Abstract

Many structures such as biological, chemical, social and data structures can be graphically represented by trees. Therefore, the concept that is represented by a tree structure may have applications in many branches of human knowledge. For example, in computer science, data structures are an important way of organising information in a computer.

A tree is a mathematical structure that can be viewed either as a graph or as a data structure. The two views are equivalent, since a tree data structure contains not only a set of elements, but also connections between the elements, resulting in a tree graph.

The first part of this research contains a description of a mathematical object that consists of an arrangement of various mathematical objects. This mathematical object is called a 'plenix' (plural 'plenices'). A plenix is like a tree structure in which every branch is a mathematical object. In other words, a plenix is a sequence of zero or more mathematical objects, where the term mathematical object is taken to include a plenix.

This definition of a plenix is similar to the definition of a list in a data structure, that is, a list is an ordered collection of values, where the same value may occur more than once. A list may have other lists as elements. Therefore, a plenix may be considered as an object that has some characteristics of a tree and some characteristics of a list.

As an example, Fig 1 represents a plenix. It consists of an arrangement of three numbers, a vector, a matrix, two sets and a Boolean entity. Actually, as illustrated by Fig 1, a plenix consists of a sequence of elements each of which consists of a sequence of elements and so on.

The above plenix may be represented as follows:
The symbols > and < are used as plenix brackets. This plenix consists of four principal elements as follows:

\[ <<<\{8, 0, 1\}, T, \{a, b\}>, \frac{3}{5}>, [4, 7, 1, 3], \sqrt{137}, \begin{bmatrix} 6 & 2 \\ 5 & 7 \end{bmatrix}, 7.4> \]

In the case of the plenix of Fig 1, the first and the third elements themselves are plenices.

The main aim of this research is to create an algebraic structure on the set of plenices and also to investigate the structure of a plenix from the point of view of pure mathematics. The Thesis is subdivided into seven chapters. Chapter one includes the introduction and the literature review. Chapters two, three and four contain the definition of an algebraic structure on the set of plenices, as well as the study of this structure. In Chapter two, the fundamental concepts of the theory of plenices are defined. Chapter three is devoted to the description of the concepts of plenix relations. A plenix may act as an operator or be an operand. A plenix may act as a function or be an argument of a function. Also, a plenix may be involved in various processes of calculus. These aspects of plenices are discussed in Chapter four.

The structure of a plenix (connection between elements) is at the very heart of the notion of a plenix. The second part of this research, in Chapters five and six, contains the study of the structure of a plenix. The structure of a plenix is called a ‘nexus’. In Chapter five, a nexus is defined as a set of sequences of numbers with some conditions. Then, the properties of this set are investigated. In Chapter six, an important type of subnexus is defined that has some resemblance to the concept of prime numbers. These subnexuses are called prime subnexuses. It is then shown that a nexus is equal to the intersection of some of its prime subnexuses. Chapter seven contains the conclusions of the work.

I am confident that the material presented in this Thesis will, in due course, find many applications in various branches of human knowledge. However, this Thesis is really a pure mathematical work and does not include any actual applications (other than a reference to the use of the concept of a plenix as a data structure in the first Chapter). The point is that to find
practical applications for a mathematical idea, in a field of knowledge, requires in-depth familiarity with that field and this is normally done by an expert in that application rather than the person who has founded the mathematical idea.

A final point that needs clarification is that there are only a few publications related to the idea of a plenix, as listed in the references at the end of the Thesis. Therefore the literature survey for this work has had a very limited scope.
I dedicate this Thesis to
my wife Behnaz,
my son Sadra
and
my parents
Ali and Malihe
for their constant support
and unconditional love
Acknowledgments

I would like to express my sincere gratitude to my PhD advisors, Professor Hoshyar Nooshin, Dr Jonathan Deane, and Dr Peter Disney for supporting me during this period. Professor Hoshyar Nooshin has helped remind me that all big questions are wrapped up in small packages and to always think small to achieve great things. Professor Nooshin’s helpful suggestions, important advice, and constant encouragement, and his logical way of thinking throughout the duration of the research have been of great value to me. The PhD thesis as such plays but a small part in the worldly knowledge and experience brought to my attention by Professor Hoshyar Nooshin. I also wish to express my appreciation to Dr Jonathan Deane, an extremely nice and helpful person in general, who gave constructive scientific advice and made valuable suggestions that improved the quality of this study. I especially thank Dr Peter Disney for his sympathetic nature, he has always been very supportive, particularly in relation to technical matters.

I will forever be thankful to my loving wife Behnaz who took over the family responsibilities and helped me concentrate on my research proposal. She supported me both mentally and physically during the course of this work. Without her help and encouragement, this study would not have been completed. I gratefully thank, my son Sadra. His positive attitude helped push me forward in my hardest moments. I love him dearly and he is my symbol of hope. My special appreciation goes to my parents to whom I am forever indebted to for their understanding, endless patience, and encouragement when it was most required.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the project.
# Theory of Plenices

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1.1 Introduction

The Space Structures Research Centre of the University of Surrey was founded by Professor Z S Makowski as a part of the Department of Civil Engineering in 1963. The aim of the Centre is to carry out research into the design and analysis of space structures. Space structures include structural forms such as single and double layer grids, barrel vaults, domes, shells and various forms of tension structures. Some examples of space structure configurations are shown in Figs 1.1 to 1.4.

The inception of the Space Structures Research Centre was in the era of the initial rapid developments of electronic digital computers. One consequence of these developments was the start of a global revolution in the matrix and computer methods of structural analysis. This may
be seen from the material presented during the First International Conference on Space Structures in September 1966, organised by the Space Structures Research Centre of the University of Surrey, see Ref 1 (The Centre has organised another four such conferences in 1975, 1984, 1993 and 2002).

A major interest of the Centre, from the early days of its inception, was the development of convenient methods for the generation and processing of information about structural configurations. The work in this area eventually resulted in the creation of an algebra, called 'formex algebra', see Refs 2 to 6.

The concepts of formex algebra are general and can be used in many different fields. In particular, the ideas may be employed for generation of geometric information about various aspects of structural systems such as element connectivity, nodal coordinates, load and support positions. The information generated may be used for different purposes such as graphic visualisation and input data for structural analysis.

The practical use of formex algebra is through a programming language called 'Formian'. The origins of Formian date back to the late seventies and the current state of the language is
A useful characteristic of formex algebra is that it allows configurations to be formulated parametrically. For instance, a formex formulation for the dome of Fig 1.1 is as follows:

\[
E = \text{genit}(1,m,1,1,0,1)|\{[1,0,0; 1,0,1],
\{[1,0,0; 1,1,1], [1,0,1; 1,1,1]\}
\]

\[
F = \text{bd}(30,360/n,360/m)|E
\]

\[
\text{Dome} = \text{pex}\{\text{rosad}(0,0,n,360/n)|F\}
\]

Fig 1.5 Dome for m= 9 and n= 3
Fig 1.6 Dome for m= 8 and n= 4
Fig 1.7 Dome for m= 7 and n= 7
Fig 1.8 Dome for m= 6 and n= 8

To understand the details of this formulation, one has to study Refs 4 to 6. However, it may be noticed that the formulation involves two parameters m and n, where m is the frequency of the subdivisions in a sector of the dome and n is the number of sectors of the dome. (See Section 2.5 of Ref 5). The configuration of Fig 1.1 corresponds to m=6 and n=6. However, another four dome configurations for some other values of m and n are shown in Figs 1.5 to 1.8. It may be seen that by using parameters, a simple formex formulation can give rise to myriads of possible configurations.
Formex algebra has proved to be an elegant and convenient tool for generating and processing geometric information. However, there is another development that can be of great help in the systematic processing of all kinds of information. This development involves the use of a mathematical object called a ‘plenix’ (The word plenices is the plural of the word plenix). A plenix is a mathematical object consisting of an arrangement of mathematical objects. For instance, a plenix may consist of an arrangement of two numbers, a vector, a matrix, three sets and a Boolean entity, as shown in Fig 1.9.

![Diagram of a plenix](image)

**Fig 1.9** A schematic representation of a plenix

The manner in which a plenix is employed in conjunction with ‘formices’ (plural of formex) is illustrated in terms of an example of data specification for structural analysis. Consider the double layer grid shown in Fig 1.10 and suppose that it is required to create a plenix that contains all the data necessary for the analysis of the grid.

To begin with, it is required to specify the coordinates of all the nodal points of the grid. Also, it is necessary to provide information about all the elements of the grid. The coordinates of all the nodes can be easily formulated and represented by a formex, say N (see Refs 4 to 6). As far as the elements are concerned, suppose that it is required to divide the elements into three groups, with each group of elements having the same cross-sectional properties. Then, each group of members can be formulated and represented by a formex (see Refs 4 to 6). For instance, let formices E1, E2 and E3 represent the three groups of elements. Each one of these formices provides a complete geometric description of the elements of a group of elements and specifies the manner in which the elements are interconnected at the nodes. The formices N, E1, E2 and E3 are placed in a plenix at positions shown in Fig 1.11.
Now, suppose that the grid has 12 support points at the corners and suppose that two of these supports are fixed supports and the rest are roller supports. The positions of the supports can be represented by formices $S_1$ and $S_2$, where $S_1$ specifies the positions of the fixed supports and $S_2$ specifies the positions of the roller supports. The formices $S_1$ and $S_2$ are placed in the data plenix at the positions shown in Fig 1.11.

It is then necessary to provide information about the loads applied to the nodal points of the grid. Suppose that there are three different types of nodal loads applied to the grid. The positions of
the nodes that are under the application of the first type of load are represented by a formex L1 (see Refs 4 to 6) and the groups of nodes under the other types of loads are represented by formices L2 and L3. The formices L1, L2 and L3 are placed in the data plenix at the positions shown in Fig 1.11.

There are four remaining items in the data plenix of Fig 1.11 that have not been explained so far. These are IA, EQ, SQ and LQ. IA is an array containing general information, including such items of information as the type of analysis that is required to be performed. EQ is an array containing information about the cross-sectional and material properties of the three groups of elements. SQ is an array containing information about the constrained degrees of freedom for the two groups of supports. Finally, LQ is an array containing information about the magnitudes of loads for the three types of loads.

The data plenix of Fig 1.11 is for a specific structure with specific arrangements of supports and loads. However, a data plenix of a similar pattern may be used for any structural system. An interface (software) may then be used to transform a data plenix into actual data suitable for various analysis and design programs. A detailed description of the idea of a data plenix is given in Ref 7.

A data plenix allows the process of data generation and organisation to be carried out in a convenient and systematic manner. In particular, the use of a data plenix is of value when the formulation of data is carried out in a parametric fashion. In this case, the problem is formulated once and then the complete data for alternative designs in automatically generated by simply choosing the required parameter values.

Plenices as data organisers have been used in dealing with space structures for a number of years. Detailed studies about plenices and their application as data organisers can be found in Refs 8 and 9.

1.2 The Aim of this Research

Mathematical modelling can be a powerful tool for understanding some aspects of phenomena which cannot be understood by verbal reasoning alone. Often when experts analyse a system to be controlled or optimised, they use a mathematical model. A mathematical model usually describes a system by a set of variables and a set of equations that establish the relationships between the variables. The variables represent some properties of the system. For mathematical modelling, one needs a mathematical object. A mathematical object is, loosely speaking,
anything one can "do mathematics on". More formally, it is an object that has a definition, obeys certain laws, and can be the target of certain operations.

The invention of the computer influenced mathematics and mathematical modelling like the other branches of human knowledge. For example, numerical methods became powerful tools for calculations in mathematics after the advent of modern computers. Also, a mathematical object that contains a collection of mathematical objects has became useful for computer modelling provided that the location of a particular piece of information can be easily found. In other words, computers need a mathematical object with an inner reference system. For example lists, two dimensional arrays and tree structures have became more and more useful for programming. However, these mathematical objects are for the storage and classification of data. But for mathematical modelling one needs a mathematical object like two dimensional arrays or tree structures with some useful operations and functions.

The aim of this research is to create a mathematical object that can represent various features of a phenomenon at the same time. This mathematical object is defined in Chapter 2 and is called a plenix. Conceptually, plenices have similarities to sets or matrices, in that plenices are collections of objects with operations, functions and relations defined on them. The graphical representation of a plenix, such as those in Fig 1.9 and 1.11, may give the impression that a plenix is similar to a tree structure or a graph. However, this similarity only relates to the 'structure' of a plenix rather than the plenix itself although a tree structure or a graph may be an element of a plenix, just as a set or a matrix can.

The basic idea of a plenix was developed in the late nineteen seventies and early nineteen eighties, as reflected in Refs 8 and 9. However, these pioneering works were mainly concerned with plenices as data structures. The purpose of the present Thesis is to extend the early ideas and to given mathematical rigour to the theory of plenices. The work has given rise to a rich algebra with extensive scope for applications in many fields of human knowledge. Also, it has been shown that this new theory has many connections with the traditional branches of mathematics.

1.3 Overview of Thesis

The Thesis is organised in seven chapters.

Chapter 1 is the introduction. It contains a brief description of the background and the aims of this research.
Chapter 1 Introduction

Chapter 2 includes the basic definitions of plenices. In this chapter the structure of a plenix is discussed. Also, an inner reference system for plenices is introduced. Furthermore, a graphical representation, that is, a dendrogram, is defined for plenices. An important notion, the address set, is defined in this chapter. The address set is an algebraic representation for the constitution of a plenix.

Furthermore, the notion of value is extended for all mathematical objects including plenix. Finally, an operation is defined for composition of two plenices and is called the duplus operation.

Chapter 3 is devoted to the description of the concepts of plenix relations. The first part of this chapter is about relationships between the panels of a plenix. In the context of the graphical representation of a plenix, a panel is a branch of its dendrogram. The second part of this chapter is about the relationships between constitutions of two plenices as well as a plenix and its substructures. In this chapter, the notion of nexus, that is, the constitution of a plenix is defined.

Chapter 4 includes the description of an aspect of major importance in theory of plenices, that is, plenix operations and functions. The first part of this chapter is about a binary operation between two plenices with different constitutions. Later, the notion of operator plenix is introduced. The term ‘operator plenix’ is used to refer to a plenix all of whose elements are operators. Examples of two operator plenices are shown in Fig 1.12

Another important notion described in Chapter 4 is a plenix function. Also, discussed in this chapter are situations when a plenix may be the argument of a function or when a plenix will act as a collection of functions.

Chapters 5 and 6 are concerned with the constitution of a plenix. In fact, the material of the Thesis may be considered to fall into two main parts. The first part consists of Chapters 2, 3 and
Chapter 1 Introduction

The first part consists of Chapters 1 and 2 and is concerned with the theory of plenices. The second part consists of Chapters 5 and 6 and includes an in-depth investigation of various interesting aspects of the constitution of plenices. This leads to a number of new concepts in plenix theory, providing an original contribution to knowledge in this field.

Chapter 7 contains a summary and conclusions of this research, followed by a number of suggestions for future work.
CHAPTER 2
Fundamentals of Plenix Algebra

2.1 Introduction

In this chapter a mathematical object that consists of an arrangement of various mathematical objects is defined. This mathematical object is called a 'plenix' (plural 'plenices'). A plenix is like a tree structure in which every branch is a mathematical object. Also, in this chapter the fundamental concepts of the theory of plenices are defined.

2.2 What Is a Plenix?

Consider the construct

\[ <2, -4.5, \text{TRUE}, \{3, 6, 9\}> \]
This is an example of a plenix. It consists of a sequence of mathematical objects that are enclosed in 'angle brackets'. The above plenix consists of numbers 2 and -4.5, the Boolean entity TRUE and the set {3, 6, 9}. In general, any mathematical object may be an element of the sequence that forms a plenix. In this context, the term 'mathematical object' is used to refer to such items as numbers, matrices, tensors, Boolean entities, sets, ... In particular, an element of a plenix may be another plenix and any element of this plenix may, in turn, be a plenix, and so on.

For example,
\[
\langle 7, \begin{bmatrix} 1 & 0 \\ 4 & 6 \end{bmatrix}, 5, 3 \rangle
\]

and
\[
\langle 8, -1, \text{TRUE}, \langle 5, \{3, 0, 2\}\rangle \rangle
\]

are plenices. In the plenix
\[
\langle 7, \begin{bmatrix} 1 & 0 \\ 4 & 6 \end{bmatrix}, 5, 3 \rangle
\]

the part
\[
\langle 5, 3 \rangle
\]
is itself a plenix. Similarly, in the plenix,
\[
\langle 8, -1, \text{TRUE}, \langle 5, \{3, 0, 2\}\rangle \rangle
\]

the part
\[
\langle \text{TRUE}, \langle 5, \{3, 0, 2\}\rangle \rangle
\]
is a plenix that contains the plenix
\[
\langle 5, \{3, 0, 2\}\rangle
\]
as one of its elements.

The term 'plenix' is derived from the Latin word 'plenus' that means full and plenices is the plural for plenix.

The essence of the above discussion may be formalised in the following definition of a plenix

Definition 2.2.1 A 'plenix' is a sequence of zero or more mathematical objects, where the term 'mathematical object' is taken to include a plenix and where, each of these mathematical objects that constitute the plenix, is referred to as a 'principal panel' of the plenix. 'Angle brackets' are used to enclose the sequence of principal panels in a plenix.

Definition 2.2.2 Amongst the set of all plenices there is a plenix that does not contain any elements. This plenix is referred to as the 'empty plenix' and is denoted by <>.
Further details regarding the concept of a plenix together with the related terminology are discussed in terms of a number of examples in the sequel.

**Example 2.2.3** The construct

\[
\langle 2, \langle 4, 5 \rangle, 1 \rangle
\]

is a plenix. This plenix has three primary elements each of which is called a 'principal panel'. In the above plenix

2,
\[
\langle 4, 5 \rangle
\]

and

1

are the principal panels. Here, the second principal panel itself is a plenix. A principal panel of a principal panel of a plenix is called a 'subsidiary panel' of the plenix. In the above example

4

and

5

are the principal panels of the plenix

\[
\langle 4, 5 \rangle
\]

and the subsidiary panels of the plenix

\[
\langle 2, \langle 4, 5 \rangle, 1 \rangle
\]

**Example 2.2.4** Consider the plenix

\[
\langle \langle 6, \langle 7, 3 \rangle \rangle, 8, \langle \rangle, 8 \rangle
\]

This plenix has four principal panels with the first and third principal panels themselves being plenices. The first principal panel, that is, the plenix

\[
\langle 6, \langle 7, 3 \rangle \rangle
\]

has, in turn, two principal panels, namely, the number

6

and the plenix

\[
\langle 7, 3 \rangle
\]

These are the principal panels of the plenix

\[
\langle 6, \langle 7, 3 \rangle \rangle
\]

and the subsidiary panels of the original plenix

\[
\langle \langle 6, \langle 7, 3 \rangle \rangle, 8, \langle \rangle, 8 \rangle
\]

The second principal panel of the plenix

\[
\langle 6, \langle 7, 3 \rangle \rangle
\]
is itself a plenix, namely,

\(<7, 3>\)

This plenix has two principal panels, namely, the numbers

7

and

3

which are subsidiary panels of the plenices

\(<6, <7, 3>>\)

and

\(<<6, <7, 3>>, 8, <>, 8>\)

**Definition 2.2.5** The term ‘panel’ is used to refer to a principal panel or a subsidiary panel of a plenix.

For instance, the plenix

\(<<6, <7, 3>>, 8, <>, 8>\)

contains the following panels:

\(<6, <7, 3>>\)

8

<>  

8  

6  

\(<7, 3>\)

7

and

3

Note that two of the above panels are the same entity, namely the integer 8. In fact, any number of panels of a plenix may be identical.

**Definition 2.2.6** The term ‘primion’ is used to refer to a mathematical object that is not a plenix and the term ‘primion panel’ is used to refer to a panel of a plenix that is not a plenix.

In the above example, the panels

6

7

3
and

8

are primion panels and the panels

<6, <7, 3>>
<7, 3>

and

<>

are ‘nonprimion’ panels.

Note that an empty plenix may appear as a panel of a plenix. A panel of a plenix that is an empty
plenix is referred to as an ‘empty panel’.

A mathematical object may appear as a primion panel of a plenix provided that it has a clear
definition of equality. The term ‘primitive panel’ is used to refer to either a primion panel or an
empty panel. The term ‘primion plenix’ is used to refer to a plenix whose primitive panels are all
primion panels. For instance, the plenix

<4, 5, <<8>, TRUE>, {2, 1}>

is a primion plenix.

Example 2.2.7 Consider the plenix

<<>, <<>, >>>

This plenix has two principal panels, namely,

<>

and

<>, <>

where the second principal panel, in turn, has two principal panels each of which is an empty
plenix. The plenix

<<>, <<>, >>>

does not contain any primion panel. A plenix of this type is referred to as a ‘blank plenix’. Every
panel of a blank plenix is either an empty plenix or a blank plenix.

2.3 Address of Panel

Definition 2.3.1 Every panel of a plenix may be associated with a sequence of integers that
indicates the position of the panel within the plenix. This sequence of integers is referred to as the
‘address’ of the panel.
For instance, consider the plenix
\[<7, 9, <5, 3, <2, 4>, o>\]

The addresses of the panels of this plenix are given in the following table.

<table>
<thead>
<tr>
<th>Panel</th>
<th>Address</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>(1)</td>
</tr>
<tr>
<td>9</td>
<td>(2)</td>
</tr>
<tr>
<td>(&lt;5, 3, &lt;2, 4&gt;, o&gt;)</td>
<td>(3)</td>
</tr>
<tr>
<td>5</td>
<td>(3, 1)</td>
</tr>
<tr>
<td>3</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>(&lt;2, 4&gt;)</td>
<td>(3, 3)</td>
</tr>
<tr>
<td>o</td>
<td>(3, 4)</td>
</tr>
<tr>
<td>2</td>
<td>(3, 3, 1)</td>
</tr>
<tr>
<td>4</td>
<td>(3, 3, 2)</td>
</tr>
</tbody>
</table>

An address \((i, j, k)\) refers to the \(k^{th}\) principal panel of the \(j^{th}\) principal panel of the \(i^{th}\) principal panel of the plenix. For example, the address of the empty panel is \((3, 4)\), referring to the \(4^{th}\) principal panel of the \(3^{rd}\) principal panel of the plenix.

An address that refers to a primitive panel of a plenix is called a ‘primitive address’. For example, in relation to the above plenix, all the addresses are primitive except for \((3)\) and \((3, 3)\).

The set of the addresses of all the panels of a plenix is called the ‘address set’ of that plenix. For instance, the set
\[\{(1), (2), (3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 3, 1), (3, 3, 2)\}\]

is the address set of the above plenix.

The address set of a plenix \(P\) is denoted by \(A_P\). The address set of the empty plenix is the empty set.
Definition 2.3.2 Let \( p \) and \( q \) be two panels of plenices \( P \) and \( Q \), respectively. Then, \( p \) and \( q \) are said to ‘correspond’ to each other provided that the address of \( p \) in \( P \) is identical to the address of \( q \) in \( Q \).

For instance, consider the plenices
\[
<4, <7, 2, <5, 1>>, 3>
\]
and
\[
<<<6, 0>>, 8>>, 9>
\]
The panels
\[
4
\]
and
\[
<7, 2, <5, 1>>
\]
in the first plenix correspond to the panels
\[
<<<6, 0>>, 8>>
\]
and
\[
9
\]
of the second plenix.

2.4 Constitution of a Plenix

Definition 2.4.1 Two plenices \( P \) and \( Q \) are said to have the same ‘constitution’ provided that \( P \) and \( Q \) have the same number of principal panels and the number of principal panels of every principal panel of \( P \) is the same as that of the corresponding principal panel of \( Q \) and so on.

For example, consider two plenices
\[
<<1, 5>>, 4, <3, 7, 2>>
\]
and
\[
<<124, 0>>, <><<1, 1, 1>>
\]
Every panel of the first plenix has a corresponding panel in the second plenix and vice versa. Therefore, these plenices have the same constitution.

The constitution of a plenix is the arrangement of its panels. This arrangement can be explicitly represented by the address set of the plenix. Therefore, two plenices have the same constitution provided that they have the same address set. For example, both the above plenices have the same address set which is
\[
\{(1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2), (3, 3)\}
\]
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This address set provides a complete description of the arrangement of the panels of the above plenices. To elaborate, this address set shows that the plenices have three principal panels the first one of which has two principal panels and the third one of which has three principal panels.

Example 2.4.2 Consider the plenices
\(<5, 1, 3, <2, 7>>, <4, 8>>\)

and
\(<5, <1, 3>, <2, 7>, <4, 8>>\)

The address sets of the first and second plenices are
\(\{(1), (2), (3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (2, 3, 1), (2, 3, 2, 1)\}\)

and
\(\{(1), (2), (3), (4), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}\)

respectively. These plenices involve the same primion panels but they do not have the same address sets. Therefore they have different constitutions.

Example 2.4.3 The plenices
\(<2>>\)

and
\(<<<2>>>\)

are of different constitutions because their address sets, namely,
\(\{(1)\}\)

and
\(\{(1), (1, 1)\}\)

are different. To elaborate, the first plenix has one principal panel, that is, 2

which is a primion panel, but the second plenix has one principal panel which is itself a plenix with one principal panel.

2.5 Equality of Plenices

Definition 2.5.1 Two plenices \(P\) and \(Q\) are said to be ‘equal’ provided that they have the same constitution and that every primion panel in \(P\) is equal to the corresponding primion panel in \(Q\) and vice versa. Equality of plenices is indicated by the usual equality symbol ‘=’.

A consequence of the above definition of equality is that if \(P\) and \(Q\) are two equal plenices then every empty panel in \(P\) corresponds to an empty panel in \(Q\) and vice versa.
Example 2.5.2 Consider the plenices
\[ <8, \frac{(2 \times 3) + 4}{2}, \triangledown, \{1, 2, 3\}, \triangledown> \]
and
\[ <2^3, 5, \triangledown, \{3, 1, 2\}, \triangledown> \]
In these plenices, the first and second principal panels are numerical, the third and the last principal panels are empty plenices and the fourth principal panel is a set. The above plenices are equal because they have the same constitution and their corresponding primion panels are equal. To wit,
\[ 8 = 2^3 \]
\[ \frac{(2 \times 3) + 4}{2} = 5 \]
and
\[ \{1, 2, 3\} = \{3, 1, 2\} \]
This equation is correct since two sets are equal provided that they have the same elements, irrespective of the order of appearance of the elements.

An alternative definition of equality for plenices may be given as follows.

Theorem 2.5.3 Let \( P \) and \( Q \) be two plenices. Then \( P \) is equal to \( Q \) if and only if they have the same number of principal panels and every principal panel in \( P \) is equal to the corresponding principal panel in \( Q \).

Proof: \((\Rightarrow)\) Suppose that \( P \) is equal to \( Q \). Therefore, \( P \) and \( Q \) have the same number of principal panels and every principal panel in \( P \) is equal to the corresponding principal panel in \( Q \).

\((\Leftarrow)\) One must show that \( P \) and \( Q \) have the same constitution and every primion panel in \( P \) is equal to the corresponding primion panel in \( Q \). By hypothesis, \( P \) and \( Q \) have the same principal panels and every principal panel in \( P \) is equal to the corresponding principal panel in \( Q \). Now, consider two cases.

Case 1: The corresponding principal panels of \( P \) and \( Q \) are primion panels. In this case, by hypothesis, every primion panel in one plenix is equal to the corresponding primion panel in the other plenix.

Case 2: The corresponding principal panel of \( P \) and \( Q \) are not primion panels, that is, plenices. In this case, by hypothesis, these principal panels have the same number of principal panels.

By this process, \( P \) and \( Q \) must have the same constitution and every primion panel in one plenix is equal to the corresponding primion panel in the other plenix. Therefore, \( P \) and \( Q \) are equal. ■
Example 2.5.4 Consider the plenices
\[ <9, \sqrt{25}, 7> \]
and
\[ < \int_0^3 x^2 \, dx, 5, \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} > \]
where
\[ \int_0^3 x^2 \, dx \]
is a definite integral and
\[ \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \]
is a determinant. The above two plenices are equal because
\[ \int_0^3 x^2 \, dx = 9, \quad \sqrt{25} = 5 \quad \text{and} \quad \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 \]

Example 2.5.5 Consider the plenices
\[ <x, <<3, 2>, y+z>, w, 1> \]
and
\[ <4, <<3, 2>, 8>, <> , 1> \]
These plenices are equal provided that
\[ x = 4, \]
\[ y + z = 8, \]
and
\[ w = <> \]

Example 2.5.6 Consider the constructs
\[ P = \langle \rangle \]
and
\[ S = \{ <> \} \]
Here, \( P \) is a plenix that has just one principal panel which is the empty set and \( S \) is a set containing a single element which is the empty plenix. Also, \( P \) and \( S \) may be panels of a plenix. For instance,
\[ <\langle <> \rangle, \{ <> \}, <\{ <> \}>, <\{ <> \}>, <\{ <> \}>> \]
is a plenix that has five principal panels each of which is equal to either \( P \) or \( S \).
2.6 The Value of a Plenix

For a long time in mathematics, the concept of 'value' was defined only for natural numbers. In more recent times, the scope of the concept of value has been extended to encompass real numbers and Boolean entities. In this work, the concept of value is further extended to apply to other mathematical objects. To begin with, the basic idea of a 'numerical value' is described. Consider the constructs

- \( IV \) (number 4 in Roman numerals),
- \( 4 \) (number 4 in Arabic numerals),
- four (number 4 in written English)
- and
- 100 (number 4 in the binary number system)

Every one of the above constructs represents the same abstract mathematical object. Namely, the unique element of the set of numbers which may be represented by 4.

Note that 4 itself is a symbol representing a 'concept' that only exists in the minds of human beings. This concept is the 'value' of 'number four'. All the above constructs represent this unique value. In fact, the concept of value for a number is unique but many different forms may be used to represent it. For instance, consider the following constructs

- \( 1 + 1 + 1 + 1 \)
- \( 2 \times 2 \)
- \( 8 \sin \left( \frac{\pi}{6} \right) \)

and

- \( \int_{0}^{2} x \, dx \)

These constructs again represent the value of number four. The simplest form for the representation of this value is the symbol 4.

However, it is important to emphasise that the symbol 4 is not the value itself but it represents the value. In other words, the value is not intrinsic to the symbol 4 but it is associated with the symbol. However, it is a long established tradition that one treats a constant number as though it is the value itself. For instance, one may say that the value of
\[ \sqrt{16} \]

is

4

whereas one really means that the value which is represented by

\[ \sqrt{16} \]

and the value which is represented by

4

are the same.

Now, consider the following sentence:

'2 is an odd number'

This is a 'proposition'. A proposition is a statement making an assertion which is either TRUE or FALSE. The state of 'truth' or 'falseness' of a proposition is considered to be the 'value' of the proposition. For example, the value of the above proposition is FALSE. However, note that similar to the case of a number discussed above, the 'values' representing the Boolean states of truth and falseness are concepts that only exist in the minds of human beings.

As another example, consider the following compound propositions

'(4 is less than 1) or (4 is an even number)'

and

'(2 is an integer) and (5 divides 30)'

The above two compound propositions have the same value, namely, TRUE.

Now, consider the ordered triple

(3, 1, 4)

This represents a 'list of numerical values' and it is logical to accept this list of numerical values as the 'value' of the above ordered triple. The most direct way of representing the value of an ordered triple is by a list of three numerical constants. For instance, one may say that the value of

(1 + 7, 2^2, \sqrt{64})

is

(8, 4, 8)

Example 2.6.1 Consider the ordered triples

(8, 4, 1)

and

(1, 8, 4)
These ordered triples consist of the same elements but the elements are ordered differently. Therefore, these ordered triples are not equal and they do not have the same value.

Note that there exists a close relationship between the definition of equality for an object and the value of the object. Indeed, two objects have the same value if they are equal. Also, an object has a clear definition of value if it has a clear definition of equality.

The idea of value for an ordered triple can be extended to any sequence of objects provided that every element of the sequence has a clear definition of equality and, therefore, all the elements have clearly defined values. Thus, the value of a sequence of objects is the sequence of the values of its elements. Indeed, two sequences of objects have the same value provided that they have the same number of elements and their corresponding elements have the same values. However, it is important to emphasise that, similar to the case of numbers and Boolean entities, the value of a sequence of objects is a concept that only exists in the minds of human beings and this is true for the value of any mathematical object.

Note that an ordered pair or an ordered triple may be interpreted as a vector. One may then say that the value of a vector is the list of the numerical values of its components.

As another example of the value of a mathematical object, consider the matrix

\[
\begin{bmatrix}
3 & 9 \\
1 & 7 \\
\end{bmatrix}
\]

This matrix is a two dimensional array of numbers. The ‘value’ of this matrix may be regarded as a 2×2 array of numerical values. In general, two matrices have the same value provided that they have the same number of rows and the same number of columns and their corresponding elements have the same numerical values. For instance, the above matrix and the matrix

\[
\begin{bmatrix}
\sqrt{9} & 3^2 \\
1 & 5+2 \\
\end{bmatrix}
\]

have the same value.

Now, consider the following set

\[
\{5, \begin{bmatrix} 8 & 0 \\ 3 & 5 \end{bmatrix}, (7, 2, 4)\}
\]

This set has three elements, namely, a number, a matrix and an ordered triple. Every element of this set has a ‘value’. To elaborate, the first element has a ‘numerical value’, the second element has a ‘matrix value’ and the last element has an ‘ordered triple value’. Therefore, one may say
that the above set represents a ‘set of values’. It is then logical to accept this set of values as the ‘value of the set’.

This way of interpreting the value of a set may be applied to any set whose elements have a clear definition of equality and, therefore, a clear definition of value. As far as the empty set is concerned, its value may be defined as the concept of a set that has no elements. Note that, the concept of the value of a set (including that of the empty set) only exists in the minds of human beings. Symbolically, the empty set is represented by {} or ∅. For instance, consider the set

\[ A = \{ x : x < -2 \text{ and } x > 7 \} \]

This set has no elements since a number cannot be both less than -2 and larger than 7. One may then say that the value of \( A \) is (the same as that of) the empty set.

Now, attention is turned to the value of a plenix. The concept of ‘value’ for a plenix is described in terms of some examples in what follows.

**Example 2.6.2** Consider the plenix

\[ <6, 10, 2.7> \]

Every panel of this plenix has a ‘value’. Therefore, the plenix represents a sequence of values. This sequence of values may be considered as the ‘value’ of the plenix. In general, the arrangement of the values of the primitive panels of a plenix is considered to be the ‘value’ of the plenix. As far as the empty plenix is concerned, its value is the concept of a plenix that has no principal panels.

**Example 2.6.3** Consider the plenices

\[ <2, <>, <1 + 1, \frac{(4 \times 2) - 6}{2} >, 9> \]

and

\[ \frac{10}{5}, <>, <2, 1>, 3^2> \]

These plenices have the same value because they have the same constitution and their corresponding primitive panels have the same values.

**Example 2.6.4** The plenices

\[ <6, 2, <8, 1>, <> \]

and

\[ <6, 2, 8, 1, <> \]
have the same primitive panels but they do not have the same value because they have different constitutions.

Example 2.6.5 If plenices
\[<a, b, c, d>\]
and
\[<9, 7, <=>, 5>\]
are known to have the same value then
\[a = 9, \quad b = 7, \quad c = <=> \quad \text{and} \quad d = 5\]

2.7 Panel Identification

Let \(P\) be a plenix with \(n\) principal panels. The principal panels of \(P\) can be denoted by

\[P_i \quad \text{or} \quad P(i) \quad \text{for} \quad i=1, 2, \ldots, n\]

That is, \(P_i\) or \(P(i)\) represents the \(i^{th}\) principal panel of \(P\). Thus, \(P\) can be written as

\[P = <P_1, P_2, \ldots, P_i, \ldots, P_n>\]

or

\[P = <P(1), P(2), \ldots, P(i), \ldots, P(n)>\]

In other words, each one of the principal panels of a plenix can be denoted using one ‘index’ after the name of the plenix. However, each one of the principal panels of \(P\) may itself be a plenix. For example suppose that \(P(i)\) is a plenix with \(m\) principal panels. In this case, each one of the principal panels of \(P(i)\) may be denoted by

\[P_{ij} \quad \text{or} \quad P(i,j) \quad \text{for} \quad j=1, 2, \ldots, m\]

That is, \(P(i, j)\) represents the \(j^{th}\) principal panel of \(P(i)\) or, what is the same thing, the \(j^{th}\) principal panel of the \(i^{th}\) principal panel of \(P\). Therefore, \(P_i\) can be written as

\[P_i = <P_{i1}, P_{i2}, \ldots, P_{ij}, \ldots, P_{in}>\]

or

\[P(i) = <P(i, 1), P(i, 2), \ldots, P(i,j), \ldots, P(i, m)>\]

Thus, each one of the principal panels of a principal panel of a plenix is denoted using two indices after the name of the plenix. This process may be continued and, therefore, each one of the principal panels of a principal panel of a principal panel of a plenix may be denoted using three indices after the name of the plenix. For instance, the \(k^{th}\) principal panel of the \(i^{th}\) principal panel of the \(j^{th}\) principal panel of \(P\) is denoted by

\[P_{ijk} \quad \text{or} \quad P(i,j,k)\]

The following diagram displays the above discussed method of representation.
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\[ P = \langle P_1, P_2, \ldots, P_n \rangle \]

\[ P_i = \langle P_{i1}, P_{i2}, \ldots, P_{in} \rangle \]

\[ P_j = \langle P_{j1}, P_{j2}, \ldots, P_{jn} \rangle \]

\[ P_{jk} = \langle P_{jk1}, P_{jk2}, \ldots, P_{jkn} \rangle \]

\[ \ldots \]

Example 2.7.1 Consider the plenix

\[ P = \langle 3, \langle 4, 5, \langle 2, 1 \rangle \rangle \rangle \]

The panels of this plenix may be represented as follows:

- \( P(1) \) denotes the first principal panel of \( P \) (that is, 3)
- \( P(2) \) denotes the second principal panel of \( P \) (that is, \( \langle 4, 5, \langle 2, 1 \rangle \rangle \))
- \( P(2,1) \) denotes the first principal panel of the second principal panel of \( P \) (that is, 4)
- \( P(2,2) \) denotes the second principal panel of the second principal panel of \( P \) (that is, 5)
- \( P(2,3) \) denotes the third principal panel of the second principal panel of \( P \) (that is, \( \langle 2, 1 \rangle \))
- \( P(2,3,1) \) denotes the first principal panel of the third principal panel of the second principal panel of \( P \) (that is, 2)
- \( P(2,3,2) \) denotes the second principal panel of the third principal panel of the second principal panel of \( P \) (that is, 1)

2.8 The Order and the Level of a Plenix

Definition 2.8.1 The number of principal panels of a plenix is called the ‘order’ of the plenix. Since the empty plenix has no principal panels, it is of order zero.

Example 2.8.2 Consider the plenix

\[ \langle \text{TRUE, FALSE, } \langle 4, 3 \rangle \rangle \]
where the first two principal panels are Boolean entities. This plenix is of order 3 and its third principal panel is of order 2.

**Definition 2.8.3** A panel of a plenix $P$ is said to be of ‘level’ one provided that it is a principal panel of $P$ and is said to be of level two provided that it is a principal panel of a principal panel of $P$ and so on. In other words, any panel of $P$ whose address has one index is of level one and any panel of $P$ whose address has two indices is of level two and so on.

**Example 2.8.4** Consider the plenix

$$P = \langle\langle 4, 3\rangle, \langle\rangle, \langle\langle 2, 0\rangle, \langle 9, 7\rangle\rangle\rangle$$

In this plenix

$$P_1 = \langle 4, 3\rangle$$

$$P_2 = \langle\rangle$$

and

$$P_3 = \langle\langle 2, 0\rangle, \langle 9, 7\rangle\rangle$$

are of level one and

$$P_{11} = 4$$

$$P_{12} = 3$$

$$P_{31} = \langle 2, 0\rangle$$

and

$$P_{32} = \langle 9, 7\rangle$$

are of level two and

$$P_{311} = 2$$

$$P_{312} = 0$$

$$P_{321} = 9$$

and

$$P_{322} = 7$$

are of level three.

The highest level of the panels of a plenix is referred to as the ‘rise’ of the plenix. The rise of the empty plenix is considered to be zero. For instance, in the above example, the rise of $P$ is three, the rise of $P_1 = \langle 4, 3\rangle$ is one, the rise of $P_2 = \langle\rangle$ is zero and the rise of $P_3 = \langle\langle 2, 0\rangle, \langle 9, 7\rangle\rangle$...
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\[ P_3 = \langle \langle 2, 0 \rangle, \langle 9, 7 \rangle \rangle \]

is two.

2.9 Finite and Infinite Plenices

A plenix may contain an infinite number of principal panels. For instance

\[ X = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \rangle \]

is a plenix with an infinite number of principal panels. A plenix such as \( X \) is referred to as a plenix of ‘infinite order’.

**Definition 2.9.1** A plenix \( P \) is said to be of infinite order provided that there is a one-to-one correspondence between the principal panels of \( P \) and the set of natural numbers, that is, \( \mathbb{N} \).

Another example of a plenix of infinite order is

\[ Y = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2^n}, \frac{1}{2^n+1}, \frac{1}{2^n+2}, \ldots \rangle \]

The \( n^{th} \) principal panel of this plenix is

\[ \langle 1, \frac{1}{2^n}, \ldots, \frac{1}{n} \rangle \]

There is a one-to-one correspondence between the principal panels of \( Y \) and the set of natural numbers \( \mathbb{N} \). So, \( Y \) is a plenix of infinite order.

**Definition 2.9.2** In contrast with a plenix of infinite order, a plenix that has a finite number of principal panels is called a plenix of ‘finite order’.

Note that a plenix of finite order may contain an infinite number of panels. For instance, consider the plenix

\[ Z = \langle X, Y \rangle \]

where \( X \) and \( Y \) are the plenices given above, that is,

\[ X = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \rangle \]

and

\[ Y = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2^n}, \frac{1}{2^n+1}, \ldots \rangle \]

The plenix \( Z \) has two principal panels, \( X \) and \( Y \) and, therefore, is a plenix of finite order but each of its principal panels has an infinite number of panels. In other words, \( Z \) has an infinite number of subsidiary panels.
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Also, consider the plenix

\[ W = <2, 9i, 9> \]

where \( 9i \) is the set of all real numbers. \( W \) has three principal panels and so it is a plenix of finite order. However, \( W \) has a principal panel \( 9i \) with an infinite number of elements.

**Example 2.9.3** Consider the following plenix of infinite order

\[ S = <S_1, S_2, \ldots, S_m, \ldots> \]

where

\[
\begin{align*}
S_1 &= 6 \\
S_2 &= <6, 6> \\
S_3 &= <6, <6, 6>> \\
S_4 &= <6, <6, <6, 6>>> \\
&\vdots \\
\end{align*}
\]

The level of all the principal panels of \( S \), namely, \( S_1, S_2, S_3, \ldots \) is one. The rise of

\[ S_2 = <6, 6> \]

is one, the rise of

\[ S_3 = <6, <6, 6>> \]

is two and so on. Since plenix \( S \) has an infinite number of principal panels, so the rise of this plenix (as well as its order) is infinite. As far as the first principal panel of \( S \) is concerned, since it is a primion (non-plenix), it does not have a rise.

A plenix of infinite rise may be formally defined as follows:

**Definition 2.9.4** A plenix \( P \) is said to be of ‘infinite rise’ provided that for any given natural number \( n \), there exists a panel of \( P \) whose level is greater than \( n \).

**Example 2.9.5** Consider the plenix

\[ T = <T_1, T_2, T_3, \ldots, T_n, \ldots> \]

where

\[
\begin{align*}
T_1 &= <5, 8> \\
T_2 &= <T_1, T_1> = <<5, 8>, <5, 8>> \\
T_3 &= <T_2, T_2> = <<<5, 8>, <5, 8>>, <<<5, 8>, <5, 8>>> \\
&\vdots \\
\end{align*}
\]
The rise of $T_1$ is one, the rise of $T_2$ is two and so on. In general, the rise of $T_n$ is $n$. Also, it is clear that for any given $n$, there will be panels of $T$ whose levels are greater than $n$. Therefore, in accordance with the above definition, the rise of $T$ is infinite.

**Example 2.9.6** Consider the plenices

\[ M = \langle 8, \langle 8, 8 \rangle, 8 \rangle \]
\[ N = \langle 8, T \rangle \]
\[ R = \langle 8, 8, 8, \ldots \rangle \]

and

\[ W = \langle 8, \langle 8, 8 \rangle, \langle 8, \langle 8, 8 \rangle \rangle, \langle 8, \langle 8, 8 \rangle \rangle, \ldots \rangle \]

where $T$ is as given in Example 2.9.5. The rise of plenix $M$ is two and its order is three. Plenix $N$ is of order two and its rise is infinite. Plenix $R$ is of infinite order but its rise is one. Finally, plenix $W$ is of infinite order and rise.

**Definition 2.9.7** A plenix whose order and/or rise are/is infinite is referred to as an ‘infinite plenix’.

The above plenices $N, R$ and $W$ are infinite plenices.

**Definition 2.9.8** A plenix whose order and rise are both finite is referred to as a ‘finite plenix’.

The following notation is adopted:

**Notation 2.9.9**

- The set of all plenices is denoted by $\mathcal{P}$.
- The set of all plenices of finite order is denoted by $\mathcal{P}_f$.
- The set of plenices of finite rise is denoted by $\mathcal{P}^f$.
- The set of all finite plenices (plenices of finite order and rise) is denoted by $\mathcal{P}_{f}$.
- The set of all plenices of order $n$ is denoted by $\mathcal{P}_n$.

**2.10 Concept of a Dendrogram**

A plenix may be represented graphically in various ways. In particular, a plenix may be graphically represented using a ‘tree-like’ diagram called a ‘dendrogram’.

**Example 2.10.1** Consider the plenix

\[ P = \langle 5, \langle 7, 2 \rangle, 8, 3 \rangle, \langle 4, 1 \rangle, 6, 1 \rangle, \langle 9, 0 \rangle \]
A dendrogram representing plenix $P$ is shown in Fig 2.10.1 As may be seen from the figure, the dendrogram contains a number of 'branches' representing the panels of $P$. The first level branches of the dendrogram represent the principal panels of $P$, the second level branches of the dendrogram represent the principal panels of the principal panels of $P$ and so on. Therefore, there exists a one-to-one correspondence between the panels of $P$ and the branches of the dendrogram of $P$. In Fig 2.10.1, the panel indicator corresponding to each branch of the dendrogram is written near it.

The address set of the above plenix $P$ is

$$A_P = \{(1), (2), (3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (2, 1, 1), (2, 1, 2), (3, 1, 1), (3, 1, 2), (3, 1, 3), (3, 2, 1), (3, 2, 2), (3, 1, 1, 1), (3, 1, 1, 2)\}$$

Note that, every address in $A_P$ corresponds to a branch of the dendrogram in Fig 2.10.1.

**Example 2.10.2** Consider plenices $S$ and $T$ of Examples 2.9.3 and 2.9.5, namely,

$$S = \langle 6, \langle 6, 6 \rangle, \langle 6, \langle 6, 6 \rangle \rangle, \langle 6, \langle 6, \rangle \rangle \rangle, ...$$

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Fig 2.10.2 Dendrogram of $S$

Fig 2.10.3 Dendrogram of $T$

and

$$T = \langle \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle 5, 8 \rangle, \langle 5, 8 \rangle \rangle, \ldots \rangle$$

These plenices are of infinite order and rise and their dendrograms are shown in Figs 2.10.2 and 2.10.3.
In Figs 2.10.1, 2.10.2 and 2.10.3 each one of the branches of the dendrograms has a panel indicator. However, the presence of these panel indicators is not essential and they may be omitted from the dendrograms. For example, the plenix \( P \) in Example 2.10.1, that is,
\[
P = <5, <<7, 2>>, 8, 3>, <<4, 1>>, 6, 1>, <9, 0>>
\]
is represented in Fig 2.10.4 by a dendrogram without panel indicators.

Example 2.10.3 Consider the plenices
\[
<5>
\]
\[
<<5>>
\]
and
\[
<<<<5>>>>
\]

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The dendrograms of these plenices are shown in Figs 2.10.5a, 2.10.5b, and 2.10.5c.

Example 2.10.4 Consider the plenices

\[ P = \emptyset \]

and

\[ Q = \langle 8, \emptyset, \langle 4, \emptyset, 1 \rangle \rangle \]

The dendrograms of these plenix are shown in Figs 2.10.6a and 2.10.6b

(a) Dendrogram of the empty plenix

(b) Dendrogram of Q

As may be seen from Fig 2.10.6, the graphical representation of the empty plenix in a dendrogram is the empty plenix itself.

2.11 Composition of Plenices

Definition 2.11.1 Suppose that,

\[ P = \langle P_1, P_2, ..., P_n \rangle \]

and

\[ Q = \langle Q_1, Q_2, ..., Q_m \rangle \]

are two plenices of finite order. The ‘composition’ of \( P \) and \( Q \) is defined as

\[ P \# Q = \langle P_1, P_2, ..., P_n \rangle \# \langle Q_1, Q_2, ..., Q_m \rangle \]

\[ = \langle P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_m \rangle \]

The symbol \# is referred to as the ‘duplus symbol’ and read as ‘duplus’. To elaborate, the composition of \( P \) and \( Q \) is defined as a plenix that consists of all the principal panels of \( P \), in the same order as in \( P \), followed by all the principal panels of \( Q \), in the same order as in \( Q \).

For instance, if

\[ P = \langle 2, \langle 3, \langle 4, 1 \rangle \rangle, \langle 5, 1 \rangle \rangle \]

and

\[ Q = \langle 8, \emptyset, \langle 4, \emptyset, 1 \rangle \rangle \]
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\[ Q = \langle\langle 2, \langle 3, 1\rangle\rangle, 4, 7\rangle \]
then
\[ P \# Q = \langle\langle 2, \langle 3, 4, 1\rangle\rangle, \langle 5, 1\rangle, \langle\langle 2, \langle 3, 1\rangle\rangle, 4, 7\rangle \]
and
\[ Q \# P = \langle\langle 2, \langle 3, 1\rangle\rangle, 4, 7, 2, \langle 3, \langle 4, 1\rangle\rangle, \langle 5, 1\rangle\rangle \]

It may be noted that, \( P \# Q \) is not equal to \( Q \# P \).

