Analytical Techniques for Modelling the Laminated Waveguide

Ioannis D. Stamatopoulos

Submitted for the Degree of Doctor of Philosophy from the University of Surrey

MSRG
School of Electronics and Physical Sciences
University of Surrey
Guildford, Surrey GU2 7XH, U.K.

May 2003

© Ioannis D. Stamatopoulos 2003
...SUCCESS IS A MIND GAME...

James Allen
Summary

This thesis is focused on a study on the dispersion formulation of the recently appeared "laminated waveguide". In this framework a new method for the analysis of a wide variety of microwave components like posts, circular junctions or corners in rectangular waveguides is developed. This method can be used to model the posts in the laminated waveguide geometry and this was the main motivation of the present work. The method gives the impedance or admittance matrix of the microwave component by applying discrimination between the localised and accessible modes on the indirect mode matching (IMM) formulation. In all cases the method proved to be fast, very accurate and easy to implement.

The application of the IMM technique to the dispersion formulation of the laminated waveguide is undertaken by implementing the Transverse Resonance Method. It is the first time that effort is paid, so that every possible polarisation of both bounded and unbounded modes is considered. In the process a new analytic solution is given for the modelling of the transverse bifurcation. Its significance relies on the fact that the method of moments is not to be used in the analysis and thus its overall speed is increased. The total procedure is expected to be straightforward and flexible.

Key words: Indirect mode matching, transverse resonance method, singular integral equations, laminated waveguide.
Acknowledgements

Initially, I would like to thank my supervisor and Head of MSRG Prof. Ian D. Robertson that I really appreciate in personal and professional level. I would like also to thank the ATI Director Prof. Mike Kearney for his kind support through my research and EPSRC for the funding.

I would also like to thank all my colleagues in Microwaves and Systems Research Group at the University of Surrey. The work environment is very positive for research to flourish. I also thank my colleagues for supporting me when in need for answers.

I would also like to thank Dr. G. Gentili, Politecnico di Milano, Italy and Prof. Yuri A. Antipov, Department of Mathematics, Louisiana State University for their useful comments on the subject of chapter 3 and 4.

A major experience of mine during my years in England is that I realised the meaning of the true friendship. I thank all my friends for that and more specifically Stelios Choulis, Parisis Flegkas, Stelios Gouveris, Giorgos Lazaridis, Dimitris Mavrakis, Christos Politis, and Jonathan Rodriguez.

Many thanks to my cousin Marianna Petra for being to me a constant invaluable support throughout my life abroad. Without her, I really doubt if I would ever manage to complete the thesis.

Finally, I would like to thank my family whose support and understanding was more than I could ever ask for. This thesis is dedicated to them.
Contents

Contents ............................................................................................................................ ii
List of Figures ................................................................................................................ iv

1 Introduction ................................................................................................................... 1
1.1 Introductory Remarks ............................................................................................. 1
1.2 Objectives ............................................................................................................... 6
1.3 Achievements ......................................................................................................... 7
1.4 Publications .......................................................................................................... 8
1.5 Structure of the Thesis ......................................................................................... 8

2 Background Theory ................................................................................................... 10
2.1 Maxwell's Equations ............................................................................................. 11
2.2 Solution Techniques for Mathematical Problems in Electromagnetics ................. 14
   2.2.1 Variational methods ...................................................................................... 16
   2.2.2 Integral equation method .......................................................................... 23
2.3 Summary ............................................................................................................... 29

3 An Improved Indirect Mode Matching Technique ................................................. 30
3.1 Introduction .......................................................................................................... 31
3.2 Basic Formulation ................................................................................................. 34
3.3 Numerical Implementation .................................................................................... 41
   3.3.1 H-plane abrupt transition between a rectangular and a radial waveguide ....... 41
   3.3.2 H-plane step transition between a rectangular and a radial waveguide .......... 44
   3.3.3 Scalar formulation for a circular junction between rectangular waveguides .... 45
   3.3.4 Vectorial formulation for a circular junction between rectangular waveguides ... 46
| Contents |
|------------------|------------------|
| 3.4 Results and Discussion | 47 |
| 3.5 Conclusion | 53 |
| 4.1 Introduction | 55 |
| 4.2 Basic Formulation | 56 |
| 4.2.1 The Internal Bifurcation | 57 |
| 4.2.2 The Post Region and The External Bifurcation | 63 |
| 4.3 Conclusion | 65 |
| 5 General Conclusions and Future Work | 66 |
List of Figures

1.1 Most common millimetre waveguides (from [3]) ........................................ 3
1.2 The Laminated Waveguide in LTCC (from [15]) ................................. 6

3.1 Geometries considered in the present chapter. (a) A guide to guide taper. (b) A circular junction. An abrupt (c) and a step (d) transition from a rectangular to a radial waveguide. (e) A post in a rectangular waveguide. (f) A circular junction between two rectangular waveguides 34
3.2 A guide to guide taper. Region III is a waveguide with known modal expansion ......................................................... 35
3.3 A circular junction. The thick line indicates the path where the Green's identity applies .................................................. 39
3.4 Equivalent impedance representation of the junction in Fig. 3.3. ......... 40
3.5 An H-plane transition from a rectangular to radial waveguide .......... 43
3.6 An H-plane transition from a rectangular to a radial waveguide ...... 47
3.7 An H-plane step transition from a rectangular to a radial waveguide. 49
3.8 Real part of $S_{11}$ of the $TE_{10}$ mode of the rectangular waveguide, against the number of expansion functions when IMM (Subsection 3.3.2) is applied for the structure of Fig. 3.7 ........................................... 49
3.9 Imaginary part of $S_{11}$ of the $TE_{10}$ mode of the rectangular waveguide, against the number of expansion functions when IMM (Subsection 3.3.2) is applied for the structure of Fig. 3.7. There is not significant difference between HFSS and this method ........................................ 50
3.10 Magnitude of reflection coefficient of the $TE_{10}$ mode scattered by an H-plane centred post in a rectangular waveguide. ............... 51
3.11 Phase of reflection coefficient of the $TE_{10}$ mode scattered by an H-plane centred post in a rectangular waveguide. ................. 51
3.12 Magnitude of reflection coefficient of the $TE_{10}$ mode scattered by a circular junction with a centred post .................. 52
3.13 Phase of reflection coefficient of the $TE_{10}$ mode scattered by a circular junction with a centred post ................ 53
List of Figures

4.1 A cell of the Laminated Waveguide. .................... 56
4.2 Equivalent multimode network representation of a cell of the Laminated Waveguide. .................... 58
4.3 Cross section of the internal bifurcation discontinuity with \( p - 1 \) number of horizontal strips. .................... 60
Chapter 1

Introduction

1.1 Introductory Remarks

This thesis is concentrated on the development of a "dispersion formulation" for modelling the recently developed "laminated waveguide". In the work described here, it has turned out that the development of an efficient formulation has two main stages. The first stage has been the development of an improved method for analyzing a variety of individual waveguide components/features, such as posts, circular junctions of corners (this is applied here to the modelling of the posts in the laminated waveguide). The second stage has been the development of an improved "dispersion" formulation for describing the electromagnetic interactions between different essential features of a laminated waveguide.

This thesis has completed the first stage of this development, in the sense that an improved method of "Indirect Mode Matching" (IMM) has been formulated mathematically and implemented as computer code, and the calculated results have been compared successfully with an existing method that is much slower and more computationally intensive. In the second stage, a general method of combining individual "feature" analyses has been completely formulated mathematically, but in the time available it has not been possible to implement it as a computer code or compare with existing methods.

The details of this work are described in the following chapters. Here the intro-
Introduction serves the need to familiarize the reader with some fundamental concepts and to answer the question why the "laminated waveguide" is a prudent choice for millimetre wave applications.

Millimetre waves are electromagnetic waves with a wavelength of about 1 to 10 mm and with corresponding frequency range from 30 to 300 GHz. The frequencies above the 300 GHz are often called "far infra-red" and by referring to the wavelength the corresponding waves are called submillimetre waves (below 1 mm). There are many military as well as civilian applications of millimetre-wave technology and they all share a number of common desirable features that are attributed mainly to the small wavelength. Since the dimensions of the components used are wavelength dependent, the operation in the millimetre-wave frequency range results in compact systems, small and high-gain antennae and narrow beamwidth. Millimetre waves can provide also wideband capability for communication systems, high resolution of radar, large Doppler shift and low interference. On the other hand there are some inherent drawbacks on the use of millimetre waves because they experienced high attenuation peaks in clear or foggy air and in rainy weather. This is a result of physical laws and cannot be changed. Another decisive disadvantage is the high cost of the millimetre wave systems which prevents their mass production. However, since commercial interest in the millimetre-wave frequency range is increasing for a variety of applications including automotive radar, point-to-point communications, passive imaging and remote sensing [1],[2] there is ongoing research worldwide towards the design and fabrication of low cost subsystems.

Millimetre waveguides [3]- [7]

Concentrating attention specifically on waveguiding media it is observed that there are two distinctly different situations where waveguides structures are used. The first one is associated with point-to-point transport of energy, and the second with the use as a circuit element for interconnecting components or serving as an integral part of these components. Since the first application involves the energy over large or moderate distances as for example for the connection between communication centres or antennas and transmitter, the major functional criteria are low energy...
losses, low signal distortion and adequate power handling capability. In the second
application the waveguide sections are short and thus the demands for high yield,
low leakage (high field confinement) and small cross sectional dimensions become
predominant. Other criteria on millimetre waveguides are the Q factor, package
density, design flexibility and the ability to integrate with solid state devices. What
it should be pointed out is that in the range of the millimetre waves no waveguide
structure has found recognition as being superior to all others and a compromise
has to be made according to the specific needs.

The available millimetre waveguides are the rectangular waveguide, the oversized
rectangular waveguide, the circular waveguide, the dielectric waveguide and its modi-
fications like the image guide, inset dielectric guide, trapped image line and inverted
strip guide, the H type guides like the groove guide, dielectric groove guide, non-
radiative dielectric guide (NRD), fence guide and artificial dielectric guide (slow
wave structure) and the planar and quasi planar guides like the microstrip line,
inverted microstrip line, slotline, coplanar waveguide (CPW), coplanar strip line

Figure 1.1: Most common millimetre waveguides (from [3])

Introductory Remarks
(CPS), triplate line and the finline. Some of them are shown in Fig. 1.1.

It is far beyond the scope of this introduction and definitely not necessary at this point to go into a detailed description of every millimetre waveguide. References [4]-[6] give detailed description and analyses of the various types of guide. There are some general information however that need to be stated:

- The more complicated the structure the higher the cost. This is simply because it is difficult and expensive to machine conventionally a millimetre waveguide owing to its small size and consequently tight manufacturing tolerances. This is the case of the rectangular and circular waveguides as well of some of the dielectric type and H type guides like the inset dielectric guide, trapped image line, groove guide, dielectric groove guide and artificial dielectric guide. Among the other guides it is kept aside that the fence guide [8], being an idea of the early 80's, is an H type guide where the solid metallic side walls have been replaced by two rows of parallel conducting posts and it can be constructed easily even with simple machinery. It is thus amenable for inexpensive mass production.

- Apart of the rectangular and circular waveguides the rest belong to the class of the open waveguides. They suffer, then, radiation and field leakage at the abrupt discontinuities, sharp bends and excitation points. This produces crosstalk phenomena between nearby components and thus degrade the performance of the device. The mechanism and the amount of leakage is not the same for all the open waveguides. For example, the NRD guide [9] and dielectric groove guide with deep grooves [10] show very promising performance. In any case, care is taken to minimize the leakage but inevitably the design flexibility is lowered and the cost is increased. Needless to say that the rectangular and circular waveguides, being totally shielded, have excellent line-to-line isolation. On the other hand, the shielding produces extra conduction losses which however, as it has been mentioned above, is an important factor only for long distance transport of energy.

- As an alternative to the expensive workshop machining, techniques can be used for the fabrication of 3D waveguides. These include micromachining, spin-coating multilayer processing [11] and low-temperature cofired ceramic (LTCC) technol-
ogy [12]. Micromachining is a selective etching technique while the multilayer processing is a deposition of dielectric layers one upon the other. Although micromachining can produce very high performance structures [13], [14], it has the limitations in design flexibility, the poor yield and, ultimately, the high production cost. Multilayer spincoating technology, on the other hand, is cheaper but its application still needs to be developed for thick structures, such as waveguides. LTCC technology, consisting of multiple layers of patterned dielectrics and conductors, is a multilayer packaging approach that has proved practical and economical for a range of microwave components [15], [16].

- The microstripe, CPW, slotline and CPS lines suffer from poor Q-factor, low package density, poor current crowding control and poor line-to-line isolation (open structures) [11]. As a result, they are unattractive for applications above the GHz and H type guides may be preferred. Considering the line-to-line isolation it should be pointed out that the planar lines have additionally the inherent drawback of coupling energy into the substrate in a mechanism similar to shock wave radiation [17]. This leakage is independent of the appearance of the discontinuities or excitation points and can be suppressed by reducing the thickness of the substrate. Within this framework, the University of Michigan developed membrane-supported coplanar waveguides (MS-CPW) realised with the use of micromachined cavities [14]. The result was a high performance transmission line under the aforementioned limitations of micromachining though.

Conventional GaAs monolithic technology has been used in 1996 [18] to fabricate a 4-μm-high metal pipe single layer rectangular waveguide. Due to the small height of the dielectric the structure exhibited high loss. In 1998 a consortium of leading UK universities (Surrey, Kent and Glasgow) was formed to pioneer a new low cost terahertz multi-chip module (T-MCM) technology. The ultimate task was the fabrication of metal pipe dielectric filled rectangular waveguides consisting of the maximum of six 50-100 μm thick dielectric layers in a sandwich arrangement. It was expected then, that the excellent isolation of the rectangular waveguide will be combined, in an inexpensive way, with the advantages of the multilayer technology (high design flexibility, ease of integration with solid state devices, high packaging
Objectives

6

Figure 1.2: The Laminated Waveguide in LTCC (from [15])

density). It is regrettable however that this task has not yet been completed.

The promising characteristics (design flexibility and high field confinement) of the
fence waveguide have been already demonstrated [19]. The idea has been reinvented
in the so called post-wall waveguide [20] where better field confinement is provided
by metallic covering on the wide surfaces of the waveguide. This type of artificial
dielectric made by rows of posts are known in optics as photonic band-gap structures
(see for example [21] and references therein). A further step is to use multilayer
LTCC technology and the result is a rectangular like multilayer waveguide where
the side walls are formed by a mesh of vertical via-holes and horizontal conducting
strip, the so called laminated waveguide [15] shown in Fig. 1.2. The laminated
waveguide has been demonstrated to have high line-to-line isolation and high design
flexibility since it can be easily integrated with other types of millimetre waveguides
as well as solid state circuits [16].

1.2 Objectives

For the present project, the dominant criteria of the optimum millimetre waveguide are the low cost (high yield), low leakage, high package density and ability
Achievements

The formulation toward the dispersion analysis of the laminated waveguide is constructed for the first time. Under this framework, a new efficient method is developed to integrate solid state devices and to form complex multi-chip modules (MCM). Additionally, the operation frequency is assumed to be above 75 GHz. The laminated waveguide can fulfill these requirements and thus an in-depth analysis of its behaviour is an important task.

There are efficient commercial software packages for electromagnetic simulation. However, it should be emphasized that experience with simulators does not replace the insight or the precise details that are obtained by a mathematical analysis of the electromagnetic structure. Simulators should be used for supplementing the knowledge attained by mathematical analysis and/or getting the feel of the performance of complicated physical structures for which mathematical analysis is not easily tractable. Consequently, the questions set on the behaviour of the laminated waveguide can be fully answered only with rigorous electromagnetic analysis.

There are few works concerning the theoretical analysis of the post-wall waveguide and the waveguide-based photonic band-gap structures [22]-[26] but none on the laminated waveguides. The only study on the laminated waveguide is that in [15], where the analysis is based on electromagnetic simulations (HFSS\textsuperscript{TM}) and experimental results. In that case, the HFSS\textsuperscript{TM} simulation is solely devoted to obtaining the insertion loss of the laminated waveguide where the incident field is the dominant mode of a rectangular waveguide. All the aforementioned questions still remain unanswered.

