READ(0,700) R(1)
WRITE(0,700)
070 FORMAT('NUMBER OF POINTS IS.....')
READ(0,600) M1
WRITE(0,600)
C NOW CALCULATE THE CARTESIAN COORDINATES OF THE POINTS
C ON THE INTERMEDIATE FIELD CIRCLES
XC(I,J) = R(I) * COS(2*PI * J/M1)
YC(I,J) = R(I) * SIN(2*PI * J/M1)
100 CONTINUE
C NOW CALCULATE GEOMETRICAL OPTICS FIELD
DO 110 I=1,3
DO 120 J=1,M1
Y = YC(I,J)
X = XC(I,J)
C TEST FOR POINTS WHICH RECEIVE ONLY DIRECT RADIATION
IF (X.GT.A) GO TO 130
C TEST FOR POINTS WHICH RECEIVE REFLECTED RADIATION
TEST = ABS(Y) - A + (A+B) / A * X
IF (TEST.GT.0.0) GO TO 140
C TEST FOR POINTS WHICH RECEIVE ONLY DIRECT RADIATION
TEST = ABS(Y) - A + (A+B) / A * X
IF (TEST.GT.0.0) GO TO 130
C REMAINING POINTS LIE IN THE SHADOW REGION
FRC(I,J) = 0.0
FIC(I,J) = 0.0
GO TO 120
C CALCULATION OF DIRECT RADIATION
130 X1 = (X - A - B) ** 2 + Y ** 2
X1 = 2 * PI * SQRT(X1)
FRC(I,J) = S17AEF(X1,0)
FIC(I,J) = S17ACF(X1,0)
GO TO 120
C FOLLOWING IS CALCULATION OF DIRECT AND REFLECTED RADIATION
140 X1 = (X - A - B) ** 2 + Y ** 2
X1 = 2 * PI * SQRT(X1)
C YIM IS THE Y-COORDINATE OF THE GEOMETRICAL OPTICS IMAGE BY
C REFLECTION OF THE SOURCE IN THE SQUARE
IF (Y.LT.0.0) YIM = +B
IF (Y.GT.0.0) YIM = -B
X2 = Y - YIM + (X - A) ** 2
X2 = 2 * PI * SQRT(X2)
FRC(I,J) = S17AEF(X1,0) - S17AEF(X2,0)
FIC(I,J) = S17ACF(X1,0) - S17ACF(X2,0)
120 CONTINUE
110 CONTINUE
WRITE(0,150)
150 FORMAT('INTERMEDIATE FIELDS CALCULATED')
NOW START USER ORIENTED PART OF PROGRAM

C

160 ROBS=99.9
WRITE(1,190)
190 FORMAT('ANGLE OF OBSERVATION IN DEGREES, PLEASE')
READ(1,200)AOBS
200 FORMAT(F7.2)
WRITE(1,030)
XOBS=ROBS*COS(2*PI*AOBS/360)
YOBS=ROBS*SIN(2*PI*AOBS/360)

NOW PERFORM THE HUYGENS-KIRCHOFF INTEGRAL
DIFFERENTIANDS ARE THE REAL AND IMAGINARY PARTS OF THE INTEGRAND
DPR=0.0
DFI=0.0
DO 210 I=1,M1

CALCULATE XI, DISTANCE FROM OBSERVATION POINT TO MIDDLE CIRCLE
XI=(XOBS-XC(2,I)**2+(YOBS-YC(2,I)**2
XI=2*PI*I*SQRT(X1)

CALCULATE GR2, GREEN'S FUNCTION ASSOCIATED WITH MIDDLE CIRCLE
GR2=S17AEF(X1,0)
G12=S17AEF(X1,0)
XI=(XOBS-XC(1,I)**2+(YOBS-YC(1,I)**2
XI=2*PI*I*SQRT(X1)
GR1=S17AEF(X1,0)
G11=S17AEF(X1,0)
XI=(XOBS-XC(0,I)**2+(YOBS-YC(0,I)**2
XI=2*PI*I*SQRT(X1)
GR3=S17AEF(X1,0)
G13=S17AEF(X1,0)

NOW CALCULATE NORMAL DERIVATIVE OF G
GGR=0.5*(GR3-GR1)/DR
GG1=0.5*(G13-G11)/DR

NOW CALCULATE GRADIENT OF FIELD
GFR=0.5*(FRC(3,I)-FRC(1,I))/DR
GFI=0.5*(FIC(3,I)-FIC(1,I))/DR

NOW CALCULATE CONTRIBUTION TO INTEGRAL OF POINTS OF ORDER 'I'
DFR=DFR+FRC(2,I)*GR-(FIC(2,I))/G1-GR2*GFR+G12*GFI
DFI=DFI+FIC(2,I)*GR+(FIC(2,I))/G1-GR2*GFI-G12*GFR
210 CONTINUE

DL IS ARC LENGTH ALONG CIRCLE
DL=(2*PI*R(2))/M1
POW=(DFR*2+DFI*2)*(DL**2)
DCB=10.0 ALOG10(POW)
DCB=DCB+21.47
WRITE(1,220)ROBS
220 FORMAT('DISTANCE OF OBSERVATION POINT=',F7.2,'WAVELENGTHS')
WRITE(1,230)AOBS
230 FORMAT('ANGLE OF OBSERVATION=',F7.2,'DEGREES')
WRITE(1,240)DCB
240 FORMAT('SCATTERED POWER=',F7.4,'DB')
WRITE(1,250)
250 FORMAT('INPUT 1 TO CONTINUE')
READ(1,260)I
260 FORMAT(I1)
IF (1.LE.1) GO TO 160
WRITE(1,270)
270 FORMAT('FAREWELL ')
APPENDIX K

Brief Description of Program POSQ

This program in its original form derives the problem of the square cylinder by 'Physical Optics'. It assumes that the field on the 'illuminated' surface of the square is due to the source and its Geometrical Optics images, and on this basis calculates the normal derivative of the field on that surface. The normal derivative is integrated to obtain the scattered far field, to which a direct far field from the source is added.

The Real and Imaginary parts of the normal field derivatives are held in the arrays GUR and GUI. The real and imaginary parts of the Green function are held in the arrays GR and GI. The real and imaginary parts of the integrand are held in the arrays DR and DI. A number NUM of equally spaced points a distance H apart along the length SIDE of the square are set up. The distance between the source and the corner of the square is D2. The distance of the source to the centre of the square is D5. The perpendicular distance of the source from the upper face is PER. If the upper face is extended and a perpendicular dropped on it from the source, then the distance from the corner of the square to the intersection of the perpendicular and the extension of the side is called EXT.

A call is made to the subroutine FLDGRD which calculates the normal derivative of the field at the points on the upper face of the square. The program then accepts a value THETA for the angle of observation measured in degrees. The subroutine GREEN is called to calculate the Green's function for points on the upper face. The integrand is calculated, and then integrated using the subroutine
SIMP. The same calculations are performed for the lower face of the square by the simple device of changing the sign of THETA. (This is permitted by the symmetry of the situation). Then the contribution to the field of the direct radiation from the source is added in, and the total power is calculated.

Notes

(i) Since the radial dependence of the Green's function and the source are the same, the factor has been omitted for simplicity. (This would be implicit in the calculation of the far field in any case).

(ii) In the subroutine FLDGRD the functions S17AFF and S17ADF are from the NAG library and calculate the Bessel functions $J_1$ and $Y_1$ respectively.

(iii) Physical Optics proper calculates a surface current density on the scatterer by assuming that the scatterer is locally an infinite plane, and that the incident field is locally a plane wave. Because we have a source which is very close to the scatterer there will be a region on the surface for which this latter assumption would not be valid. This difficulty has been surmounted by using image theory which uses the first assumption but not the second. It is likely that 'straight' physical optics would give an even worse result in this case than the present method.

(iv) In order to perform the CNFM calculation, the subroutine FLDGRD was replaced by a subroutine which calculated the normal derivative of the field from the expression given in the text.
C PROGRAM DOES USES PHYSICAL OPTICS TO DO THE SQUARE

DIMENSION CUR(501), GUI(501), CR(501), GI(501), D1(501), PI(501)
SIDE=1.0
NUM=501
M=SIDExNUM-1
D2=1.3-(1.8/SORT(2,0))
D3=1.3
PER=D2/SORT(2,0)
EXT=D2/SORT(2,0)
CALL FLDGRD(PER,EXT, SIDE, NUM, CUR, GUI)

THIS HAS CALCULATED THE FIELD GRADIENT ON ONE SIDE OF THE SQUARE

NOW INPUT AN ANGLE OF OBSERVATION

WRITE (1, 100)
100 FORMAT ('OBSERVATION ANGLE IN DEGREES PLEASE')
READ (1, 200) THETA
THETA=THETA/57.29

CALCULATE GREENS FUNCTION AND CONTRIBUTION TO INTEGRAL FROM UPPER FACE

PHI=THETA
CALL GREEN(PHI, SIDE, NUM, CR, GI)
DO 030 I=1, 501
DR(I)=GR(I)*CUR(I)-CI(I)*GUI(I)
DZ(I)=GR(I)*GUI(I)-CI(I)*CUR(I)
030 CONTINUE
CALL SIMP(DR, H, NUM, FRU)
CALL SIMP(DZ, H, NUM, FRI)

NOW CALCULATE CONTRIBUTION FROM LOWER FACE

PHI=-THETA
CALL GREEN(PHI, SIDE, NUM, CR, GI)
DO 040 I=1, 501
DR(I)=GR(I)*CUR(I)-CI(I)*GUI(I)
DZ(I)=GR(I)*GUI(I)-CI(I)*CUR(I)
040 CONTINUE
CALL SIMP(DR, H, NUM, FRL)
CALL SIMP(DZ, H, NUM, FIL)

NOW CALCULATE PHASE OF DIRECT FIELD CONTRIBUTION

PH=2.0*3.14*5907*PT*CR(THETA)
FLDR=CO(S(PH)-FRU-FPL
FLBI=SI(PH)-FIB-FIL
POWER=FLDR*2+FLB1*2
POWER=10.0*ALOG10(POWER)
POWER=POWER+3.6
WRITE(1, 300) POWER
GO TO 005
END

SUBROUTINE SIMP(D, H, NUM, AREA)

DIMENSION D(PI)
NN=1
I=2
AREA=0.0
DO 100 I=2, NN, 2
AREA=AREA+4.0*D(I)
100 CONTINUE
DO 200 I=2, LL, 2
AREA=AREA+4.0*D(I)
200 CONTINUE
AREA=H*(AREA+D(I)+D(MH))/3.0
RETURN
END

SUBROUTINE GREEN(PHI, SIDE, NUM, CR, GI)

THIS SUBROUTINE CALCULATES THE GREENS FUNCTION FOR AN ANGLE OF OBSERVATION PHI, SQUARE OF SIDE LENGTH SIDE, NUM OF EVENLY SPACED POINTS, IMPORTANT PHI MUST BE IN RADIANS

REAL KAY
DIMENSION CR(NUM), GI(NUM)
SIDE=SIDE/SORT(2,0)
KAY=2.0*3.14*5907
C PROGRAM POSO USES PHYSICAL OPTICS TO DO THE SQUARE

SPACE=SIDE/(NUM-1)
DO 100 I=1,NUM
X=(I-1)*SPACE
D3=D1-D4
PH=KAY*(D3*COS(PHI)+D4*SIN(PHI))
GUR(I)=0.25*SIN(PHI)
GUI(I)=0.25*COS(PHI)
100 CONTINUE
RETURN
END

SUBROUTINE FLDGRD(PER,EXT,SIDE,NUM,GUR,GUI)
C THIS SUBROUTINE CALCULATES THE FIELD
C GRADIENT ON A PLANE REFLECTOR OF LENGTH
C 'SIDE' DUE TO A LINE SOURCE OF PERPENDICULAR
C DISTANCE 'EXT' FROM END. REAL AND
C IMAGINARY PARTS ARE STORED IN GUR AND GUI
C FOR NUM EQUALLY SPACED POINTS.
DIMENSION GUR(NUM),GUI(NUM)
DOUBLE PRECISION S17AFF,S17ADF,XX,XX1
REAL KAY
IFAIL=0
KAY=2.0*3.1415927
A=PER
B=EXT
SPACE=SIDE/(NUM-1)
D3=I*SPACE
XX=EXT+(I-1)*SPACE
XX=DSORT(XX**2+A**2)
XX1=KAY*XX
GUR(I)=(2.0*KAY*A*S17AFF(XX1,IFAIL))/XX
GUI(I)=(2.0*KAY*A*S17ADF(XX1,IFAIL))/XX
0100 CONTINUE
RETURN
END
Figure A1: geometry for diffraction by a slit.

Figure A2: diffraction by a half plane.
Figure E1: geometry for transformation of coordinate systems and addition theorems for spherical wave functions.
Figure F1: unit vectors in spherical polar coordinates
Figure J1.

Geometry for program GOLD-2.
Figure K1: geometry for program POSQ.
Figure K2: geometry for program ROSQ.
(References to)

ELECTROMAGNETIC SCATTERING BY
PERFECTLY CONDUCTING BODIES

A thesis submitted to the Faculty of Mathematical and Physical Sciences of the University of Surrey, for the degree of Doctor of Philosophy

by

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University of Surrey

Radiation from Antennas Mounted on Irregularly Shaped Bodies

R. J. Chignell
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Progress Report No. 1.

R. J. Chignell
July 1975.
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The work presented in this report has been concerned with investigating possible new approaches to the problems of predicting the radiation patterns of antennas mounted on aircraft. Successful techniques exist for both the regions in which the aircraft is small and very large compared to a wavelength. However, the validity of the techniques in the intermediate frequency range is not established; this was the region considered in this study.

The low frequency technique uses a wire grid model of the aircraft structure. The current on all the wire segments is found using the method of moments. The radiation pattern is obtained from the current integral. The number of current elements is limited to about 300 by the constraints of computer storage and the technique is limited to aircraft under a wavelength long.

The high frequency technique is the so-called Geometrical Theory of Diffraction (G.T.D.) in which the composite problem is broken down into a sequence of canonical problems which can be corrected by drawing rays. For this technique to be successful, the so-called "flash point" must be separated by a sufficient distance and the technique is limited to aircraft with fuselage several wavelengths in diameter. As yet, penumbra correction to torsional rays can only be considered in the principal planes and the technique is most successful in the roll and pitch planes.

Radiation patterns in these and adjacent (± 15°) planes can be accurately computed but the technique may break down for other angles.

G.T.D. has been combined with other methods of analysis to give hybrid approaches. The most successful of these is the combination
of G.T.D. and modal expansion, as pursued by Burnside (Reference 1) and later Marconi Research Laboratories (Reference 2). A modal expansion of a source on a cylinder is used for the field close to the fuselage and G.T.D. for the effects of such things as wing tips. The technique enables G.T.D. to be extended to lower frequencies; without use of the modal expansion the method fails at higher frequencies.

2. Horizontally polarised yaw plane radiation patterns

Initially the horizontally polarized yaw plane radiation patterns presented in Reference 3 were considered. At 10 MHz the aircraft considered was approximately half a wavelength long and its wingspan was of similar dimensions. The aircraft could to a first crude approximation be considered as crossed dipoles as shown in figure 1. If the aircraft was considered to be symmetrical in all respects then the currents induced in the wings were also symmetrical. This implied that the current in both wings was either moving towards or away from the fuselage. The wings were not excited in the usual dipole mode. Jackson (Reference 4) considers the excitation of the various multipoles by a normal dipole. He shows that for a dipole half a wavelength long the \( Y_{10} \) multipole is excited. By similar arguments it is possible to show that a dipole excited like the aircraft wings will radiate the \( Y_{20} \) multipole.

The multipole expansion of a field in spherical harmonics employs functions \( Y_{\ell m}(\theta,\phi) \) which are specified functions of the angular variable \( \theta \) and \( \phi \). Instead of separating the angular variables \( \theta \) and \( \phi \) in spherical coordinates the angle dependence is treated as a single expansion. The \( Y_{\ell 0} \) multipoles are cylindrically symmetric and can be used to expand
the fields of dipoles. The $Y_{10}$ multipole has a $\cos \theta$ dependence and the $Y_{20}$ multipole a $\sin \theta \cos \theta$ dependences.

For the aircraft it has been shown that the $Y_{10}$ and $Y_{20}$ multipoles can be excited. The radiation pattern is thus principally given by:

$$\cos \theta + a \sin \theta \cos \theta$$

where $a$ is a complex excitation coefficient. If the current distribution on the fuselage is undisturbed by the wings then no current will be directly injected into the wing. It can then be shown that the wings are reactively coupled to the fuselage. The wings currents are then phase retarded relative to the fuselage by $\frac{\pi}{2}$. Under these circumstances the radiation pattern obtained is symmetrical about $\theta = 0$. The radiation pattern (figure 10 of reference 3) of a fighter aircraft at 4 MHz shows this symmetry. As the frequency is raised this symmetry is destroyed as the 10 MHz pattern (figure 11 of reference 3) shows. The peak of the radiation is moved towards the nose.

The expression (1) was programmed for evaluation on an interactive computer terminal. By judiciously changing $a$ the radiation pattern of the aircraft as measured at 10 MHz was reproduced (figure 11 of reference 3). This corresponded to a phase retardation of the wings of 85° relative to the fuselage.

It has been shown that the radiation patterns of aircraft in the low frequency range are simple combinations of a few multipoles. However the coefficient of the multipole expansion cannot be predicted without resort to wire grid modelling. Numerical errors can rapidly accumulate using this technique so that additional small lobes appear
in the radiation patterns predicted (figures 14 and 17, Reference 3).
It is proposed that these small extraneous lobes may be removed by
first expanding the predicted radiation pattern in terms of multipoles.
For the pattern shown in figure 17, Reference 3, the first two multipoles
would be strongly excited in the correct manner but other excitation
coefficients would be enhanced due to the smaller lobe. It should be
obvious which coefficient the smaller lobe is exciting and these can
then be set to zero. In this manner the extraneous lobes may be removed.

3. Multipole expansions and dipole representations

It has been shown that radiation patterns can be considered as a
combination of a few multipoles. The problem of predicting the radiation
patterns is then reduced to that of finding the complex excitation co-
efficient of each multipole. In order to investigate a particular
problem that of a short monopole on an infinite cylinder was considered.
Wait (Reference 5) has shown that for this geometry shown diagrammatically
in figure 2 the two components of electric field are:

\[
T(\phi) = \sum_{m=1}^{\infty} \frac{i^m \sin m(\phi - \frac{\pi}{2})}{(\kappa \sin \theta)^2 H_{m}^{(2)}(\kappa \sin \theta)}
\]

\[
T(\theta) = \cos \sum_{m=1}^{\infty} \frac{i^m \cos m(\phi - \frac{\pi}{2})}{(\kappa \sin \theta) H_{m}^{(2)}(\kappa \sin \theta)}
\]

where \( k = \frac{2\pi}{\lambda} \)

\( = 2\pi \) for a normalised geometry

\( a \) is the cylinder radius.

\( H_{m}^{(2)}(\ ) \) is a Hankel function of the 2nd type of order \( m \). Similarly
$H_m^{(2)}(\ )$ is the derivative of the function with respect to the argument.

These expressions were derived from the Green's function with the assumption being made of a vanishing short monopole. Only a few terms of each series are required to accurately specify the field but if the series is truncated after a few terms it may be rearranged as a multipole expansion. For example if the series for $T(\phi)$ is truncated after five terms it may be rearranged to give:

$$T(\phi) = a_{1R} Y_{10}(\cos\phi) + a_{2R} Y_{21}(\cos\phi) + a_{3R} Y_{30}(\cos\phi)$$
$$+ a_{4R} Y_{41}(\cos\phi) + a_{5R} Y_{50}(\cos\phi) + \ldots$$  \hspace{1cm} (3)

where:

$$a_{1R} = -\frac{1}{315} \left[ \frac{315}{H_1^{(2)}(\alpha)} - \frac{567}{H_3^{(2)}(\alpha)} - \frac{225}{H_5^{(2)}(\alpha)} \right]$$

$$a_{2R} = \frac{1}{105} \left[ \frac{140}{H_2^{(2)}(\alpha)} + \frac{1040}{H_4^{(2)}(\alpha)} \right]$$

$$a_{3R} = -\frac{1}{315} \left[ \frac{1512}{H_3^{(2)}(\alpha)} - \frac{1400}{H_5^{(2)}(\alpha)} \right]$$

$$a_{4R} = -\frac{64}{35} \frac{1}{H_4^{(2)}(\alpha)}$$

$$a_{5R} = -\frac{1640}{63} \frac{1}{H_5^{(2)}(\alpha)}$$
and the \( Y_{\ell m} \) are multipoles.

It is important to realise that the \( Y_{10} \), \( Y_{30} \) and \( Y_{50} \) depend only on \( \phi \) while \( a_{1R} \), \( a_{3R} \) and \( a_{5R} \) depend only on \( \theta \). The \( Y_{21} \) and \( Y_{41} \) terms depend on both \( \phi \) and \( \theta \) but it is relatively easy to remove the \( \theta \) dependence to the \( a_{2R} \) and \( a_{4R} \) terms, so that the multipoles depend only on \( \phi \) and the coefficients on \( \theta \). Thus within a plane specified by a constant \( \theta \), the multipole excitation coefficients need only be determined once and the complete radiation pattern can be generated as a function of \( \phi \).

It is however desirable, to have a physical interpretation, of this expansion. Jackson (Reference 4) shows that for a linear dipole carrying a sinusoidal current only the odd multipoles are excited with vector spherical harmonic excitation:

\[
a(\xi, \phi) = \frac{4\pi}{c d_1} \left[ \frac{j_{2l+1}(k d)}{\xi^{2l+1}} \right] \left[ \left( \frac{1}{2} \right)^{2l} j_{\xi} \left( \frac{k d}{\xi} \right) \right]
\]

where \( j_{\xi} \) is the peak current and the dipole of length \( d_1 \) and \( j(\xi) \) is a spherical Bessel function.

Thus a dipole of length \( d_1 \) carrying a sinusoidal current distribution can be used to represent the first and third multipoles excited by the \( \phi \) component by setting the ratios of the coefficients equal and solving for \( d_1 \). Some approximation is necessary because the excitation of the multipoles given by (4) is complex whereas that given by (5) is real. The degree of approximation is small for a sufficiently small cylinder and will be discussed later. Similarly a dipole of length \( d_2 \) carrying a current distribution \( l_2 \) as shown in figure 3 can be used to represent the even multipoles. The correct ratio of the even to odd multipoles is obtained by setting \( l_2/l_1 \) to the correct magnitude. The
The radiation pattern of a short monopole on a cylinder in a particular plane specified by \( \theta \) can be generated from a knowledge of \( d_1, d_2 \) and \( l_2/l_1 \). It should be noted that these quantities are independent of \( \phi \) but vary with \( \theta \).

It is significant that when infinite "wings" are added to the cylinder so the geometry becomes that of an \( \infty \) semi-cylindrical boss on an infinite ground plane, the even multipoles are no longer excited. The \( \phi \) radiation pattern in any plane specified by constant \( \theta \) is then determined simply by \( d_1 \). This is likely to be true near the roll plane for finite wings at a sufficiently low frequency when diffraction from the wing edges is not an important constraint. At higher frequencies it is possible that diffraction effects could be added by using G.T.D..

The multipole expansion for \( T(\theta) \) can be obtained relatively simply but the asymptotic form of the Hankel function must be used:

\[
H_\nu^{(2)}(a) = \frac{i}{\pi} r(\nu) (\frac{a}{2})^{-\nu}
\]

When this low frequency form is used the multipole expansion is readily obtained as:

\[
T(\theta) = a_1 Y_{10} + a_2 Y_{20} + a_3 Y_{30} + a_4 Y_{40} + a_5 Y_{50} + \ldots
\]

where

\[
a_1 = \frac{\pi}{2} \{ \cos(\phi - \frac{\pi}{2}) + \frac{\pi a^2}{5} \cos(\phi - \frac{\pi}{2}) + \frac{\pi^4 b^4}{105} \cos(\phi - \frac{\pi}{2}) \}
\]

\[
a_2 = \frac{i \pi^2 a}{6} \{ \cos(\phi - \frac{\pi}{2}) - \frac{2\pi a^2}{21} \cos(\phi - \frac{\pi}{2}) \}
\]
In this case the multipoles are functions of $\theta$ and the excitation coefficients of $\phi$. Again the expansion can be approximated by two dipoles but in this case the ratio of the excitation coefficients is real as the asymptotic form of the Hankel function has already been used. The short monopole on a cylinder can thus be represented at some frequencies by the system of four dipoles as shown in figure 4. Providing no diffraction effects become important this representation is likely to give the correct radiation pattern in any plane.

4. The range of validity and numerical results

The representation of the field of a short monopole on a cylinder by a system of four dipoles depended on making two important steps. First, the field had to be expressed as a series of multipoles and then the excitation coefficients were used to determine the dipole dimensions. For the two orthogonal field components different approximations were involved in making these steps. If the $\mathcal{T}(\phi)$ component of field could be accurately described by $n$ terms in Wait's original expansion then this component could be identically described by $n$ multipoles. No approximation is involved in this step. However the excitation coefficients of the multipoles are now complex, while the dipole representation can only cope with real excitations. However for sufficiently small
odd multipoles are real as shown in figure 6. In fact $a_{1R}$ is real when $\alpha = 0$ or $\sqrt{3}$. For $\alpha$ less than $\sqrt{3}$ the maximum angle $a_{1R}$ makes with the real axis is $23^\circ$, while for higher values the coefficient rapidly becomes imaginary in nature. When $\alpha$ is less than $\sqrt{3}$ the error can be minimised by modifying the phase angle of $l_1$ so that the first two multipoles are in the right relative phase and all the error is associated with the excitation of the third multipole. The excitation of this component is small for $\alpha$ less than $\sqrt{3}$ but for greater values the third multipole is more strongly excited than the first multipole. For $T(\theta)$ it is likely that the dipole representation should only be used for values of $\alpha$ less than $\sqrt{3}$. The dipole representation breaks down initially in the roll plane where $\sin \theta$ achieves its maximum value. In the roll plane the representation is likely to be valid for cylinders less than $\sqrt{3}/n$ in diameter. When applied to an aircraft the representation is likely to be correct for fuselage diameters less than half a wavelength. There is thus a range of frequencies for which the new representation provides unique information. This range lies between the fuselage length being a wavelength and its diameter being half a wavelength. For a typical modern aircraft this is a three to one frequency range.

For the $T(\theta)$ component similar arguments apply but the approximation must be made to the Hankel functions in Wait's expressions. Only an approximate multipole expansion is obtained but the coefficients are real. The dipole representation can then identically represent the multipole expansion.

Three expansions have been discussed:
1. Wait's original results
2. A multipole expansion
3. A dipole representation

For the $T(\theta)$ component methods (1) and (2) are identical and an approximation is involved in (3), while for $T(\phi)$ methods (2) and (3) are identical but approximate. All three methods have been programmed for computer evaluation and the principal plane radiation patterns obtained for various cylinder diameters are shown in figures 6 and 7. The results obtained bear out the previous comments. They are not compared with experimental results because with such small cylinder diameters a quarter wave monopole can not be considered as short.

5. A scattering representation of the effect of structures near antennas

A scattering representation of the effect of structures near antennas is proposed. The method has been developed for problems that are invariant in one dimension such as a line source near a cylinder. The problem is thus reduced to two dimensions and much simplification is achieved. This is not a necessary constraint and it is hoped that it will be removed.

The structure to be investigated must scatter in a known manner relative to some origin or be capable of decomposition into such scatterers. The field incident upon the structure is expanded as a series of incoming cylindrical waves:

$$\sum_{n} a_n H_n^{(1)} (kr_2)$$

where $H_n^{(1)} ( )$ is a Hankel function of the first kind and $r_2$ is defined
relative to the origin of the scatterer (Figure 8). Similarly the scattered field is expanded as a series of outgoing cylindrical waves:

$$\sum_n b_n H_n^{(2)}(kr_2)$$

It is assumed that the scattering matrix connecting these series is known. ie.,

$$\begin{bmatrix} b_n H_n^{(2)}(kr_2) & S_{00} & \ldots & S_{0N} & \ldots & a_n H_n^{(1)}(kr_2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_N H_N^{(2)}(kr_2) & S_{0N} & \ldots & S_{NN} & \ldots & a_N H_N^{(1)}(kr_2) \end{bmatrix}$$

with $S$ known. This is reasonable for some scatterers such as a cylinder. A cylindrically symmetric wave symmetrically incident upon a cylinder will be scattered as a cylindrically symmetric wave. This means the matrix is diagonal for a cylinder and the boundary conditions give the elements.

For the two dimensional problem a line source is considered which radiates a cylindrically symmetric field $H_0^{(2)}(kr_1)$. This is then expanded about the origin of the scatterer by means of addition theorems (Reference 6). ie.,

$$H_0^{(2)}(kr_1) = \frac{H_0^{(2)}(kD)}{2} [H_0^{(1)}(kr_2) + H_0^{(2)}(kr_2)]$$

$$+ \sum_{n=1}^\infty H_n(kD) [H_n^{(1)}(kr_1) + H_n^{(2)}(kr_1)] \cos n(\theta - \psi)$$
where \( D \) is the separation of the line source and the origin of the scatterer, \( \psi \) is the direction of the scatterer from the line source.

However, the properties of the scatterer are known so if fields \( H_n^{(1)}(kr) \) are incident upon it the fields scattered \( H_n^{(2)}(kr) \) are known. The required modifications to account for the presence of the scatterer are made to both sides of this last equation.

If the modification to the left hand side is re-expanded about the line source its reflection coefficient may be calculated in the presence of the scatterer. It is assumed the source would be matched in free space. The radiation pattern of the line source may be calculated in a similar manner but care is needed because the original addition theorem was only strictly accurate over a limited range of \( r \). However, it is believed that if the modification is made to the outgoing waves it is invariant under the change \( kD + kr \) and applies to all of space.

One problem considered using this method was a line source above an infinite semi-cylindrical boss on a ground plane as shown in figure 9. This corresponded to the roll plane behaviour of a wire aerial above an aircraft fuselage fitted with infinite wings. The problem was solved considering a cylinder with a source and an image.

The reflection coefficient of the aerial was calculated as:

\[
\left\{ \begin{array}{l}
\left(1 + \frac{H_0^{(1)}(kp)}{H_0^{(2)}(kp)} \right) \frac{[H_0^{(2)}(kD)]^2}{4} + \frac{H_0^{(2)}(2kD)}{2} \\
1 - \left(1 + \frac{H_0^{(1)}(kp)}{H_0^{(2)}(kp)} \right) \frac{[H_0^{(2)}(kD)]^2}{4} - \frac{H_0^{(2)}(2kD)}{2}
\end{array} \right.
\]
and the radiation pattern as:

\[
\sum_{n=1,3,5}^{\infty} e^{i\frac{\pi}{n^2}} \left[ 2J_n(kD) - \frac{H_n^{(1)}(k\rho)}{H_n^{(2)}(k\rho)} \right] \cos n\theta
\]

where \( p \) is the radius of the fuselage and \( D \) the height of the line source above the wings.

A more complicated problem which has been considered is that of a line source above one semi-cylindrical boss on a ground plane with two other semi-cylindrical bosses on either side. This corresponds to the case of a line source above a fuselage with infinite wings and engines and is shown in figure 11. The problem is more complicated as multiple reflections should be considered. These interactions have not been included. The expression given below includes energy reflected in one curved surface and one flat surface but does not include energy reflected in two curved surfaces. By analogy with G.T.D. this is likely to be a small effect.

The expression derived for the radiation pattern is:

\[
\sum_{n=1,3,5}^{\infty} (i)^n \cos n\theta \left( 4J_n(kD) - 2H_n^{(2)}(kD) \left( 1 + \frac{H_n^{(1)}(k\rho)}{H_n^{(2)}(k\rho)} \right) \right.
\]

\[
+ 2H_n^{(2)}(kF) \left( 1 + \frac{H_n^{(1)}(k\rho)}{H_n^{(2)}(k\rho)} \right) \cos n\psi \left[ 2^{n+1} \Gamma(n) \chi \right]
\]

\[
\sum_{\ell=0}^{\infty} (n+\ell) i^{n+\ell} \frac{J_{n+\ell}(kE)}{(kE)^n} C_{\ell}(\cos \theta - \frac{\pi}{2}) \right)
\]

\[
+ \sum_{n=2,4,6}^{\infty} (i)^n \sin n\theta \sin n\psi \chi (kF) \left( 1 + \frac{H_n^{(1)}(k\rho)}{H_n^{(2)}(k\rho)} \right) \sin \psi \left[ 2^{n+2} \Gamma(n) \chi \right]
\]

\[
\sum_{\ell=0}^{\infty} (n+\ell) i^{n+\ell} \frac{J_{n+\ell}(kE)}{(kE)^n} C_{\ell}(\cos \theta + \frac{\pi}{2}) \]
where \( p \) and \( R \) are the radii of the fuselage and engines. \( \theta, \psi, D, E \) and \( F \) are shown in the figure and \( C^{(n)}_k(\ ) \) are ultraspherical or Gegenbauer polynomials.

If \( R \to 0 \) all the terms containing it vanish and the without-engines expression is recovered. Similarly if \( p \to 0 \) in this result the expression for a source over a ground plane is left.

As yet none of these expressions have been programmed for computer evaluation, so that the validity of the expressions in the field has not been fully established. It is however thought that the technique is substantially correct as in many respects the argument is similar to that developed by Kahn and Wasylkiwskyj (Reference 7) for mutual coupling among a certain class of antennas.

The technique has initially been developed for two dimensional problems with line source excitation because only one polarisation is involved and only two coordinates are required. For three dimensional problems the second polarisation must be considered, with polarisation conversion and the third space dimension. It is hoped that problems of this type will eventually be handled but first other two dimensional problems will be investigated.

The method considers the composite problem as a set of simple scatterers, the properties of which are known. If the radar cross-section of each scattering component is known, it may be possible, by a technique similar to that described, to determine the composite radar cross-section. It is felt that the scattering approach may enable the performance of an aircraft antenna, and the aircraft radar cross-section, to be linked.
6. Conclusions and Future Work

The problem of predicting the performance of aircraft antennas in the H.F. and V.H.F. regions have been considered from several points of view. Initially it was shown that a measured radiation pattern of a particular aircraft could be generated simply by adding two multipole fields in the correct amplitude and phase. This suggested a method of removing accumulated numerical errors from the radiation patterns computed using wire grid modelling.

The particular problem considered next was that of a source on an infinitely long cylinder. It was shown that the field could be expanded in terms of a few multipoles. From this expansion an equivalent dipole representation of the problem emerged which has been shown to be valid for cylinders less than about half a wavelength in diameter. Although only valid for this limited range of cylinder sizes it is felt that this type of representation is valuable as it re-frames the problem in terms that are readily understood by all antenna engineers. It is felt that a similar type of representation could be derived for many different problems and some of these problems will be considered. The representation is particularly useful in considering the properties of antennas on aircraft as the introduction of "infinite wings" simply removes a dipole.

Finally a scattering approach has been adopted which due to its complicated nature has initially been used to solve two dimensional problems. This method is particularly useful for problems which can be decomposed into a system of sources and cylinders. In this context the problem of a wire antenna over the fuselage of a "twin engined aircraft" has been solved. This method is likely to be extremely
powerful and it is hoped that it will be further developed, so that much more complicated problems can be tackled. In some respects the method is like G.T.D. in that the difficult problem is broken down into simpler problems.
References


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FIGURE 1  The approximation of an aircraft by crossed dipoles.
FIGURE 2  The geometry of a short monopole on a cylinder.

FIGURE 3  The pair of dipoles used to represent a multipole series.
FIGURE 4 The geometry of the four dipoles used to represent the fields of a short monopole on a cylinder.
FIGURE 5  The phase angle of $a_{2R}/a_{1R}$
FIGURE 6 The roll plane radiation patterns of a short monopole on a cylinder

Relative power (dB)

Dipole Representation
Wait's "exact" result

\[ a = 0.159 \]

FIGURE 7 The pitch plane radiation pattern of a short monopole on a cylinder

Relative power (dB)

Approximate series result
Wait's "exact" result
FIGURE 8  The geometry of a line source near a generalised scatterer.

FIGURE 9  The geometry of a line source above an infinite semi-cylindrical boss.
FIGURE 10  The geometry of a line source near three infinite semi-cylindrical bosses on a ground plane. The geometry is equivalent to a line source above the fuselage of a twin engined aircraft.
University of Surrey

RADIATION FROM ANTENNAS MOUNTED ON IRREGULARLY SHAPED BODIES

R. J. Chignell

November 1975
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ON IRREGULARLY SHAPED BODIES

Progress Report No 2

R. J. Chignell

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Figures
1. Introduction

This report is concerned with possible new approaches to the problem of predicting the radiation patterns of antennas mounted on aircraft. The previous report (Reference 1) indicated that successful techniques exist for both the low and high frequency ranges; this work is concerned with the intermediate or resonant range.

The main approaches which were investigated in the previous report (Reference 1) were a dipole representation of a source on a cylinder and a scattering description of two dimensional problems.

It was shown from Wait's description (Reference 2) of a source on a cylinder that a set of simple dipoles could be used to give the same radiation patterns. Unfortunately this simple description is only valid for cylinders less than half a wavelength in diameter. The technique could be extended by including dipoles with complex dimensions but then the essential simplicity of the method is lost. The results obtained by this approach are presented in this report but further work on this idea has ceased.

The major effort since the last report has been devoted to the scattering description of the problem. The method for two dimensional problems involves inward and outward travelling cylindrical waves which are connected by a scattering matrix. This method has been extended so that both the radiation patterns of antennas near an object and the radar cross section of the object may be calculated. In addition for two dimensional objects the method has been extended to bodies of arbitrary cross-section. The method is currently being investigated for three dimensional objects. It is believed that the essential translation theorem required by the method is most readily
available for computation from a plane wave spectrum description. In this manner it is thought that the method will naturally link with the Geometrical Theory of Diffraction.

2. A Scattering Description of Radiation Problems

A scattering description of some radiation problems was introduced in the last report (Reference 1). The derivation given then contains some points which though correct as defined did not help in the extension of the method. To introduce the extensions of the method and the alterations in the technique the model will be presented anew. The method will first be considered in two dimensions as it is conceptually much simpler but the development in three dimensions is in progress and will be indicated.

a) The concept of a two dimensional scattering model

The use of a scattering description of radiation problems is not new as for instance Kahn and Wasylkiwskyj (Reference 3) used multiple scattering to predict the mutual impedance between two line antennas. Referring to Figure 1 they considered two line antennas with antenna 1 radiating. This antenna radiates a cylindrical wave described by:

$$E_z = a_1 \frac{\sqrt{wH}}{2} H_0^{(2)}(kr_1)$$  \hspace{1cm} (1)

but by translation theorems (Reference 4) this can be expanded about the origin of the second antenna.
\[ a_1 \frac{\sqrt{w\mu}}{2} H_0^{(2)}(kr_1) = \frac{a_1\sqrt{w\mu}}{2} H_0^{(2)}(kd)J_0(kr_2) + a_1\sqrt{w\mu} \sum_{n=1}^{\infty} H_n^{(2)}(kd)J_n(kr_2)\cos\theta_2 \]  

\[ (2) \]

which converges for \( r_2 < D \). The Bessel functions can be expressed as cylindrical waves so that:

\[ a_1 \frac{\sqrt{w\mu}}{2} H_0^{(2)}(kr_1) = \frac{a_1\sqrt{w\mu}}{4} H_0^{(2)}(kd) \left[ H_0^{(1)}(kr_2) + H_0^{(2)}(kr_2) \right] + \frac{a_1\sqrt{w\mu}}{2} \sum_{n=1}^{\infty} H_n^{(2)}(kd) \left[ H_n^{(1)}(kr_2) + H_n^{(2)}(kr_2) \right] \cos\theta_2 \]

\[ (3) \]

The field at \( \theta_2 \) is then interpreted as inward \( H_n^{(1)}(kr_2) \) and outward \( H_n^{(2)}(kr_2) \) travelling cylindrical waves. Kahn and Wasylkiwskyj made the explicit assumption that the line sources could be regarded as canonically minimum scattering. This is a reasonable assumption and is implicit in most attempts at deriving mutual impedance such as the induced E.M.F. method. When stated explicitly the modal properties of the antenna follow immediately and in this case indicate that the incident \( H_0^{(1)}(kr_2) \) wave would be totally absorbed. If this incident mode is absorbed the \( H_0^{(2)}(kr_2) \) wave will not be radiated and the total field should be appropriately modified i.e.

\[ a_1 \frac{\sqrt{w\mu}}{2} H_0^{(2)}(kr_1) - \frac{a_1\sqrt{w\mu}}{4} H_0^{(2)}(kd)H_0^{(2)}(kr_2) = \frac{a_1\sqrt{w\mu}}{4} H_0^{(2)}(kd)H_0^{(1)}(kr_2) + \frac{a_1\sqrt{w\mu}}{2} \sum_{n=1}^{\infty} H_n^{(2)}(kd) \left[ H_n^{(1)}(kr_2) + H_n^{(2)}(kr_2) \right] \cos\theta_2 \]

\[ (4) \]
In this manner by multiple scattering between the two antennas the scattering matrix may be derived. However using an impedance description the matrix may be directly derived; from this any other description can be obtained.

In this case

\[ S_{11} = S_{22} = \frac{-c^2}{1-c^2} \]  \hspace{1cm} (5)

and

\[ S_{12} = S_{21} = \frac{c}{1-c^2} \]  \hspace{1cm} (6)

where:

\[ C = \frac{\mathcal{H}^{(2)}_0 (kd)}{2} \]  \hspace{1cm} (7)

Once the mutual impedance has been found the derivation of the far field radiation pattern with one active and one passive line source becomes trivial. Although the original translation theorem (equation (2)) was only valid for the range \( r_2 < D \) this does not prevent the far field being found.

The proposed method of finding the radiation patterns of antennas in the vicinity of conducting bodies involves a generalisation of Kahn and Wasylkiwskijy's method. In the problem they solved, one of the incident modes was simply absorbed. For more complicated scatterers, a more general description of the modification to the field is required, and a matrix description seems most suitable.

The field incident upon the origin of the scatterer can be expressed as a sum of incoming cylindrical waves:

\[ \sum_{n} a_n (1) (kr) \cos \theta \cos 2\pi z \]  \hspace{1cm} (8)
of amplitude (complex) \( a_n \). Similarly the field scattered away from the object may be expressed as:

\[
\sum b_n H_n^{(2)}(kr) \cos n \theta_k
\]  

(9)

The amplitudes of the incident and scattered cylindrical waves may then be connected via a scattering matrix.

\[
\begin{bmatrix}
  b_0 \\
  \vdots \\
  b_n \\
\end{bmatrix}
= 
\begin{bmatrix}
  S_{00} & \cdots & S_{0n} \\
  \vdots & \ddots & \vdots \\
  S_{n0} & \cdots & S_{nn} \\
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  \vdots \\
  a_n \\
\end{bmatrix}
\]  

(10)

If the scatterer is simply free space the matrix is diagonal and unitary. Similarly for the line source considered earlier the matrix is diagonal with all the elements except \( S_{00} \) equal to one. \( S_{00} \) is zero. In general the matrix will not be diagonal and the major problem will be to derive the elements: this will be considered later. However one case of practical interest for which the scattering matrix is particularly simple is that of a cylinder with the origin at its centre. By virtue of the symmetry of the object a cylindrical wave retains its character after scattering so that the matrix is diagonal, with all non-diagonal elements zero. The non-zero elements are readily determined by application of the boundary conditions. For a wave polarized parallel to the axis of the cylinder of radius \( \rho \) the elements are:

\[
S_{nn} = - \frac{H_n^{(1)'(k\rho)}}{H_n^{(2)'(k\rho)}}
\]  

(11)

where the ' indicates the derivative with respect to the total argument.
Similarly for the orthogonal polarization,

\[
S_{nn} = - \frac{H_n^{(1)}(kr_2)}{H_n^{(2)}(kr_2)} \tag{12}
\]

b) The calculation of the field radiated by a line source in the presence of a scatterer.