Plenix composition has the following fundamental properties:

(i) Composition of two plenices is always a plenix. That is, the set of all finite plenices is closed under plenix composition.

(ii) If plenix \( P \) is of order \( n \) and plenix \( Q \) is of order \( m \) then the plenices \( P \# Q \) and \( Q \# P \) are of the order \( n + m \).

(iii) In general, plenix composition is not commutative, that is,
\[ P \# Q \neq Q \# P \]

(iv) Plenix composition is associative, that is,
\[ (P \# Q) \# R = P \# (Q \# R) \]

(v) For any plenix \( P \)
\[ P \# \langle\rangle = \langle\rangle \# P = P \]

Theorem 2.11.2 The set of all finite plenices with plenix composition is a noncommutative 'monoid'

Proof: Recall from abstract algebra that a monoid is a set with an associative binary operation and an identity element [see Ref 10]. According to the properties of plenix composition, as listed above, plenix composition is an associative binary operation and the empty plenix, that is, \( <\rangle \), is the identity element.

2.12 Composition of Infinite Plenices

Plenix composition may be used for some infinite order plenices. For example, consider the finite plenix
\[ P = \langle 1, 0, \langle 0, 2\rangle \rangle \]
and the infinite order plenix
\[ Q = \langle 0, \langle 0, 1\rangle, \langle 0, 1, 2\rangle, \langle 0, 1, 2, 3\rangle, \ldots \rangle \]
The composition of \( P \) and \( Q \) is obtained as follows:
\[ P \# Q = \langle 1, 0, \langle 0, 2\rangle, 0, \langle 0, 1\rangle, \langle 0, 1, 2\rangle, \langle 0, 1, 2, 3\rangle, \ldots \rangle \]
However, plenix composition involving certain plenices of infinite order cannot be defined. For example, the composition
\( Q \# P \)

is not well defined because for an infinite order plenix the notion of last principal panel cannot be clearly defined. So, the above composition cannot be performed.

A composition involving infinite rise plenices of finite order is well defined. For example, consider the infinite rise plenices

\[ R = <6, S> \]

and

\[ N = <T, 8> \]

where

\[ S = <6, <6, 6>, <6, <6, 6>>, <6, <6, <6, 6>>>, ... > \]

and

\[ T = <<8, 8>, <<8, 8>, <8, 8>>, <<<8, 8>, <8, 8>>, <8, 8>>, <<<8, 8>, <8, 8>>, ... > \]

Plenices \( R \) and \( N \) are both of order two and of infinite rise. The compositions \( R \# N \) and \( N \# R \) can be carried out as follows:

\[ R \# N = <6, S> \# <T, 8> \]

\[ = <6, S, T, 8> \]

and

\[ N \# R = <T, 8> \# <6, S> \]

\[ = <T, 8, 6, S> \]

Note that the result of the composition of a finite number of plenices that involve one or more infinite rise plenices is an infinite rise plenix. Furthermore, if \( P \) is an infinite plenix and \( Q \) is any plenix such that the composition of \( P \) and \( Q \) is meaningful, then \( P \# Q \) and \( Q \# P \) are both infinite plenices.

### 2.13 Using Dendrograms for Composition of Plenices

If the dendrograms of two plenices \( P \) and \( Q \) are given, then the dendrogram of the composition of \( P \) and \( Q \) may be constructed from the dendrograms of \( P \) and \( Q \). For example, consider the dendrograms shown in Figs 2.13.1a and 2.13.1b and let these represent two plenices

\[ P = <1, <0, 2, <3, 8>> > \]

and

\[ Q = <4, <7, 9>> > \]

respectively. For clarity, the dendrogram of \( Q \) is shown by thick lines.
Fig 2.13.1 Dendrogram for composition of two plenices $P$ and $Q$

The dendrogram of the plenices

$P \# Q$

and

$Q \# P$
are shown in Figs 2.13.1c and 2.13.1d, respectively.

### 2.14 Composition of Parts of Plenices

Plenix composition may involve parts of plenices. For example consider the plenices

\( P = \langle 4, 2, <0, 1>, 7, <2, 6, 3> \rangle \)

and

\( Q = <3, <5, 1, <8, <2, 9>, 4>> \)

Then

\( P(1) \# P(1,3) = \langle 4, 2, <0, 1>, 7 \rangle \# <0, 1> = \langle 4, 2, <0, 1>, 0, 1 \rangle \)

and

\( P(3) \# Q(2,3,2) = \langle 2, 6, 3, 2, 9, 9 \rangle \)

but

\( P(2) \# Q(1) \)

or

\( P(3,1) \# Q(2,3) \)

are not considered meaningful because two primions or one primion and one plenix cannot be composed. But each one of the primion panels of a plenix is a mathematical object and so it may be involved in various mathematical operations. For example, consider the plenices

\( S = \langle \text{true}, \langle [5, 1] \rangle, 3, \{4, 8, 9\} \rangle \)

and

\( T = \langle [3, 1], <4, \{2, 4\}> \rangle, \text{false} \rangle \)

Some mathematical operations between primion panels of plenices \( S \) and \( T \) are given below:

\( S(2,2) \times T(2,1) = 3 \times 4 = 12, \)

\( S(2,1) + T(1) = \begin{bmatrix} 5 & 1 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix}, \)

\( S(3) \cap T(2,2) = \{4, 8, 9\} \cap \{2, 4\} = \{4\}, \)

\( S(1) \lor T(3) = \text{true}, \)

and

\( T(3) \land S(1) = \text{false} \)

However,

\( S(3) \lor T(3) \)

or
$S(2,1) + T(2,1)$

do not represent meaningful operations.
CHAPTER 3

Relations Between Plenices

3.1 Introduction

This chapter is devoted to the description of the concepts of plenix relations. However, to begin with it is necessary to introduce the concept of the image of a panel of a plenix.

3.2 The Concept of Image

Definition. 3.2.1 Let $P$ and $Q$ be any two plenices and let $p^*$ and $q^*$ denote any two panels of $P$ and $Q$, respectively. The panel $p^*$ is said to be a 'full image' of $q^*$ if and only if $p^*$ and $q^*$ have identical addresses in $P$ and $Q$. If $p^*$ is a full image of $q^*$, then $q^*$ is also a full image of $p^*$, that is, the relationship of being full images of each other is mutual. With the same assumptions as above, if $p^*$ is itself a plenix and $p^{**}$ denotes a panel of $p^*$, then $p^{**}$ is said to be a 'partial
image* of $q^{*}$ and $q^{*}$ is said to be a ‘hyper image*’ of $p^{**}$. In other words, suppose that, the address of $p^{*}$ is $(i_{1}, i_{2}, \ldots, i_{n})$ where $i_{j}$ is a positive integer for $j = 1, 2, \ldots, n$. Then a panel in $Q$ is a partial image of $p^{*}$ provided that its address has more than $n$ items with the first $n$ terms of the address being $i_{1}, i_{2}, \ldots, i_{n}$.

For example, consider the plenices

$P = \langle 1, 2, \langle 4, 9 \rangle, 0 \rangle$

and

$Q = \langle 7, 3, \langle 8, 5 \rangle, \langle 2, \langle 4, 6 \rangle \rangle \rangle$

with their dendrograms shown in Fig 3.2.1. Then, the following panels of $P$ and $Q$ are full images of each other:

$P(1) = 1$ and $Q(1) = 7$

$P(2) = 2$ and $Q(2) = 3$

$P(3) = \langle 4, 9 \rangle$ and $Q(3) = \langle 8, 5 \rangle$

$P(4) = 0$ and $Q(4) = \langle 2, \langle 4, 6 \rangle \rangle$

$P(3, 1) = 4$ and $Q(3, 1) = 8$

$P(3, 2) = 9$ and $Q(3, 2) = 5$

Also,

$P(4) = 0$

is the hyper image of

$Q(4, 1) = 2$

$Q(4, 2) = \langle 4, 6 \rangle$

$Q(4, 2, 1) = 4$

and

$Q(4, 2, 2) = 6$
Furthermore,
\[ \mathcal{Q}(4, 1) = 2 \]
\[ \mathcal{Q}(4, 2) = <4, 6> \]
\[ \mathcal{Q}(4, 2, 1) = 4 \]

and
\[ \mathcal{Q}(4, 2, 2) = 6 \]
are partial images of
\[ \mathcal{P}(4) = 0 \]

Some relationships between two plenices are defined in the sequel.

### 3.3 Pernate Plenices

**Definition 3.3.1** Consider two finite level plenices \( P \) and \( Q \). Let every primion panel \( p \) of \( P \) have one or more primion panels of \( Q \) as images (full image, partial images or hyper image) and vice versa. Then, the plenix \( P \) is said to be a ‘pernate’ of \( Q \) and the plenix \( Q \) is said to be a pernate of \( P \).

Practically, to establish the relationship of being pernates, every primion panel \( p \) of \( P \) should be examined with respect to the primion panels of \( Q \) and every primion panel of \( Q \) which is a full image, partial image or hyper image of \( p \) be marked off. This process should be carried out for all the primion panels of \( P \). If in this process every primion panel of \( Q \) is marked off one or more times then the plenix \( P \) is a pernate of \( Q \) or vice versa, namely, take the panels of \( Q \) and compare them with \( P \).

For example, consider two plenices

\[ P = <1, 3, 5>, <7, 2> \]

and

\[ Q = <4, <8, <6, 0, 9>> \]

The dendrograms of the plenices \( P \) and \( Q \) are shown in Fig 3.3.1.

The plenices \( P \) and \( Q \) are pernates of each other because the primion panels

\[ P(1, 1) = 1 \]
\[ P(1, 2) = 3 \]

and

\[ P(1, 3) = 5 \]

have a hyper image in \( Q \), that is,

\[ Q(1) = 4 \]
Also,

\[ P(2, 1) = 7 \]

has a full image in \( Q \), that is,

\[ Q(2, 1) = 8 \]

Furthermore,

\[ P(2, 2) = 2 \]

has three partial images in \( Q \), that are,

\[ Q(2, 2, 1) = 6 \]
\[ Q(2, 2, 2) = 0 \]

and

\[ Q(2, 2, 3) = 9 \]

Therefore, \( P \) is a pernate of \( Q \).

As another example, consider two plenices

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\[ R = \langle 5, 8 \rangle, \langle 4, 3, 7 \rangle \]

and

\[ S = \langle 0, 9 \rangle, \langle 4, 2 \rangle \]

The dendrograms for \( R \) and \( S \) are shown in Fig 3.3.2.

The plenices \( R \) and \( S \) are not pernates of each other, because the primion panel

\[ R(2, 3) = 7 \]

has no primion panel in \( S \) as an image.

It should be noted that the pernate relationship is not transitive. That is, if \( P \) is a pernate of \( Q \) and \( Q \) is a pernate of \( T \), it does not necessarily follow that \( P \) is a pernate of \( T \).

\[ P = \langle 4, 1, 5 \rangle \]
\[ Q = \langle 8, 3 \rangle \]
\[ T = \langle 6, 0, 2, 7 \rangle \]

To show this, consider three plenices

The plenix \( P \) is a pernate of \( Q \) and the plenix \( Q \) is a pernate of \( T \) but the plenix \( P \) is not a pernate of \( T \). The dendrograms for \( P, Q \) and \( T \) are shown in Fig 3.3.3

An alternative definition of two plenices that are pernates of each other may be given as follows:

Two finite level plenices \( P \) and \( Q \) are pernates of each other provided that they have the same numbers of principal panels, that is, they are of the same order and every two nonprimion panels in \( P \) and \( Q \) that are full images of each other have the same orders.

This definition of two plenices that are pernates of each others may be proved to be equivalent to the initial definition as follows:
**Theorem 3.3.2** Two finite level plenices $P$ and $Q$ are pernates of each other if and only if they have the same number of principal panels, that is, they are of the same order, and every two nonprimion panels in $P$ and $Q$ that are full images of each other have the same order.

**Proof:** One may first prove that if $P$ and $Q$ are pernates of each other in accordance with Definition 3.3.1 then $P$ and $Q$ also are pernates of each other in accordance with the alternative definition. One may do this by assuming that $P$ and $Q$ are not pernates of each other in accordance with the alternative definition and by using this assumption to show that $P$ and $Q$ are not pernates of each other in accordance with Definition 3.3.1 (an implication and its contrapositive are equivalent). To do this, suppose that $p^*$ and $q^*$ are two nonprimion panels of $P$ and $Q$ that are full images of each other and they are of different orders $m$ and $n$, respectively. Without loss of generality, one can suppose that $m$ is greater than $n$. Therefore,

$$ p^{**} = p^*(n + k) \quad \text{where} \quad 0 < k < m - n \quad \text{and} \quad k \in \mathbb{N} $$

is a principal panel of $p^*$ that has no full image in $Q$. For simplicity, let $p^*$ be a nonprimion panel of $P$ of order three and $q^*$ be a nonprimion panel of $Q$ of order two and also $p^{**}$ be a primion panel of $p^*$ (also $P$). The dendrograms for panels $p^*$ and $p^{**}$ of $P$ and $q^*$ of $Q$ are shown in Fig 3.3.4.

![Dendrograms for panels $p^*$ and $p^{**}$ of $P$ and $q^*$ of $Q$](image)

The panels $p^*(1)$ and $p^*(2)$ are full images of $q^*(1)$ and $q^*(2)$ respectively and $p^{**}$ has no full image in $Q$. Also $q^*$ is a hyper image of $p^{**}$. Since $p^{**}$ has no primion panel in $Q$ as a full image so $p^{**}$ has no primion panel in $Q$ as a partial image. Also, since $q^*$ is a nonprimion panel of $Q$ therefore, $p^{**}$ has no primion panel in $Q$ as an image which means that $P$ is not a pernate of $Q$.

It should be noted that if $p^{**}$ is a nonprimion panel of $P$ then, since $P$ is a finite level plenix, so there exists one primion panel of $p^{**}$, namely, $p^{***}$ that has no primion panel in $Q$ as an image because $p^{**}$ has no primion panel in $Q$ as an image. The dendrograms for the panels $p^*$, $p^{**}$ and $p^{***}$ of $P$ and $q^*$ of $Q$ are shown in Fig 3.3.5. In this figure a dashed line means that $p^{**}$ may contain other panels.
The proof is completed by showing that if $P$ and $Q$ are pernates of each other in accordance with the alternative definition then $P$ and $Q$ are pernates of each other in accordance with Definition 3.3.1. The strategy of proof here is to assume that $P$ is not a pernate of $Q$ in accordance with Definition 3.3.1 and to deduce a contradiction from this assumption. Let's assume that $P$ is not a pernate of $Q$ in accordance with Definition 3.3.1. Therefore, there exists at least one primion panel, namely

$$p^* = P(i_1, i_2, \ldots, i_n)$$

where $i_1, i_2, \ldots, i_n \in \mathbb{N}$, of $P$ that has no primion panels in $Q$ as images. In particular $p^*$ has no primion panel in $Q$ as a hyper image. Therefore, $Q$ has the panels

$$Q(i_1, i_2, \ldots, i_{n-1})$$

$$Q(i_1, i_2, \ldots, i_{n-2})$$

$$\vdots$$

$$Q(i_1)$$

This means that the order of $P$ is at least one more than the order of $Q$. This contradicts the fact that the orders of $P$ and $Q$ are the same.

**Definition 3.3.3** Two plenices are said to be ‘abnates’ of each other provided that they are not pernates.
3.4 Subnate and Supernate Plenices

**Definition 3.4.1** Let $R$ and $S$ be two plenices. The plenix $R$ is said to be a 'subnate' of $S$ provided that $R$ and $S$ have the same number of principal panels (that is, they are of the same order) and every nonprimitive panel of order $n$ of $R$ has a nonprimitive panel of order $n$ of $S$ as full image. If $R$ is a subnate of $S$ then $S$ is said to be a 'supernate' of $R$. The fact that $R$ is a subnate of $S$ is denoted by $R \subset S$ and the fact that $S$ is a supernate of $R$ is denoted by $S \supset R$. Note that the symbols for indicating the plenix relationships of being subnates or supernates of each other are the same as the symbols for the set relationships of being subsets or supersets of each other.

As an example, consider two plenices

$$R = \langle 3, <5, 2 \rangle$$

and

$$S = \langle <8, <> , <4, <1, 6 > \rangle$$

The plenix $R$ is a subnate of $S$. Also, the plenix $S$ is a supernate of $R$. The dendrograms for plenices $R$ and $S$ are shown in Fig 3.4.1.

Note that, according to the definition of subnate/supernate relation, a plenix $P$ is both a subnate and a supernate of itself.

![Fig 3.4.1 Dendrograms for plenices $R$ and $S$](image)

It is an interesting fact that if the dendrogram of $S$ is put on the dendrogram of $R$ then all the branches of the dendrogram of $R$ are covered by the branches of the dendrogram of $S$. This shows that every panel of $R$ has a full image in $S$. However, this does not imply that if all the panels of $R$ have full images in $S$ then $R$ is a subnate of $S$. In terms of the address sets of $R$ and $S$, that is, $A_R$ and $A_S$, the above fact may be stated as follows: If $R$ is a subnate of $S$ then $A_R$ is a subset of $A_S$. However, this does not imply that if $A_R$ is a subset of $A_S$ then $R$ is a subnate of $S$. 

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To illustrate this, consider the plenices

\[ P = \langle 2, 6, 0 \rangle \]

and

\[ Q = \langle 3, 7, 1, \langle\rangle, 8 \rangle \]

with their address sets given by

\[ A_P = \{(1), (2), (3)\} \]

and

\[ A_Q = \{(1), (2), (3), (4), (5)\} \]

The dendrograms of \( P \) and \( Q \) are shown in Fig 3.4.2. It is seen that plenix \( P \) is not a subnate of \( Q \), because they are not of the same order, in spite of the fact that \( A_P \subset A_Q \) and that the dendrogram of \( Q \) can cover the dendrogram of \( P \).

Let \( P \) be a plenix. Then the subnate (supernate) consisting of \( P \) itself is said to be the ‘improper subnate (supernate)’ of \( P \) and all the other subnates (supernates) of \( P \) are said to be ‘proper subnates (supernates)’ of \( P \). The fact that \( Q \) is a proper subnate (supernate) of \( P \) may be denoted by \( Q \subset P \) (\( Q \supseteq P \)). The empty plenix as a subnate of \( P \) is said to be the ‘trivial subnate’ of \( P \). All other subnates are ‘nontrivial’.

It should be noted that if \( R \) is a subnate of \( S \) then \( R \) and \( S \) have the same order and every nonprimion panel in \( R \) has a nonprimion panel in \( S \) as a full image and these nonprimion panels in \( R \) and \( S \) are of the same orders. This implies that if \( R \) is a subnate of \( S \) then \( R \) and \( S \) are pernates of each other but the converse is not true.

For instance, suppose that

\[ R = \langle 2, \langle 7, 5 \rangle, 0 \rangle \]

and

\[ S = \langle \langle 6, 4 \rangle, 9 \rangle \]
are two plenices. The dendrograms of $R$ and $S$ are shown in Fig 3.4.3. The plenices $R$ and $S$ are pernates of each other. However, neither of them is a subnate of the other.

Now, let $r^*$, $s^*$ and $t^*$ be panels of plenices $R$, $S$ and $T$, respectively. Also, let $r^*$ be a full image of $s^*$ and $s^*$ be a full image of $t^*$. Therefore, $r^*$, $s^*$ and $t^*$ have the same address. This implies that $r^*$ is a full image of $t^*$. Considering this fact together with the definition of subnate, it is easy to see that the relationship of being a subnate (supernate) is transitive. In other words, if $R$ is a subnate (supernate) of $S$ and $S$ is a subnate (supernate) of $T$ then $R$ is a subnate (supernate) of $T$.

![Fig 3.4.3 Dendrograms of plenices $R$ and $S$.](image)

**Theorem 3.4.2** Let $P$ be a finite plenix and let $R$ and $S$ be two subnates of $P$. Then $R$ and $S$ are pernates of each other.

**Proof:** Since $R$ and $S$ are subnates of $P$, so, they have the same order as $P$. Now, suppose that $r$ and $s$ are full image nonprimion panels in $R$ and $S$, respectively. $R$ is a subnate of $P$. So, $r$ has a full image nonprimion panel $p$ in $P$. Furthermore, the orders of $r$ and $p$ are the same. Since $s$ is a full image of $r$ and $r$ is a full image of $p$, so, $s$ is a full image of $p$. By hypothesis, $S$ is a subnate of $P$. Therefore, $s$ and $p$ have the same order. Thus, $r$ and $s$ have the same order. Consequently, by Theorem 3.3.3, $R$ and $S$ are pernates of each other. ■

**Theorem 3.4.3:** Suppose that, $P$ and $Q$ are two finite plenices and they are pernates of each other. Also, suppose that $R$ and $S$ are subnates of $P$ and $Q$, respectively. Then $R$ and $S$ are pernates of each other.

**Proof:** Since, $P$ and $Q$ are pernates of each other and $R$ and $S$ are subnates of $P$ and $Q$, $R$ and $S$ have the same order. Now, one must show that if $r$ and $s$ are two nonprimion full image panels in $R$ and $S$, respectively, then $r$ and $s$ have the same order. To do this, suppose that $r$ and $s$ have order $m$ and $n$ respectively. Since $R$ is a subnate of $P$, $P$ has a nonprimion panel $p$ of order $m$, that is a full image of $r$. By the same argument, $Q$ has a nonprimion panel $q$ of order $n$, that is a full
image of $s$. By hypothesis, $P$ and $Q$ are pernates of each other. This means that $m$ must be equal to $n$. Consequently, $R$ and $S$ are pernates of each other.

### 3.5 Cognate Plenices

**Definition.** 3.5.1 Two plenices $P$ and $Q$ are said to be ‘cognates’ of each other provided that every panel of $P$ has a full image in $Q$ and every panel of $Q$ has a full image in $P$. In other words, two plenices $P$ and $Q$ are said to be ‘cognates’ of each other if and only if they are of the same constitution and this, in turn, means that they have the same address set. The relationship of being cognates is mutual. That is, if $P$ is a cognate of $Q$ then $Q$ is a cognate of $P$

**Theorem 3.5.2** Two plenices $R$ and $S$ are cognates of each other provided that $R$ is a subnate of $S$ and $S$ is a subnate of $R$.

**Proof:** Suppose that $R$ and $S$ are cognates of each other. So, they have the same constitution. Therefore, every nonprimitive panel of order $n$ of $R$ has a nonprimitive panel of order $n$ of $S$ as full image and vice versa. Thus, $R$ is a subnate of $S$ and $S$ is a subnate of $R$.

Now, assume that $R$ is a subnate of $S$ and $S$ is a subnate of $R$. Therefore, $A_R$ is a subset of $A_S$ and $A_S$ is a subset of $A_R$. This means that $A_R = A_S$. Consequently, the plenices $R$ and $S$ have the same constitution. In other words, $R$ and $S$ are cognates of each other.

It should be noted that the relationships introduced in the above sections, namely, the relationships of being pernates, abnates, cognates and subnate/supernate relate to the constitutional aspects of plenices, irrespective of the values of their primitive panels.

### 3.6. Equivalence Classes of Plenices

From the discussion in the previous section it may be said that the relationship of being cognates is:

- reflexive, that is, $P$ is a cognate of $P$,
- symmetric, that is, if $P$ is a cognate of $Q$ then $Q$ is a cognate of $P$ and
- transitive, that is, if $P$ is a cognate of $Q$ and $Q$ is a cognate of $R$, then $P$ is a cognate of $R$.

The above three properties imply that the relationship of being cognates is an ‘equivalence relation’ on the set of all plenices $P$.

If an equivalence relation is defined on a set, then the set can be partitioned into subsets by the condition that two elements of the set belong to the same subset if and only if they are equivalent. These non-intersecting subsets are referred to as ‘equivalence classes’.

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If \( P \) is a plenix then the subset of \( P \) (set of all plenices) defined by
\[
\{ T : T \text{ is a cognate of } P \}
\]
is called the ‘equivalence class’ of \( P \). The equivalence class of \( P \) is denoted by \([P]\). The equivalence class of \( P \) is the set of all plenices which are ‘equivalent’ to \( P \), that is, the set of all plenices which are cognates of \( P \). If \( P \) is equivalent to \( Q \) then this relationship is denoted by
\[
P \sim Q
\]
where \( \sim \) is referred to as the ‘equivalence symbol’. Two equivalent plenices are of the same constitution. For example, consider the plenix
\[
P = \langle 4, <7, 2>, <3, <4, 6> \rangle
\]
The equivalence class of \( P \), that is, \([P]\) is the set containing all the plenices that have the same constitution as \( P \). Included in this set are plenices
\[
Q = \langle 12, <TRUE, 8>, <FALSE, \sqrt{2}, 5> \rangle
\]
\[
R = \langle \frac{3}{4}, \{1, 4\}, 15 \rangle, <0, \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}, 6 \rangle
\]
and
\[
S = \langle \langle \rangle, \langle \langle \rangle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle \rangle
\]
where \( S \) is a blank plenix. It is clear that the equivalence class of \( P \) consists of an infinite number of plenices. The equivalence class of \( P \) may be denoted by \([P]\) or by the equivalence class of any one of the plenices that is a cognate of \( P \). For example, referring to the above example, the equivalence class of \( P \), that is, \([P]\) may also be represented by \([Q]\), \([R]\) or \([S]\).

Now, consider the equivalence class of \( \langle \rangle \), that is,
\[
[\langle \rangle]
\]
This equivalence class consists of precisely one plenix, that is, the empty plenix.

### 3.7 Quotient Spaces of Plenices

The quotient space of the set of all finite order plenices is discussed in this section, but first the concept of quotient space will be reviewed. Let \( S \) be a set for which an equivalence relation is defined and let \( S \) be divided into equivalence classes. If an operation is defined for elements of \( S \), then it may be possible to define this operation for the equivalence classes in such a way that the set of equivalence classes becomes a space (ie, a set of mathematical objects that satisfy a number of postulates) of the same type as \( S \). In this case, the set of equivalence classes is said to be a ‘quotient space’ or ‘factor space’ of \( S \).
Lemma 3.7.1 Let $P$, $Q$, $R$ and $S$ be four plenices and let $P$ be a cognate of $R$ and $Q$ be a cognate of $S$. Then

$$[P \# Q] = [R \# S]$$

Proof: Suppose that, the orders of $P$ and $Q$ are $m$ and $n$ respectively. So, the orders of $R$ and $S$ are also $m$ and $n$ respectively. Therefore, the orders of $P \# Q$ and $R \# S$ are equal. By definition of duplus operation, every nonprimion panel with $k$ principal panels in $P \# Q$ has a corresponding nonprimion panel with $k$ principal panels in $R \# S$ and vice versa. Therefore, $P \# Q$ is a subnate of $R \# S$ and $R \# S$ is a subnate of $P \# Q$. Consequently, $P \# Q$ and $R \# S$ have the same constitution. That is, $[P \# Q] = [R \# S]$. ■

Theorem 3.7.2 Consider the set of all equivalence classes in $P_f$. A binary operation is defined for these equivalent classes, namely, the operation $\#^*$

$$[P] \ #^* \ [Q] = [P \ # \ Q]$$

where $\#$ is the usual operator for the composition of plenices. The set of all equivalence classes in $P_f$ with the binary operation $\#^*$ is a monoid.

Proof: By Lemma 3.7.1, the operation $\#^*$ is well defined, that is, the operation does not depend on the particular choice of representatives. Now, suppose that $P$, $Q$ and $R$ are three plenices. Then

$$([P] \ #^* \ [Q]) \ #^* \ [R] = ([P \ # \ Q]) \ #^* \ [R]$$

$$= ([P \ # \ Q] \ # \ R)$$

Since the duplus operation is associative, so,

$$[(P \ # \ Q) \ # \ R] = [P \ # (Q \ # \ R)]$$

$$= [P] \ #^* \ [Q \ # \ R]$$

$$= [P] \ #^* ([Q] \ #^* \ [R])$$

Therefore,

$$([P] \ #^* \ [Q]) \ #^* \ [R] = [P] \ #^* ([Q] \ #^* \ [R])$$

Consequently, the operation $\#^*$ is associative. It is easy to see that the equivalence class of $\lhd$, that is, $[\lhd]$, is the identity element for this monoid. ■

Fundamentally, the operations $\#$ and $\#^*$ are not identical but, it is convenient to use the same symbol $\#$ for both of the operations and this convention will be used henceforth.

3.8 The Concept of Nexus

The equivalence class of $P$, that is $[P]$, may be viewed in two different ways. Firstly, it may be viewed as a set that contains all the plenices whose constitution is the same as that of $P$, as
discussed above. Alternatively, \([P]\) may be viewed as a mathematical object that represents the constitution of \(P\). Such a mathematical object is called a ‘nexus’. The nexus of \(P\) (that is, \([P]\)) represents the constitution of \(P\) as well as that of all the plenices that are cognates of \(P\). On the other hand, the most direct representation of the constitution of a plenix is the address set of the plenix. Therefore, the ‘value’ of a nexus \([P]\) is the address set of \(P\). Or, more accurately, the value of a nexus \([P]\) is the same as that of the address set of \(P\). The address set of a nexus \([P]\) is denoted by \(A[P]\). However, note that the address set of a nexus \([P]\) is the same as that of plenix \(P\). That is, \(A[P] = A_P\).

Nexuses are mathematical objects that represent plenix constitutions. Correspondingly, all the concepts and terminology that relate to the constitutional and relational aspects of plenices can also be used for nexuses with obvious implications. For example, one may say:

- nexuses \([P]\) and \([Q]\) are permates of each other,
- nexus \([R]\) is of order 5,
- plenix \(T\) is a subnate of nexus \([S]\),
- a panel of a nexus is the full image of a panel of a plenix,

and so on.

If \(P\) and \(Q\) are cognates of each other then \([P]\) and \([Q]\) are equal. This is denoted by \([P] = [Q]\). However, strictly speaking, it will not be incorrect to say that \([P]\) and \([Q]\) are cognates of each other.

Now, consider the plenix

\[T = <9, <0, S>, <<>, <7, 8>>\]

![Fig 3.8.1 Dendrogram of \([T]\) that is, the dendrogram of the nexus of \(T\)](image)

Fig 3.8.1 shows a configuration which is similar to the dendrogram of \(T\) with all the values of the primitive panels removed from it. This configuration is a graphical representation of the nexus of \(T\). This configuration is a graphical representation of the nexus of \(T\).
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7, that is, \([T]\). Also, one may refer to this configuration as the ‘dendrogram of \([T]\)’. There is a one-to-one correspondence between the address set of a nexus and the branches of its dendrogram. A branch of a dendrogram that corresponds to a primitive address is called a ‘primitive branch’.

Another way of representing the nexus of \(T\) is:

\[ [T] = <\square, <<\square, \square>>, \square, <\square, \square>> \]

where ‘\(\square\)’ is called the ‘porta’ symbol. A porta symbol represents a primitive address, as also a primitive branch of a dendrogram.

The nexus that represents the constitution of the empty plenix is called the ‘empty nexus’. The empty nexus is denoted by \([<>]\).

Two useful operations between nexuses are the operations of ‘union’ and ‘intersection’. These operations will be discussed in what follows.

3.9 Union of Nexuses

The concept of union of nexuses is discussed in this section. To provide a feel for this concept, one may imagine that the dendrograms of two nexuses are placed on top of each other with the result being the dendrogram of the union of the nexuses. The union of two nexuses \([R]\) and \([S]\) is denoted by \([R] \cup [S]\). For example, consider two nexuses

\[ [R] = <<\square, \square, \square>>, \square \]

and

\[ [S] = <\square, <\square, \square>> \]

The dendrograms of \([R]\), \([S]\) and \([R] \cup [S]\) are shown in Fig 3.9.1.

Therefore, the nexus \([R] \cup [S]\) is

\[ [R] \cup [S] = <<\square, \square, \square>>, \square \cup <\square, <\square, \square>> \]

\[ = <<\square, \square, \square>>, <\square, \square>> \]

**Definition 3.9.1** The union of \([R]\) and \([S]\), that is, \([R] \cup [S]\) is a nexus where all the panels of \([R]\) and \([S]\) have full images in \([R] \cup [S]\) and all the panels of \([R] \cup [S]\) have full images in \([R]\) or \([S]\) or both.

**Lemma 3.9.2** Let \([R]\) and \([S]\) be two nexuses. Then

\[ A_{[R] \cup [S]} = A_{[R]} \cup A_{[S]} \]
That is, the address set of the union of two nexuses is the union of their address sets.

Overlapping of $[R]$ and $[S]$
(For clarity, the dendrograms of $[R]$ and $[S]$ are shown by different line styles.)

$[R] \cup [S]$
(Two overlapping branches are considered as a single branch.)

**Proof:** Suppose that, $p$ is a panel in $[R] \cup [S]$ and $a$ is the address of $p$ in $A_{[R] \cup [S]}$, that is, $a \in A_{[R] \cup [S]}$. By Definition 3.9.1, $P$ has full image in $R$ or $S$. Therefore, $A$ is in $A_{[R]}$ or $A_{[S]}$. Thus, $a \in A_{[R]} \cup A_{[S]}$. This means that,

$$A_{[R] \cup [S]} \subseteq A_{[R]} \cup A_{[S]} \quad (*)$$

Now, suppose that $a$ is an address in $A_{[R]} \cup A_{[S]}$. This means that a panel $p$ with address $a$ is in $[R]$ or $[S]$. Therefore, $p$ is in $[R] \cup [S]$. Consequently, $a \in A_{[R] \cup [S]}$. Thus,

$$A_{[R]} \cup A_{[S]} \subseteq A_{[R] \cup [S]} \quad (***)$$

By using $(*)$ and $(***)$, it is clear that,

$$A_{[R] \cup [S]} = A_{[R]} \cup A_{[S]} \quad \blacksquare$$

Note that, since every address set represent a nexus uniquely and every nexus has only one address set, so, the above lemma can be used as an alternative definition for the union of two nexuses as follows:

**Definition 3.9.3** The union of two nexuses $[P]$ and $[Q]$ is the union of their address sets.

For instance, consider the above nexuses $[R]$ and $[S]$ whose address sets are

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$A_{[R]} = \{(1), (2), (1, 1), (1, 2), (1, 3)\}$

and

$A_{[S]} = \{(1), (2), (2, 1), (2, 2)\}$

The address set of $[R] \cup [S]$ is

$A_{[R] \cup [S]} = A_{[R]} \cup A_{[S]}
= \{(1), (2), (1, 1), (1, 2), (1, 3)\} \cup \{(1), (2), (2, 1), (2, 2)\}
= \{(1), (2), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\}$

This address set represents uniquely the nexus $[R] \cup [S]$, that is,

$[R] \cup [S] = \langle \square, \square, \square \rangle, \langle \square, \square \rangle$

The dendrogram of $[R] \cup [S]$ is shown in Fig 3.9.1.

Now, consider the nexuses

$[P] = \langle \square, \square, \square \rangle, \langle \square, \square \rangle$

and

$[Q] = \langle \square, \square \rangle, \langle \square, \square \rangle$

The union of $[P]$ and $[Q]$ is

$[P] \cup [Q] = \langle \square, \square, \square \rangle, \langle \square, \square \rangle \cup \langle \square, \square \rangle, \langle \square, \square \rangle
= \langle \square, \square, \square \rangle, \langle \square, \square \rangle$

The dendrograms of $[P]$, $[Q]$ and $[P] \cup [Q]$ are shown in Fig 3.9.2.
As may be deduced from this example, generally, neither \([P]\) nor \([Q]\) is a subnate of \([P] \cup [Q]\). But following theorem show that if \([P]\) is a permate of \([Q]\) then both \([P]\) and \([Q]\) are subnates of \([P] \cup [Q]\).

**Theorem 3.9.4** Let \([P]\) and \([Q]\) be permate of each other. Then, \([P]\) and \([Q]\) are subnate of \([P] \cup [Q]\).

**Proof:** One may first show that \([P]\) is a subnate of \([P] \cup [Q]\). It suffices to show that \([P]\) and \([P] \cup [Q]\) have the same order and every nonprimion panel of order \(n\) of \([P]\) has a nonprimion panel of order \(n\) of \([P] \cup [Q]\) as full image. Since \([P]\) and \([Q]\) are permates of each other they have the same order, so by the definition of the union of two nexuses, it is clear that \([P]\) and \([P] \cup [Q]\) have the same order. Now, let \(p^*\) be a nonprimion panel of \([P]\). Since every primion panel of \([P]\) has a full image in \([P] \cup [Q]\), therefore, there exists one panel of \([P] \cup [Q]\), namely, \(k^*\) such that \(p^*\) and \(k^*\) are full images of each other. Since \([P]\) and \([Q]\) are permates of each other, so either \(p^*\) has no full image in \([Q]\) or \(p^*\) has a full image in \([Q]\) and if \(p^*\) has a full image in \([Q]\) then this full image is either a primion panel or a nonprimion panel. Therefore, there are three cases:

Case 1: \(p^*\) has no full image in \([Q]\), see Fig 3.9.3 By the definition of the union of nexuses \(p^*\) appears without any change in \([P] \cup [Q]\) and this is denoted by \(k^*\).

![Fig 3.9.3](image)

Fig 3.9.3 \(p^*\) has no full image in \([Q]\)

Case 2: \(p^*\) has a primion panel in \([Q]\) as a full image, that is \(q^*\), see Fig 3.9.4. As in case 1, by the definition of the union of nexuses, \(p^*\) appears in \([P] \cup [Q]\) without any change.

Case 3: \(p^*\) has a nonprimion panel in \([Q]\) as full image, that is \(q^*\), see Fig 3.9.5. By the definition of two nexuses that are permates of each other, \(p^*\) and \(q^*\) have the same order. So by the
definition of the union of nexuses, the full image of $p^*$ in $[P] \cup [Q]$, that is, $k^*$ and $p^*$ have the same order. Thus, the proof is complete. $lacksquare$

Fig 3.9.4 $p^*$ has a primion panel in [$Q$] as a full image

Fig 3.9.5 $p^*$ has a nonprimion panel in [$Q$] as a full image

The most convenient way of thinking about the union of two nexuses is to visualise it as the 'union of the branches' of their dendrograms.

**Theorem 3.9.5 (Properties of union of nexuses)** Let $[P]$, $[Q]$ and $[R]$ be three nexuses. Then
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(1) \([P] \cup [<>] = [<>] \cup [P] = [P]\)
That is, the union of a nexus and the empty nexus is equal to the original nexus.

(2) \([P] \cup [Q] = [Q] \cup [P]\)
That is, the union of nexuses is commutative.

(3) \([P] \cup ([Q] \cup [R]) = ([P] \cup [Q]) \cup [R]\)
That is, the union of nexuses is associative.

(4) \([P] \cup [P] = [P]\)
That is, the union of a nexus with itself is equal to itself. The term ‘idempotent’ is usually used to refer to an operation that has this property.

(5) \([P] \subset [Q] \Rightarrow [P] \cup [Q] = [Q]\)
That is, if \([P]\) is a subnate of \([Q]\) then \([P] \cup [Q]\) is equal to \([Q]\).

Proof: By using Lemma 3.9.2, the proof is straightforward.

Note that, in the above theorem, the converse of implication (5), that is,
\([P] \cup [Q] = [Q] \Rightarrow [P] \subset [Q]\)
is not true, as the following example shows. Consider two nexuses
\([P] = \langle\triangle, \triangle, \triangle\rangle\)
and
\([Q] = \langle\triangle, \triangle, \triangle, \square\rangle\)
The union of \([P]\) and \([Q]\), that is,
\([P] \cup [Q] = \langle\triangle, \triangle, \triangle, \square\rangle\)
is equal to \([Q]\) but \([P]\) is not a subnate of \([Q]\).

Theorem 3.9.6 Suppose that nexuses \([P]\) and \([Q]\) are pernates of each other. Then,
\([P] \subset [Q] \iff [P] \cup [Q] = [Q]\)

Proof: Theorem 3.9.5 (5) shows that, generally,
\([P] \subset [Q] \Rightarrow [P] \cup [Q] = [Q]\)
Now, one must show that
\([P] \cup [Q] = [Q] \Rightarrow [P] \subset [Q]\)
By Definition. 3.4.1, one must show that the nexuses \([P]\) and \([Q]\) have the same number of principal panels (that is, they are of the same order) and every nonprimitive panel of order \(n\) of \(P\) has a nonprimitive panel of order \(n\) of \(Q\) as full image. To do this, suppose that, \(p^*\) is a nonprimitive panel of order \(n\) of a nexus \(P\). Since, \([P] \cup [Q] = [Q]\), so, the constitution of \([P] \cup [Q]\) and \([Q]\) are the same. This implies that \(p^*\) has a full image panel \(q^*\) in \(Q\), whose order
The operation of union for nexuses has many points of similarity with another nexus operation, that is, 'intersection'. Consider two nexuses \( [P] \) and \( [Q] \). The 'intersection' of \( [P] \) and \( [Q] \) is a nexus that contains all the panels of \( [P] \) and \( [Q] \) that are full images of each other. The intersection of \( [P] \) and \( [Q] \) is denoted by \( [P] \cap [Q] \). The symbol \( \cap \) used for the intersection of two nexuses is the same as the symbol that is used for the intersection of two sets. As an example of intersection of two nexuses, consider the nexuses
\[
[P] = \langle \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle
\]
and
\[
[Q] = \langle \Box, \Box, \Box, \Box \rangle
\]
The panels in \( [P] \) and \( [Q] \) that are full images of each other are as follows:
\[
[P](1) \quad \text{and} \quad [Q](1)
\]
\[
[P](2) \quad \text{and} \quad [Q](2)
\]
\[
[P](2, 1) \quad \text{and} \quad [Q](2, 1)
\]
and
\[
[P](2, 2) \quad \text{and} \quad [Q](2, 2)
\]
Therefore, the nexus \( [P] \cap [Q] \) is
\[
[P] \cap [Q] = \langle \Box, \Box, \Box \rangle
\]
The dendrograms for \( [P] \), \( [Q] \) and \( [P] \cap [Q] \) are shown in Fig 3.10.1

The formal definition of the intersection of two nexuses may be given as follows:

**Definition 3.10.1** The intersection of two nexuses \( [P] \) and \( [Q] \) is a nexus that is denoted by \( [P] \cap [Q] \), where every panel of \( [P] \cap [Q] \) has full images in both \( [P] \) and \( [Q] \) and, every pair of panels of \( [P] \) and \( [Q] \) that are full images of each other has a full image in \( [P] \cap [Q] \).

**Lemma 3.10.2** The address set of the intersection of two nexuses \( [P] \) and \( [Q] \) is the intersection of the address sets of \( [P] \) and \( [Q] \). That is,
\[
A_{[P] \cap [Q]} = A_{[P]} \cap A_{[Q]}
\]
**Fig 3.10.1** Dendrograms of \([P]\), \([Q]\) and \([P] \cap [Q]\). Panels in \([P]\) and \([Q]\) that are full images of each other are shown by thick lines.

**Proof:** Let \(a\) be an address in \(A_{[P] \cap [Q]}\). This means that there is a panel \(p\) in \([P] \cap [Q]\) that has \(a\) as its address. Since \(p\) is in \([P] \cap [Q]\), \(p\) has a full image in \(P\) and \(Q\). In turn, this means that \(a\) is an address in \(A_{[P]}\) and \(A_{[Q]}\). Therefore, \(a \in A_{[P]} \cap A_{[Q]}\). Thus,

\[
A_{[P] \cap [Q]} \subseteq A_{[P]} \cap A_{[Q]}
\]

Now, suppose that \(a\) is an address in \(A_{[P]} \cap A_{[Q]}\). Therefore, \(a\) is an address in \(A_{[P]}\) and \(A_{[Q]}\). This means that there are a panel in \(P\) and a panel in \(Q\) that are full images of each other. By Definition 3.10.1, this panel appears in \([P] \cap [Q]\). So, \([P] \cap [Q]\) has a panel of address \(a\). Thus, \(a \in A_{[P] \cap [Q]}\). This implies that \(A_{[P]} \cap A_{[Q]} \subseteq A_{[P] \cap [Q]}\). Consequently,

\[
A_{[P] \cap [Q]} = A_{[P]} \cap A_{[Q]}
\]

By using the above lemma, the operation of 'intersection' between two nexuses \([P]\) and \([Q]\) may be defined in terms of their address sets as follows:

The intersection of two nexuses \([P]\) and \([Q]\) is the nexus with \(A_{[P]} \cap A_{[Q]}\) as the address set.

For example, consider the nexuses

\([P] = \langle\langle\langle\square, \square\rangle, \square\rangle, \langle\square, \square, \square\rangle\rangle\)

and

\([Q] = \langle\langle\langle\square, \square, \square\rangle, \langle\square, \langle\square, \square\rangle\rangle\rangle\)

with their address sets being

\(A_{[P]} = \{(1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (1, 1, 1), (1, 1, 2)\}\)

and

\(A_{[Q]} = \{(1), (2), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 2, 1), (2, 2, 2)\}\)
The address set of the intersection \([P] \cap [\mathcal{Q}]\) is given by

\[
A_{[P] \cap [\mathcal{Q}]} = A_{[P]} \cap A_{[\mathcal{Q}]}
= \{(1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (1, 1, 1), (1, 1, 2)\} \cap
\{(1), (2), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 2, 1), (2, 2, 2)\}
= \{(1), (2), (1, 1), (1, 2), (2, 1), (2, 2)\}
\]

Therefore, the nexus \([P] \cap [\mathcal{Q}]\) is

\([P] \cap [\mathcal{Q}] = \langle\langle\Box, \Box\rangle, \langle\Box, \Box\rangle\rangle\)

The dendrograms for \([P]\), \([\mathcal{Q}]\) and \([P] \cap [\mathcal{Q}]\) are shown in Fig 3.10.2.

Fig 3.10.2 Dendrograms of \([P]\), \([\mathcal{Q}]\) and \([P] \cap [\mathcal{Q}]\), where the branches in the dendrograms of
\([P]\) and \([\mathcal{Q}]\) that are full images of each other are shown by thick lines.

Note that, as may be seen from the above example, in general, \([P] \cap [\mathcal{Q}]\) is not necessarily a subnate of \([P]\) or \([\mathcal{Q}]\). However, the next theorem shows that if \([P]\) and \([\mathcal{Q}]\) are pernates of each other then \([P] \cap [\mathcal{Q}]\) will be a subnate of both \([P]\) and \([\mathcal{Q}]\).

**Theorem 3.10.3** Let \([P]\) and \([\mathcal{Q}]\) be two nexuses and let \([P]\) and \([\mathcal{Q}]\) be pernates of each other. Then \([P] \cap [\mathcal{Q}]\) is a subnate of \([P]\) and \([\mathcal{Q}]\).
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Proof: To prove this fact, consider two pertain nexuses \([P]\) and \([Q]\). One may first show that \([P] \cap [Q]\) is a subnate of \([P]\). It suffices to show that \([P]\) and \([P] \cap [Q]\) have the same order and every nonprimion panel of order \(n\) of \([P]\) has a nonprimion panel of order \(n\) of \([P] \cap [Q]\) as full image. Since \([P]\) and \([Q]\) are pertaines of each other they have the same order so by the definition of the intersection of two nexuses, \([P]\) and \([P] \cap [Q]\) have the same order. Now, suppose that \(k^*\) is a nonprimion panel of order \(n\) of \([P] \cap [Q]\). By the definition of the intersection of two nexuses there exist two panels in \([P]\) and \([Q]\), namely, \(p^*\) and \(q^*\) respectively such that \(p^*\) and \(q^*\) are full images of each other. Again, by the definition of the intersection of two nexuses, \(p^*\) and \(q^*\) are nonprimion panels because \(k^*\) is a nonprimion panel. By the definition of two nexuses that are pertaines of each other, \(p^*\) and \(q^*\) have the same order so \(k^*, p^*\) and \(q^*\) have the same order and the proof is complete. ■

Theorem 3.10.4 (Properties of intersection of nexuses). Let \([P]\), \([Q]\) and \([R]\) be three nexuses. Then

(1) \([P] \cap [\emptyset] = [\emptyset]\)
That is, the intersection of any nexus and the empty nexus is equal to the empty nexus. Furthermore, the intersection of two nexuses \([P]\) and \([Q]\), that is, \([P] \cap [Q]\) can only be the empty nexus if either \([P]\) or \([Q]\) is the empty nexus. That is,
\([P] \cap [Q] = [\emptyset] \Rightarrow [P] = [\emptyset] \quad \text{or} \quad [Q] = [\emptyset]\)

(2) \([P] \cap [Q] = [Q] \cap [P]\)
That is, the intersection of nexuses is commutative.

(3) \([P] \cap ([Q] \cap [R]) = ([P] \cap [Q]) \cap [R]\)
That is, the intersection of nexuses is associative.

(4) \([P] \cap [P] = [P]\)
That is, the intersection of a nexus with itself is equal to itself. Therefore, the operation is idempotent.

(5) \([P] \subseteq [Q] \Rightarrow [P] \cap [Q] = [P]\)
That is, if \([P]\) is a subnate of \([Q]\) then \([P] \cap [Q]\) is a cognate of \([P]\).

Proof: By using Lemma 3.10.2, the proof is straightforward. ■

Note that, in the above theorem the converse of (5) implication, that is,
\([P] \cap [Q] = [P] \Rightarrow [P] \subseteq [Q]\)
is not true. For example, consider the nexuses
\([P] = \langle \Omega, \langle \Omega, \Box \rangle \rangle\)

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and

$$[Q] = \langle \Box, \Box, \Box, \Box \rangle$$

The intersection of \([P]\) and \([Q]\) is

$$[P] \cap [Q] = \langle \Box, \Box, \Box \rangle = [P]$$

However, \([P]\) is not a Subnate of \([Q]\).