The objective of this thesis is to provide the "dispersion formulation" for modelling the laminated waveguide. This is the first step to close the gap and give all the required answers pertaining to the electromagnetic characteristics of the laminated waveguide. What remains as a future work, is the direct implementation of the proposed formulation so that the dispersion behavior of the laminated waveguide could be predicted without further ado.

1.3 Achievements

The formulation toward the dispersion analysis of the laminated waveguide is constructed for the first time. Under this framework, a new efficient method is developed
to analyse the post. The new method is based on the admittance matrix formulation and the concept of the Indirect Mode Matching (IMM). The method is simpler than previous attempts of IMM and can be applied for the analysis of a wide variety of microwave components like the abrupt transition of a rectangular to a radial waveguide, centred posts in a rectangular waveguide and circular junctions.

The application of the IMM technique to the dispersion formulation of the laminated waveguide is proposed by implementing the Transverse Resonance Method. In the process a new analytic solution is given for the modelling of the transverse bifurcation. Its significance relies on the fact that the method of moments is not to be used in the analysis and thus its overall speed is increased.

1.4 Publications


1.5 Structure of the Thesis

The thesis is divided into five chapters. The first is the introduction and summarises the motivation of the research, the objectives and the achievements of the thesis.

Chapter 2 summarizes a number of concepts from electromagnetic field theory encountered throughout in the thesis. The emphasis is on the computational methods for electromagnetic field analysis in the frequency domain for microwave structures. The purpose is to present an up to date summary of the subject related only to the laminated waveguide.

Chapter 3 presents a new method for the analysis of a wide variety of microwave
components like posts, circular junctions or corners in rectangular waveguides. This method also can be used to model the posts in the laminated waveguide and exactly this was the motivation of the present work. The method gives the impedance or admittance matrix of the microwave component by applying discrimination between the localised and accessible modes [27] on the indirect mode matching (IMM) formulation. In this way, the method is simpler and more versatile than previous modelling techniques where IMM is involved. In all cases, the method proved to be fast, very accurate and easy to implement.

In chapter 4, the mathematical formulation required for the dispersion analysis of the laminated waveguide is given. The method adopted is the rigorous Transverse Resonance Method. In order for the technique to be applied successfully, the rigorous indirect mode matching technique is used to characterise the post, while the bifurcations are modelled by the known method of the rigorous multimode network representation. An improvement is provided for the latter method, by giving an analytic (instead of the numerical) solution. As a result, it is the first time that the laminated waveguide is characterised in a rigorous manner which allows for the calculation of both bounded (guided waves) and unbounded (radiation field) modes in any possible polarisation.

Finally, chapter 5 presents the conclusions and suggestions for future work on the analysis and design of laminated waveguide modules.
Chapter 2

Background Theory

This chapter summarizes a number of concepts from electromagnetic field theory that will be encountered throughout the thesis. The emphasis is on the computational methods for electromagnetic field analysis of microwave\(^1\) structures in the frequency domain.

It has seemed useful for the review to cover computational methods used for the modelling of microwave passive components like junctions and step discontinuities and, in general, the techniques followed in various scattering problems. It is not the intention here, to furnish an exhaustive survey of all the electromagnetic methods that appeared in the literature nor to simply copy information from one of the several excellent textbooks [28]- [31] devoted to the field. Besides, there is no space for this. The aim is to present an updated and complete coverage of the subject related only in so far as it influences the modelling of the laminated waveguide. This coverage consists of a selection of key techniques, concepts and ideas that are categorised in a unified way and shown clearly, logically and without gaps. Whenever possible, care will be taken to simplify matters without compromise on rigour. The goal is to help the reader to realise, without effort, the advantages, disadvantages and domain of applicability of every technique described so he can proceed to the following chapters with any misconceptions resolved.

\(^{1}\)The methods mentioned apply equally to both microwave and millimetre wave bands.
2.1 Maxwell’s Equations

The assumption of time harmonic fields is adopted throughout the thesis, so that fields are assumed to vary sinusoidally with time at a fixed temporal frequency \[\omega\] \cite{32,33}. Fields are represented by complex quantities, or phasors, and their dependence on time is via the factor \(e^{j\omega t}\) (\(\omega\) is the angular frequency) which will now be omitted henceforth. With phasor representation of field and sources, Maxwell’s equations take the forms:

\[
\nabla \times \vec{E} = -j\omega \mu \vec{H} - \vec{M}
\]

(2.1)

\[
\nabla \times \vec{H} = j\omega \varepsilon \vec{E} + \vec{J} + \vec{J}^{\text{ext}}
\]

(2.2)

\[
\nabla (\varepsilon \vec{E}) = \rho
\]

(2.3)

\[
\nabla (\mu \vec{H}) = 0
\]

(2.4)

where \(\vec{E}, \vec{H}\) are the electric and magnetic field vectors, \(\vec{M}\) is the magnetic current density vector, \(\vec{J}\) is the macroscopic-conduction current density vector, \(\vec{J}^{\text{ext}}\) is the electric current density vector induced by external sources and \(\rho\) is the electric charge density. \(\varepsilon\) and \(\mu\) are called the permittivity and permeability of the medium. They are scalars for isotropic media and tensors for anisotropic media and are often given as \(\varepsilon_r \varepsilon_0\) and \(\mu_r \mu_0\) where \(\varepsilon_0, \mu_0\) are the permittivity and permeability of the free space and \(\varepsilon_r, \mu_r\) are space dependent dimensionless constants. For nonmagnetic material as discussed in this thesis, the permeability \(\mu\) is equal to the value \(\mu_0\). Additionally, for linear, isotropic, (dielectric-) inhomogeneous media without sources, by eliminating either the electric or magnetic fields from equations (2.1) and (2.2), applying the identity \(\nabla \times (\nabla \times \phi) = \nabla (\nabla \phi) - \nabla^2 \phi\) and using (2.3) and (2.4), it can be shown that the electric and magnetic field intensity vectors satisfy the inhomogeneous wave equation, namely:

\[
\nabla^2 \vec{E} + k^2 \vec{E} = \nabla \left( \frac{\vec{E} \cdot \nabla \varepsilon_r}{\varepsilon_r} \right)
\]

(2.5)

\[
\nabla^2 \vec{H} + k^2 \vec{H} = \frac{\nabla \varepsilon_r}{\varepsilon_r} \times (\nabla \times \vec{H})
\]

(2.6)

where \(k^2 = \omega^2 \mu \varepsilon\). From (2.5) and (2.6), it is apparent that the different field components are coupled through the inhomogeneity of the material. For linear, isotropic, homogeneous media without sources, equations (2.5) and (2.6) reduce to
the known form of the Helmholtz equation:
\[
\nabla^2 \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} + \frac{k^2}{c^2} \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} = 0
\]  \hfill (2.7)

In the electromagnetic field theory equations can be set up in several different ways. Fields can be represented and determined by solving Maxwell’s equations directly. Equations can also be constructed by changing from the vector identities \( \vec{E} \) and \( \vec{H} \) to the magnetic and electric vector potentials \( \vec{A} \) and \( \vec{D} \) [34] (sec. 3.12) or to the magnetic and electric Hertz potentials \( \vec{\Pi} \) and \( \vec{\Psi} \) [35] (sec 1.11); in a mixed form, extra magnetic and electric scalar potentials \( \Phi_m \) and \( \Phi_e \) are entered in the equations [29] (sec. 1.4); in special cases Debye magnetic and electric scalar potentials \( \pi_m \) and \( \pi_e \) are used [35] (sec 7.11). All these versions of electromagnetic theory are equivalent to each other and can be transformed one to another.

It must be noticed that the vector and scalar potentials satisfy the Helmholtz equation or its variant for inhomogeneous waveguides - see (2.16), (2.17) below. Considering a uniform tube structure free of sources (\( \vec{M} = 0 \), \( \vec{J} = 0 \) and \( \vec{J}^{ext} = 0 \)), it can be shown that the electric and magnetic field vectors can be expressed in terms of electric and magnetic field components on a single coordinate axis only. The natural choice is the longitudinal axis of the tube structure.

Let the axis of a uniform tube structure be the z direction, and let a given wave, guided in the structure, be decomposed into longitudinal (subscript z) and transverse to z (subscript t) components. Since the dependence of the field in the positive z direction is given by the factor \( e^{-j\beta z} \) where \( \beta \) is the propagation constant of the guided wave, the Maxwell’s equations become:

\[
\nabla_t \times \vec{E}_t = -j\omega \mu \vec{H}_z \\
\nabla_t \times \vec{H}_t = j\omega \varepsilon \vec{E}_z \\
\nabla_t \times \vec{E}_z + j\beta \left( \vec{z} \times \vec{E}_t \right) = -j\omega \mu \vec{H}_t \\
\nabla_t \times \vec{H}_z + j\beta \left( \vec{z} \times \vec{H}_t \right) = j\omega \varepsilon \vec{E}_t
\]  \hfill (2.8) \hfill (2.9) \hfill (2.10) \hfill (2.11)

where \( \vec{z} \) is the unit vector in z direction. Equations (2.10) and (2.11) can be combined with each other to give:

\[
\vec{E}_t = -\frac{j\omega \mu}{k^2 - \beta^2} \left( \nabla_t \vec{H}_z \times \vec{z} - \frac{\beta}{\omega \mu} \nabla_t \vec{E}_z \right)
\]  \hfill (2.12)
Maxwell's Equations

\[
\vec{H}_t = -\frac{j\omega\varepsilon}{k^2 - \beta^2} \left( \hat{z} \times \nabla_t E_z - \frac{\beta}{\omega \varepsilon} \nabla_t H_z \right) \quad (2.13)
\]

Equations (2.12) and (2.13) show that the transverse electric and transverse magnetic field components can be expressed in terms of the longitudinal components of the electromagnetic field.

The z components also satisfy the Helmholtz equation (2.7) in a scalar form. Consequently, a given vectorial electromagnetic problem involving guided waves can be simplified to a scalar one. For example, this is the case when the dispersion characteristics of a uniform waveguide need to be determined, i.e. for example when finding the propagation constant \( \beta \). This is an example of an eigenvalue problem in electromagnetic theory.

Other forms of Maxwell's equations that have been proved useful for eigenvalue problems are given below. By putting (2.8) into (2.11) and (2.9) into (2.10) then, in the absence of sources, the pair of Telegrapher's equation is obtained \[36\] (pp 187):

\[
\frac{\partial \vec{E}_t}{\partial z} = -j\omega \mu \left( \vec{I} + \nabla_t \frac{1}{k_0^2} \nabla_t \right) \left( \vec{H}_t \times \hat{z} \right) \quad (2.14)
\]

\[
\frac{\partial \vec{H}_t}{\partial z} = -j\omega \varepsilon \left( \vec{I} + \nabla_t \frac{1}{k_0^2} \nabla_t \right) \left( \hat{z} \times \vec{E}_t \right) \quad (2.15)
\]

where \( \vec{I} \) is the dyadic identity.

Alternatively by taking \( \varepsilon_r \) as a translationally invariant in the z direction, it can be shown that in a source free medium the Helmholtz equations for the transverse components are given in the form \[32\] (pp. 595):

\[
(\nabla_t^2 + k_0^2) \vec{E}_t + \nabla_t (\vec{E}_t \nabla \ln \varepsilon_r^2) = \beta^2 \vec{E}_t \quad (2.16)
\]

\[
(\nabla_t^2 + k_0^2) \vec{H}_t - (\nabla_t \times \vec{H}_t) \times \nabla_t \ln \varepsilon_r = \beta^2 \vec{H}_t \quad (2.17)
\]

Equations (2.14) and (2.15) have been used in \[37\] and \[38\] for the calculation of the bounded modes of a dielectric waveguide. Equations (2.16) and (2.17) have similar use in \[39\]. The significance of this is that they lead directly to a nonlinear equation for \( \beta^2 \), for the following reason. If \( \vec{E}_t \) and \( \vec{H}_t \) are approximated by a linear combination of basis functions (Galerkin's method), both pairs of the differential equations above transform to a linear matrix eigenvalue problem, i.e. \( \beta \) is determined explicitly. Further, if the frequency dependence is restricted only to the unknown
coefficients of the series representation [38], [39], then the integrals (involving the basis functions) in Galerkin's algorithm are frequency independent and need only to be calculated once for a given frequency range. Moreover both pairs of equations can be used, in an efficient algorithm recently published [40], to calculate the dispersion of inhomogeneously loaded waveguides.

2.2 Solution Techniques for Mathematical Problems in Electromagnetics

Purpose of the section 2.2 is to discuss the problems of electromagnetics studied in the present work and the solution techniques used. From the mathematical point of view, these problems are boundary-value ones for partial differential equations. They can be expressed in the form:

\[ Lu = g \]  

(2.18)

where \( L \) is an integrodifferential operator, \( u \) is the function sought and \( g \) is a given excitation (this is commonly done in diffraction or scattering problems). The boundary conditions may be of Dirichlet, Neumann or mixed type.

In this thesis, mathematically rigorous solution methods are applied i.e. methods with no assumptions based on physical intuition. The only approximations are those resulting from the use of numerical methods, namely the discretization of a continuum, and from some parameters being small i.e. perturbation methods.

There are many solution methods for electromagnetic field problems. A systematic classification of the techniques used in computational electromagnetics is outside the scope of this chapter. However, a few general aspects can be considered. Generally, integral and differential equation techniques are the basis for many of the computational methods. We can distinguish them on the basis of the compactness of the solution obtained. Thus, they can be classified as exact analytical (method of separation of variables, Wiener-Hopf method), analytical-numerical (variational method, boundary-element method, mode-matching method, transverse resonance method, method of fictitious sources, spectral-domain approach) or fully numerical (finite-
difference method FDM, finite-elements method FEM, finite-difference-time-domain method FDTD, transmission-line matrix method TLM, method of lines ML).

First of all, it should be made clear that each method has its own advantages and disadvantages and more than one may be suitable for the solution of a given problem. Often, similarities can be recognized among the various solution techniques, so in many instances two apparently different approaches lead to the same result (for example, mode-matching and integral-equation techniques). An initial selection can be made in the context of the structure or problem to be solved. Choice must then be made in terms of other criteria, such as, accuracy, computing efficiency, memory requirements, flexibility and versatility. Roughly speaking, the computing efficiency increases with the amount of analytical effort required, but, at the same time, the versatility is generally reduced.

In the present work the requirements are high computing efficiency, accuracy, and simplicity of the algorithm. The last of these guarantees in the implementation as a computing code will be an easy job. Since the effort here is focused on a very specific geometry (the laminated waveguide) versatility is not a predominant criteria. However, it will be shown in the following chapters that the methods proposed are versatile enough to tackle a wide variety of electromagnetic field problems. Fully numerical methods like FEM, FDM and TLM will not be considered, for following two reasons. Although they are very flexible, the effort required for their implementation is not justified in the present problem. Additionally, the need to investigate the theoretical aspects of the laminated waveguide led the author to use analytical and analytical-numerical techniques only.

The analytical methods have limited applicability. The method of separation of variables is applied only to the so-called co-ordinate structures where the boundaries of areas considered consist of parts of co-ordinate lines (surfaces) of any classical system of orthogonal co-ordinates (Cartesian, cylindrical, spherical, elliptical, parabolic etc.) [41]. The Wiener-Hopf method is also applied to very specific configurations of the structure [42]. In this thesis the methods used are integral equation techniques, mode matching and the transverse resonance method. Details will be given when they are introduced. The variational method and integral-equation techniques are
described here, as they are generally related to the numerical methods encountered in this thesis.

2.2.1 Variational methods

General principles

Variational techniques find many uses in engineering and mathematics. They are applied to numerous problems in electrodynamics, such as scattering from periodic structures [43], modelling of waveguide discontinuities [28], determination of propagation constants in waveguides or resonant frequencies in cavities [34] and dispersion analysis in open structures [7]. A particularly important characteristic is the close relation to direct numerical methods as discussed below. In the present thesis the variational method is related to the calculation of losses in a lossy waveguides.