Consider a line source adjacent to a general two dimensional scatterer as shown in Figure 2. As with the two line sources considered earlier the field radiated by the line source may be expanded about the origin of the scatterer: (equation 3)

\[
\frac{a_1^{1/2} \nu}{2} H_0^{(2)}(kr_1) = \frac{a_1^{1/2} \nu}{4} H_0^{(2)}(kD) \left[ H_0^{(1)}(kr_2) + H_0^{(2)}(kr_2) \right] \\
+ \frac{a_1^{1/2} \nu}{2} \sum_{n=1}^{\infty} H_n^{(2)}(kD) \left[ H_n^{(1)}(kr_2) + H_n^{(2)}(kr_2) \right] \cos n\theta_2
\]

When the scatterer is introduced at origin 2 the unitary scattering matrix of free space must be replaced by the more complicated matrix associated with the scatterer. With free space:

\[
b_0 = a_0 \tag{13}
\]

but with the scatterer present:

\[
b_0 = \sum_{n=0}^{\infty} S_{0n} a_n \tag{14}
\]

so that the modification to the lowest order mode is:
\[
\{ \sum_{n=0}^{\infty} S_{on} a_n a_{0}^* \} H_0^{(2)}(kr_2)
\]

(15)

and for the general \(m\)th order mode:

\[
\{ \sum_{n=0}^{\infty} S_{mn} a_n a_{m}^* \} H_m^{(2)}(kr_2) \cos m \theta_m
\]

(16)

The total change in the field is then

\[
\sum_{m=0}^{\infty} \{ \sum_{n=0}^{\infty} S_{mn} a_n a_{m}^* \} H_m^{(2)}(kr_2) \cos m \theta_m
\]

(17)

For the special case of a cylindrical scatterer the modification to the field for an electric line source is:

\[
- \left[ \begin{array}{c}
1+ \frac{H_0^{(1)'}(kp)}{H_0^{(2)'}(kp)} \\
H_0^{(2)}(kp)
\end{array} \right] \frac{H_0^{(2)}(kd)}{2} H_0^{(2)}(kr_2)
+ \sum_{n=1}^{\infty} \left[ \begin{array}{c}
1+ \frac{H_N^{(1)'}(kp)}{H_N^{(2)'}(kp)} \\
H_N^{(2)}(kp)
\end{array} \right] H_n^{(2)}(kd) H_n^{(2)}(kr_2) \cos n(\theta_2 - \theta_1)
\]

(18)

When this modification is re-expanded about the line source the energy received by it may be calculated. Thus the voltage reflection coefficient \(\Gamma\) of the line source in the presence of the cylinder may be calculated as:

\[
- \frac{c^2 d}{(1-c^2 d)}
\]

(19)

where \(c = \frac{H_0^{(2)}(kd)}{2}\) and \(d = \left[ 1+ \frac{H_0^{(1)'}(kp)}{H_0^{(2)'}(kp)} \right] \)

(20)

In the far field the radiation from the line source may be expanded about the origin \(O_2\) as proportional to:
\[ H_0^{(2)}(kr_2)J_0(kD) + 2 \sum_{n=1}^{\infty} H_n^{(2)}(kr_2)J_n(kD) \cos n(\theta - \pi) \]  

so that the total field including the modification is for an electric line source near a cylinder:

\[ H_0^{(2)}(kr_2) \left[ J_0(kD) - \frac{H_0^{(2)}(kD)}{2} \left( 1 + \frac{H_0^{(1)'}(kp)}{H_0^{(2)'}(kp)} \right) \right] + 2 \sum_{n=1}^{\infty} H_n^{(2)}(kr_2) \cos n(\theta - \pi) J_n(kD) - \frac{H_n^{(2)}(kD)}{2} \left( 1 + \frac{H_N^{(1)'}(kp)}{H_N^{(2)'}(kp)} \right) \]  

but in the far field as \( r_2 \to \infty \)

\[ H_n^{(2)}(kr_2) \to \sqrt{ \frac{2}{\pi kr_2} } e^{-i(kr_2 - \frac{n\pi}{2} - \frac{\pi}{4})} \]

so that the far field is given by:

\[ \left[ J_0(kD) - \frac{H_0^{(2)}(kD)}{2} \left( 1 + \frac{H_0^{(1)'}(kp)}{H_0^{(2)'}(kp)} \right) \right] + 2 \sum_{n=1}^{\infty} e^{i\frac{\pi}{2}} \cos n(\theta - \pi) \left[ J_n(kD) - \frac{H_n^{(2)}(kD)}{2} \left( 1 + \frac{H_N^{(2)'}(kp)}{H_N^{(2)'}(kp)} \right) \right] \]

For a more general scatterer the far field is given by:

\[ \left[ J_0(kD) + \frac{H_0^{(2)}(kD)}{2} \left( S_{00} - 1 \right) + \sum_{n=1}^{\infty} S_{0n} H_n^{(2)}(kD) \right] \]

\[ + \sum_{n=1}^{\infty} e^{i\frac{\pi}{2}} \cos n\theta \left[ 2J_n(kD) + \frac{H_n^{(2)}(kD)}{2} S_{n0} H_N^{(2)}(kD) \right] \]

\[ + \sum_{m=1}^{\infty} H_m^{(2)}(kD) S_{nm} \]
c) The two dimensional radar cross-section

In two dimensions the radar cross-section of objects may be calculated using the generalised scattering matrix. The bistatic radar cross-section of an object gives a measure of the energy scattered in the direction $\theta$ when a plane wave is incident upon it from direction $\phi$. In most cases the monostatic radar cross-section is of primary interest and in this case $\theta$ and $\phi$ correspond to the same direction.

Consider a plane wave incident upon an origin in the direction $\phi$. This may be specified by:

$$e^{ikr\cos\phi}$$

which may be expanded into incoming and outgoing cylindrical waves:

$$e^{ikr\cos\phi} = J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\phi$$

$$= \frac{1}{2} [H_{10}^{(1)}(kr) + H_{10}^{(2)}(kr)] + \sum_{n=1}^{\infty} i^n [H_{n}^{(1)}(kr) + H_{n}^{(2)}(kr)] \cos n\phi$$

The amplitudes of the incoming waves are given by:

$$a_0 = \frac{1}{2} \quad a_n = i^n \frac{\cos n\phi}{\cos \theta}$$

so that:

$$b_0 = \frac{S_{00}}{2} + \sum_{n=1}^{\infty} i^n s_{ON} \frac{\cos n\phi}{\cos \theta}$$

$$b_\rho = \frac{S_{\rho0}}{2} + \sum_{n=1}^{\infty} i^n s_{\rho n} \frac{\cos n\phi}{n \cos \theta}$$
The modification to the total field or the scattered field is then given by

\[ H^{(2)}_0(kr) \left\{ \frac{S_{OO}^{-1}}{2} + \sum_{n=1}^{\infty} \frac{i^n S_{ON} \cos n\phi}{\cos n\theta} \right\} \]

\[ + \sum_{n=1}^{\infty} \left\{ \frac{S_{NO}}{2} - \frac{i^n \cos n\phi}{\cos n\theta} + \sum_{m=1}^{\infty} \frac{i^m S_{nm} \cos m\phi}{\cos m\theta} \right\} \frac{H^{(2)}_n(kr) \cos n\theta}{\cos n\theta} \quad (30) \]

For the monostatic radar cross-section

\[ \theta = \pi + \phi \]

or

\[ \cos n\phi = \cos n\pi \cos n\theta \]

and the monostatic R.C.S. is:

\[ H^{(2)}_0(kr) \left\{ \frac{S_{OO}^{-1}}{2} + \sum_{n=1}^{\infty} (-i)^n S_{ON} \right\} \]

\[ + \sum_{n=1}^{\infty} H^{(2)}_n(kr) \cos n\theta \left\{ \frac{S_{NO}}{2} - (-i)^n + \sum_{m=1}^{\infty} (-i)^m (i)^n S_{nm} \right\} \quad (32) \]

which becomes as \( r \to \infty \)

\[ \left\{ \frac{S_{OO}^{-1}}{2} + \sum_{n=1}^{\infty} (-i)^n S_{ON} \right\} \]

\[ + \sum_{n=1}^{\infty} \frac{\sin \pi}{n \cos n\theta} \left\{ \frac{S_{NO}}{2} - (-i)^n + \sum_{m=1}^{\infty} (-i)^m (i)^n S_{nm} \right\} \quad (33) \]
It has been shown that the radiation pattern of a line source near a two dimensional object and the radar cross-section of that object are essentially known once the elements of the scattering matrix are determined. For certain simple objects the matrix is easy to find. The obvious example is a cylinder, its matrix is simply determined as it can be made to coincide with a coordinate surface. In a similar manner the scattering matrix of an elliptical cylinder could be determined if the problem was considered in terms of incident and scattered elliptical waves. Although this would be a valid solution it would not be of much use in more complicated situations such as aircraft with an elliptical fuselage and cylindrical engines. The coordinate transformations would not only involve a shift in origin but a change of coordinate type. A method is required to determine the scattering matrix of an arbitrary two dimensional body in terms of incident and scattered cylindrical waves. Fortunately such an approach can be made by a slight modification to the recent work of Wilton and Mittra (Reference 5). They were specifically interested in finding the two dimensional radar cross-section of a scatterer with arbitrary shape.

The field outside a cylinder totally enclosing the scatterer must be continuous in the absence of additional sources. However the field only diverges at points on or inside the surface of the object. Wilton and Mittra analytically continue the field into the region between the enclosing cylinder and the object. In this way the field is continued to the surface of the scatterer where the appropriate boundary conditions are applied.
Wilton and Mittra considered the scattered field to be of the form:

\[ E_Z^S(\rho) = \sum_{n=-\infty}^{\infty} a_n \, j^{-n} H_n^{(2)}(k\rho) \exp(jn\phi) \]  

(34)

with the geometry as specified in Figure 3. They let \( n \) be both positive and negative but it is a simple matter to recast the problem with \( n \) only positive. The problem could be solved in the translated coordinate system such that:

\[ E_Z^S(\rho_o + \rho') = \sum_{m=-\infty}^{\infty} a^m \, j^{-m} H_m^{(2)}(k\rho') \exp(jm\phi') \]  

(35)

Equation (34) can be put into the form of equation (35) and retain its original form via the translation formula:

\[ H_n^{(2)}(k\rho) \exp(jn\phi) = \sum_{m=-\infty}^{\infty} H_m^{(2)}(k\rho_o) \exp[j(n-m)\phi] \]  

(36)

so that

\[ E_Z^S(\rho_o + \rho') = \sum_{n=0}^{\infty} a_n \, j^{-n} \sum_{m=-\infty}^{\infty} \int \left[ j(n-m)\phi \right] J_m^{(2)}(k\rho') \exp(jm\phi') \]  

(37)

If the coordinates \( \rho, \rho_o \) and \( \rho' \) are designated by the subscript \( i \) to denote their value at the \( i \)th point there is a set of equations for the different points which may be written in matrix form as

\[ E_i^S = [B_{in}] \, [a_n] \]  

(38)
or applying the boundary condition that

\[ E_i^{\text{inc}} + E_i^S = 0 \]

(39)

\[- E_i^{\text{inc}} = [B]\begin{bmatrix} a_n \end{bmatrix} \]

(40)

\[ [a_n] = - [B]^{-1}[E_i^{\text{inc}}] \]

The scattered field is thus determined in terms of the incident field. However if \( E_i^{\text{inc}} \) is taken as a field of form \( H_0^{(1)}(kp) \) the \( a_n \)'s calculated correspond to the \( S \) elements of the scattering matrix. Thus if the \( B \) matrix is first set up at a sufficient number of points and inverted all the elements of the scattering matrix may be calculated by multiplying by the mode functions in turn.

Using this analytical continuation method the scattering matrices and then the other radiation properties of bodies of arbitrary cross-section can be investigated. The chief advantages of the method are that

1) the calculation uses the electric field rather than the current so that problems due to the current discontinuities at corners and edges are avoided.

2) the Hankel functions involved behave in a well defined manner.

The point at which the infinite summation can be truncated can be predicted before the calculation is started. For \( H_m^{(2)}(kp) \) the most important terms occur when \( m \approx kp \) so the summation can be stopped after including a few terms with \( m \) larger than \( kp \)

3) no integration is necessary.
e) Composite problems and numerical results

The first problem that was investigated numerically was that of a line source above a semi-cylindrical boss on an infinite ground plane as shown in Figure 4. The ground plane was regarded as an image plane so that the problem actually considered was that of a cylinder with two line sources. The field of both sources was expanded about the centre of the cylinder. The scattering matrix was then used to work out the modifications to the field and all component summed. The result for the radiation pattern with an electric line source was:

\[
\sum_{n=1,3,5}^\infty \exp\left(\frac{i\pi n}{2}\right) [2J_n(kD) - \left(1 + \frac{H_n^{(1)}(kp)}{H_n^{(2)}(kp)}\right) \frac{H_n^{(2)}(kD)}{J_n^{(2)}(kD)}] \cos n\theta
\]

where \( \rho \) is the radius of the fuselage and \( D \) the height of the line source above the wings. Similarly for a magnetic line source the radiation pattern is given by:

\[
\sum_{n=1,3,5}^\infty \exp\left(\frac{i\pi n}{2}\right) [2J_n(kD) - \left(1 + \frac{H_n^{(2)}(kp)}{H_n^{(2)}(kp)}\right) \frac{H_n^{(2)}(kD)}{J_n^{(2)}(kD)}] \cos n\theta
\]

These expressions have been programmed for computer evaluation. The series was truncated when \( n \) was greater than \( k \) and the amplitude of the excitation of the cosine was less than 2% of that of the most important term. The patterns obtained together with the number of terms used and the voltage reflection coefficient of the source (assumed matched in free space) are shown in Figures 4 to 9. The Figures show that in two dimensions the radiation patterns obtained by scattering in quite large cylinders can be calculated with only
a few terms. The predominant behaviour of the cylindrical functions when \( n \approx k\phi \) is shown explicitly in Figure 10 but may be deduced implicitly from the number of lobes in each radiation pattern.

This computer program was further developed to include the effect of cylindrical engines. The expression developed for the radiation pattern of an electric line source above a "twin engined aircraft" was:

\[
\sum_{n=1,3,5}^\infty \left( i^n \cos \theta \right) \left\{ 4J_n(kD) - 2H_n^{(2)}(kD) \right\} \left\{ 1 + \frac{H_n^{(1)'}(k\rho)}{H_n^{(2)'}(k\rho)} \right\} \\
+ 2H_n^{(2)}(kF) \left\{ 1 + \frac{H_n^{(1)'}(kR)}{H_n^{(2)'}(kR)} \right\} \cos \phi_k \\
\left[ 2^{n+1} \Gamma(n) \sum_{\ell=0}^\infty (n+\ell) i^\ell \frac{J_{n+\ell}(ke)}{ke^n} C_{\ell}^{(n)} \left( \cos^2 \frac{\pi}{2} \right) \right] \\
+ \sum_{n=2,4,6}^\infty \left( i^n \sin \theta \right) \left\{ 1 + \frac{H_n^{(1)'}(kR)}{H_n^{(2)'}(kR)} \right\} H_n^{(2)}(kF) \sin \phi \times \\
\left[ 2^{n+2} \Gamma(n) \sum_{\ell=0}^\infty (n+\ell) i^\ell \frac{J_{n+\ell}(ke)}{ke^n} C_{\ell}^{(n)} \left( \cos + \frac{\pi}{2} \right) \right]
\]

where \( R \) is the radius of the engines, \( E \) the separation of the engine and fuselage centres, \( F \) the separation of the source and the engine centres. \( \phi \) was the angle of the engine centres from the source and the \( C_{\ell}^{(n)} \) were ultraspherical or Gegerbauer polynomials.

Numerically the modification to the patterns caused by the engines was dominated by the behavior of the term:

\[
-H_n^{(2)}(kF) \left\{ 1 + \frac{H_n^{(1)'}(kR)}{H_n^{(2)'}(kR)} \right\}
\]
With the fuselage diameter of 4 wavelengths the diameter of 4 wavelengths so that it might be expected that terms with $n$ approximately forty would be required. In fact the terms in $kR$ are decaying so rapidly in this region that it entirely dominates the behaviour of the expression. The terms with $n$ approximately $kR$ are by far the most important. Thus in the $H_n^{(2)}(kF)$ term the $kF$ is much larger than $n$ and the asymptotic form can be used

$$H_n^{(2)}(kF) \sim \sqrt{\frac{2}{\pi kF}} e^{-i(kF - \frac{2\pi}{2} - \frac{\pi}{4})}$$

Physically this corresponds to a plane wave being incident upon the engines and corresponds exactly with the description used in the Geometrical Theory of Diffraction. Thus the scattering method naturally turns into the Geometrical Theory of Diffraction at sufficiently high frequencies.

The radiation patterns of the "twin engined aircraft" are shown in Figures 6 to 9 and 11 to 13.

Currently the radiation properties of a two dimensional square object are being investigated numerically to test the analytic continuation method.

f) Extension of the method to three dimensional problems

In principle the extension of the method to three dimensions does not pose any major problems, however the increase in computation is such that major difficulties do arise. Corresponding to the cylindrical waves in two dimensions are the multipoles in three dimensions. As it requires two numbers to designate a multipole
there are many more of them than the simple cylindrical waves. The major difficulty is the translation theorems in three dimensions. The usual ones (References 6 and 7) which have probably come from quantum angular momentum theory require the calculation of terms involving products of Legendre functions:

\[ p_n^m(x)p_{-m}^m(x) = \sum_p a(m,n-m,v,p)p_p(x) \]

where the coefficients are products of two 3j - Wigner groups:

\[ a(m,n-m,v,p) = (2p+1) \left[ \frac{(n+m)!}{(n-m)!} \frac{(v-m)!}{(v+m)!} \right] \]

where the 3j Wigner groups are defined to be (Reference 8):

\[ \left[ \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right] = \frac{(-1)^{j_1-j_2-m_3}}{2(j_1+j_2+j_3)!} \frac{(j_1-m_1)!}{j_1-j_1-m_1+k} \frac{(j_2-m_2)!}{j_2-j_2-m_2+k} \frac{(j_3-m_3)!}{j_3-j_3-m_3+k} \]

\[ \sum_k \frac{(-1)^k}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j_3-j_1-m_2+k)!} \]

\[ x (j_1+m_1+k)(j_2-m_2+k)(j_3+m_3+k) \]
Obviously many calculations of this type are going to take a lot of computer time. Fortunately however Bruning and Lo (Reference 9) have recently developed a recursive relationship in \( p \) for the \( a \) coefficients. If this type of calculation proves necessary this will reduce the computer time drastically. Currently a plane wave spectrum representative of the translation is being considered as used by Wasylkiwskyj and Kahn (Reference 10). The translation will require an integration but it is thought that this might be the most straightforward approach, as the functions involved are simple.

In calculating the properties of a real three dimensional aircraft at all but the lowest frequencies the problem is to limit the number of modes required. The analytic continuation required may be preformed as indicated in Figure 14 but this does not take advantage of the major symmetric present in the structure. The longest single dimension in an aircraft is the fuselage length. This will influence the number of modes required. However it is thought that a major reduction in the number of terms required may result if the fuselage can be regarded as a long thin object with cylindrical symmetry. The wings and tail can then be added to make a composite scattering body. If each wing is treated separately the translation to two scattering centres symmetrically placed relative to the fuselage reduces the number of terms involved. The reduction is two fold as a single wing is a smaller object than a pair of wings it requires fewer terms and the symmetry of the pair also reduces the number of terms.

The first multipole the fuselage can excite is the \( (1,0) \) one and the first one the wings can excite is the \( (2,0) \) so that in the
yaw plane at low frequencies the radiation pattern will be given by:

\[ \sin \theta + a \sin \theta \cos \theta \]

as indicated in the first report (Reference 1).

3. The Dipole Representation of a Source on a Cylinder

It was shown in the previous progress report (Reference 1) that the concise multipole expansion of Wait (Reference 2) for a source on a cylinder could be truncated. If the truncation included only four terms then the source on a cylinder could be replaced by a system of four dipoles as shown in Figure 15. The appropriate equations have now been solved numerically so that the dimensions of the dipoles concerned can be given. The results obtained are shown in Figures 15.1 and 15.2. Dipoles one and two approximate the \( \phi \) component of the field while the third and fourth dipoles approximate the \( \theta \) component. The length of the first dipole increases progressively with cylinder size up to a value of \( \pi (2 \pi \sin \theta) \) of approximately 1.4. At this point the dipole length jumps to a larger value. This is accompanied by a discontinuity in the amplitude and phase of the current. The phase of the current can no longer be regarded as approximately real and the simplicity of the approach is lost. Discontinuities occur in both the lengths and currents of the third and fourth dipoles. However the field remains bounded as infinite current occurs in dipoles of zero length and zero current flows in dipoles of infinite length.
4. Comparison of some Computed and Experimental Results for a Monopole on a Cylinder

In the previous report (Reference 1) the results of Wait's (Reference 2) analysis of a short monopole on a cylinder were used to derive an approximate series for the electric field. This enabled the dipole representation to be derived. Some experimental radiation patterns of a quarter wavelength long monopole on a cylinder, were provided for the University of Surrey by the Royal Aircraft Establishment. Wait's concise analysis for a short monopole assumed that the cylinder was of infinite length however the experimental cylinder was of necessity finite; its ends were covered in radar absorbent material. Figures 22 to 30 show the comparison of the theoretical results with experimental results for the three principal planes at various frequencies. In some cases the assumption of a short monopole has been removed by integrating along the length of the monopole assuming a suitable sinusoidal current distribution.

In general there is acceptable rather than good agreement between theory and experiment in the roll plane. In the two other major planes which include the cylinder axis the agreement is not good. This is thought to be due to the assumption of an infinite cylinder in the theory compared to the finite one used in the experiments.
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Figure 1. The geometry of two line antennas

Figure 2. The geometry of a line source near a scatterer
Figure 3. The geometry for inside expansion using analytic continuation.
Cylinder diameter: 1 wavelength

Voltage reflection coefficient $\Gamma = 0.06507 + j0.1037$

5 terms were used in the expansion; (up to $\cos 9\theta$)

Figure 4. The radiation pattern of an electric line source a quarter of a wavelength above an infinite semi-cylindrical boss on a ground plane
Voltage reflection coefficient $\Gamma = 0.0544 + j0.0801$

7 terms used in the expansion (up to $\cos 13.6\degree$)

Figure 5. The radiation pattern of an electric line source a quarter of a wavelength above an infinite semi-cylindrical boss on a ground plane
Figure 6. The radiation pattern of an electric line source a quarter of a wavelength above an infinite semi-cylindrical boss on a ground plane.

Voltage reflection coefficient $\Gamma = 0.0433 + j0.0585$

11 terms are used in the expansion (up to $\cos 21 \theta$)

Cylinder diameter: 4 wavelengths
The voltage reflection coefficient = -0.2081 + j0.1303

(wi without engines)

Terms in cos 7 θ are included

Fuselage diameter: 1 wavelength
Engine diameter: 0.33 wavelength
Source height above wings: 0.6 wavelength
Separation of engine and fuselage centres: 1.5 wavelength

Figure 7. The radiation pattern of a magnetic line source above "an aircraft"
The voltage reflection coefficient $\rho = 0.0348 + j0.1235$

Terms in $\cos 13 \theta$ are included

Fuselage diameter: 2 wavelengths
Engine diameter: 0.66 wavelengths
Source light above wings: 1.2 wavelengths
Separation of engine and fuselage centres: 3 wavelengths

Figure 8: The radiation pattern of a magnetic line source above "an aircraft"
The voltage reflection coefficient $= 0.0438 - j0.0714$

(without engines)

Terms in $\cos 2\theta$ are included

Figure 9. The radiation pattern of a magnetic line source above "an aircraft"

Fuselage diameter: 4 wavelengths

Engine diameter: 1.33 wavelengths

Source height above wings: 2.4 wavelengths

Separation of engine and fuselage centres: 6 wavelengths
Figure 10. The relative amplitudes of the excitation coefficients for a line source above a cylinder.
Fuselage diameter: 1 wavelength
Engine diameter: 0.33 wavelengths
Source height above wings: 0.6 wavelengths
Separation of engine and fuselage centres: 1.5 wavelengths

Figure 11. The radiation pattern of an electric line source above a "twin engined aircraft"
Figure 12. The radiation pattern of an electric line source above a "twin-engined aircraft"

- Fuselage diameter: 2 wavelengths
- Engine diameter: 0.66 wavelengths
- Source height above wings: 1.2 wavelengths
- Separation of engine and fuselage centres: 3 wavelengths
Fuselage diameter: 4 wavelengths
Engine diameter: 1.33 wavelengths
Source height above wings: 2.4 wavelengths
Separation of engine and fuselage: 6 wavelengths

Figure 13. The radiation pattern of an electric line source above a "twin engined aircraft"
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Figure 15. The geometry of the four dipoles used to represent the fields of a short monopole on a cylinder.
Figure 16. The lengths of dipoles one and two
Figure 17. The amplitude and phase of the current in dipole one
Figure 18. The amplitude and phase of the current in dipole two.
Figure 19. The length of the third and fourth dipoles
(Im (I_3) = 0)

Figure 20. The amplitude of the current in dipole three
Figure 21. The amplitude of the current in dipole four.
The roll plane radiation pattern of a quarter-wave monopole on a cylinder

+ 90° (Cylinder radius = 0.159 λ)  

θ = 90°

Composed using a concise expansion and assuming a short monopole

θ = 0°

Computed using a concise expansion and integrating along the length of the monopole

- 30°

Measured patterns

x Computed using a concise expansion and assuming a short monopole

θ Computed using a concise expansion and integrating along the length of the monopole

Figure 22
The yaw plane radiation patterns of a λ/4 monopole on a cylinder

(Cylinder radius = 0.159 λ)

---

θ = 0° (Cylinder axis)

φ = 0°

---

- Measured patterns
- x Computed using a concise expansion and assuming a short monopole
- Θ Computed using a concise expansion and integrating along the length of the monopole

Figure 23
The pitch plane radiation pattern of a quarter-wave monopole on a cylinder (radius = 0.159 λ).

Measured and computed using a concise expansion and integrating along the length of the monopole.
The roll plane radiation pattern of a quarter-wave monopole on a cylinder

(Cylinder radius = 0.249 λ)

90°

θ = 90°

+ 60°

+ 30°

+ 0°

- 30°

- 60°

- 90°

Figure 25

Ω Computed using a concise expansion and integrating along the length of the monopole

x Computed using a concise expansion assuming a short monopole

- Measured
The yaw plane radiation pattern of a quarter-wave monopole on a cylinder

(Cylinder radius = 0.249 λ)

\( \phi = 0^\circ \)

\( \theta = 0^\circ \) (Cylinder axis)

- Measured patterns
- Computed using a concise expansion and integrating along the length of the monopole
- Computed using a concise expansion assuming a short monopole

Figure 26
The roll plane radiation pattern of a quarter-wave monopole on a cylinder

(Cylinder radius = 0.316 λ)

Figure 27

- Measured

x Computed using a concise expansion assuming a short monopole

O Computed using a concise expansion and integrating along the length of the monopole
The yaw plane radiation pattern of a quarter-wave monopole on a cylinder

(Cylinder radius = 0.316 λ)

φ = 0°

θ = 0° (Cylinder axis)

— Measured
x Computed using a concise expansion assuming a short monopole
O Computed using a concise expansion and integrating along the length of the monopole

Figure 28
Figure 29

- Measured patterns

θ Computed using a concise expansion and integrating along the length of the monopole.
The yaw plane radiation pattern of a quarter-wave monopole on a cylinder

(Cylinder radius = 0.498 λ)

θ = 0° (Cylinder axis)  φ = 0°

— Measured patterns
θ Computed using a concise expansion and integration along the length of the monopole

Figure 30
RADIATION FROM ANTENNAS MOUNTED
ON IRREGULARLY SHAPED BODIES

R. J. Chignell

April 1976
Contract No. AT/2064/037

RADIATION FROM ANTENNAS MOUNTED
ON IRREGULARLY SHAPED BODIES

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1. **INTRODUCTION**

This report is concerned with predicting the radiation patterns of antennas near scattering objects such as aircraft. The previous report (Reference 1) indicated that a scattering matrix was useful in the two dimensional problem of a line source near a cylinder. This report presents the work concerned with considering a line source near a more general scatterer such as a conducting square and the extension of the method to three dimensional problems.
2. A SCATTERING DESCRIPTION OF RADIATION PROBLEMS

A Scattering Description of Radiation Problems was introduced in a previous report (Reference 2) and further developed in the last report (Reference 1). The derivation presented then involved an angular dependence given by functions like \( \cos n\theta \). This form of the angular function implies a certain symmetry in the problems that can be considered. This symmetry is present in the circular cylinder, but limits the problems that can be considered because a complete orthogonal set of expansion functions is not being used.

The complete set of functions includes terms involving \( \sin n\theta \) as well as \( \cos n\theta \). Rather than using two matrices it is better to work with one set of functions expressed as \( \exp(jn\theta) \) with \( n \) positive and negative. An outline of the modified method will be presented, in terms of a \( \sigma \) matrix to avoid confusion with the \( S \) matrix previously used.

a) The general two dimensional scattering model.

The field incident upon the origin of a scatterer can be expressed as the sum of incoming cylindrical waves

\[
\sum_{n=-\infty}^{\infty} \alpha_n j^{-n_H(1)}(kr) \exp(jn\theta)
\]

of complex amplitude \( \alpha_n \).

Similarly the field scattered away from the object can be specified as:

\[
\sum_{n=-\infty}^{\infty} \beta_n j^{-n_H(2)}(kr) \exp(jn\theta)
\]
The amplitudes of the incident and scattered fields may be connected via a scattering matrix after the infinite sums have been suitably truncated.

| $\beta_{-N}$ | $\sigma_{-N,-N}$ | $\sigma_{-N,-n}$ | $\ldots$ | $\sigma_{-N,N}$ | $\alpha_{-N}$ |
| $\vdots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\cdot$ |
| $\beta_{-n}$ | $\sigma_{-n,-N}$ | $\sigma_{-n,-n}$ | $\ldots$ | $\sigma_{-n,N}$ | $\alpha_{-n}$ |
| $\vdots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\cdot$ |
| $\beta_{0}$ | $\sigma_{0,-N}$ | $\sigma_{0,-n}$ | $\ldots$ | $\sigma_{0,N}$ | $\alpha_{0}$ |
| $\vdots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\cdot$ |
| $\beta_{n}$ | $\sigma_{n,-N}$ | $\sigma_{n,-n}$ | $\ldots$ | $\sigma_{n,N}$ | $\alpha_{n}$ |
| $\vdots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\ddots$ | $\cdot$ |
| $\beta_{+N}$ | $\sigma_{N,-N}$ | $\sigma_{N,-n}$ | $\ldots$ | $\sigma_{N,N}$ | $\alpha_{N}$ |

b) The calculation of the field radiated by a line source in the presence of a general scatterer.

Consider a line source adjacent to a two dimensional scatterer. The field radiated is proportional to:

$$H_{o}^{(2)}(kr_{1})$$

but may be expanded about the origin of the scatterer by

$$H_{o}^{(2)}(kr_{1}) = H_{o}^{(2)}(kd)J_{0}(kd) + 2 \sum_{n=1}^{\infty} H_{n}^{(2)}(kd)J_{n}(kr_{1}) \cos n\theta$$

$$H_{o}^{(2)}(kr_{1}) = \sum_{n=-\infty}^{\infty} H_{n}^{(2)}(kd)J_{n}(kr_{1}) \exp(jn\theta)$$

The incident field upon the origin is then
\[ \sum_{n=-\infty}^{\infty} \frac{H_{2}^{(2)}(kD)}{2} \frac{H_{n}^{(1)}(kr)}{2} \exp(jn\theta) \]

ie.

\[ \alpha_n = j^n \frac{H_{2}^{(2)}(kD)}{2} \]

From the scattering matrix:

\[ \beta_0 = \sum_{n=-N}^{+N} \sigma_{on}\alpha_n \]

So that the modification to the Zeroth order mode is:

\[ \sum_{n=-N}^{+N} \sigma_{on}\alpha_n \cdot H_{2}^{(2)}(kr) \]

and for the general \( m \) the mode

\[ \sum_{n=-N}^{+N} \sigma_{mn}\alpha_n \cdot j^{-m}H_{2}^{(2)}(kr) \exp(jm\theta) \]

The total change in the field is then:

\[ \sum_{m=-N}^{+N} \sum_{n=-N}^{+N} \sigma_{mn}\alpha_n \cdot j^{-m}H_{2}^{(2)}(kr) \exp(jm\theta) \]

This change in the field is valid for all regions.

In the far field

\[ H_{2}^{(2)}(kr) = \sum_{n=-N}^{N} H_{n}^{(2)}(kr) J_{n}(kD) \exp(jn\theta) \]

The total field is then:
\[
\sum_{m=-N}^{N} \left\{ \sum_{n=-N}^{N} \sigma_{mn} \alpha_n - \alpha_m + j^m j_m (kD) \right\} j^{-m} \exp(jm\theta) H_n^{(2)}(kr).
\]

\[
= \sum_{m=-N}^{N} \left\{ \sum_{n=-N}^{N} \sigma_{mn} \alpha_n - \alpha_m + j^m j_m (kD) \right\} j^m H_n^{(2)}(kr) \exp(jm\theta)
\]

\[
H_m^{(2)}(kr) = (-1)^m H_m^{(2)}(kr)
\]

\[
\text{but} \quad H_m^{(2)}(z.) = \frac{\sqrt{2}}{\sqrt{\pi z}} \exp[-i(z - \frac{m\pi}{2})]
\]

\[
\alpha \exp[i\frac{\pi}{2}]
\]

\[
\sum_{m=-N}^{N} \left\{ \sum_{n=-N}^{N} \sigma_{mn} \alpha_n - \alpha_m + j^m j_m (kD) \right\} j^{-m} \exp[jm(\theta + \frac{\pi}{2})]
\]

\[\text{c) Calculation of the two dimensional R.C.S.}\]

Consider a plane wave incident upon the origin of a scatterer in the direction $\phi$. This is given by

\[
\exp(jkrcos\phi)
\]
which may be expanded into cylindrical waves.

\[
\exp(\mathrm{j}kr\cos \phi) = J_0(\mathrm{kr}) + 2 \sum_{n=1}^{\infty} j^n J_n(\mathrm{kr}) \cos n\phi
\]

\[
= J_0(\mathrm{kr}) + \sum_{n=1}^{\infty} j^n J_n(\mathrm{kr}) \left[ e^{jn\phi} + e^{-jn\phi} \right]
\]

\[
= \sum_{n=-\infty}^{+\infty} j^n J_n(\mathrm{kr}) \exp(jn\phi)
\]

\[
= \sum_{n=-\infty}^{+\infty} \frac{j^n}{2} \left[ \mathcal{H}_n^{(1)}(\mathrm{kr}) + \mathcal{H}_n^{(2)}(\mathrm{kr}) \right] \exp(jn\phi)
\]

ie the inward travelling field is given by:

\[
\sum_{n=-\infty}^{\infty} \frac{j^n}{2} \mathcal{H}_n^{(1)}(\mathrm{kr}) \exp(jn\phi)
\]

and this equals

\[
\sum_{n=-\infty}^{\infty} \alpha_n j^{-n} \mathcal{H}_n^{(1)}(\mathrm{kr}) \exp(jn\phi)
\]

so that

\[
\alpha_n j^{-n} \exp(jn\theta) = \frac{j^n}{2} \exp(jn\phi)
\]

\[
\alpha_n = \frac{(-1)^n}{2} \exp jn(\phi - \theta)
\]

Now

\[
\beta_\circ = \sum_{n=-N}^{+N} \sigma_{\circ,n} \alpha_n
\]

and

\[
\beta_m = \sum_{n=-N}^{+N} \sigma_{mn} \alpha_n
\]
So that the change in the zeroth mode is

\[ \sum_{n=-N}^{+N} \sigma_{on} \alpha_n - \alpha_o H_0^{(2)}(kr) \]

and in the \( m \) th mode:

\[ \sum_{n=-N}^{+N} \sigma_{mn} \alpha_n - \alpha_m H_m^{(2)}(kr) \exp(jn\theta) \]

So that the total change in the field is:

\[ \sum_{m=-N}^{+N} \sum_{n=-N}^{+N} \sigma_{mn} \alpha_n - \alpha_m H_m^{(2)}(kr) \exp(jm\theta) \]

Substituting for \( \alpha \)

\[ \sum_{m=-N}^{+N} \sum_{n=-N}^{+N} \sigma_{mn} (-1)^{n} \exp[jn(\phi-\theta)] - \frac{(-1)^n}{2} \exp[jm(\phi-\theta)] \]

\[ j^{-m} H_m^{(2)}(kr) \exp(jm\theta) \]

For the monostatic case

\[ \phi = \theta + \pi \]

So that

\[ \text{R.C.S.} \sum_{m=-N}^{+N} \sum_{n=-N}^{+N} (-1)^{n} j^{-m} H_m^{(2)}(kr) \exp(jm\theta) \]

which in the far field becomes:
for the bistatic case the R.C.S. is given by:

\[
\sum_{m=-N}^{+N} \sum_{n=-N}^{+N} \sigma_{mn} (-1)^n \exp[jn(\theta-\phi)] - (-1)^m \exp[jm(\phi-\theta)] \]
\[
\times j^{-m} \exp[jm(\theta + \frac{\pi}{2})]
\]

d) Properties of the scattering matrix

The concept of a scattering matrix was originally introduced by Montgomery, Dicke and Purcell (Reference 3) in terms of incoming and outgoing spherical modes. More generally the fields can be expanded in terms of arbitrary basis functions, for example an elliptic cylinder could be analysed in terms of elliptical functions. However for arbitrary shapes it is most useful to consider two dimensional problems in terms of cylindrical waves and three dimensional problems in terms of spherical waves.

In general if the characteristic fields \( E_n \) are outgoing waves and chosen to be the basis functions then:

\[
E_{\text{out}} = \sum_n \beta_n E_n
\]

Their conjugates are incoming waves so that:

\[
E_{\text{in}} = \sum_n \alpha_n E^*_n
\]
and \[ |\bar{\beta}| = |\sigma| |\bar{\alpha}| \]

When no object is present \( \bar{\beta} = \bar{\alpha} \). This is so because \( E_n + E_n^* \) represents a free space source free field.

Harrington (Reference 4) shows that for perfectly conducting bodies the \( \sigma \) matrix is diagonal. He also shows that these eigenvalues are simply related to the eigenvalues of the characteristic modal current matrix. The eigenvalues or diagonal elements of the \( \sigma \) matrix are given by:

\[
\frac{(1-j\lambda_n^*)}{(1+j\lambda_n^*)}
\]

where \( \lambda_n \) are the real eigenvalues of the modal current matrix.

This relationship provides valuable numerical checks when calculating a particular \( \sigma \) matrix. In the modal current matrix description important modes have \( \lambda_n \) small while for unimportant modes \( \lambda_n \) large. For the \( \sigma \) matrix the eigenvalues should always be unity in magnitude but variable in phase. Modes important in the scattering have eigenvalues near \(-1\) while unimportant modes have eigenvalues close to \(+1\). By finding the eigenvalues of the \( \sigma \) matrix, an important numerical check is available. The difference between the modulus of the eigenvalue and unity indicates the numerical accuracy while the phase of the eigenvalue indicates if all the important modes have been included.

e) Numerical computation

It has been shown that the scattering description of radiation problems can be applied to bodies of arbitrary shape. However when
performed in the manner indicated the method is essentially a numerically
technique and an appreciation of the likely errors is required. To
implement the technique a computer program has been developed to analyse
the behaviour of a two dimensional conducting scatterer. A full descrip-
tion of the program together with the listing and namelists appears in
Appendix A. A simple review of the procedures involved will be given
here so that the important features can be indicated.

The program has been developed with a square conducting object.
This was regarded as a severe test of the method with its four fold
symmetry and sharp corners. However the program developed is suitable
for much more general objects. The geometry of the scatterer is set
up by a short section of one sub-routine and could simply be re-programmed
to give other shapes. In addition the subroutine could be re-written so
that a geometry defined by a set of given points could be evaluated.

When considering different shaped objects the problem is one of
geometry. The object is simply enclosed in the circle of smallest
radius centred at the most suitable origin. The points where the
boundary conditions are to be applied must be selected. This is rather
an arbitrary procedure but it must be remembered that if the symmetry
of the object is to be preserved into the scattering matrix this can
only be done through the points where the boundary conditions are
applied. For the case of the square, the coordinate axes were selected
to coincide with lines of symmetry. The angular variable was then
divided up into equal divisions, as shown in figure 1. The number of
divisions was divisible exactly by four, to preserve the symmetry.
Point on the enclosing circle were then specified by \( r_{\text{min}} \) and the
angular coordinate \( \phi \). From these points perpendiculare to the
Figure 1  The Geometry used to Apply the Boundary Conditions to a Square Conductor
squares were constructed and where these crossed the square the boundary conditions were applied. To complete the geometry the length and direction of $\vec{p}'$ were calculated.

It was found that the scattering matrix could not be found if a point where the boundary conditions was to be applied lay on the enclosing circle. To overcome this numerical problem it was found necessary either to increase the size of the enclosing circle by a very small but finite amount or to slightly rotate the square.

Before the calculation can proceed, the size of the scattering matrix required for accurate calculation must be determined. To do this the properties of the Hankel functions used for the expansion must be considered. Hankel functions are the complex sum of Bessel function of the first and second kind

$$H_{\nu}^{(2)}(z) = J_{\nu}(z) + iY_{\nu}(z)$$

Now for sufficiently large $\nu$, the Bessel function of the first kind is negligibly small for values of $z$ up to about $\nu$. When $z$ approaches $\nu$, the $J$ function increases in amplitude and starts oscillating. Similarly for the Bessel function of the second kind when $z$ is small $Y_{\nu}(z)$ is very large and negative.

As $z$ approaches $\nu$ the function decreases in amplitude and starts oscillating. For sufficiently large $z$ both parts of the Hankel function tend to attenuating sinusoidal form.

Physically with $z$ much larger than $\nu$ the Hankel function represents as propagating cylindrical or spherical wave. When $z$
is much smaller than \( v \) the Hankel function represents a large reactive stored field. In the neighbourhood of \( z \sim v \) this changes to a free space propagating field. It is thus reasonable to suppose that if an incoming cylindrical wave is symmetrically scattered by a cylindrical scatterer of radius \( \rho_o \) wavelengths, the most important change will occur when \( v \sim 2\pi \rho \). This effect was demonstrated in the last report (Reference 1 figure 10). From this the probable size of the required scattering matrix can be calculated. If \( \rho_{\text{min}} \) is the radius of the enclosing circle then modes of the order of \( n \sim 2\pi \rho_{\text{min}} \) are likely to be important. In fact the importance of the modes drops rapidly for larger values of \( n \) so that \( n_{\text{max}} \sim 2\pi \rho_{\text{min}} + 4 \) was initially thought to be sufficient. However since that time it has been indicated from computing experience that the edge elements of the matrix tend to accumulate errors. Currently it is thought desirable to calculate the scattering matrix with \( n_{\text{max}} \sim 2\pi \rho_{\text{min}} + 6 \) but to disregard the edge elements and use the matrix with \( n_{\text{max}} \sim 2\pi \rho_{\text{min}} + 4 \).