**Theorem 3.9.5** Suppose that \([P]\) and \([Q]\) are two nexuses and that they are also pernates of each other. Then,

$$[P] \subseteq [Q] \iff [P] \cap [Q] = [P]$$

**Proof:** By Theorem 3.9.4 (5),

$$[P] \subseteq [Q] \Rightarrow [P] \cap [Q] = [P]$$

Now, one must show that

$$[P] \cap [Q] = [P] \Rightarrow [P] \subseteq [Q]$$

Since, \([P]\) and \([Q]\) are pernates of each other, so, by Theorem 3.10.3,

$$[P] \cap [Q] \subseteq [Q]$$

Therefore,

$$[P] = [P] \cap [Q] \subseteq [Q] \Rightarrow [P] \subseteq [Q] \quad \blacksquare$$

Note that if nexus \([P]\) is a subnate of a nexus \([Q]\) then the address set of \([P]\) is a subset the address set of \([Q]\), that is, \(A_{[P]} \subseteq A_{[Q]}\). However, the converse of this statement is not true. That is, \(A_{[P]} \subseteq A_{[Q]}\) does not imply that \([P]\) is a subnate of \([Q]\). But if a nexus \([P]\) is equal to a nexus \([Q]\) then the address set of \([P]\) is equal to the address set of \([Q]\), that is, \(A_{[P]} = A_{[Q]}\) and if \(A_{[P]} = A_{[Q]}\) then the nexus \([P]\) is equal to the nexus \([Q]\), that is,

$$[P] = [Q] \iff A_{[P]} = A_{[Q]}$$

Therefore, the best way for proving the equality between two nexuses is to prove the equality of their address sets.

Two interesting facts about the combinations of the operations of union and intersection of nexuses are presented in next theorem

**Theorem 3.10.6** (distributive laws) Let \([P], [Q]\) and \([R]\) be three nexuses. Then,

$$[P] \cap ([Q] \cup [R]) = ([P] \cap [Q]) \cup ([P] \cap [R])$$

and

$$[P] \cup ([Q] \cap [R]) = ([P] \cup [Q]) \cap ([P] \cup [R])$$
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Proof: By the above discussion one must show that the nexuses that are both sided of equality sign have the same address sets. That is,

\[ A[P \cap (Q \cup R)] = A[(P \cap Q) \cup (P \cap R)] \]

and

\[ A[P \cup (Q \cap R)] = A[(P \cup Q) \cap (P \cup R)] \]

The first equality will be proven and the proof of the second equality is similar. As shown in lemmas 3.9.2 and 3.10.2


and


Therefore,


and


On the other hands,


and


\[ = A[(P \cap Q) \cup (P \cap R)] \]

Since, \( A[P], A[Q] \) and \( A[R] \) are sets and the distributive laws is hold in set theory, so,


Therefore, by \((\star)\) and \((** \star)\),

\[ A[P] \cap (Q \cup R) = A[(P \cap Q) \cup (P \cap R)] \]

And the proof is completed. ■

For example, consider the nexuses

\[ [P] = <\Box, \Box, \Box, \Box> \]

\[ [Q] = <\Box, \Box, \Box, \Box, \Box> \]

and

\[ [R] = <\Box, \Box, \Box, \Box, \Box> \]

The dendrograms of \([P], [Q]\) and \([R]\) are shown in Fig3.10.3. The address sets of these nexuses are
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\[ A_{[P]} = \{(1), (2), (2, 1), (2, 2), (2, 3)\} \]
\[ A_{[Q]} = \{(1), (2), (1, 1), (2, 1), (2, 1), (2, 2)\} \]
and
\[ A_{[R]} = \{(1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\} \]
The address sets of nexuses
\[ [P] \cap ([Q] \cup [R]) \]
and
\[ ([P] \cap [Q]) \cup ([P] \cap [R]) \]
are
\[ A_{[P]} \cap ([Q] \cup [R]) = A_{[P]} \cap A_{[Q]} \cup A_{[R]} \]
\[ = A_{[P]} \cap (A_{[Q]} \cup A_{[R]}) \]
\[ = \{(1), (2), (2, 1), (2, 2), (2, 3)\} \cap \]
\[ (\{(1), (2), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\} \cup \]
\[ \{(1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\}) \]
\[ = \{(1), (2), (2, 1), (2, 2)\} \cap \]
\[ (\{(1), (2), (3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2)\}) \]
\[ = \{(1), (2), (2, 1), (2, 2)\} \]
and
\[ A_{([P] \cap [Q]) \cup ([P] \cap [R])} = A_{[P]} \cap [Q] \cup A_{[P]} \cap [R] \]
\[ = (A_{[P]} \cap A_{[Q]}) \cup (A_{[P]} \cap A_{[R]}) \]
\[ = (\{(1), (2), (2, 1), (2, 2), (2, 3)\} \cap (\{(1), (2), (1, 1), (1, 2),
(1, 3), (2, 1), (2, 2)\}) \cup (\{(1), (2), (2, 1), (2, 2), (2, 3)\}) \cap
\{(1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\}) \]
\[ = \{(1), (2), (2, 1), (2, 2)\} \cup \{(1), (2)\} \]
\[ = \{(1), (2), (2, 1), (2, 2)\} \]
Therefore,
\[ A_{[P] \cup ([Q] \cap [R])} = A_{[P] \cup [Q]} \cap ([P] \cup [R]) \]
Consequently,
\[ [P] \cap ([Q] \cup [R]) = ([P] \cap [Q]) \cup ([P] \cap [R]) \]
3.11 The Number Nontrivial Subnates of a Nexus

In this section, a method will be introduced for counting the number of nontrivial subnates of a nexus.

Consider two nexuses

\[ [P] = \langle \langle \langle \langle \emptyset , \emptyset , \emptyset >, \langle \emptyset , \emptyset > \rangle, \langle \emptyset , \emptyset > \rangle \rangle \]
and

\[ \Box = \langle \Box, \Box \rangle, \Box, \langle \Box, \Box \rangle, \Box \rangle \]

The first nexus has four nexuses as nontrivial subnates where

\[ \langle \Box, \Box \rangle \]
\[ \langle \Box, \Box, \Box \rangle, \Box \rangle \]
\[ \langle \Box, \Box \rangle \]

and

\[ \langle \Box, \Box, \Box \rangle, \langle \Box, \Box \rangle \]

The dendrograms for \([P]\) and its subnates are shown in Fig 3.11.1.

The nexus \([\Box]\) has six nexuses as nontrivial subnates. These are

\[ \langle \Box, \Box, \Box \rangle \]
\[ \langle \Box, \Box, \Box, \Box \rangle \]
\[ \langle \Box, \Box, \Box, \Box, \Box \rangle \]
\[ \langle \Box, \Box, \Box, \Box, \Box \rangle \]
\[ \langle \Box, \Box, \Box, \Box, \Box \rangle \]

and

\[ \langle \Box, \Box, \Box, \Box, \Box, \Box \rangle \]

The dendrograms for nexuses \([\Box]\) and their subnates are shown in Fig 3.11.2.
Fig 3.11.2 Dendrograms for $[Q]$ and its subnates
**Definition 3.11.1** The set of all nontrivial subnates of nexus \([P]\) is denoted by \(S([P])\). Also, the set of all nontrivial subnates of the empty plenix has just one element, that is, the empty plenix. Therefore,

\[
S([<>]) = \{[<>]\}
\]

For instance, in the above example

\[
S([P]) = \{<[\square, \square], <\square, \square, \square>], <\square, <\square, \square>],
\quad<[\square, \square], \square, <\square, \square>],
\quad<[\square, \square], \square, <\square, \square>],
\quad<[\square, \square], \square, <\square, \square>],
\quad<[\square, \square], \square, <\square, \square>],
\quad<[\square, \square], \square, <\square, \square>]
\]

and

\[
S([\varnothing]) = \{<[\square, \square, \square], <\square, \square, \square>],
\quad<[\square, \square, \square], \square, <\square, \square>],
\quad<[\square, \square, \square], \square, <\square, \square>],
\quad<[\square, \square, \square], \square, <\square, \square>],
\quad<[\square, \square, \square], \square, <\square, \square>],
\quad<[\square, \square, \square], \square, <\square, \square>]
\]

In the next three theorems the properties of the set of nontrivial subnates of a nexus will be investigated.

**Theorem 3.11.2** Suppose that \([P]\) and \([\varnothing]\) are two nexuses and the nexus \([P]\) is a subnate of \([\varnothing]\), then the set of all nontrivial subnates of \([P]\) is a subset of the set of all nontrivial subnates of \([\varnothing]\). That is,

\[
[P] \subset [\varnothing] \Rightarrow S([P]) \subset S([\varnothing])
\]

Note that, here the symbol \(\subset\) is used to imply that \([P]\) is a subnate of \([\varnothing]\) and to imply that \(S([P])\) is a subset of \(S([\varnothing])\).

**Proof:** To prove this statement, let \([T]\) be an element of \(S([P])\). That is, \([T]\) is a nontrivial subnate of \([P]\). Also, \([P]\) is a subnate of \([\varnothing]\). So, \([T]\) is a subnate of \([\varnothing]\) because the relationship of being subnates is transitive. Therefore, \([T]\) is an element of \(S([\varnothing])\) and this means that \(S([P])\) is a subset of \(S([\varnothing])\). \(\blacksquare\)

**Theorem 3.11.3** Suppose that \([P]\) and \([\varnothing]\) are two nexuses. Then the intersection of the set of all nontrivial subnates of \([P]\) and the set of all nontrivial subnates of \([\varnothing]\) is a subset of the set of all nontrivial subnates of the intersection of \([P]\) and \([\varnothing]\). That is,

\[
(S([P]) \cap S([\varnothing])) \subset S([P] \cap [\varnothing])
\]

Note that, the symbol \(\cap\) on the left-hand side of the above relation implies intersection of two sets but the same symbol on the right-hand side of the relation implies intersection of nexuses. Furthermore, if \([P]\) and \([\varnothing]\) are pernates of each other then
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\[(S([P]) \cap S([Q])) = S([P] \cap [Q])\]

**Proof:** Let \([T]\) be an element of \(S([P]) \cap S([Q])\). This means that \([T]\) is an element of both \(S([P])\) and \(S([Q])\). Therefore, \([T]\) is a nontrivial subnate of both \([P]\) and \([Q]\). In other words, \([T]\) is a nontrivial subnate of \([P] \cap [Q]\). So, \([T]\) is an element of \(S([P] \cap [Q])\).

Now, the equality

\[(S([P]) \cap S([Q])) = S([P] \cap [Q])\]

will be proven. This is an equality between two sets. Recalling from set theory, if \(A\) and \(B\) are two sets, then a necessary and sufficient condition for \(A = B\) is that both \(A \subseteq B\) and \(B \subseteq A\). Also, the number of elements of set of \(A\) is denoted by \(|A|\) [see Ref 11].

Now, referring to the first part of the theorem, for any two nexuses \([P]\) and \([Q]\) (not necessarily pernates of each other) it is proved that

\[(S([P]) \cap S([Q])) \subseteq S([P] \cap [Q])\]

So, to prove the above equality, it suffices to show that

\[S([P] \cap [Q]) \subseteq (S([P]) \cap S([Q]))\]

To do this, let \([R]\) be an element of \(S([P] \cap [Q])\). This implies that \([R]\) is a nontrivial subnate of \([P] \cap [Q]\), that is,

\[[R] \subseteq [P] \cap [Q]\]

Since \([P]\) and \([Q]\) are pernates of each other then \([P] \cap [Q]\) is a subnate of both \([P]\) and \([Q]\). Therefore,

\[[R] \subseteq [P] \cap [Q] \subseteq [P]\]

and

\[[R] \subseteq [P] \cap [Q] \subseteq [Q]\]

Since the relationship of being subnate is transitive, \([R]\) is a subnate of both \([P]\) and \([Q]\). Thus, \([R]\) is an element of both \(S([P])\) and \(S([Q])\). This implies that, \([R]\) is an element of \(S([P]) \cap S([Q])\) and the proof is complete. ■

**Theorem 3.11.4** Suppose that \([P]\) and \([Q]\) are two nexuses and also, \([P]\) and \([Q]\) are pernates of each other. Then

\[(S([P]) \cup S([Q])) \subseteq S([P] \cup [Q])\]

That is, the union of the set of all nontrivial subnates of \([P]\) and the set of all nontrivial subnates of \([Q]\) is a subset of the set of all nontrivial subnates of the union of \([P]\) and \([Q]\).

**Proof:** The proof is similar to Theorem 3.11.3. ■
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**Theorem 3.11.5** Any two elements of the set of all nontrivial subnates of a nexus are pernates of each other. In other words, if \([R]\) and \([S]\) are nontrivial subnates of \([P]\), that is, if \([R]\) and \([S]\) are elements of \(S([P])\), then \([R]\) and \([S]\) are pernates of each other. Also, if \([Q]\) and \([T]\) are pernates of each other then any element of \(S([Q])\) and \(S([T])\) are pernates of each other.

**Proof:** the proof follows from Theorems 3.4.2 and 3.4.3. ■

**Theorem 3.11.6** The set of all nontrivial subnates of a nexus is closed under union and intersection. In other words, if \([R]\) and \([S]\) are nontrivial subnates of \([P]\) then

\[ [R] \cup [S] \quad \text{and} \quad [R] \cap [S] \]

are also nontrivial subnates of \([P]\).

**Proof:** The proof follows from definitions of subnate, union and intersection of two plenices. ■

**Definition 3.11.7** Consider two sets \(A\) and \(B\) whose elements are nexuses. Define \(A \# B\) as the set

\[ \{[R] \# [S] : [R] \in A \text{ and } [S] \in B\} \]

For example, let

\[ A = \{<\square, \square>, <\square, \square>, \square\} \]

and

\[ B = \{<\square, \square, <\square, \square>, \square>, <\square, \square>, <\square, \square, \square>, \} \]

be two sets of nexuses. For clarity, in set \(B\), the primion panels are shown by the symbol \(\square\). The set \(A \# B\) is given by:

\[ A \# B = \{<\square, \square> \# <\square, \square, <\square, \square>, \square>, <\square, \square> \# <\square, \square>, \square>, \}

\[ <\square, \square> \# <\square, \square>, <\square, \square>, <\square, \square>, \square> \# <\square, \square>, <\square, \square>, \}

\[ <\square, \square, <\square, \square>, \square>, <\square, \square>, \square> \# <\square, \square>, <\square, \square>, \}

\[ = \{<\square, \square, <\square, \square>, \square>, <\square, \square>, <\square, \square>, \square>, \}

\[ <\square, \square, <\square, \square>, <\square, \square>, \square>, <\square, \square>, \square>, \}

\[ <\square, \square>, \square, <\square, \square>, \square>, <\square, \square>, \square, \}

\[ <\square, \square>, \square, <\square, \square>, \square>, <\square, \square>, \square, \}

\[ <\square, \square>, \square, <\square, \square>, \square>, <\square, \square>, \square, \}

\[ <\square, \square>, <\square, \square>, <\square, \square>, \square>, <\square, \square>, <\square, \square>, \}

Now, consider two nexuses

\[ [P] = <\square, \square>, <\square, \square>, \square> \]

and

\[ [Q] = <\square, <\square, \square>, <\square, \square>, \square>, <\square, \square>, <\square, \square>, \square> \]

The dendrograms for \([P]\) and \([Q]\) are shown in Fig 3.11.3.
The nexus $[P]$ has four nexuses as nontrivial subnates. More specifically, $\mathcal{S}([P])$ has four elements, as follows:

$$\mathcal{S}([P]) = \{ \langle \Box, \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle, \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle, \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle, \Box, \Box \rangle \}$$

Also, $\mathcal{S}([Q])$ has six elements, as follows:

$$\mathcal{S}([Q]) = \{ \langle \Box, \Box \rangle, \langle \Box, \Box \rangle, \langle \Box, \Box \rangle, \langle \Box, \Box \rangle, \langle \Box, \Box \rangle, \langle \Box, \Box \rangle \}$$

and

$$\mathcal{S}([P]) \# \mathcal{S}([Q]) = \{ \langle \Box, \Box, \Box \rangle \# \langle \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle \# \langle \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle \}$$

Now, consider $[P] \# [Q]$, that is

$$[P] \# [Q] = \{ \langle \Box, \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle \}$$

The dendrogram of $[P] \# [Q]$ is shown in Fig 3.11.4.

Also, consider the set of all nontrivial subnates of $[P] \# [Q]$, that is,

$$\mathcal{S}([P] \# [Q]) = \{ \langle \Box, \Box, \Box, \Box \rangle, \langle \Box, \Box, \Box, \Box \rangle, \}$$

As one can see, the set of all nontrivial subnates of $[P] \# [Q]$, namely, $\mathcal{S}([P] \# [Q])$ is equal to the set $\mathcal{S}([P]) \# \mathcal{S}([Q])$. The material of the next theorem is the essence of the above example.
Theorem 3.11.8 Let $[P]$ and $[Q]$ be two nexuses. Then the set of all nontrivial subnates of $[P] \# [Q]$ is equal to the set $S([P]) \# S([Q])$. In other words,

$$S([P]) \# S([Q]) = S([P] \# [Q])$$

Proof: This is an equality between two sets. So, a necessary and sufficient condition for this equality is that

$$S([P]) \# S([Q]) \subseteq S([P] \# [Q])$$

and

$$S([P] \# [Q]) \subseteq S([P]) \# S([Q])$$

Considering the fact that the relationship of being subnate is compatible, it is clear that the set

$$S([P]) \# S([Q])$$

is a subset of the set of all nontrivial subnates of $[P] \# [Q]$, that is

$$S([P]) \# S([Q]) \subseteq S([P] \# [Q])$$

Conversely, let $[T]$ be an element of $S([P] \# [Q])$. Therefore, $[T]$ is a nontrivial subnate of $[P] \# [Q]$, so $[T]$ and $[P] \# [Q]$ have the same order. $[T]$ may be written down as

$$[T] = [R] \# [S]$$

where $[R]$ has the same order as $[P]$ and $[S]$ has the same order as $[Q]$. One may state that, $[R]$ is a subnate of $[P]$ and $[S]$ is a subnate of $[Q]$. To prove this statement, suppose that $[R]$ is not a subnate of $[P]$, then there exists a panel of order $n$ of $[R]$, namely, $r^*$ that has no full image of order $n$ in $[P]$. So, $r^*$ as a panel of

$$[T] = [R] \# [S]$$

has no full image of order $n$ in $[P] \# [Q]$

and this means that $[T]$ is not a subnate of $[P] \# [Q]$ which contradicts the assumption that $[T] \in S([P] \# [Q])$.
Theorem 3.11.9 Let \([P]\) and \([Q]\) be two nexuses. Now, consider the Cartesian products of two sets \(S([P])\) and \(S([Q])\), that is
\[
S([P]) \times S([Q]) = \{(R, S) : R \in S([P]) \text{ and } S \in S([Q])\}
\]
There exists a one to one correspondence between this set and the set
\[
S([P]) \# S([Q])
\]
Proof: For the proof of this theorem, one may show that there is a one to one function \(f\) from \(S([P]) \times S([Q])\) onto \(([P]) \# ([Q])\). To do this, one can define \(f\) by
\[
 f([R], [S]) = [R] \# [S]
\]
where \([R]\) and \([S]\) are nontrivial subnates of \([P]\) and \([Q]\), respectively. It is clear that \(f\) is one to one and onto. Therefore, the cardinal numbers of \(S([P]) \times S([Q])\) and \(S([P]) \# S([Q])\) are the same.

The fact just established together with the equality
\[
S([P]) \# S([Q]) = S([P] \# [Q])
\]
implies that the number of nontrivial subnates of \([P] \# [Q]\) is equal to product of the number of nontrivial subnates of \([P]\) and the number of nontrivial subnates of \([Q]\).

For instance, in the above example \(S([P])\) has four elements and \(S([Q])\) has six elements. Therefore, \(S([P] \# [Q])\) has twenty-four elements.

The remainder of this section consists of several examples illustrating techniques of counting of the number of all nontrivial subnates of a nexus.

Example 3.11.10 Consider the nexus
\[
[R] = \langle\bigcirc, \bigcirc, \bigcirc, \bigcirc\rangle
\]
This nexus has two nontrivial subnates, that is
\[
\langle\bigcirc, \bigcirc, \bigcirc\rangle
\]
and
\[
\langle\bigcirc, \langle\bigcirc, \bigcirc\rangle, \bigcirc\rangle
\]
Now, consider the nexus
\[
[S] = \langle[R]\rangle = \langle\langle\bigcirc, \langle\bigcirc, \bigcirc\rangle, \bigcirc\rangle\rangle
\]
This nexus has three nontrivial subnates, that is,
\[
\langle\bigcirc\rangle
\]
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$\langle \Box, \Box, \Box \rangle$

and

$\langle \Box, \langle \Box, \Box, \Box \rangle, \Box \rangle$

So, $|S([R])|$ has just one more element than $|S([S])|$. The dendrograms of $[R]$ and $[S]$ and their nontrivial subnates are shown in Fig 3.11.5.

![Dendrograms of [R], [S], and their nontrivial subnates](image)

Fig 3.11.5 Dendrograms for $[R]$, $[S]$, and their nontrivial subnates

Similarly, if

$[T] = \langle [S] \rangle = \langle [R] \rangle = \langle \langle \Box, \langle \Box, \Box, \Box \rangle, \Box \rangle \rangle$

then $|S([T])|$ has just one more element than $|S([S])|$ and has two more elements than $|S([R])|$.

**Example 3.11.11** Consider the nexus

$[P] = \langle \langle \langle \Box, \Box, \Box \rangle, \Box, \Box \rangle, \langle \Box, \Box \rangle, \langle \Box, \Box, \Box \rangle \rangle$

The dendrogram of $[P]$ is shown in Fig 3.11.6.
This nexus may be decomposed in terms of its principal panels as follows:

\[ [P] = [R] \# [S] \# [T] \]

where

\[ [R] = <<<\Box, \Box>, \Box, \Box>> \]
\[ [S] = <<\Box, \Box>> \]

and

\[ [T] = <<\Box, <\Box, \Box>>> \]

The dendrograms for \([R], [S]\) and \([T]\) are shown in Fig 3.11.7.

The sets of all nontrivial subnates of \([R], [S]\) and \([T]\) have three, two and three elements, respectively. That is,

\[ |S([R])| = 3 \]
\[ |S([S])| = 2 \]

and

\[ |S([T])| = 3 \]

So, the number of the nontrivial subnates of \([P]\) is equal to
\[ |S([P])| = |S([R])| \times |S([S])| \times |S([T])| \]
\[ = 3 \times 2 \times 3 \]
\[ = 18 \]

Example 3.11.12 Consider the nexus

\[[\mathcal{Q}] = \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle \rangle \]

The dendrogram of \([\mathcal{Q}]\) is shown in Fig 3.11.8.

The nexus \([\mathcal{Q}]\) may be written as

\[[\mathcal{Q}] = [P] \# [R] \]

where

\[[P] = \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle \rangle \]

and

\[[R] = \langle \square, \langle \square, \langle \square, \square \rangle \rangle, \langle \square, \langle \square, \square \rangle \rangle, \langle \square, \langle \square, \square \rangle \rangle, \langle \square, \langle \square, \square \rangle \rangle \rangle \rangle \]

Also,

\[ |S([\mathcal{Q}])| = |S([P])| \times |S([R])| \]

Now, consider the nexus \([P]\). This nexus may be written as

\[[P] = \langle [P_1] \rangle \]

where

\[[P_1] = \langle \square, \langle \square, \square \rangle, \square \rangle, \langle \square, \langle \square, \square \rangle, \square \rangle \rangle \]

By Example 3.11.10,

\[ |S([P])| = |S([P_1])| + 1 \]

and also,
Chapter 3 Relations Between Plenices

\[ [P_1] = \langle \square \rangle \# [T] \# \langle \square \rangle \]

where

\[ [T] = \langle \langle \langle \square, \square \rangle, \square \rangle \rangle \]

So,

\[
|S([P_1])| = |S(\langle \square \rangle)| \times |S([T])| \times |S(\langle \square \rangle)|
\]

\[ = 1 \times |S([T])| \times 1 \]

\[ = |S([T])| \]

Furthermore,

\[ [T] = \langle [T_1] \rangle \]

where

\[ [T_1] = \langle \langle \langle \square, \square \rangle, \square \rangle \rangle \]

and

\[ |S([T])| = |S([T_1])| + 1 \]

Finally,

\[ [T_1] = \langle \langle \langle \square, \square \rangle, \square \rangle \rangle \# \langle \square \rangle \]

and

\[
|S([T_1])| = |S(\langle \langle \langle \square, \square \rangle, \square \rangle \rangle)| \times |S(\langle \square \rangle)|
\]

\[ = |S(\langle \langle \langle \square, \square \rangle, \square \rangle \rangle)| \times 1 \]

Since nexus \( \langle \square, \square \rangle \) has just one nontrivial subnate then, by Example 3.11.10, the nexus \( \langle \langle \langle \square, \square \rangle, \square \rangle \rangle \) has two nontrivial subnate. Therefore,

\[ |S([T_1])| = 2 \times 1 \]

\[ = 2 \]

Also

\[ |S([T])| = |S([T_1])| + 1 \]

\[ = 2 + 1 \]

\[ = 3 \]

Furthermore,

\[
|S([P_1])| = |S(\langle \square \rangle)| \times |S([T])| \times |S(\langle \square \rangle)|
\]

\[ = 1 \times 3 \times 1 \]

\[ = 3 \]

Finally,

\[ |S([P])| = |S([P_1])| + 1 \]

\[ = 3 + 1 \]
Similarly, one may calculate the number of elements of the set of nontrivial subnates of $[R]$ and
this will be found to be 5. Therefore, by the above equation, that is,

$$|\mathcal{S}([Q])| = |\mathcal{S}([P])| \times |\mathcal{S}([R])|$$

the nexus $[Q]$ has 20 nexuses as nontrivial subnates. The results of the above calculation are
shown on the dendrogram in Fig 3.11.9.

**Example 3.11.13** In this example the number of nontrivial subnates of a nexus will be found by
using its dendrogram. Consider the nexus

$$[P] = <<<\Box, \Box, <<<\Box, \Box>, \Box>, \Box>, <<<\Box, \Box, \Box>, \Box, \Box>,
<<<\Box, \Box, <<<\Box, \Box>, \Box>, \Box>, \Box>, \Box>$$

The dendrogram of $[P]$, together with the calculation for finding the number of nontrivial subnates,
is shown in Fig 3.11.10. For the first principal panel of $[P]$ the full calculations are shown but for the other two principals panels the calculations are shown in an abbreviated form.

![Dendrogram](image)

Fig 3.11.9 Dendrogram is used for calculating the number of nontrivial subnates of $[Q]$
Fig 3.11.10 Dendrogram is used for calculating the number of nontrivial subnates of \([Q]\)
4.1 Introduction

A plenix may act as an operator or be an operand. A plenix may act as a function or be an argument of a function. Also, a plenix may be involved in various processes of calculus. These aspects of plenices are discussed in this chapter.

4.2 Binary Operations between Two Cognate Plenices

In this section some binary operations between two cognate plenices are defined. Now, consider the plenices

\[ P = \langle 2, \langle 4, 7 \rangle, 1 \rangle \]

and
These plenices have the same constitutions, that is, $P$ and $Q$ are cognates. The ‘sum’ of the above plenices is defined as follows

\[ P + Q = \langle 2, <4, 7>, 1 \rangle + \langle 0.2, <0.4, 0.7>, 0.1 \rangle \]
\[ = \langle 2 + 0.2, <4 + 0.4, 7 + 0.7>, 1 + 0.1 \rangle \]
\[ = \langle 2.2, <4.4, 7.7>, 1.1 \rangle \]

To elaborate, the sum of plenices $P$ and $Q$ is a plenix that is of the same constitution as $P$ and $Q$ and in which every primion panel is obtained as the sum of the corresponding primion panels of $P$ and $Q$. A good way of visualising the summation of two cognate plenices is to use their dendrograms as in Fig 4.2.1. Through the dendrograms, the relationships between the corresponding panels can be easily seen.

![Dendrogram representation of the summation of two plenices](image)

If two corresponding primion panels in plenices $P$ and $Q$ are not the same type of mathematical objects, then the composition of $P$ and $Q$ is not meaningful. For example, if

\[ P = \langle \{4, 7\}, 8 \rangle \]

and

\[ Q = \langle 2, \{5, 9\} \rangle \]

then

\[ \langle \{4, 7\}, 8 \rangle + \langle 2, \{5, 9\} \rangle \]

does not represent a meaningful operation because the first principal panel of $P$ is a set but the first principal panel of $Q$ is a number.

To generalise the definition of addition to any other binary operation, let $A$ be a set and let ‘$\cdot$’ be a binary operation on the set $A$. Then, $a_1 \cdot a_2$ is an element of $A$, where $a_1$ and $a_2$ are elements of $A$. Now, consider two cognate plenices $P$ and $Q$ and assume that all the primion panels of $P$ and $Q$ are members of $A$. For example, let

\[ P = \langle a_1, <a_2, a_3> \rangle \]

and
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\[ Q = \langle a_4, <a_5, a_6> \rangle \]

where \( a_1, a_2, a_3, a_4, a_5 \) and \( a_6 \) are contained in \( A \). The binary operation \( \bullet \) may be extended to the plenices \( P \) and \( Q \) as follows:

\[
P \circ Q = \langle a_1, <a_2, a_3> \rangle \circ <a_4, <a_5, a_6> >
\]

Here the operation \( \circ \) is the extended version of operation \( \bullet \), where the operation \( \bullet \) applies to the elements of set \( A \) and where the operation \( \circ \) applies to any two plenices \( P \) and \( Q \) that are in the set of all plenices whose primion panels are contained in \( A \) and where \( P \) and \( Q \) are of the same constitution. The result of the operation \( \circ \) on two cognate plenices is obtained by the application of the operation \( \bullet \) on every pair of full image primion panels of the two plenices. The plenices \( P, Q \) and \( R \circ Q \) are cognates. Also, if the binary operation \( \bullet \) is commutative then the operation \( \circ \) is commutative. Furthermore, if the operation \( \bullet \) is associative then the operation \( \circ \) is associative. Fundamentally, the operations \( \bullet \) and \( \circ \) are not the same. However, it is convenient to use the same symbol \( \bullet \) for both of the operations and this convention will be used henceforth. In fact, this convention was used for the previous example involving the addition operation. This dual usage the symbol \( \bullet \) will not create any ambiguity since the type of usage of the symbol \( \bullet \) will always be clear from the context.

The next example shows that if \( A \) is a set with algebraic structure with some properties, and \( B \) is a set of finite plenices with the same constitution whose primion panels are the elements of \( A \), then the algebraic structure of \( A \) may be extended to \( B \). In this case, the set \( B \) has the same structure and properties as \( A \).

Before the next example, it will be useful to recall some of the principles of propositional calculus which was founded by the Stoics. Chrysippus (about 281-208 B.C.) gave the definition 'A proposition is what is true or false'. This means that the content of the proposition is ignored, and so is its construction, attention is paid only to its truth value, true or false. Here, true and false will be represented by \( T \) and \( F \), respectively.

There are combinations of propositions for which the truth value of the combined proposition depends only on the truth values of the constituents. Such combinations form the subject matter of propositional calculus. Some combined propositions are familiar from everyday language and were investigated by the Stoics:

'Conjunction': \( a \) and \( b \), written \( a \wedge b \), is defined as the proposition that is true if and only if \( a \) is true and \( b \) is true.

'Disjunction': \( a \) or \( b \), written \( a \vee b \), is defined as the proposition that is true if \( a \) is true or \( b \) is true or both are true.

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These combinations are therefore defined by prescribing their truth values for every possible choice of the truth values of $a$ and $b$. This is illustrated in the table below.

Truth table of classical logic

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$a \land b$</th>
<th>$a \lor b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Two composite propositions $p$ and $q$ are equivalent, written as $p = q$, if they always have the same truth value. The commutative laws

$$a \lor b = b \lor a$$
$$a \land b = b \land a$$

are immediate consequences of the definitions of conjunction and disjunction. It may also be noticed that the words 'and' and 'or' are used commutatively in everyday language. In fact, the above truth table shows at a glance that when the values in the columns for $a$ and $b$ are interchanged, the values in the columns for $a \land b$ and $a \lor b$ do not change.

Using appropriate truth tables, one can also prove the associative laws, that is,

$$a \land (b \land c) = (a \land b) \land c$$

and

$$a \lor (b \lor c) = (a \lor b) \lor c$$

The 'absorption' laws, that is,

$$a \land (a \lor b) = a$$

and

$$a \lor (a \land b) = a$$

may also be proved using truth tables. Truth tables may also be used to deduce the distributive laws,

$$a \land (b \lor c) = (a \land b) \lor (a \land c)$$

and

$$a \lor (b \land c) = (a \lor b) \land (a \lor c)$$

Now, consider a set $L$ with two operations $\land$ and $\lor$. Suppose that these operations are commutative, associative and obey the absorption laws, then $(L, \land, \lor)$ is called a 'lattice'. If a
lattice is such that the distributive laws hold for any triplets of its elements, then it is called a 'distributive lattice'.

Returning to the discussion of the propositional calculus, for every proposition $a$ a 'complement' $a^c$ can be defined by the truth table

<table>
<thead>
<tr>
<th>$a$</th>
<th>$a^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

With the help of a truth table one can see that $a \land a^c$ has the value $F$ for every $a$. It is customary to denote such a proposition (that is, a proposition whose value is always $F$) by $n$. If $n_1$ and $n_2$ are two propositions that are 'always false', then $n_1 = n_2$. Therefore, one can write

$$a \land a^c = n$$  \((*)\)

This is the 'law of contradiction', that is, $a$ and not-$a$ cannot both be true simultaneously.

In contrast, a proposition that is always true is denoted by $e$. Therefore, one can write

$$a \lor a^c = e$$ \((**)\)

This is the 'law of the excluded middle', that is, at least one of the propositions $a$ and not-$a$ is true.

Furthermore,

$$a \land n = n$$

$$a \lor n = a$$

$$a \land e = a$$

and

$$a \lor e = e$$

A lattice in which there are two elements $n, e$ with the above properties and in which for every element $a$ there is an element $a^c$ with the properties (\(\ast\)) and (\(\ast\ast\)) is called a 'complemented lattice'. A distributive complemented lattice is called a 'Boolean lattice' (or 'Boolean algebra').

As an example, consider the nexus

$$[P] = <\Box, \ll\Box, \Box, \Box>, \Box>$$

The dendrogram of $[P]$ is shown in Fig 4.2.2.

Now, let $\mathcal{B}$ be the set of all plenices that are cognates of $[P]$ and the values of their primion panels are $T$ or $F$. The set $\mathcal{B}$ has 32 elements. Some of the elements of $\mathcal{B}$ are as follows:

$$\mathcal{Q} = <T, \ll T, F, T>, F>$$
\[ R = \langle F, \langle F, T, F \rangle, T \rangle \]

and

\[ S = \langle T, \langle F, T, T \rangle, F \rangle \]

The dendrograms of \( Q, R \) and \( S \) are shown in Fig 4.2.3.

For every element of \( B \), such as \( Q, R \) and \( S \), a complement can be defined by changing the value of every primion panel of that element. For example

\[ Q^c = \langle F, \langle F, T, F \rangle, T \rangle \]

\[ R^c = \langle T, \langle T, F, T \rangle, F \rangle \]

and

\[ S^c = \langle F, \langle T, F, F \rangle, T \rangle \]

where \( Q^c, R^c \) and \( S^c \) are complements of \( Q, R \) and \( S \), respectively. The dendrograms of \( Q^c, R^c \) and \( S^c \) are shown in Fig 4.2.4.
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Note that, the relationship of being complements is mutual. That is, if $Q^C$ is a complement of $Q$ then $Q$ is a complement of $Q^C$ and both $Q$ and $Q^C$ are the elements of $B$. In other words, set $B$ is closed under the operation of finding complement. For instance, in the above example $R^C = Q$ and $R = Q^C$.

Among the elements of set $B$, there exist two special plenices. One of these is the plenix in which the values of all its primion panels are true and the other is the plenix in which the values of all its primion panels are false. These plenices are denoted by $E$ and $N$, respectively. The dendrograms of $E$ and $N$ are shown in Fig 4.2.5.

One can see that for every element $R$ of $B$

\[ R \land R^C = N \]
\[ R \lor R^C = E \]

Also,

\[ R \land N = N \]
\[ R \lor \overline{N} = R \]
\[ R \land E = R \]

and

\[ R \lor E = E \]

The dendrograms for two of the above relations are shown in Fig 4.2.6.

\[ \begin{array}{ccc}
F & T & F \\
F & T & F \\
\land & \land & \land \\
R & R^C & N \\
\end{array} \]

\[ \begin{array}{ccc}
F & T & F \\
F & T & F \\
\lor & \lor & \lor \\
R & E & E \\
\end{array} \]

Fig 4.2.6 Dendrograms for relations \( R \land R^C = N \) and \( R \lor E = E \)

It is easy to see that, both \( \land \) and \( \lor \) are idempotent, commutative and associative and they satisfy the absorption laws and distributives laws. So \((B, \land, \lor)\) is a Boolean lattice (or Boolean algebra).
4.3 Binary Operations between two Plenices, Where One is a Subnate of The Other

Now, consider the plenices

\[ S = \langle 2, 4 > \]

and

\[ T = \langle 0.3, 0.6, 0.8 > \]

The plenix \( S \) is a subnate of \( T \). The full image of every primion panel of \( S \) in \( T \) is either a primion panel or a plenix. For example, the full image of the first principal panel of \( S \) is a primion panel in \( T \) and the full image of the first principal panel of the second principal panel of \( S \) namely,

\[ S_{21} = 7 \]

is a plenix in \( T \), namely,

\[ T_{21} = \langle 0.6, 0.1 > \]

The principal panels of this plenix are partial images of

\[ S_{21} = 7 \]

Now, let it be required to obtain the sum of plenices \( S \) and \( T \). A dendrogram for this sum is shown in Fig 4.3.1

For the panels of \( S \) whose corresponding panels in \( T \) are their full images, the process of addition is straightforward. That is, the summation operation acts between a primion panel of \( S \) and a primion panel in \( T \). Thus, the value of \( S_1 = 2 \) is added to the value of \( T_1 = 0.3 \) and the value of \( S_{22} = 4 \) is added to the value of \( T_{22} = 0.8 \). In the case of the panel \( S_{21} = 7 \) of \( S \), the full image in \( T \) is a plenix, namely

\[ T_{21} = \langle 0.6, 0.1 > \]
In this case the summation operation acts between the primion panels $S_{21}$ and its partial images in $T$, individually. Thus, the value of $S_{21} = 7$ is added to the value of $T_{211} = 0.6$ and $T_{212} = 0.1$, individually. Therefore, the sum of $S$ and $T$ may be defined as

$$S + T = \{2, 4\} + \{0.3, 0.6, 0.1\}, 0.8 >\$$
$$= \{2+0.3, \{0.6, 7+0.1\}, 4 + 0.8\}$$
$$= \{2.3, \{7.6, 7.1\}, 4.8\}$$

It is important to notice that, as may be seen from the above example,

- The operation of addition of two plenices is commutative as well as associative,
- If one of the summands is a supernate of the other summand then the constitution of the sum will be the same as that of the summand which is the supernate of the other one.

As another example, consider the plenices

$$R = \{(1, 2\}, \{3, 7\}, \{5, 6, 2\}\$$

and

$$S = \{(3\}, \{(9\}, \{8, 4, 7\}, \{2, 6\}\}$$

The plenix $R$ is a subnate of $S$ and all the primion panels involved are sets. The union of plenices $R$ and $S$ may be written as

$$R \cup S = \{(1, 2\}, \{3, 7\}, \{5, 6, 2\} \cup \{(9\}, \{8, 4, 7\}, \{2, 6\}$$
$$= \{(1, 2\} \cup \{3\}, \{(3, 7) \cup \{9\}, \{3, 7\} \cup \{8, 4, 7\}, \{5, 6, 2\} \cup \{2, 6\}\}$$
$$= \{(1\}, \{2\}, \{3, 7\}, \{3, 7, 8, 4\}, \{5, 6, 2\}\$$

A dendrogram representing the union of plenices $R$ and $S$ is shown in Fig4.3.2.

Note that, the union operation on the set of plenices whose panels are sets is both commutative, associative and idempotent, that is, $P \cup P = P$. 

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4.4 Binary Operations between Two Plenices that are Pernates of Each Other

Consider two plenices

\[ R = \langle 1, 5, 2 \rangle, 8, \langle 4, 7 \rangle \]

and

\[ S = \langle 12, 34, 71 \rangle, \langle 62, 29 \rangle, 54 \]

For clarity, in plenix \( R \), the primion panels are printed in bold. The plenices \( R \) and \( S \) are pernates of each other. The dendrograms for \( R \) and \( S \) are shown in Fig 4.4.1.

![Dendrograms of R and S](image)

Let it be required to obtain the sum of the plenices \( R \) and \( S \), that is,

\[ R + S \]

Since the plenices \( R \) and \( S \) are pernates of each other, for every primion panel \( r \) of \( R \) there are three possibilities as follows:

Case 1: There is a primion panel in \( S \) that is a full image of \( r \). For example, the primion panels

\[ R_{11} = 1 \]
\[ R_{12} = 5 \]

and

\[ R_{13} = 2 \]

of \( R \) have three primion panels in \( S \) as full images, namely,

\[ S_{11} = 12 \]
\[ S_{12} = 34 \]

and

\[ S_{13} = 71 \]
In this case the summation operation acts between a primion panel of $R$ and its full image in $S$. Therefore, the value of $R_{11} = 1$ is added to the value of $S_{11} = 12$ to give rise to the corresponding panel of $R + S$. Also the value of $R_{12} = 5$ is added to the value of $S_{12} = 34$ and so on.

Case 2: There is a nonprimion panel in $S$ that is a full image of $r$. For example, the primion panel $R_2 = 8$

has a nonprimion panel in $S$ as a full image, that is,

$S_2 = <62, 29>$

In this case the summation operation acts between the primion panel of $R$ and its partial images in $S$, individually. So, the value of $R_2 = 8$ is added to the values of $S_{21} = 62$ and $S_{22} = 29$, individually.

Case 3: There is a primion panel in $S$ as a hyper image of $r$. For example, the primion panels $R_{31} = 4$

and $R_{32} = 7$

have a primion panel in $S$ as hyper image, that is,

$S_3 = 54$

Note that, $R_{31}$ and $R_{32}$ have no partial image or full image in $S$. In this case the summation operation acts between the primion panels $R_{31}$ and $R_{32}$ and their hyper image in $S$, that is, $S_3$.

The sum of $R$ and $S$ is a plenix, whose constitution is the same as the constitution of the union of nexuses $[R]$ and $[S]$, that is,

$[R] \cup [S]$

and the values of the primion panels of $R + S$ may be obtained as explained in the above three cases. Thus,

\[
R + S = <<1, 5, 2>, 8, <4, 7>>, <<12, 34, 71>, <62, 29>, 54>
\]

\[
= <<1 + 12, 5 + 34, 2 + 71>, <8 + 62, 8 + 29>, <4 + 54, 7 + 54>>
\]

\[
= <<13, 39, 73>, <70, 37>, <58, 61>>
\]

The dendrogram of $R + S$ is shown Fig 4.4.2.

In general, if $P$ and $Q$ are two plenices whose primion panels are members of a set $A$ and if $'*'$ is a binary operation on $A$, then

(1) The operation $P * Q$ is possible if $P$ and $Q$ are not abnates of each other.

(2) If $P$ and $Q$ are cognates then $P * Q$ will also be a cognate of $P$ and $Q$.

(3) If $P$ is a subnate of $Q$ then $P * Q$ will be a cognate of $Q$. Also, if $P$ is a supernate of $Q$ then $P * Q$ will be a cognate of $P$. 

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(4) If $P$ and $Q$ are pernates of each other then $P \ast Q$ will be cognate of $[P] \cup [Q]$.

![Dendrogram for summation of two plenices that are pernates of each other](image)

Fig 4.4.2 Dendrogram for summation of two plenices that are pernates of each other

### 4.5 Operator Plenices

In the above three sections, operation between two plenices that are cognates, subnates or pernates of each other are discussed. In all these cases, operation plenices are meaningful provided that all the primion panels of the operands are of the same mathematical type. Now, one may require operating between two plenices with different types of mathematical objects as primion panels. To do this, the concept of ‘operator plenix’ must be introduced.
**Definition 4.5.1** The term ‘operator plenix’ is used to refer to a plenix every primion panel of which is a binary operation.

For examples

\[(O) = <<+, +>>, <+>, +>>\]

and

\[(O) = <<+, \cup>, \lor>>\]

are operator plenices, where \(\cup\) is the symbol of ‘union operation’ for sets and \(\lor\) is the symbol for the logical ‘or’ operation for Booleans. Here, a convention is adapted where the symbol for an operator plenix contains opening and closing parentheses. The dendrograms for the above plenices are shown in Fig 4.5.1.

![Dendrograms for operator plenices (O) and (O)](image)

Note that each of the symbols +, \(\cup\) and \(\lor\) represents a ‘rule for a process’ and this rule has a clear definition of equality, see section 2.2. Therefore, due to the clarity of the definition of an operation (or a function) it may appear as a primion panel of a plenix, see section 2.2. The application of an operator plenix for operation between two plenices that have different mathematical objects as primion panels, is introduced through an example. Consider the plenices

\[R = <<5, \{4, 7\}>, T>\]

and

\[S = <<9, \{7, 8\}>, F>\]

The primion panels of these plenices involve different types of mathematical objects, namely, numbers, sets and Boolean entities. Now, consider the construct

\[R(O)S\]

where \((O)\) is the operator plenix in Fig 4.5.1, namely,

\[(O) = <<+, \cup>, \lor>>\]
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The dendrogram of the plenices $R$, $(O)$, $S$ and $R(0)S$ are shown in Fig 4.5.2.

![Dendrogram](image)

Fig 4.5.2 Dendrograms for the operation $R(0)S$

The construct $R(0)S$ represents a plenix that is obtained through a series of operations between the primion panels of $R$ and $S$ as follows:

$\begin{align*}
R(0)S &= <<5, \{4, 7\}>, T> <<+, \cup>, \lor> <<9, \{7, 8\}>, F>
= <<5 + 9, \{4, 7\} \cup \{7, 8\}>, T \lor F>
= <<14, \{4, 7, 8\}>, T>
\end{align*}$

In the above example, the plenices $R$, $(O)$, $S$ and $R(0)S$ are cognates. In this case every primion panel of $R(0)S$ is obtained by an operation between the corresponding primion panels of $R$ and $S$ as operands and with the corresponding primion panel of $(O)$ as operator.

4.6 Constitution of an Operator Plenix

An operator plenix can be a nontrivial subnate of either of its operands. For example, consider the plenices

$P = <<5, <<3, 1>>, 7>>$

and

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The dendrograms of $P$ and $Q$ are shown in Fig 4.6.1. An operator plenix for these operands can be a nontrivial subnate of $[P]$ or $[Q]$. To elaborate, the nexus of this operator plenix can be

\[
\langle □, \langle □, □ \rangle, □ \rangle
\]

or

\[
\langle □, □ \rangle
\]

Some possible operator plenices for $P$ and $Q$ are

\[
(T) = <+, \langle<<, +>, \times\rangle>
\]

\[
(S) = <\times, \langle\times, +\rangle>
\]
and

\((R) = <+ , \times >\)

The results of the operations between plenices \(P\) and \(Q\) using the above operator plenices are shown in Fig 4.6.3.

In Fig 4.6.3 (a), (b) and (c), every primion panel of an operator plenix either has two primion panels in \(P\) and \(Q\) as full images, see Fig 4.6.3 (a), or has more than two primion panels in \(P\) and \(Q\) as partial images, in Fig 4.6.3 (c). To elaborate, the second principal panel of operator plenix \((R)\) has three partial images in \(P\), namely, \(P(2, 1, 1), P(2, 1, 2)\) and \(P(2, 2)\) and three partial images in \(Q\), namely, \(Q(2, 1, 1), Q(2, 1, 2)\) and \(Q(2, 2)\).

In the case when a primion panel of the operator plenix has a full image in \(P\) and a full image in \(Q\), the primion panel of the operator plenix operates between its full images in \(P\) and \(Q\) as operand.

In the case when there are partial images, the primion panel of the operator plenix operates between its partial images in \(P\) and \(Q\) as operands, individually.

As far as an operation (or a function) is concerned, its ‘value’ is a ‘rule’ for the mapping of the elements of a set that is called the ‘domain’ of the operation (or function) into the elements of a set that is called the ‘range’ of the operation (or function). Two operations (or functions) may then be said to be ‘equal’ provided that they have the same value. Therefore, an operation (or a function) has a clear definition of equality and this will allow it to be a primion panel of a plenix.

Now, consider the operator plenix

\((O) = << \cap, +\rangle, \cup \rangle\)

This operator plenix has the operations of intersection, addition and union as primion panels. All of these three operations are commutative and associative. So, the above operator plenix may be said to be ‘commutative’ and ‘associative’.

In general, if all the primion panels of an operator plenix share a property then the operator plenix is said to have that property. Conversely, if an operator plenix is said to have a property then it is expected that all the primion panels of the operator plenix have that property.
(a) Dendrogram for operation $P(T)Q$

(b) Dendrogram for operation $P(S)Q$

Figure continued
An operator plenix may operate between two identical plenices. For example, let $P$ be a plenix and let $(O)$ be an operator plenix. The operation between $P$ and itself may be written as

$$P(O)P = P(O)^2$$

In addition, if the operator plenix is associative then

$$P(O)^n = (P(O)^{n-1})(O)P$$

for positive integer $n>1$

and

$$P(O)^n(O)P(O)^m = P(O)^{n+m}$$

for positive integers $m>1$ and $n>1$
4.7. Operations between Subnate/Supernate Plenices, Using Operator Plenices

Consider plenices
\[ P = \langle \{7, 5\}, <3, 9> \rangle \]
And
\[ Q = \langle \{4, 7\}, \{2, 6\}, <2, 8> \rangle \]
The plenix \( P \) is a subnate of \( Q \). The dendrograms of \( P \) and \( Q \) are shown in Fig 4.7.1.

![Dendrograms of P and Q](image)

**Fig 4.7.1 Dendrograms of \( P \) and \( Q \)**

Suppose that one requires to operate between \( P \) and \( Q \) with an operator plenix. The operator plenix that can operate between \( P \) and \( Q \) must be a nontrivial subnate of \( Q \).