A variational method is, in few words, an approximation method that converges fast to the correct result, due to the special (stationary) character of the functional used. Consider the functional \( F(y, x) \) where \( y(x) \) is a function of the independent variable \( x \). The goal is to find the conditions under which the function \( y(x) \) minimises (or maximises) the functional \( F \), and the minimum (or maximum) itself. The function \( y(x) \) is perturbed by a quantity \( \delta y(x) = \epsilon \eta(x) \), where \( \epsilon \) is an arbitrary constant and \( \eta(x) \) is an unknown function that satisfies given boundary conditions. If the perturbed function \( y_p(x) = y(x) + \delta y(x) \) is entered into the functional \( F \) and it is required that the extremum be obtained for the function \( y(x) \) (i.e., \( \delta y(x) \equiv 0 \)), then it is required [44]:

\[
\frac{dF}{d\epsilon} \bigg|_{\epsilon=0} = 0
\]  

Equation (2.19) leads to the Euler-Lagrange equation that gives the conditions the function \( y(x) \) should satisfy in order to minimise (or maximise) the functional \( F \). Various forms of the Euler-Lagrange equation can be obtained, according to the number of unknown functions, the number of independent variables and the boundary conditions imposed; however the basic idea remains the same. If there are subsidiary constraints (as in isoperimetric problems) then the desired function \( y(x) \) minimises (or maximises) the functional \( F_1 = F + \Phi(\lambda) \) where \( \Phi \) is the constraint
functional with a known value and \( \lambda \) is the Lagrange multiplier that needs to be found [44].

Often in the literature, the Euler-Lagrange equation is obtained via the calculus of variations. In this framework, the first variation of \( F \) is set equal to zero:

\[
\delta F = F(y + \delta y) - F(y) = 0
\]

This is a necessary condition for \( F \) to have an extremum. In this case the functional \( F \) is called \emph{stationary}. Equation (2.20) also leads to the Euler-Lagrange equation for the function \( y(x) \). The beauty of a stationary formula is that any approximation of \( y(x) \), entered in the functional \( F \), does not need to satisfy the strict conditions imposed by (2.20). \( F \) still converges fast to its min or max value, which is obtained when \( y(x) \) exactly satisfies the Euler-Lagrange equation and the corresponding boundary conditions. The stationary functional \( F \) exhibits an error of only \( O(\varepsilon^2) \) as \( \varepsilon \to 0 \), when \( y(x) \) has an error of \( O(\varepsilon) \) [34].

In the majority of cases in electromagnetic field problems, the requirements on \( y(x) \) (namely the Euler-Lagrange equation) are already known. What is required is for the functional \( F \) to be found. There are some general rules [45] that help the functional to be determined when the Euler-Lagrange equation is given.

- Consider the linear, positive definite operator \( L \) dense in the complex Hilbert space. If the Euler-Lagrange equation is of the form (2.18) where \( u \) and \( g \) belong to the domain \( D_L \) and range \( R_L \) of \( L \) respectively, then the functional:

\[
F(u) = \langle Lu, u \rangle - \langle u, g \rangle - \langle g, u \rangle
\]

(2.21)

takes its minimum value \( u = u_0 \) if and only if \( u_0 \) is the unique solution of (2.18). In equation (2.21), "\( \langle, \rangle \)" denotes the inner product.

In the case of a real Hilbert space the functional reduces to the form \( F(u) = \langle Lu, u \rangle - 2 \langle u, g \rangle \). Obviously, instead of solving (2.18) directly, \( u \) is determined by minimising the functional \( F(u) \). This is obtained by adopting approximation techniques like the Rayleigh-Ritz method (see below). Modifications of the above functional emerge according to the boundary conditions imposed. If for example non-homogeneous conditions are assumed on \( u \), then
the corresponding functional is found by considering the equivalent problem
\[ L\nu = g\nu \] where \( \nu = u - \psi \) and \( g\nu = g - L\psi \). If \( \psi \) satisfies the non homogeneous conditions then \( \nu \) does not, and the functional is found by proceeding as before.

- General procedures for establishing stationary functionals for various problems in electromagnetic theory can be obtained by using Hamilton's principle [46] and the reaction concept [47]. Hamilton's principle requires that an electrodynamic system is in a state where its energy has, at every time, a minimum value. Mathematically this is expressed as the following:

\[ \delta I = \delta \int_{t_0}^{t_1} L \, dt = 0 \] (2.22)

where

\[ L(t) = \int_{V} \left( \frac{1}{2} \varepsilon \vec{E}^2 - \frac{1}{2} \mu \vec{H}^2 + \vec{J} \cdot \vec{A} - \Phi_{\rho} \right) \, dv \] (2.23)

\( L(t) \) is called the Lagrange function. Equations (2.22) and (2.23) are alternative expressions of Maxwell's equations. It is apparent that \( I \) in (2.22) is the stationary functional while the unknown functions are subject to the Maxwell's equations. Assuming time harmonic quantities, various expressions for \( I \) can be given [45], [48]. For example in a medium free of sources, by making use of Gauss's identities it is found that \( I = -\frac{1}{2} [\vec{E} \times \vec{H}^*, \vec{n}] - \frac{1}{2} < \vec{E}, \nabla \times \vec{H}^* + io\vec{D}^* > \) where the inner products \(< o, o >\) and \([ o, o ]\) are volumic and surface integrals respectively and \( \vec{n} \) indicates the unit vector normal the boundary surface. Further constraints can be imposed by the use of Lagrange multipliers.

Last, note a Legendre transformation can be used to convert the Lagrange function into a Hamiltonian function. The resulting Hamilton's equations are equivalent to Euler-Lagrange equations. This is stated to prevent confusion arising because, sometimes in the literature, variational techniques are implemented via Hamilton's equation rather than Euler-Lagrange equations. For example, this is the case in [49], which deals with the electromagnetic scattering problem from periodically arranged cylinders.

In the reaction concept the functional [47] is given simply by the reaction of
field \( a \) on source \( b \) i.e:

\[
F(a, b) = \langle a, b \rangle = \int (\vec{E}_a \cdot d\vec{J}_b - \vec{H}_a \cdot d\vec{M}_b) dv
\]  

(2.24)

where \( \vec{J} \) and \( \vec{M} \) denote electric and magnetic current sources respectively.

The corresponding Euler-Lagrange equation is the Lorenz reciprocity equation. The reaction concept can lead to various stationary formulae for problems associated with waveguides and cavities [34].

Specific variational methods

The stationary equation, once set up, can be solved by the use of numerical methods. The most widely used are the Rayleigh-Ritz method, the Kantorovitz method, Galerkin’s method (Methods of Moments-MoM) and the method of Least Squares. Information is now given about these numerical techniques and their relation with the variational concept is shown. The variational method is also directly related to other numerical techniques, such as the finite difference method (FDM) and the finite elements method (FEM), which will not considered in the present thesis.

**Rayleigh-Ritz method** [46]. Consider the functional \( F(u) \). Assume that the unknown function \( u(x) \) can be approximated by a series of known linearly independent basis functions \( \varphi_m(x) \) (\( m = 1, \ldots, n \)) with unknown coefficients:

\[
u(x) = \sum_{m=1}^{n} c_m \varphi_m(x)
\]  

(2.25)

Substituting for \( u(x) \) into the functional \( F(u) \) results in a function of \( n \) independent variables i.e. \( F = F(c_1, \ldots, c_n) \). Therefore, following standard variational procedure, \( F \) obtains its extremum when:

\[
\frac{\partial F}{\partial c_m} = 0 \quad (m = 1, \ldots, n)
\]  

(2.26)

It is obvious that the Rayleigh-Ritz method is based on fundamental variational principles. It is also related to Galerkin’s method and the MoM described below. With respect to the Galerkin’s method the relation is established if \( L \) is a self-adjoint\(^2\) positive definite operator dense in the real Hilbert space. These conditions

\(^2\) The adjoint \( L^a \) of \( L \), with respect to a given product, satisfies the relation

\[
\langle Lu, g \rangle = \langle u, L^a g \rangle.
\]

If \( L = L^a \) then \( L \) is called self-adjoint
are relaxed for the case of MoM. The relation between Galerkin's and Rayleigh-Ritz methods is shown by considering the problem of minimising the functional

\[ F(u) = \langle Lu, u \rangle - 2 \langle u, g \rangle, \]

which corresponds to the Euler-Lagrange equation (2.18). Then, implementation of (2.26) results in the following system of equations:

\[ \sum_{k=1}^{n} c_k \langle L \varphi_m, \varphi_k \rangle = \langle g, \varphi_m \rangle \quad (m = 1, \ldots, n) \quad (2.27) \]

The form of equations (2.27) shows the direct relation between the Rayleigh-Ritz method and Galerkin's method (see below). After the \( c_m \) have been found from (2.27), then \( u(x) \) is determined from (2.25). Moreover, if \( L \) is a lower bounded operator, then the solution given by the Rayleigh-Ritz method converges to the solution given by (2.18) \(^3\). The relation of Rayleigh-Ritz with the MoM is shown by considering the problem of finding the extremum of the functional \( F(u) = \langle u, g^a \rangle \) where \( g^a \) is a chosen excitation and \( u \) is required to satisfy (2.18). Then it can be shown that this leads to the MoM solution [45]. Indeed, by entering the constraint (2.18) into \( F(u) \) the problem becomes finding the extremum of the functional \( F_1(u) = \langle u, g^a \rangle + \lambda \langle g - Lu \rangle \) where \( \lambda \) is the Lagrange multiplier. By imposing the condition (2.20), \( F_1(u) \) takes the final form:

\[ F_1(u) = \langle u, g^a \rangle + \lambda \langle g^a, g \rangle - \langle u^a, Lu \rangle \quad (2.28) \]

while the Euler-Lagrange equations are found to be \( Lu = g \) and \( L^a u^a = g^a \) (\( L^a \) is the adjoint operator). Then, the Rayleigh-Ritz procedure is applied, i.e. equations (2.25) and (2.26), lead directly to a MoM solution.

Kantorovitz method [45]. Sometimes this is called Galerkin's incomplete method [33]. It can be thought of as a generalisation of the Rayleigh-Ritz method. The difference from the latter is that the coefficients \( c_m \), instead of being constants, are now unknown functions of one of the independent variables. Application of (2.26) results in a system of differential equations. The approximate solution of the Kantorovitz method is generally more accurate, compared with that obtained by the Rayleigh-Ritz method with the same finite number of unknown coefficients.

\(^3\)The solution given by the Rayleigh-Ritz method converges in energy to the solution of (2.18). The positive definiteness implies that the convergence in energy means convergence, if \( Lu = f \) has a solution of finite energy. The later is guaranteed if \( L \) is a lower bounded operator.
and the same set of basis functions. However, if the number of the unknowns tends to infinity than it is not straightforward to prove that the result converges to the correct one.

**Galerkin’s method** [50]. Galerkin’s method is a very general method and it is not related only to variational problems (actually, it is is not considered a variational technique). A brief description of Galerkin’s method and the generalized Galerkin’s method (Method of Moments MoM) follows. Consider again problem (2.18). \( u \) is approximated by an expression of the form (2.25). The functions \( \varphi_m \) are called basis functions. Then (2.18) becomes:

\[
\sum_{k=1}^{n} c_k L \varphi_k \simeq g \quad \text{or} \quad \sum_{k=1}^{n} c_k L \varphi_k - g = e
\]

(2.29)

where \( e \) is the error due to the approximation of \( u \). This error is minimised, in a mean sense, by projecting it on a set of weighting functions \( w_m \). Thus:

\[
< e, w_m > = 0 \quad (m = 1, \ldots, n)
\]

(2.30)

which results in:

\[
\sum_{k=1}^{n} c_k < L \varphi_k, w_m > = < g, w_m > \quad (m = 1, \ldots, n)
\]

(2.31)

If the set of basis and weighting functions coincides then the result is:

\[
\sum_{k=1}^{n} c_k < L \varphi_k, \varphi_m > = < g, \varphi_m > \quad (m = 1, \ldots, n)
\]

(2.32)

(2.32) corresponds to the Galerkin’s method for the solution of the problem (2.18), while (2.31) corresponds to what is called the method of weighted residuals or, more common in electromagnetics, the method of moments (MoM). It has been mentioned, already that Galerkin’s method is equivalent to the Rayleigh-Ritz method when \( L \) is a self-adjoint, definite positive operator dense in the real Hilbert space. It has been shown, also, that MoM is with no restriction a variational process. Due to this stationary character, it is not necessary for each basis function \( \varphi_m \) to satisfy the boundary conditions exactly.

Let’s cast more light on the choice of the basis and weighting functions \( \varphi_m \) and \( w_m \) respectively. (2.25) implies that the functions \( \varphi_m \) belong to domain \( D_L \) of
the operator $L$. However, $\varphi_m$ do not have to be linearly independent as long as
the set $L\varphi_m$ is linearly independent [51]. Concerning the weighting functions $w_m$, (2.30) implies that they form a set of independent variables in the range $R_L$ of $L$.
From $<Lu,w_m> = <u,L^aw_m>$ it is apparent that they belong also to the domain $D_{L^a}$ of the adjoint operator $L^a$. Since $R_L \subset D_{L^a}$ ($D_{L^a} = N_{L^a} \oplus \{\text{closure of } R_L\}$)\footnote{$N_{L^a}$ denotes the null space of $L^a$, (i.e., if $w_m \in N_{L^a}$ then $L^aw_m = 0$), $\oplus$ denotes the direct sum and the subspaces are orthogonal complements} then the necessary condition for (2.18) to have a meaningful solution is that the weighting functions must be in $D_{L^a}$ [52] (usually it is easier to find $D_{L^a}$ than $R_L$).
In Galerkin's method, the set of basis and weighting functions coincide and so $\varphi_m$ should, simultaneously, belong to $D_L$, $D_{L^a}$ and $R_L$. The later can be satisfied if $L$
is a self-adjoint operator. However, this is an adequate but not necessary condition for Galerkin's method to be applied.

Another issue in the Galerkin's and MoM solution is the convergence of the results. It
has been mentioned already, that if $L$ is a self-adjoint, positive definite operator the
results do converge to a solution of finite energy of (2.18) (if there is any). Note, that
there is no constraint on the orthogonality of $\varphi_m$ (if $\varphi_m$ are not orthogonal then the
convergence is proved by use of the projection operators [53]). Also if $L$ is indefinite,
then convergence can be proved, in some cases, by other means [54]. Generally if $L$
is neither positive nor positive definite the convergence cannot be established in
Galerkin's method. Also, nothing can be said about the convergence when basis and
weighting functions are different i.e., MoM solutions [50]. Therefore, the most of the
problems in electromagnetics lack any sort of mathematical convergence criteria.
The results, however, are still accurate after a prudent choice of basis and weighting
functions [55] and the error rate can be shown to be again of the order of $O(\epsilon^2)$ [56]. Last, it is pointed out that the point-matching method, which is the simplest
implementation of MoM ($w_m(x) \equiv \delta(x_m)$), can lead to instabilities [51], [57].

Method of Least Squares [50]. In this case the problem is the minimisation of the functional:

$$F(u) = <Lu - g, Lu - g>$$ (2.33)

If $L$ is dense in a real Hilbert space and $g$ belongs to $D_{L^a}$, then the problem of
minimisation of \( F'(u) \) can give [58]:

\[
\sum_{k=1}^{n} c_k < L \varphi_k, L \varphi_m > = < g, L \varphi_m > \quad \text{or} \\
\sum_{k=1}^{n} c_k < L^a L \varphi_k, \varphi_m > = < L^a g, \varphi_m > \quad (m = 1, \ldots, n) \tag{2.34}
\]

(2.34) is a Galerkin's solution for the problem \( L^a L u = L^a g \). If MoM is used for the same problem, then the result is:

\[
\sum_{k=1}^{n} c_k < L^a L \varphi_k, w_m > = < L^a g, w_m > \quad (m = 1, \ldots, n) \tag{2.35}
\]

If \( L^a L \) is a positive definite operator and the sequence \( \{ \varphi_m \} \) is complete in \( D_{L^a L} \) then the solution given by (2.34) converges in energy to the solution of (2.18). If \( L^a L \) is not a positive definite operator nothing can be said about the convergence of the method of Least Squares. However, if the family of functions \( L \varphi_m \) is a topological base\(^5\) in \( R_L \) then the method of Least Squares converges to the correct result [59]. Generally the method of Least Squares is the safest technique to utilize when very little is known about the nature of the operator and the exact solution.