For each incident cylindrical wave there are thus \((2n_{\text{max}} + 1)\) complex unknowns to be found, each corresponding to the amplitude of a scattered mode. To determine these unknowns at least \((2n_{\text{max}} + 1)\) different complex equations are found. A different equation is generated at each point where the boundary conditions are applied, but in the case of the square the boundary conditions are best applied a multiple of four or eight times. This means that more equations may be available than unknown, and some sort of least squares fit can be used.

The program that has been developed is not currently capable of making full use of the eigenvalues of the scattering matrix. It is hoped that future development will correct this. A library routine
is used to check the scattering matrix but it gives the eigenvalues in an arbitrary order and has not been used to diagonalise the matrix. Its main value is that it checks the numerical accuracy of the calculated eigenvalues. Manual checking of the phase enables convergence to be established. Experience suggests that the eigenvalues are relatively unaffected by the numerical errors that appear to accumulate around the edges of the scattering matrix. However the scattered field patterns are most sensitive to any errors in the scattering matrix and it is thought that the most correct result will be obtained if the diagonal form of the matrix can be used. This is however a computing rather than an electromagnetic problem.

The error accumulates at the edge of the matrix when the perodicity in the position of the points where the boundary conditions are applied corresponds with that of the functions considered. Small errors can then add in phase. An alternative method of eliminating the error is to apply the boundary conditions at more points.
The extension of the matrix method to three dimensions does not present many conceptual problems but does pose major problems in handling the total number of modes involved. For two dimensional problems a cylindrical wave is completely specified by the order of the Hankel function and the polarisation. The two polarisations can in general be considered separately as no mechanism exists for conversion from one polarisation to the other.

In three dimensions, spherical modes must be specified in terms of multipoles which require two numbers to be designated. In addition the modes can be either electric or magnetic in nature and either even or odd relative to a coordinate axis. For each Hankel function index $n$, there are $(4n + 2)$ spherical modes as opposed to 4 cylindrical modes. Unlike the cylindrical case all the modes have to be considered simultaneously as in three dimensional case it is possible to obtain polarisation conversion in scattering. This means that the required number of modes increases dramatically with the size of the scatterer as shown in figure 2. If for instance 50 modes can be calculated then for two dimensional problems $|n_{\text{max}}|$ is approximately twelve and objects two wavelengths in radius or four wavelengths extent, can be handled. However in three dimensions $|n_{\text{max}}|$ is reduced to four and objects enclosed by a sphere only two thirds of a wavelength in radius can be considered. If the number of modes is greater the discrepancy between the two and three dimensional cases with regard to the maximum size of the scatterer becomes larger. If the number of modes is doubled, in two dimensions, an object of twice the size can be investigated, but in three dimensions the object size can only be increased by fifty percent.

3. APPLICATION TO THREE DIMENSIONAL PROBLEMS
Figure 2: The Total Number of Modes Required for Different Values of $n_{\text{max}}$ ($\sqrt{2\pi \rho_{\text{min}}}$)

The total no of modes

Three dimensional problems

Two dimensional problems
(Both polarisations)

(One polarisation)
It is unlikely that the full three dimensional scattering matrix is very useful. For objects larger than a wavelength in extent the number of modes is likely to be prohibitive. For smaller objects just the complexity of the coordinates transformations, required for the process of analytically continuing the field, makes the full three dimensional matrix difficult to program. A summary of the coordinate transformations required are given in Appendix B. For the smaller objects wire grid modelling is probably the most successful technique.

This does not however mean that the scattering matrix technique is not useful. On the contrary there are several types of problem for which the matrix approach is likely to be most successful. The most important type of problem involves finite bodies with cylindrical symmetry or bodies of revolution. This is relevant to the problem of aircraft antennas as to a first approximation at least aircraft fuselages are cylindrical.

It was shown earlier that the scattering matrix technique was equivalent to Harrington's model current technique. Indeed he has already investigated bodies of revolution with cylindrical excitation (Reference 5). The scattering matrix technique however is more readily adaptable to the problem of a body of revolution with non-cylindrical excitation, for instance from a dipole or slot. The field in each plane must be computed individually but this is the price paid for being able to work with a two dimensional scattering matrix. The prediction of the field radiated by a dipole adjacent to a finite cylindrical object is essentially that of an aircraft antenna near a fuselage. To complete the problem wings and a tail must be added. If infinite wings are added as a ground plane through the cylinder axis
then they act as a mode filter so that certain modes are not excited. In many situations these modes will still not be excited if the wings are mode finite, however the diffraction caused by the wing edges must not be included. This may perhaps be done by G.T.D.

a) The modes excited by a source near a finite cylindrical body

The use of the two dimensional scattering matrix to solve the problem of a source near a finite cylindrical object, involves the use of a dyadic Green's function to predict the incident field. Consider an electric dipole for instance at $P$ as shown in figure 3. The transverse electric field at $Q$ can be expressed in terms of an infinite set of spherical modes relative to $o$ via the dyadic Green's function $\overline{Z}$ (Reference 6) where

$$\overline{Z}(\mathbf{r}, \mathbf{r}') = \frac{j k}{4\pi} \left( \frac{V}{k} \right)^2 \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \sum_i \xi_i \left\{ \begin{array}{ll}
  e_i^{(+)}(\mathbf{r}') e_i^{(1)}(\mathbf{r}) + e_i^{(+)}(\mathbf{r}') e_i^{(1)}(\mathbf{r}), & r<r' \\
  e_i^{(1)}(\mathbf{r}') e_i^{(+)}(\mathbf{r}) + e_i^{(1)}(\mathbf{r}') e_i^{(+)}(\mathbf{r}), & r>r'
\end{array} \right.$$

where

$$r e_i^{(+)}(\mathbf{r}) = \hat{N}_n^{(2)}(kr) e_i^{(0,\phi)} - j \eta Z_s n_i^{(2)}(kr) e_i^{(0,\phi)}$$

$$r e_i^{(+)}(\mathbf{r}) = \hat{N}_n^{(2)}(kr) e_i^{(0,\phi)}$$

$$e_i^{(1)}(\mathbf{r}) = n_i^{(1)}(\mathbf{r}) + e_i^{(+)}(\mathbf{r})$$

$$n_i^{(1)}(\mathbf{r}) = \mathcal{H}_n^{(1)}(kr) e_i^{(0,\phi)} + e_i^{(+)}(\mathbf{r})$$

$$\xi_i = \mathcal{H}_n^{(1)}(kr) e_i^{(0,\phi)} + e_i^{(+)}(\mathbf{r})$$
Figure 3  An Electric Dipole with Arbitrary Position and Orientation
\[ e'_{ri}(\theta, \phi) = r_o \sqrt{n(n+1)} \phi_{nm} \]

\[ z'_{si} = -j \frac{\sqrt{n(n+1)}}{r \omega e} \]

\[ \mathbf{e}'_1 = \mathbf{e}_1 \]

\[ \mathbf{e}'_1 = \mathbf{e}_1 \times r_o \]

and

\[ \mathbf{e}_1(\theta, \phi) = \mathbf{e}_{nm} = -r_t \nabla \phi_{nm} \]

\[ = \frac{1}{N_{nm}} \left( -r \mathbf{P}^m_n(\cos \theta) \sin \phi \right) \]

where

\[ N_{nm} = \frac{4 \pi (n+m)! n(n+1)}{e_{nm}(2n+1)(m-n)!}; \quad e_{nm} = 1, \quad m = 0 \]

\[ e_{nm} = 2, \quad m > 0 \]

\( P^m_n(\cos \theta) \) are associated Legendre polynomials,

\[ t \nabla = \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi} \]

\( \hat{\theta} \) and \( \hat{\phi} \) are unit vectors in the \( \theta \) and \( \phi \) directions.

Using the dyadic Green's function the transverse electric field at \( Q \) is expressed as:

\[ \mathbf{rE}_t = -r_u \overline{Z(r, r')} \mathbf{l} \]
which for \( r<r' \) can be expressed as:

\[
\sqrt{\frac{s}{\pi r_k}} \sum_j \hat{h}_n^{(2)}(kr')e^{i(\theta',\phi')}u_o
\]

\[
= \sum_n n(n+1)H_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o e^{i(\theta',\phi')}.u
\]

\[
= \sum_n \hat{h}_n^{(2)}(kr')e^{i(\theta',\phi')}u_o
\]

To indicate the interpretation of each mode the field for \( n=1 \)

will be fully expanded so that:

\[
x_t = \sqrt{\frac{3\pi}{j \sqrt{r}}}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
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\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]

\[
\frac{1}{J_1(kr)} \frac{\hat{H}_1^{(2)}(kr')e^{i(\theta',\phi')}u_o}{r'k} - \frac{2n\hat{h}_n^{(2)}(kr')\phi_n(\theta',\phi')r_ou_o}{k^2r'^2\omega e} e^{i(\theta',\phi')}
\]
When expanded in this manner, the first three terms can be interpreted as open circuited electric dipoles at 0 oriented along the x, y and z axes respectively. Similarly the last three terms represent magnetic dipoles.

It can be seen from this that if a dipole with arbitrary location and orientation is considered all the modes will be excited at the origin and the situation will be very complex. However in most cases of practical interest the problem is much simpler. If for instance P can be made to lie in a principal plane the number of modes is halved. Similarly if $\mathbf{u}_o$ can be made to coincide with the direction of a principal axis then the number of modes is drastically reduced. Alternatively if only one polarisation is considered at a time the number of modes is reduced. In most practical situations one or all these situations can exist greatly simplifying the original problem. In some cases it will be simpler to resolve the dipole orientation into the coordinate directions and superimpose the results of the three greatly simplified problems.

To indicate how the number of modes is reduced various tables have been prepared indicating under different conditions which modes are excited. For an arbitrary location and orientation all modes are excited. However if the dipole is made z oriented and positioned at $\phi' = \frac{\pi}{2}$ then only the following modes are excited.

For the electric modes, for every allowed combination of $n$ and $m$ one mode is excited with the same symmetry as $m$. For the magnetic modes, for every allowed combination of $n$ and $m$ except $m = 0$ one mode is excited with the opposite symmetry of $m$. 
**Electric Modes** (z oriented dipole at $\phi' = \frac{\pi}{2}$)

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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Excited</td>
<td>Odd</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Excited</td>
<td>Odd</td>
<td>Even</td>
<td></td>
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</tr>
<tr>
<td>3</td>
<td>Excited</td>
<td>Odd</td>
<td>Even</td>
<td>Odd</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Excited</td>
<td>Odd</td>
<td>Even</td>
<td>Odd</td>
<td>Even</td>
</tr>
</tbody>
</table>

**Magnetic Modes**

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>Even</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>Even</td>
<td>Odd</td>
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</table>

The number of modes excited is halved by specifying a dipole orientation corresponding with a coordinate direction. The modes are further drastically reduced in number if the dipole approaches one of the coordinate axes. eg. $\theta' \to 0$.

**The Electric Modes Excited** (z oriented dipole $\phi' = \frac{\pi}{2}$ $\theta' \to 0$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>1</td>
<td>Excited</td>
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<td>4</td>
<td>Excited</td>
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</tr>
</tbody>
</table>

No magnetic modes are excited
For a z oriented dipole approaching the z axis there is one mode excited at the origin for each value of n. This represents a major reduction in the required number of modes to less than the number required for an arbitrary two dimensional object. This case is also relevant to the prediction of the radiation patterns of H.F. aircraft antennas. For an aircraft with a long thin fuselage and a tail mounted aerial these conditions apply and may be used to predict the effect of the fuselage on the radiation pattern.

If the dipole approaches the other axis \( (\theta' \rightarrow \pi/2) \) then a different set of modes are selected:

**Electric Modes** \((z \text{ oriented dipole } \phi' = \pi/2, \theta' = \pi/2)\)

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>1</td>
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</table>

**Magnetic Modes**

<table>
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<tr>
<th>m</th>
<th>n</th>
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</table>
For a $y$ oriented dipole at a general position $\theta'$, $\phi' = \frac{\pi}{2}$, the same modes are excited as for a $z$ oriented dipole. However for the extreme positions different modes remain.

**Electric Modes (y oriented dipole at $\theta' = 0$, $\phi' = \frac{\pi}{2}$)**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
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**Magnetic Modes**

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</tbody>
</table>

For the other extreme case when $\theta'$ becomes $\frac{\pi}{2}$.

**Electric Modes (y oriented dipole at $\theta' = \frac{\pi}{2}$, $\phi' = \frac{\pi}{2}$)**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
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</tbody>
</table>

No magnetic modes are excited
This is also a case relevant to an aircraft and could be used to investigate the effect of the fuselage on a wing root slot antenna.

It has been shown that for specific geometrics, some of which are relevant to aircraft antenna problems, a great reduction can be made in the number of modes that have to be considered. The reduction is summarised for certain cases in figure 4.

b) The calculation of the radiation pattern of a source near a finite cylindrical scatterer

Once the modes excited by the source at the origin of the cylindrical object have been calculated one can proceed to the calculation of radiation patterns. Here the assumption of a cylindrical scatterer or quasi-cylindrical scatterer is most important. The point is that if a mode is incident upon the origin then the energy incident in a particular \( \phi \) plane is scattered in that plane. This means that the problem can essentially be reduced to two dimensions and the two dimensional matrix used.

However each \( \phi \) plane must be considered separately. This is not necessarily a disadvantage because if means that the number of incident modes may be further reduced; in the principal planes some of the modes have zero intensities. The number of modes may also be reduced if the two principal polarisations are considered separately, polarisation conversion is not likely to be an important mechanism for cylindrical objects, symmetrically excited.

It is also important to realise that different approximations might be required in different planes. An important example of this occurs for the tail mounted aircraft antenna and scattering from the
Figure 4 The Total Number of Modes Required for Various Three Dimensional Situations as a Function of $|n_{\text{max}}|$ ($\sim 2\pi r_{\text{min}}$)
In the yaw plane the assumption that the antenna approaches the fuselage axis is likely to give accurate results, however this assumption may lead to errors in predicting the pitch plane radiation patterns.

The analysis to give, the modification to the radiated field, caused by the scatterer, is in essence identical to the two dimensional case presented earlier. The only important difference is that the field is given in terms of SPHERICAL Hankel functions rather than cylindrical Hankel functions. This difference is necessary to give the correct radial distribution.

In applying the method as indicated to the problem of aircraft antennas it is necessary to remember that for the scattering matrix method to apply, the source must be outside the sphere enclosing the scatterer. If the analogy with the impedance matrices is to be carried further the enclosing sphere is the "reference plane". This is not so for most aircraft antenna but does not mean that the method can not be used, because the original $\beta_{in}$ matrix is still valid. However if calculations are to be made using this matrix it is recommended that the scattering matrix and its eigenvalues should be calculated as a numerical check of convergence. The field of the dipole must be expanded to give the incident electric field at the scatterer.
4. AN AIRCRAFT MODEL FOR RADIATION PATTERN CALCULATIONS

The method that has been developed in this work appears to be most useful in determining the scattering from arbitrary bodies of revolution in the resonant range. The main interest in this frequency region is in the prediction of radiation from antennas on aircraft. In most cases the fuselages are approximately cylindrical but the other sections such as the wings and tail are not. However in other antenna problems an equivalent cylindrical structure is used to replace the physical structure. The most common example of this occurs for flat strip or elliptical dipoles. Much data, with regard to impedances and current distributions, exists for dipoles of circular cross-sections but not for other cross-sections. If an equivalent circular dipole can be used all the required information is available. Fortunately such an equivalence has been established and appears in King's book (Reference 7). A flat strip dipole of width $a$ is replaced by a circular dipole of the same length but of diameter $a/2$.

The wings of many aircraft can to at least a first order approximation be regarded as flat strip dipoles and by this equivalence may be regarded as circular dipoles. A scattering model of an aircraft thus consists of a cylinder with cylindrical wings and tail as indicated in figure 5.
Figure 5 A Possible Representation of an Aircraft in Terms of Cylinders
The results so far obtained from the computer program have been encouraging but continuing development is required before new problems can be tackled. The changes needed will now be listed in approximately the correct order.

1) Careful consideration needs to be given to the computer which is to be used. Currently the program runs on Surrey Universities ICL machine. This computer was originally used because of its location so that a good service would be obtained. However if the program is made any larger, although still being a relatively small job, this advantage will disappear. It might then be more advantageous to use a different machine.

2) The size of the matrices that are used needs to be increased if really meaningful results are to be obtained. Currently n runs from -7 to +7 and this allows calculations to be made on nearly circular cylinders enclosed by a circle of a little over half a wavelength. However when symmetries are imposed on the body many terms in the matrix become zero and more higher order terms are required. The size of the matrices needed requires increasing to cope with the symmetries and later to allow for larger bodies. It is envisaged eventually that the program will cope with bodies four wavelengths in length. These will require an enclosing circle two wavelengths in radius so that important terms will have $|n|$ approximately twelve. For this situation 35 x 35 matrices will be required. However it is not proposed at this stage that the program should be made that large, and probably an intermediate size of 21 x 21 or 25 x 25 is sufficient until the program is fully developed.
3) If the number of elements in the matrix is increased the number of points where the boundary conditions apply must also be increased. However unless there is a lot of redundant information errors accumulate at the edges of the scattering matrix. Currently the 15 x 15 matrix is calculated using 16 points. The periodicity of the outer elements of the matrix is similar to that of the boundary conditions points and small errors at each point add in phase. The only way to combat these errors is to increase the number of points where the boundary conditions are applied making the set of equations even more over determined. Probably at least 20 points should be used to determine the 15 x 15 matrix and possibly as many as 24.

4) One problem which is specifically associated with the computing concerns the diagonalisation of the scattering matrix. Currently the eigenvalues of the matrix are calculated, but in a random order. This means that if the diagonal form is to be used the eigenvalues must be sorted into the correct order. This could be done using the eigenvectors but it is thought that other library routines probably exist for diagonalisation of matrices. The diagonal form will save much time in the calculation of the field parameters particularly in three dimensions when each plane is calculated separately.

5) In order that the program can be rapidly used for any shaped two dimensional object it is desirable that the program can automatically check that all the required terms have been included. It would be best if the size of the matrix could be reset but this is not essential if a warning is printed. The order of the Bessel functions |n| must exceed $2\pi \times$ the radius of the enclosing circle of wavelengths. The other test concerns the phase of the eigenvalues. This is basically a modification
method and the change in each mode is proportional to unity minus the eigenvalue. Thus the result has converged when the phase angle of the eigenvalue tends to $0^\circ$.

6) So far the program is essentially two dimensional. The scattering part of the problem is simply made three dimensional by using spherical rather than cylindrical Bessel Hankel functions. It is thought that this change only requires an index in the call statements to be changed.

7) The last change enabled the radiation patterns to be calculated for bodies of revolution with a dipole source on the axis and oriented at right angles to it. The pattern in the plane in which the dipole could be regarded as a line source is calculated only. For other planes the dyadic Green's function must be used, to calculate the field incident upon the scatterer. Initially it is proposed that only the principal plane patterns be calculated and the required modes can be obtained from the tables contained elsewhere in this report. The tables can be summarised so that for each value of $m$ and $n$ and each situation an algorithm can be used to choose the required modes.

Note: At this point it should be possible to make comparisons with the experimental measurements being made by the Royal Aircraft Establishment on a finite cylinder excited by a slot. It may be possible by superimposing solutions to include wings at right angles to the fuselage axis.

More developments of the program may then be pursued:

8) The program may then be modified to predict non-principal plane patterns. In general these are weighted sums of the principal plane results. This will allow the effects of swept back equivalent
cylindrical wings to be investigated by superimposing solutions assuming no coupling between cylinders.

9) More complicated structures may then also be considered, made up of many cylinders. Unfortunately before this point is reached it is likely that the effects of coupling between the cylinders will have to be considered. When this happens the scattering matrix which has been used so far becomes a sub-matrix of a more general scattering matrix.
6. POSSIBLE FURTHER DEVELOPMENTS

In this work it has been shown that the radiation patterns and scattering properties associated with objects of arbitrary cross-section can be calculated. The object actually considered was of square cross-section and this provided one of the most severe tests of the method. The four fold symmetry of the square compelled three quarters of the elements of the $S$ matrix to be zero. This meant that in applying the boundary conditions only a quarter of the degrees of freedom were left and the fit among the remaining variables was more difficult. The extension of the method to the scattering of three dimensional bodies of rotation has been shown and only requires the development of suitable computer programs. In principle it should be possible to get routines to automatically sort out the required modes.

The inclusion of infinite wings is essentially a trivial problem particularly if their plane passes through the cylinder axis. The wings will then simply act as mode filters again reducing the total number of modes required. The inclusion of finite wings presents a more difficult problem. There are various ways in which this might be coped with. One method would be to regard the modes filtered by infinite wings as being scattered by the wing edges and fuselage profile. Some method of allowing for coupling to other $\phi$ planes might be required. More conventional techniques such as G.T.D. or the use of a wing shape edge current might also be fruitful.

It is also interesting to speculate on the possible future developments of the method. In pursuing this work to the present position the author found the early work of Wasylkiwskyj and Kahn of great value. They were mainly interested in the mutual impedance
between antennas with scattering properties that could be readily expressed in terms of modal matrices. They were initially interested in dipoles which respond to only one mode. The author's development of this work followed their important arguments with one antenna replaced by a more general scatterer. In trying to predict the further development of this work it is interesting to look at the more recent papers of these two authors (References 8 & 9). Like this work theirs depends largely on the use of suitable coordinate transformations. In their more recent work they take the Fourier transform of the field and obtain a different set of coordinate transformations. This is essentially an angular spectrum of plane waves. They manage to identify the model coefficient and obtain the transformed field as an integral of plane waves, rather than a sum of modes. This is an important step as it enables arbitrary antennas to be used, ones for which the model coefficients can not easily be found. The details of the method are presented in Appendix C for completeness.

The other essential feature of this method is that at high frequencies the plane wave spectrum approach naturally becomes the asymptotic method generally known as the Geometrical Theory of Diffraction. In this manner the modal scattering description of the resonant frequency range naturally turns into the high frequency optical technique.
APPENDIX A

The Computer Program

This appendix is a description of a computer program developed at the University of Surrey to investigate the scattering from conducting objects. The program calculates the scattering matrix of the object and then uses it to calculate the monostatic radar cross-section. A line source is then introduced at various positions and the radiation patterns are calculated.

Results from the program are not included in this report because as yet sufficiently large matrices have not been used. The behaviour of the monostatic radar cross-section of a small square object indicates that as $|N|$ increases to five a substantially correct result is obtained. Small errors of sixth order remain. Currently however when the scattering matrix is calculated with an $|N|$ of six, large errors occur and the result is dominated by the error giving a $\cos6\theta$ pattern. This error will be removed by increasing the number of points where the boundary conditions are applied.

There now follows a brief description of each routine.
1) The Main Routine

The main routine transfers control between the subroutines. It sets up the principal matrix equations calls the manipulating routines and then reads the values into the scattering matrix. It then calls the routines to evaluate the radiated fields.
**Namelist Main Routine**

- **JJ**: An array used to store the values of Bessel functions of the first type.
- **Y**: An array used to store the values of Bessel functions of the second type.
- **RB**: An array used to store values prior to printing.
- **THS**: An array used to store the angular coordinate of the points where the boundary conditions are applied.
- **RIS**: An array used to store the radial coordinate of the points where the boundary conditions are applied.
- **AR**: A real array containing the elements of the $\beta$ matrix partitioned into real and imaginary components.
- **AAR**: An array used to store the results of the least square solution of the AR array.
- **BR**: A real array containing the values of the incident field partitioned into real and imaginary components.
- **QR**: An array required by the Householder routine in F04AMF.
- **ALPHA**: An array required by F04AMF.
- **WKSP**: One dimensional arrays used by F04AMF as workspace.
- **WY**
- **WZ**
- **WR**
AA  The transposed scattering matrix.
S   The scattering matrix.
CI  The square root of minus one.
CS  A complex dummy variable.
A   Half the length of a side of the square.
PI  \pi
SQ  The square root of two.
RM  The minimum radius circle to enclose the square.
NMAX The number of points of which the boundary conditions are applied less one.
NMP The number of points at which the boundary conditions are applied.
MMAX The order of the highest mode considered.
MM  The total number of modes included (Odd).
MMP MMAX plus one.
I   Dummy variables used in DO loops.
J   M   MM
IM  Number of complex equations used to solve for x.
IPIV   An array required by FO4AMF.
SR     Contains the real part of the S matrix for entry to FO2AKF.
SI     Contains the imaginary part of the S matrix for entry to FO2AKF.
WWR    Contains the real part of the eigenvalues on exit from FO2AKF.
WI     Contains the imaginary part of the eigenvalues on exit from FO2AKF.
VR     Contains the real part of the eigenvectors on exit from FO2AKF.
VI     Contains the imaginary part of the eigenvectors on exit from FO2AKF.
INT    A one dimensional integer array used as a work area by FO2AKF.
K      The normalised wavenumber.
BIN    The Beta matrix which is inverted to give the scattering matrix.
HI     An array used to store the values of Hankel functions of the first type.
E      An array used to store the values of the incident field at the points where the boundary conditions are applied.
JM Complex unknowns.

ETA Sets the accuracy of F04AMF.

IFAIL Used to detect failure in the NAG subroutines.

IM2 Twice the number of complex equations.

JM2 Twice the number of complex unknowns.
The Flow Chart of the MAIN ROUTINE

START

SET DIMENSIONS OF ALL ARRAYS
SET A, PI, SQ, RM, K, RMK,
NMAX, NMP

WRITE NMAX, MMAX, SQUARE SIZE,
MINIMUM ENCLOSING RADIUS

CALL SUBROUTINE ABIN

WRITE BIN

CALL SUBROUTINE BESSEL

CALCULATE AND WRITE INCIDENT
FIELD MATRIX

PARTITION MATRIX EQUATION

CALL LIBRARY ROUTINE
FO4AMF

LIBRARY ROUTINE
FO4AMF
ESTABLISH AA MATRIX (COMPLEX) 
TRANSPOSE TO GIVE S MATRIX 

CALL LIBRARY ROUTINE F02AKF 

LIBRARY ROUTINE F02AKF 

WRITE EIGENVALUES OF S MATRIX 

CALL SUBROUTINE MONOSTAT 

SUBROUTINE MONOSTAT 

CALL SUBROUTINE PAT SCAT 

SUBROUTINE PAT SCAT 

STOP
The Listing of the Main Routine of the Program

MASTER JOAN

C SCATTERING FROM A SQUARE CONDUCTING OBJECT

REAL JJ(101)
DIMENSION R8(20), TH<(20), RIS(20), Y(90)
DIMENSION WKP(33)
DIMENSION AA(33,33), RR(33,33), ALPHAS(33)
DIMENSION WY(33), WR(33), IDIV(33)
DIMENSION SR(20,20), SI(20,20), WR(20,20), WI(20,20), VU(20,20), WI(20,20), VU(20,20)

DIMENSION IWT(20)
REAL K
COMPLEX B1N(20,20)
COMPLEX H1(20,20), E(20,20), AA(20,20)
COMPLEX S(20,20)
COMPLEX CI
COMPLEX CS
COMMON//, A, PI
A=0.2
PI=3.1415926536
SQ=SQR(2.)
RM=SQR(A)
K=2.*PI
RMK=K*RM
NMAX=15
NMP=NMAX+1
WRITE (3, 99) NMAX
FORMAT (1X, 5HMMAYS, J?)

IF (NMP .GE. 20) GOTO 100
MMAX=7
MMM=2*MMAX+1
MMP=MMAX+1
WRITE (3, 96) MMAX
FORMAT (1X, '5HMMAX=, 12)
WRITE (3, 310) RM
FORMAT (1X, 13HSQUARE SIZE =, E6.2)
WRITE (3, 330) A
WRITE (3, 330) RM
CALL ABIM(RIS, THS, B1N, MMAX, NMP)

WRITE THE BIN MATRIX

WRITE (3, 320)
FORMAT (1X, 31HTHE REAL PART OF THE BIN MATRIX)
WRITE (3, 320)
DO 80 I=1, NMP
DO 81 J=1, MMM
RB(J)=REAL(B1N(I, J))
WRITE (3, 320) (RB(J), J=1, MMM)
CONTINUE

WRITE (3, 330) (RB(J), J=1, MMM)

WRITE (3, 340) (RB(J), J=1, MMM)
CONTINUE

WRITE (3, 330) (RB(J), J=1, MMM)
CONTINUE

C END OF WRITING THE BIN MATRIX
C \text{I} = \text{CMPLX}(0.0, 1.0) \\
\text{MAXM} = \text{MMAX} + 1 \\
\text{DO 67} \text{ I} = 1, \text{NMP} \\
\text{CALL BESSEL(K*RI(S(I)), MAXM, JJ, Y)} \\
\text{DO 68} \text{ M} = 1, 2*\text{MMAX} + 1 \\
\text{MM} = \text{M} - \text{MMAX} - 1 \\
\text{IF} (\text{MM}, \text{LT}, 0) \text{ GO TO 110} \\
\text{H1} (\text{I, M}) = \text{CMPLX}(\text{JJ} (\text{MM} + 1), \text{Y} (\text{MM} + 1)) \\
\text{GO TO 68} \\
\text{110} \text{ MM} = \text{IABS} (\text{MM}) \\
\text{H1} (\text{I, M}) = ((-1)^{\text{MM}}) * \text{CMPLX} (\text{JJ} (\text{MM} + 1), \text{Y} (\text{MM} + 1)) \\
\text{68} \text{ CONTINUE} \\
\text{69} \text{ CONTINUE} \\
\text{DO 70} \text{ MM} = 1, 2*\text{MMAX} + 1 \\
\text{MM} = \text{MM} - \text{MMAX} - 1 \\
\text{DO 71} \text{ I} = 1, \text{NMP} \\
\text{CS} = \text{CEXP} (\text{CMPLX}(0.0, \text{M} \times \text{THS(I)})) \\
\text{F} (\text{I, MM}) = -(\text{H1} (\text{I, MM}) \times \text{CS}) / (\text{CI} \times \text{MM}) \\
\text{71} \text{ CONTINUE} \\
\text{70} \text{ CONTINUE} \\
\text{C WRITE THE E MATRIX} \\
\text{WRITE (3, 97)} \\
\text{WRITE (3, 340)} \\
\text{340 FORMAT} (1X, 35H THE REAL PART OF THE INCIDENT FIELD) \\
\text{WRITE (3, 97)} \\
\text{DO 111} \text{ I} = 1, \text{NMP} \\
\text{DO 120} \text{ J} = 1, \text{MMM} \\
\text{RB} (\text{J}) = \text{REAL} (E (\text{I, J})) \\
\text{120} \text{ CONTINUE} \\
\text{WRITE (3, 98)} (\text{RB} (\text{J}), \text{J} = 1, \text{MMM}) \\
\text{111} \text{ CONTINUE} \\
\text{WRITE (3, 97)} \\
\text{WRITE (3, 350)} \\
\text{WRITE (3, 97)} \\
\text{350 FORMAT} (1X, 40H THE IMAGINARY PART OF THE INCIDENT FIELD) \\
\text{DO 130} \text{ I} = 1, \text{NMP} \\
\text{DO 140} \text{ J} = 1, \text{MMM} \\
\text{RB} (\text{J}) = \text{AIMAG} (E (\text{I, J})) \\
\text{140} \text{ CONTINUE} \\
\text{WRITE (3, 98)} (\text{RB} (\text{J}), \text{J} = 1, \text{MMM}) \\
\text{130} \text{ CONTINUE} \\
\text{WRITE (3, 97)} \\
\text{C END OF WRITING E MATRIX}
**MOD TO SOLVE IM COMPLEX EQUATIONS IN IM unknowns (IM .GT. JM) BY
C LEAST SQUARES METHOD (MMM is the no. of right-hand sides)**

IM = 16
JM = 2 * MM + 1
DO 50 I = 1, IM
DO 51 J = 1, JM
AR(I, J) = REAL(BIN(I, J))
AR(I*J) = IMAG(BIN(I, J))
AR(I, JM+J) = -AR(I*J)
AR(I+1, JM+J) = AR(I, J)
51 CONTINUE
50 CONTINUE

IM = 1
JM = 2 * MM + 1
DO 52 MM = 1, MMM
DO 53 J = 1, IM
AR(I, MM) = REAL(E(I, MM))
AR(I+1, MM) = IMAG(E(I, MM))
53 CONTINUE
52 CONTINUE

FAI = 2. ** (-37)
IFAIL = 0
IM = 2 * IM
JM = 2 * JM
CALL F04AMF(AR, 33, AAR, 33, BR, 33, IM, MM, ETA, QR, 53, ALPHA,
WKSP, WY, WZ, WR, PIV, IFAIL)
IF (IFAIL .EQ. 0) GOTO 55
WRITE(3, 59) IFAIL
STOP
59 FORMAT(/ / / / / 1X, 8H IFAIL=, I1 )
55 CONTINUE

DDD = 1, MMM
AA(J, MM) = CMPLX(AAR(J, MM), AAR(JM+1, MM))
57 CONTINUE
56 CONTINUE

***END OF LEAST SQUARES MOD***

WRITE(3, 95)
95 FORMAT(/ / / / 1X, 8H MATRIX,///)
DO 201 I = 1, MMM
DO 202 J = 1, MMM
IF (CABS(AA(I, J)) .GT. 0.000001) GO TO 202
AA(I, J) = CMPLX(0.0, 0.0)
202 CONTINUE
201 CONTINUE

A IS TRANSPOSED TO GIVE S MATRIX

DO 85 J = 1, MMM
DO 86 I = 1, MMM
S(I, J) = AA(J, I)
RB(I) = REAL(AA(I, J))
SR(I, J) = REAL(S(I, J))
SI(I, J) = IMAG(S(I, J))
84 CONTINUE
85 CONTINUE

WRITE(3, 98)(RB(I), I = 1, JM)
88 CONTINUE
87 CONTINUE

WRITE(3, 97)
DO 87 J = 1, MMM
DO 88 I = 1, MMM
RB(I) = IMAG(AA(I, J))
88 CONTINUE
87 CONTINUE

WRITE(3, 98)(RB(I), I = 1, JM)
C    CALC. THE EIGENVALUES OF S

WRITE(3,499)
! IFAIL=0
CALL F02AKF(SR,20,SI,20,MMI,WWR,VI,VR,20,VI,20,INT,IFAIL)
IF (IFAIL .EQ. 0) GOTO44
WRITE(3,498) IFAIL
STOP
!
FORMAT(///1X,7I1)
CONTINUE
WRITE(3,47)(WWR(I),VI(I),I=1,MMI)
FORMAT(10(/1X,2E13.5))
C END OF CALCULATING EIGENVALUES

98 FORMAT(1X,10E12.5)
97 FORMAT(///)
C CALL THE VARIOUS SUBROUTINES TO CALCULATE THE SCATTERED FIELD

MMAX=MMAX-2
CALL NONOSTAT(MMAX,S)
CALL PATSCAT(MMAX,S)
100 CONTINUE
STOP
END
2) **Subroutine ABIN**

This routine establishes the geometry of the square object and evaluates the Beta matrix used for analytic continuation.
### Additional Namelist Subroutine ABIN

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>H2</td>
<td>The Hankel function of the second type.</td>
</tr>
<tr>
<td>BN</td>
<td>A dummy variable for the elements of the β array.</td>
</tr>
<tr>
<td>CJ</td>
<td>The square root of minus one raised to the power N.</td>
</tr>
<tr>
<td>J</td>
<td>Arrays used to store Bessel functions of the first kind.</td>
</tr>
<tr>
<td>JD</td>
<td></td>
</tr>
<tr>
<td>RTW</td>
<td>The square root of two divided by two.</td>
</tr>
<tr>
<td>RDMX</td>
<td>The maximum radial distance between the square and its smallest enclosing circle.</td>
</tr>
<tr>
<td>PHI</td>
<td>The angular coordinate of points on the smallest enclosing circle.</td>
</tr>
<tr>
<td>P2</td>
<td>π/2</td>
</tr>
<tr>
<td>P4</td>
<td>π/4</td>
</tr>
<tr>
<td>RD</td>
<td>The minimum distance between a point on the enclosing circle and the square.</td>
</tr>
<tr>
<td>PD</td>
<td>Angular direction of point on the square from the point on the circle RD away.</td>
</tr>
<tr>
<td>XIC</td>
<td>The x, y coordinates of a point on the circle.</td>
</tr>
<tr>
<td>YIC</td>
<td></td>
</tr>
<tr>
<td>XIS</td>
<td>The x, y coordinates of the point on the square.</td>
</tr>
<tr>
<td>YIS</td>
<td></td>
</tr>
</tbody>
</table>
Variables used to ensure the correct sign for Bessel functions of negative order.
The Flow Chart of SUBROUTINE ABIN

INPUT MMAX, NMP

SET ARRAY DIMENSIONS
SET VARIABLES: CI, SQ, RTW,
RM, RDMX

SET I = I + IMAX

SET PHI, RD, PD, XIC, YIC, XIS, YIS

IF PHI IS NOT IN THE FIRST
QUADRANT RESET RD, PD, XIS, YIS

SET THS(I), RIS(I)

CALL SUBROUTINE
BESSEL

CALL SUBROUTINE
BESJN

SUBROUTINE
BESSEL

SUBROUTINE
BESJN
31 SET NN = 1
   → 2*MMAX+1

32 SET N
   & EN = 0

SET M, JM, JMM,
   JC, JR & H2

BN = BN+CJ*H2*RJ*CEXP
   (CI*(N-M)*PHI+M*PD))

NO

JM = 2*MMAX+1?

YES

SET BIN(I, NN)

NO

NN = 2*MMAX+1?

YES

NO

I = IMAX?

YES

RETURN
C THIS SUBROUTINE SETS UP THE TRI MATRIX

REAL K
COMMON/ K1, K2, P1
COMPLEX, DIMENSION(Z, 2), W, G, C, D
COMMON/ D1, D2

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

IF (D1, GT, 1.0, AND, D1, IF, PI) GO TO 42
IF (D1, GT, 1.0, AND, D1, IF, 5. * P2) GO TO 43
IF (D1, GT, 1.0, AND, D1, IF, 2. * PI) GO TO 44
GO TO 40

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

ON (1, D1)
S0 = D011(D2)
RT = 1.5 D0
X = D0

CONTINUE

I0 = 1
J0 = 1
GO TO 1

IF (M, GT, 1.0, AND, M, IF, PI) GO TO 52

CONTINUE

GO TO 1
N.A.G. Routines F04AMF and F02AKF

These are two routines used by the main program. They are part of the Nottingham Algorithms Group Library and are implemented on the University of Surrey's computer.

**F04AMF**

Gives the accurate least squares solution of an overdetermined set of m linear equations in n unknowns, m ≥ n with multiple right hand sides AX = B.

The m x n matrix A is reduced to upper right triangular form, R, by applying Householder transformations, Q, with pivoting so that QA = R. The right hand side matrix B is transformed into the matrix C by applying the same transformation Q, QB = C, and an approximate solution X is calculated by back substitution in the equation RX = C. The residual matrix P = B - AX is calculated and a correction D to X is found by solving the linear least squares problem AD = P, ie. RX = QP. X is replaced by X + P and the correction process is repeated until D becomes negligible. Double precision accumulation of innerproducts are used throughout the calculation.

**F02AKF**

Calculates all the eigenvalues and eigenvectors of a complex matrix.

The complex matrix is reduced to upper Hessenberg form by stabilised elementary similarity transformations. The eigenvalues and eigenvectors of the Hessenberg matrix are calculated using the LR algorithm. The eigenvectors of the Hessenberg matrix are multiplied
by the transformation matrix to give the eigenvectors of the original matrix.
Subroutines BESSEL, BESJN and GAMMA

These are a set of subroutines which were developed within the Physics Department of the University of Surrey. They are used for calculating Bessel functions.
The Listing of the Subroutine BESSEL

SUBROUTINE BESSEL(X,M,J,Y)
REAL J(101), Y(90)
CALL F4JOV0ACC(X, VALUE,Y(1))
CALL F4J1Y1ACC(X, VALUE,Y(2))
DO 71 I=3,M+1
    V(I)=2.*(I-2)*Y(I-1)/X-Y(I-2)
    CONTINUE
71 - CALL BESJN(X,0.,M,8,J)
RETURN
END
The Listing of the Subroutine BESJN

SUBROUTINE BESJN(X, A, NNN, D, J)
DIMENSION RR(101)
REAL L, LAMDA, M1, N1, J(101), J1(101)
INTEGER D
LOGICAL LL
IF ((A .LT. 0.) .OR. (A .GE. 1.) .OR. 
1. (X .LT. 0.) .OR. (IABS(NNN) .GT. 100)) GO TO 6666
IF (X .EQ. 0.) GO TO 999.
nmax = nnn
ll = (nmax .GE. 0)
IF (ll) GOTO 10
IF (A .EQ. 0.) nnmax = 1
nmax = nmax + 1
10

fps = .5 * (10. ** (-D))
d1 = 2.3025 * FLOAT(D) + 1.3863
sum = ((X/2.)**A)/GAMMA(1.+ A)
k = 1
t2 = 0.
nmax = nmax + 1
do 11
11
j1(k) = 0.0
if (nmax .EQ. 0.) goto 50
v = (d1 * 0.5)/FLOAT(nmax)
20 IF (V .GT. 10.) GOTO 30
p = y * 5.77941E-5 - 1.76148E-3
p = y * 5.77941E-5 - 1.76148E-3
p = y * 2.08645E-2
p = y * 1.29013E-1
p = y * 1.29013E-1
p = y * 8.57770E-1
p = y * 8.57770E-1
p = y * 1.01250E+0
20 GOTO 40
40 y = ALOG(V) - 0.775
p = (0.775 - ALOG(Q))/(1.+ D)
p = 1./(1.+ p)
t2 = y * p/q
60 IF (KK - 2) 50, 60, 60
50 t1 = t2 * FLOAT(NMAX)
v = (d1 * 0.73576)/X
kk = kk + 1
GOTO 20
60 t2 = 1.3591 * k * t2
if (t1 - t2) 61, 61, 62
61 nu = (1.+ t2)
GOTO 70
62 nu = (1.+ t1)
70 l = 1.0
4 = 0
lim = nu/2
80 m = m + 1
m1 = FLOAT(m)
l = l * ((m1 + A)/(m1 + 1.0))
1f (m = lim) 80, 81, 81
r1 n = m + 1

The Listing of the Function GAMMA

FUNCTION GAMMA(X)  
DIMENSION C(13)  
DATA C  
1/ 0.00053 94989 '58808, 0.00261 93072 82746, 0.02044 96308 23597,  
2 0.07309 48364 14370, 0.2796 4 3915 78538, 0.55338 75923 85752,  
3 0.99999 99999 99999, -0.00083 27247 08684, 0.00463 86580 77922,  
4 0.02252 38347 47260, -0.17044 79328 74746, -0.05681 03350 85194,  
5 1.13060 33572 86565  
Z=X  
IF(X .GT. 0.0) GO TO 1  
IF(X .EQ. AINT(X)) GO TO 5  
Z=1.0/Z  
IF(Z .LE. 1.0) GO TO 4  
F=1.0  
IF(Z .LT. 2.0) GO TO 3  
F=F*Z  
GO TO 2  

3 Z=Z-1.0  
4 GAMMA=  
1 F*((((((C(1)Z+C(2))Z+C(3))Z+C(4))Z+C(5))Z+C(6))Z+C(7))/  
2 ((((((C(8)Z+C(9))Z+C(10))Z+C(11))Z+C(12))Z+C(13))Z+1.0)  
IF(X .GT. 0.0) RETURN  
GAMMA=3.141592653589793/(SIN(3.141592653589793*X)*GAMMA)  
RETURN  
5 GAMMA=0.  
WRITE(2,10) X  
RETURN  
10 FORMAT(1X,GAMMA ... ARGUMENT IS NON-POSITIVE INTEGER = 10E20.5)  
END
The Subroutine MONOSTAT

This subroutine evaluates the monostatic R.C.S. of the scatterer investigated using the relationship that:

The Monostatic R.C.S. $\alpha$

$$\sum_{m=-N}^{+N} \left[ \sum_{n=-N}^{+N} \sigma_{mn} \right] j^{-m} \exp\left[ jm(\theta + \frac{\pi}{2}) \right]$$

The result is normalised relative to the value obtained when $\theta = 0$. 
Additional Namelist for Subroutine MONOSTAT

SC  An array used for \[ \sum_{n=-N}^{N} \sigma_{mn} \cdot l \]

PS  The normalizing constant.