Some operator plenices that can be used in this case are as follows:

\( (O) = \langle \cup, \cap, <+, \times> \rangle \)
\( (R) = \langle \cap, \cap, <+, \neg> \rangle \)
\( (S) = \langle \cup, \times, +\rangle \)
\( (T) = \langle \cap, \cup, \times\rangle \)

and
\( (U) = \langle \cup, \pm\rangle \)

The dendrograms for these operator plenices are shown in Fig 4.7.2. The first and second of the above operator plenices, that is, \( (O) \) and \( (R) \) are cognates of \( Q \) and the remaining three are nontrivial subnates of \( Q \). For example, the operator plenix \( (T) \) will operate between \( P \) and \( Q \) as follows:

\[
P(T)Q = \langle \{7, 5\}, <3, 9> \rangle \langle \cap, \cup, \times\rangle \langle \{4, 7\}, \{2, 6\}, <2, 8> \rangle \\
= \langle \{7, 5\} \cap \{4, 7\}, \{7, 5\} \cup \{2, 6\}, <3 \times 2, 9 \times 8> \rangle \\
= \langle \{7\}, \{7, 5, 2, 6\}, <6, 72> \rangle 
\]

The result is a plenix that is a cognate of \( Q \). The dendrogram for the above operation is shown in Fig 4.7.3.
Fig 4.7.2 Dendrograms for operator plenices \((O), (R), (S), (T)\) and \((U)\)

Fig 4.7.3 Dendrogram for operation \(P(T)Q\)
4.8 Operations between Plenices that are Pernates of Each Other, Using Operator Plenices

The use of operator plenices for operations between plenices that are pernates of each other is discussed in this section. Consider the plenices

\[ R = \langle\{1, 7\}, \langle5, 3, 9\rangle\rangle \]

and

\[ S = \langle\langle1, 2\rangle, \langle8, 4\rangle\rangle, \langle6\rangle\rangle \]

The plenices \(R\) and \(S\) are pernates of each other. The dendrograms of these plenices are shown in Fig 4.8.1.

The constitution of an operator plenix that can operate between \(R\) and \(S\) must be a nontrivial subnate of the union of \([R]\) and \([S]\), that is,

\[ [R] \cup [S] = \langle\square, \langle\square, \square, \square\rangle\rangle \cup \langle\langle\square, \square\rangle, \square\rangle \]

\[ = \langle\langle\square, \square\rangle, \langle\square, \square, \square\rangle\rangle \]

The dendrograms of nontrivial subnates of \([R] \cup [S]\) are shown in Fig 4.8.2

Now, consider the operator plenix

\[ (O) = \langle\langle\cup, \cap\rangle, \langle+, \times, \_\rangle\rangle \]

This operator plenix is a cognate of \([R] \cup [S]\) and can operate between plenices \(R\) and \(S\) as follows:

\[ R(O)S = \langle\{1, 7\}, \langle5, 3, 9\rangle\rangle \cup \langle\langle\cup, \cap\rangle, \langle+, \times, \_\rangle\rangle \langle\{1, 2\}, \langle8, 4\rangle\rangle, \langle6\rangle\rangle \]

\[ = \langle\langle\{1, 7\} \cup \{1, 2\}, \{1, 7\} \cap \{8, 4\}\rangle, \langle5 + 6, 3 \times 6, 9 - 6\rangle\rangle \]

\[ = \langle\langle\{1, 7, 2\}, \{}\rangle, \langle11, 18, 3\rangle\rangle \]
Here the symbol {} denotes the empty set. The result is a plenix that is a cognate of \([R] \cup [S]\).

The dendrogram of the operation \(R(0)S\) is shown in Fig 4.8.3.

![Dendrogram for \(R(0)S\)](image)

Note that, in the sections 4.5, 4.7 and 4.8 three modes of operation between two plenices with an operator plenix are defined. In the first mode (section 4.5), operands are cognates of each other. In the second mode (section 4.7), one operand is a subnate of the other. In the third mode...
(section 4.8), the operands are pernates of each other. Generally, in all the above modes the constitution of the result of the operation is the same as the constitution of the union of the operands and the constitution of the operator plenix is the same as the constitution of a nontrivial subnate of the union of the operands. To elaborate, in the first mode, since two operands are cognates of each other so the union of the operands is a cognate of either one of the operands. In this mode, the constitution of the operator plenix is the same as a nontrivial subnate of each operand. In the second mode, since one operand is a subnate of the other, so the constitution of the union of the operands is the same as the constitution of the operand which is the supernate of the other. In this mode, the operator plenix should be a nontrivial subnate of the operand which is the supernate of the other.

As an example, consider the plenices

\[ P = \langle\langle 3, 2, 5\rangle, 7\rangle \]

and

\[ Q = \langle 8, \langle 1, 4, 9\rangle \rangle \]

and the operator plenix

\[ (T) = \langle\langle \times, \times, \times\rangle, \langle \times, \langle +, \rightarrow, \times\rangle \rangle \]

The dendrograms of \( P \), \( Q \) and \( (T) \) are shown in Fig 4.8.4.

![Dendrograms of plenices P and Q and (T)](image)

Fig 4.8.4 Dendrograms of plenices \( P \) and \( Q \) and \( (T) \)
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The operator plenix \((T)\) is a supernate of both plenices \(P\) and \(Q\) and it is also a supernate of \(P \cup Q\). Since the primion panels \(P_2 = 7\) and \(Q_{22} = 4\) correspond to two operations, namely, \((T)_{221} = +\) and \((T)_{222} = -\), so the operation \(P(T)Q\) is not meaningful.

However, this fact that the constitution of an operator plenix must be a nontrivial subnate of the union of the operands is a necessary condition but it is not a sufficient condition. For example, consider plenices

\[ S = <<4, F>, \{3, 7\>> \]

and

\[ R = <<8, 2>, T>, \{1, 9\}> \]

The dendrogram of \(S\) and \(R\) are shown in Fig 4.8.5.

![Dendrogram of S and R](image)

Fig 4.8.5 Dendrograms of \(S\) and \(R\)

The nexus

\([Q] = <\Box, \Box>\)

is a nontrivial subnate of \(R\) but there is no operator plenix with the constitution as \([Q]\) such that the operation between \(S\) and \(R\) is meaningful. This is because the first primion panel of \([Q]\) has primion panels of different type of mathematical object in \(S\) and \(R\) as partial images. To elaborate, the first primion panel of \([Q]\) has two primion panels in \(S\) as partial images, that is, the number 4 and the Boolean entity \(F\) and also it has three primion panels in \(R\) as partial images, that is, the numbers 8 and 2 and the Boolean entity \(T\). These primion panels of \(S\) and \(R\) are not the same type of mathematical object and corresponding to these different types of mathematical objects there is only one operation.

4.9 Operations between a Plenix and a Primion

In this section, an operation between a plenix and a primion (non-plenix mathematical object) is discussed. This kind of operation is illustrated by some examples.
Now, consider the number
\[ k = 4 \]
and the plenix
\[ T = <<2, 7>, 1> \]
The 'summation' between \( k \) and \( T \) may be defined as follows:
\[
\begin{align*}
    k + T &= 4 + <<2, 7>, 1> \\
    &= <<4 + 2, 4 + 7>, 4 + 1> \\
    &= <<6, 11>, 5>
\end{align*}
\]
In other words, the summation operation acts between the primion \( 4 \) and all the primion panels of \( T \), individually. The plenices \( T \) and \( k + T \) are cognates of each other.

Now, let \( k \) be a real number and let \( P \) be a plenix. Then \( kP \) is a plenix that is obtained by multiplying each primion panel of \( P \) by \( k \). The plenices \( P \) and \( kP \) are cognates of each other. For example suppose that
\[
k = 10
\]
and
\[
P = <4, \begin{bmatrix} 2 & 7 \\ 9 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 8 & 5 \end{bmatrix}>
\]
then
\[
kP = 10 \times <4, \begin{bmatrix} 2 & 7 \\ 9 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 8 & 5 \end{bmatrix}>
\]
\[
= <10 \times 4, <10 \times \begin{bmatrix} 2 & 7 \\ 9 & 1 \end{bmatrix}, 10 \times \begin{bmatrix} 3 & 8 & 5 \end{bmatrix}>
\]
\[
= <40, \begin{bmatrix} 20 & 70 \\ 90 & 10 \end{bmatrix}, \begin{bmatrix} 30 & 80 & 50 \end{bmatrix}>
\]
In the above example the construct \( kP \) may be considered as a definition of scalar multiplication for a plenix. In other words, one may consider this operation as multiplying a plenix by a number \( k \). Note that, this scalar multiplication is defined only for any plenix in which scalar multiplication is defined for all its primion panels.

As another example for an operation between a primion and a plenix, let
\[
k = \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix}
\]
be a matrix and let
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$Q = <<2, \begin{bmatrix} 9 & 2 \\ 0 & 3 \end{bmatrix}>, <8, \begin{bmatrix} 6 \\ 6 \end{bmatrix}>>$

be a plenix whose primion panels are either a scalar or a matrix. Then

$$kQ = \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times <<2, \begin{bmatrix} 9 & 2 \\ 0 & 3 \end{bmatrix}>, <8, \begin{bmatrix} 6 \\ 6 \end{bmatrix}>>$$

$$= <<\begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times 2, \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times 0, \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times 8, \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times 6 >>$$

$$= <<8, 14, 36, 29>, <32, 56, 66>, 8, 40, 36>>$$

The dendrogram of $kQ$ is shown in Fig 4.9.1.

In the above constructs, namely,

$$k + T$$

$$kP$$

and

$$kQ$$

one can change the order of the operands and the constructs may be written as follows:

$$T + k$$

$$Pk$$

and

$$Qk$$

However, in general, changing the order of the operands is only acceptable provided that the operation is defined for the new order of operands. To elaborate, consider the following four examples.

Example 4.9.1: let

$k = 8$

be a number and let

$P = <<3, 6>, 2>$

be a plenix. Then

$$k + P = 8 + <<3, 6>, 2>$$

$$= <<8 + 3, 8 + 6>, 8 + 2>$$

$$= <<11, 14>, 10>$$

and

$$P + k = <<3, 6>, 2> + 8$$

$$= <<3 + 8, 6 + 8>, 2 + 8>$$
Thus, \( k + P \) is equal to \( P + k \) In other words, the summation operation between a number and a plenix whose primion panels are numbers is commutative.

\[
kQ = \begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 0 & 3 \end{bmatrix}
\]

\[
kQ = \begin{bmatrix} 8 & 14 \\ 2 & 10 \end{bmatrix}
\]

Fig 4.9.1 Dendrograms for \( Q \) and \( kQ \)

**Example 4.9.2:** let

\[
k = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}
\]
be a matrix and let
\[
R = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
6 & 4 \\
1 & 7
\end{bmatrix}
\]
be a plenix whose primion panels are matrices, then
\[
k \times R = \begin{bmatrix}
2 & 0 \\
1 & 5
\end{bmatrix} \times \begin{bmatrix}
2 & 0 \\
0 & 1 \\
6 & 4 \\
1 & 7
\end{bmatrix} = \begin{bmatrix}
4 & 0 \\
2 & 5
\end{bmatrix}, \begin{bmatrix}
0 & 6 \\
30 & 23
\end{bmatrix} \times \begin{bmatrix}
18 & 2
\end{bmatrix}
\]
and
\[
R \times k = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
6 & 4 \\
1 & 7
\end{bmatrix} \times \begin{bmatrix}
2 & 0 \\
1 & 5
\end{bmatrix} = \begin{bmatrix}
4 & 0 \\
3 & 15
\end{bmatrix}, \begin{bmatrix}
0 & 6 \\
16 & 20
\end{bmatrix} \times \begin{bmatrix}
19 & 5
\end{bmatrix}
\]
It can be seen that both \(k \times R\) and \(R \times k\) are defined but they are not equal. In other words, this operation is not commutative.

**Example 4.9.3:** let
\[
k = \begin{bmatrix}
2 \\
1
\end{bmatrix}
\]
be a column vector and let
\[
S = \begin{bmatrix}
4 & 0 \\
5 & 3 \\
8 & 6
\end{bmatrix}
\]
be a plenix whose primion panels are row vectors, then
\[
k \times S = \begin{bmatrix}
2 \\
1
\end{bmatrix} \times \begin{bmatrix}
4 & 0 \\
5 & 3 \\
8 & 6
\end{bmatrix} = \begin{bmatrix}
8 & 0 \\
4 & 0
\end{bmatrix}, \begin{bmatrix}
10 & 6 \\
5 & 3
\end{bmatrix} \times \begin{bmatrix}
16 & 12
\end{bmatrix}
\]
and
\[ S \times k = <\begin{bmatrix} 4 & 0 \\ 5 & 3 \\ 8 & 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} > \]

\[ = <\begin{bmatrix} 4 & 0 \\ 5 & 3 \\ 8 & 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} > \]

It can be seen that both \( k \times S \) and \( S \times k \) are defined. However, the result of \( k \times S \) is a plenix whose primion panels are matrices but the result of \( S \times k \) is a plenix whose primion panels are numbers.

**Example 4.9.4:**

Let

\[ k = \begin{bmatrix} 1 & 5 \end{bmatrix} \]

be a row vector and let

\[ T = <\begin{bmatrix} 1 & 0 \\ 4 & 7 \\ 0 & 2 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 0 \end{bmatrix} > \]

be a plenix then

\[ k \times T = \begin{bmatrix} 1 & 5 \end{bmatrix} \times <\begin{bmatrix} 1 & 0 \\ 4 & 7 \\ 0 & 2 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 0 \end{bmatrix} > \]

\[ = <\begin{bmatrix} 1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 4 & 7 \\ 0 & 2 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 5 \end{bmatrix} \times \begin{bmatrix} 8 & 6 \\ 5 & 0 \end{bmatrix} > \]

\[ = <\begin{bmatrix} 21 & 35 \end{bmatrix}, \begin{bmatrix} 15 & 47 \end{bmatrix}, \begin{bmatrix} 33 & 6 \end{bmatrix} > \]

Now, consider the construct

\[ T \times k = <\begin{bmatrix} 1 & 0 \\ 4 & 7 \\ 0 & 2 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 0 \end{bmatrix} > \times \begin{bmatrix} 1 & 5 \end{bmatrix} \]

Since the matrix

\[ \begin{bmatrix} 1 & 5 \end{bmatrix} \]

is not compatible for multiplication with any primion panel of \( T \), then the above operation is not meaningful.

Considering the above four examples, some facts about an operation between a primion \( k \) and a plenix \( P \) are as follows:

1- The result of an operation between a primion \( k \) and a plenix \( P \), or a plenix \( P \) and a primion \( k \) is a plenix.
2- If $\otimes$ represents any operation between a primion and a plenix, then the plenices $k \otimes P$, $P \otimes k$ and $P$ are cognates of each others, that is,

$$[k \otimes P] = [P \otimes k] = [P]$$

3- The familiar operations such as $+$, $\times$, $\cup$, $\cap$, $\lor$, $\ldots$ can be used between a primion $k$ and a plenix $P$ (or between a plenix $P$ and a primion $k$) provided that the operations between $k$ and all the primion panels of $P$ are defined. In this case $k$ and $P$ ($P$ and $k$) are said to be conformable for the operation. If the operation (between two primion) is commutative then the operation between $k$ and $P$ ($P$ and $k$) is also commutative (as seen in Example 4.9.1).

4- If the operation is conformable but not commutative then the value of corresponding primion panels in $k \otimes P$ and $P \otimes k$ are not necessarily equal. In this case the full image primion panels in $k \otimes P$ and $P \otimes k$ either may be the same type of mathematical object (as seen in Example 4.9.2) or may not be of the same type (as seen in Example 4.9.3).

5- If the operation is not commutative then it is possible that the operation between a primion $k$ and a plenix $P$ is meaningful but the operation between a plenix $P$ and a primion $k$ is not meaningful (as seen in Example 4.9.4).

6- If an expression involves more than one operation between a primion and a plenix, then the result of the combination of the objects may be dependent on the way in which the objects are grouped. In other words, the order of combination is important. For example, consider the expressions

$$\left( \begin{bmatrix} 8 & 3 \\ 0 & 7 \end{bmatrix} \times <2, <5, 1>> \right) + \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 8 & 3 \\ 0 & 7 \end{bmatrix} \times ( <2, <5, 1>> + \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix})$$

These expressions have the same terms and operations, but they have different orders of association (combination). The result of the first expression may be obtained as follows:

$$\left( \begin{bmatrix} 8 & 3 \\ 0 & 7 \end{bmatrix} \times <2, <5, 1>> \right) + \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix} = \left( <16, 6> \times \begin{bmatrix} 40 & 15 \\ 0 & 35 \end{bmatrix}, \begin{bmatrix} 8 & 3 \\ 0 & 7 \end{bmatrix} \right) + \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 \\ 4 & 20 \end{bmatrix} <49, 17>, \begin{bmatrix} 17 & 5 \\ 4 & 13 \end{bmatrix}$$

However, as far as the second expression is concerned, the term
\[
(\langle 2, <5, 1> \rangle + \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix})
\]

involves summation of scalars and a matrix which is not meaningful. Therefore, the second expression is not a valid expression.

4.10 Plenix Functions

In this section the notion of a plenix function is discussed. Starting with an illustrative example, consider the plenix

\[ R = \langle<\pi/2, \pi/6>, <\pi, \pi/4, \pi/3>\rangle \]

The primion panels of \( R \) are angles measured in radians. The dendrogram of \( R \) is shown in Fig 4.10.1a. The construct

\[ \sin R \]

is a ‘plenix function’ where \( R \) is the argument of the function. The construct \( \sin R \) represents a plenix that has the same constitution as \( R \), every primion panel of which will be the sine of its corresponding primion panel in \( R \) as follows:

\[
\sin(R) = \sin(\langle<\pi/2, \pi/6>, <\pi, \pi/4, \pi/3>\rangle)
\]

\[
= \langle<\sin(\pi/2), \sin(\pi/6)>, <\sin(\pi), \sin(\pi/4), \sin(\pi/3)>\rangle
\]

\[
= \langle<1, 1/2>, <0, \sqrt{2}/2, \sqrt{3}/2>\rangle
\]

The dendrogram of \( \sin R \) is shown in Fig 4.10.1b.

Generally, let \( f \) be a function and let \( D \) be the domain of \( f \). Suppose that \( D^* \) is the set of all plenices whose primion panels belong to \( D \). Then \( f \) can be used on any plenix that is a member of \( D^* \) by acting on every primion panel of this plenix.

For example, let \( g \) be a function defined by

\[ g(x) = \sqrt{9 - x^2} \]

The domain of \( g \) is the set of all real numbers that are less than or equal to 3 and greater than or equal to -3, that is,

\[ D = [-3, 3] \]

Therefore, \( D^* \) is the set of all plenices whose primion panels belong to \( D \). For instance, the plenix

\[ S = \langle<2, <0, 3, -1>, <1, -2>\rangle \]

is a member of \( D^* \). The dendrogram of which is shown in Fig 4.10.2a. The function \( g \) may act on \( S \) as follows:
\[ g(S) = g(<2, <0, 3, -1>, <1, -2>>) \]
\[ = <g(2), <g(0), g(3), g(-1)>, <g(1), g(-2)>> \]
\[ = <\sqrt{5}, <3, 0, \sqrt{8}>, <\sqrt{8}, \sqrt{5}>> \]

The constitutions of \( S \) and \( g(S) \) are the same. The dendrogram for \( g(S) \) is shown in Fig 4.10.2b.

Fig 4.10.1a Dendrogram of \( R \)

Fig 4.10.1b Dendrogram of \( \sin R \)
Fig 4.10.2a Dendrogram of $S$

From now on, the set of all plenices whose primion panels are real numbers is denoted by $P_{91}$ and the members of this set are called 'real plenices'. Now, let

$$M = \langle 4, 7, 1 \rangle, \langle 3, 5 \rangle$$

be a real plenix and $h$ be a function from $P_{91}$ to $P_{91}$ denoted by
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\[ h(P) = \begin{cases} 
2P & \text{if } [P] \subseteq [M] \\
P + M & \text{if } [P] \supseteq [M] \\
\langle \rangle & \text{otherwise}
\end{cases} \]

For instance, if

\[ P = \langle 8, 3, 8 \rangle, 6 \rangle \]
then \([P] \subseteq [M]\) and therefore

\[ h(P) = h(\langle 8, 3, 8 \rangle, 6) \]
\[ = 2(\langle 8, 3, 8 \rangle, 6) \]
\[ = \langle 16, 6, 16 \rangle, 12 \rangle \]

Also, if

\[ P = \langle 40, 70, 10 \rangle, <30, <50, 50> \rangle \]
then \([P] \supseteq [M]\) and therefore

\[ h(P) = P + M \]
\[ = \langle 40, 70, 10 \rangle, <30, <50, 50> \rangle + \langle 4, 7, 1 \rangle, <3, 5 \rangle \]
\[ = \langle 44, 77, 11 \rangle, <33, <55, 55> \rangle \]

Furthermore, if

\[ P = \langle 8, <6, 9>, 0 \rangle, 2 \rangle \]
then plenices \(P\) and \(M\) are supernates of each other, so

\[ h(P) = h(\langle 8, <6, 9>, 0 \rangle, 2 \rangle) \]
\[ = \langle \rangle \]

As another example of a plenix function, consider the operator plenix

\[ (O) = \langle +, \times, + \rangle \]

Now, let \(f\) be a function from \(P\) to \(P\) denoted by

\[ f(P) = P(O)^2 + 3P \]

The domain of \(f\) is the set of all real plenices that are supernates of \([O]\). For instance, the real plenix

\[ P = \langle 7, <2, 5>, <9, 6> \rangle \]
is a member of the domain of \(f\). The result of the application of \(f\) on \(P\) is a plenix whose primion panels may be obtained as follows:
\[ f(P) = P(O)^2 + 3P \]
\[ = \langle 7, 5, 9 \rangle \langle 9, 6 \rangle \langle +, \times \rangle \langle 7, 5, 9 \rangle \langle 9, 6 \rangle \]
\[ + 3(\langle 7, 5, 9 \rangle \langle 9, 6 \rangle) \]
\[ = \langle 7 + 7, 5 \times 5, 9 + 9, 6 + 6 \rangle \]
\[ + \langle 3 \times 7, 3 \times 5, 3 \times 9, 3 \times 6 \rangle \]
\[ = \langle 14, 25 \rangle \langle 18, 12 \rangle + \langle 21, 15 \rangle \langle 27, 18 \rangle \]
\[ = \langle 14 + 21, 4 + 6, 25 + 15, 18 + 27, 12 + 18 \rangle \]
\[ = \langle 35, 10, 40 \rangle, <45, 30> \]

For any \( P \) that is in the domain of \( f \), \( P \) and \( f(P) \) are cognates of each other. The dendrogram for \( f(P) \) is shown in Fig 4.10.3.
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Figure continued from previous page

$$3P = 3 \times P$$

$$P(O)^2 + 3P = P(O)^2 + 3P$$

$$= P(O)^2 + 3P$$

$$f(P) = P(O)^2 + 3P$$
4.11 Plenix Functions of Real Variables

Now, let \( f_1, f_2 \) and \( f_3 \) be three real-valued functions of a real variable \( x \), defined by

\[
\begin{align*}
  f_1(x) &= \sqrt{25 - x^2} \\
  f_2(x) &= e^x \\
  f_3(x) &= \frac{1}{x^2 - 1}
\end{align*}
\]

and

The plenix

\[
F(x) = \langle f_1(x), f_2(x), f_3(x) \rangle
\]

is a 'plenix-valued' function of \( x \) or, for short, a plenix function of \( x \). In other words, the argument of \( F(x) \), that is, \( x \) is a real number and the value of \( F(x) \) is a plenix. This function maps a real number into a plenix whose constitution is the same as the plenix \( F(x) \). The value of every primion panel of \( F(x) \) is determined by the values of \( f_1(x), f_2(x) \) and \( f_3(x) \). For example, if

\[ x = 0 \]

Then

\[
F(0) = \langle f_1(0), f_2(0), f_3(0) \rangle
\]

\[ = \langle 5, 1, -1 \rangle \]

The domain of \( F(x) \) is the intersection of the domains of \( f_1(x), f_2(x) \) and \( f_3(x) \). To elaborate, the domain of \( f_1(x) \) is the set of all real numbers that are greater than or equal to \(-5\) and less than or equal to \(5\), that is, \([−5, 5]\), the domain of \( f_2(x) \) is the set of all real numbers, that is, \(\mathbb{R}\) and the domain of \( f_3(x) \) is the set of all real numbers except for the numbers \(-1\) and \(1\), that is, \(\mathbb{R} \setminus \{-1, 1\}\).

Therefore, the domain of \( F(x) \) is

\[
[-5, 5] \cap \mathbb{R} \cap (\mathbb{R} \setminus \{-1, 1\}) = [-5, -1) \cup (-1, 1) \cup (1, 5] = [-5, 5] \setminus \{-1, 1\}
\]

Alternatively, in the above example, the plenix function \( F(x) \) may be written as follows:

\[
F(x) = \langle \sqrt{25 - x^2}, e^x, \frac{1}{x^2 - 1} \rangle
\]
The function $F(x)$ is a plenix function of one variable. The symbols used for plenix functions are $F, H, G,$ etc. Such a symbol represents a rule for production of a plenix value. The function values corresponding to $x$, are denoted by $F(x), H(x), G(x),$ etc. These are customarily called the $F$ function, $H$ function, $G$ function, etc, of $x$.

Talking in general about functions, sometimes, an expression may be considered as a function. For example, the expression

$$y = 4x^2 + 9$$

defines $y$ as a function of $x$. When it is specified that the domain is (for example) the set of real numbers, $y$ is then a function of $x$, that is, a value $y$ is associated with each real value $x$ by multiplying the square of $x$ by 4 and adding 9 to the result.

A function $f$ can be regarded as a set of ordered pairs $(x, y)$. Each ordered pair is said to be an element of the function, where the domain of the function is the collection of all objects that occur as the first members of the elements and the range is the collection of all objects that occur as the second members of the elements.

### 4.12 Plenix Functions of Several Variables

A 'plenix valued function of several independent variables' is a function which takes on a value or values corresponding to every set of values of several independent variables. For example, consider the plenix valued function $H$ of two variables $m$ and $n$ defined by

$$H(m, n) = \begin{cases} m^2 + n & \text{if } m \leq n \\ m - n & \text{if } m > n \end{cases}$$

where $m$ and $n$ can be any real numbers. This function may be regarded either as a function of two variables $m$ and $n$ or as a function of points $(m, n)$. In the latter case, the domain is said to be the entire plane. In other words,

$$H : \mathbb{R}^2 \to \mathbb{R}$$

For instance, if $m = 2$ and $n = 1$ then

$$H(2, 1) = \begin{cases} 2 + 1 & \text{if } 2 \leq 1 \\ 2 - 1 & \text{if } 2 > 1 \end{cases}$$

$$= 3, <1, 2>$$

Also, if $m = 1$ and $n = 2$ then

$$H(1, 2) = \begin{cases} 1 + 2 & \text{if } 1 \leq 2 \\ 1 - 2 & \text{if } 1 > 2 \end{cases}$$

$$= 3, <3, 2>$$

Furthermore, if $m = n = 0$ then...

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\[ H(0, 0) = \langle 0, 0, 0 \rangle \]

As another example of a plenix function of more than one variable, consider a plenix function \( G \) of three variables \( m, n \) and \( k \) defined by

\[
G(m, n, k) = \begin{cases} 
  \langle m + n, \frac{2n + k}{n}, m^2 - n, m + n + k \rangle & \text{if } m + n > k + 2 \\
  \langle m, \frac{m}{n}, n, k \rangle & \text{if } m + n \leq k \\
  \langle n, k \rangle & \text{otherwise}
\end{cases}
\]

where \( m, n \) and \( k \) are real numbers. For instance, if \( m = 3, n = 1 \) and \( k = 4 \) then

\[ m + n = 3 + 1 = 4 = k \]

Therefore,

\[
G(3, 1, 4) = \langle 3 + (2 \times 1) + 4, \frac{3^2 - 1}{3 + 1 + 4}, 1 + 4, 3 + 1 \rangle >
\]

\[ = \langle 9, 8, 4, 7 \rangle > \]

Also, if \( m = 3, n = 4 \) and \( k = 6 \) then

\[ k < m + n < k + 2 \]

Therefore,

\[
G(3, 4, 6) = \langle 3 + (2 \times 4) + 6, \frac{3}{4}, 4 + 6, 3 + 4 \rangle >
\]

\[ = \langle 17, 4, 6, 10 \rangle > \]

Generally, a plenix valued function \( F \) of \( n \) independent variables \( x_1, x_2, \ldots, x_n \) is a collection of rules that assigns a plenix

\[ P = F(x_1, x_2, \ldots, x_n) \]
to each \( n \)-tuple \((x_1, x_2, \ldots, x_n)\). The domain of \( F \) consists of the objects which are sequences \((x_1, x_2, \ldots, x_n)\) of \( n \) numbers.

### 4.13 Limits of Plenix Functions of Real Variables

Let

\[ f_1(x) = x^3 + 2x + 5 \]
\[ f_2(x) = \sqrt{x^2 + 1} \]

and

\[ f_3(x) = \frac{1}{x + 4} \]

be three real-valued functions of a single variable \( x \). The limits of \( f_1(x) \), \( f_2(x) \) and \( f_3(x) \) as \( x \) approaches zero are the numbers 5, 1 and \( 1/4 \), respectively. Now, suppose that \( F(x) \) is a plenix-valued function of variable \( x \) defined by

\[ F(x) = \langle f_1(x), f_2(x), f_3(x) \rangle \]

Then, the limit of \( F(x) \) as \( x \) approaches zero may be defined as follows:

\[
\lim_{x \to 0} F(x) = \lim_{x \to 0} (\langle x^3 + 2x + 5, \sqrt{x^2 + 1}, \frac{1}{x + 4} \rangle)
\]

\[
= \langle \lim_{x \to 0} (x^3 + 2x + 5), \lim_{x \to 0} \sqrt{x^2 + 1}, \lim_{x \to 0} \frac{1}{x + 4} \rangle
\]

\[
= \langle 5, 1, \frac{1}{4} \rangle
\]

Generally, if \( F(x) \) is a plenix function, then the limit of \( F(x) \) as \( x \) approaches \( x_0 \) exists if and only if the limit of every primion panel of \( F(x) \) as \( x \) approaches \( x_0 \) exists. Furthermore, the plenix function \( F(x) \) is said to be continuous at the point \( x = x_0 \) if and only if

\[
\lim_{x \to x_0} F(x) = F(x_0)
\]

In other words, the plenix function \( F(x) \) is continuous at point \( x = x_0 \) if and only if every primion panel of \( F(x) \) is continuous at point \( x = x_0 \). For instance, in the above example

\[
\lim_{x \to 0} F(x) = F(0) = \langle 5, 1, \frac{1}{4} \rangle
\]

Therefore, \( F(x) \) is continuous at \( x = 0 \).

Now, consider two permute plenix functions

\[ F(x) = \langle f_1(x), f_2(x), f_3(x) \rangle \]

and
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\[ G(x) = \langle g_1(x), g_2(x), g_3(x) \rangle \]

where \( f_1(x), f_2(x), f_3(x), g_1(x), g_2(x) \) and \( g_3(x) \) are real-valued functions. Let all these functions have limits as \( x \) approaches \( x_0 \). Then

\[
\lim_{x \to x_0} (F(x) + G(x)) = \lim_{x \to x_0} \langle f_1(x), f_2(x), f_3(x) \rangle + \langle g_1(x), g_2(x), g_3(x) \rangle
\]

\[
= \langle \lim_{x \to x_0} (f_1(x) + g_1(x)), \lim_{x \to x_0} (f_2(x) + g_2(x)), \lim_{x \to x_0} (f_3(x) + g_3(x)) \rangle
\]

On the other hand,

\[
\lim_{x \to x_0} F(x) + \lim_{x \to x_0} G(x) = \langle \lim_{x \to x_0} f_1(x), \lim_{x \to x_0} f_2(x), \lim_{x \to x_0} f_3(x) \rangle + \langle \lim_{x \to x_0} g_1(x), \lim_{x \to x_0} g_2(x), \lim_{x \to x_0} g_3(x) \rangle
\]

Therefore,

\[
\lim_{x \to x_0} (F(x) + G(x)) = \lim_{x \to x_0} F(x) + \lim_{x \to x_0} G(x)
\]

Similarly, it can be shown that

\[
\lim_{x \to x_0} k \times F(x) = k \times \lim_{x \to x_0} F(x) \quad \text{(where } k \text{ is a scalar)}
\]

\[
\lim_{x \to x_0} (F(x) \times G(x)) = \lim_{x \to x_0} F(x) \times \lim_{x \to x_0} G(x)
\]

and

\[
\lim_{x \to x_0} (F(x) \# G(x)) = \lim_{x \to x_0} F(x) \# \lim_{x \to x_0} G(x)
\]

Considering the definition of continuity together with the above facts, if \( F(x) \) and \( G(x) \) are continuous at \( x = x_0 \) then all of the following combinations are also continuous at \( x = x_0 \)

\[ F(x) + G(x) \]

\[ k \times F(x) \quad \text{(where } k \text{ is a scalar)} \]

\[ F(x) \times G(x) \]

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4.14 Differentiation of Plenix Functions of Real Variables

One of the principal notions in the universe is motion and change. Calculus is the mathematics of motion and change. Differential calculus deals with rates of change and integral calculus deals with the problem of determining a function from the information about the function’s rate of change. Attention is now turned to the consideration of two concepts of calculus, that is, differentiation and integration in the context of plenices.

Consider three real-valued functions $h_1(x)$, $h_2(x)$ and $h_3(x)$ and suppose that these functions are differentiable with respect to $x$. The plenix function

$$H(x) = \langle h_1(x), h_2(x), h_3(x) \rangle$$

is defined as the ‘derivative’ of the plenix function

$$H'(x) = \frac{dH}{dx} = \langle \frac{dh_1(x)}{dx}, \frac{dh_2(x)}{dx}, \frac{dh_3(x)}{dx} \rangle$$

with respect of $x$. The domain of $H'(x)$ is the intersection of the domains of $h_1'(x)$, $h_2'(x)$ and $h_3'(x)$. A plenix function $H(x)$ is said to be differentiable at $x = x_0$ if and only if $h_1(x)$, $h_2(x)$ and $h_3(x)$ are differentiable at $x = x_0$. A plenix function $H(x)$ is said to be differentiable if it is differentiable at every $x$ value in its domain. For example, if

$$H(x) = \langle \sin x, \langle \ln x, e^x \rangle, \tan x \rangle$$

then

$$H'(x) = \frac{dH}{dx} = \langle \cos x, \frac{1}{x}, e^x \rangle, 1 + \tan^2 x \rangle$$

and $\frac{dH}{dx}$ at $x = \frac{\pi}{3}$ is

$$H'(\frac{\pi}{3}) = \langle \frac{1}{2}, \frac{3}{\pi}, e^\frac{\pi}{3} \rangle, 4 \rangle$$

The convention is adopted that, wherever there is no possible ambiguity, the terms $f$, $g$, $F$, $G$, ... are used instead of $f(x)$, $g(x)$, $F(x)$, $G(x)$, ...

Now, consider two plenix functions

$$F = \langle f_1, f_2 \rangle, f_3 \rangle$$
These plenix functions are pernates of each other. If \( f_1, f_2, f_3, g_1, g_2 \) and \( g_3 \) are differentiable with respect to \( x \) then, \( F \) and \( G \) will also be differentiable with respect to \( x \). In this case

\[
\frac{d}{dx}(F \times G) = \frac{d}{dx}( \langle f_1, f_2, f_3 \rangle \times \langle g_1, g_2, g_3 \rangle )
\]

\[
= \frac{d}{dx}( \langle f_1 \times g_1, f_2 \times g_2, f_3 \times g_3 \rangle )
\]

\[
= \langle \frac{df_1}{dx} \times g_1 + f_1 \times \frac{dg_1}{dx}, \frac{df_2}{dx} \times g_2 + f_2 \times \frac{dg_2}{dx}, \frac{df_3}{dx} \times g_3 + f_3 \times \frac{dg_3}{dx} \rangle
\]

On the other hand,

\[
\left( \frac{d}{dx} F \right) \times G + F \times \left( \frac{d}{dx} G \right) = \left( \frac{d}{dx} \langle f_1, f_2, f_3 \rangle \times \langle g_1, g_2, g_3 \rangle \right) + \left( \langle f_1, f_2, f_3 \rangle \times \left( \frac{d}{dx} \langle g_1, g_2, g_3 \rangle \right) \right)
\]

\[
= \langle \langle \frac{df_1}{dx}, \frac{df_2}{dx}, \frac{df_3}{dx} \rangle \times g_1 + f_1 \times \frac{dg_1}{dx}, \langle \frac{df_1}{dx}, \frac{df_2}{dx}, \frac{df_3}{dx} \rangle \times g_2 + f_2 \times \frac{dg_2}{dx}, \langle \frac{df_1}{dx}, \frac{df_2}{dx}, \frac{df_3}{dx} \rangle \times g_3 + f_3 \times \frac{dg_3}{dx} \rangle
\]

Thus

\[
\frac{d}{dx}(F \times G) = \left( \frac{d}{dx} F \right) \times G + F \times \left( \frac{d}{dx} G \right)
\]

Also, it can be shown that
\[ \frac{d}{dx}(kF) = k \left( \frac{d}{dx}F \right) \]
\[ \frac{d}{dx}(PQ) = \left( \frac{d}{dx}P \right) + \left( \frac{d}{dx}Q \right) \]

and
\[ \frac{d}{dx}(H \# T) = \left( \frac{d}{dx}H \right) \# \left( \frac{d}{dx}T \right) \]

where \( k \) is a real number, \( P \) and \( Q \) are plenix functions that are pernates of each other and are differentiable with respect to \( x \) and \( F, H \) and \( T \) are any three plenix functions that are differentiable with respect to \( x \).

4.15 Partial Derivatives of Plenix Functions

In this section, the notion of partial derivative of functions of several variables is extended to plenix functions of several variables. Let \( H(x, y) \) be a plenix function of two independent variables \( x \) and \( y \) defined by
\[ H(x, y) = \langle h_1(x, y), h_2(x, y) \rangle, \langle h_3(x, y), h_4(x, y) \rangle \]
where \( h_1(x, y), h_2(x, y), h_3(x, y) \) and \( h_4(x, y) \) are functions of independent variables \( x \) and \( y \). The 'partial derivative' of \( H \) with respect to \( x \) may be defined as follows:
\[ \frac{\partial H}{\partial x} = \langle \frac{\partial h_1}{\partial x}, \frac{\partial h_2}{\partial x}, \frac{\partial h_3}{\partial x}, \frac{\partial h_4}{\partial x} \rangle \]
where \( \frac{\partial h_1}{\partial x}, \frac{\partial h_2}{\partial x}, \frac{\partial h_3}{\partial x} \) and \( \frac{\partial h_4}{\partial x} \) are partial derivatives of \( h_1, h_2, h_3 \) and \( h_4 \) with respect to \( x \), respectively. Similarly, the partial derivative of \( H \) with respect to \( y \) may be defined as
\[ \frac{\partial H}{\partial y} = \langle \frac{\partial h_1}{\partial y}, \frac{\partial h_2}{\partial y}, \frac{\partial h_3}{\partial y}, \frac{\partial h_4}{\partial y} \rangle \]

Furthermore, the 'mixed partial derivatives' of \( H \) are defined by
\[ \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial y} \right) \]
\[ \frac{\partial^2 H}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial H}{\partial x} \right) \]
\[ \vdots \]
\[ \frac{\partial^4 H}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial x} \right) \right) \right) \]
\[ \vdots \]
Note that, the constitution of a plenix function is not changed by a derivative or a partial derivative operation.

4.16 Integrals of Plenix Functions

In what follows the integral of a plenix function is described. But first, recall from integral calculus that there are two classes of integrals, namely, definite integrals and indefinite integrals [see Ref 12].

The definite integral of a real function of a single variable $f(x)$ from $a$ to $b$, where $a$ and $b$ are any real numbers, may be defined as follows:

$$
\lim_{n \to \infty} \left( \frac{b-a}{n} \sum_{k=1}^{n} f\left[a + \frac{k}{n}(b-a)\right] \right)
$$

Note that, a necessary and sufficient condition that a bounded function have a integral on a given interval is that the function be continuous ‘almost everywhere’. The definite integral from $a$ to $b$ of $f(x)$ is denoted by

$$
\int_{a}^{b} f(x) \, dx
$$

Geometrically, the value of the definite integral from $a$ to $b$ of $f(x)$ may be considered as being represented by the area between the curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$.

The indefinite integral of $f(x)$ with respect to $x$ is a function $g(x)$ such that

$$
\frac{dg(x)}{dx} = f(x)
$$

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The function \(g(x)\) is also called an 'antiderivative' of \(f(x)\). From the definition of a derivative, it is clear that if \(g(x)\) is an indefinite integral of \(f(x)\) then \(g(x) + c\), where \(c\) is any constant, is also an indefinite integral of \(f(x)\). The constant \(c\) is called the 'constant of integration'. The set of all indefinite integrals (antiderivatives) of \(f(x)\) with respect to \(x\) is denoted by

\[
\int f(x) \, dx = g(x) + c
\]

The first fundamental theorem of calculus allows definite integrals to be computed in terms of indefinite integrals, since if \(g(x)\) is the indefinite integral for \(f(x)\), then

\[
\int_{a}^{b} f(x) \, dx = g(b) - g(a)
\]

Now, let \(F(x)\) be a plenix function. The indefinite integral of \(F(x)\) with respect to \(x\) is a plenix function \(G(x)\) such that

\[
\frac{dG(x)}{dx} = F(x)
\]

An indefinite integral of \(F(x)\), that is, \(G(x)\) and \(F(x)\) are cognates of each other.

Now, let

\[
G(x) = <g_1(x), g_2(x)>, <g_3(x), g_4(x)>
\]

be an indefinite integral, with respect to \(x\), of

\[
F(x) = <f_1(x), f_2(x)>, <f_3(x), f_4(x)>
\]

that is,

\[
\frac{dG(x)}{dx} = F(x)
\]

The dendrograms of \(G(x)\) and \(F(x)\) are shown in Fig 4.16.1.

By definition of derivative

\[
\frac{dG(x)}{dx} = \frac{d}{dx} \left(<g_1(x), g_2(x)>, <g_3(x), g_4(x)>\right)
\]

\[
= <\frac{d}{dx} (g_1(x)), \frac{d}{dx} (g_2(x)>), <\frac{d}{dx} (g_3(x)), \frac{d}{dx} (g_4(x))>
\]

\[
= F(x)
\]

\[
= <<f_1(x), f_2(x)>, <f_3(x), f_4(x)>>
\]

The dendrograms of \(\frac{dG(x)}{dx}\) and \(F(x)\), are shown in Fig 4.16.2.
By definition of equality of plenices, the following equations are valid:

\[ \frac{d}{dx} g_1(x) = f_1(x) \]

\[ \frac{d}{dx} g_2(x) = f_2(x) \]

\[ \frac{d}{dx} g_3(x) = f_3(x) \]

and

\[ \frac{d}{dx} g_4(x) = f_4(x) \]

Therefore,

\[ \int f_1(x) \, dx = g_1(x) + c_1 \]

\[ \int f_2(x) \, dx = g_2(x) + c_2 \]
where \( c_1, c_2, c_3 \) and \( c_4 \) are constants of the integration. Thus, the indefinite integral of \( F(x) \), which is denoted by

\[
\int F(x)\,dx = \langle \int f_1(x)\,dx, \int f_2(x)\,dx, \int f_3(x)\,dx, \int f_4(x)\,dx \rangle
\]

is equal to

\[
\langle g_1(x) + c_1, g_2(x) + c_2, g_3(x) + c_3, g_4(x) + c_4 \rangle = G(x) + C
\]

where \( C \) is a plenix constant whose primion panels are \( c_1, c_2, c_3 \) and \( c_4 \). That is,

\[
C = \langle c_1, c_2 \rangle, \langle c_3, c_4 \rangle
\]

The plenices \( C \) and \( G(x) \) are cognates of each other.

Since the derivative of a plenix whose primion panels are constants, (that is, a plenix constant) is a plenix whose primion panels are zero, a plenix constant may be added to a plenix indefinite integral and will still correspond to the same integral. For this reason, indefinite integrals may be written in the form

\[
\int F(x)\,dx = G(x) + C
\]

where \( C \) is a plenix constant which is the constant of integration. Note that, the plenix constant \( C \) must be a subnate of \( G(x) \).

Now, let \( F(x) \) be a plenix function of finite level and finite order and let all the primion panels of \( F(x) \) be continuous on the interval \([a, b]\), where \( a \) and \( b \) are real numbers. The definite integral of \( F(x) \) from \( a \) to \( b \) may be defined as follows:

\[
\lim_{n \to \infty} \left( \frac{b-a}{n} \sum_{k=1}^{n} F \left[ a + \frac{k}{n} (b-a) \right] \right)
\]

The value of a definite plenix integral is a plenix constant (a plenix all whose primion panels are constants) and is denoted by

\[
\int_{a}^{b} F(x)\,dx
\]

Now, consider the plenix function

\[
F(x) = \langle f_1(x), f_2(x), f_3(x) \rangle
\]

where \( f_1(x), f_2(x) \) and \( f_3(x) \) are continuous on \([a, b]\), then

\[
\int_{a}^{b} F(x)\,dx = \lim_{n \to \infty} \left( \frac{b-a}{n} \sum_{k=1}^{n} F \left[ a + \frac{k}{n} (b-a) \right] \right)
\]
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{b-a}{n} \left< f_1[a + \frac{k}{n} (b-a)], <f_2[a + \frac{k}{n} (b-a)], f_3[a + \frac{k}{n} (b-a)]> \right>
\]

\[
= \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f_1[a + \frac{k}{n} (b-a)], \quad \sum_{k=1}^{n} f_2[a + \frac{k}{n} (b-a)], \sum_{k=1}^{n} f_3[a + \frac{k}{n} (b-a)]> \]

Now, let \( g_1(x), g_2(x) \) and \( g_3(x) \) be antiderivatives of \( f_1(x), f_2(x) \) and \( f_3(x) \), respectively. Also, let

\[
G(x) = <g_1(x), g_2(x), g_3(x)>
\]

be a plenix function. It is clear that \( G(x) \) is an antiderivative of

\[
F(x) = <f_1(x), f_2(x), f_3(x)>
\]

The first fundamental theorem of calculus can be used for all the primion panels of

\[
< \int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \int_a^b f_3(x)dx>
\]

The result may be obtained as follows:

\[
< \int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \int_a^b f_3(x)dx> = <g_1(b) - g_1(a), g_2(b) - g_2(a), g_3(b) - g_3(a)>
\]

\[
= G(b) - G(a)
\]

Thus, the first fundamental theorem of calculus is also applicable to plenix functions. That is,

\[
\int_a^b F(x)dx = G(b) - G(a)
\]
In what follows some familiar properties of definite integrals are described for definite plenix integrals of plenix functions.

1. \[ \int_{a}^{b} F(x)\,dx \]

is equal to a plenix all whose primion panels are zero and it has the same constitution as that of \( F(x) \).

2. \[ \int_{a}^{b} F(x)\,dx = -\int_{b}^{a} F(x)\,dx \]

This says that reversing the order of plenix integration bounds changes the sign of all the primion panels of the plenix that is the value of the integral.

3. \[ \int_{a}^{b} kF(x)\,dx = k \int_{a}^{b} F(x)\,dx \quad \text{(where } k \text{ is a scalar)} \]

That is, the plenix integral of \( k \) times a plenix function is \( k \) times the plenix integral of the plenix function.

4. \[ \int_{a}^{b} (F(x) + G(x))\,dx = \int_{a}^{b} F(x)\,dx + \int_{a}^{b} G(x)\,dx \]

That is, the plenix integral of summation of plenix functions is the summation of plenix integrals of the plenix functions.

The last two properties 3 and 4 are also hold for indefinite plenix integrals. That is,

\[ \int kF(x)\,dx = k \int F(x)\,dx \quad \text{(where } k \text{ is a scalar)} \]

and

\[ \int (F(x) + G(x))\,dx = \int F(x)\,dx + \int G(x)\,dx \]
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4.17 Plenix Functions of Plenix Variables

Now, the attention is turned to plenix functions whose independent variables are plenices. Normally, this type of plenix function is a function plenix, that is, a plenix whose primion panels are functions. Plenix functions come in many varieties. In what follows, some types of plenix functions are discussed by means of some examples.

Example 1- Consider the plenix function
\[ F(P) = \langle f_1(P_1), <f_2(P_{21}), f_3(P_{22}) \rangle \]
where \( P \) is a plenix which is independent variable of \( F \) and \( P \) must be a supernate of the nexus \( \langle \square, \square, \square \rangle \)

This nexus and the function plenix \( F \) are cognates of each other.

\( f_1, f_2 \) and \( f_3 \) are either plenix functions or other functions, namely, real functions, vector functions and so on. \( P_1, P_{21} \) and \( P_{22} \) are panels of \( P \). These panels are either primion panels or nonprimion panels (that is, a plenix) of \( P \). The functions \( f_1, f_2 \) and \( f_3 \) act on their full images in \( P \). To elaborate, \( f_1 \) is the first principal panel of function plenix \( F \) so, \( f_1 \) acts on the first principal panel of \( P \), that is, \( P_1, f_2 \) is the first principal panel of the second principal panel of \( F \) so, \( f_2 \) acts on \( P_{21} \) and so on. For instance, consider three real valued functions of one variable as follows:

\[ f_1(m) = m + 2 \]
\[ f_2(m) = m^2 + 7 \]
and
\[ f_3(m) = \frac{m + 3}{5} \]

and suppose that,
\[ P = \langle <3, 8, 1>, <2, 7>, 4 \rangle \]
In this case
\[ P_1 = <3, 8, 1> \]
and
\[ P_{21} = <2, 7> \]
are nonprimion panels, that is, plenices and
\[ P_{22} = 4 \]
is a primion panel, that is a number.