**Perturbation method**, also, belongs to the same schema of variational techniques. It is derived, in practice, by considering the perturbation of the electrodynamic system due to small changes of a parameter in question. However, often the final expression is a simplification of that is given by a variational process. This is, for example, the case for the power-loss method used for the calculation of the losses in a rectangular waveguide. As a consequence, the perturbation method has a smaller domain of applicability than the variational one.

### 2.2.2 Integral equation method

An integral equation method is one where a specific electromagnetic problem (scattering problem, dispersion problem etc) is reduced in a rigorous manner to the solution of a linear integral equation or a system of coupled linear integral equations.

So, rather than solving a partial differential equation with boundary conditions, the

\(^5\)The sequence \( \{ w_k \} \) is topological base in a topological space \( R_L \), if every \( g \in R_L \) can be represented as \( g = \lim_{n \to \infty} \sum_{k=0}^{n} \alpha_k(n) w_k \).
problem is formulated as:

\[ u(s) = v(s) + \lambda \int_{a}^{b} W(s,t)u(t)\,dt \quad (2.36) \]

where the kernel \( W(s,t) \) is a function of two variables \( s \) and \( t \) defined and continuous in the interval \([a, b]\), \( \lambda \) is a parameter and \( u \) and \( v \) are functions continuous in the same interval. The problem is to determine \( u \) when \( W(s,t) \), and \( v \) are given. Note, under some conditions, \( W(s,t) \), \( u \) and \( v \) can be piecewise continuous or even singular. The integral equation in (2.36) is of the second kind. One of the first kind may be derived from (2.36) by replacing the left-hand side term with zero. An homogeneous equation is obtained when \( v \) is set equal to 0, i.e. no excitation. Integral equation methods are the ones most commonly used in surface scattering problems (boundary integral methods) and, generally, they can be considered as a form of generalised MoM technique [55].

There are various ways of deriving integral equations for electromagnetic field problems. The common approach is to express the fields in terms of the magnetic vector potential \( \mathbf{A} \) and the electric vector potential \( \mathbf{F} \). Then the integral equation is derived when convolution is used to relate \( \mathbf{A} \) and \( \mathbf{F} \) with equivalent electric and magnetic currents \( \mathbf{J} \) and \( \mathbf{M} \), respectively, on the surface of the scatterer [34]. Seeking for expressions more amenable for numerical treatment, mixed-forms are obtained by adding scalar electric and magnetic potentials \( \Phi_e \) and \( \Phi_m \) respectively [29]. Another approach is by the use of the second Green's identity and the Green's function [35]. All these theories can be derived from the theory of distribution [59]. The latter approach is demonstrated below.

For simplicity the problem is considered 2-D and thus the field is given in scalar form. Consider the nonphysical problem where both regions \( R_- \) and \( R_+ \) (inside and outside the scatterer respectively) have the same wavenumber \( k_0 \) and the field \( u^d \):

1. satisfies the outgoing wave conditions OWC, namely that it is finite far from the scatterer,

2. satisfies the edge conditions at the edges of the scatterer (integrable singularity)
[60], i.e.

\[ \int_{\delta S} (k^2|u|^2 - |\nabla u|^2) \, ds < \infty \]  

where \( \delta S \) is a bounded area containing the edge and

(3) experiences jump discontinuities on the boundary \( B \) of the scatterer of the type:

\[ \tau = u_+^d - u_-^d = u_+ - u_- \]  
\[ \eta = \frac{du_+^d}{dn} - \frac{du_-^d}{dn} = v_+ - v_- \]

where \( v_\pm = \frac{du_\pm}{dn} \). "+" ("-" indicates outside (inside) the scatterer.

In other words, since there is no excitation field, the jump discontinuities are associated with the presence of sources at the boundary. In terms of source distributions the Helmholtz equation is given as [59]:

\[ \nabla^2 u^d + k_0^2 u^d = \eta \delta_B + \nabla(\vec{n}\tau \delta_B) \]  

where \( \delta_B \) is the Dirac distribution on \( B \). Consider the Helmholtz equation for a Green’s function \( G \) (\( G \) is the field produced by a point source of unit amplitude) i.e.

\[ \nabla^2 G + k_0^2 G = \delta_r \]  

where \( \delta_r \) is a Dirac \( \delta \)-function of position \( r \) in space: by applying the convolution \( u^d(r) = u^d * \delta_r \), the following equation is obtained:

\[ u^d(r) = \int_B \eta(l') G(r, l') \, dl' + \int_B \tau(l') \frac{dG(r, l')}{dn} \, dl' \]  

where \( \frac{dG(r, l')}{dn} = \vec{n}\nabla G(r, l') \). Equation (2.42) is valid everywhere apart from the points on \( B \). When \( r \) approaches \( B \) by applying the classical theorem about the value of a discontinuous Fourier series on its discontinuity the sought integral equations are obtained:

\[ u_\pm^d(l) = \pm \frac{\tau}{2} + \int_B \eta(l') G(l, l') \, dl' + \int_B \tau(l') \frac{dG(l, l')}{dn} \, dl' \]  

\[ \frac{du_\pm^d}{dn}(l) = \pm \frac{\eta}{2} + \int_B \eta(l') \frac{dG(l, l')}{dn} \, dl' \]
(2.44) is obtained from (2.43) provided that \( \tau = 0 \) and differentiating both sides with respect to \( \eta \). The \( \pm \) sign corresponds to the approach from the region \( R_{\pm} \) and "\( f^* \)" indicates principal value integral\(^6\).

Concerning the physical problem, the object is illuminated by a given excitation \( u^i \) (the incident field), and the total field \( u \) is given by \( u = u^d + u^i \). \( u^d \) is the diffracted field and since, by (2.38) and (2.39), its value on \( B \) is related to \( \tau \) and \( \eta \), equations (2.43) and (2.44) constitute the integral-equation formulation of the problem at hand. If \( u \) represents the electric or magnetic field, the equation is called the electric or magnetic field integral equation respectively (EFIE or MFIE). These equations can take various forms, according to the boundary condition on the scatterer, the choice of the Green’s function, the polarisation of the field and the definitions of \( \tau \) and \( \eta \). For example, for scattering from a perfect conducting material, if \( u \) represents tangential electric field, equation (2.43) can be used by setting \( \tau = 0 \) (EFIE). Alternatively, (2.43) can also be applied to the case of the tangential magnetic field problem (MFIE).

For scattering from cylinders a pair of coupled integral equations of the form of (2.43) is used [61]. Coupled equations are also obtained for imperfectly conducting cylinders by considering impedance boundary conditions [62], [63]. Note, that \( \tau \) and \( \eta \) are defined quite arbitrarily. This permits the establishment of more efficient equations as described below. Expressions are, also, obtained by considering vectorial field problems (Stratton-Chu equations); the procedure, however, remains the same.

After \( \tau \) and \( \eta \) have been defined, \( u^d \) is described everywhere through equation (2.42).

By considering the total field to vanish in \( R_{+} \) (\( R_{-} \)), i.e. \( \tau = u_{+} \ (\tau = -u_{-}) \) and \( \eta = v_{+} \ (\eta = -v_{-}) \) the following equations are obtained:

\[
0 = u^i(r) + \int_{B} v_{+}(l')G(r, l') \, dl' + \int_{B} u_{+}(l') \frac{dG(r, l')}{dn} \, dl' \quad \text{for } r \in R_{-} \quad (2.45)
\]

\[
0 = \int_{B} u_{-}(l')G(r, l') \, dl' + \int_{B} u_{-}(l') \frac{dG(r, l')}{dn} \, dl' \quad \text{for } r \in R_{+} \quad (2.46)
\]

Equations (2.45) and (2.46) belong to the form of the extended integral equations or

\(^6\)The integral \( \int_{B} \tau(l') \frac{dG(r,l')}{dn} \, dl' \) is a Cauchy type singular integral due to the singularity of \( \frac{dG(r,l')}{dn} \).

EIE\textsuperscript{7} [64]. Their significance is that the kernels are regular, a desirable feature for numerical solutions. They are the cornerstone of the method proposed in the next chapter and more details will be given therein.

The integral equation formulation is one of the most rigorous methods for solving electromagnetic field problems and sometimes is the only way for achieving results with reasonable accuracy (as, for example, with 3D scattering from randomly rough surfaces [65]). However, there are some drawbacks on its use. First of all, for bounded scatterers, the relevant integral equations are known to be not uniquely solvable for a discrete set of frequencies, called irregular frequencies, that correspond to the resonant frequencies of a cavity with the same geometry, but filled with the dielectric of the original problem and with perfectly conducting walls. From the mathematical point of view, irregular frequencies are the eigenvalues of the operator of the homogeneous integral equation [66]. To ensure uniqueness, one should consider more complicated integral equations like the combined-field integral equation (CFIE) [67], [68], combined-source integral equation (CSIE) [69]-[71], augmented electric- and magnetic- field integral equations (AEFIE and AMFIE) [72] and the dual-surface integral equations [73]. CFIE is derived by observing that EFIE and MFIE have different eigenvalues and thus a unique solution can be obtained if they are linearly combined. In the CSIE formulation the unknown surface densities are no longer the tangential components of the electromagnetic field, but some fictitious surface currents which radiate the true scattered field. In other words the formulation is based on the arbitrariness on the definition of $\tau$ and $\eta$ introduced before. However, the solution is still not unique for spherical scatterers. AEFIE or AMFIE is established by involving both the tangential and normal components of a single type of field. Its advantage over CFIE is that it is only as complicated as the original EFIE or MFIE, but the solution is again not unique for spherical scatterers.

The dual-surface integral formulation is based on the extended integral equation. EIE also suffers from non-uniqueness at resonant frequencies [74]. It can be shown that uniqueness can be obtained if both EFIE and MFIE are applied in the interior of the scatterer [75]. If only EFIE (or MFIE) is used then uniqueness is preserved

\textsuperscript{7}The use of EIE is named in literature extinction theorem or extended boundary conditions method (EBCM).
with a deformation of the internal surface i.e. EIE is applied in the deformed surface and is combined linearly with the EFIE on the surface of the scatterer. This is the dual surface equation [73]. Furthermore, the deformation distance should be less than a quarter of a wavelength and the combination coefficient should be a real number. In contrast to CFIE or CSIE formulations, the dual-surface equations only require one type of integral operator (i.e., EFIE type or MFIE type), which may simplify the programming and the overall accuracy.

Note that the phenomenon of non-uniqueness of the solution of the integral equation is observed whether the scatterer in the original problem is a dielectric obstacle or a perfectly-conducting one. The situation for a dielectric obstacle is more complicated because, mathematically, the problem is reduced to a pair of coupled integral equations for a pair of unknown scalar or vector fields. Some pairs have irregular frequencies, others do not. Referring to the CSIE for dielectric obstacles, a single integral equation can be established but it contains hypersingular kernels, which are more difficult to implement numerically. In this case, generalised rooftop functions are used to tackle the problem [76].

All the procedures for reducing the pair of equations to a single one first appeared for periodical structures [77], but were reconsidered much later [78]-[80]. The corresponding formulae for the sophisticated integral forms quoted above will not to be given here; the interested reader is advised to consult the references for more information.

Another drawback on the use of an integral equation relates to the way it is solved. Integral equations, apart from some special forms, are solved numerically, thanks to MoM. Unlike the FEM, the resulting linear system is dense, with no symmetry property, so the inversion of the corresponding matrix is an intense problem when large-size scatterers are considered. A lot of work is currently being devoted to speeding up the solution of these systems (involving the use of iterative methods like the conjugate-gradient method, gradient minimal residual method, Neuman series etc.) [29] (ch. 4 and references therein). However, the modelling of millimetre-wave components involves scatterers of small size compared with the wavelength, and thus the aforementioned problem is not encountered in the present thesis.
2.3 Summary

The Variational method and Integral Equation techniques have been described in detail. Their broad use in the electromagnetic field problems has been discussed. It is hoped that this chapter give the reader the background to follow the discussion of the rest of the thesis.
Chapter 3

An Improved Indirect Mode Matching Technique

This chapter presents a new method for the analysis of a wide variety of microwave structures such as posts, circular junctions or corners in rectangular waveguides. This method also can be used to model the posts in the laminated waveguide and this was the main motivation of the present work. The method gives the impedance or admittance matrix of the structure by applying discrimination between the localised and accessible modes [27] in the indirect mode matching (IMM) formulation. In this way, the method is simpler and more versatile than previous modelling techniques where IMM is involved, since no use of Green’s function is required for the coupling between different ports and additional obstacles. For example, posts at the centre of a circular junction, can be treated using different terminations of the corresponding transmission lines. The formulation is given for a two- as well as a multi-port system and numerical demonstration is provided for an abrupt H-plane transition from a rectangular to a radial waveguide, a centrally located H-plane circular post in a rectangular waveguide and a two-port circular junction between rectangular waveguides. The results are compared with HFSS™ and give excellent agreement. In all the aforementioned cases the method proved to be fast, very accurate and easy to implement.
3.1 Introduction

The network representation of waveguide components of unusual shape such as posts and junctions has been received considerable attention at the past, and many contributions have been devoted to this subject [27], [81]-[86]. According to this type of modelling, the waveguide modes are considered as distinct transmission lines ending at the obstacle, which is modelled as a scattering, impedance or admittance matrix. This approach provides an efficient tool for characterizing more complicated structures like filters where many components have to be interconnected. Although this review is not meant to be exhaustive, among the available methods, we distinguish for their computational efficiency the boundary contour mode matching method (BCMM) [81], the boundary integral resonant mode matching method (BIRMEM)[82, 83], the multimode network representation in the form presented in [27], and the generalised admittance matrix (GAM) [84]-[86]. A comprehensive review can be found in [87].

The BCMM and BIRMEM are the most general techniques and they have been applied for arbitrarily shaped discontinuities. The BCMM is a mode matching method in the least square sense. The BIRMEM method gives a Foster's network representation of the component by making use of the generic expression of cavity structures [88] and in contrast with BCMM this is performed only once for the whole frequency range. In [27] a very efficient method for multimode network representation is provided by explicitly distinguishing between the localised and accessible modes. This method has been applied before to many types of waveguide discontinuities [89], [90]-[93] and in [27] is extended for cubic junctions with the use of the Green’s function of the cubic resonator. The GAM is based on the mode matching technique and has recently been demonstrated with the use of the indirect mode matching (IMM) [94] to characterize post obstacles [84] and the transition region between a circular cavity and a number of rectangular waveguides [85, 86]. The IMM (also called the Waterman or Extinction-Theorem method) is a simple mode matching technique between non-coincident surfaces based on the second Green's identity and it has been applied in scalar or vectorial form for scattering problems in gratings [94]-[96], non-eccentric spheres or non-eccentric cylinders [97]-[99] and for the determination
of the eigenmodes in arbitrarily shaped optical fibre cores [100, 101]. Despite its simplicity it is apparent that the use of IMM for microwave modelling is somehow limited to the recent papers of [84]-[86].

While very capable, the above techniques [27], [81]-[86] have some disadvantages. BCMM needs to be performed repeatedly for every frequency point. In the BIRMEM a number of the eigenfuctions of the cavity need to be firstly found. This number can be kept small, but still the implementation of the algorithm on the computer is a quite demanding task. The method in [27] has not been until now generalised without the use of Green’s functions which complicates the formulation. Finally, the IMM in the form presented in [84]-[86], is not able to give the coupling between different ports and this problem is either left as an open issue [84] (other techniques such as modal analysis or the unimoment method are invoked) or is addressed again by the use of a Green’s function [86] that adds to the complexity of the total approach. This is equivalent to saying that the IMM, in the form of [84]-[86], cannot model directly a step discontinuity such as the step discontinuity between a rectangular and a radial waveguide (Fig. 3.1.d). Indeed, it is demonstrated in this chapter that if the IMM is applied in a traditional way ([84]-[86]) on the aforementioned structure, relative convergence phenomena are observed (the solution is correct only for a specific number of expansion functions). This goes back to the well known problem of relative convergence, that occurs when mode matching is applied to planar step discontinuities [55].