SS  The radar cross-section.

PAD The amplitude of the radar cross-section.

NX  The number of rows of columns of the S matrix which are disregarded.
The Flow Chart of Subroutine MONOSTAT

INPUT PARAMETERS
MMAX, S

WRITE OUTPUT HEADINGS

SET I = 1, 2*MMAX+1

SET M & SC(I) = -1^m

SET K = 1, 2*MMAX+1

SET N INCREASE SC(I)

K = 2*MMAX+1?

NO

I = 2*MMAX+1?

YES

I = 2*MMAX+1?

NO

INCREASE PS

I = 2*MMAX+1

YES
SET I = 1, 2\times \text{MMAX} + 1

SET N INCREASE SS

\text{I = 2\times \text{MMAX} + 1?}

WRITE ANGLE & RADAR CROSS-SECTION

\text{L = 46?}

\text{NO}

\text{RETURN}

\text{YES}
The Listing of Subroutine MONOSTAT

SUBROUTINE MONOSTAT(MX,S)

THIS SUBROUTINE CALCULATES THE MONOSTATIC R.C.S. OF THE SCATTERER

COMPLEX S(20,20),SC(20)
COMPLEX CI,PS,SS
MX=1
MMAX=MX-MX
* PI=3.14159265358
* P2=PI/2.0
* CI=COMPX(0.0,1.0)
* DO 10 I=1,2*MMAX+1
* M=I-1
* SC(I)=-(COMPX(-1.0,0.0))*M
* DO 20 K=1,2*MMAX+1
* S=K-1
* SC(I)=SC(I)*((-1.0)**K)*S(I+NX,K+NX)
* CONTINUE
10 CONTINUE
* WRITE (5,101)
* WRITE (5,111)
* WRITE (5,101)
* PS=COMPX(0.0,0.0)
* DO 50 I=1,2*MMAX+1
* PS=PS+SC(I)
* CONTINUE
50 CONTINUE
* PS=PS*TEST(PS)
* DO 60 L=1,45
* TH=2.0*(FI04(L)-1.0)
* THR=PI*TH/180.0
* PH=THR+P2
* SS=COMPX(0.0,0.0)
* DO 60 I=1,2*MMAX+1
* S=(-1.0)*M
* SS=SS+SC(I)*EXP(COMPX(0.0,0)**PH)*CI*(-M)
* CONTINUE
60 CONTINUE
* WRITE (5,121) TH,PS
* WRITE (5,121) TH,SS
* WRITE (5,121) TH,PS
* WRITE (5,121) TH,SS
* CONTINUE
100 FORMAT (/)
110 FORMAT (1X,2E12.6)
120 FORMAT (1X,2E12.6)
RETURN
END
The Flow Chart of Subroutine PATSCAT
(- used to evaluate the radiation patterns)

10
INPUT MMMAX, S

20
SET J = 1, MMM

CALCULATE AND WRITE D

30
SET Z, MAXM, MMM

CALL SUBROUTINE BESSEL

40
SET I = 1, MMM

CALCULATE H2(I)

50
SET J = 1, MMM

SET SM(I) ZERO

60
SM(I) = SM(I) + S(J, I)

70
SM(I) = SM(I) * H2(I)

80
I = MMM?

90
YES

...
SET KK = 1, MMM

SET KKK & SS

NO

KK = MMM?

YES

SET K = 1, 91

SET TH, THD, PS

SET L = 1, MMM

SET CS & PS

NO

L = MMM?

YES

WRITE ANGLE & POWER

NO

K = 91?

YES

NO

J = S?

YES

RETURN
The Listing of Subroutine PATSCAT

SUBROUTINE PATSCAT (X, S)

DIMENSION JJ(101), Y(90)

COMPLEX S(24,90), H2(20), SM(20)

COMPLEX SS, JS, CS

COMPLEX CJ

DX = 2

NY = 12X-PX

CJ = COMPLEX (0.9, 1.0)

P1 = 3.1415926536

P2 = P1/2.0

I0 = 10, J1 = 1, IS

D = 1.0 + 3.5 * J

F1TF (5, 10) I

Z = 2.0 * P1 * 0

MX = 414X+1

M0 = 2.1 * K14X+1

F1 = 2.4145X+1

CALL AESSFL (Z, NY, DJ, J)

DO 20 J = 1, 14

I1 = J - 1

IF (S(I1,J) .GT. 0) GO TO 20

H2(I1) = COMPLEX (JJ(I1+1), Y(I1+1))

GO TO 28

28 CONTINUE

29 CONTROL

DO 40 J = 1, 14

S(I1,J) = CI1KX ((1.0, 1.0))

I2 = I1 + I1 - 1

DO 40 J = 1, 14

S(I1,J) = S(I1,J) + S(I1,JX) * (CI * (N-I) )* 0.5 * H2(JX)

40 CONTROL

S(I1,J) = S(I1,J) + 0.5 * H2(1)

IF (S(I1,J) .GT. 90) GO TO 29

41 CONTINUE

42 CONTROL

SS = COMPLEX (0.0, 0.0)

K1 = 1

K2 = K1 + 1

K3 = K2 + 1

KX = K2 * K2 - 1

SS = SS + S(KX) * CEXP (COMPLEX (0.0, FLOAT(KX) * P2))

50 CONTROL

DO 60 K = 1, 91

TH = P1 * (FLOAT(K) - 1.0)

T2 = T1 + P1 * 1.0

PS = COMPLEX (0.0, 0.0)

DO 60 J = 1, 14

I3 = G24VY - 1

I4 = I12 * P1 (COMPLEX (0.0, FLOAT(T2) ) * (T2 + P1))

I5 = SS * PS / (SS * CONJG(SS))

I6 = T2 + P1 * 1.0

I7 = T12 * (S12 * P1) / PS

60 CONTINUE

10 CONTROL

11 CONTROL

FOR AT (1X, J, K, S, Z)

120 FOR AT (1X, J, K, 17.4)

END
a) Characterisation of the vector wave equation (Reference 10)

The electric and magnetic fields in a source-free homogeneous medium are divergenceless and each satisfies a vector wave equation:

\[ \nabla \times \nabla \times \mathbf{A} - k^2 \mathbf{A} = 0 \]  

(B1)

It is well known (Reference 1) that independent solutions of this equation can be constructed as:

\[ \mathbf{M}_{mn} = \nabla U_{mn} \times \mathbf{R} \]
\[ \mathbf{N}_{mn} = \frac{1}{k} \nabla \times \mathbf{M}_{mn} \]  

(B2)

with the added relation:

\[ \mathbf{M}_{mn} = \frac{1}{k} \nabla \times \mathbf{N}_{mn} \]

where \( \mathbf{R} \) is a position vector from the origin \( O \), and the potentials \( U_{mn} \) are a corresponding complete set of solutions of the scalar wave equations

\[ \nabla^2 u + k^2 u = 0 \]  

(B3)

In spherical coordinates such a set of characteristic equations is given by:
$u_{mn} = Z_n(kR)P_n^m(\cos \theta) \exp(i m \phi),$ 

$\text{as } n \leq m \leq n$ (B4)

$Z_n(kR)$ is generally any spherical Bessel function. The associated Legendre function $P_n^m(\cos \theta)$ is taken as in Stratton.

$$P_n^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \quad |m| \leq n$$ (B5)

Other forms can differ by a $(-)^m$ factor.

The specific forms for the vector wave function are:

$$\overrightarrow{M}_{mn} = Z_n(kR) \left[ i_2 \frac{\im(P_{n}^m(\cos \theta))}{\sin \theta} \exp(i m \phi) - i_3 \exp(i m \phi) \frac{d}{d\theta} P_n^m(\cos \theta) \right]$$ (B6)

$$\overrightarrow{N}_{mn} = \overrightarrow{1}_1 \frac{Z_n(kR)}{kR} n(n+1) P_n^m(\cos \theta) \exp(i m \phi)$$

$$+ \frac{1}{kR} \frac{d}{dR} [RZ_n(kR)] \left[ i_2 \exp(i m \phi) \frac{d}{d\theta} P_n^m(\cos \theta) + i_3 \frac{\im(P_{n}^m(\cos \theta))}{\sin \theta} \exp(i m \phi) \right]$$

where $\overrightarrow{1}_1, \overrightarrow{i}_2$, and $\overrightarrow{i}_3$ are unit vectors in directions of increasing $R, \theta, \phi$ respectively.

Note that Maxwell's equations for harmonic time dependence $\exp(-i \omega t)$ and for sourceless regions are:

$$\nabla \times \overrightarrow{E} = i \omega \overrightarrow{H}$$

$$\nabla \times \overrightarrow{H} = -i \omega \overrightarrow{E}$$ (B7)
where \( k^2 = \omega^2 \mu c \). From (B2) it follows that for an E-field contribution of the form \( \overline{M}_{mn} \), there is associated exactly an H-field term \( (k/\omega \mu) \overline{N}_{mn} \). Similarly for an E-field contribution \( \overline{N}_{mn} \), there is associated an H-field term \( (k/\omega \mu) \overline{M}_{mn} \). Furthermore the detailed forms in (B6) show that \( \overline{M}_{mn} \) has no radial component and hence all radial fields components must be represented by \( \overline{N}_{mn} \) solely, i.e., E type modes in which only the electric field has a radial component and H type modes similarly.

For a generalised field both types of modes are present:

\[
\overline{E} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [A \overline{M}_{mn} + B \overline{N}_{mn}]
\]

\[
\overline{H} = \left( \frac{k}{\omega \mu} \right) \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [A \overline{N}_{mn} + B \overline{M}_{mn}]
\]

It is noted that if a translation or rotation or both of the coordinates is made then a new set of scalar and vector wave functions can be analogously defined with respect to the new coordinates. Since any one of the previous vector wave functions defines a perfectly valid vector field it must be expandable over the new set. That is under coordinate rotations and or translations there MUST exist addition theorems of the form:

\[
\overline{M}_{\mu \nu}(R, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [C(\mu, \nu | m, n) \overline{M}_{mn}(R; \theta, \phi')] + D(\mu, \nu | m, n) \overline{N}_{mn}(R; \theta, \phi')]
\] (B9)
\[ N_{\mu\nu}(R,\theta,\phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [C(\mu,\nu|m,n)N_{mn}(R;\theta;\phi')] + D(\mu,\nu|m,n)M_{mn}(R;\theta;\phi')] \]

where \((R;\theta;\phi')\) is the new set of spherical coordinates.

b) Coordinate Rotation

It is relatively simple to describe the addition theorems for \(M_{mn}\) and \(N_{mn}\) under coordinate rotation.

The second set of coordinates is \((R,\theta;\phi')\) centred at 0 but with respect to the original \((R,\theta,\phi)\) system. The radial coordinate is invariant under rotation.

The addition theorem for the spherical SCALAR wave function is first found. These theorems are well known in quantum mechanics as a special case in the study of the 3-dimensional rotation group. In terms of Euler's angles of rotation, the addition theorem appears as

\[ Y_{jm}(\theta,\phi) = \sum_{m'=-j}^{j} Y_{jm'}(\theta;\phi')D_{m'm}(\alpha\beta\gamma) \]  

\((B10)\)

and

\[ Y_{jm}(\theta,\phi) = (-)^m \left[ (\frac{2j+1}{4\pi}) \frac{(j-m)!}{(j+m)!} \right] \frac{1}{2} r_j^m(\cos\theta) \exp(\im\phi) \]  

\((B11)\)

The coordinates \(\theta, \phi\) being the original set while \(\theta', \phi'\) are the new set, obtained by a rigid rotation of the first set of axes through the Euler angle \(\alpha, \beta, \gamma\). The convention adopted is that the angles \(\alpha, \beta, \gamma\) are positive in the sense of rotation of a
right-handed screw in the right handed frame of axes. The coefficients
\( D_{m'n}(\alpha, \beta, \gamma) \) are the matrix elements of a unitary transformation and are
given by:

\[
D_{m'm}(\alpha, \beta, \gamma) = [\exp i m'] [\exp i m] D_{m'm}(\beta) [\exp i m] (j, m') \alpha \beta \gamma
\]

where:

\[ d_{m'm}(\beta) = \frac{(-1)^{j} (j+m')!(j-m')!}{(j+m)!(j-m)!} \sum_{\sigma}^{|j+i-j-m'|} (j+m') (j-m') \]

Both sides of equation (B10) involve surface harmonics of the same
order \( j \) and hence it is a trivial step to introduce additional
radial wave function factors \( z_{j}(kR) \) on both sides to obtain a relation
on total wave functions. Using equation (B4), (B10) can be written as:

\[
u_{m}(R, \theta, \phi) = \sum_{m=-m}^{m} \beta(\mu, m, n) u_{m}(R, \theta, \phi) \]

where from (B11)

\[
\beta(\mu, m, n) = (-)^{m+n} [\frac{(n+m)!}{(n-m)!}] \frac{1}{2} D_{m'm}(\alpha, \beta, \gamma)
\]

So far the addition theorems for the spherical SCALAR wave functions
have been found.

The vector operator \( \mathbf{v} \) is defined independently of the coordinate
system and due to the common origin the same \( R \) is the position vector
for both. It follows immediately from (B14) and the basic definitions
of the vector wave function that:
\[ M_{\mu n}(R, \theta, \phi) = \nabla u_{\mu n}(R, \theta, \phi) \times R \]
\[ = \sum_{m=-n}^{n} \beta(\mu, m, n) \bar{M}_{mn}'(R, \theta; \phi') \]

(c) **Rigid Coordinate Translation**

Suppose a second coordinate origin is taken at point O' whose coordinates are \((R', \theta', \phi')\) with respect to the first origin O. The set of spherical coordinates \((r; \theta', \phi')\) is introduced with respect to O' such that the axes \(\theta' = 0\) and \(\phi' = 0\) are parallel to the axes \(\theta = 0\) and \(\phi = 0\). This is then a rigid translation of the coordinate system. The known addition theorem for the spherical scalar wave function is

\[ u_{\mu \nu}(R, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a(\mu, \nu|m, n\mu_{mn}^\prime(R'; \theta'; \phi') \]

[It should be noted that depending on the value of \(R\) relative to \(R'\), the \(u_{mn}\) may involve a spherical Bessel function \(Z'_n(kR)\) which is generally not of the same type as the \(Z_n(kR)\) involved in \(u_{\mu \nu}\). This distinction is not important in the derivation below, which involves only the local properties of \(u_{mn}'\).]

Since the grad operator is invariant of coordinate system:

\[ M_{\mu \nu}(R, \theta, \phi) = \nabla u_{\mu \nu} \times R \]
\[ = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a(\mu, \nu|m, n\nu_{mn}' \times R] \]

(B17)
The problem therefore is to expand terms of the form $\mathbf{V}_m \times \mathbf{R}$
in terms of $M_{mn}$ and $N_{mn}$ vector wave functions.

If $\mathbf{R}$ is the position vector of $O'$ relative to $O$ then:

$$\mathbf{R} = \mathbf{R}_o + \mathbf{R}'$$  \hspace{1cm} (B19)

So that:

$$\mathbf{V}_m \times \mathbf{R} = (\mathbf{V}_m \times \mathbf{R}_o) + (\mathbf{V}_m \times \mathbf{R}')$$  \hspace{1cm} (B20)

but

$$\mathbf{V}_m \times \mathbf{R}' = M_{mn}$$  \hspace{1cm} (B21)

So the remaining problem is to determine an expansion for $\mathbf{V}_m \times \mathbf{R}_o$.

The simplest approach to the problem appears to be to first identify the part which is represented by the $N_{mn}'$ function since this involves only the vector components in the $i_1'$ direction. That is using equation (B16) an expansion of the form:

$$i_1'.\mathbf{V}_m \times \mathbf{R}_o = \sum_{p=1}^{\infty} \sum_{q=-p}^{p} A_{qp} i_1'.N_{qp}$$

$$= \sum_{p=1}^{\infty} \sum_{q=-p}^{p} A_{qp} \frac{Z}{p(p+1)} \frac{kR'}{p} p(q) \cos q' \exp(iq') \hspace{1cm} (B21)$$

is sought.

It may readily be ascertained that a fixed vector field of magnitude and direction equal to $\mathbf{R}_o$ is represented at the general point $(R;\theta';\phi')$ by components
The derivation then consists in using \((B22)\) with trigometric, Legendre and Bessel functions identities. The computations are tedious.

For the radial part comparison of the expansions on the two sides of \((B21)\) yields:

\[
\overline{R}_o \cdot \overline{l}_1 = R_o [\sin \theta \sin \phi' \cos (\phi - \phi') + \cos \theta \cos \phi']
\]

\[
\overline{R}_o \cdot \overline{l}_2 = R_o [\sin \theta \cos \phi' \cos (\phi - \phi') - \cos \theta \sin \phi']
\]

\[
\overline{R}_o \cdot \overline{l}_3 = R_o [\sin \theta \sin (\phi - \phi')]
\]

\((B22)\)

This completely identifies the radial component and hence all possible \(N'_{mn}\) which enter the expansion of \(V\overline{u}'_m x \overline{R}_o\). If these terms are subtracted from \(V\overline{u}'_m x \overline{R}_o\) the remaining expression contains terms in \(\overline{M}'_{mn}\) only. These can be found by considering either the \(\overline{l}_2\) or \(\overline{l}_3\) components.

The result may be summarised as follows:
\[ n = m = 0; \]
\[ V'_\infty \times \vec{R}_o = kR_o \cos \theta \bar{M}_o \hat{1} + \frac{kR_o}{2} \sin \theta \, \exp(-i\phi) \bar{M}_1 \]
\[ - kR_o \sin \theta \, \exp(i\phi) \bar{M}_1 \]
\[ \text{(B24a)} \]

and for \( n \neq 0 \)
\[ V_{mn}' \times \vec{R}_o = \frac{kR_o}{n(n+1)} \left\{ \right. \]
\[ \int \cos \theta \left[ \frac{n+m}{n} \bar{M}_{m,n-1} + \frac{n-m+1}{n+1} \bar{M}_{m,n+1} \right] \]
\[ + \frac{\sin \theta \, \exp(-i\phi)}{2} \left[ - \frac{1}{n} \bar{M}_{m+1,n-1} + \frac{1}{n+1} \bar{M}_{m+1,n+1} \right] \]
\[ + \frac{\sin \theta \, \exp(i\phi)}{2} \left[ \frac{(n+m)(n+m-1)}{n} \bar{M}_{m-1,n-1} \right] \]
\[ - \frac{(n-m+1)(n-m+2)}{n+1} \bar{M}_{m-1,n+1} \}
\[ \text{(B24b)} \]

Referring back to the general form of expansion in (B9) equations (B17, B18, B20, B21 and B24) can be summarised as:
\[ \bar{M}_{\mu \nu}(R, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} C(\mu, \nu|m,n)M'_{mn} \]
\[ + D(\mu, \nu|m,n)N'_{mn} \]
\[ \text{(B25a)} \]
where \( C(\mu, v|m,n) = \alpha(\mu, v|m,n) \)

\[
\begin{align*}
&+ \frac{kR \cos \theta}{2n+3} \cdot \frac{n+m+1}{n+1} \cdot \alpha(\mu, v|m,n+1) \\
&+ \frac{kR \cos \theta}{(2n-1)} \cdot \frac{n-m}{n} \cdot \alpha(\mu, v|m,n-1) \\
&- \frac{kR \sin \theta \exp(-i\phi)}{2(2n+3)(n+1)} \cdot \alpha(\mu, v|m-1,n+1) \\
&+ \frac{kR \sin \theta \exp(i\phi)}{2(2n-1)n} \cdot \alpha(\mu, v|m-1,n-1) \\
&+ \frac{kR \sin \theta \exp(i\phi)}{2(2n+3)} \cdot \frac{(n+m+2)(n+m+1)}{n+1} \cdot \alpha(\mu, v|m+1,n+1) \\
&- \frac{kR \sin \theta \exp(i\phi)}{2(2n-1)} \cdot \frac{(n-m)(n-m)}{n} \cdot \alpha(\mu, v|m+1,n-1)
\end{align*}
\]

\( (B25b) \)

\[
D(\mu, v|m,n) = \frac{ikR \cos \theta}{n(n+1)} \cdot \alpha(\mu, v|m,n)
\]

\[
\begin{align*}
&- \frac{ikR \sin \theta \exp(-i\phi)}{2n(n+1)} \cdot \alpha(\mu, v|m-1,n) \\
&- \frac{ikR \sin \theta \exp(i\phi)}{2n(n+1)} \cdot \frac{(n+m+1)(n-m)}{n} \cdot \alpha(\mu, v|m+1,n)
\end{align*}
\]

\( (B25c) \)

and

\[
\alpha(\mu, v|m,n) = \frac{(\nu+2m+1)!}{\nu! (m+\nu)!} \left[ \begin{array}{c c c}
\nu+\nu & m & n \\
0 & 0 & 0 \\
\mu & -\mu & 0
\end{array} \right] \left[ \begin{array}{c c c}
\nu+\nu & m & n \\
0 & 0 & 0 \\
\mu & -\mu & 0
\end{array} \right]
\]

\( (B26) \)

where the 3j Wigner coefficients are defined to be (Reference 12)
\[
\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & m_3 
\end{bmatrix} = (-1)^{j_1-j_2-m_3} \times
\]

\[
\frac{(j_1+j_2-j_3)! (j_1-j_2+j_3)! (-j_1+j_2+j_3)! (j_1+m_1)!}{(j_1-m_1)! (j_2+m_2)! (j_2-m_2)! (j_3+m_3)! (j_3-m_3)!}
\]

\[
\frac{1}{(j_1+j_2+j_3+1)!}
\]

\[
\times \sum_k \frac{(-1)^k}{k! [(j_1+j_2-j_3-k)! (j_1-m_1-k)! (j_2+m_2-k)! \times (j_3-j_2+m_1+k)! (j_3-j_1-m_2+k)!]} \quad (B27)
\]
APPENDIX C

THE TRANSFORMATION OF COORDINATES VIA FOURIER TRANSFORMS

![Figure C1](image)

The transverse electric field at P due to antenna 1 at origin O₁ is in terms of \( r₁, \theta₁, \phi₁ \) (see figure C1)

\[
\overline{E}(r₁) = \Theta^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \{ s₁ⁿ_ⁿm(\theta₁, \phi₁) \overline{H}(2)(kr₁) \overline{e}_nⁿ \overline{e}_nm(\theta₁, \phi₁) \}
\]

where

\[
\overline{e}_{nm}(\theta, \phi) = -\frac{1}{N_{nm}} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \phi} P_m^n(\cos \theta) \cos \phi \frac{1}{\sin \phi}
\]

By means of addition theorems developed by Stein and Cruzan (Reference the field could be expanded about origin O₂ in terms of coordinates \( r₂, \theta₂, \phi₂ \) and since the fields must be finite at O₂ the resulting expressions for the transverse electric field must be of the form:
where \( a_{l,n}^{(e,0)} \) are the amplitudes of the incoming and outgoing spherical modes and the \( J_n \) are spherical Bessel functions. The addition theorems enable the amplitudes to be found but in general the procedure is cumbersome. It is better to express the field of antenna 1 as a superposition of plane waves and compare the resulting expression with the corresponding expansion in spherical modes. The \( a_{l,n}^{(e,0)} \) can then be identified as integrals involving the far field pattern of antenna 1.

The field at \( P \) due to the antenna at \( O \) with nothing visible at \( O_2 \) may be written as two-dimensional Fourier transforms:

\[
\begin{align*}
\bar{E}(\mathbf{r}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \bar{E}(k_x, k_y) \exp(-j\mathbf{k}\cdot\mathbf{r}) \\
\bar{H}(\mathbf{r}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \bar{H}(k_x, k_y) \exp(-j\mathbf{k}\cdot\mathbf{r})
\end{align*}
\]

where \( \mathbf{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \)

The fields will be decomposed into \( E \) and \( H \) modes along \( z_{01} \).

Let \( \pi' \) and \( \pi'' \) be respectively the scalar \( E \) and \( H \) mode Hertz potentials. Then:

\[
\bar{E}(\mathbf{r}) = \nabla \times \nabla \times \bar{z}_{01} \pi' - j\mu_0 \nabla \times \bar{z}_{01} \pi''
\]
\[\vec{H}(r_1) = j\omega_0 \nabla \times \vec{z}_0 \cdot \vec{w} \times \nabla \times \vec{z}_0 \cdot \vec{w}^*\]  

(C9)

Denote by \(f', f''\) the Fourier transforms of the Hertz potentials defined by:

\[f'(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi'(x_1, y_1, 0) \cdot \exp[j(k_x x_1 + k_y y_1)] \cdot dx_1 dy_1\]  

(C10)

\[f''(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi''(x_1, y_1, 0) \cdot \exp[j(k_x x_1 + k_y y_1)] \cdot dx_1 dy_1\]  

(C11)

The transformed fields \(\vec{E}, \vec{H}\) in (C4 and C5) are related to \(f'\) and \(f''\) by

\[\vec{E} = -k \times (k \times \vec{z}_0) \cdot f' - \omega \vec{z}_0 \cdot k \times \vec{z}_0 \cdot f''\]  

(C12)

\[\vec{H} = \omega \vec{z}_0 \cdot \vec{z}_0 \cdot k \times (k \times \vec{z}_0) \cdot f''\]  

(C13)

The connection between the plane wave representation and the spherical mode representation can be made through the radial components of the fields. The electromagnetic fields can also be derived from potential functions (Debye potentials) \(\vec{r}_1 \cdot \vec{E}\) and \(\vec{r}_1 \cdot \vec{H}\) the former yielding E modes in spherical coordinates, the latter \(H\) modes.

Using the identity:

\[\vec{r}_1 \cdot \vec{E} = x_1 E_x + y_1 E_y + z_1 E_z\]  

(C14)

with (C4) and (C12) gives
where

\[ \psi = \exp(-jkr) \]  

(C16)

Expressing \( \bar{k} \) in spherical \((k, \alpha, \beta)\) rather than rectangular coordinates this becomes.

\[
\bar{r}_1 \cdot \bar{E}(\bar{r}_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \left[ \psi_1 x_1 x_y \left( k_z k'_z - w\mu_0 k_z f'' \right) \right]
\]

(C15)

After evaluating (C4) and (C5) by the method of stationary phase and employing (C12) and (C13) \( f' \) and \( f'' \) may be related to the far fields radiated by antenna 1.

Using the definition of the far field pattern:

\[
\overline{F}(\theta, \phi) = F_0 \overline{\theta}_0 + F_\phi \overline{\phi}_0
\]

\[
= \lim_{r \to \infty} \frac{1}{r^2} \left[ \exp jkr \right] \overline{E}_r (r, \theta, \phi)
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ (j)^n a_{nm}(e,0) - j e_{nm}(e,0) (\theta, \phi) \right\}
\]

\[
+ (j)^{n+1} a_{nm}(e,0) - j e_{nm}(e,0) (\theta, \phi) \}
\]

(C18)

the result is:
\[ f'\left(\theta_1, \phi_1\right) = \frac{\theta_0^2 F_{\theta_0}^{\left(\theta_0, \phi_0\right)}}{-jk^3 \sin \theta_1 \cos \theta_1} \]  
(C19)

\[ f''\left(\theta_1, \phi_1\right) = \frac{\theta_0^2 F_{\phi_0}^{\left(\theta_0, \phi_0\right)}}{jk^3 \sin \theta_1 \cos \theta_1} \]  
(C20)

Substituting (C19) and (C20) into (C17) and changing the variables to \( \alpha, \beta \) gives:

\[ \overline{r}_1 \cdot \overline{E}(\overline{r}_1) = \frac{k^2}{\pi} \int_0^{2\pi} d\beta \int_\pi^0 d\alpha [F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_1}{\partial \alpha} - \sin \alpha F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_1}{\partial \alpha}] \]  
(C21)

and similarly:

\[ \overline{r}_1 \cdot \overline{H}(\overline{r}_1) = \frac{n^2}{\pi} \int_0^{2\pi} d\beta \int_\pi^0 d\alpha [F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_1}{\partial \alpha} + \sin \alpha F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_1}{\partial \alpha}] \]  
(C22)

The contour of integration in the complex \( \alpha \) plane runs from \( (0,0) \) to \( (\pi,0) \) as shown in figure C2.
The integral representations in (C21) and (C22) express the Debye potentials in terms of the coordinate system \((r_1', \theta_1, \phi_1)\). These potentials can also be expressed in terms of the translated coordinate system \((r_2', \theta_2, \phi_2)\) by noting that the translation of coordinates corresponds to a multiplication of all far field quantities by \(\exp[-j k \cdot \vec{D}]\). So that upon changing \(\vec{r}_1\) into \(\vec{r}_2\) in (21) and (22)

\[
\overline{r_2}.E(\overline{r}_2) = \frac{k}{2\pi} \int_0^{2\pi} d\beta \int_0^{\pi} d\alpha \exp(-j k \cdot \overline{r}_2)
\]

\[
[F_{1\beta}(\alpha, \beta) \frac{\partial \psi_2}{\partial \beta} + \sin \alpha F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_2}{\partial \alpha}]
\]

\[
\overline{r_2}.H(\overline{r}_2) = \frac{\mu}{2\pi} \int_0^{2\pi} d\beta \int_0^{\pi} d\alpha \exp(-j k \cdot \overline{r}_2)
\]

\[
[F_{1\alpha}(\alpha, \beta) \frac{\partial \psi_2}{\partial \alpha} - \sin \alpha F_{1\beta}(\alpha, \beta) \frac{\partial \psi_2}{\partial \beta}]
\]

where

\[
\psi_2 = \exp(-j k \cdot \overline{r}_2)
\]

Choosing \(\vec{D} = D_x \overline{r}_2 + D_y \overline{r}_2 + D_z \overline{r}_2\) with \(D_z > 0\) each integral in (C23) and (C24) contains a decaying exponential for \(r_2 = 0\) since \(\text{Im} \cos \alpha < 0\) or \(k\). Consequently the representations (C23) and (C24) yeild a set of potential functions that are finite at \(r_2 = 0\). All field components derived from these potentials remain finite and the transverse fields must possess series expansions in terms of spherical vector mode functions as in (C3).

The spherical mode description gives
\[
\begin{align*}
\overline{E}_t &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \{ V_{nm}^{(e)}(x) \overline{e}_{nm}^{(e)}(\theta, \phi) + V_{nm}^{(O)}(x) \overline{e}_{nm}^{(O)}(\theta, \phi) \\
+ V_{nm}^{(e)}(x) [\overline{e}_{nm}^{(e)}(\theta, \phi) \times \overline{x}_0] + V_{nm}^{(O)}(x) [\overline{e}_{nm}^{(O)}(\theta, \phi) \times \overline{x}_0] \} \quad (C26)
\end{align*}
\]
and

\[
\begin{align*}
\overline{H}_t &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \{ I_{nm}^{(e)}(x) [\overline{r}_0 \times \overline{e}_{nm}^{(e)}(\theta, \phi)] \\
+ I_{nm}^{(O)}(x) [\overline{r}_0 \times \overline{e}_{nm}^{(O)}(\theta, \phi)] \\
+ I_{nm}^{(e)}(x) \overline{e}_{nm}^{(e)}(\theta, \phi) + I_{nm}^{(O)}(x) \overline{e}_{nm}^{(O)}(\theta, \phi) \} \quad (C27)
\end{align*}
\]
and:

\[
\begin{align*}
\overline{j} \omega \overline{E}(\overline{r}, \overline{E}) &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ \frac{n(n+1)}{x} \right] I_{nm}^{(e,O)}(\theta, \phi) \overline{\phi}_{nm}^{(e,O)}(\theta, \phi) \quad (C28)
\end{align*}
\]

\[
\begin{align*}
\overline{j} \omega \overline{H}(\overline{r}, \overline{H}) &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ \frac{n(n+1)}{x} \right] V_{nm}^{(e,O)}(\theta, \phi) \overline{\phi}_{nm}^{(e,O)}(\theta, \phi) \quad (C29)
\end{align*}
\]
where

\[
\overline{\phi}_{nm}^{(e,O)}(\theta, \phi) = \left( \frac{1}{N_{nm}} \right) \overline{r}_n^m (\cos \theta) \cos m \phi \sin n \phi
\]

After comparing the mode voltage in (C26) - (C29) with the expansion coefficients in (C3) the result is:

\[
\begin{align*}
\overline{r}_2.\overline{E}(\overline{r}_2) &= -2 \xi_2 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{n(n+1)}{x} a_{nm}^{(e,O)} (kD) \\
\overline{r}_2.\overline{H}(\overline{r}_2) &= \overline{\phi}_{nm}^{(e,O)}(\theta_2, \phi_2) \left[ \frac{\overline{r}_n^m (k\overline{r}_2)}{k\overline{r}_2} \right] \quad (C31)
\end{align*}
\]
\[ \mathbf{r}_2 \mathbf{H} (r_2) = -j 2(n) \sum_{n=1}^{\infty} \sum_{m=0}^{n} n(n+1)a_{nm}^{(e,0)}(\mathbf{kd}) \]

Equations (C24) (C25) (C31) and (C32) are alternative representations of the Debye potentials. Using the expansion of \( \exp(-jkr_2) \) in terms of scalar mode functions (Stratton p409-410) in (C24) and (C25) one can solve for the spherical mode amplitudes \( a_{nm} \). The result is:

\[ a_{1nm}^{(e,0)}(\mathbf{kr}_2) = \int_0^{2\pi} d\beta \int_{\Gamma} d\alpha (-j)^n \exp(-jkr_2) \]

\[ \sin \alpha F_{lt}^{(\alpha, \beta)} (e) e^{(e,0)}(\alpha, \beta) \] (C33)

\[ a_{2nm}^{(e,0)}(\mathbf{kr}_2) = \int_0^{2\pi} d\beta \int_{\Gamma} d\alpha (-j)^{n+1} \exp(-jkr_2) \]

\[ \sin \alpha F_{lt}^{(\alpha, \beta)} (e) e^{(e,0)}_{nm}(\alpha, \beta) \] (C34)
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RADIATION FROM ANTENNAS MOUNTED
ON IRREGULARLY SHAPED BODIES

Progress Report No 4

D. H. Munro
February 1977
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REFERENCES
1. INTRODUCTION

This report is concerned with the radiation patterns of antennas near scattering objects such as aircraft. The previous report (ref. 1) gave the theory for two dimensional scattering and indicated the conversion to three dimensions. This report applies the two dimensional theory to programming and presents the results with experimental comparison.

2. A SCATTERING DESCRIPTION OF RADIATION PROBLEMS

2.1 The Calculation of the Scattering Matrix

The method used to obtain the scattering matrix is described in the second report (ref. 2) and Wilton and Mittra (ref. 3) but the mathematics used are in parts incorrect and confusing. Thus the method will be summarised below with suitable changes to standardise the notation.

The scattered field in the coordinate system, 0, of the scatterer can be described, as shown before, by the equation

$$E^S(\rho) = \sum_{n=\infty}^{\infty} \beta_n j^n_H(2)(k\rho)\exp(jn\phi) \tag{1}$$

with geometry as in fig. 1. However, this solution will not converge for $$\rho < \rho_{\min}$$ as that region contains singularities, although the scattered field at P can be described in the translated coordinate system, 0', for $$\rho < \rho_{\max}$$, such that $$\rho'$$ is normal to the surface, by
**Fig. 1**
Geometry for Outside Expansion using Analytic Continuation

**Fig. 2**
Geometry for Field Calculation with an Arbitrary Line Source
\[
E^S(p_o + p') = \sum_{m=-\infty}^{\infty} \beta_m j^{-m} J_m(kp') \exp(jm\phi')
\]  
(2)

In order to express the field at \(P\) in the translated coordinate system using the original coefficients, \(\beta_n\), the addition theorem for Hankel functions is invoked from Watson (ref. 4), noting that \(\rho_o > \rho'\), which gives

\[
E^S(p_o + p') = \sum_{n=\infty}^{\infty} \beta_n j^{-n} \sum_{m=-\infty}^{\infty} h^{(2)}_{n-m}(kp_o) \exp[j(n-m)\phi_o] J_m(kp') \exp(jm\phi')
\]  
(3)

Thus, for every point \(P\) on the surface of the cylinder, there is an equation which describes the field as cylindrical waves around a translated origin and will converge, but which still retains the original expansion coefficients which are common to all points on the surface. For \(I\) points, this can be written in matrix notation as

\[
[B_i] = [B_{in}] [\beta_n]
\]  
(4)

where

\[
B_{in} = j^{-n} \sum_{m=-\infty}^{\infty} h^{(2)}_{n-m}(kp_{oi}) \exp[j(n-m)\phi_{oi}]
\]  

\[
J_m(kp_i') \exp(jm\phi_i')
\]  
(5)

It is now possible to apply the boundary condition that on the surface
$E_i^s = -E_i^{inc}$

If $E_i^{inc}$ is taken to be

$$E_i^{inc} = a_j^t j^{-t} H_t^i (k_0) \exp(jt\phi_i)$$  \hspace{1cm} (6)

with $a_j^t = 1$ and remembering that

$$\beta_n = \sigma_n a_t$$

then for an incident field of order $t$

$$\beta_n = \sigma_n t$$  \hspace{1cm} (7)

Therefore to find the scattering matrix, it is necessary to solve

$$- [E_i^{inc}] = [B_{in}] [\sigma_n]$$  \hspace{1cm} (8)

Finally, the limits at which the infinite summations over $m$ and $n$ in $[B_{in}]$ can be truncated must be found. It has been shown in the second report (ref. 2) that

$$|n_{max}| > k_p min$$  \hspace{1cm} (9)

is a necessary condition for the summation to converge but Wilton and Mittra (ref. 3) have shown that the summation will only converge when
2.2 The Calculation of the Field Radiated by an Arbitrary Line Source in the Presence of an Arbitrary Two-Dimensional Scatterer

The last report (ref. 1) described the calculation of the radiated field due to a line source on the x-axis. With a slight alteration, the field due to an arbitrary line source can be found. Consider a line source at \( O_1 \) and an arbitrary two-dimensional scatterer with an origin at \( O_2 \) as shown in fig. 2. The field radiated at \( P \) is proportional to

\[
H_0^{(2)} (kr_1)
\]

but may be expanded about the origin of the scatterer by using the addition theorem given by Watson (ref. 4) as

\[
H_0^{(2)} (kr_1) = \sum_{n=-\infty}^{\infty} H_n^{(2)} (kD) J_n (kr_2) \exp(jn(\theta_2 - \phi)) \quad (11)
\]

provided \( r_2 < D \).

The field incident on the scatterer is then given as

\[
\sum_{n=-\infty}^{\infty} \frac{H_n^{(2)} (kD)}{2} H_n^{(1)} (kr_2) \exp(jn(\theta_2 - \phi)) \quad (12)
\]

but it must be written in the form
to use the scattering matrix. It follows that

\[ \alpha_n = \frac{j^{n_H(2)}(kD)}{2} \exp(-jn\phi) \]  (14)

It has been shown in the last report (ref. 1) that the change in the field is given by

\[ \sum_{m=-N}^{N} \sum_{n=-N}^{N} \sigma_{mn} \alpha_n \alpha_m j^{-m_H(2)}(kr_2) \exp(jm\theta_2) \]  (15)

but the far field due to the source is now given by Watson (ref. 4) as

\[ H_o^{(2)}(kr_1) = \sum_{n=-N}^{N} H_n^{(2)}(kr_2) J_n(kD) \exp(jn(\theta_2 - \phi)) \]  (16)

for \( r_2 > D \). Finally, the total field is given by the sum of equations (15) and (16) to be

\[ \sum_{m=-N}^{N} \sum_{n=-N}^{N} \sigma_{mn} \alpha_n \alpha_m + j^{m_H(2)}(kD) \exp(-jm\phi) \exp(jm\theta_2) \]  (17)

as \( r \rightarrow \infty \).

2.3 The Calculation of the Two Dimensional R.C.S.

This section in the last report (ref. 1) was found to be incorrect, and has been adjusted but the results are still dubious
so this section is only included for completeness as it may not yet be correct.

A plane wave incident upon the origin 0 of a scatterer in the direction \( \theta \), as in fig. 3, is given by

\[
\exp(jk \cdot r) = \exp(jkr \cos(\phi - \theta)) \quad (18)
\]

This can be expanded into incoming and outgoing cylindrical waves which gives the incident waves as

\[
\sum_{n=-\infty}^{\infty} j^n \frac{H_n^{(1)}(kr)}{2} \exp(jn(\phi - \theta)) \quad (19)
\]

The form of the incident field must be

\[
\sum_{n=-\infty}^{\infty} \alpha_n j^{-n} H_n^{(1)}(kr) \exp(jn\phi) \quad (20)
\]

to allow the use of the scattering matrix. Equating both wave descriptions gives

\[
\alpha_n = \frac{(-1)^n}{2} \exp(-jn\theta) \quad (21)
\]

Now the R.C.S. is given as proportional to

\[
\sum_{m=-N}^{N} \sum_{n=-N}^{N} \sigma_{mn} \alpha_{n-m} j^{-m} H_m^{(2)}(kr) \exp(jm\phi) \quad (22)
\]

Taking \( r \) to infinity gives, for the bistatic case
Fig. 3
Geometry for an Incident Plane Wave.
\[ \sum_{m=-N}^{N} \sum_{n=-N}^{N} \sigma_{mn} \alpha_{n} \exp(jm\phi) \]  \hspace{1cm} (23)

and for the monostatic case the substitution \( \phi = \theta + \pi \) is made.

3. PROGRAM DEVELOPMENT

3.1 Introduction

In the last report (ref. 1), a plan for the development of the computer program was put forward. This has been used as a rough guideline for the work completed since the report. However, some of the suggestions have proved to be unnecessary since corrections in the mathematics have eliminated most of the errors.

3.2 Choice of Machine

The first suggestion was to change the computer used. At first, Surrey Universities' ICL 1905 machine was best for program development but for production runs with large scattering matrices, the London University CDC 7600 machine was used. This machine gave a hundred fold decrease in mill time and a four fold increase in core space available which allowed several runs in one job.

With this increase in machine capabilities, the sizes of the matrices have increased to give a maximum of a 51x51 scattering matrix using up to 80 boundary points. This allows the program to handle bodies up to 6 wavelengths in length using
only 28 seconds of mill time. It is also thought to be possible to increase the size of the scatterer up to 8 wavelengths using the full core space of the CDC machine.