Now, consider the above plenix function, that is,
\[ F(P) = \langle f_1(P_1), <f_2(P_{21}), f_3(P_{22}) \rangle \]
The independent variables of $f_1$, $f_2$ and $f_3$, namely, $P_1$, $P_{21}$ and $P_{22}$ are either primion panels or nonprimion panels of $P$. In both cases; the functions $f_1$, $f_2$ and $f_3$ are well defined. To elaborate, in the above example, the independent variable of $f_1$, that is, 

$$P_1 = <3, 8, 1>$$

is a plenix. In this case, the function $f_1$ acts on all the primion panels of $P_1$, individually. Note that, the primion panels of $P_1$ must be in the domain of $f_1$.

Therefore, the plenix function $F$ may act on the plenix $P$ as follows:

$$F(P) = \langle f_1(P_1), f_2(P_{21}), f_3(P_{22})\rangle$$

$$= \langle f_1(<3, 8, 1>), f_2(<2, 7>), f_3(4)\rangle$$

$$= \langle f_1(3), f_1(8), f_1(1)\rangle, \langle f_2(2), f_2(7), f_3(4)\rangle$$

$$= \langle 3 + 2, 8 + 2, 1 + 2\rangle, \langle 4 + 7, 49 + 7, \frac{4 + 3}{5}\rangle$$

$$= \langle 5, 10, 3\rangle, \langle 11, 56\rangle, \frac{7}{5}\rangle$$

Example 2- Suppose that

$$g_1(m) = \det(m)$$

$$g_2(m) = m!$$

and

$$g_3(m) = ||m||$$

are three functions. The independent variable of $g_1$ is a matrix, the independent variable of $g_2$ is an integer and the independent variable of $g_3$ is a vector. The function $g_1$ returns the determinant of a matrix, $g_2$ returns the factorial of an integer and $g_3$ returns the Euclidean norm of a vector.

Now, consider the plenix function $G(T) = \langle g_1(T_{11}), g_2(T_{12}), g_3(T_2)\rangle$ where $T$ is a plenix which is a supernate of the function plenix $G(T)$. For example, suppose that

$$T = \langle\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 0 \\ 7 & 2 & 9 \end{bmatrix}\rangle, \langle [3, 8, 2, 5], [6, 4, 1]\rangle$$

The plenix function $G$ may act on the plenix $T$ as follows:

$$G(T) = \langle g_1(T_{11}), g_2(T_{12}), g_3(T_2)\rangle$$

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Note that, in the above examples the results, namely, \( F(P) \) and \( G(T) \) are plenices. Also, the plenices \( F(P) \) and \( G(T) \) have the same constitution as \( P \) and \( T \), respectively. Generally, the constitution of the independent variable may be changed by the plenix function. (See the next example)

Example 3- Consider two plenix functions

\[
F_1(m) = \langle m + 2, \sqrt{m+1}, m^2 + 7, \frac{m+1}{m-1} \rangle
\]

and

\[
F_2(P) = \langle f_1(P_1), f_2(P_2) \rangle
\]

where

\[
f_1(x) = 3x^2 + 5
\]

and

\[
f_2(x) = e^x
\]

are two real functions.

Now, let

\[
H(S) = \langle F_1(S_1), F_2(S_2) \rangle
\]

be a plenix function. The domain of \( H \) is the set of all real plenices which are supernates of the function plenix \( H \). Also they have a plenix as the second principal panel because the independent variable of \( F_2 \) is a plenix. For instance, the plenix

\[
S = \langle 3, \langle 1, 2 \rangle, 0 \rangle
\]

is a member of the domain of \( H \). This plenix has two principal panels. The first principal panel of \( S \), that is,

\[
S_1 = 3
\]

is a number and the second principal panel of \( S \), that is,

\[
S_2 = \langle 1, 2 \rangle, 0 \rangle
\]
is a plenix.

Therefore, the plenix function $H$ may act on $S$ as follows:

$$H(S) = <F_1(S_1), F_2(S_2)>
= <F_1(3), F_2(<1, 2>, 0)>$$

$$= <<3+2, \sqrt{3+1}>, <3^2+7, \frac{3+1}{3-1}>, <f_1(<1, 2>), f_2(0)>$$

$$= <<5, 2>, <16, 2>>, <<(3 \times 1^2)+5, (3 \times 2^2)+5>, e^0>>$$

$$= <<5, 2>, <16, 2>>, <<8, 17>, 1>>$$

The result, that is, $H(S)$ is a plenix but this plenix has not necessarily the same constitution as $S$, that is the independent variable of $H$. But it is important to know that the plenices $S$ and $H(S)$ are supernates of the function plenix $H$ and generally, they are pernates of each other. These facts hold because in the above three examples there exist two strong conditions, namely, the independent variable of the plenix function was a supernate of the function plenix and every primion panel of the function plenix acts on the full image panel of its independent variable. In the next examples these conditions are omitted.

Example 4- Consider the plenix function $H$ that is explained in example 3 and change its independent variable $S$ to the plenix

$S = <3, <<1, 2>, 0>, <8, 5>>$

This plenix is not a supernate of the function plenix $H$. Indeed, $H$ and $S$ are abnates of each other. The plenix function $H$ may act on $S$ as follows:

$$H(S) = <F_1(S_1), F_2(S_2)>
= <F_1(3), F_2(<1, 2>, 0)>$$

$$= <<3+2, \sqrt{3+1}>, <3^2+7, \frac{3+1}{3-1}>, <f_1(<1, 2>), f_2(0)>$$

$$= <<5, 2>, <16, 2>>, <<(3 \times 1^2)+5, (3 \times 2^2)+5>, e^0>>$$

$$= <<5, 2>, <16, 2>>, <<8, 17>, 1>>$$

It can be seen that, $H(S)$ and $S$ are abnates of each other. Note that, $H(S)$ is equal to $H(S)$ but $S$ and $S$ are abnates of each other. This fact indicates that the plenix function $H$ is not a one to one function.

Example 5- Consider the plenix function

$$K(R) = <K_1(R_2), K_2(R), K_3(R_1)>$$

where
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\[ K_1(P) = <> \]
\[ K_2(P) = 2P \]

and

\[ K_3(m) = \langle \langle m + 2, 2m^2 \rangle, 7 \rangle \]

are plenix functions and \( m \) is a real number. Now, suppose that

\[ R = \langle <4, <1, 0>>, <5, 2>>, 3 \rangle \]

The plenix function \( K \) may act on \( R \) as follows:

\[ K(R) = \langle K_1(R_2), <K_2(R), K_3(R_{12})> \rangle \]
\[ = \langle K_1(<5, 2>), <K_2(<4, <1, 0>>, <5, 2>>, 3>>, K_3(<1, 0>>) \rangle \]
\[ = \langle <>, <<8, <2, 0>>, <10, 4>>, <<<3, 2>>, 7>>, <<2, 0>>, 7 >>> \]

The dendrograms for \( K(R) \) and \( R \) are shown in Fig 4.17.1.

---

Fig 4.17.1. Dendrograms of \( K(R) \) and \( R \)
In this example the plenix $R$ and the function plenix $K$ are abnates of each other. Also, the plenix $R$ and the plenix $K(R)$ are abnates of each other. But it is important to know that, if $F$ is any plenix function and $P$ is any plenix then the plenix $F(P)$ is always a supernate of $F$.

### 4.18 Operator Plenices

An operator plenix is a plenix whose primion panels are operators. For example, the plenix

$$ D = \langle \langle \frac{d}{dx}, \frac{\partial}{\partial y}, \frac{d^2}{dx^2} \rangle \rangle $$

is an operator plenix. Now, consider the plenix function

$$ F(x, y) = \langle \langle f_1(x, y), f_2(x, y), f_3(x, y), f_4(x), f_5(x) \rangle \rangle $$

where

$$ f_1(x) = x^2 + 7x $$
$$ f_2(x) = 5x^3 - \cos x $$
$$ f_3(x, y) = xy^2 + e^{xy} $$
$$ f_4(x) = 8x + \ln x $$
$$ f_5(x) = e^{\sin x} $$

and

The operator $D$ may act on the plenix function $F$ as follows:

$$ D(F(x, y)) = \langle \langle \frac{d}{dx}, \frac{\partial}{\partial y}, \frac{d^2}{dx^2} \rangle \rangle (\langle \langle f_1(x), f_2(x), f_3(x, y), f_4(x), f_5(x) \rangle \rangle) $$

$$ = \langle \langle \frac{d}{dx} f_1(x), \frac{d}{dx} f_2(x), \frac{\partial}{\partial y} f_3(x, y), \frac{d^2}{dx^2} f_4(x), \frac{d^2}{dx^2} f_5(x) \rangle \rangle $$

$$ = \langle \langle \frac{d}{dx} (x^2 + 7x), \frac{d}{dx} (5x^3 - \cos x), \frac{\partial}{\partial y} (xy^2 + e^{xy}), $$

$$ \frac{d^2}{dx^2} (8x + \ln x), \frac{d^2}{dx^2} (e^{\sin x}) \rangle \rangle $$

$$ = \langle \langle 2x + 7, 15x^2 + \sin x, 2xy + xe^{xy}, $$

$$ \frac{1}{x^2}, -\sin x e^{\sin x} + \cos x e^{\sin x} \rangle \rangle $$
5.1 Introduction

In the previous chapters, the theory of plenices was discussed. The reader may have realised that the constitution of a plenix is an important aspect of the concept of plenices. Also, many terms in the theory of plenices relate to the constitution of a plenix, such as cognate, pernate, abnate, subnate. Furthermore, the constitutions of the operands play an important role in plenix operations. For example, two plenices can be added provided that their constitutions would allow the operation. In fact, the plenix constitution, by itself, is an interesting mathematical entity and has been given some attention in the previous chapters. In particular, in Chapter 3 an equivalence relation is defined on the set of plenices as follows:

\[ P \sim Q \iff P \text{ and } Q \text{ have the same constitution} \]
This defines the equivalence class of $P$ as the set of all plenices whose constitution is the same as that of $P$. In fact, when dealing with equivalence classes, it is customary to consider all the plenices of the same constitution as a single mathematical object. So, the study of equivalence classes of plenices is effectively the study of the constitution of plenices.

So far, the concept of 'nexus' that represents the constitution of a plenix has been discussed in the context of the theory of plenices. However, in this chapter the concept of nexus is evolved in a manner that is independent of the idea of plenices. In Chapter 2, two ways of showing the constitution of a plenix were presented, that is, using a dendrogram or the porta symbol.

Fig 5.1.1 Dendrogram of $[P]$, that is, the dendrogram of the nexus of $P$

For example, consider the plenix

$P = <9, <<0, 5>>, <7, 8>>$

Then, the constitution of $P$, that is, $[P]$ can be represented using the dendrogram of Fig 5.1.1 or using the porta symbol $\Box$ in

$[P] = <\Box, <<\Box, \Box>>, \Box, <\Box, \Box>>$

In both cases, representation of the nexus $[P]$ is dependent on the definition of plenix $P$. However, using the concept of an address set (see Section 2.3), a nexus can be defined quite independently of a plenix. This approach in defining nexuses is used in the present chapter.

5.2 Structure of a Nexus

**Definition 5.2.1** An address is a sequence of $\mathbb{N}^*$ such that $a_k = 0$ implies that $a_i = 0$, for all $i \geq k$, where, $\mathbb{N}^*$ is the set of all non-negative integers, that is,

$\mathbb{N}^* = \mathbb{N} \cup \{0\}$

A typical address is of the form

$(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$

where $a_i$ and $n$ belong to $\mathbb{N}$. Such an address will hereafter be denoted by
The sequence of all zeros is called the 'empty address' and is denoted by ( ).

**Definition 5.2.2** A nexus $N$ is a nonempty set of addresses where for a finite nexus

$$(a_1, a_2, \ldots, a_n) \in N \Rightarrow (a_1, a_2, \ldots, a_{n-1}, t) \in N \quad \forall 0 \leq t \leq a_n \quad (*)$$

and for an infinite nexus

$$\{a_i\}^\infty_{i=1} \in N, \quad a_i \in \mathbb{N} \Rightarrow \forall n \in \mathbb{N}, \quad \forall 0 \leq t \leq a_n \quad (a_1, a_2, \ldots, a_{n-1}) \in N \quad (**)$$

Note that condition (**) does imply condition (*).

For example, the set

$$N = \{(1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\}$$

is a nexus.

A nexus containing an infinite number of addresses is referred to as 'infinite nexus'. Also, the empty set as a nexus is regarded as a nexus with no address. This is written as \{\} and is referred to as the 'empty nexus'.

**Remark 5.2.3 (i)** Definition 5.2.2 implies that

$$(a_1, a_2, \ldots, a_{n-1}, a_n) \in N \Rightarrow (a_1, a_2, \ldots, a_{n-1}) \in N \quad \forall n \geq 1$$

(ii) The set of all addresses is a nexus. Also, the set of all finite addresses is a nexus.

A nexus may be represented graphically using a dendrogram. For instance, consider the nexus

$$N = \{(1), (2), (3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 1, 1), (3, 1, 2), (3, 2, 1), (3, 2, 2)\}$$

A dendrogram representing nexus $N$ is shown in Fig 5.2.1.

The dendrogram contains a number of 'stalks' representing the addresses of $N$. Here, the term 'stalk' is used to refer to each vertical line segment of the dendrogram. There exists a one-to-one correspondence between the addresses of $N$ and the stalks of the dendrogram of $N$. In Fig 5.2.1, the address of $N$ corresponding to each stalk of the dendrogram is written next to it.

As another example, consider the nexuses

$$M = \{(1)\}$$
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and

\[ T = \{( (), (1), (1, 1)\} \]

The dendrograms of \( M \) and \( T \) are shown in Fig 5.2.2

Now, consider the following sets

\[ A_0 = \{(), (1), (2)\} \]
\[ A_1 = \{(2, 1), (2, 2)\} \]
\[ A_2 = \{(2, 2, 1), (2, 2, 2)\} \]
\[ \vdots \]
\[ A_n = \{(2,2,\ldots,2,1), (2,2,\ldots,2)\} \]
\[ \vdots \]

The union of all the above sets is an infinite nexus, that is,

\[ N = \bigcup_{n=0}^{\infty} A_n \]
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The dendrogram of $N$ is shown in Fig 5.2.3.

![Dendrogram for the infinite nexus $N$](image)

**Definition 5.2.4** Let $N$ be a nexus and $a$ be an address and let $a \in N$. If

$$a = (a_1, a_2, \ldots, a_n)$$

for some $a_n \in \mathbb{N}$,

then $a$ is said to be of 'level' $n$. If $a$ is an infinite sequence of $\mathbb{N}$, then $a$ is said to be of 'level' $\infty$. If $a = ()$, then $a$ is said to be of 'level' 0 (zero).

The level of $a$ is denoted by $l(a)$. The set of all addresses of $N$ whose level is $k$ will be denoted by $N^k$. That is,

$$N^k = \{a \in N : l(a) = k\}$$

In particular, $N^0 = \{(\ )\}$.

Suppose that $M$ is a subset of a nexus $N$. The highest level of the elements of $M$ is referred to as the 'rise' of $M$ and is denoted by $\text{rise}(M)$. If $M$ contains an element of infinite level, then the rise of $M$ is also infinite. In particular, the highest level of the elements of a nexus $N$ is referred to as the 'rise' of a nexus $N$ and denoted by $\text{rise}(N)$.

There are two important types of relations which often arise in mathematics, order relations and equivalence relations. In Chapter 3, by using an equivalence relations, the notion of nexus was defined. However, the concept of nexus also involves some order relation. It will be shown later that every nexus is a partially ordered set and also is a semilattice. A partial order relation in a set $P$ is a relation which is symbolised by $\leq$ and assumed to have the following properties:

1. $a \leq a$ for every $a$ (reflexivity)
2. $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetry)
(3) \(a \leq b \text{ and } b \leq c \Rightarrow a \leq c\) (transitivity).

A nonempty set for which there is defined a partial order relation is called a partially ordered set. Two elements \(a\) and \(b\) in a partially ordered set are called ‘comparable’ if one of them is less than or equal to the other. The adverb ‘partially’ in the phrase ‘partially ordered set’ is intended to emphasise that there may be pairs of elements in the set which are not comparable. If any two elements in a set are comparable, then the set is called ‘totally ordered set’ or a ‘chain’.

A lattice is a partially ordered set \(L\) in which each pair of elements has a greatest lower bound, that is, a ‘supremum’ and a least upper bound, that is, a ‘infimum’. A semilattice is a partially ordered set in which each pair of elements has a supremum (or infimum).

**Definition 5.2.5** Let \(a = \{a_i\}_{i \in \mathbb{N}}\) and \(b = \{b_i\}_{i \in \mathbb{N}}\) be addresses. Then \(a \leq b\)

- if \(l(a) = 0\), that is, \(a = ()\),
- or if \(l(a) = 1\), that is, \(a = (a_1)\), \(b\) is a nonempty address and \(a_1 \leq b_1\),
- or if \(1 < l(a) < \infty\) and \(l(a) \leq l(b)\) and \(a_i = b_i\) for all \(1 \leq i < l(a)\) and \(a_{l(a)} \leq b_{l(a)}\),
- or if \(l(a) = \infty\), then \(a\) can only be equal to \(b\).

To elaborate, consider two addresses \(a = (a_1, a_2, \ldots, a_n)\) and \(b = (b_1, b_2, \ldots, b_m)\). The necessary condition for \(a \leq b\) is \(l(a) \leq l(b)\), so, firstly one must check the levels of \(a\) and \(b\). If \(n \leq m\), then the necessary condition is fulfilled. In the next step, one must check the sufficient condition for \(a \leq b\). That is, the first \(n-1\) terms of \(a\) must be equal to the first \(n-1\) terms of \(b\) and the last term of \(a\), that is \(a_n\), must be less than or equal to the \(n^{th}\) term of \(b\), that is \(b_n\). In other words, \(a_i = b_i\) for \(i = 1, 2, \ldots, n-1\) and \(a_n \leq b_n\).

As an example, consider the nexus

\[N = \{((), (1), (2), (1, 1), (1, 2), (1, 3), (1, 2, 1), (1, 3, 1), (1, 3, 2),\]
\[(1, 3, 1, 1), (1, 3, 2, 1), (1, 3, 2, 2)\}\]

whose dendrogram is shown in Fig 5.2.4. Consider the addresses \(a = (1, 3, 1)\) and \(b = (1, 3, 2, 2)\) of \(N\). Since \(l(a) \leq l(b)\), the necessary condition for \(a \leq b\) holds. Also, the first two terms of \(a\), that is, \(1\) and \(3\), are equal to the first two terms of \(b\) and the last term of \(a\), that is \(1\) is less than the third term of \(b\), that is \(2\). So the sufficient condition also, holds. Therefore, \(a \leq b\).

Now, suppose that, \(a = (1, 3, 1, 1)\) and \(b = (1, 3, 2, 1)\). Since \(l(b) = l(a)\), the necessary condition for \(a \leq b\) or \(b \leq a\) holds. But the first three terms of \(a\) and \(b\) are not equal. Specifically, they have different values for the third term. Therefore, they are not comparable.
Now, consider the address (1, 3, 2, 2) of \( N \). The list of all addresses of \( N \) that are less than or equal to (1, 3, 2, 2) are

\[
() < (1) < (1, 1) < (1, 2) < (1, 3) < (1, 3, 1) < (1, 3, 2, 1) < (1, 3, 2, 2)
\]

The addresses in the above list are indicated by thick lines in Fig 5.2.4.

![Dendrogram of nexus \( N \), every stalk of the dendrogram whose address is less than or equal to (1, 3, 2, 2) is indicated by a thick line](image)

Note that, by Definition 5.2.5, in any nexus \( N \), the empty address is less than all other addresses. That is,

\[
() < a \quad \forall a \in N
\]

**Theorem 5.2.6** Let \( N \) be a nexus, then, \((N, \leq)\) is a partially ordered set.

**Proof:** To prove the theorem, it must be shown that the relation \( \leq \) is reflexive, antisymmetric and transitive. The reflexivity is obviously true since \( a \leq a \). Let \( a, b \in N \), \( a \leq b \) and \( b \leq a \). One can show that \( a = b \). To this end, consider the following four cases:

Case 1: If \( l(a) = 0 \), then \( l(b) \leq l(a) = 0 \), therefore, \( b = (a) = a \).

Case 2: If \( l(a) = 1 \), then \( 1 \leq l(b) \leq l(a) \) and hence, \( l(b) = 1 \). Suppose that \( b = (b_1) \) and \( a = (a_1) \), for some \( a_1, b_1 \in \mathbb{N} \). Thus, \( a_1 \leq b_1 \) and \( b_1 \leq a_1 \). Therefore, \( a_1 = b_1 \) and \( a = b \).

Case 3: Let \( n = l(a) > 1 \), that is, \( a = (a_1, a_2, \ldots, a_n) \). Then \( l(b) = n \) (see the third case in Definition 5.2.5). Assume that \( b = (b_1, b_2, \ldots, b_n) \). Therefore, \( a_i \leq b_n, b_n \leq a_i \) and \( a_i = b_i \) for all \( 1 \leq i < n \). Hence, \( a_i = b_j \) for all \( 1 \leq j \leq n \). In other words \( a = b \).

Case 4: If \( l(a) = \infty \), then \( a = b \), see the last case in Definition 5.2.5.

Finally, it is necessary to prove that \( \leq \) is transitive. For this, let \( a, b, c \in N \), \( a \leq b \) and \( b \leq c \). If \( l(c) = 0 \), then \( l(a) = l(b) = 0 \) and hence \( a = c = (a) \). If \( l(c) = 1 \), then \( l(b) = 0 \) or \( l(b) = 1 \). First
assume that \( l(b) = 0 \), this implies that \( l(a) = 0 \), therefore, \( a = (\ ) \leq c \). Now, suppose that, \( l(b) = 1 \) and \( l(a) \neq 0 \). Then, \( a = (a_1), b = (b_1), \) and \( c = (c_1) \) for some \( a_1, b_1, c_1 \in \mathbb{N} \), and \( a_1 \leq b_1 \leq c_1 \). Therefore, \( a \leq c \). Now, suppose that \( 1 < l(c) = k < \infty \) and \( c = (c_1, c_2, \ldots, c_k) \), for some \( c_i \in \mathbb{N} \). Then, \( b = (b_1, b_2, \ldots, b_m) \) and \( a = (a_1, a_2, \ldots, a_n) \) where \( b_i, a_i \in \mathbb{N}, n \leq m \leq k, a_n \leq b_n, b_m \leq c_m, a_i = b_i \) and \( b_j = c_j \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Therefore, \( a_n \leq c_n \) and \( a_r = c_r \) for all \( 1 \leq r \leq n \). Consequently, \( a \leq c \). Finally, if \( l(c) = \infty \), then from case 3 of Definition 5.2.5 it may be concluded that \( a = b = c \). ■

**Theorem 5.2.7** Let \( N \) be a nexus, then, \( (N, \leq) \) is a lower semilattice.

**Proof:** For the proof of this theorem, it must be shown that the greatest lower bound of \( a \) and \( b \), that is, \( a \wedge b \) belongs to \( N \) for any address \( a = \{a_i\}_{i \in K} \) and \( b = \{b_i\}_{i \in K} \) in \( N \). Now, consider three cases:

**Case 1:** If \( a \) or \( b \) is the empty address, then \( a \wedge b = (\ ) \).

**Case 2:** If \( a \) and \( b \) are not the empty address and \( a_1 \neq b_1 \) then \( a \wedge b = (a_1 \wedge b_1) \).

**Case 3:** Assume that, \( a \) and \( b \) are not the empty address and \( a_1 = b_1 \). In this case suppose that \( n \) is the largest element of \( K \) for which \( a_n = b_n \). Let \( c = (a_1, a_2, \ldots, a_n, a_{n+1} \wedge b_{n+1}) \). This implies that \( a \wedge b = c \) and \( c \) belongs to \( N \). ■

To illustrate the idea, consider the nexus

\[
N = \{ (\ ), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2), (3, 1, 1), (3, 1, 2), (3, 2, 1), (3, 1, 1, 1), (3, 1, 1, 2), (3, 1, 2, 1), (3, 1, 2, 2) \}
\]

The dendrogram of \( N \) is shown in Fig 5.2.5. Suppose that one requires to find the infimum of two addresses \( a = (1, 2) \) and \( b = (3, 2, 1) \) of \( N \). Since, the first terms of \( a \) and \( b \), that is, 1 and 3, are not equal, so, by case 2 of the above theorem,

\[
\text{inf} \{a, b\} = (1 \wedge 3) = (1)
\]

as indicated in Fig 5.2.5.

Now, consider two addresses \( c = (3, 1, 1, 1) \) and \( d = (3, 1, 2, 1) \). Since the first and second terms of \( c \) and \( d \) are equal, by case 3 of the above theorem,

\[
\text{inf} \{c, d\} = (3, 1, 1 \wedge 2) = (3, 1, 1)
\]

as indicated in Fig 5.2.5.

Note that, every address that is less than \( (3, 1, 1) \), namely, \( (3, 1), (3), (2), (1) \) or \( (\ ) \), is a lower bound of \( (3, 1, 1, 1) \) and \( (3, 1, 2, 1) \). However, \( (3, 1, 1) \) is the greatest lower bound of \( c \) and \( d \), that is, the infimum of \( c \) and \( d \).
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Note that, in general, two addresses of a nexus need not necessarily have a least upper bound.

For example, consider the nexus

\[ N = \{(\), (1), (2), (1, 1), (1, 2)\} \]

The least upper bound of the addresses (1, 1) and (2), that is, \( \text{sup} \{(1, 1), (2)\} \) does not exist.

**Theorem 5.2.8** Suppose that \( S \) is a subset of the set of all finite addresses. Then \( S \) is a nexus if and only if, \( a \in S \) and \( b < a \) imply that \( b \in S \).

**Proof.** Suppose that \( S \) is a nexus, one must show that if \( a = (a_1, a_2, \ldots, a_n) \in S \) and if \( b \leq a \), then \( b \in N \). Since \( b \leq a \), so \( b \) is of the form

\[ b = (a_1, a_2, \ldots, a_k, t) \]

where \( k < n \) and \( t \leq a_{k+1} \). By definition of a nexus and Remark 5.2.3, since,

\[ a = (a_1, a_2, \ldots, a_n) \in S \]

so,

\[ b = (a_1, a_2, \ldots, a_k, t) \in S \]

Conversely, suppose that,

\[ a = (a_1, a_2, \ldots, a_n) \in S \]

One must show that

\[ (a_1, a_2, \ldots, a_{n-1}, t) \in S \quad \forall 0 \leq t \leq a_n \]

By Definition 5.2.5

\[ (a_1, a_2, \ldots, a_{n-1}, t) \leq (a_1, a_2, \ldots, a_n) \quad \forall 0 \leq t \leq a_n \]
So,

\[(a_1, a_2, \ldots, a_{n-1}, t) \in \mathcal{N} \quad \forall 0 \leq t \leq a_n \]

**Definition 5.2.9** (i) Every address of a nexus \( \mathcal{N} \) that has only one term is called a ‘principal address’ of \( \mathcal{N} \). In other words, a principal address is of the form \((a)\) where \( a \) is a positive integer.  

(ii) If \((a_1, a_2, \ldots, a_n)\) is an address of \( \mathcal{N} \), then all the addresses of the form

\[(a_1, a_2, \ldots, a_n, t) \quad t \in \mathbb{N}\]

of \( \mathcal{N} \) are called principal addresses of \((a_1, a_2, \ldots, a_n)\).

For example, consider the nexus

\[\mathcal{N} = \{(), (1), (2), (1, 1), (1, 2)\}\]

The addresses (1) and (2) are principal addresses of \(\mathcal{N}\) and (1, 1) and (1, 2) are principal addresses of (1).

**Definition 5.2.10** The number of principal addresses of a nexus \( \mathcal{N} \) is called the ‘order of the nexus’ and is denoted by \( \text{Ord}(\mathcal{N}) \).

**Definition 5.2.11** Let \( \mathcal{N} \) be a nexus and let \( a = (a_1, a_2, \ldots, a_n) \) be an address of \( \mathcal{N} \). The first term \( a_1 \) is said to be the ‘stem’ of \( a \) and is denoted by \( \text{stem}(a) \). The set of all addresses of \( \mathcal{N} \) whose stem is \( k \) is denoted by \( \mathcal{N}_k \). That is,

\[\mathcal{N}_k = \{a \in \mathcal{N} : \text{stem}(a) = k\}\]

In particular, \( \mathcal{N}_0 = \{()\} \).

For example, consider the nexus

\[\mathcal{M} = \{(), (1), (2), (1, 1), (1, 2), (3, 1), (3, 2), (3, 2, 1)\}\]

The number of principal addresses of \( \mathcal{M} \) is three. So, \( \text{Ord}(\mathcal{N}) = 3 \). The stem of address ( ) is zero, the stem of addresses (1), (1, 1) and (1, 2) is 1, the stem of address (2) is 2 and the stem of addresses (3), (3, 1), (3, 2) and (3, 2, 1) is 3.

**Definition 5.2.12** An address \((a_1, a_2, \ldots, a_n)\) of a nexus \( \mathcal{N} \) is called a ‘primitive address of \( \mathcal{N} \)’ provided that \((a_1, a_2, \ldots, a_n, a_{n+1})\) does not belong to \( \mathcal{N} \), for any \( a_{n+1} \neq 0 \). The set of all primitive addresses of a nexus \( \mathcal{N} \) is denoted by \( P(\mathcal{N}) \).

For example, consider the nexus

\[\mathcal{N} = \{(), (1), (2), (3), (1, 1), (1, 2), (1, 3), (3, 1), (3, 2), (3, 1, 1), (3, 1, 2)\}\]

The addresses (1, 1), (1, 2), (1, 3), (2, 3), (1, 1), (3, 1, 1), (3, 1, 2) and (3, 2) are primitive addresses of \( \mathcal{N} \). The dendrogram of \( \mathcal{N} \) is shown in Fig 5.2.6 in which the stalks corresponding to primitive addresses are shown by thick lines.

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Definition 5.2.13 Let $N$ be a nexus and let $a = (a_1, a_2, \ldots, a_k)$ be an address of $N$. The set

$$\{(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n) \in N : a_{k+i} \in \mathcal{N} \text{ for } i = 1, 2, \ldots, n-k\}$$

is called the 'panel' of $a$ and is denoted by $q_a$. In other words, if $a = (a_1, a_2, \ldots, a_k)$, then every address $b$ of $N$ is an address in $q_a$ provided that the first $k$ terms of $b$ are the same as the corresponding terms of $a$. Note that, the 'panel' of $a$ does not include $a$. Also, $q(\cdot)$ includes all the addresses of $N$ except for the empty address itself.

Definition 5.2.14 Let $N$ be a nexus and let $a$ be an address of $N$. The set

$$\{b \in N : a \leq b\}$$

is called the 'quasi panel' of $a$ and is denoted by $Q_a$. The 'quasi panel' of $a$ includes $a$ and the 'quasi panel' of $a$ excluding $a$ is denoted by $Q_a^- = Q_a - \{a\}$.

For example, consider the nexus

$$N = \{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 2, 1), (2, 2, 2), (2, 3, 1), (2, 3, 2)\}$$

The dendrogram of nexus $N$ is shown in Fig 5.2.7.

Now, consider the address $(2, 2)$ of $N$. Then

$$q_a = \{(2, 2, 1), (2, 2, 2)\}$$

and

$$Q_a = \{(2, 2, 1), (2, 2, 2), (2, 3, 1), (2, 3, 2)\}$$
are the panel and the quasi panel of the address \((2, 2)\) of \(N\), respectively. The elements of the panel and the quasi panel of the address \((2, 2)\), are shown by thick lines in Fig 5.2.7 (a) and 5.2.7 (b), respectively.

![Dendrogram of nexus N](image)

**Fig 5.2.7 Dendrogram of nexus \(N\).** The addresses in \(q_a\) and \(Q_a\) are shown by thick lines in \(a\) and \(b\) respectively.

**Definition 5.2.15** Let \(S\) be a nonempty subset of \(N\), then

\[
q_S = \bigcup_{a \in S} q_a \quad \text{and} \quad Q_S = \bigcup_{a \in S} Q_a
\]

Note that all addresses of \(q_a\) are greater than \(a\), but the converse of this statement is not true. For instance, in the above example, the addresses \((2, 3), (2, 3, 1), (2, 3, 2)\) are greater than \((2, 2)\), but none of these are in \(q_a\). However, the converse of the statement is true for \(Q_a\), that is, all the addresses of nexus \(N\) that are greater than or equal to \(a\) belong to \(Q_a\). Therefore,

\[
b \in Q_a \Leftrightarrow a \leq b \quad \forall b \in N
\]

Two necessary and sufficient conditions for an address to be in \(q_a\) are formalised in the following lemma.

**Lemma 5.2.16** Let \(N\) be a nexus and let \(a = (a_1, a_2, \ldots, a_k)\) and \(b\) be addresses in \(N\). Then
Proof: (i) (\(\Rightarrow\)) Suppose that \(b \in q_a\). Then one can set \(b = c\). So \(a < b \leq c\).

(\(\Leftarrow\)) Conversely, suppose that, \(b \in N\) and there exists an address \(c \in q_a\) such that \(a < b \leq c\). Now, since \(a < b\), so \(b\) is of the form

\[b = (a_1, a_2, ..., a_k, b_1, b_2, ...b_n)\]

or

\[b = (a_1, a_2, ..., a_k + t, b_1, b_2, ...b_n)\]

for some \(t \in \mathbb{N}\). But \(c \in q_a\), so

\[c = (a_1, a_2, ..., a_k, c_1, c_2, ..., c_m)\]

Now, if

\[b = (a_1, a_2, ..., a_k + t, b_1, b_2, ...b_n)\]

then, by definition of relation \(\leq\), \(b \geq c\). This is a contradiction of the hypothesis (that is, \(a < b \leq c\)). Therefore,

\[b = (a_1, a_2, ..., a_k, b_1, b_2, ...b_n)\]

This implies that, \(b \in q_a\).

(ii) \(\Rightarrow\) Suppose that \(b \in q_a\). Then one can set \(b = c\). So \(a < b \leq c\).

(\(\Leftarrow\)) Conversely, suppose that, \(c \in q_a\) and \(a < c \leq b\) for some \(b \in N\). Since \(c \in q_a\) and \(a < c\), so

\[c = (a_1, a_2, ..., a_k, a_{k+1}, ..., a_n)\]

Also, \(c \leq b\) implies that, \(l(c) \leq l(b)\) and the first \(n-1\) terms of \(c\) are equal to \(n-1\) terms of \(b\). Consequently, at least, the first \(k\) terms of \(b\) are \(a_1, a_2, ..., a_k\). Therefore, by definition of \(q_a\) (Definition 5.2.13), \(b \in q_a\).

(iii) Suppose that \(c \in Q_b\). Therefore, \(b \leq c\). By hypothesis, \(b \in q_a\), so, \(a < b \leq c\), by part (ii), \(c \in q_a\).

Thus \(Q_b \subseteq q_a\). 

The main properties of \(q_a\) and \(Q_a\) are discussed in the following theorems.

**Theorem 5.2.17** Let \(N\) be a nexus, then

(i) \(Q(\emptyset) = N\) and \(q(\emptyset) = N - \{\emptyset\}\).

(ii) \(a\) is a primitive address of \(N\) if and only if \(q_a = \emptyset\). Also, if \(S\) is a subset of the set of all primitive addresses of \(N\), then \(q_S = \emptyset\).

(iii) \(a\) is a maximal address of \(N\) if and only if \(Q_a = \{a\}\). Also, if \(S\) is a subset of the set of all maximal addresses of \(N\), then \(Q_S = S\).
(iv) For every address in $N$, $q_a \subseteq Q_a$.

**Proof:**

(i) By using the definition of $Q_a$ and letting $a = ()$,

$$Q(()) = \{a \in N: () \leq a\} = N$$

Since $(()) \notin q(())$, then $q(()) \subseteq N - \{()\}$. Now, suppose that $b \in N - \{()\}$. Therefore, $(()) < b \leq b$, and by Theorem 5.2.14 part (i), $b \in q(())$. Thus, $q(()) = N - \{()\}$.

(ii) The proof follows from the definition of a primitive address.

(iii) Suppose that $a$ is a maximal address in $N$. This implies that for any address $b \in N$, if $b \geq a$, then $b = a$. Therefore, the set of all addresses that are greater than or equal to $a$ contain only $a$. This implies that $Q_a = \{a\}$. Also, if $S$ is a subset of the set of all maximal addresses of $N$, then

$$Q_S = \bigcup_{s \in S} Q_s = \bigcup_{s \in S} \{s\} = S$$

(iv) Let $b \in q_a$, then $a < b$. Therefore, $b \in Q_a$ which implies that $q_a \subseteq Q_a$. $\blacksquare$

Note that every maximal address is a primitive address of $N$, but the next example shows that the converse of this statement is not true.

Consider the nexus

$$N = \{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\}$$

The dendrogram of $N$ is shown in Fig 5.2.8. The addresses $(1, 1), (1, 2), (2, 1), (2, 2)$ and $(2, 3)$ are primitive addresses. Amongst these addresses only $(1, 2)$ and $(2, 3)$ are maximal. To elaborate, $(1, 1), (2, 1)$ and $(2, 2)$ are not maximal addresses, because $(1, 1) < (1, 2)$ and $(2, 1) < (2, 2) < (2, 3)$.

![Fig 5.2.8 Dendrogram of $N$. The maximal addresses are represented by thick lines.](image)

**Theorem 5.2.18** Let $a$ be an address of a nexus $N$ and let $S$ be a nonempty subset of $N$. Then

(i) $q_a$ and $Q_a$ are closed under the meet operation $\wedge$. Furthermore, if $N$ is considered as a semilattice, then $q_a$ and $Q_a$ are subsemilattices of $N$. 

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(ii) $S \subseteq Q_s$ but $S \not\subseteq q_S$. Furthermore, if any two elements of $S$ are either not comparable, or are comparable with the same level, then $S \cap q_S = \emptyset$

(iii) If $T \subseteq S$ then $q_T \subseteq q_S$ and $Q_T \subseteq Q_s$.

(iv) $Q_{qa} = Q_a$ and $q_{qa} \subseteq q_a$. More precisely, if $a = (a_1, a_2, \ldots, a_n)$, then

$$q_{qa} = q_a - \{(a_1, a_2, \ldots, a_n, k) \in N : k \in \mathbb{N}\}$$

In other words, if $a$ is of level $n$, then the addresses in $q_a$ are of level $n+1$ and

$$q_{qa} = \{b \in q_a : \text{level}(b) = n+2\}$$

(v) If $a = (a_k)$ is a principal address of $N$ and $\text{Ord}(N) = n$, then

$$q_{(a_k)} = N_k - \{a_k\} \quad \text{and} \quad Q_{(a_k)} = \bigcup_{i=k}^{n} N_i$$

Proof: (i) Suppose that $a = (a_1, a_2, \ldots, a_k)$ and $b, c \in q_a$. Therefore, $b$ and $c$ are of the form

$$b = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n)$$

and

$$c = (a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_m)$$

Now, consider two cases:

Case 1: $b_1 \neq c_1$, then

$$b \land c = (a_1, a_2, \ldots, a_k, b_1 \land c_1)$$

the result of $b \land c$ is either $b_1$ or $c_1$. Therefore,

$$(a_1, a_2, \ldots, a_k, b_1 \land c_1) \in q_a$$

Case 2: $b_i = c_i$ for $i = 1, 2, \ldots, t$, where $t \leq (n \land m)$. Then

$$b \land c = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_t, b_{t+1} \land c_{t+1}) \in q_a$$

So, in both cases $b \land c \in q_a$. Consequently, $q_a$ is closed under the meet operation. The proof for $Q_a$ is similar.

(ii) By Definition 5.2.15 $Q_S = \bigcup_{a \in S} Q_a$. Since $a \in Q_a$, so, $S \subseteq Q_S$. Now, using a counterexample one can show that, $S \not\subseteq q_S$. To this end, consider the nexus $N = \{((), (1), (2))\}$. Suppose that $S = \{(1), (2)\}$. Since the addresses in $S$ are primitive addresses, by Theorem 5.2.17 (ii), $q_S = \emptyset$. So, $S \not\subseteq q_S$. Now, suppose that $S$ is a subset of $N$ and any two elements of $S$ are not comparable or comparable with the same level. In this case, one can show that, $S \cap q_S = \emptyset$. By using contradiction, suppose that $S \cap q_S \neq \emptyset$. So, there is $a \in S \cap q_S$. Therefore, $a \in q_S = \bigcup_{b \in S} q_b$. This implies that $a \in q_b$ for some $b \in S$. Since $a \in q_b$, so, $b < a$. This means that $a$ and $b$ are comparable and the level of $a$ is less than the level of $b$. This contradicts the hypothesis. Consequently $S \cap q_S = \emptyset$. 

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(iii) Let \( a \in q_T \). Since \( q_T = \bigcup_{t \in T} q_t \), there exists \( t' \in T \) such that \( a \in q_{t'} \). But \( T \subseteq S \). Therefore, \( t' \in S \).

Since \( q_S = \bigcup_{s \in S} q_s \), then, \( q_{t'} \subseteq q_S \). Thus \( a \in q_S \). This implies that \( q_T \subseteq q_S \). By the same argument, one can show that \( Q_T \subseteq Q_S \).

(iv) By using part (ii), \( Q_a \subseteq Q_{Q_a} \). So, one must show that \( Q_{Q_a} \subseteq Q_a \). To this end, suppose that, \( c \in Q_{Q_a} = \bigcup_{b \in Q_a} q_b \). Therefore, \( c \in Q_b \) for some \( b \in Q_a \). So, \( a \leq b \leq c \). This implies that \( c \in Q_a \).

Thus \( Q_{Q_a} = Q_a \).

By the same argument, and by changing the inequality \( a \leq b \leq c \) to \( a < b < c \), one can show that \( Q_{Q_a} = Q_a \).

Now, for the rest of the proof of part (iv) suppose that \( c \in Q_a \) and also suppose that \( c \) is of the form \((a_1, a_2, \ldots, a_n, k)\) where \( k \neq 0 \). Consider
\[
Q_{Q_a} = \bigcup_{b \in Q_a} q_b = \bigcup_{b \in \{ (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m) : k \neq 0 \}} q_b
\]
By using contradiction, assume that
\[
c \in \bigcup_{b \in \{ (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m) : k \neq 0 \}} q_b
\]
Therefore, \( c \in q_b \) where \( b = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m) \). So, \( b < c \). This implies that \( c \) is of the form
\[
c = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_i)
\]
where \( b_1, b_2 \neq 0 \). This contradicts the hypothesis (that is, \( c = (a_1, a_2, \ldots, a_n, k) \) where \( k \neq 0 \)).

Using the above argument, one can show that every address \( b \) of \( Q_a \) of the form
\[
b = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)
\]
where \( b_1, b_2 \neq 0 \), is an address of \( Q_{Q_a} \). Therefore,
\[
Q_{Q_a} = Q_a - \{(a_1, a_2, \ldots, a_n, k) : k \in \mathbb{N} \}
\]
(v) By definition of \( Q_a \), all addresses in \( N \) that have \( k \) as the stem are in \( q_{(k)} \). Since, \( (k) \notin q_{(k)} \), so \( q_{(k)} = N_k - \{(k)\} \). Now, consider \( Q_{(k)} = \{ b \in N : (k) \leq b \} \). The addresses in \( Q_{(k)} \) are of the form \( b = (b_1, b_2, \ldots, b_n) \in N \), where \( b_1 \geq k \). Therefore, \( b \in N_i \) for some \( i \geq k \). On other hand, every address in \( \bigcup_{i=k}^n N_i \) is the form \( d = (d_1, d_2, \ldots, d_m) \) where \( d_1 \geq k \). This implies that \( d \in Q_{(k)} \).

Therefore, \( Q_{(k)} = \bigcup_{i=k}^n N_i \). \( \blacksquare \)

**Theorem 5.2.19** Suppose that \( N \) is a nexus and \( a \) is an address of \( N \).
(i) \( a = \inf Q_a \), that is, \( a \) is the greatest lower bound of \( Q_a \). Therefore,

\[ Q_a = \{ b \in N : b \land a = a \} \]

(ii) If \( a = (a_1, a_2, \ldots, a_n) \) is a nonprimitive address, then \( q_a \) has a greatest lower bound, which is,

\[ \inf q_a = (a_1, a_2, \ldots, a_n, 1) \]

(iii) For every nonprimitive address \( a \) of \( N \), there exists an address \( b \) of \( N \) such that \( q_a = Q_b \).

**Proof:** (i) By definition of \( Q_a \), all addresses in \( Q_a \) are greater than or equal to \( a \), so \( a \) is a lower bound of \( Q_a \) and also \( Q_a \) contains \( a \). Recall from real analysis that if \( S \) is a set and \( a \) is a lower bound of \( S \) and it is also in \( S \), then \( a \) is the greatest lower bound of \( S \) [see Ref 13]. Therefore, \( a \) is the greatest lower bound of \( Q_a \).

(ii) Let \( a = (a_1, a_2, \ldots, a_n) \). Since all addresses in \( q_a \) have their first \( n \) terms the same as the first \( n \) terms of \( a \), so, any address \( b \) in \( q_a \) is of the form

\[ b = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m) \]

Since \( a \) is a nonprimitive address, so, \( b_1 \geq 1 \). Now consider the address

\[ c = (a_1, a_2, \ldots, a_n, 1) \]

of \( q_a \). For every \( b \) in \( q_a \), \( b \land c = c \). Therefore, \( c \) is a lower bound of \( q_a \). Also, \( c \) is in \( q_a \). Therefore, \( c = (a_1, a_2, \ldots, a_n, 1) \) is the greatest lower bound of \( q_a \).

(iii) Suppose that \( a \) is a nonprimitive address of \( N \), and \( b = \inf q_a \). Now, to show that \( q_a = Q_b \), suppose that \( c \in q_a \). Since \( b = \inf q_a \), so, \( b \leq c \). Therefore, \( c \in Q_b \). Thus \( q_a \subseteq Q_b \). Now, let \( c \in Q_b \). Then \( b \leq c \). Since \( b \in q_a \) and \( b = \inf q_a \), therefore, \( a < b \leq c \). By Lemma 5.2.14 (ii), \( c \in q_a \). Thus, \( Q_b \subseteq q_a \). Since we also have \( q_a \subseteq Q_b \) then \( q_a = Q_b \).

**Definition 5.2.20** Let \( a = (a_1, a_2, \ldots, a_n, \ldots) \) be an address. The function \( \text{shift}(k, a) \), where \( k \) is a positive integer, transforms address \( a \) into a new address by adding \( k \) to the stem of \( a \). In other words, if

\[ b = \text{shift}(k, a) \]

then

\[ b = (a_1 + k, a_2, \ldots, a_n, \ldots) \]

For example, consider the address \( a = (5, 2, 9, 1, 3, 7) \) and the integer \( k = 4 \). The effect of function \( \text{shift} \) on \( a \) is as follows:

\[ \text{shift}(4, a) = (5 + 4, 2, 9, 1, 3, 7) \]

\[ = (9, 2, 9, 1, 3, 7) \]
Now, suppose that $S$ is a set of addresses. Then the effect of function $\text{shift}$ on $S$ is defined as follows:

$$\text{shift}(k, S) = \{ \text{shift}(k, s) : s \in S \}$$

That is, the $\text{shift}$ function is applied to all the elements of $S$.

For example, consider the nexus

$$N = \{(\ ), (1), (2), (2, 1), (2, 2)\}$$

The effect of function $\text{shift}(4, N)$ is as follows:

$$\text{shift}(4, N) = \{ \text{shift}(4, a) : a \in N \} = \{(4), (1+4), (2+4, 1), (2+4, 2)\} = \{(4), (5), (6), (6, 1), (6, 2)\}$$

**Definition 5.2.21** Let $N$ and $M$ be two nexuses and let the order of $N$ be $n$. The ‘composition’ of $N$ and $M$ is denoted by $N \# M$ and is defined as

$$N \# M = N \cup \text{shift}(n, M)$$

In this context, the symbol $\#$ is referred to as ‘duplus’.

For example, consider the nexuses

$$N = \{(\ ), (1), (2), (3), (2, 1), (2, 2)\}$$

and

$$M = \{(\ ), (1), (2), (1, 1), (1, 2), (1, 3)\}$$

The dendrograms of nexuses $N$ and $M$ are shown in Fig 5.2.9. The order of $N$ is three, therefore,

$$\text{shift}(3, M) = \{(3), (4), (5), (4, 1), (4, 2), (4, 3)\}$$

The composition of $N$ and $M$ is

$$N \# M = N \cup \text{shift}(3, M) = \{(\ ), (1), (2), (3), (2, 1), (2, 2)\} \cup \{(3), (4), (5), (4, 1), (4, 2), (4, 3)\} = \{(\ ), (1), (2), (3), (4), (5), (2, 1), (2, 2), (4, 1), (4, 2), (4, 3)\}$$

The dendrogram of nexus $N \# M$ is shown in Fig 5.2.9.

The Definition 5.2.21, may be generalised in terms of another function as follows:

**Definition 5.2.22** Let $a$ and $b$ be two addresses. The function $\text{annex}(a, b)$, transforms the address $b$ into a new address by putting all the terms of $a$, in the same order as in $a$, before the first term of $b$. In other words, if $a = (a_1, a_2, ..., a_n)$ and $b = (b_i)_{i\in k}$, then

$$c = \text{annex}(a, b)$$
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will be

\[(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots)\]

The term \(a\) in an annex function \(\text{annex}(a, b)\) must be a finite address but \(b\) can be of infinite level.

For instance, consider two addresses \(a = (6, 3, 8, 2, 5)\) and \(b = (7, 1, 4, 6)\). The effect of writing

\[c = \text{annex}(a, b)\]

is

\[c = \text{annex} ((6, 3, 8, 2, 5), (7, 1, 4, 6))\]

or

\[c = (6, 3, 8, 2, 5, 7, 1, 4, 6)\]

Now, suppose that \(S\) is a set of addresses and \(a\) is a finite address. The effect of function \(\text{annex}(a, S)\) on \(S\) is as follows:

\[\text{annex}(a, S) = \{\text{annex}(a, s) : s \in S\}\]

In other words, the function \(\text{annex}\) transforms \(S\) into a new set of addresses by putting all the terms of address \(a\) before the first term of every address in \(S\).