Trying to be fair with the literature review, the group of numerical techniques, such as, the finite element method [102], the boundary element method [103], the method of lines [104], the finite difference time domain method [105] and the transmission line matrix method [106], need to be quoted here. All these techniques can be used in the analysis (although their functionality to provide a network representation of a microwave component is not always warranted), but they do require a considerable computational effort.

In this chapter it is shown that there is a wide variety of relatively simple structures that can be characterised rigorously if an implicit discrimination between the localised and accessible modes, like the one presented in [27], is applied in the IMM.
formulation. Although both formulations have been extensively used for years, for the best of the author’s knowledge, a general unified approach combining both of them for modelling waveguide components is not available. The main advantage of the method presented in this chapter compared with the others mentioned above [27], [81]-[86], [102]-[106] is that it is simple, yet accurate and versatile. This is because, in contrast with [84]-[86], the present method accounts directly for the interaction between higher order modes at different ports of the structure without the need for a Green’s function and the overall admittance/impedance matrix of the structure is obtained in a single step. Modification of the structure, like for example, additional posts at the centre of a junction can be easily entered in the algorithm since they can be considered as different terminations of the corresponding transmission lines. Furthermore, the method allows for the correct edge conditions to be incorporated in the process. All these, however, are not without a cost. Due to the direct coupling between different ports, the resulting matrix is larger than previous attempts of IMM, although the inversion of the matrix can still be performed successfully (at least for all the cases investigated by the author). Structures that can be modelled with the proposed technique are those where the IMM easily applies, i.e. they are composed of individual so-called co-ordinate structures where the boundaries of areas considered consist of parts of co-ordinate lines (surfaces) of any classical system of orthogonal co-ordinates (Cartesian, cylindrical, spherical, elliptical, parabolic etc.). Such structures can be inductive or capacitive posts in rectangular waveguides, waveguide transitions, circular junctions and bends in rectangular waveguides to cite a few. It should be made clear at this point that the purpose of the method in this chapter is to simplify matters whenever possible, rather than adding another method to the plethora of the existing ones.

The problems considered in this chapter are shown in Fig. 3.1. In Fig. 3.1.a a structure of three cascaded waveguides is depicted (the transition region is a waveguide with known modal expansion). In Fig. 3.1.b a circular junction between a number of rectangular waveguides is depicted. A circular post is also centrally located in the junction and the problem is equivalent with the one dealt in [86]. The basic formulation, with respect to the structures shown in Fig. 3.1.a and Fig. 3.1.b is given in Section 3.2. In Section 3.3 a numerical demonstration follows by applying
Figure 3.1: Geometries considered in the present chapter. (a) A guide to guide taper. (b) A circular junction. An abrupt (c) and a step (d) transition from a rectangular to a radial waveguide. (e) A post in a rectangular waveguide. (f) A circular junction between two rectangular waveguides

the method to the structures of Fig. 3.1.c, 3.1.d, 3.1.e and 3.1.f.

3.2 Basic Formulation

Consider the structure in Fig. 3.2. The middle waveguide - with a known modal expansion (region III) - acts as a transition between the other waveguides (region I,II) and needs to be modelled as an admittance matrix. The first step in the formulation is to write the second Green's identity in region II for the electric field intensity $\vec{E}$ and the weighting functions $\vec{E}_l$ [97]:

$$\int_{V_3} (\vec{E} \cdot \nabla \times \nabla \times \vec{E}_l - \vec{E}_l \cdot \nabla \times \nabla \times \vec{E}) \, dv =$$

$$= \int_{S} (\vec{n} \times \vec{E}_l - \vec{E}_l \times \nabla \times \vec{E}) \, \vec{n} \, dv = 0 \quad (3.1)$$

$V_3$ is the volume enclosed by the surface $S$ ($S = S_1 + S_2 + S_p$) as shown in Fig. 3.2. By $\vec{n} \times \vec{E}_l = 0$ and $\vec{n} \times \vec{E} = 0$ on $S_p$ and Maxwell's equations, (3.1) is given in the
Figure 3.2: A guide to guide taper. Region III is a waveguide with known modal expansion form:

\[ \int_{S_1} [\vec{H}_i(\vec{n}_1 \times \vec{E}^{(1)}) - \vec{E}_l(\vec{H}^{(1)} \times \vec{n}_1)] \, ds_1 + \]

\[ + \int_{S_2} [\vec{H}_i(\vec{n}_2 \times \vec{E}^{(2)}) - \vec{E}_l(\vec{H}^{(2)} \times \vec{n}_2)] \, ds_2 = 0 \quad (3.2) \]

It should be mentioned that (3.2) is still valid when there are corners on the surface \( S \) since the field has integrable singularity there. \( \vec{E}_l \) (\( \vec{H}_l \)) satisfies the Helmholtz equation in volume \( V_3 \) namely:

\[ \nabla \times \nabla \times \vec{E}_l - \omega^2 \varepsilon_0 \mu_0 \vec{E}_l = 0 \quad (3.3) \]

and the boundary condition \( \vec{n} \times \vec{E}_l = 0 \) \( (\vec{n} \times (\vec{n} \times \vec{H}_l) = 0) \) on the surface \( S_p \). Consequently \( \vec{E}_l \) and \( \vec{H}_l \) correspond to the electric and magnetic field of the \( l \) propagating mode of the waveguide III. According to the direction of the propagating modes there are two sets of linearly independent solutions of (3.3) (moving toward to \( S_1 \) or \( S_2 \) ) which when applied to (3.2) produces a pair of independent families of equations. If \( S_1 \) and \( S_2 \) are equiphase surfaces of the modes propagating in waveguides I and II respectively, then the electric and magnetic field intensities are given by (according to standard waveguide textbooks [43]):

\[ \vec{E}^{(\nu)} = \sum_{k=1}^{\infty} V_k^{(\nu)} \vec{e}_k^{(\nu)} \quad (3.4) \]
where the normalised transverse eigenfunctions $\tilde{e}_k^{(v)}$ and $\tilde{l}_k^{(v)}$ satisfy the orthogonality equation:

$$\int_{S_v} \left( \tilde{n}_\nu \times \tilde{e}_p^{(v)} \right) \tilde{l}_q^{(v)} \, ds = \int_{S_v} \left( \tilde{n}_q \times \tilde{n}_\nu \right) \tilde{e}_p^{(v)} \, ds = \delta_{pq}$$  \hspace{1cm} (3.6)

with $\nu = 1, 2$ for the waveguide I and II respectively. Obviously $V_k^{(r)}$ and $I_k^{(r)}$ are given by the equations:

$$V_k^{(v)} = \int_{S_v} \left( \tilde{n}_\nu \times \tilde{E}^{(r)} \right) \tilde{H}_k^{(v)} \, ds$$  \hspace{1cm} (3.7)

$$I_k^{(v)} = \int_{S_v} \left( \tilde{H}^{(v)} \times \tilde{n}_\nu \right) \tilde{e}_k^{(v)} \, ds$$  \hspace{1cm} (3.8)

From (3.4) and (3.5) it is apparent that an infinite number of modes are excited at the surfaces $S_1$ and $S_2$. However only a small number of them (accessible) are interacting with adjacent discontinuities while the rest (localised) vanish before they reach any nearby component. As has been demonstrated in the literature [27], the localised modes can be distinguished from the accessible ones in (3.4). More specifically, if the waveguide modes are viewed as distinct transmission lines, the localised modes are terminated by their characterising impedance and (3.4) is given in the form:

$$\tilde{E}^{(v)} = \sum_{k=1}^{a_v} V_k^{(v)} \tilde{e}_k^{(v)} + (-1)^\nu \sum_{k=a_v+1}^{\infty} Z_k^{(v)} I_k^{(v)} \tilde{e}_k^{(v)}$$  \hspace{1cm} (3.9)

or by (3.8)

$$\tilde{E}^{(v)} = \sum_{k=1}^{a_v} V_k^{(v)} \tilde{e}_k^{(v)} + (-1)^\nu \sum_{k=a_v+1}^{\infty} Z_k^{(v)} I_k^{(v)} \int_{S_v} \left( \tilde{H}^{(v)} \times \tilde{n}_\nu \right) \tilde{e}_k^{(v)} \, ds$$  \hspace{1cm} (3.10)

where $a_\nu$ is the number of accessible modes of the waveguide $\nu$ ($\nu = 1, 2$). The difference with [27] is that now the field matching will be performed with the use of the IMM, i.e. (3.2), rather than the use of a Green's functions. Under this framework, (3.10) is put into (3.2) to give:

$$\sum_{r=1}^{a_1} V_r^{(1)} \int_{S_1} \left[ \tilde{H}_1(s_1) \left( \tilde{n}_1(s_1) \times \tilde{e}_r^{(1)}(s_1) \right) \right] \, ds_1 +$$

$$+ \sum_{r=1}^{a_2} V_r^{(2)} \int_{S_2} \left[ \tilde{H}_1(s_2) \left( \tilde{n}_2(s_2) \times \tilde{e}_r^{(2)}(s_2) \right) \right] \, ds_2 = \int_{S_1} \left[ \tilde{H}^{(1)}(s_1) \times \tilde{n}_1(s_1) \right]$$
It is reasonable now for the magnetic field to be written as a linear combination of unknown vector functions weighted with the modal voltages of the accessible modes. Thus

\[
\bar{H}^{(\nu)} \times \bar{n}_\nu = \sum_{k=1}^{a_1} V_k^{(1)} \bar{M}_k^{(\nu,1)} + \sum_{k=1}^{a_2} V_k^{(2)} \bar{M}_k^{(\nu,2)}
\]

(3.12)

where again \( \nu = 1, 2 \) for the surface \( S_{1,2} \) respectively. Note that, from (3.12) and (3.8) the following is obtained:

\[
I_{p_\nu}^{(\nu)} = \sum_{k=1}^{a_1} V_k^{(1)} \int_{S_\nu} \bar{M}_k^{(\nu,1)} \bar{e}_p^{(\nu)} ds_\nu + \sum_{k=1}^{a_2} V_k^{(2)} \int_{S_\nu} \bar{M}_k^{(\nu,2)} \bar{e}_p^{(\nu)} ds_\nu
\]

(3.13)

or

\[
Y_{p_\nu q_m}^{(\nu,m)} = \int_{S_\nu} \bar{M}_k^{(\nu,m)} \bar{e}_p^{(\nu)} \bar{e}_q^{(m)} ds_\nu
\]

(3.14)

where \( p_\nu, q_m \) are running from 1 to \( a_{\nu,m}, \nu, m = 1, 2 \) and \( Y_{p_\nu q_m}^{(\nu,m)} \) are the seeking admittance parameters of a \( (a_1 + a_2) \times (a_1 + a_2) \) finite dimensional admittance matrix. To determine \( \bar{M}_k^{(\nu,m)} \), (3.12) is used with (3.11) to give:

\[
\sum_{\nu=1}^{2} \sum_{r=1}^{a_\nu} V_r^{(\nu)} \int_{S_\nu} \left[ \bar{H}_l(s_\nu) \times \bar{\nu}_\nu(s_\nu) \right] \bar{e}_p^{(\nu)}(s_\nu) ds_\nu =
\]

\[
= \sum_{\nu=1}^{2} \sum_{r=1}^{a_\nu} V_r^{(\nu)} \sum_{m=1}^{a_m} \int_{S_m} \bar{M}_k^{(\nu,m)}(s_m)
\]

\[
\left[ (-1)^m \int_{S_m} K^{(m)}(s_m, s'_m) \left[ \bar{H}_l(s'_m) \times \bar{\nu}_m(s'_m) \right] ds'_m + \bar{E}_l(s_m) \right] ds_m
\]

(3.15)

where the dyadic kernel \( K^{(\nu)}(s, s') \) is given by the equation:

\[
K^{(\nu)}(s, s') = \sum_{k=0}^{a_0+1} Z_k^{(\nu)}(s) \bar{e}_k^{(\nu)}(s')
\]

(3.16)
Since equation (3.15) must be satisfied for every linear combination of the linearly independent coefficients $V_r^{(1)}$ and $V_r^{(2)}$, the following is imposed:

$$\int_{\Sigma_v} [\vec{H}_v(s_v) \times \vec{n}_v(s_v)] e^{(v)}_{\theta} e^{(v)}_{\phi} ds_v = \sum_{m=1}^{2} \int_{\Sigma_m} \bar{M}_{p,v}^{(m,v)}(s_m) \left[ (-1)^m \int_{\Sigma_m} \bar{K}^{(m)}(s_m, s'_m) \vec{H}_v(s'_m) \times \vec{n}_v(s'_m) ds'_m + \bar{E}_v(s_m) \right] ds_m \quad (3.17)$$

The two families of $\vec{E}_l$, $\vec{H}_l$ modes (pertaining to the direction of their propagation) when they are applied to (3.17) give four families of equations for successive values of $l$. These equations then are solved with the Galerkin's algorithm after having expanded $\bar{M}_{p,v}^{(1,\nu)}$ and $\bar{M}_{p,v}^{(2,\nu)}$ in a suitable set of basis vector functions. To simplify the solution process, different combinations of the weighting modes $\vec{E}_l$ and $\vec{H}_l$ (as to give zero on one of the surfaces $S_1$ and $S_2$) can also be used. This is demonstrated with a numerical example in the following section. Equation (3.17) is the new formula proposed in the present work and together with (3.14) constitutes the rigorous admittance representation solution of the problem at hand. All the localised modes are implicitly taken into account and the only complexity arises from the dyadic kernels $\bar{K}_{p,v}^{(w)}(s, s')$ which fortunately sometimes can be summed in closed form.

The solutions above can be generalized for multiport structures like the one in Fig 3.3. A number ($N$) of waveguides are connected to a circular junction with a centred post. The surface $S_c$ denotes a circular port from where cylindrical wavemodes emerge. The impedance matrix of it will be now given. The second Green's identity is applied in the volume enclosed by the surface $S (S = S_p + S_0^{(1)} + \ldots + S_0^{(N)} + S_1 + \ldots + S_N)$ (Fig 3.3). Since the contribution from the surface $S_p$ shrinks to zero, (3.2) now becomes:

$$\sum_{m=1}^{N} \int_{S_0^{(m)}} \left[ \vec{H}_l^{(m)}(s_0) \times \vec{n}_0(s_0) \right] ds_0 = \sum_{m=1}^{N} \int_{S_m} \left[ \vec{H}_l^{(0)}(s_m) \times \vec{n}_m(s_m) \right] ds_m \quad (3.18)$$

where $\vec{E}^{(0)}$, $\vec{H}^{(0)}$ and $\vec{E}^{(m)}$, $\vec{H}^{(m)}$ ($m = 1, \ldots, N$) are the electric and magnetic field of the circular junction and the $m$th rectangular waveguide respectively and
Figure 3.3: A circular junction. The thick line indicates the path where the Green's identity applies.