3.3 Investigation of the Required Number of Boundary Points

Now before the number of modes required for convergence can be investigated, the number of boundary points necessary to produce a sufficiently overdetermined set of equations must be found. In the case of the square, with the difficulties due to the four fold symmetry, it has been found that to carry through the symmetry using symmetric boundary points, the number of points must be at least double the total number of modes. However, if an odd number of points are used then less are required although the solution tends to lose the neat symmetry of symmetric points.

This can be seen in graphs 1 and 2 where the total number of modes is 31. The plot for 40 boundary points is very divergent from the others in both graphs while those for 60, 70 and 80 are virtually identical as expected. Also, the scattering pattern plot for 41 points is more stable than that for 50 points; although, in the radar cross-section, the plot for 41 can be seen diverging at the symmetry points of $\frac{\pi}{2}$ and $\pi$.

3.4 The Investigation of the Required Number of Modes

Having determined the number of boundary points, the number of modes to give the scattering matrix for a given size of object must be decided. As has been shown, the important modes are those
where the order of the mode, \( m \), is such that \(|m| < k \rho_{\min}\). After the highest mode in this range has been included, the solution starts to converge rapidly as shown in graph 3. This is a graph of the modulus of the expansion coefficients of the change in the field, \( \beta_m - \alpha_m \). It can be seen that the solution will have converged after about 5 modes more than \( k \rho_{\min} \) have been included. The rise at 25 modes is due to edge element errors which are discussed later.

As the outside elements of the scattering matrix become small, the errors gain in effect in this area. This introduces a special effect in the case of the square, where the four fold symmetry causes every order divisible by four to have a greater importance than the adjacent orders. This causes a greater degree of error where the important orders are near the outside of the matrix. This is shown in graph 4 where the experimental shape shows a peak at approximately 155°. The closest to this case are the plots with the important mode deepest in the scattering matrix.

However every fourth order stabilises towards an individual form as can be seen in graphs 5-8. This effect is purely dependent on the highest order of mode used in finding the scattering matrix, not the highest order used in finding the scattering pattern. This is shown in graph 9 where it can be seen that the pattern only becomes unstable when too few modes are used.

If many more modes are used than necessary, the edge element errors begin to build up rapidly, causing the pattern to become
GRAPH 3

Demonstrates convergence of the solution for a predicted number of modes.
SCATTERING PATTERN

Relative Gain (dB)

0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180

Effect on accuracy
Demonstrating the four fold

Antenna Distance = 1.30 1.50 1.70 1.90 2.10 2.30 2.50 2.70 2.90 3.10
Angle of marker = 1.40 1.60 1.80 2.00 2.20 2.40 2.60 2.80 3.00 3.20
No. of Modes = 17 19 21 23 25 27 29 31 33 35
No. of boundary points = 70 80 90 100 110 120 130 140 150 160

Graph 4
very unstable again. This can be eliminated by ignoring the edge elements and just using the interior of the matrix to give the scattering pattern. This is shown in graph 10 where the pattern returns to the correct form for the number of modes used in finding the matrix when sufficient edge elements are ignored.

3.5 The Use of the Eigenvalues

In the previous report (ref. 1), it was suggested that if the eigenvalues were used to give the scattered field, then the errors would be avoided and the calculation time decreased. Unfortunately this is not the case. The eigenvalues are the elements in the diagonal %E matrix in the following matrix equation from Sokolnikoff and Redheffer (ref. 4)

\[
[C^{-1}][\sigma][C] = [E]
\]

where \( C \) is the matrix of eigenvectors and \( C^{-1} \) is the inverse. The eigenvalues would be used to give the field in the following matrix equation

\[
[\beta] = [C][E][C^{-1}][\alpha]
\]

instead of

\[
[\beta] = [\sigma][\alpha]
\]

Obviously, using the eigenvalues to give the field is a time-consuming process with only the possibility of avoiding errors being the reason to use them to give the field. Computing
experience has shown that the eigenvalues are relatively free from edge element errors but the errors are carried through to the eigenvectors and will still affect the solution. Thus the eigenvalues are only useful in showing the accuracy and convergence of the solution.

3.6 Experimental Comparison

At this point it was possible to make a comparison between experiment and theory. The Royal Aircraft Establishment constructed a square section cylinder with a dipole source near to a corner and measured the radiation in the azimuthal plane. This is therefore an approximation of the two dimensional square with a line source near a corner which is the computational model. The result can be seen in graph 11. The comparison is very good and only diverges at angles where there may be large experimental errors. The computed plot was calculated with 60 boundary points, 29 modes in the total for the scattering matrix and used only 27 modes to calculate the field. The square side is 1.8 wavelengths long and the line source is on the corner and 1.3 wavelengths away from the centre of the square.

4. FUTURE DEVELOPMENT

Having successfully developed the two dimensional case for the square cylinder, in agreement with experiment, the program can be extended to compute the three-dimensional case.
GRAPH 11

Experimental Comparison
for an arbitrary non-re-entrant body of rotation. As the theory is being limited to finding the scattering matrices for bodies of rotation, only two dimensional scatterers need be considered. Therefore the program is being developed to give the geometry involved in encircling an object defined by points joined by straight lines. A problem which will arise at this stage is the positioning of the source. The theory finds the smallest circular cylinder around the scatterer and assumes that the field is incident from outside this cylinder. This will not be the case for a large number of aircraft but in the last report it was suggested that the solution would still be applicable with the source inside the cylinder if the eigenvalues were checked for convergence. It is intended to investigate the behaviour of the solution with the source in several positions and to increase the enclosing cylinder to include an antenna position for a geometry with a known solution.

At this stage the program is still two-dimensional but a three-dimensional pattern can be obtained by converting the cylindrical Bessel functions into spherical Bessel functions. This fairly simple conversion will give the pattern in the principal plane for which a dipole source is on the axis and orthogonal to the axis. For other principal planes and orientations of the dipole, a much more complex conversion is required using the Dyadic Green's functions to give the required modes. Non-principal planes could then be found by the weighted sum of the principal planes and the effect of wings could be
included. At this stage however, coupling becomes an important effect and any further development will have to include coupling which will require significant alterations to the theory.
APPENDIX A

THE COMPUTER PROGRAM

This appendix is a description of the latest revision to the computer program described in the previous report. The program does not calculate any more than is claimed in the previous report but it now functions correctly apart from the subroutine MONOSTA which gives dubious results. All the routines which have been altered are described and the others are standard routines.

The program has been condensed to allow runs with realistic data on the machine. Thus the program has a maximum of 80 boundary points and 51 modes with more core space available.

A.1 The Main Program

The main program controls the flow of computation and calls the various subroutines when necessary. The only major calculation is to find the incident field matrix for which the equation

\[ F_{\text{inc}}^{(1)} = \mathbf{j} t H_{\text{t}}(\mathbf{k}) \exp(jt\mathbf{t}) \]

is used.
The Flowchart of the Main Program PMUA

1. Dimension matrices
2. Read variables
3. Assign constants
4. Print important variables
5. Call ABIN to find \([B]\) matrix
6. Find incident field matrix \([E]\)
7. Call FO4AMF to solve \([-E] = [B][\sigma]\)
8. Transpose \([\sigma]\) and print \([\sigma]'\)
9. Call FO2AKF to find eigenvalues of \([\sigma]'\)
10. Print eigenvalues with modulus and phase
11. Call MONOSTA to find monostatic R.C.S.
12. Call PATSCAT to find scattered field
13. Print data
PROGRAM PMAC INPUT, OUTPUT, PUNCH, TAPE5=INPUT, TAPE6=OUTPUT,  
TPE7=PUNCH)
C SCATTERING FROM A SQUARE CONDUCTING OBJECT
INTEGER SUPPS
REAL K
DOUBLE PRECISION JJ(101), JD(101), Y(90)
DIMENSION AR(160,102), BR(160,51), AAR(102,51), OR(160,102),
DIAGET(102), WR(160), TPV(102), INT(51), EVPHA(51), FVMOD(51),
ER(51), THS(80), RIS(80), WKSP(102), WR(192), WZ(102),
SR(51,51), ST(51,51), WR(51), VR(51,51), VI(51,51)
& DATA(6,92),
INTEGER DN(80,51), H(80,51), F(80,51), AA(51,51), S(51,51), CI, CS
& H2A(51), SM(51), SC(51), ALPHA(51), BETA(51)
EQUIVALENCE (AA(1,5), JJ), (AA(1,9), JD)
E, (OR, VI, THS), (WKSP, WR), (WR, WI), (WZ, INT), (AR, SR, EVPHA)
E, (BR, S1, EVMOD), (AA, H2A)
E, (AAR, VR, RIS), (BIN, H1, E, S), (AA(1,2), SC), (AA(1,3), SM)
COMMON /BN1/K, A, P4, RM, SQ/BM3/P2, Nx/BN3/CI
& COMMON /BN4/AR, AAR, BR, GR, WKSP, WY, WZ, DIAGE, WR, IPIV, BIN
&/BN5/ANTD, NAD
LEVEL 2. AR, AAR, BR, OR, WKSP, WY, WZ, SR, VR, SI, VI, WWR, WI, INT, EVPHA, R
IF VMOD, THS, DIAGE, WR, IPIV, BIN, H1, E, S
M1DIM=160
M2DIM=51
M3DIM=M1DIM/2
M4DIM=2*M2DIM
READ. (5,5) NMP, MMAX, NX, SUPPS, A
5 FORMAT (4I4, ER, 2)
PI=ATAN2(0.0, -1.)
P=PI/2.6
P4=PI/4.
CT=(0., 1.)
SQ=SQRT(2.)
RM=(SQRT(0.001)*A
K=2.*PI
RMK=K*R
GAP=0.072792
ANTD=RM+GAP
NAD=2
MMH=2*MMAX+1
WRITE (6,10) NMP
10 FORMAT (6,10) NMP
IF (NMP .GT. M3DIM) GOTO 130
WRITE (6,15) MMAX
WRITE (6, 20) NX
WRITE (6, 25) A
WRITE (6, 30) RMK
15 FORMAT (1X, 4HNMPP=, I2)
20 FORMAT (1X, 3HNAME=, I2)
25 FORMAT (1X, 13HSQURE SIZE =, F8.2)
30 FORMAT (1X, 28H TIMES THE MINIMUM RADIUS =, F8.2)
CALL ABINR(RIS, THS, BIN, MMAX, NMP, MMH, M2DIM, M3DIM, JJ, JD, Y)

C PREPARE A REAL SUBSTITUTE
C OF THE COMPLEX BIN MATRIX FOR FUTURE USE

DO 35 I=1, NMP
DO 35 J=1,MM
ELEMENT=REAL(BIN(I,J))
AR(I,J)=ELEMENT
AR(NMP+I,MM+J)=ELEMENT
ELEMENT=AIMAG(BIN(I,J))
AR(NMP+I,J)=ELEMENT
AR(I,MM+J)=-ELEMENT
35 CONTINUE

CALL CALCULATE THE INCIDENT FIELD MATRIX
DO 45 I=1,NMP
X=K*RIS(I)
CALL BESSEL(X,MAX, JJ,Y)
DO 45 M=1,MM
MM=M+1
IF (MM .LT. 0) GOTO 40
HI(I,M)=CMPLX(SNGL(JJ(MM+1)),SNGL(Y(MM+1)))
GOTO 45

40 MM=MM
HI(I,M)=((-1)**MM)*CMPLX(SNGL(JJ(MM+1)),SNGL(Y(MM+1)))
45 CONTINUE
DO 50 MM=1,MM
MM=MM
HI(I,M)=((-1)**MM)*CMPLX(SNGL(JJ(MM+1)),SNGL(Y(MM+1)))
50 CONTINUE

DO 55 I=1,NMP
DO 55 J=I,MM
S(I,J)=CMPLX(AAR(I,J),AAR(MM+J,I))
RB(J)=REAL(S(I,J))
55 CONTINUE

CALL SOLVE THE COMPLEX EQUATION [BIN][BETA] = -[E]
BY THE LEAST SQUARES METHOD USING THE REAL VERSIONS
OF THE MATRICES AR, AAR AND BR RESPECTIVELY
ETA=2.**(37)
IFAIL=0
I=2*NMP
J=2*MM
CALL F04AMF(AR,M1DIM,AAR,M4DIM,BR,M1DIM,M2,MM2,MM,ETA,OP,MAIN
IDTAGE,WKSP,WY,WZ,WR,IPIV,IFAIL)

C ***END OF LEAST SQS: MOD***
C BETA IS TRANSPOSED TO GIVE THE SCATTERING MATRIX, S
C AND S IS PRINTED.
IF (SUPPS.EQ.1) WRITE (6,60)
60 FORMAT (1X,RHS MATRIX,///)
DO 70 J=1,MM
DO 70 J=1,MM
S(J,J)=CMPLX(AAR(J,I),AAR(MM+J,I))
RB(J)=REAL(S(J,J))
70 CONTINUE
SR(I,J)=RR(J)
IF (ABS(SR(J)) .LE. 0.000001) RR(J)=0.0

65 CONTINUE
IF (SUPPS .LE. 1) WRITE (6,120) (RR(J),J=1,MMM)

70 CONTINUE
IF (SUPPS .LE. 1) WRITE (6,125)
DO 80 I=1,MMM
DO 75 J=1,MMM
RR(J)=AIMAG(S(I,J))
SI(I,J)=RR(J)
IF (ABS(SR(J)) .LE. 0.000001) RR(J)=0.0

75 CONTINUE
IF (SUPPS .LE. 1) WRITE (6,120) (RR(J),J=1,MMM)
80 CONTINUE

C CALCULATE THE EIGENVALUES OF S
C AND THE MODULUS AND PHASE OF THE EIGENVALUES

WRITE (6,85)
85 FORMAT (6,R5)
85 FORMAT (///10X,16HEIGENVALUES OF S,10X,7HM0DULUS,9X,5HPHASE///)

IF (!FAIL) CALL FZ2AKF(SR,M2DIM,SI,M2DIM,MMM,WWR,VI,VR,M2DIM,VI,M2DIM,INT,

1 IFAIL)
DO 90 , = 1, MMM
FVPHA(I)=ATAN2(WI(I),WWR(I))
EVMD(I)=SQR(T(WI(I))**2+WWR(I)**2)

90 CONTINUE
WRITE (6,95) (WWR(I),WI(I),EVMD(I),FVPHA(I),I=1,MMM)
95 FORMAT (10(/1X,4F15.5))

C END OF CALCULATING EIGENVALUES

C CALL THE VARIOUS SUBROUTINES TO CALCULATE THE SCATTERED FIELD

CALL MONOSTA(MMAX,M2DIM,SC,DATA,ALPHA,BETA)
CALL PATSCAT(MMAX,S,M2DIM,DATA,ALPHA,BETA)
NADI=NAD+1
WRITE (6,105)
WRITE (6,110) (DATA(I,92),I=2,NADI)
WRITE (6,125)
DO 100 I=1,91
J=(I-1)*2
WRITE (6,115) J,(DATA(L,I),I=1,NADI)
100 CONTINUE
DO 103 I=1,NADI
NMAX=MMAX-NY
IF (I .LE. 1) WRITE (7,118) A,NMP,MMAX,NMAX
IF (I .NE. 1) WRITE (7,118) A,NMP,MMAX,NMAX,DATA(I,92)
WRITE (7,117) (DATA(I,J),J=1,91)
103 CONTINUE
105 FORMAT(///1X,5HANGLE,5X,6HR.E,8.,26X,36HS C A T T E R I N G P A T T E R N,751X,19H(ANTENNA DISTANCEFS)///)
110 FORMAT (29X,6E12.2)
115 FORMAT (7F11.4)
118 FORMAT ((F7.3,315,F7.3)
120 FORMAT (1X,10E12.5)

31
125 FORMAT (///)
130 CONTINUE
STOP
END
A.2 Subroutine ABIN

This routine establishes the geometry of the square object and calculates the [B] matrix using the equation

\[
B_{in} = \sum_{m=-\infty}^{\infty} H_{n-m}^{(2)}(x_{o1}) \exp[j(n-m)\phi_{o1}]
\]

\[
J_{m}(x_{o1}) \exp(jm\phi_{o1})
\]
A.3 The Flowchart of Subroutine ABIN

1. ABIN
2. Define summation limits
3. Call BESSEL to find $J_n(kp_o)$ and $Y_n(kp_o)$
   - Do 30 I = 1, No. of boundary points
   - Calculate geometry of boundary point
   - Call BESJN to find $J_n(kp')$
   - Calculate $[B_{in}]$ elements for Ith point
4. RETURN
SUBROUTINE ABIN(RIS, THS, BIN, MMAX, NMP, MMM, N1, J, JD, Y)
C THIS SUBROUTINE SETS UP THE BIN MATRIX
COMPLEX B(N1, M1), HP, RN, CI, CJ, H1
REAL K
DOUBLE PRECISION J(101), JD(101), Y(90)
DIMENSION THS(N1), RIS(N1)
COMMON /BN1/K, A, P4, RM, SR/RN2/PI, P2, NX/RN3/CT
LEVEL 2, THS, RIS, BIN
RTW=0.5*SQ
MMAX=2*MMAX
MMM=2*MMAX
X=K*RM
CALL BESSEL(X, MMAX, J, Y)
DO 30, I=1, NMP
PHI=2.*PI*(I-1.)/NMP+0.0001
XIC=RM*COS(PHI)
YIC=RM*SIN(PHI)
IF (PHI.GT.P2, AND PHI.LT.P2) GO TO 5
IF (PHI.LT.5.*PI, AND PHI.LE.3.*PI) GO TO 10
YIC=RTW*RM*
GOTO 20
5 RD=RM*(COS(PHI-3.*PI)-RTW)
PD=7.*PI
YIC=YIC+RTW*RD
GOTO 20
10 RD=RM*(COS(PHI-5.*PI)-RTW)
PD=4.*PI
YIC=YIC+RTW*RD
GOTO 20
15 RD=RM*(COS(PHI-7.*PI)-RTW)
PD=3.*PI
YIC=YIC+RTW*RD
YIS=YIC+RTW*RD
GOTO 20
20 CONTINUE
THS(I)=ATAN2(YIS, XIS)
RTS(I)=SORT((XIS*XIS+YIS*YIS))
X=K*RD
CALL BESSEL(X, MMAX, J, JD)
DO 30, NN=1, MMAX
NN=NN+1
DO 25, JH=1, MMHD
JH=JH+1
JH=JH+1
H2=CMPLX(SNGL(JC), -SNGL(YJC))
IF ((N+M) .LT. 0) H2=H2*H2
R1=JD(JR)
35
IF (M LT 0) RJ = JHH* RJ
RN = RN + (-1)**M*RJ*M2*EXP(CI*((N+M)*PHI-M*PD))
25 CONTINUE
BIN(I,NN) = CI**(N)*BN
30 CONTINUE
RETURN
END
This subroutine evaluates the monostatic R.C.S. of the scatterer investigated using the equation

\[
R.C.S. = \sum_{n=-N}^{N} \left( \sum_{m=-N}^{N} \sigma_{mn} a_{-m} \right) \exp(jm(\theta+\pi))
\]

where

\[
a_n = \frac{(-1)^n}{2} \exp(-jn\theta)
\]
A.5 The Flowchart of the Subroutine MONOSTA

1. Define summation limits with edge errors ignored

2. Do 20 TH = 0, 180

3. Calculate incident field amplitudes $[\alpha_m]$ for $0 \leq M \leq M_{\text{MAX}}$

4. Do 10 $M = M_{\text{MAX}}$

5. Calculate scattered field amplitudes $[\beta_m]$

6. Calculate R.C.S., normalize and store in data file

7. Return to the main program
SUBROUTINE MONOSTA(MX, S,M1, SC, DATA, ALPHA, BETA)

THIS SUBROUTINE CALCULATES THE MONOSTATIC RCS OF THE SCATTERER

DIMENSION DATA(6,92)
COMPLEX S(M1,M1), SC(M1), CI, PS, SS, ALPHA(M1), BETA(M1)
COMMON /BN2/ PI, P?, NX/BN3/ CI

LFVFL 2, S
MMAX=MX-NX
MM=2*MMAX+1
DO 20 L=1,91
TH=2.*FLOAT(L)-1.
THR=PI*TH/180.0
DO 5 T=1,MM
N=I-MMAX-1
ALPHA(I)=CMPLX((-1.)**N*0.5,0.)*CEXP(CMPLX(0.,FLOAT(N)*THR))
5 CONTINUE
DO 10 I=1,MM
BFTA(I)=(0.,0.)
DO 10 K=1,MM
N=K-MMAX-1
BETA(I)=BFTA(I)+S(I+NX,K+NX)*ALPHA(K)
10 CONTINUE
SS=(0.,0.)
DO 15 I=1,MM
M=J-MMAX-1
SC(I)=BETA(I)-ALPHA(I)
SS=SS+SC(I)*CEXP(CMPLX(0.,FLOAT(M)*THR+PI))
15 CONTINUE
IF (L.EQ.1) PS=SS*CONJG(SS)
DATA(1,L)=RFAL((SS*CONJG(SS))/PS)
20 CONTINUE
RETURN
END
A.6 The Subroutine Patscat

This subroutine calculates the scattering pattern for a variable number of source distances using the equation

\[ \sum_{m=-N}^{N} \sum_{n=-N}^{N} \sigma \alpha_{mn} - \alpha_{m} + j^{m} J_{m} (kD) \exp(-jm\phi) \exp(jm\phi_{2}) \]

where \[ \alpha_{n} = \frac{J_{n}^{H}(2)(kD)}{n} \exp(-jn\phi) \]
The Flowchart of the Subroutine PATSCAT

PATSCAT

Define summation limits with edge errors ignored

DO 50 J=1, No. of source distances

Calculate source distance D

Call BESSEL to find $J_n(kD)$ and $Y_n(kD)$

Calculate $H_n^{(2)}(kD)$

Calculate incident and scattered field amplitudes, $\alpha_m$ and $\beta_m$

Calculate the modulus of the total field

DO 50 K=0, 180

Calculate total field for Kth angle, normalize and store in DATA

50 continue

return
SUBROUTINE PATSCAT(MX, S, M1, H2, SM, JJ, Y, DATA, ALPHA, BETA)
DOUBLE PRECISION JJ(101), Y(90)
DIMENSION DATA(6, 92)
COMPLEX S(M1, M1), H2(M1), SM(M1), SS, PS, CS, CI, ALPHA(M1), BETA(M1)
COMMON /BN2/ PI, P2, NX/BN3/ CI

THANT = 0.
MMAX = MX - NX
MAXM = MMAX + 1
MM = 2 * MMAX + 1
DO 50 J = 1, MMAX
   J = J + 1
   DATA(J + 1, 92) = D
   Z = 2.0 * PI * D
   CALL BESSEL(Z, MAXM, JJ, Y)
   DO 10 I = 1, MMAX
      H2(I) = CMPLX(SNGL(JJ(I)), -SNGL(Y(I)))
   CONTINUE
   DO 15 I = 1, MMAX
      ALPHA(I) = CI**N**5 * H2(I) * CEXP(CMPLX(0., FLOAT(-N) * THANT))
   CONTINUE
   DO 20 JX = 1, MMAX
      JX = JX + 1
      HTA(T) = HTA(T) + S(I + NX, JX + NX) * ALPHA(JX)
   CONTINUE
      SM(I) = (HTA(I) - ALPHA(I)) * CI**(-N)
      IF (M . LT. 0) GOTO 25
   SM(T) = SM(T) + SNGL(JJ(I))
   GOTO 30
25 JH = (-1.)**M
   SM(T) = SM(T) + JH*SNGL(JJ(JH))
30 CONTINUE
   SM(T) = SM(T) * (CI**M)
5 CONTINUE
   SS = (0., 0.)
   DO 40 KK = 1, MMAX
      KKK = KK - MMAX + 1
      SS = SS + SH(KK)
   CONTINUE
   DO 50 K = 1, 91
      TH = 2.0 * (FLOAT(K) - 1.0)
      THD = TH * PI / 180.0
      PS = (0., 0.)
      DO 45 L = 1, MMAX
         N = L - MMAX + 1
         CS = CEXP(CMPLX(0., 0., FLOAT(N) * THD))
PS=PS+CS*SM(L)
45 CONTINUE
PAD=REAL((PS*CONJG(PS))/(SS*CONJG(SS)))
DATA(J+1,K)=10.*A10G10(ABS(PAD))
50 CONTINUE
RETURN
END
A.7 The Standard Routines

Two of the other routines are N.A.G. routines ie. FO4AMF and FO2AKF. FO4AMF is used to find the least squares solution of the following matrix equation

$$- [E_{nt}] = [b_{in}][\sigma_{nt}]$$

and FO2AKF is used to find the eigenvalues of the solution $[\sigma_{nt}]$.

The other routines, BESSEL, BESJN and GAMMA originate in the CERN program library and are used to give the Bessel functions $Y_n(x)$ and $J_n(x)$.
REFERENCES


University of Surrey

DEPARTMENT OF PHYSICS

RADIATION FROM ANTENNAS MOUNTED
ON IRREGULARLY SHAPED BODIES

The Scattering Matrix for a Body of Revolution

D. MUNRO

1978
The calculation of a scattering matrix for a body of revolution.

The method used to calculate the scattering matrix is similar to that used by Wilton and Mittel (Ref.) for an infinite cylinder. The scattered field is given by Klain and Wasylikowski (Ref.) as

\[ \mathbf{E}_{\text{scatt}} = \sum_i \sqrt{E} \mathbf{b}_i \hat{H}_n^{(2)}(kr) \mathbf{e}'_i(\theta, \phi) + \sqrt{E} \mathbf{b}_i \hat{H}_n^{(1)}(kr) \mathbf{e}''_i(\theta, \phi) \]

in the coordinate system \( \mathcal{O} \), shown in Fig. 1. The summation over \( i \) is a simplification of

\[ \sum_{i = 1}^{N} \sum_{n = 0}^{\infty} \sum_{m = -n}^{n} \sum_{l = 0}^{n} \]

and \( \mathbf{e}'_i(\theta, \phi) \) and \( \mathbf{e}''_i(\theta, \phi) \) are defined in appendix 1.

The scattering body is a body of revolution, the scattering in each plane containing the axis of revolution will be identical, and only the scattering matrix in one of those planes need be considered. The scattering in any other plane is discussed later. Thus, we shall only deal with the \( \varphi \) plane with \( \varphi = \frac{\pi}{2} \).
\[ E_{\text{scat}} = \sum_{n=1}^{\infty} \left( \mathbf{H}^{(2)}_{m} (\mathbf{r}_n) \left( \cos m \Phi, \sin m \Phi \right) (A_i \cos \Theta_0) \right. \\
+ \sqrt{5} \mathbf{B}_{n}^{(2)} (\mathbf{r}_n) \left( \cos m \Phi, \sin m \Phi \right) (B_i \cos \Theta_0) \left. \right) \cos m \Phi, \sin m \Phi \right) (-A_i \cos \Theta_0) \]

where \( A_i \) and \( B_i \) are given in appendix 1.

The scattered field coefficients for a particular incident field can be found on the surface of the scatterer using the boundary conditions, but the equation for the field in coordinates of \( r \) and \( \Phi \) will not converge when \( r < r_i \). This is due to the discontinuity of the scatterer in the space outside of the field, defined by \( E \) over which the field must converge, and is discussed for the 3-D situation by M. W. Liu and M. Mittra (Ref. 1). We must therefore redefine the field in coordinates \( r', \Phi', \) and \( \Phi \), which describe an area of space free of singularities.

To do this, the Bessel function addition theorem from Abramowitz and Stegun (Ref. 2) is used to write the expansion below as shown in appendix 2.

\[ \mathbf{H}^{(2)}_{m} (\mathbf{r}_n) = \left( \begin{array}{cc} \cos m \Phi & \cos (m+\frac{1}{2}) \Phi \\ \sin m \Phi & \sin (m+\frac{1}{2}) \Phi \end{array} \right) \]

\[ \sum_{n=0}^{\infty} \mathbf{H}^{(2)}_{m} (\mathbf{r}_n) \mathbf{J}_n (\mathbf{r}_n) \left( \begin{array}{c} \cos (n+\frac{1}{2}) \Phi' \\ \sin (n+\frac{1}{2}) \Phi' \end{array} \right) \left( \begin{array}{c} \cos \Phi' \\ \sin \Phi' \end{array} \right) \]
\( \hat{H}_n^{(2)}(R) \) is given in appendix 3 to be
\[
\hat{H}_n^{(2)}(R) = (n+1) \left[ \hat{H}_n^{(2)}(R) - \hat{H}_{n+1}^{(2)}(R) \right]
\]
Using these expansions, we arrive at a convergent matrix equation for the 2nd boundary point from equation 16.
\[
[ -E^{\text{scat}} ] = \left[ \begin{array}{cc} \Delta^{E \text{inc}} & \Delta^{E \text{inc}} \\ \Delta^{E \text{inc}} & \Delta^{E \text{inc}} \end{array} \right] \left[ \begin{array}{c} E^{\text{inc}} \\ i \end{array} \right]
\]
Now, the boundary condition on the surface
\[
-r E^{\text{inc}} = r E^{\text{inc}}
\]
can be applied where \( r E^{\text{inc}} \) is given as
\[
3 \ r E^{\text{inc}} = \sum \sqrt{\alpha} \frac{\hat{H}_n^{(1)}(R)}{c_t(\Theta, \Phi)} + \sqrt{\alpha} \frac{\hat{H}_{n+1}^{(1)}(R)}{c_t(\Theta, \Phi)}
\]
As the scatterer has an axis of rotation on the \( X \)-axis, only the energy incident in the \( XY \) plane will be scattered into the \( XY \) plane and into no other plane. So again we can set \( \Theta = 0 \) and \( T = R \). We must also expand \( \hat{H}_n^{(1)}(R) \) using the same expansion from appendix 2, and \( \hat{H}_n^{(1)}(R) \) using the expansion from appendix 3. If we set \( a_t = 1 \) for one mode and all others to zero then using the equations linking incident and scattered coefficients given by Rahn & Wasylikowski.
we have
\[ \frac{E_i}{E_{in}} = \frac{S_{in}}{S_{in}} a_i = S_{in} \]

Therefore, to find the scattering matrix we must solve

\[ 5a \quad -[\vec{\Theta} E_{inc}] = [\vec{\Theta} \Sigma_{in}] [S_{in}] \]

As is shown in Appendix 1, Ai and Bi are alternatively zero so the incident modes are either $\Theta$ or $\Phi$ phases. If we take one element of the incident field matrix we have

\[ -[\vec{\Theta} E_{inc}] = \frac{E_{inc}}{E_{in}} \Theta = \frac{\sum_{i=1}^{n} S_{in}}{i E_{in}} \Theta = \frac{\sum_{i=1}^{n} S_{in}}{i E_{in}} \Theta \]

The second summation must be zero and the same will be true for a $\Phi$ phase incident mode so $S_{in}$ and $S_{in}$ it will all be zero. Thus we can reduce equation 5a to

\[ 5b \quad -[\vec{\Phi} E_{inc}] = [\vec{\Phi} \Sigma_{in}] [S_{in}] \]

giving an independent scattering matrix for each phase.

The number of scattering matrix elements necessary can be further reduced as the angular part of some scattered modes are zero at all $\Phi$ and these modes...
Even more incident modes will be zero for special source positions and this will reduce the number of scattering matrix elements further, but this is discussed later.
The calculation of a scattering matrix for an infinite cylinder.

The 2-D scattering matrix is found in an identical way to the 3-D matrix using only simpler equations. The scattered field is given by Wilton and Matthaei (ref.) to be

$$E_{\text{scat}} = \sum_{n=-\infty}^{\infty} \beta_n H_n^{(2)}(kr) \cos \phi$$

where the geometry is identical to the projection onto the $xy$ plane of the 3-D case as in Fig. 1. The Bessel function addition theorem is used to produce the expansion also given in appendix 2 as

$$H_n^{(2)}(kr) \cos \phi = \sum_{m=-\infty}^{\infty} H_n^{(2)}(kr_1) J_m(kr_1) i^{m+n} \phi' e^{im\phi}$$

for $r_1 > r$.

Once again we have the matrix equation for the $\phi$ boundary point

$$[E_{\text{inc}}] = [X_{\text{scat}}] [\beta_n]$$

and using the same boundary conditions we have to solve for the scattering matrix $[\beta]$ in the equation

$$-[E_{\text{inc}}] = [X_{\text{scat}}] [\beta]$$

where $E_{\text{inc}} = \frac{1}{4\pi} \alpha + H_0^{(2)}(kr) \cos \phi$ with $\alpha = 1$ and $H_0^{(2)}(kr) \cos \phi$ extended.
The calculation of the spherical modes incident on a sphere containing the scatterer due to a source outside the sphere.

The sphere $S$ is defined by $r = a$ as shown in Fig. 2. Outside $S$ in a whole with orientation $\omega$, at $r = \infty$. The field at any point $P$ can be expanded about $C$, the origin, using the Dyadic Green's function as shown by Wynn and Wainwright, Jones.

\[
10 \quad \mathbf{E} = \mathbf{E}_0 \int \frac{d^2 \mathbf{R}}{4 \pi} \left\{ \sum_{n=1}^{\infty} \frac{i}{n} \mathbf{H}_n^{(2)}(kr)v(r) \mathbf{e}_n^1(\theta, \phi) \cdot \mathbf{U}_0 \right\}
\]

\[
= -n(n+1) \mathbf{H}_n^{(2)}(kr) \mathbf{e}_n^1(\theta, \phi) \mathbf{U}_0 \frac{e_n^1(\theta, \phi)}{(kr)^2} \left[ \sum_{n=1}^{\infty} \frac{i}{n} \mathbf{H}_n^{(2)}(kr) \mathbf{e}_n^1(\theta, \phi) \cdot \mathbf{U}_0 \right]
\]

where $\mathbf{H}_2^{(2)}$ becomes consistent for normalizing with $\mathbf{H}_2^{(1)}(kr)$.

For $r > a$, all the Green's functions $\mathbf{H}_n^{(1)}(kr)$ become $\mathbf{H}_n^{(2)}(kr)$ and $\mathbf{H}_n^{(3)}(kr)$ respectively.

The fields can now be put in the form suitable for the scattering matrix as

\[
11 \quad \mathbf{E} = \sum_{n=1}^{\infty} \frac{\mathbf{H}_n^{(2)}(kr) \mathbf{e}_n^1(\theta, \phi) \cdot \mathbf{U}_0}{kn^2} - n(n+1) \mathbf{H}_n^{(2)}(kr) \mathbf{e}_n^1(\theta, \phi) \mathbf{U}_0
\]

where $\mathbf{e}_n^1 \cdot \mathbf{U}_0$.
The purely scattered field coefficients are given by Kahn and Wassijewsky (Ref 1) as

\[ b_i = \sum \phi_s \{ e^{i(\theta-\phi)} - 1 \} \]

giving the total field coefficients as

\[ b_i = \text{scat } b_i + \text{out } b_i \]

where \( \text{out } b_i \) is found using the Dyadic Green's function with \( F \). This gives the field as

\[ E = \frac{1}{i} \sum b_i \hat{\mathbf{e}} \cdot \mathbf{H}^{(2)}(r) e^{i(\theta, \phi)} + \frac{1}{4} \sum \text{out } b_i \hat{\mathbf{e}} \cdot \mathbf{H}^{(2)}(r) e^{i(\theta, \phi)} \]

If this is taken to the limit of \( r \to \infty \)

\[ \mathbf{H}^{(2)}(r) \propto \frac{1}{r} \]

As the scattered field, considered in only in the XY plane, \( \theta = \frac{\pi}{2} \) giving

\[ E = \sum b_i \hat{\mathbf{e}} \cdot \mathbf{H}^{(2)}(r) e^{i(\theta, \phi)} \]

\[ E = \sum b_i \hat{\mathbf{e}} \cdot (\cos \theta, \sin \theta, \sin \theta, -\cos \theta) \]
As each phase is treated separately in producing the scattering matrices, each phase is treated separately in producing the scattering pattern.
The calculations of the cylindrical modes incident on a circular cylinder containing the scatterer due to a source outside the circle.

The cylinder $C$ is defined by $r = a$, as shown in Fig. 3. Outside $C$ is a line source at $r$, and the field due to the source at any point $P$, in the coordinate system $O_1$, is given as

17. \[ E \propto H_0^0 (kr_1) \]

This can be expanded about $O_1$, noting that the expansion is from $O_1$ to $O$, the reverse of that in calculating the scattering matrix. Using the expansion from the Bessel function addition theorem in Appendix 2 to give

\[ E \propto \sum_{n=-\infty}^{\infty} H_n^0 (kr') J_n (kr) e^{\pm i n (\phi - \phi')} \]

for $r' > r$

\[ \propto \sum_{n=-\infty}^{\infty} \alpha_n \left[ H_n^0 (kr) J_n (kr') e^{\pm i n (\phi - \phi')} \right] \]

in the form

18. \[ E_{inc} \propto \sum_{n=-\infty}^{\infty} \alpha_n \left[ H_n^0 (kr) e^{\pm i n \phi} \right] \]

where \[ \alpha_n = \int \frac{H_n^0 (kr') e^{\pm i n \phi'}}{2} \]

and we can find the scattered field coefficient.
\[ B_{\text{inc}} = \frac{E_{\text{inc}}}{m} \cdot \beta_{\text{inc}} \]

where \( \beta_{\text{inc}} \) is the outgoing coefficient of the incident field and \( \beta_{\text{inc}} = \alpha \). The outgoing field coefficient is found in a similar fashion to be

1a. \[ B_{\text{out}} = \beta_{\text{inc}} \cdot J_n(k_r r) \cdot e^{-i\alpha} \]

We now have the total field given as

20a. \[ E_{\text{tot}} = \frac{E_{\text{inc}}}{m} \cdot \beta_{\text{tot}} \cdot J_n(k_r r) \cdot e^{-i\alpha} \]

where \[ \beta_{\text{tot}} = \beta_{\text{inc}} + \beta_{\text{circ}} \]

As \( r \to \infty \), \( J_n(k_r r) \to 0 \) \( m \), so we have

20b. \[ E_{\text{tot}} \to \frac{E_{\text{inc}}}{m} \cdot \beta_{\text{tot}} \cdot e^{-i\alpha} \]
The calculation of the modes incident on a sphere containing the scatterer due to a source inside the sphere

The sphere $S$ is defined by $r = R$, as shown in Fig. 4. Inside $S$ is a slit with orientation $\gamma = \theta$, at $A$. The field at any point $P$ can be expanded about a point $O$, outside $S$ and on the $x_1$ plane where the projection of $P$ on that plane is perpendicular to the surface of the scatterer. This is given by Helmholtz's expansion for the field incident on the scatterer as

$$ F_i = \frac{\sin \gamma}{\gamma} \int \frac{e^{i \gamma r_i}}{r_i} d \gamma $$

$$ \sim \sum \frac{\sin \omega_i^2 \rho}{\omega_i^2 \rho} \frac{e^{i \omega_i \rho}}{\rho} \phi_{\omega_i} $$

In this case, the exponential form of the vector modes is used as

$$ F_{\omega_i} = \frac{1}{\gamma} \frac{e^{i \gamma r_i}}{r_i} $$

$$ \sim \sum \frac{\sin \omega_i^2 \rho}{\omega_i^2 \rho} \frac{e^{i \omega_i \rho}}{\rho} \phi_{\omega_i} $$

We now have a field about $O'$ outside the sphere defined by $r > R$ in the form

$$ F_i = \frac{1}{r_i} \frac{e^{i \gamma r_i}}{r_i} \frac{1}{\gamma} $$

$$ \sim \sum \frac{\sin \omega_i^2 \rho}{\omega_i^2 \rho} \frac{e^{i \omega_i \rho}}{\rho} \phi_{\omega_i} $$

where

$$ \phi_{\omega_i} = -\frac{1}{\gamma} \frac{e^{i \gamma r_i}}{r_i} \frac{e^{i \omega_i \rho}}{\rho} \phi_{\omega_i} $$
\[ b_{in}(r''') = \sqrt{\frac{6\pi}{\rho}} \left( \sum_i \left( \frac{J_i(kr)}{k} \right) e^{-i(\theta'' - \phi'')} \right) \]

Now, as we are only interested in the field incident on \( S \) in the \( xy \) plane, we can set \( \Theta = \frac{\pi}{2} \) to give

\[ E_x = \sum_i \left( i \hat{E}_n(r''') \right) \hat{A}_n(r''') e^{i m \phi} \left( A_i \Theta + B_i \phi \right) \]

\[ + \sum_i \left( i \hat{E}_n(r''') \right) \hat{A}_n(r''') e^{i m \phi} \left( B_i \Theta - A_i \phi \right) \]

Each mode \( \hat{A}_n(r''') \) can be expanded about 0 in terms of \( r \) and \( \phi \), noting that the expansion is from 0' to 0 using

\[ \hat{A}_n(r''') = (n+1) \frac{1}{\hat{E}_n(r''')} \hat{A}_n(r''') - \hat{A}_n(r''') \]

from appendix 3 and

\[ \hat{E}_n(r''') e^{i m \phi} = \delta^{(n-\pi \frac{1}{2})} \phi \frac{\hat{E}_n(r''')}{\hat{E}_n(r''')} \]

for \( r''' > r \)

\[ \sum_{n=0}^{\infty} \hat{A}_n(r''') J_{\pi}(r') e^{i(n-\pi \frac{1}{2}) \phi} e^{i \pi \phi} \]

\[ = \sum_{n=0}^{\infty} \hat{E}_n(r''') \right( \frac{\phi}{\hat{E}_n(r''')} \right) J_{\pi}(r') e^{i \pi \phi} \]

from appendix 2 and with \( \xi = \frac{\pi}{\hat{E}_n(r''')} \)

If we substitute these equations into equation 2.3 and reverse the order of summation, we have
\[
\begin{align*}
t_2 E = \sum_i \sqrt{\frac{\nu_i}{2}} \left\{ \sum_j b_i^E (r^\prime) \left[ \frac{(n+1)^2}{k_2^2} \right] A_i + \sum_j b_i^H (r^\prime) \left[ \frac{(n+1)^2}{k_2^2} \right] B_i \right\} \\
+ \sum_j b_i^H (r^\prime) \left[ \frac{(n+1)^2}{k_2^2} \right] A_i + \sum_j b_i^E (r^\prime) \left[ \frac{(n+1)^2}{k_2^2} \right] B_i \right\} \\
\text{for } r'' > r
\end{align*}
\]

This can be put in the form of incoming and outgoing cylindrical modes about \( \theta \) as

\[
\begin{align*}
t_2 E = \sum_i \frac{1}{2} \left[ A_i \epsilon_0 \left( r'' \right) H_{10}^-(r) + B_i \epsilon_0 \left( r'' \right) H_{10}^+(r) \right] e^{j \phi} \\
+ \sum_i \frac{1}{2} \left[ A_i \epsilon_0 \left( r'' \right) H_{10}^-(r) + B_i \epsilon_0 \left( r'' \right) H_{10}^+(r) \right] e^{-j \phi} \\
\text{for } r'' > r \text{ where}
\end{align*}
\]

\[
\begin{align*}
A_i &= \frac{1}{2} \left[ A_i \epsilon_0 \left( r'' \right) H_{10}^-(r) + B_i \epsilon_0 \left( r'' \right) H_{10}^+(r) \right] \\
B_i &= \frac{1}{2} \left[ A_i \epsilon_0 \left( r'' \right) H_{10}^-(r) + B_i \epsilon_0 \left( r'' \right) H_{10}^+(r) \right]
\end{align*}
\]

and similarly for the \( \phi \) coefficients.