For instance, consider the nexus

\[
\begin{align*}
\text{N} & \quad \text{M} \\
(2, 1) & \quad (1, 2) \\
(1) & \quad (1) \\
\text{N} \# \text{M} & \quad \text{N} \# \text{M}
\end{align*}
\]

Fig 5.2.9 Dendrograms of nexuses \(N, M\) and \(N \# M\)

For instance, consider the nexus

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\[ N = \{(), (1), (2), (1, 1), (1, 2)\} \]

and an address \( a = (3, 2, 4, 1) \), then,

\[
\text{annex}(a, N) = \{\text{annex}(a, b) : b \in N\} = \{(3, 2, 4, 1), (3, 2, 4, 1, 1), (3, 2, 4, 1, 2), (3, 2, 4, 1, 1, 1), (3, 2, 4, 1, 2)\}
\]

The following example shows that, by using the function \( \text{annex} \), one is able to add a nexus to a primitive address of another nexus.

![Dendrograms of N, M and N \cup \text{annex}((3, 1), (M))](image)

For instance, consider the nexuses

\[ N = \{(), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\} \]

and

\[ M = \{(), (1), (2), (2, 1), (2, 2)\} \]

The dendrograms of the nexuses involved in this example are shown in Fig 5.2.10.

Suppose that it is required to add the nexus \( M \) to the primitive address \((3, 1)\) of \( N \). To do this, firstly one must apply the function \( \text{annex} \) to the nexus \( M \):

\[
\text{annex}((3, 1), M) = \{(3, 1), (3, 1, 1), (3, 1, 2), (3, 1, 2, 1), (3, 1, 2, 2)\}
\]

The required action may now be archived as the union of \( N \) and \( \text{annex}((3, 1), M) \), that is,
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\[ N \cup annex((3, 1), M) = \{(\ ), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\} \cup \{(3, 1), (3, 1, 1), (3, 1, 2), (3, 1, 2, 1), (3, 1, 2, 2)\}\]
\[ = \{(\ ), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2), (3, 1, 1), \]
\[ (3, 1, 2), (3, 1, 2, 1), (3, 1, 2, 2)\} \]

**Definition 5.2.23** Consider the function

\[ f: P(N) \rightarrow V \]

where \( P(N) \) is a set of primitive addresses of a nexus \( N \) and \( V \) is a set of values. \( P \) is the set of all \((a, f(a))\). That is,

\[ P = \{(a, v) \mid a \in P(N), v \in V\} \]

In other words, plenix \( P \) is the set of all ordered pairs \((a, v)\), where \( a \in P(N) \) and \( v \in V \), that is,

\[ P = \{(a, v) \mid a \in P(N), v \in V\} \]

One can represent the above statement graphically. Thus, consider the nexus

\[ N = \{(\ ), (1), (2), (1, 1), (1, 2)\} \]

and the set

\[ V = \{1/7, 1/5, \int x^3 \, dx\} \]

The set of the primitive addresses of \( N \) is

\[ P(N) = \{(1, 1), (1, 2), (2)\} \]

Suppose that

\[ f: P(N) \rightarrow V \]

is defined as follows:

\[ f((1, 1)) = 1/7 \quad f((1, 2)) = 1/5 \quad f((2)) = \int x^3 \, dx \]

Then the plenix \( P \) is obtained as follows.

\[ P = \{((1, 1), 1/7), ((1, 2), 1/5), ((2), \int x^3 \, dx)\} \]

The dendrogram of \( P \) is shown in Fig 5.2.11.

Using the usual plenix notation (see Chapter 2), \( P \) is represented by

\[ P = \langle \langle 1/7, 1/5 \rangle, \int x^3 \, dx \rangle \]

Note that, in the above example, one may create some different plenices by changing the definition of function \( f \), although the allocation of different values will not affect the constitution of the resulting plenices.

In chapter 2, the notion of plenices was defined and their properties were discussed. In establishing the theory of plenices the idea of a nexus emerged as an object representing the
constitution of a plenix. However, in the present chapter a nexus was defined independently of the concept of a plenix. However, in the above discussion, the relationship between plenices and nexuses is highlighted again, although, here, it is a plenix that is defined in terms of a nexus.

5.3 Subnexus of a Nexus

In this section the concept of a subset of a nexus is introduced.

Definition 5.3.1 Let $N$ be a nexus. A nonempty subset $S$ of $N$ is called a subnexus of $N$ provided that $S$ itself is a nexus. The set of all subnexuses of $N$ is denoted by $\text{SUB}(N)$. The subnexus $\{()\}$ of $N$ is said to be the trivial subnexus of $N$.

For example, consider the nexus

$$N = \{(,), (1), (2), (3), (1,1), (1,2), (3,1), (3,2), (3,3)\}$$

The following sets are some subnexuses of $N$.

$$M = \{(), (1), (2), (3,1), (3,2)\}$$
$$S = \{(), (1), (2), (1,1), (1,2)\}$$
$$T = \{()\}$$

The dendrogram of $N$, $M$, $S$ and $T$ are shown in Fig 5.3.1.

Theorem 5.3.2. Let $N$ be a nexus and let $S$ be a subset of $N$. $S$ is a subnexus of $N$ if and only if $a \in S$, $b \in N$ and $b \leq a$ implies that $b \in S$.

Proof. The proof follows from Theorem 5.2.8. ■

From Theorem 5.3.2 the following results may be obtained.
Corollary 5.3.3 Let $N$ be a nexus and let $\{S_\alpha | \alpha \in \Lambda\}$ be a family of $SUB(N)$, where $\Lambda$ is an index set. Then, $\bigcup_{\alpha \in \Lambda} S_\alpha$ and $\bigcap_{\alpha \in \Lambda} S_\alpha$ are subnexuses of $N$.

Proof: Let $a \in \bigcup_{\alpha \in \Lambda} S_\alpha$ and $b \in N$ and let $b \leq a$. By using Theorem 5.3.2, one must show that $b \in \bigcup_{\alpha \in \Lambda} S_\alpha$. Since $a \in \bigcup_{\alpha \in \Lambda} S_\alpha$, so, there exist $\alpha_0 \in \Lambda$ such that $a \in S_{\alpha_0}$, where $S_{\alpha_0} \in SUB(N)$.

However, $S_{\alpha_0}$ is a subnexus of $N$, therefore, $b \in S_{\alpha_0}$. This implies that, $b \in \bigcup_{\alpha \in \Lambda} S_\alpha$. The proof for $\bigcap_{\alpha \in \Lambda} S_\alpha$ is similar. ■

Before the next corollary, recall from lattice theory that a lattice $L$ is called a 'complete lattice' if every subset $M$ of $L$ has a least upper bound, that is, supremum and a greatest lower bound, that is, infimum. Also, a bounded lattice is a lattice $L$, with two constant elements that are represented by 0 and 1 and are called upper bound and lower bound, respectively [see Ref 14]. The constants 0 and 1 satisfy the following conditions:

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For all \( x \in L \) \( x \land 1 = x \) and \( x \lor 1 = 1 \)

For all \( x \in L \) \( x \land 0 = 0 \) and \( x \lor 0 = x \)

**Corollary 5.3.4** \((\text{SUB}(N), \subseteq)\) is a bounded complete lattice.

**Proof:** Two elements ( ) and \( N \) of \( \text{SUB}(N) \) can be considered as zero and unit elements of lattice \((\text{SUB}(N), \subseteq)\), respectively. So, \((\text{SUB}(N), \subseteq)\) is a bounded lattice. Now, if one shows that every subset \( H = \{S_\alpha \mid \alpha \in \Lambda\} \) of \( \text{SUB}(N) \) has a supremum and an infimum, that is \( \text{sup} H \) and \( \text{inf} H \), respectively, then \( \text{SUB}(N) \) is a complete lattice. One may define \( \text{sup} H = \bigcup_{\alpha \in \Lambda} S_\alpha \) and

\[
\text{inf} H = \bigcap_{\alpha \in \Lambda} S_\alpha.
\]

The proof follows from Theorem 5.3.2. ■

**Theorem 5.3.5** If \( T \) is a subnexus of \( N \), then \( Q_T = N \) and \( q_T = N - \{()\} \)

**Proof:** Since \( T \) is a subnexus of \( N \), so, ( ) \( \in T \). Therefore, \( Q(()) \subseteq Q_T \). By Theorem 5.2.17 part (i), \( Q(()) = N \). Thus, \( Q_T = N \). By the same theorem, \( q(()) = N - \{()\} \). By a similar argument, \( q_T = N - \{()\} \). ■

5.4 Generating Nexuses

**Definition 5.4.1** Let \( N \) be a nexus and \( \emptyset \neq \Lambda \subseteq N \). Then the smallest subnexus of \( N \) containing \( \Lambda \) is called the 'subnexus of \( N \) generated by \( \Lambda \)' and is denoted by \( \langle \Lambda \rangle \). If \( \Lambda = \{a_1, a_2, \ldots, a_n\} \), then instead of \( \langle \Lambda \rangle \) one can write \( \langle a_1, a_2, \ldots, a_n \rangle \). If \( \Lambda \) has only one element \( a \), then the subnexus \( \langle a \rangle \) is called a 'cyclic subnexus of \( N \)'. If \( N = \langle a \rangle \), then \( N \) is called a 'cyclic nexus'.

For example, consider the nexus

\[
N = \{(), (1), (2), (3), (2, 1), (2, 2), (3, 1)\}
\]

The subnexus of \( N \) generated by two addresses (3) and (2, 1) is

\[
\langle(3), (2, 1)\rangle = \{(), (1), (2), (3), (2, 1)\}
\]

Also, the subnexus

\[
\langle(2, 2)\rangle = \{(), (1), (2), (2, 1), (2, 2)\}
\]

is a cyclic subnexus of \( N \) generated by \( \{2, 2\} \).

**Theorem 5.4.2** Let \( N \) be a nexus and \( \emptyset \neq \Lambda \subseteq N \), and let \( a \) be an address in \( N \). Then

(i) \( \langle \Lambda \rangle = \{b \in N \mid \exists \ n \in \Lambda, \ b \leq a\} \)

(ii) \( \langle a \rangle = \{b \in N \mid b \leq a\} \)

**Proof:** (i) Suppose that

\[
B = \{b \in N \mid \exists \ n \in \Lambda, \ b \leq a\}
\]
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From Theorem 5.3.2, \( B \subseteq \text{SUB}(N) \). The relation \( \leq \) is reflexive so, \( A \subseteq B \). Now, one must show that \( B \) is the smallest subnexus of \( N \) which contains \( A \). To do this, suppose that \( S \) is an arbitrary subnexus of \( N \) and \( A \subseteq S \). Now, let \( b \) be an element of \( B \), then \( b \leq a \) for some \( a \in A \) and hence \( b \in S \) (by Theorem 5.3.2). Therefore, \( B \subseteq S \).

(ii) The proof follows directly from part (i).

**Theorem 5.4.3** Let \( N \) be a nexus and let \( \emptyset \neq A \subseteq N \). Then

\[
\langle A \rangle = \bigcup_{a \in A} \langle a \rangle
\]

**Proof:** By Corollary 5.3.3, \( \bigcup_{a \in A} \langle a \rangle \) is a subnexus of \( N \). Also, \( A \subseteq \bigcup_{a \in A} \langle a \rangle \). Now, suppose that \( b \in \langle A \rangle \). By Theorem 5.4.2, \( b \leq a \) for some \( a \in A \). This implies that \( b \in \langle a \rangle \). So, \( b \in \bigcup_{a \in A} \langle a \rangle \).

Therefore,

\[
\langle A \rangle \subseteq \bigcup_{a \in A} \langle a \rangle
\]  

(*)

Now, suppose that \( b \in \bigcup_{a \in A} \langle a \rangle \). Therefore, \( b \in \langle a \rangle \) for some \( a \in A \). Thus, \( b \leq a \), so, by Theorem 5.4.2, \( b \in \langle A \rangle \). Therefore,

\[
\bigcup_{a \in A} \langle a \rangle \subseteq \langle A \rangle
\]  

(**)

By (*) and (**) \( \langle A \rangle = \bigcup_{a \in A} \langle a \rangle \). ■

**Corollary 5.4.4** The set of all primitive addresses of a nexus \( N \) generates the nexus, that is, \( N = \langle P(N) \rangle \).

**Proof:** Let \( a \) be an address of \( N \). Since there exists \( p \in P(N) \) such that \( a \leq p \), therefore from Theorem 5.4.2 the required proof is established. ■

**Corollary 5.4.5.** Suppose that \( a \) and \( b \) are two addresses of a nexus \( N \). Then,

\[
a \leq b \iff \langle a \rangle \subseteq \langle b \rangle
\]

**Proof:** (\( \Rightarrow \)) Suppose that \( c \) is an address in \( \langle a \rangle \). By part (i) of the present corollary, \( c \leq a \) and by hypothesis \( a \leq b \). So, \( c \leq b \). Therefore, \( c \in \langle b \rangle \). Consequently, \( \langle a \rangle \subseteq \langle b \rangle \).

(\( \Leftarrow \)) Since \( \langle a \rangle \subseteq \langle b \rangle \) and \( a \in \langle a \rangle \), so, \( a \in \langle b \rangle \). Thus \( a \leq b \).

**Corollary 5.4.6.** Let \( a = \{a_i\}_{i \in \mathbb{N}} \) be an infinite address and let \( N \) be a cyclic nexus generated by \( a \), that is, \( N = \langle a \rangle \). Then,
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Hence \((N, \leq)\) is a chain. In particular, if \(a = (a_1, a_2, ..., a_n)\) is a non-empty finite address. Then

\[
<\alpha> = \{(\alpha), (\alpha_1, \alpha_2, \alpha_3), ..., (\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n) | 0 \leq \tau_n \leq a_n, \forall n \in \mathbb{N}\}
\]

and hence the number of addresses of \(N\) is

\[
|<a>| = (\sum_{i=1}^{n} a_i)^{+} - 1
\]

**Proof:** Suppose that

\[
A = \{(\alpha), (\alpha_1, \alpha_2, \alpha_3), ..., (\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n) | 0 \leq \tau_n \leq a_n, \forall n \in \mathbb{N}\}
\]

and \(b \in <\alpha>\), therefore, \(b \leq a\). If \(l(b) = \infty\), then \(b = \alpha\). Now, let \(n = l(b) < \infty\), since \(b \leq a\), so,

\[
b = (a_1, a_2, ..., a_{n-1}, \tau_n)
\]

where \(0 \leq \tau_n \leq a_n\). Therefore, \(b \in A\). Conversely, suppose that \(b \in A\), then \(b \leq a\) and hence \(b \in <\alpha>\). Consequently, \(A = <a>\).

**Corollary 5.4.7** Let \(N = <a>\) be a cyclic nexus where \(a \neq (\) and \(M \in \text{SUB}(N)\). Then \(M\) is a cyclic nexus and hence \((\text{SUB}(N), \subseteq)\) is a chain. In particular, if \(a = (a_1, a_2, ..., a_n)\) is a nonempty finite address, then

\[
|\text{SUB}(N)| = (\sum_{i=1}^{n} a_i)^{+} - 1
\]

**Proof:** Let \(M \in \text{SUB}(N)\). If there is an infinite address in \(M\), namely \(b\), then, since \(b \leq a\), \(M = N\).

Now, assume that all the elements of \(M\) are finite and that \(t \in M\) is of highest level. Then, clearly 

\(M = <\alpha>\).

**Theorem 5.4.8** Consider a cyclic nexus \(N = <a>\), where \(a = (a_1, a_2, ..., a_n)\). The number of primitive elements of \(N\) is

\[
(\sum_{i=1}^{n} a_i) - (n - 1)
\]

where \(n\) is the level of \(a\), that is, \(l(a)\).

**Proof:** The nonprimitive elements of \(N\) are

\[
(\), (a_1), (a_1, a_2), (a_1, a_2, a_3), ..., (a_1, a_2, ..., a_{n-1})
\]
Therefore, the number of nonprimitive elements of $N$ is $n$. Now, by Corollary 5.4.6, the number of addresses of $N = \langle a \rangle$ is $\left( \sum_{i=1}^{n} a_i \right) + 1$. Therefore, the number of primitive elements of $N$ is

$$\left( \sum_{i=1}^{n} a_i \right) + 1 - n = \left( \sum_{i=1}^{n} a_i \right) - (n - 1)$$

For example, consider the nexus

$$N = \{(\,), (1), (2), (2, 1), (2, 2), (2, 3), (2, 4), (2, 4, 1), (2, 4, 2), (2, 4, 3), (2, 4, 3, 1), (2, 4, 3, 2)\}$$

The dendrogram of $N$ is shown in Fig 5.4.1.

Fig 5.4.1 Dendrogram of cyclic nexus $N = \langle(2, 4, 3, 2)\rangle$. The nonprimitive addresses are represented by thick lines.

The nexus $N$ is a cyclic nexus generated by $a = (2, 4, 3, 2)$. So $N = \langle(2, 4, 3, 2)\rangle$. By Corollary 5.4.6, the number of addresses of $N$ is

$$|\langle a \rangle| = \left( \sum_{i=1}^{n} a_i \right) + 1$$

where $a_1 = 2$, $a_2 = 4$, $a_3 = 3$ and $a_4 = 2$. Therefore,

$$|\langle a \rangle| = (2 + 4 + 3 + 2) + 1 = 12$$

Since $l(a) = 4$, so, by Theorem 5.4.8, the number of nonprimitive addresses of $N$ is 4 and they are $\{(\,), (2), (2, 4), (2, 4, 3)\}$

In Fig 5.4.1, nonprimitive addresses are shown by thick lines. Now, by Theorem 5.4.8, the number of primitive elements of a nexus $N$ is

$$\left( \sum_{i=1}^{n} a_i \right) - (n - 1)$$
Thus, the number of primitive elements of the nexus in the present example, namely, 
\(N = \langle (2, 4, 3, 2) \rangle\) is
\[(2 + 4 + 3 + 2) - (4 - 1) = 8\]

**Definition 4.4.9** (i) Suppose that \(a\) and \(b\) are two addresses of a nexus \(N\) and that \(a\) and \(b\) are noncomparable in \((N, \leq)\), then one can say that \(a\) and \(b\) are ‘independent’.

(ii) If \(\emptyset \neq \beta \subseteq N\) and \(N = \langle \beta \rangle\) and every pair of the elements of \(\beta\) are independent, then \(\beta\) is called a ‘base’ for \(N\).

**Theorem 5.4.10** Let \(\beta = \{b_1, b_2, \ldots, b_n\}\) be the set of all ‘maximal addresses’ of a finite nexus \(N\). Then

(i) \(\beta\) is a base for \(N\).

(ii) If \(K\) is any set of generators of \(N\), then \(|\beta| \leq |K|\)

(iii) If \(|\beta| = |K|\), then \(\beta = K\). Therefore, every finite nexus has a unique base, that is, \(\beta\).

**Proof:** (i) Suppose that \(b_1, b_2 \in \beta\). If \(b_1\) and \(b_2\) are comparable, then \(b_1 \leq b_2\) or \(b_2 \leq b_1\). In both cases, the relation is a contradiction of the property of maximality of elements of \(\beta\). So the elements of \(\beta\) are independent. Now, suppose that an address \(a \in N\), then there exists one maximal address \(b\) in \(\beta\) such that, \(a \leq b\). This means that any address \(a\) in \(N\) is less than or equal to one element of \(\beta\). Therefore, the proof follows from Theorem 5.4. 2.

(ii) Suppose that 
\[K = \{k_1, k_2, \ldots, k_t\}\]

By Theorem 5.4.3, \(N = \bigcup_{i=1}^{t} < k_i >\). Since \(\beta \subseteq N\), so, every element of \(\beta\) is in one of the \(< k_i >\) for \(i = 1, 2, \ldots, t\). Also, since the elements of every \(< k_i >\) constitute a chain and also the elements of \(\beta\) are independent, so there cannot be more than one element of \(\beta\) in \(< k_i >\) for \(i = 1, 2, \ldots, t\). Therefore \(|\beta| \leq |K|\)

(iii) Suppose that \(b_1 \in < k_i >\). It has already been shown in Corollary 5.4.6 that the elements of \(< k_i >\) constitute a chain and \(k_t\) is the largest element in \(< k_i >\). Since \(b_t\) is a maximal element of \(N\), therefore, \(b_1 = k_t\). By hypothesis, \(|\beta| = |K|\), thus \(\beta = K\). 

**Definition 5.4.11** The unique number mentioned in the above theorem (that is, the number of the elements of the base of a nexus), is called the dimension of a nexus.

For example, consider the nexuses
\[N = \{( ), (1), (2), (3), (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}\]

and
\[ M = \{(), (1), (2), (2, 1), (2, 2), (2, 2, 1), (2, 2, 2), (2, 2, 2, 1), (2, 2, 2, 2), (2, 2, 2, 1, 1), (2, 2, 2, 2, 1, 2)\} \]

The dendrograms of \( N \) and \( M \) are shown in Fig 5.4.2. The maximal elements of \( N \) are \((1, 2), (2, 2)\) and \((3, 2)\). The maximal elements of \( M \) are \((2, 2, 2, 2)\) and \((2, 2, 2, 1, 2)\). Therefore, the dimensions of \( N \) and \( M \) are 3 and 2 respectively.

Fig 5.4.2: Dendrograms of nexuses \( N \) and \( M \) the elements of the bases are shown by thick lines.

**Corollary 5.4.12** The dimension of all cyclic nexuses is one.

**Proof:** The proof follows the definition of a cyclic nexus. ■

**Theorem 5.4.13** Let \( N \) be a cyclic nexus generated by a finite address \( a \), that is, \( N = \langle a \rangle \). Then every subnexus of \( N \) is a cyclic subnexus.

**Proof:** Suppose that \( K \) is a subnexus of \( N \). Recall that every subset of a totally ordered set is a totally ordered set. So, \( K \) is a totally ordered set. This means that \( K \) has only one maximal element \( b \). Now, by Theorem 5.4.10, \( K \) is generated by \( b \), that is, \( K = \langle b \rangle \). ■

The following example shows that, generally, the converse of Theorem 5.4.13 is not true. That is, if every proper subnexus of \( N \) is cyclic then \( N \) is not necessarily a cyclic nexus. Consider the nexus

\[ N = \{(), (1), (2), (1, 1)\} \]

\( N \) has two maximal elements, that is, \((2)\) and \((1, 1)\). Therefore by Theorem 5.4.10,

\[ N = \langle 2, (1, 1) \rangle \]
as may be seen from the dendrogram of $N$ in fig 5.4.3. So, $N$ is not cyclic, but all the proper subnexuses of $N$, namely, {()}, {{()}}, {((), (1)}}, and {{{()}, (1), (1)}} are cyclic and equal to $<()>$, $<(1)>$, $<(2)>$ and $<(1, 1)>$, respectively.

![Dendrogram of $N$. The maximal elements represented by thick lines.](image)

In the next theorem, a topology is defined on the set of subnexuses of a nexus. Recall that a topological space is a set $X$ together with a set $T$ consisting of some of subsets of $X$ such that $T$ satisfies the following three conditions:

1. The empty set $\emptyset$ and $X$ are in $T$.
2. The intersection of any finite number of sets in $T$ is also in $T$.
3. The union of any number of sets in $T$ is also in $T$.

Let $X$ be a set. A ‘basis’ for a topology on $X$ is a collection $B$ of subsets of $X$ (called basis elements) satisfying the following properties:

(i) For every $x \in X$, there is at least one basis element $B$ containing $x$.

(ii) If $x$ belongs to the intersection of two basis elements $B_1$ and $B_2$, then there is a basis element $B_3$ containing $x$ such that $B_3 \subseteq (B_1 \cap B_2)$ [see Ref 15].

**Theorem 5.4.11** Let $T = \text{SUB}(N) \cup \{\emptyset\}$. Then, $(N, T)$ is a topological space for which $\{<a> \mid a \in N\}$ is a basis.

**Proof:** By Corollary 5.3.3, the union and the intersection of an arbitrary subnexuses of a nexus is a subnexus. So $(N, T)$ is a topological space. Now, suppose that, $x \in N$. Therefore, $x \in \langle x \rangle$. Furthermore, let $x \in \langle a \rangle \cap \langle b \rangle$. So, $x \leq a, b$. Thus, by Corollary 5.4.5, $\langle x \rangle \subseteq \langle a \rangle$ and
This implies that, $x \in <x> \subseteq <a> \cap <b>$. Hence, \{<a> | a \in N\} is a basis for $(N, T)$. 

5.5 The Number of Subnexuses of a Finite Nexus

In this section the number of subnexuses of a finite nexus is determined. To do this, some introductory ideas will be discussed. In this section all the nexuses considered are finite.

**Definition 5.5.1** Let $N$ be a nexus and let $\text{SUB}(N)$ be the set of all subnexuses of $N$. The following relation may be defined on $\text{SUB}(N)$:

$$S_1 \sim S_2 \iff \text{Ord}(S_1) = \text{Ord}(S_2) \quad S_1, S_2 \in \text{SUB}(N)$$

The relation $\sim$ is an equivalence relation on $\text{SUB}(N)$. The equivalence class $S \in \text{SUB}(N)$ is denoted by $[S]$. Therefore, the set

$$\{[S] : S \in \text{SUB}(N)\}$$

is a partition of the set of nontrivial subnexuses of $N$.

**Definition 5.5.2** Let $N$ be a nexus. Suppose that the relation $\sim$ is defined on $N$ as follows:

$$a \sim b \iff \text{stem}(a) = \text{stem}(b) \quad a, b \in N$$

The relation $\sim$ is an equivalence relation on $N$. The equivalence class $a \in N$ is denoted by $[a]$. Therefore, the set

$$\{[a] : a \in N\}$$

is a partition of $N$. Note that

$$N_{a_1} = \{a \in N : \text{stem}(a) = a_1\}$$

By the above notation

$$[a] = N_{a_1}$$

where $a_1$ is the stem of $a$. Therefore,

$$\{N_k : 1 \leq k \leq \text{Ord}(N)\}$$

is a partition of $N$.

**Remark 5.5.3** Consider an equivalence class, in accordance with Definition 5.5.1, and assume that every element of the class has $k$ principal addresses. This class is denoted by $P_k$. Therefore, $\{P_k | 1 \leq k \leq \text{Ord}(N)\}$ is a partition of the set of all nontrivial subnexuses of a nexus $N$.

This remark is of value in counting the subnexuses of a nexus. To illustrate the point, consider the nexus

$$N = \{(,), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4)\}$$

The dendrogram of $N$ is shown in Fig.5.5.1.
This nexus has three subnexuses with one principal address, namely,
\(<(1)>, <(1, 1)>, <(1, 2)>\)

So,
\[P_1 = \{<(1)>, <(1, 1)>, <(1, 2)>\}\]

The subnexuses are shown by their generator. Now, consider the subnexuses of \(N\) with two principal addresses. All the addresses that have 2 as the stem, namely,
\((2), (2, 1), (2, 2), (2, 3), (2, 4),\)
can generate a subnexus with two principal addresses. Their subnexuses are
\(<(2)>, <(2, 1)>, <(2, 2)>, <(2, 3)>, <(2, 4)>\)

Furthermore, the union one of these subnexus and the subnexuses with one principal address is also a subnexus with two principal addresses. These subnexuses are the elements of \(P_2\), as follows:
\[P_2 = \{<(1), (2)>, <(1, 1), (2)>, <(1), (2, 1)>, <(1), (2, 2)>, <(1), (2, 3)>, <(1), (2, 4)>, <(1, 1), (2, 4)>, <(1, 2), (2)>, <(1, 2), (2, 1)>, <(1, 2), (2, 2)>, <(1, 2), (2, 3)>, <(1, 2), (2, 4)>\}\]

According to remark 5.5.2, \(P_1\) and \(P_2\) constitute a partition of the set of all nontrivial subnexuses of \(N\). So, every nontrivial subnexus of \(N\) is in \(P_1\) or \(P_2\) (not in both). Therefore, to count the nontrivial subnexuses of \(N\), one can add the number of elements of \(P_1\) to that of \(P_2\). Consequently, the number of nontrivial subnexuses of \(N\) will be 3 + 15 = 18.

**Lemma 5.5.4** Suppose that all the addresses of nexus \(N\) are principal, that is,
\[N = \{((), (1), (2), \ldots, (n))\} \text{ for some } n \in \mathbb{N}\]
The number of subnexuses of \(N\) is \(n+1\), that is, \(|\text{SUB}(N)| = n+1\)

**Proof:** The subnexuses of \(N\) are
\[\{(), \{((), (1)\}, \{(), (1), (2)\}, \ldots, \{(), (1), (2), \ldots, (n)\}\} = N\]
amounting to $n+1$ subnexuses. ■

**Lemma 5.5.5** Suppose that $N$ is a nexus with three principal addresses having $n$, $m$ and $r$ principal addresses, respectively. That is,

$$N = \{(1), (1), (2), (3), (1, 1), (1, 2), \ldots, (1, n), (2, 1), (2, 2), \ldots, (2, m), (3, 1), (3, 2), \ldots, (3, r)\}$$

Then, the number of nontrivial subnexuses of $N$ is

$$|\text{SUB}(N)| = (r+1) + (n+1)(m+1) + (n+1)(m+1) (r+1)$$

**Proof:** The nexus $N$ has $n+1$ subnexuses with only one principal address, namely,

$$<(1)>, <(1, 1)>, <(1, 2)>, \ldots, <(1, n)>$$

Also, $N$ has $(n+1)(m+1)$ subnexuses with two principal addresses. These subnexuses may be shown by their generators as follows:

$$<(2)>, \ldots, <(2, m)>$$

Furthermore,

$$<(2), (1, 1)>, \ldots, <(2, m), (1, 1)>$$

$$<(2), (1, 2)>, \ldots, <(2, m), (1, 2)>$$

$$\vdots$$

$$<(2), (1, n)>, \ldots, <(2, m), (1, n)>$$

$$<(3)>, \ldots, <(3, r)>$$

The union of every one of these subnexuses with one of the above $(n+1)(m+1)$ subnexuses is a subnexus with three principal addresses. The number of these subnexuses is $(n+1)(m+1) (r+1)$.

All these subnexuses are shown in Table 5.5.2 with their generators. This table has $(n+1)(m+1)$ rows and $(r+1)$ columns. Therefore, by Remark 5.5.3, $N$ has

$$(n+1) + (n+1)(m+1) + (n+1)(m+1) (r+1)$$

nontrivial subnexuses. ■

The results of Lemma 5.5.5 may be generalised as follows:

**Theorem 5.5.6** Suppose that $N$ is a nexus having $k$ principal addresses with the $i^{th}$ principal address having $n_i$ principal addresses for $i = 1, 2, \ldots, k$. The number of nontrivial subnexuses of $N$ is

$$\sum_{j=1}^{k} \prod_{i=1}^{j} (n_i + 1) = (n_1+1) + (n_1+1)(n_2+1) + \ldots + (n_1+1)(n_2+1) \ldots (n_k+1)$$
Table 5.5.2 Subnexuses on \( N \) with three principal addresses

<table>
<thead>
<tr>
<th>(&lt;(2), (3)&gt;)</th>
<th>(&lt;(2), (3, 1)&gt;)</th>
<th>(\ldots)</th>
<th>(&lt;(2), (3, r)&gt;)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt;(2), (3)&gt;)</td>
<td>(&lt;(2), (3, 1)&gt;)</td>
<td>(\ldots)</td>
<td>(&lt;(2), (3, r)&gt;)</td>
</tr>
<tr>
<td>(&lt;(2), (3)&gt;)</td>
<td>(&lt;(2), (3, 1)&gt;)</td>
<td>(\ldots)</td>
<td>(&lt;(2), (3, r)&gt;)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(&lt;(2), (3)&gt;)</td>
<td>(&lt;(2), (3, 1)&gt;)</td>
<td>(\ldots)</td>
<td>(&lt;(2), (3, r)&gt;)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

Theorem 5.5.7 Let the order of nexus \( N \) be \( k \), and let

\[ N_i = \{ (a_1, a_2, \ldots, a_n) \in N \mid a_1 = i \} \] for \( 1 \leq i \leq k \)

In other words, \( N_i \) is the set of all addresses of \( N \) whose stem is \( i \). Suppose that the number of nontrivial subnexuses of \( <N_i> \) is \( n_i \). Then the number of nontrivial subnexuses of \( N \) is equal to

\[ n_1 + n_1 n_2 + n_1 n_2 n_3 + \ldots + n_1 n_2 n_3 \ldots n_k \]
Proof: Let $S$ be a nontrivial subnexus of $<N_1>$. Then $S \cup T$, for any non-trivial subnexus $T$ of $<N_2>$ is a subnexus of $N$ with two principal addresses and conversely, every subnexus of $N$ with two principal addresses is of the form $S \cup T$, where $S$ and $T$ are nontrivial subnexus of $<N_1>$ and $<N_2>$, respectively. Therefore, the number of all subnexuses of $N$ with two principal addresses is equal to $n_1n_2$. Similarly, one can show that the number of all subnexuses with $i$ principal addresses is equal to $n_1n_2...n_i$. Now, by Remark 5.5.3, the proof of the theorem is complete.

For example, consider the nexus

$N = \{(,), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 3, 1), (2, 3, 2)\}$

The dendrogram of $N$ is shown in Fig 5.5.2.

In this case

$N_1 = \{(1), (1, 1), (1, 2)\}$

and

$N_2 = \{(2), (2, 1), (2, 2), (2, 3), (2, 3, 1), (2, 3, 2)\}$

Now, consider the nexuses

$<N_1> = \{(,), (1), (1, 1), (1, 2)\}$

and

$<N_2> = \{(,), (1), (2), (2, 1), (2, 2), (2, 3), (2, 3, 1), (2, 3, 2)\}$

$<N_1>$ and $<N_2>$ are cyclic nexuses, that is,

$<N_1> = <(1, 2)>$

and

$<N_2> = <(2, 3, 2)>$

By corollary 5.4.7, $<N_1>$ and $<N_2>$ have three and seven nontrivial subnexus, respectively. That is,
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\[ | \text{SUB} < N_1 > | = n_1 = 3 \]

and

\[ | \text{SUB} < N_2 > | = n_2 = 7 \]

By Theorem 5.5.7, the number of nontrivial subnexuses of \( N \) is

\[ n_1 + n_1 n_2 = 3 + (3 \times 7) = 24 \]

5.6 Homomorphisms of Nexuses

Definition 5.6.1 Let \( M \) and \( N \) be two nexuses and let \( f \colon M \to N \) be a function. Then \( f \) is called a ‘meet-homomorphism’ if

\[ f(()) = () \]

and

\[ f(a \land b) = f(a) \land f(b) \quad \forall a, b \in M \]

In addition, if \( f \) is onto then it is called a ‘meet-epimorphism’, if \( f \) is one-to-one then it is called a ‘meet-monomorphism’ and if \( f \) is bijective then it is called a ‘meet-isomorphism’. If \( f \) is a meet-isomorphism, then \( f^{-1} \) is also a meet-isomorphism. Furthermore, the ‘kernel’ of \( f \) is defined by

\[ \ker f = \{ a \in M \mid f(a) = () \} \]

For example, let \( k \in \mathbb{N}^* \) and let \( N \) be a nexus. Also, let the function \( f_k \) be defined from \( N \) to \( N \) as

\[ f_k(a_1,a_2,\ldots,a_n) = \begin{cases} (a_1,a_2,\ldots,a_k) & k < n \\ (a_1,a_2,\ldots,a_n) & k \geq n \end{cases} \]

By definitions of \( f_k \) and relation \( \leq \) on a nexus, \( f_k(a) \leq a \) and also, \( a \leq b \) implies that \( f_k(a) \leq f_k(b) \).

For example, consider the nexus

\[ N = \{ (), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2), (2,2,1), (2,2,2) \} \]

The result of the function \( f_2 \) on \( N \) is

\[ f_2(N) = \{ (), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (1,1), (1,1), (1,1), (1,2), (1,2), (2,2), (2,3), (1,2) \} \]

Removing the repeated elements

\[ f_2(N) = \{ (), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3) \} \]

The dendrograms of \( N \) and \( f_2(N) \) are shown in Fig 5.6.1.

One can see that for any address \( a \) of \( N \), \( f_2(a) \leq a \). For instance,

\[ f_2((2,1,3)) = (2,1) \leq (2,1,3) \]

Also, for any pair of addresses \( a \) and \( b \) of \( N \), if \( a \leq b \) then \( f_2(a) \leq f_2(b) \). For instance, since, \( (2,1) \leq (2,2,1) \) then
Now, one can show that $f_k$ is a meet-homomorphism. This requires the satisfaction of two conditions, as explained in Definition 5.6.1. The first condition is $f_k(()) = ()$ which is satisfied in the present case. Now, the second condition of meet-homomorphism, that is,

$$f_k(a \land b) = f_k(a) \land f_k(b) \quad \forall a, b \in N$$

needs to be proven. Since, $a \land b \leq a$ and $a \land b \leq b$, so, by the property of $f_k$, namely,

$$a \leq b \Rightarrow f_k(a) \leq f_k(b)$$

$f_k(a \land b) \leq f_k(a)$ and $f_k(a \land b) \leq f_k(b)$. Therefore, $f_k(a \land b) \leq f_k(a) \land f_k(b)$. This means that $f_k(a \land b)$ is a lower bound for $f_k(a)$ and $f_k(b)$. Now, one can show that

$$f_k(a \land b) = \inf\{f_k(a), f_k(b)\}$$

To this end, one must show that if there exists any address $c$ of $N$ such that $c$ is a lower bound for the set $\{f_k(a), f_k(b)\}$, then $c \leq f_k(a \land b)$. Now, suppose that $c$ is a lower bound for the set $\{f_k(a), f_k(b)\}$, then $c \leq f_k(a)$ and $c \leq f_k(b)$. This implies that $l(c) \leq k$. Therefore, $f_k(c) = c$. Thus,

$$c = f_k(c) \leq f_k(a) \leq a \quad c = f_k(c) \leq f_k(b) \leq b$$

So, $c \leq a \land b$. This implies that $f_k(c) \leq f_k(a \land b)$. But $f_k(c) = c$, then $c \leq f_k(a \land b)$. Consequently,

$$f_k(a \land b) = f_k(a) \land f_k(b)$$
The function \( f_k \) is a meet-homomorphism, and clearly
\[ \text{Ker} f_k = \{ a \in N \mid f_k(a) = () \} = N \]
Also, \( \text{Ker} f_k = \{ () \} \) for all \( k \neq 0 \). Furthermore, if \( k \geq \text{rise}(N) \) then \( f_k \) is an identity. Furthermore, if \( k < \text{rise}(N) \) then \( f_k \) is not a meet-monomorphism.

Note: in the remainder of this chapter and throughout the next chapter the word 'meet' is omitted from meet-homomorphism, meet-epimorphism, meet-monomorphism and meet-isomorphism.

**Theorem 5.6.2**: Let \( N \) and \( M \) be two nexuses and let \( f : N \rightarrow M \) be a monotone map (also called isotone map), that is, \( a \leq b \) implies that \( f(a) \leq f(b) \), then
(i) For every \( a, b \in N \nabla \)
\[ f(a \wedge b) \leq f(a) \wedge f(b) \]
(ii) For every comparable \( a, b \in N \)
\[ f(a \wedge b) = f(a) \wedge f(b) \]
(iii) If \( N \) is a cyclic nexus, then \( f \) is a homomorphism.
(iv) If \( K \) is a subnexus of \( M \), then \( f^{-1}(K) \) is a subnexus of \( N \).

**Proof**: (i) Since \( a \wedge b \leq a \), \( a \wedge b \leq b \) and \( f \) is monotone, then
\[ f(a \wedge b) \leq f(a) \; \text{and} \; f(a \wedge b) \leq f(b) \]
However, \( f(a) \wedge f(b) \) is greatest lower bound (infimum) of \( f(a) \) and \( f(b) \), therefore,
\[ f(a \wedge b) \leq f(a) \wedge f(b) \]
(ii) Let \( a, b \in N \) and let \( a \leq b \). Therefore, \( a = a \wedge b \) and \( f(a) = f(a \wedge b) \), also
\[ f(a) \wedge f(b) \leq f(a) = f(a \wedge b) \]
It has already been shown in (i) that \( f(a \wedge b) \leq f(a) \wedge f(b) \). Therefore, \( f(a \wedge b) \leq f(a) \wedge f(b) \).
(iii) If \( N \) is a cyclic nexus, then \( N \) is a totally ordered set. This means that all the addresses in \( N \) are comparable. Therefore, for every \( a, b \in N \)
\[ f(a \wedge b) = f(a) \wedge f(b) \]
This shows that \( f \) is a homomorphism.
(iv) Suppose that, \( a \in f^{-1}(K) \) and \( b \in N \) such that, \( b \leq a \). Since \( f \) is a monotone map, so \( f(b) \leq f(a) \). Also \( f(a) \in K \) and \( K \) is a subnexus of \( M \) (by hypothesis). Therefore, \( f(b) \in K \). This means that \( b \in f^{-1}(K) \). This implies that \( f^{-1}(K) \) is a subnexus of \( N \). ■

**Theorem 5.6.3** Let \( M \) and \( N \) be nexuses and let \( f \) be a homomorphism from \( M \) to \( N \), then
(i) \( a \leq b \Rightarrow f(a) \leq f(b), \; \forall \; a, b \in M \)
(ii) \( \text{Ker} f \in \text{SUB}(M) \)
(iii) If \( B \in \text{SUB}(N) \), then \( f^{-1}(B) \in \text{SUB}(M) \)
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(iii) If $f$ is an epimorphism and $A \subseteq SUB(M)$, then $f(A) \subseteq SUB(N)$

**Proof.** (i) Let $a \leq b$, then $a \wedge b = a$ and $f(a \wedge b) = f(a)$. However, since $f$ is a homomorphism then, $f(a) \wedge f(b) = f(a)$. Hence, $f(a) \leq f(b)$.

(ii) Since $f((x)) = (x)$, then $f((x)) \in Ker f$. Now, let $a \leq b$ and $b \in Ker f$. Then, $f(a) \leq f(b) = (x)$ and hence $f(a) = (x)$. Thus $a \in Ker f$.

(iii) Let $a \leq b$ and $b \in f^{-1}(B)$. Then, $f(a) \leq f(b) \wedge f(b) \in B$, which implies that $f(a) \in B$ and hence $a \in f^{-1}(B)$.

(iv) Let $a \leq b$ and $b \in f(A)$. Since $f$ is an epimorphism, there exist $c \in A$ and $d \in M$ such that $b = f(c)$ and $a = f(d)$. Hence,

$$\quad f(d) = a = a \wedge b = f(d) \wedge f(c) = f(d \wedge c).$$

Now, since $d \wedge c \leq c$, then $d \wedge c \in A$, which implies that $a \in f(A)$. ■

**Corollary 5.6.4** Let $f : M \to N$ be an epimorphism of nexuses, and let $\emptyset \neq X \subseteq M$. Then,

$$\quad f(X) = f(X)$$

**Proof:** By Theorem 5.6.3 (iv), $f(X)$ is a subnexus of $N$. Clearly, $f(X) \subseteq f(X)$. Now, let $A \subseteq SUB(N)$ and $f(X) \subseteq A$. If $b \in f(X)$ then $b = f(a)$ for some $a \in X$. Hence, $a \leq x$, for some $x \in X$ which implies that $f(a) \leq f(x)$ and hence $b = f(a) \in A$. That is, $f(X) \subseteq A$. ■

**Corollary 5.6.5** Let $f : M \to N$ be an isomorphism, and let $\beta$ be a base for $M$. Then, $f(\beta)$ is a base for $N$.

**Proof:** Since $M = \langle \beta \rangle$, then by Corollary 6.3, $N = f(M) = \langle f(\beta) \rangle$. Now, it can be shown that $f(\beta)$ is a subnexus of $N$. To do this, let $a \in f(\beta)$, $c \in M$ and $c \leq a$. Then, there exist $b \in \beta$ and $d \in N$ such that $a = f(b)$ and $c = f(d)$. Therefore, $f(d) \leq f(b)$. Now, since $f^{-1}$ is a homomorphism from $N$ into $M$, then $f^{-1}(f(d)) \leq f^{-1}(f(b))$. Thus, $d \leq b$, which implies $d \in \beta$, and hence $c \in f(\beta)$. Consequently, $f(\beta)$ is a subnexus of $N$. so, $\langle f(\beta) \rangle = f(\beta)$. Now, if $f(b)$ and $f(b')$ are two elements of $f(\beta)$, then clearly, if $f(b) \leq f(b')$ or $f(b') \leq f(b)$ then $b \leq b'$ or $b' \leq b$, respectively. Therefore, $f(b)$ and $f(b')$ are independent. ■

5.7 Direct product and Direct sum of Nexuses

**Definition 5.7.1** Let $N$ be a nexus and let $M$ be a finite nexus. Then, the set of addresses

$$\quad \{a_j\}_{j \in \mathbb{N}} : \forall k \in \mathbb{N}, (a_1, a_2, \ldots, a_k) \in M, \{a_j\}_{j \leq k, j \in \mathbb{N}} \in N$$

is called the direct product of $M$ and $N$, and is denoted by $M \times N$.  

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For example, consider two nexuses
\[ M = \{ (\cdot), (1), (1, 1) \} \]
and
\[ N = \{ (\cdot), (1), (2), (2, 1) \} \]
Then
\[ M \times N = \{ (\cdot), (1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 1, 2, 1) \} \]
Note that, if \( M \) and \( N \) are finite nexuses, both \( M \times N \) and \( N \times M \) are meaningful. In the above example,
\[ N \times M = \{ (\cdot), (1), (2), (1, 1), (2, 1), (2, 1, 1), (2, 1, 1, 1) \} \]
In this example, the bold and non-bold digits are used to show belonging to \( M \) or \( N \).

**Theorem 5.7.2** If \( M \) and \( N \) are as described in Definition 5.7.1, then their direct product, that is, \( M \times N \) is a nexus.

**Proof:** Let \( \{a_j\}_{j \in \mathbb{N}} \) be an infinite address belonging to \( M \times N \) and let \( n \) be an arbitrary element of \( \mathbb{N} \). Also, let \( k \in \mathbb{N} \) such that \( (a_1, a_2, ..., a_k) \in M \) and \( \{a_j\}_{j=k+1}^{\infty} \in N \). If \( n > k \), since \( (a_{k+1}, a_{k+2}, ..., a_n) \in N \), then \( (a_1, a_2, ..., a_n) \in M \times N \). If \( n \leq k \), since \( (a_1, a_2, ..., a_n) \in M \), then \( (a_1, a_2, ..., a_n) \in M \times N \). Hence, Definition 5.2.1 holds. Therefore, every element of \( M \times N \) is an address. Now, let \( (a_1, a_2, ..., a_n) \) be an address of \( M \times N \) and \( k \in \mathbb{N} \) such that
\[ (a_1, a_2, ..., a_k) \in M \], \( (a_{k+1}, a_{k+2}, ..., a_n) \in N \) and \( 0 \leq t < a_n \)
Then, since \( (a_{k+1}, a_{k+2}, ..., a_n - t) \in N \), it can be concluded that
\[ (a_1, a_2, ..., a_{n-1}, a_n - t) \in M \times N \]
Hence, the condition of Definition 5.2.2 holds and therefore, \( M \times N \) is a nexus.

**Definition 5.7.3** Let \( N \) and \( M \) be two finite nexuses. Then the set \( (M \times N) \cup (N \times M) \) is called the direct sum of \( M \) and \( N \) and is denoted by \( M \oplus N \).

**Theorem 5.7.4** If \( M \) and \( N \) are two finite nexuses, \( M \oplus N \) is a nexus.

**Proof:** Clearly, \( M \oplus N \) is finite. The proof is similar to the end part of the proof of Theorem 5.7.2.

For instance, consider \( M \) and \( N \) in the example in Definition 5.7.1, then
\[ M \oplus N = (M \times N) \cup (N \times M) \]
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\[\{(1), (1), (1, 1), (1, 2), (1, 1, 1), (1, 1, 2), (1, 2, 1, 1, 2, 1, 1)\} \cup \\{(1), (2), (1, 1, 1), (2, 1), (1, 1, 2), (1, 1, 2, 1, 1), (1, 1, 2, 1, 1, 1)\} \]

\[\{(1), (2), (1, 1, 1), (2, 1), (1, 1, 2), (1, 1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 1, 1, 2), (1, 2, 1, 2, 1, 1)\}\]
CHAPTER 6
Prime and Maximal Subnexuses of a Nexus

6.1 Introduction

In this chapter two important subnexuses of a nexus, namely, prime and maximal subnexuses are introduced and the properties of these subnexuses are investigated. Then, it will be proven that every subnexus of a nexus is equal to the intersection of some prime subnexuses of the nexus. Finally, a metric will be defined on a nexus.

6.2 Prime Subnexuses of a Nexus

Definition 6.2.1 A proper subnexus $P$ of a nexus $N$ is said to be a prime subnexus of $N$ if $a \land b \in P$ implies that $a \in P$ or $b \in P$ for any $a, b \in N$. 

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For example, consider a nexus \( N \). It may be shown that the trivial subnexus of \( N \), that is, \( \{()\} \) is a prime subnexus of \( N \). To show this, suppose that \( a \) and \( b \) are two addresses in \( N \) such that, \( a \wedge b \in \{()\} \). Also, suppose that \( a \) and \( b \) are not in \( \{()\} \). This implies that, \( () \neq a \geq (1) \) and \( () \neq b \geq (1) \). This, in turn, implies that \( () \neq a \wedge b \geq (1) \). But, this creates a contradiction with the hypothesis \( a \wedge b \in \{()\} \). Therefore, \( \{()\} \) is a prime subnexus of any nexus.

As another example, consider the nexus
\[
N = \{(), (1), (2), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\}
\]
The dendrogram of \( N \) is shown in Fig 6.2.1.