\( \vec{E}_l^{(m)}, \vec{H}_l^{(m)} \) are the electric and magnetic field respectively of the testing\(^1\) mode in volume \( V_m \). By considering that \( \vec{E}_l^{(m)}, \vec{H}_l^{(m)} \) are defined only in volume \( V_m \) and vanish elsewhere then the testing modes corresponding to different volumes are independent of each other and so the equation in (3.18) can split into \( N \) equations (one for each volume \( V_m \)). A process similar to that followed for the two port case is applied to every equation and the final result can be written as a linear combination of the \( N \) individual formulations to give:

\[
\begin{align*}
  c_{\nu} \int \left[ \vec{n}_\nu(s_\nu) \times \vec{E}_l^{(\nu)}(s_\nu) \right] \vec{h}_p^{(\nu)}(s_\nu) \, ds_\nu &= -c_{\nu} \int \left[ \vec{h}_p^{(\nu)}(s_\nu) \right] \vec{E}_l^{(\nu)}(s_\nu) \, ds_\nu = -c_{\nu} \int \vec{M}_p^{(\nu)}(s_\nu) \, ds_\nu \\
  \left[ \vec{n}_\nu(s'_\nu) \times \vec{E}_l^{(\nu)}(s'_\nu) \right] \, ds'_\nu - \vec{H}_l^{(\nu)}(s_0) \, ds_\nu - \int \vec{M}_p^{(\nu)}(s_0) \, ds_0 \\
  \sum_{m=1}^N c_m \left[ \vec{n}_0(s'_0) \times \vec{E}_l^{(m)}(s'_0) \right] \, ds'_0 - \sum_{m=1}^N c_m \vec{H}_l^{(m)}(s_0) \, ds_0
\end{align*}
\]

\( (3.19) \)

\(^1\)The term "testing" refers to the Galerkin's algorithm.
Figure 3.4: Equivalent impedance representation of the junction in Fig. 3.3.

for \( \nu = 1, \ldots, N \) and

\[
\begin{align*}
\int_{S_0}^{N} & \sum_{m=1} c_m \left[ \vec{r}_0(s_0) \times \vec{E}_l^{(m)}(s_0) \right] \vec{h}_p^{(0)}(s_0) ds_0 = - \sum_{m=1}^{N} c_m \int_{S_m}^{N} \vec{M}_{p_0}^{(m,0)}(s_m) \\
\left[ \int_{S_m}^{N} \vec{K}^{(m)}(s_m, s') \left[ \vec{r}_m(s_m) \times \vec{E}_l^{(m)}(s') \right] ds' - \vec{H}_l^{(m)}(s_m) \right] ds_m - \\
- \int_{S_0}^{N} \vec{M}_{p_0}^{(0,0)}(s_0) \left[ \int_{S_0}^{N} \vec{K}^{(0)}(s_0, s') \sum_{m=1}^{N} c_m \\
\left[ \vec{r}_0(s_0) \times \vec{E}_l^{(m)}(s') \right] ds'_0 - \sum_{m=1}^{N} c_m \vec{H}_l^{(m)}(s_0) \right] ds_0 \quad (3.20)
\end{align*}
\]

for \( \nu = 0 \), where \( \vec{h}_k^{(0)} \) are the cylindrical transverse eigenfunctions of the cylindrical junction, \( \vec{r}_k^{(m)} \) \((m = 1, \ldots, N)\) are the transverse eigenfunctions of the mth rectangular waveguide, \( a_\nu \) the number of accessible modes on surface \( S_\nu \) \((\nu = 0, \ldots, N)\), \( p_\nu, q_\nu = 1, \ldots, a_\nu \) and \( S_0 = S_0^{(1)} + \ldots + S_0^{(N)} \). The kernels \( \vec{K}^{(\nu)}(s_\nu, s'_\nu) \) are again given by equation (3.16). \( c_m (m = 1, \ldots, N) \) are arbitrary coefficients weighting the N individual formulations. The desired impedance parameters are now given by:

\[
Z^{(\nu,m)}_{p_\nu,q_\nu} = \int_{S_\nu} M^{(\nu,m)}_{p_\nu,q_\nu} \vec{h}_p^{(\nu)} ds_\nu \quad (3.21)
\]

With equations (3.19)-(3.21) the impedance parameters of the multiport cylindrical junction can be obtained, which is now modelled as shown in Fig. 3.4. \( \vec{E}_l \) and \( \vec{H}_l \) are chosen to be the modes of the rectangular waveguides and different combinations of them give the solution to the equation (3.19) as it will be demonstrated in the
next section. In contrast with [85] and [86] the solution does not involve the use of a Green's function and so it is considered simpler. Also, in [85] and [86] the transition regions and the cylindrical body of the junction have been considered separately while in the present work they are manipulated in a single step.

3.3 Numerical Implementation

3.3.1 $H$-plane abrupt transition between a rectangular and a radial waveguide

The formulations of the previous section will be demonstrated for the simple structure shown in Fig. 3.5. It can be recognized as a constituent part of other multiport systems and the analysis described here shares common features with that followed for more complicated structures. From the design point of view, successive transitions of this type can be used to approximate more complicated structures like transitions of arbitrary shape [107]. Various analysis methods have been proposed [108]-[111] but none of them is as simple and rigorous as the one proposed here.

The structure is symmetric and it is assumed that only the dominant $TE_{10}$ mode propagates into the rectangular waveguide. Thus, there is only a $y$ component of the electric field on the surfaces $S_1$ and $S_2$ and so the problem can be simplified to a scalar one, i.e the equation (3.1) reduces to:

$$\int_{C_1} (\psi_1 \frac{\partial E_y^{(1)}}{\partial n_1} - E_y^{(1)} \frac{\partial \psi_1}{\partial n_1}) \, dc_1 + \int_{C_2} (\psi_1 \frac{\partial E_y^{(2)}}{\partial n_2} - E_y^{(2)} \frac{\partial \psi_1}{\partial n_2}) \, dc_2 = 0 \tag{3.22}$$

where $C_1$ and $C_2$ are the lines shown in Fig 3.5. Also, according to the depicted coordinate system $n_1 = -z$ and $n_2 = r$ where $r$ is the radial coordinate in a shifted coordinate system. Equation (3.17) of Section 3.2 appears in a scalar form too:

$$\int_{C_{\nu}} \psi^{(\nu)}(x) \frac{\partial \psi_1(x)}{\partial n_{\nu}} \, dc_{\nu} = \sum_{n=1}^{2} \int_{C_m} M^{(m,\nu)}(x) \left[ (-1)^m \int_{C_m} K^{(m)}(x, x') \frac{\partial \psi_1(x')}{{\partial n}_m} \, dx' + j \omega \mu \psi_1(x) \right] dx \tag{3.23}$$

($p_{\nu} = 1, \ldots, a_{\nu}, \nu = 1, 2$).
The testing functions $\psi_l(x, z)$ are a linear combination of the wave-functions

$$\psi_l^\pm(x, z) = \sqrt{\frac{2}{\alpha}} \cos \left[ (2l - 1) \frac{\pi}{\alpha} x \right] e^{\pm j\beta_l z} \tag{3.24}$$

of the rectangular waveguide, where $\beta_l = \frac{\omega}{\sqrt{\mu \varepsilon}} \left( \frac{(2l-1)\pi}{\alpha} \right)^2$, $k_0 = \omega \sqrt{\mu \varepsilon}$, $l = 1, 2, \ldots$ and $\alpha$ is the width of the rectangular waveguide. To solve for $M_{p_v}^{(m, \nu)}(m, \nu = 1, 2)$ the testing functions $\psi_l^-(x, z)$ and

$$\psi^+_l(x, z) = \psi^+_l(x, z) - \psi^-_l(x, z) \quad \text{and} \quad \psi^-_l(x, z) = \psi^+_l(x, z) + \psi^-_l(x, z) \tag{3.25} \tag{3.26}$$

are used in (3.23). Note that $\psi^+_l(x, z) = 0$ and $\frac{\psi^+_l(x, z)}{\theta n_1} = 0$ on the line $c_1$. With the above choice of testing functions the functions $M_{p_v}^{(\nu, 1)}(x)$ do not need to be found. Indeed since for $l \leq a_1$ the quantity $\int_{C_1} K^{(1)}(x, x') \frac{\psi^+_l(x')}{\theta n_1} dx'$ is 0 then the only unknown in equation (3.23) is $M_{p_v}^{(2, \nu)}(\nu)$. For $l \geq a_1$ the contribution of $M_{p_v}^{(1, \nu)}(x)$ in (3.23) again vanishes if $\psi^-_l(x, z)$ are used as testing functions. After $M_{p_v}^{(2, \nu)}(\phi)$ being found the admittance parameters $Y_{p_v}^{(1, \nu)}$ are derived directly by applying the testing functions $\psi^+_l(x)$ ($l \leq a_1$) on equation (3.23). Summarising the above, one gets:

$$\int_{C_1} e^{(\nu)}(c_v) \frac{\psi^+_l(c_v)}{\theta n_1} dc_v = \int_{C_2} M_{p_v}^{(2, \nu)}(\phi) \left[ \int_{C_2} K^{(2)}(\phi, \phi') \frac{\psi^+_l(\phi')}{\theta n_2} d\phi' + j \omega \mu \psi^+_l(\phi) \right] d\phi \tag{3.27}$$

where the testing function $t_\theta(\phi)$ is defined as:

$$t_\theta(\phi) = \begin{cases} 
\psi^+_l(\phi) & \text{for } l = 1, \ldots, a_1 \\
\psi^-_l(\phi) & \text{for } l = a_1 + 1, \ldots, m_0
\end{cases} \tag{3.28}$$

and

$$Y_{l, q_v}^{(1, \nu)} = \frac{(-1)^\nu}{2j \omega \mu} \left\{ - \int_{C_1} e^{(\nu)}(c_v) \frac{\psi^+_l(c_v)}{\theta n_1} dc_v + \int_{C_2} M_{q_v}^{(2, \nu)}(\phi) \left[ \int_{C_2} K^{(2)}(\phi, \phi') \frac{\psi^+_l(\phi')}{\theta n_2} d\phi' + j \omega \mu \psi^+_l(\phi) \right] d\phi \right\} \tag{3.29}$$

$$Y_{l, q_v}^{(2, \nu)} = (-1)^\nu \int_{C_2} M_{q_v}^{(2, \nu)}(\phi) e^{(2)}(\phi) d\phi \tag{3.30}$$
Numerical Implementation

for \( l = 1, \ldots, a_1, q_\nu = 1, \ldots, a_\nu \) and \( \nu = 1, 2 \). \( m_0 \) is determined in (3.31). Equation (3.27) reveals that \( M_r^{(2,\nu)} \) is projected to the space of functions spanned by the basis
\[
E_1 K^{(2)}(\varphi, \varphi') \frac{\partial e_{\nu}(\varphi')}{\partial \nu_2} d\varphi' + jw_0 \mu e_0(\varphi).
\]
The best approximation of \( M_r^{(2,\nu)} \) on this basis can be found as described in [50]. However, the Galerkin’s algorithm is faster. Following this approach, \( M_r^{(2,\nu)} \) is expanded in a number, \( m_0 \), of suitable expansion functions:
\[
M_r^{(2,\nu)}(\varphi) = \sum_{m=1}^{m_0} c_m g_m(\varphi) \tag{3.31}
\]
\( M_r^{(2,\nu)}(\varphi) \) pertains to the unknown magnetic field on \( C_2 \) and thus experiences singularities at the edges. It then comes to mind that the singularity could be incorporated into the expansion functions. However, since the basis functions \( b_l(\varphi) \), defined previously, do not satisfy the aforementioned boundary conditions (they are 0 at the corners) and what is required is the projection of \( M_r^{(2,\nu)} \) on sinus functions, i.e. the \( Y_{l_r}^{(2,\nu)} \), to be found rather than \( M_r^{(2,\nu)} \) itself, then the expansion functions can be relaxed by the singularity conditions. Indeed, the author experimented with both types of the expansion function, for the structure of Fig. 3.5, and found exactly the same result. For the rest of the structures encountered in this chapter, \( b_l(\varphi) \) are not 0 any more at the edges, and an insignificant difference due to the type of expansion functions used does exist. As suggested above, \( g_m(\varphi) \) is given by:
\[
g_m(\varphi) = \frac{1}{N_2} \cos \left[ \frac{(2m - 1)\pi}{\varphi_0} \varphi \right] = e_{m}^{(2)}(\varphi) \tag{3.32}
\]
i.e. \( g_m(\varphi) \) is the transverse eigenfunction of the radial guide \( (N_2 = \sqrt{\frac{2\pi}{k}} \) and \( \varphi_0 \) is shown in Fig. 3.5). This means that the series form of the kernel \( K^{(2)} \) does not apply and so the algorithm runs faster.

Another point is that the kernel \( K^{(\nu)}(x, x') \) and the testing functions \( \psi^\nu(\varphi) \) are frequency dependent. While the frequency dependence of the kernels is kept out of the integrals in (3.23) (frequency is restricted to the terms \( Z^{(\nu)}_k \)), this is not the case for the testing functions due to the term \( e^{\pm \imath \beta z} \). Frequency is decoupled from \( e^{\pm \imath \beta z} \) as follows (note that on \( S^{(\nu)} \), \( z \) depends on \( \varphi \)): For the propagating modes \( (\beta_l \text{ real}) \), \( e^{\pm \imath \beta z} \) is approximated by a series of Bessel functions [35]. However, for the evanescent modes \( (\beta_l \text{ imaginary}) \) the aforementioned series converges very slowly and a Taylor series expansion until the second order term has been proved successful.

More specifically, after some simple algebra \( e^{\pm \imath \beta z} \) is given by:

\[
e^{\pm \imath \beta z} = \pm e^{\pm \imath \beta^{(0)}_l z} \left\{ 1 + \frac{jjz k^{(0)}_0}{\beta^{(0)}_l} (k_0 - k^{(0)}_l) - \left[ \frac{(xk^{(0)}_0)^2}{(\beta^{(0)}_l)^2} + \frac{jjz(2l - 1)\pi^2 x_2}{(\beta^{(0)}_l)^3 x^2} \right] (k_0 - k^{(0)}_l) \right\}
\]

(3.33)

where \( \beta^{(0)}_l = \beta_l(f_0), k^{(0)}_0 = k_0(f_0) \) and \( f_0 \) is the central frequency in the range of interest. The integrals now need to be calculated only once for all the frequency range. In the Galerkin's algorithm, the matrix inversion is again repeated for every frequency point but this requires negligible time. After \( M^{(2,\nu)} \) has being found, (3.29) and (3.30) give the admittance parameters of the structure of Fig. 3.5.

### 3.3.2 \( H \)-plane step transition between a rectangular and a radial waveguide

The structure under analysis is shown in Fig. 3.1.d. The impedance matrix will now be given. The process is in this subsection the same as that in the previous subsection. The same expansion functions are used. The kernel \( K^{(2)} \) is now different and so no further simplification is possible. The validity of the present method against previous implementations of the IMM can now be made clear.

If the IMM is used in the conventional way then by following the procedure described in [84], \( E^{(\nu)}_0 \) and \( \frac{\partial E^{(\nu)}_r}{\partial n} \) in (3.22) are expanded as a linear combination of orthogonal
functions (note that (3.22) is equivalent to (1) in [84], although the notation is different). These orthogonal functions can be considered to be the wavefunctions of the modes of the corresponding guide. The impedance/admittance matrix is obtained by simple matrix manipulation. This is adequate to characterise the structure of the previous subsection. However, this is not the case for the problem of Fig. 3.1.d. Indeed, if the expansion functions correspond to the wavefunctions of the radial waveguide then they are not, any more, orthogonal on the port $C_2$ and the results are erroneous. In fact, the solution is correct only for a specific number of expansion functions. The situation is similar to the relative convergence problem which has been reported before, when mode-matching is applied on planar discontinuities. It can be attributed to the different oscillations of the testing and expansion functions (see [55] pp.284-91 for more details). Note that, by following the methodology in [84], the expansion functions are set orthogonal on $C_2$ even for the structure in Fig. 3.1.d, which is equivalent to considering a transition similar to the one depicted in Fig. 3.5 if now the radial guide has zero length. The admittance/impedance matrix is, subsequently, obtained by following techniques like modal analysis or FEM.

In contrast, the method of the present work still models the structure of Fig. 3.1.d in a single step and thus extends the capability of the IMM formulation.