Thus we have the incoming field about 0 in the correct form to use the scattering matrix for 2 - D cylindrical modes. The scattered field coefficients are found by
27 \quad \text{sect} \theta, \phi = \frac{1}{x^2} \left( \text{inc} \theta, \phi - \text{inc} \beta, \phi \right)

and the scattered field is given by

\[ r \tilde{E}_n = \frac{1}{x^2} \left( \text{sect} \theta, \phi \right) \tilde{E}_0 + \frac{1}{x^2} \left( \text{sect} \beta, \phi \right) \tilde{E}_0. \]

as \( r \to \infty \) and \( \hat{r} \to 1 \), this allows the coefficient \( \beta, \theta, \phi \) to be reduced to

\[ \beta, \theta = \alpha, \theta = \frac{1}{2} \left\{ \frac{x^2}{i} \text{sh} \left[ \frac{\pi}{x^2} \left( \frac{i}{x^2} \text{sh} \left( \frac{i}{x^2} \right) \right) \right] \tilde{A} \right\} \]

and similarly for the \( \phi \) components, allowing the \( \tilde{E}_0 \) in Equations 28 and 28 to cancel. The total field is then

\[ r \tilde{E}_n = r \tilde{E}_\text{sect} + r \tilde{E}_\text{out} \]

where \( r \tilde{E}_\text{out} \) is found in the same way as for a source outside the sphere \( S \).
The calculation of the modes incident on a sphere cylinder containing the scatterer due to a source inside the cylinder.

The cylinder \( C \) is defined by \( \text{Emin} \) as shown in fig. 5. The field \( \text{Einc} \) due to the source \( \text{Esource} \) is given in the coordinate system \( A \) as \( \text{Einc} = \text{Esource} \). This can be expanded about a point, outside the circle defined by \( \text{Emin} \), say \( \text{O} \) where \( t'' \) is perpendicular to the surface of the object, as

\[ \text{Esource} = H_n^2(\text{Rt}) \]

\[ 30 \quad E_{inc} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(\text{Rt}) e^{i(\text{n-m}\phi)} H_m^2(\text{Rt}) e^{i\text{m}\phi} \text{ for } t'' < t_2 \]

\[ 31 \quad E_{inc} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \beta_n^{in} \xi_n \xi_m^2(\text{Rt}) e^{i(\text{n-m}\phi)} J_m(\text{Rt}) e^{i\text{m}\phi} \text{ for } t'' > t \]

and \( E_{out} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(\text{Rt}) e^{i(\text{n-m}\phi)} H_m^2(\text{Rt}) e^{i\text{m}\phi} \text{ for } t'' < t \)

where \( \beta_n^{in} = J_n(\text{Rt}) e^{i\text{n}\phi} \)

\[ 32 \quad E_{inc} = \sum_{m=-\infty}^{\infty} [\alpha_m H_m^1(\text{Rt}) + \beta_m H_m^2(\text{Rt})] e^{i\phi} \text{ for } t'' > t \]

Reversing the order of summation gives the field in the form that is required by the scattering matrix.
and \( E_{\text{out}} = \sum_{m=\infty}^{\infty} \beta_{m} \phi_{m}(r) e^{im\phi} \) for \( r'' < r \)

where \( \alpha_{m} = \beta_{m} = \sum_{n=\infty}^{\infty} \beta_{n} \phi_{n}(r) e^{im\phi} \)

and \( \beta_{m} = \sum_{n=\infty}^{\infty} \beta_{n} \phi_{n}(r) e^{im\phi} \)

Now, as previously shown, the scattered field is found using

33 \[ \beta_{\text{scat}} = \sum_{m=\infty}^{\infty} \alpha_{m} - \beta_{m} \]

and the total field is found using

34 \[ \beta_{m} = \beta_{\text{scat}} + \beta_{\text{out}} \]

giving the total field as

35 \[ E_{\text{tot}} = \sum_{m=\infty}^{\infty} \beta_{m} e^{im\phi} \]
The radar cross-section of a cylindrical body.

1. Plane wave incident on the origin O of a scatterer in the direction θ, as in Fig. 6, is given by

\[ \exp(i k \cdot l) = \sum_{n=-\infty}^{\infty} \frac{H_n(i k r)}{n!} \exp(in(\phi - \Theta)) \]

This gives an incident field coefficient suitable for using in a scattering matrix of

\[ \alpha_n = \frac{(-1)^n \exp(-in \Theta)}{2} \]

The R.C.S. is given as proportional to

\[ \sum_{m=-N}^{N} \left( \sum_{n=-N}^{N} \alpha_n \alpha_m \right) \delta^{m-n} H_m^{(2)}(kr) \exp(i m \phi) \]

Taking \( r \) to infinity we have for the bistatic case

\[ \sum_{m=-N}^{N} \left( \sum_{n=-N}^{N} \alpha_n \alpha_m \right) \exp(i m \phi) \]

and for the monostatic case, the substitution \( \phi = \Theta + \pi \) is made.
A discussion of the limit required on the summations.

One of the attractions of finding the scattered field by this method is that the number of modes required can be calculated before any work is done. The process of finding the limits is identical in either 2-D or 3-D and therefore only the 2-D case need be considered. This has been shown by Wilton and Mittra (ref 7) for the limits required on the scattering matrix to be $|\mathbf{S}| < \min \left( R^+ \right)$ (see equation 6 and fig. 1), however experience has shown that in most cases the limit is dependent on the radius of a circle of the same area as the scatterer. Thus we calculate $J_n(R)$ and use all the modes until $n = n_{\text{max}}$ such that $J_{n_{\text{max}}}(R) < 0.01$

Wilton and Mittra (ref 7) also stated that the limit on the internal summation over $t$ (see equation 6 fill) is dependent on $R^+ R_{\text{min}}$. As $t = \min$ in our calculations we simply take $n_{\text{max}} = 2 n_{\text{max}}$ and this has proved adequate.

The limits required for the field due to a source outside the enclosing circle are determined by the convergence of the outgoing field coefficients which are proportional to $J_n(R^-)$ (see eqn 19). Thys we find $n_{\text{max}}$ such that $J_{n_{\text{max}}}(R^-) < 0.001$ and $n_{\text{max}} > n_{\text{max}}$ where $n_{\text{max}}$ is the limit on the orders required for the scattering matrix. Where $n > n_{\text{max}}$ we can see that the scattering matrix tends to
that for free space and the scattered field
coefficients \( m \) can be taken to be zero for
in \( r_{\text{max}} \). But for a source outside the
enclosing cylinder, the number of modes
required for the outgoing field can be greater
than the number of modes for the scattered
field. This allows the source to be moved away
from the object without significantly increasing
the difficulty of the problem. An example of the
application of this solution is the field due
to the tail fins of an aircraft, excited by a
nose antenna.

This solution is also applied when the
source is inside the enclosing cylinder. We
are using the Bessel function addition theorem
where the inside summation is twice-out limit
and twice the outside summation limits, but
in both 2-D and 3-D cases, we reverse the summation. The new outside summation limit
the maximum order for the convergence of the
new outside summation is dependant on \( R_1 + R_2 \)
and \( r_{\text{max}} \) is found using \( \text{Tr}_{\text{max}} \) < \( 0.01 \).
This is the limit on the order of incident modes
and will be greater than the order used in the
scattering matrix. Once again, the scattering
matrix is assumed to tend towards a free
space scattering matrix for orders outside
the known matrix and the scattered field
coefficients are set to zero.

Having decided on the number of modes
necessary, we must now decide on the
number of boundary points required for a
stable scattering matrix. This number is
dependent on the number of scattered modes in the scattering matrix and must be greater than this number. Although it is possible to calculate a matrix with the same number of points as modes, it was found that twice the number of modes was a suitable ratio to produce a stable matrix. This was found empirically by increasing the number of points until there was no significant change in the total field pattern.
A discussion of the incident modes necessary

In the 3-D case, a large number of incident modes are possible as was shown in previous reports. However, it is possible to reduce this number for incident modes although all the scattered modes must be considered. The incident field is given in Section 3.1 equation 11 and each coefficient is dependent on

\[ e^i(\theta, \phi) \cdot \mathbf{U}_0 \text{ and } \mathbf{\Phi}_i(\theta', \phi') \text{ to } \mathbf{U}_0 \]

or

\[ e^o(\theta, \phi) \cdot \mathbf{U}_0 \]

for any mode, if the coefficient is zero, there will be no incident field and this mode need not be considered. The dot products given above are the factors which will decide whether a coefficient is zero or not. If these are calculated before calculating the scattering matrix, we will know which matrix elements we need to consider. Then we will be left with a rectangular scattering matrix and this causes problems in checking for convergence in the scattering matrix. We are no longer able to calculate the eigenvalues although it may be possible to use the

reciprocity check.
Program development

scatterer shapes and the calculation of the boundary
front

At present, the program will calculate the
scattering matrix, in the 3-D case, for a
body of revolution with the axis of revolution
defined as the x-axis. In the 2-D case,
the body is a cylinder with the constant
cross-section up the z-axis. The cross-
section of the object in the xz plane is
non-reentrant although Wilton and Mittra (ref)
give the method for dealing with reentrant shapes.
Any object within these limits can be dealt with
although most will need some approximation.
The cross-section is defined by vertex
coordinates and the coordinates of the centre
of a circle for each vertex which defines
the outline to the next vertex. Straight-
lines are defined using a circle of very
large radius if it should be possible for
effects to be included without great difficulty.
All the dimensions are in wavelengths.
The boundary points are found by dividing
the perimeter of the object into equal lengths.
The corners do not raise any problem by this
method as in most cases, having boundary
points on the corners and therefore trying to
field the field at the corners can be avoided.
If, as in the program, a do not have as much
symmetry as the object, then only when
sufficient modes and points have been taken
does the final pattern show the symmetry of the
object. This is very useful in checking for stability of the scattering matrix.
Maximum scatter size

The limiting factor in determining the maximum size of the scatterer that the program can handle is the size of the scattering matrix. The program has been written to make the most efficient use of all the core space available in solving the simultaneous equation to find the scattering matrix. The factor that determines the amount of core required is the number of scattered modes required. This number is dependent on the size of the scatterer alone and is proportional to the maximum radius of the smallest circle which encloses the area of the cross-section in the flow of the scatterer. In the 2-D case, the number of modes is twice the order of the highest mode required, but in the 3-D case, the number of modes is proportional to the square of the highest order. The maximum order is proportional to the radius of the smallest circle which encloses the area of the cross-section of the scatterer in the X-Y plane in both 2-D and 3-D cases. Thus the maximum size of the scatterer can vary with the shape of the scatterer but a rough guide is that in the 2-D case, the maximum size is an object with a maximum dimension of 1.0 wavelengths, and in the 3-D case, of 0.5 wavelengths.

Although the maximum size in the 2-D case is acceptable, in the 3-D case, obvious ways of getting around this problem...
first is to use a computer with an extremely large working space although as the amount of space required increases by the fourth power as the size of the scatterer is increased, a machine with conventional hard core memory would might not give a sufficient increase and new machines with virtual storage might be the answer. The second route is to use and array processing might be the answer. The second way is to not use the straight 3-D solution to the problem but the method required when the source is inside the enclosing sphere. This uses a 2-D scattering matrix. The problem is really sidestepped but at the cost of increased computer time although this is a less restrictive limitation. The third way requires more theoretical work but present program performance indicates that more scattered modes are being used than are necessary and many might be eliminated. This is discussed further in the next section.
Scattering matrix stability

The most difficult problem in finding the scattering pattern of an object is in knowing when the scattering matrix is stable and gives the correct pattern. It is possible to deduce that the pattern is stable with repeated runs with different limits on the summations involved, but this is time-consuming. It is therefore desirable to have a method for testing the stability of the scattering matrix on only one run. Various tests have been devised to give some indication of the stability but these are only qualitative and only by repeated runs can the stability be confirmed when a practical change to the program is made. In the 2-D situation, the instability is well understood and is discussed below but in the 3-D case, there are still problems although the same methodology applies to a certain extent.

There are two levels to the instability, the first, when by simply studying the scattering matrix itself, the instability is obvious. The stable matrix will have a central core of elements of low orders where $N < k_{min}$, of the order of magnitude of unity. All the other elements will decay to zero for large $N$ except the diagonal elements which will go to one for large $N$. Also the matrix is symmetric about the diagonal although rounding errors mean that the outer elements in the sides of the matrix decay slower than in the top and bottom. The matrix is then

2-D case to avoid this problem affecting the final pattern although the reason for this being necessary is not known. If there is any large deviation from this form then the matrix will be unstable and the scattering pattern is highly suspect. The usual form of this deviation is that some of the outer elements, instead of decreasing to zero, become very large. This is generally due to taking too low or too high a limit on the various summations. Too low a limit means that not enough modes have been included to calculate the outer elements accurately and too high a limit means that modes of very high order allow rounding errors to assume a far greater importance than normal and distort the outer elements.

A second, more subtle method of showing instability is to calculate the eigenvalues and find their modulus and phase. The modulus should be of each eigenvalue should be one and an accuracy of one significant figure is sufficient to indicate stability. The phase should decay to zero for large orders but this is awkward to see with the present program as it mixes up the eigenvalues. However, if the scattering matrix passes the visual check the smallest phases are those for the highest orders and these should be less than 0.00001.

An alternative check on the scattering matrix is possible by using the principle of reciprocity. This says that, given an incident field on the scatterer and the concomitant scattered field,
the an incident field, with the same coefficient as the scattered field, will cause a scattered field with the same coefficient as the original incident field. This is expressed in the program using the equation
\[ B_m = \sum_{n} S_n^* (S_{pn} x_p) \]
and \( B_m \) should equal \( x_p \). This test has turned out to be much more sensitive to errors than using eigenvalues and will probably require greater accuracy in calculating the scattering matrix before it becomes useful.

There is also a fast way of checking the stability of the scattering and the source are symmetrical across an axis. If the object and source are symmetrical, the scattering pattern should be symmetrical, but it is possible to select the boundary points, to compute a non-symmetrical arrangement of points. This fake object will produce a symmetrical pattern only if the scattering matrix is sufficiently stable to produce the correct pattern. This is the finest check of the matrix stability, but it is only available when the pattern should be symmetrical.

In the 3-D case, there is an additional cause of instability to be considered. As shown previously, some of the possible incident and scattered modes need not be included as they are zero in the plane we are considering. If the scattering matrix includes any elements
for these modes, then they will fluctuate wildly as the routine to solve the simultaneous equations cannot converge to a meaningful solution for them and this can affect the convergence of the matrix as a whole. A solution for the matrix can sometimes be found by using fairly inaccurate field matrices and under these conditions unnecessary elements in the matrices as rows or columns of very small or very large numbers. Also, the scattering matrix found this way produces a fairly stable pattern, indicating again the inimportance of these extra modes. The interesting point for any future work is that although all the modes predicted as unnecessary by the present theory are excluded, there still appear to be more scattered modes which contribute nothing to the scattering matrix. Perhaps if these modes could be eliminated, great savings in space and increase in scatter size should be feasible.

Checking the 3-D matrix for stability tends to be done by inspection of the matrix or using a symmetrical object and source as the matrix is usually rectangular and neither the eigenvalues nor the reciprocal coefficients can be calculated using the rectangular matrix.
Experimental Comparison

In the previous report, the theoretical and experimental plots of the scattering pattern of a square cylinder with the source just outside the enclosing circle were shown to be very close. However, there was still a problem with moving the source inside the enclosing circle and in calculating the scattering matrix for a long, thin shape such as a rectangle. The program can now produce reliable results for these cases as shown in Fig. (a). Here the scattering object is as shown in Fig. (b).

This situation includes both programs and Fig. (c) shows that the theoretical curve is very close to the experimental curve. In the region behind the rectangle from the source, the experimental curve is flatter than the theoretical and this is thought to be due to range effects in the experimental system that smooth out any noise.

In the 3-D case, there has been no chance for experimental comparison. The program that calculates the pattern for a source outside the enclosing sphere can only handle small objects and these do not give very conclusive patterns. One is shown in Fig. (d) and the form of scattering can be seen to be as expected. However, the program that calculates the pattern for a source inside the enclosing sphere has not yet produced a sensible pattern, and further work will be required on this program, as it is the...
one which has the most flexibility and will be the most used of the two programs.
Future Development

Stabilizing the scattering matrix

Although the 2-D scattering matrix is now fully understood, the 3-D scattering matrix is still giving problems, as explained previously. This is evident in that there is some difficulty in finding the scattering matrix, especially for objects of high symmetry, such as a sphere where the 2-D equivalent is the easiest and most accurate. When a matrix is found, some elements are several orders of magnitude larger than the expected value. During this problem will be necessary before this 3-D solution can be used on any real object.

In fact, this sort of problem indicated that unnecessary modes are being included in the system of linear equations used to find the scattering matrix. Also, it is worthwhile studying those elements which are divergent. They affect in fixed patterns dependent on the order of the modes and this indicates that the solution is to be found on the theoretical side of the problem rather than the computing.

A reassuring fact is that the scattering pattern that is found is stable to within one order of the instability of the scattering matrix. This suggests that the divergent elements of the scattering matrix, although individually...
significant, produce no overall effect in the final pattern and could be eliminated from the calculations. However, further investigation is necessary to prove or disprove these ideas.

Another aspect of scattering matrix stability that requires attention is the accuracy of the eigenvalue and reciprocity tests of stability. As was stated previously, these tests are very susceptible to errors as the scattering object deviates from a highly symmetric shape such as a circle. It could be worthwhile finding the cause of this as a smaller scattering matrix could be used and the final scattered pattern will be more reliable. The first place to look is in the calculation of the boundary points' geometry as this seems to be the greatest source of error.
The 3-D program is only set up to calculate the elevation and azimuth planes at present and a roll plane will be required. The elevation and azimuth planes are those containing the axis of rotation of the scatterer, and both use the same method by simply rotating the source about the axes of rotation. The roll plane requires each point to be calculated using a different incident field as the source is moved around the object. This requires greater computation than the other planes and substantial modifications to the program but there are no theoretical problems envisaged.
The 3-D radar cross-section.

In the 2-D case, the R.C.S. has been calculated and was very useful in confirming the stability of the scattering matrix for symmetrical objects. In the 3-D case, the R.C.S. has not been calculated. The R.C.S. could be as useful as in the 2-D case but the associated problems are much greater. An individual scattering matrix will be required and although this would seem to invalidate any test of scattering matrix stability, the differences between the R.C.S. matrix and the scattered pattern matrix are expected to be minimal.
Scattering from more than one body of revolution.

Finally, the program can be modified to attempt to find a scattering pattern for a real aircraft. A simple approximation was suggested in the third report (Ref. 4) which is to treat fuselage, wings, tail and engines as independent bodies of revolution. But the wings and tail can be treated as bodies of revolution is shown in Fig. 1 where a flat strip of width a is replaced by a cylindrical strip of diameter a/2. Thus, as a first approximation, the aircraft can be treated as a collection of cylindrical bodies and the individual scattering patterns combined to produce the total scattering. This should be sufficient for simply shaped aircraft but a more complex shape will require the coupling to be taken into account.
Appendix 1

The spherical vector modes

The spherical modes used are those given by Hahn and Weisylkiwskyj (ref.) as

\[ \mathbf{e} \equiv (\Theta, \Phi) = \frac{1}{N_{nm}} \left\{ -f_{+} D_{nm} \left( \frac{\cos m \Phi}{\sin m \Phi} \right) \right\} \]

where \[ \frac{N_{nm}^2}{E_{nm}} = \frac{4\pi (n+m)! (n+1) (2n+1)}{E_{nm} (2n+1) (n-m)!} \] \[ E_{nm} = 1, \quad m = 0 \]
[2, \quad m \neq 0]

\[ f_{+} = \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} + \Phi \frac{1}{\sin \Theta} \frac{\partial}{\partial \Phi} \]

Giving

\[ \mathbf{e} \equiv (\Theta, \Phi) = -\frac{1}{N_{nm}} \left\{ \frac{\partial D_{nm}}{\partial \Theta} \left( \frac{\cos m \Phi}{\sin m \Phi} \right) \right\} \]

\[ \Phi \frac{1}{\sin \Theta} \left( \frac{\cos m \Phi}{\sin m \Phi} \right) \frac{\partial}{\partial \Theta} \left( \frac{\cos m \Phi}{\sin m \Phi} \right) \]

Also \[ \mathbf{e} = \mathbf{e} \times \mathbf{e} \]

Vectors modes using complex exponentials are also used in the report and these are calculated by substituting in equation 1

\[ e^{im \Phi} \quad \text{for} \quad (\cos m \Phi) \]

\[ \text{and} \quad E_{nm} = 2 \quad \text{for} \quad E_{nm} = 1, \quad m = 0 \]
For the special case where \( \Theta = \frac{\pi}{2} \) we have
\[
\phi_\Theta(\phi) = (\cos m\phi) A_i \Theta_0 + (\sin m\phi) B_i \Theta_0
\]
where \( A_i = -\frac{1}{N_{nm}} \left[ \frac{\partial P_n^m(\cos \Theta)}{\partial \Theta} \right] \Theta = \frac{\pi}{2} \)
and \( B_i = (\pm, -) \frac{m}{N_{nm}} P_n^m(0) \)

\[
\frac{\partial P_n^m(\cos \Theta)}{\partial \Theta} \bigg|_{\Theta = \frac{\pi}{2}} \quad \text{and} \quad P_n^m(0)
\]
are found from Magnus, Oberhochtrager and Leoni (ref. 1) to be
\[
\frac{\partial P_n^m(\cos \Theta)}{\partial \Theta} \bigg|_{\Theta = \frac{\pi}{2}} = (-2)^{m+1} \frac{\pi}{2} \sin(n+m) \frac{\pi}{2} \frac{\Gamma \left( \frac{n+m+1}{2} \right)}{\Gamma \left( \frac{n-m+1}{2} \right)}
\]

\[
P_n^m(0) = (-2)^{m} \frac{\pi}{2} \cos(n+m) \frac{\pi}{2} \frac{\Gamma \left( \frac{n+m+1}{2} \right)}{\Gamma \left( \frac{n-m+1}{2} \right)}
\]

\[
\frac{\partial P_n^m(\cos \Theta)}{\partial \Theta} \quad \text{and} \quad \left[ \frac{m}{\sin \Theta} P_n^m(\cos \Theta) \right] \sin \Theta = 0
\]
are given in appendix 4.

\( \text{note 1: As the associated Legendre polynomials given by the subroutine ALEGF have a factor of } (-1)^m \)
\( \text{difference from those given by Magnus et al. all the equations are adjusted accordingly.} \)
Appendix 2

The Bessel function expansion

We start with Graf's theorem as stated in Attamoriity and Stegun (ref.) as

\[ b_\nu (u) (\cos \nu x) = \sum_{k=-\infty}^{\infty} b_{\nu+k} (u) J_k (\nu) (\cos kx) \]

\( \nu \) unrestricted real number
\( k \) integer number
\( \mu, \nu \) positive

If we convert to the symbol conventions used in the report, we have for \( \nu' > \nu \),

\[ b_{\nu} (e) (\cos \nu (\phi - \phi')) = \sum_{k=-\infty}^{\infty} b_{\nu+k} (e') J_k (\nu) (\cos k(\phi - \phi' - \pi)) \]

Changing to exponential functions we have

\[ b_{\nu} (e) e^{i\nu(\phi - \phi')} = \sum_{k=-\infty}^{\infty} b_{\nu+k} (e') J_k (\nu) e^{i\nu(\phi - \phi' - \pi)} \] for \( \nu' > \nu \)

If the left-hand side is changed to the desired
form we have
\[ B_n(e) \exp[i\phi] = \sum_{k=-\infty}^{\infty} B_{n-k}(e') \exp[i(k\phi)] J_k(p') \exp[i\phi] \exp[i(k\phi)] \]
for \( e' > e \).

If we set \( k = -l \) we have the first useful form of the expansion.

\[ B_n(e) \exp[i\phi] = \sum_{k=-\infty}^{0} B_{n-k}(e') \exp[i(k\phi)] J_k(p') \exp[i\phi] \]
for \( e' > e \).

The expansion for \( e' < e \) is found in the same way to be

\[ B_n(e) \exp[i\phi] = \sum_{k=-\infty}^{0} B_{n-k}(e') \exp[i(k\phi)] J_k(p') \exp[i\phi] \]
for \( e' < e \).

This is still not the desired form for the 2-D problem but if we restrict \( k \) to integer values \( k = l \) and set \( m = n - k \) we have

\[ B_n(e) \exp[i\phi] = \sum_{m=-\infty}^{n} J_m(p') \exp[i(m\phi)] b_m(e') \exp[i\phi] \]
for \( e' < e \).

Equations 1 and 2 can be written in trigonometric modes as

\[ B_n(e) \exp[i\phi] = \sum_{k=-\infty}^{\infty} B_{n-k}(e') \exp[i(k\phi)] J_k(p') \exp[i\phi] \]

for \( e' > e \).
3. \( b_{n}(e)(\cos \phi, \sin \phi) = \lim_{k \to \infty} b_{n-k}(e') J_{k}(p) / J_{k}(p) = \frac{\cos(n-m)\phi - \sin(n-m)\phi'}{(\sin(n-m)\phi', \cos(n-m)\phi)} \frac{\sin k \phi}{\sin k \phi'} \) for \( e' > e \).

4. \( b_{n}(e)(\cos n \phi) = \lim_{k \to \infty} J_{n-m}(p) b_{n-k}(e')/J_{k}(p) = \frac{\cos(n-m)\phi - \sin(n-m)\phi'}{(\sin(n-m)\phi', \cos(n-m)\phi)} \frac{\sin k \phi}{\sin k \phi'} \) for \( e' < e \).

It is possible to use spherical Bessel functions in equations (1) and 3 as \( \eta \) can be a half integer as shown below in the desired forms.

5. \( b_{n}(e) e^{i \eta \phi} = e^{i(n-\frac{1}{2})\phi} \frac{1}{\sqrt{2}} \leq \hat{b}_{n-k}(e') J_{k}(p) e^{i(n-k+\frac{1}{2})\phi} \) for \( e' > e \).

6. \( \hat{b}(n)(\cos m \phi) = \frac{\cos m \phi}{\cos(n+\frac{1}{2})\phi} \frac{1}{\sqrt{2}} \lim_{k \to \infty} \hat{b}_{n-k}(e') J_{k}(p) \) for \( e' < e \).

where \( \hat{b}_{n}(e) = \sqrt{\frac{1}{2 \pi}} b_{n+\frac{1}{2}}(e) \)
Appendix 3

The spherical Bessel function derivation with argument

Starting with the equation from Abramowitz and Stegun

$$\left( \frac{d}{dz} \right)^k \left\{ z^{-\nu} J_\nu (z) \right\} = (-1)^k z^{-v-k} J_{\nu+k} (z)$$

Let $k = 1$, $v = n + \frac{2}{3}$ and $\hat{b}_n (z) = \sqrt{\frac{\pi}{2}} b_{n+\frac{2}{3}} (z)$

to give

$$\left( \frac{d}{dz} \right)^{-n-1} \hat{b}_n (z) = -3^{-n-2} \hat{b}_{n+1} (z)$$

Differentiating with $\hat{b}_n (z) = \frac{d}{dz} (b_n (z))$

we have

$$\frac{1}{3} \left\{ \frac{d}{dz} \right\}^{-n} \hat{b}_n (z) - b_n (z)(n+1) z^{-n-2} \right\} = -3^{-n-2} \hat{b}_{n+1} (z)$$

which becomes

$$\hat{b}_n (z) = (n+1) \frac{1}{3} \hat{b}_n (z) - \hat{b}_{n+1} (z)$$
Appendix 4

Some contiguous relations for associated Legendre polynomials.

All the equations are taken from Magnus et al. (ref.) but the associated Legendre polynomials are multiplied by a factor of \(-1\) to make the equations consistent with the subroutine which generates the polynomials.

To evaluate \( \frac{dP^\mu_\nu(x)}{dx} \)

We have from Magnus

\(1\)

\[(1-x^2) \frac{dP^\mu_\nu(x)}{dx} = \nu P^\mu_{\nu+1}(x) - (\nu+1) P^\mu_{\nu-1}(x)
\]

and

\(2\)

\[(\nu+1) P^\mu_{\nu+1}(x) - (\nu+1) P^\mu_{\nu-1}(x) = - (1-x^2)^{\frac{1}{2}} P^\mu_{\nu+1}(x)
\]

Equation 2 gives

\[\nu x P^\mu_\nu(x) + (\nu+1) x P^\mu_{\nu+1}(x) - (\nu+1) P^\mu_{\nu+1}(x) = (1-x^2)^{\frac{1}{2}} P^\mu_{\nu+1}(x)
\]

Substituting this into \(1\) equation 1 we have

\[\frac{dP^\mu_\nu(x)}{dx} = (1-x^2)^{\frac{1}{2}} P^\mu_{\nu+1}(x) - \nu x P^\mu_\nu(x)
\]

Also from Magnus we have

\[P^\mu_{\nu+2}(x) - 2(\nu+1) x (1-x^2)^{\frac{1}{2}} P^\mu_{\nu+1}(x) + (\nu+1)(\nu+2) P^\mu_\nu(x) = 0
\]
If we let $\mu + 1 = \nu$, we have

4. \[ z \omega x (1-x)^{\frac{1}{2}} P_\nu^\mu(x) = P_\nu^\mu(x) + (\nu - \mu + 1)(x + \omega) P_\nu^\mu(x) \]

Dividing equation 3 by $(1-x)^{\frac{1}{2}}$ and substituting in equation 4, we have

\[ (1-x)^{\frac{1}{2}} \frac{d P_\nu^\mu(x)}{dx} = \frac{1}{2} \left( P_\nu^{\mu+1}(x) + (\nu - \mu + 1)(x + \omega) P_\nu^{\mu-1}(x) \right) \]

\[ = \frac{1}{2} \left\{ P_\nu^\mu(x) - (\nu - \mu + 1)(x + \omega) P_\nu^\mu(x) \right\} \]

let $x = \cos \theta$
\[ dx = -\sin \theta \, d\theta \]

and \[ \frac{d P_\nu^\mu(\cos \theta)}{d\theta} = \frac{1}{2} \left\{ (\nu - \mu + 1)(x + \omega) P_\nu^{\mu-1}(\cos \theta) - P_\nu^{\mu+1}(\cos \theta) \right\} \]

To evaluate \[ \frac{\mu P_\nu^\mu(\cos \theta)}{\sin \theta} \] at \[ \sin \theta = 0 \]

If we take equation 4 and substitute $x = \cos \theta$ we have

\[ \frac{\mu P_\nu^\mu(\cos \theta)}{\sin \theta} = \frac{1}{2} \left\{ P_\nu^{\mu+1}(\cos \theta) + (\nu - \mu + 1)(x + \omega) P_\nu^{\mu-1}(\cos \theta) \right\} \]
Appendix 5

The geometry of the scatterer

The object can be either an infinite cylinder as in the 2-D problem or a body of revolution with the x-axis defined as the axis of revolution as in the 3-D problem. In either case the object is fully defined in the XY plane by giving the vertices and the centres of each circle forming producing an arc between each vertex and the next as shown in Fig A.1. The boundary points are all defined to be equidistant on the surface and an even number of points are always chosen for computational reasons. The object is enclosed by a cylinder or a sphere in the 2-D case or a sphere in the 3-D case with the smallest radius rmin with each boundary point having values for &r, r, and rmin calculated and stored for use in calculating the scattering matrix.

If the problem has a source which is inside the sphere defined by rmin, then the field is expanded using the geometry in Fig A.2. The object is as defined previously with a source at a point A given by the vector r, r, and a are the projections of A and r, respectively, on the XY plane with h as the z coordinate of r. The point O' is chosen to be on the surface of the enclosing sphere in the XY plane such that the angle of f, the vector joining O' and A', is chosen to be perpendicular to the surface of the object. This angle is found using
the following equations
\[ CC^2 = x_0^2 + y_0^2 \]
\[ 0C^2 = (x_{i-1} - x_0)^2 + (y_{i-1} - y_0)^2 \]
\[ a = \pm \sin \Theta \]
\[ \phi_0 = \tan^{-1} \frac{y_{i-1}}{x_{i-1}} \]
\[ \hat{\phi} = \phi - \phi_0 \]
\[ CA = \phi_0 - \hat{\phi} \]
\[ CA^2 = OC^2 + a^2 - 2OCa \cos \hat{\phi} \]
\[ a = \sin^{-1} \left( \frac{a}{CA} \sin \hat{\phi} \right) \]
\[ \phi'' = \phi_0 - \hat{\phi} \]

where \( \hat{\phi}, CA \) are angles and \( OC, a, CA \) are sides in the triangle \( OCA \).

Having found \( \phi'' \), \( \phi_0 \) is known and the other coordinates can be found using the following equations
\[ \tau'' = \tau_{\text{min}} \]
\[ \phi_{\text{tanh}} = \tau_{\text{min}} - \phi'' + \phi' \]
\[ \phi'' = \phi'' - \sin^{-1} \left( \frac{a}{\tau_{\text{min}} \sin \phi_{\text{tanh}}} \right) \]
\[ b = a \sin \left( \phi_{\text{tanh}} + \phi'' - \phi'' \right) \]
\[ \sin \left( \phi'' - \phi'' \right) \]
\[ r = r' \cos \theta' \]
\[ r'' = \sqrt{r'^2 + l^2} \]
\[ \theta'' = \cos^{-1}\left( \frac{r'}{r''} \cos \theta' \right) \]

Although these equations are for the 3-D situation, by setting \( \theta' = \frac{\pi}{2} \), the 2-D equations are found.
University of Surrey

DEPARTMENT OF PHYSICS

RADIATION FROM ANTENNAE MOUNTED ON IRREGULARLY SHAPED BODIES

J.J. GRIBBLE

DECEMBER 1939
The work described in this report forms part of the research studies of the author towards the degree of Ph.D. in the University of Surrey and supported by a studentship of S.R.C.

Presented in this form, it is a report of progress, up to December 1979, for the information of the Radio Department, R.A.E. Farnborough under MoD contract AT/2064/O37.
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RADIATION FROM ANTENNAE MOUNTED
ON IRREGULARLY SHAPED BODIES

December 1979

J.J. GRIBBLE
Department of Physics
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INTRODUCTION

This report is concerned with the theoretical and computational development of numerical methods for calculating the electromagnetic scattering from certain classes of irregularly shaped, perfectly conducting bodies. Particular attention is given to a method which was first described in a paper by Wilton and Mittra in 1972 [1]. Historically, this paper has its roots in earlier work on inverse scattering published in 1970 [2]. Further work was done by Chignell and Munro, and this report is partly an introduction to, an critique of, Munro's later work. Mentions made in the final part of the report of an alternative method.
DESCRIPTION OF PRINCIPLES

The Principle of the method is simple: The incident and scattered fields are expressed as the sum of a series of fundamental modes (e.g. $H^{(2)}(kp)\exp(jn\phi)$) for a scattered field $n$th mode) of which the coefficients of the scattered field are the unknowns. The series are truncated at a point dictated by the characteristic size of the scattering body, so that the number of [significant] unknowns is finite. A system of simultaneous linear equations is obtained for the unknown coefficients by applying the appropriate boundary condition at a number of points on the surface of the scattering body. The system of equations can be solved for the unknown coefficient hence determining the scattered field. There is of course, a problem. The method as described above can only be used (in two dimensions) for a scatterer whose cross-section does not depart significantly from being circular (see [1] 'Mode Matching of Fields'). In the case of a general cross-section of scatterer, it has been shown [3] that one cannot be sure that the formal series representing the scattered field will converge inside a circle of minimum radius; centred at the origin which encloses the scatterer. Because of this it is necessary to use a Bessel function addition theorem to express the scattered field in terms of a series expanded about a new origin, so that this new series can be guaranteed to converge at the surface of the scatterer. This is necessary so that one can apply the Boundary condition at the surface of the body.

The chief interest of the report is scattering from perfectly conducting bodies, so that the method is in principle
The chief interest of the report is scattering from perfectly conducting bodies, so that the method is in principle applicable to aircraft or other structures made of good-conducting material. The boundary condition used is the appropriate special case of the Leontovich condition:

\[ E - (E \cdot n) \hat{n} = 0 \quad \text{or} \quad E_{\text{tangential}} = 0 \quad (1) \]

at all points of the scattering surface, where \( n \) is the unit normal vector. This condition should be noted, as it is incorrectly applied by Munro.

The results of the calculations for a particular body can be stated in the form of a 'Scattering Matrix'. This was justified in a paper by Kahn and Wasylkiwskyj [4] wherein they showed:

'... the coefficients \( d_1 \) and \( r_1 \) can be associated with waves incident from infinity, and \( \beta_1 \) and \( \delta_1 \) with waves from the scatterer. The scatterer is assumed linear.

We will show that if the scatterer is linear and passive then there exists a transformation between the two sets of coefficients \( \{ d_1, r_1 \} \) and \( \{ \beta_1, \delta_1 \} \) which is of the form \( \{ d_1, \delta_1 \} = \{ \beta_1, \delta_1 \} \). This relationship will be phrased in terms of an infinite dimensional scattering matrix.'

The utility of the scattering matrix approach is that once the Scattering Matrix has been calculated for a particular scatterer, then thereafter the scattering of any incident field by that same scatterer can be obtained by a simple multiplication of the column vector representing the incoming field coefficients
by the scattering matrix.

There is an important point to be noted about the use of the scattering Matrix which is not made clear in Munro's work. Any incident field may be split into an outgoing part and an incoming part. This corresponds in the two dimensional case, with an assumed time dependence of the fields of \( \exp(+jwt) \), to Hankel functions of the second and first kind respectively. The Total Outgoing field is obtained by operating with the scattering matrix on the incoming part only of the incident field. In short:

\[
[\text{Total Outgoing field}] = + [\text{Scattering Matrix}] \\
\times [\text{Incoming field from source}]
\]

To obtain the pure scattered field one must subtract the outgoing part of the Incident field from the total outgoing field. In Munro's work \( E^{inc} \) must be understood to mean the incoming part of the incident field, rather than the total incident field.

The method as presented in [1] was applicable to infinite cylinders of uniform cross-section. Munro has proposed an extension to finite Three-Dimensional bodies of revaluation. This is based upon a presently unjustifiable assumption about scattering from such a body, which effectively transforms the three dimensional problem into a two dimensional problem. Now because (a) the infinite uniform cylinder can be solved exactly for an oblique incidence with any additional hypothesis and (b) since previous work has considered only an incident field of transverse magnetic (TM) polarisation in two dimensions, it
is felt that it would be useful to present the formal solution for the case of an incident Transverse Electric (TE) field. This also has some bearing on the misapplication of boundary by Munro. In addition, derivations of the matrix equation which have previously appeared have been rather obscure. It is felt that the one presented here is reasonably clear.

**TE INCIDENCE IN TWO DIMENSIONS**

The Scatterer is an infinite, irregularly shaped 'cylinder' with a cross-section which is invariant under translation along the Z-axis (see Fig. 1). Accordingly, we work with the Z component of the magnetic field, H, which we regard as a scalar potential for the problem. With an assumed time dependence of \( \exp(j \omega t) \) Maxwell's equations become:

\[
(v^2 + k^2)H = 0 \quad \text{with} \quad k = \omega/c
\]

This is justified because the problem is invariant under Z-translation and hence the equations for the polarisation perpendicular and parallel to the Z-axis decouple so that the TE and TM cases may be treated separately. As an aside, classes of scatterers for which decoupling occurs even for oblique incidence have been given by Uslenghi [5].

In the XY plane the \( \rho \) and \( \phi \) components of the Electric Field can be obtained, using \( E = \frac{1}{j \omega \varepsilon_0} \nabla \times H \) thus

\[
H_z = \sum_{m=-\infty}^{+\infty} j^{-m} (a_m H_m^{(1)}(k\rho) + b_m H_m^{(2)}(k\rho)) \exp(jm\phi) \quad (2.a)
\]

using the curl equation, and absorbing \( 1/j \omega \varepsilon_0 \) into the coefficients:
\[
F_\rho = \frac{1}{\rho} \frac{\partial H_\rho}{\partial \rho} \quad \quad E_\phi = -\frac{\partial H_\rho}{\partial \rho}
\]

So that
\[
E_\rho = \sum_{m=-\infty}^{+\infty} \frac{j m}{\rho} \{ a_m H_m^{(1)}(k\rho) + b_m H_m^{(2)}(k\rho) \} \times \exp(jm\phi)
\]
\[
E_\phi = \sum_{m=-\infty}^{+\infty} -k j m \{ a_m H_m^{(1)}(k\rho) + b_m H_m^{(2)}(k\rho) \} \times \exp(jm\phi)
\]

where the dot indicates differentiation with respect to the Hankel function argument.

Note the boundary condition carefully: It is that the tangential component of the total electric field at the surface of the scatterer is zero, not the transverse component.

Now:

Unit Tangent Vector \( \vec{\tau} = \frac{\hat{\rho}}{\rho} + j \hat{\phi} \)

So the boundary condition becomes

\[
E_\rho \frac{\partial E_\rho}{\partial \phi} + E_\phi = 0
\]

If the scatterer were 'large' in terms of wavelengths, and not sharply curved, then \( \frac{1}{\rho} \frac{\partial \rho}{\partial \phi} \ll 1 \) and we can use the simpler boundary condition \( E_\phi = 0 \). However, in general:

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial \phi} \sum_{m=-\infty}^{+\infty} \frac{j m}{\rho} \{ a_m H_m^{(1)}(k\rho) + b_m H_m^{(2)}(k\rho) \} \exp(jm\phi)
\]
\[
+ \sum_{m=-\infty}^{+\infty} -\frac{k}{2} \{ a_m (H_m^{(1)}(k\rho) - H_{m+1}^{(2)}(k\rho)) + b_m (H_{m-1}^{(2)}(k\rho) - H_{m+1}^{(2)}(k\rho)) \} \times \exp(jm\phi) = 0
\]
where we have used

\[ \hat{H}_m^{(1)}(2)(k\rho) = \frac{1}{2} \left( H_{m-1}^{(1)}(k\rho) - H_{m+1}^{(1)}(k\rho) \right) \]

Strictly, we should now use the Hankel function addition theorem \([1]\) equation 4 to transform the Hankel functions of 2nd kind into formal series which will always converge at the scatterer surface. However in the interests of keeping the algebra simple we will leave them untransformed.

Now define the column vectors

\[
\begin{align*}
\mathbf{a} &= \begin{bmatrix} a_0 & a_{-1} \end{bmatrix}^T \\
\mathbf{b} &= \begin{bmatrix} b_0 & b_{-1} \end{bmatrix}^T
\end{align*}
\]

and defining the matrix elements

\[
A_{im} = \left[ \frac{1}{\rho_i} \frac{d\rho_i}{d\phi_i} \right] \frac{im}{\rho_i} H_m^{(1)}(k\rho_i) - \frac{k}{2} \left( H_{m-1}^{(1)}(k\rho_i) - H_{m+1}^{(1)}(k\rho_i) \right) \exp(im\phi_i)
\]

and

\[
B_{im} = \left[ \frac{1}{\rho_i} \frac{d\rho_i}{d\phi_i} \right] \frac{im}{\rho_i} H_m^{(2)}(k\rho_i) - \frac{k}{2} \left( H_{m-1}^{(2)}(k\rho_i) - H_{m+1}^{(2)}(k\rho_i) \right) \exp(im\phi_i)
\]

where the subscript \(i\) indicates that the subscripted quantity is to be evaluated at the \(i\)th boundary point.