The subnexus
\[
P = \{(), (1), (2), (2,1), (2, 2)\}
\]
is a prime subnexus of \( N \), but the subnexus
\[
K = \{(), (1), (2), (1, 1), (2, 1)\}
\]
is not, because
\[
(1, 2) \wedge (2, 2) = (1) \in K
\]
but neither \((1, 2)\) nor \((2, 2)\) belong to \( K \).

Now, to prove that \( P \) is a prime subnexus of \( N \), it has to be shown that if \( a \wedge b \in P \), then, \( a \in P \) or \( b \in P \) for any \( a \) and \( b \) in \( N \). Suppose that, \( a \wedge b \in P \) but neither \( a \) nor \( b \) belong to \( P \). Therefore, \( a, b \in N - P = \{(1, 1), (1, 2), (1, 3)\} \). However, \( N - P \) is a chain, because,
\[
(1, 1) < (1, 2) < (1, 3)
\]
Since \( N - P \) is a chain, so, if \( a, b \in N - P \) then \( a \wedge b \) must be equal to \( a \) or \( b \). In both cases, \( a \wedge b \) must be in \( N - P \). Therefore, if \( a \wedge b \in P \) then \( a \) or \( b \) must be in \( P \). This implies that \( P \) is a prime subnexus of \( N \).

The results of the above two examples are given in the following two theorems.
Theorem 6.2.2 Let \( N \) be a nexus. The trivial subnexus of \( N \), that is, \{{()}\}, is a prime subnexus of \( N \).

Theorem 6.2.3 Suppose that \( N \) is a nexus and \( P \) is a subnexus of \( N \). Also, suppose that, \( N - P \) is a chain. Then, \( P \) is a prime subnexus of \( N \).

Theorem 6.2.4 Let \( N \) be a nexus generated by an address \( a \), that is, \( N = \langle a \rangle \). Then every proper subnexus \( P \) of \( N \) is a prime subnexus of \( N \).

Proof: Since \( N = \langle a \rangle \), it implies that \( N \) is a cyclic nexus and since every cyclic nexus is a chain and every subset of the chain set is a chain, so \( N - P \subseteq N \) is a chain. Now by Theorem 6.2.3, \( P \) is a prime subnexus of \( N \). \( \square \)

Now, recall from Chapter 5 that the set of all addresses of a nexus \( N \) whose stem is \( k \) is denoted by \( N_k \). That is,
\[
N_k = \{ a \in N : \text{stem}(a) = k \}
\]
In particular, \( N_0 = \{{()}\} \).

The subnexus generated by \( N_k \) \((k = 1, 2, \ldots, \text{Ord}(N)) \), is
\[
\langle N_k \rangle = N_k \cup \{{()}, (1), (2), \ldots, (k-1)\}
\]
If \( \text{Ord}(N) = n \), then
\[
N = \bigcup_{k=0}^{n} \langle N_k \rangle = \bigcup_{k=0}^{n} N_k
\]
Note that, \( N_k \cap N_t = \emptyset \) for all \( k \neq t \).

Also, if \( t < k \), then
\[
\langle N_k \rangle \cap \langle N_t \rangle = \{{()}, (1), \ldots, (t)\}
\]
Furthermore, if \( a \in N_t \) and \( b \in N_k \), then \( a \land b \in N_n \), more specifically, \( a \land b = (t) \).

To illustrate the above discussed points, let \( k = 3, t = 2 \) and \( N \) be the nexus
\[
N = \{{()}, (1), (2), (3), (2, 1), (2, 2), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2)\}
\]
The dendrogram of \( N \) is shown in Fig 6.2.2.

Therefore,
\[
N_1 = \{{()}\}
\]
\[
N_2 = \{(1), (2, 1), (2, 2)\}
\]
and
\[
N_3 = \{(2), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2)\}
\]
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The subnexus generated by $N_3$ is

$$<N_3> = \{((),(1), (2), (3), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2))\}$$

This subnexus may also be obtained as

$$<N_3> = N_3 \cup \{((),(1), (2))\}$$

$$= \{(3), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2)\} \cup \{((),(1), (2))\}$$

$$= \{((),(1), (2), (3), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2))\}$$

Also,

$$N_3 \cap N_2 = \{(3), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2)\} \cap \{(2), (2, 1), (2, 2)\}$$

$$= \emptyset$$

and

$$<N_3> \cap <N_2> = \{((),(1), (2), (3), (3, 1), (3, 2), (3, 2, 1), (3, 2, 2))\} \cap \{((),(1), (2), (2, 1), (2, 2))\}$$

$$= \{((),(1), (2))\}$$

Furthermore, suppose that $a = (2, 1) \in N_2$ and $b = (3, 2, 1) \in N_3$, then

$$a \wedge b = (2, 1) \wedge (3, 2, 1)$$

$$= (2) \in N_2$$

**Theorem 6.2.5** Let $N$ be a nexus and let $\text{Ord}(N) = n$. Then, the subnexus

$$P = \bigcup_{i=1}^{k} <N_i> \quad 1 \leq k < n$$

is a prime subnexus of $N$.

**Proof:** Suppose that $a \wedge b \in P$ and $a \notin P$. Now, if $b \notin P$ then $a \in N_t$ and $b \in N_t$ for some $t > k$ and $l < n$. So, $a \wedge b \in N_t \cap N_l$. Therefore, $a \wedge b \notin P$ and this creates a contradiction. Consequently, the assumption of $b \notin P$ cannot be true. ■
Theorem 6.2.6 Let \( N \) be a nexus and let \( m \) be an address of \( N \). Then, \( m \) is a maximal address of \( N \) if and only if the subnexus \( P = <N - \{m\}> \) is a prime subnexus.

Proof: Suppose that, \( m \) is a maximal address of \( N \). In this case,

\[
P = <N - \{m\}> = N - \{m\}
\]

Now, if \( a, b \in N \) and \( a \neq b \), then \( a \in P \) or \( b \in P \). So, \( P \) is a prime subnexus of \( N \). Conversely, suppose that,

\[
P = <N - \{m\}> = N - \{m\}
\]

is a prime subnexus of \( N \). If \( m \) is not a maximal address of \( N \), then there exists an address \( a \) of \( N \) such that \( m < a \). Since, \( P = N - \{m\} \) and \( m \neq a \), so \( a \in P \). Also, \( m < a \) and this implies that \( m \in <a> \). Therefore, \( P = <N - \{m\}> = N \) and this creates a contradiction with the fact that \( P \) is a proper subnexus of \( N \).

Theorem 6.2.7 Let \( \{P_i\}_{i \in I} \) be a family of prime subnexus of a nexus \( N \) and let \( P = \bigcup_{i \in I} P_i \) be a proper subset of \( N \). Then, \( P \) is a prime subnexus of \( N \).

Proof: By Corollary 5.3.3 \( P = \bigcup_{i \in I} P_i \) is a proper subnexus of \( N \). Now, suppose that \( a \land b \in P = \bigcup_{i \in I} P_i \), then, \( a \land b \in P_i \) for some \( i \in I \). Since every \( P_i \) is a prime subnexus, so \( a \in P_i \) or \( b \in P_i \). This means that \( a \in \bigcup_{i \in I} P_i = P \) or \( b \in \bigcup_{i \in I} P_i = P \).

Note that, the intersection of two prime subnexus of a nexus \( N \) can be a prime or a non-prime subnexus of \( N \). The following two examples illustrate this fact.

Consider the nexus

\[
N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2)\}
\]

The dendrogram of \( N \) is shown in Fig 6.2.3. The subnexuses

\[
P_1 = \{(), (1), (2), (1,1), (2), (2,1), (2,2)\}
\]

and

\[
P_2 = \{(), (1), (2), (1,1), (1,2), (2,1)\}
\]

are prime subnexuses (see Theorem 6.2.5) but

\[
P_1 \cap P_2 = \{(), (1), (2), (1,1), (2,1)\}
\]

is not prime, because

\[
(1,2) \land (2,2) = (1) \in P_1 \cap P_2
\]

but neither \((1,2)\) nor \((2,2)\) belong to \(P_1 \cap P_2\).
Therefore, in this case, the intersection of two prime subnexuses of $N$ does not result in a prime subnexus of $N$.

For the second example, consider the nexus $N = \{(\), (1), (2)\}$

Since, (2) is a maximal address of $N$, then

$$P_1 = N - \{(2)\} = \{(\), (1)\}$$

is a prime subnexus of $N$ (see Theorem 6.2.5). Also, by Lemma 6.2.2,

$$P_2 = \{(\)\}$$

is a prime subnexus of $N$.

In this case, the intersection of $P_1$ and $P_2$ is a prime subnexus of $N$. To wit,

$$P = P_1 \cap P_2 = P_2$$

is a prime subnexus of $N$.

**Theorem 6.2.8** Let $N$ be a nexus and let $a, b \in N$. Then

$$\langle a \rangle \cap \langle b \rangle = \langle a \land b \rangle$$

**Proof:**

$$x \in \langle a \rangle \cap \langle b \rangle$$

$$\iff x \in \langle a \rangle \text{ and } x \in \langle b \rangle$$

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\[ \forall x \leq a \text{ and } x \leq b \]
\[ \Leftrightarrow x \leq a \land b \]
\[ \Leftrightarrow x \in (a \land b) \]

**Definition 6.2.9** Let \( S \) be a subset of a nexus \( N \) and let \( a \) be an address in \( N \). Then,
\[
a \land S = \{ a \land s : s \in S \}\]

**Lemma 6.2.10** Let \( K \) be a subnexus of a nexus \( N \) and let \( a \) be an address in \( N \). Then,
\[
a \land K = <a> \cap K
\]
Therefore, \( a \land K \) is a subnexus of \( N \) and \( a \land K \subseteq <a> \).

**Proof:** Let \( x \in a \land K \). Therefore, \( x = a \land k \) for some \( k \in K \). By the definition of the meet operation \( x \leq a, k \). Since, \( K \) and \( <a> \) are the subnexuses of \( N \), so, \( x \in <a> \) and \( x \in K \). Thus \( x \in <a> \land K \), and this, in turn, implies that \( a \land K \subseteq <a> \land K \).

Now, suppose that \( x \in <a> \land K \). So, \( x < a \) and \( x \in K \). Therefore, \( x \leq a \) and \( x = k \) for some \( k \in K \). Also, since \( x \leq a \), so,
\[
x = a \land x = a \land k \in a \land K
\]
This implies that \( <a> \land K \subseteq a \land K \). Thus, \( a \land K = <a> \land K \). Since, \( <a> \) and \( K \) are subnexuses of \( N \), so, \( <a> \land K \) is a subnexus of \( N \). Therefore, \( a \land K \) is a subnexus of \( N \), and \( a \land K = <a> \land K \subseteq <a> \).

**Theorem 6.2.11** Let \( T \) and \( K \) be two subnexuses of a nexus \( N \). Then,

(i) \[ T \cap K = \bigcup_{t \in T, k \in K} <t \land k> \]

(ii) \[ T \cap K = T \land K = \{ t \land k : t \in T, k \in K \} \]

**Proof:**

(i) \[ X \in T \cap K \]
\[ \Rightarrow x \in T \text{ and } x \in K \]
\[ \Rightarrow x \leq t \text{ and } x \leq k \]
\[ \Rightarrow x \leq t \land k \]
\[ \Rightarrow x \in <t \land k> \]
\[ \Rightarrow x \in \bigcup_{t \in T, k \in K} <t \land k> \]

(ii) Since every element of \( T \land K \) is of the form \( t \land k \), so, \( T \land K \subseteq T \land K \). Now, suppose that \( x \in T \cap K \) so \( x < t \land k > \). This implies that \( x \leq t \land k \), so, \( x \leq t \) and \( x \leq k \). Also, since \( T \) and \( K \) are
subnexuses of \( N \), so, \( x \in T \) and \( x \in K \). Therefore, \( x = t_1 \) and \( x = k_1 \) for some \( t_1 \in T \) and \( k_1 \in K \). Thus

\[
x = t_1 \land k_1 \in \{ t \land k : t \in T, k \in K \} = T \land K
\]

and this implies that \( T \cap K \subseteq T \land K \). Therefore, \( T \cap K = T \land K \). \( \blacksquare \)

**Corollary 6.2.12** Let \( K \) be a subnexus of a nexus \( N \) and let \( a \) and \( b \) be two addresses of \( N \). Then \( a \land b \in K \) implies that

\[
K = \langle K \cup \{a\} \rangle \cap \langle K \cup \{b\} \rangle
\]

**Proof:** It is easy to see that

\[
K \subseteq \langle K \cup \{a\} \rangle \cap \langle K \cup \{b\} \rangle
\]

By Theorem 6.2.9

\[
\langle K \cup \{a\} \rangle \cap \langle K \cup \{b\} \rangle = \bigcup_{x \in K \cup \{a\}, y \in K \cup \{b\}} \langle x \land y \rangle
\]

Since \( k \land b, k \land a \leq k \) and since, \( a \land b \leq k \) for some \( k \in K \), so

\[
\langle x \land y \rangle \subseteq K \quad \forall x \in \langle K \cup \{a\} \rangle \quad \forall y \in \langle K \cup \{b\} \rangle
\]

Therefore,

\[
\bigcup_{x \in K \cup \{a\}, y \in K \cup \{b\}} \langle x \land y \rangle \subseteq K.
\]

Thus

\[
K = \langle K \cup \{a\} \rangle \cap \langle K \cup \{b\} \rangle. \quad \blacksquare
\]

**Theorem 6.2.13** Let \( N \) be a nexus and let \( P \) be a subnexus of \( N \). Then, the following statements are equivalent:

(i) \( P \) is a prime subnexus of \( N \),

(ii) \( K_1 \cap K_2 \subseteq P \) implies \( K_1 \subseteq P \) or \( K_2 \subseteq P \) for any subnexuses \( K_1 \) and \( K_2 \) of \( N \),

(iii) \( \langle a \rangle \cap \langle b \rangle \subseteq P \) implies \( a \in P \) or \( b \in P \) for any \( a, b \in N \),

(iv) \( K_1 \cap K_2 = P \) implies \( K_1 = P \) or \( K_2 = P \) for any subnexuses \( K_1 \) and \( K_2 \) of \( N \).

**Proof:** (i) \( \Rightarrow \) (ii) Suppose that (ii) does not hold, then there exist some subnexuses \( K_1 \) and \( K_2 \) such that \( K_1 \cap K_2 \subseteq P \) but \( K_1 \nsubseteq P \) and \( K_2 \nsubseteq P \). So there exist addresses \( a \) and \( b \) such that \( a \in K_1 - P \) and \( b \in K_2 - P \). Since \( a \land b \leq a \) and \( a \land b \leq b \) then \( a \land b \in K_1 \) and \( a \land b \in K_2 \) and therefore \( a \land b \in K_1 \cap K_2 \subseteq P \). However, this contradicts the primeness of \( P \).

(ii) \( \Rightarrow \) (iii) Since, \( \langle a \rangle \) and \( \langle b \rangle \) are subnexuses of \( N \) and since (ii) is true for any subnexuses \( K_1 \) and \( K_2 \) of \( N \) and also, \( a \in \langle a \rangle \) and \( b \in \langle b \rangle \), therefore, (iii) must be true.
(iii) ⇒ (iv) If (iv) does not hold, then there exist subnexuses $K_1$ and $K_2$ such that $P = K_1 \cap K_2$ but $P \neq K_1$ and $P \neq K_2$. Hence, there exist addresses $a$ and $b$ such that $a \in K_1 - P$ and $b \in K_2 - P$. This means that, $a, b \notin P$. However, by (iii), since $a, b \notin P$, so, $\langle a \rangle \cap \langle b \rangle \subsetneq P$.

On the other hand, $a \land b \in K_1$ and $a \land b \in K_2$. Then, $a \land b \in K_1 \cap K_2$. This implies that, $\langle a \land b \rangle \subseteq K_1 \cap K_2$. By Theorem 6.2.7,

$$\langle a \rangle \cap \langle b \rangle = \langle a \land b \rangle \subseteq K_1 \cap K_2 = P$$

This creates a contradiction with $\langle a \rangle \cap \langle b \rangle \subsetneq P$.

(iv) ⇒ (i) Suppose that $a \land b \in P$ for some $a, b \in N$. By Corollary 6.2.12,

$$P = \langle P \cup \{a\} \rangle \cap \langle P \cup \{b\} \rangle$$

It follows that $P = \langle P \cup \{a\} \rangle$ or $P = \langle P \cup \{b\} \rangle$ and therefore $a \in P$ or $b \in P$. Hence, $P$ is a prime subnexus of $N$.

**Theorem 6.2.14** Let $N$ and $M$ be two nexuses and let $f: N \to M$ be a homomorphism between $N$ and $M$. Then,

(i) If $P$ is a prime subnexus of $M$ and $f^{-1}(P) \neq N$ then $f^{-1}(P)$ is a prime subnexus of $N$. This, in turn, implies that the kernel of $f$ is a prime subnexus of $N$.

(ii) If $f$ is an epimorphism and $P$ is a prime subnexus of $N$ and if also $f(P) \neq M$, then $f(P)$ is a prime subnexus of $M$.

**Proof:** (i) Let $a, b \in N$ and let $a \land b \in f^{-1}(P)$. Also, let $a \notin f^{-1}(P)$. Therefore, since $f$ is a homomorphism

$$f(a \land b) = f(a) \land f(b) \in P$$

and

$$f(a) \notin P$$

Since $P$ is prime, so $f(b) \in P$. This implies that $b \in f^{-1}(P)$. Furthermore, since $\{(), ()\}$ is a prime subnexus of $M$, the kernel of $f$ is a prime subnexus of $N$.

(ii) Let $c, d \in M$ and let $c \land d \in f(P)$ and $c \notin f(P)$. Since $f$ is an epimorphism, so there exist $a, b \in N$ such that $f(a) = c$ and $f(b) = d$. Now,

$$c \land d = f(a) \land f(b) = f(a \land b) \in f(P)$$

and

$$c = f(a) \notin f(P)$$

Therefore, $a \land b \in P$ and $a \notin P$, however, since $P$ is prime then $b \in P$. This implies that $f(b) = d \in f(P)$.

**Theorem 6.2.15** Let $N$ be a nexus and let $P$ be a subnexus of $N$. Suppose that
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Let $f: N \rightarrow \{((), (1))$ be a function such that $f(P) = ()$ and $f(N - P) = (1)$ Then, is a homomorphism if and only if $P$ is a prime sublattice of $N$.

Proof: By Theorem 6.2.14, if $f$ is a homomorphism then, $P = \ker f$ is prime. Now, suppose that $P$ is a prime and $a, b \in N$. One may consider two cases:

Case 1: $a \in P$. Since $P$ is a sublattice of $N$ and $a \wedge b \leq a$, so, $a \wedge b \in P$. Therefore,

$$f(a \wedge b) = () = () \wedge f(b) = f(a) \wedge f(b)$$

Case 2: $a, b \not\in P$. Since $P$ is a prime sublattice of $N$, so, $a \wedge b \not\in P$. Therefore,

$$f(a \wedge b) = (1) = (1) \wedge (1) = f(a) \wedge f(b)$$

\[\text{Theorem 6.2.16: Let } N \text{ be a nexus and let } P \text{ be a sublattice of } N. \text{ Then, } P \text{ is a prime sublattice of } N \text{ if and only if } N - P \text{ is closed under the meet operation.}\]

Proof: Let $P$ be a prime sublattice of $N$ and let $a, b \in N - P$. Then $a \wedge b \in N - P$ because of the primeness of $P$. Now, let $a \wedge b \in P$, and let $a \not\in P$. Since $N - P$ is closed under meet operation, so, $b \in P$. Therefore, $P$ is a prime sublattice of $N$.

\[\text{Theorem 6.2.17: Let } N \text{ be a nexus and let } \{(()) \} \neq S \text{ be a nonempty subset of } N. \text{ Then } N - Q_S \text{ and } N - q_S \text{ are sublattices of } N. \text{ In particular, } N - Q_a \text{ and } N - q_a \text{ are sublattices of } N \text{ for some } (()) \neq a \in N. \text{ Here, the terms } q_S, Q_S, q_a \text{ and } Q_a \text{ refer to panels and quasipanels relating to } S \text{ and } a, \text{ see Definitions 5.2.13, 5.2.14 and 5.2.15.}\]

Proof: Let $a \in N - Q_S$, $b \in N$ and let $b \leq a$. If $b \in Q_S$ then there exists $s \in S$ such that $s \leq b$ and since $b \leq a$, then, $s \leq a$. However, by definition, $Q_S$ contains all the addresses that are larger than one of the address $s$ in $S$. Then, $s \leq a$ implies that $a \in Q_S$. But this creates a contradiction with the hypothesis $a \in N - Q_S$ (imply that $a \not\in Q_S$). Therefore, $b \in N - Q_S$ and $N - Q_S$ is a sublattice of $N$. The proof for $N - q_S$ is similar.

\[\text{Theorem 6.2.18 Let } N \text{ be a nexus and let } a \text{ be a nonempty address of } N. \text{ Then, } N - Q_a \text{ and } N - q_a \text{ are prime sublattices of } N.\]

Proof: By Theorem 5.3.2, $N - Q_a$ and $N - q_a$ are sublattices of $N$. Now, suppose that

$x, y \in N$, $x \wedge y \in N - Q_a$ and $x \not\in N - Q_a$.

If $y \not\in N - Q_a$, since $Q_a$ is closed under the meet operation so, $x \wedge y \in Q_a$. However, this creates a contradiction with the hypothesis. Therefore, $y \in N - Q_a$ and $N - Q_a$ is a prime sublattice of $N$. The proof of the primeness of $N - q_a$ is similar.
Lemma 6.2.19 Let $S$ be a subset of a nexus $N$ and let $S$ be closed under the meet operation. Then $Q_S$ is closed under the meet operation.

**Proof:** Let $x, y \in Q_S$. By definition of $Q_S$, there exist $a, b \in S$ such that $a \leq x$ and $b \leq y$. Since, $S$ is closed under the meet operation, so, $a \land b \in S$. Also, $a \land b \leq a \leq x$ and $a \land b \leq b \leq y$. Therefore, $a \land b \leq x \land y$ and this implies that $x \land y \in Q_S$. ■

The following example shows that the converse of the above lemma is not true. That is, if $Q_S$ is closed under the meet operation, it does not necessarily imply that $S$ is also closed under the meet operation. Consider the nexus

$$N = \{(), (1), (2), (3), (2, 1), (2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

The dendrogram of $N$ is shown in Fig 6.2.4.

![Dendrogram of N](image)

Fig 6.2.4 Dendrogram of $N$. The addresses in $S$ are shown by thick lines.

The subset

$$S = \{(2), (2, 1, 2), (2, 2, 2)\}$$

is not closed under the meet operation. To wit,

$$(2, 1, 2) \land (2, 2, 2) = (2, 1) \notin S$$

but

$$Q_S = Q(2) \cup Q(2, 1, 2) \cup Q(2, 2, 2)$$

$$= \{(2), (3), (2, 1), (2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

$$\cup \{(2, 1, 2) \cup (2, 2, 2)\}$$

$$= \{(2), (3), (2, 1), (2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

is closed under the meet operation.

Theorem 6.2.20 Let $S$ be a nonempty subset of a nexus $N$, and let $\{()\} \neq S$ be closed under the meet operation. Then $N - Q_S$ is a prime subnexus of $N$.

**Proof:** Suppose that $S$ is closed under the meet operation. Then, by Lemma 6.2.19, $Q_S$ is closed under the meet operation. Now, the proof is complete using Theorem 6.2.16. ■
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Note that, if $S$ is a subset of $N$ and also closed under the meet operation, then $qs$ is not necessarily closed under the meet operation and therefore, $N-qs$ is not prime. For example, consider the nexus

$$N = \{(), (1), (2), (2, 1), (2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

The dendrogram of $N$ is shown in Fig 6.2.5.

Fig 6.2.5 Dendrogram of $N$. The addresses in $S$ are shown by thick lines and the addresses in $qs$ are shown by dotted lines.

Suppose that

$$S = \{(2, 1), (2, 2)\}$$

Since $\langle 2, 1 \rangle \wedge \langle 2, 2 \rangle = \langle 2, 1 \rangle$, $S$ is closed under the meet operation. Now, consider

$$qs = q\langle 2, 1 \rangle \cup q\langle 2, 2 \rangle$$

$$= \{(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

Here, $qs$ is not closed under the meet operation, because

$$\langle 2, 1, 1 \rangle \wedge \langle 2, 2, 2 \rangle = \langle 2, 1 \rangle \notin qs$$

**Theorem 6.2.21** Let $N$ be a nexus. Then, every prime subnexus of $N$ is of the form $N-Q_a$ for some $(\not= a \in N$.

**Proof:** By Theorem 6.2.18, $N - Q_a$ for any $(\not= a \in N$ is a prime subnexus of $N$. Now, suppose that $P$ is a prime subnexus of $N$. One must show that $P = N - Q_a$ for some $a \in N$. Alternatively, one may show that $N - P = Q_a$ for some $a \in N$. By definition of a prime subnexus, $N \not= P$. So there must exist an address $b$ of $N$ such that $b \notin P$. Now, consider the subnexus $\langle b \rangle$ of $N$, where, $P \cap \langle b \rangle \not= \emptyset$ because both $P$ and $\langle b \rangle$ have at least one common address which is the empty address $(\not=)$. Suppose that $a_0$ is the smallest address of $\langle b \rangle$ such that $a_0 \notin P$. Now, consider $Q_{a_0} = \{d \in N : a_0 \leq d\}$. Suppose that $d$ is an address in $Q_{a_0}$ and also, suppose that $d$ is in $P$. Since
P is a subnexus of \( N \) and \( d \) is in \( P \) and since \( a_0 \leq d \), then, \( a_0 \in P \). However, this creates a contradiction with the hypotheses \( a_0 \notin P \). Therefore,

\[
Q_{a_0} = \{d \in N : a_0 \leq d\} \subseteq N - P. \quad (*)
\]

Now, one must show that \( N - P \subseteq Q_{a_0} \). To do this, suppose that \( N - P \notin Q_{a_0} \), then, there exists an address \( k \) such that \( k \in (N - P) - Q_{a_0} \). This means that \( k \notin Q_{a_0} \) and \( k \notin P \). Now, consider the cyclic subnexus \( <k> \) of \( N \) and suppose that \( c_0 \) is the smallest address in \( <k> \), such that, \( c_0 \notin Q_{a_0} \). This means that, \( c_0 \leq k \) and also, \( c_0 \) is not greater than or equal to \( a_0 \). Furthermore, \( c_0 \notin P \). If \( c_0 \) is in \( P \) then every address \( t \) which is less than \( c_0 \) will be in \( P \). Also, \( c_0 \) is in \( <k> \). So, every address \( t \) which is less than \( c_0 \) is in \( <k> \). Consequently, \( t \leq c_0 \leq k \) and \( t \notin Q_{a_0} \), otherwise, if \( t \in Q_{a_0} \), then \( a_0 \leq t \in P \), and so \( a_0 \in P \). This creates a contradiction with the fact that, \( a_0 \notin P \). So, \( t \) is in \( <k> \), \( t \leq c_0 \) and \( t \notin Q_{a_0} \). This creates a contradiction with the fact that \( c_0 \) is the smallest address in \( <k> \) such that \( c_0 \notin Q_{a_0} \). Therefore, \( c_0 \notin P \). By definition of the meet operation, \( a_0 \wedge c_0 \) must be less than or equal to \( a_0 \) and \( c_0 \). But \( a_0 \wedge c_0 \) is not equal to \( a_0 \) or \( c_0 \), because, if \( a_0 \wedge c_0 = a_0 \), then \( a_0 \leq c_0 \) and also, \( c_0 \leq k \). Therefore, \( a_0 \leq k \) and this means that \( k \in Q_{a_0} \). But, it is known that \( k \notin Q_{a_0} \). So, \( a_0 \wedge c_0 \) is not equal to \( a_0 \). Now, if \( a_0 \wedge c_0 \) is equal to \( c_0 \), then \( c_0 \leq a_0 \) and \( a_0 \notin P \). However, this creates a contradiction with the above mentioned property of \( c_0 \). Therefore, \( a_0 \wedge c_0 \) is not equal to \( a_0 \) or \( c_0 \) and is less than \( a_0 \) and \( c_0 \). This means that \( a_0 \wedge c_0 \in P \). Since \( a_0 \), \( c_0 \notin P \), so this creates a contradiction with the primeness of \( P \). Thus,

\[
N - P \subseteq Q_{a_0} \quad (**)
\]

To conclude, the Eqns (*) and (**) imply that \( N - P = Q_{a_0} \). ■

**Corollary 6.2.22** Suppose that \( N \) is a nexus. Then, the number of prime subnexuses of \( N \) is equal to the number of addresses in \( N \) minus 1, that is, \( |N| - 1 \).

**Proof:** By the above theorem, there is a one to one correspondence between the nonempty addresses in \( N \) and the prime subnexuses of \( N \). ■

**Lemma 6.2.23** Let \( N \) be a nexus and let \( K \) be a subnexus of \( N \). Then \( K \cap Q_a = \emptyset \) for all \( a \in N - K \). Also, \( K \cap \bigcup_{a \in N - K} Q_a = \emptyset \) and \( K = N - \bigcup_{a \in N - K} Q_a \).

**Proof:** Suppose that \( K \cap Q_a \neq \emptyset \) for some \( a \in N - K \). Then, there must exist one or more addresses \( b \in K \cap Q_a \). So, \( b \in K \) and \( b \in Q_a \). Since \( b \in Q_a \), so, \( a \leq b \). But, \( K \) is a subnexus of \( N \) and so \( b \in K \). Therefore, \( a \in K \). However, this creates a contradiction with the hypothesis \( a \in N - K \). Thus, \( K \cap Q_a = \emptyset \) and since, \( K \cap Q_a = \emptyset \) for all \( a \in N - K \), so
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\[
K \cap \left( \bigcup_{a \in N-K} Q_a \right) = \bigcup_{a \in N-K} (K \cap Q_a) = \bigcup_{a \in N-K} (\emptyset) = \emptyset.
\]

Also, \( K \cup \left( \bigcup_{a \in N-K} Q_a \right) = N \), therefore,

\[
K = N - \bigcup_{a \in N-K} Q_a. \tag*{\blacksquare}
\]

**Theorem 6.2.24** Let \( N \) be a nexus and let \( K \) be a subnexus of \( N \). Then \( K = \bigcap_{a \in N-K} (N - Q_a) \). In other words, every subnexus of a nexus \( N \) is an intersection of some prime subnexuses of \( N \).

**Proof:** By Lemma 6.2.23, \( K = N - \left( \bigcup_{a \in N-K} Q_a \right) \). Now, one can prove that

\[
\bigcap_{a \in N-K} (N - Q_a) = N - \left( \bigcup_{a \in N-K} Q_a \right)
\]

To do this, suppose that

\[
b \in \bigcap_{a \in N-K} (N - Q_a).
\]

\[
\iff b \in (N - Q_a) \quad \forall a \in N - K
\]

\[
\iff b \notin Q_a \quad \forall a \in N - K
\]

\[
\iff b \notin \bigcup_{a \in N-K} Q_a
\]

\[
\iff b \in N - \left( \bigcup_{a \in N-K} Q_a \right)
\]

So, \( K = \bigcap_{a \in N-K} (N - Q_a) \). By Theorem 6.2.18, \((N - Q_a)\) is a prime subnexus of \( N \) for any \((\not=) a \in N\). Therefore, \( K \) is equal to the intersection of prime subnexuses of \( N \). \( \tag*{\blacksquare} \)

**Lemma 6.2.25** Suppose that \( a \) and \( b \) are addresses in a nexus \( N \). Then,

(i) \( a \) and \( b \) are not comparable \( \iff Q_a \cap Q_b = \emptyset \)

(ii) \( a \) and \( b \) are comparable and \( a \leq b \) \( \iff Q_b \subseteq Q_a \)

(iii) \( a \) and \( b \) are comparable and \( a \leq b \Rightarrow q_b \subseteq Q_a \)

(iv) \( a \) and \( b \) are not comparable \( \Rightarrow a \land b = x \land y \quad \forall x \in Q_a, \forall y \in Q_b \)

**Proof:** (i) Suppose that \( a \) and \( b \) are not comparable and \( c \in Q_a \cap Q_b \). Then \( c \in Q_a \) and \( c \in Q_b \), so, \( a \leq c \) and \( b \leq c \). Therefore, \( a, b \in \langle c \rangle \). This means that, \( a \) and \( b \) are comparable. However, this contradicts the hypothesis, that is, \( a \) and \( b \) are comparable. Conversely, let \( Q_a \cap Q_b = \emptyset \). If \( a \) and
b are comparable, without loss of generality, suppose that \( a \leq b \), so \( b \in Q_a \). This implies that 
\( Q_a \cap Q_b \neq \emptyset \). This creates a contradiction with this assumption that \( Q_a \cap Q_b = \emptyset \).

(ii) Suppose that \( a \leq b \) and \( c \in Q_b \), therefore, \( a \leq b \leq c \). This implies that \( c \in Q_a \). Thus, \( Q_b \subseteq Q_a \).
Conversely, suppose that \( Q_b \subseteq Q_a \). Since, \( b \in Q_b \), so, \( b \in Q_a \) and this implies that \( a \leq b \). This, in turn, implies that \( a \) and \( b \) are comparable.

(iii) The proof is similar to that for (ii).

(iv) Let \( x \) and \( y \) be two addresses of \( Q_a \) and \( Q_b \), respectively. Now, consider two cyclic subnexuses \( \langle x \rangle \) and \( \langle y \rangle \) of \( N \). Since \( x \in Q_a \) and \( y \in Q_b \), then, \( a \leq x \) and \( b \leq y \). Therefore, \( a \in \langle x \rangle \) and \( b \in \langle y \rangle \). Since,
\[
 a \wedge b \leq a \in \langle x \rangle \quad \text{and} \quad a \wedge b \leq b \in \langle y \rangle,
\]
so,
\[
 a \wedge b \in \langle x \rangle \quad \text{and} \quad a \wedge b \in \langle y \rangle.
\]
Consequently, by Theorem 6.2.8
\[
 a \wedge b \in \langle x \rangle \cap \langle y \rangle = \langle x \wedge y \rangle.
\]
Therefore,
\[
 a \wedge b \leq x \wedge y \tag{*}
\]
Now, one can show that \( x \wedge y \leq a \). To do this, if \( x \wedge y > a \), then \( a < x \wedge y \leq y \) and this implies that \( y \in Q_a \). Since it is assumed above that \( x \) and \( y \) are in \( Q_a \) and \( Q_b \), then \( y \in Q_a \cap Q_b \). but, by (i) \( Q_a \cap Q_b = \emptyset \). Therefore, \( x \wedge y \leq a \). Similarly, \( x \wedge y \leq b \). Consequently,
\[
 x \wedge y \leq a \wedge b \tag{**}
\]
and by (*) and (**) \( a \wedge b = x \wedge y \).■

It should be noted that, if \( a \) and \( b \) are two non-comparable addresses, then it can be proved, in a manner similar to that Lemma 6.2.25 part (i), that \( q_a \cap q_b = \emptyset \). But, the next example shows that, generally, the converse of this statement is not true. Namely, \( q_a \cap q_b = \emptyset \) does not necessarily imply that \( a \) and \( b \) are non-comparable.

Consider, the nexus
\[
 N = \{(1), (2), (1, 1), (1, 2), (2, 1), (2, 2)\}
\]
The dendrogram of \( N \) is shown in Fig 6.2 6. Now, suppose that \( a = (1) \) and \( b = (2) \), then 
\[
 q_a = \{(1, 1), (1, 2)\} \quad \text{and} \quad q_b = \{(2, 1), (2, 2)\}.
\]
Therefore, \( q_a \cap q_b = \emptyset \) but \( (1) < (2) \), so \( a \) and \( b \) are comparable. The next lemma formalises this example.
Lemma 6.2.26 Suppose that, \( a \) and \( b \) are two addresses of a nexus \( N \). Then,

(i) If \( a \) and \( b \) are not comparable or comparable with the same level, then \( a \cap b = \emptyset \).

(ii) If \( a \) and \( b \) are not comparable or comparable with the same level, then \( a \wedge b = x \wedge y \) for all \( x \in q_a \) and \( y \in q_b \). More precisely, if \( a < b \) then \( x \wedge y = a \) for all \( x \in q_a \) and \( y \in q_b \).

Proof: (i) If \( a \) or \( b \) is a primitive address, then obviously, \( a \cap b = \emptyset \). If \( a \) and \( b \) are two non-primitive and non-comparable addresses, then the proof is the same as that in lemma 6.2.25 part (i). Now, suppose that \( a \) and \( b \) are comparable addresses with the same level. Without loss of generality, assume that, \( a < b \). If \( a \cap b \neq \emptyset \), then there exists an address \( d \) of \( N \) such that \( d \in q_a \cap q_b \). By definition of \( q_a \), the level of \( d \) must be greater than both the levels \( a \) and \( b \). Since \( a \) and \( b \) have the same level and \( a < b \), so, if \( a = (a_1, a_2, \ldots, a_n) \) then \( b = (a_1, a_2, \ldots, a_n+t) \), where \( t \) is a natural number. Since, \( a < d \) and the level of \( a \) less is than the level of \( d \) so,

\[
d = (a_1, a_2, \ldots, a_n, k_1, k_2, \ldots, k_m)
\]

Also, \( b < d \) and the level of \( b \) is less than the level of \( d \), thus

\[
d = (a_1, a_2, \ldots, a_n+t, k_1, k_2, \ldots, k_m)
\]

Therefore, \( a_n = a_n+t \) so, \( t \) must be equal to zero. This means that \( a = b \). However, this contradicts the hypothesis \( a < b \). Thus \( q_a \cap q_b = \emptyset \).

(ii) If \( a \) and \( b \) are not comparable, then the proof is the same as that in Lemma 6.2.25 part (ii). □

It should be noted that, if \( a < b \) and \( a = (a_1, a_2, \ldots, a_n) \), then, \( b \) should be equal to

\[
b_1 = (a_1, a_2, \ldots, a_n, k_1, k_2, \ldots, k_m) \quad \text{or} \quad b_2 = (a_1, a_2, \ldots, a_n+t, k_1, k_2, \ldots, k_m)
\]

Now, if

\[
b_1 = (a_1, a_2, \ldots, a_n, k_1, k_2, \ldots, k_m)
\]
then $b_1 \in q_a$. Therefore, $q_{b_1} \subseteq q_a$. However, if

$$b_2 = (a_1, a_2, ..., a_{n+m}, k_1, k_2, ..., k_m)$$

then, by the same argument as in Lemma 6.2.26 (i), one can show that $q_a \cap q_{b_2} = \emptyset$. Therefore, if $a$ and $b$ are two comparable addresses with different levels, then $q_a \cap q_b$ can be empty or nonempty.

To illustrate the point discussed, consider the example:

Suppose that

$N = \{(1), (1), (2), (1, 1), (1, 2), (1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 1), (1, 3, 2), (1, 2, 3, 1), (1, 2, 3, 2), (1, 3, 2, 1), (1, 3, 2, 2)\}$

The dendrogram of $N$ is shown in Fig 6.2.7.

![Dendrogram of N](image)

**Fig 6.2.7** Dendrogram of $N$. The addresses $b_1$ and $b_2$ are greater than address $a$

Now, consider the addresses $a = (1, 2), b_1 = (1, 2, 3)$ and $b_2 = (1, 3, 2)$ of $N$. $b_1$ and $b_2$ are greater than $a$, because,

$$a = (1, 2) < (1, 2, 3) = b_1$$ and $$a = (1, 2) < (1, 2+1, 2) = (1, 3, 1) = b_2$$

Now,

$$q_a = \{(1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 2, 3, 1), (1, 2, 3, 2)\}$$

and

$$q_{b_1} = \{(1, 2, 3, 1), (1, 2, 3, 2)\}$$

So, $b_1 \in q_a$ and $q_{b_1} \subseteq q_a$. But, if one considers $b_2 = (1, 3, 2)$, then,

$$q_{b_2} = \{(1, 3, 2, 1), (1, 3, 2, 2)\}$$

So, $b_2 \notin q_a$ and $q_a \cap q_{b_2} = \emptyset$. 

Masoud Bolourian
Theorem 6.2.27 Let $a$ and $b$ be two addresses of a nexus $N$ and let $P_1 = N - Q_a$ and $P_2 = N - Q_b$ be two prime subnexuses of $N$. Then,

(i) If $a$ and $b$ are two comparable addresses, then $P_1 \cap P_2$ is a prime subnexus of $N$.

(ii) If $a$ and $b$ are not comparable addresses, then $P_1 \cap P_2$ is not a prime subnexus of $N$ and $P_1 \cup P_2 = N$.

Proof: (i) Without loss of generality, suppose that $a \leq b$. So, $Q_b \subseteq Q_a$ and $P_1 \subseteq P_2$. Therefore, $P_1 \cap P_2 = P_1$.

(ii) Suppose that $x \in Q_a$ and $y \in Q_b$, so, $x \not\in P_1$ and $y \not\in P_2$. Therefore, $x, y \not\in P_1 \cap P_2$. Since, $a$ and $b$ are not comparable, so, $a \land b < a$ and $a \land b < b$. Consequently,

$$a \land b \not\in Q_a = N - P_1 \quad \text{and} \quad a \land b \not\in Q_b = N - P_2.$$ 

Therefore, $a \land b \in P_1$ and $a \land b \in P_2$. This means that $a \land b \in P_1 \cap P_2$. By Lemma 6.2.26, $a \land b = x \land y$. Thus, $x \land y \in P_1 \cap P_2$ and $x, y \not\in P_1 \cap P_2$. Consequently, $P_1 \cap P_2$ is not a prime subnexus of $N$.

It is clear that $P_1 \cup P_2 \subseteq N$. Now, suppose that $c \in N$ and $c \not\in P_1$, so $c \in Q_a = N - P_1$. Therefore, $a \leq c$. If $c \in Q_b$ then $b \leq c$, therefore, $a, b \in <c>$ and this implies that $a$ and $b$ are comparable. However, this contradicts the hypothesis. Therefore, $c \not\in Q_b$ and $c \in P_2 = N - Q_b$. Thus, $c \in P_1 \cup P_2$ and this implies that $N \subseteq P_1 \cup P_2$. Consequently, $N = P_1 \cup P_2$. 

Theorem 6.2.28 Let $a$ and $b$ be two addresses of a nexus $N$ and let $P_1 = N - Q_a$ and $P_2 = N - Q_b$ be two prime subnexuses of $N$. Then, $P = N - Q_{a \land b}$ is the largest prime subnexus of $N$ which is a subset of $P_1 \cap P_2$.

Proof: There are two cases to be considered:

Case one: $a$ and $b$ are two comparable addresses. Without loss of generality, suppose that $a \leq b$. So, $a \land b = a$ and $P = N - Q_{a \land b} = N - Q_a = P_1$. Also, $P_1 \cap P_2 = P_1$ (see proof of Theorem 6.2.27 part (i)). Therefore, $P = P_1 \subseteq P_1 = P_1 \cap P_2$.

Case two: $a$ and $b$ are not comparable. Clearly, $P$ is a prime subnexus of $N$. Now, one must show that $P \subseteq P_1 \cap P_2$. To do this, suppose that $x$ is an address of $P$. So, $x \not\in Q_{a \land b}$. This implies that $x < a \land b \leq a$ and $x < a \land b \leq b$. Thus, $x \not\in Q_a$ and $x \not\in Q_b$. Therefore, $x \in P_1$ and $x \in P_2$, that is, $x \in P_1 \cap P_2$. Consequently, $P \subseteq P_1 \cap P_2$. Now, one must show that $P$ is the largest prime subnexus of $N$, such that $P \subseteq P_1 \cap P_2$. To do this, suppose that $P^*$ is a prime subnexus of $N$ such that $P^* \subseteq P_1 \cap P_2$. By Theorem 6.2.21, $P^* = N - Q_d$, where $d$ is an address of $N$. Since $P^* \subseteq P_1 \cap P_2$, so, $P^* \subseteq P_1 = N - Q_a$ and $P^* \subseteq P_2 = N - Q_b$. Therefore, $Q_a \subseteq Q_d$ and $Q_b \subseteq Q_d$. This implies that $d \leq a$ and $d \leq b$. Thus, $d \leq a \land b$. Consequently, $a \land b \in Q_d$ and $P^* \subseteq P$. Since every
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P* is a subset of P then P must be the largest prime subnexus of N which is a subset of

\[ P_1 \cap P_2. \]

The above theorem can be extended to a finite set of prime subnexuses of N as follows:

**Theorem 6.2.29** Suppose that \( P_1, P_2, \ldots, P_n \) are prime subnexuses of a nexus \( N \) such that

\[ P_i = N - Q_{a_i} \text{ for } i = 1, 2, \ldots, n. \]

Also, suppose that \( a = \bigwedge a_i \text{ for } i = 1, 2, \ldots, n. \) Then, \( P = N - Q_a \) is the largest prime subnexus of \( N \) which is a subset of \( \cap P_i \text{ for } i = 1, 2, \ldots, n. \)

**Proof:** proof follows the same manner as in the above theorem.

**Theorem 6.2.30** Let \( S \) be a subset of a nexus \( N \) and let \( a = \bigwedge s_i \text{ for all } s_i \in S. \) Then, \( Q_a \) is the smallest quasipanel of \( N \) that contains \( S. \)

**Proof:** By definition of \( a, \) it may be seen that \( S \subseteq Q_a. \) Now, suppose that \( b \in N \) and \( S \subseteq Q_b. \) Therefore, \( b \leq s_i \text{ for all } s_i \in S. \) Thus, \( b \) is a lower bound of \( S. \) But, \( a \) is the greatest lower bound of \( S, \) so \( b \leq a. \) This implies that \( Q_a \subseteq Q_b. \)

**Corollary 6.2.31** Let \( S \) be a subset of a nexus \( N \) and let \( a = \bigwedge s_i \text{ for all } s_i \in S. \) Then \( P = N - Q_a \) is the largest prime subnexus of \( N \) disjoint of \( S \) (that is \( P \cap S = \emptyset). \)

**Proof:** By the above theorem, \( Q_a \) is the smallest quasipanel of \( N \) that contains \( S. \) Thus, \( P \cap Q_a = \emptyset. \) Therefore, \( P = N - Q_a \) is the largest prime subnexus of \( N \) disjoint of \( S. \)

**Theorem 6.2.32** Let \( S \) be a subset of a nexus \( N \) and let \( S \) be closed under the meet operation. Now, if \( a = \bigwedge s_i \text{ for all } s_i \in S \) then \( Q_S = Q_a \) and conversely. That is, if \( Q_S = Q_a \) then \( a \) is the greatest lower bound of \( S. \)

**Proof:** Since \( S \) is close under the meet operation, \( a \in S. \) Now, suppose that

\[ x \in Q_S = \bigcup_{s_i \in S} Q_{s_i}. \]

Therefore, \( x \in Q_{s_i} \text{ for some } s_i \in S. \) This implies that \( s_i \leq x. \) Since \( a = \bigwedge s_i, \) so,

\[ a \leq s_i \leq x. \]

Therefore, \( x \in Q_a \) and

\[ Q_S \subseteq Q_a \quad (*) \]

Now, suppose that \( y \in Q_a \text{ thus } a \leq y. \) Since \( a \in S, \) so \( y \in \bigcup_{s_i \in S} Q_{s_i}. \) Therefore,

\[ Q_a \subseteq Q_S \quad (**). \]

By (*) and (**), \( Q_S = Q_a. \)

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Conversely, suppose that $Q_s = Q_a$. Since $a \in Q_a$, so $a \in Q_s = \bigcup_{s_i \in S} Q_{s_i}$. Therefore, $a \in Q_{s_i}$ for some $s_i \in S$. Thus $s_i \leq a$ for some $s_i \in S$. Also, $Q_s = \bigcup_{s_i \in S} Q_{s_i} = Q_a$, so, $Q_{s_i} \subseteq Q_a$ for any $s_i \in S$.

Therefore, $s_i \in Q_a$ for any $s_i \in S$. Thus, $a \leq s_i$ for any $s_i \in S$. This implies that $a$ is a lower bound of $S$. Since $s_i \leq a$ for some $s_i \in S$, so, $a \in S$. This, in turn, implies that $a$ is the greatest lower bound of $S$. ■

**Theorem 6.2.33** Let $P$ be a prime subnexus of a nexus $N$ and let $\text{Ord}(N) = \text{Ord}(P) = n$. Then $(N - P) \subseteq N_k$ for some $k \leq n$, where, $N_k$ is the set of all addresses of $N$ whose stem is $k$ (see the discussion following theorem 6.2.4). The fact that $(N - P) \cap N_k$ may alternatively be expressed as $(N - P) \cap N_t = \emptyset$ for all $t \neq k$.

**Proof:** Since $\text{Ord}(N) = \text{Ord}(P)$, so $P$ contain all principal addresses of $N$. By Theorem 6.2.21, there exists $a \in N$ such that $N - P = Q_a$. Now, one must show that, $Q_a \cap N_t = \emptyset$ for all $t \neq k$. If $Q_a \cap N_t \neq \emptyset$ for some $t \neq k$, then there exist $x, y \in N$ such that $x \in Q_a \cap N_k$ and $y \in Q_a \cap N_t$. Without loss of generality, suppose that $k \leq t$ as discussed after Theorem 6.2.4, $x \land y = (k)$. Since $(k)$ is a principal address, so, $x \land y \in P$, and this creates a contradiction with primeness of $P$.