### 3.3.3 Scalar formulation for a circular junction between rectangular waveguides

This is the case of Fig. 3.1.e. The problem is scalar as soon as the junction and the rectangular waveguides have the same height (the $TE_{10}$ mode is incident in the junction and the impedance matrix is calculated in terms of the $E_y$ component only). The problem is considered as a 3-port one (two rectangular waveguide ports and one radial guide port). The testing functions in every sub-region are used in the way have been described in subsection 3.3.1.
3.3.4 Vectorial formulation for a circular junction between rectangular waveguides

This is the case of Fig. 3.1.f. The height of the junction and that of the rectangular waveguides are different. As a result, an incident $TE_{10}$ mode excites the full spectrum of TE and TM modes in both the rectangular and radial guides. The problem is considered again to be a 3-port one. The formulation is given by (3.19)-(3.21) for $N = 2$. The impedance matrix is given in terms of the unknown vectors $\tilde{M}_{q_{0},r}^{(m,\nu)}$ of $\tilde{E}$, and testing vectors in every subregion are now both the TE and TM modes of the rectangular waveguide. Note that, the testing vectors are used again in the way is described in subsection 3.3.1 and thus only the kernel of the radial guide is involved in the process. A series representation similar to that described in subsection 3.3.1 is implemented for the testing wave-vectors. In the Galerkin’s algorithm $\tilde{M}_{q_{0},r}^{(0,\nu)}$ is expanded as follows:

$$\tilde{M}_{q_{0},r}^{(0,\nu)}(\phi, y) = \sum_{r=1}^{N} \tilde{M}_{q_{0},r}^{(0,\nu)}(\phi, y)$$  \hspace{1cm} (3.34)

where $\tilde{M}_{q_{0},r}^{(0,\nu)}$ is everywhere but on $S_{0}^{(r)}$ ($r = 1, \ldots, N$). On $S_{0}^{(r)}$, $\tilde{M}_{q_{0},r}^{(0,\nu)}$ is defined as:

$$\tilde{M}_{q_{0},r}^{(0,\nu)}(\phi, y) = \sum_{m=1}^{m_0} \sum_{n=1}^{n_0} c_f(q_{0,\nu}, r) g_{m}(\phi) h_{m}(y) \phi_0 + \sum_{m=1}^{m_0} \sum_{n=1}^{n_0} c_y(\nu, q_{0,\nu}, r) g_{m}(\phi) h_{m}(y) \phi_0$$  \hspace{1cm} (3.35)

where $c_f$ and $c_y$ are unknown coefficients to be found and $g_f$, $h_f$, $g_y$, $h_y$ are suitable expansion functions, i.e:

$$g_{m}(\phi) = \cos((2m - 1)\pi \phi / \phi_0)$$  \hspace{1cm} (3.36)
$$h_{m}(y) = \cos(2n\pi y / h)$$  \hspace{1cm} (3.37)
$$g_{m}(\phi) = \sin((2m - 1)\pi \phi / \phi_0)$$  \hspace{1cm} (3.38)
$$h_{m}(y) = \sin(2n\pi y / h)$$  \hspace{1cm} (3.39)

where $\phi_0$ is the angular length of $S_{0}^{(r)}$ and $h$ the height of the circular guide. Since the components of $\tilde{M}_{q_{0,m}}^{(\nu,m)}$ are coupled in the formulation, the resulting matrix is considerably larger $(2m_0(2n_0 + 1) \times 2m_0(2n_0 + 1))$ than in the scalar case. The method can still be used successfully though.
The components of the dyadic kernel of the radial guide are in a double infinite series form (due to the oscillations in the vertical direction). This degrades the speed of the method. However, it is shown in the next section that the results are accurate even if a reasonable, small number of summation terms is used so the algorithm still runs reasonably fast.

3.4 Results and Discussion

The convergence of the method depends on:

- The number of accessible modes: It should be ensured that all the necessary accessible modes have taken into account, which is equivalent to say that all the localised modes have died out before reaching any nearby obstacle. -20dB was a safe threshold in all cases.

- The number of expansion functions: If $m_0$ is the number of expansion functions
in the azimuthal direction ($\varphi$) and $a^{(0)}$ the maximum number of azimuthal oscillations reserved for the radial accessible modes then in all cases it should be the case that $m_0 > a^{(0)} + 1$. In the vertical direction ($y$), the number of expansion functions is unrelated to the number of accessible modes.

- The number of terms required in the sum of the kernel: It should, definitely, be larger than the number of expansion functions. The oscillations in the vertical direction are less than those in the azimuthal.

Generally, all these numbers are kept reasonably small, giving a high efficiency for the method.

The algorithm has been implemented on a PC Pentium 4, 2.5 GHz, 256 MB RAM. The code was written in MathCAD™ 8 Professional and the results are compared with a finite-element method (HFSS™). For all the cases, the dimensions of the rectangular waveguide are the same ($a = 2512$ um and $b = 1000$ um) and the frequency range is 65-85 GHz.

In Fig. 3.6, $\varphi_0 = 60^\circ$. The real and imaginary part of the $S_{11}$ of the dominant waveguide mode are shown. Since radial ports cannot be defined in HFSS™, a radial short circuit at a distance $r_1 = 3410$ um is realised (Fig. 3.6). 3 radial accessible modes and only 6 expansion functions have been considered. Even in this case the agreement with HFSS™ is excellent. Computational time is about 4 seconds. As has been reported in the previous subsection, IMM in the conventional [84] way may, also, be used to analyse the structure of Fig. 3.5 (3.6). The results exactly coincide for the same number of expansion functions.

Considering the structure in Fig. 3.7, the short circuit is at distance $r_1 = 4455$ um. Also, $\varphi_0 = 45^\circ$ and $\varphi_1 = 60^\circ$. The number of radial accessible modes is again 3, the number of expansion functions is 5 and only 8 oscillations have been preserved for the radial kernel. Computational time is now 7 seconds and the agreement with HFSS™ is good. Real and imaginary part of $S_{11}$ against the number of expansion functions is depicted in Fig. 3.8 and 3.9 respectively. $S_{11}$ now is obtained by applying IMM in the traditional way (subsection 3.3.2). Frequency is chosen to be 70 GHz. It is shown that the results are acceptable only when the number of expansion functions
Figure 3.7: An H-plane step transition from a rectangular to a radial waveguide.

Figure 3.8: Real part of $S_{11}$ of the $TE_{10}$ mode of the rectangular waveguide, against the number of expansion functions when IMM (Subsection 3.3.2) is applied for the structure of Fig. 3.7.
Results and Discussion

Figure 3.9: Imaginary part of $S_{11}$ of the $TE_{10}$ mode of the rectangular waveguide, against the number of expansion functions when IMM (Subsection 3.3.2) is applied for the structure of Fig. 3.7. There is not significant difference between HFSS and this method.

is set to 9.

The radius of the post shown in Fig. 3.10 and 3.11, is $r_p = 100$ um. 3 radial accessible modes have been used and 4 expansion functions. The number of the oscillations in the circular kernel is set to 10. Computational time is 5 secs. The agreement with HFSS™ is good, although HFSS™ converges rather slowly. This means that HFSS™ requires many repetitions and quite much computer sources (RAM, Hard-Disc space) to give the result for the same accuracy. HFSS™ was performed in a slower computer so time-comparison between the methods is not directly possible. For the sake of completeness it is mentioned that the simulation with HFSS™ requires 10 minutes to be executed.

The structure in Fig. 3.12 (and 3.13) is the most demanding one from the computational-time point of view. The dimensions are: $h = 6000$ um, $r_0 = 1677$ um, $r_p = 100$ um and $\varphi_0 = 97^\circ$. Four oscillations in the azimuthal and 3 in the vertical direction are kept for both the TE and TM accessible modes of the radial guide. Consequently, 28 accessible modes, in total, have been considered in the radial guide. With respect to the number of the expansion functions, $m_0$ and $n_0$, in (3.35), are set to 5 and
Results and Discussion

Figure 3.10: Magnitude of reflection coefficient of the $TE_{10}$ mode scattered by an H-plane centred post in a rectangular waveguide.

Figure 3.11: Phase of reflection coefficient of the $TE_{10}$ mode scattered by an H-plane centred post in a rectangular waveguide.
Figure 3.12: Magnitude of reflection coefficient of the $TE_{10}$ mode scattered by a circular junction with a centred post.

3 respectively. This number of $n_0$ is adequate in all cases. 12 and 8 oscillations in azimuthal and vertical direction have been preserved for the kernel representation. The computational time is about 50 secs. The agreement is good with HFSS™, but again the convergence of HFSS™ was quite slow (about an hour).

What needs to be mentioned, also, is that the number of expansion functions should have an upper limit. Beyond this limit the results are starting to become erroneous and eventually the algorithm fails (the matrix in the Galerkin’s method is singular and cannot be inverted). This is an inherent feature of the IMM and it is attributed to the special behaviour of the testing functions on the contour integrals. More specifically, due to the term $e^{j\beta z(r,\phi)}$, the testing functions tend to a delta distribution, on the contours, for increasing number of oscillations [112]. In the present work, the phenomenon is apparent, for example, for more than 33 expansion functions in the modelling of the step discontinuity. For the vectorial formulation the phenomenon is more intense (the limits are: 23, 19, 16, 12 and 12 azimuthal oscillations with 3, 4, 5, 6 and 7 vertical oscillations respectively for the structure in Fig. 3.12). In any case the phenomenon is considered of minor significance, since it is
observed well beyond the number of the expansion functions required for accurate results to be obtained.

3.5 Conclusion

A new, rigorous yet simple and versatile method for the analysis of a wide variety of microwave components has been presented using the admittance matrix formulation and the concept of indirect mode matching (IMM). The method is simpler than previous works on IMM and can be used directly for the dispersion analysis of the laminated waveguide as it is presented in the next chapter. In this chapter, the method has been numerically implemented for a number of cases, like the abrupt transition of a rectangular to a radial waveguide, centred posts in a rectangular waveguide and a circular junction. It can generally be used for any other multiport system components as long as the IMM easily applies. The method is fast and accurate and it has been compared with simulated results by a FEM (HFSS™).
Chapter 4

Formulation for the Dispersion Analysis of the Laminated Waveguide

In the present chapter, the mathematical formulation required for the dispersion analysis of the laminated waveguide is given. The method adopted is the rigorous Transverse Resonance Method. A single cell of the laminated waveguide (which is a periodic structure) is segmented into three individual parts, namely the internal and external bifurcations and the vertical post. Rigorous indirect mode matching, as described in the previous chapter, is used to characterise the post, while the bifurcations are modelled by a known procedure, namely a rigorous multimode network representation [90], [93], suitably modified for the present problem.

Concerning the latter method, it is shown here that the best approach is to use an analytic (rather than a numerical) solution of the singular integral equations involved in the method. This yields an improvement, as compared with previous implementations of the method. As a result, the laminated waveguide has, for the first time, been characterised in a rigorous manner that allows for the calculation of both bounded modes (guided waves) and unbounded modes (leaky modes, radiation field modes) with any possible polarisation.
Introduction

4.1 Introduction

The Laminated Waveguide is a promising waveguide medium in the mm-wave region. It was first discussed in [15] and, as noted in ch.1, is a natural evolution of the post-wall waveguide. The similarity between the two structures relates to the vertical-post arrangement, which is responsible for reflecting the vertical components of the electric field. However, the horizontal components of the electric field can be reflected back only in the laminated waveguide, due to its horizontal strip geometry. As a result, better field confinement is introduced, and the behaviour of the laminated waveguide is expected to resemble that of the rectangular waveguide. Previous theoretical analyses of the post-wall waveguide and waveguide-based band-gap structures [22]-[26] are not adequate to show this horizontal confinement effect, so a new dispersion analysis of the laminated waveguide is needed. The only known previous study on the laminated waveguide is that in [15], where the analysis is based on electromagnetic simulations (HFSS™) and experimental results. In that case, the HFSS™ simulation gave the insertion loss of the laminated waveguide only in the case where the incident field is the dominant mode of a rectangular waveguide.

The Laminated Waveguide can be viewed as a periodic structure. Analytical methods for periodic structures range from integral equation techniques and the Waterman method [64], to differential techniques and coupled-mode methods (see [59] for more details and references). Other work devoted solely to the analysis of periodic waveguides can be found in [113]-[115], where various boundary integral methods are used, in [116] where a spectral-domain approach is utilised, in [117] where the formulation is based on coupled integral equations, and in [118] where Hahn's method is implemented. The last of these is a promising technique based on modal field matching along transverse planes, but is restricted to periodic waveguides with constant circular cross section.

In the present work the Transverse Resonance Method [119] is chosen as the most suitable method to analyse the given waveguide. This is for the following reasons: (1) the method is rigorous and simple; (2) most importantly, it allows for the segmentation of the structure, so that subsequent variations (for example, different lengths of the strips, or changed radius of the posts, or additional lines of posts), can
be incorporated without much effort.

In this framework, a cell of the periodic waveguide is considered as a rectangular waveguide in which the partitioning walls experience a Floquet phase difference, i.e pseudoperiodicity (see Fig. 4.1).

In section 4.2, a rigorous formulation is established that covers bounded and unbounded modes. The difference between the two cases is that for bounded modes the external bifurcation is not taken into account (because the field dies out before reaching the discontinuity) and the propagation constant is real (rather than complex). Comments are also made on the calculation of ohmic losses and on precautions to be taken when a perturbation method is used. But generally the procedure appears to be straightforward and flexible.

4.2 Basic Formulation

laminated waveguide replaces a be 77 GHz and the dielectric material as glass.

The structure of Fig. 4.1 can be analysed in three parts, i.e. the internal bifurcation, the circular post region and the external bifurcation. Every part will be characterised by its equivalent multimode impedance or admittance matrix. With this scheme, care should be taken to simplify matters by taking advantage of the uniformity of the
discontinuities and thus avoiding unnecessary coupling between modes of different polarisation. More specifically:

- $TE_z$ and $TM_z$ waves are not coupled at the internal bifurcation. Also there is no coupling of modes with different $z$-oscillation.

- $TE_y$ and $TM_y$ waves are not coupled in the post region. Also there is no coupling of modes with different $y$-oscillation. Only a subregion consisting of the post and two successive strips is analysed. The impedance matrix obtained is exactly the same for all the subregions.

- $TE$ and $TM$ coupling does occur at the external bifurcation, since all six components of the electromagnetic field appear simultaneously on the discontinuity [120]. However, there is no coupling for modes with different $z$-oscillation. Note the need to take into account the interaction between modes discussed above and the continuous spectrum of standing-wave modes that surround the whole device.

Adjacent discontinuities are connected via the "accessible" modes. Since the modal bases differ from one region to another the connection between the relevant impedance or admittance matrices is provided by means of matrices representing the transformation of the accessible modes from one modal base into another one.

Summarising the above, the overall matrix representation of the cell, is as shown in Fig. 4.2. $p$ is the number of strips and $s = s_1 + s_2 + s_3$. By requiring the electric (magnetic) field to vanish on the perfect electric (perfect magnetic) wall, a non-linear eigenvalue equation is obtained for the determination of the propagation factor. The solution can be obtained by using the secant method.

4.2.1 The Internal Bifurcation

To deal with the internal and external bifurcation, the electric and magnetic fields near the discontinuity are expanded in a set of $TM_z$ and $TE_z$ modes. The modal base can be derived from the electric and magnetic vector potentials, using a method described in [34] (sec. 3-12). For the reader's convenience, this is outlined below.

The electromagnetic fields are given in terms of electric vector potential $\vec{A}$ and
Figure 4.2: Equivalent multimode network representation of a cell of the Laminated Waveguide.
magnetic vector potential \( \vec{F} \). If we take \( \vec{F} = 0 \) and \( \vec{A} = \psi \hat{z} \), where \( \psi \) is a scalar and \( \hat{z} \) the unit vector in \( z \) direction, then by Maxwell's equations, it is obtained:

\[
\vec{E} = -j \omega \mu \vec{A} + \frac{1}{j \omega \varepsilon} \nabla (\nabla \cdot \vec{A}) \quad \vec{H} = \nabla \times \vec{A}
\]

This can be expanded in rectangular coordinates as:

\[
E_x = \frac{1}{j \omega \varepsilon} \frac{\partial^2 \psi}{\partial y \partial z} \quad H_x = \frac{\partial \psi}{\partial y} \\
E_y = \frac{1}{j \omega \varepsilon} \frac{\partial^2 \psi}{\partial x \partial z} \quad H_y = -\frac{\partial \psi}{\partial z} \\
E_z = \frac{1}{j \omega \varepsilon} \left( \frac{\partial^2 \psi}{\partial z^2} + k^2 \right) \psi \quad H_z = 0
\]

where \( k^2 = \omega^2 \mu \varepsilon \). This gives rise to a transverse magnetic to \( z \) (TM\(_z\)) solution.