Then eq. (3) can be written

\[ [A] \mathbf{a} + [B] \mathbf{b} = 0 \quad (4) \]

Square brackets indicate a matrix, \(0\) is the null column vector.
Kahn and Wasylkiwskyj's Paper [4] implies that there exists a scattering matrix \[ \sigma \] such that

\[ b = [\sigma] a \quad (5) \]

for arbitrary \( a \). Substitute (5) into (4).

\[ [A] a + [B][\sigma] a = 0 \]

If this is true for arbitrary \( a \), then:

\[ [\sigma] = -[B]^{-1}[A] \quad (6) \]

Once again, the scattering matrix is expressed in terms of geometrical quantities. The principal physical difference from the TM case is that the curvature of the scattering surface i.e. \( \frac{1}{p} \frac{dp}{d\phi} \) appears in the analysis.

How significant is the effect of the curvature. From the definitions of the Matrix elements \( A_{im} \) and \( B_{im} \) it can be seen that a measure of its effect is the dimensionless quantity

\[ P_{im} = \frac{2m}{k \rho_i^2} \frac{d\rho_i}{d\phi_i} \frac{H^{(1)}(2)(k \rho_i)}{H^{(1)}(2)(k \rho_i) - H^{(1)}(2)(k \rho_i)} \]

Estimation of the magnitude of this term will be discussed later.

THE THREE DIMENSIONAL DEVELOPMENT

Electromagnetic Scattering from bodies of completely arbitrary shape is a complicated problem. To simplify the extension of the Wilton-Mittra method to 3-dimensions, attention has been restricted to bodies of revolution. This has involved the use of a certain hypothesis in order to convert the Three-Dimensional
problem into a two-dimensional one. It is stated twice in the draft of Munro's notes that:

'As the scattering body is a body of revolution, the scattering in each plane containing the axis of revolution will be identical, and only the scattering matrix in one of those planes need be considered.'

and again:

'As the scatterer has an axis of revolution on the X-axis, only the energy incident in the XY plane will be scattered into the XY plane and into no other plane-----'

Neither of these is an exact statement of the assumption. All that one can say is: because the scatterer is invariant under rotation about the X-axis, and since the scattering matrix is dependent upon only the geometry of the scatterer, then the scattering matrix will not depend on the appropriate angular co-ordinate.

To talk of energy-scattering is also misleading (one cannot talk about energy scattering in a plane, only power scattering). Indeed, the variation of field in say, the XY plane, will in general be associated with a Poynting Vector which does not lie in that plane. To say that, for example, an incident Poynting vector which is contained in a plane containing the axis of revolution, gives rise to a scattered Poynting vector which also lies in that plane, may even be true, but it is certainly not relevant.

What is done is to find the expansion of the field in modal form and to apply the boundary conditions only for points lying in the XY plane. As the axis of revolution is taken to
lie along the X-axis, these points lie on the intersection of
the scattering surface with a plane containing the axis of
revolution. It is assumed that this gives the correct
scattered field for field points lying in the XY plane. At
present there is no analytical justification of this assumption.
It was suggested by Dr. Cornbleet that heuristic assessment of
the validity of this assumption can be made by comparison of
exact solutions of 'Canonical problems' with solutions obtained
using only the restricted set of boundary points. This is
easily done in the case of the sphere.

In the following, the notation of Kahn and Wasylkiwskyj
is used.

\[ \sum_{nmp} \text{ is written for } \sum_{n=1}^{\infty} \sum_{m=0}^{even, odd} \]

and \( \hat{e}_{nmp}(\theta \phi) \) are transverse vectors - i.e. a linear combination
of \( \hat{\theta} \) and \( \hat{\phi} \) in spherical polar co-ordinates.

The total transverse electric field can be written

\[
\mathbf{E}_t = \sum_{nmp} j \sqrt{\varepsilon} (b_{nmp}^E \hat{H}_n^{(2)}(kr) + a_{nmp}^E \hat{H}_n^{(1)}(kr)) \times \hat{e}_{nmp}(\theta \phi) + (b_{nmp}^H \hat{H}_n^{(2)}(kr) + a_{nmp}^H \hat{H}_n^{(1)}(kr)) \times (\hat{e}_{nmp}(\theta \phi) \times \hat{r})
\]

where \( \hat{e}_{nmp} \) is an angular vector mode. Since in this case the
boundary condition is \( \mathbf{E}_t = 0 \) and since for all \( \theta \) and \( \phi \) the
surface is defined by \( r = a \), the solution is
and this is true, no matter what part of the surface is sampled.

However, the validity (or otherwise) of the assumption for more general bodies of revolution is not yet established. Because of the high degree of symmetry of the sphere, I do not regard it as a severe test. The basis for the assumption seems to be a statement by Chignell in [7] on p. 23.

'Here the assumption of a cylindrical scatterer or quasi-cylindrical scatterer is most important. The point is that if a mode is incident upon the origin then the energy incident in a particular \( \phi \) plane is scattered on that plane'.

The justification for this statement lies in the work of Garbacz and Turpin, and Harrington and Mautz ([8] and [9]), where it is shown that there exists a set of surface currents and corresponding radiated fields which are characteristic of a given perfectly conducting scatterer. If the outgoing field is expanded in terms of these radiated fields, and the incoming field in terms of the complex conjugates of the characteristic radiated fields, then, the scattering matrix is diagonal, and the non-zero elements are complex numbers of unit modulus. So, if the \( n \)th incoming mode is \( f_n^\phi \), then the associated scattered field, will be of form \( e^{i\phi} f_n \). The last quoted statement of the cylindrical scattering hypothesis can be seen as justified if the field is
being expanded in characteristic modes of the scatterer, but since we are restricted to Spherical Harmonics, this is not the case.

In the future, it is hoped to investigate other canonical problems to test the validity of the hypothesis, but this investigation will be handicapped by the lack of exact solutions for finite bodies of revolution. Reference [6] lists only the Sphere, Prolate and Oblate spheroids, the wire and the disc. In the meantime all one can say is that the hypothesis will hold only for bodies which are 'almost' spherical in some sense. One should also note that it is impossible even to definitely prove the validity of the assumption by this method. Logically, one could only definitely disprove it.

BOUNDARY CONDITIONS AND CROSS POLARISATION

A more serious objection to the extension of the method to three-dimensions as proposed by Munro is misapplication of the boundary conditions. For convenience he has followed Kahn and Wasylkiwskyj in expanding in Spherical Modes only the Transverse electric field. This is perfectly permissible as the radial part of the field can easily be obtained from applying Maxwell's equations. However, in Munro's notes, the radial field is never mentioned. Along the contour of the scatterer in the XY plane, he applies the boundary conditions

\[ rE^{\text{scat}} = - r E^{\text{inc}} \]

where \( E^{\text{scat}} \) and \( E^{\text{inc}} \) are transverse electric fields, but the correct boundary condition
is that the total tangential field is zero at the surface.
Symbolically:

\[ E_{\text{total}}^\theta = 0. \]

\[ \frac{1}{\rho} \frac{d}{d\phi} \rho E_{\text{total}}^\rho + E_{\text{total}}^\phi = 0 \]

Munro omits the \( \frac{1}{\rho} \frac{d}{d\phi} \) term. Since \( E_{\rho} \) depends on both \( E_{\text{total}}^\theta \) and \( E_{\text{total}}^\phi \) this omission means that his equations provide no way of 'mixing' \( E_{\theta} \) and \( E_{\phi} \). As he then goes on to derive, quite consistently the \( \hat{\theta} \) and \( \hat{\phi} \) equations in the XY plane decouple, so that there is no mechanism for cross-polarisation between polarisations, lying in, and perpendicular to, the XY plane. Qualitatively, this agrees with measurements made at RAE of the field of a Monopole mounted on a cylinder.

An explicit correction of the equations is not presented, but it would proceed along lines similar to the TE case which has already been described, in that factor like \( \frac{1}{\rho} \frac{d\rho_{i}}{d\phi_{i}} \) would be introduced to account for the effect of the surface curvature.

At this stage it would probably be more useful to have an estimate of the order of magnitude of the error which is introduced by the wrong boundary conditions. Unfortunately this is a complicated problem. An approach which was tried was to treat the quantity

\[ P_{\text{im}} = \frac{2m}{k \rho_{i}^2} \frac{d\rho_{i}}{d\phi_{i}} \frac{H^{(1)}(2)(k \rho_{i})}{H^{(1)}(2)(k \rho_{i}) - H^{(1)}(2)(k \rho_{i})} \]
as a 'small' perturbation to the matrix elements. Unfortunately the value of \(|m|\) goes from zero, to values comparable to \(k\rho_i\). Different approximations for the Hankel functions are applicable for these two regimes, and this has proved a great stumbling block. Work is proceeding on this point. The best that can be offered at this stage is the following (to be taken with a large pinch of salt):

(i) Assume that \(\left| \frac{H_m^1(2)}{H_{m-1}^1(2)} - H_m^1(2) \right| = 1\)

(ii) Since the \(\rho_i\) are on the boundary of the scatterer, all the \(k\rho_i\) will be of the same order.

(iii) \(|m|\) goes from 0 to about \(k\rho_i\). So, assume the average value of \(|m|\) is \(\frac{1}{2} k\rho_i\). This gives:

\[
p_{im} = \frac{1}{\rho_i} \frac{d\rho_i}{d\phi_i}
\]

Since power depends on the square of field strength, say that the discrepancy in power measurements is given by the mean square on the contour of the scatterer on the XY plane, i.e.

\[
\text{Power Discrepancy} \sim \frac{1}{2\pi} \int \frac{1}{\rho^2} \left( \frac{d\rho}{d\phi} \right)^2 d\phi
\]

It is suggested that if this quantity is very small, then one can safely neglect the cross-polarisation. If one evaluates this quantity for the rectangular contour corresponding to RAE cylinder then the value 2.3 is obtained. (See appendix I). Estimates of the experimental figure derived from polar power plots supplied by RAE give 0.14 or 1.4 for two different configurations. Whilst the theoretical and experimental figures are of the same sort of
size, there is not really enough data (in the opinion of this author) to say conclusively whether or not the proposed explanation for the discrepancy in the polarisation is complete. For example, restricting the boundary points to the XY plane contour probably also contributes to the effect. Two other points that have been glossed over are:

(i) Because the antenna was mounted on the cylinder, the scatterer was effectively part of the source, and there was no clear cut distinction between the incident and the scattered fields. Thus for 'condition 4', for which the agreement between theory and experiment is fair, since the system was invariant under reflection in the horizontal plane, one could assent with some degree of confidence, that there was no polarisation component perpendicular to that plane in the incident field. This is only strictly true for condition 1 if one takes the 'incident field' to be that of the 'dipole by neglection' which is seen for observation angles differing by little from the \( \theta = 90^\circ \).

(ii) The polar plots show the total fields from the system, so that if the 'incident' fields are pure, one would expect that the 'actual' cross-polarisation for the 'scattered' fields would be larger than the experimental values quoted above, improving the agreement between experiment and theory.

LOW FREQUENCY CONSIDERATIONS

The method of expressing the electromagnetic field as a sum of modes, and using translation theorems ('analytic continuation') to sidestep difficulties due to divergence of series, is in
principle valid at all frequencies. In this section we shall show that the modal representation can be transformed into a form which agrees with the equations derived from a simple physical model of a cylindrical scatterer in the low frequency limit. The significant feature is that the 'analytic continuation' introduces a number of arbitrary parameters which can be identified with a similar set of parameters which appear in the 'Physical' model. We will consider only the TM case in detail, but the TE case will be discussed briefly at the end.

Suppose the scattered field is: (Fig. 3)

$$E_{\text{escat}} = \sum_{K=-\infty}^{+\infty} B_{K} H_{K}^{(2)}(k\rho) \exp(jK\phi)$$  \hspace{1cm} (1f1)

We already know that strictly, this is only valid outside the smallest circle centered at the origin which encloses the scatterer (Fig. 3).

We can choose to change the variables and represent the scatterer by a set of line images which lie inside the enclosing circle, but whose positions are otherwise arbitrary. This indicates that at this stage, the change of variables is purely a mathematical device. If the $j^{th}$ image has strength $A_{j}$, position vector $s_{j}$ and angular co-ordinates $\sigma_{j}$, then the new representation is:

$$E_{\text{escat}} = \sum_{j=1}^{N} A_{j} H_{0}^{(2)}(k|\rho - s_{j}|)$$  \hspace{1cm} (1f2)

Again this is only strictly valid outside the enclosing circle. To reconcile (1f1) with (1f2), we note that $\rho > s_{j}$, so that we can apply Graf's addition theorem (Abramovitz and Stegun 9.1.79),
Substitution of (1f3) into (1f2), reversal of summation signs and comparison with (1f1) gives:

\[
\beta_K = \sum_{j=1}^{N} A_j J_K(k s_j) \exp(-j K s_j) \tag{1f4}
\]

In practice, (1f1) will be truncated so that only a finite number of \( \beta \) coefficients and hence only a finite number of \( A \)s are required. So, given a set of \( \beta \)s, a representation of the scatterer in terms of images can be found, subject only to the condition that

\[
\det(J_K(k s_j) \exp(-j K s_j)) \neq 0 \tag{1f5}
\]

In order to solve for the \( A \)s we would have to apply (1f2) at the boundary of the scatterer, outside the range of validity. To overcome this, follow Wilton and Mittra, and relate the field to a new origin (Fig. 4). We consider the field at a boundary point on the surface of the scatterer, i.e. at the point \((\rho_j, \phi_j)\) and apply Gray's addition theorem again to the quantity

\[H_{K}^{(2)}(k \rho_j) \exp(j K \phi_j)\]. Assuming \( \rho_{0j} > \rho_j \),

\[
H_{K}^{(2)}(k \rho_j) \exp(j K (\phi_{0j} - \phi_j)) = \sum_{\Lambda=-\infty}^{+\infty} H_{K+\Lambda}^{(2)}(k \rho_{0j}) J_{\Lambda}(k \rho_j) \exp(j \Lambda \alpha_j) \tag{1f6}
\]

Using (1f6), (1f4) and (1f1) we obtain an expression for \( E_{\text{Scat}}(\rho_j) \):

\[
E_{\text{Scat}}(\rho_j) = \sum_{K=-\infty}^{+\infty} \exp(j K (2 \phi_j - \phi_{0j})) \sum_{\Lambda=-\infty}^{+\infty} H_{K+\Lambda}^{(2)}(k \rho_{0j}) J_{\Lambda}(k \rho_j) \exp(j \Lambda \alpha_j)
\]

\[
\times \prod_{i=1}^{N} A_i J_K(k s_i) \exp(-j K s_i) \tag{1f7}
\]
\[
\sum_{i=1}^{N} A_i \sum_{A=-\infty}^{+\infty} J_A(k\rho_j') \exp(jA\alpha_j) \sum_{K=-\infty}^{+\infty} H_{\frac{2}{k+k'}}^{(2)}(k\rho_j) J_K(ks_i) \exp jK(2\phi_j - \phi_{0j} - \sigma_i)
\]

We examine the last angular factor in (1f8), and we see that it would be advantageous to choose

\[
2\phi_j - \phi_{0j} - \sigma_i = \phi_{0j} - \sigma_i
\]

\[
\rightarrow \phi_j = \phi_{0j} \rightarrow \alpha_j = 0
\]

If we now assume that \(\rho_{0j} > s_i\), then:

\[
E_{\text{Scat}}(\rho_j) = \sum_{j=1}^{N} A_i \sum_{A=-\infty}^{+\infty} J_A(k\rho_j') \exp(jA\alpha_j) H_{\frac{2}{k+k'}}^{(2)}(k|\rho_{0j} - s_i|) \exp(jA\beta_j)
\]

Replacing \(A\) by \(-A\) in (1f9), if \(\rho_{0j} - s_i > \rho_j\), then

\[
E_{\text{Scat}}(\rho_j) = \sum_{i=1}^{N} A_i H_{\frac{2}{k+k'}}^{(2)}(k|\rho_{0j} - s_i|)
\]

This is just (1f2) but we have continued the range of its validity by the Wilton-Mittra analytic continuation process. In doing so we have used or derived the following conditions:

\[
\rho_{0j} > \rho_j' \quad \phi_j = \phi_{0j} \quad |\rho_{0j} - s_i| > \rho_j'
\]
The first assumption was that we expanded the field about an origin outside the scatterer at which the field is 'well behaved'. The second indicates a 'preferred' origin associated with each boundary point. The third condition is the most interesting. It provides the most essential condition for a representation of a scatterer by a collection of point scatterers, i.e. that no point scatterer may coincide with a boundary point. Although this can be seen intuitively, on physical grounds, here the significant factor is that the condition has come out of the mathematics. A more general interpretation follows from noting that $|\rho_j'| = \text{constant}$ is a circle centred on $O'$, so the interior of this circle is free from point scatterers. Compare with Wilton and Mittra. p. 312:

'... we expand in terms of the cylindrical wave function which are valid inside the largest circle of radius $\rho_{\text{max}}'$ which excludes the scatterer'.

Finally, we note that $\rho_j'$ is essentially arbitrary. In particular, as $\rho_j' \to \infty$, the part of the circumference of the circle centred on $O'$ at the $j$th boundary point, tends to an infinite straight line passing through the boundary point, and perpendicular to $\rho_j$. Thus, in order to have positions of point scatterers which are permissible for all possible choices of $\rho_j'$ they must lie inside the 'Interior Polygon' defined by the set of boundary points chosen. Currently it is not believed that there is any deepen physical significance to the 'interior polygon', since $\rho_j'$ is after all, still arbitrary, and the 'interior polygon' arises as a result of considering an extreme case.

Now we will attempt to tie up the above considerations with
a simple, 'more physical' model. We attempt to represent the
infinite cylindrical scatterer by a collection of infinite, parallel perfectly conducting, circular cylindrical wires.

Suppose we have, as in Fig. (5) a collection of such wires, whose centres are at $s_1, s_2, \text{etc}$. They have radii $a_1, a_2 \text{etc}$. Physically we can see that the field at a point not inside one of the wires can be written as:

$$E(\rho) = \sum_{i=1}^{N} \sum_{m=-\infty}^{+\infty} A_{i,m} H_0^{(2)}(k|\rho - s_i|) \exp(jm\theta_i) \quad (1f12)$$

+ External field

Subject to the boundary condition $E_{total} = 0$ for $|\rho - s_i| = a_1 \forall i$. We consider these equations at low frequencies, assuming that the incident electric field does not vary significantly over any individual wire, i.e. $ka_i << 1$ for all $i$. We follow through the steps of an approximate solution of these equations.

To the first order of approximation, it is assumed that for all $m \neq 0 A_{i,m} = 0$. Then the boundary conditions at the surface of the $j$th wire give

$$A_{j,0} H_0^{(2)}(ka_j) + \sum_{i \neq j} A_{i,0} H_0^{(2)}(k|s_j - s_i|) = -E_{inc}(s_j) \quad (1f13)$$

Since the wires are small $\rho - s_i = s_j - s_i$ and writing $|s_j - s_i| = s_{ji}$ which is constant, we have

$$A_{j,0} H_0^{(2)}(ka_j) + \sum_{i \neq j} A_{i,0} H_0^{(2)}(ks_{ji}) = -E_{inc}(s_j) \quad (1f14)$$
Applying (lfl4) to each of the wires gives a system of simultaneous linear equations which can in principle be solved for the $A_{j0}$s. Since the equations are essentially symmetric in the $A_{j0}$s, they will all have approximately the same magnitudes, but will differ significantly in their arguments, i.e. we can write

$$|A_{j0}| \approx A \quad \text{for all } j.$$ 

We now estimate the magnitudes of the $A_{jm} \neq 0$ relative to $A$. We write as a correction to (1f13):

$$+ \sum_{K=-\infty}^{+\infty} A_{jk} H_K^{(2)}(ka_j) \exp(jK\theta_j) + \sum_{i \neq j} A_{i0} H_0^{(2)}(k|\rho - z_i|) = E^{inc}(z_j)$$

(1f15)

For $A_{j0}$, the equations derived from (1f15) are the same as before, but for $A_{jk} \neq 0$, we rewrite (1f15) using the addition theorem for $H_0^{(2)}(k|\rho - z_i|)$:

$$H_0^{(2)}(k|\rho - z_i|) = + \sum_{K=-\infty}^{+\infty} H_K^{(2)}(ks_{ij}) J_K(ka_j) \exp j K(\beta_{ij} - \theta_j)$$

$$= + \sum_{K=-\infty}^{+\infty} H_K^{(2)}(ks_{ij}) J_K(ka_j) \exp j K(\beta_{ij} - \theta_j)$$

(1f16)

where the last line follows from replacement of $K$ by $-K$ in the summation. Substitution of (1f16) into (1f15) gives, for $K \neq 0$:

$$A_{jk} H_K^{(2)}(ka_j) + \sum_{i \neq j} A_{i0} H_K^{(2)}(ks_{ij}) J_K(ka_j) \exp(-j K \beta_{ij}) = 0$$
so \[ A_{jK} = - \sum_{i \neq j} A_{i0} \frac{H_{K}^{(2)}(ks_{ij})}{H_{K}^{(2)}(ka_{j})} J_{K}(ka_{j}) \exp(-jKs_{ij}) \]

i.e. \[ \left| \frac{A_{jK}}{A} \right| < \sum_{i \neq j} \left| \frac{H_{K}^{(2)}(ks_{ij})}{H_{K}^{(2)}(ka_{j})} \right| \left| J_{K}(ka_{j}) \right| \]

We replace \( s_{ij} \) by an average \( D \), which gives the size of the scatterer as a whole, and assume that for all \( i = a_{i} = a \).

Then since \( ka << 1 \)

\[ \left| \frac{A_{jK}}{A} \right| < \pi N \frac{H_{K}^{(2)}(kD)}{K!(K-1)!} \left( \frac{ka}{2} \right)^{2K} \]

From (1f17) is obvious that for fixed \( N \) and \( D \), \( |A_{jK}/A| \) can always be made small by choosing \( ka \) small enough. This is consistent with the critical assumption that only the \( A_{j0} \) are important, and so our perturbation procedure is valid. So for a thin wire model at low frequencies the scattered field can be written as

\[ E^{\text{Scat}}(\rho) = \sum_{i=1}^{N} A_{j}H_{0}^{(2)}(k|\rho - s_{i}|) \]  

(1f18)

We see that (1f18) and (1f2) are identical. That is, the scattered field form a collection of wires with finite radii, has the same form as the field from a collection of line sources. This, in retrospect is fairly obvious. The significant point is this: Take a scatterer, represent it geometrically by a collection of discrete points which coincide with the boundary of the physical scatterer. Now represent the scattering properties (at least at low frequencies) by a collection of wires placed so that each boundary point lies on the circumference of a wire. This simulates the actual boundary condition problem at the cost of
introducing a number of arbitrary (but non-zero) parameters - the wire radii. If we abandon the idea of any 'physical' model and use a collection of point sources, then the location of the point sources is arbitrary, except for the constraint:

\[ |p_{ij} - s_j| > \rho_j \]  

(1f11)

If the source position nearest the jth boundary point is at \( s_j \), then we can see that there are parameters \( |p_{ij} - s_j| \) which must be greater than zero, but otherwise arbitrary. These can be identified with the wire radii in the 'physical' model. Hence the two approaches are entirely equivalent at low frequencies. Such wire models are usually treated by setting up integral equations which are solved by the method of moments.

The above analysis has also been performed for the case of Transverse-Electric incidence in two dimensions. As the details are very similar, only certain points will be noted here. The most significant change is due to the intrusion of the vector nature of the electric field into the analysis. We now have to assume that the magnitude and direction of the field over the region of the wire both remain constant. This means that when we consider the scattered field from a wire, it is no longer independent of angle, and in fact the field is:

\[
H_{\text{scat}}(\rho) = \sum_{i=1}^{N} \beta_i H_1^{(2)}(k|\rho - s_i|) \exp(j\theta_i)
\]

\[
+ \beta_i H_{-1}^{(2)}(k|\rho - s_i|) \exp(-j\theta_i)
\]

(1f19)
\[ H_{\text{scat}}(\rho) = \sum_{i=1}^{N} \left\{ \beta_i H_1^{(2)}(k|\rho - s_i|) \exp(j\theta_i) + \beta_i H_{-1}^{(2)}(k|\rho - s_i|) \exp(-j\theta_i) \right\} \]  

(1f19.a)

where \(|\beta_i/\beta_i| = 1\). We choose to work with the Z component of the magnetic field which we regard as a scalar potential for the problem. The Transverse electric field components are easily obtainable by differentiation. As before we use analytic continuation of (1f19) and show that the form is valid, subject to the constraint that no image can coincide with a boundary point. As before, the 'new origin' is associated with the \(j^{th}\) boundary point must lie on the radius passing through that point. A more interesting point which came out was that the angles \(\theta_j\) must also be measured from that line. This did not emerge from the Transverse-Magnetic case, because the scattered field did not depend on the azimuthal angles measured at the wires.

The field (1f19) corresponds to a circular wire carrying a circumferentially varying magnetic current. This is not a physical model in the strict sense, since no-one has yet found magnetic monopoles, which could carry such a current; however, to talk of magnetic current is fairly common practice, and we feel that the use of the adjective 'Physical' is justified.

A serious objection to the line of reasoning could be this: a priori, one would expect to be able to represent \(H\) as a sum of fields 'point' sources, as in the transverse magnetic case on the lines of (1f2) which is not the same as (1f19) however the two are essentially the same - consider figure 7 - two 'point' sources close to each other, which are of strengths \(\pm A\).
\[
\text{Field} = A\left\{H_0^{(2)}(kr_1) - H_0^{(2)}(kr_2)\right\}
\]

but \[
H_0^{(2)}(kr_1) = \sum_{K=-\infty}^{+\infty} H_K^{(2)}(kr)J_K\left(\frac{kd}{2}\right)\exp(jK\alpha)
\]

and \[
H_0^{(2)}(kr_2) = \sum_{K=-\infty}^{+\infty} H_K^{(2)}(kr)J_K\left(\frac{kd}{2}\right)\exp jK(\pi - \alpha)
\]

Where we have assumed that \( r > \frac{d}{2} \), i.e. the field point lies outside the circle just enclosing the sources.

\[
\text{Field} = \sum_{K=-\infty}^{+\infty} H_K^{(2)}(kr)J_K\left(\frac{kd}{2}\right)\exp(jK\alpha)\left\{1 - (-1)^K\right\}
\]

The \( K = 0 \) term vanishes, and if \( kd \) is small:

\[
\text{Field} = 2A\left\{H_1^{(2)}(kr)J_1\left(\frac{kd}{2}\right)\exp j\alpha + H_{-1}^{(2)}(kr)J_{-1}\left(\frac{kd}{2}\right)\exp -j\alpha\right\}
\]

\[
\text{Field} = \frac{kd}{4} 2A\left\{H_1^{(2)}(kr)\exp j\alpha - H_{-1}^{(2)}(kr)\exp -j\alpha\right\}
\]

If we measure angle from some other line than the 'axis' of the dipole, this amounts to making the change \( \alpha \to \alpha + \gamma \). With this substitution, the field has the required form, with

\[
\beta = \left(\frac{k\alpha d}{2}\right)\exp j\gamma \quad \text{and} \quad \beta = \left(\frac{k\alpha d}{2}\right)\exp -j\gamma
\]

So once again, the Wilton and Mittra Modal expansion can be expressed in a form which is identical with a 'physical' model at low frequency.
FUTURE WORK

The following are future lines of possible development:

1. Further investigation of Munro's extension of the Wilton and Mittra modal method to three dimensions. As mentioned earlier, this can be done for 'canonical problems' for which comparison between exact solutions obtained by applying the boundary conditions at the intersection of the scattering surface and a plane containing the axis of revolution, can be made.

2. Investigation of the modal method in the high frequency limit, from which one may hope to extract expressions for 'reflected rays' 'creeping rays' etc. on the lines of G.T.D. For this purpose, the appropriate sections in a Paper by Clemmow and Weston [10] can perhaps be used as a model.

3. Further development of the idea of representation of scattered fields in terms of a sum of 'point source' fields. It has already been shown that these representations of the scattered field are essentially equivalent, and the change of unknowns in (1f4) can be made at any frequency. Convergence of the scattered field at the boundary points is guaranteed, provided no point sources coincide with the boundary. As the positions can be chosen a prior, this is no problem. Such a method which partially sidesteps the conversion of large matrices, is as follows: We consider the TM case, in two dimensions.

* i.e., not just at low frequencies. At higher frequencies however, the comparison with 'real' current carrying wires is no longer valid.
Represent (Fig. 8) a scatterer geometrically by a set of boundary points located at positions $P_1, P_2, ..., P_M$. Place inside the boundary point sources of strength $A_i$ at positions $S_1, S_2, ..., S_N$. $N$ is less than or equal to $M$. If the incident field is $E_{\text{inc}}(P)$, then the quantity

$$E_{\text{inc}} = (E_{\text{inc}}(P_1) E_{\text{inc}}(P_2) ... E_{\text{inc}}(P_M))$$

can be considered to form a vector. Similarly for the quantities

$$F_j = (H_0^{(2)}(k|P_1 - S_j|)H_0^{(2)}(k|P_2 - S_j|) ... H_0^{(2)}(k|P_N - S_j|)).$$

The Boundary value problem may be stated as finding coefficients $A_j$ such that

$$\sum_{j=1}^{N} A_j F_j = -E_{\text{inc}}$$

(1f20)

We define a scalar product by $A \cdot B = \sum_{i=1}^{N} A_i^* B_i$. Conventional practice would be to convert (1f20) to a set of simultaneous linear equations for the unknown coefficients by taking the scalar product of (1f20) with $F_R$ thus:

$$\sum_{j=1}^{N} A_j (F_R^* F_j) = -F_R^* E_{\text{inc}}$$

(1f21)

This is similar to the conventional method of moments, where the 'F's are taken to be both 'basis' and 'test' functions. The disadvantages of this method are that it requires the inversion of a large matrix whose elements are $F_R^* F_j$, and also, if one wants numerically to estimate the reliability of the solution by, say, increasing the number of boundary points, then the whole tedious
calculation has to be gone through again at great cost in computer time.

The following modification is proposed:- From the Fs, construct an orthonormal set of vectors, using the Gram-Schmidt Orthonormalisation process. Thus:

\[ u_1 = N_1 F \]
\[ u_2 = N_2 \{ F - (u_1 \cdot F)u_1 \} \]
\[ u_3 = N_3 \{ F - (u_1 \cdot F)u_1 - (u_2 \cdot F)u_2 \} \]

etc.

The \( N_s \) are normalising coefficients. Equation (1f20) becomes

\[ \sum_{j=1}^{N} B_j u_j = -E_{inc} \quad \text{or} \quad B_k = -u_k \cdot E_{inc} \]

From the Bs, the coefficients of the Fs can be determined, and hence the far field. At no single stage is there the inversion of a large matrix. More importantly, the orthonormal vectors \( u \) do not have to be all worked out at once. It may be, especially at low frequencies, that only the first few of them are important. Because the \( u_s \) are orthogonal, calculation of \( u_{n+1} \), and hence \( B_{n+1} \), will not affect the already calculated values of the first \( n \) \( u_s \) and Bs. By inspecting successive values of the Bs, we can check for numerical convergence and also get some idea of the accuracy of the solution. If after some value of \( n \), the Bs become so small that they can be neglected, then the process can be halted there, and we are saved the labour of calculating the remaining \( u_s \).
It is hoped to start writing a program to implement the above method in the near future.

The extension to three dimensions for scalar diffraction problems is trivial - this is affected by making the replacement

$$H_0^{(2)}(k|\rho - s_i|) \rightarrow \frac{\exp(-jk|\rho - s_i|)}{|\rho - s_i|}$$

i.e. a replacement of the two dimensional scalar Green's function by the three-dimensional scalar Green's function. In the full 3 dimensional vector diffraction problem, the question of polarisation rears its ugly head and dyadic Green's functions will have to be introduced, and the point source will have a direction as well as a magnitude. However, no changes in principle are presently envisaged.
INTRODUCTION AND CRITIQUE: REFERENCES


APPENDIX I

Evaluation of.

\[ \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{\rho^2} \left( \frac{d\rho}{d\phi} \right)^2 \right) d\phi \]

round a rectangular contour, refer to Fig. 2 and consider a rectangle of sides \(a\) and \(b\). Consider one of the 'b' sides; on it

\[ \rho = \frac{a}{2 \cos \phi} \]

\[ \frac{d\rho}{d\phi} = \frac{a \sin \phi}{2 \cos^2 \phi} \]

The contribution to the integral from that side is

\[ \frac{1}{2\pi} \int_{-\tan^{-1}(b/a)}^{\tan^{-1}(b/a)} \tan^2 \phi \, d\phi = \frac{1}{2\pi} \int_{-\tan^{-1}(b/a)}^{\tan^{-1}(b/a)} \sec^2 \phi - 1 \, d\phi \]

\[ = \frac{1}{2\pi} \left[ \frac{2b}{a} - 2 \tan^{-1} \left( \frac{b}{a} \right) \right] \]

Consideration of symmetry give the final value to be

\[ \frac{2}{\pi} \left| x + \frac{1}{x} - \frac{\pi}{2} \right| \quad \text{with } x = \frac{b}{a} \]

for \( x = 5 \) this is approximately 2.3.
Measurements were made by RAE of the field of a monopole mounted on a cylinder of length 56 cms and diameter 12 cms. The measurements were taken in the horizontal plane of polarisations parallel and perpendicular to the plane. We consider here only two configurations or conditions: 'Condition 1' with the antenna pointing vertically and 'Condition 4' with the antenna lying in the horizontal plane. In these two cases one would expect that the incident polarisations would be purely vertical and horizontal respectively. Using the polar plots supplied by RAE, the ratio of the cross-polarised power to the power contained in the incident polarisation was measured at 10° intervals and the mean found. The results are summarised in the following table:

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<th>SERIAL #</th>
<th>FREQUENCY MHz</th>
<th>POWER RATIO</th>
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</tr>
<tr>
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<td>1.4</td>
</tr>
</tbody>
</table>

The relative inconsistency of condition 4 measurements compared with condition 1 is probably due to the type of average used.
Fig 3
Fig 4
Fig 7

'Dipole' formed of two 'point' sources

±A

Fig 8
The calculation of a scattering matrix for a body of revolution.

The method used to calculate the scattering matrix is similar to that used by William and Mittie (Ref. 3) for an infinite cylinder. The scattered field is given by Hahn and Wasylkiewskys (Ref. 4) as

\[ E_{\text{scat}} = \sum_{i} \left[ \hat{E}_{i}^{(1)}(r) e^{i(\theta, \phi)} + \sqrt{\frac{5}{6}} \hat{E}_{i}^{(2)}(r) e^{i(\theta, \phi)} \right] \]

in the coordinate system, O, shown in Fig. 1. The summation over \( i \) is a simplification of

\[ \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{2}{a^{2}} \]

and \( E_{i}^{(1)}(\theta, \phi) \) and \( E_{i}^{(2)}(\theta, \phi) \) are defined in appendix I.

If the scattering body is a body of revolution, the scattering in each plane containing the axis of revolution will be identical and only the scattering matrix in one of those planes need be considered. The scattering in any other plane is discussed later. Thus we shall only deal with the XZ plane with \( \theta = \frac{\pi}{2} \).
The scattered field coefficients for a particular incident field can be found on the surface of the scatterer using the boundary conditions, but the equation for the field in coordinates of $r$ and $\phi$ will not converge when $r < r_0$. This is due to the discontinuity of the scatterer in the space outside, the field defined by $\vec{E}_s$, over which the field must converge and is discussed for the 2-D situation by Wilton and Mittig (Ref. 1). We must therefore define the field in coordinates $r$, $\phi$, $\rho$, $\phi_1$, which describe an area of space free of inhomogeneity.

To do this, the Bessel function addition theorem from Abramovitch and Stegun (Ref. 2) is used to give the expansion below as shown in appendix 1.

\[
\hat{H}^{(2)}(\rho_1, \phi) = \frac{\cos(m \phi)}{\sqrt{\cos^2(m \phi) + \sin^2(m \phi)}} \hat{H}^{(0)}(\rho_1, \phi)
\]

\[
\sum_{n=0}^{\infty} \hat{H}^{(2)}_{\pm n}(\rho_1) \times \hat{H}^{(1)}(\rho_1) \left( \cos \left( \frac{\pi n}{2} - \frac{\pi}{2} \phi \right), \sin \left( \frac{\pi n}{2} - \frac{\pi}{2} \phi \right) \right) \left( \cos \phi, \sin \phi \right)
\]

for $r^+ > r_1$
\( \hat{A}_n^{(2)}(R_\tau) \) is given in appendix 3 to be
\[
\hat{A}_n^{(2)}(R_\tau) = (n+1) \frac{1}{R_{\tau}} \hat{A}_n^{(2)}(R_\tau) - \hat{A}_{n+1}^{(2)}(R_\tau)
\]

Using these expansions, we arrive at a convergent matrix equation for the 3D boundary point from equation 1b:
\[
\begin{bmatrix} r E \text{scat} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} X \neq 1 \\ \frac{\partial}{\partial \phi} \end{bmatrix} \begin{bmatrix} F \neq 1 \\ \frac{\partial}{\partial \phi} \end{bmatrix}
\]

Now the boundary condition on the surface
\[
r E \text{scat} = -E_{\text{inc}}
\]
can be applied where \( r E_{\text{inc}} \) is given as
\[r E_{\text{inc}} = E \sqrt{\frac{\hat{A}_n^{(1)}(R_\tau)}{\hat{A}_n^{(2)}(R_\tau)}} e^1(\theta, \phi)
\]
\[+ \sqrt{\frac{\hat{A}_n^{(2)}(R_\tau)}{\hat{A}_n^{(2)}(R_\tau)}} e^2(\theta, \phi)
\]

As the scatterer has no axes of revolution on the XY plane, only the energy incident in the XY plane will be scattered into the XY plane and into no other plane, so again we can set \( \theta = \frac{\pi}{2} \). We must also expand \( \hat{A}_n^{(1)}(R_\tau) \) and \( \hat{A}_n^{(2)}(R_\tau) \) using the same expansion from appendix 2 and appendix 3. Each \( a_{n+1} \) is zero for one mode and all others to zero then using the equation linking incident and scattered coefficients given by Kohn and Wannier.
ref.) we have

$$b_i = \frac{\gamma}{2} E_{in} - S_{it} - \frac{\gamma}{2} E_{in}$$

Therefore to find the scattering matrix $S$, we must solve

$$S = \left[ -\frac{E_{in}}{E_{st}} \right] = \left[ X_{en} \right] \left[ S_{it} \right].$$

As is shown in Appendix I, $A_i$ and $B_i$ are alternatively zero, so the incident modes are either $\phi_0$ or $\phi_0$ phases. If we take one element of the incident field matrix we have

$$-\frac{E_{in}}{E_{st}} = X_{en} S_{it} + \frac{\gamma}{2} E_{in}$$

The second summation must be zero and the same will be true for a $\phi_0$ phase incident mode so $S_{it}$ and $S_{it}$ will all be zero. Thus we can reduce equation 5a to

$$\left[ -\frac{E_{in}}{E_{st}} \right] = \left[ X_{en} \right] \left[ S_{it} \right]$$

$$\left[ -\frac{E_{in}}{E_{st}} \right] = \left[ \phi X_{en} \right] \left[ \phi S_{it} \right]$$

Giving an independent scattering matrix for each phase.

The number of scattering matrices elements necessary can be further reduced as the angular part of some scattered modes are zero at all $\phi$ and these modes will play no part in the scattering.
Even more incident modes will be zero for special source positions, and this will reduce the number of scattering matrix elements further, but this is discussed later.
The calculation of a scattering matrix for an infinite cylinder.

The 2-D scattering matrix is found in an identical way to the 3-D matrix using only simpler equations. The scattered field is given by Wilton and Mittra (ref.) to be

\[ E_{\text{scat}} = \sum_{n=-\infty}^{\infty} \beta_n H_n^{(2)}(kr) e^{i\phi} \]

where the geometry is identical to the projection onto the XY plane of the 3-D case as in Fig. 1. The Bessel function addition theorem is used to produce the expansion also given in Appendix 2 as

\[ H_n^{(2)}(kr) e^{i\phi} = \sum_{n'=0}^{\infty} H_{n+n'}^{(2)}(kr') J_n(r') e^{i(n'-\phi)} e^{i\phi}, \]

for \( r' > r \).

Once again, we have the matrix equation for the 2-D boundary point

\[ [\mathbf{E}_{\text{scat}}] = [\mathbf{X}_{\text{sm}}] [\mathbf{\beta}_n] \]

and using the same boundary condition, we have to solve for the scattering matrix \([\mathbf{\alpha}]\) in the equation

\[ -[\mathbf{E}_{\text{inc}}] = [\mathbf{X}_{\text{sm}}] [\mathbf{\alpha n}] \]

where \( E_{\text{inc}} = \mathbf{r} \times H_+^{(2)}(kr) e^{i\phi} \) with \( \alpha_1 = 1 \) and \( H_+^{(2)}(kr) e^{i\phi} \) expands...
The calculation of the spherical modes incident on a sphere containing the scatterer due to a source outside the sphere.

The sphere $S$ is defined by radius $a$ as shown in Fig. 2. Outside $S$ is a sphere with orientation $Q_0$ at +1. The field at any point $P$ can be expanded about $O$, the origin, using the Dyadic Green's function as shown by Klein and Werginowski (ref) to give

\[ E = \frac{c_0^2}{2\pi} \sum_{n} \left[ \frac{\hat{J}_n(a)}{R_0} \int \frac{\hat{a}^2(\theta', \phi')}{R'} \cos \theta' e^{i(\theta, \phi)} \, d\Omega' \right] n(a) \frac{\hat{H}_n^{(2)}(a)}{R_0} \Phi_n(\theta', \phi') \, d\Omega' \]

where $A = \frac{-2c_0^2}{a^2 n(n+1)}$ to become consistent for normalizing with $K+W$ for $t \geq r$, all the Bessel functions $J_n$ and $H_n^{(2)}$ become $J_n$ and $J_n^{(2)}$ respectively.

The field can now be put in the form suitable for the scattering matrix as

\[ E = \sum_{n} \left[ \hat{J}_n(a) \hat{a} \cos \theta \right] \cos \theta e^{i(\theta, \phi)} + \hat{J}_n(a) \hat{a} \sin \theta \right] \sin \theta e^{i(\theta, \phi)} + \int \frac{\hat{a}^2(\theta', \phi')}{} \cos \theta' e^{i(\theta, \phi)} \, d\Omega' \]

where

\[ n \frac{\hat{H}_n^{(2)}(a)}{R_0} \cos \theta e^{i(\theta, \phi)} \, d\Omega' \]

The calculation of the spherical modes incident on a sphere containing the scatterer due to a source outside the sphere.

The sphere $S$ is defined by radius $a$ as shown in Fig. 2. Outside $S$ is a sphere with orientation $Q_0$ at +1. The field at any point $P$ can be expanded about $O$, the origin, using the Dyadic Green's function as shown by Klein and Werginowski (ref) to give

\[ E = \frac{c_0^2}{2\pi} \sum_{n} \left[ \frac{\hat{J}_n(a)}{R_0} \int \frac{\hat{a}^2(\theta', \phi')}{} \cos \theta' e^{i(\theta, \phi)} \, d\Omega' \right] n(a) \frac{\hat{H}_n^{(2)}(a)}{R_0} \Phi_n(\theta', \phi') \, d\Omega' \]

where $A = \frac{-2c_0^2}{a^2 n(n+1)}$ to become consistent for normalizing with $K+W$ for $t \geq r$, all the Bessel functions $J_n$ and $H_n^{(2)}$ become $J_n$ and $J_n^{(2)}$ respectively.