Therefore, $Q_a \cap N_t = \emptyset$ for all $t \neq k$. ■

**Theorem 6.2.34** Let $N$ and $M$ be nexuses and let $f: N \rightarrow M$ be a homomorphism. Also, let $b$ be an address in the image of $N$ in $M$, that is, $b \in f(N) \subseteq M$. Then

$$f^{-1}(Q_b) = Q_{f^{-1}(b)}$$

**Proof:** Consider $Q_b$. By Theorem 6.2.18 $P = M - Q_b$ is a prime subnexus of $M$. By Theorem 6.2.14, $f^{-1}(P)$ is a prime subnexus of $N$. So, there exists $a \in N$ such that $f^{-1}(P) = N - Q_a$. Since $a \in f^{-1}(Q_b)$, so, $f(a) \in Q_b$, thus

$$b \leq f(a) \quad (\ast)$$

Also, $f^{-1}(b) \subseteq Q_a$, therefore, $a \leq d$ for any $d \in f^{-1}(b)$. Since $f$ is a homomorphism, so

$$f(a) \leq f(d) = b \quad \forall \, d \in f^{-1}(b) \quad (\ast\ast)$$

By $(\ast)$ and $(\ast\ast)$, $f(a) = b$. Now, consider $Q_{f^{-1}(b)} = \bigcup_{d \in f^{-1}(b)} Q_d$. Since $d \in f^{-1}(b)$, so $Q_d \subseteq Q_a$ for any $d \in f^{-1}(b)$. This means that, $\bigcup_{d \in f^{-1}(b)} Q_d \subseteq Q_a$. Thus $f^{-1}(Q_b) \subseteq Q_a$. Now, one must show that $Q_a \subseteq f^{-1}(Q_b)$. Suppose that $x \in Q_a$. Therefore, $a \leq x$ and $f(a) \leq f(x)$. Since $b = f(a)$, so $f(x) \in Q_b$ and this means that $x \in f^{-1}(Q_b)$. Therefore,
\[ f^{-1}(Q_b) = Q_a = \bigcup_{d \in f^{-1}(b)} Q_d = Q_{f^{-1}(b)}. \]  

**Theorem 6.2.35** Let \( N \) and \( M \) be nexuses and let \( f: N \rightarrow M \) be an epimorphism and let \( a \) be an address of \( N \). Then
\[ f(Q_a) = Q_{f(a)}. \]

**Proof:** By Theorem 6.2.18 \( P = N - Q_a \) is a prime subnexus of \( N \) and by Theorem 6.2.14 (ii) \( f(P) \) is a prime subnexus of \( M \). By Theorem 6.2.21 there exists \( b \in M \) such that \( f(P) = M - Q_b \). This implies that since \( f \) is an epimorphism then \( f(Q_a) = Q_b \). Now, one must show that, \( b = f(a) \). By hypothesis \( f \) is an epimorphism, so, there exists \( d \in Q_a \) such that \( f(d) = b \). Since \( d \in Q_a \), then, \( a \leq d \) and
\[ f(a) \leq f(d) = b \]  

(*)

Also, \( f(a) \in Q_b \) thus,
\[ b \leq f(a) \]  

(***)

By (*) and (***) \( b = f(a) \). Therefore, \( f(Q_a) = Q_{f(a)}. \)  

**Corollary 6.2.36** Let \( N \) and \( M \) be nexuses and let \( f: N \rightarrow M \) be an epimorphism. Also, let \( m \) be a maximal address of \( N \). Then,
(i) \( f(m) \) is a maximal address of \( M \).
(ii) \( f(\ ) = \ ) \)

**Proof:** (i) In Theorem 6.2.35 assume that \( a = m \). Therefore, \( f(Q_m) = Q_{f(m)} \). Since \( m \) is a maximal address, so \( Q_m = m \) and \( f(m) = Q_{f(m)} \). This means that \( f(m) \) is a maximal address of \( M \).

(ii) Similarly, assume that \( a = \ ) \) and use the fact that \( Q(\ ) = N \). Therefore, \( M = Q_{f(\ )} \). Since \( \ ) \in M \), then \( f(\ ) \leq \ ) \). Thus \( f(\ ) = \ ).

It should be noted that in the above corollary, part (ii) may also be proved by the monotone property of homomorphism \( f \), that is, \( a \leq b \) implies that \( f(a) \leq f(b) \), as follows:
The monotone property implies that if \( a \leq b \) then \( f(a) \leq f(b) \). Since \( f \) is an epimorphism, so there exists \( a \in N \) such that \( f(a) = \ ) \). Also, since \( f \) is monotone and \( \ ) \leq a \), therefore, \( f(\ ) \leq f(a) = \ ). Thus, \( f(\ ) = \ ).

**Theorem 6.2.37** Let \( N \) be a nexus and let \( f \) be a function from \( N \) to itself, that is, \( f: N \rightarrow N \) defined by
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Then \( f \) is a homomorphism. This function is called 'pruning function' and is denoted by \( f_{qa} \).

**Proof:** One must show that,

\[
f_{qa}(x \land y) = f_{qa}(x) \land f_{qa}(y)
\]

for all \( x, y \in N \).

Since \( qa \) is closed under the meet operation (see Theorem 6.2.18). Then if \( x, y \in qa \), then

\[
x \land y \in qa
\]

and

\[
f_{qa}(x \land y) = a = a \land a = f_{qa}(x) \land f_{qa}(y)
\]

If \( x, y \notin qa \), then \( x \land y \notin qa \) and

\[
f_{qa}(x \land y) = x \land y = f_{qa}(x) \land f_{qa}(y)
\]

If \( x \in qa \) and \( y \notin qa \) then \( x \land y \notin qa \) since, if \( x \land y \in qa \), then \( a \leq (x \land y) \leq y \). This means that \( y \in qa \) and this creates a contradiction with the hypothesis \( y \notin qa \). Now, consider \( Q_y \) and \( Q_a \).

Since \( y \in Q_y \) and \( x \in Q_a \), then by Lemma 6.2.25 part (iv),

\[
x \land y = a \land y.
\]

Thus,

\[
f_{qa}(x \land y) = x \land y = a \land y = f_{qa}(x) \land f_{qa}(y)
\]

Therefore, \( f \) is a homomorphism. \( \blacksquare \)

Note that, in the above theorem, if \( qa \) is replaced by \( Q_a \), the theorem would still be valid. In this case, the homomorphism related to \( Q_a \) is denoted by \( f_{Q_a} \). Now, suppose that \( N \) is a nexus and \( P \) is a prime subnexus of \( N \). By Theorem 6.2.21 \( P = N - Q_a \) for some \( a \in N \). Now, consider the homomorphism \( f_{Q_a} : N \rightarrow N \). The prime subnexus \( P \) is the inverse image of \( f_{Q_a} \). That is, \( P = f_{Q_a}^{-1}(N) \). Therefore, every prime subnexus of a nexus \( N \) is a inverse image of \( a f_{Q_a} \) for some \( a \in N \).

### 6.3 Maximal Subnexus of a Nexus

**Definition 6.3.1** A maximal subnexus of a nexus \( N \) is a subnexus \( U \), not equal to \( N \), such that there are no subnexuses between \( U \) and \( N \). In other words, if \( T \) is a subnexus which contains \( U \) as a subset, then either \( T = U \) or \( T = N \).

For example, consider the nexus

\[
N = \{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\}
\]

and a subnexus

\[
U = \{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2)\}
\]

The dendrograms of \( N \) and \( U \) are shown in Fig 6.3.1.
Fig 6.3.1 Dendrograms of $N$, $U$, $T$ and $K$. In the case of $U$, $T$ and $K$, the dendrograms on the left show the deleted parts by dotted lines.

Now, suppose that $V$ is another subnexus such that $U \subset V$. This implies that $V = N$. Therefore, $U$ is a maximal subnexus of $N$. But the subnexus

$$T = \{((), (1), (2), (2,1), (2,2), (2,3))\}$$

is not a maximal subnexus, because there exists a subnexus

$$K = \{() , (1), (2), (1,1), (2,1), (2,2), (2,3)\}$$
such that, \( T \subset K \) and \( K \neq N \). The dendrograms of \( T \) and \( K \) are shown in Fig 6.3.1.

**Theorem 6.3.2** Let \( N \) be a nexus and let \( m = \{ m_i : i \in \mathbb{N} \} \) be the set of all maximal addresses of \( N \). Then, every maximal subnexus \( U \) of \( N \) is of the form \( U = N - \{ m_i \} \) for some \( m_i \in m \).
Furthermore, the number of maximal subnexuses of \( N \) is equal to the number of maximal addresses of \( N \), that is, \( |m| \).

**Proof:** It is clear that \( U = N - \{ m_i \} \) is a maximal subnexus of \( N \). Now, suppose that \( U \neq N \) is a maximal subnexus of \( N \). Since \( N = \cup <m> \), so \( U \) does not contain all the maximal addresses of \( N \). Also, \( U \) must contain all the maximal addresses except for one of them, because if \( m_i \) and \( m_j \) are two maximal addresses of \( N \) such that \( m_i, m_j \notin U \). Then \( U \subset U \cup <m_k> \subset N \), for \( k = i \) or \( j \). So, \( U \) is not a maximal subnexus. Now, suppose that \( a < m_k \) and \( a \notin U \) then \( U \subset U \cup <a> \subset N \). So, \( U \) is not a maximal subnexus of \( N \). Consequently, \( U \) contains all the addresses of \( N \) except for \( m_k \), that is, \( U = N - \{ m \} \). ■

**Corollary 6.3.3** Let \( N \) be a nexus. Every maximal subnexus of \( N \) is a prime subnexus of \( N \).

**Proof:** Let \( U \) be a maximal subnexus of \( N \). By Theorem 6.3.2, \( U = N - \{ m \} \), where \( m \) is a maximal address of \( N \). Since, \( m \) is a maximal address, so \( Q_m = \{ m \} \). Therefore, \( U = N - \{ m \} = N - Q_m \).

The rest of proof follows from Theorem 6.2.18. ■

**Corollary 6.3.4** Let \( N \) be a nexus. Every prime subnexus of \( N \) is a maximal subnexus of \( N \) if and only if

\[ N = \{(\), (1)\} \]

**Proof:** Suppose that \( N \) is a nexus and that every prime subnexus of \( N \) is a maximal subnexus of \( N \). Now, consider a nonempty address \( a \) of \( N \). The subnexus \( P = N - Q_a \) is a prime subnexus of \( N \). So, \( P \) is a maximal subnexus of \( N \). By Theorem 6.3.2, \( P = N - \{ m \} \), where \( m \) is a maximal address of \( N \). Therefore, \( Q_a = \{ m \} \). However, for every address \( a \) of \( N \), \( a \in Q_a \), so, \( a \) is a maximal address of \( N \). Since \( a \) is an arbitrary address of \( N \), then, every nonempty address of \( N \) is a maximal address of \( N \). This means that, \( N = \{(\), (1)\} \). Conversely, if \( N = \{(\), (1)\} \), then, \( N \) has just one prime subnexus, that is, \( \{(\)\} \), which is also maximal. ■

**Theorem 6.3.5** Let \( N \) be a nexus. \( N \) is cyclic if and only if \( N \) has just one maximal subnexus.

**Proof:** If \( N \) is a cyclic nexus generated by \( a \), that is, if \( N = <a> \), then \( N \) has just one maximal address, namely, \( a \). By Theorem 6.3.2, \( U = N - \{ a \} \) is the only maximal subnexus of \( N \). Conversely, suppose that \( N \) has just one maximal subnexus \( U \). By Theorem 6.3.2, \( U = N - \{ a \} \),
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where a is a maximal address of N. Now, consider the cyclic subnexus <a> of N. Obviously, <a> \subseteq N. Now, one must show that N \subseteq <a>. To do this, suppose that b \in N. If a and b are comparable then b \leq a and, so, b \in <a>. If a and b are not comparable then there exists a maximal address m such that b \leq m. This implies that T = N - {m} is another maximal subnexus of N. However, this contradicts the uniqueness of the maximal subnexus of N. Therefore, all addresses of N are comparable with a. Thus, N is cyclic.

Theorem 6.3.6 Let N and M be two nexuses and let f be a homomorphism from N to M

(i) If f is an epimorphism and U \subset N is a maximal subnexus of N and f(U) \neq M then f(U) is a maximal subnexus of M.

(ii) If f is a monomorphism and V \subset M is a maximal subnexus of M and f^{-1}(V) \neq N then f^{-1}(V) is a maximal subnexus of N.

Proof: (i) Since U is a maximal subnexus of N, so, by Theorem 6.3.2, U = N - {m} where m is a maximal address of N. By Theorem 6.2.14, f(U) is a prime subnexus of M and by Theorem 6.2.21, f(U) = M - Q_b where b is an address of M - f(U). Since f is an epimorphism and the only address outside U is m, then f(m) = b. By Theorem 6.2.34, f^{-1}(Q_b) = Q_{f^{-1}(b)} = Q_m = \{m\}.

Since f is a function, Q_b must contain only one element, that is, b. In other words, Q_b = \{b\}. Therefore, b is a maximal address of M. This implies that f(U) = M - \{b\}, and f(U) is a maximal subnexus of M.

(ii) Since V is a maximal subnexus of M, V = M - \{m\}, where m is a maximal address of M. By Corollary 6.3.3 V is a prime subnexus of N. Also, by Theorem 6.2.14, f^{-1}(V) is a prime subnexus of M, so, f^{-1}(V) = N - Q_a, where a is an address of N - f^{-1}(V). Now, one must show that Q_a = \{a\}. Suppose that Q_a contains another address say b. Since f^{-1}(V) \neq N, f(Q_a) = m implies that f(a) = f(b). But, f is a monomorphism, therefore, a = b. Thus, f^{-1}(V) = N - \{a\} is a maximal subnexus of N.

6.4 Nexuses as Metric Spaces

In this section a metric on a nexus will be defined. However, at first, it is necessary to define some concepts and prove some lemmas.

Recall that, in mathematics, a metric space is a set where a notion of 'distance' (called a metric) between elements of the set is defined. The metric space which most closely corresponds to our intuitive understanding of space is the three dimensional Euclidean space. In fact, the notion of
'metric' is a generalization of the Euclidean metric arising from the four long known properties of the Euclidean distance, that is,

1. The distance from point \( A \) to point \( B \) is always nonnegative.
2. The distance from point \( A \) to point \( B \) is zero if and only if \( A \) and \( B \) are identical.
3. The distance from \( A \) to \( B \) is the same as from \( B \) to \( A \).
4. The distance from \( A \) to \( B \) plus that from \( B \) to \( C \) is greater than or equal to the distance from \( A \) to \( C \) (the triangle inequality).

The Euclidean metric defines the distance between two points as the length of the straight line connecting them.

The geometric properties of the space depend on the metric chosen, and by using different metrics one can construct interesting non Euclidean geometries such as those used in the theory of general relativity. Now, the formal definition of a metric space is introduced [see Ref 13]:

A metric space is an ordered pair \((M, d)\) where \( M \) is a set and \( d \) is a metric on \( M \), that is, \( d \) is a function

\[
d : M \times M \rightarrow \mathbb{R}
\]

such that for any \( x, y \) and \( z \) in \( M \)

1. \( d(x, y) \geq 0 \) (nonnegativity).
2. \( d(x, y) = 0 \) if and only if \( x = y \) (identity of indiscernibles).
3. \( d(x, y) = d(y, x) \) (symmetry).
4. \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality).

The function \( d \) is also called a 'distance function' or simply 'distance'. Often \( d \) is omitted from \((M, d)\) and one just writes \( M \) for a metric space if it is clear from the context what the metric used is.

**Definition 6.4 1:** Let \( N \) be a nexus and let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_m) \) be two comparable addresses of \( N \). The distance between \( a \) and \( b \) is denoted by \( d(a, b) \) and defined as follows:

\[
d(a, b) = \left| \sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j \right|
\]

For example consider two addresses

\[
a = (3, 4, 2, 5, 6)
\]

and

\[
b = (3, 4, 2, 5, 8, 3, 8, 6, 2)
\]

The address \( a \) is less than \( b \) and
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\[ d(a, b) = \left| (3 + 4 + 2 + 5 + 6) - (3 + 4 + 2 + 5 + 8 + 3 + 8 + 6 + 2) \right| \]
\[ = \left| 20 - 41 \right| \]
\[ = \left| -21 \right| \]
\[ = 21 \]

**Definition 6.4.2:** Let \( N \) be a nexus and let \( a = (a_1, a_2, ..., a_n) \) be an address in \( N \). The distance between \( a \) and the empty address ( ) is called the ‘extent’ of \( a \) and is denoted by \( E(a) \) or \( Ea \). Therefore,
\[ Ea = \sum_{i=1}^{n} a_i \]
The extent of the empty address ( ) is equal to zero, that is, \( E( ) = 0 \). Note that, by the above notation
\[ d(a, b) = \left| Ea - Eb \right| \]
where \( a \) and \( b \) are two comparable addresses of a nexus.

**Lemma 6.4.3:** Let \( N \) be a nexus and let \( a, b \) and \( c \) be three comparable addresses in \( N \). Then
(i) \( a \leq b \iff Ea \leq Eb \)
(ii) \( d(a, a) = 0 \)
(iii) \( d(a, b) = d(b, a) \)
(iv) \( d(a, b) = 0 \iff a = b \)
(v) \( d(a, b) \leq d(a, c) + d(c, b) \)
(vi) \( d(a, b) = d(a, c) + d(c, b) \iff a \leq c \leq b \)

**Proof:**
(i) The proof follows by definition of \( \leq \) on a nexus (Definition 5.2.5), definition of extent of an address (Definition 6.4.2), and the fact that \( a \) and \( b \) are comparable.
(ii) \( d(a, a) = \left| Ea - Ea \right| = 0 \)
(iii) \( d(a, b) = \left| Ea - Eb \right| = \left| Eb - Ea \right| = d(b, a) \)
(iv) If \( d(a, b) = \left| Ea - Eb \right| = 0 \), then \( Ea = Eb \). Since \( a \) and \( b \) are comparable, so, \( a = b \). Now, suppose that \( a = b \). By part (ii), \( d(a, b) = 0 \)
(v) \( d(a, b) = \left| Ea - Eb \right| \)
\[ = \left| (Ea - Ec) + (Ec - Eb) \right| \leq \left| Ea - Ec \right| + \left| Ec - Eb \right| \]
Where the above inequality holds because of the ‘triangle inequality’ property of absolute value (i.e., \( |a + b| \leq |a| + |b| \)).
(vi) Suppose that \( a \leq c \leq b \). Therefore, \( Eb - Ea \), \( Ec - Ea \) and \( Eb - Ec \) are nonnegative integers. So,
\[ d(a, b) = \left| Ea - Eb \right| \]
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\[
\begin{align*}
&= Eb - Ea \\
&= Eb - Ea + Ec - Ec \\
&= (Ec - Ea) + (Eb - Ec) \\
&= | Ea - Ec | + | Ec - Eb |
\end{align*}
\]
\[
= d(a, c) + d(c, b).
\]

**Lemma 6.4.4** Suppose that \( N \) is a nexus. The set 
\[
K = \{ a \in N : Ea \leq n \}
\]
where \( n \) is a nonnegative integer, is a subnexus of \( N \).

**Proof:** It must be shown that if \( a \in K \) and \( b \in N \) such that \( b \leq a \), then \( b \in K \). Since \( b \leq a \), so \( Eb \leq Ea \). Therefore, \( Eb \leq n \) and this means that \( b \in K \). ■

**Lemma 6.4.5** Let \( N \) be a nexus and let \( a, b \) and \( c \) be three noncomparable addresses of \( N \). Then \( a \wedge b, a \wedge c \) and \( b \wedge c \) are comparable and two of them are equal to \( a \wedge b \wedge c \). Therefore, these two are equal.

**Proof:** It is true that \( a \wedge b < a \) and \( a \wedge c < a \) (where, the inequality signs are \(<\) rather than \(\leq\) since \(a, b\) and \(c\) are noncomparable). So, \( a \wedge b \) and \( a \wedge c \) are two addresses in \( <a> \). Therefore, \( a \wedge b \) and \( a \wedge c \) are comparable. Similarly, it can be shown that \( a \wedge b \) and \( b \wedge c \) are comparable and so are \( a \wedge c \), and \( b \wedge c \). Thus, \( a \wedge b, a \wedge c, \) and \( b \wedge c \) are comparable. Now without loss of generality, suppose that, \( a \wedge b \leq a \wedge c \leq b \wedge c \). Consider the three equations
\[
\begin{align*}
&= (a \wedge b) \wedge (a \wedge c) = (a \wedge b) \\
&= (a \wedge b) \wedge (b \wedge c) = (a \wedge b) \\
&= (a \wedge c) \wedge (b \wedge c) = (a \wedge c)
\end{align*}
\]
Since the left sides of the above equations are equal, so, the right sides of them are also equal. Therefore,
\[
( a \wedge b \wedge c = (a \wedge b) = (a \wedge c)
\]
and the proof is complete. ■

**Definition 6.4.6** Let \( N \) be a nexus and let \( a \) and \( b \) be two addresses in \( N \) where \( a \) and \( b \) may or may not be comparable. Then the distance between \( a \) and \( b \) is denoted by \( D(a, b) \) and defined as follows:
\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b)
\]
This definition is, in fact, the generalisation of Definition 6.4.1.
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To provide a feel for the concept of distance between two comparable addresses \( a \) and \( b \), one can imagine that if there are no addresses between \( a \) and \( b \) and if \( a < b \) then the distance between \( a \) and \( b \) is 1.

For example, if \( a = (4, 6, 3) \) and \( b = (4, 6, 4) \), then \( d(a, b) = 1 \). See, Fig 6.4.1

\[
\begin{align*}
a = (4, 6, 3) & \quad \text{Distance is one unit} \quad b = (4, 6, 4)
\end{align*}
\]

Fig 6.4.1

Now, if \( a = (4, 3) \) and \( b = (4, 5, 2) \), then \( a < b \) and there are three addresses between \( a \) and \( b \), that is,

\[
a = (4, 3) < (4, 4) < (4, 5) < (4, 5, 1) < (4, 5, 2) = b
\]

If one denoted a 'unit distance ' by the symbol \( \leftrightarrow \), then one may write

\[
a = (4, 3) \leftrightarrow (4, 4) \leftrightarrow (4, 5) \leftrightarrow (4, 5, 1) \leftrightarrow (4, 5, 2) = b
\]

It can be seen that there are four unit distances between \( a \) and \( b \). Therefore,

\[
d(a, b) = 4
\]

This distance may also be obtained using the extents of \( a \) and \( b \), that is,

\[
Eb - Ea = (4 + 5 + 2) - (4 + 3) = 4
\]

Now, consider two noncomparable addresses \( a = (2, 1, 2) \) and \( b = (3, 2, 3) \) of the nexus

\[
N = \{(), (1), (2), (3), (2, 1), (2, 2), (3, 1), (3, 2), (2, 1, 1), (2, 1, 2), (3, 2, 1), (3, 2, 2), (3, 2, 3)\}
\]

The dendrogram of \( N \) is shown in Fig 6.4.2. In this figure, the addresses \( a, b \) and \( a \wedge b \) are shown by thick lines.

Fig 6.4.2 Dendrogram of \( N \). The addresses \( a, b \) and \( a \wedge b \) are shown by thick lines.
By Definition 6.4 6, the distance between \( a \) and \( b \) is the distance between two comparable addresses \( a \) and \( a \wedge b \) plus the distance between two comparable addresses \( b \) and \( a \wedge b \). Therefore, the distance between \( a \) and \( b \) may be obtained as follows:

\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b)
\]
as illustrated in Fig 6.4.3

As shown in Fig 6.4.3,

\[
a \wedge b = (2, 1, 2) \wedge (3, 2, 3) = (2).
\]

So,

\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b)
= |E_a - E(a \wedge b)| + |E_b - E(a \wedge b)|
= |5 - 2| + |8 - 2|
= 9
\]

The distance between two addresses may also be shown graphically in a dendrogram. For example, the information conveyed by Fig 6.4.3 may also be shown by the dotted path in Fig 6.4.2.
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Note that, if \( a \) and \( b \) are comparable, then
\[
D(a, b) = d(a, b)
\]
Also, note that
\[
D(a, a) = d(a, a) = 0
\]
Furthermore,
\[
D(a, ( )) = d(a, ( )) = Ea
\]
Finally, suppose that \( a \) and \( b \) are noncomparable. Since \( a \wedge b \leq a \) and \( a \vee b \leq b \), therefore,
\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b)
= |E_a - E(a \wedge b)| + |E_b - E(a \wedge b)|
= E_a - E(a \wedge b) + E_b - E(a \wedge b)
= E_a + E_b - 2E(a \wedge b)
\]

**Theorem 6.4 7** Let \( N \) be a nexus and let \( a, b \) and \( c \) be three addresses in \( N \). Then
(i) \( D(a, b) = d(a, b) \iff a \) and \( b \) are comparable
(ii) \( D(a, b) = 0 \iff a = b \)
(iii) \( D(a, b) = D(b, a) \)
(iv) \( D(a, b) \leq D(a, c) + D(c, b) \)

Therefore, \( N \) is a metric space with respect to the distance function \( D \).

**Proof:** (i) Suppose that \( D(a, b) = d(a, b) \). By the Definitions 6.4.1 and 6.4.6 one can obtain the equation
\[
E_a - E(a \wedge b) + E_b - E(a \wedge b) = |E_a - E_b|
\]
If \( E_a > E_b \), then,
\[
E_a - E(a \wedge b) + E_b - E(a \wedge b) = E_a - E_b
\]
Therefore,
\[
E(a \wedge b) = E_b
\]
Since \( a \wedge b \) and \( b \) are comparable, \( a \wedge b = b \). This implies that \( a \) and \( b \) are comparable and \( b \leq a \).
If \( E_a < E_b \), by a similar argument one can show that \( a \) and \( b \) are comparable and \( a \leq b \). Now, suppose that \( a \) and \( b \) are comparable. Without loss of generality, assume that, \( a \geq b \). Then,
\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b)
= d(a, b) + d(b, b)
= d(a, b)
\]
(ii) suppose that
\[
D(a, b) = d(a, a \wedge b) + d(b, a \wedge b) = 0
\]
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Since the value of $d(a, a \land b)$ and $d(b, a \land b)$ are not negative, so, the value of $d(a, a \land b)$ and $d(b, a \land b)$ must both be equal to zero. Therefore,

$$d(a, a \land b) = E_a - E(a \land b) = 0$$

and

$$d(b, a \land b) = E_b - E(a \land b) = 0$$

By Lemma 6.4.3, $a = a \land b$ and $b = a \land b$. Thus, $a \leq b$ and $b \leq a$. So, $a = b$.

The proof of the converse is trivial.

(iii) $D(a, b) = d(a, a \land b) + d(b, a \land b)$

$$= d(b, a \land b) + d(a, a \land b)$$

$$= D(b, a)$$

(iv) Let $a$, $b$ and $c$ be three addresses in $N$. By Lemma 6.4.5 $a \land b$, $a \land c$, and $b \land c$ are comparable and two of them are equal to $a \land b \land c$. So, there are three different cases that are to be considered:

$$a \land b = a \land c \leq b \land c,$$

$$a \land b = b \land c \leq a \land c$$

and

$$a \land c = b \land c \leq a \land b.$$

Consider the first case, that is,

$$a \land b = a \land c \leq b \land c.$$

Therefore,

$$D(a, b) = d(a, a \land b) + d(b, a \land b)$$

$$\leq d(a, a \land b) + d(b, a \land b) + 2d(c, b \land c)$$

Since, $a \land b = a \land c$ and $a \land c \leq b \land c \leq b$, so, by part (v) of Lemma 6.4.3,

$$d(b, a \land b) = d(b, b \land c) + d(b \land c, a \land c)$$

Therefore,

$$D(a, b) \leq d(a, a \land b) + d(b, a \land b) + 2d(c, b \land c)$$

$$= d(a, a \land c) + d(b, b \land c) + d(b \land c, a \land c) + 2d(c, b \land c)$$

Again, by using part (v) of Lemma 6.4.3,

$$d(c, b \land c) + d(b \land c, a \land c) = d(c, a \land c)$$

Thus,

$$D(a, b) \leq d(a, a \land c) + d(b, b \land c) + d(b \land c, a \land c) + 2d(c, b \land c)$$

$$= d(a, a \land c) + d(c, a \land c) + d(b, b \land c) + d(c, b \land c)$$

$$= D(a, c) + D(c, b)$$
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The proofs of cases two and three can be done in a similar fashion.

In a metric space \((N, D)\), one can define an 'open ball' of radius \(r\) and centre \(a\) which is denoted by \(B_r(a)\) and given by:

\[
B_r(a) = \{ b \in N : D(a, b) < r \}
\]

**Theorem 6.4**

In a metric space \((N, D)\), every open ball \(B_r(a)\) is closed under the meet operation. Therefore, \(B_r(a)\) is a semilattice.

**Proof:** Let \(b, c \in B_r(a)\). One must show that \(b \wedge c \in B_r(a)\), that is, \(D(a, b \wedge c) < r\).

\[
D(a, b \wedge c) = d(a, a \wedge b \wedge c) + d(b \wedge c, a \wedge b \wedge c)
= E_a - E(a \wedge b \wedge c) + E(b \wedge c) - E(a \wedge b \wedge c) \quad (*)
\]

By Lemma 6.4 5 there are three cases, namely:

\[
a \wedge b \wedge c = a \wedge b = a \wedge c \leq b \wedge c,
\]

\[
a \wedge b \wedge c = a \wedge b = b \wedge c \leq a \wedge c
\]

and

\[
a \wedge b \wedge c = a \wedge c = b \wedge c \leq a \wedge b
\]

The first case will be proven and the proofs of the other two are similar.

Now, suppose that

\[
a \wedge b \wedge c = a \wedge b = a \wedge c \leq b \wedge c
\]

Therefore, from (*)

\[
D(a, b \wedge c) = E_a - E(a \wedge b \wedge c) + E(b \wedge c) - E(a \wedge b \wedge c)
= E_a - E(a \wedge b) + E(b \wedge c) - E(a \wedge b)
\leq E_a - E(a \wedge b) + E_b - E(a \wedge b)
= d(a, a \wedge b) + d(b, a \wedge b)
= D(a, b) < r
\]

**Lemma 6.4 9** Let \(N\) and \(M\) be two nexuses and let \(f : N \to M\) be a monomorphism.

(i) If \(a, b \in N\) and \(a \leq b\) then

\[
d(a, b) \leq d(f(a), f(b))
\]

(ii) For every \(a \in N\)

\[
E_a \leq E_f(a)
\]

**Proof:** (i) Suppose that \(d(a, b) = n\). Therefore, there exist \(n-1\) addresses between \(a\) and \(b\), namely,

\[
a < k_1 < k_2 < \ldots < k_{n-1} < b
\]

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where \( k_i \) is an address for \( i = 1, 2, \ldots, n-1 \). Since \( f \) is a monomorphism,
\[
f(a) < f(k_1) < f(k_2) < \ldots < f(k_{n-1}) < f(b)
\]
This means that there exist at least \( n-1 \) addresses between \( f(a) \) and \( f(b) \). Therefore,
\[
d(f(a), f(b)) \geq n
\]
Thus
\[
d(a, b) \leq d(f(a), f(b))
\]
(ii) By using part (i) and assuming that \( b = () \), one can show that
\[
d(a, ()) \leq d(f(a), f(()))
\]
Since \( f \) is a monomorphism and \( () \leq a \), so, \( f(() \leq f(a) \) and \( Ef(() \leq Ef(a) \). Therefore
\[
E_a \leq Ef(a) - Ef(() \leq Ef(a)
\]
In the above lemma \( a \) and \( b \) are considered to be comparable addresses. However, the idea can be generalised. This is done in the next theorem.

**Theorem 6.4 10** Let \( N \) and \( M \) be two nexuses and let \( f: N \rightarrow M \) be a homomorphism. Then, \( f \) is a monomorphism if and only if for all \( a, b \in N \), \( D(a, b) \leq D(f(a), f(b)) \).

**Proof:** Suppose that \( f: N \rightarrow M \) is a monomorphism. By Definition 6.4 6,
\[
D(a, b) = d(a, a \land b) + d(b, a \land b)
\]
Now, using Lemma 6.4 9 part (i),
\[
d(a, a \land b) + d(b, a \land b) \leq d(f(a), f(a \land b)) + d(f(b), f(a \land b))
\]
\[
= d(f(a), f(a) \land f(b)) + d(f(b), f(a) \land f(b))
\]
\[
= D(f(a), f(b))
\]
That is,
\[
D(a, b) \leq D(f(a), f(b))
\]
To prove the converse, suppose that, for all \( a, b \in N \), \( D(a, b) \leq D(f(a), f(b)) \). One must show that if \( f(a) = f(b) \), then \( a = b \).

By hypothesis,
\[
D(a, b) \leq D(f(a), f(b))
\]
\[
= 0
\]
Since \( D(a, b) \) is non-negative, \( D(a, b) = 0 \) implies that \( a = b \). ■

**Definition 6.4 11** Let \( N \) be a nexus and let \( a \) and \( b \) be two addresses in \( N \). Also, let \( E_a \neq E_b \). The ratio \( D(a, b)/(E_a - E_b) \) is called the 'comparability index' and is denoted by \( CI(a, b) \).

**Theorem 6.4 12** Suppose that \( N \) is a nexus and \( a \) and \( b \) are two addresses in \( N \) with different extents. Then
(i) $a < b \iff \text{CI}(a, b) = -1$

(ii) $a > b \iff \text{CI}(a, b) = 1$

**Proof:** Suppose that, $a < b$

\[
\text{CI}(a, b) = \frac{D(a, b)}{(Ea - Eb)} = \frac{(Ea + Eb - 2E(a \land b))}{(Ea - Eb)} = \frac{(Eb - Ea)}{(Ea - Eb)} = -1
\]

Now, suppose that, $\text{CI}(a, b) = -1$. So,

\[
(Ea + Eb - 2E(a \land b))/(Ea - Eb) = -1
\]

Therefore,

\[
Ea + Eb - 2E(a \land b) = -Ea + Eb
\]

and

\[
Ea = E(a \land b)
\]

Since, $a$ and $a \land b$ are comparable ($a \land b \leq a$), so, $a = a \land b$. This implies that $a < b$.

The proof of (ii) is similar. ■

In the next corollary both parts of the above theorem will be used for comparability check of two addresses.

**Corollary 6.4 13** Suppose that $N$ is a nexus and $a$ and $b$ are two addresses in $N$ with different extents. Then, $a$ and $b$ are comparable if and only if $|\text{CI}(a, b)| = 1$.

**Proof:** The proof follows from Theorem 6.4 12. ■

**Theorem 6.4 14** Suppose that $N$ is a nexus and $a$ is an address in $N$. Then, $N$ is a cyclic nexus with $a$ as the generator, that is, $N = \langle a \rangle$, if and only if for every address $b$ of $N$, the comparability index of $a$ and $b$ is equal to 1, that is,

\[
\text{CI}(a, b) = 1 \quad \forall b \in N
\]

**Proof:** Suppose that $N$ is a cyclic nexus with $a$ as the generator. Therefore, for every address $b$ in $N$, $b \leq a$. Now, by Theorem 6.4 12 (ii), for every address $b$ of $N$, $\text{CI}(a, b) = 1$.

Conversely, suppose that for every address $b$ in $N$, the comparability index of $a$ and $b$ is equal to 1, that is, $\text{CI}(a, b) = 1$. Therefore, by Theorem 6.4 12 (ii), for every address $b$ in $N$, $b \leq a$. Thus, $N$ is cyclic with $a$ as the generator. ■

**Lemma 6.4 15** Let $N$ be a nexus and let $a$ be an address in $N$. Then
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\[ a = () \iff \text{Cl}(a, b) = -1 \text{ for all } b \in N \]

**Proof:** Suppose that \( a = () \). Then,
\[
\text{Cl}((), b) = \frac{D((), b)}{E(()) - Eb} = \frac{Eb}{-Eb} = -1
\]
Conversely, suppose that \( \text{Cl}(a, b) = -1 \) for all \( b \in N \). By Theorem 6.4 12 (i), \( a < b \) for all \( b \in N \). Therefore, \( a = () \). ■

**Theorem 6.4 16** Suppose that \( N \) is a nexus and \( a \) is an address in \( N \). Then
\[ \mathcal{Q}_a = \{ b \in N : \text{Cl}(a, b) = -1 \} \]
**Proof:** Let \( K = \{ b \in N : \text{Cl}(a, b) = -1 \} \). Now, suppose that \( b \in \mathcal{Q}_a \). Therefore, \( a \leq b \) and by Theorem 6.4 12 (i), \( \text{Cl}(a, b) = -1 \). Thus, \( \mathcal{Q}_a \subseteq K \).

Conversely, suppose that \( b \in K \). So, \( \text{Cl}(a, b) = -1 \) and by Theorem 6.4 12 (i), \( a \leq b \). This implies that \( b \in \mathcal{Q}_a \) and so, \( K \subseteq \mathcal{Q}_a \). Therefore, \( \mathcal{Q}_a = \{ b \in N : \text{Cl}(a, b) = -1 \} \). ■

**Corollary 6.4 17** Let \( N \) be a nexus and let \( K \) be a subnexus of \( N \). Suppose that there exists \( a \in N - K \) such that \( \text{Cl}(a, b) = -1 \) for all \( b \in N - K \). Then, \( K \) is a prime subnexus of \( N \).
**Proof:** By Theorem 6.4 16, \( N - K = \mathcal{Q}_a \) and so, \( K = N - \mathcal{Q}_a \). Then, by Theorem 6.2.21, \( K \) must be a prime subnexus. ■

**Theorem 6.4 18:** Let \( N \) be a nexus and let \( a \) be an address in \( N \). Then \( \mathcal{Q}_a \cup <a> \) is the subnexus generated by \( \mathcal{Q}_a \), that is, \(<Q_a>\). In other words, \( \mathcal{Q}_a \cup <a> \) is the smallest subnexus containing \( \mathcal{Q}_a \).
**Proof:** Suppose that \( K = <Q_a> = \bigcup_{b \in Q_a} <b> \). One must show that \( K = \mathcal{Q}_a \cup <a> \). To do this,
assume that \( c \in K \). Therefore, \( c \in <b> \) for some \( b \in Q_a \). This implies that \( c \leq b \) and \( a \leq b \).
Therefore, \( a \) and \( c \) are comparable. If \( a \leq c \) then \( c \in \mathcal{Q}_a \) and if \( c \leq a \) then \( c \in <a> \). So, in both cases \( c \in \mathcal{Q}_a \cup <a> \).

Now, suppose that \( d \in \mathcal{Q}_a \cup <a> \). Therefore, \( d \in \mathcal{Q}_a \) or \( d \in <a> \). If \( d \in \mathcal{Q}_a \), since \( \mathcal{Q}_a \subseteq <Q_a> \), so, \( d \in <Q_a> \). If \( d \in <a> \), since \( a \in \mathcal{Q}_a \), so, \( d \in <Q_a> \). Thus, in both cases \( d \in <Q_a> \). ■

**Theorem 6.4 19** Suppose that \( N \) is a nexus and \( a \) is an address in \( N \). Let,
\[ K = \{ b \in N : |\text{Cl}(a, b)| = 1 \} \]
Then \( K \) is a subnexus of \( N \). Particularly, \( K \) is a subnexus generated by \( \mathcal{Q}_a \).
Proof: \(|\text{CI}(a, b)| = 1\) implies that: \(\text{CI}(a, b) = 1\) or \(\text{CI}(a, b) = -1\). Therefore,
\[
K = \{b \in N : |\text{CI}(a, b)| = 1\} = \{b \in N : \text{CI}(a, b) = 1\} \cup \{b \in N : \text{CI}(a, b) = -1\}.
\]
Using Theorems 6.4.14 and 6.4.16, one can show that,
\[
<a> = \{b \in N : \text{CI}(a, b) = 1\}
\]
and
\[
Q_a = \{b \in N : \text{CI}(a, b) = -1\}
\]
Therefore
\[
K = \{b \in N : |\text{CI}(a, b)| = 1\} = <a> \cup Q_a
\]
Now, by Theorem 6.4.18, one can show that \(K\) is a subnexus of \(N\).

For an example of the application of Theorem 6.4.19, suppose that,
\[
N = \{(\), (1), (2), (3), (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2),
(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2), (3, 2, 1), (3, 2, 2), (2, 2, 1, 1),
(2, 2, 1, 2), (2, 2, 2, 1), (2, 2, 2, 2)\}
\]
Now, consider the address \(a = (2, 2)\). The dendrogram of \(N\) is shown in Fig 6.4.4, where the subnexus
\[
K = \{b \in N : |\text{CI}(a, b)| = 1\}
\]
is shown by thick lines.

Fig 6.4.4 Dendrogram of \(N\) with the subnexus \(K\) is shown by thick lines
CHAPTER 7

Conclusion and Summary

with Suggestions for Future Work

7.1 Overview

The roots of the concept of a plenix go back to the late 1970s. At that time, an extensive programme of research was undertaken in the Space Structures Research Centre of the University of Surrey, with the aim of finding convenient ways of generating data for analysis and design of space structures consisting of many thousands of elements. The geometry of such a structural system often involves many types of symmetries that can be used to simplify the generation of information for analysis and design purposes. However, in addition to geometric information, it is necessary to produce information about the properties the material(s) of the elements, positions and particulars of the supports and the nature and magnitudes of the external
Information about the external loads should include the details of dead weights, snow loads, wind effects, earthquake forces, temperature changes and so on.

In order to be able to conveniently handle the vast amount of varied data that defines a space structure, a sophisticated form of data base was evolved which was called a ‘plenix’ (plural ‘plenices’). A plenix had to be capable of containing any type of information either in explicit constant form, or in a ‘generic form’, that is as a ‘parametric formulation’. The term ‘plenix’ comes from the Latin word ‘plenus’ meaning ‘full’. This choice was a reflection of the intention of a plenix being capable of representing the ‘full spectrum’ of mathematical objects. The generic nature of a plenix as a data base places the concept in a class of its own with capabilities which are far beyond any normal data base.

In the years that followed the invention of the concept of a plenix, extensive use of it was made as a generic data base in the research projects of the Space Structure Research Centre. The aim of the present research is to create an algebra based on plenices, that is, to define meaningful relations, operations and functions for plenices and turn the concept of a plenix into a proper mathematical system with potential for applications in all branches of science and technology.

The concepts that constitute the ‘algebra of plenices’ are described in Chapters 2 to 4 of the present Thesis, with each chapter focusing on an aspect of the algebra. The Thesis contains a large number of theorems which give an indication of the richness of this mathematical area. However, if one wants to just cover the essentials of ‘plenix algebra’, a much smaller number of theorems could serve the purpose.

Plenix algebra, as a mathematical system, is a new addition to the vast universe of mathematical ideas and it is hoped that plenix algebra will, in time, find its right place in mathematics. In what follows, a number of points regarding the material of the Thesis are highlighted in order to bring out their significance.

7.2 The Concept of a Plenix

The ideas of a plenix together with its basic particulars are described in Chapter two. In a way, a plenix is the most general mathematical object, in the sense that a plenix can contain any other mathematical object. This may prove to be of value, in frequently encountered situations, where the solution of a problem involves more than one type of mathematical object. Then, the use of plenices may allow a whole collection of different operations to be achieved through a single plenix operation. This may be likened to the effect of a determinant of a matrix. To elaborate,
any problem whose solution involves a determinant may, of course, be handled by avoiding the use of the concept of determinant. However, a determinant often allows a more elegant solution by avoiding detailed numerical operations.

Similar comments may be made regarding the use of vectors and matrices. Therefore, mathematical objects that contain a collection of other mathematical objects allow one to formulate the problems conceptually and avoid involvement in detailed numerical work. This aspect is of particular significance in the present time of widely available highly sophisticated digital computers. In today’s mathematical applications it is desirable to use approaches when, as far as possible, one can solve the problems conceptually and leave the detailed computations to be carried out by a computer.

In defining the concept of a plenix, as in the case of any other algebraic object, the concept of ‘equality’ is of central importance. In the case of a plenix, this becomes even more crucial, because the concept of equality for a plenix involves the definitions of equality for the whole gamut of mathematical objects, since every element of a plenix must have a clear definition of equality.

The concept of equality has a closely associated concept, namely, the concept of ‘value’. Normally, it is believed that only scalars have values and that a matrix or a set does not have an associated value. However, it may be argued that any mathematical object that has a clear definition of equality may be considered to have a ‘value’. In fact, the very ‘thing’ that acts as the ‘measure’ for equality may be regarded as the ‘value’. The value of a vector is a ‘list of numerical values’, the value of a set is a ‘set of values’ and so on. This is the view that forms one of the fundamental pillars of the concept of a plenix.

Another important concept that plays a crucial role in the theory of plenices is the concept of an ‘address’. An address is simply a sequence of integers that represents the position of an element of a plenix. However, it turns out that the idea of an address provides a convenient way of representing the structure of a plenix. To elaborate, a set of addresses becomes a means of describing the constitution of a plenix and various aspects of constitution find interesting reflections on the ‘address set’ of the plenix.

To represent the constitution of a plenix, in addition to the address set, a graphical method is employed. A graphical representation of a plenix is referred to as a ‘dendrogram’. This is a tree-like graphical object that has proved to be highly effective in helping to visualise the problems.
Chapter 7 Conclusion and Summary with Suggestions for Future Work

7.3 Relations in Theory of Plenices

Algebraic structures in mathematics involve concepts that are termed ‘relations’. In scalar algebra, these are the relations of equality, being less than, being larger than and so on. In set theory there are the relations of being a subset or a superset and so on. Normally, the most fundamental of all the relations in an algebra is the relation of equality that defines the character of the objects on which the algebra operates.

With regard to plenix algebra, the definition of equality is described in Chapter two, but all the other relations are defined in Chapter three. Plenix algebra is rich in relations. Firstly, there are the relations between various parts of a plenix. Namely, the relations of being subnate and supernate. Also, between the panels of two plenices, there are the relations of being full image, partial image and hyper image. Furthermore, between two plenices, there are the relations of being cognates and pernates of each other.

These concepts govern the correspondence between (parts of) plenices, necessary for carrying out operations between them. A relation of major importance in plenix algebra is the relation of being pernates of each other. Two plenices that are not pernates of each other cannot be operands in any kind of plenix operation.

Plenix relations other than equality, are all related to the constitution of a plenix and are independent of the values of primion panels of plenices.

With the constitution of a plenix in mind, a major part of Chapter three is concerned with ‘equivalence classes of plenices’. Such an equivalence class consists of all plenices that have the same constitution, irrespective of the values of their primion panels. A related idea is the concept of ‘nexus’. This simply represents the constitution of a plenix and may be represented by an address set. The idea of a nexus turns out to be full of interesting properties and, in addition to the material related to nexuses in Chapter three, Chapters five and six are also devoted to the exploration of the concept of nexuses. Furthermore, in Chapters five and six, the correspondence between different aspects of nexuses and various fundamental and well established mathematical concepts is discussed.

7.4 Operations and Functions in Plenix Algebra

The most fundamental operation between plenices is that of ‘composition’ which is represented by the ‘duplus’ symbol #. This operation has a concatenation effect and is the basic operation of plenix algebra.

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However, any other unary or binary operation can be applied to plenices, provided that the operation is meaningful in relation to the values of the 'primion panels' and that the constitutions of the operand plenices (in the case of binary operations) allow the operation to take place. For this to happen, in general, it is necessary for the operand plenices to be 'pernates of each other'. The relationship of being pernates of each other is essential for two plenices to be the operands for any binary operation. This innovative concept of collection opens up many possibilities for operating on a collection of objects in a single go.

A further step in providing collective operation on a variety of objects is when a plenix acts as a unary or binary operator. There would then be one or two plenix operands. This will allow a complex combination of operations of all kinds to be performed on various objects, all at the same time. This is thought to be of enormous value in the context of an electronic computing environment, where it is desirable to formulate the problems conceptually, as far as possible, and leave the detailed computation to a computer to deal with. This may be likened to the use of matrix algebra to allow the formulation of problems to be achieved in simple and elegant forms and the usually huge amount of numerical computation to be performed by a computer.

Another concept of value that is evolved in Chapter four of the Thesis is the use of a plenix as the argument of a function. To elaborate, a plenix may be the argument of any mathematical function provided that all the primion panels of the plenix can meaningfully be arguments of the function. A generalisation of this concept allows a plenix to act as a function. In this case, every primion panel of the plenix is a function. These functions may consist of any variety of mathematical functions. The function plenix will then have an argument as a plenix. The primion panels of the argument plenix will be the arguments of the functions in the function plenix. For the concept to be applicable, it is necessary that the function and argument plenices are pernates of each other and that all the primion function-argument pairs are meaningful. The value of this collective application of functions, in particular, for problem solving in the context of electronic computing is considerable.

7.5 Nexus as an Abstract Algebraic structure

As may be seen in Chapters two, three and four, the constitution of a plenix plays an important role in the theory of plenices. As a result, one of the interesting domains for research in plenix theory is the constitution of a plenix, irrespective of the values of its primion panel. The mathematical object that represents the constitution of a plenix is called a nexus. The notion of a nexus is introduced in Chapter three.
The concept of the nexus, as an abstract algebraic structure, is certainly worthy of attention. As a result, in Chapter five a nexus is defined axiomatically, by using the concept of the address set. Also, an interesting fact about a relationship between plenix and nexus is shown in this chapter, that is, the concept of plenix is defined via of the concept of nexus.

In Chapters five and six, the properties of nexuses are investigated from the view point of pure mathematics. Many familiar concepts in an abstract algebra such as substructures, cyclic substructures, generators of an algebra, homomorphism of an algebra, direct product and direct sum of an algebra, metric space, decomposition theorem and so on, are studied deeply in the context of nexus algebra. The material involves a number of original concepts, including those of ‘extent’ and ‘compatibility index’. This means that nexus algebra has great potential as an algebraic structure.

7.6 Suggestions for Future Work

For practical applications of plenices, it will be essential to have the capability of generating and processing plenices using a computer. To achieve this, it will be necessary to have suitable instructions for plenix processing included in general programming languages. Therefore, it will be of value if a basic framework is created for instructions that can effect various tasks in plenix processing.

In further advancing the plenix theory, it will be useful to develop the concept of a plenix that can include operands of all kinds, as well as operators and functions that can work on the operands in the plenix.

Nexus algebra has a strong structure and it has a great potential for development in many directions. For example, fractions of nexuses are an interesting subject for investigation. Also, the theory of modules is an important part of modern algebra. This theory can be developed in the context of nexus algebra.

In mathematics, the category theory deals, in an abstract way, with mathematical structures and relationships between them. Category theory is a powerful conceptual framework allowing one to see the universal components of a family of structures of a given kind, and how these structures are interrelated. Therefore, it is useful to study nexus algebra through category theory.
Publications

The following materials are published:


3. Bolourian, M, Structure of Nexuses, accepted for publication in: International Journal of Algebra. ISSN 1312-8868
REFERENCES