In the dual sense, if we choose \( \vec{A} = 0 \) and \( \vec{F} = \psi \hat{z} \) then

\[
\vec{E} = -\nabla \times \vec{F} \quad \vec{H} = -j \omega \varepsilon \vec{F} + \frac{1}{j \omega \mu} \nabla (\nabla \cdot \vec{F})
\]

Expanded in rectangular coordinates, this is

\[
E_x = -\frac{\partial \psi}{\partial y} \quad H_x = \frac{1}{j \omega \mu} \frac{\partial^2 \psi}{\partial x \partial z} \\
E_y = \frac{\partial \psi}{\partial z} \quad H_y = \frac{1}{j \omega \mu} \frac{\partial^2 \psi}{\partial y \partial z} \\
E_z = 0 \quad H_z = \frac{1}{j \omega \mu} \left( \frac{\partial^2 \psi}{\partial z^2} + k^2 \right) \psi
\]

This gives rise to a transverse electric to \( z \) (TE\(_z\)) solution.

In our problem the potentials are:

\[
\vec{F}(y, z) = \cos(m \pi y / l) e^{j \beta_n z} \hat{z} \quad \text{and} \quad \vec{A}(y, z) = \sin(m \pi y / l) e^{j \beta_n z} \hat{z}
\]

Pseudoperiodicity is introduced by the term \( e^{j \beta_n z} \). \( \beta_n = \beta_0 + 2 \pi n / d \), where \( \beta_0 \) is the propagation constant of the dominant Floquet harmonic, \( \beta_n \) is the propagation constant of the \( n \)th Floquet harmonic, \( l \) is the height of the waveguide (\( l_0 \) and \( l_p \) according to Fig. 4.1) and \( d \) is the period, i.e. the distance between successive posts. The impedance/admittance matrix is obtained in a fashion similar to that described in [93] where a rigorous multimode network representation is provided. The interested reader is advised to consult the relevant reference for more information.

Here, attention is focused on an important feature of the procedure described in this chapter. The fundamental integral equation for TM\(_z\) polarisation is given in
Figure 4.3: Cross section of the internal bifurcation discontinuity with \( p - 1 \) number of horizontal strips.

the form\(^1\):

\[
\sin(m\pi y/l_0) = \int_{L} f(y') K(y, y') \, dy'
\]  

(4.7)

where the symbol \( f \), as before, indicates a principal value integral and the integral is taken over the open arc \( L \) that is the union of all open arcs \( L_r \) connecting adjacent strips as shown in Fig. 4.3. Note that \( l_0 \) in Fig. 4.3 denotes the cell height. It is not equal to the open arc \( L \), unless zero-width horizontal strips are assumed. The kernel \( K(y, y') \) is:

\[
K(y, y') = \frac{j\omega \varepsilon}{2K^2_n} \sum_{r=0}^{P} \frac{\sin \left( \frac{\pi(y-y_r)}{l_r} \right)}{\cos \left( \frac{\pi(y'-y_r)}{l_r} \right) - \cos \left( \frac{\pi(y-y_r)}{l_r} \right)} u_r(y) u_r(y')
\]  

(4.8)

Also \( K^2_n = \omega^2 \varepsilon \mu - \beta^2_n \), \( y_r \) denotes the end points of the arcs \( L_r \), and \( u_r(y) \) is a function having the value zero everywhere but in arc \( L_r \) where it is 1 (\( y_0 = 0 \) and \( u_0(y) = 1 \) everywhere on \( L \)). \( f(y') \) is the unknown function we are looking for. Since \( K(y, y') \) is singular for \( y = y' \), (4.7) is a Cauchy singular integral equation.

Now consider the following integral:

\[
\Phi(\zeta) = \int_{L} \sum_{r=0}^{P} \frac{P_r(\zeta, y')}{Q_r(\zeta, y')} \, dy'
\]  

(4.9)

where

\[
P_r(\zeta, y') = f(y') \sin \left( \frac{\pi(y-y_r)}{l_r} \right) u_r(y') u_r(\zeta)
\]  

(4.10)

\(^1\)A similar equation applies for the \( T E_s \) case
Basic Formulation

\[
Q_r(\zeta, y') = \cos \left[ \frac{\pi (y' - y_r)}{l_r} \right] - \cos \left[ \frac{\pi (\zeta - y_r)}{l_r} \right] \quad (4.11)
\]

and \( \zeta \) is a complex variable that does not take up values on \( L \). \( P_r(\zeta, y) \) and \( Q_r(\zeta, y) \) satisfy all the conditions of analyticity stated in [121] (sec. 9.1), namely:

1. \( P_r(\zeta, y) \) and \( Q_r(\zeta, y) \) are entire\(^2\) analytic functions with respect to the complex variable \( z \) for all values \( \zeta \in L \).

2. \( P_r(\zeta, y) \) satisfies the Holder\(^3\) condition with respect to the complex variable \( \zeta \).

3. \( Q_r(\zeta, y) \) has continuous derivatives with respect to \( \zeta \), which satisfies the Holder condition.

4. At the points where \( Q_r(\zeta, y) = 0 \), \( Q'_r(\zeta, y) = 0 \), and \( Q''_r(\zeta, y) = 0 \), are distinct from zero. This condition is satisfied due to the functions \( u_r \).

Note that \( P_r \) is Holder everywhere on \( L \) with respect to the variable \( y' \), apart possibly at the ends of the arcs where an integrable singularity may be introduced by the unknown function \( f(y') \).

By the same reasoning as in [121] sec. 9.2, the integral (4.9) is given in the form:

\[
\Phi(\zeta) = \int_{L} \frac{\varphi(\zeta, y')}{y' - \zeta} \, dy' \quad (4.12)
\]

where

\[
\varphi(\zeta, y') = \sum_{r=0}^{p} \frac{P_r(\zeta, y')}{\Omega_r(\zeta, y')} \quad (4.13)
\]

where \( \Omega_r(\zeta, y') \) is an analytic function of \( \zeta \) and \( y' \) that need not be defined at this stage.

It is now obvious that (4.12) is a Cauchy-type singular integral equation. Its relation with (4.9) can be deduced directly by taking the limit of \( \zeta \) on \( L \) i.e

\[
\Phi(y)^+ + \Phi(y)^- = \int_{L} \sum_{r=0}^{p} \frac{P_r(y, y')}{Q_r(y, y')} \, dy' = c \int_{L} f(y') K(y, y') \, dy' = g(y) \quad (4.14)
\]

\(^2\)For complete definitions see [121] (Ch. 1).

\(^3\)A function \( P(t) \) satisfies the Holder condition on two points \( t_1 \) and \( t_2 \) on an arc \( L \) when \(|P(t_2) - P(t_1)| \leq M|t_2 - t_1|^\mu\), with \( M \) and \( \mu \) real positive constants. In this thesis the exponent \( \mu \) is always \( 0 < \mu \leq 1 \).
Basic Formulation

\[ \Phi(y)^+ - \Phi(y)^- = \pi j f(y) \sum_{r=0}^{P} \frac{F_r(y)}{Q_r'(y, y')} \]  

(4.15)

where

\[ F_r(y) = \sin \left[ \frac{\pi (y - y_r)}{L_r} \right] u_r(y) \]  

(4.16)

\[ Q_r'(y, y) = \frac{\partial Q(y', y)}{\partial y'} \bigg|_{y'} \]  

(4.17)

c is the coefficient \(-\frac{j \omega}{2k_c}\), and \(g(y) = c \sin(m\gamma/\lambda_0)\). \(\Phi(y)^\pm\) indicates the value of \(\Phi(\zeta)\) when \(\zeta\) moves toward \(y\) above or below the real axis in the complex plane. Equations (4.14) and (4.15) have the form of a typical Riemann-Hilbert problem. The details of its solution can be found in [121] and will not be repeated here. The solution is given in the form of a singular integral. Numerically, Cauchy-type integrals can be evaluated by partial integration. However, the author of the present thesis found that it is faster for the evaluation to be performed using equation (4.14) where \(\Phi(y)^\pm\) is approximated by the value of \(\Phi(y)\) obtained when \(y\) is a complex number with a very small positive/negative imaginary part (in practice \(\pm j \times 10^{-5}\)).

This the first time that an analytic solution has been outlined for this kind of problem. In the previous attempts, Guglielmi et al. [90],[93] implemented a numerical solution of (4.7) by means of the method of moments (MoM). However, Arnt et.al. [87] found objections on the efficiency of Guglielmi et.al. method because it proved five times slower than the Arnt et.al. generalised s-matrix technique. Although no practical comparison of techniques has been carried out here, the present analytical solution should speed up the Guglielmi et al. method and thus may eliminate this claimed drawback.

What also needs to be pointed out is that the expansion functions used in the MoM in [90] have an imposed singularity at the ends of the arcs. This singularity is chosen a priori in [90] in order to resemble the behaviour, at the metallic edges, of the true quantity under approximation (electric or magnetic field). This seems reasonable from the physical point of view. However, it is shown here that this is not justified mathematically, since by equation (4.14) it is observed that the behaviour of \(f(y')\) at the ends of the arcs is of the type \((y' - y_r)^{\pm \frac{1}{2}}\). Actually, it is known that for a homogeneous singular equation of the type (4.14) the only non-trivial solution \(\Phi(\zeta)\)
in the class $L_p$ ($p > 1$) belongs to the class $L_2$ \[122\] (p. 176). Furthermore the equation (4.14) is of the type:

$$\Phi(y)^+ + G(y)\Phi(y)^- = g(y)$$

(4.18)

where $G(y) \equiv 1$ everywhere on $L$ (including the ends), and $g(y)$ is Holder on $L$ (including the ends). If any other $G(y)$ is used in an attempt to compensate for various end conditions, then the two problems (4.14) and (4.18) are not equivalent and the choice is wrong.

Summarising, a singular integral equation of type (4.7) implies that the unknown quantity $f(y')$ has singularities, at the ends of the arc, of the type $(y' - y_\infty)^{\pm \frac{1}{2}}$. It follows that the implementation of MoM in \[90\] was inappropriate because the result converges non-uniformly to the correct end behaviour.

4.2.2 The Post Region and The External Bifurcation

The problem of modelling the post region has been tackled in detail in the previous chapter, where a new multimode indirect mode matching method has been developed. This method can be applied to characterize the post in a cell of the laminated waveguide, by noting that the modal base is $y$ oriented ($TM_y$ and $TE_y$) and that the testing functions must be Floquet harmonics moving in the $-z$ direction. Note that the vector potentials in the cylindrical region do not contain the Floquet phase term because the cylindrical waves do not experience pseudoperiodicity.

Modelling of the internal bifurcation and of the post region is adequate to determine the bounded modes of the laminated waveguide. This is because: (a) the width of the strips is considered adequately large for higher order modes to die-out before reaching the external bifurcation; and (b) the spacing between posts is adequately small for no leakage of the dominant mode $TE_{10}$ to occur. If either (a) or (b) is not true, then leakage appears in form of unbounded modes. In this case the propagation constant is complex and the external bifurcation must be included in the analysis. The external bifurcation can be modelled in a fashion similar to that described in \[7\], \[123\]. The complete modal base consists of a number of discrete modes plus a

---

4A function $\Phi(\zeta)$ belongs to the class $L_p$ if the inequality $\int_L f(y) \, dy < \infty$ holds. $L$ is a given arc.
continuous spectrum of modes that correspond to radiation. Note that the complete spectrum is responsible for the external coupling between the sides of the laminated waveguide.

In all previous treatments, all metal components have been assumed perfectly conducting. If, however, finite conductivity is considered then ohmic losses are present and the propagation constant is complex even for the bounded modes. The analysis now is much more elaborate than has been described, mainly because perturbation techniques can be used to calculate only the losses of the dominant mode. For the rest of the modes (mixed polarisation) perturbation techniques are inapplicable due to the strip edges. Mathematically, this is due to the breaking of the analyticity, in the vicinity of the edges, of the reflection coefficient of a mixed polarised wave [33] (pp.140-141). This phenomenon means that the bifurcations must be modelled by taking explicitly into account the new boundary conditions. A simplification has been developed by treating the strips as constant impedance walls [33], where the edge peculiarity is incorporated in the impedance wall modelling. Even in this case, application of the multimode network requires the propagation constants of a large number (theoretically infinite) of parallel-plate modes to be found. For the post region the method of the previous chapter again can be applied. Perturbation can be considered in this case (because no sharp edges appear), so the localised and accessible modes are those of the perfectly conducting case. However, the second Green's identity is successfully implemented, i.e. the contribution of the walls is eliminated, if testing functions are now the modes that correspond to a waveguide consisting of parallel constant-impedance walls. As a result the entries in the impedance matrix are now complex numbers (instead of pure imaginary). What is obvious, finally, is that the analysis of losses should be executed with caution and only if there is a practical interest. For example, it is impractical to find the losses for a high-order mode since its domain of appearance is of no practical interest. For the dominant $TE_{10}$ mode the perturbation can be applied even in the vicinity of the edges, if the width of the laminated waveguide is larger than its height. So the impedance matrices for the bifurcations (but not for the post) are the same in both the perfect and imperfect conducting cases. The final dispersion equation is obtained by connecting the corresponding matrices with lossy transmission lines (see. Fig. 4.2).
4.3 Conclusion

The formulation required for the dispersion analysis of the laminated waveguide has been described in detail in this chapter. For the first time, attention has been paid to having every possible polarisation of both bounded and unbounded modes analysed and formulated rigorously. The analysis is based on the Transverse Resonance Method and the technique developed in the chapter 3 has been used. Further, an improvement of the multimode network method has been obtained by implementing the analytic solution of the fundamental singular integral equation. Comments have been made on the calculation of ohmic losses and on precautions that should be taken when the perturbation method is used. Generally the procedure defined in this chapter seems to be straightforward and flexible.
Chapter 5

General Conclusions and Future Work

The achievements of the work described here have been as follows. A formulation of a dispersion analysis of the laminated waveguide has been constructed for the first time. In this framework, a new efficient method is developed to model the post discontinuity. The new method is based on the admittance matrix formulation and the concept of Indirect Mode Matching (IMM). The application of the IMM technique to the dispersion formulation of the laminated waveguide is undertaken by implementing the Transverse Resonance Method. In the process a new analytic solution is given for the modelling of the transverse bifurcation; thus the overall speed of the analysis is increased.

The future development of the work in this thesis is more or less obvious. The technique of chapter 4 must be applied in order to implement the dispersion analysis of the laminated waveguide. The analysis should show in a concise way: (1) the useful frequency region of the modes guided in the laminated waveguide; (2) the coupling mechanism between them; (3) the surface waves emerging; and (4) the radiation leakage of the unbounded modes. This information is required in the process of designing waveguide modules. This should an important step towards an in-depth understanding of the behaviour of the laminated waveguide. Other observed phenomena, like associate waves [124] and excess of ohmic losses [59], if
any, could be investigated since they tend to appear in irregular waveguides.

Implementation of the formulation in this thesis, would also allow the present model to be compared with simple models and with more complicated analysis techniques like the Finite-Element-Method (FEM). As mentioned in chapter 1, previous attempts to model the post-wall waveguide should be regarded as simplified models because no complete model of the laminated waveguide has been available in the literature. (When only the the dominant mode is guided into a laminated waveguide, this behaves as post-wall waveguide).

The FEM is used by HFSS™. Although its domain of applicability seems unlimited, it is a time and memory intensive method. So other techniques must be developed and this is the context of the work in the present thesis. Obviously, in the work described here a comparison of methods has been performed for the new IMM method of chapter 3, but not for the Transverse Resonance Method described in chapter 4.

Generally, LTCC technology and laminated waveguide modules have created great interest amongst engineers in the field. The techniques described in the present thesis can be used, with little modification, on the modelling of structures consisting of arrays of metallic rods or strips. These structures can be used to build filters, lenses and artificial dielectrics, in the millimetre and sub-millimetre ranges. The main contribution of the present thesis is exactly this: it provides the basis for tools that may make the design of LTCC modules easier. Hopefully, engineers may eventually take advantage of these tools and proceed to the design of components, such as couplers, filters, transitions and cavities, in a more efficient way.
Bibliography


Bibliography


[85] G. G. Gentili and A. Melloni, “Analysis of the X-Junction Between Two Rect-
angular Waveguides and a Circular Waveguide”, *IEEE Microwave and Guided Wave Letters*, vol. 7, no. 8, August 1997.