The field can now be put in the form suitable for the scattering matrix as

\[ E = \sum_{n} \left[ \hat{J}_n(a) \hat{a} \cos \theta \right] \cos \theta e^{i(\theta, \phi)} + \hat{J}_n(a) \hat{a} \sin \theta \right] \sin \theta e^{i(\theta, \phi)} + \int \frac{\hat{a}^2(\theta', \phi')}{} \cos \theta' e^{i(\theta, \phi)} \, d\Omega' \]

where

\[ n \frac{\hat{H}_n^{(2)}(a)}{R_0} \cos \theta e^{i(\theta, \phi)} \, d\Omega' \]
\[
\text{inc} \; H = \text{inc} \; E = \sqrt{\frac{1}{2\pi} \left\{ \frac{\hat{A}^{(2)}(k_r)}{n} \right\} e^{i \Phi'}} - U_0
\]

The fully scattered field coefficients are given by Nahm and Wassilliewsky (ref.)

12 \begin{equation}
\text{scat} \; E_i = \sum \text{scat} \; E_i \text{inc} \; E_i \text{inc} \; E_i - \text{inc} \; E_i,
\end{equation}

where \( \text{out} \; E_i \) is found using the Dyadic Green's function with \( \tilde{\mathbf{G}} \).

13 \begin{equation}
\text{tot} \; E_i = \text{scat} \; E_i + \text{out} \; E_i
\end{equation}

This gives the field as

14 \begin{equation}
\tilde{E}_i = \sum \text{tot} \; E_i \hat{\mathbf{n}}(R_i) e^{i(\theta, \phi)}
\end{equation}

If this is taken to the limit of \( r \to \infty \)

15 \begin{equation}
\hat{\mathbf{n}}(R_i) \propto \frac{1}{r}
\end{equation}

16 \begin{equation}
\tilde{E}_i = \sum \text{tot} \; E_i \hat{\mathbf{n}}(R_i) e^{i(\theta, \phi)}
\end{equation}

As the scattered field considered is only in the \( xy \) plane, \( \theta = \frac{\pi}{2} \) giving

16 \begin{equation}
\tilde{E}_i = \sum \text{tot} \; E_i \hat{\mathbf{n}}(R_i) e^{i(\theta, \phi)}
\end{equation}
As each phase is treated separately in producing the scattering matrices, each phase is treated separately in producing the scattering pattern.
The calculations of the cylindrical modes incident on a circular cylinder containing the scatterer due to a source outside the circle

The cylinder C is defined by \( r \), as shown in Fig. 3. Outside C is a line source at \( r \), and the field due to the source at any point \( P \) in the coordinate system \( O' \) is given as

\[
E = \mathbf{H}_0^{(2)} (kr) \]

This can be expanded about \( 0 \), noting that the expansion is from \( 0 \) to \( 0 \), the reverse of that in calculating the scattering matrix using the expansion from the Bessel function addition theorem in Appendix 2 to give

\[
E = \sum_{n=-\infty}^{\infty} \mathbf{H}_n^{(2)} (kr') J_n (kr) e^{in(\phi - \phi')} \quad \text{for} \quad r > r'
\]

The field incident on \( C \) can be written in the form

\[
E_{\text{inc}} = \sum_{n=-\infty}^{\infty} \mathbf{H}_n^{(1)} (kr) e^{in\phi}
\]

where

\[
\mathbf{H}_n^{(1)} (kr) = \frac{\mathbf{H}_n^{(2)} (kr) e^{-in\phi'}}{2}
\]

and we can find the scattered field coefficient.
\[ \beta_{n} = \leq 0 \text{ inc} \text{ m inc} - \beta_{n} \text{ inc} \text{ m inc} \]

where \( \beta_{n} \text{ inc} \) is the outgoing coefficient of the incident field and \( \beta_{n} = 2i\text{inc} \). The outgoing field coefficient is found in a similar fashion to be

\[ \beta_{n}^{\text{out}} = \int J_{n}(Kr) e^{-im\phi} \]

we now have the total field given as

\[ E_{\text{tot}} = \sum_{m} \beta_{n}^{\text{tot}} - \int J_{m}(Kr) e^{-im\phi} \]

where \( \beta_{n}^{\text{tot}} = \beta_{n}^{\text{inc}} + \beta_{n} \)

As \( r \to \infty \), \( J_{m}(Kr) \to \frac{i^{m}}{m} \), so we have

\[ E_{\text{tot}} \to \sum_{m} \beta_{n}^{\text{tot}} e^{im\phi} \]
The calculation of the modes incident on a sphere containing the scatterer due to a source inside the sphere.

The sphere $S$ is defined by $r = r_0$ as shown in Fig. 4. Inside $S$ is a cylinder with orientation $y$ at $A$. The field at any point $P$ can be expanded about a point $O$ outside $S$ and on the $XY$ plane, where the transverse projection of $r$ on that plane is perpendicular to the surface of the scatterer. This is given by

$$r = \mathbf{E} = \sum \frac{\hat{A}^{(2)}(r_2)}{r_2} \left\{ \frac{\hat{R}^{(2)}(r_2)}{r_2} \right\} \hat{n}(\theta, \phi) \cdot \mathbf{U}_0$$

In this case, the exponential form of the vector modes is used. We now have a field about $O'$ outside the sphere defined by $r'$ in the form

$$r' = \mathbf{E} = \sum \frac{\hat{A}^{(2)}(r_2)}{r_2} \left\{ \frac{\hat{R}^{(2)}(r_2)}{r_2} \right\} \hat{n}(\theta, \phi) \cdot \mathbf{U}_0$$

where

$$\mathbf{E} = -\sum \frac{\hat{A}^{(2)}(r_2)}{r_2} \left\{ \frac{\hat{R}^{(2)}(r_2)}{r_2} \right\} \hat{n}(\theta, \phi) \cdot \mathbf{U}_0$$

and

$$\mathbf{E} = \sum \frac{\hat{A}^{(2)}(r_2)}{r_2} \left\{ \frac{\hat{R}^{(2)}(r_2)}{r_2} \right\} \hat{n}(\theta, \phi) \cdot \mathbf{U}_0$$
Now as we are only interested in the field incident on $S$ in the $XY$ plane, we can set $\Theta = \frac{\pi}{2}$ to give

$$\mathbf{E}_2 = \sum \frac{\hat{\mathbf{E}}_1(\mathbf{r}_1')}{r_1'} \mathbf{H}^\text{inc}_n(\mathbf{r}_2) e^{i m \phi} (A_1 \cos \phi_0 + B_1 \sin \phi_0) + \frac{\hat{\mathbf{E}}_2(\mathbf{r}_1')}{r_1'} \mathbf{H}^\text{inc}_n(\mathbf{r}_2) e^{i m \phi} (B_1 \cos \phi_0 - A_1 \sin \phi_0)$$

Each mode $\mathbf{H}^\text{inc}_n(\mathbf{r}_2)$ and $\mathbf{H}^\text{inc}_{n+1}(\mathbf{r}_2)$ can be expanded about 0 in terms of $r$ and $\phi$ noting that the expansion is from 0 to 0 using

$$\mathbf{H}^\text{inc}_n(\mathbf{r}_2) = (n+1) \mathbf{H}^\text{inc}_{n+1}(\mathbf{r}_2) - \mathbf{H}^\text{inc}_n(\mathbf{r}_2)$$

From appendix 3 and

$$\mathbf{H}^\text{inc}_n(\mathbf{r}_2) e^{i m \phi} = e^{i(m-n-\frac{1}{2})} \sqrt{\frac{kr_2}{4\pi}} J_{m-n-\frac{1}{2}}(kr_2)$$

for $r > r_1$,

$$\sum_{n=-\infty}^{\infty} \frac{\hat{\mathbf{E}}_1(\mathbf{r}_1')}{{r}_1'} J_{m-n-\frac{1}{2}}(kr_2) e^{i(m-n-\frac{1}{2})} \phi^m e^{i \phi}$$

From appendix 2 and will yield

$$\sum_{n=-\infty}^{\infty}$$

If we substitute these equations into equation 2.3 and reverse the order of summation we have
\[ r_2 \frac{E}{n} = \sum_{i} \frac{\lambda_{i} E_{i}(r'')} \left( \frac{\lambda_{i}}{r_{2}} \right)^{2} \left\{ [A_{i} e^{i\phi_{0}} + B_{i} e^{-i\phi_{0}}] \right\} \]

for \( r'' \geq r \)

This can be put in the form of incoming and outgoing cylindrical modes about \( \phi \) as

\[ r_2 \frac{E}{n} = \sum_{i} \left\{ \frac{\lambda_{i} E_{i}(r'')} \left( \frac{\lambda_{i}}{r_{2}} \right)^{2} \right\} \left\{ [\alpha_{i} e^{i\phi_{0}} + \beta_{i} e^{-i\phi_{0}}] \right\} \]

\[ \text{for } r'' \geq r \text{ where} \]

\[ \alpha_{i} = \frac{\lambda_{i} E_{i}(r'')} \left( \frac{\lambda_{i}}{r_{2}} \right)^{2} \]

\[ \beta_{i} = \frac{\lambda_{i} E_{i}(r'')} \left( \frac{\lambda_{i}}{r_{2}} \right)^{2} \]

\[ \text{and similarly for the } \phi \text{ coefficients, thus we have the incoming field about } 0 \text{ in the correct form to use the scattering matrix for 2-D cylindrical modes, the scattered field coefficients are found by} \]
27 \quad \beta_{\pm} = e^{\pm \alpha t} \phi_{\pm} \quad \text{and the scattered field is given by}

28 \quad r \mathbf{E} = \frac{\varepsilon \mathbf{E}_0}{r} \left[ e^{+i\phi} \mathbf{E}_0 \right]_{\text{inc}} + \frac{\varepsilon \mathbf{E}_0}{r} \left[ e^{-i\phi} \mathbf{E}_0 \right]_{\text{tot}}

as \ r \to \infty \text{ and } \mathbf{E}_2 \to \mathbf{E}_0 \text{ this allows the coefficients } \beta_{\pm} \text{ to be reduced to}

29 \quad \beta_{\pm} = \alpha \pm \frac{1}{i} \frac{1}{r} \left[ \sum_{j} b_j \mathbf{b}_j \left( \mathbf{E}_2 \right)_j \right] A_i

+ \frac{1}{i} \frac{1}{r} \left[ \sum_{j} b_j \mathbf{b}_j \left( \mathbf{E}_2 \right)_j \right] B_j

and similarly for the \phi \ \text{component , allowing the } \mathbf{E}_2 \text{ in equations 28 and 28 to cancel. The total field is then}

\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{tot}}

where \mathbf{E}_{\text{out}} \text{ is found in the same way as for a source outside the sphere } S.
The calculation of the modes incident on a cylinder containing the scatterer due to a source inside the cylinder.

The cylinder is defined by the axis of the source at \( r = r_2 \). The field at \( r \), due to the source at \( r' \), is given in the coordinate system \( A \) as \( H_0(\theta, z) \). This can be expanded about a point outside the circle defined by \( r_2 \), say \( r' = r_2 + z \), where \( z \) is perpendicular to the surface of the object, as

\[
E_{\text{source}} = H_0(\theta, r')
\]

\[
\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{2}{\pi} J_n(r_1) e^{-im\phi} H_n^m(\theta) e^{im\phi} \quad \text{for } r < r_2
\]

\[
\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{2}{\pi} J_n(r_1) e^{-im\phi} H_n^m(\theta) e^{im\phi} \quad \text{for } r > r_2
\]

using the Bessel function addition theorem as shown in appendix 2.

This field can then be expanded, using the same addition theorem, about \( O \) as

\[
E_{\text{inc}} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{2}{\pi} J_n(r_1) e^{-im\phi} J_n(\theta) e^{im\phi} \quad \text{for } r < r_2
\]

and

\[
E_{\text{out}} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{2}{\pi} J_n(r_1) e^{-im\phi} H_n^m(\theta) e^{im\phi} \quad \text{for } r > r_2
\]

where \( b_n^m = J_n(r_1) e^{-im\phi} \).

Reversing the order of summation gives the field in the form that is required by the scattering matrix as

\[
E_{\text{inc}} = \sum_{m=-\infty}^{\infty} \left[ a_m H^m(\theta) + b_m H_m^m(\theta) \right] e^{im\phi} \quad \text{for } r < r_2
\]
and \( E_{\text{out}} \equiv \sum_{m=-\infty}^{\infty} B_m \tilde{H}_m(r) e^{i\phi} \) for \( r'' < r \)

where \( \alpha_m = B_m \equiv \sum_{n=-\infty}^{\infty} \beta_{m-n} \tilde{H}_n(R-r') e^{i(n-m)\phi''} \)

and \( B_m \equiv \sum_{n=-\infty}^{\infty} \beta_{m-n} \tilde{J}_n(\Pi r') e^{i(n-m)\phi''} \)

Now, as previously shown, the scattered field is found using

33 \( B_{\text{scat}} \equiv \sum_{m=-\infty}^{\infty} \alpha_m - B_m \)

and the total field is found using

34 \( B_m^{\text{tot}} = B_m^{\text{scat}} + B_m^{\text{out}} \)

giving the total field as

35 \( E_{\text{tot}} = \sum_{m=-\infty}^{\infty} B_m^{\text{tot}} e^{i\phi} \)
The radar cross-section of a cylindrical body.

If plane wave incident on the origin \( O \) of a scatterer in the direction \( \theta \), as in Fig. 6, is given by

\[
\mathcal{E}(j k, l) = \sum_{n=0}^{\infty} \frac{H_n^1(k r)}{n!} \mathcal{E}(j n(\Phi - \phi))
\]

This gives an incident field coefficient suitable for using in a scattering matrix of

\[
\alpha_n = \frac{(-1)^n}{2} \mathcal{E}(j n(\Phi - \phi))
\]

The R.C.S. is given as proportional to

\[
\sum_{m=-N}^{N} \sum_{n=-N}^{N} (\xi - \alpha_n - \alpha_m) \delta^{mn} H_n^2(k r) \mathcal{E}(j m\Phi)
\]

Taking \( r \) to infinity, we have for the bistatic case

\[
\sum_{m=-N}^{N} (\xi - \alpha_m) \mathcal{E}(j m\Phi)
\]

and for the monostatic case, the substitution \( \Phi = \phi + \pi \) is made.
A discussion of the limit required on the summations.

One of the attractions of finding the scattered field by this method is that the number of modes required can be calculated before any work is done. The process of finding the limits is identical in either 2-D or 3-D and therefore only the 2-D case need be considered. This has been shown by Wilton and Mittra (ref. 6) for the limits required on the scattering matrix to be \( |n| > k r_{\text{min}} \) (see equation 6 and fig. 1). However, experience has shown that in most cases, the limit is dependent on the radius, \( r_c \), of a circle of the same area as the scatterer. Thus, we calculate \( J_n(k r_c) \) and use all the modes until \( n = n_{\text{max}} \) such that \( J_{n_{\text{max}}}(k r_c) < 0.01 \).

Wilton and Mittra (ref. 6) also stated that the limit on the internal summation over \( t_f \) (see equation 6), \( \beta_f + \beta_f + \min \) in our calculations we simply take \( r_{\text{max}} = 2 r_{\text{max}} \) and this has proved adequate.

The limits required for the field due to a source outside the enclosing circle are determined by the convergence of the outgoing field coefficients which are proportional to \( J_n(k r_f) \) (see eqn. 19). Thus we find \( n_{\text{max}} \) such that \( J_{n_{\text{max}}}(k r_f) < 0.001 \) and \( n_{\text{max}} > n_{\text{max}} \) where \( n_{\text{max}} \) is the limit on the orders required for the scattering matrix. Where \( m = n_{\text{max}} \), we can see that the scattering matrix tends to
that for free space and the scattered field coefficients can be taken to be zero for maximum for a source outside the enclosing cylinder, the number of modes required for the outgoing field can be greater than the number of modes for the scattered field. This allows the source to be moved away from the object without significantly increasing the difficulty of the problem. An example of the application of this solution is the field due to the tail fins of an aircraft excited by a nose antenna.

This solution is also applied when the source is inside the enclosing cylinder. We are using the Bessel function addition theorem where the inside summation is twice the outside summation. In both 2-D and 3-D cases, we reverse the summation. The maximum order for the convergence of the new outside summation is dependant on $n_{\text{max}}$ and $\gamma$ and $n_{\text{max}}$ is found using $J_{n_{\text{max}}}(k_{\text{inc}}r) < 0.001 where $k_{\text{inc}}$ is the limit on the order of incident modes and will be greater than the order used in the scattering matrix. Once again, the scattering matrix is assumed to tend towards a free-space scattering matrix for orders outside the known matrix, and the scattered field coefficients are set to zero.

Having decided on the number of modes necessary, we must now decide on the number of boundary points required for a stable scattering matrix. This number is...
dependent on the number of scattered modes in the scattering matrix and must be greater than this number. Although it is possible to calculate a matrix with the same number of points as modes, it was found that twice the number of modes was a suitable ratio to produce a stable matrix. This was found empirically by increasing the number of points until there was no significant change in the total field pattern.
A discussion of the incident modes necessary

In the 3-D case, a large number of incident modes are possible as was shown in previous reports. However, it is possible to reduce this number for incident modes although all the scattered modes must be considered. The incident field is given in section 3.1 equation 17 and each coefficient is dependant on

\[ e_i(\theta', \phi') \cdot \psi_0 \] and \[ \Phi_i(\theta', \phi') \psi_0 \]

or

\[ e_i(\theta', \phi') \cdot \psi_0 \]

For any mode, if the coefficient is zero, there will be no incident field and this mode need not be considered. The dot products given above are the factors which will decide whether a coefficient is zero or not. If these are calculated before calculating the scattering matrix, we will know which matrix elements we need to consider. Thus we will be left with a rectangular scattering matrix and this causes problems in checking for convergence in the scattering matrix. If we are no longer able to calculate the eigenmode, although it may be possible to use the reciprocity check.
Program development

Scatterer shapes and the calculation of the boundary point

At present, the program will calculate the scattering matrix, in the 3-D case, for a body of revolution with the axes of revolution defined as the $x$-axis. In the 2-D case, the body is a cylinder with the constant cross-section in the $z$-axis. The cross-section of the object in the $xy$ plane is non-rectangular although Wilton and Maitra [58] give the method for dealing with rectangular shapes. Any object within these limits can be dealt with although most will need some simplification.

The cross-section is defined by vertex coordinates and the coordinates of the centre of a circle for each vertex which defines the outline to the next vertex. Straight lines are defined using a circle of very large radius. It should be possible for ellipses to be included without great difficulty.

All the dimensions are in wavelengths. The boundary points are found by dividing the perimeter of the object into equal lengths. The corners do not raise any problems by the method as in most cases having boundary points on the corners and therefore trying to hold the field at the corners can be avoided if, as in the program, it does not have as much symmetry as the object, then only when sufficient modes and points have been taken does the final pattern show the symmetry of the
object. This is very useful in checking for stability of the scattering matrix.
Maximum scatterer size

The limiting factor in determining the maximum size of the scatterer that the program can handle is the size of the scattering matrix. The program has been written to make the most efficient use of all the core space available in solving the simultaneous equation to find the scattering matrix. The factor that determines the amount of core required is the number of scattered modes required. This number is dependent on the size of the scatterer alone, and is proportional to the maximum radius of the smallest circle which encloses the area of the cross-section in the XY plane of the scatterer. In the 2-D case, the number of modes is twice the order of the highest mode required, but in the 3-D case, the number of modes is proportional to the square of the highest order. The maximum order is proportional to the radius of the smallest circle which encloses the area of the cross-section of the scatterer in the XY plane in both 2-D and 3-D cases. Thus, the maximum size of the scatterer can vary with the size of the scatterer, but a rough guide is that in the 2-D case, the maximum size is an object with a maximum dimension of 10 wavelengths, and in the 3-D case, of 0.5 wavelengths. Although the maximum size in the 2-D case is acceptable in the 3-D case, obviously it is not acceptable. There are three possible ways of getting around this problem.
first is to use a computer with an extremely large working space although, as the amount of space increases by the fourth power, as the size of the scatterer is increased, a machine will be conventional hard core memory would not give a sufficient increase and new machines with virtual storage might be the answer. The second route is to use and array processing might be the answer. The second way is to not use the straight 3-D solution to the problem but the method requires when the source is inside the enclosing sphere. As this uses a 2-D scattering matrix, the problem is really sidestepped but at the cost of increased computer time although this is a less restrictive limitation. The third way requires more theoretical work but present program performance indicates that more scattered modes are being used than are necessary and more might be eliminated. This is discussed further in the next section.
scattering matrix stability

The most difficult problem in finding the scattering pattern of an object is in knowing when the scattering matrix is stable, and gives the correct pattern. It is possible to deduce that the pattern is stable with repeated runs with different limits on the summations involved, but this is time-consuming. It is therefore desirable to have a method for testing the stability of the scattering matrix on only one run. Various tests have been devised to give some indication of the stability but these are only qualitative and only by repeated runs can the stability be confirmed. When a crucial change to the program is made, the instability is well understood and is discussed below but in the 3-D case, there are still problems although the same methodology applies to a certain extent.

There are two levels to the instability, the first, when by simply studying the scattering matrix itself, the instability is obvious. The stable matrix will have a central core of elements of low orders where all other elements will decay to zero for large $n$. Also the matrix is symmetric about the diagonal although rounding errors mean that the outer elements in the sides of the matrix decay slower than in the top and bottom. The matrix is transformed in the
Z-D case to avoid this problem affecting the final pattern although the reason for this being necessary is not known. If there is very large deviation from this form then the matrix will be unstable and the scattering pattern is highly suspect. The usual form of the deviation is that some of the outer elements, instead of decaying to zero, become very large. This is generally due to taking too few or too high a limit on the various summations. Too low a limit means that not enough modes have been included to calculate the outer elements accurately and too high a limit means that modes of very high order allow rounding errors to assume a far greater importance than normally and distort the outer elements.

A second, more subtle method of testing stability is to calculate the eigenvalues and find their moduli and phase. The modulus of each eigenvalue should be one and an accuracy of one significant figure is sufficient to indicate stability. The phase should decay to zero for lower orders but this is awkward to see with the present program as it mixes up the eigenvalues. However, if the scattering matrix passes the visual check the smallest phases are those for the highest orders and these should be less than 0.0001.

An alternative check on the scattering matrix is possible by using the principle of reciprocity. This says that given an incident field on the scatterer and the concomitant scattered field
As an incident field, with the same coefficient as the scattered field, will cause a scattered field with the same coefficient as the original incident field. This is expressed in the program using the equation

\[ B_m = S^* S_m (S S_n^*) \]

and \( B_m \) should equal \( S \). This test has turned out to be much more sensitive to errors than using eigenvalues and will probably require greater accuracy in calculating the scattering matrix before it becomes useful.

There is also a far swifter method of checking the stability of the scatterer and the source are symmetrical across an axis. As the object and source are symmetrical, the scattering pattern should be symmetrical, but it is possible in selecting the boundary points, to confute a non-symmetrical arrangement of points. This pseudo-object will produce a symmetrical pattern only if the scattering matrix is sufficiently stable to produce the correct pattern. This is the finest check of the matrix stability, but it is only available when the pattern should be symmetrical.

In the 3-D case, there is an additional cause of instability to be considered. As shown previously, some of the possible incident and scattered modes need not be included as they are zero in the plane we are considering. If the scattering matrix includes any elements
for these modes, then they will fluctuate wildly as the routine to solve the simultaneous equations cannot converge to any value for them and this can upset the convergence of the matrix as a whole. A solution for the matrix can sometimes be found by using fairly inaccurate field matrices and under these conditions, unnecessary modes affect as rows or columns of very small or very large number. Also, if the scattering matrix found this way produces a fairly stable pattern, indicating again the unimportance of these extra modes. An interesting point for any future work is that although all the modes predicted as unnecessary by the present theory are excluded, these still appear to be more scattered modes which contribute nothing to the scattering matrix. If these modes could be eliminated, great savings in work space and increase in scatter size should be possible. Checking the 3-D matrix for stability tends to be done by inspection of the matrix or using a symmetrical object and source as the matrix is usually rectangular and neither the eigenvalues nor the reciprocal coefficients can be calculated using the rectangular matrix.
Experimental Comparison

In the previous report, the theoretical and experimental plots of the scattering pattern of a square cylinder with the source just outside the enclosing circle were shown to be very close. However, there was still a problem with placing the source inside the enclosing circle and in calculating the scattering matrix for a long, thin shape such as a rectangle. This can now produce reliable results for these cases as shown in Fig. (Here the scattering object is as shown in Fig. 1.)

This situation includes both problems and Fig. 2 shows that the theoretical curve is very close to the experimental curve in the region behind the rectangle from the source, the experimental curve is flatter than the theoretical and this is thought to be due to range effects in the experimental system that smooth out any nodes.

In the 3-D case, there has been no chance for experimental comparison. The program that calculates the pattern for a source outside the enclosing sphere can only handle small objects and these do not give very conclusive patterns. One is shown in Fig. 3 and the form of scattering can be seen to be as expected. However, the program that calculates the pattern for a source inside the enclosing sphere has not yet produced a sensible pattern and further work will be required on this program as it is the
one which has the most flexibility and will be the most used of the two programs.
Future Development

Stabilizing the scattering matrix

Although the 2-D scattering matrix is now fully understood, the 3-D scattering matrix is still giving problems as explained previously. This is evident in that there is some difficulty in finding the scattering matrix, especially for objects of high symmetry such as a sphere where the 2-D equivalent is the easiest and most accurate. When a matrix is found, some elements are several orders of magnitude larger than the expected value. Given this problem will be necessary before this 3-D solution can be used on any real object.

![A figure or diagram indicating](image)

In the past, this sort of problem indicated that unnecessary modes are being included in the system of linear equations used to find the scattering matrix. Also, it is worthwhile studying those elements which are divergent and appear in set patterns dependent on the order of the modes and this indicates that the solution is to be found on the theoretical side of the problem rather than the computing.

A reassuring fact is that the scattering pattern that is found is stable. In spite of the instability of the scattering matrix, this suggests that the divergent elements of the scattering matrix, although individual
significant, produce no overall effect in the final pattern and could be eliminated from the calculations. However, further investigation is necessary to prove or disprove these ideas.

Another aspect of scattering matrix stability that requires attention is the accuracy of the eigenvalue and reciprocity tests of stability. As was stated previously, these tests are very susceptible to errors as the scattering object departs from a highly symmetric shape such as a circle. It could be worth while finding the cause of this as a smaller scattering matrix could be used and the final scattered pattern will be more reliable. The first place to look is in the calculation of the boundary points geometry as this seems to be the greatest source of error.
The roll plane

The 3-D program is only set up to calculate the elevation and azimuth planes at present and a roll plane will be required. The elevation and azimuth planes are those containing the axis of rotation of the scatterer and both use the same method by simply rotating the source about the axes of rotation. The roll plane requires each point to calculated using a different incident field as the source is moved around the object. This requires greater computation than the other planes and substantial modifications to the program but there are no theoretical problems envisaged.
In the 2-D case, the R.C.S. has been calculated and was very useful in confirming the stability of the scattering matrix for symmetrical objects. In the 3-D case, the R.C.S. has not been calculated. The R.C.S. could be as useful as in the 2-D case but the associated problems are much greater. An individual scattering matrix will be required and although this would seem to invalidate any test of scattering matrix stability, the differences between the R.C.S. matrix and the scattered pattern matrix are expected to be minimal.
Finally, the program can be modified to attempt to find a scattering factor for a real aircraft. A simple approximation was suggested in the third report (ref. 7), which is to treat the fuselage, wings, tail, and engines as independent bodies of revolution. The wings and tail can be treated as bodies of revolution, as shown in Fig. (7), where a flat strip of width \( a \) is replaced by a cylindrical strip of diameter \( a/2 \). Thus, as a first approximation, the aircraft can be treated as a collection of cylindrical bodies and the individual scattering patterns combined to produce the total scattering. This should be sufficient for simply shaped aircraft, but a more complex shape will require the coupling to be taken into account.
Appendix 1.

The spherical vector modes

The spherical modes used are those given by Helmholtz and Weyl, and are defined as

\[ \phi_i (\theta, \phi) = \frac{1}{N_{nm}} \left\{ \sum (-1)^{m} P_n^m(\cos \theta)(\sin m\phi) \right\} \]

where \( \xi = 1 \), \( \epsilon \), \( \eta \), and \( \zeta \) as \( n = 1 \), \( m = 0 \) \( \cos \theta \)

\[ N_{nm} = \frac{4\pi (n+m)! \cdot n(n+1)}{\varepsilon_{nm} (2n+1) (n-m)!} \]

\[ \epsilon_{nm} = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0 \end{cases} \]

\[ r_\theta \vec{V} = \phi_0 \frac{\partial \phi_0}{\partial \theta} + \phi_0 \sin \theta \frac{\partial \phi_0}{\partial \phi} \]

giving

\[ z \phi_i (\theta, \phi) = \frac{1}{N_{nm}} \left\{ \sum \frac{\partial P_n^m(\cos \theta)}{\partial \theta} (\cos m\phi) \right\} \theta_0 + \sin \theta \frac{\partial P_n^m(\cos \theta)}{\partial \phi} (\cos m\phi) \phi_0 \]

Also \( \epsilon'' = \epsilon' \times \epsilon_0 \).

Vectors modes using complex exponentials are also used in the report and these are calculated by substituting in equation 1

\[ e^{im\phi} \text{ for } \cos m\phi \]

\[ e^{im\phi} \text{ for } \sin m\phi \]

\[ \varepsilon \text{ for } \varepsilon \leq \varepsilon \]

\[ \varepsilon \text{ for } \varepsilon \leq \varepsilon \]

\[ m = -n \text{ for } m = 0 \cos 1 \]

and \( \epsilon_{nm} = 2 \text{ for } \epsilon_{nm} = 1, m = 0 \)
For the special case where $\Theta = \frac{\pi}{2}$ we have

$$e^{i(\mathbf{v} \cdot \hat{r})} = \left( \cos m\phi \right) A_i \Theta + \left( \sin m\phi \right) B_i \phi,$$

where

$$A_i = -\frac{1}{N_{nm}} \left[ \frac{\partial P^m_n(\cos \Theta)}{\partial \Theta} \right]_{\Theta = \frac{\pi}{2}}$$

and

$$B_i = (+, -, \frac{m}{N_{nm}} P^m_n(0))$$

and $P^m_n(0)$ are found from Magnus, Oberhettinger, and Soni (ref.) to be

$$\left[ \frac{\partial P^m_n(\cos \Theta)}{\partial \Theta} \right]_{\Theta = \frac{\pi}{2}} = (-2)^m \pi^{-\frac{3}{2}} \sin(nm) \frac{1}{2} \left[ \frac{n + m + 1}{n - m + \frac{1}{2}} \right]$$

$$P^m_n(0) = (-2)^m \pi^{-\frac{1}{2}} \cos(nm) \frac{1}{2} \left[ \frac{n + m + 1}{n - m + \frac{1}{2}} \right]$$

$\frac{\partial P^m_n(\cos \Theta)}{\partial \Theta}$ and $\left[ \frac{m}{\sin \Theta} P^m_n(\cos \Theta) \right]_{\sin \Theta = 0}$ are given in appendix 4.

Note 1: As the associated Legendre polynomials given by the subroutine ALEGF have a factor of $2^m$ difference from those given by Magnus et al., all the equations are adjusted accordingly.
Appendix 2
The Bessel Function Expansion

We start with Eq. 3.2 which is stated in Abramowitz and Stegun (Ref. 5) as

\[ b_{n}(w) \left( \cos \nu X \right) \left( \sin \nu X \right) = \sum_{k=-\infty}^{\infty} b_{n+k}(w) J_{k}(\nu) \left( \cos k \alpha \right) \left( \sin k \alpha \right) \]

\( \nu \) unrestricted real number
\( k \) integer number
\( w, \omega, \alpha, \chi \) are positive
\( u > v \)

If we convert to the symbol conventions used in the report, we have for \( \nu' > \nu \)

\[ b_{n}(\nu') \left( \cos \nu' \left( \phi - \phi' \right) \right) \left( \sin \nu' \left( \phi - \phi' \right) \right) = \sum_{k=-\infty}^{\infty} b_{n+k}(\nu') J_{k}(\nu') \left( \cos k \left( \phi' - \phi - \pi \right) \right) \left( \sin k \left( \phi' - \phi - \pi \right) \right) \]

Changing to exponential functions we have

\[ b_{n}(\nu') e^{i \nu' \left( \phi - \phi' \right)} = \sum_{k=-\infty}^{\infty} b_{n+k}(\nu') J_{k}(\nu') e^{i k \left( \phi' - \phi - \pi \right)} \text{ for } \nu' > \nu \]

If the left-hand side is changed to the desired
form we have
\[ b_{2n}(e) e^{i\mu} = \sum_{k=\infty}^{\infty} b_{2n-k}(e') e^{i\mu} \text{J}_k(p_1) e^{i\phi} e^{iK(\phi' - \phi - \pi)} \]
for \( e' > e \).

If we set \( k = -k \) we have the first useful form of the expansion
\[ b_{2n}(e) e^{i\mu} = \sum_{k=\infty}^{\infty} b_{2n-k}(e') e^{i\mu} \text{J}_k(p_1) e^{i\phi} \]
for \( e' > e \).

The expansion for \( e' < e \) is found in the same way to be
\[ b_{2n}(e) e^{i\mu} = \sum_{k=\infty}^{\infty} b_{2n-k}(p_1) e^{i\mu} \text{J}_k(p_1) e^{i\phi} \]
for \( e' < e \).

This is still not the desired form for the 2-D problem but if we restrict \( k \) to integer values \( k = n \) and set \( m = n-k \) we have
\[ b_{2n}(e) e^{i\mu} = \sum_{m=-\infty}^{\infty} J_{n-m}(e') e^{i(n-m)\phi} b_m(p_1) e^{i\phi} \]
for \( e' < e \).

Equations 1 and 2b can be written in trigonometric modes as
\[ b_{2n}(e) e^{i\mu} = \sum_{k=\infty}^{\infty} b_{2n-k}(e') e^{i\mu} \text{J}_k(p_1) e^{i\phi} \]
\[ \begin{align*}
\text{(3) } & \quad \hat{B}_{n}(e)(\cos \phi, \sin \phi) = \sum_{k=-\infty}^{\infty} \hat{B}_{n-k}(e) \text{J}_k(p_i) \left( \cos(n-m+\frac{1}{2})\phi, \sin(n-m+\frac{1}{2})\phi \right) / \sin k\phi, \\
& \quad \text{for } e' > e, \\
\text{(4) } & \quad \hat{B}_{n}(e)(\cos n\phi, \sin n\phi) = \sum_{k=-\infty}^{\infty} \text{J}_{n-m}(p_i) \hat{B}_{n-k}(e) \left( \cos(n-m)\phi, \sin(n-m)\phi \right) / \cos k\phi, \\
& \quad \text{for } e' < e, \\
\text{It is possible to use spherical Bessel functions in equations (1) and (3) as } \nu \text{ can be a half integer as shown below in the desired forms:} \\
\text{(5) } & \quad \hat{B}_{n}(e) \hat{B} \left( \cos \nu \phi, \sin \nu \phi \right) = \hat{B}_{n-k}(e) \text{J}_k(p_i) \left( \cos(n-k+\frac{1}{2})\phi, \sin(n-k+\frac{1}{2})\phi \right) / \sin k\phi, \\
& \quad \text{for } e' > e, \\
\text{(6) and } & \quad \hat{B}_{n}(e)(\cos \nu \phi, \sin \nu \phi) = \left( \begin{array}{cc}
\cos \nu \phi \\
\sin \nu \phi
\end{array} \right) \left( \begin{array}{c}
\cos \phi \\
\sin \phi
\end{array} \right) \sum_{k=-\infty}^{\infty} \hat{B}_{n-k}(e) \text{J}_k(p_i) \left( \cos(n-k+\frac{1}{2})\phi, \sin(n-k+\frac{1}{2})\phi \right) / \sin k\phi, \\
& \quad \text{for } e' > e, \\
\text{where } & \quad \hat{B}_{n}(e) = \sqrt{\frac{1}{n+\frac{1}{2}}} \hat{B}_{n+\frac{1}{2}}(e).
\end{align*} \]
Appendix 3.

The spherical Bessel function derivation with argument

Starting with the equation from Abramowitz
and Stegun

\[
\left( \frac{1}{Z^2} \frac{d}{dZ} \right)^{k} \left( Z^{n+k} \hat{J}_n(z) \right) = (-1)^k Z^{-n-k} \hat{J}_{n-k}(z)
\]

dlet \( k = 1 \), \( \nu = n + \frac{1}{2} \) and \( \hat{J}_n(z) = \sqrt{\frac{n+rac{1}{2}}{\pi}} J_{n+\frac{1}{2}}(z) \)

to give

\[
\left( \frac{1}{Z^2} \frac{d}{dZ} \right) \left( Z^{n-1} \hat{J}_n(z) \right) = -Z^{-n} \hat{J}_{n+1}(z)
\]

Differentiating with \( \hat{J}_n(z) = \frac{d}{dz} \hat{J}_n(z) \)

we have

\[
\frac{1}{Z} \left( Z^{n-1} \hat{J}_n(z) - \hat{J}_n(z)(n+1) Z^{-n-2} \right) = -Z^{-n-2} \hat{J}_{n+1}(z)
\]

which becomes

\[
\hat{J}_n(z) = (n+1) \frac{1}{Z} \hat{J}_n(z) - \hat{J}_{n+1}(z)
\]
Appendix 4.

Some contiguous relations for associated Legendre polynomials.

All the equations are taken from Magnus et al. (ref.) but the associated Legendre polynomials are multiplied by a factor of \(-1/2u\) to reach the equations consistent with the subroutine which generates the polynomials.

To evaluate \( \frac{dP_{2}^{\mu}(\cos \theta)}{d\theta} \)

We have from Magnus

1. \( \frac{(1-x^2)}{dx} \frac{dP_{2}^{\mu}(x)}{dx} = (\mu+1) \times P_{2}^{\mu}(x) - (\mu+1) \times P_{2+1}^{\mu}(x) \)

and

2. \( (\mu+1) \times P_{2+1}^{\mu}(x) - (\mu+1) \times P_{2}^{\mu}(x) = -(1-x^2)^{\frac{1}{2}} \times P_{2}^{\mu+1}(x) \)

Equation 2 gives

\( \mu \times P_{2}^{\mu}(x) + (\mu+1) \times P_{2}^{\mu}(x) - (\mu+1) \times P_{2+1}^{\mu}(x) = -(1-x^2)^{\frac{1}{2}} \times P_{2}^{\mu+1}(x) \)

Substituting this into equation 1 we have

\( \frac{(1-x^2)}{dx} \frac{dP_{2}^{\mu}(x)}{dx} = (1-x^2)^{\frac{1}{2}} \times P_{2}^{\mu+1}(x) - \mu \times P_{2}^{\mu}(x) \)

Also from Magnus we have

\( P_{2}^{\mu+2}(x) - 2(\mu+1)x(1-x^2)^{\frac{1}{2}} \times P_{2}^{\mu+1}(x) + (\mu+1)(\mu+1) \times P_{2}^{\mu}(x) = 0 \)
If we let \( \mu + 1 = \omega \), we have

\[ z \omega x (1-x^2)^{-\frac{1}{2}} P_{\frac{\mu}{2}} (x) = P_{\frac{\mu}{2}+1} (x) + (\omega - \frac{1}{2})(\omega - 1) x P_{\frac{\mu}{2}} (x) \]

Dividing equation 3 by \((1-x^2)^{\frac{1}{2}}\) and substituting in equation 4, we have

\[ (1-x^2)^{\frac{1}{2}} \frac{d P_{\frac{\mu}{2}} (x)}{dx} = \frac{1}{2} \{ P_{\frac{\mu}{2}+1} (x) - (\omega - \frac{1}{2})(\omega - 1)x P_{\frac{\mu}{2}} (x) \} \]

let \( x = \cos \theta \)

\[ dx = -\sin \theta \, d\theta \]

and \( d P_{\frac{\mu}{2}} (\cos \theta) = \frac{1}{2} \{ (\omega - \frac{1}{2})(\omega - 1)x P_{\frac{\mu}{2}+1} (\cos \theta) - P_{\frac{\mu}{2}} (\cos \theta) \} \]

To evaluate \( \frac{\mu P_{\frac{\mu}{2}} (\cos \theta)}{\sin \theta} \) at \( \sin \theta = 0 \)

If we take equation 4 and substitute \( x = \cos \theta \) we have

\[ \frac{\mu P_{\frac{\mu}{2}} (\cos \theta)}{\sin \theta} = \frac{1}{2} \{ (\omega - \frac{1}{2})(\omega - 1)x P_{\frac{\mu}{2}+1} (\cos \theta) + (\omega - \frac{1}{2})(\omega - 1)x P_{\frac{\mu}{2}} (\cos \theta) \} \]
Appendix 5

The geometry of the scatterer.

The object can be either an infinite cylinder or in the 2-D problem or a body of revolution with the x-axis defined as the axis of revolution as in the 3-D problem. In either case, the object is fully defined by the centres of each circle producing an arc between each vertex and the rest shown in Fig A.1. The boundary points are all defined to be equidistant on the surface and an even number of points are always chosen for computational reasons. The object is enclosed by a spherical body in the 2-D case or a sphere in the 3-D case. Each boundary point having values for \( T_1 \), \( T_2 \), and \( T_3 \) calculated and stored for use in calculating the scattering matrix.

If the problem has a source which is inside the sphere defined by \( T_{\min} \), then the field is expanded using the geometry in Fig A.2. The object is as defined previously, with a source at a point \( A \) given by the vector \( T_1 \), \( T_2 \), and \( T_3 \) in the projections of \( A \) and \( \hat{r} \) respectively on the \( xy \) plane with \( \hat{z} \) as the 3rd coordinate of \( \hat{r} \). The point \( 0' \) is chosen to be on the surface of the enclosing sphere in the \( xy \) plane such that the angle of \( 0' \) to the vector joining \( 0' \) and \( A \) is chosen to be perpendicular to the surface of the object. This angle is found using...
the following equations
\[ OC^2 = x_{o_i}^2 + y_{o_i}^2 \]
\[ CX^2 = (x_{i-1} - x_{o_i})^2 + (y_{i-1} - y_{o_i})^2 \]
\[ \alpha = \pm \sin \theta \]
\[ \phi_0 = \tan^{-1} \frac{y_{o_i}}{x_{o_i}} \]
\[ \hat{\alpha} = \phi' - \phi_0 \]
\[ CA'^2 = OC^2 + a^2 - 2OCa \cos \hat{\alpha} \]
\[ \hat{\alpha} = \sin^{-1} \left( \frac{a}{CA'} \sin \hat{\alpha} \right) \]
\[ \phi'' = \phi_0 - \hat{\alpha} \]

where \( \hat{\alpha}, \hat{\alpha}' \) are angles and \( OC, a, CA' \) are sides in the triangle \( OCA' \).

Having found \( \phi'' \), \( O' \) is known and the other coordinates can be found using the following equations
\[ t'' = t_{\min} \]
\[ \phi_{\text{tan}} = 2\pi \phi'' + \phi' \]
\[ \phi'' = \phi'' - \sin^{-1} \left( \frac{r''}{r_{\text{tan}} \sin \phi_{\text{tan}}} \right) \]
\[ \beta = \frac{a \sin (\phi_{\text{tan}} + \phi'' - \phi''')}{\sin (\phi'' - \phi''')} \]
$h'' = r'\cos\theta$

$r'' = \sqrt{r^2 + l^2}$

$\Theta'' = \cos^{-1}\left(\frac{r'}{r''\cos\Theta'}\right)$

Although these equations are for the 3-D situation, by setting $\Theta' = \pi/2$, the 2-D equations are found.